

# NELSON QMATHS

## MATHEMATICAL METHODS

YEAR

11

Ross Brodie  
Regina Edwards  
Stephen Swift

4TH EDITION





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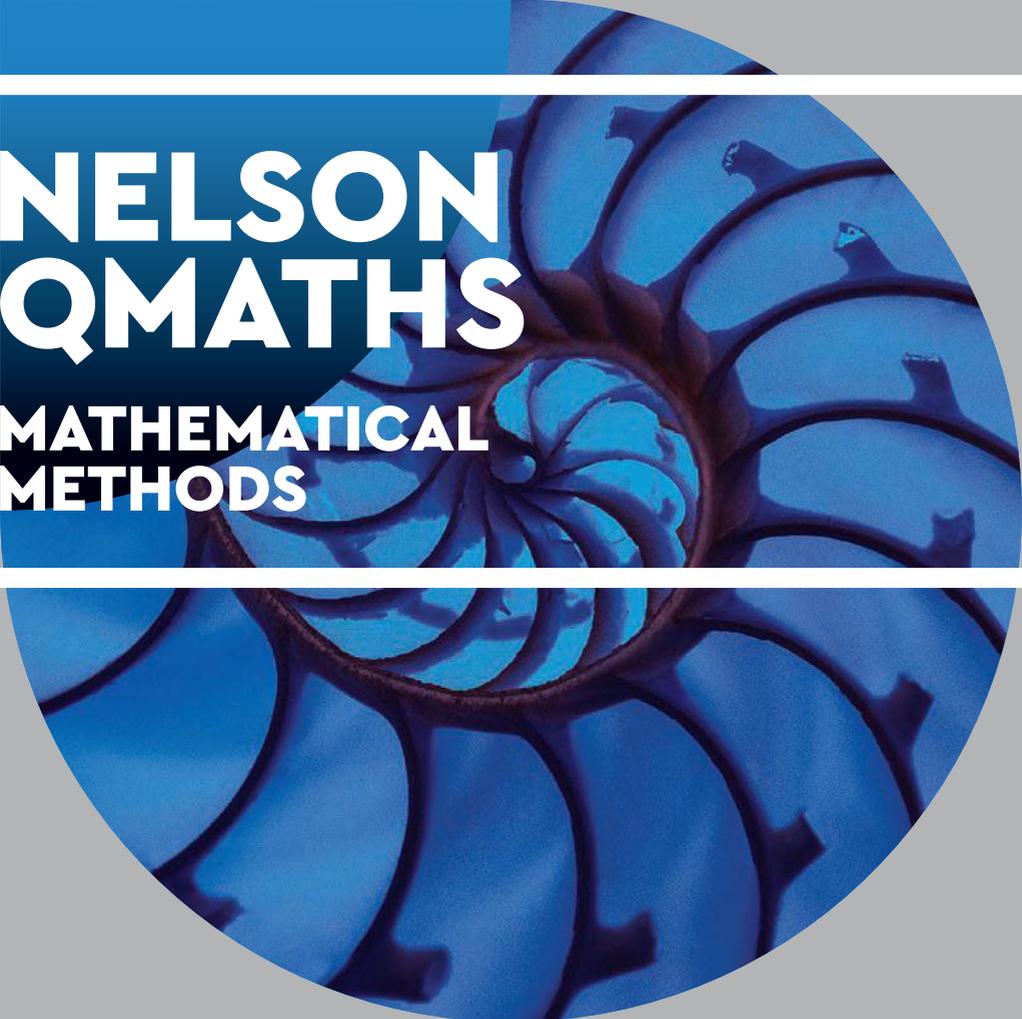
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**Nelson QMaths 11 Mathematical Methods**

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This syllabus forms part of a new senior assessment and tertiary entrance system in Queensland. Along with other senior syllabuses, it is still being refined in preparation for implementation in schools from 2019.

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# PREFACE

*Nelson QMaths*, Queensland's longest-running senior mathematics series, has been rewritten for the new syllabuses and assessment procedures for implementation from 2019. Based on the Australian Curriculum, four new senior mathematics courses have been introduced into Queensland schools.

- Essential Mathematics
- General Mathematics
- Mathematical Methods
- Specialist Mathematics

With the introduction of external QCE examinations, *Nelson QMaths* will have a renewed focus on assessment, and will include features such as topic tests, practice examinations, *ExamView* question banks and chapter quizzes, video tutorials and worked solutions to all exercise questions. In this book, teachers will find familiar features such as clear worked examples, graded exercises, strong syllabus coverage, Investigations, Technology, chapter summaries, chapter reviews and a glossary/index. We wish all teachers and students using this book every success in embracing the new mathematics courses.

## ABOUT THE AUTHORS

**Ross Brodie** has taught mathematics and been Head of Mathematics at a variety of regional and urban schools in Queensland. He is an experienced teacher and author, having co-written *Nelson QMaths 7-10*, *New QMaths 11-12* and *Nelson Senior Maths 11-12* (Australian Curriculum).

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**Scott Smith** and **Cherylanne Saywell** created the video tutorials.

**Tania Eastcott** and **Elizabeth Nabhan** wrote the topic tests.

**Sharon Kirkby** and **Dina Antoniou** wrote the worked solutions to all exercise sets.



# SYLLABUS REFERENCE GRID

Topics and subtopics	<i>Nelson QMaths 11 Mathematical Methods chapter</i>
<b>UNIT 1</b>	
<b>Arithmetic and geometric sequences and series 1</b>	
Arithmetic sequences	1 Arithmetic sequences and series
<b>Functions and graphs</b>	
Functions	4 Functions and graphs
Review of quadratic relationships	2 Algebra and polynomials
Inverse proportions	2 Algebra and polynomials
Powers and polynomials	2 Algebra and polynomials
Graphs of relations	4 Functions and graphs
<b>Counting and probability</b>	
Language of events and sets	3 Basic probability
Review of the fundamentals of probability	3 Basic probability
Conditional probability and independence	5 Combinations and conditional probability
Binomial expansion	5 Combinations and conditional probability
<b>Exponential functions 1</b>	
Indices and the index laws	2 Algebra and polynomials
<b>Arithmetic and geometric sequences and series 2</b>	
Geometric sequences	6 Geometric sequences and series
<b>UNIT 2</b>	
<b>Exponential functions 2</b>	
Introduction to exponential functions	9 Exponential functions and logarithms
<b>The logarithmic function 1</b>	
Introduction to logarithms	9 Exponential functions and logarithms
<b>Trigonometric functions 1</b>	
Circular measure and radian measure	8 Trigonometric functions
Introduction to trigonometric functions	8 Trigonometric functions
<b>Introduction to differential calculus</b>	
Rates of change and the concept of derivatives	7 Rates of change and the derivative
Properties and computation of derivatives	10 Properties of derivatives
Applications of derivatives	12 Applications of derivatives
<b>Further differentiation and applications 1</b>	
Differentiation rules	10 Properties of derivatives
<b>Discrete random variables 1</b>	
General discrete random variables	11 Discrete random variables

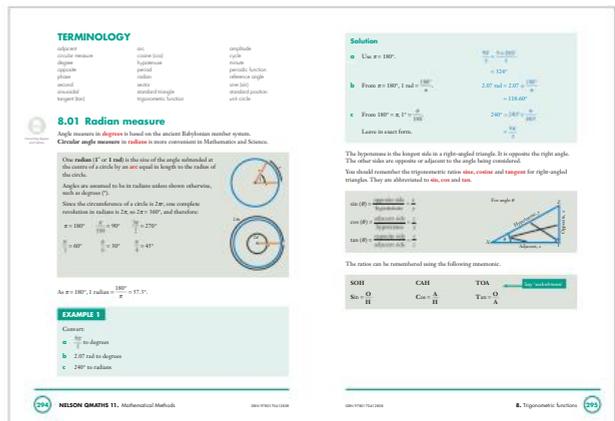
# ABOUT THIS BOOK

## AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of syllabus subject matter (© Queensland Curriculum and Assessment Authority 2017)



- Terminology** previews the key words and phrases from within the chapter.



# IN EACH CHAPTER

- Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand-side.
- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the glossary at the back of the book.
- Graded exercises include **Problem solving** questions, are linked to worked examples and include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, through group work, discovery and modelling activities.
- **Technology** promotes ICT in the classroom, using spreadsheets, graphics calculators and the Internet.
- Instructions are provided for the TI-84 Plus CE, Casio fx-CG20AU and TI-Nspire CX calculators.

**EXAMPLE 10**

Sketch the graph of  $y = \sin(x) - 2$  in the domain  $0 \leq x \leq 2\pi$ .

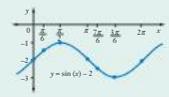
**Solution**

Construct a table of values for  $y = \sin(x) - 2$

$0 \leq x \leq 2\pi$ . You know that  $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$  so it's convenient to use values such as  $x = \frac{\pi}{6}$  and  $x = \frac{7\pi}{6}$ .

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\pi$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$2\pi$
y	-2	-1.5	-1	-2	-2.5	-3	-2

Plot the points and join them with a smooth curve.



**Translation of trigonometric functions**

The graphs of  $y = \sin(x) + D$ ,  $y = \cos(x) + D$  and  $y = \tan(x) + D$  are translated  $D$  units vertically compared with the graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$  respectively.

If  $D > 0$ , the graph is translated up.

If  $D < 0$ , the graph is translated down.

**Reflection of trigonometric functions in the x-axis**

The graphs of  $y = -\sin(x)$ ,  $y = -\cos(x)$  and  $y = -\tan(x)$  are reflections in the x-axis of the graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$  respectively.

**Exercise 8.04 The graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$**

- Sketch the graph of each function in the domain  $-\frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$ .
  - $f(x) = \sin(x)$
  - $f(x) = \cos(x)$
  - $f(x) = \tan(x)$
- Sketch the graph of each function in the domain  $0 \leq x \leq 2\pi$ .
  - $f(x) = \cos(x) - 5$
  - $f(x) = \tan(x) + 3$
  - $f(x) = \sin(x) + 4$

ISBN 9780170412858 8. Trigonometric functions **311**

**INVESTIGATION**

**THE GRAPHS OF  $y = A \sin(Bx + C)$  AND  $y = A \cos(Bx + C)$**

Use your CAS or graphics calculator for this investigation.

**TI-84 Plus CE**

Make sure that the angle measure is set to radians. Press **WINDOW** and set the domain and range values as  $-3\pi \leq X \leq 3\pi$  and  $-4 \leq Y \leq 4$ .

Press **Y=** and enter the function  $Y_1 = \sin(x)$  into the function line using **(X,T,θ,)/** to enter the variable.

Press **ENTER**.

Then press **ZOOM** to display the graph of the function.

Now press **Y=** and enter the function  $Y_2 = 3 \sin(x)$  into the function line.

Press **ENTER**.

Then press **ZOOM** to display the graph of the function.

Now follow the same procedure to enter  $Y_3 = \sin(x)$  and display the graph.

You can use the **ZOOM** button or the **WINDOW** button to modify the domain and range values if you want to see more detail.

**Casio fx-CG20AU**

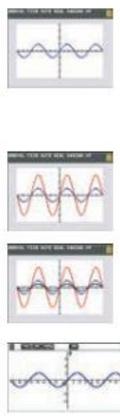
Press **MODE** and select the **Graph** page.

Press **SHIFT** **F3** (**V-Window**) and set the domain and range values as  $-3\pi \leq X \leq 3\pi$  and  $-4 \leq Y \leq 4$ .

Enter the function  $Y_1 = \sin(x)$  into the function line using **(X)** to enter the variable.

Press **ENTER**.

Then press **F6** (**DRAW**) to display the graph of the function.



You can use your graphics calculator to find approximate solutions for trigonometric equations.

**TECHNOLOGY**

**Solution of trigonometric equations**

Solve  $2 \sin(x) = \sqrt{3}$ ,  $0 \leq x \leq 2\pi$ , correct to 2 decimal places.

Rearrange the equation:  $\sin(x) = \frac{\sqrt{3}}{2}$

**TI-84 Plus CE**

Make sure that the angle measure is set to radians.

Press **Y=** and enter the functions

$Y_1 = \sin(x)$

$Y_2 = \frac{\sqrt{3}}{2}$

Then press **ZOOM** to display the graphs of the functions.

Press **WINDOW** and change the window settings to:

$0 \leq X \leq 2\pi$  and  $-1 \leq Y \leq 1$

to make the intersections clearer.

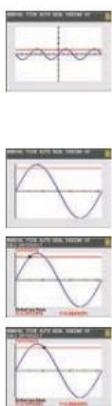
Press **2ND** **F2** (**TRACE**) **5** and select **5: intersect**.

Move the cursor to position the cross on  $Y_1$  close to the first point of intersection of the graphs. Press **ENTER**. Then move the cursor to position the cross on  $Y_2$  close to the point of intersection of the graphs. Press **ENTER**. Finally move the cursor to position the cross close to your guess for the point of intersection of the graphs. Press **ENTER**.

Repeat the procedure to find the second intersection.

State the result.

$x = 1.05, 2.09$



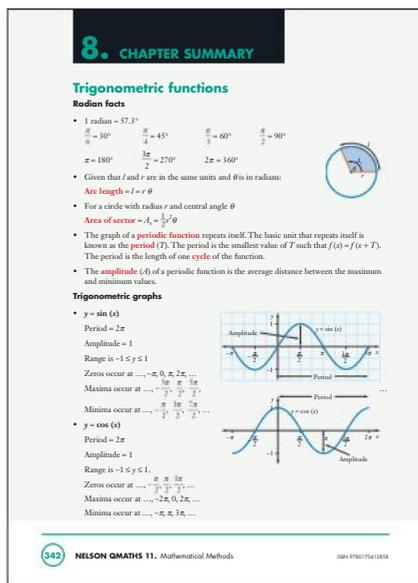


## AT THE END OF EACH CHAPTER

- **Chapter summary** summarises the key concepts, skills and formulas of the chapter.
- **Chapter review** contains revision exercises and are linked to worked examples.
- **Practice examinations** after every three chapters revise the skills and knowledge of those chapters.

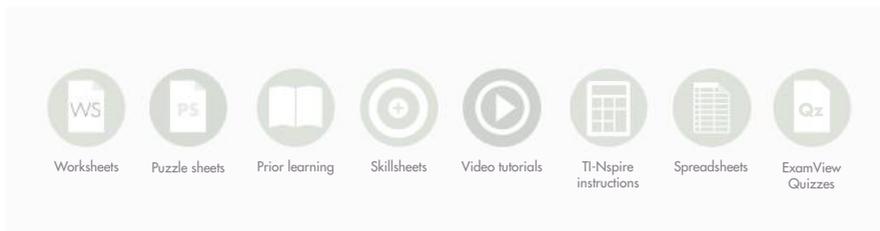
## AT THE END OF THE BOOK

- **Glossary and index** includes a comprehensive dictionary of course terminology.
- **Answers** (with worked solutions on the teacher website).



## NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, [www.nelsonnet.com.au](http://www.nelsonnet.com.au). These include:



- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Prior learning** exercises that revise prerequisite skills for the chapter
- **Skillsheets** of examples and exercises of prerequisite skills and knowledge
- **Video tutorials**: selected worked examples from each chapter explained online
- **Graphics calculator instructions**: for TI-Nspire CX calculators.
- **Spreadsheets**: *Excel* files
- **ExamView quizzes**: interactive and self-marking



# NELSONNET TEACHER WEBSITE

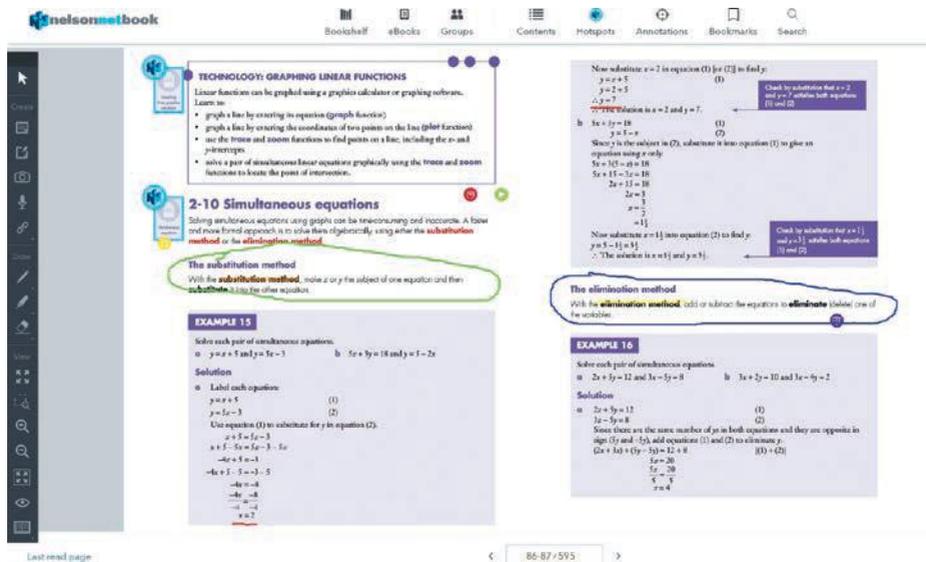
The *NelsonNet* teacher website, also at [www.nelsonnet.com.au](http://www.nelsonnet.com.au), contains:

- **Teaching plan**, in Microsoft Word and PDF formats
- **Topic tests**, in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- **ExamView** exam-writing software and question banks
- **Resource Finder**: search engine for *NelsonNet* resources.

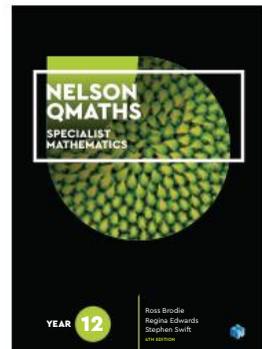
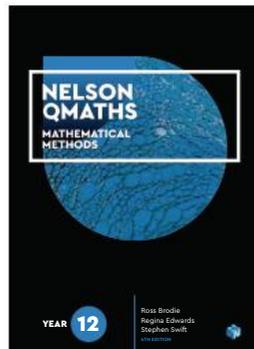
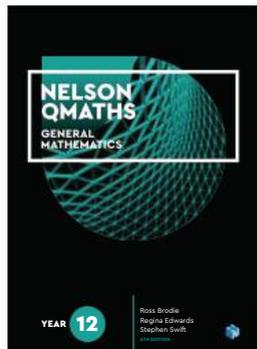
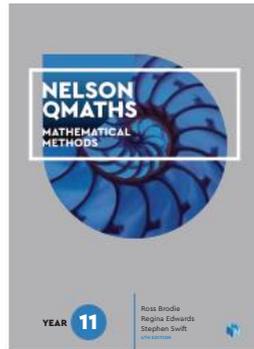
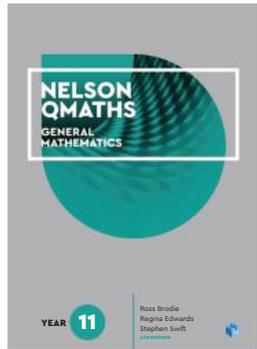
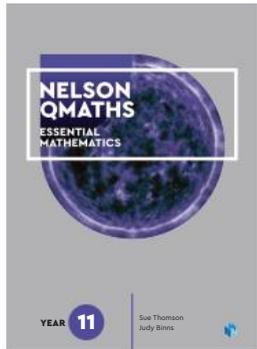
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- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students.



# NELSON QMATHS 11-12 SERIES



# LIST OF SYMBOLS AND ABBREVIATIONS

$=$	is equal to	$\mathbb{Z}, \mathbb{J}$	the integers	$(x, y)$	Cartesian coordinates, ordered pair
$\equiv$	is identically equal to	$\mathbb{Z}^+$	the positive integers	${}^n C_r, \binom{n}{r}$	the number of combinations of $r$ objects from $n$ objects
$\neq$	is not equal to	$\mathbb{N}$	the natural numbers	$n!$	factorial $n$
$\approx$	is approximately equal to	$\circ$	degree	$\%$	percentage
$<$	is less than	$\infty$	infinity	$\lim_{h \rightarrow 0}$	the limit as $h \rightarrow 0$
$>$	is greater than	$\Rightarrow$	implies	$\rightarrow$	gets closer to
$\leq$	is less than or equal to	$\Leftrightarrow$	if and only if	$\uparrow$	gets closer to from below
$\geq$	is greater than or equal to	$\infty$	is proportional to	$\downarrow$	gets closer to from above
AP	arithmetic progression	$\Delta$	the discriminant	$\frac{dy}{dx}, f'(x)$	the derivative of $y, f(x)$
$t_n, a_n$	$n$ th term of a sequence	QED	quod erat demonstrandum, statement that a proof is finished	$\left. \frac{d}{dx} f(x) \right _{x=c}$	the derivative at $x = c$
$S_n$	the sum to $n$ terms	$\sqrt{\quad}$	square root, radical sign	wrt	with respect to
GP	geometric progression	$\overline{A, A'}$	not $A$ , the complement of $A$	$s$	displacement
$S_\infty$	sum to infinity	$\aleph_0$	aleph 0, the cardinality of $\mathbb{N}$	$v$	speed, velocity
$FV$	future value	:	such that	$a$	acceleration
$PV$	present value	$S$	sample space	$\log_{10}$	common logarithm
$i$	interest rate	$n(A)$	the number of elements in $A$	$\ln$	natural logarithm
$S_{\overline{n} i}, S_{\overline{n} }$	future value annuity factor	$P(A)$	the probability of $A$	$\log_a b$	the logarithm of $b$ to the base $a$
$a_{\overline{n} i}, a_{\overline{n} }$	present value annuity factor	$P(A B)$	the probability of $A$ given $B$	$\cos$	trigonometric cosine ratio
$\in$	is a member of	$ A $	the cardinality of set $A$	$\sin$	trigonometric sine ratio
$\cup$	union	$1:1$	one-to-one	$\tan$	trigonometric tangent ratio
$\cap$	intersection	$1:n$	one to $n$	$\mu = E(X)$	expected value of the random variable $X$
$\pm$	plus or minus	$n:1$	$n$ to one	$\text{Var}(X)$	variance of the random variable $X$
$\pi$	$\text{pi} \approx 3.14$	$n:n$	$n$ to $n$	$\sigma_X = \text{SD}(X)$	standard deviation of the random variable $X$
$\subseteq$	is a subset of	LHS	left-hand side		
$\subset$	is a proper subset of	RHS	right-hand side		
$\notin$	is not a member of	$ a $	absolute value of $a$		
$\Sigma$	the sum of	$m$	gradient of a straight line		
...	and so on	$c$	$y$ -intercept of a straight line		
$\{\}, \emptyset$	the null set				
$\mathbb{U}, \mathbb{E}$	the universal set				
$\mathbb{R}$	the real numbers				
$\mathbb{R}^+$	the positive real numbers				

# LIST OF MATHEMATICAL VERBS

## A glossary of 'doing words' commonly found in mathematics problems

**analyse:** study and state in detail the relationship of parts of a situation

**apply:** use knowledge or a procedure in a given situation

**calculate:** find a numerical value

**comment:** express an opinion or judgement about a statement or calculation

**communicate:** transmit information to others

**compare:** state similarities and differences and their significance

**consider:** take into account

**construct:** draw an accurate diagram or logically arrange items or ideas

**convert:** change from one form to another

**define:** give the meaning of or identify in exact terms

**demonstrate:** show to be correct

**describe:** state the features of a situation, object, pattern, event, etc

**determine:** find the answer or make a decision using evidence

**discuss:** give reasons based on evidence for and against a position or proposal

**establish:** introduce and develop a result

**evaluate:** find the value of or state the application, strengths and limitations of a solution

**examine:** state the details and assumptions of a situation

**expand:** remove brackets or change a product to an extended sum of terms

**explain:** state the meaning in logical detail

**explore:** examine or state the details and assumptions of a situation

**factorise:** convert to factors or change a sum of terms to a product of factors

**formulate:** give mathematical expression to an idea or situation

**hence find/prove:** find an answer or prove a result using previous answers or information supplied

**identify:** state the type, name or distinguishing feature of an item or situation

**interpret:** state a conclusion or trend from given information

**investigate:** establish facts, trends or conclusions from collected information

**justify:** give reasons or evidence for an answer or conclusion

**modify:** change to accommodate different information

**obtain:** find an answer or conclusion

**prove:** use logical steps to establish the truth of

**recall:** remember (and state)

**recognise:** use knowledge to identify features of a situation

**rationalise:** make rational, remove surds

**show that:** in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

**simplify:** reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

**sketch:** draw a diagram that shows the general shape and includes relevant features

**solve:** find the answer or explanation for a problem, particularly the values of variables

**substitute:** replace a variable by a number to calculate an answer

**translate (to mathematical form):** express a situation as mathematical relationships

**verify:** check a solution or result, usually referring to the given situation

**write/state:** give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

# 1

## ARITHMETIC SEQUENCES AND SERIES

Traffic lights in Australia follow the time sequence: red, green, amber, red, green, ... In some countries there is an amber signal between the red and green, so they follow a different sequence. The letters on a normal keyboard follow the sequence q, w, e, r, t, y, u, ... This sequence stopped keys jamming in the first typewriters by placing common letters on the same fingers. The numbers 5, 6, 8, 11, 15, 20, ... also follow a sequence.

- 1.01 Sequences and recursive definitions
  - 1.02 Arithmetic sequences
  - 1.03 Sum of an arithmetic sequence
  - 1.04 Applications of arithmetic sequences and series
- Chapter summary  
Chapter review

## SYLLABUS SUBJECT MATTER

### Arithmetic sequences

- recognise and use the recursive definition of an arithmetic sequence:  $t_{n+1} = t_n + d$
- use the formula  $t_n = t_1 + (n - 1)d$  for the general term of an arithmetic sequence and recognise its linear nature
- use arithmetic sequences in contexts involving discrete linear growth or decay, such as simple interest
- establish and use the formula for the sum of the first  $n$  terms of an arithmetic sequence

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Prior learning

# TERMINOLOGY

AP	arithmetic sequence	arithmetic series
common difference	first term	general term
$n$ th term	recursive	sequence
series	square numbers	sum of an AP
term	triangular numbers	

## 1.01 Sequences and recursive definitions

A **mathematical sequence** is an ordered list of numbers. Every natural number has a member of the sequence associated with it. Some sequences have simple rules that allow you to work out a member of the sequence from its place in the sequence. The **Fibonacci** sequence, 1, 1, 2, 3, 5, 8, 13, 21, 35, ... is one of the most famous because it occurs frequently in nature.

A **sequence** is a list of numbers in a particular order. Each member of the sequence is called a **term**, usually written as  $t_1, t_2, t_3, t_4, \dots$ . The subscript tells you where it is in the sequence. Sometimes the **first term** is written as  $a$ , so  $a = t_1$ . The  **$n$ th term** is written as  $t_n$ .

### EXAMPLE 1

A sequence is given by 9, 6, 3, 0, -3, ...

- a** What is the eighth term of the sequence?
- b** Which term is equal to -18?

### Solution

- a** The sequence goes down by 3 between each term. Write more terms. 9, 6, 3, 0, -3, -6, -9, -12, ...

Write the answer.

The eighth term is -12.

- b** Keep writing terms until you get to  $t_{10}$ .  $t_8 = -12, t_9 = -15, t_{10} = -18$

Write the answer.

-18 is the 10th term.

You can write the rule for the sequence in Example 1 as:  $t_1 = 9, t_{n+1} = t_n - 3$ .

You could also write the rule as  $t_n = 12 - 3n$  so that you could work out terms from the term number.

A **recursive** definition of a sequence gives terms of the sequence as a rule that shows how each term is calculated from the preceding term.  $t_1$  is usually given as well.

The **general term** of a sequence is a rule that gives the  $n$ th term in terms of  $n$ .

## EXAMPLE 2

A sequence is given by  $t_1 = 2$ ,  $t_n = (t_{n-1} - 3)^2$

- a Write the first 5 terms of the sequence.
- b Write the general term.

### Solution

- a Substitute 2 in the recursive rule.

$$t_2 = (t_1 - 3)^2$$

$$= (2 - 3)^2 = 1$$

Substitute  $t_1$  and calculate the answer.

Substitute to find  $t_3$ .

$$t_3 = (1 - 3)^2 = 4$$

Repeat to find the first 5 terms.

$$2, 1, 4, 1, 4, \dots$$

- b After the first term, the sequence alternates between 1 and 4.

$$t_1 = 2, t_2 = 1, t_4 = 1, t_6 = 1, \dots$$

$$\text{and } t_3 = 4, t_5 = 4, t_7 = 4, \dots$$

Use  $(-1)^n$ ; it alternates between  $-1$  and  $1$ .  $t_1 = 2$ ,  $t_n = 2.5 - (-1)^n \times 1.5$  for  $n > 1$ .

The general term of a sequence is usually easier to find than the one in Example 2.

## INVESTIGATION

### FIBONACCI NUMBERS

- Who is the Fibonacci sequence named after? What was his actual name and when did he live?

Look at the numbers of petals on common flowers. What do you find?

Find some instances of Fibonacci sequences that occur in nature.

See if you can find a general term for the Fibonacci sequence.



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## Exercise 1.01 Sequences and recursive definitions

Example  
1

- 1** A sequence is given by 3, 5, 7, 9, 11, ...
- a** What are the next 4 terms?
  - b** What is  $t_{12}$ ?
  - c** Which term is equal to 29?
- 2** Find the indicated term for each sequence.
- a** 4, 8, 12, 16, ... ( $t_{11}$ )
  - b** 3, -2, -7, -12, ... ( $t_{10}$ )
  - c** 1, 9, 25, 49, ... ( $t_8$ )
  - d** 3, 12, 48, 192, ... ( $t_7$ )
- 3** Find which term is equal to the given number for each sequence.
- a** 5, 7, 9, 11, ... (49)
  - b** 12, 5, -2, -9, ... (-58)
  - c** 1, 3, 9, 27, ... (2187)
  - d** 0.25, 0.05, 0.01, 0.002, ... (0.000 016)

Example  
2

- 4** A sequence is given by  $t_1 = 5, t_{n+1} = t_n + 3$ .
- a** Write the first 5 terms of the sequence.
  - b** Write the general term.
- 5** Write the first 5 terms for each sequence.
- a**  $t_1 = 12, t_{n+1} = t_n - 4$
  - b**  $t_1 = 12, t_n = t_{n-1} \div 2$
  - c**  $t_1 = 4, t_{n+1} = 2t_n - 1$
  - d**  $t_1 = 5, t_{n+1} = \frac{t_n + 4}{2}$
  - e**  $t_1 = \frac{3}{16}, t_n = 4t_{n-1}$
- 6** Find the general term for each sequence.
- a**  $t_1 = 7, t_{n+1} = t_n + 3$
  - b** 80, 20, 5, 1.25, ...
  - c**  $t_1 = 8, t_n = \frac{2t_{n-1} + 5}{2}$
  - d** 1, 8, 27, 64, ...
  - e**  $t_1 = 6, t_{n+1} = \frac{t_n + 4}{2}$

### Problem solving

- 7** Mila saves \$20 a week. She currently has \$240 saved.
- a** How much will she have in 8, 10 and 20 weeks from now?
  - b** How long will it take her to reach her target of \$800?
- 8** Taj has been practising for the 100 m race in the inter-house carnival three weeks from now. Three weeks ago he ran 24 seconds, a fortnight ago it was 20 seconds, last week it was 18 seconds and this week it was 17 seconds.
- a** If he keeps improving at this rate, what time could he expect to run in the carnival?
  - b** Assuming nothing changes, what is the best time he could hope for in the future?

## 1.02 Arithmetic sequences

The sequence 12, 16, 20, 24, ... increases by 4 each time. Its recursive definition  $t_1 = 12, t_{n+1} = t_n + 4$  has a number added to each term to give the next. The recursive definition of 19, 16, 13, 10, ... has  $-3$  added to each term to give the next;  $t_1 = 19, t_{n+1} = t_n + (-3)$ .



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An **arithmetic sequence** (or **arithmetic progression** [AP]), is defined by  $t_{n+1} = t_n + d$ , where  $d$  is a constant called the **common difference**;  $d = t_{n+1} - t_n$ .

You can find any term from the first term and common difference.

### EXAMPLE 3

Find the common difference of any of the following that are arithmetic sequences.

- a** 2, 5, 9, 14, ...                      **b** 8, 2,  $-4$ ,  $-10$ , ...  
**c** 1, 11, 21, 31, ...                    **d** 16, 8,  $-8$ ,  $-16$

### Solution

- a** Check the differences.

State your conclusion.

$$t_2 - t_1 = 3, t_3 - t_2 = 4$$

The difference of successive terms is not a constant, so it is not an arithmetic sequence.

- b** Check the differences.

State your conclusion.

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = -6$$

There is a common difference of  $-6$  for successive terms, so it is an AP.

- c** Check the differences.

State your conclusion.

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = 10$$

There is a common difference of 10 for successive terms, so it is an AP.

- d** Check the differences.

State your conclusion.

$$t_2 - t_1 = -8, t_3 - t_2 = -16$$

The difference of successive terms is not a constant, so it is not an arithmetic sequence.

The arithmetic sequence given by  $a = 5$ ,  $t_{n+1} = t_n + 7$  has the terms:

$$t_1 = a = 5$$

$$t_2 = t_1 + 7 = 12 = 5 + 1 \times 7$$

$$t_3 = t_2 + 7 = 19 = 5 + 2 \times 7$$

$$t_4 = t_3 + 7 = 26 = 5 + 3 \times 7$$

and so on.

Each term is equal to the first term added to 7 times one less than the term number.

The **general term** of an arithmetic sequence with first term  $a$  and common difference  $d$  is given by  $t_n = a + (n - 1)d$ .

#### EXAMPLE 4

A sequence is given by 2.5, 7.9, 13.3, 18.7, ...

- a Write the general term.
- b What is the fourteenth term?
- c What is the first term to exceed 165?

#### Solution

- a Check differences.

There is a common difference of 5.4.

Write the general term.

$$\begin{aligned}t_n &= a + (n - 1)d \\ &= 2.5 + (n - 1) \times 5.4\end{aligned}$$

Simplify.

$$= 5.4n - 2.9$$

- b Substitute  $n = 14$ .

$$t_{14} = 5.4 \times 14 - 2.9$$

Simplify.

$$= 72.7$$

- c Substitute  $t_n = 165$  and solve for  $n$ .

$$165 = 5.4n - 2.9$$

$$5.4n = 167.9$$

$$n = 31.092\dots$$

Check  $t_{31}$  and  $t_{32}$ .

$$t_{31} = 164.5, t_{32} = 169.9$$

Write the answer.

$t_{32} = 169.9$  is the first term to exceed 165.

You can find the terms of an arithmetic sequence from any two terms, as long as you know which terms they are.

### EXAMPLE 5

The fifth term of an arithmetic sequence is 54 and the ninth term is 40.

What is the 20th term?

#### Solution

State the symbols you will use.

Let  $t_1 = a$  and common difference =  $d$ .

Write the terms using  $a$  and  $d$ .

$$a + 4d = 54$$

$$a + 8d = 40$$

Make sure you define the variables.

Solve to find  $d$ .

$$(a + 8d) - (a + 4d) = 40 - 54$$

$$4d = -14$$

$$d = -3\frac{1}{2}$$

Find  $a$ .

$$a + 4 \times \left(-3\frac{1}{2}\right) = 54$$

$$a - 14 = 54$$

$$a = 68$$

Find the required term.

$$t_{20} = a + 19d$$

$$t_{20} = 68 + 19 \times \left(-3\frac{1}{2}\right) = 1\frac{1}{2}$$

Write the answer.

The 20th term is  $1\frac{1}{2}$ .

## Exercise 1.02 Arithmetic sequences

1 State if each sequence is an AP, and if it is, find the common difference.

**a** 5, 3, 1, -1, ...

**b**  $3 + x, 4 - x, 5 - 3x, 6 - 5x, \dots$

**c** 2, 2.2, 2.22, 2.222, ...

**d** 3, 5, 8, 12, ...

**e**  $\frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \dots$

**f** 6, 12, 24, 48, ...

2 Find the general term of each sequence.

**a** 3, 7, 11, 15, ...

**b** 102, 99, 96, 93, ...

**c**  $2\frac{1}{2}, 1\frac{1}{3}, \frac{1}{6}, -1, \dots$

**d** 1.8, 3.9, 6, 8.1, ...

**e** -15, -13, -11, -9

**f**  $2x + 3y, 4x + 2y, 6x + y, 8x, \dots$

Example

3

Example

4

- 3** Find the nominated terms of each sequence.
- a**  $2, -1, -4, -7, \dots$  ( $t_7$  and  $t_{20}$ )      **b**  $3, 11, 19, 27, \dots$  ( $t_{11}$  and  $t_{16}$ )
- c**  $3x, 5x - 2, 7x - 4, 9x - 6, \dots$  ( $t_9$  and  $t_{32}$ )      **d**  $-2\frac{1}{2}, -1\frac{3}{4}, -1, -\frac{1}{4}, \dots$  ( $t_6$  and  $t_{12}$ )
- e**  $0.\bar{4}, 0.\bar{6}, 0.\bar{8}, 1.\bar{1}, \dots$  ( $t_9$  and  $t_{15}$ )      **f**  $7, 1, -5, -11, \dots$  ( $t_{20}$  and  $t_{30}$ )

Example  
5

- 4** The 12th term of an AP is 16 and the 20th term is 5. What is the 30th term?
- 5** The 7th and 9th terms of an AP are 2.9 and 4.7 respectively. What is the 16th term?
- 6** The 4th term of an AP is  $x$  and the 9th term is  $y$ . What is the 16th term?
- 7** What is the first term of the sequence  $3, 7.1, 11.2, 15.3, \dots$  to exceed 200?

### Problem solving

- 8** The sum of the 5th and 10th terms of an AP is equal to the 20th term, which is 80. What is the 30th term?
- 9** Show that the general term of an AP with  $p$ th term  $x$  and  $q$ th term  $y$  is  $\frac{x(n-q) + y(p-n)}{p-q}$ .
- 10** The 4th term of an AP is 28 and the 17th term is the first to exceed 50. What are the possible values of the first term?



## 1.03 Sum of an arithmetic sequence

Notice the pattern and shortcut in calculating each sum:

$$\begin{aligned} 3 + 7 + 11 + 15 + 19 + 23 &= (3 + 23) + (7 + 19) + (11 + 15) \\ &= 3 \times 26 \\ &= 78 \end{aligned}$$

$$\begin{aligned} 2 + 4 + 6 + 8 + 10 + 12 + 14 &= (2 + 14) + (4 + 12) + (6 + 10) + 8 \\ &= 3\frac{1}{2} \times 16 \\ &= 56. \end{aligned}$$

Both are sums of APs and in both cases,

total  $= \frac{n}{2} \times (\text{first} + \text{last})$ , where  $n$  is the number of terms. It is obvious from the examples that this is always true for an AP.

When Carl Friedrich Gauss was 10 years old his teacher asked the class to add all the numbers from 1 to 100 to keep them busy. Inventing the method above in his head, he found the answer straight away and wrote it on his slate. The teacher thought he was joking and told him to do it properly. However, Gauss was right, as shown when other students finally finished.



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## EXAMPLE 6

Evaluate each series.

**a**  $28 + 25 + 22 + 19 + 16 + 13 + 10 + 7 + 4$       **b**  $7.9 + 11.6 + 15.3 + 19 + \dots + 93$

### Solution

**a** Count the terms.

There are 9 terms.

Find the sum of the first and last terms.

$$\text{First} + \text{last} = 32$$

Use Gauss's method to find the total.

$$\text{Sum} = \frac{9}{2} \times 32 = 144$$

**b** Find the common difference.

Common difference  $d = 3.7$

Write the formula of the general term.

$$t_n = a + (n - 1)d$$

Substitute values to find  $n$ .

$$93 = 7.9 + 3.7(n - 1)$$

$$n = 24$$

Find the sum.

$$\text{Sum} = 12(7.9 + 93) = 1210.8$$

The partial sums of a sequence make another sequence called a **series**. An **arithmetic series (sum of an AP)** is the sum of the terms of an arithmetic progression.

The partial sum  $S_n$  is often referred to as the **sum to  $n$  terms**.

The general term of an **arithmetic series** is given by

$$S_n = \frac{n}{2}(t_1 + t_n) = \frac{n}{2}[2a + (n - 1)d],$$

where  $a$  is the first term and  $d$  is the common difference of the sequence.

The last ( $n$ th) term of the sequence is often written as  $l$ , giving  $S_n = \frac{n}{2}(a + l)$ .

You can prove the formula by writing the sequence backwards and adding:

$$S_n = t_1 + t_2 + t_3 + \dots + t_{n-2} + t_{n-1} + t_n$$

$$S_n = t_n + t_{n-1} + t_{n-2} + \dots + t_3 + t_2 + t_1$$

$$S_n + S_n = [t_1 + t_n] + [t_2 + t_{n-1}] + [t_3 + t_{n-2}] + \dots + [t_{n-2} + t_3] + [t_{n-1} + t_2] + [t_n + t_1]$$

Now substitute  $t_1 = a$ ,  $t_2 = a + d$ ,  $t_3 = a + 2d$ , ...,  $t_{n-1} = a + (n - 2)d$  and  $t_n = a + (n - 1)d$

$$2S_n = [a + a + (n - 1)d] + [a + d + a + (n - 2)d] + \dots + [a + (n - 1)d + a]$$

$$= [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \dots + [2a + (n - 1)d]$$

$$= n[2a + (n - 1)d]$$

$$\text{Dividing by 2, } S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}(t_1 + t_n).$$

QED

### EXAMPLE 7

Evaluate each arithmetic series.

- a**  $2\frac{1}{2}, 1\frac{1}{3}, \frac{1}{6}, -1, \dots$  to 50 terms  
**b**  $2x + 3y, 4x + 2y, 6x + y, 8x, \dots$  to 40 terms

### Solution

- a** Find the common difference. Common difference,  $d = -1\frac{1}{6}$   
Write the formula. Sum to  $n$  terms,  $S_n = \frac{n}{2}[2a + (n-1)d]$   
Substitute values.  $S_{50} = \frac{50}{2}\left[2 \times 2\frac{1}{2} + 49 \times \left(-1\frac{1}{6}\right)\right]$   
Calculate the answer.  $= -1304\frac{1}{6}$
- b** Find the common difference. Common difference  $d = 2x - y$   
Write the formula. Sum to  $n$  terms,  $S_n = \frac{n}{2}[2a + (n-1)d]$   
Substitute values.  $S_n = \frac{40}{2}[2(2x + 3y) + 39(2x - y)]$   
Calculate the answer.  $= 1640x - 660y$



Arithmetic series

### EXAMPLE 8

How many terms does it take for the series  $7 + 9 + 11 + 13 + \dots$  to first exceed 456?

### Solution

- Choose variables. Let first term  $a = 7$  and common difference  $d = 2$ .
- Write the sum to  $n$  terms. Use  $S_n = \frac{n}{2}[2a + (n-1)d]$ .
- Substitute values.  $456 = \frac{n}{2}[14 + 2(n-1)]$
- Simplify.  $14n + 2n^2 - 2n = 912$   
 $2n^2 + 12n - 912 = 0$   
 $n^2 + 6n - 456 = 0$

Use the quadratic formula to solve the equation.

$$\begin{aligned} n &= \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times (-456)}}{2} \\ &= \frac{-6 \pm \sqrt{1860}}{2} \\ &= \frac{-6 \pm 2\sqrt{465}}{2} \\ &= -3 \pm \sqrt{465} \\ &\approx -24.6 \text{ or } 18.6 \end{aligned}$$

But  $n$  must be a positive whole number, so try  $n = 18$  and  $19$ .

$$S_{18} = 9(14 + 2 \times 17) = 432$$

$$S_{19} = 9.5(14 + 2 \times 18) = 475$$

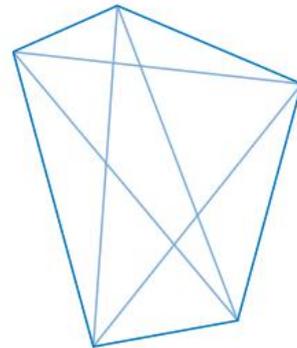
Write the answer.

19 terms are needed for the series to first exceed 456.

## INVESTIGATION

### DIAGONALS

- Count the number of diagonals in a convex pentagon.
- How many diagonals does a convex quadrilateral have?
- How many diagonals are there in a convex hexagon?
- Can you find a formula for the number of diagonals in a convex polygon with  $n$  sides?
- Can you prove the formula?



## Exercise 1.03 Sum of an arithmetic sequence

- 1 Find the sum of each series.
  - a  $23 + 26 + 29 + 32 + 35 + 38 + 41 + 44 + 47 + 50$
  - b  $82 + 80 + 78 + \dots + 16$
  - c  $8\frac{1}{2} + 10\frac{1}{3} + 12\frac{1}{6} + \dots + 36$
  - d  $1\frac{3}{8} + 1\frac{1}{8} + \frac{7}{8} + \dots - 3\frac{3}{8}$
  - e  $(9x - 3) + (7x - 2) + (5x - 1) + \dots + (5 - 7x)$
  - f  $16.3 + 21.2 + 26.1 + \dots + 94.7$

Example  
6

Example

7

- 2** Evaluate each series.
- a**  $-35 - 32 - 29 - \dots$  to 30 terms
- b**  $21 + 17\frac{1}{5} + 13\frac{2}{5} + \dots$  to 40 terms
- c**  $12x + (11x + 2) + (10x + 4) + \dots$  to 20 terms
- d**  $1.8 + 3.1 + 4.4 + \dots$  to 60 terms
- e**  $-2.5 - 5.3 - 8.1 - \dots$  to 40 terms
- f**  $\frac{14}{15} + 4\frac{2}{15} + 7\frac{1}{3} + \dots$  to 50 terms

Example

8

- 3** An arithmetic sequence has first term 9 and common difference 4.5. How many terms must be summed for it to reach at least 1000?
- 4** How many terms of the series  $-45 - 39 - 33 - \dots$  does it take to first exceed 155?
- 5** What is the largest value reached by the series  $46 + 40 + 34 + \dots$ ?

### Problem solving

- 6** An arithmetic sequence has a sum to 20 terms of 46 and a first term of 8. What is the sum to 30 terms?
- 7** The 5th term of an AP is 72 and the 30th term is  $-38$ . What is the minimum number of terms needed for the sum to be negative?
- 8** The sum to 16 terms of an AP is 92 and the sum to 20 terms is 175. What is the sum to 100 terms?
- 9** The sum of an arithmetic sequence to 40 terms is the same as the 40th term, and the 7th term is 26. What is the sum to 25 terms?
- 10** The tenth term of an arithmetic sequence is 18 and the sum to 20 terms is 280. How many terms are required for the corresponding series to exceed 500?



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## 1.04 Applications of arithmetic sequences and series



Arithmetic series assignment



Arithmetic series extension

You can apply arithmetic sequences and series in many situations involving steady increases or decreases. That is, when the domain of a linear function is the positive integers.

### EXAMPLE 9

A park has some trapezoidal garden beds bordered by paths radiating out from a central point. In one bed, there are 25 snapdragons planted in the first row and 4 more in each successive row. The last row has 89 plants.

- a How many rows of snapdragons are there?
- b How many snapdragons were planted altogether?

### Solution

- a Choose variables.

Use the general term of an AP.

Substitute values to find the number of rows.

Write the answer.

- b Use the sum of an AP.

Write the answer.

Let  $a = 25$  and  $d = 4$ .

The number of snapdragons in the  $n$ th row is given by  $t_n = a + (n - 1)d$ .

$$89 = 25 + 4(n - 1)$$

$$4(n - 1) = 64$$

$$n = 17$$

There are 17 rows of snapdragons.

The number of snapdragons altogether is given by

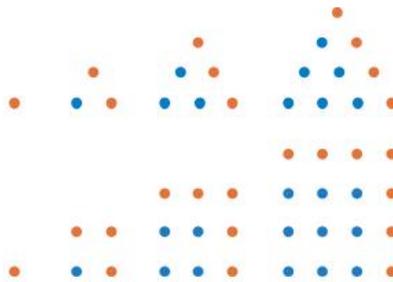
$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$= 8.5(2 \times 25 + 16 \times 4)$$

$$= 1394$$

1394 snapdragons were planted.

The numbers of dots in triangles of dots with sides of length 1, 2, 3, 4, ... are called **triangular numbers**.



**Square numbers** count the numbers of dots in squares of side lengths 1, 2, 3, 4, ...

Similarly, **pentagonal numbers** and **hexagonal numbers** count the numbers of dots in pentagons and hexagons of sides 1, 2, 3, 4, ...

### EXAMPLE 10

Find a formula for the  $n$ th triangular number.

#### Solution

Look at the pattern of numbers.

Starting from 1, the  $n$ th triangular number  $T_n$  is given by

$$\begin{aligned} T_n &= T_{n-1} + n \\ &= T_{n-2} + (n-1) + n \\ &= 1 + 2 + \dots + (n-1) + n \end{aligned}$$

Choose the variables.

This is the sum of  $n$  terms of an AP; first term  $a = 1$ , last term  $l = n$ .

Write the sum.

$$T_n = \frac{n}{2}(a+l) = \frac{n(n+1)}{2}$$

Write the answer.

The  $n$ th triangular number is given by

$$T_n = \frac{n(n+1)}{2}.$$

## Exercise 1.04 Applications of arithmetic sequences and series

- 1 Treated pine power poles are stacked in rows at a depot so that each row of poles has one less than the one below. The bottom row has 20 power poles and the top row has 8. How many power poles are stacked altogether?
- 2 **a** Describe the numbers added to give square numbers.  
**b** Find a formula for the sum of the odd numbers from 1 to  $n$ .
- 3 Find a formula for the pentagonal numbers.
- 4 Find a formula for the hexagonal numbers.

Example  
9

Example  
10

### Problem solving

- 5 Julie saved \$6000 each year for 8 years and got 6% simple interest on her savings.
  - a** What would the first \$6000 have grown to after 8 years?
  - b** What would the second \$6000 have grown to after 8 years?
  - c** What would she have altogether after 8 years?
- 6 A water tank can hold 7900 litres, but after a long dry spell has only 1300 litres in it. It started raining steadily at 10 a.m. and by 11:30 a.m. the tank was up to 2500 litres.
  - a** How much water was added each hour?
  - b** Assuming it continued to rain steadily, how much was in the tank at 3 p.m.?
  - c** What time would the tank be full if it kept raining?
- 7 A fuel tanker costing \$130 000 decreased in value by \$8500 each year. Write a formula for its value after  $n$  years and calculate how long it takes for the value to drop to \$28 000.
- 8 A sheepdog runs back and forth behind a mob of sheep to drive them towards a gate in the corner of a large paddock. At the start the dog runs 147 m along the diagonal between the fences, but the distance the dog runs decreases by 9 m for each run between the fences.

How far does the dog run altogether to drive the sheep through the 3 m gate?

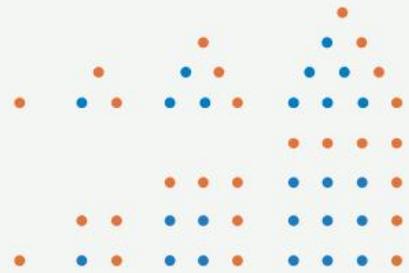


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# 1. CHAPTER SUMMARY

## Arithmetic sequences and series

- A **sequence** is a list of numbers in a particular order. Each member of the sequence is called a **term**, usually written as  $t_1, t_2, t_3, t_4, \dots$ . The subscript tells you where it is in the sequence. Sometimes the **first term** is written as  $a$ , so  $a = t_1$ . The  **$n$ th term** is written as  $t_n$ .
- A **recursive** definition of a sequence gives terms of the sequence as a rule that shows how each term is calculated from the preceding term.  $t_1$  is usually given as well.
- The **general term** of a sequence is a rule that gives the  $n$ th term in terms of  $n$ .
- An **arithmetic sequence** (or **arithmetic progression** [AP]) is defined by  $t_{n+1} = t_n + d$ , where  $d$  is a constant called the **common difference**;  $d = t_{n+1} - t_n$ .
- The **general term** of an arithmetic sequence with first term  $a$  and common difference  $d$  is given by  $t_n = a + (n - 1)d$ .
- The partial sums of a sequence make another sequence called a **series**.
- The general term of an **arithmetic series** is given by  $S_n = \frac{n}{2}(t_1 + t_n) = \frac{n}{2}[2a + (n - 1)d]$ , where  $a$  is the first term and  $d$  is the common difference of the sequence. The last ( $n$ th) term of the sequence is often written as  $l$ , so  $S_n = \frac{n}{2}(a + l)$ .
- The numbers of dots in triangles of dots with sides of length 1, 2, 3, 4, ... are called **triangular numbers**.
- **Square numbers** count the numbers of dots in squares of side lengths 1, 2, 3, 4, ...
- Similarly, **pentagonal numbers** and **hexagonal numbers** count the numbers of dots in pentagons and hexagons of sides 1, 2, 3, 4, ...



# 1. CHAPTER REVIEW

## Arithmetic sequences and series

- 1** A sequence is given by 1, 3, 7, 15, ...
- a** What is the 7th term?                      **b** Which term is equal to 1023?
- 2** A sequence is given by  $t_1 = 1$ ,  $t_n = 2t_{n-1} + 4$ .
- a** Write the first 5 terms of the sequence.   **b** Write the general term.
- 3** Find the common difference of any of the following that are arithmetic sequences.
- a** 2, 4, 8, 16, ...    **b** 5, 9, 13, 17, ...    **c** 9, 7, 4, 0, -5, ...    **d** 11, 9, 7, 5, ...
- 4** A sequence is given by  $14, 10\frac{5}{7}, 7\frac{3}{7}, 4\frac{1}{7}, \dots$
- a** Write the general term.                      **b** What is the 10th term?
- c** What is the first term below -30?
- 5** The ninth term of an AP is 28 and the 15th term is -6. What is the third term?
- 6** Evaluate  $21.9 + 25 + 28.1 + 31.2 + \dots + 65.3$ .
- 7** Find the sum of  $2p + 10q, 3p + 8q, 4p + 6q, 5p + 4q, \dots$  to 30 terms.
- 8** How many terms are required for  $21.5 + 19.6 + 17.7 + 15.8 + \dots$  to first exceed 130?
- 9** A stack of canned baked beans in a supermarket is made by arranging 4 rows of cans, one behind the other. Each of the first rows is 35 cans long. Another 4 rows are stacked on top, but each row has one less can than the first row. This is repeated until the stack is 15 rows high. How many cans are in the display?
- 10** Find a formula for the sum of the first  $n$  positive numbers with remainder 1 when divided by 3.

Example

1

Example

2

Example

3

Example

4

Example

5

Example

6

Example

7

Example

8

Example

9

Example

10

### Problem solving

- 11** The eighth term of an AP is half the first term and the 22th term is -20. What is the 14th term?
- 12** What is the sum of the numbers from 100 to 1000 inclusive that are not divisible by 11?
- 13** The sum of an AP to 21 terms is the same as the sum to 11 terms. The 63rd term is -12. What is the sum to 94 terms?
- 14** There are 1500 leaves per metre of branch on a tree. A tree 20.6 metres high has branches every 0.6 metres up the trunk, starting 2 m above the ground. The lengths of the branches follow an AP. The first branch is 5 m long and the last is 0.2 m long. Calculate the number of leaves on the tree.



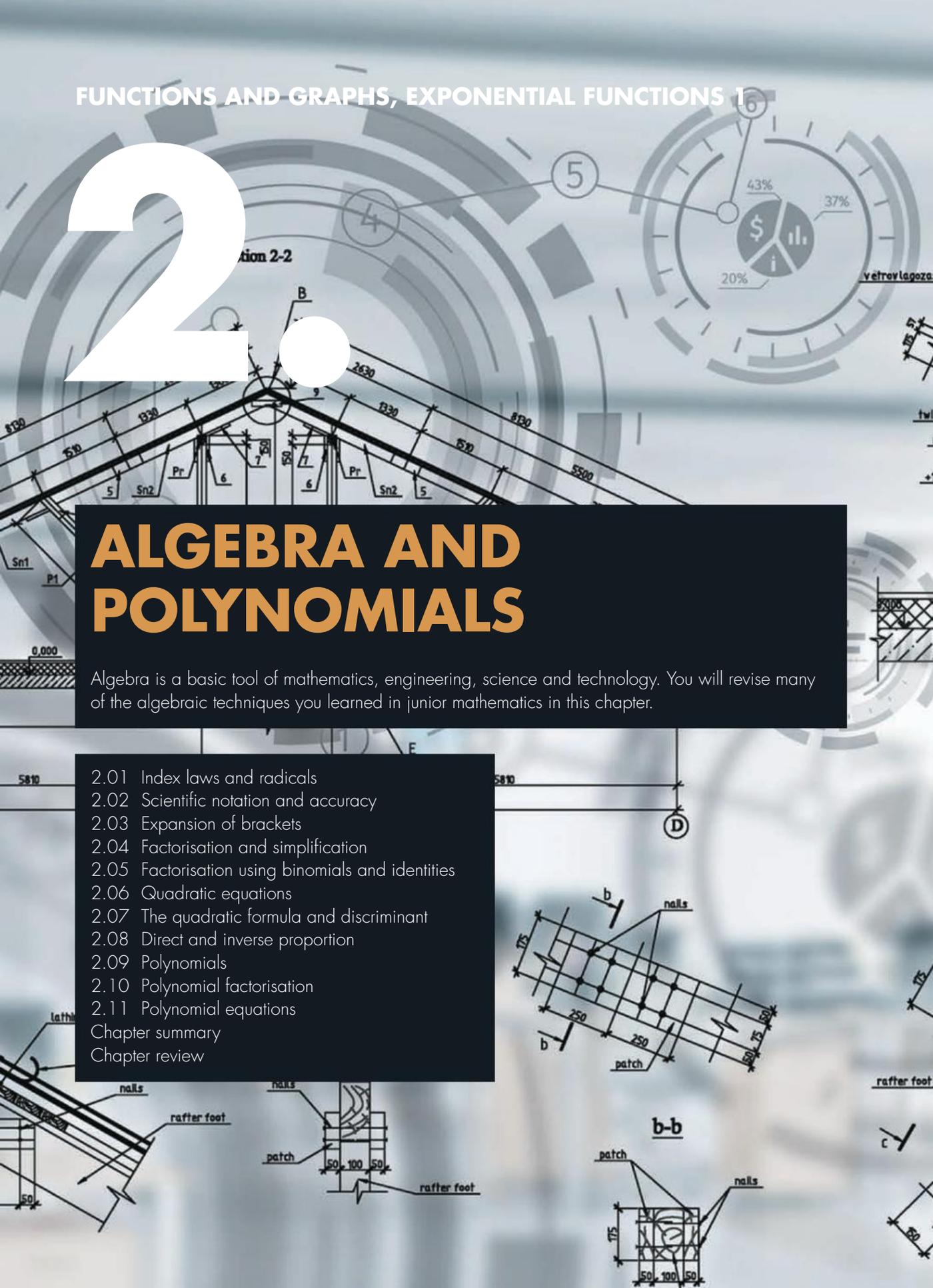
Practice quiz

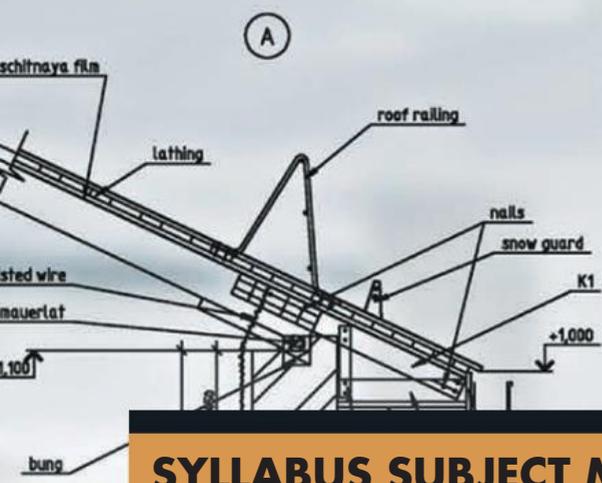
# 2

## ALGEBRA AND POLYNOMIALS

Algebra is a basic tool of mathematics, engineering, science and technology. You will revise many of the algebraic techniques you learned in junior mathematics in this chapter.

- 2.01 Index laws and radicals
- 2.02 Scientific notation and accuracy
- 2.03 Expansion of brackets
- 2.04 Factorisation and simplification
- 2.05 Factorisation using binomials and identities
- 2.06 Quadratic equations
- 2.07 The quadratic formula and discriminant
- 2.08 Direct and inverse proportion
- 2.09 Polynomials
- 2.10 Polynomial factorisation
- 2.11 Polynomial equations
- Chapter summary
- Chapter review





## SYLLABUS SUBJECT MATTER

### Review of quadratic relationships

- examine examples of quadratically related variables
- solve quadratic equations algebraically using factorisation, the quadratic formula (both approximate and simplified surd form), completing the square and using technology
- understand the role of the discriminant to determine the number of solutions to a quadratic equation

### Inverse proportions

- examine examples of inverse proportion

### Powers and polynomials

- identify the coefficients and the degree of a polynomial
- expand quadratic and cubic polynomials from factors
- use the factor theorem to factorise cubic polynomials in cases where a linear factor is easily obtained
- solve cubic equations using technology, and algebraically in cases where a linear factor is easily obtained

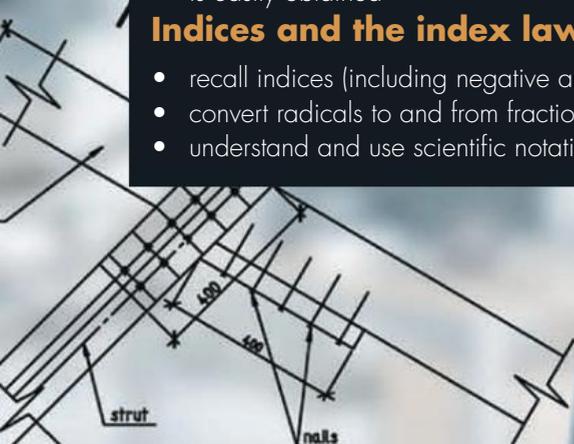
### Indices and the index laws

- recall indices (including negative and fractional indices) and the index laws
- convert radicals to and from fractional indices
- understand and use scientific notation

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Prior learning



# TERMINOLOGY

base	binomial	coefficient
constant	cubic	direct proportion
discriminant	exponent	factor theorem
highest common factor	identity	index
inverse proportion	leading term	linear
monic	null factor law	polynomial
power	quadratic	quartic
radical	remainder theorem	root
scientific notation	significant figures	variable



Review of  
index laws



Fractional indices  
and radicals

## 2.01 Index laws and radicals

Index notation is a shorthand way of expressing repeated multiplication.

$a^n$  is called a **power** where:  $a^n = a \times a \times a \times \dots \times a$  (to  $n$  factors)

$n$  is called the **index** or **exponent** and  $a$  is called the **base**.

Index laws simplify calculations with indices. Each index law is proved using the basic definition above.

### The index laws

If  $m$  and  $n$  are positive integers and  $a$  is any number:

First law:  $a^m \times a^n = a^{m+n}$

Second law:  $a^m \div a^n = a^{m-n}$

Third law:  $(a^m)^n = a^{mn}$

Fourth law:  $(ab)^n = a^n b^n$

Strictly speaking, the second law only works for  $m > n$ , as there is no meaning for a zero or negative index. However, the definition is extended for these cases as *demonstrated* below.

$$3^4 \div 3^4 = 3^{4-4} = 3^0$$

$$\text{But } 3^4 \div 3^4 = 81 \div 81 = 1$$

To be consistent  $3^0 = 1$ .

This can be repeated for any base *except* 0, so  $a^0 = 1$  when  $a \neq 0$ .

Similarly, a meaning for a negative index can be demonstrated.

$$a^0 = 1$$

Dividing by  $a^n$ :

$$\frac{a^0}{a^n} = \frac{1}{a^n} \Rightarrow a^{-n} = \frac{1}{a^n}$$

Finally, consider  $a^{\frac{1}{q}}$  with a fractional index. Then  $\left(a^{\frac{1}{q}}\right)^q = a^{\left(\frac{1}{q} \times q\right)} = a^1 = a$ .

So,  $a^{\frac{1}{q}}$  to the power  $q$  would be  $a$ . Then,  $a^{\frac{1}{q}}$  would have to be the  $q$ th root of  $a$ .

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

$$a^{\frac{p}{q}} = \sqrt[q]{\left(a^p\right)} = \left(\sqrt[q]{a}\right)^p$$

$$a^0 = 1 \quad \text{where } a \neq 0$$

$$a^{-n} = \frac{1}{a^n} \quad \text{where } a \neq 0$$

$$a^{\frac{p}{q}} = \sqrt[q]{\left(a^p\right)} = \left(\sqrt[q]{a}\right)^p \quad \text{where } p \text{ is an integer, } q \text{ is a positive integer and } a \text{ is positive.}$$

$\sqrt{\quad}$  is called the **radical** symbol.

### EXAMPLE 1

Simplify and express with positive indices.

**a**  $\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5}$

**b**  $\frac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m}$

#### Solution

- a** Simplify the numerical part.  
Apply index laws.

$$\frac{4h^{-6} \times 3h^2}{h^{-3} \times 6h^5} = \frac{12}{6} \times h^{(-6+2-(-3)-5)}$$

Evaluate and simplify.

$$= 2h^{-6}$$

Write with a positive exponent.

$$= \frac{2}{h^6}$$

- b** Express bases as products of prime factors.

$$\frac{2^{m-n} \times 3^{m-n} \times 6^n}{12^m} = \frac{2^{m-n} \times 3^{m-n} \times (2 \times 3)^n}{(2^2 \times 3)^m}$$

Apply index laws.

$$= \frac{2^{m-n} \times 3^{m-n} \times 2^n \times 3^n}{2^{2m} \times 3^m}$$

$$= 2^{(m-n+n-2m)} \times 3^{(m-n+n-m)}$$

Evaluate exponents.

$$= 2^{-m} \times 3^0$$

Apply definitions of powers.

$$= \frac{1}{2^m} \times 1$$

Evaluate and simplify.

$$= \frac{1}{2^m}$$

## EXAMPLE 2

Simplify and express with positive indices.

**a**  $x^3 \times \sqrt{x}$

**b**  $6\sqrt{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$

### Solution

**a** Apply the definition of a fractional index.  $x^3 \times \sqrt{x} = x^3 \times x^{\frac{1}{2}}$

Apply index laws.

$$= x^{3+\frac{1}{2}}$$

Evaluate exponents.

$$= x^{\frac{7}{2}}$$

**b** Express bases as products of prime factors.  $\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}} = \sqrt[6]{2^3 a^4} \times \sqrt{2^3} \times a^{-\frac{8}{3}}$

Apply the definition of a fractional index.

$$= (2^3 a^4)^{\frac{1}{6}} \times (2^3)^{\frac{1}{2}} \times a^{-\frac{8}{3}}$$

Apply index laws.

$$= 2^{\frac{3}{6}} a^{\frac{4}{6}} \times 2^{\frac{3}{2}} \times a^{-\frac{8}{3}}$$

$$= 2^{\frac{1}{2} + \frac{3}{2}} \times a^{\frac{2}{3} + \left(-\frac{8}{3}\right)}$$

Evaluate exponents and simplify.

$$= 2^2 a^{-2}$$

Write with a positive exponent.

$$= \frac{4}{a^2}$$

## Exercise 2.01 Index laws and radicals

Round answers to 4 decimal places where appropriate.

1 Express with positive indices.

**a**  $a^{-2}$

**b**  $4^{-3}$

**c**  $g^{-7}$

**d**  $5^{-4}$

**e**  $y^{-6}$

**f**  $(cd^2)^{-4}$

**g**  $4h^{-4}$

**h**  $(4h)^{-4}$

**i**  $2k^{-1}$

**j**  $7p^{-4}$

**k**  $8q^{-6}$

**l**  $(4v^{-2})^{-1}$

Example  
1

2 Express without denominators.

**a**  $\frac{1}{y^4}$

**b**  $\frac{1}{4^3}$

**c**  $\frac{1}{b^9}$

**d**  $\frac{t^4}{k^6}$

**e**  $\frac{y^2}{p^6}$

**f**  $\frac{4j^8}{i^6}$

**g**  $\frac{3y^{-2}}{m^5}$

**h**  $\frac{2m^{-1}}{z^4}$

**i**  $\frac{1}{x^{-8}}$

**j**  $\frac{2x^4}{y^{-4}}$

**k**  $\frac{3k^{-3}}{b^{-1}}$

**l**  $\frac{20r^{-9}}{5w^{-6}}$

3 Express as roots (using radical signs).

**a**  $6^{\frac{1}{2}}$

**b**  $z^{\frac{1}{4}}$

**c**  $216^{\frac{1}{3}}$

**d**  $y^{\frac{3}{5}}$

**e**  $49^{\frac{3}{2}}$

**f**  $v^{\frac{7}{3}}$

**g**  $8^{\frac{2}{3}}$

**h**  $16^{\frac{1}{4}}$

**i**  $m^{\frac{2}{2}}$

**j**  $u^{\frac{1}{4}}$

**k**  $\left(\frac{1}{10}\right)^{\frac{1}{2}}$

**l**  $(p^{-3})^{\frac{1}{5}}$

Example  
2

4 Express the following as powers.

**a**  $\sqrt{k}$

**b**  $\sqrt[4]{3}$

**c**  $\sqrt{5}$

**d**  $\sqrt[5]{4}$

**e**  $\sqrt[3]{k}$

**f**  $\sqrt{c^3}$

**g**  $(\sqrt[3]{6})^4$

**h**  $\sqrt[4]{e^3}$

**i**  $\sqrt[2]{28^4}$

**j**  $\sqrt[5]{q^6}$

**k**  $(\sqrt{8})^5$

**l**  $(\sqrt[6]{16})^7$

5 Simplify and express with positive indices.

**a**  $r^{-6} \times r^4$

**b**  $4y^{-5} \times 3y^{-4}$

**c**  $\frac{12p^6 \times 6p^{-2}}{8p^{-1} \times 3p}$

**d**  $\frac{4j^5 \times 5j^{-4}}{10j^7}$

**e**  $\frac{2m^3 \times m^{-10}}{12m^6}$

**f**  $\frac{5k^{-3}}{2k^4} \div \frac{10k^5}{k^{-2}}$

**g**  $3g^{-3}h^4 \times 5g^4h^{-6}$

**h**  $5k^3m^{-6} \times (k^{-3}m^{-4})^{-5}$

**i**  $\frac{8v^{-4}w^2 \times 4v^{10}w^{-8}}{2vw^{-9} \times v^{-6}w^3}$

**j**  $\frac{12p^{-4}q^{-3} \times (2pq^4)^2}{(3p^{-1}q^4)^{-3} \times (18p^4)^2}$

**k**  $\left(\frac{e^2f^{-4}}{e^{-5}f}\right)^3 \times \frac{4e^{-3}f^4}{3e^6f^6}$

**l**  $\frac{c^7d^8}{d^{-5}} \div \frac{(8c^{-2}d^3)^{-1}}{4^{-2}c^{-2}d^3}$

6 Simplify and express with positive indices.

a  $a^4 \times \sqrt{a}$

b  $b^2 \times \sqrt{b^3}$

c  $3n^{-3} \times 2\sqrt{n^5}$

d  $6g^5 \times 2\sqrt[3]{g^4}$

e  $4\sqrt[3]{w^2} \times 3\sqrt{p^5 w^6}$

f  $5\sqrt[3]{m^3 n^{-2}} \times 2\sqrt{m^5 n^6}$

g  $7\sqrt[4]{s^2 u^6} \times \sqrt{9s^{-8} u^3}$

h  $8\sqrt[3]{p^{12} q^{-8}} \times 3\sqrt[4]{p^{-10} q^9}$

7 Evaluate:

a  $16^{\frac{3}{4}}$

b  $9^0$

c  $8^{\frac{1}{3}}$

d  $36^0$

e  $36^{\frac{1}{2}}$

f  $27^{\frac{2}{3}}$

g  $17^0$

h  $81^{\frac{1}{2}}$

i  $49^{\frac{3}{2}}$

j  $36^{\frac{6}{4}}$

k  $64^{\frac{2}{3}}$

l  $32^{\frac{6}{5}}$

m  $8^{-\frac{2}{3}}$

n  $\left(\frac{3}{4}\right)^{-2}$

o  $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$

p  $18^0$

q  $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$

r  $216^{-\frac{2}{3}}$

s  $25^{\frac{1}{2}}$

t  $\left(6\frac{1}{4}\right)^{-\frac{1}{2}}$

u  $\left(2\frac{1}{4}\right)^{\frac{3}{2}}$

v  $4^{\frac{1}{3}} \times \sqrt[3]{16}$

w  $(0.01)^{-\frac{5}{2}}$

x  $1000^{-\frac{4}{3}}$

8 Simplify and express with positive indices.

a  $\frac{2^{n+2} \times 2^{n-3}}{2^{2n-5}}$

b  $\frac{5^{2k} \times 5^{k-1}}{5^{2k-2}}$

c  $\frac{3^{j+1} \times 2^{j+3}}{6^j}$

d  $\frac{6^{2n-m} \times 9^{n+m}}{36^{n-m}}$

e  $\frac{15^{4i-5} \times 21^{2i+1}}{35^{2i-2} \times 45^{2i-3}}$

f  $\frac{8^{2f-3} \times 16^{4-f}}{4^{f+2}}$



Scientific notation and significant figures

## 2.02 Scientific notation and accuracy

Very large and very small numbers are hard to read. They have many digits.

**Scientific notation** or **standard notation** avoids this problem.

Numbers in **scientific notation** have a **mantissa** from 1 to 10 multiplied by a power of ten. This power is the **order** of the number.

$$\begin{array}{ccc}
 25\,000\,000 = 2.5 \times 10^7 & & \\
 \begin{array}{cc} \nearrow & \uparrow \\ \text{MANTISSA} & \text{ORDER} \\ \searrow & \downarrow \end{array} & & \\
 0.006\,37 = 6.37 \times 10^{-3} & & 
 \end{array}$$



NASA/JPL-Caltech/OCIW

NOTE:

$$10^3 = 1000$$

$$10^2 = 100$$

$$10^1 = 10$$

$$10^0 = 1$$

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{100} = 0.01$$

$$10^{-3} = \frac{1}{1000} = 0.001$$

### EXAMPLE 3

Write in scientific notation:

**a** 61 000

**b** 764 000 000 000

**c** 0.0008

**d** 0.000 000 472

### Solution

**a** There is one number before the decimal part in scientific notation. Insert a decimal point after the 6. The number of digits (including zeros) after the 6 tells you the power of 10 by which 6.1 must be multiplied.  $61\ 000 = 6.1 \times 10^4$

**b** There are 11 digits after the 7. Insert a decimal point after the 7. Multiply by  $10^{11}$ .  $764\ 000\ 000\ 000 = 7.64 \times 10^{11}$

**c** The 8 is in the 4th decimal place. Write down 8. Multiply it by  $10^{-4}$ .  $0.0008 = 8 \times 10^{-4}$

**d** The 4 is in the 7th decimal place. Write the decimal point after the 4. Multiply by  $10^{-7}$ .  $0.000\ 000\ 472 = 4.72 \times 10^{-7}$

### EXAMPLE 4

Write as ordinary numbers.

**a**  $6.25 \times 10^4$

**b**  $9.1 \times 10^{-2}$

### Solution

**a** The order shows you how many places to move the decimal point. The order is positive. Move the decimal point 4 places to the right.  $6.25 \times 10^4 = 62\,500$

**b** The order is  $-2$ . Move the decimal point two places to the left.  $9.1 \times 10^{-2} = 0.091$

The *accuracy* of a measurement is clear when written in scientific notation. The number of **significant figures** is the length of the mantissa.

In any number:

- non-zero digits are always significant
- zeros are never significant except:
  - between two non-zero digits
  - at the end of a decimal number

### EXAMPLE 5

How many significant figures are in these numbers?

**a** 57 800

**b** 0.0065

**c** 3.001 25

**d** 620 035

**e** 3

**f** 3.200

### Solution

**a** Zero digits are at the end of the whole number.  $57\,800$   
There are three non-zero digits at the start. **3 significant figures.**

**b** There are two non-zero digits.  $0.0065$   
**2 significant figures.**

- |          |  |                                    |
|----------|--|------------------------------------|
| <b>c</b> | There is a non-zero digit at the start and three decimal places after the zeros. All digits are significant. | 3.001 25<br>6 significant figures. |
| <b>d</b> | Zeros are between whole numbers so all digits are significant.   | 620 035<br>6 significant figures.  |
| <b>e</b> | All whole numbers are significant.   | 3<br>1 significant figure.         |
| <b>f</b> | Zeros are at the end of a decimal number. They are significant.  | 3.200<br>4 significant figures.    |

## Exercise 2.02 Scientific notation and accuracy

1 Write in scientific notation.

- |          |                  |          |             |          |                 |
|----------|------------------|----------|-------------|----------|-----------------|
| <b>a</b> | 6300             | <b>b</b> | 142 000 000 | <b>c</b> | 0.0075          |
| <b>d</b> | 0.000 000 938    | <b>e</b> | 500 000     | <b>f</b> | 0.0006          |
| <b>g</b> | 210              | <b>h</b> | 0.79        | <b>i</b> | 31 090 000      |
| <b>j</b> | 0.040 62         | <b>k</b> | 9 623 000   | <b>l</b> | 0.000 000 002   |
| <b>m</b> | 40               | <b>n</b> | 5.6         | <b>o</b> | 0.5             |
| <b>p</b> | 4000             | <b>q</b> | 0.007       | <b>r</b> | 302 000 000 000 |
| <b>s</b> | 0.000 000 000 74 | <b>t</b> | 102 000     | <b>u</b> | 0.000 053       |
| <b>v</b> | 27 100           | <b>w</b> | 58          | <b>x</b> | 0.095           |

Example  
3

2 Write as ordinary numbers.

- |          |                    |          |                      |          |                        |          |                        |
|----------|--------------------|----------|----------------------|----------|------------------------|----------|------------------------|
| <b>a</b> | $4.3 \times 10^5$  | <b>b</b> | $2.1 \times 10^{-3}$ | <b>c</b> | $9.84 \times 10^6$     | <b>d</b> | $3.19 \times 10^{-1}$  |
| <b>e</b> | $6 \times 10^{-9}$ | <b>f</b> | $2 \times 10^4$      | <b>g</b> | $4.2 \times 10^1$      | <b>h</b> | $5.8 \times 10^{-2}$   |
| <b>i</b> | $1.6 \times 10^0$  | <b>j</b> | $9.5 \times 10^{-5}$ | <b>k</b> | $4.64 \times 10^2$     | <b>l</b> | $2.1 \times 10^3$      |
| <b>m</b> | $5 \times 10^4$    | <b>n</b> | $3 \times 10^{-1}$   | <b>o</b> | $4.08 \times 10^{-6}$  | <b>p</b> | $9.842 \times 10^{-2}$ |
| <b>q</b> | $7 \times 10^4$    | <b>r</b> | $6.03 \times 10^7$   | <b>s</b> | $5.106 \times 10^{-3}$ | <b>t</b> | $9 \times 10^{-7}$     |
| <b>u</b> | $3.24 \times 10^8$ | <b>v</b> | $1.63 \times 10^4$   | <b>w</b> | $2.95 \times 10^3$     | <b>x</b> | $8.3 \times 10^{-1}$   |

Example  
4

3 Simplify and write in scientific notation.

- |          |   |          |  |
|----------|---|----------|--|
| <b>a</b> | $4.6 \times 10^{-8} \times 3.2 \times 10^9$                           | <b>b</b> | $6.1 \times 10^9 \div (8.34 \times 10^{-2})$                     |
| <b>c</b> | $5.64 \times 10^7 \times 3.9 \times 10^{11}$                          | <b>d</b> | $8 \times 10^6 \times 2 \times 10^{-9} \div (4 \times 10^3)$     |
| <b>e</b> | $7.1 \times 10^5 \div (1.6 \times 10^{-9}) \times 2.3 \times 10^{-4}$ | <b>f</b> | $4.15 \times 10^9 + 3.84 \times 10^7$                            |
| <b>g</b> | $8.63 \times 10^{-8} - 1.96 \times 10^{-12}$                          | <b>h</b> | $4.6 \times 10^{11} \times 3.1 \times 10^{-5} + 5.8 \times 10^7$ |

4 Simplify and write in scientific notation.

<b>a</b>	$\frac{4861 \times 2 \times 10^6}{3.58 \times 10^{-5}}$	<b>b</b>	$\frac{3.1 \times 10^6 \times 8.2 \times 10^{12}}{5.8 \times 10^{11} \times 0.067}$	<b>c</b>	$\frac{0.5 \times 1.8 \times 10^{-3}}{2.4 \times 10^{-6} \times 3.2 \times 10^{11}}$
<b>d</b>	$\frac{9.85 \times 10^9 + 9.84 \times 10^9}{9.85 \times 10^{-9} - 9.84 \times 10^{-9}}$	<b>e</b>	$\frac{4.71 \times 10^6 + 2.95 \times 10^7}{3.5 \times 10^9 \times \pi}$	<b>f</b>	$\frac{\pi \times 5.3 \times 10^{-6}}{2 \times 10^9 \times 3.604 \times 10^{18}}$

Example

5

5 State the number of significant figures.

<b>a</b>	24 700	<b>b</b>	0.000 35	<b>c</b>	700	<b>d</b>	0.000 005
<b>e</b>	10	<b>f</b>	9.65	<b>g</b>	3.047	<b>h</b>	105 030 000
<b>i</b>	510 000	<b>j</b>	0.050 63	<b>k</b>	0.000 087 5	<b>l</b>	1.8
<b>m</b>	0.359	<b>n</b>	0.059	<b>o</b>	76 060 000	<b>p</b>	0.000 300 45
<b>q</b>	100 047	<b>r</b>	0.063	<b>s</b>	1.063	<b>t</b>	2000
<b>u</b>	0.000 000 005 08	<b>v</b>	2 030 405 000	<b>w</b>	19	<b>x</b>	908

6 Round off the following numbers to the number of significant figures (s.f.) in brackets.

<b>a</b>	42 (1 s.f.)	<b>b</b>	475 (2 s.f.)	<b>c</b>	247.6 (3 s.f.)
<b>d</b>	0.000 006 03 (2 s.f.)	<b>e</b>	24 641 (1 s.f.)	<b>f</b>	3.6084 (2 s.f.)
<b>g</b>	63 480 (3 s.f.)	<b>h</b>	18 946 724 965 (4 s.f.)	<b>i</b>	36.000 000 478 (3 s.f.)

## 2.03 Expansion of brackets

The **distributive law** is used to expand and factorise expressions.

### Distributive law

For any real numbers  $a$ ,  $b$  and  $c$ ,

$$a(b + c) = ab + ac$$

The term outside the brackets is multiplied by each term inside the brackets.

You progressively remove brackets using the distributive law to **expand** an algebraic expression. Then you **simplify** and collect like terms.

You can use the **FOIL** method as a shortcut for two brackets with two terms each. This is called a **binomial product**.

### FOIL method

First terms, Outer terms, Inner terms, Last terms

$$(a + b)(c + d) = ac + ad + bc + bd$$



Expanding algebraic expressions



Algebra using diagrams



Factorising using diagrams



Expanding binomial products

## EXAMPLE 6

Expand and simplify:

**a**  $4(3x - 5)$

**b**  $-2(6a - 3)$

**c**  $(2c - 3)(4c + 5)$

**d**  $(2x - 1)(3x + 4y - 5)$

### Solution

**a** Expand using the distributive law.

$$4(3x - 5) = 4 \times 3x + 4 \times (-5)$$

Simplify terms.

$$= 12x - 20$$

**b** You can use brackets around negatives.

$$-2(6a - 3) = (-2) \times 6a + (-2) \times (-3)$$

Simplify terms.

$$= -12a + 6$$

**c** *Method 1:* Distributive law

Expand using the distributive law.

$$(2c - 3)(4c + 5)$$

$$= 2c(4c + 5) + (-3)(4c + 5)$$

$$= 2c \times 4c + 2c \times 5 + (-3) \times 4c + (-3) \times 5$$

Simplify terms.

$$= 8c^2 + 10c - 12c - 15$$

Collect like terms.

$$= 8c^2 - 2c - 15$$

*Method 2:* FOIL

Expand using FOIL.

$$(2c - 3)(4c + 5)$$

$$= 2c \times 4c + 2c \times 5 + (-3) \times 4c + (-3) \times 5$$

Simplify terms.

$$= 8c^2 + 10c - 12c - 15$$

Collect like terms.

$$= 8c^2 - 2c - 15$$

**d** Expand using the distributive law.

$$(2x - 1)(3x + 4y - 5)$$

$$= 2x(3x + 4y - 5) + (-1)(3x + 4y - 5)$$

Expand again and simplify.

$$= 6x^2 + 8xy - 10x - 3x - 4y + 5$$

Collect like terms.

$$= 6x^2 + 8xy - 13x - 4y + 5$$

You should already know the special binomial products below.



Special  
binomial  
products

### Perfect squares

$$(a + b)^2 = a^2 + 2ab + b^2$$

### Difference of two squares

$$(a + b)(a - b) = a^2 - b^2$$

### EXAMPLE 7

Expand and simplify:

**a**  $(2x + 1)^2$

**b**  $(y - 3)^2$

**c**  $(3a - 4)(3a + 4)$

**d**  $(x + 5 + y)(x - 5 - y)$

### Solution

**a** Use perfect squares.

$$(2x + 1)^2 = (2x)^2 + 2(2x \times 1) + (1)^2$$

Simplify terms.

$$= 4x^2 + 4x + 1$$

**b** Use perfect squares.

$$(y - 3)^2 = (y)^2 + 2(-3 \times y) + (-3)^2$$

Simplify terms.

$$= y^2 - 6y + 9$$

**c** Use the difference of squares.

$$(3a + 4)(3a - 4) = (3a)^2 - (4)^2$$

$$= 9a^2 - 16$$

**d** Use the difference of squares.

$$(x + 5 + y)(x - 5 - y)$$

$$= [x + (5 + y)][x - (5 + y)]$$

$$= (x)^2 - (5 + y)^2$$

Use perfect squares.

$$= x^2 - [5^2 + 2(5 \times y) + y^2]$$

$$= x^2 - 5^2 - 10y - y^2$$

$$= x^2 - 10y - y^2 - 25$$

### Perfect cubes

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

## EXAMPLE 8

Expand  $(2x - 3)^3$ .

### Solution

Write the expression.

$$(2x - 3)^3$$

Use the cubic identity.

$$= (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3$$

Simplify.

$$= 8x^3 - 36x^2 + 54x - 27$$

You can also expand this by using the perfect squares identity and multiplying through.

$$\begin{aligned}(2x - 3)^3 &= (2x - 3)(2x - 3)^2 \\ &= (2x - 3)(4x^2 - 12x + 9) \\ &= 8x^3 - 36x^2 + 54x - 27\end{aligned}$$

## Exercise 2.03 Expansion of brackets

1 Expand and simplify:

**a**  $5(3h + 6)$

**c**  $-3q(4q - 8) + 2q(q + 1)$

**e**  $4t(5 - 2x + 7y)$

**g**  $5y(y - 2) - 6(y - 3)$

**i**  $4(2bk + 7) - 3bk(2b - bk)$

**k**  $2f(g - f) - 3g(g - f^3)$

**b**  $-3n(2m - 4)$

**d**  $7h(-3g^2 + 7h)$

**f**  $3hj(2j - 5h + 1)$

**h**  $-2y^4(3 - x - 2b - 6z^2)$

**j**  $-5yz^2(2x^4 - 5y + 3xy)$

**l**  $3p(p - q^3) - 2q(q - 4p^4)$

2 Expand and simplify:

**a**  $(x + y)(2x + 4 + 3y)$

**c**  $(2a - 1)(3a + 4b + 6)$

**e**  $-(2v - 4w)(v + w + c)$

**b**  $(2x - 3y)(3x - 4y + 5)$

**d**  $(3c + d)(2c - d - 1)$

**f**  $-3b(b + a)(b + 3a - 5)$

3 Expand and simplify:

**a**  $(3r - 4)(3r + 2)(r - 1)$

**c**  $(m + 5f)^2(m - 3f)$

**e**  $5(3v + 7)(v - 4)^2$

**g**  $(4q - 5p + 8)(q - p + 4)$

**i**  $(6y - 3z + 1)(2y + 3z - 1)$

**k**  $(c^2 + 5c + 7)(c^3 - 2c^2 + 4c - 1)$

**b**  $(2t - g)(t + g)(3t - 4g)$

**d**  $(5k - 2a)^2(k - 3a)$

**f**  $7b(2b + 1)(b - 4)(3b - 6)$

**h**  $(2g - h - 4)(4g + 3h - 9)$

**j**  $(2z^2 - 3z + 4)(z^2 + 5z + 2)$

**l**  $3cd^2(5c - 4d)(3c^2 - 2d^3 + 2)$

Example  
6

**4** Expand and simplify each expression.

**a**  $(a + 1)(3a - 4)$

**c**  $(4h + 3)(7h + 5)$

**e**  $(7m - 8)(6m + 7)$

**g**  $(2p - 5g)(5p - 3g)$

**i**  $(5u + 3w)(7u - 6w)$

**k**  $(5e - 6f)(5e - 7f)$

**b**  $(3v + 4)(5v - 8)$

**d**  $(4g - 7)(9g + 2)$

**f**  $(a - b)(5a - 3b)$

**h**  $(3t + j)(7t + 3j)$

**j**  $(8x - 5y)(8x + 9y)$

**l**  $(5z - 7u)(7z + 4u)$

Example  
**7**

**5** Expand and simplify:

**a**  $(a - b)^2$

**d**  $(3y - 4)^2$

**g**  $(4h + 3m)^2$

**j**  $(2f - 9)^2$

**b**  $(k - m)^2$

**e**  $(5f + 4)^2$

**h**  $(2p - 5j)^2$

**k**  $(8v + e)^2$

**c**  $(x + y)^2$

**f**  $(2a - 7)^2$

**i**  $(7a + 5b)^2$

**l**  $(6f - 10c)^2$

**6** Expand and simplify:

**a**  $(a - b)(a + b)$

**c**  $(g + 3h)(g - 3h)$

**e**  $(3q - 4)(3q + 4)$

**g**  $(3p + 7q)(3p - 7q)$

**i**  $(5 - x)(5 + x)$

**b**  $(x + y)(x - y)$

**d**  $(y - 6)(y + 6)$

**f**  $(4r - 5s)(4r + 5s)$

**h**  $(8i - 3j)(8i + 3j)$

**j**  $(8 + 5k)(8 - 5k)$

**7** Expand the following.

**a**  $(x + y + 1)(x + y - 1)$

**c**  $(3t + s + 2)(3t - s - 2)$

**b**  $(a + 2b + 4)(a + 2b - 4)$

**d**  $(4w + 5 + z)(4w - z - 5)$

Example  
**8**

**8** Expand the following.

**a**  $(x + a)^3$

**e**  $(3v - 4w)^3$

**i**  $(2x + 1)^3$

**b**  $(x - 5)^3$

**f**  $(x + 3)^3$

**j**  $(x + 7)^3$

**c**  $(3x - 2)^3$

**g**  $(2m + 3n)^3$

**k**  $(x + 2y)^3$

**d**  $(x + 1)^3$

**h**  $(x - 4)^3$

**l**  $(x^2 - 3y^2)^3$

**9** Expand the following.

**a**  $(a + b)(a^2 - ab + b^2)$

**c**  $(a - b)(a^2 + ab + b^2)$

**e**  $(z + 1)(z^2 - z + 1)$

**g**  $(p + 5)(p^2 - 5p + 25)$

**i**  $(3u - 4v)(9u^2 + 12uv + 16v^2)$

**b**  $(m + n)(m^2 - mn + n^2)$

**d**  $(x - y)(x^2 + xy + y^2)$

**f**  $(d - 3)(d^2 + 3d + 9)$

**h**  $(2 + 3k)(4 - 6k + 9k^2)$

**j**  $(5e - 2i)(25e^2 + 10ei + 4i^2)$

## 2.04 Factorisation and simplification

The opposite of expansion of brackets is **factorisation**. To factorise an expression, look for the **highest common factor**. This is the largest term that divides into all terms. Put it outside the brackets. Then put the other factors of each term inside the brackets.

### EXAMPLE 9

Factorise by finding common factors.

**a**  $8x + 12$

**b**  $6xy - 9y^2$

**c**  $3(x + 6) - 2a(x + 6)$

**d**  $6x - 2z + xz - 12$

### Solution

**a** Highest common factor is 4.

$$8x + 12 = 4 \times 2x + 4 \times 3$$

Take it outside the brackets.

$$= 4(2x + 3)$$

**b** Highest common factor is  $3y$ .

$$6xy - 9y^2 = 3y \times 2x + 3y \times (-3y)$$

Take it outside the brackets.

$$= 3y(2x - 3y)$$

**c** Highest common factor is  $(x + 6)$ .

$$3(x + 6) - 2a(x + 6)$$

Take it outside the brackets.

$$= (x + 6) \times 3 + (x + 6) \times (-2a)$$

$$= (x + 6)(3 - 2a)$$

**d** Group the terms into pairs with common factors.

$$6x - 2z + xz - 12 = 6x - 12 + xz - 2z$$

Highest common factor of  $6x - 12$  is 6.

$$= 6(x - 2) + z(x - 2)$$

Highest common factor of  $xz - 2z$  is  $z$ .

$x - 2$  is the highest common factor.

$$= (x - 2)(6 + z)$$

Take it outside the brackets.

### EXAMPLE 10

Factorise each expression.

**a**  $15ab - 3a^2b^2 + 4b^3$

**b**  $12p^3 - 8p^2q^2$

### Solution

**a**  $b$  is the largest factor of all terms.

$$15ab - 3a^2b^2 + 4b^3$$

$$= b(15a - 3a^2b + 4b^2)$$

**b**  $4p^2$  is the largest common factor.

$$12p^3 - 8p^2q^2 = 4p^2(3p - 2q^2)$$

You simplify algebraic fractions in the same way as ordinary fractions. Cancel common factors of the numerator and denominator. You might need to factorise the numerator or denominator first.

### EXAMPLE 11

Simplify each expression.

a  $\frac{7a-14}{56}$

b  $\frac{5m-3}{3} - \frac{2m+8}{4}$

c  $\frac{4}{2v-3} - \frac{3}{4v+6}$

d  $\frac{3p}{p+3} + \frac{3p-3}{2p^2+6p}$

### Solution

- a Highest common factor is 7. Factorise the top and cancel.

$$\begin{aligned}\frac{7a-14}{56} &= \frac{7^1(a-2)}{56^8} \\ &= \frac{a-2}{8}\end{aligned}$$

- b Lowest common multiple of 4 and 3 is 12. Express both fractions with a denominator of 12.

$$\begin{aligned}\frac{5m-3}{3} - \frac{2m+8}{4} \\ &= \frac{4(5m-3)}{12} - \frac{3(2m+8)}{12}\end{aligned}$$

Expand brackets in the numerator and simplify.

$$= \frac{20m-12-6m-24}{12}$$

$$= \frac{14m-36}{12}$$

Factorise and cancel.

$$\begin{aligned}&= \frac{2^1(7m-18)}{2^6} \\ &= \frac{7m-18}{6}\end{aligned}$$

- c** Express each fraction with the lowest common multiple (the product) of each denominator  $(2v - 3)(4v + 6)$ .

Expand brackets in the numerator and simplify.

Factorise and cancel.

You can use the difference of squares identity in the denominator.

- d** Invert the second fraction and multiply.

Factorise each fraction and cancel common factors.

Write the simplified fractions.

Multiply.

$$\begin{aligned} & \frac{4}{2v-3} - \frac{2}{4v+6} \\ &= \frac{4(4v+6)}{(2v-3)(4v+6)} - \frac{2(2v-3)}{(4v+6)(2v-3)} \\ &= \frac{16v+24-4v+6}{(2v-3)(4v+6)} \\ &= \frac{12v+30}{(2v-3)(4v+6)} \\ &= \frac{\cancel{6}^3(2v+5)}{\cancel{2}^1(2v-3)(2v+3)} \\ &= \frac{3(2v+5)}{4v^2-9} \end{aligned}$$

$$\begin{aligned} & \frac{3p}{p+3} \div \frac{3p-3}{2p^2+6p} \\ &= \frac{3p}{p+3} \times \frac{2p^2+6p}{3p-3} \\ &= \frac{\cancel{3}p}{\cancel{p+3}} \times \frac{2p(\cancel{p+3})}{\cancel{3}(p-1)} \\ &= \frac{p}{1} \times \frac{2p}{(p-1)} \\ &= \frac{2p^2}{p-1} \end{aligned}$$

## Exercise 2.04 Factorisation and simplification

- 1** Factorise the following.

**a**  $6a + 8$

**d**  $x^2 + x$

**g**  $a^2b + ab$

**j**  $2a^2b - 4ab$

**b**  $18m - 16$

**e**  $m^2 + 4m$

**h**  $cd^2 - c^2d$

**k**  $10a^2b + 12ab^2$

**c**  $-14x - 7y$

**f**  $2x^2 - 4x$

**i**  $24m^2 + 8mn$

**l**  $36ab^2 + 48a^2b$

Example  
**9**

**2** Factorise the following.

**a**  $2(a + b) + x(a + b)$

**c**  $(2 + a) - 3(2 + a)^2$

**e**  $ac + bc - a - b$

**g**  $15mp + 9mq - 10np - 6nq$

**i**  $gi - hj + gi - hi$

**b**  $3a^2(x + y) - 12a(x + y)$

**d**  $12a^2c(3b - 2)^2 + 4a(3b - 2)$

**f**  $5gi - 2gj - 5hi + 2hj$

**h**  $8cd + 20c - 12d - 30$

**j**  $km^2 - 2nh^2 - 2nm^2 + kh^2$

**3** Factorise by grouping.

**a**  $ac + bc - a - b$

**c**  $15mp + 9mq - 10np - 6nq$

**e**  $x^3 - x^2 + x - 1$

**g**  $6x^3 + 2xy^2 - 9x^2y - 3y^3$

**i**  $gi - hj + gj - hi$

**b**  $3gi - 2gj - 6hi + 4hj$

**d**  $8cd + 20c - 12d - 30$

**f**  $8gh^2 - 4g^2 - 6h^3 + 3gh$

**h**  $3h^3 - 9hi + 2h^2i - 6i^2$

**j**  $a^3c - 3ab + a^4 - 3bc$

**4** Factorise:

**a**  $6a - 15b$

**c**  $12a^2c^2 - 4abc$

**e**  $12m^4n^2 - 18m^2n^3 + 30mn^2$

**b**  $12c^2 - 20c$

**d**  $12a^2b - 8ab^2$

**f**  $40x^3y - 24x^2y^2 + 64x^2y$

**5** Factorise and simplify each expression.

**a**  $\frac{3p - 12q}{24}$

**c**  $\frac{8c^2 + 24cd}{24cd}$

**e**  $\frac{2v - 3}{2} + \frac{v - 4}{3}$

**g**  $\frac{v}{v+1} + \frac{3}{v}$

**i**  $\frac{2w}{w+1} + \frac{3w}{w-1}$

**b**  $\frac{10b - 35}{40}$

**d**  $\frac{6p^2 - 9pq}{8pq - 12q^2}$

**f**  $\frac{6q - 5}{4} - \frac{2q - 3}{5}$

**h**  $\frac{3}{2f - g} + \frac{5}{3f + g}$

**j**  $a + \frac{a^2}{a+5}$

**6** Factorise and simplify each expression.

**a**  $\frac{a}{2(a+b)} \times \frac{a+b}{a}$

**b**  $\frac{2x+8}{3x-9} \times \frac{7x-21}{5x+10}$

**c**  $\frac{m-1}{5m} \div \frac{m^2-1}{m^2}$

**d**  $\frac{b^2-16}{b+4} \times \frac{7}{2b-8}$

**e**  $\frac{4t^2-6t}{6t-12} \div \frac{8t-12}{2t^2-4t}$

**f**  $\frac{dx^2-9d}{x^2+3x} \times \frac{5x}{2dx-60d}$

**g**  $\frac{q^2-4q}{10q-2} \div \frac{q^2-16}{3q+12}$

**h**  $\frac{3p}{2p+3} \div \frac{3p-3}{4p^2+6p} \times \frac{p-1}{4p^3}$

Example  
**10**

Example  
**11**

## 2.05 Factorisation using binomials and identities



Factorising algebraic expressions



Factorising quadratic expressions



Factorising quadratic expressions (Advanced)



Excel spreadsheet: Factorising trinomials



Excel worksheet: Factorising trinomials

### EXAMPLE 12

Factorise these quadratics:

a  $x^2 - 11x + 24$

b  $20x^2 - 31x - 9$

### Solution

#### a *Decomposition method*

Consider the quadratic  $x^2 - 11x + 24$ . The coefficient of the middle term is  $-11$ .

The product of the coefficients of the first and last terms is  $+24$ .

Two numbers are needed that add to  $-11$  and multiply to  $+24$ . Since the product is positive, the numbers must have the same sign. As they add to give  $-11$ , both must be negative.

Find numbers whose product is 24.

$$-24 \times (-1) \quad \text{or} \quad -12 \times (-2) \quad \text{or}$$

$$-3 \times (-8) \quad \text{or} \quad -6 \times (-4)$$

Select the pair that adds to  $-11$ .

$$-3 + (-8) = -11$$

Write the expression.

$$x^2 - 11x + 24$$

Decompose  $-11x$  using  $-3 + (-8) = -11$ .

$$= x^2 - 3x + -8x + 24$$

Factorise. Be careful with signs!

$$= x(x - 3) - 8(x - 3)$$

Factorise again: common factor is  $(x - 3)$ .

$$= (x - 8)(x - 3)$$

#### *Cross method*

Consider the quadratic  $x^2 - 11x + 24$ .

Possible factors of 1 are  $1 \times 1$  and  $-1 \times (-1)$ .

Possible factors of 24 are  $24 \times 1$ ,  $-24 \times (-1)$ ,  $8 \times 3$ , and so on.

Set up the 'cross' with  $1 \times 1$  on the left, and try the factors of the second term, on the right. The coefficients of the first terms are 1 and 1. The coefficients of the last terms are  $-8$  and  $(-3)$ .

Write the result.

1	<del>6</del>	-8	
1	-10	-11	That's it!
1	<del>4</del>	-3	
1	-6	-8	→ $(1x + -8)$
1	-10	-11	
1	<del>4</del>	-3	→ $(1x + -3)$

$$x^2 - 11x + 24 = (x - 8)(x - 3)$$

**b Decomposition (fraction) method**

Consider the quadratic  $20x^2 - 31x - 9$ . The coefficient of the middle term is  $-31$ .

The product of the coefficients of the first and last terms is  $-180$ .

$a + b = -31$  and  $ab = -180$  are needed.  $a$  and  $b$  must have opposite signs.

$$180 = 180 \times 1 = 90 \times 2 = 60 \times 3 = 45 \times 4 = 36 \times 5 = 30 \times 6 = 20 \times 9 = 18 \times 10 = 15 \times 12$$

Choose the pair that gives the correct sum.

$$5 + (-36) = -31$$

Write the fraction using 5 and  $-36$  with  $20x$  (from  $20x^2$ ) and denominator 20.

$$20x^2 - 31x - 9 = \frac{(20x - 36)(20x + 5)}{20}$$

Factorise each factor on the top.

$$= \frac{4(5x - 9) \times 5(4x + 1)}{20}$$

Cancel.

$$= (5x - 9)(4x + 1)$$

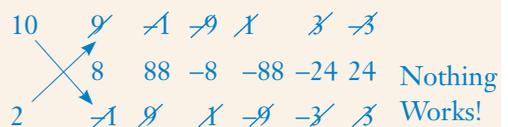
**Cross method**

Consider the quadratic  $20x^2 - 31x - 9$ .

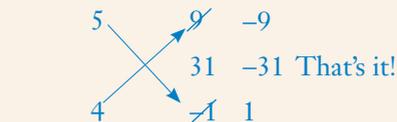
Possible factors of 20 are  $20 \times 1$ ,  $-20 \times (-1)$ ,  $10 \times 2$ ,  $-10 \times (-2)$ ,  $5 \times 4$  and  $-5 \times (-4)$ .

Possible factors of  $-9$  are  $9 \times -1$ ,  $-9 \times 1$  and  $3 \times -3$ .

Set up a 'cross' with  $10 \times 2$  on the left, and try factors of the second term on the right.



Try a new 'cross' with  $5 \times 4$  on the left. The binomials are  $[5x + (-9)]$  and  $(4x + 1)$ .



Write the result.

$$20x^2 - 31x - 9 = (5x - 9)(4x + 1)$$

To factorise **perfect squares** or **difference of squares**, use the identities:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - b^2 = (a + b)(a - b)$$

### EXAMPLE 13

Factorise each expression.

**a**  $4k^2 - 12k + 9$

**b**  $16m^2 - 25$

#### Solution

**a** Write in the form  $a^2 + 2ab + b^2$ .

Take care with the signs.

Use the perfect square identity.

$$4k^2 - 12k + 9 = (2k)^2 + 2(2k)(-3) + (-3)^2$$

$$= (2k - 3)^2$$

**b** Write in the form  $a^2 - b^2$ .

Use the difference of squares identity.

$$16m^2 - 25 = (4m)^2 - (5)^2$$

$$= (4m + 5)(4m - 5)$$

To factorise a **sum of cubes** or **difference of cubes** use these identities:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

### EXAMPLE 14

Factorise each expression.

**a**  $27x^3 + 1$

**b**  $x^3 - 8$

#### Solution

**a** Write in the form  $a^3 + b^3$ .

Use the cubic identity.

Simplify.

$$27x^3 + 1 = (3x)^3 + (1)^3$$

$$= [(3x) + (1)][(3x)^2 - (3x)(1) + (1)^2]$$

$$= (3x + 1)(9x^2 - 3x + 1)$$

**b** Write in the form  $a^3 - b^3$ .

Use the cubic identity.

Simplify.

$$x^3 - 8 = (x)^3 - (2)^3$$

$$= [(x) - (2)][(x)^2 + (x)(2) + (2)^2]$$

$$= (x - 2)(x^2 + 2x + 4)$$



Sum and  
difference of  
cubes

## Exercise 2.05 Factorisation using binomials and identities

Example  
12

1 Factorise the following.

**a**  $3k^2 - 11k + 10$

**b**  $v^2 + 6v - 16$

**c**  $5m^2 + 6mn + n^2$

**d**  $7p^2 - 24p + 9$

**e**  $7x^2 + xy - 6y^2$

**f**  $2z^2 - 11z + 12$

**g**  $3m^2 + 4my - 32y^2$

**h**  $j^2 + jk - 12k^2$

**i**  $7v^2 + 19vw - 6w^2$

**j**  $3c^2 + 19c + 28$

**k**  $3x^2 - 8xy + 5y^2$

**l**  $5e^2 - ef - 4f^2$

**m**  $5z^2 + 36z + 7$

**n**  $3c^2 - 14cd - 49d^2$

**o**  $2n^2 + 3n - 35$

**p**  $3y^2 - 23y + 30$

**q**  $7g^2 - 52gh - 32h^2$

**r**  $5m^2 - 44mn + 32n^2$

**s**  $3a^2 - 31a + 56$

**t**  $7x^2 - 9xy + 2y^2$

**u**  $7m^2 - 30mp + 27p^2$

**v**  $3z^2 + 4yz - 15y^2$

**w**  $c^2 - 4cd - 12d^2$

**x**  $3r^2 - rs - 4s^2$

2 Factorise the following.

**a**  $3x^2 - 11x + 10$

**b**  $x^2 + 6x - 16$

**c**  $7x^2 + x - 6$

**d**  $2x^2 - 11x + 12$

**e**  $4x^2 - 13x - 12$

**f**  $8x^2 + 18x + 9$

**g**  $5x^2 + 36x + 7$

**h**  $3x^2 - 14x - 49$

**i**  $7x^2 - 52x - 32$

**j**  $5x^2 - 44x + 32$

**k**  $20x^2 - 37x - 18$

**l**  $25x^2 - 30x - 16$

Example  
13

3 Factorise each expression.

**a**  $x^2 + 8xy + 16y^2$

**b**  $4b^2 - 20bc + 25c^2$

**c**  $n^2 - 49$

**d**  $25p^2 + 70pq + 49q^2$

**e**  $49b^2 - 56b + 16$

**f**  $81v^2 - 121w^2$

**g**  $36g^2 - 12g + 1$

**h**  $h^2 - 64$

**i**  $100 - 81g^2$

**j**  $144m^2 + 120em + 25e^2$

**k**  $9y^2 - 48y + 64$

**l**  $81a^2 - 126ab + 49b^2$

**m**  $100p^2 + 260pr + 169r^2$

**n**  $64g^2 - 121$

Example  
14

4 Factorise each expression.

**a**  $t^3 - u^3$

**b**  $64t^3 + 27u^3$

**c**  $8j^3 - m^3$

**d**  $f^3 + 8$

**e**  $h^3 - 64$

**f**  $8k^3 + 27$

**g**  $216j^3 - 125k^3$

**h**  $w^3 + 343$

**i**  $64d^3 + e^3$

**j**  $1 - 27m^6$

## 2.06 Quadratic equations

In a quadratic equation, the highest power of the **variable** is 2.

### Standard form of the quadratic equation

The standard (general) form of the quadratic equation is

$$ax^2 + bx + c = 0 \text{ where } a \neq 0.$$

You can solve the quadratic equation  $ax^2 + bx + c = 0$  by factorising and using the **null factor law**.

### Null factor law

If  $ab = 0$ , then either  $a = 0$ ,  $b = 0$  or  $a = b = 0$ .



### EXAMPLE 15

Solve the following quadratic equations.

**a**  $(x - 3)(x + 4) = 0$     **b**  $5p^2 + 2p - 10 = 2p^2 + 14p - 1$     **c**  $\frac{m-12}{m+5} - \frac{m}{2} = 2m$

### Solution

**a** Write the equation.

$$(x - 3)(x + 4) = 0$$

Apply the null factor law.

$$(x - 3) = 0 \text{ or } (x + 4) = 0$$

Solve the resulting equations.

$$x = 3 \text{ or } x = -4$$

**b** Write the equation.

$$5p^2 + 2p - 10 = 2p^2 + 14p - 1$$

Rearrange and express in standard form.

$$3p^2 - 12p - 63 = 0$$

Take out the common factor.

$$3(p^2 - 4p - 21) = 0$$

Factorise the quadratic.

$$3(p - 7)(p + 3) = 0$$

Solve using the null factor law.

$$p = 7 \text{ or } p = -3$$

**c** Write the equation.

$$\frac{m-12}{m+5} - \frac{m}{2} = 2m$$

Eliminate the denominators.

$$2 \cancel{(m+5)} \times \frac{m-12}{\cancel{m+5}} - \cancel{2}(m+5) \times \frac{m}{\cancel{2}} = 2(m+5) \times (2m)$$

$$2 \times (m - 12) - (m + 5) \times m = 2(m + 5) \times (2m)$$

Simplify.	$2m - 24 - m^2 - 5m = 4m^2 + 20m$
Collect like terms.	$5m^2 + 23m + 24 = 0$
Factorise.	$(5m + 8)(m + 3) = 0$
Solve for $m$ .	$m = -1.6$ or $m = -3$



Completing  
the square

**Completing the square** is an alternative method of solving quadratic equations if factorisation fails. It relies on the factorisation  $x^2 + 2ax + a^2 = (x + a)^2$  to make a perfect square of the variable parts of the equation.

### Completing the square

- 1 Start with the standard form of the quadratic equation.
- 2 If necessary, divide to make the coefficient of  $x^2$  equal to 1.
- 3 Put the constant term on the right-hand side.
- 4 Find  $a$  [from the pattern  $x^2 + 2ax + a^2 = (x + a)^2$ ] by halving the coefficient of  $x$ .
- 5 Add  $a^2$  to both sides.
- 6 Write the left-hand side as a perfect square.
- 7 Take the square root, writing two equations to show the positive and negative cases.
- 8 Find the final solutions.

### EXAMPLE 16

Solve  $3p^2 = 12 - 15p$  correct to 2 decimal places.

#### Solution

Write the equation.

$$3p^2 = 12 - 15p$$

Express with the constant only on the RHS.

$$3p^2 + 15p = 12$$

Divide throughout by the common factor 3.

$$p^2 + 5p = 4$$

$$a = 5 \div 2 = \frac{5}{2}, \text{ so } a^2 = \frac{25}{4}$$

$$p^2 + 5p + \frac{25}{4} = 4 + \frac{25}{4}$$

Add  $a^2$  to both sides.

Use  $x^2 + 2ax + a^2 = (x + a)^2$  to factorise the LHS, and simplify the RHS.

$$\left(p + \frac{5}{2}\right)^2 = \frac{41}{4}$$

Take the square roots.

$$p + \frac{5}{2} = \pm \frac{\sqrt{41}}{2}$$

Solve the equations.

$$p = \frac{\sqrt{41} - 5}{2} \quad \text{or} \quad p = \frac{-\sqrt{41} - 5}{2}$$

Evaluate correct to 2 decimal places.

$$p \approx 0.70 \quad \text{or} \quad p \approx -5.70$$

A **root**, or solution, of an equation is a value of the variable that makes the equation true. A root is a value of  $x$  where the graph of a function crosses the  $x$ -axis, i.e.  $f(x) = 0$ .

### EXAMPLE 17

Use a graph to find the roots of  $x^2 - 3x - 5 = 0$ .

#### Solution

Write the equation.

$$x^2 - 3x - 5 = 0$$

Express as a function.

$$f(x) = x^2 - 3x - 5$$

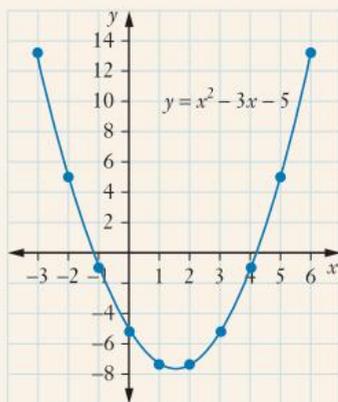
Construct a table of values for the function.

$x$	-3	-2	-1	0	1	2	3	4	5	6
$y$	13	5	-1	-5	-7	-7	-5	-1	5	13

Use the values to sketch the function, joining points with a smooth curve.

The roots of the quadratic are the values of  $x$  where  $y = f(x) = 0$ .

Use the graph to estimate where  $f(x) = 0$ .



Write the result.

The roots of  $x^2 - 3x - 5 = 0$  are  $x \approx -1.2$  and  $x \approx 4.2$ .

## TECHNOLOGY

### Finding roots with a graphics calculator

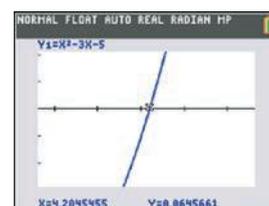
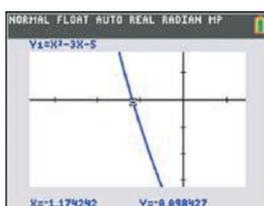
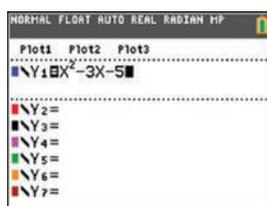
All graphics calculators have a **Trace** function. You can use this to find approximate roots of quadratics.

#### TI-84 Plus CE

Press  $\boxed{Y=}$  and enter  $Y_1 = x^2 - 3x - 5$  using  $\boxed{X,T,\theta,n}$ , press  $\boxed{\text{enter}}$  and  $\boxed{\text{graph}}$ .

Use  $\boxed{\text{window}}$  or  $\boxed{\text{zoom}}$  and **Zbox** to find the part needed.

As you scroll across the graph using **Trace** ( $\boxed{F4}$ ), the value of  $Y$  changes from negative to positive and vice versa. The value of one root is between  $x = -1.2121\dots$  and  $x = -1.1742\dots$ . Repeat to find the other root  $4.1666\dots < x < 4.2045\dots$ . Therefore  $x \approx -1.2$  or  $x \approx 4.2$ .

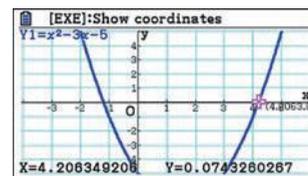
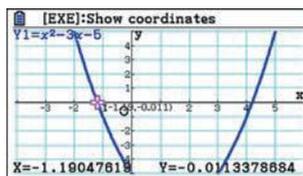
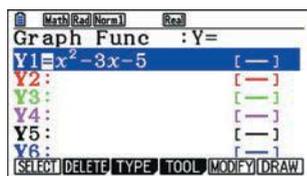


#### Casio fx-CG20AU

Choose **5 Graph** and enter  $Y1 = x^2 - 3x - 5$  using  $\boxed{X,\theta,T}$ , press  $\boxed{\text{EXE}}$  and  $\boxed{F6}$ .

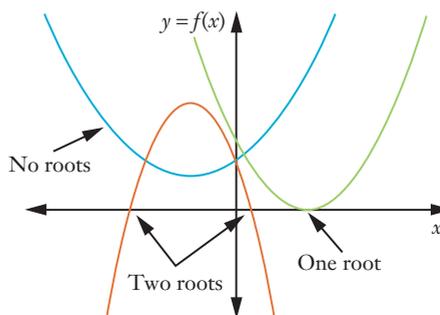
Use **V-Window** ( $\boxed{\text{SHIFT}} \boxed{F3}$ ) or **Zoom** ( $\boxed{\text{SHIFT}} \boxed{F2}$ ) and **BOX** to find the part needed.

As you scroll across the graph using **Trace** ( $\boxed{F1}$ ), the value of  $Y$  changes from negative to positive and vice versa. The value of one root is between  $x = -1.2301\dots$  and  $x = -1.1904\dots$ . Repeat to find the other root  $4.1666\dots < x < 4.2063\dots$ . Therefore  $x \approx -1.2$  or  $x \approx 4.2$ .

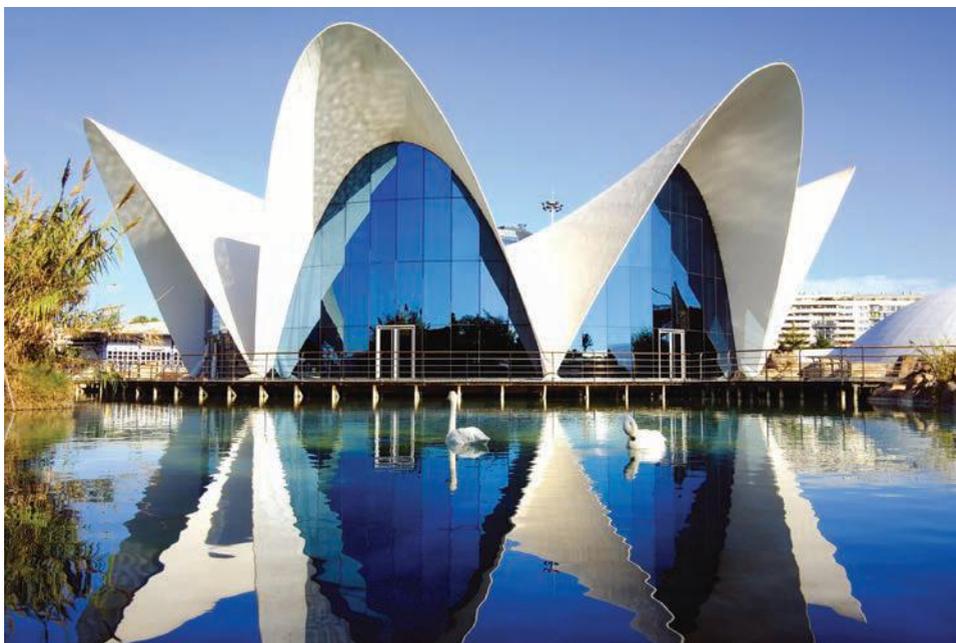
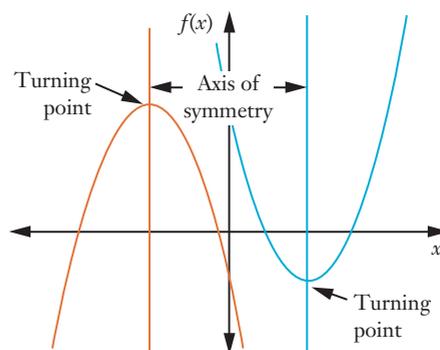


The shape of the quadratic is called a **parabola**. Graphs of all quadratics have this distinctive shape. Quadratics are also known as **parabolic functions**.

A quadratic may have as many as two roots.



A distinctive feature of a parabola of the form  $f(x) = ax^2 + bx + c$  is that it is symmetrical about a vertical axis passing through its **turning point**.



Alamy Stock Photo/lorenzo codacci

## Exercise 2.06 Quadratic equations

Example  
15

1 Solve each equation by factorisation where necessary.

**a**  $2x(x - 12) = 0$

**b**  $(x + 2)(x - 5) = 0$

**c**  $x^2 - 8x = 0$

**d**  $4x^2 + 6x = 0$

**e**  $x^2 + 9x + 14 = 0$

**f**  $x^2 - 8x + 15 = 0$

2 Solve each equation by factorisation.

**a**  $3x^2 + 5x - 12 = 0$

**b**  $2x^2 - 24x + 22 = 0$

**c**  $3x^2 + 15x + 12 = 0$

**d**  $4x^2 - 28x - 32 = 0$

**e**  $2x^2 + 14x + 20 = 0$

**f**  $x^2 - 2x = 3$

3 Solve each equation by factorisation.

**a**  $x - 4 = \frac{5}{x}$

**b**  $x^2 - x = 6$

**c**  $x^2 - x = 12x$

**d**  $x^2 - 6x = -9$

**e**  $2x^2 + 5x + 2 = 0$

**f**  $3x^2 + 10x + 3 = 0$

4 Solve each equation by factorisation.

**a**  $5a^2 + 8a - 4 = 0$

**b**  $2v^2 - 19v + 42 = 0$

**c**  $n^2 - 64 = 0$

**d**  $4c^2 - 28c + 49 = 0$

**e**  $14e^2 + 33e + 18 = 0$

**f**  $9p^2 + 10p - 16 = 0$

**g**  $5h^2 - 31h - 28 = 0$

**h**  $x^2 + 2x - 48 = 0$

**i**  $9u^2 + 68u + 35 = 0$

**j**  $4r^2 + 8r - 5 = 0$

**k**  $16y^2 + 30y - 25 = 0$

**l**  $32g^2 - 18g - 45 = 0$

5 Solve each equation by factorisation.

**a**  $2b^2 = 11b - 14$

**b**  $10d^2 + 10d + 10 = 3d^2 - 9d$

**c**  $6x^2 - 5x - 50 = 5x^2 - 5x - 1$

**d**  $5a^2 = 4(a^2 + 2a - 4)$

**e**  $4g(g + 6) = g - 15$

**f**  $16(x^2 - 1) = 14x - 19$

**g**  $4f = 2f(3f - 1) - 36$

**h**  $(3e + 2)(5e - 3) = 34$

**i**  $2z^2 - 6 = 39z - 3z^2 - 60$

**j**  $3(t - 1) = 2t(t - 1)$

**k**  $(2m - 1)(1 + 3m) = 4$

**l**  $(3 - p)(p - 1) = 2p(p - 3)$

6 Solve each equation by factorisation.

**a**  $x^2 - 2x + 2 = \frac{1 + 3x^2}{4}$

**b**  $\frac{x^2 + 9x}{4} + 5 = \frac{4x + 1}{2} - 2x$

**c**  $\frac{3x^2}{2} + 2x = 1\frac{1}{3} - \frac{x}{3}$

**d**  $\frac{x^2}{4} + \frac{x}{3} = \frac{x^2}{6} + \frac{x + 2}{4}$

**e**  $16x = \frac{2x - 54}{x - 4}$

**f**  $4x - \frac{48}{x - 3} = x + 2$

**g**  $\frac{x - 6}{x + 1} = \frac{2}{x + 9}$

**h**  $\frac{35x}{2x - 3} - 3 = 2x$

**i**  $4 - \frac{42}{2x^2 - 3x} = \frac{4}{2x - 3}$

**j**  $\frac{2(x + 2)}{15} - \frac{2}{x - 2} = 1$

**k**  $12x - \frac{36x - 3}{2x + 2} = 15$

**l**  $\frac{5x + 5}{2x - 3} = \frac{3x + 5}{x + 1} - \frac{2x^2}{2x^2 - x - 3}$

**7** Solve each equation by completing the square. Where necessary, express answers in radical form and correct to 2 decimal places.

**a**  $k^2 + 4k - 5 = 0$

**b**  $v^2 + 2v - 35 = 0$

**c**  $j^2 = 9j - 8$

**d**  $x^2 = x + 132$

**e**  $m^2 = \frac{8}{9} - \frac{2m}{3}$

**f**  $b^2 + 4b - 6 = 0$

**g**  $a^2 = 1 - a$

**h**  $z^2 = \frac{3z}{2} + 2$

**i**  $v^2 = 3(3v - 1)$

**j**  $8x(x - 1) + 40 = 7x(x + 1)$

**k**  $2x^2 + x - 6 = 0$

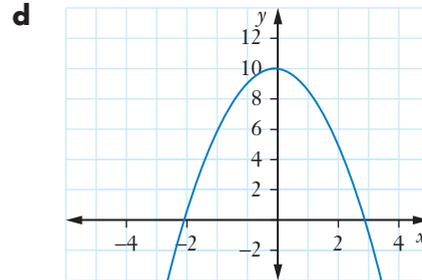
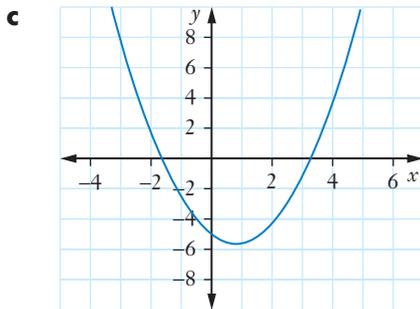
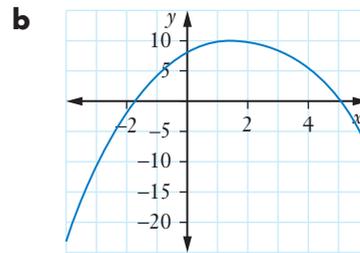
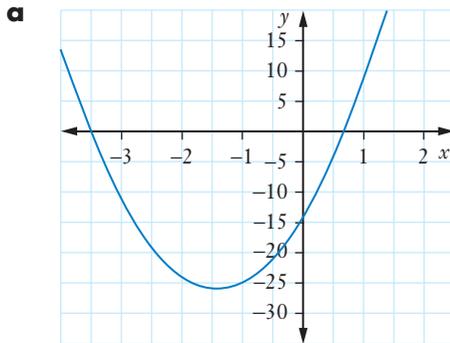
**l**  $3c^2 = 10 - c$

**m**  $2m(m - 3) + 1 = 0$

**n**  $7r^2 = 12r - 3$

**o**  $h(5h - 7) = 1\frac{1}{2}$

**8** For each graph, estimate the roots of the quadratic equation correct to 1 decimal place.



**9** Solve each equation correct to one decimal place by using a graph.

**a**  $x^2 + x - 12 = 0$

**b**  $5 - 4x - x^2 = 0$

**c**  $x^2 - 4x - 6 = 0$

**d**  $3x + 7 - 2x^2 = 0$

**e**  $2x^2 + 3x - 8 = 0$

**f**  $\frac{x^2}{2} - 4x - 7 = 0$

**10** Use a graphics calculator to graph and solve each equation correct to two decimal places.

**a**  $2x^2 - 5x - 10 = 0$

**b**  $8 - \frac{x^2}{2} - 4x = 0$

**c**  $5x^2 - 14x - 11 = 0$

**d**  $6x + 31 - 3x^2 = 0$

**e**  $4x^2 - 8x - 26 = 0$

**f**  $\frac{x^2}{5} + 7x - 15 = 0$

## Problem solving

- 11** The length of a rectangular lawn is 0.9 m more than its width, and its area is  $16.2 \text{ m}^2$ . Write an equation and solve it to find the dimensions of the lawn.
- 12** A square room is 3 m high and 10 L of paint is needed to paint all the walls and ceiling. If 1 L of the paint will cover  $16 \text{ m}^2$ , write an equation and find the length of the room.
- 13** The length of a closed box is three times its height, and the box is 4 cm wide. If the total surface area is  $88 \text{ cm}^2$ , find the dimensions of the box.



Quadratic formula



Solving algebraic equations

## 2.07 The quadratic formula and discriminant

### Quadratic formula

For any quadratic equation  $ax^2 + bx + c = 0$  where  $a \neq 0$ , real solutions are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{when } b^2 - 4ac \geq 0$$



The quadratic formula



Excel spreadsheet: The quadratic formula



Excel worksheet: The quadratic formula

### Proof of the quadratic formula

$$ax^2 + bx + c = 0.$$

$$a\left(x^2 + \frac{b}{a}x\right) = -c$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) = \frac{b^2}{4a} - c \quad \text{(Complete the square)}$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{(Take the square root of both sides)}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(Make } x \text{ the subject)} \quad \text{QED}$$

**EXAMPLE 18**

Solve  $3x^2 = 4x + 12$  using the quadratic formula.

**Solution**

Express in standard form.

$$3x^2 - 4x - 12 = 0$$

Compare with  $ax^2 + bx + c = 0$ .

$$a = 3, b = -4, c = -12$$

Use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute for  $a$ ,  $b$  and  $c$ .

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times (3) \times (-12)}}{2 \times (3)}$$

Simplify.

$$= \frac{4 \pm \sqrt{160}}{6}$$

$$= \frac{4 \pm 4\sqrt{10}}{6}$$

Factorise.

$$= \frac{4(1 \pm \sqrt{10})}{6}$$

Simplify.

$$= \frac{2(1 \pm \sqrt{10})}{3}$$

Write as two solutions.

$$= \frac{2(1 + \sqrt{10})}{3} \quad \text{or} \quad \frac{2(1 - \sqrt{10})}{3}$$

If approximate solutions are needed, evaluate.

$$\approx 2.77 \quad \text{or} \quad -1.44$$

Write the approximate solutions.

$$x \approx 2.77 \quad \text{or} \quad x \approx -1.44$$



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For  $ax^2 + bx + c = 0$ , the expression  $\Delta = b^2 - 4ac$ , called the **discriminant**, is useful for determining the number of roots a quadratic equation has.

If  $b^2 - 4ac > 0$ , then  $ax^2 + bx + c = 0$  has two real roots.

If  $b^2 - 4ac = 0$ , then  $ax^2 + bx + c = 0$  has one real root.

If  $b^2 - 4ac < 0$ , then  $ax^2 + bx + c = 0$  has no real roots.

### EXAMPLE 19

Use the discriminant to determine the number of real roots of each equation.

**a**  $-4x^2 + 2x + 5 = 0$

**b**  $5x^2 - 8x + 3 = 0$

#### Solution

**a** Write the equation.

$$-4x^2 + 2x + 5 = 0$$

Determine  $a$ ,  $b$ , and  $c$ .

$$a = -4, b = 2, c = 5$$

Evaluate  $b^2 - 4ac$ .

$$b^2 - 4ac = (2)^2 - 4(-4)(5)$$

$$= 4 + 80$$

$$= 84$$

$84 > 0$ . Therefore  $-4x^2 + 2x + 5 = 0$  has two real roots.

**b** Write the equation.

$$5x^2 - 6x + 3 = 0$$

Determine  $a$ ,  $b$ , and  $c$ .

$$a = 5, b = -6, c = 3$$

Evaluate  $b^2 - 4ac$ .

$$b^2 - 4ac = (-6)^2 - 4(5)(3)$$

$$= 36 - 60$$

$$= -24$$

$-24 < 0$ . Therefore  $5x^2 - 6x + 3 = 0$  has no real roots.

## Exercise 2.07 The quadratic formula and discriminant

- 1** Solve each quadratic equation using the quadratic formula. Where appropriate, give both exact and approximate answers correct to 2 decimal places.

**a**  $x^2 + 14x + 45 = 0$

**b**  $x^2 + 20x + 51 = 0$

**c**  $x^2 = 7x + 294$

**d**  $x^2 + 4x = 7$

**e**  $x^2 - 6x = 21$

**f**  $x^2 = 13x + 140$

- 2** Solve each quadratic equation using the quadratic formula, correct to 2 decimal places where necessary.

**a**  $3f^2 + f - 14 = 0$

**b**  $5t^2 = 9t + 18$

**c**  $6x^2 = 5x + 1$

**d**  $x^2 - x - 5 = 0$

**e**  $9p^2 = 15p + 5$

**f**  $2d^2 + 3d = 6$

- 3** Solve each quadratic equation, using the most appropriate method.

**a**  $x^2 + 6x - 10 = 0$

**b**  $18x^2 + 15x = 12$

**c**  $x^2 - 4x - 13 = 0$

**d**  $x^2 - 6.146x + 9.328 = 0$

**e**  $1 - \frac{x}{2} + \frac{x^2}{24} = 0$

**f**  $\frac{1}{x+1} - \frac{4}{x} = \frac{1}{x+3}$

**g**  $x^2 + 29x + 6 = 0$

**h**  $3x^2 - 5.6x + 0.56 = 0$

- 4** Use the discriminant to determine the number of roots of the following.

**a**  $3x^2 + 5x - 1 = 0$

**b**  $x^2 - 4x + 4 = 0$

**c**  $2x^2 - 3x + 5 = 0$

**d**  $-x^2 + 5x - 8 = 0$

**e**  $-3x^2 - 6x - 3 = 0$

**f**  $-x^2 - 2x + 5 = 0$

Example  
**18**

Example  
**19**

### Problem solving

- 5** Hans can walk 1 km/h faster than Oscar. He completes a 20 km hike 1 hour before him. Write an equation and solve it to find their walking speeds.
- 6** A bath toy bought for \$ $x$  is sold for \$24, making  $x\%$  profit. What was the cost of the toy?



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- 7 A tank can be filled in 6 hours using two pipes. The larger pipe alone would fill it 5 hours sooner than the smaller pipe alone. How long would each pipe alone take?
- 8 A cyclist travelled at a certain speed to a town 30 km away, then reduced his speed by 4 km/h and came back. If the cyclist had travelled at a steady 8 km/h for the whole trip, the total time would have been half an hour less. What was the initial speed of the cyclist?
- 9 The total resistance  $R$  (in ohms) of two resistors  $R_1$  and  $R_2$  in parallel is given by
- $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$
- A particular resistor is placed in parallel with one 20 ohms greater in resistance. The total resistance is 24 ohms. What are the resistances of the two resistors?

## 2.08 Direct and inverse proportion

When two quantities vary in the same ratio, then one variable is said to **vary directly** with the other. Consider the case where a car is moving at a constant speed of 60 km/h. In one hour it will cover 60 km and in 2 hours it will cover 120 km. Therefore speed is the ratio of distance to time and distance varies directly with time. This can be written as  $d = 60t$ , where  $d$  is distance and  $t$  is time and 60 is the constant of variation.



Direct proportion

$d$  is proportional to  $t$  is written as  $d \propto t$

If two quantities ( $y$  and  $x$ ) vary directly then:

- 1  $y$  and  $x$  both increase or both decrease
- 2 the **ratio** of the quantities is **constant**

$$y \propto x \rightarrow \frac{y}{x} = k \quad \text{or} \quad y = kx \quad (k \text{ is the constant of variation})$$

### EXAMPLE 20

For gas at a constant temperature, the volume  $V$  varies directly with the temperature  $T$  (in Kelvins). If there are 2000 cm<sup>3</sup> of gas when the temperature is 400 K,

- a what volume is present if the temperature is reduced to 300 K?
- b at what temperature would 500 cm<sup>3</sup> be present?

## Solution

- a Find  $k$  using  $V = kT$ .

$$k = \frac{V}{T}$$

$$\begin{aligned}k &= \frac{2000}{400} \\ &= 5\end{aligned}$$

Use  $k$  to calculate  $V$  when the temperature is 300 K.

$$\begin{aligned}V &= kT \\ &= 5 \times 300 \\ &= 1500 \text{ cm}^3\end{aligned}$$

Write the answer as a statement.

1500 cm<sup>3</sup> is present at 400 K.

- b Rearrange  $V = kT$  to express in terms of  $T$ .

$$T = \frac{V}{k}$$

Solve for  $T$ .

$$\begin{aligned}T &= \frac{500}{5} \\ &= 100 \text{ K}\end{aligned}$$

Write the answer as a statement.

500 cm<sup>3</sup> is present when  $T = 100$  K.

When the increase in one variable causes a decrease in the other and vice versa, then the variables are **inversely proportional**. For example, if it took one painter six hours to paint a room, then two painters would take three hours to paint the same room. So the time taken to paint the room **varies inversely** with the number of painters. This can be written as

$t = \frac{6}{p}$ , where  $t$  is time taken and  $p$  is the number of painters.

If two quantities ( $y$  and  $x$ ) are inversely proportional:

- 1 when  $y$  increases  $x$  decreases, and when  $y$  decreases  $x$  increases
- 2 the **product** of the quantities is **constant**

$$y \propto \frac{1}{x} \rightarrow y = \frac{k}{x} \rightarrow xy = k$$

### EXAMPLE 21

If  $y = 12$  when  $x = 15$  and  $y$  varies inversely with  $x$ , find:

**a**  $x$  when  $y = 24$

**b**  $y$  when  $x = 10$

#### Solution

**a** Find  $k$  using  $xy = k$ .

$$\begin{aligned}k &= xy \\ &= 15 \times 12 \\ &= 180\end{aligned}$$

Rearrange  $xy = k$  in terms of  $x$ .

$$x = \frac{k}{y}$$

Solve for  $x$ .

$$\begin{aligned}x &= \frac{180}{24} \\ &= 7.5\end{aligned}$$

**b** Rearrange  $xy = k$  in terms of  $y$ .

$$y = \frac{k}{x}$$

Solve for  $y$ .

$$\begin{aligned}y &= \frac{180}{10} \\ &= 18\end{aligned}$$

Inverse proportion occurs frequently in nature. Gravitational force is inversely proportional to the square of the distance between the centres of two masses. If the distance is doubled, the force is decreased by a factor of 4.

### EXAMPLE 22

The electrostatic force ( $F$ ) acting between two charged particles varies inversely as the square of the distance ( $d$ ) between them. If  $F = 36$  when  $d = 4$ , find  $d$  when  $F = 64$ .

#### Solution

Find  $k$  using  $Fd^2 = k$ .

$$\begin{aligned}k &= Fd^2 \\ &= 36 \times 4^2 \\ &= 576\end{aligned}$$

Rearrange  $Fd^2 = k$  in terms of  $d$ .

$$d^2 = \frac{k}{F}$$

Solve for  $d$  when  $F = 64$ .

$$d = \sqrt{\frac{k}{F}}$$
$$d = \sqrt{\frac{576}{64}}$$
$$= \pm 3$$

Distance is positive.

$$d = 3 \text{ (only, distance } > 0)$$

Write the answer as a statement.

When the force  $F = 64$ , the distance between the particles will be 3 units.

## Exercise 2.08 Direct and inverse proportion

- 1** The mass ( $m$ ) of an object varies directly with its volume ( $V$ ).
- Write an equation to express this proportion.
  - If  $100 \text{ cm}^3$  of a substance has a mass of  $150 \text{ g}$ , find the mass of  $180 \text{ cm}^3$  of the same substance.

Example  
20

- 2** The distance ( $d$ ) travelled is directly proportional to the number of pedal turns ( $p$ ) on a bicycle. If the bike travels  $48 \text{ m}$  per  $20$  pedal turns, find:
- the distance travelled after  $16$  pedal turns
  - the number of pedal turns needed to travel  $72 \text{ m}$ .



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- 3** The safe working load ( $L \text{ kg}$ ) that can be lifted by a cable is directly proportional to the square of its diameter ( $d \text{ cm}$ ). If a cable with a  $4 \text{ cm}$  diameter can safely lift a load of  $2500 \text{ kg}$ , find:
- the safe working load of a  $6 \text{ cm}$  diameter cable
  - the smallest diameter cable capable of lifting a  $625 \text{ kg}$  load safely.
- 4** The time taken for a pendulum to swing varies directly as the square root of its length. If one swing of a  $98 \text{ cm}$  pendulum takes  $2$  seconds, find:
- the time taken for one swing of a  $50 \text{ cm}$  pendulum (correct to one decimal place)
  - the length of a pendulum (to the nearest centimetre) if it takes  $4$  seconds for  $1$  swing.
- 5** If  $y = 12$  when  $x = 4$  and  $y$  varies inversely with  $x$ , find:
- |                            |                           |                            |
|----------------------------|---------------------------|----------------------------|
| <b>a</b> $x$ when $y = 16$ | <b>b</b> $x$ when $y = 8$ | <b>c</b> $x$ when $y = 24$ |
| <b>d</b> $y$ when $x = 12$ | <b>e</b> $y$ when $x = 8$ | <b>f</b> $y$ when $x = 4$  |

Example  
21

- 6 If  $a = 3$  when  $b = 5$  and  $b$  varies inversely with  $a$ , find:  
**a**  $a$  when  $b = 3$                       **b**  $a$  when  $2b = 10$                       **c**  $b$  when  $3a = 15$
- 7 The time needed to fill a petrol tank is inversely proportional to the diameter of the fuel hose. If it takes 8 minutes to fill a tank with a 2 cm diameter hose, determine:  
**a** how many minutes and seconds it would take with a 2.5 cm diameter hose  
**b** what diameter hose would be needed to fill the tank in 2 minutes.
- 8 The fuel consumption of a car varies inversely as the square of the mass of the car. A 1200 kg car has a consumption of 8 km/L. Find:  
**a** the fuel consumption of a 1500 kg car (correct to 2 decimal places)  
**b** the mass of a car (to the nearest kg) if its fuel consumption is 10 km/L.
- 9 The force of attraction between two bodies varies inversely as the square of the distance between them. When the distance is 24 units, the force is  $\frac{3}{4}$  of a unit. Find:  
**a** the distance (correct to 2 decimal places) if the force is 36 units  
**b** the force when the distance is 4 units.

## 2.09 Polynomials

A **polynomial** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0 \text{ for integers } n \geq 0$$

where  $a_0, a_1, a_2, \dots, a_n$  are real numbers called the **coefficients** of the polynomial. If  $a_n \neq 0$ , then the polynomial is of **degree**  $n$ , the highest power of  $x$ , and has **leading term**  $a_n x^n$ .

$a_0$  is called the **constant term**.

A polynomial is called a **monic** polynomial when the leading coefficient,  $a_n$ , is 1.

Examples of polynomials are:

- |                         |  |
|-------------------------|--|
| 3                       | A polynomial of degree 0. It could be written as $3x^0$ . The leading term is 3.   |
| $4x - 8$                | A polynomial of degree 1, with leading term $4x$ and a constant of $-8$ . It is a linear polynomial.   |
| $3x^2 + 4\sqrt{5}x + 2$ | A polynomial of degree 2. The leading term is $3x^2$ , the constant is 2 and the coefficient of $x$ is $4\sqrt{5}$ . It is a quadratic polynomial. |
| $x^4$                   | A monic polynomial of degree 4. The leading term is $x^4$ and the constant is zero. This is a quartic polynomial.                                  |

### EXAMPLE 23

For each expression, determine whether it is a polynomial, giving reasons. If it is a polynomial, state its degree, leading term and identify the coefficients.

**a**  $-2x + 3x^2$

**b**  $x^5 - \frac{x^2}{3} + 4 + 10x^9$

**c**  $x^3 - \frac{5}{x} + x^2$

**d**  $x^2 + 3\sqrt{x} - 10$

#### Solution

- a** Write the expression with powers of  $x$  in descending order.

$$3x^2 - 2x$$

All of the powers are non-negative integers.

This is a polynomial.

Write the degree, leading term and coefficients.

$$\text{Degree} = 2$$

$$\text{Leading term} = 3x^2$$

$$\text{Coefficients} = 3, -2$$

- b** Write the expression with powers of  $x$  in descending order.

$$10x^9 + x^5 - \frac{x^2}{3} + 4$$

All of the powers are non-negative integers.

This is a polynomial.

Write the degree, leading term and coefficients.

$$\text{Degree} = 9$$

$$\text{Leading term} = 10x^9$$

$$\text{Coefficients} = 10, 1, -\frac{1}{3}, 4$$

- c** Write the expression with powers of  $x$  in descending order.

$$x^3 + x^2 - 5x^{-1}$$

The power of the last term is a negative integer. This is not a polynomial.

- d** Write the expression with powers of  $x$  in descending order.

$$x^2 + 3x^{\frac{1}{2}} - 10$$

The second term has a positive fractional power. This is not a polynomial.

### EXAMPLE 24

If  $f(x) = -2x^3 + 3x^2 - x - 4$ , evaluate:

**a**  $f(0)$

**b**  $f(-1)$

**c**  $f(0.5)$

**d**  $f(\sqrt{2})$

### Solution

**a** Write  $f(x)$ .

$$f(x) = -2x^3 + 3x^2 - x - 4$$

Substitute 0 for  $x$  and evaluate.

$$\begin{aligned} f(0) &= -2(0)^3 + 3(0)^2 - (0) - 4 \\ &= -4 \end{aligned}$$

**b** Substitute  $-1$  for  $x$  and evaluate.

$$\begin{aligned} f(-1) &= -2(-1)^3 + 3(-1)^2 - (-1) - 4 \\ &= 2 + 3 + 1 - 4 \\ &= 2 \end{aligned}$$

**c** Substitute  $0.5$  for  $x$  and evaluate.

$$\begin{aligned} f(0.5) &= -2(0.5)^3 + 3(0.5)^2 - (0.5) - 4 \\ &= -0.25 + 0.75 - 0.5 - 4 \\ &= -4 \end{aligned}$$

**d** Substitute  $\sqrt{2}$  for  $x$  and evaluate.

$$\begin{aligned} f(\sqrt{2}) &= -2(\sqrt{2})^3 + 3(\sqrt{2})^2 - (\sqrt{2}) - 4 \\ &= -2(2\sqrt{2}) + 3(2) - \sqrt{2} - 4 \\ &= 2 - 5\sqrt{2} \end{aligned}$$

## Exercise 2.09 Polynomials

Example  
23

**1** State which of the following are polynomials, giving reasons.

**a**  $-3x^4 - 3x + 5 + \frac{1}{x^2}$

**b**  $5x^2 - 2x + 8$

**c**  $x^2 + 2^x + 7$

**d**  $2x - 1$

**e**  $10x^4 + 9x^2 - 2\sqrt{x} + 1$

**f**  $2x^2 - \frac{3}{4}x + 8$

**g**  $\frac{1}{x-1} + x^2$

**h**  $3x^5 - 2x^2 + \sqrt{x^2 - 2x + 1}$

**2** For each of the following polynomials, state its leading term and degree.

**a**  $3 - x^2 - x^4 - 5x$

**b**  $3x^7 - x^5 + 4x^4 - 2x$

**c**  $5x + 2$

**d**  $t^9 - 9t^7 + t^5 + 15t^3$

**e**  $5$

**f**  $7y^3 - y^{10}$

- 3** For the polynomial  $P(x) = 2x^6 - 3x^5 + \frac{x^4}{4} - 2x^2 + 7$ , write down:
- |                               |                                   |
|-------------------------------|-----------------------------------|
| <b>a</b> the degree of $P(x)$ | <b>b</b> the coefficient of $x^4$ |
| <b>c</b> the constant term    | <b>d</b> the coefficient of $x^2$ |
- 4** If  $S(x) = 2x^4 - x^3 + 5x^2 + 6x - 4$ , evaluate:
- |                  |                 |                        |                         |
|------------------|-----------------|------------------------|-------------------------|
| <b>a</b> $S(3)$  | <b>b</b> $S(1)$ | <b>c</b> $S(0)$        | <b>d</b> $S(-2)$        |
| <b>e</b> $S(-3)$ | <b>f</b> $S(4)$ | <b>g</b> $S(\sqrt{2})$ | <b>h</b> $S(-\sqrt{3})$ |
- 5** If  $P(x) = 2x^4 - x^2 + 7$ , show that  $P(x) = P(-x)$ .
- 6** Use your calculator to evaluate  $P(-6)$  when  $P(x) = -3x^5 + 8x^4 - 4x^2 - 8$ .
- 7** For the polynomial  $Q(x) = (a + 2)x^4 + (b - 1)x^3 + (c - 3)x^2 + d + 4$ , find the values of
- |   |   |
|---|---|
| <b>a</b> $b$ if the coefficient of $x^3$ is 5           | <b>b</b> $c$ if $x^2$ has a coefficient of $-\frac{1}{3}$ |
| <b>c</b> $a$ if the leading term has a coefficient of 5 | <b>d</b> $a$ and $b$ if the polynomial has degree 2       |
| <b>e</b> $d$ if the constant is $-2$                    | <b>f</b> $a$ if $Q(x)$ is monic                           |
- 8** Given  $f(x) = x^3 + 2x^2 + x$ , find:
- |                                  |                              |
|----------------------------------|------------------------------|
| <b>a</b> $f(x)$ when $x = -2$    | <b>b</b> $x$ when $f(x) = 0$ |
| <b>c</b> $f(x)$ when $x = a + 2$ | <b>d</b> $f(-2)f(x + h)$     |

## 2.10 Polynomial factorisation

You will find it useful to write polynomials in factored form to solve equations and draw graphs. You can factorise using **long division**.

### Remainder theorem

If a polynomial  $P(x)$  is divided by  $(x - a)$ , then  $Q(x)$  and  $R$  exist such that  $P(x) = (x - a)Q(x) + R$ , and the remainder  $R$  is given by  $R = P(a)$ .

### Proof of the remainder theorem

The division identity states that for polynomials  $P(x)$  and  $D(x) = x - a$ , there exist polynomials  $Q(x)$  and  $R(x)$  such that  $P(x) = Q(x)D(x) + R(x)$ , and that the degree of  $R(x)$  is less than the degree of  $D(x)$ .

Since the degree of  $D(x) = (x - a)$  is 1, its leading term will divide every term in  $P(x)$  except the constant. Thus the remainder will be a real number and  $P(x)$  can be written as

$$P(x) = (x - a)Q(x) + R.$$

Substituting the value  $x = a$  gives

$$\begin{aligned} P(a) &= (a - a)Q(x) + R \\ &= 0 + R \end{aligned}$$

so  $R = P(a)$       QED

### EXAMPLE 25

Find  $Q(x)$  and  $R$  such that  $P(x) = (x + 3)Q(x) + R$ , where  $P(x) = x^4 - 2x^2 + 7$ .

#### Solution

Divide  $(x + 3)$  into  $P(x)$ .

Make sure that terms of the same order remain in the same column.

Note:  $P(x)$  does not have terms of order 1 or 3, but it is convenient to write them in with a 0 coefficient.

$$\begin{array}{r} x^3 - 3x^2 + 7x - 21 \\ x + 3 \overline{) x^4 + 0x^3 - 2x^2 + 0x + 7} \\ \underline{x^4 + 3x^3} \phantom{+ 7x - 21} \\ -3x^3 - 2x^2 \phantom{+ 0x + 7} \\ \underline{-3x^3 - 9x^2} \phantom{+ 0x + 7} \\ 7x^2 + 0x \phantom{+ 7} \\ \underline{7x^2 + 21x} \phantom{+ 7} \\ -21x + 7 \\ \underline{-21x - 63} \\ 70 \end{array}$$

Write the final answer.

Therefore  $Q(x) = x^3 - 3x^2 + 7x - 21$   
and  $R = 70$

### Factor theorem

$(x - a)$  is a factor of the polynomial  $P(x)$  if and only if  $P(a) = 0$ .

#### Proof of the factor theorem

If  $(x - a)$  is a factor of  $P(x)$ , then there must be a polynomial  $Q(x)$  such that

$$P(x) = (x - a)Q(x)$$





Factorising  
polynomials

## EXAMPLE 27

Factorise  $P(x) = x^3 - 7x^2 - 73x - 65$ .

### Solution

Substitute  $x = 1$  into  $P(x)$ .

$$P(x) = x^3 - 7x^2 - 73x - 65$$

$$P(1) = (1)^3 - 7(1)^2 - 73(1) - 65$$

$$= 1 - 7 - 73 - 65$$

$$= -144$$

$P(1) \neq 0$  so  $(x - 1)$  is not a factor.

Substitute  $x = -1$  into  $P(x)$ .

$$P(-1) = (-1)^3 - 7(-1)^2 - 73(-1) - 65$$

$$= -1 - 7 + 73 - 65$$

$$= 0$$

$P(-1) = 0$  so  $(x + 1)$  is a factor of  $P(x)$ .

Divide  $(x + 1)$  into  $P(x)$  to obtain its other factor  $Q(x)$ .

$$\begin{array}{r} x^2 - 8x - 65 \\ x+1 \overline{) x^3 - 7x^2 - 73x - 65} \\ \underline{x^3 + x^2} \phantom{- 73x - 65} \phantom{0} \\ -8x^2 - 73x \phantom{- 65} \phantom{0} \\ \underline{-8x^2 - 8x} \phantom{- 65} \phantom{0} \\ -65x - 65 \\ \underline{-65x - 65} \\ 0 \end{array}$$

Write  $P(x)$  as a product of  $(x + 1)$  and  $Q(x)$ .

$$P(x) = (x + 1)(x^2 - 8x - 65)$$

$Q(x)$  factorises as  $(x - 13)(x + 5)$ . Write  $P(x)$  as a product of three linear factors.

$$P(x) = (x + 1)(x - 13)(x + 5)$$

## Exercise 2.10 Polynomial factorisation

Example  
25

- For the polynomials  $P(x) = 3x^4 - 2x^2 + x - 5$  and  $D(x) = x^2 - x - 1$ , find  $Q(x)$  and  $R(x)$  such that  $P(x) = Q(x)D(x) + R(x)$ .
- For the polynomials  $P(x) = 3x^5 - 11x^4 + 10x^3 - 10x^2 - 9x - 12$  and  $D(x) = x^2 - 3x - 1$ , find  $Q(x)$  and  $R(x)$  such that  $P(x) = Q(x)D(x) + R(x)$ .

**3** For the polynomials  $P(x) = 8x^4 - 12x^3 - 6x^2 + 19x - 7$  and  $D(x) = 2x - 3$ , find  $Q(x)$  and  $R(x)$  such that  $P(x) = Q(x)D(x) + R(x)$ .

**4** Show that  $x^3 - 2x^2 - 16x + 32$  is divisible by  $(x^2 - 16)$  and hence factorise the expression.

**5** Factorise the following.

**a**  $2x^3 + x^2 - 7x - 6$

**b**  $x^3 + x^2 - 5x + 3$

**c**  $x^3 - 2x^2 - x + 2$

**d**  $2x^3 - x^2 - 8x + 4$

**e**  $3x^3 + 16x^2 + 23x + 6$

**f**  $2x^3 + 3x^2 - 17x - 30$

**g**  $3x^3 + 11x^2 + 5x - 3$

**h**  $3x^3 + x^2 - 22x - 24$

**i**  $4x^3 + 21x^2 + 14x - 24$

Example  
**26**

Example  
**27**

## 2.11 Polynomial equations

You can use the factor theorem to solve an equation written as  $P(x) = 0$ , where  $P(x)$  is a polynomial.

### EXAMPLE 28

Solve  $2x^3 + 5x^2 - x - 6 = 0$

#### Solution

Let  $P(x) = 2x^3 + 5x^2 - x - 6$ .

$$P(x) = 2x^3 + 5x^2 - x - 6$$

By inspection,  $P(1) = 0$  and  $P(-2) = 0$ . Therefore  $(x - 1)$  and  $(x + 2)$  are factors. Use polynomial division to find the remaining factor.

$$P(x) = (x - 1)(x + 2)(2x + 3)$$

Rearrange in the form  $P(x) = 0$ .

$$(x - 1)(x + 2)(2x + 3) = 0$$

Apply the null factor law. Write the solutions.

$$x = 1 \text{ or } x = -2 \text{ or } x = -\frac{3}{2}$$

### EXAMPLE 29

Solve  $\frac{x-1}{2} = \frac{(x+4)^2 - 22}{3x^2}$

#### Solution

Write the equation.

$$\frac{x-1}{2} = \frac{(x+4)^2 - 22}{3x^2}$$

Cross multiply to remove the fractions.

$$3x^2(x-1) = 2[(x+4)^2 - 22]$$

$$3x^3 - 3x^2 = 2(x^2 + 8x + 16 - 22)$$



Algebra and  
polynomials  
assignment

Collect like terms.

$$3x^3 - 3x^2 = 2x^2 + 16x - 12$$

Rearrange.

$$3x^3 - 5x^2 - 16x + 12 = 0$$

Let  $P(x) = 3x^3 - 5x^2 - 16x + 12$

$$P(x) = 3x^3 - 5x^2 - 16x + 12$$

By inspection,  $P(-2) = 0$  and  $P(3) = 0$ . Use division to find the remaining factor,  $(3x - 2)$ .

$$P(x) = (x + 2)(x - 3)(3x - 2)$$

$$(x + 2)(x - 3)(3x - 2) = 0$$

Write the solutions.

$$x = -2 \text{ or } x = 3 \text{ or } x = \frac{2}{3}$$

## Exercise 2.11 Polynomial equations

Your teacher may tell you to use your graphics calculator for *some* of these questions.

1 Solve each equation.

**a**  $2x^3 + x^2 - 4x - 3 = 0$

**c**  $x^3 - 3x^2 - 13x + 15 = 0$

**e**  $x^4 + x^3 - 11x^2 - 9x + 18 = 0$

**g**  $3x^4 + 13x^3 - 16x^2 - 68x + 48 = 0$

**b**  $3x^3 + 4x^2 - 12x - 16 = 0$

**d**  $x^3 + 2x^2 - 20x + 24 = 0$

**f**  $2x^4 - x^3 - 9x^2 + 4x + 4 = 0$

**h**  $18x^4 - 39x^3 - 54x^2 + 21x + 18 = 0$

2 Solve each equation.

**a**  $x^3 - 24 = 10x - 3x^2$

**c**  $10x^3 = (11x + 6)(x^2 + 1)$

**b**  $4x^3 - 10x + 5 = 2x^3 - x^2 - 2x$

**d**  $\frac{x^2 + 5}{6} = x - \frac{2}{x}$

Example  
28

Example  
29



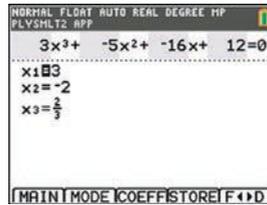
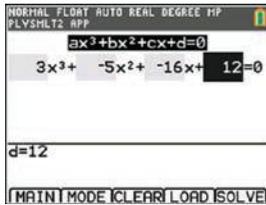
## TECHNOLOGY

### Solving polynomial equations using a calculator

Polynomial equations such as  $\frac{x-1}{2} = \frac{(x+4)^2 - 22}{3x^2}$  can be solved using a graphics calculator.

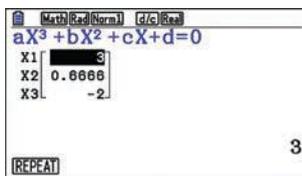
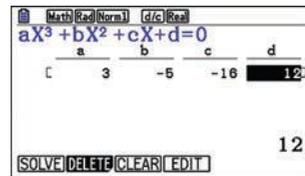
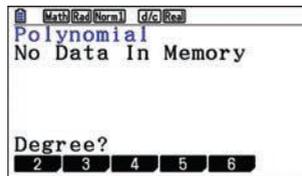
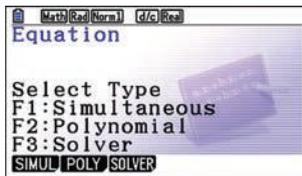
#### TI-84 Plus CE

Express as  $3x^3 - 5x^2 - 16x + 12 = 0$ . Press the **APPS** button and select the polynomial solver (item 7). Select the polynomial root finder (item 1).  $P(x)$  is a 3rd degree polynomial, so set the order to 3. Press **GRAPH** and then enter the values for  $a$ ,  $b$ ,  $c$  and  $d$ . Press solve (**F5**) to obtain solutions for  $P(x)$ . Therefore  $x = 3$  or  $x = -2$  or  $x = \frac{2}{3}$ .



#### Casio fx-CG20AU

Express as  $3x^3 - 5x^2 - 16x + 12 = 0$ . Select **Equation** (item 10) from the menu and select **Polynomial**.  $P(x)$  is a 3rd degree polynomial, so set the degree to 3. Enter the values for  $a$ ,  $b$ ,  $c$  and  $d$ . Press **solve** to obtain solutions for  $P(x)$ . Therefore  $x = 3$  or  $x = 0.666 = \frac{2}{3}$  or  $x = -2$ .



## 2. CHAPTER SUMMARY

### Algebra and polynomials

#### Index laws

- For positive integers  $m$  and  $n$  and any number  $a \neq 0$ :

$$\text{First law: } a^m \times a^n = a^{m+n}$$

$$\text{Second law: } a^m \div a^n = a^{m-n}$$

$$\text{Third law: } (a^m)^n = a^{mn}$$

$$\text{Fourth law: } (ab)^n = a^n b^n$$

- Zero, negative and fractional indices:

$$a^0 = 1, \text{ where } a \neq 0$$

$$a^{-n} = \frac{1}{a^n}, \text{ where } a \neq 0$$

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

#### Scientific notation

- Numbers in **scientific notation** have a **mantissa** (base) from 1 to 10 multiplied by a power of ten. This power is the **order** of the number.

#### Expansion and factorisation

- The distributive law: For any real numbers  $a$ ,  $b$  and  $c$ ;  $a(b + c) = ab + ac$
- The FOIL method (First, Outer, Inner, Last)  $(a + b)(c + d) = ac + ad + bc + bd$
- Perfect squares:  $(a + b)^2 = a^2 + 2ab + b^2$
- Difference of squares:  $(a + b)(a - b) = a^2 - b^2$
- Perfect cubes:  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
- To factorise an expression, look for the **highest common factor** and place it outside the brackets.
- Factorising a sum or product of cubes:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

## Quadratic equations

- The standard form of the quadratic equation is  $ax^2 + bx + c = 0$ , where  $a \neq 0$
- The **null factor law**: If  $ab = 0$ , then either  $a = 0$ ,  $b = 0$  or  $a = b = 0$
- A quadratic equation can also be solved by completing the square by arranging the expression in the form  $x^2 + 2ax + a^2 = (x + a)^2$
- A **root**, or solution, of an equation is a value of the variable that makes the equation true.
- Roots of a function are the values for which  $f(x) = 0$ .

## The quadratic formula and discriminant

- For any quadratic equation  $ax^2 + bx + c = 0$  where  $a \neq 0$ , real solutions are given by 
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ when } b^2 - 4ac \geq 0$$
- The discriminant,  $\Delta = b^2 - 4ac$ 
  - If  $b^2 - 4ac > 0$ , then  $f(x) = ax^2 + bx + c$  has two real roots
  - If  $b^2 - 4ac = 0$ , then  $f(x) = ax^2 + bx + c$  has one real root
  - If  $b^2 - 4ac < 0$ , then  $f(x) = ax^2 + bx + c$  has no real roots

## Direct and inverse proportion

- When two quantities vary in the same ratio, then one variable is said to **vary directly** with the other. This is known as **direct proportion**, shown as  $y \propto x \rightarrow \frac{y}{x} = k$  or  $y = kx$
- When the increase in one variable causes a decrease in the other and vice versa, then the variables are **inversely proportional**, shown as  $y \propto \frac{1}{x} \rightarrow y = \frac{k}{x} \rightarrow xy = k$

## Polynomials and polynomial factorisation

- A **polynomial** is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0 \text{ for integers } n \geq 0$$

where  $a_0, a_1, a_2, \dots, a_n$  are real numbers called the **coefficients** of the polynomial and  $n$  is the **degree**, the highest power of  $x$ .

- A **monic** polynomial has leading coefficient,  $a_n = 1$
- **Remainder theorem**: If a polynomial  $P(x)$  is divided by  $(x - a)$ , then  $Q(x)$  and  $R$  exist such that  $P(x) = (x - a)Q(x) + R$ , and the remainder  $R$  is given by  $R = P(a)$ .
- **Factor theorem**:  $(x - a)$  is a factor of the polynomial  $P(x)$  if and only if  $P(a) = 0$ .

# 2. CHAPTER REVIEW

## Algebra and polynomials

Examples  
1, 2

1 Simplify and express with positive indices where appropriate.

a  $\frac{24p^3q \times 3p^5q^{-3}}{24p^{-6}q^4}$

b  $\frac{9^4 \times 12^{-2} \times 8^2}{18^3}$

c  $81^{\frac{3}{4}}$

d  $\sqrt[3]{8^2}$

e  $(\sqrt[3]{27})^4$

f  $\frac{(\sqrt{2})^{-1}}{4p^{-3}}$

Example  
3

2 Write in scientific notation.

a 0.002 064

b 36 320

c 95

d 0.022

Example  
4

3 Write as ordinary numbers.

a  $6.87 \times 10^4$

b  $3.21 \times 10^7$

c  $4.27 \times 10^2$

d  $8.9 \times 10^{-4}$

Example  
5

4 State the number of significant figures in each number.

a 202 063

b 0.084

c 3.097

d 7000

Examples  
6-8

5 Expand and simplify:

a  $3x(x + y)$

b  $5(3h - 6)$

c  $3h^2j(2j^2 - 5j + 1)$

d  $(2p + 3)(4p^2 + 9 - 6p)$

e  $3p(p^3 - q^3) - 2q(q^3 - 4p^4)$

f  $4t^2(5t^2 - 2t + 7)$

g  $(3x - 5)^2$

h  $(2a - 7)^2$

i  $(a + 5)^2(2a - 3)$

j  $(8v + 2e)^2$

k  $(k + 2h - 3)(2k - h + 5)$

l  $(x + y)(x - y)$

m  $(y - 6)(y + 6)$

n  $(8 + 5k)(8 - 5k)$

o  $(5 - x)(x + 5)$

p  $(a + b)^3$

q  $(2x - 4y)^3$

r  $2de(4d + 8e)^3$

Examples  
9, 10

6 Factorise:

a  $5m^2n^2 - 5m^3n^2 - 5m^2n$

b  $6hi - 3h^2 - 2hg + 4gi$

c  $3m^2 - 16m - 12$

d  $8v^2 + 26vw + 15w^2$

e  $3r^7s - 12r^5s^3 + 3r^4s^4 - 12r^2s^6$

Example  
11

7 Simplify:

a  $\frac{6a^2 - 54b^2}{2a^2 + 12ab + 18b^2}$

b  $\frac{p^2 - 1}{4p^2 - 4p - 24} \div \frac{p^2 - 2p + 1}{2p^2 - p - 15}$

c  $\frac{3r + 1}{5r - 2} - \frac{2r^2 - 6}{15r^2 + 4r - 4} + \frac{2r + 5}{3r + 2}$

8 Factorise each expression.

a  $7p^2 - 24pq + 9q^2$

b  $3y^2 - 23y + 30$

c  $3a^2 - 31a + 56$

d  $3x^2 - 8xy + 5y^2$

e  $5e^2 - ef - 4f^2$

f  $16z^2 + 58az - 63a^2$

9 Factorise:

a  $x^2 + 8xy + 16y^2$

b  $h^2 - 64$

c  $25p^2 + 70pq + 49q^2$

d  $9y^2 - 48y + 64$

e  $16m^2 - 9f^4$

f  $27p^3 + 1$

g  $16y^2 - 24y + 9$

h  $8 - 125q^6$

i  $p^6 - 64$

j  $9g^2 + 25f^2 + 30fg$

k  $f^3 + 8$

l  $1 - 27m^6$

Example  
12

Examples  
12-14

Example  
15

Example  
15

Example  
16

Example  
17

Example  
18

Example  
19

Example  
23

10 Solve each quadratic equation, correct to two decimal places where necessary.

a  $3x^2 + 2x = 8$

b  $6g^2 + 27g + 30 = 0$

c  $5m^2 = 2m + 9$

d  $\frac{h+2}{h-2} = 3h+4$

11 Use factorisation to find the roots of each equation.

a  $3x^2 - 18x + 15 = 0$

b  $-2x^2 - 18x - 28 = 0$

c  $5x^2 + 30x + 40 = 0$

d  $10x - x^2 - 24 = 0$

12 Solve  $x^2 + 2 = 8x$  by completing the square.

13 Use the **Trace** function on a graphics calculator to find approximate roots of each equation, correct to three decimal places.

a  $x^2 + 3x - 9 = 0$

b  $x^2 + 5x + 2 = 0$

c  $x^2 - 3x + 1 = 0$

14 Use the quadratic formula to solve each equation correct to 3 decimal places.

a  $x^2 - x - 5 = 0$

b  $9p^2 = 15p + 5$

c  $5(x^2 + 1) = 7(x + 1)$

d  $z^2 - \frac{z}{5} - \frac{3}{5} = 0$

15 Use the discriminant to determine whether each equation has two real roots, one real root or no real roots.

a  $5x^2 + 2x - 4 = 0$

b  $x^2 - 4x + 9 = 0$

c  $2x^2 + 4x + 2 = 0$

16 State whether each expression is a polynomial and why.

a  $4x - 3$

b  $3x^2 + 2^x + 9$

c  $5x^2 - 2\sqrt{3x} + 6$

d  $4x^2 - \frac{2}{3}x + 4$

e  $\frac{6}{x - \sqrt{5}} - 2x^2$

f  $3x^5 - 2x^2 + \sqrt{x^2 - 8x + 16} + 3$

Example  
23

**17** Write down the leading term and degree for each polynomial.

**a**  $5 - x^2 - 6x^3 - 5x$

**b**  $6x^2 - x^5 + 3x^4 - 4x$

**c** 4

**d**  $t^9 - t^7 + t^5 + 15t^3$

Example  
24

**18** If  $f(x) = 5x^2 - 2x + 4$  and  $g(x) = (4x - 3)^3$ , evaluate:

**a**  $f(2)$

**b**  $f(-3)$

**c**  $f(\sqrt{6})$

**d**  $g(-4)$

**e**  $g(2)$

**f**  $g\left(\frac{1}{2}\right)$

**g**  $f[g(x)]$

**h**  $f[g(-2)]$

Example  
25

**19** Find each quotient  $Q(x)$  and remainder  $R$ .

**a** Divide  $x^3 + 5x - 7$  by  $x - 4$ .

**b** Divide  $2x^6 - x^5 - 2x^4 + x^3 + 4x - 2$  by  $x - 1$ .

**c** Divide  $9x^4 + 5x^3 - x^2 + 4x - 5$  by  $x^3 + x^2 - x + 1$ .

Examples  
26, 27

**20** Factorise each polynomial.

**a**  $x^3 - 2x^2 - x + 2$

**b**  $4x^3 + 21x^2 + 14x - 24$

**c**  $3x^3 + 16x^2 + 23x + 6$

Examples  
28, 29

**21** Solve each equation by factorisation.

**a**  $3x^3 + 2x^2 - 11x - 10 = 0$

**b**  $2x^3 - 3x^2 - 11x + 6 = 0$

**c**  $4x^5 + 22x^4 + 12x^3 - 90x^2 - 108x = 0$

**d**  $20x^6 - 100x^5 + 35x^4 + 405x^3 - 450x^2 = 0$

**e**  $(5x + 2)(x + 3) + 2x = 3x^2(x + 1)$

**f**  $\frac{25x - 25}{2x} + \frac{10}{2x^2 + 2x} = \frac{3x^2 + 8}{x + 1}$

Example  
20

**22** The compression,  $C$ , of a car spring is directly proportional to the force applied. A force of 2000 N causes a compression of 1.6 cm.

**a** Determine the constant of proportionality.

**b** Write the equation in terms of force ( $F$ ).

**c** What compression would be caused by a 5000 N force?

**d** What force would cause a compression of 4.5 cm?

Examples  
21, 22

**23** The time taken for a journey is inversely proportional to the speed. A trip takes 2 hours at 60 km/h.

**a** What is the constant of proportionality?

**b** Write the equation in terms of speed.

**c** How long would the trip take at 80 km/h?

**d** What speed would be necessary to do the trip in 30 minutes?

## Problem solving

- 24** A rectangle has an area of 108 square units. Its length is 3 units longer than its width. Find the dimensions of the rectangle.
- 25** A rectangle has an area of 36 square units and a perimeter of 26 units. Find the length of the sides.
- 26** A taxi fare shared equally amongst a group of tourists costs a total of \$72. If two more tourists were in the group, the same fare would cost each person \$3 less. How many tourists were in the group and how much did each pay?
- 27** A shipping container is proportioned to have its width twice the size of its height, and its length twice the size of its width. It has a total volume of  $64 \text{ m}^3$ . Find the dimensions of the container.



# 3.

## BASIC PROBABILITY

How much should you be charged to insure a second-hand car? What about house or travel insurance? Should your doctor prescribe tablets for a disease? The cost of insurance depends on how likely you are to make a claim. Your doctor needs to know whether the disease is riskier than the medication. In this chapter you will learn how mathematical probability helps to answer questions like these.

3.01 The language of events and sets

3.02 Fundamental probability

3.03 Probabilities of combined events

3.04 Relative frequencies

Chapter summary

Chapter review

## SYLLABUS SUBJECT MATTER

### Language of events and sets

- recall the concepts and language of outcomes, sample spaces and events as sets of outcomes
- use set language and notation for events, including  $\bar{A}$  or  $A'$  for the complement of an event,  $A \cap B$  for the intersection of events  $A$  and  $B$ , and  $A \cup B$  for the union, and recognise mutually exclusive events
- use everyday occurrences to illustrate set descriptions and representations of events, and set operations, including the use of Venn diagrams

### Review of the fundamentals of probability

- review probability as a measure of 'the likelihood of occurrence' of an event
- recall the probability scale:  $0 \leq P(A) \leq 1$  for each event  $A$ , with  $P(A) = 0$  if  $A$  is an impossibility and  $P(A) = 1$  if  $A$  is a certainty
- recall the rules:  $P(\bar{A}) = 1 - P(A)$  and  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- use relative frequencies obtained from data as point estimates of probabilities

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Prior learning

# TERMINOLOGY

and	cardinality	certain
complement	countable	disjoint
element	elementary outcome	empty set
event	event space	expected frequency
favourable outcome	finite	grid
impossible	infinitely countable	intersection
mutually exclusive	not	null set
or	outcome	probability
proper subset	random	relative frequency
replacement	sample space	set
simplified tree diagram	subset	table
tree diagram	trials	union
universal set	Venn diagram	



Set operations

## 3.01 The language of events and sets

When you roll a normal die, there are 6 possible results: 1, 2, 3, 4, 5 or 6.



Venn diagrams

When you toss two coins there are 4 possible results: both heads (HH), head and tail (HT), tail and head (TH) or two tails (TT). HT and TH are different, depending on which coin comes down heads.



Venn diagrams

The possible results are usually written as the **sets**  $\{1, 2, 3, 4, 5, 6\}$  and  $\{HH, HT, TH, TT\}$ .



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A **set** is a **well-defined** collection of objects. Each object is called an **element** of the set. To be **well-defined** you must be able to tell whether or not an object is in the set and to tell one element from another.

A set is usually named by a capital letter. The number of elements in a set is called the **cardinality** of the set. The cardinality of  $A$  is written as  $|A|$ .

The symbol  $\in$  means 'is a member of', so  $2 \in E$  means 2 is in the set  $E$  and  $3 \notin E$  means that 3 is not in the set  $E$ .

The **universal set**  $U$  or  $E$  is the set that contains all the possible elements of sets in a particular discussion. The **null** or **empty** set  $\{\}$  or  $\emptyset$  has no elements.

You can write the set of positive even numbers less than 10 as  $E = \{2, 4, 6, 8\}$ ,

$E = \{x \in \mathbf{Z}^+ : x < 10 \text{ and } \frac{x}{2} \in \mathbf{Z}^+\}$  or  $E = \{0 < x < 10 : x = 2n, n \in \mathbf{Z}\}$ . ‘:’ means ‘such that’.

The cardinality of the counting numbers (natural numbers  $\mathbf{N}$ ) is written as  $\aleph_0$ . (aleph 0).

A set with 3 elements can be **counted** by matching the elements with 1, 2 and 3.

A **countable** set is a set whose elements can be matched with the counting numbers in the same way.

A countable set is either **finite** or **infinitely countable** so also has cardinality  $\aleph_0$ . You can write the elements of a countable set as a sequence.

### EXAMPLE 1

- a** List the elements of the sets  $A = \{x : x \text{ is a prime number less than } 15\}$ ,  
 $B = \{y \in A : 3y = 12\}$ ,  $C = \{n : n \text{ is a positive even number}\}$ ,  $D = \{\text{days of the week}\}$ .
- b** State the cardinality of  $A, B, C$  and  $D$ .

#### Solution

- a** Prime numbers have exactly 2 factors.  $A = \{2, 3, 5, 7, 11, 13\}$   
 $y = \frac{12}{3} = 4$ , but  $4 \notin A$ .  $B = \{\}$   
Use ... to show that a pattern continues.  $C = \{2, 4, 6, 8, 10, 12, \dots\}$   
Give the names.  $D = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$
- b** There are six elements in  $A$ .  $|A| = 6$   
 $B$  is the null set.  $|B| = 0$   
Write a pairing of even numbers and counting numbers.  $2 \leftrightarrow 1, 4 \leftrightarrow 2, 6 \leftrightarrow 3, 8 \leftrightarrow 4, 10 \leftrightarrow 5, \dots$   
Write the cardinality of  $C$ .  $|C| = \aleph_0$   
Count the days of the week.  $|D| = 7$

You can compare and combine the elements of different sets.

Two sets are **equal** if they have exactly the same elements.

The set  $A$  is a **subset** of  $B$  if every element of  $A$  is also in  $B$ . This is shown as  $A \subseteq B$ .

$A$  is a **proper subset** of  $B$  if it is also smaller than  $B$ . This is written as  $A \subset B$ .

The set of all the elements not in  $A$  is called the **complement** of  $A$ , shown as  $\bar{A}$  or  $A'$ .

The **union** of  $A$  and  $B$ ,  $A \cup B$ , is the set that contains all the elements in  $A$  or  $B$ .

The **intersection** of  $A$  and  $B$ ,  $A \cap B$ , is the set that contains the elements in both  $A$  and  $B$ .

Two sets  $A$  and  $B$  are **mutually exclusive (disjoint)** if they have no common elements.

It is usual to exclude the trivial case where one of the sets is the empty set.  $A$  and  $B$  are mutually exclusive if  $A \cap B = \{\}$ ,  $A \neq \{\}$  and  $B \neq \{\}$ .

The union of  $A$  and  $\bar{A}$  must include every element so  $A \cup \bar{A} = U$ .

The elements of  $\bar{A}$  are not in  $A$  so  $A \cap \bar{A} = \{\}$ .

Clearly,  $\bar{\bar{A}} = A$ ,  $\bar{U} = \{\}$  and  $\bar{\{\}} = U$ .

## EXAMPLE 2

$A = \{2, 4, 6, 8\}$ ,  $B = \{0, 1, 2, 3, 4\}$ ,  $C = \{5, 6, 7, 8\}$ ,  $D = \{3, 6, 9\}$  and  $E = \{0, 2, 4\}$  are all sets of digits. Find:

- |          |                           |          |                             |          |            |
|----------|---------------------------|----------|-----------------------------|----------|------------|
| <b>a</b> | $\bar{A}$                 | <b>b</b> | $A \cap B$                  | <b>c</b> | $C \cup D$ |
| <b>d</b> | $ (B \cup C) \cap D $     | <b>e</b> | all the subsets of $D$      |          |            |
| <b>f</b> | the proper subsets of $E$ | <b>g</b> | the mutually exclusive sets |          |            |

### Solution

- |          |  |   |
|----------|--|---|
| <b>a</b> | Write all the digits not in $A$ .  | $\bar{A} = \{0, 1, 3, 5, 7, 9\}$  |
| <b>b</b> | Write the elements in both $A$ and $B$ .   | $A \cap B = \{2, 4\}$   |
| <b>c</b> | Write the elements in $C$ or $D$ (or both).  | $C \cup D = \{3, 5, 6, 7, 8, 9\}$   |
| <b>d</b> | The inner bracket must be done first.<br>Now do the intersection.<br>Count the elements. | $B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$<br>$(B \cup C) \cap D = \{3, 6\}$<br>$ (B \cup C) \cap D  = 2$ |

**e** Do all the subsets systematically so you don't miss any.

**f** Don't include the set itself.

**g** Mutually exclusive sets have no common elements.

The subsets are  $\{3, 6, 9\}$ ,  $\{3, 6\}$ ,  $\{3, 9\}$ ,  $\{6, 9\}$ ,  $\{3\}$ ,  $\{6\}$ ,  $\{9\}$ ,  $\{\}$ .

The proper subsets are  $\{0, 2\}$ ,  $\{2, 4\}$ ,  $\{0, 4\}$ ,  $\{0\}$ ,  $\{2\}$ ,  $\{4\}$ ,  $\{\}$ .

$B \cap C = \{\}$ ,  $C \cap E = \{\}$  and  $D \cap E = \{\}$ , so there are three mutually exclusive pairs of sets:  $B, C$ ;  $C, E$ ;  $D, E$

## INVESTIGATION

### POWER SETS

The **power set** of a given set is the set of all subsets of the set.

The power set of the set  $A = \{a, b, c\}$  is

$$P(A) = \{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \{\}\}.$$

- 1 What is the cardinality of  $A$ ?
- 2 What is the cardinality of  $P(A)$ ?
- 3 What is the cardinality of the power set of the null set?
- 4 What is the cardinality of the power set of the set  $B = \{b\}$ ?
- 5 Investigate the cardinality of the power sets of sets of cardinality 2, 4 and 5.
- 6 Can you write a rule for the cardinality of a power set of any finite set?

There are 6 possible results from rolling a normal die: 1, 2, 3, 4, 5 and 6.

There are 52 possible results for a card drawn from a normal pack, including the queen of clubs, the 3 of hearts, the ace of spades and so on.

In a card game, you might be interested in the likelihood of getting any ace, rather than a particular card.

There are 8 possible results from tossing three coins: HHH, HHT, HTH, THH, HTT, THT, TTH and TTT.

You might be interested in the likelihood of getting two heads, rather than just one of the 8 possible results.

In a probability situation:

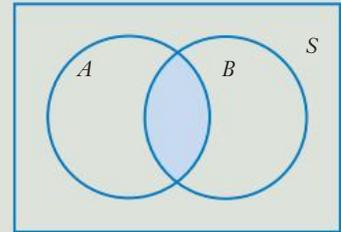
- each possible individual result is called an **outcome**
- the set of all outcomes is called the **sample space** and is written as  $S$
- an **event** is a subset of the sample space
- $A$  **and**  $B$  means  $A \cap B$ : both  $A$  and  $B$  occur
- $A$  **or**  $B$  means  $A \cup B$ : at least one of  $A$  and  $B$  occur
- not  $A$  means  $\bar{A}$

The sample space is the universal set for the situation.

In probability,  $n(A)$  is often used for  $|A|$ , the number of elements in  $A$ .

You can use a **Venn diagram** to show events in a sample space. The sample space is shown by a rectangle and events are shown as circles or ovals within it.

The intersection of two sets is shown by the overlapping part of their circles.



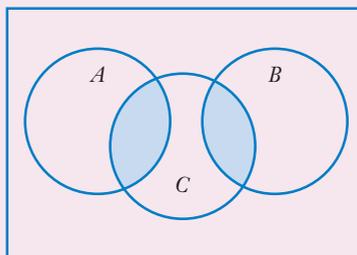
### EXAMPLE 3

The three-handed game of 1000 uses only the 9, 10, Jack, Queen, King and Ace of each suit (hearts, diamonds, clubs and spades). The order of cards in each suit is 9, J, Q, K, 10, A, with the ace as the highest. Each player is dealt 7 cards, with the remaining 3 going into the kitty. You must have the king and queen of a suit and lead one to declare trumps.

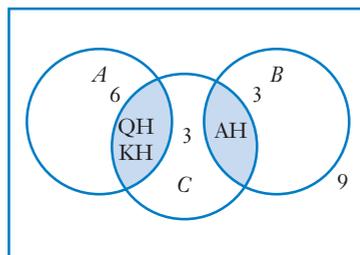
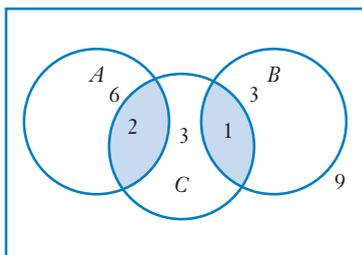
- Use the symbols 9, J, Q, K, 10, A and H, D, C and S to write the sample space.
- Write the event  $A =$  'cards to declare trumps'.
- Write the event  $B =$  'ace'.
- Write the event  $C =$  'hearts'.
- Write  $n(S)$ ,  $n(A)$  and  $n(C)$ .
- Find  $n(A$  and  $C)$ .
- Find  $n(A$  or  $B)$ .
- Draw a Venn diagram showing  $A$ ,  $B$  and  $C$ .

## Solution

- a** Write all possible outcomes.  $S = \{9H, 10H, JH, QH, KH, AH, 9D, 10D, JD, QD, KD, AD, 9C, 10C, JC, QC, KC, AC, 9S, 10S, JS, QS, KS, AS\}$
- b** Write all the kings and queens.  $A = \{QH, KH, QD, KD, QC, KC, QS, KS\}$
- c** Write the aces.  $B = \{AH, AD, AC, AS\}$
- d** Write the hearts.  $C = \{9H, 10H, JH, QH, KH, AH\}$
- e** Count the numbers of elements.  $n(S) = 24, n(A) = 8$  and  $n(C) = 6$
- f**  $(A \text{ and } C) = A \cap C$ .  
Count the set.  $A \cap C = \{QH, KH\}$   
 $n(A \text{ and } C) = 2$
- g**  $(A \text{ or } B) = A \cup B$ .  
Count the set.  $A \cup B = \{QH, KH, AH, QD, KD, AD, QC, KC, AC, QS, KS, AS\}$   
 $n(A \text{ or } B) = 12$
- h**  $A$  and  $B$  are mutually exclusive, so show them with no intersection.  $C$  intersects with both  $A$  and  $B$ .



You can show some or all the individual outcomes or the numbers of elements in a Venn diagram if it helps you solve a problem. You could redraw the diagram in Example 3h as shown below.



### Exercise 3.01 The language of events and sets

Example

1

- 1 List the elements of each set.
  - a  $A = \{\text{even numbers between 13 and 33}\}$
  - b  $B = \{n \in \mathbf{N}: 21 < 2n - 1 < 51\}$
  - c  $C = \{x = 2n + 1: n \in \mathbf{N}, 4 \leq n \leq 12\}$
  - d  $D = \{\text{subsets of } \{1, 3, 5, 7\}\}$
  - e  $F = \{\text{3-letter abbreviations of months of the year}\}$

2 State the cardinality of each set in question 1.

3 How many elements does each set have?

- a  $A = \{\text{odd numbers between 2 and 20}\}$
- b  $B = \{x = 2n: n \in \mathbf{N}: 8 \leq n \leq 16\}$
- c  $C = \{\text{result of tossing two coins}\}$
- d  $D = \{\text{possible 'word' arrangements formed from the letters p, q and r}\}$
- e  $F = \{\text{numbers that are both even and odd}\}$

Example

2

4  $A = \{a, b, c\}$ ,  $B = \{b, d, e\}$ ,  $C = \{m, p, r\}$ ,  $D = \{p, r, t\}$  and  $F = \{\text{non-vowels}\}$ . Find:

- a  $U$
- b  $A \cap B$
- c  $C \cup D$
- d  $\overline{F}$
- e  $A \cup C$
- f  $A \cap C$
- g  $n(F)$
- h any disjoint sets
- i the subsets of  $D$
- j the proper subsets of  $C$

5  $M = \{3, 7, 9\}$ ,  $P = \{2, 4, 6, 8\}$ ,  $Q = \{1, 2, 3, 4, 5\}$ ,  $T = \{5, 6, 7, 8\}$  and  $V = \{0, 3, 6, 9\}$ . Find:

- a  $U$
- b  $M \cup P$
- c  $\overline{Q \cup T}$
- d  $\overline{Q \cap T}$
- e  $\overline{Q \cup T}$
- f  $\overline{Q \cap T}$
- g  $\overline{\overline{V}}$
- h any disjoint sets
- i the subsets of  $P$
- j the proper subsets of  $T$

Example

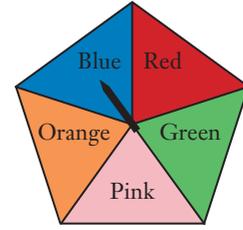
3

6 Two coins are tossed. Use H and T for each of the following.

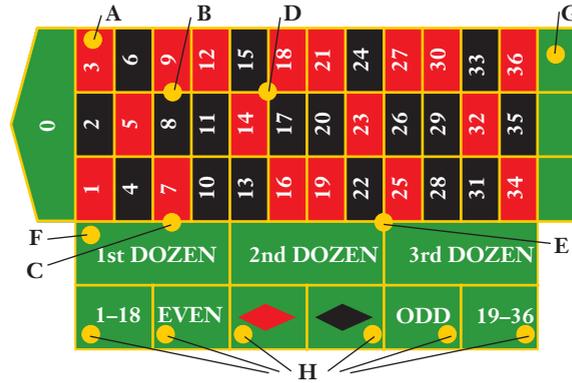
- a List the sample space.
- b Write the event  $A$  'the first coin is heads'.
- c Write the event  $B$  'the second coin is heads'.
- d Write the event  $A$  and  $B$ .
- e Write the event  $A$  or  $B$ .
- f Write the event  $C$  'At least one coin is tails'.
- g Write the event  $D$  'Exactly one coin is tails'.
- h Write the event  $E$  'Two tails'.
- i Write the event  $F$  'No tails'.
- j Write the event not  $C$ .

## Problem solving

- 7** A spinner made in the shape of a regular pentagon is spun. The equal-sized sectors are coloured blue, red, green, pink and orange.



- Write the sample space.
  - Write the event 'a primary colour'.
  - Write the event 'not a primary colour'.
  - Write the event 'black or white'.
  - Write the event 'not black or white'.
  - How many elements are there in each of events **b–e**?
- 8** A roulette wheel in a casino has numbers 0 through 36. Bets are placed on a table marked as shown. Eight different types of bets are possible. They are indicated in the diagram by the letters A to H, which show the positions at which chips would be placed to indicate the bets. You cannot bet on 0. The house wins all bets if 0 comes up.



The bets are:

- A: a single number—'straight up'  
 B: two numbers—'split' (either of the numbers the chip touches wins)  
 C: three numbers—'street' (any of the three numbers in the row wins)  
 D: four numbers—'corner' (any of the four numbers the chip touches wins)  
 E: six numbers—'six line' (any of the numbers in the two rows wins)  
 F: first, second or third dozen—'dozen'  
 G: left, middle or right twelve-number column—'column'  
 H: odd/even, red/black, first/second half—'even chances'.

Find  $n$  for each of the possible bets.



Dashboard  
probabilities



Venn diagrams

## 3.02 Fundamental probability

Many situations have a finite sample space in which the outcomes are equally likely.

When you toss 3 coins the sample space is {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}. Consider the events  $A = \{\text{exactly 2 heads}\}$  and  $B = \{\text{exactly 3 heads}\}$ .

$n(A) = 3$  and  $n(B) = 1$ , so you would say that  $A$  is more likely than  $B$ . Real experience shows that you get 2 heads about 3 times as often as 3 heads when you toss three coins.

The **probability** of an event  $E$  in a finite sample space  $S$  with equally likely outcomes is

$$P(E) = \frac{n(E)}{n(S)}.$$

Equally likely outcomes are referred to as **random**.

### EXAMPLE 4

From a class of 24 Maths Methods students, 9 study both Physics and Chemistry and 17 take Physics. Two students do not do either Physics or Chemistry. What is the probability that a randomly selected student takes Chemistry but not Physics?

#### Solution

Identify the sample space.

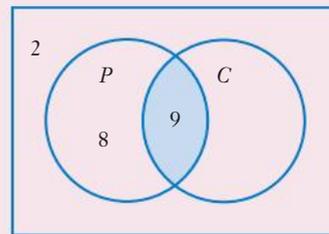
$S = \{\text{students in the class}\}$

Choose names for events.

Let  $P = \{\text{students doing physics}\}$  and

$C = \{\text{students doing chemistry}\}$

Draw a Venn diagram.



Put in the numbers known.

$$n(\overline{P \cup C}) = 2.$$

$$n(P \cap C) = 9 \text{ and } n(P) = 17, \text{ so}$$

$$n(P \cap \overline{C}) = 8.$$

Find the number studying only chemistry.

$$n(C \cap \overline{P}) = 24 - (2 + 8 + 9) = 5$$

Calculate the required probability.

$$P(\text{Chemistry but not physics}) = \frac{n(C \cap \overline{P})}{n(S)} = \frac{5}{24}$$

Write the answer.

The probability that a student does chemistry but not physics is  $\frac{5}{24}$ .

When you toss a coin, throw a die, draw a card or make any random selection, one of the outcomes of the sample space occurs. This means that in a probability situation, one of the outcomes occurs. Events that include this outcome also occur.

The empty set has no outcomes, so cannot occur. It is **impossible** and has probability 0.

The sample space has all outcomes, so must occur. It is **certain** and has probability 1.

The **complement** of  $A$ , 'not  $A$ ', has probability given by  $P(\bar{A}) = 1 - P(A)$ .

The above results are easily proven.

$$P(\{\}) = \frac{n(\{\})}{n(S)} = \frac{0}{n(S)} = 0 \text{ and } P(S) = \frac{n(S)}{n(S)} = 1.$$

Suppose that the event  $A$  has  $k$  outcomes. Then  $n(A) = k$ .

Then  $\bar{A}$  consists of all the outcomes not in  $A$ , so has  $n(S) - k$  outcomes.

$$P(\bar{A}) = \frac{n(\bar{A})}{n(S)} = \frac{n(S) - k}{n(S)} = \frac{n(S)}{n(S)} - \frac{k}{n(S)} = 1 - P(A). \quad \text{QED}$$

### EXAMPLE 5

A box contains 3 red, 4 green and 5 yellow marbles. A marble is chosen at random. What is the probability that it is not green?

#### Solution

Write the probability of getting a green marble.  $P(\text{green}) = \frac{n(\text{green})}{n(S)} = \frac{4}{12} = \frac{1}{3}$

Use the rule for complementary events.  $P(\text{not green}) = 1 - P(\text{green}) = 1 - \frac{1}{3} = \frac{2}{3}$

What do you do if the outcomes of a probability situation are not equally likely?

To qualify as a probability situation, it still has to be true that the probability of the sample space is 1. This means that the sum of probabilities of all the outcomes is 1.

### EXAMPLE 6

A die is weighted so that the probability of rolling a 6 is increased to  $\frac{5}{16}$  and the probability of a 1 decreases to  $\frac{1}{16}$ . The probabilities of 2 to 5 are all the same. What is the probability of rolling:

- a** a 3?      **b** a number more than 3?      **c** a number less than 3?

## Solution

- a** Write the probability of {2, 3, 4, 5}.

$$P(\{2, 3, 4, 5\}) = 1 - P(\{1, 6\})$$

Substitute the known values.

$$= 1 - \left( \frac{1}{16} + \frac{5}{16} \right)$$

Simplify.

$$= 1 - \frac{6}{16} = \frac{5}{8}$$

Use the equal probabilities of 2–5.

$$P(3) = P(\{2, 3, 4, 5\}) \div 4$$

Simplify.

$$= \frac{5}{8} \div 4 = \frac{5}{32}$$

Write the answer.

The probability of a 3 is  $\frac{5}{32}$ .

- b** Add the probabilities of each outcome.

$$P(\{4, 5, 6\}) = \frac{5}{32} + \frac{5}{32} + \frac{5}{16}$$

Simplify.

$$= \frac{5}{8}$$

Write the answer.

The probability of more than 3 is  $\frac{5}{8}$ .

- c** Add the probabilities of each outcome.

$$P(\{1, 2\}) = \frac{1}{16} + \frac{5}{32}$$

$$= \frac{7}{32}$$

Write the answer.

The probability of less than 3 is  $\frac{7}{32}$ .

## Exercise 3.02 Fundamental probability

Example  
4

- 1** A rugby squad of 24 students has 5 students with a Pacific island background. Three of those with islander backgrounds are 17 years old. There are ten 17-year-olds in the team altogether. What is the probability that a randomly selected student from the squad is:

- a** an islander who is not 17?
- b** a non-islander who is 17?
- c** neither an islander nor 17?

- 2** A bag contains 5 red marbles, 8 green marbles and 7 black marbles. A marble is taken at random from the bag.
- a** What is the sample space?
  - b** What is the probability of taking a black marble?
  - c** What is the probability of taking a white marble?
  - d** What is the probability of taking a green marble?
  - e** Which is more likely: a green marble or a red marble?
- 3** A bag in the sports store contains 5 white hockey balls, 6 red cricket balls and 4 white cricket balls. A ball is taken at random.
- a** What is the probability that the ball is red?
  - b** What is the probability that the ball is white?
  - c** Which is more likely: a red ball or a white ball?
  - d** What is the probability that a hockey ball is selected?
  - e** What is the probability that a cricket ball is chosen?
- 4** A sack contained 6 venomous snakes and 5 harmless snakes. One snake found a hole in the sack and escaped.
- a** List the sample space for the escaped snake, using  $V$  for venomous and  $H$  for harmless. Assume that the snakes are equally able to escape.
  - b** What is the probability that a venomous snake escaped?
  - c** What is the probability that a harmless snake escaped?
  - d** What is the probability that the bag now contains equal numbers of venomous and harmless snakes?
  - e** What is the probability that the bag now contains 10 snakes?
- 5** A die is weighted so that the probability of a number is proportional to the number. For example, a six has twice the probability of a three.
- a** Find the probability of each element of the sample space (i.e. 1, 2, 3, 4, 5 and 6).
  - b** What is the probability of an even number?
  - c** What is the probability of a prime number?
  - d** Find the probability of an odd number.

### Problem solving

- 6** A baby born in Australia is selected at random. Assume that babies are equally likely to be born at any time of the year. Find the probability that the baby was born:
- a** after 30 June
  - b** on the weekend
  - c** in July
  - d** between midnight and 4 a.m.
  - e** on a Monday.
- 7** Three horses  $A$ ,  $B$  and  $C$  are in a race.  $A$  is twice as likely to win as  $B$  and  $B$  is twice as likely to win as  $C$ . Calculate their respective probabilities of winning, assuming that other horses in the race have a combined 79% chance of winning.

- 8 An Australian navy patrol vessel has 21 personnel, of whom 5 are women. Two of the women have served more than 5 years and 1 of these is an officer. There are 5 officers altogether on the vessel, not including the captain, who is a man. All except 1 of the officers have served more than 5 years and 16 of all the people on board have served more than 5 years.



Fairfax-Indication/Dallas Kiljonen

Find the probability that a person randomly selected from the vessel is:

- a an officer (including the captain)
- b a female officer
- c a man who has served 5 years or less
- d a woman who has served more than 5 years
- e a female captain
- f a man who has served more than 5 years.



Tree diagrams

### 3.03 Probabilities of combined events

A sample space where you can think of events one after the other is often organised as a branching diagram.

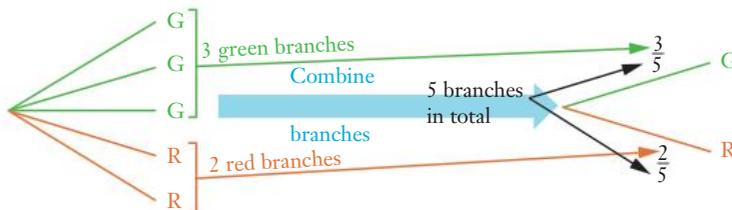


Tree diagrams

A **tree diagram** shows the successive stages of a compound sample space as branches. Draw multiple forks at each stage to show the possible outcomes. The last 'twigs' show the last outcomes of the sample space.

You can combine the branches with the same outcome to make a **simplified tree diagram**. Use a fraction to label the combined branches. Put the number of combined outcomes on the top and the total number of outcomes in the stage on the bottom.

Consider taking a marble from a bag with 3 green and 2 red marbles. A basic tree diagram has 5 branches. You can combine the same outcomes to make just 2 branches instead.



When several objects are taken from a group of objects, you can **replace** each one before taking the next. Otherwise, you can select **without replacement**, taking them all at once.

## EXAMPLE 7



Tree diagrams

A bag contains 3 green and 2 red marbles. Two marbles are taken out at random. What is the probability they are both red, given that:

- a the first marble is replaced before the second is taken?
- b the marbles are taken out together?

### Solution

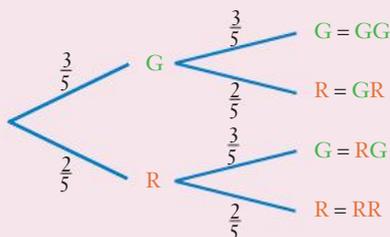
- a Choose abbreviations.

Draw the simplified tree diagram.

The first marble can be any of the 5 marbles: G, G, G, R, R.

The second marble can also be any of the 5 marbles because the first is replaced.

Use G for green and or R for red.



You can show combined outcomes as GG, GR, RG and RR if you like.

Count the sample space.

$$n(S) = 5 \text{ branches} \times 5 \text{ branches} = 25$$

Count the event.

$$n(RR) = 2 \text{ branches} \times 2 \text{ branches} = 4$$

Find the probability.

$$P(\text{both red}) = \frac{4}{25} = 0.16$$

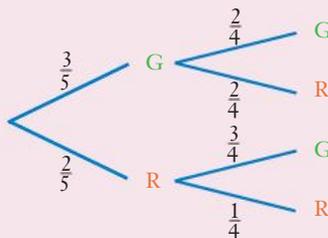
Write the answer.

The probability of both marbles being red is 0.16.

- b Draw the simplified tree diagram.

The first marble can be any of the 5 marbles: G, G, G, R, R.

The second marble can only be one of the remaining 4.



You don't need to show the combined outcomes at the end.

Count the sample space.

$$n(S) = 5 \text{ branches} \times 4 \text{ branches} = 20$$

Count the event.

$$n(RR) = 2 \text{ branches} \times 1 \text{ branch} = 2$$

Find the probability.

$$P(\text{both red}) = \frac{2}{20} = \frac{1}{10} = 0.1$$

Write the answer.

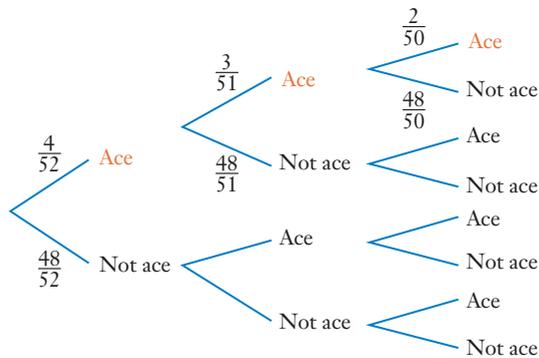
The probability of both marbles being red is 0.1.

Notice in Example 7 that for the probability of different colours, you have to use two final ‘twigs’, GR and RG. For both **a** and **b** you get  $3 \times 2$  and  $2 \times 3$  outcomes for different colours.

For part **a**,  $P(\text{different colours}) = \frac{12}{25} = 0.48$ . For part **b**,  $P(\text{different colours}) = \frac{12}{20} = 0.6$ .

The probability of different colours is greater without replacement.

What is the probability of being dealt three cards that are all aces? A tree diagram that has branches for every card is very big, so it is better to draw a simplified diagram using *Ace* and *Not ace*. This is shown below.



From the diagram,  $P(3 \text{ aces in a row}) = \frac{4 \times 3 \times 2}{52 \times 51 \times 50} = \frac{1}{5525} \approx 0.0002$ .

Copy and complete the diagram with the rest of the fractions.

A **table** or **grid** shows a sample space you can think of as a 2D combination of outcomes. One outcome is shown in the first row and the other in the first column. Each cell of the table shows a combined outcome.

## EXAMPLE 8

Two tetrahedral dice with faces numbered 1 to 4 are thrown. The sum of the faces that the dice rest on is then worked out.

What is the probability of a sum of 6?



### Solution

Draw a table showing the outcomes.

		First die			
		1	2	3	4
Second die	1	$1 + 1 = 2$	$1 + 2 = 3$	$1 + 3 = 4$	$1 + 4 = 5$
	2	$2 + 1 = 3$	$2 + 2 = 4$	$2 + 3 = 5$	$2 + 4 = 6$
	3	$3 + 1 = 4$	$3 + 2 = 5$	$3 + 3 = 6$	$3 + 4 = 7$
	4	$4 + 1 = 5$	$4 + 2 = 6$	$4 + 3 = 7$	$4 + 4 = 8$

Count the sample space.  $n(S) = 16$

Count the event.  $n(6) = 3$

Calculate the probability.  $P(6) = \frac{3}{16} = 0.1875$

Write the answer. The probability of a total of 6 is  $\frac{3}{16}$ .

Two events in a sample space may have outcomes in common. In this case they will have an intersection. You can show this as a Venn diagram.

From the diagram,

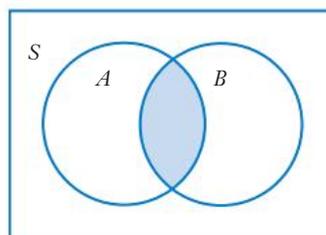
$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Dividing by  $n(S)$ ,

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} - \frac{n(A \cap B)}{n(S)}.$$

But  $\frac{n(A \cup B)}{n(S)} = P(A \cup B) = P(A \text{ or } B)$ , etc., so you get a rule

for combining probabilities.



$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \text{ or } P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

For **mutually exclusive events**,  $P(A \cap B) = 0$ , so  $P(A \cup B) = P(A) + P(B)$ .

## EXAMPLE 9



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- a** In the card-game of poker, a straight is a hand with cards in order, like 8-9-10-J-Q. A flush is a hand with all cards the same suit, like 5 spades. In draw poker, players can discard some cards from their hand and get new ones from a re-shuffled pack. What is the probability that a player who keeps the 7-8-9-10 of clubs will draw a card to get a flush or a straight?
- b** A full house is a hand with 3 of one card and 2 of another, like Q-Q-6-6-6. The probability of being dealt a full house is about 0.144% and the probability of being dealt a straight is about 0.392%. What is the probability of being dealt a full house or a straight?

### Solution

- a** Choose names. Let  $F$  = getting a flush and  $St$  = getting a straight  
Find  $n$ (sample space). There are 52 cards and 48 left, so  $n(S) = 48$ .  
Find  $n$ (flush). There are 13 clubs and 9 left, so  $n(F) = 9$ .  
Find  $P$ (flush).  $P(F) = \frac{9}{48} = \frac{3}{16}$   
Find  $n$ (straight). There are 4 sixes and 4 jacks, so  $n(St) = 8$ .  
Find  $P$ (straight).  $P(St) = \frac{8}{48} = \frac{1}{6}$   
Find  $n$ (straight and flush). The 6 and jack of clubs are left, so  $n(St \cap F) = 2$ .  
Find  $P$ (straight and flush).  $P(St \cap F) = \frac{2}{48} = \frac{1}{24}$

Use the rule.

$$\begin{aligned}P(S \cup F) &= P(S) + P(F) - P(S \cap F) \\&= \frac{1}{6} + \frac{3}{16} - \frac{1}{24} \\&= \frac{5}{16}\end{aligned}$$

Write the answer.

The probability of getting a straight or flush is  $\frac{5}{16}$ .

**b** Note the relationship.

You can't have a full house and a straight.

Write the rule.

The events are disjoint, so

$$P(A \cup B) = P(A) + P(B).$$

Calculate the probability.

$$\begin{aligned}P(\text{full house} \cup \text{straight}) &= 0.144\% + 0.392\% \\&= 0.536\%\end{aligned}$$

Write the answer.

The probability of being dealt a full house or a straight is about 0.536%.

### Exercise 3.03 Probabilities of combined events

- A bag contains three \$1 coins and two \$2 coins. A coin is randomly selected, its value is noted and it is then returned to the bag. This is done twice more. Assuming that the coins are equally likely to be selected, calculate the probability that:
  - all 3 coins selected are \$2 coins
  - at least two \$2 coins are selected.
- Find the probability that the total of the 2 numbers rolled on a pair of normal dice is greater than 4.
- Find the probability that a card selected at random from a well-shuffled pack of normal playing cards is:
  - a club
  - a court card (J–Q–K)
  - a court card that is also a club
  - a court card or a club.
- A jar contains 4 blue and 2 yellow balls. Another jar contains 6 blue and 3 yellow balls. A jar is randomly selected and a ball is chosen at random from that jar. Find the probability that it is yellow.
- Two cards are drawn without replacement from a well-shuffled normal deck of 52 playing cards. Find the probability that the cards are a king and an ace.
- A container holds 12 identical pieces of equipment, of which 4 are defective. Three pieces of equipment are randomly selected from the container, one after the other and without replacement. Find the probability that:
  - all 3 pieces of equipment are defective
  - at least 1 of the pieces of equipment is defective.

Example  
7

Example  
8

Example  
9

- 7** There are 3 containers of light bulbs.
- Container 1 has 10 light bulbs, of which 4 are defective.
  - Container 2 has 6 light bulbs, of which 1 is defective.
  - Container 3 has 8 light bulbs, of which 3 are defective.

Use a tree diagram to find the probability that a light bulb selected at random is defective.

- 8** A counter is placed at zero on a number line. A die is rolled and the counter is moved according to the following rules.

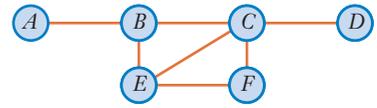
- 1 unit to the left for an odd number on the die
- 1 unit to the right for an even number on the die.

The game is over after 5 rolls or when the counter reaches 3 or  $-2$ .

**a** Draw a tree diagram to describe all the possible paths that the counter may take.

**b** Find the probability that the game concludes in under 5 rolls.

- 9** In the diagram, the letters  $A, B, C, D, E$  and  $F$  represent benches in a park and the lines represent paths connecting them. Natalie begins at  $A$  and walks from bench to bench. She stops when she cannot continue to walk without going over the same path twice.



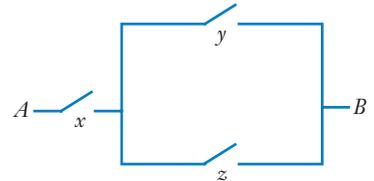
**a** Find the probability that Natalie stops on bench  $B$ .

**b** What is the probability that she stops on bench  $E$ ?

### Problem solving

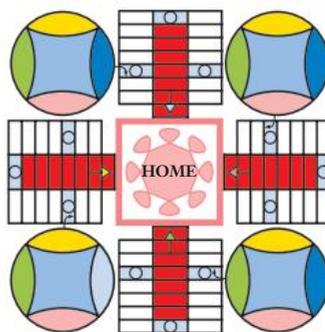
- 10** Monica and Ernesto, who are of roughly equal ability, play each other in a squash tournament. The first person to win 2 games in a row or who wins a total of 3 games wins the tournament. What is the probability that Monica wins the tournament in 3 games or less?

- 11** An electric current may flow from  $A$  to  $B$  in the circuit shown. The current will flow from  $A$  to  $B$  only if the circuit is completed. That is, the switch at  $x$ , plus at least one of the switches at  $y$  and  $z$ , must be down. The switches all operate independently and are just as likely to be up as down.



- a** Given any arrangement of switches at random, what is the probability that:
- the current will flow from  $A$  to  $B$ ?
  - the current will not flow from  $A$  to  $B$ ?
- b** Draw another circuit from  $A$  to  $B$  so that the probability that the current will flow from one side to the other is  $\frac{7}{8}$ .

- 12** In the game of parcheesi, a player may move a token out of the starting base if, when a pair of dice is rolled, either at least one die shows a five or the sum of the uppermost faces is equal to 5. What is the probability that a player can move a token out of the starting base on the first roll of the dice?



## 3.04 Relative frequencies

The past is usually a guide to the future. When you say that a coin is equally likely to land with heads or tails up, you are using your past experience. The sample space and probability are actually based on experience, even though you don't say so.

The equality of outcomes in throwing dice or tossing coins is so well-established that it is considered obvious. In cases where the outcomes are not so well-established you *estimate* the probability using the data available.

For example, 1467 people out of 9861 surveyed said they were left-handed. This can be written as  $\frac{1467}{9861} \approx 0.15$ . This suggests that the probability of someone being left-handed is about 15%. The number of people surveyed means this would be a good estimate, but it doesn't mean that 15 out of every hundred people *must* be left-handed. You would expect *about* 15 in every hundred, but it could be any number. This is called a **point estimate** because only one value is involved, not a range of values.

**Relative frequency** is used to estimate probability from data.

The result characteristic being estimated is called the **favourable outcome**.

The number of data items used is called the number of **trials**.

The estimated probability is given by

$$\text{Relative frequency} = \frac{\text{Frequency of favourable outcome}}{\text{Number of trials}}$$

Conversely, the **expected frequency** of an outcome is given by

$$\text{Expected frequency} = \text{Probability} \times \text{Number of trials}$$



Basic probability assignment



Basic probability extension

## EXAMPLE 10

In a survey of newspaper readers, 297 preferred the *Courier Mail*, 263 the *Australian*, 176 the *Australian Financial Review* and 104 another paper. What is the probability of choosing someone at random who prefers the *Courier Mail*?

### Solution

State the number of favourable outcomes.

*Courier Mail* frequency = 297

State the number of trials.

Number of trials =  $297 + 263 + 176 + 104$   
= 840

Calculate the relative frequency.

Relative frequency =  $\frac{297}{840} = \frac{99}{280} \approx 0.354$

Write the answer.

The probability of choosing someone at random who prefers the *Courier Mail* is about 0.35 or 35%.



Heads and tails



Rolling dice

## INVESTIGATION

### FLIPPING BOXES

For this investigation you will need a number of small rectangular cardboard boxes, such as matchboxes, biscuit boxes, gift boxes or headache tablet boxes.

- 1 Measure the faces of each box and calculate the area of each face. Number the faces from 1 to 6 and express the area of each face as a percentage of the total surface area of the box.
- 2 Predict the relative frequency of landing on each face when the box is flipped into the air.
- 3 Flip each box in the air at least 100 times and each time note the number of the side it lands on. Work out the relative frequency for each side. Present these results in a table.
- 4 Compare the area of each face with the relative frequency of landing on that face.
- 5 Compare your results with those of others.
- 6 Make a conclusion about the relationship between the area of each face and the probability of landing on that face.

Write a report of your findings, including all data, calculations and conclusions.



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## EXAMPLE 11

Weather bureau records show that the probability of rain in Tully in November is about 35% on any particular day. How many wet days would you expect in a school week in Tully in November?

### Solution

State the number of trials.

There are 5 school days in a week.

Use the formula.

$$\begin{aligned}\text{Expected frequency} &= P(E) \times n \\ &= 0.35 \times 5 \\ &= 1.75\end{aligned}$$

Round appropriately for the answer.

There will be about 2 wet days in a school week in Tully in November.

## Exercise 3.04 Relative frequencies

- 1 A packaging machine fills 100 g packets of potato chips. As part of a quality assurance program, the contents of 40 packets are weighed. The results are shown in the table.

Weight of contents (g)	90–94	95–99	100–104	105–109	110–114
Number of packets	3	8	21	7	1

Estimate as a decimal the probability that the next packet filled by the machine will:

- a contain less than 95 g of chips
- b contain at least 110 g of chips
- c be underweight ( $< 100$  g)

A carton of chips holds 500 packets.

- d How many underweight packets would you expect to find in a carton?

- 2 People walking down George Street were asked how they had travelled to the city.
- a Find correct to 2 decimal places the relative frequency for each method of transport.
  - b What is the probability as a fraction that the next person spoken to would have travelled by car?
  - c If you asked another 20 people, how many would you expect to have travelled by train?
  - d For 30 people, how many could you expect to travel by bus?

Main method of transport	Number of people
Taxi	18
Train	101
Bus	78
Car	86
Bicycle	6
Other	11

Examples  
10, 11

- 3** Ten identical pieces of cardboard are tossed into a wastepaper basket and the number of cards that make it into the bin is counted. The results for many attempts are:

0 1 1 1 1, 2 3 3 3 3, 1 1 1 1 2, 2 2 2 2 2, 1 1 1 1 1,  
 4 4 5 5 2, 2 2 6 5 1, 2 2 5 5 2, 2 2 2 2 2, 2 2 4 6 6  
 2 2 3 3 3, 3 2 2 2 2, 2 0 0 0 3, 3 4 4 4 4, 0 2 2 3 3  
 3 3 3 3 0, 0 2 2 6 7, 2 3 3 3 3, 3 3 3 4 4, 4 4 6 6 9

- a** How many trials were conducted in this experiment?  
**b** Construct a frequency distribution table for this data.  
**c** If 10 cards are again thrown into the same wastepaper basket in the same way, what is the probability that exactly 2 will land in the basket?

### Problem solving

- 4** The reasons that Queensland drivers had their licences cancelled or were disqualified from holding a driver's licence during a particular year are shown in the table.

Reason for cancellation	No. of drivers
Alcohol related:	
• blood alcohol > 0.05%	3357
• refusal to provide breath sample	417
• other	525
Dangerous driving	1589
Speeding	3792
Driving under disqualification	1516
Driving without a licence	1413
Demerit points suspension	3159
Other	2846
<b>Total</b>	<b>18 614</b>



Fairfax Information/Rohan Thomson

About 300 disqualifications occurred during a 5-day blitz on Queensland highways. Estimate the number of cancellations from:

- a** speeding  
**b** the accumulation of demerit points  
**c** an alcohol-related cause.

- 5** The clock on the mobile phones of a group of students were checked against the actual time. The approximate amounts they were out (in seconds) were:

+10	0	0	+20	0	-10	-50	+40	+30	0	0	+20
0	0	0	0	-30	-10	0	0	+30	-20	+50	+50
-50	0	-30	0	+20	0	-10	-30	+10	0	0	-10
-20	+20	0	0	+10	0	-10	-20	0	0	+20	0

- What was the number of trials?
  - What was the frequency of the correct time (0)?
  - What was the relative frequency of the correct time?
  - What was the relative frequency of being 20 seconds fast (+20)?
  - If you checked a group of 50 students at the school, how many would you expect to have the correct time?
- 6** In cases of the measles, complications are relatively common. From 4000 cases in children in a recent outbreak, 54 developed complications of some sort, resulting in death for 3 children. In 5 cases where children survived, they suffered serious brain damage, and in 2 other cases the children became deaf.

Complications from the measles vaccination also occur. From 50 000 vaccinations, 80 children had some complications. In 75 of these, it was a fever that subsided after a few days. In 1 case, the child suffered some loss of hearing, and the other 4 cases developed mild cases of measles that did not result in any permanent loss of hearing.

- What is the relative frequency of complications from measles?
- What is the probability of complications from the vaccine?
- What is the probability of serious complications from measles?
- What is the probability of serious complications from the vaccine?
- If 1000 children were vaccinated, how many would you expect to suffer complications?



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# 3. CHAPTER SUMMARY



Probability  
find-a-word

## Basic probability

### Sets

- A **set** is a **well-defined** collection of objects. Each object is called an **element** of the set.
- The number of elements in a set is called the **cardinality**  $|A|$  of the set.
- The symbol  $\in$  means 'is a member of', so  $2 \in E$  means 2 is in the set  $E$  and  $3 \notin E$  means that 3 is not in the set  $E$ .
- The **universal set**  $U$  or  $E$  is the set that contains all the possible elements of sets in a particular discussion. The **null** or **empty** set  $\{\}$  or  $\emptyset$  has no elements.
- The cardinality of the counting numbers (natural numbers  $\mathbb{N}$ ) is written as  $\aleph_0$  (aleph 0). A **countable** set is one such that you can pair all the elements with the counting numbers. It is either **finite** or **infinitely countable**.
- Two sets are equal if they have exactly the same elements.

### Subsets and combinations of sets

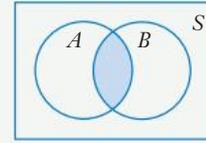
- The set  $A$  is a **subset** of  $B$  if every element of  $A$  is also in  $B$ . This is shown as  $A \subseteq B$ . Set  $A$  is a **proper subset** of  $B$  if it is also smaller than  $B$ . This is written as  $A \subset B$ .
- The set of all the elements not in  $A$  is called the **complement** of  $A$ , shown as  $\bar{A}$  or  $A'$ .
- The **union** of  $A$  and  $B$ ,  $A \cup B$ , is the set that contains all the elements in  $A$  or  $B$ . The **intersection** of  $A$  and  $B$ ,  $A \cap B$ , is the set that contains the elements in both  $A$  and  $B$ .
- **Mutually exclusive (disjoint)** sets have no common elements.  $A$  and  $B$  are mutually exclusive if  $A \cap B = \{\}$ ,  $A \neq \{\}$  and  $B \neq \{\}$ .
- $A \cup \bar{A} = U$       •  $A \cap \bar{A} = \{\}$       •  $\bar{\bar{A}} = A$ ,  $\bar{U} = \{\}$  and  $\bar{\{\}} = U$

### Events and sample spaces

In a probability situation:

- Each possible individual result is called an **outcome**.
- The set of all outcomes is called the **sample space** and is written as  $S$ .
- An **event** is a subset of the sample space.
- $A$  **and**  $B$  means  $A \cap B$ : both  $A$  and  $B$  occur.
- $A$  **or**  $B$  means  $A \cup B$ : at least one of  $A$  and  $B$  occur.
- not  $A$  means  $\bar{A}$ .
- In probability,  $n(A)$  is often used for  $|A|$ , the number of elements in  $A$ .

- A **Venn diagram** shows events in a sample space. The sample space is shown by a rectangle and events are shown as circles or ovals within it. The intersection of two sets is shown by the overlapping part of their circles.



## Basic probability

- The **probability** of an event  $E$  in a sample space  $S$  with equally likely outcomes is

$$P(E) = \frac{n(E)}{n(S)}.$$

- Equally likely outcomes are referred to as **random**.
- The empty set has no outcomes, so cannot occur. It is **impossible** and has probability 0.
- The sample space has all outcomes, so must occur. It is **certain** and has probability 1.
- The **complement** of  $A$ , 'not  $A$ ', has probability given by  $P(\bar{A}) = 1 - P(A)$ .

## Combined probabilities

- A **tree diagram** shows successive stages of a compound sample space as branches. Combining branches with the same outcome makes a **simplified tree diagram**. A fraction on the combined branches shows the number of combined outcomes on the top and the total number of outcomes in the stage on the bottom.
- When several objects are taken from a group of objects, you can **replace** each one before taking the next. Otherwise, you can select **without replacement**, taking them all at once.
- A **table** or **grid** shows a sample space you can think of as a 2D combination of outcomes. Each cell of the table shows a combined outcome.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$  or  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ .
- For **mutually exclusive events**,  $P(A \cap B) = 0$ , so  $P(A \cup B) = P(A) + P(B)$ .

## Relative frequency

- **Relative frequency** is used to estimate probability from data.
- The result characteristic being estimated is called the **favourable outcome**.
- The number of data items used is called the number of **trials**.
- The estimated probability is given by

$$\text{Relative frequency} = \frac{\text{Frequency of favourable outcome}}{\text{Number of trials}}.$$

- Conversely, the **expected frequency** of an outcome is given by

$$\text{Expected frequency} = \text{Probability} \times \text{Number of trials}.$$

# 3. CHAPTER REVIEW

## Basic probability

Example  
1

- 1 **a** List the elements of the sets  $A = \{3n + 1 : n \in \mathbf{Z}, 2 \leq n \leq 6\}$  and  $B = \{\text{prime numbers between 20 and 40}\}$ .
- b** What are the cardinalities of  $A$  and  $B$ ?

Example  
2

- 2  $P = \{2, 3, 7, 11\}$ ,  $Q = \{7, 11, 13, 17\}$  and  $R = \{3, 5, 7, 13, 17\}$  are sets of prime numbers less than 20. Find:
- a**  $\bar{P}$       **b**  $Q \cap R$       **c** proper subsets of  $Q$       **d**  $Q \cup R$
- Draw a Venn diagram of the sets  $P$ ,  $Q$  and  $R$ .

Example  
3

- 3 One of each of white, black, pink, and yellow snooker balls are placed with 2 of the red balls in a box. Two are taken out at random without replacement.
- a** Use the symbols W, B, R, P and Y to write the sample space.
- b** Write the event 'at least one of the balls is black'.
- c** Write the event 'at least one of the balls is red'.
- d** Write the event 'a black or a yellow ball is taken'.
- e** Write the event 'a black and a yellow ball are drawn'.
- f** Write the event 'both balls drawn are black'.

Example  
4

- 4 There are 23 students in a class. Five of the 13 girls have blue eyes and there are 16 students who do not have blue eyes. What is the probability of randomly selecting a blue-eyed boy from the class?

Example  
5

- 5 A coin is chosen at random from a purse containing 4 twenty-cent coins, 6 \$1 coins and 5 fifty-cent coins. What is the probability that it is a fifty-cent coin?

Example  
6

- 6 An octahedral die numbered 1 to 8 is weighted so that the probabilities of 1 and 4 are 0.15 and the probabilities of 5 and 8 are reduced to 0.1. The probabilities of the other numbers are all the same. What is the probability of rolling:
- a** a prime number?      **b** a number less than 5?

Example  
7

- 7 A box contains 3 blue, 1 black and 2 red marker pens. Two pens are taken at random.
- a** Draw a tree diagram of the sample space.
- b** Find  $P(2 \text{ blue pens})$ .
- c** Find  $P(2 \text{ different-coloured pens})$ .

Example  
8

- 8 Find the probability that the difference of the 2 numbers rolled on a pair of dice is less than 3.

9 In a group of 30 Year 11 students, 18 do Maths Methods and the probability that a randomly selected student from the group takes Maths Methods and Biology is 0.2. There are 12 students who do Biology. A student is randomly selected from the group. What is the probability the student takes Maths Methods or Biology?

- 10 When people in a shopping mall were asked their main reason for being there, 20 said to buy food, 15 said to buy clothes, 25 said they were 'just looking' and 10 said they had come in to cool down.
- What is the probability of randomly choosing someone who is there to buy clothes?
  - From 200 people, how many would you expect to be there to buy food?

### Problem solving

11 The students in a Year 11 class wrote their names in a table on the board showing what they wanted to do when they left school and the highest education level of their parents. The results are shown in the table below.

		Student school leaving preference		
		Work	Uni	Other education
Parent education	School only	Jenny, Kathy, Andrew, William	Jane	Barbara
	Degree	Scarlet	Brad, Jason. Lisa, Merryl	Tony, Peter, Debbie
	Trade/Certificate	Mary, Samantha, Paul	Ross, Zoe	Alan, David, Tammy, Wayne, Skye

What are the following events from the sample space when a student is chosen at random?

- Parent with degree.
- Parent with trade or certificate.
- Student wanting to go to uni  $\cap$  Parent with trade or certificate.
- Parent who has only school education  $\cap$  Student wants to start work straight after school.

Consider the numbers of students in each category.

e Do you think the table shows any relationships? Explain your answer.

- 12 32 of the teachers at a school have no children at home and of these, 19 live with a partner. There are 6 teachers who have children at home but do not live with a partner. Altogether, there are 63 teachers at the school. How many teachers live with children and a partner?
- 13 What is the probability of getting 3 or more heads in a row at some time when you toss a coin 4 times?
- 14 Sebastian and Georgia take turns to shake a marble out from a bottle at random. There are 7 green and 5 red marbles in the bottle. Sebastian goes first. The first person to get two marbles of the same colour wins.

What is the probability that Sebastian wins?



Practice quiz

# Practice examination 1



**Time: 90 minutes**  
**Perusal time: 5 minutes**  
**Marks: 50**

## Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

## Question 1 (2 marks)

- a** Find the next 3 terms of the sequence 6, 13, 20, 27, ...
- b** Find the 15th term of an arithmetic sequence with first term 3 and common difference  $-0.4$ .

## Question 2 (2 marks)

- a** Find the sum to 20 terms of the arithmetic sequence 20, 22, 24, 26, ...
- b** How many terms of  $2.4 + 4 + 5.6 + 7.2 + \dots$  are required to first exceed 2000?

## Question 3 (5 marks)

Simplify each expression and express with positive indices.

**a** 
$$\frac{4^3 \times 8^{-3}}{16^{-5}}$$

**b** 
$$\frac{3^{\frac{7}{2}} (\sqrt{3})^{-3}}{2 \times 9^{\frac{1}{2}}}$$



Year 11  
formulas

**Question 4** (2 marks)

Simplify each expression.

**a**  $2a(b^2 - a^3) - 5a^2(b - 3a^2)$

**b**  $(3m - 5p)^2$

**Question 5** (5 marks)

Solve each equation by the indicated method.

**a**  $10x^2 - 25x + 15 = 0$  by factorisation

**b**  $4w^2 = 6w - 1$  using the quadratic formula

**Question 6** (2 marks)

$f(x) = 5(x + 1)$  and  $g(x) = \frac{2x - 5}{x}$ .

Simplify  $g(f(4))$ .

**Question 7** (2 marks)

Use the discriminant to determine whether the quadratic equation  $9x^2 - 6x - 1 = 0$  has any real roots. If it does, determine their nature.

**Question 8** (2 marks)

**a** What is the degree of  $P(x) = 3x^2 - 4x^4 - 17x^3 + 2x^5 + 6x - 8$ ?

**b** Show that  $x + 2$  is a factor of  $P(x)$ .

**Question 9** (2 marks)

$A = \{\text{odd numbers between 0 and 20}\}$ ,  $B = \{\text{numbers divisible by 3 between 0 and 20}\}$ .

Find  $A \cap B$  and  $A \cup B$ .

**Question 10** (1 mark)

A pencil case has 3 blue, 2 black and 2 red pens in it. What is the probability that a pen taken out at random is black?

**Question 11** (1 mark)

The probability of a student in a Year 11 class having blue eyes is 0.3.

The probability of having fair hair is 0.35.

The probability of having both blue eyes and fair hair is 0.25.

What is the probability of a student having either fair hair or blue eyes?

**Question 12** (2 marks)

There are 24 students in a Mathematical Methods class.

8 of the students do Physics and 11 do Chemistry.

8 do neither Physics nor Chemistry.

Draw a Venn diagram and find the probability of a student from the class doing both Physics and Chemistry.

**Question 13** (2 marks)

The ages of cars in a saleyard are shown below.

Age of car	Number available
Under 2 years	7
2 or 3 years	9
4 or 5 years	18
6 or more	6

The cars are equally likely to be sold.

Estimate the probability that the next car sold is:

- a** under 4 years old
- b** over 3 years old.

**Question 14** (1 mark)

The 4th term of an arithmetic sequence is 1.1 and the 11th term is  $-1$ .

What is the first term?

**Question 15** (3 marks)

Two marbles are taken out together from a bag containing 5 red and 7 blue marbles.

Use a tree diagram to find the probability that they are of different colours.

**Question 16** (3 marks)

Solve  $3p^2 + 2 = 6p$  by completing the square.

**Question 17** (3 marks)

Solve  $2x^4 - 10x^3 + 16x^2 - 8x = 0$  without using a graphics calculator.

**Question 18** (2 marks) (Technology-active)

What is the sum of the numbers from 15 to 375 inclusive that are not divisible by 3 or 7?

**Question 19** (2 marks) (Technology-active)

Two line repairers can repair five power lines in 3 hours. After a storm, 50 lines need repairing. How many repairers are needed to repair the lines in one working day (8 hours)?

**Question 20** (4 marks) (Technology-active)

**a** Show that the volume of a cylindrical tank is given by

$$V(r) = r \left( \sqrt{\frac{S}{2}} + \sqrt{\pi r} \right) \left( \sqrt{\frac{S}{2}} - \sqrt{\pi r} \right),$$

where  $r$  is the radius and  $S$  is amount of sheet metal used.

**b** What is the maximum volume possible using  $50 \text{ m}^2$  of stainless steel, correct to 2 decimal places, excluding off-cuts?

**Question 21** (2 marks) (Technology-active)

Any goods produced by a factory that have some minor fault are called factory 'seconds'. About 6% of a factory's production of microwave ovens are seconds. These are retained, and the rest are sent to retailers for selling. Retailers return about 4% of their microwave ovens and these also become seconds.

About 40% of all the seconds have surface blemishes.

What is the percentage probability of an item selected at random from production becoming a second with surface blemishes?

**END OF EXAMINATION**

# 4.

## FUNCTIONS AND GRAPHS

Much of mathematics is about rules. Functions and relations are one of the most important ways to write down rules. Functions use letters and symbols to show how different quantities are related and graphs to picture the relationships. Functions are used in most of modern society: science, business, engineering, manufacturing, computing, building, economics and so on. In this chapter, you will learn more about quadratics, hyperbolas and polynomials to examine general methods that apply to all functions.

- 4.01 Relations and graphs
- 4.02 Functions and graphs
- 4.03 Graphs of quadratic functions
- 4.04 Models and problems involving quadratic functions
- 4.05 Graphs of quadratic relations
- 4.06 Hyperbolic functions
- 4.07 Graphs of polynomials
- 4.08 Transformations of functions
- 4.09 Equations involving functions
- 4.10 Piecewise functions
- Chapter summary
- Chapter review



## SYLLABUS SUBJECT MATTER

### Functions

- understand the concept of a function as a mapping between sets, a graph and as a rule or a formula that defines one variable quantity in terms of another
- recognise the distinction between functions and relations, and use the vertical line test to determine whether a relation is a function
- use function notation, domain and range, independent and dependent variables
- examine transformations of the graphs of  $f(x)$  including dilations and reflections and the graphs of  $y = af(x)$  and  $y = f(bx)$ , translations and the graphs of  $y = f(x + c)$  and  $y = f(x) + d$ ;  $a, b, c, d \in \mathbf{R}$
- recognise and use piecewise (hybrid) functions as a combination of multiple sub-functions with restricted domains
- identify contexts suitable for modelling piecewise functions and use them to solve practical problems (taxation, taxis, the changing velocity of a parachutist)

### Review of quadratic relationships

- recognise and determine features of the graphs of  $y = x^2$ ,  $y = ax^2 + bx + c$ ,  $y = a(x - b)^2 + c$ , and  $y = a(x - b)(x - c)$ , including their parabolic nature, turning points, axes of symmetry and intercepts
- identify contexts suitable for modelling with quadratic functions and use models to solve problems with and without technology; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis
- determine turning points and zeros of quadratics with and without technology

### Inverse proportions

- recognise features of the graphs of  $y = \frac{1}{x}$  and  $y = \frac{a}{x - b}$ , including their hyperbolic shapes, their intercepts, their asymptotes and behaviour as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$

### Powers and polynomials

- recognise and determine features of the graphs of  $y = x^3$ ,  $y = a(x - b)^3 + c$  and  $y = k(x - a)(x - b)(x - c)$ , including shape, intercepts and behaviour as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$
- recognise and determine features of the graphs  $y = a(x - b)^4 + c$  including shape and behaviour
- solve equations involving combinations of the functions above using technology where appropriate

### Graphs of relations

- recognise and determine features of the graphs of  $x^2 + y^2 = r^2$  and  $(x - a)^2 + (y - b)^2 = r^2$ , including their circular shapes, their centres and their radii
- recognise and determine features of the graph of  $y^2 = x$ , including its parabolic shape and its axis of symmetry



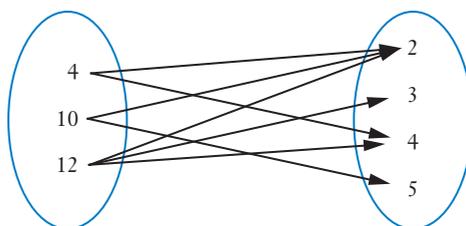
# TERMINOLOGY

1 : 1	asymptote	axis of symmetry
codomain	concave downwards	concave upwards
cubic	dependent variable	dilation
discontinuity	domain	equation of a function
function	gradient	hyperbola
inclination	independent variable	maximum
minimum	ordered pair	parabola
piecewise	point of inflection	quadratic function
quadratic relation	range	relation
simultaneous equations	turning point	vertex
vertical line test	y-intercept	zero

## 4.01 Relations and graphs

You can use Cartesian coordinates and the Cartesian plane to show connections between pairs of numbers. A mapping goes from the first numbers to the second numbers in the pairs.

The diagram shows some of the pairs in the mapping of natural numbers ‘is exactly divisible by’.



You could write the pairs shown above as  $(4, 2)$ ,  $(4, 4)$ ,  $(10, 2)$ ,  $(10, 5)$ ,  $(12, 2)$ ,  $(12, 3)$  and  $(12, 4)$ . They are called **ordered pairs** or **couples** as the order is part of the information.

The sequence  $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$  maps the natural numbers to rational numbers between 0 and 1.

The rule  $|y| < 4 - |x|$  maps real numbers to real numbers.

A **relation** is a set of ordered pairs. The relation can be shown by a list of the ordered pairs, a rule, a mapping or part of the Cartesian plane.

The first element of the ordered pairs is called the **independent variable** and the second is called the **dependent variable**.

The **domain** is the set of first numbers in the relation.

The **range** is the set of second numbers in the relation.

The **codomain** is the set of numbers allowed as second numbers.

## EXAMPLE 1

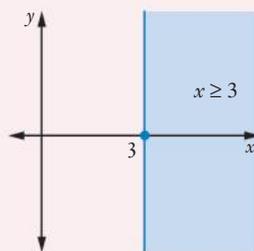
- a What is the domain of  $x \geq 3$ ?
- b What is the range?
- c Sketch the graph.

### Solution

- a The  $x$ -value can be any real number from 3 upwards.
- b Any  $y$ -values are allowed.
- c The points on the line  $x = 3$  are included, so it is a solid line.  
Show the intercept and the inequality.

The domain is  $x \geq 3$ .

The range is  $\mathbf{R}$ .



The equation of a circle with radius  $r$  and centre at the origin is  $x^2 + y^2 = r^2$ .

## EXAMPLE 2

- a What shape is the graph of  $x^2 + y^2 = 20$ ?
- b Sketch  $x^2 + y^2 > 20$ .

### Solution

- a Write in the form  $x^2 + y^2 = r^2$ .

$$x^2 + y^2 = (\sqrt{20})^2$$

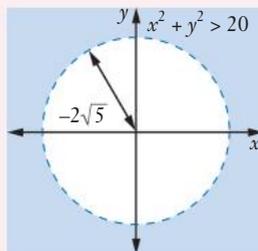
Simplify  $\sqrt{20}$ .

$$x^2 + y^2 = (\sqrt{4 \times 5})^2 = (2\sqrt{5})^2$$

Write the answer.

$x^2 + y^2 = 20$  is a circle of radius  $2\sqrt{5}$  with centre at the origin.

- b The points on the line  $x^2 + y^2 = 20$  are not included, so it is a dashed line.  
The points are on the outside of the circle, so shade outside.



You could test points to work out the shading.

(1, 1) is not in the relation because  $1^2 + 1^2 = 2 < 20$ .

(0, 5) is in because  $0^2 + 5^2 = 25 > 20$ .

The **equation of a circle** with radius  $r$  and **centre at**  $(a, b)$  is  $(x - a)^2 + (y - b)^2 = r^2$ .

### EXAMPLE 3

Find the centre and radius of  $4x^2 + 24x + 4y^2 = 60y - 161$

#### Solution

Make the coefficients of  $x^2$  and  $y^2$  both 1.  $x^2 + 6x + y^2 = 15y - 40\frac{1}{4}$

Put the variable terms on the LHS.  $x^2 + 6x + y^2 - 15y = -40\frac{1}{4}$

Complete the squares.  $x^2 + 2 \times 3x + 3^2 + y^2 - 2 \times \frac{15}{2}y + \left(\frac{15}{2}\right)^2$   
 $= -40\frac{1}{4} + 9 + 56\frac{1}{4}$

Write perfect squares and simplify.  $(x + 3)^2 + (y - 7\frac{1}{2})^2 = 25$

Write in standard form.  $(x + 3)^2 + (y - 7\frac{1}{2})^2 = 5^2$

Write the answer. The centre is  $(-3, 7\frac{1}{2})$  and the radius is 5.



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## Exercise 4.01 Relations and graphs

Example  
1

- 1 a** What is the domain of  $x < 4$ ?  
**b** What is the range?  
**c** Sketch the graph.
- 2 a** What is the domain of  $y \leq -1$ ?  
**b** What is the range?  
**c** Sketch the graph.
- 3 a** What is the domain of  $x \geq -2$ ?  
**b** What is the range?  
**c** Sketch the graph.
- 4 a** What is the domain of  $y > 5$ ?  
**b** What is the range?  
**c** Sketch the graph.
- 5 a** What is the domain of  $2 < x < 4$  and  $-3 \leq y \leq 1$ ?  
**b** What is the range?  
**c** Sketch the graph.
- 6 a** What is the domain of the relation  $y =$  the length  $x$ , rounded to the nearest metre?  
**b** What is the range?  
**c** Sketch the graph for  $0 \leq x \leq 5$ , where  $x$  is in metres.
- 7** Describe and sketch each of the following.
- |                                       |                                       |
|---------------------------------------|---------------------------------------|
| <b>a</b> $x^2 + y^2 = 36$             | <b>b</b> $(x + 3)^2 + (y - 5)^2 = 9$  |
| <b>c</b> $(x + 2)^2 + (y + 1)^2 = 10$ | <b>d</b> $(x - 6)^2 + (y - 4)^2 = 52$ |
- 8** Describe and sketch each of the following.
- |   |                                       |
|---|---------------------------------------|
| <b>a</b> $x^2 + y^2 \leq 24$            | <b>b</b> $(x - 1)^2 + (y + 2)^2 > 16$ |
| <b>c</b> $(x + 4)^2 + (y - 3)^2 \geq 1$ | <b>d</b> $(x - 2)^2 + (y - 5)^2 < 48$ |
- 9** Find the centre and radius of each of the following.
- |   |   |
|---|---|
| <b>a</b> $x^2 + y^2 - 6x + 8y = 24$     | <b>b</b> $x^2 + y^2 - 2x - 6y = 26$     |
| <b>c</b> $2x^2 + 2y^2 - 10x + 14y = 13$ | <b>d</b> $x^2 + y^2 + 4x + 8y + 11 = 0$ |
| <b>e</b> $x^2 + y^2 + 8x - 4y = 61$     |   |

Example  
2

Example  
3

### Problem solving

- 10** Describe and sketch the graph of  $|x - 1| + |y + 2| \leq 3$ .

## 4.02 Functions and graphs

The most useful relations are those that have a rule so that you can work out a unique  $y$ -value from any given  $x$ -value, like  $y = 3x$  or  $y = 2x^2 - 4$ .



Shutterstock.com/Africa Studio

A **function** is a  $1 : n$  relation. Every number in the domain of a function is in at most one ordered pair. Different ordered pairs must have different first numbers.

Ordered pairs of a function  $f$  are often written as  $(x, f(x))$  or just  $f(x)$ .  $x$  represents the domain value and  $f(x)$  is the corresponding range value.

If  $a = b$ , then  $f(a) = f(b)$ .

Every first number of a function belongs in only one ordered pair, so if  $a = b$  then  $f(a) = f(b)$ . The circle  $x^2 + y^2 = 25$  is not a function because  $(3, 4)$  and  $(3, -4)$  are both ordered pairs in the function.

$y = 2x^2 - 4$  is a function because any  $x$ -number has only one  $y$ -number.  $(1, -2)$  and  $(-1, -2)$  don't matter because the first numbers are different.

## EXAMPLE 4

Which of the following are functions?

- a**  $(3, 5), (4, 5), (5, 8), (7, 9), (7, 10)$
- b**  $(2, 3), (3, 4), (4, 4), (5, 4), (6, 5)$
- c**  $f(x) = 3x^4 - 5$
- d**  $(x, \pm\sqrt{x})$ , where  $x$  is positive

### Solution

- a** Do any pairs have the same first number?

Write the answer.

- b** Do any pairs have the same first number?

Write the answer.

- c** Do any pairs have the same first number?

Write the answer.

- d** Do any pairs have the same first number?

Write the answer.

$(7, 9)$  and  $(7, 10)$  are different pairs with the same first number.

**a** is not a function.

Every pair has a different first number

**b** is a function.

$3x^4 - 5$  is unique for any  $x$ -value

**c** is a function.

$(9, 3)$  and  $(9, -3)$  have the same first number.

**d** is a not function.

However,  $(x, \sqrt{x})$  is a function because  $\sqrt{x}$  is taken to mean the positive root.

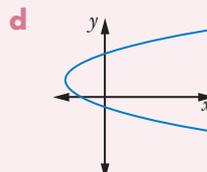
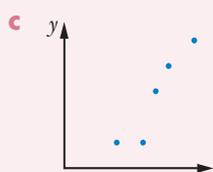
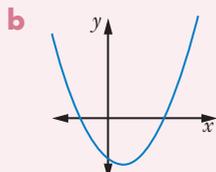
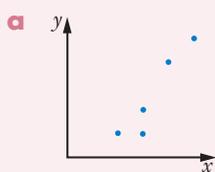
In a function, every  $x$  value has only one  $y$  value. Thus the graph of a function will have no points in a vertical line.

### The vertical line test

A graph with any points in a vertical line is not the graph of a function.

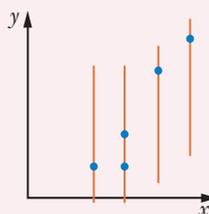
### EXAMPLE 5

State whether or not each of the following graphs represent functions.



### Solution

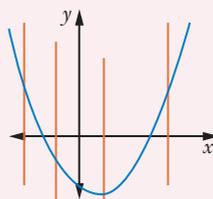
**a** Try the vertical line test.



Write the answer.

Two points are in a vertical line, so **a** is not a function.

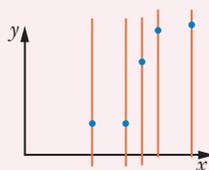
**b** Try the vertical line test.



Write the answer.

No two points are in a vertical line, so **b** is a function.

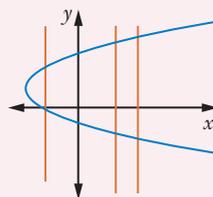
**c** Try the vertical line test.



Write the answer.

No two points are in a vertical line, so **c** is a function.

**d** Try the vertical line test.



Write the answer.

Some points are in a vertical line, so **d** is not a function.

## Exercise 4.02 Functions and graphs

1 State whether or not each of the following is a function.

**a**  $(4, 1), (5, 7), (-1, -6), (0, 2), (3, 8)$

**b**  $(-3, 7), (2, 5), (6, -3), (8, 5), (4, -7)$

**c**  $(11, -1), (11, 4), (11, 0), (11, -3)$

**d**  $(7, 16), (-2, -2), (10, 22), (-5, -8), (10, 0)$

**e**  $y = 4x - 11$

**f**  $xy = 1$

**g**  $5x^2 = 2y$

**h**  $2y^2 = 5x$

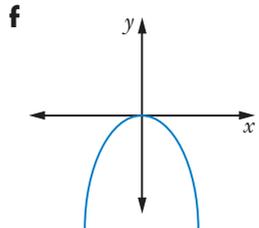
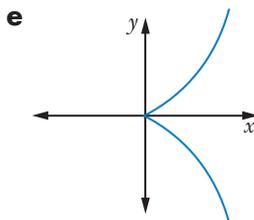
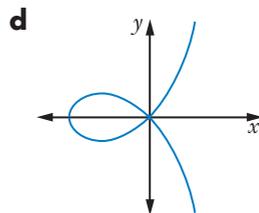
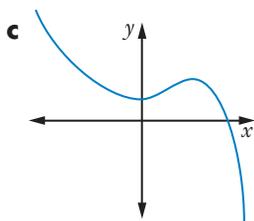
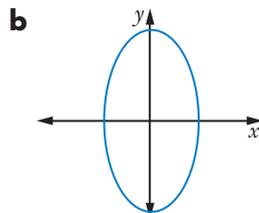
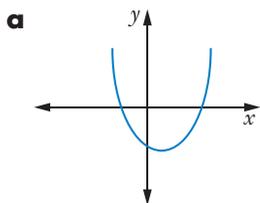
**i**  $f(x) = 7x - 10$

**j**  $f(x) = 10$

Example

4

2 State whether or not each graph represents a function.



Example

5

## 4.03 Graphs of quadratic functions



Sketching quadratic functions



Features of a parabola



Excel spreadsheet: Investigating parabolas 1



Excel worksheet: Investigating parabolas 1



Excel spreadsheet: Curve sketcher



Excel worksheet: Curve sketcher



Graphing non-linear equations

- The graph of a **quadratic function** is called a **parabola**. It has a **turning point** where the slope changes from negative to positive, or positive to negative for a negative coefficient of  $x^2$ .
- The graph has a vertical **axis of symmetry** that passes through the turning point.
- For a positive coefficient of the  $x^2$  term:
  - the function decreases to a **minimum** value at the turning point, and then increases
  - as  $x \rightarrow \infty, f(x) \rightarrow \infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow \infty$ .
- For a negative coefficient of the  $x^2$  term:
  - the function increases to a **maximum** value at the turning point, and then decreases
  - as  $x \rightarrow \infty, f(x) \rightarrow -\infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow -\infty$ .
- The turning points of  $y = \pm x^2$  are at the origin,  $(0, 0)$ .
- The **domain** is the set of real numbers  $\mathbf{R}$  and the **range** is  $y \geq$  minimum or  $y \leq$  maximum depending on the coefficient of  $x^2$ .
- Compared to  $y = x^2$ , the graph of  $f(x) = a(x - b)^2 + c$  is **dilated** by  $a$  and inverted if  $a$  is negative. The graph appears narrower if  $|a| > 1$  and wider if  $|a| < 1$ . All points are shifted  $b$  units to the right and  $c$  units higher, so the turning point is  $(b, c)$ .

When you are asked to *sketch* a graph, show the general shape and important points. Always find the  $y$ -intercept. You can use it to check if you've drawn the graph the right way up.



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## EXAMPLE 6

Sketch the graph of:

a  $f(x) = 2(x + 3)^2 + 1$

b  $y = -\frac{(x-1)^2}{3} - 2$

### Solution

a How is the graph changed from  $y = x^2$ ?

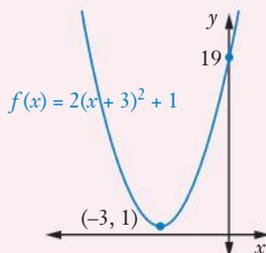
The graph is vertically stretched by a factor of 2 from  $y = x^2$  and the minimum point is 3 units left and 1 unit up, so it is  $(-3, 1)$ .

Find the  $y$ -intercept.

$$f(0) = 19$$

Sketch the graph.

You do not need to show axis scales but try to get the relative positions of the points in the right place.



Identify the graph with its equation.

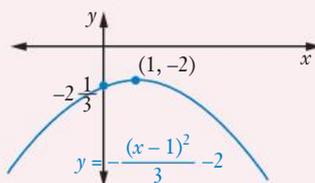
b How is the graph changed from  $y = x^2$ ?

The graph is vertically compressed by a factor of 3 from  $y = x^2$  and inverted. The maximum point is 1 unit right and 2 down, at  $(1, -2)$ .

Find the  $y$ -intercept.

$$\text{At } x = 0, y = -2\frac{1}{3}.$$

Sketch the graph.



You can express some quadratic functions in factored form, like  $f(x) = (x + 3)(x - 2)$ . The roots of the quadratic equation correspond to the **zeros** of the function. By symmetry, the turning point must be halfway between the zeros.

A quadratic function in the form  $y = a(x - b)(x - c)$  has **zeros** at  $x = b$  and  $x = c$ . The turning point is halfway between  $b$  and  $c$ .

Compared to  $y = x^2$ , if  $|a| > 1$ , the graph is vertically stretched by the factor  $|a|$ .

If  $|a| < 1$ , it is vertically compressed.

### EXAMPLE 7

Sketch the graph of  $f(x) = (x + 3)(x - 2)$ .

#### Solution

It is in factored form, so give the zeros.

$$f(x) = 0 \text{ at } x = -3 \text{ or } x = 2.$$

Find the turning point.

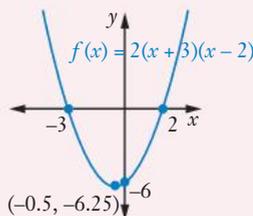
$$\text{The turning point is at } x = \frac{-3 + 2}{2} = -\frac{1}{2}.$$

$$f\left(-\frac{1}{2}\right) = -6\frac{1}{4}, \text{ a minimum.}$$

Find the  $y$ -intercept.

$$f(0) = -6$$

Sketch the graph.



When you are given a quadratic function in general form, find any zeros. You might have to use the quadratic formula. If there are no zeros, find the turning point to sketch the function.

The zeros of  $f(x) = ax^2 + bx + c$  are at  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ .

The turning point is at  $x = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = \frac{-b}{2a}$ .

The turning point of  $f(x) = ax^2 + bx + c$  is at  $x = \frac{-b}{2a}$ .

### EXAMPLE 8

Graph each quadratic function and state its range.

**a**  $y = 7x - 2x^2 + 15$

**b**  $y = -x^2 + x - 2, -3 \leq x < 3$

#### Solution

**a** Try to factorise.

$$\begin{aligned} y &= 7x - 2x^2 + 15 \\ &= -1(2x^2 - 7x - 15) \\ &= -1(x - 5)(2x + 3) \end{aligned}$$

State the zeros.

There are zeros at  $x = -1\frac{1}{2}$ ,  $x = 5$

Find the turning point.

The turning point is at  $x = \frac{-1\frac{1}{2} + 5}{2} = 1\frac{3}{4}$ .

At  $x = 1\frac{3}{4}$ ,  $y = 21\frac{1}{8}$

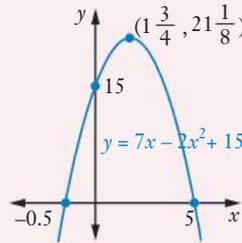
State the turning point.

$-2x^2$  is a negative, so there is a maximum at  $(1\frac{3}{4}, 21\frac{1}{8})$ .

State the  $y$ -intercept.

At  $x = 0$ ,  $y = 15$

Sketch the graph.



State the range.

The range is  $y \leq 21\frac{1}{8}$

**b** Try to factorise.

Can't factorise. Use the quadratic formula.

$a = -1$ ,  $b = 1$ ,  $c = -2$

$\Delta = 1^2 - 4(-1)(-2) = -7 < 0$ , so no zeros

Use  $x = \frac{-b}{2a}$  to find the turning point.

$x = \frac{-1}{2(-1)} = \frac{1}{2}$

When  $x = \frac{1}{2}$ ,  $y = -(\frac{1}{2})^2 + \frac{1}{2} - 2 = -1\frac{3}{4}$

Turning point at  $(\frac{1}{2}, -1\frac{3}{4})$

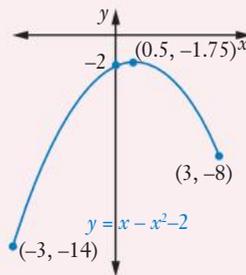
State the turning point.

$-x^2$  is negative so there is a maximum at  $(\frac{1}{2}, -1\frac{3}{4})$ .

State the  $y$ -intercept.

At  $x = 0$ ,  $y = -2$ .

Sketch the graph, giving the exact values of the endpoints.



State the range.

The range is  $-14 \leq y < -1\frac{3}{4}$ .

## TECHNOLOGY

### Graphing quadratics

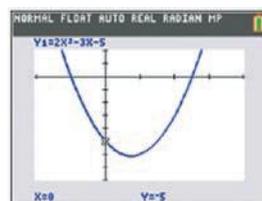
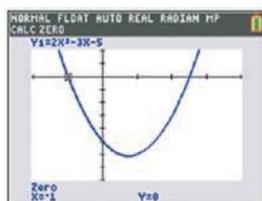
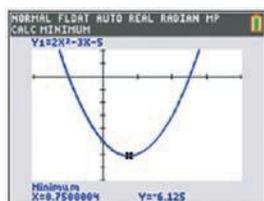
You can use your calculator to find or check key points of a graph such as  $f(x) = 2x^2 - 3x - 5$ .

#### TI-84 Plus CE

Press  $\boxed{Y=}$  and enter  $Y_1 = 2x^2 - 3x - 5$  using  $\boxed{X,T,\theta,n}$ , press  $\boxed{\text{enter}}$  and  $\boxed{\text{graph}}$ .

Use  $\boxed{\text{window}}$  or  $\boxed{\text{zoom}}$  and **Zbox** to find the part needed. Use  $\boxed{\text{calc}}$  ( $\boxed{2\text{nd}}$   $\boxed{\text{trace}}$ ) to find the zeros, minimum and y-intercept.

Use the left and right cursor controls to set left, right and a guess in-between to find the minimum and zeros. Use value and set  $X = 0$  to find the y-intercept.



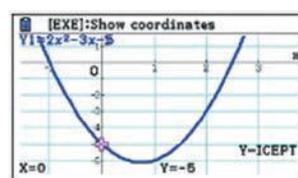
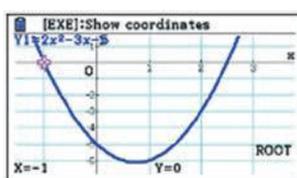
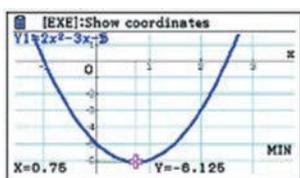
#### Casio fx-CG20AU

Choose **5 Graph** and enter  $Y1 = 2x^2 - 3x - 5$  using  $\boxed{X,\theta,T}$ , press  $\boxed{\text{EXE}}$  and  $\boxed{\text{F6}}$ .

Use **V-Window** ( $\boxed{\text{SHIFT}}$   $\boxed{\text{F3}}$ ) or **Zoom** ( $\boxed{\text{SHIFT}}$   $\boxed{\text{F2}}$ ) and **BOX** to find the part needed.

Use **G-Solv** ( $\boxed{\text{SHIFT}}$   $\boxed{\text{F5}}$ ) to find the zeros, minimum and y-intercept.

Press  $\boxed{\blacktriangleright}$  to get the second zero (**ROOT**).



You will not get full marks in a test if you just copy a graph from your calculator when you are asked to sketch the graph. However, you should use it to check your work if possible.

## Exercise 4.03 Graphs of quadratic functions

1 Sketch the graph of each function.

**a**  $y = (x + 1)^2 - 2$

**c**  $y = 9 + (x - 1)^2$

**e**  $y = 12 - 4(x + 2)^2$

**b**  $f(x) = 3(x - 2)^2 + 3$

**d**  $f(x) = 1 - 2(x + 3)^2$

**f**  $f(x) = -2 - 4(x - 4)^2$

Example  
6

2 Sketch the graph of each function.

**a**  $f(x) = (x - 1)(x - 4)$

**c**  $f(x) = (2x - 1)(x + 3)$

**b**  $y = -2(x + 2)(x + 5)$

**d**  $y = (2 - x)(3x + 5)$

Example  
7

3 Sketch the graph of each function.

**a**  $y = x^2 - 3x - 4$

**c**  $y = 3x - 2x^2 + 20$

**b**  $f(x) = 15 - x^2 - 2x$

**d**  $f(x) = 14x - 4x^2 - 6$

Example  
8

4 Sketch the graph of each function.

**a**  $f(x) = 9x^2 + 12x - 1$

**c**  $f(x) = 8x^2 - 16x + 7$

**b**  $y = -12x^2 - 12x - 2$

**d**  $y = -2x^2 + 12x - 17$

5 Complete the square to sketch the graph of each function.

**a**  $y = x^2 - 2x + 2$

**c**  $y = x^2 - 4x + 1$

**b**  $f(x) = 2x^2 - 8x + 5$

**d**  $f(x) = -x^2 - 2x - 5$

6 Sketch the graph of each function and state its range.

**a**  $y = x^2 + 6x + 19, -4 < x < 1$

**c**  $y = 2x^2 + 8x + 5, -4 < x \leq 1$

**e**  $y = -4x^2 - 8x - 1$

**g**  $y = 4x^2 - 20x + 41$

**i**  $y = 2x^2 + 5x + 3$

**k**  $y = -x^2 - 8x - 15, -6 < x < 1$

**b**  $f(x) = 4x^2 - 4x - 3, -1 < x \leq 1$

**d**  $f(x) = 8 - 10x - 3x^2, -3 < x \leq 2$

**f**  $f(x) = 4x - 2x^2 - 6$

**h**  $f(x) = -3x^2 - 24x - 5$

**j**  $f(x) = 3x^2 - 8x - 3$

**l**  $f(x) = 12x - 3x^2 + 15, -2 < x \leq 7$

7 Use your graphics calculator to find the key points of:

**a**  $y = x^2 + 6x + 2$

**c**  $y = 23x - 4x^2 - 1$

**b**  $f(x) = 9 - 2x^2 - 5x$

**d**  $f(x) = 7 - 5x + 2x^2$

### Problem solving

8 Find the equation of each parabola in the form  $f(x) = ax^2 + bx + c$ .

**a**  $y$ -intercept of  $-6$  and turning point at  $(2, -2)$

**b**  $y$ -intercept of  $-3$  and  $x$ -intercepts of  $-1$  and  $2$

**c**  $x$ -intercepts of  $-2$  and  $3$  and a maximum of  $5$

9 Find the distance between the turning points of  $y = 10x - x^2$  and  $2y = 75 + 10x - x^2$ .

## 4.04 Models and problems involving quadratic functions

You can solve problems involving the intersections of straight lines with quadratic functions using the graph. You should remember straight lines from Year 10 mathematics.

- The **equation of a straight line** is  $y = mx + c$ ,  $y - y_1 = m(x - x_1)$  or  $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ , where  $m$  is the **gradient** of the line,  $c$  is the **y-intercept**,  $(x_1, y_1)$  and  $(x_2, y_2)$  are points on the line.
- The gradient is given by  $m = \tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$ , where  $\theta$  is the **inclination** of the line (the angle made with a horizontal line).

### EXAMPLE 9

Find the length of the line  $y = 2x + 2$  cut off by the graph of the function  $f(x) = 2x^2 - 3x - 5$ .

#### Solution

Find the zeros of  $f(x)$ .

$$\begin{aligned} f(x) &= 2x^2 - 3x - 5 \\ &= (2x - 5)(x + 1) \end{aligned}$$

The zeros are at  $x = -1, 2\frac{1}{2}$ .

Find the turning point.

$$\begin{aligned} \text{Turning point is at } x &= \frac{-1 + 2\frac{1}{2}}{2} = \frac{3}{4} \\ f\left(\frac{3}{4}\right) &= -6\frac{1}{8} \end{aligned}$$

Since  $2x^2$  is positive there is a minimum at  $\left(\frac{3}{4}, -6\frac{1}{8}\right)$ .

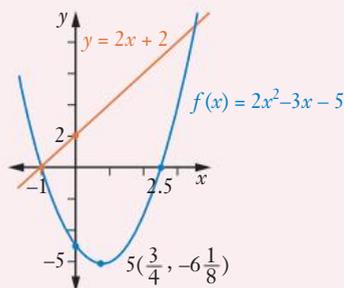
State the y-intercept.

$$f(0) = -5$$

State the intercepts of  $y = 2x + 2$ .

$y = 2x + 2$  has intercepts at  $(0, 1)$  and  $(-1, 0)$ .

Sketch the graphs on the same axes.



Find the intersections.

The intersections are about  $(-1, 0)$  and  $(3\frac{1}{2}, 9)$ .

Check in the equations.

$-1$  is the  $x$ -intercept for both.

$$f(3\frac{1}{2}) = 2 \times 12\frac{1}{4} - 3 \times 3\frac{1}{2} - 5 = 9$$

$$y(3\frac{1}{2}) = 2 \times 3\frac{1}{2} + 2 = 9$$

Find the distance between the intersections.

$$\begin{aligned} d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= \sqrt{(3\frac{1}{2} - (-1))^2 + (9 - 0)^2} \\ &= \frac{\sqrt{405}}{2} = \frac{9\sqrt{5}}{2} \end{aligned}$$

State the answer.

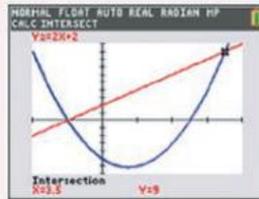
$\frac{9\sqrt{5}}{2}$  units is cut off  $y = 2x + 2$  by  $f(x)$ .

### TI-84 Plus CE

Use  $\boxed{y=}$  to enter the functions in  $Y_1$  and  $Y_2$ .

Press  $\boxed{\text{graph}}$  to see the graph and set  $\boxed{\text{window}}$  appropriately.

Use  $\boxed{\text{calc}}$ , **5: intersect** to find the intersection.

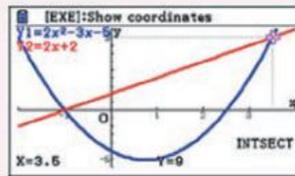


### Casio fx-CG20AU

Choose **5 Graph** and enter the functions in  $Y1$  and  $Y2$ .

Press  $\boxed{\text{F6}}$  to see the graph and set **V-Window** appropriately.

Use **G-Solv** and  $\boxed{\text{F5}}$ : **INTSECT** to find the intersection.



TI-Nspire CX  
Chapter 4

Some problems involving maxima or minima can be solved using quadratic functions.

## EXAMPLE 10

A farmer wants to set up a rectangular pen with 120 m of temporary fencing against an existing straight fence.

Use a model to find the dimensions she should use to get the maximum area for her sheep. Verify and evaluate the usefulness of the model.

### Solution

Draw a diagram and choose the variables.



Find the other dimension.

Let the width =  $x$  m

Length =  $120 - 2x$  m

Write the area.

Area,  $A = x(120 - 2x) = 2x(60 - x)$

Find the zeros.

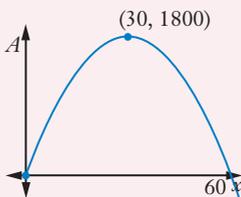
$A(x) = 0$  at  $x = 0, x = 60$

Find the turning point.

Midpoint of zeros,  $x = 30$

Since  $2x^2$  is negative, there is a maximum at  $(30, 1800)$ .

Sketch the graph to check.



State the answer.

She should have a pen 30 m wide by 60 m parallel to the existing fence to get the maximum area.

Verify the model.

Length of fencing used =  $2 \times 30 + 60$  m = 120 m ✓ OK

Area =  $30$  m  $\times$   $60$  m =  $1800$  m<sup>2</sup>

Check the areas of pens with lengths close to 60 m.

61 m long: width = 29.5, area =  $29.5$  m  $\times$   $61$  m  
=  $1799.5$  m<sup>2</sup>

59 m long: width = 30.5, area =  $30.5$  m  $\times$   $59$  m  
=  $1799.5$  m<sup>2</sup>

State your findings.

The model uses all the fencing and small changes to the dimensions reduces the area, so it is a useful model.

You could also use your graphics calculator in Example 10 to check the maximum.

Only three points are needed to determine a quadratic function. If the turning point is known, then only one other point is needed.

## EXAMPLE 11

A function passes through (1, 8), (3, 16) and (5, 20). Model it with a quadratic function, verify that your function is correct and evaluate the usefulness of the method.

### Solution

Write a general quadratic function.

$$\text{Let } f(x) = ax^2 + bx + c$$

Use the values to write equations.

$$f(1) = a + b + c = 8$$

$$f(3) = 9a + 3b + c = 16$$

$$f(5) = 25a + 5b + c = 12$$

Eliminate  $c$ .

$$9a + 3b + c - a - b - c = 16 - 8 \Rightarrow 8a + 2b = 8$$

$$25a + 5b + c - 9a - 3b - c = 20 - 16 \Rightarrow 16a + 2b = 4$$

Solve for  $a, b$ .

$$16a + 2b - 8a - 2b = 4 - 8 \Rightarrow a = -\frac{1}{2}$$

$$8 \times \left(-\frac{1}{2}\right) + 2b = 8 \Rightarrow b = 6$$

Find  $c$ .

$$-\frac{1}{2} + 6 + c = 8 \Rightarrow c = 2\frac{1}{2}$$

Write the function.

$$f(x) = -\frac{1}{2}x^2 + 6x + 2\frac{1}{2}$$

Check values.

$$f(1) = -\frac{1}{2} + 6 + 2\frac{1}{2} = 8 \text{ OK}$$

$$f(3) = -\frac{1}{2} \times 9 + 6 \times 3 + 2 = -18 + 36 - 2 = 16 \text{ OK}$$

$$f(5) = -\frac{1}{2} \times 25 + 6 \times 5 + 2\frac{1}{2} = -12\frac{1}{2} + 30 + 2\frac{1}{2} = 20 \text{ OK}$$

Try a simpler model.

A straight line through the first two points has gradient  $\frac{8}{2} = 4$ .

A straight line through the last two points has gradient  $\frac{4}{2} = 2$ .

State the result.

There cannot be a polynomial of power 1 through the points.

Write the answer.

The function  $f(x) = -\frac{1}{2}x^2 + 6x + 2\frac{1}{2}$  passes through the points and is the simplest polynomial that works, so using a quadratic is very useful for three points.

## Exercise 4.04 Models and problems involving quadratic functions

Example  
9

1 Find the length of the line  $2x - y - 5 = 0$  cut off by the curve  $f(x) = 3x - x^2 + 1$ .

Example  
10

2 Find the length of  $y = x - 2$  cut off by  $f(x) = 3x^2 + 3x - 7$ .

3 Longer fence sections need fewer posts, but the rails need to be thicker to provide rigidity. The labour and materials cost for paling fence panels of length  $x$  metres is about  $\$(22x^3 - 110x^2)$  per panel. You also pay about  $\$210x$  per panel for transport and equipment use/hire for a job in an average suburb with standard front gates.

- What is the cost of a paling fence per panel, with panels of length  $x$  m?
- What is the panel length (to the nearest 0.1 m) for the lowest cost?
- What is the lowest possible cost to fence a block 20 m by 30 m?

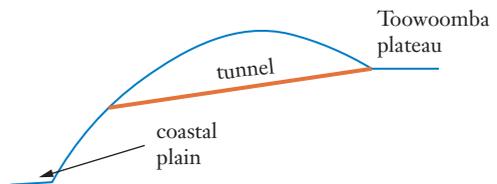
Example  
11

4 A quadratic function passes through  $(0, 50)$ ,  $(10, 30)$  and  $(20, 60)$ . Find the quadratic function.

5 A quadratic function has values of 20, 30 and 50 at  $x = 2$ ,  $x = 3$  and  $x = 6$  respectively. Find the function.

### Problem solving

6 The height of the escarpment at the edge of a plateau is modelled by  $h = \frac{x(16\,000 - x)}{200\,000}$ , from  $x = 0$  at the base of the escarpment to  $x = 10\,000$  at the start of the plateau. The steepest gradient that a railway line can have is 1 in 40.



It is proposed to build a railway tunnel from the edge of the plateau to the other side of the escarpment as shown. Then the line will zigzag down to the coastal plain. How far horizontally and vertically from the edge of the plain will the tunnel emerge? Give your answers to the nearest metre. Verify and evaluate the model.

7 The cost of making calculator chips in a factory decreases according to the equation  $c = \$850 - 0.016995n$  per chip, where  $n$  is the number of chips, up to 50 000 chips. The cost per chip then remains constant. The initial cost of setting up the masks for the chips is  $\$21\,000\,000$ . The chips are sold for  $\$2$  each. Find an expression for the total cost of making  $n$  chips and determine the minimum number that must be sold to make a profit. Is there a maximum profit?



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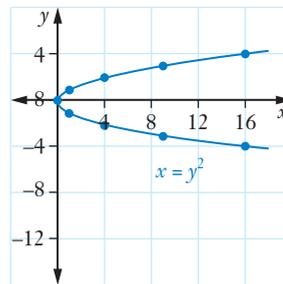
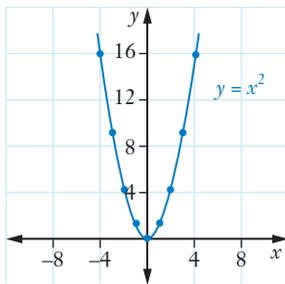
- 8** Resistance wire is used in heating elements. When a current is passed through the wire, its resistance increases with the temperature, further raising the temperature. A particular heating element using Nichrome 60 wire has a resistance of  $80 \Omega$  at  $0^\circ\text{C}$ ,  $83 \Omega$  at  $300^\circ\text{C}$  and  $85.5 \Omega$  at  $600^\circ\text{C}$ . Model the situation with a quadratic function to find the resistance at the operating temperature of  $1200^\circ\text{C}$ . Verify and evaluate the model.
- 9** The water solubility of a particular compound changes with temperature as follows.  
Maximum amount dissolved ( $\text{g/L}$ ) =  $25 - T + 0.0125T^2$  where  $T$  is measured in  $^\circ\text{C}$ . Sketch a graph of this relationship, and hence find the temperature with the minimum solubility.
- 10** The height of a ball thrown upwards at  $25 \text{ m/s}$  is approximately given by the equation  $H(t) = 2 - 5t^2 + 25t$ . Find the turning point and intercepts, and hence the time taken to reach the maximum height and the time taken to reach the ground. Verify the result.

## 4.05 Graphs of quadratic relations

The ordered pairs of the quadratic relation  $y^2 = x$  are the reverse of the ordered pairs of the quadratic function  $y = x^2$ . The graph of  $y^2 = x$  is turned  $90^\circ$  clockwise from that of  $y = x^2$ .

$x$	-4	-3	-2	-1	0	1	2	3	4
$y^2$	16	9	4	1	0	1	4	9	16

$y$	-4	-3	-2	-1	0	1	2	3	4
$x^2$	16	9	4	1	0	1	4	9	16



- A **quadratic relation** in the form  $x = a(y - b)(y - c)$  has  **$y$ -intercepts** at  $y = b$  and  $y = c$ . the graph is reversed if  $a$  is negative. The **vertex** is halfway between  $b$  and  $c$ . There is a horizontal **axis of symmetry** that passes through the vertex.
- Compared to  $y = x^2$ , if  $|a| > 1$ , the graph is horizontally stretched by the factor  $|a|$ , and if  $|a| < 1$ , it is horizontally compressed by  $|a|$ .

You can sketch a quadratic relation similarly to a quadratic function. Instead of finding zeros, you find the  $y$ -intercepts. Instead of finding the  $y$ -intercept, you find the  $x$ -intercept.

### EXAMPLE 12

Sketch the graph of  $x = (y + 3)^2 - 2$ .

#### Solution

State the turning point.

The vertex is at  $(-2, -3)$ .

Find the  $y$ -intercepts.

At  $x = 0$ ,  $(y + 3)^2 - 2 = 0$

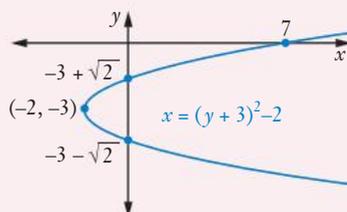
$$y + 3 = \pm \sqrt{2}$$

$$y = -3 \pm \sqrt{2}$$

Find the  $x$ -intercept.

At  $y = 0$ ,  $x = 7$

Sketch the graph.



You might have to rearrange a quadratic relation to sketch it. If there are no  $y$ -intercepts, you need to complete the square.

### EXAMPLE 13

Sketch the graph of  $8y - x + 3 = 2y^2 + 12$ .

#### Solution

Rearrange.

$$x = -2y^2 + 8y - 9$$

Find any  $y$ -intercepts.

$\Delta = 64 - 72 = -8$ , so there are no  $y$ -intercepts.

Complete the square.

$$\begin{aligned} x &= -2(y^2 - 4y + 4) - 9 + 8 \\ &= -2(y - 2)^2 - 1 \end{aligned}$$

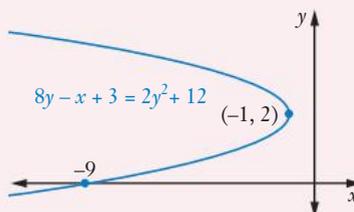
State the vertex.

The vertex is at  $(-1, 2)$ .

Find the  $x$ -intercept.

At  $y = 0$ ,  $x = -9$ .

Sketch the graph.



## Exercise 4.05 Graphs of quadratic relations

Sketch the graph of each quadratic relation.

1  $x = (y + 2)(y - 4)$ .

2  $x = (y - 2)^2 + 5$ .

3  $x = -y^2 + 4$ .

4  $y^2 = x + y + 6$ .

5  $y - x = 5y - 2y^2 + 8$ .

6  $2x - 5y = 4y^2 - 6 + 5y$ .

### Problem solving

7 Find the quadratic relation that passes through the points  $(-8, 0)$ ,  $(-11, 3)$  and  $(-23, -3)$ .

8 Find the quadratic relation that has  $y$ -intercepts 2 and 6 and  $x$ -intercept 24.

## 4.06 Hyperbolic functions

Hyperbolic functions are based on the inverse proportion function  $y = \frac{1}{x}$ .

### INVESTIGATION

#### THE HYPERBOLIC FUNCTION

- 1 Calculate the values of  $\frac{1}{x}$  for  $x = -20, -10, -5, -2, -1, -0.5, -0.2, -0.1, -0.05, 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10$  and  $20$ .
- 2 Is there a value at  $x = 0$ ?
- 3 Draw a graph of the function  $y = \frac{1}{x}$  on squared paper.
- 4 What happens to the value of  $y$  as  $x \rightarrow -\infty$ ?
- 5 What happens to the value of  $y$  as  $x \uparrow 0$  ( $x \rightarrow 0$  from below)?
- 6 What happens to the value of  $y$  as  $x \downarrow 0$ ?
- 7 What happens to the value of  $y$  as  $x \rightarrow \infty$ ?
- 8 Use your calculator to draw the graphs of  $y = \frac{2}{x}$ ,  $y = \frac{0.5}{x}$ ,  $y = \frac{1}{x+3}$ ,  $y = \frac{1}{x-3}$ ,  $y = \frac{1}{x} + 1$ ,  $y = \frac{1}{x} - 4$  and  $y = -\frac{1}{x}$ .
- 9 What do you find about the graphs of the related functions?

Example  
12

Example  
13



Excel spreadsheet:  
Curve sketcher



Excel worksheet:  
Curve sketcher



Graphing  
non-linear  
equations



Shutterstock.com/ Alf Ribeiro

The shape of the function  $y = \frac{1}{x}$  is called a (rectangular) **hyperbola**.

It has a **discontinuity** at  $x = 0$ . The line  $x = 0$  is a **vertical asymptote**. As  $x \uparrow 0$ ,  $y \rightarrow -\infty$  and as  $x \downarrow 0$ ,  $y \rightarrow \infty$ .

The line  $y = 0$  is a **horizontal asymptote**. As  $x \rightarrow \pm\infty$ ,  $y \rightarrow 0$  but never reaches 0.

Compared to  $y = \frac{1}{x}$ :

- the graph of  $y = -\frac{1}{x}$  is inverted
- the graph of  $y = \frac{a}{x}$  is **vertically dilated** by the factor  $a$ ; stretched if  $|a| > 1$  and compressed if  $|a| < 1$
- the graph of  $y = \frac{1}{x-b}$  is **translated horizontally** by  $b$ ; the vertical asymptote is moved to  $x = b$
- the graph of  $y = \frac{1}{x} + c$  is **translated vertically** by  $c$ ; the horizontal asymptote is moved to  $y = c$

You can use dilation and translations from  $y = \frac{1}{x}$  to sketch the graphs of hyperbolic functions. Show the positions that  $(-1, -1)$  and  $(1, 1)$  are moved to so that the scale is indicated.

**EXAMPLE 14**

Sketch the graph of:

**a**  $y = -\frac{3}{x}$

**b**  $y = \frac{1}{x} + 2$

**c**  $y = \frac{1}{x-4}$

**Solution**

- a**
- What is the effect of the negative sign?

The  $-$  inverts the graph from  $\frac{1}{x}$ .

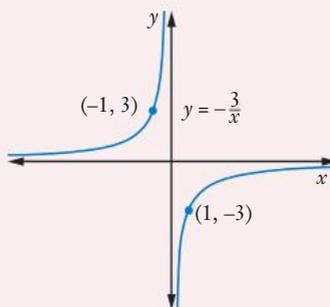
What is the effect of the 3?

The 3 vertically stretches the graph from  $\frac{1}{x}$ .State the effect on  $(-1, -1)$  and  $(1, 1)$ . $(1, 1) \rightarrow (1, -3)$  and  $(-1, -1) \rightarrow (-1, 3)$ 

State the asymptotes.

The vertical asymptote is  $x = 0$  and the horizontal asymptote is  $y = 0$ , the axes.

Sketch the graph, showing important points.



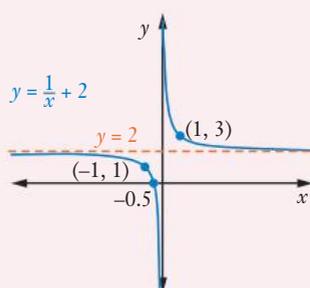
- b**
- What is the effect of the 2?

The graph is moved up 2 units.

State the asymptotes.

The vertical asymptote is  $x = 0$  and the horizontal asymptote is  $y = 2$ .State the effect on  $(-1, -1)$  and  $(1, 1)$ . $(-1, -1) \rightarrow (-1, 1)$  and  $(1, 1) \rightarrow (1, 3)$ 

Sketch the graph, showing important points and the asymptote.



c What is the effect of the 4?

The graph is moved 4 units right.

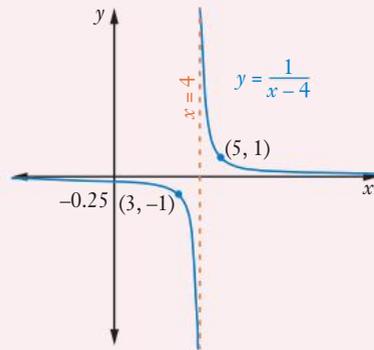
State the effect on  $(-1, -1)$  and  $(1, 1)$ .

$(-1, -1) \rightarrow (3, -1)$  and  $(1, 1) \rightarrow (5, 1)$

State the asymptotes.

The vertical asymptote is  $x = 4$  and the horizontal asymptote is  $y = 0$ .

Sketch the graph, showing important points and asymptotes. The  $x$ -axis is already shown.



When you have to combine effects, the new positions of  $(1, 1)$  and  $(-1, -1)$  are a good way to check your work.

### EXAMPLE 15

Sketch the graph of  $f(x) = \frac{2}{x+1} - 3$ .

#### Solution

State the effect of the 2.

The 2 vertically stretches the graph from  $\frac{1}{x}$ .

State the effect of the 1.

The graph is moved 1 left from  $\frac{1}{x}$ .

The vertical asymptote is  $x = -1$ .

State the effect of the 3.

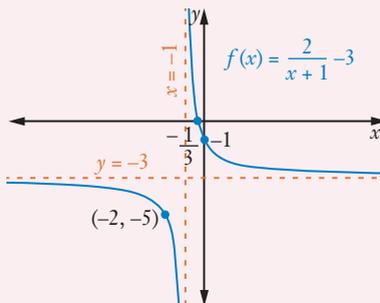
The graph is moved 3 down from  $\frac{1}{x}$ .

The horizontal asymptote is  $y = -3$ .

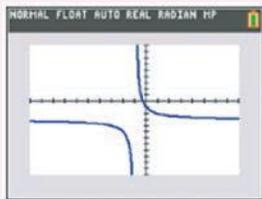
State the effect on  $(-1, -1)$  and  $(1, 1)$ .

$(-1, -1) \rightarrow (-1, 1)$  and  $(1, 1) \rightarrow (1, 3)$

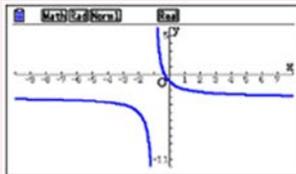
Sketch the graph, showing important points and asymptotes.



### TI-84 Plus CE



### Casio fx-CG20AU



You have to show working and give exact values for intercepts, points and so on in an exam, but you can use your calculator to find them in a technology-active exam.

## Exercise 4.06 Hyperbolic functions

1 Sketch the graph of each function.

**a**  $y = \frac{1}{2x}$

**b**  $f(x) = \frac{3}{x}$

**c**  $y = -\frac{1}{3x}$

**d**  $f(x) = \frac{1}{x} - 1$

**e**  $y = 3 + \frac{1}{x}$

**f**  $f(x) = 2 - \frac{1}{x}$

**g**  $y = \frac{1}{x+2}$

**h**  $f(x) = \frac{1}{3-x}$

**i**  $y = -\frac{1}{x+4}$

2 Sketch the graph of  $f(x) = \frac{1}{x-2} - 1$ .

3 Sketch the graph of  $y = \frac{3}{4-x}$ .

4 Sketch the graph of  $f(x) = 2 + \frac{1}{2x}$ .

5 Sketch the graph of  $y = \frac{3}{x+4} + 2$ .

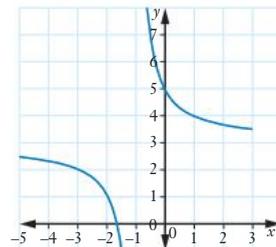
6 Sketch the graph of  $f(x) = 3 - \frac{1}{4(x-3)}$ .

Example  
**14**

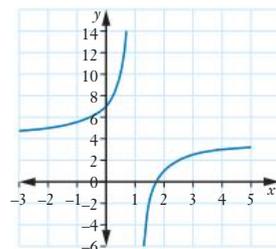
Example  
**15**

## Problem solving

- 7 The graph of a hyperbola has asymptotes of  $x = 3$  and  $y = 2$  and passes through  $(4, 6)$ . Write a possible equation for the function and verify your solution.
- 8 The graph of a hyperbola has asymptotes of  $x = -2$  and  $y = 4$  and a  $y$ -intercept of  $2.5$ . Write a possible equation for the function and verify your solution.
- 9 Write a possible equation for the function with the graph shown here and verify your solution.



- 10 Write a possible equation for the function with the graph shown here and verify your solution.



Graphing  
non-linear  
equations



Polynomials

## 4.07 Graphs of polynomials

You can work out the shapes of the graphs of polynomials from the shapes of quadratics and simple polynomials.

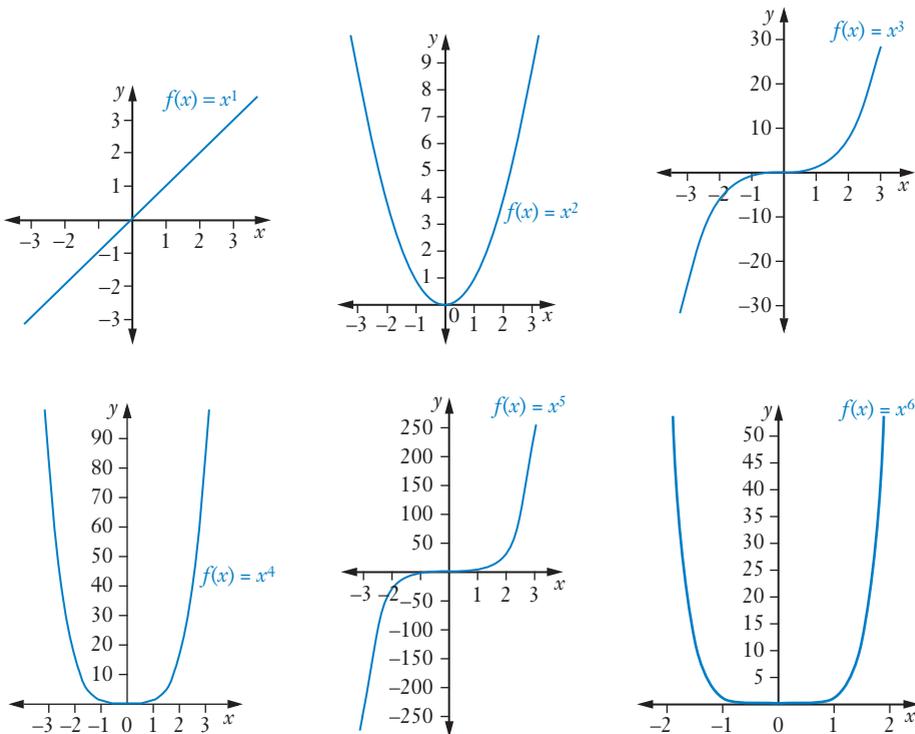
### TECHNOLOGY

#### Polynomials of the form $y = x^n$

The website has a spreadsheet called 'Polynomials' that you can use to examine the shapes of simple polynomials. Use your graphics calculator or the spreadsheet to look at the polynomials  $y = x^n$  for  $n = 1$  to  $5$ .

What do you find?

The graphs of  $y = x^n$  (where  $n$  is a positive integer) are shown below for  $n = 1$  to 6. Note that the vertical scales are different so that the general shape of each can be seen.



Recall the comparison of graphs of quadratic functions expressed as  $f(x) = a(x - b)^2 + c$  with the graph of  $y = x^2$ . You can compare the graphs of  $y = a(x - b)^n + c$  with the graphs of  $y = x^n$  in the same way.

- The graph of a power function of the form  $y = x^n$  for  $n > 1$  has a turning point at  $(0, 0)$ .
- For even powers of  $n$  the turning point is a **minimum** and the graph is **concave upwards**. This means that the inside of the curve is above the outside of the curve.
- For odd powers of  $n$ , the graph changes from **concave downwards** to **concave upwards** at the turning point. This is called a **point of inflection**.
- The graphs of even powers of  $n$  have an axis of symmetry through the turning point.
- Compared to  $y = x^n$ , the graph of  $f(x) = a(x - b)^n + c$  is **dilated** by  $a$  and inverted if  $a$  is negative. The graph appears narrower if  $|a| > 1$  and wider if  $|a| < 1$ . All points are shifted  $b$  later (to the right) and  $c$  higher, so the turning point is  $(b, c)$ .
- The **domain** is the set of real numbers  $\mathbf{R}$ .
- The **range** of odd powers is  $\mathbf{R}$ .
- The **range** of even powers is  $y \geq b$  for  $a > 0$  and  $y \leq b$  for  $a < 0$ .

You should show the new position of  $(-1, -1)$  or  $(1, 1)$  or both when you sketch a cubic of the form  $f(x) = a(x - b)^n + c$  so that the scale is clear. You should also show the  $y$ -intercept. Together with the turning point, this is sufficient to show the graph.

### EXAMPLE 16

Sketch the graph of:

**a**  $f(x) = 3(x + 1)^3 - 4$

**b**  $y = 3 - 0.5(x - 2)^3$

### Solution

**a** State the changes from  $y = x^3$ .

The graph is shifted 1 unit to the left, 4 down and vertically stretched by 3 times.

Where is the turning point?

The turning point is at  $(-1, -4)$ .

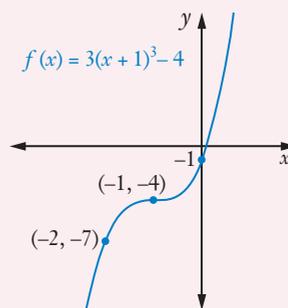
Find the  $y$ -intercept.

$$f(0) = -1$$

State the effect on  $(-1, -1)$  and  $(1, 1)$ .

$$(-1, -1) \rightarrow (-2, -7) \text{ and } (1, 1) \rightarrow (0, -1)$$

Sketch the graph, showing important points.



**b** State the changes from  $y = x^3$ .

The graph is shifted 2 units to the right, 3 units up and vertically compressed by 0.5. It is also inverted.

Where is the turning point?

The turning point is at  $(2, 3)$ .

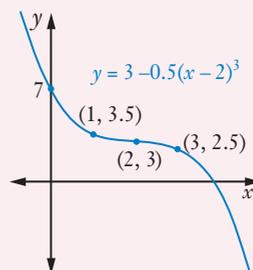
Find the  $y$ -intercept.

$$\text{At } x = 0, y = 7$$

State the effect on  $(-1, -1)$  and  $(1, 1)$ .

$$(-1, -1) \rightarrow (1, 3.5) \text{ and } (1, 1) \rightarrow (3, 2.5)$$

Sketch the graph, showing important points.



When you sketch a quartic, show the turning point and the new positions of  $(-1, 1)$  and  $(1, 1)$ .

### EXAMPLE 17

Sketch the graph of:

**a**  $y = 1 - (x - 2)^4$

**b**  $f(x) = 2(x + 3)^4 - 4$

### Solution

**a** State the changes from  $y = x^4$ .

The graph is shifted 2 units to the right, 1 up and inverted.

Where is the turning point?

The turning point is at  $(2, 1)$ .

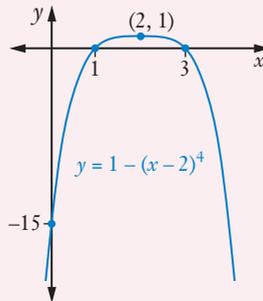
Find the  $y$ -intercept.

At  $x = 0, y = -15$

State the effect on  $(-1, 1)$  and  $(1, 1)$ .

$(-1, 1) \rightarrow (1, 0)$  and  $(1, 1) \rightarrow (3, 0)$

Sketch the graph, showing important points.



**b** State the changes from  $y = x^4$ .

The graph is shifted 3 units to the left, 4 down and vertically stretched by 2.

Where is the turning point?

The turning point is at  $(-3, -4)$ .

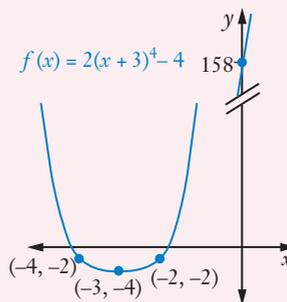
Find the  $y$ -intercept.

$f(0) = 158$

State the effect on  $(-1, 1)$  and  $(1, 1)$ .

$(-1, 1) \rightarrow (1, 0)$  and  $(1, 1) \rightarrow (3, 0)$

Sketch the graph, showing important points. You can use a break in the axis to show the  $y$ -intercept.



A quadratic of the form  $y = a(x - b)(x - c)$  has zeros at  $b$  and  $c$ . By extension, a cubic of the form  $y = k(x - a)(x - b)(x - c)$  must have zeros at  $a, b$  and  $c$ . For this to be the case, the shape must be different to the shape of  $y = x^3$ .

## INVESTIGATION

$$y = k(x - a)(x - b)(x - c)$$

Use your graphics calculator to draw the graphs of the following functions to investigate their shapes.

**1**  $y = (x - 2)(x - 4)(x - 5)$

**2**  $y = -(x - 2)(x - 4)(x - 5)$

**3**  $y = 3(x - 2)(x - 4)(x - 5)$

**4**  $y = 2(x + 1)(x - 3)(x + 4)$

**5**  $y = -2(x + 1)(x + 1)^2$

What can you say about the shape of a function of the form  $y = k(x - a)(x - b)(x - c)$ ?

What happens if one of the roots is repeated?

A **cubic** of the form  $y = k(x - a)(x - b)(x - c)$  has zeros at  $a$ ,  $b$  and  $c$ . The function has a minimum point between two zeros and a maximum point between the others unless two zeros are the same. If  $a \neq b = c$ , then either the minimum or maximum point will be at  $(b, 0)$ .

For  $k < 0$ , as  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$  and as  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ .

For  $k > 0$ , as  $x \rightarrow -\infty$ ,  $y \rightarrow -\infty$ ; as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ .

When you sketch functions of the form  $y = k(x - a)(x - b)(x - c)$ , you should check the signs between the zeros and find the  $y$ -intercept to check the shape.

If  $a = b = c$ , then the function  $y = k(x - a)^3$  and you can sketch the function as a simple cubic.

## EXAMPLE 18

Sketch the graph of:

**a**  $f(x) = -2(x + 1)(x - 1)(x - 4)$

**b**  $y = 0.5(x + 3)(x - 2)^2$

### Solution

**a** State the zeros.

The graph has zeros at  $x = -1, 1$  and  $4$ .

Find the  $y$ -intercept.

$$f(0) = -8$$

Check the signs between the zeros.

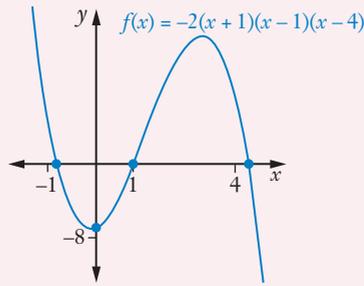
For  $-1 < x < 1$ ,  $f(x) < 0$

For  $1 < x < 4$ ,  $f(x) > 0$

State the shape of  $f(x)$ .

$f(x)$  has a minimum between  $-1$  and  $1$ , a maximum between  $1$  and  $4$ , as  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$  and as  $x \rightarrow \infty$ ,  $y \rightarrow -\infty$ .

Sketch the graph, showing important points. Try to avoid showing the maximum or minimum as halfway between the zeros, as you do not know exactly where they are.



**b** State the zeros.

Find the  $y$ -intercept.

Check the sign between the zeros.

State the shape of  $f(x)$ .

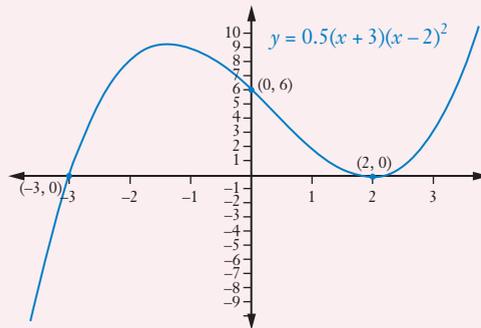
Sketch the graph, showing important points.

The graph has zeros at  $x = -3$  and  $2$ .

At  $x = 0, y = 6$

For  $-3 < x < 2, y > 0$

$y$  has a maximum between  $-3$  and  $2, y > 0$  near  $x = 2$ , so  $y$  has a minimum at  $x = 2$ , as  $x \rightarrow -\infty, y \rightarrow -\infty$  and as  $x \rightarrow \infty, y \rightarrow \infty$ .



## Exercise 4.07 Graphs of polynomials

**1** Sketch the graph of each function.

**a**  $f(x) = (x-1)^3 + 3$

**b**  $y = 4 - (x+2)^3$

**c**  $f(x) = 3 - (x-1)^3$

**d**  $y = -3 - 0.4(x+2)^3$

**e**  $f(x) = 2(x-4)^3 + 1$

**f**  $y = 2 + 3(x+1)^3$

**2** State the range of each function.

**a**  $y = (x+2)^3 - 3$  for  $-4 \leq x \leq 1$

**b**  $f(x) = 2 - (x+1)^3$  for  $-3 \leq x < 3$

**c**  $y = (x-2)^3 - 3$  for  $-2 < x \leq 5$

**d**  $y = 1 - (x+1)^3$  for  $-4 \leq x \leq 4$

**e**  $f(x) = -1 - 0.4(x-2)^3$  for  $-3 < x < 5$

**f**  $y = 2(x-3)^3 + 5$  for  $-1 \leq x \leq 4$

**3** Sketch the graph of each function.

**a**  $y = (x+2)^4 - 1$

**b**  $f(x) = 3 + (x-1)^4$

**c**  $f(x) = -(x-4)^4 + 2$

**d**  $y = 1 - 2(x+1)^4$

**e**  $y = -1 - 3(x+3)^4$

**f**  $f(x) = -0.5(x-2)^4 + 7$

Example  
**16**

Example  
**17**

**4** State the range of each function.

**a**  $f(x) = -(x - 1)^4 - 3$  for  $-2 < x \leq 2$

**b**  $y = 4 - (x + 3)^4$  for  $-5 < x < 1$

**c**  $f(x) = 0.6(x - 4)^4 + 6$  for  $2 \leq x < 6$

**d**  $y = -1 - 3(x - 1)^4$  for  $-2 \leq x \leq 3$

**e**  $f(x) = -1 - 4(x - 3)^4$  for  $1 < x < 4$

**f**  $y = 2(x - 5)^4 - 3$  for  $3 < x < 6$

**5** Sketch the graph of each function.

**a**  $f(x) = (x - 3)(x + 2)(x + 1)$

**b**  $y = -(x + 1)(x - 4)(x - 5)$

**c**  $f(x) = (5 - x)(5 + x)(x + 1)$

**d**  $y = (x - 2)(x + 2)(x - 2)$

**e**  $f(x) = 0.5(x + 2)(x - 3)(x + 1)$

**f**  $y = -2(x - 1)^2(2x - 5)$

### Problem solving

**6** The graph of a cubic function passes through the  $x$ -axis when  $x = -1$ ,  $x = 0$  and  $x = 3$ . If the graph also passes through the point  $(-3, 36)$ , find the function.

**7** The graph of a cubic function has a point of inflection at  $(3, 2)$ .

**a** What is the equation for the family of functions that satisfy this condition?

**b** From this family, find the particular cubic function that passes through the point  $(-1, 34)$ .

**8** A fourth power function has a maximum at  $(2, 5)$  and a  $y$ -intercept of 1. It has no other maxima or minima. Find a possible function.

**9** The concentration of live algae after  $D$  days in a reservoir, where the water has a high phosphate content, is modelled by  $m(D) = 0.39D^2 - 0.01D^3 + 0.75D$ , where  $m$  is measured in grams/litre.

**a** Use your graphics calculator to sketch  $m(D)$ .

**b** Find the  $x$ -intercepts of  $m(D)$ .

**c** What mass of algae would be present after 10, 20 and 30 days?

**d** Use a graphics calculator to find the maximum mass of algae present. How would this be related to the initial amount of nutrient present in the water?

**e** From your graph, suggest the domain of  $D$  for which the model is accurate.

**10** A specimen of bacteria is smeared on a round agar plate of diameter 12 cm and left for testing. The area of the bacterial colony is given by

$$A = 6t^2 - t^3 + 16t + 0.5 \text{ cm}^2, \text{ where } t \text{ is the number of weeks after the culture is started.}$$

**a** What is the initial size of the colony?

**b** Sketch the graph for the first 10 weeks.

**c** When does the model of growth become inadequate?

Assume that the model is accurate for the first 7 weeks.

**d** What percentage of the plate is covered after 4 weeks?

**e** What is the maximum percentage of the plate that is covered?



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## 4.08 Transformations of functions

You have already seen in this chapter how the basic functions  $f(x) = x^2$ ,  $f(x) = \frac{1}{x}$ ,  $f(x) = x^3$  and  $f(x) = x^4$  are affected by adding a number to  $x$ , adding a number to the function or multiplying the function by a number to give functions of the form  $g(x) = af(x + c) + d$ .

What happens if you change the coefficient of  $x$ ?

### INVESTIGATION

#### The effect of $b$ on $f(bx)$

##### Part 1

- 1 Use your graphics calculator to draw the functions  $Y1 = x^3 + 1$ ,  $Y2 = (2x)^3 + 1$  and  $Y3 = (3x)^3 + 1$ .
- 2 What happens to the zero at  $(-1, 0)$ ? Trace the functions if necessary.
- 3 What is the effect of  $b$  on  $f(bx)$  for  $b > 1$ ?

##### Part 2

- 1 Use your graphics calculator to draw the functions  $Y1 = x^3 + 1$ ,  $Y2 = (0.5x)^3 + 1$  and  $Y3 = (0.25x)^3 + 1$ .
- 2 What happens to the zero at  $(-1, 0)$ ? Trace the functions if necessary.
- 3 What is the effect of  $b$  on  $f(bx)$  for  $0 < b < 1$ ?

##### Part 3

- 1 Use your graphics calculator to draw the functions  $Y1 = x^3 + 1$ ,  $Y3 = (-x)^3 + 1$ ,  $Y3 = (-3x)^3 + 1$  and  $Y4 = (-0.5x)^3 + 1$ .
- 2 What happens to the zero at  $(-1, 0)$ ? Trace the functions if necessary.
- 3 What is the effect of  $b$  on  $f(bx)$  for  $b < 0$ ?

### Transformations of functions

Compared to the graph of  $y = f(x)$ , the graph of  $y = af(bx + c) + d$  is:

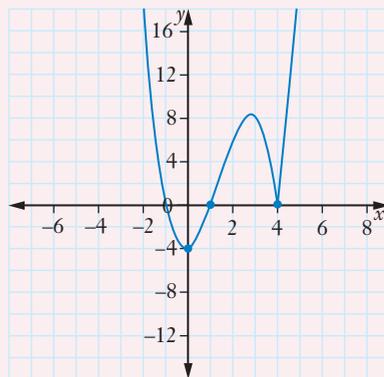
- reflected in the  $x$ -axis for  $a$  negative, vertically stretched for  $|a| > 1$ , vertically compressed for  $|a| < 1$
- reflected in the  $y$ -axis for  $b$  negative, horizontally compressed for  $|b| > 1$ , horizontally stretched for  $|b| < 1$
- moved horizontally by  $c$ , left for  $c > 0$ , right for  $c < 0$
- moved vertically by  $d$ , up for  $d > 0$ , down for  $d < 0$

### EXAMPLE 19

The graph of the function  $f(x)$  is shown.

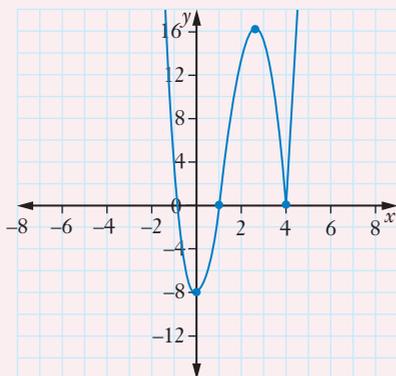
Use the same scale to draw the graph of:

- a**  $2f(x)$
- b**  $-f(x)$
- c**  $-0.5f(x)$
- d**  $f(2x)$
- e**  $f(-x)$
- f**  $f(0.5x)$
- g**  $f(x+2)$
- h**  $f(x-1)$
- i**  $f(x)+1$
- j**  $f(x)-2$

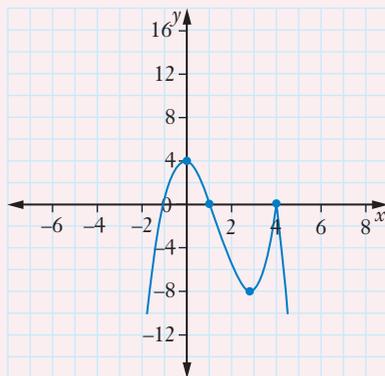


### Solution

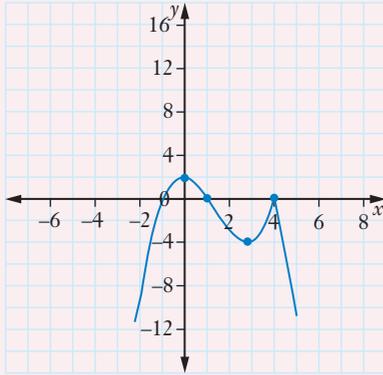
- a** Stretch vertically by the factor 2; double the distances of all points from the  $x$ -axis. Join the points.



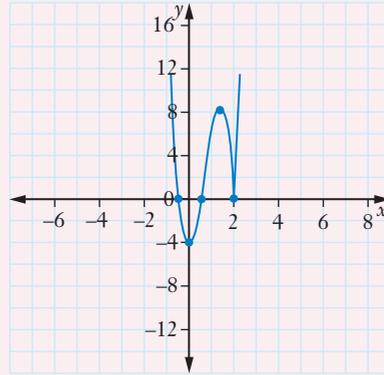
- b** Reflect all points in the  $x$ -axis; invert. Join the points.



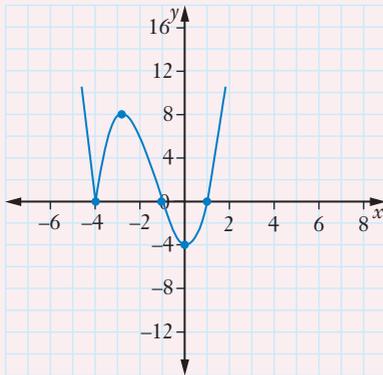
- c** Reflect in the  $x$ -axis and compress vertically by 0.5; invert and halve the distances from all points from the  $x$ -axis. Join the points.



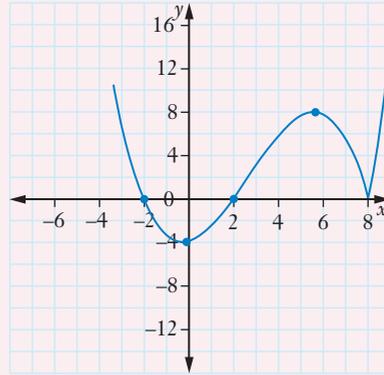
- d** Compress horizontally by the factor 2; halve the distances from all points from the  $y$ -axis. Join the points.



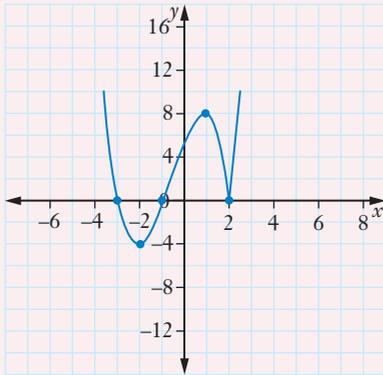
- e** Reflect all points in the  $y$ -axis. Join the points.



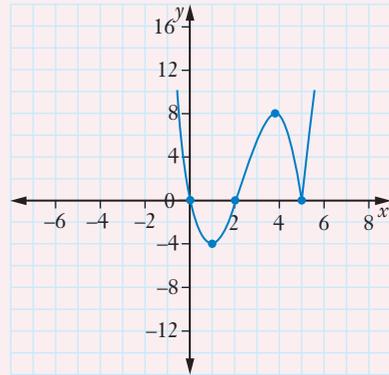
- f** Stretch horizontally by the factor 0.5; double the distance of all points from the  $y$ -axis. Join the points.



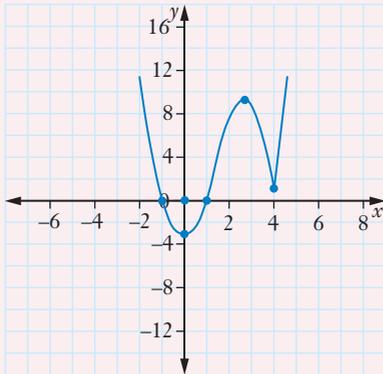
- g** Move all points left by 2.  
Join the points.



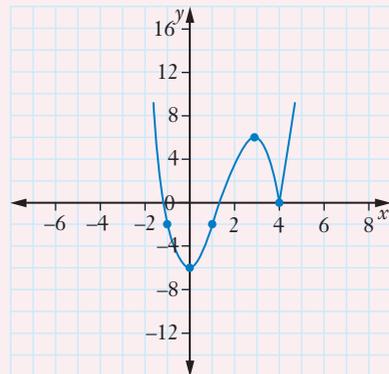
- h** Move all points right by 1.  
Join the points.



- i** Move all points up by 1.  
Join the points.



- j** Move all points down by 2.  
Join the points.

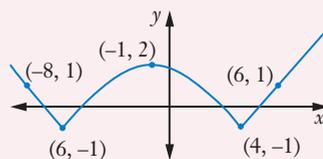


When you have combined transformations of the form  $y = af(bx + c) + d$ , do any horizontal and/or vertical stretches or compressions and reflections before the translations. If you are given points on the graph, use the points to work out the new points and join them to make the new graph.

## EXAMPLE 20

The graph of the function  $f(x)$  is shown on the right.

Use the same scale to draw the graph of  $3f(2x + 3) - 4$ .



### Solution

State the transformations.

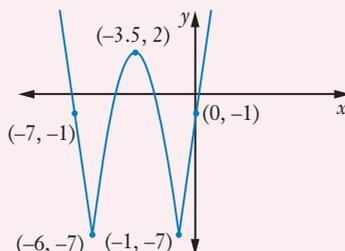
Stretch vertically by a factor of 3, compress horizontally by a factor of 2, translate 3 to the left and 4 down.

Apply to the points. Triple the  $y$ -coordinate, halve the  $x$ -coordinate, take 3 from the  $x$ -coordinate and take 4 from the  $y$ -coordinate.

$(-8, 1) \rightarrow (-8, 3) \rightarrow (-4, 3) \rightarrow (-7, 3) \rightarrow (-7, 1)$   
 $(-1, 2) \rightarrow (-1, 6) \rightarrow (-0.5, 6) \rightarrow (-3.5, 6) \rightarrow (-3.5, 2)$   
 $(6, 1) \rightarrow (6, 3) \rightarrow (3, 3) \rightarrow (0, 3) \rightarrow (0, -1)$   
 $(4, -1) \rightarrow (4, -3) \rightarrow (2, -3) \rightarrow (-1, -3) \rightarrow (-1, -7)$   
 $(-8, 1) \rightarrow (-8, 3) \rightarrow (-4, 3) \rightarrow (-7, 3) \rightarrow (-7, 1)$

Sketch the graph.

If you are required to do multiple transformations of a function without specific points, choose some points to calculate the changes.



## Exercise 4.08 Transformations of functions

1 The graph of the function  $f(x)$  is shown.

Draw the graph of:

**a**  $0.5f(x)$

**b**  $-f(x)$

**c**  $-2f(x)$

**d**  $f(-x)$

**e**  $f(-0.5x)$

**f**  $f(-2x)$

**g**  $f(x+1)$

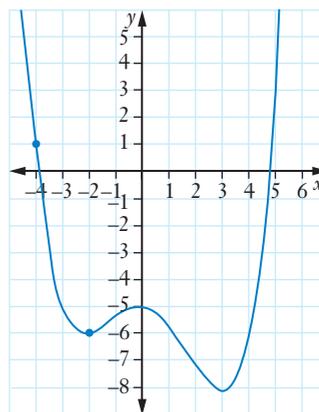
**h**  $f(x-3)$

**i**  $f(x+3)$

**j**  $f(x)+2$

**k**  $f(x)-1$

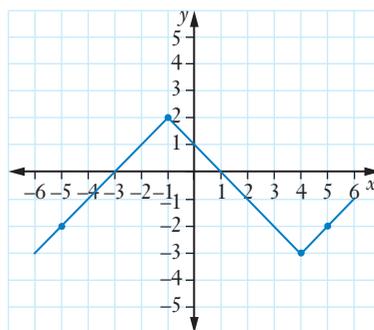
**l**  $f(x)-3$



Example  
19

- 2** The graph of the function  $f(x)$  is shown.  
Draw the graph of:

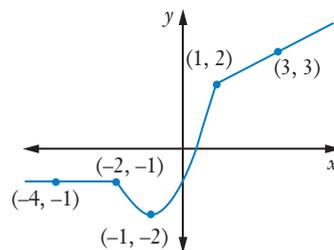
- |                     |                     |
|---------------------|---------------------|
| <b>a</b> $1.5f(x)$  | <b>b</b> $-0.4f(x)$ |
| <b>c</b> $2f(x)$    | <b>d</b> $f(-x)$    |
| <b>e</b> $f(-2.5x)$ | <b>f</b> $f(1.5x)$  |
| <b>g</b> $f(x-4)$   | <b>h</b> $f(x+1)$   |
| <b>i</b> $f(x+2)$   | <b>j</b> $f(x)+3$   |
| <b>k</b> $f(x)-2$   | <b>l</b> $f(x)-4$   |



Example  
**20**

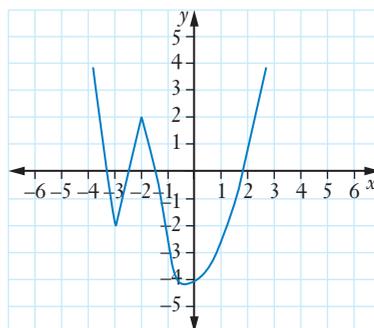
- 3** The graph of the function  $f(x)$  is shown.  
Draw the graph of:

- a**  $2f(0.5x-2)+1$   
**b**  $-f(4-x)-3$   
**c**  $0.5f(2x+3)+4$   
**d**  $2-3f(0.5x+1)$



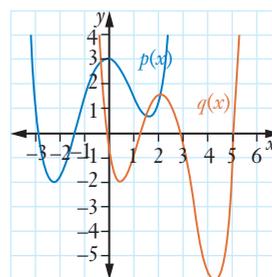
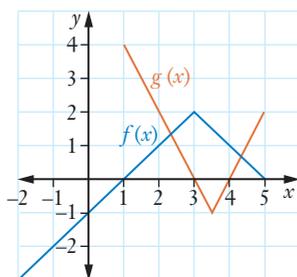
- 4** The graph of the function  $f(x)$  is shown.  
Draw the graph of:

- a**  $2-0.5f(x+1)$   
**b**  $-f(-x)$   
**c**  $2f(-2x-3)-1$   
**d**  $0.5f(0.5x+3)+2$



### Problem solving

- 5** The graphs of  $f(x)$  and  $g(x)$  are shown on the same axes below. Write  $g(x)$  as a transformation of  $f(x)$ .



- 6** The graphs of  $p(x)$  and  $q(x)$  are shown on the same axes above.  
Write  $q(x)$  as a transformation of  $p(x)$ .
- 7** Use the graph from question 5 to write  $f(x)$  as a transformation of  $g(x)$ .
- 8** Use the graph from question 6 to write  $p(x)$  as a transformation of  $q(x)$ .

## 4.09 Equations involving functions

Many real-world relationships are written as functions. A situation with different relationships might give an equation using two functions.

If you want to find the temperature that a heated object reaches, you need to find the temperature at which the heat lost is equal to the heat gained. If you want to find the natural price of a resource, you need to find the price at which the demand and supply will balance.

These are **simultaneous equations**. You can solve two simultaneous equations using algebra or graphs. You can often see *how many solutions* there are by sketching the graphs. You can also see whether you should use technology to solve the equations.

### EXAMPLE 21

Solve  $x^3 = 3x^2 + 2x - 1$  graphically, correct to two decimal places.

#### Solution

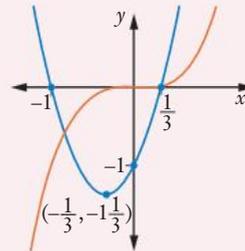
Write the functions to be graphed.

$$y = x^3 \text{ and } y = 3x^2 + 2x - 1.$$

To sketch the functions, you need the zeros of the quadratic.

$y = 3x^2 + 2x - 1 = (3x - 1)(x + 1)$  has zeros at  $-1$  and  $\frac{1}{3}$ . The turning point is halfway between, at  $(-\frac{1}{3}, -1\frac{1}{3})$ .

Sketch the graphs on the same axes.



Write the number of solutions.

There are two intersections, so there are two solutions.

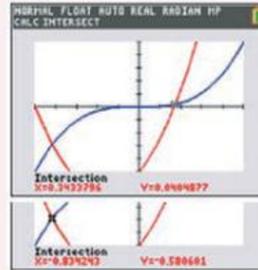
Decide the solution method.

There is no simple linear factor, so use technology.

### TI-84 Plus CE

Use  $\boxed{Y=}$  to enter the functions in  $Y_1$  and  $Y_2$ , set the  $\boxed{\text{window}}$  to  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$  and draw the  $\boxed{\text{graph}}$ .

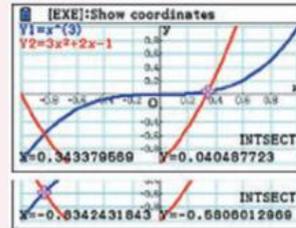
Use  $\boxed{\text{calc}}$ , **5: intersect** twice to find the intersections. Use the cursor keys to set the left and right bounds and the estimate.



### Casio fx-CG20AU

Choose **5 Graph** and enter the functions in  $Y1$  and  $Y2$ . Set the **V-Window** to  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$  and press  $\boxed{\text{F6}}$  to draw the graph.

Use  $\boxed{\text{F5}}$ : **G-Solv** and  $\boxed{\text{F5}}$ : **INTSECT** to find the intersections. Use the cursor keys to swap between the intersections.



Write the solutions.

$$x \approx -0.83 \text{ or } x \approx 0.34$$

You may find it simpler to solve an equation by finding the zero(s) of a combined function.

### EXAMPLE 22

Solve  $6x + 2 = 2x^3 + 3x - 1$ , correct to 2 decimal places.

#### Solution

Rearrange to a single function.

The function is  $f(x) = 2x^3 - 3x - 3 = 0$

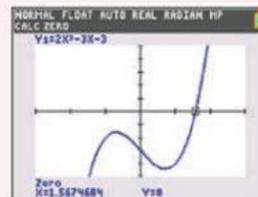
Decide on the method.

There is no simple linear factor, so use technology.

### TI-84 Plus CE

Use  $\boxed{Y=}$  to enter the functions in  $Y_1$  and  $Y_2$ , draw the  $\boxed{\text{graph}}$ , set the  $\boxed{\text{window}}$  appropriately.

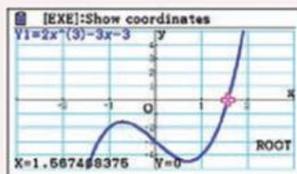
Draw the  $\boxed{\text{graph}}$  again and use  $\boxed{\text{calc}}$ , **2: zero** to find the zero(s). Use the cursor keys to set the left and right bounds and the estimate.



### Casio fx-CG20AU

Choose **5 Graph**, enter the function in Y1, press **F6** to draw the graph and set the **V-Window** appropriately.

Press **F6** again, **F5**: **G-Solv** and **F1**: **ROOT** to find the zeros.



Write the solution(s).

$$x \approx 1.57$$

In some cases you may be able to solve an equation to get exact roots.

### EXAMPLE 23

$$\text{Solve } 9x + 10 = x^3$$

#### Solution

Rearrange to equal zero.

$$0 = x^3 - 9x - 10 = p(x)$$

Decide on the method.

$$p(-2) = 0, \text{ so } (x + 2) \text{ is a factor.}$$

Factorise.

$$(x + 2)(x^2 - 2x - 5) = 0$$

Use the null factor law.

$$x + 2 = 0 \text{ or } x^2 - 2x - 5 = 0$$

Solve the equations.

$$x = -2 \text{ or } x = \frac{2 \pm \sqrt{24}}{2} = \frac{2 \pm 2\sqrt{6}}{2}$$

Simplify and write the answers.

$$x = -2, \text{ or } x = \frac{1 - 2\sqrt{6}}{1} \text{ or } x = \frac{1 + 2\sqrt{6}}{1}$$

### Exercise 4.09 Equations involving functions

Solve each equation, correct to two decimal places where appropriate.

1  $x^3 = 2x^2 + x - 15$

2  $x^3 = x^2 + 6x - 5$

3  $x^3 = 4x - 2x^2 + 6$

4  $(x - 3)(x - 1)(x + 2) = x^2 - x - 2$

5  $x^4 = (x + 2)(7 - x)$

6  $x^4 = 4x^2 + 3x + 1$

7  $x^4 = (x - 2)(x - 4)(x + 5)$

8  $x^3 = x^2 + 4x$

9  $2x^3 = x^2 + 7x - 6$

10  $x^3 = 4x - 3x^2 + 12$

Examples  
21,22

Example  
23



Functions  
assignment



Functions  
problems

## 4.10 Piecewise functions

A **piecewise** or **hybrid** function has different rules for different parts of the domain.

### EXAMPLE 24

$$f(x) = \begin{cases} 3x + 2, & x < 4 \\ 4x - 2, & 4 \leq x < 6 \\ 2x + 3, & x > 6 \end{cases}$$

- a** Find the values at  $x = 3, 5$  and  $9$ .  
**b** Sketch the graph for  $0 \leq x \leq 10$ .

### Solution

- a**  $3 < 4$ , so use  $3x + 2$ .

$4 \leq 5 < 6$ , so use  $4x - 2$ .

$9 > 6$ , so use  $2x + 3$ .

- b**  $3x + 2$  is a straight line.

$4x - 2$  is a straight line.

$2x + 3$  is a straight line.

$$f(3) = 3 \times 3 + 2 = 11$$

$$f(5) = 4 \times 5 - 2 = 18$$

$$f(9) = 2 \times 9 + 2 = 20$$

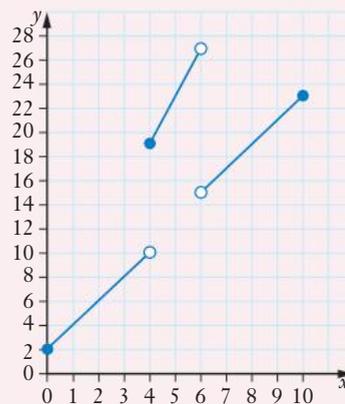
$$f(0) = 2, f(3) = 11 \text{ and } 3 \times 4 + 2 = 14$$

$$f(4) = 14, f(5) = 18 \text{ and } 4 \times 6 - 2 = 22$$

$$2 \times 6 + 3 = 15, f(7) = 17 \text{ and}$$

$$f(10) = 2 \times 10 + 2 = 22$$

Sketch the graph, using open circles for the interval ends that are not included.





### TI-84 Plus CE

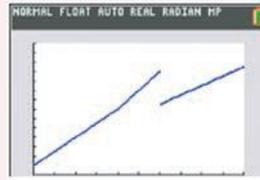
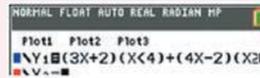
Use  $\boxed{y=}$  to enter the function in  $Y_1$  as  $(3X+2)(X<4)+$

$(4X-2)(X\geq 4)(X<6)+(2X+3)$

$(X>6)$ .

The inequality signs are in the test menu ( $\boxed{2nd}$  $\boxed{math}$ ).

Set the  $\boxed{window}$  to  $0 \leq x \leq 10$ ,  
 $0 \leq y \leq 28$  and  $\boxed{graph}$ .



### Casio fx-CG20AU

Choose **5 Graph**, enter the function parts as

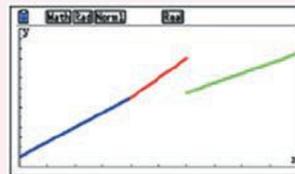
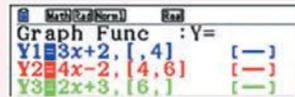
$Y1 = 3X+2, [4, ]$

$Y2 = 4X-2, [4, 6]$

$Y3 = [6, ]$

Square brackets are given by  $\boxed{SHIFT}$ +  
and  $\boxed{SHIFT}$   $\boxed{=}$ .

Set the **V-Window** to  $0 \leq x \leq 10$ ,  
 $0 \leq y \leq 28$  and draw the graph. You can  
change all the parts to the same colour  
using **FORMAT** ( $\boxed{SHIFT}$   $\boxed{5}$ ).



The function in Example 24 has a discontinuity at  $x = 6$ . Discontinuities are not common in real-life piecewise functions. They are often about money, like prices and taxation. Some are just step graphs, which have horizontal sections.

## INVESTIGATION

### PERSONAL TAX RATES

Go to the **Australian Taxation Office** website [www.ato.gov.au](http://www.ato.gov.au) for this investigation.

Taxation is worked out for a financial year. A financial year is from July 1 of a year to June 30 of the following year.

Find the taxation rates for this financial year.

Are they the same as the last financial year?

Draw a (step) graph of the personal tax rates for incomes from 0 – \$250 000. Make your vertical axis from 0 to 0.6.

Calculate the amount of income tax paid for the entire year for incomes from 0 to \$250 000. If it is different from last year's, show that on the graph as well.

Investigate personal tax rates in another country. Your teacher might assign different people to different countries.

### EXAMPLE 25

Screened top soil costs \$65/m<sup>3</sup> for 0–2 m<sup>3</sup>, \$54/m<sup>3</sup> for quantities from 3 m<sup>3</sup> to 9 m<sup>3</sup> and \$48/m<sup>3</sup> for 10 m<sup>3</sup> or more. Construct a function for the total prices of quantities up to 20 m<sup>3</sup> and sketch the graph.

#### Solution

Choose variable names.

Let the total price of  $x$  m<sup>3</sup> be  $P(x)$ .

Where does the cost change?

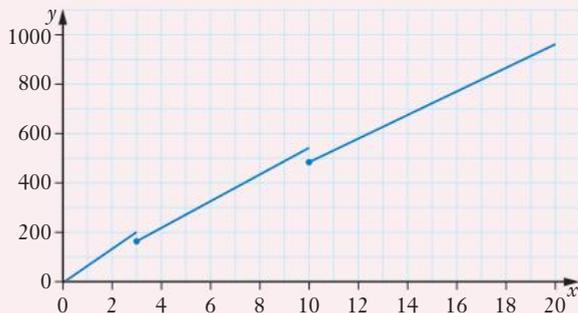
Cost changes at 1 m<sup>3</sup> and 10 m<sup>3</sup>.

Write the function using

$$P(x) = \begin{cases} 65x, & x < 3 \\ 54x, & 3 \leq x < 10 \\ 48x, & x \geq 10 \end{cases}$$

Price = cost  $\times$  quantity.

Draw the graph.



## Exercise 4.10 Piecewise functions

Example  
24

$$1 \quad f(x) = \begin{cases} 2x + 4, & x \leq 6 \\ 3x - 5, & x > 6 \end{cases}$$

- Find the values of  $f(5)$ ,  $f(6)$  and  $f(7)$ .
- Sketch the graph for  $0 \leq x \leq 10$ .

$$2 \quad f(x) = \begin{cases} x + 1, & x < 3 \\ 2x - 2, & 3 \leq x < 5 \\ 8 - x, & x \geq 5 \end{cases}$$

- Find the values of  $f(2)$ ,  $f(3)$ ,  $f(4)$ ,  $f(5)$  and  $f(7)$ .
- Sketch the graph for  $0 \leq x \leq 12$ .

$$3 \quad f(x) = \begin{cases} 3x^2 + 4, & x \leq 3 \\ 8x + 7, & x > 3 \end{cases}$$

- Find the values of  $f(1)$ ,  $f(3)$  and  $f(5)$ .
- Sketch the graph for  $0 \leq x \leq 10$ .

$$4 \quad f(x) = \begin{cases} 3 + \sqrt{4 - (x - 2)^2}, & 0 \leq x < 4 \\ 3, & 4 \leq x \leq 7 \\ 3 - \sqrt{4 - (x - 9)^2}, & 7 < x \leq 11 \end{cases}$$

- Find the values of  $f(2)$ ,  $f(6)$  and  $f(8)$ .
- Sketch the graph for  $0 \leq x \leq 11$ .

- 5 Royal Gala apples cost \$5.50/kg for up to 2 kg, \$4.50/kg for 2–5 kg and \$3.50/kg for 5 kg or more. Construct a function for the price for 0–12 kg and draw a graph.

Example  
25

### Problem solving

- 6 Transfer duty is a tax for house purchases. The rate for home purchases depends on the purchase price:
- For prices up to \$350 000, it's \$1 for each \$100 or part thereof
  - For prices over \$350 000 up to \$540 000, it's \$3500 plus \$3.50 for every \$100 or part thereof over \$350 000
  - For prices over \$540 000 up to \$1 000 000, it's \$10 150 plus \$4.50 for every \$100 or part thereof over \$540 000
  - For prices over \$1 000 000, it's \$30 850 plus \$5.75 for every \$100 or part thereof over \$1 000 000
- Construct a function for home transfer duty.
  - Calculate the duty for properties bought for \$300 000, \$600 000 and \$900 000.
  - Construct a graph for the home transfer duty for properties worth up to \$1 200 000.

- 7** The price of a taxi trip in Brisbane during the day (5 a.m. to 7 p.m.) depends on the distance of the trip and the time spent stationary (waiting, at traffic lights, at intersections, etc). You also pay a flag-fall cost of \$2.90 and a booking fee of \$1.50 if you book. The rate is \$2.17/km and \$0.81/minute for the time waiting in traffic. In light traffic, the average speed of a taxi is 45 km/h and about 5% of travel time is spent waiting. In heavy traffic, the average speed drops to 30 km/h and about 25% of the time is spent waiting.
- Show that  $C = 4.4 + 2.224d$  is the cost of a booked taxi trip of  $d$  km in light traffic.
  - Show that  $C = 4.4 + 2.575d$  is the cost of a booked taxi trip of  $d$  km in heavy traffic.
  - Draw a graph showing both functions on the same axes.
- 8** The terminal velocity of a falling person is approximately 54 m/s without a parachute and about 5–8 m/s with a parachute, depending on the parachute design. It takes a parachutist about 12 seconds to reach 54 m/s after jumping. It takes about 3.5 seconds to slow to 8 m/s after opening the parachute. A skydiver jumps from a plane and opens her parachute 35 seconds later. She reaches the ground 1 minute after jumping from the plane.
- Construct a function for her velocity  $t$  seconds after jumping, assuming that she accelerates and decelerates at constant rates.

It is more realistic to model her speed as  $56(1 - 0.75^t)$  after jumping from the plane and  $8 + 46 \times 0.25^t$  in the  $t$  seconds after opening the parachute.

- Construct a function for her velocity  $t$  seconds after jumping, assuming that she accelerates and decelerates in accordance with the above.
- Use both functions to find her speed 3, 5 and 10 seconds after jumping and comment on the difference between the functions.
- Find her speed 1, 2, 3 and 4 seconds after opening her parachute and comment on the difference between the functions.
- Graph both functions on the same axes and comment on their differences.



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# 4. CHAPTER SUMMARY

## Functions and graphs

### Relations and functions

- A **relation** is a set of **ordered pairs** like  $(3, -5)$ . The relation can be shown by a list of the ordered pairs, a rule, a mapping or part of the Cartesian plane.
- The first element of the ordered pairs is called the **independent variable** and the second is called the **dependent variable**.
- The **domain** is the set of first numbers in the relation.
- The **range** is the set of second numbers in the relation.
- The **codomain** is the set of numbers allowed as second numbers.
- A **1 : 1** relation pairs elements from the domain and range uniquely.
- The **equation of a circle** with radius  $r$  and **centre at the origin** is  $x^2 + y^2 = r^2$ .
- The **equation of a circle** with radius  $r$  and **centre at  $(a, b)$**  is  $(x - a)^2 + (y - b)^2 = r^2$ .
- A **function** is a  $1 : n$  relation. Every number in the domain of a function is in at most one ordered pair. Different ordered pairs must have different first numbers.
- Ordered pairs of a function  $f$  are often written as  $(x, f(x))$  or just  $f(x)$ .  $x$  represents the domain value and  $f(x)$  is the corresponding range value. If  $a = b$ , then  $f(a) = f(b)$ .
- A graph with any points in a **vertical line** is not the graph of a function.

### Quadratic functions

- The graph of a **quadratic function** is called a **parabola**. It has a **turning point** where the slope changes from negative to positive, or positive to negative for a negative coefficient of  $x^2$ .
- The graph has a vertical **axis of symmetry** that passes through the turning point.
- For a positive coefficient of the  $x^2$  term: the function decreases to a **minimum** value at the turning point, and then increases. As  $x \rightarrow \infty, f(x) \rightarrow \infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow -\infty$ .
- For a negative coefficient of the  $x^2$  term: the function increases to a **maximum** value at the turning point, and then decreases. As  $x \rightarrow \infty, f(x) \rightarrow -\infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow \infty$ .
- The turning points of  $y = \pm x^2$  are at the origin,  $(0, 0)$ .
- The **domain** is the set of real numbers  $\mathbf{R}$  and the **range** is  $y \geq \text{minimum}$  or  $y \leq \text{maximum}$ , depending on the coefficient of  $x^2$ .

- Compared to  $y = x^2$ , the graph of  $f(x) = a(x - b)^2 + c$ , is **dilated** by  $a$  and inverted if  $a$  is negative. The graph appears narrower if  $|a| > 1$  and wider if  $|a| < 1$ . All points are shifted  $b$  units to the right and  $c$  units higher, so the turning point is  $(b, c)$ .
- A **quadratic** function in the form  $y = a(x - b)(x - c)$  has **zeros** at  $x = b$  and  $x = c$ . The turning point is halfway between  $b$  and  $c$ . Compared to  $y = x^2$ , if  $|a| > 1$ , the graph is vertically stretched by the factor  $|a|$ ; if  $|a| < 1$ , it is vertically compressed.
- The turning point of  $f(x) = ax^2 + bx + c$  is at  $x = \frac{-b}{2a}$ .

### Straight lines and gradients

- The **equation of a straight line** is  $y = mx + c$ ,  $y - y_1 = m(x - x_1)$  or  $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ , where  $m$  is the **gradient** of the line,  $c$  is the **y-intercept**,  $(x_1, y_1)$  and  $(x_2, y_2)$  are points on the line.
- The gradient is given by  $m = \tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$ , where  $\theta$  is the **inclination** of the line (the angle made with a horizontal line).

### Quadratic relations

- A **quadratic relation** in the form  $x = a(y - b)(y - c)$  has **y-intercepts** at  $y = b$  and  $y = c$ . The graph is reversed if  $a$  is negative. The turning point (**vertex**) is halfway between  $b$  and  $c$ . There is a horizontal **axis of symmetry** that passes through the vertex.
- Compared to  $x = y^2$ , if  $|a| > 1$ , the graph is horizontally stretched by the factor  $|a|$ , and if  $|a| < 1$ , it is horizontally compressed by  $|a|$ .

### Hyperbolas (inverse proportions)

- The shape of the function  $y = \frac{1}{x}$  is called a (**rectangular**) **hyperbola**.
- It has a **discontinuity** at  $x = 0$ . The line  $x = 0$  is a **vertical asymptote**. As  $x \uparrow 0$ ,  $y \rightarrow -\infty$  and as  $x \downarrow 0$ ,  $y \rightarrow \infty$ .
- The line  $y = 0$  is a **horizontal asymptote**. As  $x \rightarrow \pm\infty$ ,  $y \rightarrow 0$  but never reaches 0.
- Compared to  $y = \frac{1}{x}$ :
  - the graph of  $y = -\frac{1}{x}$  is inverted
  - the graph of  $y = \frac{a}{x}$  is **vertically dilated** by the factor  $a$ ; stretched if  $|a| > 1$  and compressed if  $|a| < 1$
  - the graph of  $y = \frac{1}{x - b}$  is translated horizontally by  $b$ ; the vertical asymptote is moved to  $x = b$
  - the graph of  $y = \frac{1}{x} + c$  is translated vertically by  $c$ ; the horizontal asymptote is moved to  $y = c$

## Polynomial functions

- The graph of a power function of the form  $y = x^n$  for  $n > 1$  has a turning point at  $(0, 0)$ .
- For even powers of  $n$  the turning point is a **minimum** and the graph is **concave upwards**. This means that the inside of the curve is above the outside of the curve.
- For odd powers of  $n$ , the graph changes from **concave downwards** to **concave upwards** at the turning point. This is called a **point of inflection**.
- The graphs of even powers of  $n$  have an axis of symmetry through the turning point.
- Compared to  $y = x^n$ , the graph of  $f(x) = a(x - b)^n + c$ , is **dilated** by  $a$  and inverted if  $a$  is negative. The graph appears narrower if  $|a| > 1$  and wider if  $|a| < 1$ . All points are shifted  $b$  units to the right and  $c$  units higher, so the turning point is  $(b, c)$ .
- The **domain** is the set of real numbers  $\mathbf{R}$ .
- The **range** of odd power functions is  $\mathbf{R}$ .
- The **range** of even power functions is  $y \geq b$  for  $a > 0$  and  $y \leq b$  for  $a < 0$ .
- A **cubic** of the form  $y = k(x - a)(x - b)(x - c)$  has zeros at  $a, b$  and  $c$ . The function has a minimum point between two zeros and a maximum point between the others unless two zeros are the same. If  $a \neq b = c$  then either the minimum or maximum point will be at  $(b, 0)$ . For  $k < 0$ , as  $x \rightarrow -\infty, y \rightarrow \infty$  and as  $x \rightarrow \infty, y \rightarrow -\infty$ . For  $k > 0$ , as  $x \rightarrow -\infty, y \rightarrow -\infty$ ; as  $x \rightarrow \infty, y \rightarrow \infty$ .

## Transformations of functions

- Compared to the graph of  $y = f(x)$ , the graph of  $y = af(bx + c) + d$  is:
  - Reflected in the  $x$ -axis for  $a$  negative, vertically stretched for  $|a| > 1$ , vertically compressed for  $|a| < 1$
  - Reflected in the  $y$ -axis for  $b$  negative, horizontally compressed for  $|b| > 1$ , horizontally stretched for  $|b| < 1$
  - Moved horizontally by  $c$ , left for  $c > 0$ , right for  $c < 0$
  - Moved vertically by  $d$ , up for  $d > 0$ , down for  $d < 0$
- You can often use graphs to see the number of solutions of 2 **simultaneous equations**.

## Piecewise functions

- A **piecewise** or **hybrid** function has different rules for different parts of the domain.

# 4. CHAPTER REVIEW

## Functions and graphs

Example

1

- 1 a What are the domain and range of  $x \leq 2$ ?  
Sketch the graph.
- b What are the domain and range of  $y > 5$ ?  
Sketch the graph.

Example

2

- 2 a What is the shape of  $(x + 2)^2 + (y - 4)^2 < 9$ ?  
b What are the domain and range?  
c Sketch the graph.

Example

3

- 3 Find the centre and radius of:  
a  $x^2 + y^2 + 2x - 8y + 1 = 0$                       b  $x^2 + y^2 + 6x + 14y + 33 = 0$ .

Example

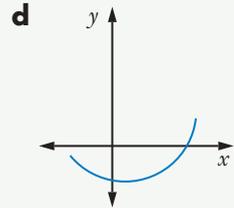
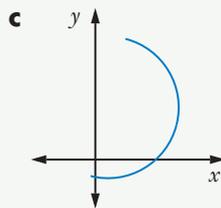
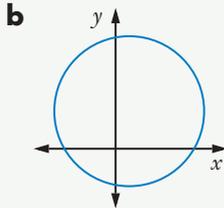
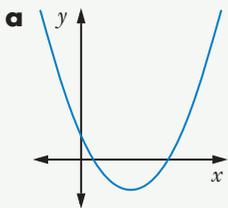
4

- 4 State whether or not each of the following is a function.
- |   |  |   |  |
|---|--|---|--|
| a | $\{(3, 5), (6, 7), (4, 1), (5, 7), (3, 1), (7, 7)\}$ | b | $\{(1, 6), (-1, 6), (3, 6), (5, 6), (2, 6)\}$  |
| c | $\{(7, 1), (7, 3), (7, 9), (7, 4), (7, 13)\}$        | d | $\{(1, 6), (2, 5), (4, 5), (3, 8), (-1, -5)\}$ |
| e | $y = 7x + 8$   | f | $x^2 + y^2 = 36$                               |
| g | $y = 16x^2$  | h | $x = 4y^2$                                     |
| i | $3x + 7y = 15$                                       | j | $y = 0$  |

Example

5

- 5 State whether or not each graph represents a function.



Examples

6, 7

- 6 Sketch the graph of each function.
- |   |                        |   |                        |   |                          |
|---|------------------------|---|------------------------|---|--------------------------|
| a | $f(x) = 4 - (x - 2)^2$ | b | $y = 0.4(x + 2)^2 - 3$ | c | $f(x) = 2(4 - x)(x + 1)$ |
|---|------------------------|---|------------------------|---|--------------------------|

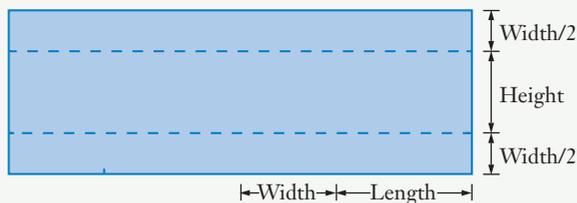
Example

8

- 7 Sketch the graph of each function and state its range.
- |   |  |
|---|--|
| a | $f(x) = 8 - x^2 - 2x, -6 < x < 3$        |
| b | $y = 3x^2 - 9x + 4$                      |
| c | $f(x) = 2x^2 - 4x + 5, -3 \leq x \leq 3$ |

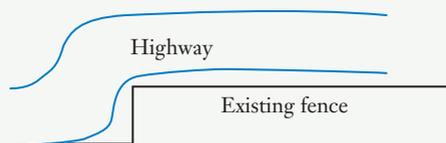


- 20** The top of an isolated hill is measured to be 240 m above the level of the surrounding plain. The hill is about 1500 m from one side to the other. It is decided to use a parabola to model the profile of the hill, which is symmetrical from one side to the other.
- Sketch the hill's shape, setting the zero of the coordinate system on its left side.
  - Find the height  $h$  as a function of the horizontal distance  $x$  across the hill.
  - Find correct to 2 decimal places the drop in height for a horizontal movement of 100 from the peak.
  - Verify and evaluate the model.
- 21** A carton with overlapping flaps is made from a rectangular piece of cardboard as shown.



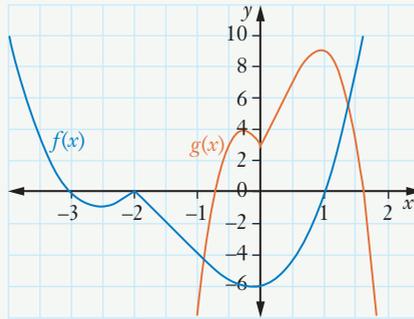
Find the dimensions of the carton that will give a maximum volume for a sheet of cardboard 150 cm by 100 cm. Verify and evaluate the model used.

- 22** A quadratic relation passes through the points  $(-3, -2)$  and  $(5, 2)$  and has an  $x$ -intercept at  $-1$ . Find the relation.
- 23** An old highway has a safety fence along the sides of a straight towards a 'dog-leg'. The fence has a right-angle that goes 6 m to one side and then turns back in the original direction. An enclosure is to be built at the corner using another 200 m of fencing.



- What dimensions should it be to get the largest area?
  - What is the largest area than can be enclosed?
- 24** The graph of a hyperbola has asymptotes of  $x = -1$  and  $y = 3$  and a  $y$ -intercept of 5. Write a possible equation for the function.

- 25** The graphs of  $f(x)$  and  $g(x)$  are shown on the same axes. Write  $g(x)$  as a transformation of  $f(x)$ .



- 26** The transfer duty rates for investment property in Queensland are as follows:

- Up to \$5000: Nil
- Over \$5000, up to \$75 000: \$1.50 for every \$100 or part thereof over \$5000
- Over \$75 000, up to \$540 000: \$1050 plus \$3.50 for every \$100 or part thereof over \$75 000
- Over \$540 000, up to \$1 000 000: \$17 325 plus \$4.50 for every \$100 or part thereof over \$540 000
- Over \$1 000 000: \$38 025 plus \$5.75 for every \$100 or part thereof over \$1 000 000

Construct a model for duty on investment property in Queensland and sketch the graph for prices up to \$1 200 000. Verify and evaluate the model.



Practice quiz

# 5.

## COMBINATIONS AND CONDITIONAL PROBABILITY

How many ways can you choose a committee of 5 people from a class?

Which is better: 2 out of 4, or 5 out of 10 choices?

How do doctors work out health risks? How do they know which drugs really work?

5.01 Combinations and Pascal's triangle

5.02 Using combinations in probability

5.03 The binomial expansion

5.04 Conditional probability

5.05 Dependent and independent events

5.06 Applications to data

Chapter summary

Chapter review

CHANCE

## SYLLABUS SUBJECT MATTER

### Conditional probability and independence

- understand the notion of a conditional probability and recognise and use language that indicates conditionality
- use the notation  $P(A|B)$  and the formula  $P(A \cap B) = P(A|B)P(B)$  to solve problems
- understand the notion of independence of an event  $A$  from an event  $B$ , as defined by  $P(A|B) = P(A)$
- establish and use the formula  $P(A \cap B) = P(A)P(B)$  for independent events  $A$  and  $B$
- use relative frequencies obtained from data as point estimates of conditional probabilities and as indications of possible independence of events

### Binomial expansion

- understand the notion of a combination as an unordered set of  $r$  objects taken from a set of  $n$  distinct objects
- recognise and use the link between Pascal's triangle and the notation  $\binom{n}{r}$
- expand  $(x + y)^n$  for small positive integers

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Prior learning

# TERMINOLOGY

and  
conditional probability  
elementary outcome  
false negative  
intersection  
probability  
sample space  
union

binomial expansion  
contingency table  
event  
false positive  
or  
random  
true negative

combination  
dependent  
factorial  
independent  
Pascal's triangle  
replacement  
true positive



Combinations

## 5.01 Combinations and Pascal's triangle

If you toss **one coin**, it can land tails or heads, so there is 1 way to have 0 heads and 1 way to have 1 head. You can write this as **1, 1** for respectively, 0 heads or 1 head.



Combinations  
calculations

If you have **two coins** you can have both tails, either of the 2 coins heads, or both heads.

You can write **1, 2, 1** for respectively, 0 heads, 1 head or 2 heads from 2 coins.

If you have **three coins**, you can have all tails, any of 3 coins for 1 head, any of 3 coins for a tail and two heads, or all 3 heads.

You can write **1, 3, 3, 1** for respectively, 0, 1, 2 or 3 heads from 3 coins.

What about if you have **four coins**?

No heads or 4 heads is easy; there is only 1 way for each of these.

1 head or 3 heads is easy; any of the 4 can be heads or any of the 4 can be tails.

For 2 heads, you can add the new coin to any of the other 3, and you already have 3 ways, so there are 6 ways.

You can write **1, 4, 6, 4, 1** for respectively, 0, 1, 2, 3 or 4 heads from 4 coins.

For **five coins**, 0 heads and 5 heads is easy; 1 way each.

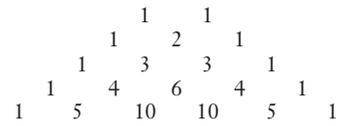
1 head or 4 heads is easy; 5 ways each for any one head or any one as a tail.

For 2 heads, you can add the extra coin to any of the 4 ways of getting 1 head, and you already have 6 ways, so there are 10 ways.

For 3 heads, you can add the extra coin to any of the 6 ways of getting 2 heads, and you already have 4 ways, so there are 10 ways.

You can write **1, 5, 10, 10, 5, 1** for respectively, 0, 1, 2, 3, 4 or 5 heads from 5 coins.

The numbers make a pattern called **Pascal's triangle**. It is usual to put an extra 1 at the top and say this is how many ways you can get no heads from no coins.



You can work out how many ways you can get 3 heads from 6 coins like this:

You can add the extra coin to any of the 10 ways of getting 2 coins from 5 coins, and there are already 10 ways from those 5 coins, so there are 20 ways altogether.

This always works, so each entry on a new row is the sum of the two entries diagonally above it in the previous row.

You should also be able to see that choosing 3 heads from 6 coins is the same as choosing 3 objects from a collection of 6 objects.

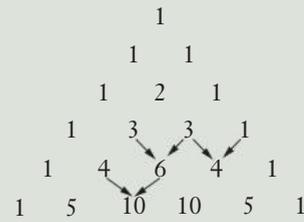
A choice of  $r$  objects from  $n$  objects is called a **combination**,

and the number of combinations is shown as  ${}^n C_r$  or  $\binom{n}{r}$ .

Combinations have no order.

The numbers of combinations are shown by **Pascal's triangle**:

for each new row, you put a 1 at each end and diagonally add the numbers of the previous row.



### EXAMPLE 1

Write the 6th and 7th rows of Pascal's triangle.

#### Solution

Write the 5th row.

1 5 10 10 5 1

Put 1s at the ends for the new row.

Add diagonally from above.

1 6 15 20 15 6 1

Repeat for the 7th row.

1 7 21 35 35 21 7 1

You can use combinations instead of tree diagrams in many situations.

## EXAMPLE 2

Five friends are going to the beach in a car and they can all drive. How many different pairs can travel in the front seats as driver and front passenger?



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### Solution

You need the combinations of 2 from 5.

$${}^5C_2 = 10$$

Choose the number from Pascal's triangle.

Write the answer.

There are 10 different pairs of friends who can travel in the front.

Pascal's triangle is good, but if you need to find  $\binom{11}{3}$  it is tedious working out the triangle before you can get the answer. There's a formula!



Factorial notation

The number of combinations of  $r$  objects from  $n$  objects is given by

$${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1 \times 2 \times 3 \times \cdots \times r} = \frac{n!}{r!(n-r)!}$$

where the **factorial**,  $n!$ , is defined by  $n! = 1 \times 2 \times 3 \times \cdots \times n$  for  $n > 1$  and  $0! = 1! = 1$ .

For row 1, the formula gives  ${}^1C_0$  and  ${}^1C_1$ .

$${}^1C_0 = \frac{1!}{0! \times 1!} = \frac{1}{1 \times 1} = 1 \text{ and } {}^1C_1 = \frac{1!}{1! \times 0!} = \frac{1}{1 \times 1} = 1, \text{ so the formula is correct for row 1.}$$

On a row, you add numbers diagonally to get the middle numbers. From row 6 to row 7, you put the 1s at the end and do  $1 + 6 = 7$ ,  $6 + 15 = 21$ ,  $15 + 20 = 35$ ,  $20 + 15 = 35$ ,  $15 + 6 = 7$ .

That is,  ${}^6C_0 + {}^6C_1 = {}^7C_1$ ,  ${}^6C_1 + {}^6C_2 = {}^7C_2$ ,  ${}^6C_2 + {}^6C_3 = {}^7C_3$ , and so on.

You can write the rule to work out Pascal's triangle as  ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ .

To see whether the formula always works, you have to check that this always works.

You need to use the fact that  $r! = r \times (r-1) \times (r-2) \times \dots \times 2 \times 1 = r \times (r-1)!$ , etc.

$$\begin{aligned}
 {}^n C_r + {}^n C_{r+1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} \\
 &= \frac{n!}{r![(n-r) \times (n-r-1)!]} + \frac{n!}{[(r+1) \times r!](n-r-1)!} \\
 &= \frac{n!}{r!(n-r-1)! \times (n-r)} + \frac{n!}{r!(n-r-1)! \times (r+1)} \\
 &= \frac{n!(r+1) + n!(n-r)}{r!(n-r-1)! \times (n-r) \times (r+1)} \\
 &= \frac{n![(r+1) + (n-r)]}{(r+1) \times r! \times (n-r) \times (n-r-1)!} \\
 &= \frac{n!(1+n)}{[(r+1) \times r! \times (n-r) \times (n-r-1)!]} \\
 &= \frac{(n+1)n!}{(r+1)!(n-r)!} \\
 &= \frac{(n+1)!}{(r+1)![n+1-(r+1)]!} \\
 &= {}^{n+1} C_{r+1}
 \end{aligned}$$

It is true, so the formula always gives the Pascal's triangle numbers.

### EXAMPLE 3

Find how many combinations of 3 objects you can get from 11 objects.

#### Solution

You need the combinations of 3 from 11.

$${}^{11} C_3$$

Use the formula.

$$\begin{aligned}
 &= \frac{11!}{3! \times 8!} \\
 &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}
 \end{aligned}$$

Cancel and calculate.

$$\begin{aligned}
 &= \frac{11 \times 10 \times 9}{3 \times 2 \times 1} \\
 &= 165
 \end{aligned}$$

Write the answer.

There are 165 combinations of 3 objects from 11 objects.

Notice that for  ${}^{11}C_3$  you multiply 3 numbers from 11 down and divide by 3!

This will always work, so  ${}^{18}C_4 = \frac{18 \times 17 \times 16 \times 15}{4 \times 3 \times 2 \times 1} = 3060$ . This is much easier than writing out

Pascal's triangle up to line 18.

You should also have noticed that Pascal's triangle (and combinations) are symmetrical about the centre so it is always true that  ${}^nC_r = {}^nC_{n-r}$ .

## TECHNOLOGY

### Combination calculations

You can use your graphics calculator to work out the combinations above.

#### TI-84 Plus CE

Press **[math]** and use the cursor to go across to **PROB**.

Press 3, put in 11, use the cursor to go to the second space, put in 3 and press **[enter]**.

Now do  ${}^{18}C_4$ .

Expression	Result
${}^{11}C_3$	165
${}^{18}C_4$	3060

#### Casio fx-CG20AU

Choose **1 Run Matrix** and press **[OPTN]**.

Press **[F6] [F3]** to get **PROB** and put in 11.

Press **[F3]** to get **C (nCr)**, put in 3 and **[EXE]**.

Now do  ${}^{18}C_4$ .

${}^{11}C_3$	165
${}^{18}C_4$	3060

## Exercise 5.01 Combinations and Pascal's triangle

Example  
1

1 Construct Pascal's triangle up to row 12 and keep it for reference.

2 Use Pascal's triangle to find each value.

<b>a</b> ${}^7C_3$	<b>b</b> ${}^8C_5$	<b>c</b> ${}^9C_4$	<b>d</b> ${}^6C_2$	<b>e</b> ${}^{10}C_6$
<b>f</b> $\begin{pmatrix} 9 \\ 6 \end{pmatrix}$	<b>g</b> $\begin{pmatrix} 8 \\ 7 \end{pmatrix}$	<b>h</b> $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$	<b>i</b> $\begin{pmatrix} 10 \\ 3 \end{pmatrix}$	<b>j</b> $\begin{pmatrix} 9 \\ 2 \end{pmatrix}$

3 Use the formula to find each value.

<b>a</b> ${}^{13}C_5$	<b>b</b> ${}^{16}C_{12}$	<b>c</b> ${}^{21}C_3$	<b>d</b> ${}^{15}C_6$	<b>e</b> ${}^{14}C_7$
<b>f</b> $\begin{pmatrix} 18 \\ 5 \end{pmatrix}$	<b>g</b> $\begin{pmatrix} 22 \\ 4 \end{pmatrix}$	<b>h</b> $\begin{pmatrix} 13 \\ 6 \end{pmatrix}$	<b>i</b> $\begin{pmatrix} 20 \\ 7 \end{pmatrix}$	<b>j</b> $\begin{pmatrix} 30 \\ 4 \end{pmatrix}$

Example  
3

- 4** How many ways can you choose 8 things from 12?
- 5** Find the number of ways to select groups of 12 objects from 20 objects.

### Problem solving

- 6** Find the value of  $n$ , given each equation.
- a**  ${}^n C_2 = 6$                       **b**  ${}^{n+1} C_3 = {}^n C_2$                       **c**  ${}^n C_2 = 45$   
**d**  ${}^{n+1} C_7 = {}^n C_6$                       **e**  ${}^n C_2 = 66$                       **f**  ${}^n C_5 = {}^n C_{11}$
- 7** A normal hand of the card game, Gin Rummy, has 7 cards from a pack of 52 cards. How many different gin rummy hands are possible?
- 8** How many triangles can be drawn using five given points on the circumference of a circle?
- 9** A simple lotto game uses the numbers 1 to 40, with 6 numbers being chosen. How many different combinations are possible?
- 10** A car has a radio with six push buttons that can be tuned to different stations. In a city with 12 radio stations, how many different selections of those stations can be chosen for the push buttons?
- 11** The menu at a small Chinese restaurant has 15 dishes. You generally order the same number of dishes as there are people in your party. How many different combinations may be chosen by a party of 6 people?
- 12** A student council of 5 is to be selected from 12 nominations. Seven girls and five boys have been nominated. The principal of the school has stated that for balance there must be either 2 or 3 girls on the council. How many possible councils are there?
- 13** A normal hand in the game of poker has 5 cards. Some people like to play with the joker as a 'wild' card. How many extra possible hands are there with the joker added?
- 14** A pet shop owner wishes to display 5 animals in the shop window. There are 6 cats and 3 dogs available for display. How many combinations of 5 display animals can be chosen so as to include:
- a** exactly one dog?                      **b** at least one dog?
- 15** A company board has 10 directors. Six of the directors are in favour of a merger proposal while 4 are opposed to it. A subcommittee of 3 directors is to be chosen to investigate the merger proposal. How many subcommittees can be selected so that exactly 2 directors are in favour of the proposal?

## 5.02 Using combinations in probability

Many probability problems are most easily solved using combinations.

### EXAMPLE 4

A committee of 5 people is to be chosen at random from a class of 20. What is the probability that Esrif and Johanna are both on the committee?

#### Solution

State the number of committees.

$$\text{Number of committees} = {}^{20}C_5$$

Write the expression.

$$= \frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5}$$

State the committees with E and J.

Committees with Esrif and Joanna have 3 others.

State the number with E and J.

$$\text{Number with Esrif and Joanna} \\ = {}^{18}C_3$$

Write the expression.

$$= \frac{18 \times 17 \times 16}{1 \times 2 \times 3}$$

State the probability.

$$P(\text{Committee with Esrif and Joanna}) \\ = \frac{{}^{18}C_3}{{}^{20}C_5}$$

Change to multiplication.

$$= {}^{18}C_3 \times \frac{1}{{}^{20}C_5}$$

Write the expression.

$$= \frac{18 \times 17 \times 16}{1 \times 2 \times 3} \times \frac{1 \times 2 \times 3 \times 4 \times 5}{20 \times 19 \times 18 \times 17 \times 16}$$

Nearly everything cancels.

$$= \frac{1}{19}$$

Write the answer.

The probability that Esrif and Joanna are both on the committee is  $\frac{1}{19}$ .

You should check any probability problem involving a selection of some of a group *without replacement* to see if combinations are useful.

## EXAMPLE 5

Four marbles are taken at random from a bag containing 5 red, 3 green and 4 yellow marbles. What is the probability that 2 are red and 2 are yellow?

### Solution

Write the number of possibilities.

$$n(4 \text{ marbles}) = {}^{12}C_4$$

Write the number with 2 red and 2 yellow.

$$n(2R \text{ and } 2Y) = {}^5C_2 \times {}^4C_2$$

Write the probability expression.

$$P(2R \text{ and } 2Y) = \frac{{}^5C_2 \times {}^4C_2}{{}^{12}C_4}$$

Change to multiplication.

$$= {}^5C_2 \times {}^4C_2 \times \frac{1}{{}^{12}C_4}$$

Write the expression.

$$= \frac{5 \times 4}{2 \times 1} \times \frac{4 \times 3}{2 \times 1} \times \frac{4 \times 3 \times 2 \times 1}{12 \times 11 \times 10 \times 9}$$

Cancel and simplify.

$$= \frac{4}{33}$$

Write the answer.

The probability of getting 2 red and 2 yellow marbles is  $\frac{4}{33}$ .

## Exercise 5.02 Using combinations in probability

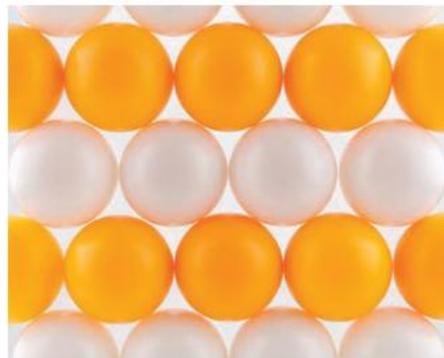
- 1 What is the probability that a committee of 5 chosen from 25 people has both David and Lee on it?
- 2 Find the probability of getting 4 red pens when you take 4 pens at random from a case containing 6 red, 6 blue and 6 black pens.

Example  
4

Example  
5

### Problem solving

- 3 A bag has 6 white and 6 yellow table tennis balls in it. What is the probability that there is exactly 1 yellow ball in the first 5 balls taken out at random?
- 4 Four faulty stopwatches have been put back into the box of stopwatches by mistake. The other 8 stopwatches in the box are OK. A group of students borrow 5 stopwatches to do some timing after school the week before the sports carnival. What is the probability that 2 of the stopwatches are faulty?



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- 5** A short test has 10 questions. What is the probability (correct to 3 decimal places) of guessing exactly half of the answers by chance if:
- a** the answers are true/false?
  - b** the questions are multiple-choice with 5 possible answers: A, B, C, D, E?
- 6** In the card game Poker, a full house consists of 3 of a kind and a pair, like KKK88. What is the probability of being dealt a full house from a normal pack of well-shuffled cards?

## 5.03 The binomial expansion

You can already do these:

$$(x + y)^0 = 1$$

$$(x + y)^1 = 1x + 1y$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

You can multiply by  $(x + y)$  to go further:

$$\begin{aligned}(x + y)^3 &= (1x^2 + 2xy + 1y^2)(x + y) \\ &= 1x^2x + 1x^2y + 2xyx + 2xyy + 1y^2x + 1y^2y \\ &= 1x^3 + 3x^2y + 3xy^2 + 1y^3\end{aligned}$$

$$\begin{aligned}(x + y)^4 &= (1x^3 + 3x^2y + 3xy^2 + 1y^3)(x + y) \\ &= 1x^3x + 1x^3y + 3x^2yx + 3x^2yx + 3xy^2x + 3xy^2y + 1y^3x + 1y^3y \\ &= 1x^4 + 4x^3y + 6x^2y^2 + 6xy^3 + 1y^4\end{aligned}$$

Each expansion starts and finishes with a coefficient of 1.

When you multiply by  $(x + y)$  to get the next power, the coefficients add in exactly the same way as Pascal's triangle. The coefficients of  $(x + y)^n$  are the  $n$ th row of Pascal's triangle.

You can work out the coefficients of  $(x + y)^4$  in a different way.

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y)$$

The  $x^4$  term has no  $y$ s, so you use 0  $y$ s from 4 brackets:  ${}^4C_0$

The  $x^3y$  term has 1  $y$ , so you use 1  $y$  from 4 brackets:  ${}^4C_1$

The  $x^2y^2$  term has 2  $y$ s, so you use 2  $y$ s from 4 brackets:  ${}^4C_2$

The  $xy^3$  term has 3  $y$ s, so you use 3  $y$ s from 4 brackets:  ${}^4C_3$

The  $y^4$  term has 4  $y$ s, so you use 4  $y$ s from 4 brackets:  ${}^4C_4$

You should be able to see the connection between the coefficients and combinations.

## The binomial expansion

$$(x + y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1}y + {}^n C_2 x^{n-2}y^2 + {}^n C_3 x^{n-3}y^3 + \dots + {}^n C_1 xy^{n-1} + {}^n C_0 y^n.$$

### EXAMPLE 6

Expand

**a**  $(x + y)^{12}$

**b**  $(x - y)^8$

**c**  $(2a - 3b)^5$

### Solution

**a** Find the coefficients.

$$\begin{aligned} {}^{12}C_0 &= 1, {}^{12}C_1 = 12, {}^{12}C_2 = 66, {}^{12}C_3 = 220, \\ {}^{12}C_4 &= 495, {}^{12}C_5 = 792, {}^{12}C_6 = 924, {}^{12}C_7 = 792, \\ {}^{12}C_8 &= 495, {}^{12}C_9 = 220, {}^{12}C_{10} = 66, {}^{12}C_{11} = 12, \\ {}^{12}C_{12} &= 1 \end{aligned}$$

Write the expansion with descending powers of  $x$ .

$$\begin{aligned} (x + y)^{12} &= x^{12} + 12x^{11}y + 66x^{10}y^2 + 220x^9y^3 + 495x^8y^4 \\ &\quad + 792x^7y^5 + 924x^6y^6 + 792x^5y^7 + 495x^4y^8 \\ &\quad + 220x^3y^9 + 66x^2y^{10} + 12xy^{11} + y^{12} \end{aligned}$$

**b** Write as  $(x + y)^8$ .

$$(x - y)^8 = (x + (-y))^8$$

Find the coefficients.

$$\begin{aligned} {}^8C_0 &= 1, {}^8C_1 = 8, {}^8C_2 = 28, {}^8C_3 = 56, {}^8C_4 = 70, \\ {}^8C_5 &= 56, {}^8C_6 = 28, {}^8C_7 = 8, {}^8C_8 = 1 \end{aligned}$$

Write the expansion with descending powers of  $x$ .

$$\begin{aligned} (x - y)^8 &= x^8 + 8x^7(-y) + 28x^6(-y)^2 + 56x^5(-y)^3 + 70x^4(-y)^4 \\ &\quad + 56x^3(-y)^5 + 28x^2(-y)^6 + 8x(-y)^7 + (-y)^8 \end{aligned}$$

Simplify.

$$\begin{aligned} &= x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 \\ &\quad - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8 \end{aligned}$$

**c** Write as  $(x + y)^8$ .

$$(2a - 3b)^5 = [(2a) + (-3b)]^5$$

Find the coefficients.

$$1, 5, 10, 10, 5, 1$$

Write the expansion with descending powers of  $x$ .

$$\begin{aligned} (2a - 3b)^5 &= (2a)^5 + 5(2a)^4(-3b) + 10(2a)^3(-3b)^2 \\ &\quad + 10(2a)^2(-3b)^3 + 5(2a)(-3b)^4 + (-3b)^5 \end{aligned}$$

Simplify the powers.

$$\begin{aligned} &= 32a^5 + 5 \times 16a^4 \times (-3b) + 10 \times 8a^3 \times 9b^2 \\ &\quad + 10 \times 4a^2 \times (-27b)^3 + 5 \times 2a \times 81b^4 + (-243b)^5 \end{aligned}$$

Simplify the terms.

$$\begin{aligned} &= 32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 \\ &\quad + 810ab^4 - 243b^5 \end{aligned}$$



The binomial expansion

Notice that the negative terms alternate in parts **b** and **c** above. You might prefer to use this instead of showing the negatives in the powers.

It may be easier in many cases to simplify some of the coefficients to get

$$(x + y)^n = x^n + nx^{n-1}y + \frac{1}{2}n(n-1)x^{n-2}y^2 + \dots$$

This is particularly useful if you only want a few terms.

### EXAMPLE 7

For the expansion of  $(x + y)^{17}$ , find:

- a** the first 3 terms      **b** the term that contains  $x^{10}$

In a technology-active exam use your calculator to work out the values of  ${}^nC_r$ .

#### Solution

- a** Write the first 3 terms using  ${}^nC_r$ .       $(x + y)^{17} = {}^{17}C_0x^{17} + {}^{17}C_1x^{16}y + {}^{17}C_2x^{15}y^2 + \dots$   
Simplify.       $= x^{17} + 17x^{16}y + 136x^{15}y^2 + \dots$
- b** The term with  $x^{10}$  has  $y^7$ .       ${}^{17}C_7x^{10}y^7$   
Work out the coefficient.       $= 19\,448x^{10}y^7$

### Exercise 5.03 The binomial expansion

Example  
**6**

**1** Expand each expression.

- a**  $(t - v)^8$       **b**  $(n - t)^{12}$       **c**  $(a + q)^{15}$       **d**  $(r - u)^{14}$       **e**  $(q + g)^{11}$

**2** Expand:

- a**  $(n + 4c)^5$       **b**  $(y - 5a)^6$       **c**  $(p - 3q)^7$       **d**  $(m + 2k)^8$       **e**  $(k - 3e)^9$

**3** Expand:

- a**  $(3v + 2n)^4$       **b**  $(3n - 4m)^4$       **c**  $(3m - 4g)^5$       **d**  $(2m + 3c)^7$       **e**  $(3x - 2y)^7$

Example  
**7**

**4** Find the first 3 terms of each of the following.

- a**  $(q - x)^9$       **b**  $(n + y)^{20}$       **c**  $(2g - v)^{12}$       **d**  $(e - 3m)^{20}$       **e**  $(4m + 5u)^7$

**5** Find the term containing  $x^{12}$  in  $(3x - 2y)^{16}$ .

**6** Find the term containing  $m^8$  in the expansion of  $(2m + 5n)^{12}$ .

## 5.04 Conditional probability

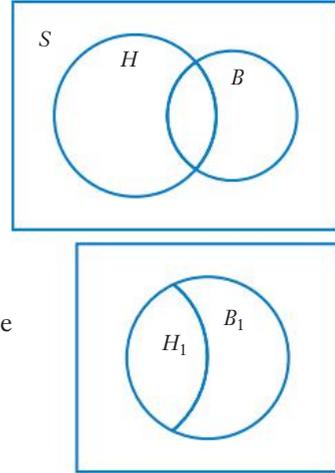
A medical researcher may be interested in the probability of developing a disease under particular circumstances, for example, the probability of developing heart disease among people who have a BMI over 30. You can show the events as a Venn diagram as shown.

The researcher is only interested in the people who have a BMI over 30, so they are not interested in the whole sample space (all people).

The restricted event is  $H_1$  and it is in the restricted sample space  $S_1$  as on the right here.

The probability is given by  $P(H_1) = \frac{n(H_1)}{n(S_1)}$ .

$$\begin{aligned} \text{But } H_1 &= H \cap B \text{ and } S_1 = B, \text{ so } P(H_1) = \frac{n(H \cap B)}{n(B)} \\ &= \frac{n(H \cap B)}{n(B)} \times \frac{n(S)}{n(S)} \\ &= \frac{n(H \cap B)}{n(S)} \times \frac{n(S)}{n(B)} \\ &= P(H \cap B) \times \frac{1}{P(B)} \\ &= \frac{P(H \cap B)}{P(B)} \end{aligned}$$



This means you can find the probability of the restricted event  $H_1$  by dividing the probability of the intersection by the probability of having a BMI over 30.

It is easy to confuse  $P(H_1)$  with  $P(H)$ , the probability of getting heart disease in the general population, so it is given a special name.

The conditional probability of  $A$  given  $B$ ,  $P(A|B)$ , is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This is often written in the form  $P(A \cap B) = P(A|B) \times P(B)$ .

The definition means that the sample space is restricted to the given event. The definition of conditional probability makes many probability calculations easier. It doesn't mean the given event has to occur first, but it often does.

### EXAMPLE 8

Two marbles are drawn from a bag containing 5 yellow and 4 red marbles without replacement. What is the probability that they are both red?

#### Solution

Find the probability that the first marble is red.

$$P(\text{1st red}) = \frac{4}{9}$$

Find the probability that the second marble is *also* red. There are 3 reds left out of 8 marbles.

$$P(\text{2nd red} | \text{1st red}) = \frac{3}{8}$$

Use conditional probability.

$$P(\text{2nd red} \cap \text{1st red}) = P(\text{2nd red} | \text{1st red}) \times P(\text{1st red})$$

Substitute values.

$$= \frac{3}{8} \times \frac{4}{9} \\ = \frac{1}{6}$$

Write the answer.

The probability that both marbles are red is  $\frac{1}{6}$ .

You will generally find that using conditional probability is easier than using other methods such as tree diagrams. Conditional probability is particularly important in medical research. You cannot ethically ask people to smoke for a long period to find out if smoking is a major factor in causing lung cancer. However, you can use statistics to estimate the probabilities of dying from lung cancer for smokers and non-smokers.

### EXAMPLE 9

In 2014, 78 341 men died in Australia. 4941 died of lung cancer and of these, 4353 were smokers. According to ABS data, 11.1% of men of the average age of those that died from lung cancer were smokers when they died. Compare the probabilities of dying of lung cancer for male smokers and non-smokers.

#### Solution

Choose abbreviations.

Let  $S$  = male smokers and  $L$  = male lung cancer death

What probabilities are needed?

$P(L|S)$  = Probability of a smoker dying of lung cancer

$P(L|\bar{S})$  = Probability of a non-smoker dying of lung cancer

Find the probabilities needed for the calculations.

$$P(L \cap S) = \frac{4353}{78\,341}$$

$$P(L \cap \bar{S}) = \frac{4941 - 4353}{78\,341} = \frac{588}{78\,341}$$

$$P(S) = 0.111$$

$$P(\bar{S}) = 0.889$$

Find the probabilities.

$$P(L|S) = \frac{P(L \cap S)}{P(S)} = \frac{4353}{78\,341} \div 0.111 = 0.5004 \dots$$

$$P(L|\bar{S}) = \frac{P(L \cap \bar{S})}{P(\bar{S})} = \frac{588}{78\,341} \div 0.889 = 0.008884$$

Write the answer.

The probability of male smokers dying from lung cancer is about 50.0% and for non-smokers about 0.8%.

Source: Australian Bureau of Statistics, 2014, Causes of Death, Australia, cat. no. 3303.0, viewed 01 Jan 2017, <<http://www.abs.gov.au/AUSSTATS/abs@.nsf/allprimarymainfeatures/E639C737904FE78FCA25803B0017FF1A?opendocument>>

From the example, the probability of a male smoker dying of lung cancer is about 60 times that of a non-smoker.

## INVESTIGATION

### THE CHALLENGER SPACE SHUTTLE DISASTER

The two solid-fuel booster rockets for the *Challenger* space shuttle each had three stages joined together. The gaps between the stages were sealed with O-rings to prevent the escape of fuel and consequent explosion of the rockets. In order to increase the reliability of the seal, a second O-ring was put in place so that, if the first failed, the second would prevent the escape of fuel. NASA assumed that failures of the O-rings would be independent. However, not long before the *Challenger* disaster in 1986 it was discovered that, if the first O-ring failed, the probability of failure of the second was quite high. In other words, failure of the back-up O-ring was not independent of failure of the first O-ring.

- Investigate the probabilities involved in the *Challenger* disaster.

The break-up of the *Columbia* space shuttle in 2003 on re-entry had a different cause.

- Find out how many space shuttle flights there were, and compare the risks of travel on a space shuttle with the risks of travel by car. You will need to consider whether to evaluate the risks per trip or per kilometre.



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## Exercise 5.04 Conditional probability

Example

8

- 1 A bag contains five yellow beads and seven red beads. Two beads are taken without replacement. What is the probability of drawing:
  - a two yellow beads?
  - b two red beads?
  - c one of each colour?
  - d If three beads are taken without replacement, what is the probability that they are all yellow?
- 2 A coin is weighted so that  $P(H) = \frac{2}{3}$ . If 'heads' appears, a number is selected at random from the numbers 1 through 9. If 'tails' appears, a number is selected at random from the numbers 1 through 5. Find the probability that an even number is selected.
- 3 A pair of normal dice are thrown. Find the probability that the sum is 10 or greater if:
  - a a 5 appears on the first die
  - b a 5 appears on at least one of the dice.
- 4 A study of the causes of death of women in Australia found that, of 54 000 deaths, 9400 were from heart attacks. Of those women who died from heart attacks, 3020 were classed as obese. A total of 5840 of the 54 000 women studied were considered to be obese. Compare the probabilities of obese and non-obese women dying of heart attacks.

Example

9

### Problem solving

- 5 Part of a comprehension test has five multiple-choice questions with four choices for each. What is the probability of guessing
  - a all answers correctly?
  - b four answers correctly?
  - c three answers correctly?
  - d no answers correctly?
- 6 If one engine fails, a twin-engine aeroplane can continue to fly on only the second engine. Unfortunately, when it is flying on only the second engine, the increased load increases the probability that this engine also will fail, so pilots will normally land as soon as possible if an engine fails. The probability of a single engine failure on a flight from Cairns to Townsville is 0.1%. The probability that both engines will fail is only 0.000 25%. What is the percentage probability of the second engine failing if the first has already failed?
- 7 At a school, 4% of the boys and 1% of the girls are taller than 180 cm (6 feet in the old measure). Given that 60% of all students at the school are female, find the probability that, if a randomly selected student is taller than 180 cm, the student is female.
- 8 The probability of a motor vehicle accident being correctly attributed to driver inattention (texting, talking, etc.) is 0.6. The probability of a motor vehicle accident being incorrectly attributed to driver inattention is 0.2. About 30% of all motor vehicle accidents are actually due to driver inattention. What is the probability that an accident attributed to driver inattention is correctly attributed?

## 5.05 Dependent and independent events

Actions that are independent do not affect each other. For example, the result of tossing a fair coin is independent of what has happened before. The coin does not know what side it previously came down on, so the probability of heads is always  $\frac{1}{2}$ .

Mathematically, for  $A$  = 'previous toss is heads' and  $B$  = 'heads on this toss',  $P(B) = P(B|A)$ .

This is usually used as a definition of independent events. Unfortunately, if  $P(A) = 0$  then  $P(B|A)$  is undefined because you cannot divide by zero. Some mathematicians prefer to use  $P(A \cap B) = P(A) \times P(B)$  instead.

Two events  $A$  and  $B$  are **independent** if and only if  $P(B) = P(B|A)$  or  $P(A \cap B) = P(A) \times P(B)$ .

Events that are not independent are called **dependent** events.

$P(B|A) \times P(A) = P(A \cap B) = P(A|B) \times P(B)$ , so for non-trivial independent events  $A$  and  $B$ ,

$P(A \cap B) = P(A) \times P(B)$ ,  $P(A) = P(A|B)$  and  $P(B) = P(B|A)$  are equivalent.

You can use the concept of independence to simplify some probability calculations.

### EXAMPLE 10

A fair coin is tossed three times. What is the probability of three heads?

#### Solution

State the conditions.

Each toss of the coin is independent.

Use the rule for independent events.

$$P(1\text{st H} \cap 2\text{nd H} \cap 3\text{rd H})$$

$$= P(1\text{st H} \cap 2\text{nd H}) \times P(3\text{rd H})$$

Use it again.

$$= P(1\text{st H}) \times P(2\text{nd H}) \times P(3\text{rd H})$$

Substitute values.

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$$

Calculate the result.

$$= \frac{1}{8}$$

Write the answer.

The probability of three heads from 3 tosses of a fair coin is  $\frac{1}{8}$ .

When you are calculating probabilities from combined events, you need to work out whether the events are dependent or independent. Then you apply the appropriate rule.

The probability of the event  $A \cap B$  is calculated using:

$P(A \cap B) = P(A) \times P(B)$  for **independent** events  $A$  and  $B$ ;

$P(A \cap B) = P(A|B) \times P(B)$  or  $P(A \cap B) = P(B|A) \times P(A)$  for **dependent** events  $A$  and  $B$ .

### EXAMPLE 11

Duchenne muscular dystrophy is a genetic disease carried on the X-chromosome. Girls have two X-chromosomes, one from each parent and boys have an X-chromosome from their mother and a Y-chromosome from their father. This means that the disease is carried by the mother but only affects boys.

Mary does not know if she is a carrier or not. What is the probability that if she has a child it will have the disease?

#### Solution

What are the conditions for Mary to have a child with muscular dystrophy?

For her child to have muscular dystrophy, Mary must be a carrier and have a boy who inherits the gene.

Choose abbreviations.

Let  $M$  = a child with Duchene muscular dystrophy,  $C$  = Mary is a carrier and  $B$  = Mary has a boy.

What genes must Mary's mother have?

Mary's mother has two X chromosomes, one of which has the muscular dystrophy gene.

What is the probability that Mary is a carrier?

Mary has one X from her mother, so  $P(C) = \frac{1}{2}$

What is the probability of a boy?

$$P(B) = \frac{1}{2}$$

What is the probability that Mary is a carrier and has a boy?

Being a carrier and having a boy are independent, so  $P(C \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

What is the probability of a child having muscular dystrophy if Mary is a carrier?

A child gets an X-chromosome from Mary, so  $P(M | C \cap B) = \frac{1}{2}$

What is the probability of having a child with muscular dystrophy?

$$\begin{aligned}P(M) &= P(M | C \cap B) \times P(C \cap B) \\ &= \frac{1}{2} \times \frac{1}{4} \\ &= \frac{1}{8}\end{aligned}$$

Write the answer.

Mary has a probability of  $\frac{1}{8}$  of having a child with Duchene muscular dystrophy.

Nowadays, it is likely that Mary would have a test to see if she is a carrier before deciding to have a baby. When she becomes pregnant the foetus could also be tested.

In some situations you can find the required probability by working out the complement.

### EXAMPLE 12

What is the probability that at least two people in a group of 7 choose the same letter of the alphabet, correct to 3 decimal places?

#### Solution

Write the complementary event.

The complementary event is 'they all pick different letters'.

Find the complementary probability.

$$P(\bar{A}) = P(2\text{nd different}) \times P(3\text{rd different}) \times \dots$$

Use conditional probability.

$$= \frac{25}{26} \times \frac{24}{26} \times \frac{23}{26} \times \frac{22}{26} \times \frac{21}{26} \times \frac{20}{26}$$

Evaluate.

$$= 0.4127$$

Find the required probability.

$$P(\text{at least 2 the same}) = 1 - 0.4127\dots = 0.5872\dots$$

Write the answer.

The probability that at least 2 people out of 7 pick the same letter of the alphabet is about 0.587.

### Exercise 5.05 Dependent and independent events

- 1 The probability that a person wears glasses is about  $\frac{1}{3}$  and the probability that a person is left-handed is about  $\frac{1}{15}$ . Find the probability that a person chosen at random is:
- left-handed and wears glasses
  - left-handed and does not wear glasses
  - right-handed and wears glasses.

Example  
10



## Problem solving

- 8** The probability that a man will live 10 more years is  $\frac{1}{4}$ , and the probability that his wife will live longer than 10 more years is  $\frac{1}{3}$ . Find the probability that:
- a** both will be alive in 10 years
  - b** at least one will be alive in 10 years
  - c** neither will be alive in 10 years
  - d** only the wife will be alive in 10 years
  - e** only the husband will be alive in 10 years
- 9** From company records it is known that 80% of all applicants for a particular position have passed a standard literacy examination. According to the records, an applicant who fails the examination on the first try has a 90% chance of passing the exam on the second try. Three applicants take the examination.
- a** What is the probability that two will pass?
  - b** What is the probability that at least one will pass?
  - c** What is the probability that an applicant will fail the examination twice?
  - d** If an applicant can take the exam only twice, what is the probability of their passing the exam in one or two trials?
- 10** A women's fitness club reviewed its accounts and found that 12% of all accounts were in arrears while the other 88% were fully paid up. The review also revealed that 66% of the accounts belonged to married women. Only 4% of the accounts were both in arrears and belonged to married women.
- a** Find the probability of an account not being in arrears and belonging to an unmarried woman.
  - b** Find the probability of an account being in arrears, given that the account belongs to an unmarried woman.
  - c** Is account status independent of marital status?
- 11** Red-green colour blindness is a recessive genetic condition carried on the X-chromosome. About 17% of women carry the gene.
- a** What is the percentage probability that a man will have the condition?
  - b** What percentage of women can be expected to have red-green colour blindness?
- 12** What is the probability that at least two people from a group of 25 have the same birthday?



Combinations  
assignment



Combinations  
problems

## 5.06 Applications to data

Tag-and-release methods of counting wild populations are often used to count populations more cheaply than could otherwise be done. The method relies on the assumption that a tagged animal has the same chance of being caught as untagged animal.

### EXAMPLE 13

Fisheries officers are concerned about the prevalence of European carp in an inland waterway. On one field trip they catch 50 carp, tag and release them. A few weeks later they make another trip to the same area of the river and catch another 90 carp, of which they find 8 are tagged. Estimate the population of carp in this section of the river.



Shutterstock.com/K Steve Cope

### Solution

Find the probability of catching a tagged carp.  $P(\text{tagged}) = \frac{8}{90} = \frac{4}{45}$

State the basic probability rule.  $P(E) = \frac{n(E)}{n(S)}$

Use the total number tagged.  $\frac{4}{45} = \frac{50}{n(S)}$

Solve for  $n(S)$ .  $n(S) = 50 \div \frac{4}{45} \approx 563$

Round appropriately and write the answer. **There are probably about 560 carp in this section of the river.**

In Example 13, the fisheries officers are making assumptions that mean their estimate may be different from the real population. What could be some weaknesses in the assumption? Which weaknesses produce an under-estimate and which an over-estimate?

Probabilities calculated across the rows or down the columns of tables are often used to decide whether two variables or events are independent or not.

A **contingency table** is a table showing two characteristics of data across rows and down columns. Probabilities as percentages may be calculated:

by **row** based on the totals at the right-hand sides of the columns

by **column** based on the totals at the bottoms of the columns

by **table total** based on the total of the whole table, usually at the bottom right-hand corner

Examination of probabilities by row or column may suggest whether variables are dependent.

### EXAMPLE 14

The results of an opinion poll of support for daylight saving in Queensland in South-east Queensland and other areas are shown.

	SEQ	Regional
For	1874	416
Against	923	772

a Use the table to calculate the probabilities of support by area.

b What can you conclude from the results?

### Solution

a Extend the table to do totals.

	SEQ	Regional	Total
For	1874	416	2290
Against	923	772	1705
Total	2797	1188	3995

	SEQ	Regional	Total
For	67%	35%	57%
Against	33%	65%	43%
	100%	100%	100%

Calculate percentages by column.

State the result.

Probability of support in SEQ = 0.67

Probability of regional support = 0.35

b Compare probabilities.

$P(\text{For}) = 0.57$  and  $P(\text{For} | \text{SEQ}) = 0.67$

Write a conclusion.

Support for daylight saving and area are not independent; support depends on where you live.

Medical pathology tests are never 100% reliable. Doctors need to assess the results together with other symptoms and may order repeat tests to confirm a diagnosis.

In pathology testing,

- a **false positive** is a positive test result for a condition that the person does not actually have
- a **false negative** is a negative test result for a condition that the person actually has.

### EXAMPLE 15

The HIV test has an accuracy of 95%. HIV has an incidence of only about 0.1% in Australia and about 14% in the vulnerable population. The vulnerable population consists of people who indulge in risky behaviour such as sharing needles.

Find the probability of:

- a a false positive result in the general population
- b a true positive result in the general population
- c a false positive result in the vulnerable population
- d a true positive result in the vulnerable population
- e two successive false positive results in the vulnerable population

### Solution

- a Choose abbreviations.

$T$  = positive test result,  $H$  = has HIV

State the probability of a positive test result given no infection (test wrong).

$$P(T | \bar{H}) = 0.05$$

State the probability of infection.

$$P(\bar{H}) = 0.999$$

Find the probability of a false positive.

$$\begin{aligned} P(T \cap \bar{H}) &= P(T | \bar{H}) \times P(\bar{H}) \\ &= 0.05 \times 0.999 \\ &= 0.04995 \end{aligned}$$

State the answer.

The probability of a false positive in the general population is about 0.05.

- b State the probability of a positive test result with an infection (test right).

$$P(T | H) = 0.95$$

State the probability of infection.

$$P(H) = 0.001$$

Find the probability of a true positive in the general population.

$$\begin{aligned}P(T \cap H) &= P(T | H) \times P(H) \\ &= 0.95 \times 0.01 \\ &= 0.0095\end{aligned}$$

State the answer.

The probability of a true positive in the general population is about 0.0095.

- c** State the probability of a positive test result given no infection (test wrong).

$$P(T | \bar{H}) = 0.05$$

State the probability of infection.

$$P(\bar{H}) = 0.86$$

Find the probability of a false positive in the vulnerable population.

$$\begin{aligned}P(T \cap \bar{H}) &= P(T | \bar{H}) \times P(\bar{H}) \\ &= 0.05 \times 0.86 \\ &= 0.043\end{aligned}$$

State the answer.

The probability of a false positive in the vulnerable population is about 0.043.

- d** State the probability of a positive test result with an infection (test right).

$$P(T | H) = 0.95$$

State the probability of infection.

$$P(H) = 0.14$$

Find the probability of a true positive in the vulnerable population.

$$\begin{aligned}P(T \cap H) &= P(T | H) \times P(H) \\ &= 0.95 \times 0.14 \\ &= 0.133\end{aligned}$$

State the answer.

The probability of a true positive in the vulnerable population is about 0.13.

- e** State the relationship of two false positives.

False positive results are independent.

Find the probability of 2 false positives in the vulnerable populations.

$$\begin{aligned}P(2 \text{ false positives}) &= 0.043 \times 0.043 \\ &= 0.0018\dots\end{aligned}$$

State the answer.

The probability of 2 false positives in the vulnerable population is about 0.002.

Since the probability of false positives is about 5 times the probability of true positives, it would not be worthwhile to run a general screening program for HIV. This is true for nearly all rare conditions. Tests for rare diseases should only be done when there is a reason to suspect the disease. Some rare genetic diseases that affect babies are routinely tested for because they profoundly affect development.

## Exercise 5.06 Applications to data

Example  
13

- 1 A farming couple with a large dam has stocked it with fingerling salmon to make extra money. After allowing time for the salmon to grow they caught 30 salmon, tagged and released them. A few days later they again caught 30 salmon and found that 3 of them were tagged. How many salmon were left in the dam?
- 2 The heights of some Year 12 students and the numbers of computers in their houses are shown below.

Example  
14

		Height		
		Short ( $< 166$ cm)	Average ( $166$ – $175$ cm)	Tall ( $> 175$ cm)
Number of computers	Few (0–2)	7	6	5
	Average (3–4)	4	7	5
	Many ( $> 4$ )	10	9	11

- a Use the table to calculate the probabilities of different numbers of computers by height.
  - b Do the results suggest dependence or not?
- 3 An over-the-counter pregnancy test kit claims to have an accuracy of 99%, but this is under ideal conditions. It is more realistic to say the accuracy is about 95%. About 30% of women who buy and use the kit are actually pregnant.
    - a What is the probability of a false positive?
    - b What is the probability of a true positive?
    - c What is the probability of a false negative?
    - d What is the ratio of false positives to true positives?

Example  
15

### Problem solving

- 4 A local council is concerned about the effects of feral pigeons in the city area. It decides to investigate the problem before deciding if any action is warranted. Officers from the National Parks and Wildlife service catch, band and release 40 pigeons in the central park. A week later they catch another 60 pigeons and find that 3 of them have bands.
  - a What is the estimated population of pigeons?
  - b What are some weaknesses in this method?

- 5** Grey nurse sharks were tagged and released in known habitats on the east coast of Australia in 2003 to estimate the population. Altogether, 22 sharks of at least 2 m in length were tagged. Divers observed sharks in habitat areas 2 weeks later and saw 313 sharks, of which 143 were 2 m or more. Of the 143 sharks 2 m or more, 16 had tags. Observations over the next year showed that 7 of the tagged sharks showed evidence of hooking since being tagged.
- Estimate the total number of grey nurse sharks of size 2 m or more.
  - Estimate the total number of grey nurse sharks altogether.
  - What are some problems with the estimates?

*Source:* Otway, N.M. and Burke, A.L., 2004. Mark-recapture population estimate and movements of Grey Nurse Sharks. Final Report to EnvironmentAustralia. Project No. 30786/87. NSW Fisheries Final Report Series No. 63. ISSN 1442-0147.

- 6** A survey of women in office jobs about their return to work plans after pregnancy had the results shown below.

	Returning to work		
	Full-time	Part-time	Not returning
<b>Professional</b>	8	10	8
<b>Administrative</b>	6	12	33
<b>Clerical</b>	29	31	68

- Use the table to calculate the probabilities of returning to work by occupation type.
  - What can you conclude about return to work plans and occupation type?
- 7** The following information was collected about the lunch-break preferences of employees in a construction firm.

	30 minutes	45 minute	1 hour
<b>Workshop</b>	14	20	12
<b>On-site</b>	20	14	6
<b>Office</b>	6	8	13

- Use the table to calculate the probabilities of lunch-break preferences by work type.
- What can you conclude about lunch break preference and work type?

- 8** The following information was collected on a website about support for an Australian republic. People were asked to respond to the statement ‘Australia should end the monarchy and become a republic.’

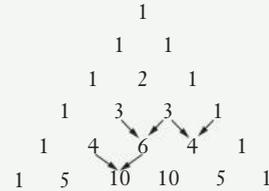
Gender	Disagree	Don't care	Agree
Male	310 979	168 623	366 904
Female	247 427	129 928	186 983

- a** Find the probability for support by gender.
- b** Find the probability of opposition by gender.
- c** What can you conclude?
- 9** An experimental *foetal blood factor* test was used to check the stain in a car in a famous Australian trial. The forensic expert who carried out the test told the jury it proved that there was blood from a baby under 1-year-old in the car. The designer of the test testified for the defence that the test was only 80% accurate, but he was not a very impressive witness. The jury convicted the woman of murdering her baby, not believing that it was taken by a dingo. About 0.9% of the population is under 1 year old.
- a** What is the probability of a false positive?
- b** What is the probability of a true positive?
- c** What other information should have been available to the jury, besides the test accuracy?
- 10** About 5% of Australians suffer from diabetes at any one time. A simple urine test for diabetes is about 99% accurate.
- a** What is the probability of a false positive?
- b** What is the probability of a true positive?

# 5. CHAPTER SUMMARY

## Combinations and conditional probability

- A choice of  $r$  objects from  $n$  objects is called a **combination**, and the number of combinations is shown as  ${}^n C_r$  or  $\binom{n}{r}$ . Combinations have no order
- The numbers of combinations are shown by **Pascal's triangle**:
- Factorial  $n$ ,  $n!$** , is defined by  $n! = 1 \times 2 \times 3 \times \dots \times n$  for  $n > 1$  and  $0! = 1! = 1$ . The number of combinations of  $r$  objects from  $n$  objects is given by



$${}^n C_r = \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots \times r} = \frac{n!}{r!(n-r)!}$$

- ${}^n C_r = {}^n C_{n-r}$
- The **binomial expansion** is given by
 
$$(x+y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1}y + {}^n C_2 x^{n-2}y^2 + {}^n C_3 x^{n-3}y^3 + \dots + {}^n C_1 xy^{n-1} + {}^n C_0 y^n.$$
- The **conditional probability** of  $A$  given  $B$ ,  $P(A | B)$  is defined by

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \text{ or } P(A \cap B) = P(A | B) \times P(B).$$

- Two events  $A$  and  $B$  are **independent** if and only if  $P(A \cap B) = P(A) \times P(B)$ .
- Events that are not independent are called **dependent** events.
- For non-trivial independent events  $A$  and  $B$ ,  $P(A \cap B) = P(A) \times P(B)$ ,  $P(A) = P(A | B)$  and  $P(B) = P(B | A)$  are equivalent.
- The probability of the event  $A \cap B$  is calculated using:
  - $P(B) = P(B | A)$  or  $P(A \cap B) = P(A) \times P(B)$  for **independent** events  $A$  and  $B$ ;
  - $P(A \cap B) = P(A | B) \times P(B)$  or  $P(A \cap B) = P(B | A) \times P(A)$  for **dependent** events  $A$  and  $B$ .
- A **contingency table** is a table showing two characteristics of data across rows and down columns. Probabilities as percentages may be calculated:
  - by **row** based on the totals at the right-hand sides of the columns
  - by **column** based on the totals at the bottoms of the columns
  - by **table total** based on the total of the whole table, usually at the bottom right-hand corner
- Examination of probabilities by row or column may suggest whether variables are dependent.
- In pathology testing, a **false positive** is a positive test result for a condition that the person does not actually have. A **false negative** is a negative test result for a condition that the person actually has.

# 5. CHAPTER REVIEW

## Combinations and conditional probability

Example  
1

1 Construct the 10th row of Pascal's triangle.

Examples  
2, 3

2 How many ways can you choose 6 batteries from a box of 10?

Example  
4

3 From 24 people in a class, 3 are left-handed. What is the probability that a random selection of 4 students from the class will have one left-handed person?

Example  
5

4 Three coloured blocks are taken at random from a pile of blocks with 6 red, 4 green and 3 blue ones. What is the probability that two will be green and one will be red?

Example  
6

5 Expand each of the following.

a  $(x - y)^9$

b  $(3x + 2y)^8$

Example  
7

6 What are the first 3 terms of  $(3p - 4q)^6$ ?

Examples  
8, 9

7 Write down the term of  $(t + 4q)^{14}$  containing  $t^4$ .

Example  
10

8 A carton of eggs contains three 55 g eggs, five 59 g eggs and two 52 g eggs. Find the probability of choosing two 55 g eggs at random.

Example  
11

9 A card is drawn from a well-shuffled pack of cards at random and the suit is noted. The card is then replaced in the pack, which is shuffled again. This is done 3 times. What is the probability that all 3 cards will be clubs?

Example  
12

10 You need to inherit the sickle-cell trait from both parents to get sickle-cell disease. If your maternal and paternal grandmothers both had brothers with sickle-cell disease but their parents didn't, what is the probability that you will have sickle-cell disease?

Example  
13

11 What is the probability that at least 2 out of 4 friends have their birthdays in the same month?

Example  
14

12 Fifty rabbits were caught in traps at night in an area in western Queensland. They had small tags attached and were then released. A few weeks later 85 rabbits were caught in the same area and 13 were found to have been previously tagged. Estimate the population of rabbits in the area.

13 Forty Year 11 girls were asked for their heights and shoe sizes. The results are shown in the table below.

	Short (< 158 cm)	Average (158-167 cm)	Tall (over 167 cm)
Small shoes ( $< 6\frac{1}{2}$ )	5	6	1
Average shoes ( $6\frac{1}{2} - 8$ )	3	11	5
Large shoes ( $> 8$ )	1	4	4

a Calculate the probabilities of different shoe sizes by height.

b Do the results suggest dependence or not?

- 14** The blood test for a disease affecting about 3% of people has an accuracy of 90%.
- a** What is the probability of a false positive in the general population?
  - b** What is the probability of a true positive in the general population?
  - c** What is the probability of two successive false positives in the general population?

### Problem solving

- 15** Find the value of  $n$ , given that  ${}^{n+2}C_4 = 3 \times {}^nC_4$ .
- 16** A badminton competition of 8 teams has 14 rounds in the season before the finals. The top 3 teams play off for the title of badminton champion team for the year. What is the probability of correctly choosing the three finalists at the start of the season?
- 17** 40% of the students surveyed about hair-washing were boys. 10% of the boys and 70% of the girls used conditioner after shampooing their hair. Find the probability that a student uses conditioner.
- 18** A euchre pack of cards consists of the ace, 7–10, jack, queen and king of each of the four suits, together with a joker. A hand consists of five cards.
- a** What is the probability of being dealt two aces in the first two cards?
  - b** What is the probability of being dealt two jacks of the same colour and the joker in the first three cards?
  - c** What is the probability of being dealt four 7s?
  - d** What is the probability of being dealt all four jacks and the joker?
- 19** Girls outnumbered boys 60% to 40% at a large Brisbane secondary school. Of the girls, 3% were classified as very tall, while 7% of the boys were classified as very tall. Find:
- a** the probability that a student selected at random is very tall
  - b** the probability that a very tall student from the school is male
  - c** the expected number of very tall male students from 3 busloads of students (180 students) going from the school to the swimming carnival.
- 20** What is the probability that at least 2 cards in a hand of 5 cards from a normal pack are the same number/letter (a pair)?
- 21** 1502 out of 6530 Australian women aged over 50 whose bone mineral density was tested have osteoporosis. Of these, 169 are smokers. 653 of the women altogether are smokers. Compare the probability of osteoporosis in women over 50 for those who smoke compared to those who do not smoke.
- 22** A medical researcher was investigating a possible genetic factor and heart disease. The task was complicated by the fact that some of the patients investigated were also obese, which is a known risk factor. Of 1400 people with heart disease, 800 were obese. Of these, 100 had the genetic factor present. 80 of the non-obese patients also had the genetic factor present. Of 1200 people without heart disease, 300 were obese. 28 of these had the genetic factor. 65 of the remaining 900 had the genetic factor present. About 30% of the population suffer from heart disease. Compare the probabilities of heart disease for those with and without the genetic factor and comment on your result.

# 6.

## GEOMETRIC SEQUENCES AND SERIES

You looked at arithmetic sequences and series in Chapter 1. A fixed number is added to each term to get the next term in an arithmetic sequence, like 1, 5, 9, 13, 17, ...

The sequence 4, 12, 36, 108, ... is not arithmetic, but it does follow a simple rule. It is a geometric sequence. How is each term obtained from the previous one?

- 6.01 Geometric sequences
- 6.02 General term of a geometric sequence
- 6.03 Sum of a geometric sequence
- 6.04 The sum to infinity
- 6.05 Present and future value
- 6.06 Annuities
- Chapter summary
- Chapter review

## SYLLABUS SUBJECT MATTER

### Geometric sequences

- recognise and use the recursive definition of a geometric sequence:  $t_{n+1} = rt_n$
- use the formula  $t_n = r^{n-1}t_1$  for the general term of a geometric sequence and recognise its exponential nature
- understand the limiting behaviour as  $n \rightarrow \infty$  of the terms  $t_n$  in a geometric sequence and its dependence on the value of the common ratio  $r$
- establish and use the formula  $S_n = t_1 \frac{r^n - 1}{r - 1}$  for the sum of the first  $n$  terms of a geometric sequence
- establish and use the formula  $S_\infty = \frac{t_1}{(1-r)}$  for the sum to infinity of a geometric sequence
- use geometric sequences in contexts involving geometric growth or decay, including compound interest and annuities

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Prior learning

## TERMINOLOGY

annuity  
 appreciation  
 Consumer Price Index, CPI  
 depreciation  
 future value  
 geometric sequence  
 immediate annuity  
 ordinary annuity  
 sum of a GP  
 worth of money

annuity factor  
 common ratio  
 converge  
 discount rate  
 general term  
 geometric series, sum of a GP  
 inflation  
 present value  
 sum to  $n$  terms

annuity-due  
 compound interest  
 deflation  
 diverge  
 geometric progression, GP  
 growth factor  
 $n$ th term  
 reducing interest loan  
 sum to infinity



Classifying  
 sequences

## 6.01 Geometric sequences

Each term of the sequence  $-192, 48, -12, 3, -0.75, \dots$  is multiplied by  $-\frac{1}{4}$  to get the next.

You can write this as the **recursive rule**  $t_{n+1} = -0.25t_n$ , where  $t_1 = -192$ .

A **geometric sequence** (or **geometric progression** [GP]) is a sequence with terms given

by  $t_{n+1} = rt_n$  for a constant  $r$  called the **common ratio**.  $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \dots$  or  $r = \frac{t_{n+1}}{t_n}$ .

The first term is frequently written as  $t_1 = a$ .



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## EXAMPLE 1

Find whether or not each of the following is a GP or not and if it is, state the common ratio.

- a** 75, 15, 3,  $\frac{3}{5}$ , ...      **b** 4, -8, -16, 32, ...      **c** 5,  $-5\sqrt{2}$ , 10,  $-10\sqrt{2}$ , ...

### Solution

- a** Find the ratios of successive terms until you run out or find a change.

$$\frac{t_2}{t_1} = \frac{15}{75} = \frac{1}{5}$$

$$\frac{t_3}{t_2} = \frac{3}{15} = \frac{1}{5}$$

$$\frac{t_4}{t_3} = \frac{\frac{3}{5}}{3} = \frac{1}{5}$$

State the answer.

$$\frac{t_{n+1}}{t_n} \text{ is constant, so it is a GP with } r = \frac{1}{5}.$$

- b** Find the ratios of successive terms until you run out or find a change.

$$\frac{t_2}{t_1} = \frac{-8}{4} = -2$$

$$\frac{t_3}{t_2} = \frac{-16}{-8} = 2$$

State the answer.

$$\frac{t_{n+1}}{t_n} \text{ is not constant, so it is not a GP.}$$

- c** Find the ratios of successive terms until you run out or find a change.

$$\frac{t_2}{t_1} = \frac{-5\sqrt{2}}{5} = -\sqrt{2}$$

$$\frac{t_3}{t_2} = \frac{10}{-5\sqrt{2}} = -\frac{10\sqrt{2}}{5 \times 2} = -\sqrt{2}$$

$$\frac{t_4}{t_3} = \frac{-10\sqrt{2}}{10} = -\sqrt{2}$$

State the answer.

$$\frac{t_{n+1}}{t_n} \text{ is constant, so it is a GP with } r = -\sqrt{2}.$$

You can find any term of a geometric progression (GP) from a given term and the common ratio. Since  $t_{n+1} = rt_n$ ,  $t_n = \frac{t_{n+1}}{r}$ . You can write these as  $t_n = rt_{n-1}$  and  $t_{n-1} = \frac{t_n}{r}$ .

## EXAMPLE 2

Find the first 4 terms of each GP.

- a** First term 8, common ratio 1.5.  
**c** Third term 6, common ratio  $-\frac{1}{3}$ .

- b** First term 12, common ratio  $\frac{1}{\sqrt{2}}$ .  
**d** Fourth term  $10x^2$ , common ratio  $2x$ .

### Solution

- a** Use the definition of a GP.

$$t_2 = rt_1 = 1.5 \times 8 = 12$$

$$t_3 = rt_2 = 1.5 \times 12 = 18$$

$$t_4 = rt_3 = 1.5 \times 18 = 27$$

Write the answer.

The GP is 6, 12, 18, 27, ...

- b** Use the definition of a GP.

$$t_2 = rt_1 = \frac{1}{\sqrt{2}} \times 12 = 6\sqrt{2}$$

$$t_3 = rt_2 = \frac{1}{\sqrt{2}} \times 6\sqrt{2} = 6$$

$$t_4 = rt_3 = \frac{1}{\sqrt{2}} \times 6 = 3\sqrt{2}$$

Write the answer.

The GP is 12,  $6\sqrt{2}$ , 6,  $3\sqrt{2}$ , ...

- c** Use the definition for  $t_4$ .

$$t_4 = rt_3 = -\frac{1}{3} \times 6 = -2$$

Work backwards for  $t_2$  and  $t_1$ .

$$t_2 = \frac{t_3}{r} = 6 \div \left(-\frac{1}{3}\right) = -18$$

$$t_1 = \frac{t_2}{r} = -18 \div \left(-\frac{1}{3}\right) = 54$$

Write the answer.

The GP is 54, -18, 6, -2, ...

- d** Work backwards.

$$t_3 = \frac{t_4}{r} = \frac{10x^2}{2x} = 5x$$

$$t_2 = \frac{t_3}{r} = \frac{5x}{2x} = 2.5$$

$$t_1 = \frac{t_2}{r} = \frac{2.5}{2x} = \frac{5}{4x}$$

Write the answer.

The GP is  $\frac{5}{4x}$ ,  $\frac{5}{2}$ ,  $5x$ ,  $10x^2$ , ...

## Exercise 6.01 Geometric sequences

Example  
1

**1** State whether or not each sequence is a GP. If it is, then find the common ratio.

**a** 2, 4, 8, 16, ...

**b**  $-x, -3x, -9x, -27x, \dots$

**c** 1, 4, 9, 25, ...

**d**  $\frac{1}{2}, -\frac{1}{6}, \frac{1}{18}, -\frac{1}{54}, \dots$

**e** 8, -4, 2, 1, ...

**f**  $5, 5\sqrt{2}x^{-2}, 10x^{-4}, 10\sqrt{2}x^{-6}$

**2** Find the common ratio of each GP.

**a** 1, 3, 9, ...

**b**  $12, -8, 5\frac{1}{3}, \dots$

**c** 5, -25, 125, ...

**d** 5, 15, 45, ...

**e** 4, 6, 9, ...

**f**  $6, -9, \frac{27}{2}, \dots$

**3** Find the first 4 terms of each GP.

**a** First term 2, common ratio -3

**b** First term -1, common ratio 5

**c** First term 2, common ratio  $\frac{3}{2}$

**d** First term  $9x$ , common ratio  $-\frac{1}{3}x$

**e** First term  $50y^{-2}$ , common ratio  $\frac{1}{5}y$

**f** First term  $a$ , common ratio  $r$ .

**4** Find the first 4 terms of each GP.

**a** Second term 10, common ratio -2

**b** Third term -4, common ratio 4

**c** Fourth term  $\frac{1}{10}$ , common ratio  $-\frac{1}{5}$

**d** Fifth term  $8x^2$ , common ratio  $-2x$

**e** First term 20, second term -5

**f** First term 20, fourth term  $67\frac{1}{2}$

Example  
2

### Problem solving

- 5** \$5400 is invested at compound interest of 6.5% p.a. over 4 years. Show that the values of the investment at the end of each year form a GP.
- 6** The value of a painting appreciates (grows) by 6% each year. Show that the values of the painting at the end of each year form a GP.
- 7** A car bought for \$28 990 loses 15% of its value (depreciates) each year. What is it worth after 3 years?
- 8** Show that the values of items that depreciate by the same percentage each year form a GP.

## 6.02 General term of a geometric sequence

Consider a GP with first term  $a$  and common ratio  $r$ .

The second term is given by  $t_2 = rt_1 = ar$ .

The third term is given by  $t_3 = rt_2 = a \times ar = ar^2$ .

The fourth term is given by  $t_4 = rt_3 = a \times ar^2 = ar^3$ .

The fifth term is given by  $t_5 = rt_4 = a \times ar^3 = ar^4$ .

The **general term** of a GP with first term  $a$  and common ratio  $r$  is given by  $t_n = t_1 r^{n-1}$  or  $t_n = ar^{n-1}$ .

The general term is often called the  **$n$ th term**.

### EXAMPLE 3

Find the indicated term for each geometric sequence.

**a**  $a = 7, r = 3, 5$ th term

**b** First term 8, common ratio  $-\frac{1}{4}, t_6$

**c**  $3\frac{1}{4}, 6\frac{1}{2}, 13, \dots, 10$ th term

**d** 36, 12, 4,  $\dots$ , 6th term

### Solution

**a** Write the formula.

$$t_n = ar^{n-1}$$

Substitute values and evaluate.

$$\begin{aligned} t_5 &= 7 \times 3^4 \\ &= 567 \end{aligned}$$

**b** Write the formula.

$$t_n = ar^{n-1}$$

Substitute values and evaluate.

$$\begin{aligned} t_6 &= 8 \times \left(-\frac{1}{4}\right)^5 \\ &= -\frac{1}{128} \end{aligned}$$

**c** State the values of  $t_1$  and  $r$ .

$$a = 3\frac{1}{4}, r = 2$$

Write the formula.

$$t_n = ar^{n-1}$$

Substitute values and evaluate.

$$\begin{aligned} t_{10} &= 3\frac{1}{4} \times (2)^9 \\ &= 1664 \end{aligned}$$

**d** State the values of  $a$  and  $r$ .

$$a = 36, r = \frac{1}{3}$$

Write the formula.

$$t_n = ar^{n-1}$$

Substitute values and evaluate.

$$\begin{aligned} t_6 &= 36 \times \left(\frac{1}{3}\right)^5 \\ &= \frac{4}{27} \end{aligned}$$

## INVESTIGATION

### LIMITING BEHAVIOUR

What happens to the value of a geometric sequence as you continue?

**1** Calculate the values of the tenth, twentieth and thirtieth terms for each of the following geometric sequences.

**a**  $a = 8, r = 3$

**b**  $a = 8, r = -3$

**c**  $a = 8, r = 0.5$

**d**  $a = 8, r = -0.5$

**2** What happens to the values of these geometric sequences as  $n$  becomes larger?

**3** Predict the behaviour as  $n \rightarrow \infty$ .

**4** Sketch graphs of the GPs given by  $t(n) = 10 \times 2^n$  and by  $t(n) = 10 \times (0.2)^n$ .

**5** What can you say about their behaviours as  $n \rightarrow \infty$ ?

**6** How does the behaviour of a GP as  $n \rightarrow \infty$  depend on the values of  $a$  and  $r$ ?

For  $-1 < r < 1, r \neq 0$ , a geometric sequence **converges** to 0 as  $n \rightarrow \infty$ .

This means that as  $n \rightarrow \infty, t_n \rightarrow 0$ .

For  $r > 1$  or  $r < -1$ , a geometric sequence **diverges** as  $n \rightarrow \infty$ .

This means that the value of the sequence increases without limit.

For the trivial case  $r = 1$ , a geometric sequence is constant. For the case  $r = -1$ , the value jumps between  $a$  and  $-a$ ; it oscillates.

You can use any two known terms to find the values of a geometric sequence. Be careful to consider all possible answers.

### EXAMPLE 4

The third and seventh terms of a geometric sequence are 405 and 5 respectively. Find the first and tenth terms.

#### Solution

Write the formulas for the terms.  $405 = ar^2$  and  $5 = ar^6$

Divide to find the common ratio.  $\frac{ar^6}{ar^2} = \frac{5}{405}$

$$r^4 = \frac{1}{81}$$

$$r = \frac{1}{3} \text{ or } -\frac{1}{3}$$

Find the first term.

$$405 = a \times \left(\frac{1}{3}\right)^2 \text{ or } a \times \left(-\frac{1}{3}\right)^2$$

$$a = 9 \times 405 = 3645$$

Find the tenth term.

$$\begin{aligned} t_{10} &= ar^9 \\ &= 3645 \times \left(\frac{1}{3}\right)^9 \text{ or } 3645 \times \left(-\frac{1}{3}\right)^9 \end{aligned}$$

$$= \frac{5}{27} \text{ or } -\frac{5}{27}$$

Write the answer in a sentence. The first term is 3645 and the tenth term is  $\frac{5}{27}$  or  $-\frac{5}{27}$ .

**Compound interest, appreciation** and **depreciation** at a constant rate are all examples of geometric sequences.

Suppose you invest \$3000 at a compound interest rate of 5%. The value grows by 5% each year, so the value after the first year is  $\$3000 \times 1.05 = \$3150$ .

After 2 years, the value is  $\$3150 \times 1.05 = \$3307.50$ .

The value after  $n$  years is  $\$3000 \times 1.05^n$ . This is the formula for a GP with a 0th term instead of a first term. Appreciation and depreciation at fixed rates are calculated in the same way.

The **growth factor** for **compound interest** and **appreciation** is  $(1 + i)$ , where  $i$  is the interest rate or appreciation rate. The value is given by  $A = P(1 + i)^n$ , where  $P$  is the initial value and  $n$  is the number of interest/appreciation periods (**rests**).

The **growth factor** for **depreciation** is  $(1 - i)$ , where  $i$  is the rate of depreciation. The value is given by  $A = P(1 - i)^n$ , where  $P$  is the initial value and  $n$  is the number of depreciation periods.

### EXAMPLE 5

Find the value of each investment.

- a An investment of \$6000 at 4.9% interest for 20 years.
- b An investment of \$8500 at 7.2% interest with monthly rests for 8 years.

#### Solution

- a Write the values of the variables.  $P = 6000, i = 0.049, n = 20$

Write the formula.  $A = P(1 + i)^n$

Substitute values and evaluate.  $= \$6000 \times 1.049^{20}$   
 $= \$15\,619.28$

Write the answer. The investment is worth \$15 619.28 after 20 years.

- b Write the values of the variables.  $P = 8500, i = \frac{0.072}{12} = 0.006, n = 8 \times 12 = 96$

Write the formula.  $A = P(1 + i)^n$

Substitute values and evaluate.  $= \$8500 \times 1.006^{96}$   
 $= \$15\,094.72$

Write the answer. The investment is worth \$15 094.72 after 8 years.

## Exercise 6.02 General term of a geometric sequence

Example

3

- 1 Find the indicated term for each GP.
  - a  $a = 10, r = 1.4$ , 6th term
  - b First term  $-3$ , common ratio  $-0.5$ ,  $t_{10}$
  - c  $4, -4\sqrt{3}, 12, \dots$ , 10th term
  - d  $80, -20, 5, \dots$ , 6th term
  - e  $\sqrt{xy}, \frac{x}{\sqrt{y}}, \frac{x\sqrt{x}}{y\sqrt{y}}, \dots$ , 12th term
  - f  $5, 8.5, 14.45, \dots$ , 15th term, correct to 2 decimal places
- 2 A GP has first term 8 and common ratio 4.5. What is the first term to exceed 1000?
- 3 A GP has first term 10 and common ratio 0.6. What is the first term to be less than 0.01?
- 4 Is 972 a term of the GP  $-4, 12, -36, \dots$ ? If it is, which one?
- 5 Is  $-448$  a term of the GP  $7, -14, 28, \dots$ ? If it is, which one?
- 6 Find the number of terms in each of the following GPs.
  - a 4096, 1024, 256,  $\dots$ , 1
  - b 243, 81, 27,  $\dots$ ,  $\frac{1}{243}$
  - c  $\frac{1}{7}, \frac{1}{\sqrt{7}}, 1, \dots, 2401\sqrt{7}$
  - d  $96, 48, 24, \dots, \frac{3}{128}$
  - e 10, 4, 1.6,  $\dots, \frac{128}{3125}$
- 7
  - a The 6th and 7th terms of a GP are 10 and 20. Find the 12th term.
  - b The 3rd and 5th terms of a GP are 8 and 4. What is the 10th term?
  - c The 3rd and 6th terms of a GP are  $-5$  and 15. What is the 10th term?
- 8 Find the value of each investment or asset.
  - a \$2840 at 6.3% for 4 years with monthly rests.
  - b \$9000 at 4.8% for 8 years with quarterly rests.
  - c An antique dresser initially worth \$1200 after 5 years at 11% appreciation.
  - d A rare coin initially worth \$450 after 6 years at 9.5% appreciation.
  - e A bulldozer bought for \$160 000 after 4 years at 8% depreciation.
  - f A car bought for \$54 900 after 3 years at 15% depreciation.

Example

4

Example

5

### Problem solving

- 9 How long will it take \$9000 to accumulate to at least \$15 000 at 6% compound interest?
- 10 An investment can be made at 5.6% with annual rests, 5.5% with quarterly rests, 5.4% with monthly rests or 5.3% with daily rests. Which gives the greatest interest?

## 6.03 Sum of a geometric sequence

In Chapter 1, you learned that a **series** is a sequence whose terms are the partial sums of another sequence.

A **geometric series (sum of a GP)** is the sum of the terms of a geometric progression. It is usual to write the **partial sums** as

$$S_1 = a,$$

$$S_2 = a + ar,$$

$$S_3 = a + ar + ar^2,$$

$$S_4 = a + ar + ar^2 + ar^3,$$

...

The partial sum  $S_n$  is often referred to as the **sum to  $n$  terms**.

The geometric sequence 2, 6, 18, 54, ... produces the geometric series 2, 8, 26, 80, ...

You can find the sum without adding them all up as shown below.

$$S_6 = 2 + 6 + 18 + 54 + 162 + 486$$

$$3S_6 = 6 + 18 + 54 + 162 + 486 + 1458$$

$$3S_6 - S_6 = 1458 - 2 = 2 \times 3^6 - 2$$

$$2S_6 = 1456$$

$$S_6 = 728$$

You can do the same thing with the general case:

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1}$$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n$$

$$rS_n - S_n = ar^n - a$$

$$S_n(r - 1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

If  $r < 1$ , it is convenient to write this as  $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(r^n - 1)}{r - 1} \times \frac{-1}{-1} = \frac{a(1 - r^n)}{1 - r}$

The sum of a GP to  $n$  terms is given by

$$S_n = \frac{t_1(r^n - 1)}{r - 1} = \frac{t_1(1 - r^n)}{1 - r} \quad \text{or} \quad S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$$



## EXAMPLE 6

Find the sum of each GP.

**a**  $4, 12, 36, \dots$ , to 8 terms

**b**  $80 + 20 + 5 + \dots$ , to 5 terms

### Solution

**a** Write the values of the variables.

$$a = 4, r = 3, n = 8$$

Write the formula.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Substitute and evaluate.

$$= \frac{4(3^8 - 1)}{3 - 1}$$

$$= 13\,120$$

Write the answer.

The sum to 8 terms is 13 120.

**b** Write the values of the variables.

$$a = 80, r = \frac{1}{4}, n = 5.$$

Write the formula.

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

Substitute and evaluate.

$$= \frac{80 \left( 1 - \left( \frac{1}{4} \right)^5 \right)}{1 - \frac{1}{4}}$$

$$= 106 \frac{9}{16}$$

Write the answer.

The sum to 5 terms is  $106 \frac{9}{16}$ .



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## EXAMPLE 7

How many terms of the series  $2\frac{1}{4} + 3 + 4 + \dots$  are needed for the sum to exceed 200?

### Solution

Check the sequence.

$$\frac{t_2}{t_1} = \frac{3}{2\frac{1}{4}} = 3 \times \frac{4}{9} = \frac{4}{3} = \frac{t_3}{t_2}, \text{ so it is a GP.}$$

Write the values of the variables.

$$a = 2\frac{1}{4}, r = \frac{4}{3}, S_n = 200$$

Write the formula.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Substitute values and solve.

$$200 = \frac{2\frac{1}{4} \left[ \left( \frac{4}{3} \right)^n - 1 \right]}{\frac{4}{3} - 1}$$

$$\left( \frac{4}{3} \right)^n - 1 = 200 \times 3 \div \frac{9}{4} = 29\frac{17}{27}$$

$$\left( \frac{4}{3} \right)^n = 30\frac{17}{27}$$

Try some values of  $n$ .

$$\left( \frac{4}{3} \right)^{11} = 23.676\dots$$

$$\left( \frac{4}{3} \right)^{12} = 31.569\dots$$

Check the sums.

$$S_{11} = \frac{2\frac{1}{4} \left[ \left( \frac{4}{3} \right)^{11} - 1 \right]}{\frac{4}{3} - 1} = 153.069\dots$$

$$S_{12} = \frac{2\frac{1}{4} \left[ \left( \frac{4}{3} \right)^{12} - 1 \right]}{\frac{4}{3} - 1} = 206.342\dots$$

Write the answer.

12 terms are need for the sum to exceed 200.

### EXAMPLE 8

The common ratio of a GP is  $\frac{5}{2}$  and the sum to 5 terms is 1031. What is the 7th term?

#### Solution

Write the known variable values.  $r = \frac{5}{2}, S_5 = 1031$

Write the appropriate formula.  $S_n = \frac{a(r^n - 1)}{r - 1}$

Substitute and solve for  $a$ .  $1031 = \frac{a \left[ \left( \frac{5}{2} \right)^5 - 1 \right]}{\frac{5}{2} - 1}$

$$\begin{aligned} 1031 \times \frac{3}{2} &= a \times \left( \frac{3125}{32} - 1 \right) \\ &= a \times \frac{3093}{32} \\ a &= \frac{1031 \times 3 \times 32}{2 \times 3093} = 16 \end{aligned}$$

Write the answer.

The first term is 16.

You can check your answer to Example 8 by calculating the sum to 5 terms using  $a = 16$  and  $r = \frac{5}{2}$ .

### Exercise 6.03 Sum of a geometric sequence

Example  
6

1 Find the sum of each geometric series.

**a**  $1\frac{1}{4} - 2\frac{1}{2} + 5 - 10 + \dots$  (to 9 terms)

**c**  $20 + 30 + 45 + \dots$  (to 15 terms)

**b**  $1 + 0.5 + 0.25 + \dots$  (to 10 terms)

**d**  $1 + 3 + 3^2 + \dots$  (to 10 terms)

2 Find the sum of each series.

**a**  $5 + 10 + 20 + \dots + 640$

**c**  $3072 + 1536 + 768 + \dots + 3$

**b**  $7 + 21 + 63 + \dots + 5103$

**d**  $4 + 4\sqrt{3} + 12 + \dots + 972$

Example  
7

3 How many terms of the sequence 3, 6, 12, ... must be added to amount to 1533?

4 How many terms of the series  $2 - 4 + 8 - 16 + \dots$  must be added for the sum to be greater than 100?



## 6.04 The sum to infinity

What happens to the value of a geometric series as  $n$  becomes large? Does it converge or diverge? Consider the sequences 7, 14, 28, 56, ... and 8, 4, 2, 1, 0.5, 0.25, ...

The first sequence diverges, so the sum must also diverge. The second converges, but what

about the sum,  $S_n = \frac{a(1-r^n)}{1-r} = \frac{8(1-0.5^n)}{0.5} = 16 - 16 \times 0.5^n$ ?

The values are 8, 12, 14, 15, 15.5, 15.75, 15.875, ...

It seems to converge to  $16 = \frac{a}{1-r}$ .

A geometric series converges to  $S_\infty = \frac{t_1}{1-r} = \frac{a}{1-r}$  for  $-1 < r < 1$  ( $r \neq 1$ ).

The value  $S_\infty$  is often called the **sum to infinity** of a GP.

### EXAMPLE 9

Find the sum to infinity of the GP 8, -4, 2, -1, ...

#### Solution

Choose the variables.

First term  $a = 8$  and common ratio  $r = -0.5$ .

Write the formula.

$$S_\infty = \frac{a}{1-r}$$

Substitute values.

$$= \frac{8}{1-(-0.5)}$$

$$= 5\frac{1}{3}$$

Write the answer.

The sum to infinity is  $5\frac{1}{3}$ .

You can write a recurring decimal as an infinite geometric series with  $0 < r < 1$ , so it converges. You can use the sum to infinity to change it to a fraction.

### EXAMPLE 10

Change  $0.5\overline{27}$  to a fraction.

#### Solution

Write the recurring part as a GP.

$$0.\overline{027} = 0.027 + 0.000\ 27 + 0.000\ 002\ 7 + \dots$$

Write the first term and common ratio.

$$a = 0.027, r = 0.01$$

Find the sum to infinity.

$$\begin{aligned} S_{\infty} &= \frac{a}{1-r} \\ &= \frac{0.027}{1-0.01} \\ &= \frac{0.027}{0.99} \\ &= \frac{0.027 \times 1000}{0.99 \times 1000} \\ &= \frac{27}{990} \\ &= \frac{3}{110} \end{aligned}$$

Add the other part of the fraction.

$$\begin{aligned} 0.5\overline{27} &= \frac{1}{2} + \frac{3}{110} \\ &= \frac{58}{110} \\ &= \frac{29}{55} \end{aligned}$$

$$0.5 = \frac{1}{2}$$

### Exercise 6.04 The sum to infinity

1 Find the sum to infinity of each series.

**a**  $18 + 6 + 2 + \dots$

**b**  $18 - 6 + 2 - \dots$

**c**  $100 + 20 + 4 + \dots$

**d**  $100 - 20 + 4 - \dots$

**e**  $0.8 + 0.4 + 0.2 + \dots$

2 Convert each recurring decimal to a fraction.

**a**  $0.4\overline{5}$

**b**  $0.3\overline{6}$

**c**  $0.5\overline{12}$

**d**  $0.24\overline{8}$

**e**  $0.120\overline{6}$

Example

9

Example

10

## Problem solving

- 3 A pump takes water from a dam to fill a large tank. As the tank fills, the back pressure makes the pump work less efficiently. In the first hour, the pump puts 500 litres of water into the tank, but the amount of water transferred decreases by 30% every hour thereafter. What is the limit of how much water will be pumped into the tank?
- 4 A ball is dropped from a height of 24 m and rebounds to a height of 16 m. The height decreases in the same ratio on every bounce. What is the total distance travelled by the ball before it comes to rest?
- 5 Achilles and a tortoise have a 2.5 km race, but the tortoise has a 1 km start because it can only move at half the speed of Achilles. When Achilles has caught up to the tortoise's starting position, the tortoise is half a kilometre ahead. When Achilles gets to that position, the tortoise is a quarter of a kilometre ahead, and so on. This means that Achilles can never catch the tortoise, so the tortoise must win the race. This is called **Zeno's paradox**. State the flaw in the argument and find when Achilles passes the tortoise.



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## 6.05 Present and future value

You hear expressions like the price of things or the value of an investment in '2010 dollars', or wages described in 'real dollars'. In this section you will learn what they mean and how they are calculated.

Suppose you borrow \$5000 from your parents and pay them back in 5 years. Instead of lending you the money, they could have invested it. Five years in the future at 6.2% p.a. compound interest, the \$5000 would grow to about \$6754. This means that lending you the money would cost them about \$1574 in **interest foregone**.

The **future value** of money is the amount it would grow to at compound interest.

The **present value** is the amount invested now at compound interest to get a future sum.

The **worth of money** (or **discount rate**) is the interest rate used in these calculations.

## EXAMPLE 11

Kam borrowed \$40 000 from a bank to invest on the stock exchange and promised to pay it back in 5 years time at 7% p.a. interest. What is the future value of the money?

### Solution

Use the formula for compound interest.

$$A = P(1 + i)^n$$

Substitute  $P = 40\,000$ ,  $i = 0.07$ ,  $n = 5$ .

$$= \$40\,000 \times 1.07^5$$

$$= \$56\,102.07$$

Write the answer.

The future value is \$56 102.07.

Of course, the interest rate that the bank charges in Example 11 is more than they think that their professional managers can make by investing the money. This means that Kam is gambling that he can do better than the professionals. Present value and future value are related by the compound interest formula.

$$FV = PV(1 + i)^N \text{ or } PV = FV(1 + i)^{-N}, \text{ where}$$

$FV$  = future value,  $PV$  = present value,  $i$  = interest rate and  $N$  is the number of years.

The first formula is just the compound interest formula with  $PV$  and  $FV$  instead of  $P$  and  $A$ .

To reverse the formula, divide  $FV = PV(1 + i)^N$  by  $(1 + i)^N$ .

$$\text{This gives } PV = \frac{FV}{(1 + i)^N} = FV(1 + i)^{-N}.$$

Notice that financial and economics formulas often have variable names that are two-letter capitals instead of single letter lower case. You have to adjust the formula for interest

calculated with non-annual rests, so for monthly rests  $FV = PV \left(1 + \frac{i}{12}\right)^{12N}$ .

## EXAMPLE 12

What is the present value of \$230 000 due in 6 years time at a discount rate of 8% p.a.?

### Solution

Use the formula.

$$PV = FV(1 + i)^{-N}$$

Substitute  $P = 230\,000$ ,  $i = 0.078$ ,  $N = 6$ .

$$= \$230\,000 \times 1.08^{-6}$$

$$= \$144\,939.01$$

Write the answer.

The present value is \$144 939.01.

## TECHNOLOGY

### TVM calculations

You can do 'Time Value of Money' calculations on your graphics calculator to find present and future values.

In these calculations money paid by you is negative and money paid to you is positive.

PMT = amount of regular payments

P/Y = number of payments each year

C/Y = calculations per year, usually the same as P/Y

#### TI-84 Plus CE

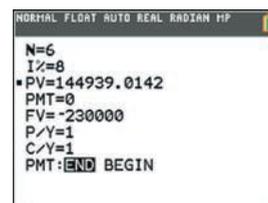
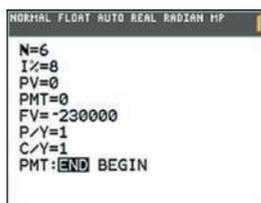
Press  $\boxed{\text{apps}}$ , **1: Finance** and **1: TVM Solver**.

Complete the values for Example 12.

The 230 000 is negative because it is 'due'.

It is the amount you will pay.

Move the cursor to PV and press  $\boxed{\text{alpha}}\boxed{\text{enter}}$  (solve).



#### Casio fx-CG20AU

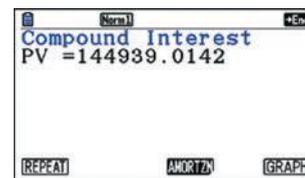
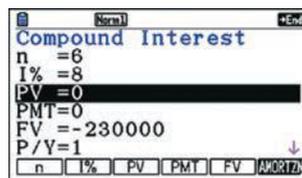
Choose **C Financial** and **F2: Compound Interest**.

Complete the values for Example 12.

The 230 000 is negative because it is 'due'.

It is the amount you will pay.

Change P/Y to 1 and press  $\boxed{\text{F3}}$  (PV).



Your parents might have told you that everything cost less when they were children. This is because there is usually a gradual increase in prices over time. This increase is called **inflation** and a decrease is called **deflation**.

The Australian Bureau of Statistics does a survey of prices every quarter. They use these surveys to construct the Consumer Price Index (**CPI**) that measures inflation. The total for the base year (2012) is standardised to 100. For June 2016, the CPI was 108.6, so prices increased by 8.6% over the 4 years from 2012 to 2016. For June 2015 it was 107.5, so prices

increased by  $\frac{1.1}{107.5} \times 100 = 1.02\%$  from June 2015 to June 2016.



TI-Nspire CX  
Chapter 6

If the CPI goes up by 10% but wages only increase by 8%, then **real** wages have decreased. This means that the amount you can buy has gone down. You can use the present and future value formulas to take inflation into account.

### EXAMPLE 13

The average rate of inflation in Australia from 1990 to 2010 was 2.7% p.a. A car cost \$25 000 in 2010. How much would this be in 1990 dollars?

#### Solution

Use the present value formula.

$$PV = FV(1 + i)^{-N}$$

Substitute values.

$$= \$25\,000 \times 1.027^{-20}$$

$$= \$14\,673.41$$

Write the answer.

\$25 000 in 2010 is \$14 673.41 in 1990 dollars.

### Exercise 6.05 Present and future value

- 1 Find the value of \$4800 in 6 years if money is worth 9.4% p.a.
- 2 Find the future value of \$8600 in 10 years time at 16.5% p.a.
- 3 Find the future value of \$45 000 due in 5 years time at 13.2% p.a. compounding monthly.
- 4 Find the present value of \$23 840 due in 4 years time at a discount rate of 6.9% p.a.
- 5 What is the present value of payments of \$3000 in 2 years and \$3000 in 6 years at 10.7% p.a.?
- 6 What is the present value of equal payments of \$1400 at the end of 1, 2 and 3 years at a discount rate of 8.75% p.a.?
- 7 What is the present value of equal payments of \$3600 at the end of 1, 2, 3, 4 and 5 years at 12.1% p.a.?
- 8 How much would a dollar in the year 2015 be worth in 2025 at an inflation rate of:
 

<b>a</b> 9%?	<b>b</b> 6%?	<b>c</b> 12%?	<b>d</b> 4%?
--------------	--------------	---------------	--------------

Example  
11

Example  
12

Example  
13

#### Problem solving

- 9 Andrea's grandmother gave her \$5000 when she turned 21. Six years later, her brother David received \$5500 when he turned 21. Was her grandmother fair in her gifts if the inflation rate is 1.8%?
- 10 A horse cost \$1300 in 2010. Assuming average inflation of 4% p.a., what would you expect a similar horse to cost in 2015?

- 11** An insurance salesman offers a contract whereby the client makes payments of \$1200 a year for 5 years, and at the end of 5 years collects \$10 000. If money is worth 10% p.a., work out whether this is fair. (*Hint*: Calculate the future value of each payment and add them.)
- 12** At 6.5% p.a. inflation, work out what a 70c ice-block from the year 1998 would cost in 2018.
- 13** In 1945 an apprentice earned £3 (3 pounds) and 9 shillings. The inflation rate since 1945 has averaged 7.5%. What is the 1945 wage in 2000 dollars? (*Note*: £1 = 20 shillings, and in 1966 Australia changed its currency so that £1 = \$2.)



Alamy Stock Photo/INTERFOTO



Geometric series assignment



Geometric series problems

## 6.06 Annuities

Your parents might be paying off their house by making payments every month. Your grandparents could be getting payments every fortnight from a fund they bought with their superannuation. For some investments you make regular payments over a long time to get a large amount at the end.

These examples all involve equal payments for a large amount at the start or finish.

A mathematical **annuity** is a series of equal payments at regular intervals with compound interest.

The **present value** of an annuity is the value at the start of the series of payments.

The **future value** of an annuity is the value at the end of the series of payments.

The **annuity factor** is the value of an annuity with payments of \$1.

For an **ordinary annuity** (or **immediate annuity**), the payments are made at the end of the interest period. For an **annuity-due**, payments are made at the beginning of the interest period.



Annuities

### EXAMPLE 14

Rebecca invests \$3000 at the end of each year over 5 years at 6.5% p.a. interest. How much will she get back at the end of the 5 years?

#### Solution

Find the future value of the first payment.  
There are 4 interest periods for this.

$$FV(\text{1st payment}) = 3000 \times 1.065^4 \\ \approx \$3859.40$$

Find the future value of the second payment.	$FV(2\text{nd payment}) = 3000 \times 1.065^3$ $\approx \$3623.85$
Find the future value of the third payment.	$FV(3\text{rd payment}) = 3000 \times 1.065^2$ $\approx \$3402.68$
Find the future value of the fourth payment.	$FV(4\text{th payment}) = 3000 \times 1.065^1$ $\approx \$3195$
Find the future value of the fifth payment.	$FV(5\text{th payment}) = 3000 \times 1.065^0$ $= \$3000$
Find the total.	Total = $\$3859.40 + \$3623.85 + \dots$ $\approx \$17\,080.93$
Write the answer.	Rebecca would get about $\$17\,080.93$ .

Instead of calculating the amounts 1 year at a time, you could use the sum of a GP to do it in one calculation.

In Example 14,  $a = 3000$  and  $r = 1.065$ , so

$$\text{Total} = \frac{3000(1.065^5 - 1)}{1.065 - 1} = 17\,080.92.$$

The amount is slightly different because of rounding errors in the individual calculations.

For examples like this, it is more common to make payments at the beginning of the interest period (year). In that case you would multiply the answer by 1.065 to get  $\$18\,191.18$  for the annuity-due. This would be paid at the end of the 5th year.

$n$  payments of  $\$1$  at interest rate  $i$  would give a sum with  $a = 1$  and  $r = (1 + i)$ , so the future value of the payments would be  $\frac{1[(1+i)^n - 1]}{(1+i) - 1} = \frac{(1+i)^n - 1}{i}$ .

The present value of this annuity is given by

$$\begin{aligned} PV &= FV(1+i)^{-n} = \frac{(1+i)^n - 1}{i} \times \frac{(1+i)^{-n}}{1} \\ &= \frac{(1+i)^n(1+i)^{-n} - (1+i)^{-n}}{i} \\ &= \frac{1 - (1+i)^{-n}}{i} \end{aligned}$$

For  $n$  payments of \$1 at interest rate  $i$ :

**Future value annuity factor,**  $S_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$

**Present value annuity factor,**  $a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i}$

Multiply by  $(1+i)$  to change to an annuity-due.

Where there is no ambiguity, the symbols may be written as  $S_{\overline{n}|}$  or  $a_{\overline{n}|}$ .

You can find the value of any annuity by multiplying the payment by the annuity factor.

It doesn't make financial sense to pay the last contribution for a savings plan when you get the lump sum. This means that savings plans are usually annuities-due.

### EXAMPLE 15

Ignoring fees and taxes, what fortnightly contributions are needed to get a lump sum of \$500 000 in 30 years time from a superannuation fund paying 11.6% p.a.?

#### Solution

State the variables.

$$i = \frac{0.12}{26}, n = 30 \times 26 = 780$$

Use the future value formula.

$$S_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

Substitute values and evaluate.

$$\begin{aligned} S_{\overline{780}|} &= \frac{\left(1 + \frac{0.116}{26}\right)^{780} - 1}{\frac{0.116}{26}} \\ &= \left[\left(1 + \frac{0.116}{26}\right)^{780} - 1\right] \times \frac{26}{0.116} \\ &\approx 6995.2219 \end{aligned}$$

Change to an annuity-due.

$$\left(1 + \frac{0.116}{26}\right) S_{\overline{780}|} = 7026.4314$$

Calculate the contribution needed.

$$\text{Contribution} = \frac{500\,000}{7026.4314} \approx \$71.16$$

Write the answer.

A lump sum of \$500 000 needs \$71.16 a fortnight for 30 years.

Relatively small contributions over a long time can give you very large lump sum payments, but you need to remember that inflation decreases the value.

You can use the present value formula in the same way as above to find the payments needed for a house loan. These are often called **reducing interest loans** because the amount of interest paid goes down as you pay off more of the loan. However, it is often better to work out how much you can afford to borrow.

### EXAMPLE 16

Abbie and Ahmed can afford payments of \$2840 a month for a house loan. Their bank is charging 6.4% p.a. interest. How much could they borrow over a period of 25 years?

#### Solution

State the variables.

$$i = \frac{0.064}{12}, n = 25 \times 12 = 300$$

Use the present value formula.

$$a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i}$$

Substitute values and evaluate.

$$\begin{aligned} S_{\overline{300}|} &= \frac{1 - \left(1 + \frac{0.064}{12}\right)^{-300}}{\frac{0.064}{12}} \\ &= \left[1 - \left(1 + \frac{0.064}{12}\right)^{-300}\right] \times \frac{12}{0.064} \\ &\approx 149.4831 \end{aligned}$$

Calculate the total.

$$\begin{aligned} \text{Amount} &= 149.4831 \times \$2840 \\ &\approx \$424\,532.05 \end{aligned}$$

Write the answer.

They could afford to borrow about \$424 000.

Abbie and Ahmed would actually need to allow money for legal fees and other costs in buying their house. In this situation it is safer to round down than to just round the answer.

## TECHNOLOGY

### Annuity calculations

You can use TVM to do annuity calculations on your graphics calculator.

Remember that money paid by you is negative and money paid to you is positive.

#### TI-84 Plus CE

Press  $\boxed{\text{apps}}$ , **1: Finance** and **1: TVM Solver**.

Complete the values for Example 15 and find the payment. You need to change P/Y to 26 (fortnights).

Complete the values for Example 16 with P/Y = 12 (months) and find the payment.

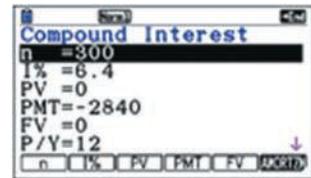
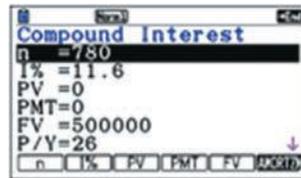


#### Casio fx-CG20AU

Choose **C Financial** and **F2: Compound Interest**.

Complete the values for Example 15 and find the payment. You need to change P/Y to 26 (fortnights).

Complete the values for Example 16 with P/Y = 12 (months) and find the payment.



## INVESTIGATION

### SUPERANNUATION AND TAXATION

- Investment decisions are often influenced by taxation rates. Governments sometimes use the taxation system to attempt to direct money into particular types of investments.
- In Australia, *negative gearing* has been permitted on investment in residential housing.
- This effectively guarantees a reasonable rate of return on money invested in rental housing through a reduction of personal tax for the landlord. In recent times the Australian government has changed the rules relating to the taxation of superannuation payouts and annuities.

- 1 Use the library or other resources to find the taxation rules relating to superannuation contributions and payouts.
- 2 In groups, discuss the possible reasons for the changes in taxation rules. Consider the relative age of the Australian population.
- 3 Discuss the merits and demerits of using the taxation system as an instrument of government policy.

### Exercise 6.06 Annuities

Use your graphics calculator as directed by your teacher in this exercise.

- 1
  - a Taylor invests \$2500 at the end of each year over 7 years at 7.1% p.a. interest. How much will she get back at the end of the 7 years?
  - b What would she get if she made payments at the start of each year?
- 2 Calculate the value of each annuity, correct to 4 decimal places.
  - a  $S_{\overline{10}|0.05}$
  - b  $a_{\overline{15}|6\%}$
  - c  $S_{\overline{240}|0.63\%}$
  - d  $a_{\overline{300}|0.8\%}$
- 3 Find the present value annuity factor of:
  - a 20 payments at 7.5%
  - b 300 payments at 0.94%
  - c 240 payments at 1.5%
- 4 Find the future value annuity factor of:
  - a 10 payments at 6%
  - b 120 payments at 1.4%
  - c 300 payments at 0.85%
- 5 Find the future value of payments of:
  - a \$300 a quarter at 12.4% p.a. over 15 years
  - b \$75 a week at 11.2% p.a. over 25 years
  - c \$150 a fortnight at 11.2% p.a. over 25 years.
- 6 Find the present value of payments of:
  - a \$210 a month at 9.2% p.a. over 8 years
  - b \$65 a week at 16% p.a. over 5 years
  - c \$130 a fortnight at 16% p.a. over 5 years.
- 7 How much would you have to pay into a superannuation fund each month to accumulate \$725 000 at 9.5% p.a. over 35 years?
- 8 How much could be borrowed on a reducing interest loan for each of the following?
  - a \$350 a fortnight at 6.1% p.a. over 25 years
  - b \$600 a month at 5.6% p.a. over 30 years
  - c \$920 a month at 7.3% p.a. over 20 years
  - d \$210 a week at 6.4% p.a. over 40 years

Example  
14

Example  
15

Example  
16

- 9** What repayments are required for each reducing interest loan?
- a** \$410 000 at 6.2% p.a. over 30 years with monthly payments
  - b** \$850 000 at 5.9% p.a. over 40 years with fortnightly payments
  - c** \$270 000 at 7.2% p.a. over 20 years with weekly repayments
  - d** \$520 000 at 5.4% p.a. over 35 years with monthly repayments

### Problem solving

- 10** Compare the results of making the following savings at 9% p.a. over 20 years, stating reasons for any significant differences:
- Plan A: \$240 a month
  - Plan B: \$120 a fortnight
  - Plan C: \$60 a week
- 11** Katherine has bought a savings plan with payments of \$800 a year for 12 years at 15% p.a. The first payment is to be made immediately.
- a** How much can she expect to collect if fees are not deducted from the investment?
  - b** Katherine discovers that an agent's fee of 20% is deducted from all payments. Explain the effect of this on her savings plan.
- 12** What fortnightly contributions would be necessary to obtain a benefit of \$300 000 in 25 years time from a superannuation fund paying 11.6% p.a. interest with fees of 5% of contributions?
- 13** Simone and Jasson want to borrow \$750 000 over 30 years to buy a house. The current interest rate is 6.4% p.a.
- a** What fortnightly payment is required?
  - b** Explain what would happen if interest rates rose by 2%.
- 14** Peter has just been retrenched at age 48. He got a payment of \$295 000, including long service, holiday pay and the retrenchment payment negotiated by the union. His parents both lived to 75 and he expects to do the same. He can purchase an immediate annuity paying 10.5% p.a. interest.
- a** What fortnightly payment would he get?
  - b** What will the payment be worth in today's dollars when he is 75, assuming inflation averages 5%?
- 15** May has a choice of two loans. She can get a flat-rate interest loan at 11% p.a. or a reducing interest loan at 16% p.a. She needs to borrow \$32 000. Which is the best loan for May? Justify any assumptions you make.

## Geometric sequences and series

- A **geometric sequence** (or **geometric progression** [GP]) is a sequence with terms given by  $t_{n+1} = rt_n$ . The constant  $r = \frac{t_{n+1}}{t_n} = \frac{t_n}{t_{n-1}}$  is called the **common ratio**. The first term is often written as  $t_1 = a$ .
- The **general term** (***n*th term**) of a GP with first term  $a$  and common ratio  $r$  is given by  $t_n = t_1 r^{n-1} = ar^{n-1}$   
For a GP for which  $-1 < r < 1$ ,  $r \neq 0$ ,  $t_n \rightarrow 0$  as  $n \rightarrow \infty$ . It **converges** to 0.  
For a GP for which  $r > 1$  or  $r < -1$ , the sequence increases without limit. It **diverges**.
- The **growth factor** for **compound interest** and **appreciation** is  $(1 + i)$ , where  $i$  is the interest rate or appreciation rate. The value is given by  $A = P(1 + i)^n$ , where  $P$  is the initial value and  $n$  is the number of interest/appreciation periods (**rests**).
- The **growth factor** for **depreciation** is  $(1 - i)$ , where  $i$  is the rate of depreciation. The value is given by  $A = P(1 - i)^n$ , where  $P$  is the initial value and  $n$  is the number of depreciation periods.
- A **geometric series** (**sum of a GP**) is the sum of the terms of a geometric progression. It is given by  $S_n = \frac{t_1(r^n - 1)}{r - 1} = \frac{t_1(1 - r^n)}{1 - r}$  or  $S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}$ .
- For a GP with  $-1 < r < 1$ ,  $r \neq 0$ , the geometric series converges to  $S_\infty = \frac{t_1}{1 - r} = \frac{a}{1 - r}$ . This is called the **sum to infinity** of the GP.
- The **future value** of money is the amount it would grow to at compound interest.
- The **present value** is the amount invested now at compound interest to get a future sum.
- The **worth of money** (or **discount rate**) is the interest rate used in these calculations.
- $FV = PV(1 + i)^N$  or  $PV = FV(1 + i)^{-N}$ , where  $FV$  = future value,  $PV$  = present value,  $i$  = interest rate and  $N$  is the number of years.
- The increase in costs over time is called **inflation** and a decrease is called **deflation**.
- The Consumer Price Index (**CPI**) measures inflation.

- A mathematical **annuity** is a series of equal payments at regular intervals with compound interest.
- The **present value** of an annuity is the value at the start of the series of payments.
- The **future value** of an annuity is the value at the end of the series of payments.
- The **annuity factor** is the value of an annuity with payments of \$1.
- For an **ordinary annuity (immediate annuity)**, the payments are made at the end of the interest period. For an **annuity-due**, payments are made at the beginning of the interest period.
- For  $n$  payments of \$1 at interest rate  $i$ :

**Future value annuity factor,**  $S_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$

**Present value annuity factor,**  $a_{\overline{n}|i} = \frac{1 - (1+i)^{-n}}{i}$

Multiply by  $(1+i)$  to change an ordinary annuity to an annuity-due.

- Where there is no ambiguity, the annuity symbols are shown as  $S_{\overline{n}|}$  or  $a_{\overline{n}|}$ .
- **Reducing interest loans** are loans at compound interest, so the interest paid goes down over time.

# 6. CHAPTER REVIEW

## Geometric sequences and series

- 1 State whether or not each of the following are GPs. If they are, then find the common ratio.  
**a** 60, 15, 3.75, ...      **b**  $3, 6\sqrt{3}, 36, \dots$       **c** 5, -10, 15, ...
- 2 Find the first 4 terms of each of the following GPs.  
**a** First term 7, common ratio -3  
**b** Third term 20, common ratio  $5^{-\frac{1}{2}}$   
**c** First term  $m$ , common ratio  $\frac{n}{m}$
- 3 Find the sixth term of a GP with first term 5 and common ratio 2.
- 4 What is the first term of the GP 2, 9, 40.5, ... to exceed 3000?
- 5 Find the tenth term of a GP with 5th term -18 and seventh term -9.
- 6 An industrial drilling machine cost \$185 000 at the beginning of 2018. It depreciates by 12% each year. What is it worth at the beginning of 2025?
- 7 Find the sum to 10 terms of  $12 + 6 + 3 + \dots$ , correct to 3 decimal places.
- 8 How many terms of the GP 0.5, 1.5, 4.5, ... must be added to exceed 4000?
- 9 The sum to 5 terms of a GP with common ratio 2 is 217. Find the first and tenth terms.
- 10 What is the sum to infinity of the series  $36 - 12 + 4 - \dots$ ?
- 11 Convert  $0.\overline{163}$  to a fraction.
- 12 Find the future value of \$6000 in 5 years time, assuming that money is worth 7% p.a.
- 13 Find the present value of \$25 000 due in 4 years time at a rate of 12% p.a.

Example  
1

Example  
2

Example  
3

Example  
3

Example  
4

Example  
5

Example  
6

Example  
7

Example  
8

Example  
9

Example  
10

Example  
11

Example  
12

Example  
13

**14** The inflation rate in Australia was an average of 9.75% p.a. during the 1970s. It was so high, the \$50 note was issued in 1973. What would a 1980 \$50 note have been worth in 1970 dollars?

Example  
14

**15** Tania invested \$4000 at the end of each year over 4 years at 8% p.a. interest. How much did she have at the end of the 4 years?

Example  
15

**16** Calculate the value of each annuity, correct to 4 decimal places.

**a**  $S_{\overline{8}|11\%}$

**b**  $a_{\overline{20}|6\%}$

Example  
15

**17** How much would you have to pay into a superannuation fund each year to get \$600 000 at 8% p.a. over 15 years?

Example  
16

**18** How much could you borrow on a reducing interest loan over 20 years at 7.2% p.a. interest with repayments of \$1500 a month?

### Problem solving

**19** A car bought for \$64 990 loses the same proportion of its value each year. It is worth only \$27 180 after 5 years. What was its value when it was 2 years old?

**20** How long will it take for \$6000 to double in value at 9.5% p.a. compound interest?

**21** A trapeze artist hanging from a trapeze is pushed by another artist and swings 0.4 m away. Every time she returns to the extreme of her swing she is pushed again. She lets go when she reaches the end of the swing from the 6th push to fly to another trapeze. She travels 20% extra away from the starting position on each push. How far (correct to 2 decimal places) does she swing altogether before letting go?



Getty Images/The Washington Post

**22** Zoe and Taj are playing on a swing in the park. Zoe is pushing Taj and on the first push he swings up 600 mm from the starting height. Every subsequent push is only 40% as effective as the previous one because he is swung back further and further. What is the sum of the heights that Taj swings up?

**23** Jan and Terry, aged 32 and 33 respectively, have a combined monthly income of \$10 800 after tax. They are currently making monthly payments on car loans of \$730, have no other debts and have saved \$84 000 towards buying a house. Their mortgage broker says they should allow \$15 000 for legal fees on buying a house. They are willing to spend up to 50% of their current income on payments. Work out how much they can afford to pay for a house at an interest rate of 6% p.a. Indicate any assumptions you make and any problems they might have.



Practice quiz

## Practice examination 2 ●●○○

**Time: 90 minutes**  
**Perusal time: 5 minutes**  
**Marks: 50**

### Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**



Year 11  
formulas

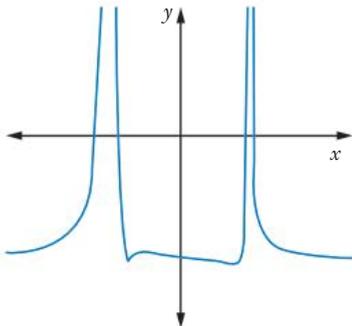
### Question 1 (2 marks)

Find the centre and radius of the graph of  $x^2 + y^2 - 6x + 4 = 3$ .  
State its domain and range.

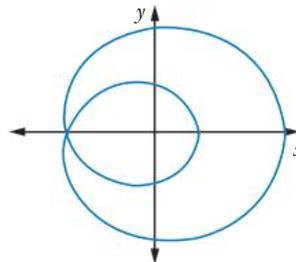
### Question 2 (1 mark)

State which of these graphs represent a function, giving reasons.

**A**



**B**



**Question 3** (2 marks)

Find the coordinates of the turning point on the graph of each equation, and state whether the one in part **a** is a maximum or minimum point.

**a**  $f(x) = (5 - x)^2 + 2$

**b**  $x = 3 - (y - 1)^2$

**Question 4** (6 marks) (Technology-active)

Sketch the graph of each equation. Label all important features.

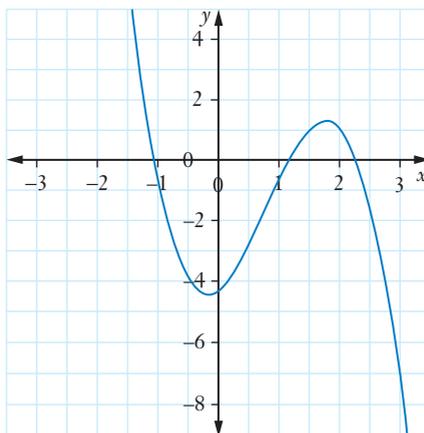
**a**  $y = \frac{1}{x-3} + 2$

**b**  $y = (x + 2)^4 - 3$

**c**  $y = 2(x - 1)(2 - x)(2 + x)$

**Question 5** (2 marks)

Copy this graph of  $f(x)$ . Sketch the graph of  $2f(x - 2) + 3$  on the same scale.

**Question 6** (1 mark)

Part of a row of Pascal's triangle is shown below.

... 105 455 1365 3003 ...

... X ...

What number should be placed in the location marked X?

**Question 7** (2 marks)

- a What is the value of  $\binom{12}{9}$ ?
- b In how many ways can you select 5 different objects from 8 objects?

**Question 8** (2 marks)

A bucket with 5 white and 6 yellow tennis balls is shaken up and 2 balls are taken out. What is the probability that both are yellow?

**Question 9** (1 mark) (Technology-active)

What are the first 3 terms of the expansion of  $(x + y)^{16}$ ?

**Question 10** (2 marks) (Technology-active)

A container has 20 beads, including 8 pearl beads and 6 clear beads. What is the probability of getting a pearl bead and then a clear bead when two beads are shaken out?

**Question 11** (1 mark)

What is the probability of getting 4 fours when 4 normal dice are rolled?

**Question 12** (2 marks)

The table below shows the self-reported weights of Year 11 students.

	Underweight	Average	Overweight
Male	8	14	5
Female	6	16	9

- a What is  $P(\text{thinks underweight} \mid \text{female})$ ?
- b What is  $P(\text{thinks underweight} \mid \text{male})$ ?

**Question 13** (2 marks)

- a Find the next 3 terms of 1, 3, 9, 27, ...
- b Find the 12th term of the geometrical sequence with first term 0.025 and common ratio  $-2$ .

**Question 14** (2 marks)

A geometric progression is given by 108,  $-36$ , 12, ...

- a Find the sum to 6 terms.
- b Find the sum to infinity.

**Question 15** (2 marks) (Technology-active)

- a Julie invests \$5000 when money is worth 7% p.a. What will its future value be (correct to the nearest cent) after 10 years?
- b It is anticipated that inflation will be about 2.5% p.a. for the foreseeable future. What will the future value of her investment be in today's dollars?

**Question 16** (2 marks) (Technology-active)

Sketch the graph of  $y = 12x - 2x^2 - 21$ , showing all important features.

**Question 17** (2 marks)

High-quality baker's flour costs \$1.30/kg for up to 10 kg. Over 10 kg, the price drops to \$1.10/kg and over 100 kg it is \$0.95/kg.

Construct a function for the price for  $x$  kg of the flour.

**Question 18** (2 marks)

What is the probability of 3 women and 2 men being randomly chosen for a focus group from a panel of 6 men and 6 women?

**Question 19** (2 marks) (Technology-active)

One thousand people were asked to state their favourite team in the NRL competition. The results are shown below.

	Queensland team	Interstate team
Born in Queensland	404	254
Born elsewhere	94	198

Is support for a Queensland team independent of birthplace? Justify your answer.

**Question 20** (2 marks) (Technology-active)

How much could you borrow at 6% p.a. compound interest if you could afford to pay \$1700 a month over 25 years?

**Question 21** (4 marks)

The yield of a commercial crop over a long period was  $y = \frac{7.6x + 7}{x + 2}$  tonne/ha for  $x$  kg/ha of fertiliser.

- a** Rearrange the equation to the form  $y = A + \frac{B}{x + C}$  to find the maximum possible yield.
- b** What fertiliser will give a yield of 6 t/ha?

**Question 22** (4 marks)

A tourist operator takes groups of 8 people snorkelling at a coral reef.

In 2 hours, at least one of the people will see an octopus emerge.

Find the probability that you will see 2 octopuses if you snorkel for 4 hours.

Justify your answer.

**Question 23** (2 marks)

The sum of a geometric progression to 5 terms is 976.

The sum to infinity is 972.

What is the 7th term?

**END OF EXAMINATION**

# 7.

## RATES OF CHANGE AND THE DERIVATIVE

You probably already know lots about how things grow and decay. Calculus allows you to construct relatively simple mathematical models of change. You can use these models to work out the ways things change.

The applications of calculus include computer-generated graphics (movies), geography (Google Earth), network visualisation (social networks like Facebook), computer vision (driverless cars), robotics, computer games, photography (panorama photographs) and artificial intelligence (optical character recognition).

- 7.01 Average rates of change
- 7.02 Instantaneous rates of change
- 7.03 Difference quotients
- 7.04 Introduction to limits
- 7.05 The derivative
- 7.06 Interpreting the derivative as a rate of change
- 7.07 The slope of a curve
- Chapter summary
- Chapter review

## SYLLABUS SUBJECT MATTER

### Rates of change and the concept of derivatives

- explore average and instantaneous rate of change in a variety of practical contexts
- use a numerical technique to estimate a limit or an average rate of change
- examine the behaviour of the difference quotient  $\frac{f(x+h) - f(x)}{h}$  as  $h \rightarrow 0$  as an informal introduction to the concept of a limit
- differentiate simple power and polynomial functions from first principles
- interpret the derivative as the instantaneous rate of change
- interpret the derivative as the gradient of a tangent line of the graph of  $y = f(x)$

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Prior learning

# TERMINOLOGY

acceleration  
chord  
derivative  
first principles  
independent variable  
rate  
slope of a curve

average rate of change  
decreasing  
difference quotient  
gradient  
instantaneous rate of change  
rate of change  
tangent

average speed  
dependent variable  
displacement  
increasing  
limit of a function  
secant  
velocity



Rates of change

## 7.01 Average rates of change

A **rate** is the ratio between two related quantities.

Rates often involve something changing per unit of time, such as speed, which is the ratio of distance to time, measured in km/h.



Rates of change -  
Gradient of secants

A **rate of change** is similar to a rate, but instead of using the quantities themselves, it is their changes that are important. The symbol  $\Delta$  (Greek capital letter delta) is used to represent the change in a quantity.

### Rate of change

The rate of change of a second quantity with respect to a first quantity is the quotient of their changes:

$$\text{Rate of change} = \frac{\text{change in quantity 2}}{\text{change in quantity 1}}$$

If the quantities are symbolised by  $x$  and  $y$ , then you can write this as:

$$\text{Rate of change of } y \text{ with respect to } x = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

'with respect to' is commonly abbreviated to 'wrt'.

You know that for the graph of a linear function  $y = mx + c$ , the gradient,  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

This means that the gradient of a linear function is the rate of change of the dependent variable wrt the independent variable.

## EXAMPLE 1

The volume of dry air measured at the same pressure, but different temperatures, is shown below.

Temperature ( $^{\circ}\text{C}$ )	10	40	90	150	190
Volume (mL)	415	459	532	620	678

- Calculate the rate of change of volume wrt temperature for each interval and comment on the results.
- Draw a graph to show the volume and temperature.

### Solution

- The rate of change can be calculated between successive pairs of measurements by extending the table as follows.

Temperature, $T$ ( $^{\circ}\text{C}$ )	10	40	90	150	190
Volume, $V$ (mL)	415	459	532	620	678
$\Delta V$		44	73	88	58
$\Delta T$		30	50	60	40
Rate of change (mL/ $^{\circ}\text{C}$ )		1.47	1.46	1.47	1.45

State the rate of change.

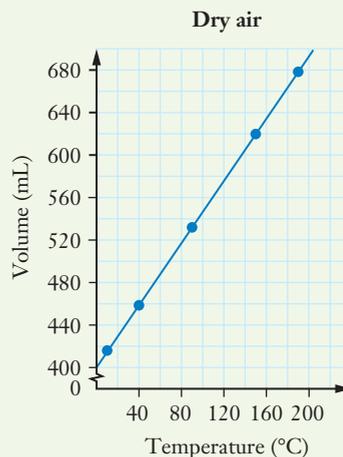
The rates of change for the intervals given are:

1.47, 1.46, 1.47 and 1.45.

Comment on the results.

The rates of change are all approximately the same. The differences could be due to experimental error or rounding errors.

- Plot the points and draw in a line of best fit.





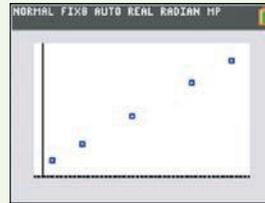
You could use a graphics calculator to draw the graph.

### TI-84 Plus CE

Press  $\boxed{\text{stat}}$  and select **1: Edit** and press  $\boxed{\text{enter}}$ . Then enter the temperature data into  $L_1$  and the volume data into  $L_2$ .



Press  $\boxed{2\text{nd}}$   $\boxed{y=}$  [stat plot] and press  $\boxed{\text{enter}}$ . Turn Plot 1 on.



Press  $\boxed{\text{zoom}}$ , then select **9: ZoonStat** to display the scatter plot of the data.

You can calculate the equation of the line of best fit or regression line for the data using the statistics function of the calculator.



Press  $\boxed{\text{stat}}$ , arrow across to **CALC**, select **4: LinReg(ax+b)**.

Set the Xlist to  $L_1$  and the Ylist to  $L_2$  if needed.

Arrow down to **Calculate** and press  $\boxed{\text{enter}}$  to display the equation of the line of best fit. The line of best fit is in the form  $y = ax + b$ .

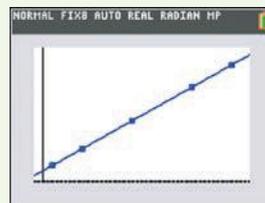
Press  $\boxed{y=}$  and enter the equation for the line of best fit as:



$$Y_1 = 1.4617X + 400.5$$

and press  $\boxed{\text{enter}}$ .

Press  $\boxed{\text{graph}}$  to display the scatter plot and the line of best fit.



## Casio fx-CG20AU

Select the **Statistics** page from the **Main Menu**.

Name List 1 'Temp' and List 2 'Vol'.

Enter the values for temperature and volume from the table of values.

Press **F6** (SET) and set the **Graph Type** to **Scatter**.

Return to the lists and press **F1** (GRAPH) to display the scatter plot.

You can calculate the equation of the line of best fit or regression line for the data using the statistics function of the calculator.

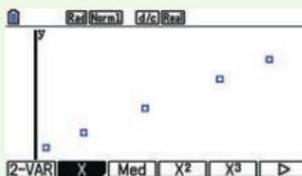
Press **F1** (CALC), **F2** (X) and then **F1** ( $ax + b$ ) to display the details of the line of best fit for the data. The line of best fit is in the form  $y = ax + b$ .

Press **F6** (DRAW) to display the scatter plot with the line of best fit.

	List 1	List 2	List 3	List 4
SUB	Temp	Vol		
1	10	415		
2	40	450		
3	90	532		
4	150	620		

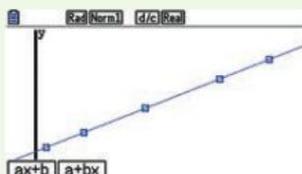
415

GRAPH CALC TEST INTR DIST >



LinearReg(ax+b)  
a = 1.46173835  
b = 400.473118  
r = 0.99999869  
r<sup>2</sup> = 0.99999738  
MSe = 0.04151732  
y = ax + b

COPY DRAW



## Both calculators

The gradient of the line of best fit (1.4617 ...) is the rate of change of volume wrt temperature.

In the previous example, the rate of change is **constant** (within experimental error), but most rates of change are not constant. In these cases, the **average rate of change** is calculated.

## Average rate of change

The average rate of change of a variable with respect to another variable between two points is given by:

$$\text{Average rate of change} = \frac{\text{change in independent variable}}{\text{corresponding change in dependent variable}}$$

If the independent variable is  $x$  and the dependent variable is  $y$ , then:

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

If no other variable is indicated, then you assume the rate of change of a variable is with respect to time.

### Rates of change by time

**Displacement** is the change in position from the starting point to the end point without regard for the path travelled.

**Speed** is the rate of change of the distance of a moving object.

**Velocity** is the rate of change of the displacement of a moving object.

**Acceleration** is the rate of change of the velocity of a moving object. (Displacement, velocity and acceleration are scientifically considered to have a direction as well as a measure.)

**Flow rate** measures the speed at which a fluid travels and is the rate of change of volume.

This meaning of ‘average’ also applies to rates of change with special names. Average speed does not mean the average of speeds.

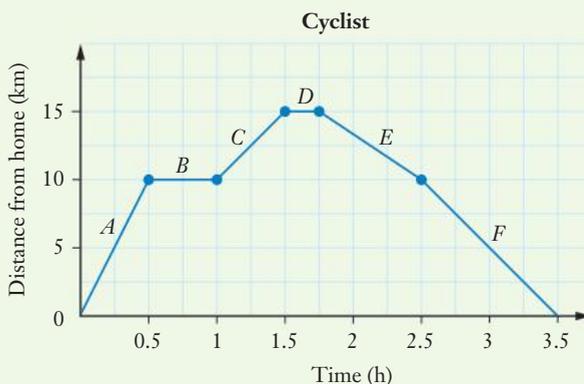
$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

$$\text{Average velocity} = \frac{\text{total displacement}}{\text{total time taken}}$$

### EXAMPLE 2

The graph shows the distance of a cyclist from her home at various times on a trip to a shop 15 km away and back again.

- When was she moving at the greatest speed?
- What was her average speed going to the shop?
- How long did she stop at the shop?
- What was her average speed on section *E*?
- What was her average speed for the whole trip?
- What was her average velocity for the whole trip?



## Solution

**a** The gradient of a distance–time graph represents speed. The gradient is steepest on section *A*.

She was moving at greatest speed on section *A*.

**b** Her speed was not constant. It took her a total of  $1\frac{1}{2}$  hours to go the 15 km to the shop.

Write the formula for average speed.

$$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$$

Substitute for distance and time.

$$= \frac{15 \text{ km}}{1.5 \text{ h}}$$

Evaluate.

$$= 10 \text{ km/h}$$

**c** Section *D* of the graph represents the period of time for which she stopped at the shop.

She stopped for a quarter of an hour.

**d** On section *E* she travelled 5 km in three-quarters of an hour.

$$\text{Average speed} = \frac{5 \text{ km}}{0.75 \text{ h}}$$

Evaluate and round off.

$$\approx 6.7 \text{ km/h}$$

**e** The whole trip was 30 km and took her 3.5 hours. Substitute for distance and time.

$$\text{Average speed} = \frac{30 \text{ km}}{3.5 \text{ h}}$$

Evaluate and round off.

$$\approx 8.6 \text{ km/h}$$

**f** Write the formula for average velocity.

$$\text{Average velocity} = \frac{\text{total displacement}}{\text{total time}}$$

She finished her trip where she started, so her displacement is 0 km.

$$= \frac{0 \text{ km}}{3.5 \text{ h}}$$

$$= 0$$



iStock.com/RyanJlan

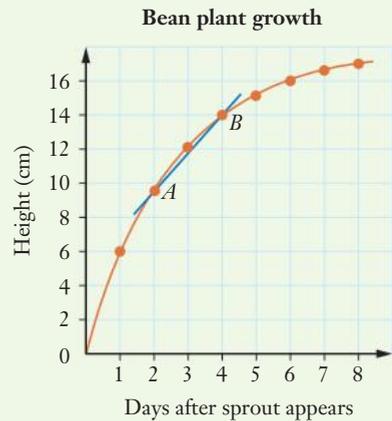
You can use a graph to calculate rates of change, even when the graph is a curve.

### EXAMPLE 3

The graph shows the average height of bean plants measured as they grow. Use the graph to calculate the average rate of growth for:

- a the first 2 days
- b the next 2 days
- c days 5 and 6.

State how the rate of growth changes.



### Solution

- a Write the formula.

$$\text{Average rate of growth} = \frac{\text{change in height}}{\text{change in time}}$$

Read the information from the graph and substitute into the formula.

$$= \frac{9.5 \text{ cm} - 0 \text{ cm}}{2 \text{ days} - 0 \text{ days}}$$

Evaluate and round off.

$$\approx 4.8 \text{ cm/day}$$

- b Substitute the information for the second 2 days.

$$\text{Average rate of growth} = \frac{14 \text{ cm} - 9.5 \text{ cm}}{4 \text{ days} - 2 \text{ days}}$$

Evaluate and round off.

$$\approx 2.3 \text{ cm/day}$$

- c Substitute the information for days 5 and 6.

$$\text{Average rate of growth} = \frac{16 \text{ cm} - 15 \text{ cm}}{6 \text{ days} - 5 \text{ days}}$$

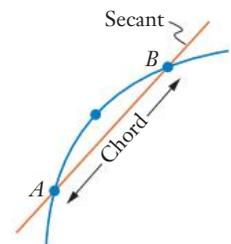
Evaluate and round off.

$$= 1 \text{ cm/day}$$

Comment on the growth rate.

The plant height increases at a decreasing rate, so the rate of growth is decreasing.

The line  $AB$  marked on the graph in Example 3 corresponds to the second 2 days. It is called a **secant**. The line segment from  $A$  to  $B$  is called the **chord**  $AB$ . The gradient of  $AB$  is the average rate of growth for these 2 days. (The diagram at right is exaggerated to show the chord and secant more clearly.)



## INVESTIGATION

### POPULATION PREDICTIONS

An estimate of the population of the world is given in the table below.

#### World population (millions)

Year	1000	1200	1400	1500	1600	1700	1750	1800	1850	1900	1950	2000
Population	350	435	465	475	490	630	800	900	1300	1600	2500	6100

- 1 Draw a smooth curve by hand to illustrate this data.
- 2 Work out the average rate of increase for each century.
- 3 Estimate the rate of increase for 2000–2100.
- 4 Estimate the likely population in the year 2050 and justify any assumptions you make.



Shutterstock.com/Awhimedes

## Exercise 7.01 Average rates of change

- 1 Express each statement as a rate.
- a It costs \$36.75 for 15 kg of potatoes.
  - b A wheel turns 180 times in 4 minutes.
  - c A car travels 3000 m in 100 s.
  - d An acceleration of  $50 \text{ m/s}^2$  is produced by a force of 8 N (newtons).
  - e AU\$800 is worth the same as US\$640.
- 2
- a An insecticide is diluted by placing 8 mL in a 20 L bucket of water. What is the rate of dilution, and how much diluted insecticide will a 100 mL bottle make?
  - b A two-stroke fuel mixture is made by adding petrol to oil at the rate of 50 L/L. How much oil must be added to 4 L of petrol?
  - c The rate at which a dripping tap uses water is about 0.2 L/h. How much water would be wasted in a month?
  - d The germination rate for parsnip seed is about 35 plants/100 seeds. How many seeds are required to produce 500 plants?
  - e Bricks are used at a rate of 53 bricks/ $\text{m}^2$  to build a wall. How many bricks are needed for a wall 15 m long and 2.4 m high?

Example  
1

- 3 Use a numerical method to determine whether the quantities in the table are related by a fixed rate.

$m$	10	20	50	100	250
$n$	1.5	3	7.5	15	37.5

- 4 Use a graphical method to determine whether the quantities in the table are related by a fixed rate.

$r$	0.125	0.5	1.25	1.875	2.25
$D$	0.5	2.0	5.0	7.5	9.0

- 5 This table shows how the length of a spring changes as different masses are hung on it.

Mass (g)	30	50	80	120	140	150
Length (cm)	15	16.8	18	19.8	22.2	23.4

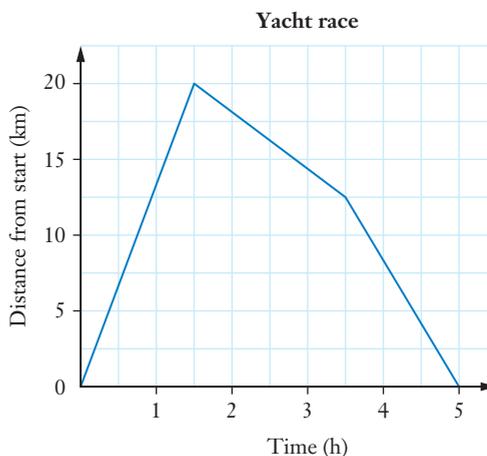
- a Draw a graph to show the information.
- b Comment on the shape of the graph.
- c What is the average rate of change of length with respect to mass, correct to 2 decimal places?
- d What is the length for a mass of 0 g, and what does it represent?

- 6** An air-conditioning system in a car has a small leak. The temperature in the car, with the air-conditioning on full, gradually increases as shown below. During this time, the external temperature remains at about  $28^{\circ}\text{C}$ .

Time	11 a.m.	11:20 a.m.	11:40 a.m.	12 p.m.	12:20 p.m.	12:40 p.m.	1 p.m.	1:20 p.m.
Temp ( $^{\circ}\text{C}$ )	18.0	19.8	21.3	22.5	23.5	24.3	25.0	25.5

- Draw a graph.
  - Comment on the rate of change of the temperature in the car.
  - What is the average rate of change of temperature during this time?
  - Will the air-conditioning be of any use at 3 p.m.?
  - Will it be of any use the next day?
- 7** The Moon orbits at an average distance of 383 000 km from the centre of the Earth. The Moon takes 27.3 days to complete an orbit. The orbit is in the same direction as the revolution of the Earth round the Sun.
- Find how far the Moon travels in 24 hours, to the nearest kilometre.
  - What is the speed of the Moon in its orbit, to the nearest km/h?
  - To an observer on Earth, why does the Moon appear to take 28 days to orbit?

- 8** The graph shows the distance of a yacht from the starting line in a race with 3 legs.



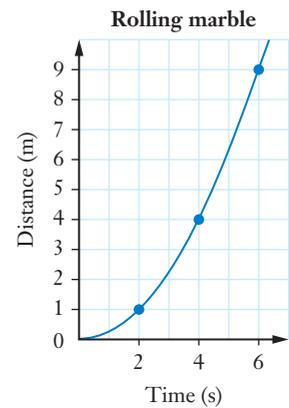
Example  
**2**

- What is the fastest speed shown on the graph?
  - On which leg would the spinnaker be most likely to be used (the fastest leg, with the wind behind)?
  - Which leg is the beat to windward (the slowest leg, with the most tacking needed)?
  - What is the average speed of the yacht shown on the graph?
- 9** A map shows the distance and time that it should take to drive a car between Bundaberg and Mackay via Gladstone and Rockhampton. The distance is 334 km and the time given is 3 hours 40 minutes. What average driving speed has been assumed, to the nearest km/h?
- 10** A car travelling at 30 m/s increases its speed to 50 m/s in 5 s. What is the average rate of change of speed?

- 11** The distance a marble rolls down a long plank at an angle of  $5^\circ$  is shown in the graph.

Find the average rate of change of distance (speed) from:

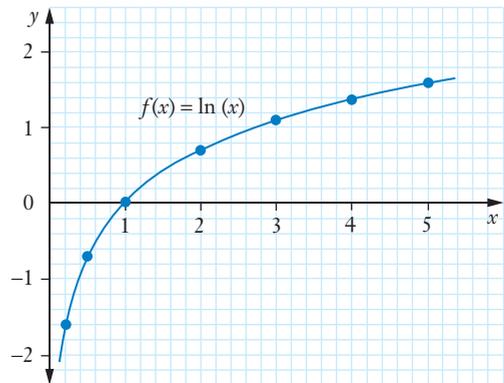
- a** 0 s to 2 s
- b** 2 s to 4 s
- c** 4 s to 6 s
- d** 0 s to 6 s



- 12** The symbol  $\ln(x)$  refers to the natural logarithm of  $x$ . You will learn about logarithms in Chapter 8 of this text.  $f(x) = \ln(x)$  is drawn on the right.

Use the graph to find the average rate of change of  $\ln(x)$  from:

- a**  $x = 0.5$  to  $x = 3$
- b**  $x = 1$  to  $x = 4$
- c**  $x = 1$  to  $x = 5$
- d**  $x = 0.2$  to  $x = 3$



### Problem solving

- 13** The following information was obtained by changing the amount of watering of young tomato plants and observing their growth over 3 weeks. Use a graph to determine whether growth is related to water by a constant rate.

<b>Water per day (mL)</b>	20	50	100	200	300
<b>Growth (cm)</b>	5	20	35	60	65

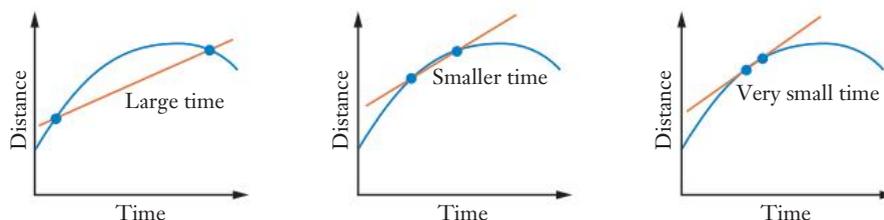
- 14** A group of students made a simple balance with a pan on one side of the fulcrum and a sliding weight on the other. They put different masses in the pan and moved the sliding weight to make it balance. The distance between the sliding weight and the fulcrum was measured. Their results are given in the table. Determine whether these measurements are consistent with a constant rate relationship.

<b>Mass in pan (g)</b>	50	100	200	250	400
<b>Distance (cm)</b>	5	12	27	29	53

## 7.02 Instantaneous rates of change

A car's speedometer ('speedo') gives the speed at an instant, without having to worry about averages. You can use chords to find the average speed on a graph of distance and time.

As more chords are used, the shape of the graph made by the chords gets closer to the shape of the actual graph. If you reduce the change to an instant, you get exactly the shape of the graph. The average speeds also become instantaneous speeds.



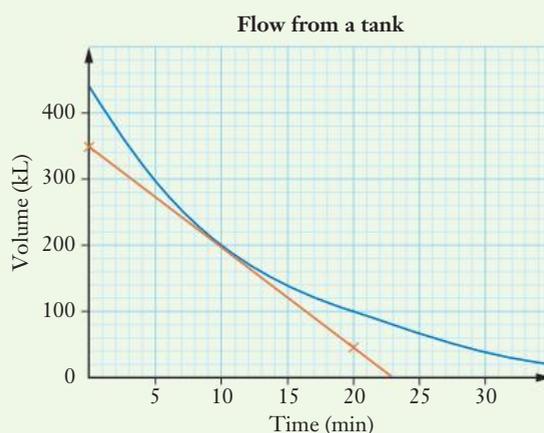
As you reduce the change in time, the secant becomes more like a tangent, which touches the curve at just one point. The slope of a secant (or chord) gives the average speed, so the gradient of a tangent gives the **instantaneous speed**. This applies to any rate of change.

### Instantaneous rate of change

The **instantaneous rate of change** of a variable at a point is the same as the **gradient of the tangent** to the graph at that point.

### EXAMPLE 4

A valve on a tank of water is opened to drain its contents. Use the tangent to the graph shown here to find the flow rate 10 minutes after the valve is opened.



## Solution

Write the formula for the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the relevant values of  $x$  and  $y$ .

$$= \frac{350 \text{ kL} - 45 \text{ kL}}{0 \text{ min} - 20 \text{ min}}$$

Evaluate.

$$\approx -15 \text{ kL/min}$$

State the result.

The flow rate at 10 minutes is the same as the gradient at 10 minutes, so the flow rate is about  $-15 \text{ kL/min}$ .

*Note:* The negative sign means that water is flowing *out* of the tank.

## EXAMPLE 5

Plot the graph of  $y = x^2 - 5$  from  $x = -3$  to  $x = 3$ , then use the graph to find the rate of change of  $y$  wrt  $x$  at  $x = 2$ .

## Solution

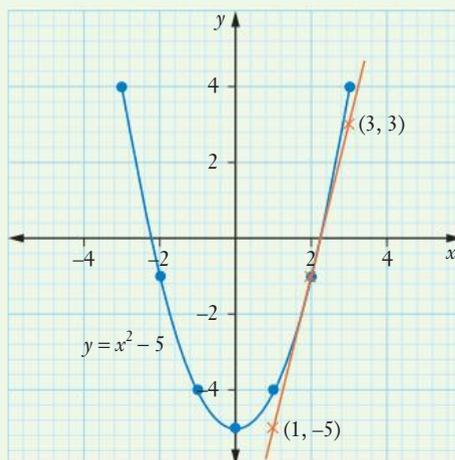
You can use a table of values to plot the graph and draw the tangent at  $x = 2$ .

$x$	-3	-2	-1	0	1	2	3
$y$	4	-1	-4	-5	-4	-1	4

Plot the points and join them with a smooth curve.

Draw in the tangent at  $(2, -1)$ .

From the graph, you can see that the tangent passes through  $(1, -5)$ ,  $(2, -1)$  and  $(3, 3)$ .



Write the formula for the gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substitute the relevant values using points (1, -5) and (3, 3).

$$= \frac{3 - (-5)}{3 - 1}$$

Evaluate.

$$= 4$$

State the result.

The gradient of the graph at  $x = 2$  is 4, so the rate of change of  $y$  wrt  $x$  at  $x = 2$  is 4.

The rate of change for a linear function such as  $y = 2x - 5$  is constant. So the average rate of change and the instantaneous rate of change are the same for a linear function.

To find the instantaneous rate of change for a non-linear function, you need to draw a tangent and calculate the gradient of the tangent. This only gives approximate answers. A graphics calculator can improve the accuracy.

## EXAMPLE 6

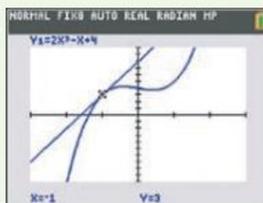
Use a graphics calculator to draw a tangent to  $f(x) = 2x^3 - x + 4$  at  $x = -1$  and find the rate of change of  $y$  wrt  $x$  at that point.

### Solution

Select the **Graph** menu and enter the function into Y1. Graph the function and set appropriate limits for  $x$  and  $y$  in the **WINDOW** (or **V-Window**), e.g.  $-3 \leq x \leq 3$  and  $-10 \leq y \leq 10$ .

### TI-84 Plus CE

Press  $\boxed{2\text{nd}} \boxed{\text{prgm}} \boxed{\text{draw}}$  and then select **5:Tangent**(. Use the arrow keys to position the cross at  $x = -1$ . Press  $\boxed{\text{enter}}$ .



From the display you can see that the tangent passes through  $(-1, 3)$ . Select another convenient point, say  $(0, 8)$  on the tangent.

Calculate the gradient of the tangent.

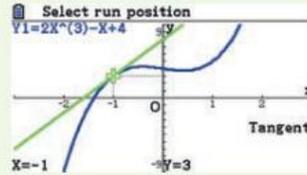
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 3}{0 - (-1)} \\ &= 5 \end{aligned}$$



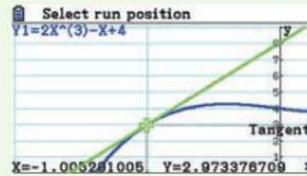
TI-Nspire CX  
Chapter 7

### Casio fx-CG20AU

Press **F4** (Sketch) and then **F2** (tangent).  
Then select  $x = -1$  as the 'run position'.



Now zoom in to display the graph in the 3rd quadrant of the Cartesian plane. You can do this by pressing **F2** (Zoom) or adjusting the V-Window settings.



Locate any two convenient points on the tangent and use the  $x$ - and  $y$ -coordinates to calculate the gradient of the tangent. For example, the points  $(-1, 3)$  and  $(0, 8)$  could be used.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{8 - 3}{0 - (-1)} \\ &= 5 \end{aligned}$$

Calculate the gradient of the tangent.

#### Both calculators

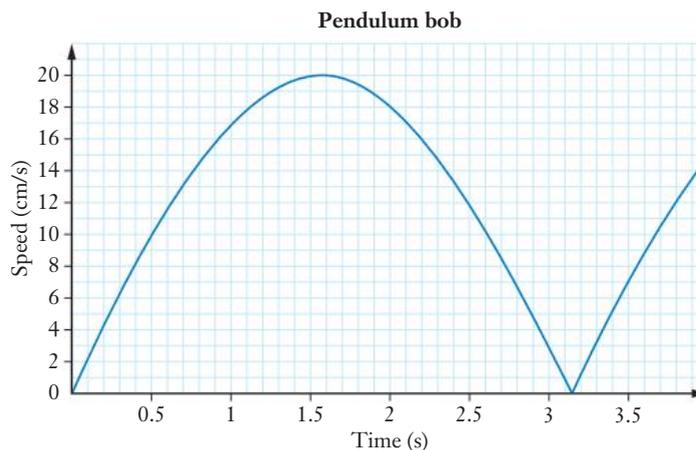
State the result.

The rate of change of  $y$  wrt  $x$  at  $x = -1$  is 5.

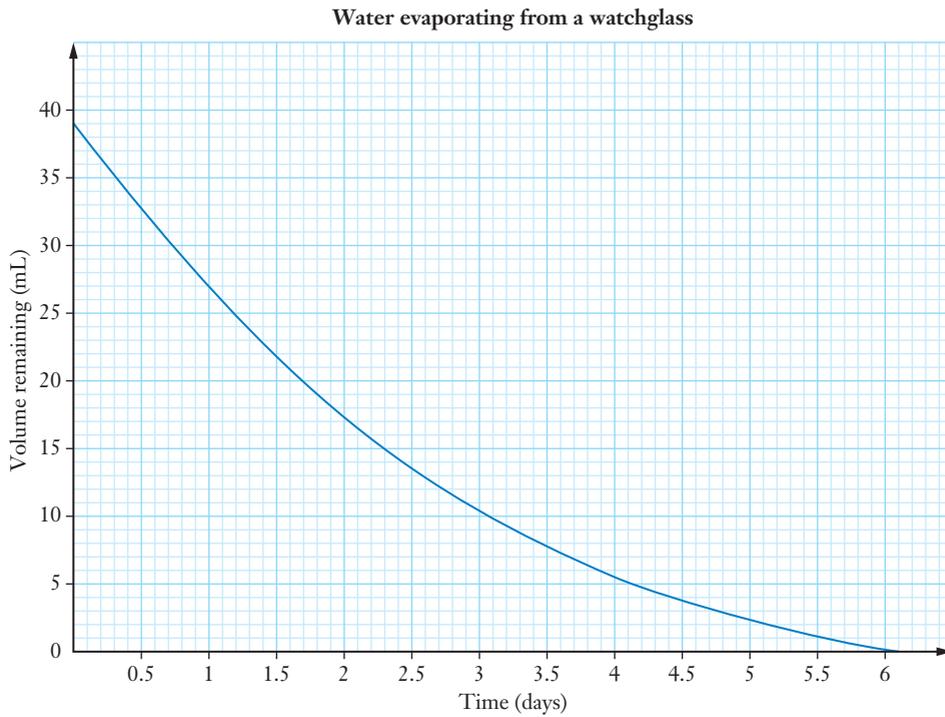
## Exercise 7.02 Instantaneous rates of change

Example  
4

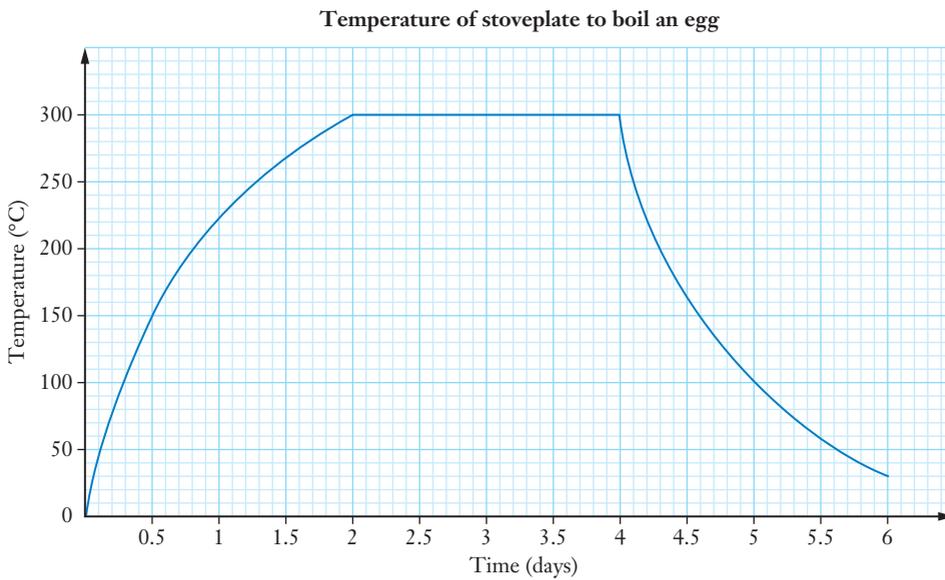
- The following graph shows the speed of the bob of a long pendulum. Use tangents to find the rate of change of speed of the bob at 1 s, 2 s and 3 s.



- 2** The amount of water left in a watchglass during evaporation is shown below. Use tangents to find the rate of evaporation at 1 day and 4 days.



- 3** The temperature of a stove plate used to boil an egg is shown below. Use tangents to find the rate of change of temperature at 1 minute, 3 minutes and 5 minutes.



Example  
**5**

- 4** What is the slope of the graph  $f(x) = 3 - 2x$ ? What is the slope of the tangent at  $x = 5$ ? What is the slope of the tangent at other points?
- 5** Plot the graph of  $f(x) = 2x^2 + x - 4$  for  $-2 \leq x \leq 3$ . Use a tangent to find the gradient at:
- a**  $x = -1$                       **b**  $x = 1$                       **c**  $x = 2$
- 6** Plot the graph of  $f(x) = 5 - 3x - x^2$  for  $-3 \leq x \leq 4$ . Use a tangent to find the gradient at:
- a**  $x = -2$                       **b**  $x = 0$                       **c**  $x = 3$

Example  
**6**

- 7** Use a graphics calculator to draw the graph of  $f(x) = x^3 - 6$  for  $-2 \leq x \leq 2$ . Use a tangent to find the gradient at:
- a**  $x = -1$                       **b**  $x = 0$                       **c**  $x = 1$
- 8** Use a graphics calculator to draw the graph of  $f(x) = (x + 4)(x - 3)$  for  $-5 \leq x \leq 4$ . Use a tangent to find the gradient at:
- a**  $x = -3$                       **b**  $x = -1$                       **c**  $x = 1$

### Problem solving

- 9** The distance travelled by a drag-racing car in the first seconds after taking off is given by the equation  $d = 5t - t^3 + 8t^2$ , where  $d$  is in metres and  $t$  is in seconds. Use a graph to find the velocity at 2 seconds and the time taken to travel 100 m. What is the velocity at 100 m?



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- 10** What is the rate of change of  $y = x^2 - 2x + 3$  at  $x = 4$ ?

## 7.03 Difference quotients

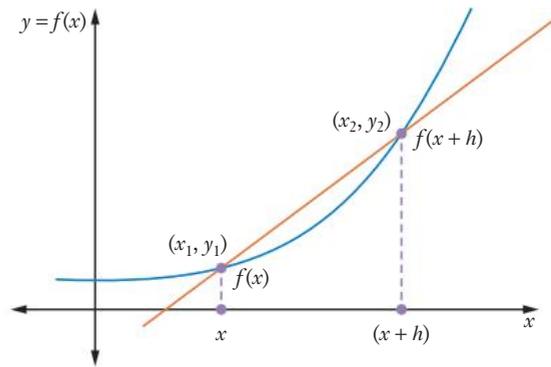
Consider the graph of the function shown.

Here,  $y = f(x)$ , so  $x_1 = x$ ,  $x_2 = (x + h)$ ,  
 $y_1 = f(x)$  and  $y_2 = f(x + h)$ .

Average rate of change of  $y$  wrt  $x = \frac{y_2 - y_1}{x_2 - x_1}$ .

So, average rate of change of  $f(x)$

wrt  $x = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h}$ .



### Difference quotient

A **difference quotient** is an expression of the form:

$$\frac{f(x + h) - f(x)}{h}$$

A difference quotient represents the average change in the value of a function between  $x$  and  $(x + h)$ .

### EXAMPLE 7

Calculate the difference quotient for the function  $f(x) = 5x - 3$ .

#### Solution

Substitute into  $f(x + h) - f(x)$ .

$$f(x + h) - f(x) = f[5(x + h) - 3] - (5x - 3)$$

Evaluate.

$$\begin{aligned} &= 5x + 5h - 3 - 5x + 3 \\ &= 5h \end{aligned}$$

Substitute into the difference quotient expression.

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{5h}{h} \\ &= 5 \end{aligned}$$

In general, you should be able to cancel out the  $h$  in the denominator of a difference quotient. If you can't, then you may have made an error.



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You could represent the arch of the Sydney Harbour Bridge as a function. What would happen to the difference quotient of the function as you moved from one side of the bridge to the other?

### EXAMPLE 8

Calculate the difference quotient for the function  $f(x) = 3x^2 + 4x - 3$ .

#### Solution

Calculate  $f(x + h)$ .

$$f(x + h) = 3(x + h)^2 + 4(x + h) - 3$$

Expand and simplify.

$$= 3(x^2 + 2xh + h^2) + 4x + 4h - 3$$

$$= 3x^2 + 6xh + 3h^2 + 4x + 4h - 3$$

Calculate  $f(x + h) - f(x)$ .

$$f(x + h) - f(x) = 3x^2 + 6xh + 3h^2 + 4x + 4h - 3 - (3x^2 + 4x - 3)$$

Expand and simplify.

$$= 6xh + 3h^2 + 4h$$

Substitute into the difference quotient expression.

$$\begin{aligned} \frac{f(x + h) - f(x)}{h} &= \frac{6xh + 3h^2 + 4h}{h} \\ &= \frac{h(6x + 3h + 4)}{h} \\ &= 6x + 3h + 4 \end{aligned}$$

**EXAMPLE 9**

← This example has more complex algebra.

Calculate the difference quotient for  $f(x) = \frac{1}{x+2}$ .

**Solution**

Calculate  $f(x+h)$ . 
$$f(x+h) = \frac{1}{x+h+2}$$

Calculate  $f(x+h) - f(x)$ . 
$$f(x+h) - f(x) = \frac{1}{x+h+2} - \frac{1}{x+2}$$

Substitute into the difference quotient expression. 
$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$$

Use the common denominator  $(x+h+2)(x+2)$  to simplify the numerator.

$$= \frac{\frac{x+2}{(x+h+2)(x+2)} - \frac{x+h+2}{(x+h+2)(x+2)}}{h}$$

Simplify.

$$= \frac{\frac{x+2-x-h-2}{(x+h+2)(x+2)}}{h}$$

$$= \frac{\frac{-h}{(x+h+2)(x+2)}}{h}$$

Cancel the  $h$ .

$$= \frac{-1}{(x+h+2)(x+2)}$$

When finding difference quotients, avoid these common mistakes:

- $f(x+h) \neq f(x) + h$
- $(x+h)^2 \neq x^2 + h^2$  instead use FOIL to obtain  $(x+h)^2 = x^2 + 2xh + h^2$
- $5(x+h)^2 \neq (5x+5h)^2$ , instead  $5(x+h)^2 = 5(x^2 + 2xh + h^2) = 5x^2 + 10xh + 5h^2$ .

### Exercise 7.03 Difference quotients

Example

7

1 Given  $f(x) = 4x + 1$ , find:

a  $f(x + h)$

b  $f(x + h) - f(x)$

c  $\frac{f(x + h) - f(x)}{h}$

2 Given  $f(x) = 7 - 3x$ , find:

a  $f(x + h)$

b  $f(x + h) - f(x)$

c  $\frac{f(x + h) - f(x)}{h}$

3 Calculate the difference quotient for each function.

a  $2x + 7$

b  $8 - 5x$

c  $6x - 4$

Example

8

4 Given  $f(x) = 3x^2$ , find:

a  $f(x + h)$

b  $f(x + h) - f(x)$

c  $\frac{f(x + h) - f(x)}{h}$

5 Given  $f(x) = x^2 - 5x$ , find:

a  $f(x + h)$

b  $f(x + h) - f(x)$

c  $\frac{f(x + h) - f(x)}{h}$

6 Calculate the difference quotient for each function.

a  $2x^2 - 4x + 3$

b  $4x^2 + 3x - 2$

c  $8 - \frac{1}{2}x^2$

Example

9

7 Given  $f(x) = \frac{1}{x}$ , find:

a  $f(x + h)$

b  $f(x + h) - f(x)$

c  $\frac{f(x + h) - f(x)}{h}$

8 Calculate the difference quotient for each function.

a  $\frac{1}{x+1}$

b  $\frac{3}{x-2}$

c  $\frac{1}{3-2x}$

#### Problem solving

9 Calculate the difference quotient for  $f(x) = x^3 - x - 9$ .

10 Calculate the difference quotient for  $f(x) = \frac{x-3}{5-x}$ .

11 Calculate the difference quotient for  $f(x) = \frac{(x-8)^2}{x}$ .

12 Use the fact that  $(a + b)(a - b) = a^2 - b^2$  to show that the difference quotient for

$$f(x) = \sqrt{x^2 - 1} \text{ is given by } \frac{2x + h}{\sqrt{(x + h)^2 - 1} + \sqrt{x^2 - 1}}.$$

## 7.04 Introduction to limits

The concept of a **limit** is significant to calculus. The limit of a function  $f(x)$  is the value that the function gets closer to as the value of  $x$  gets closer to a particular value.

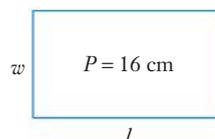


### INVESTIGATION

#### LIMITS

You are given 16 cm of wire and asked to form a rectangle whose area is as large as possible. Your task is to work out what dimensions the rectangle should have.

- 1 Let  $w$  represent the width of the rectangle and let  $l$  represent the length of the rectangle.



- 2 You know that  $P = 2(l + w)$ , so  $16 = 2(l + w)$ .  
Rearranging, you get  $l = 8 - w$
- 3 You also know that the area of a rectangle,  $A = lw$ . Substitute for  $l$  to write an expression for  $A$  in terms of  $w$  only.
- 4 Use your expression to complete the following table of values, showing how the area changes as  $w$  changes.

$w$	0	1	2	3	4	5	6	7	8
$A$									

- 5 At what value of  $w$  does the area appear to have its greatest value?
- 6 Now complete this table of values.

$w$	3	3.2	3.5	3.9	4	4.1	4.5	4.7	5
$A$									

- 7 Complete the statement:

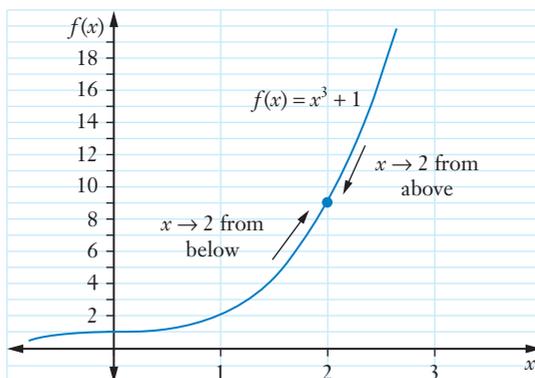
The largest area of the rectangle is ...  $\text{cm}^2$  and this occurs when the width is ... cm.

In the previous investigation, you saw that the value of  $A$  gets closer and closer to 16 as the value of  $w$  gets closer to 4. In limit terminology you say ‘the limit of  $A$  as  $w$  approaches 4 is 16.’ This is written as:

$$\lim_{w \rightarrow 4} A = \lim_{w \rightarrow 4} (8w - w^2) = 16$$

You use the arrow symbol ‘ $\rightarrow$ ’ for ‘approaches’.

Here is the graph of  $f(x) = x^3 + 1$ . You want to know the value of the limit of  $f(x) = x^3 + 1$  as  $x$  approaches 2.



The tables below show how the value of  $f(x)$  changes as  $x$  approaches 2 *from below* (from the left) and *from above* (from the right).

Approaching from below		Approaching from above	
$x$	$f(x)$	$x$	$f(x)$
1	2	3	28
1.5	4.375	2.5	16.625
1.9	7.859	2.1	10.261
1.99	8.8806	2.01	9.1206
1.999	8.988 01	2.001	9.012 01

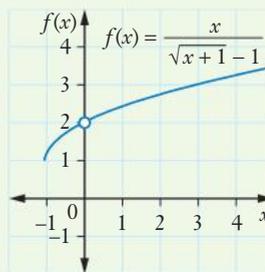
It is clear from the tables of values that  $f(x)$  approaches 9 as  $x$  approaches 2 from above or below. You write this as:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} (x^3 + 1) = 9$$

## EXAMPLE 10

The graph of  $f(x) = \frac{x}{\sqrt{x+1}-1}$  is shown here.

- a Find the value of  $f(x)$  when  $x = 0$ .
- b Use tables of values to find  $\lim_{x \rightarrow 0} f(x)$ .



### Solution

- a Substitute for  $x = 0$ .

$$\begin{aligned}\frac{x}{\sqrt{x+1}-1} &= \frac{0}{\sqrt{0+1}-1} \\ &= \frac{0}{1-1} \\ &= \frac{0}{0} \quad \text{which is undefined}\end{aligned}$$

- b Construct tables of values for  $f(x)$  as  $x$  approaches 0 from below and above.

Approaching from below		Approaching from above	
$x$	$f(x)$	$x$	$f(x)$
-1	1	1	2.414 213 ...
-0.5	1.707 106 ...	0.5	2.224 744 ...
-0.1	1.948 683 ...	0.1	2.048 808 ...
-0.01	1.994 987 ...	0.01	2.004 987 ...
-0.001	1.999 499 ...	0.001	2.000 499 ...
-0.0001	1.999 949 ...	0.0001	2.000 049 ...

Describe the results.

As  $x$  approaches 0 from below,  $f(x)$  approaches 2; and as  $x$  approaches 0 from above,  $f(x)$  approaches 2.

Write the result.

$$\lim_{x \rightarrow 0} f(x) = 2$$

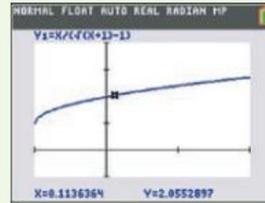
You could also use a graphics calculator to sketch the graph of the function and then use the **Trace** function to see how  $f(x)$  behaves as  $x$  approaches 0 from below and above.

### TI-84 Plus CE

Press  $\boxed{y=}$  and enter the function

$$Y_1 = \frac{x}{\sqrt{x+1}-1}$$

Press the  $\boxed{\text{graph}}$  button, then press the  $\boxed{\text{window}}$  button and set appropriate maximum and minimum values for  $x$  and  $y$ .



Press the  $\boxed{\text{window}}$  button and examine how the value of  $f(x)$  changes as  $x$  approaches 0 from below and above.

You will see that the function is undefined at  $x = 0$ .

### Casio fx-CG20AU

Select the **Graph** page.

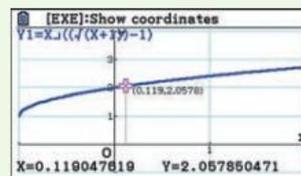
Enter the function  $Y1 = \frac{x}{\sqrt{x+1}-1}$ .

Press  $\boxed{\text{F6}}$  (**Draw**) to sketch the graph.

Press  $\boxed{\text{F3}}$  (**V-Window**) and set appropriate maximum and minimum values for  $x$  and  $y$ .

Then press  $\boxed{\text{F1}}$  (**Trace**) to examine how the value of  $f(x)$  changes as  $x$  approaches 0 from below and above.

You will see that the function is undefined at  $x = 0$ .



Example 10 shows that you can often find the limit of a function as  $x$  approaches  $c$  even if the function is undefined at  $x = c$ .

It is not always possible to find the limit of a function.

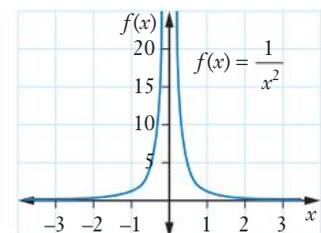
Consider the graph of  $f(x) = \frac{1}{x^2}$  shown here. You want to see if the limit exists at  $x = 0$ .

Looking at the graph, you can see that:

- as  $x \rightarrow 0$  from above,  $f(x) \rightarrow \infty$
- as  $x \rightarrow 0$  from below,  $f(x) \rightarrow \infty$

In fact  $f(x)$  keeps getting larger and larger without approaching a unique number  $L$ .

This means that the limit does not exist.



## Conditions under which limits do not exist

$\lim_{x \rightarrow c} f(x)$  does not exist if:

- $f(x)$  approaches different numbers as  $x \rightarrow c$  from below and above
- $f(x)$  increases or decreases without bound as  $x \rightarrow c$ .

Sometimes you need to use algebra to simplify an expression to find a limit. In Example 11 below you are not substituting  $-3$ , just getting close. That means you can cancel  $(x + 3)$ .

### EXAMPLE 11

Find  $\lim_{x \rightarrow -3} \left( \frac{x^2 + x - 6}{x + 3} \right)$ .

#### Solution

The expression is undefined at  $x = -3$ .

Try factorising the numerator.

Cancel and simplify.

Rewrite the limit.

Find the limit as  $x \rightarrow -3$ .

$$\frac{x^2 + x - 6}{x + 3} = \frac{(x + 3)(x - 2)}{(x + 3)}$$

$$= x - 2 \quad (x \neq -3)$$

$$\lim_{x \rightarrow -3} \left( \frac{x^2 + x - 6}{x + 3} \right) = \lim_{x \rightarrow -3} (x - 2)$$

$$= -5$$

## Exercise 7.04 Introduction to limits

1 Copy and complete each table of values and use the result to estimate the stated limit.

a  $\lim_{x \rightarrow 4} (4x - 7)$

$x$	3.9	3.99	3.999	4	4.001	4.01	4.1
$f(x)$				?			

b  $\lim_{x \rightarrow 1} (2x^2 + x - 5)$

$x$	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

Example  
10

c  $\lim_{x \rightarrow 2} \left( \frac{x+4}{x^2+2x-8} \right)$

$x$	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

d  $\lim_{x \rightarrow 5} \left( \frac{\cos(x)}{x} \right)$

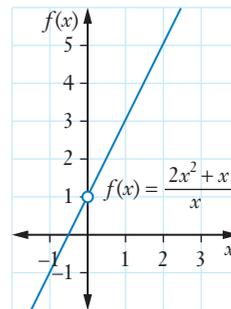
$x$	4.9	4.99	4.999	5	5.001	5.01	5.1
$f(x)$				?			

e  $\lim_{x \rightarrow 1} \left( \frac{\ln(x)}{x-1} \right)$

$x$	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

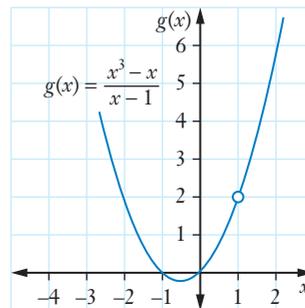
2 The graph of  $f(x) = \frac{2x^2+x}{x}$  is shown here.

- a Find the value of  $f(x)$  when  $x = 0$ .  
 b Use tables of values to find  $\lim_{x \rightarrow 0} f(x)$ .



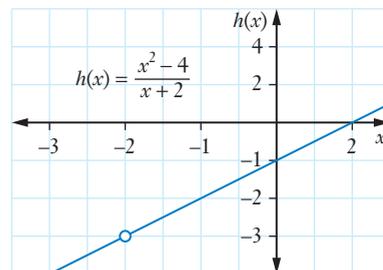
3 The graph of  $g(x) = \frac{x^3-x}{x-1}$  is shown here.

- a Find the value of  $g(x)$  when  $x = 1$ .  
 b Use tables of values to find  $\lim_{x \rightarrow 1} g(x)$ .



4 The graph of  $h(x) = \frac{x^2-4}{x+2}$  is shown here.

- a Find the value of  $h(x)$  when  $x = -2$ .  
 b Use tables to find  $\lim_{x \rightarrow -2} h(x)$ .



5 Use tables to find each limit if possible.

a  $\lim_{x \rightarrow 3} (2x + 1)$

b  $\lim_{x \rightarrow 2} (4 - 3x)$

c  $\lim_{x \rightarrow 5} (x^2 - 4)$

d  $\lim_{x \rightarrow 2} \left( \frac{x^2 - 4}{x - 2} \right)$

e  $\lim_{x \rightarrow -2} \left( \frac{x^2 - 4}{x - 2} \right)$

f  $\lim_{x \rightarrow 3} \left( \frac{5}{x - 1} \right)$

6 Use tables of values to calculate each limit.

a  $\lim_{x \rightarrow 1} \left( \frac{x^2 - 1}{x - 1} \right)$

b  $\lim_{x \rightarrow 4} \left( \frac{16 - x^2}{x - 4} \right)$

c  $\lim_{x \rightarrow -1} \left( \frac{x^2 - 2x - 3}{x + 1} \right)$

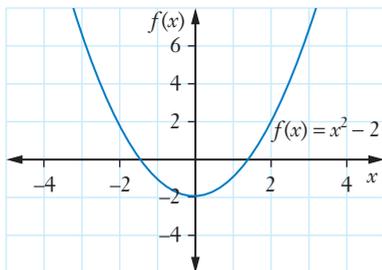
d  $\lim_{x \rightarrow -3} \left( \frac{x^2 + 7x + 12}{x + 3} \right)$

e  $\lim_{x \rightarrow 0} \left( \frac{\sqrt{x+3} - \sqrt{3}}{x} \right)$

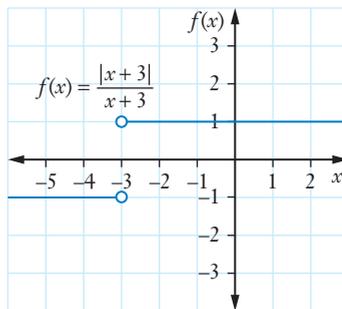
f  $\lim_{x \rightarrow 0} \left( \frac{\sin(x)}{x} \right)$

7 For each graph, find the limit if it exists. If the limit does not exist, explain why.

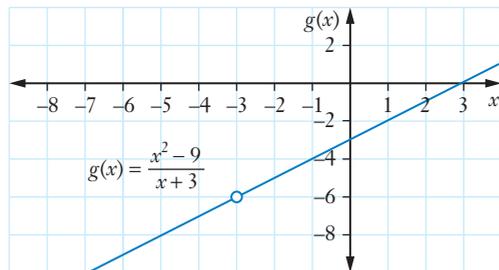
a  $\lim_{x \rightarrow 0} f(x)$



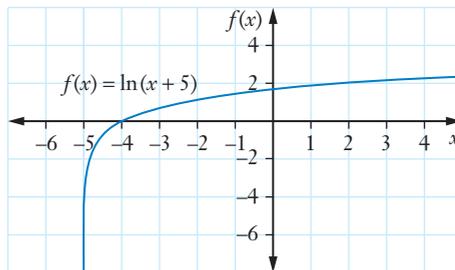
b  $\lim_{x \rightarrow -3} f(x)$



c  $\lim_{x \rightarrow -3} g(x)$



d  $\lim_{x \rightarrow -5} f(x)$



**8** Find each limit if it exists. If the limit does not exist, explain why.

**a**  $f(x) = \frac{x-1}{x+1}, \lim_{x \rightarrow -1} f(x)$

**b**  $f(x) = \frac{3x}{x^2+1}, \lim_{x \rightarrow 1} f(x)$

**c**  $g(x) = \sin\left(\frac{1}{x}\right), \lim_{x \rightarrow 0} g(x)$

**d**  $h(x) = \cos(\pi x), \lim_{x \rightarrow -1} h(x)$

**e**  $f(x) = \ln(3-x), \lim_{x \rightarrow 3} f(x)$

**f**  $g(x) = \frac{\sqrt{x+5}-4}{x-2}, \lim_{x \rightarrow 2} g(x)$

**9** Use a graphics calculator to find each limit if possible.

**a**  $\lim_{x \rightarrow 4} (7x - 3)$

**b**  $\lim_{x \rightarrow 1} \left(\frac{4}{x-1}\right)$

**c**  $\lim_{x \rightarrow 2} \left(\frac{4}{x-1}\right)$

**d**  $\lim_{x \rightarrow 5} \left(\frac{x^2-25}{x-5}\right)$

**e**  $\lim_{x \rightarrow 5} \left(\frac{x^2-16}{x-5}\right)$

**f**  $\lim_{x \rightarrow -2} \left(\frac{x^2+x-2}{x+2}\right)$

Example

**11**

**10** Use an algebraic method to calculate each limit (if it exists). Use a graphics calculator to confirm your results graphically.

**a**  $\lim_{x \rightarrow -3} \left(\frac{x^2+x-6}{x+3}\right)$

**b**  $\lim_{x \rightarrow -1} \left(\frac{x^2-x-2}{x+1}\right)$

**c**  $\lim_{x \rightarrow -5} \left(\frac{x^2+6x+5}{x+5}\right)$

**d**  $\lim_{x \rightarrow 1} \left(\frac{1-4x+3x^2}{1-x}\right)$

**e**  $\lim_{x \rightarrow 0} \left(\frac{2x^2+7x-4}{x-4}\right)$

**f**  $\lim_{x \rightarrow -1} \left(\frac{x^3-8}{x-2}\right)$

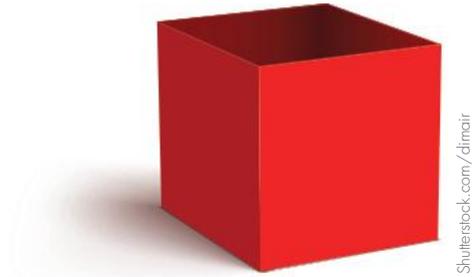
**g**  $\lim_{x \rightarrow 2} \left(\frac{x^2-2x-3}{x^2-8x+15}\right)$

**h**  $\lim_{x \rightarrow 1} \left(\frac{x-1}{x^3-x^2+x-1}\right)$

## Problem solving

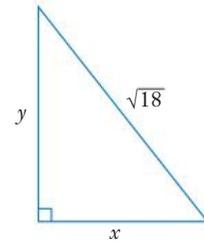
**11** Sam takes a square piece of material with side length 12 cm and creates an open box by cutting equal squares  $x$  cm by  $x$  cm from the corners and turning up the sides.

- Draw and label a diagram that represents the box.
- Verify that the volume of the box is given by  $V = 4x(6 - x)^2$ .
- Draw a graph of  $V$  for  $0 \leq x \leq 6$ .
- At what value of  $x$  does the box appear to have maximum volume?
- Use a table of values to estimate the limit of  $V$  at the value of  $x$  determined in part d.



**12** Alya wants to build a garden bed in the shape of a right-angled triangle. The length of the hypotenuse must be  $\sqrt{18}$  m and she wants the area of the triangle to be as large as possible.

- Write an equation for the area of the triangle ( $A$ ) in terms of  $x$ .
- Use limits to find the value of  $x$  at which the area of the triangle is greatest.



## 7.05 The derivative

The gradient of a linear function is constant. However, the gradient of a non-linear function varies. An approximate value for the gradient at any point on a non-linear function can be found by using a secant as shown in the following diagram. The secant passing through  $P$  and  $Q$  cuts off the chord  $PQ$ .

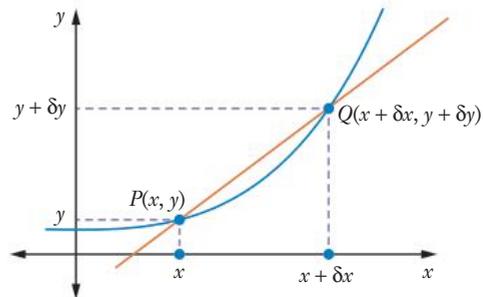
For a straight line you know that:

$$\text{Gradient} = m = \frac{y_2 - y_1}{x_2 - x_1}$$

where  $(x_1, y_1)$  and  $(x_2, y_2)$  are two points on the line.

In the diagram on the right,  $P(x, y)$  and  $Q(x + \delta x, y + \delta y)$  are two points on the graph of the function.

The symbol  $\delta$  means 'a small change'.



Finding derivatives  
from first principles

For the chord  $PQ$ :

$$\begin{aligned} m_{PQ} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{y + \delta y - y}{x + \delta x - x} \\ &= \frac{\delta y}{\delta x} \end{aligned}$$

The closer  $P$  and  $Q$  become, the closer the gradient of the chord is to the exact value of the gradient of the tangent to the curve at  $P$ .

If  $P(x, y)$  and  $Q(x + \delta x, y + \delta y)$  are points on the function  $f(x)$ , the gradient of  $PQ$  can be rewritten as:

$$\begin{aligned} m_{PQ} &= \frac{f(x + \delta x) - f(x)}{x + \delta x - x} \\ &= \frac{f(x + \delta x) - f(x)}{\delta x} \end{aligned}$$

So, as  $\delta x$  becomes smaller and smaller ( $\delta x \rightarrow 0$ ),  $m_{PQ}$  gets closer to the gradient of the tangent at  $P$ . Using limits you can write:

The gradient of the tangent at  $P = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ , provided the limit exists.

Replacing  $\delta x$  with a letter  $h$ , you have the formula for the difference quotient.

So, the gradient of a function at  $P(x, f(x)) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$ .

### EXAMPLE 12

For the function  $f(x) = 2x^2 - 13x + 15$ ,

- a** find the rule for the gradient of the function
- b** find the value of the gradient at  $x = 5$ .

### Solution

- a** Write down the rule for gradient.

$$m = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Write the function.

$$f(x) = 2x^2 - 13x + 15$$

Substitute  $(x + h)$  for  $x$  to find the numerator.

$$\begin{aligned} f(x + h) - f(x) \\ = 2(x + h)^2 - 13(x + h) + 15 - (2x^2 - 13x + 15) \end{aligned}$$

Expand and simplify.

$$\begin{aligned} &= 2x^2 + 4xh + 2h^2 - 13x - 13h + 15 - 2x^2 + 13x - 15 \\ &= 4xh - 13h + 2h^2 \end{aligned}$$

Factorise.  $= h(4x - 13 + 2h)$

Replace the numerator in the rule.  $m = \lim_{h \rightarrow 0} \frac{h(4x - 13 + 2h)}{h}$

Cancel the common factor.  $= \lim_{h \rightarrow 0} (4x - 13 + 2h)$

Find the limit as  $h \rightarrow 0$ .  $= 4x - 13$

State the result. The gradient of  $f(x) = 2x^2 - 13x + 15$  is  $4x - 13$ .

**b** Calculate the gradient at  $x = 5$ .  $4x - 13 = 20 - 13 = 7$

State the result. The gradient of  $f(x)$  at  $x = 5$  is 7.

The gradient of a function:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

is so important it is given a special name – the **derivative**.

The derivative of the function has the symbol  $f'(x)$ .

### Derivative of a function

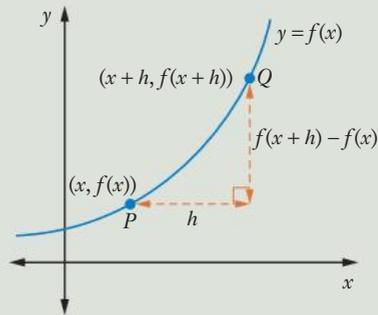
The derivative of the function  $f(x)$  is defined as:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The derivative can be written as:

$$f'(x) \text{ or } \frac{df}{dx} \text{ or } \frac{d}{dx} f(x).$$



You can also define the derivative of the function  $f(x)$  at the point where  $x = c$  as:

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

This can also be written as:  $\left. \frac{d}{dx} f(x) \right|_{x=c}$

Finding the derivative using the above definition is called **differentiating from first principles**.



### EXAMPLE 13

Find a rule for the derivative of  $f(x) = x^2 + 3x - 8$  from first principles.

#### Solution

Write the definition.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Write the function.

$$f(x) = x^2 + 3x - 8$$

Find  $f(x+h)$ .

$$f(x+h) = (x+h)^2 + 3(x+h) - 8$$

Expand the brackets.

$$= x^2 + 2xh + h^2 + 3x + 3h - 8$$

Now find  $f(x+h) - f(x)$ .

$$\begin{aligned} f(x+h) - f(x) &= x^2 + 2xh + h^2 + 3x + 3h - 8 - (x^2 + 3x - 8) \\ &= 2xh + 3h + h^2 \end{aligned}$$

Simplify.

$$= 2xh + 3h + h^2$$

Factorise.

$$= h(2x + 3 + h)$$

Replace the numerator in the definition.

$$f'(x) = \lim_{h \rightarrow 0} \frac{h(2x + 3 + h)}{h}$$

Cancel the common factor.

$$= \lim_{h \rightarrow 0} (2x + 3 + h)$$

Find the limit as  $h \rightarrow 0$ .

$$= 2x + 3$$

Write the result.

$$f'(x) = 2x + 3$$

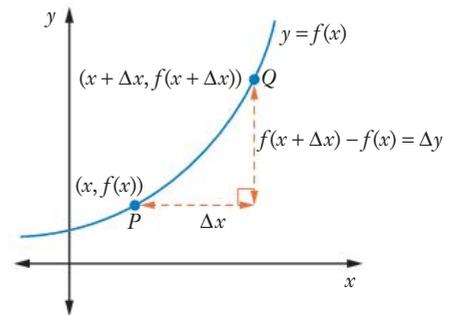
$\frac{d}{dx}$  does not mean a fraction. It is just a symbol.

Because you often make  $f(x) = y$  when graphing functions, it is also common to write the derivative in terms of  $x$  and  $y$ .

Using the diagram and the definition of the derivative stated previously, you can see that

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \text{ where } \Delta x = h$$

$$\text{So } y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$



## Exercise 7.05 The derivative

**1** For each function, find the rule for the gradient and the value of the gradient of the function at the point given.

**a**  $f(x) = 3x - 7$ ;  $(1, -4)$

**b**  $f(x) = 8 - x$ ;  $(3, 5)$

**c**  $g(x) = x^2 - 2$ ;  $(2, 2)$

**d**  $h(x) = x^2 + 5x$ ;  $(-1, -4)$

Example  
**12**

**2** Find the rule for the gradient of  $f(x) = x^3$  at any point.

**3** Calculate the value of the gradient of  $f(x) = 3x^2 - x + 7$  at  $x = 3$ .

**4** Find a rule for the derivative of each function using first principles.

**a**  $f(x) = 5x$

**b**  $g(x) = 7 - 2x$

**c**  $h(x) = 3 + x^2$

**d**  $f(x) = 3x^2 - 4x + 7$

Example  
**13**

**5** Find a rule for the derivative of  $f(x) = 2x^3 + 7$  using first principles.

**6** Find a rule for the derivative of  $g(x) = (2x + 5)^2$  using first principles.

**7** Find the derivative of  $f(x) = 12 + 6x - 5x^2$  at  $x = 1$ .

**8** Find the derivative of  $f(x) = (x - 3)^3$  at  $x = 1$ .

**9** Use the definition to find the derivative of each function at the given value of  $x$ .

**a**  $x^2 + 5x - 7$  at  $x = 3$

**b**  $x^2 + 5x + 9$  at  $x = 3$

**c**  $3x^2 + 4x + 1$  at  $x = -2$

**d**  $x^2 - 6x + 4$  at  $x = 1$

**e**  $4 - x^2 + 2x$  at  $x = -5$

**f**  $4x - 5x^2 - 6x^3$  at  $x = 8$

**10** Find the derivative of each function.

**a**  $9x + 2$

**b**  $x - 7$

**c**  $3x^2$

**d**  $9$

**e**  $10x^2 + 3x + 1$

**f**  $1 - 5x - x^2$

**g**  $3x - 4x^2 + 2$

**h**  $7x^3$

**i**  $x^4$

### Problem solving

**11** Use first principles to show that  $\frac{d}{dx}(cx^2) = 2cx$ .

**12** Use first principles to show that  $\frac{d}{dq}(7q) = 7$ .

**13** Find the derivative of  $f(x) = \frac{1}{x+2}$  using first principles.

**14** Which of the functions  $f(x) = 3x^2 - 5x + 2$  or  $g(x) = 2x^2 - x + 9$  is steeper at  $x = 3$ ?

## 7.06 Interpreting the derivative as a rate of change

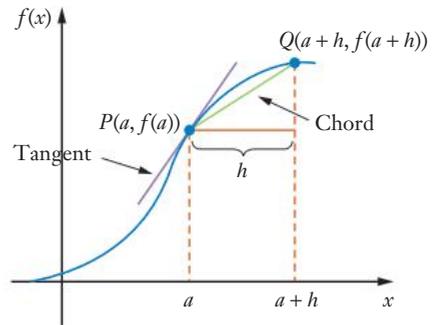
The slope of a non-linear function between different pairs of points is usually different. For non-linear functions, the slope calculated using points on the function is the average rate of change. You use a limit to find the exact rate of change at any point; the **instantaneous rate of change**.

Consider the function shown here. The average rate of change is the slope of the chord  $PQ$ , while the instantaneous rate of change at  $P$  is the slope of the tangent at  $P$ .

The average rate of change of the function is given by:

$$\frac{f(a+h) - f(a)}{h}$$

The smaller  $h$  becomes, the closer the slope of the chord  $PQ$  gets to the slope of the tangent at  $P$ .



### Instantaneous rate of change

The instantaneous rate of change of the function  $f(x)$  is given by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The instantaneous rate of change of  $f(x)$  at  $x = a$  is  $f'(a)$ .

### EXAMPLE 14

The volume of water in a tank (in litres) at  $t$  minutes is given by the rule  $V(t) = 2t^2 - 15t + 30$ .

- What is the volume of water in the tank when  $t = 6$ ?
- What is the volume of water in the tank when  $t = 8$ ?
- What is the average rate of change of the volume of water in the tank from  $t = 6$  to  $t = 8$ ?
- Calculate the instantaneous rate of change at  $t = 6$ .
- Calculate the instantaneous rate of change at  $t = 8$ .

## Solution

- a** Write the rule for volume.  $V(t) = 2t^2 - 15t + 30$
- Substitute for  $t = 6$ .  $V(6) = 2 \times 6^2 - 15 \times 6 + 30 = 12$
- State the result. The volume of water in the tank at  $t = 6$  is 12 L.
- b** Substitute for  $t = 8$ .  $V(8) = 2 \times 8^2 - 15 \times 8 + 30 = 38$
- State the result. The volume of water in the tank at  $t = 8$  is 38 L.
- c** Write the rule for average rate of change. Average rate of change  $= \frac{V(a+h) - V(a)}{h}$
- Here,  $x = 6$  and  $h = 8 - 6 = 2$ .  $= \frac{V(8) - V(6)}{2}$
- $$= \frac{38 - 12}{2}$$
- $$= 13$$
- State the result. The average rate of change from  $t = 6$  to  $t = 8$  is 13 L/min.
- d** Write the rule for the derivative.  $V'(t) = \lim_{h \rightarrow 0} \frac{V(t+h) - V(t)}{h}$
- Substitute for  $V(t)$ .  $= \lim_{h \rightarrow 0} \frac{2(t+h)^2 - 15(t+h) + 30 - (2t^2 - 15t + 30)}{h}$
- Expand the numerator.  $= \lim_{h \rightarrow 0} \frac{2t^2 + 4th + 2h^2 - 15t - 15h + 30 - 2t^2 + 15t - 30}{h}$
- Simplify.  $= \lim_{h \rightarrow 0} \frac{4th + 2h^2 - 15h}{h}$
- Factorise the numerator.  $= \lim_{h \rightarrow 0} \frac{h(4t + 2h - 15)}{h}$
- Cancel.  $= \lim_{h \rightarrow 0} (4t + 2h - 15)$
- Find the limit as  $h \rightarrow 0$ .  $= 4t - 15$
- Substitute for  $t = 6$ .  $V'(6) = 4 \times 6 - 15 = 9$
- State the result. The instantaneous rate of change at  $t = 6$  is 9 L/min.
- e** Substitute for  $t = 8$ .  $V'(8) = 4 \times 8 - 15 = 17$
- State the result. The instantaneous rate of change at  $t = 8$  is 17 L/min.

If an object moves in a straight line, its change in position from its starting position is called its **displacement**. Unlike distance, displacement can be positive or negative. On the Cartesian plane, a positive displacement is to the right and means the object is moving forward. A negative displacement is to the left and means the object is moving backwards.

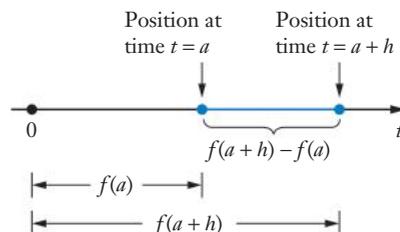
The diagram below shows the **displacement** of an object.

The average **velocity** of the object is

$$\frac{\Delta x}{\Delta t} = \frac{f(a+h) - f(a)}{h}$$

The instantaneous velocity is:

$$x' = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$



Because  $x = f(t)$ , you can write this as:

$$f'(t) = v(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

The instantaneous velocity at  $t = a$  is:

$$v(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Using a similar approach, you can show that average **acceleration** is:

$$\frac{v(a+h) - v(a)}{h}$$

and instantaneous acceleration is:

$$v'(t) = a(t) = \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h}$$

Velocity and acceleration usually mean instantaneous velocity and acceleration.



Graphs of rates of change

- **Displacement** is the change in position of an object from its initial to its final position regardless of the path taken.
- **Velocity**,  $v(t)$ , is the rate of change of displacement with respect to time.  
 $v(t) = x'(t)$ , where position =  $f(t)$
- **Acceleration**,  $a(t)$ , is the rate of change of velocity with respect to time.  
 $a(t) = v'(t)$ , where velocity =  $v(t)$

## EXAMPLE 15

A ball is thrown straight upward so that its height (in metres) relative to the ground is given by the equation:

$$x = 6 + 10t - 4.9t^2$$

Find:

- a its position after 1 second
- b its position after 2 seconds
- c its displacement after 2 seconds
- d its velocity after  $\frac{1}{2}$  a second
- e its acceleration after 2 seconds

### Solution

- a Write the equation for position.  $x = 6 + 10t - 4.9t^2$

Substitute for  $t = 1$ . 
$$= 6 + 10 \times 1 - 4.9 \times 1^2$$
$$= 11.1$$

State the result. After 1 second the ball is 11.1 m above the ground.

- b Substitute for  $t = 2$ . 
$$x = 6 + 10 \times 2 - 4.9 \times 2^2$$
$$= 6.4$$

State the result. After 2 seconds the ball is 6.4 m above the ground.

- c Calculate the initial position, i.e. when  $t = 0$ . 
$$\text{Initial position} = 6 + 10 \times 0 - 4.9 \times 0^2$$
$$= 6$$

Calculate the displacement. 
$$\text{Displacement} = x_2 - x_1$$
$$= 6.4 - 6$$
$$= 0.4$$

State the result. After 2 seconds, the ball is 0.4 m above its initial position.

- d Write the rule for velocity. 
$$x'(t) = v(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

Substitute for  $f(t)$ . 
$$= \lim_{h \rightarrow 0} \frac{6 + 10(t+h) - 4.9(t+h)^2 - (6 + 10t - 4.9t^2)}{h}$$

Expand and simplify. 
$$= \lim_{h \rightarrow 0} \frac{6 + 10t + 10h - 4.9t^2 - 9.8th - 4.9h^2 - 6 - 10t + 4.9t^2}{h}$$

- Simplify.  $= \lim_{h \rightarrow 0} \frac{10h - 9.8th - 4.9h^2}{h}$
- Factorise.  $= \lim_{h \rightarrow 0} \frac{h(10 - 9.8t - 4.9h)}{h}$
- Cancel.  $= \lim_{h \rightarrow 0} (10 - 9.8t - 4.9h)$
- Find the limit.  $= 10 - 9.8t$
- Substitute for  $t = \frac{1}{2}$ .  $v\left(\frac{1}{2}\right) = 10 - 9.8 \times \frac{1}{2} = 5.1$
- State the result. The velocity after  $\frac{1}{2}$  a second is 5.1 m/s upwards.
- e** Write the rule for acceleration.  $a(t) = v'(t) = \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h}$
- Substitute for  $v(t)$ .  $= \lim_{h \rightarrow 0} \frac{(10 - 9.8[t+h]) - (10 - 9.8t)}{h}$
- Expand and simplify.  $= \lim_{h \rightarrow 0} \frac{10 - 9.8t - 9.8h - 10 + 9.8t}{h}$
- Cancel.  $= \lim_{h \rightarrow 0} \frac{-9.8h}{h}$
- Find the limit.  $= \lim_{h \rightarrow 0} -9.8$
- State the result. The acceleration is  $-9.8 \text{ m/s}^2$ , i.e.  $9.8 \text{ m/s}^2$  downwards.
- The acceleration is constant. The acceleration after 2 seconds is  $-9.8 \text{ m/s}^2$ .

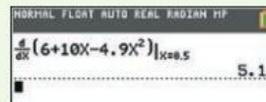
Objects accelerate downwards under gravity near the Earth at about  $9.8 \text{ m/s}^2$ .

You can use your graphics calculator for derivatives.



### TI-84 Plus CE

Press the **[math]** button and select **8:nDeriv(**. Input the required information and press **[enter]**.



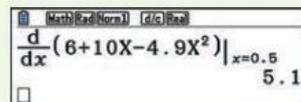
### Casio fx-CG20AU

Select the **Run-Matrix** menu.

Press **[F4]** (**MATH**). Press **[F4]** ( $d/dx$ ).

Enter the function  $y = 6 + 10x - 4.9x^2$ .

Enter the value of  $x = 0.5$ . Press **[EXE]**.



## INVESTIGATION

### ROLLER-COASTER SPEEDS

Part of the track of a roller-coaster can be modelled as

$$h = \frac{80d^3 - 3480d^2 - 7200d + 1217791}{29791}$$
$$\approx 0.002685d^3 - 0.1168d^2 - 0.2417d + 40.878$$

for  $0 \leq d \leq 35$ , where  $d$  is the horizontal distance in metres and  $h$  is the height of the track in metres.

The acceleration  $a$  of a roller-coaster car can be described as

$$a = -\left(0.2 + \frac{9.81m}{\sqrt{m^2 + 1}}\right)$$

where  $m$  is the gradient of the track.

Investigate points of interest with regard to the movement of the roller-coaster cars.

Consider:

- where the cars are accelerating
- where they are decelerating
- where they are not accelerating
- where they might obtain the greatest speed.

It will be helpful to calculate values and draw some graphs of the model.



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## Exercise 7.06 Interpreting the derivative as a rate of change

- 1** A car is bought for \$22 000 and its value over an 8-year period is shown in the table. The decrease in value over time is called depreciation.

Time (years)	Value (\$)
1	16 200
2	14 350
3	11 760
4	8980
5	7820
6	6950
7	6270
8	5060

- a** Calculate the average change in value of the car in the first year.
- b** Calculate the average change in value over the 8 years.
- c** Calculate the average change in value from year 1 to year 5.
- d** Calculate the average change in value from year 4 to year 8.
- e** Comment on the car's rate of depreciation over time.
- 2** Find the average rate of change from  $x = 1$  to  $x = 4$  for each function.
- a**  $f(x) = 8$                       **b**  $f(x) = 2x$                       **c**  $f(x) = 5 - 3x$
- d**  $f(x) = x^2$                       **e**  $f(x) = \frac{x^2}{2}$                       **f**  $f(x) = x^3$

- 3** For each function, find the average rate of change from  $x_1$  to  $x_2$ .
- a**  $f(x) = 3x + 8$ ;  $x_1 = 0$  and  $x_2 = 5$                       **b**  $g(x) = 12 - 5x$ ;  $x_1 = 1$  and  $x_2 = 4$
- c**  $f(x) = x^2 + 2x - 9$ ;  $x_1 = 2$  and  $x_2 = 7$                       **d**  $f(x) = \frac{36}{x^2} + 5$ ;  $x_1 = 1$  and  $x_2 = 3$
- e**  $f(x) = \frac{24}{x+2}$ ;  $x_1 = 0$  and  $x_2 = 4$

Example  
14

- 4** A block of metal is being heated. The temperature ( $^{\circ}\text{C}$ ) of the block is given by the equation:

$$T = 0.25t^2 + 0.5t + 12 \quad (\text{where } t \text{ is in minutes})$$

- a** What is the temperature of the block when  $t = 2$ ?
- b** What is its temperature when  $t = 4$ ?
- c** What is the average rate of change of the block's temperature from  $t = 2$  to  $t = 4$ ?
- d** Calculate the instantaneous rate of change in temperature at  $t = 1$ .
- e** Calculate the instantaneous rate of change in temperature at  $t = 6$ .
- 5** Oil is being drained from a vessel. The volume ( $V$ ) of oil in litres remaining in the vessel after  $t$  minutes is given by:

$$V = \frac{1}{60}(t - 50)^2$$

- a** What is the average rate of change of the volume from  $t = 1$  to  $t = 5$ ?
- b** Calculate the instantaneous rate of change in volume at  $t = 2$ .
- c** Calculate the instantaneous rate of change in volume at  $t = 8$ .

- 6** A stone is launched vertically upward from level ground. Its approximate height (in metres) relative to the ground is given by the equation:

$$x = 30t - 5t^2$$

Calculate:

- a** the position of the stone after 2 seconds
  - b** the position of the stone after 5 seconds
  - c** the displacement of the stone after 3 seconds
  - d** the velocity of the stone after 1 second
  - e** the velocity of the stone after 4 seconds
  - f** the acceleration of the stone after 2 seconds.
- 7** The position of a particle moving along the  $x$ -axis is given by:  
 $x = 27 + 6t - 2t^2$ , where  $x$  is in metres and  $t$  is in seconds.
- a** Calculate the velocity after 5 seconds.
  - b** Calculate the acceleration after 8 seconds.
  - c** Draw graphs of the velocity and acceleration for the first 10 seconds.
- 8** The position of a particle is given by  $x = 3t^2 + 5t + 2$ , where  $x$  is in metres and  $t$  is in seconds. Find a rule for:
- a** the velocity of the particle
  - b** the acceleration of the particle.
- 9** Use a graphics calculator to find the velocities of objects with the following positions at the times indicated.
- |   |   |
|---|---|
| <b>a</b> $x = 12t - 17$ at $t = -3$ and $t = 4$       | <b>b</b> $x = 4t^2 + 7t - 5$ at $t = 0$ and $t = 5$         |
| <b>c</b> $x = 12 + 9t - 3t^2$ at $t = 2$ and $t = -4$ | <b>d</b> $x = 2t^3 - 6t^2 - 8t + 9$ at $t = -1$ and $t = 3$ |
- 10** Use a graphics calculator to find the accelerations of objects with the following velocities at the times indicated.
- |   |   |
|---|---|
| <b>a</b> $v = 4t^3 + 3t$ at $t = -2$ and $t = 3$      | <b>b</b> $v = 0.5t^4 - 0.25t^3 + t$ at $t = -3$ and $t = 7$ |
| <b>c</b> $v = 0.2t^2 - 0.1t^5$ at $t = 1$ and $t = 3$ | <b>d</b> $v = 0.8t^3 - 0.1t^4$ at $t = -2$ and $t = 5$      |

### Problem solving

- 11** The position of an object is given by  $x = 2t^3$ . Use first principles to find a rule for the acceleration of the object.
- 12** The position of an object projected vertically upwards is given by  $x = 64t - 16t^2$ , where  $x$  is in metres and  $t$  is in seconds. When the object reaches its greatest height, the velocity is 0. Find:
- a** the greatest height reached by the object
  - b** the velocity of the object when it reaches the ground.

- 13** For the first 10 minutes of motion, the position of a freight train in the shunting yard is given by  $x = \frac{20}{t+2}$ , where  $x$  is in metres and  $t$  is in minutes.

- a** Use first principles to find a rule for the instantaneous velocity at any time.  
**b** Find the velocity of the freight train when  $t = 5$ .



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Slopes of curves



Derivatives assignment

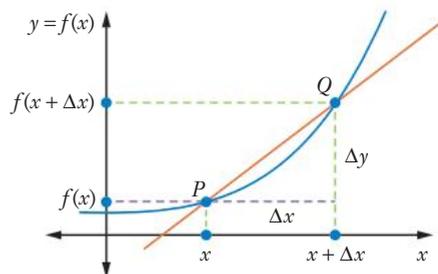
## 7.07 The slope of a curve

The diagram on the right shows a function  $y = f(x)$  that passes through  $P$  and  $Q$  whose coordinates are:

$$P(x, f(x)) \text{ and } Q(x + \Delta x, f(x + \Delta x))$$

From coordinate geometry you know the gradient of  $PQ$  is:

$$\begin{aligned} m_{PQ} &= \frac{f(x + \Delta x) - f(x)}{x + \Delta x - x} \\ &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \end{aligned}$$



### Slope of a curve

The slope of the tangent to the graph of  $y = f(x)$  at point  $P(a, f(a))$  is:

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

That is, the slope of the tangent to the graph of  $y = f(x)$  is the derivative at  $x = a$ .

## EXAMPLE 16

Find the equation of the tangent to  $f(x) = x^2$  at  $x = 3$ .

### Solution

The tangent passes through  $(3, f(3))$ .

$$f(3) = 3^2 = 9$$

State the result.

The tangent passes through  $(3, 9)$ .

Find the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Substitute for  $f(x)$ .

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

Expand the numerator.

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

Simplify.

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h}$$

Factorise.

$$= \lim_{h \rightarrow 0} \frac{h(2x+h)}{h}$$

Cancel.

$$= \lim_{h \rightarrow 0} (2x+h)$$

Find the limit.

$$= 2x$$

Find the gradient of the tangent at  $(3, 9)$ .

$$m = f'(3) = 6$$

Write the point–gradient form of the equation of a line.

$$y - y_1 = m(x - x_1)$$

Substitute for  $m = 6$  and  $(x_1, y_1) = (3, 9)$ .

$$y - 9 = 6(x - 3)$$

$$y - 9 = 6x - 18$$

$$y = 6x - 9$$

State the result using standard form.

The tangent to  $f(x) = x^2$  at  $x = 3$  is  
 $6x - y + 9 = 0$ .

## EXAMPLE 17

Given that  $f(x) = 3x^2 + 8x + 11$ , find:

- a the equation of the slope of  $f(x)$
- b where the slope of the graph is 0
- c where the slope of the graph is  $-4$ .

### Solution

- a Write the rule for the derivative.
- $$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$
- Substitute for  $f(x)$ .
- $$= \lim_{h \rightarrow 0} \frac{3(x+h)^2 + 8(x+h) + 11 - (3x^2 + 8x + 11)}{h}$$
- Expand the numerator.
- $$= \lim_{h \rightarrow 0} \frac{3x^2 + 6xh + 3h^2 + 8x + 8h + 11 - 3x^2 - 8x - 11}{h}$$
- Simplify.
- $$= \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 8h}{h}$$
- Factorise the numerator.
- $$= \lim_{h \rightarrow 0} \frac{h(6x + 3h + 8)}{h}$$
- Cancel.
- $$= \lim_{h \rightarrow 0} (6x + 3h + 8)$$
- Find the limit as  $h \rightarrow 0$ .
- $$= 6x + 8$$
- State the result. The slope of  $f(x)$  is  $f'(x) = 6x + 8$ .
- b Let  $f'(x) = 0$ .  $0 = 6x + 8$
- Solve.  $6x = -8$
- $$x = -1\frac{1}{3}$$
- State the result. The slope of the graph of  $f(x)$  is 0 when  $x = -1\frac{1}{3}$ .
- c Let  $f'(x) = -4$ .  $-4 = 6x + 8$
- Solve.  $6x = -12$
- $$x = -2$$
- State the result. The slope of the graph of  $f(x)$  is  $-4$  when  $x = -2$ .

## INVESTIGATION

### GRAPHS OF DERIVATIVES

- 1 Use a graphics calculator to draw a graph of  $f(x) = x^2 + 3x - 10$ . Adjust the **window** (or **V-Window**) to suit.
- 2 Calculate  $f'(x)$  and draw the graph of  $f(x)$  and  $f'(x)$  on the same set of axes. Adjust the **window** to show both graphs if necessary.
- 3 What is the significance of the points at which  $f(x)$  cuts the  $x$ -axis?
- 4 What is the significance of the point at which  $f'(x)$  cuts the  $x$ -axis?
- 5 Clear the graphs.
- 6 Graph  $g(x) = x^3 - 2x^2 - 5x + 8$ .
- 7 Calculate  $g'(x)$  and graph it on the same set of axes as  $g(x)$ .
- 8 What is the significance of the points at which  $g(x)$  cuts the  $x$ -axis?
- 9 What is the significance of the points at which  $g'(x)$  cuts the  $x$ -axis?
- 10 Clear the graphs.
- 11 Repeat steps 1 to 5 with other quadratic functions.
- 12 Repeat steps 6 to 10 with other cubic functions.
- 13 Copy and complete:

The graph of the derivative of a quadratic function is a \_\_\_\_\_ function.

The graph of the derivative of a cubic function is a \_\_\_\_\_ function.



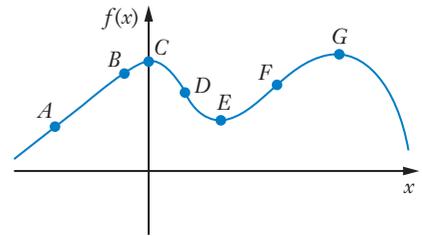
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The water height changes as the tide goes in and out. The shape repeats about every 12 hours. What happens to the shape of the graph of its derivative? What does the derivative represent?

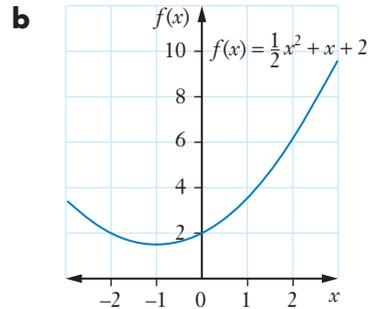
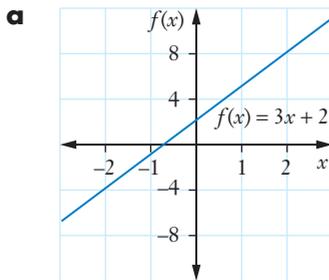
## Exercise 7.07 The slope of a curve

- 1 Given that  $f(2) = 6$  and  $f'(2) = -5$ , what is the slope of the graph of  $f(x)$  at  $(2, 6)$ ?
- 2 Use the graph of  $f(x)$  shown below to answer the questions that follow.

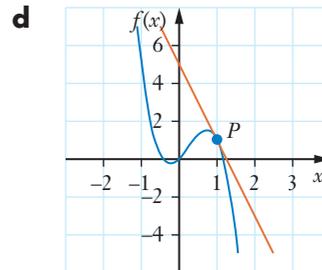
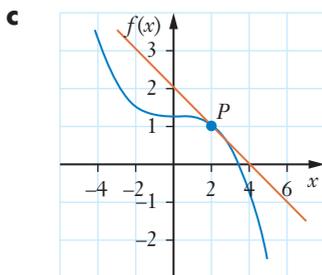
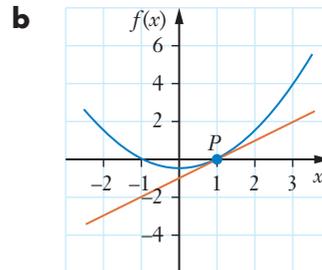
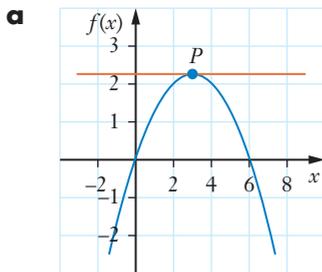
- a At which points is  $f'(x) = 0$ ?
- b Between which points is  $f'(x) > 0$ ?
- c Between which points is  $f'(x) < 0$ ?



- 3 For each graph of  $f(x)$ , sketch the graph of  $f'(x)$ .



- 4 In each graph, a tangent to the curve has been drawn at  $P$ . Calculate the value of  $f'(x)$  at  $P$ .



**5** For each function, find the equation (in standard form) of the tangent to  $f(x)$  at the point indicated.

**a**  $f(x) = \frac{1}{2}x^2$  at  $(-2, 2)$

**b**  $f(x) = x^2 + 3x$  at  $(-2, -2)$

**c**  $f(x) = 10x - 2x^2$  at  $(3, 12)$

**d**  $f(x) = 2x^2 + 5x - 4$  at  $(-1, -7)$

**e**  $f(x) = 3x^2 + 2x - 5$  at  $(-2, 3)$

**f**  $f(x) = 2x^3$  at  $(1, 2)$

**6** For each function in question 5, find the value of  $x$  where the gradient of the function is 0.

**7** Given  $f(x) = 3x^2 + 2x + 1$ , find:

**a** the derivative of  $f(x)$  at any point  $(x, f(x))$

**b** the slope of the curve when  $x = 1$

**c** the equation of the tangent to  $f(x)$  when  $x = 1$ .

**8** Given that  $f'(x) = 3$  at the point  $P(1, -1)$  which lies on  $f(x)$ , find the equation of the tangent to the graph of  $f(x)$  at  $P$ .

**9** If  $g'(x) = \frac{1}{2}$  at  $Q(-1, -2)$ , find the equation of the tangent to the graph of  $g(x)$  at  $x = -1$ .

**10** Given that  $f(x) = 2x^2 - 9x + 7$ , find:

**a** the equation of the slope of  $f(x)$

**b** where the slope of the graph is 0

**c** where the slope of the graph is  $-5$

**d** where the slope of the graph is 3.

**11** Use a graphics calculator to find the slope of each function at the  $x$  values indicated.

**a**  $f(x) = 3x^3 - 4x^2 + 2x - 9$  at  $x = -2$  and at  $x = 2$

**b**  $f(x) = 5x^2 - 7x^5$  at  $x = -1$  and at  $x = 3$

**c**  $f(x) = \frac{1}{x+2}$  at  $x = -1$  and at  $x = 4$

**d**  $f(x) = \frac{2x}{x-3}$  at  $x = -2$  and at  $x = 5$

**12** At what values of  $x$  is the tangent to the curve  $f(x) = x^3 - 2x$  equal to 0?

**13** The gradient tangent to a curve is a measure of the steepness of the curve at that point. Find which function,  $f(x) = x^2 + 3x - 5$  or  $g(x) = 4x - 2x^2 + 3$ , is steeper at  $x = 5$ .

**14** Find which function,  $f(x) = 3x^2 - 5x + 2$  or  $g(x) = 2x^2 - x + 9$ , is steeper at  $x = 3$ .

**15** Find the values of  $x$  for which  $f(x) = 2x^2 - 4x + 1$  is steeper than  $g(x) = x^2 + 8x - 3$ .

- 16** Find any point on the function  $f(x) = x^2 - 4x + 7$  where the gradient of the curve is 0.
- 17** Find all points on the function  $f(x) = x^3 - 2x^2 + x - 10$  where the gradient of the curve is 0.

### Problem solving

- 18** Is it possible for the same gradient function to be produced by two different functions? Explain your answer.
- 19** At which point on  $f(x) = 5x^2 - 2x + 3$  is the tangent to  $f(x)$  parallel to the line  $4x - 3y = 7$ ?
- 20** At which point on  $f(x) = 5x^3$  is the tangent to  $f(x)$  parallel to the line  $15x - 4y + 8 = 0$ ?
- 21** Find the values of  $x$  for which the gradient function  $f(x) = 3x^2 - 5x + 2$  is greater than the gradient function of  $g(x) = 3x^2 + x - 4$ .

# 7. CHAPTER SUMMARY

## Rates of change and the derivative

- The **rate of change** of a second quantity with respect to (wrt) a first quantity is the quotient of their changes:

$$\text{Rate of change} = \frac{\text{change in quantity 2}}{\text{change in quantity 1}}$$

$$\text{Rate of change of } y \text{ with respect to } x = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- The **average rate of change** of a variable with respect to another between two points is given by:

$$\text{Average rate of change} = \frac{\text{change in independent variable}}{\text{corresponding change in dependent variable}} = \frac{\Delta y}{\Delta x}$$

- The **instantaneous rate of change** of a variable at a point is the same as the **gradient of the tangent** to the graph at that point.
- A **difference quotient** is an expression of the form  $\frac{f(x+h) - f(x)}{h}$ .

It represents the average change in the value of a function between  $x$  and  $(x+h)$ .

- If  $f(x)$  becomes arbitrarily close to a unique number  $L$  as  $x$  approaches  $c$  from either below or above, then the **limit** of  $f(x)$  as  $x$  approaches  $c$  is  $L$ . This is written as

$$\lim_{x \rightarrow c} f(x) = L$$

- $\lim_{x \rightarrow c} f(x)$  does not exist if:
  - $f(x)$  approaches different numbers as  $x \rightarrow c$  from below and above
  - $f(x)$  increases or decreases without bound as  $x \rightarrow c$ .

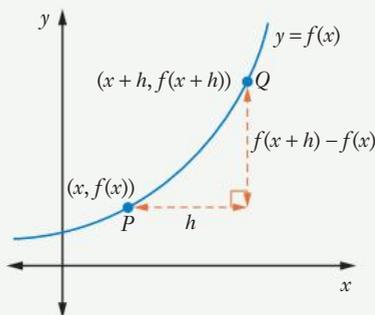
- The **derivative** of the function  $f(x)$  is:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

The derivative can be written as:

$$f'(x) \text{ or } \frac{df}{dx} \text{ or } \frac{d}{dx} f(x).$$



- The derivative of the function  $f(x)$  at the point where  $x = c$  is:

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ or } \frac{d}{dx} f(x) \Big|_{x=c}$$

- The derivative of  $y = f(x)$  may be defined as  $y' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$  and written as  $y'$  or  $\frac{dy}{dx}$ .
- The derivative of  $y$  at  $x = c$  can be written as  $\frac{dy}{dx} \Big|_{x=c}$  or  $y'(c)$ .

- The **instantaneous rate of change** of the function  $f(x)$  is given by:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

The instantaneous rate of change of  $f(x)$  at  $x = a$  is  $f'(a)$ .

- **Distance** is how far an object has travelled.
- **Displacement** is the change in position of an object from its initial to its final position regardless of the path taken.
- **Velocity**,  $v(t)$ , is the rate of change of displacement with respect to time.

$$v(t) = x'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}, \text{ where position} = f(t)$$

- **Acceleration**,  $a(t)$ , is the rate of change of velocity with respect to time.

$$a(t) = v'(t) = \lim_{h \rightarrow 0} \frac{v(t+h) - v(t)}{h}, \text{ where velocity} = v(t)$$

- The slope of the tangent to the graph of  $y = f(x)$  at point  $P(a, f(a))$  is:

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

# 7. CHAPTER REVIEW

## Rates of change and the derivative

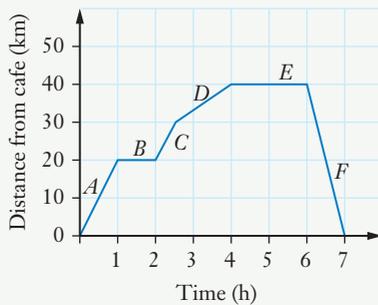
- 1** The results in the table were obtained by measuring the height of water in a rainwater tank and the flow rate from the tank. Use a graph to determine whether they are consistent with a constant rate.

Example  
**1**

Water height (cm)	250	100	150	190	70
Flow rate (L/min)	9	4	5	7	3

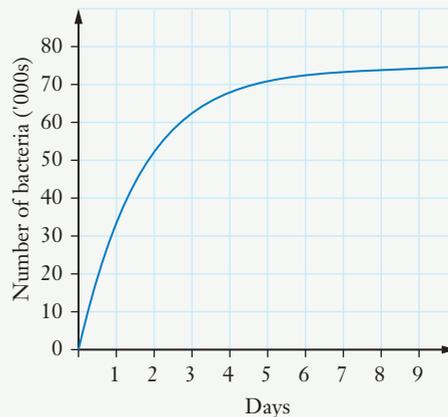
- 2** A cycling club meets at a café and leaves for their regular Sunday outing. The group stops twice during their outing. The graph shows the distance of one of the cyclists from the café at various times during the outing.

Example  
**2**



- a** When was the cyclist moving at his greatest speed?  
**b** What was the average speed of the cyclist between the two stops?  
**c** How long is their second stop?  
**d** What was the cyclist's average speed on section *D*?  
**e** What was the cyclist's average speed for the whole trip?  
**f** What was the cyclist's average velocity for the whole trip?
- 3** The graph shows the number of bacteria found in a culture over time.

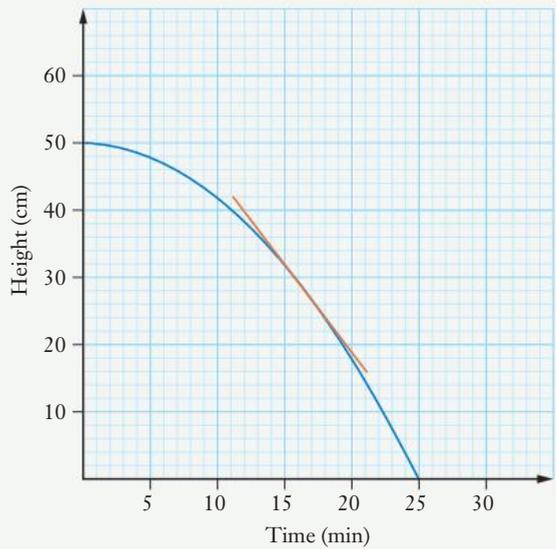
- a** Use the graph to calculate the average rate of increase:
- for the first day
  - for the first 2 days
  - between days 4 and 9
- b** Describe how the rate changes.



Example  
**3**

Example  
**4**

- 4** Liquid is drained from a tank when a release valve is opened. The graph shown here shows the height of liquid in the tank over time. Use a tangent to the graph to find the rate (correct to one decimal place) at which the height is changing 15 minutes after the valve is opened.



Example  
**5**

- 5** Sketch the graph of  $y = x^2 - 3x - 10$  from  $x = -2$  to  $x = 6$ , then use the graph to find the rate of change of  $y$  wrt  $x$  at  $x = 2$ .

Example  
**6**

- 6** Sketch the graph of  $f(x) = x^3 - 5x^2 + 4x - 8$  from  $x = -2$  to  $x = 4$ , then use the graph to find the rate of change of  $y$  wrt  $x$  at  $x = 0$ .

Examples  
**7-9**

- 7** Calculate the difference quotient for each of the following.

**a**  $f(x) = 4x - 9$

**b**  $f(x) = 12 - 3x$

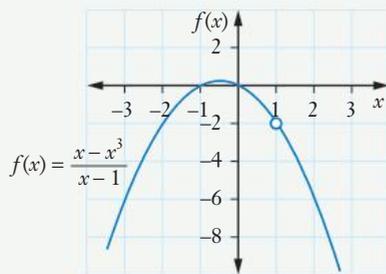
**c**  $f(x) = 5x^2 + 7x$

**d**  $f(x) = x^3$

**e**  $f(x) = \frac{1}{3-x}$

Example  
**10**

- 8** The graph of  $f(x) = \frac{x-x^3}{x-1}$  is shown here.



- a** Find the value of  $f(x)$  when  $x = 1$ .

- b** Use tables of values to find  $\lim_{x \rightarrow 1} f(x)$ .

- 9 Find each limit if it exists. If the limit does not exist, explain why. Use a graphics calculator to confirm your results.

a  $\lim_{x \rightarrow 0} \left( \frac{8}{x^2} \right)$

b  $\lim_{x \rightarrow -2} \left( \frac{x^2 - 4}{x + 2} \right)$

c  $\lim_{x \rightarrow -3} \left( \frac{2x^2 + x - 15}{x + 3} \right)$

d  $\lim_{x \rightarrow 2} \left( \frac{|x|}{x - 2} \right)$

- 10 For the function  $f(x) = 3x^2 - 2x - 9$ ,
- find the rule for the gradient of the function
  - find the value of the gradient at  $x = -2$ .

- 11 Given that  $f(x) = 4x^2 + 9x - 7$ ,
- find a rule for the derivative
  - find the derivative of the function at  $x = -3$ .

- 12 The volume of liquid in a container (in mL) at  $t$  seconds is given by the rule  $V(t) = 2t^2 + t + 3$ .
- What is the volume of liquid in the container when  $t = 2$ ?
  - What is the volume when  $t = 5$ ?
  - What is the average rate of change of the volume of liquid in the container from  $t = 2$  to  $t = 5$ ?
  - Calculate the instantaneous rate of change at  $t = 3$ .
  - Calculate the instantaneous rate of change at  $t = 5$ .

- 13 A ball is thrown straight upward so that its height (in metres) relative to the ground is given by the equation:

$$x = 125 + 8t - 5t^2$$

where  $t$  is in seconds. In terms of the ball's height, find:

- its position after 1 s
  - its position after 4 s
  - its displacement after 2 s
  - its velocity after  $\frac{1}{5}$  s
  - its velocity after 3 s
  - its acceleration after 4 s
- 14 The displacement of an object falling from a height is given by  $x = 200 - 4.9t^2$ , where  $x$  is the height in metres after  $t$  s.

Find its velocity after:

- 1 s
- 2 s
- 3 s
- 4 s

Example  
15

- 15 The position of an object moving away from its origin is given by:

$$x = \frac{t^2}{6} + t$$

where  $x$  is in metres and  $t$  is in seconds.

- a Calculate the velocity of the object after 15 seconds.
- b Calculate the acceleration after 12 seconds.
- c Draw graphs of the velocity and acceleration for the first 30 seconds.

Example  
16

- 16 A cyclist in a road race starts to move off in the first section of the race. The position of the cyclist is given by  $x = 1.5t^2 + t$ , where  $t$  is in seconds and  $x$  is in metres. Find the cyclist's displacement and velocity after 1 and 2 seconds.

Example  
16

- 17 Find the equation of the tangent to  $f(x) = x^2 - 3x + 5$  at  $x = -1$ .

Example  
17

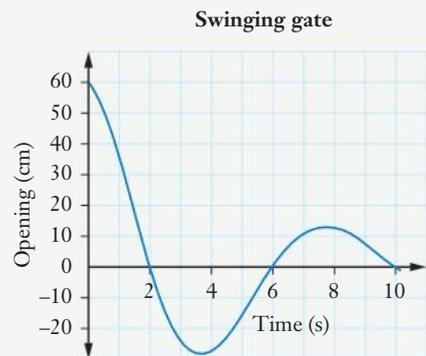
- 18 Given that  $f(x) = 2x^2 - 7x + 4$ , find:
- a the equation of the slope of  $f(x)$
  - b where the slope of the graph is 0
  - c where the slope of the graph is 5.

### Problem solving

- 19 A jogger left home at 6:00 a.m., jogging 8 km in 1 h 30 min. He jogged a further 6 km in the next 45 min. before stopping for 15 min. He then returned home, arriving back at 11:15 a.m.
- a Draw a graph of his trip.
  - b When was he jogging the fastest?
  - c When was he jogging the most slowly?

- 20 A gate has a spring that returns it to the central closed position. The opening of the gate is shown in the graph. Use a tangent to estimate the rate of closure of the gate at:

- a 2 s
- b 4 s
- c 6 s
- d 8 s



- 21** The marginal cost of the production of an item at a particular production volume is the cost of producing the last item. It is actually the rate of change of the cost of production. For a particular manufacturer, the cost of producing  $n$  items is given by the function

$$C(n) = 0.0004n^2 + 0.27n + 5, \text{ where } C(n) \text{ is in dollars.}$$

- a** Find the cost of producing:
- i** 500 items
  - ii** 501 items.
- b** Find the marginal cost of production of the 501st item.
- c** Find the cost of producing:
- i** 1000 items
  - ii** 1001 items.
- d** Find the marginal cost of production of the 1001st item.
- e** Find the cost of producing:
- i** 1500 items
  - ii** 1501 items.
- f** Find the marginal cost of production of the 1501st item.
- g** Find a rule to work out the approximate marginal cost of production of the  $n$ th item.
- 22** Find where the gradient function  $f(x) = x^2 + 7x - 3$  is greater than the gradient function of  $g(x) = 3x^2 - 5x - 8$ .
- 23** The total surface area of a cylinder is given by  $A(r) = 2\pi rh + 2\pi r^2$ . If the radius is changing but the height is constant, find a rule for the rate of change of the total surface area with respect to the radius,  $r$ .



Practice quiz

# 8

# TRIGONOMETRIC FUNCTIONS

Trigonometry has an ancient history. Ancient Greeks, Indians, Arabians and Persians began it to study circles, Earth geometry, astronomy and surveying.

Today it is used in Science, Economics, Medical science and many other fields.

Periodic phenomena and functions are easiest to understand as graphs. In this chapter you will look at many such graphs.

- 8.01 Radian measure
- 8.02 Arc length and sector area
- 8.03 The unit circle and trigonometric ratios
- 8.04 The graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$
- 8.05 Transformations of trigonometric functions
- 8.06 Trigonometric equations
- 8.07 Modelling with trigonometric functions
- Chapter summary
- Chapter review

## SYLLABUS SUBJECT MATTER

### Circular measure and radian measure

- define and use radian measure and understand its relationship with degree measure
- calculate lengths of arcs and areas of sectors in circles

### Introduction to trigonometric functions

- understand the unit circle definition of  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$  and periodicity using radians
- recall the exact values of  $\sin \theta$ ,  $\cos \theta$  and  $\tan \theta$  at integer multiples of  $\frac{\pi}{6}$  and  $\frac{\pi}{4}$
- sketch the graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$  on extended domains
- investigate the effect of the parameters  $A$ ,  $B$ ,  $C$ ,  $D$  on the graphs of  $y = A \sin(Bx + C) + D$ ,  $y = A \cos(Bx + C) + D$  with and without technology
- sketch the graphs of  $y = A \sin(Bx + C) + D$ ,  $y = A \cos(Bx + C) + D$  with and without technology
- identify contexts suitable for modelling by trigonometric functions and use them to solve practical problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis
- solve equations involving trigonometric functions with and without technology

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Prior learning

# TERMINOLOGY

adjacent  
circular measure  
degree  
opposite  
phase  
second  
sinusoidal  
tangent (tan)

arc  
cosine (cos)  
hypotenuse  
period  
radian  
sector  
standard triangle  
trigonometric function

amplitude  
cycle  
minute  
periodic function  
reference angle  
sine (sin)  
standard position  
unit circle



Converting degrees  
and radians

## 8.01 Radian measure

Angle measure in **degrees** is based on the ancient Babylonian number system.

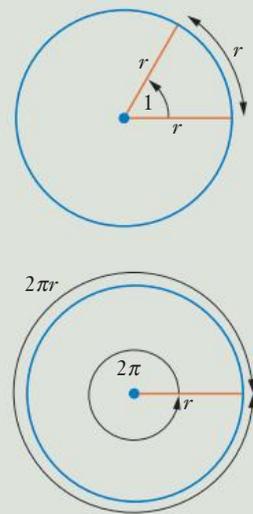
**Circular angle measure** in **radians** is more convenient in Mathematics and Science.

One **radian** ( $1^c$  or **1 rad**) is the size of the angle subtended at the centre of a circle by an **arc** equal in length to the radius of the circle.

Angles are assumed to be in radians unless shown otherwise, such as degrees ( $^\circ$ ).

Since the circumference of a circle is  $2\pi r$ , one complete revolution in radians is  $2\pi$ , so  $2\pi = 360^\circ$ , and therefore:

$$\begin{array}{lll} \pi = 180^\circ & \frac{\pi}{2} = 90^\circ & \frac{3\pi}{2} = 270^\circ \\ \frac{\pi}{3} = 60^\circ & \frac{\pi}{6} = 30^\circ & \frac{\pi}{4} = 45^\circ \end{array}$$



As  $\pi = 180^\circ$ ,  $1 \text{ radian} = \frac{180^\circ}{\pi} = 57.3^\circ$ .

### EXAMPLE 1

Convert:

- $\frac{9\pi}{5}$  to degrees
- 2.07 rad to degrees
- $240^\circ$  to radians

## Solution

**a** Use  $\pi = 180^\circ$ .

$$\begin{aligned}\frac{9\pi}{5} &= \frac{9 \times 180^\circ}{5} \\ &= 324^\circ\end{aligned}$$

**b** From  $\pi = 180^\circ$ ,  $1 \text{ rad} = \frac{180^\circ}{\pi}$ .

$$\begin{aligned}2.07 \text{ rad} &= 2.07 \times \frac{180^\circ}{\pi} \\ &\approx 118.60^\circ\end{aligned}$$

**c** From  $180^\circ = \pi$ ,  $1^\circ = \frac{\pi}{180}$ .

$$\begin{aligned}240^\circ &= 240^\circ \times \frac{\pi}{180^\circ} \\ &= \frac{4\pi}{3}\end{aligned}$$

Leave in exact form.

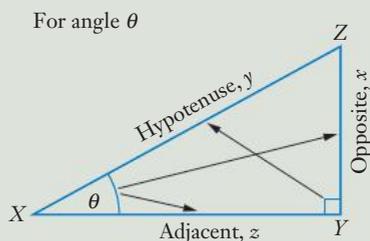
The hypotenuse is the longest side in a right-angled triangle. It is opposite the right angle. The other sides are opposite or adjacent to the angle being considered.

You should remember the trigonometric ratios **sine**, **cosine** and **tangent** for right-angled triangles. They are abbreviated to **sin**, **cos** and **tan**.

$$\sin(\theta) = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{x}{y}$$

$$\cos(\theta) = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{z}{y}$$

$$\tan(\theta) = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{x}{z}$$



The ratios can be remembered using the following mnemonic.

**SOH**

$$\text{Sin} = \frac{\text{O}}{\text{H}}$$

**CAH**

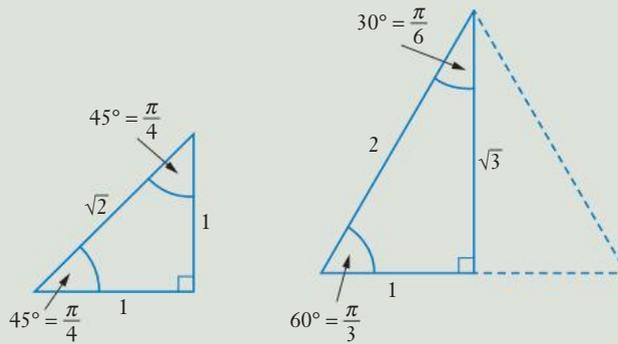
$$\text{Cos} = \frac{\text{A}}{\text{H}}$$

**TOA**

$$\text{Tan} = \frac{\text{O}}{\text{A}}$$

← Say 'sock-ah-towa'

## Exact trigonometric ratios using standard triangles



Angle	$30^\circ = \frac{\pi}{6}$	$45^\circ = \frac{\pi}{4}$	$60^\circ = \frac{\pi}{3}$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

### EXAMPLE 2

Find the exact value of:

**a**  $\sin\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right)$

**b**  $\tan^2\left(\frac{\pi}{6}\right)$

#### Solution

- a** Substitute exact values.

$$\begin{aligned} \sin\left(\frac{\pi}{4}\right) \times \cos\left(\frac{\pi}{6}\right) &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{6}}{4} \end{aligned}$$

Evaluate.

**b** Write the expression.

$$\tan^2\left(\frac{\pi}{6}\right) = \left[\tan\left(\frac{\pi}{6}\right)\right]^2$$

Substitute exact values.

$$= \left(\frac{1}{\sqrt{3}}\right)^2$$

Evaluate.

$$= \frac{1}{3}$$

You need to use your calculator for most trigonometric ratios.

### EXAMPLE 3

Calculate correct to two decimal places:

**a**  $\sin\left(\frac{4\pi}{7}\right)$       **b**  $\cos(1.12)$

#### Solution

Make sure the calculator is set to radians.

##### TI-84 Plus CE

**a** Press  $\boxed{2\text{nd}} \boxed{\text{mode}} \boxed{\text{quit}}$  if necessary. Enter the expression as shown.



State the result.

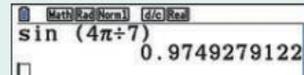
**b** Enter the expression as shown.



State the result.

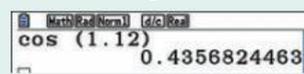
##### Casio fx-CG20AU

Select the **Run-Matrix** page. Enter the expression as shown.



$$\sin\left(\frac{4\pi}{7}\right) \approx 0.97$$

Enter the expression as shown.



$$\cos(1.12) \approx 0.44$$



TI-Nspire CX  
Chapter 8

## Exercise 8.01 Radian measure

**1** Express each angle in radians, leaving your answers in terms of  $\pi$ .

**a**  $60^\circ$       **b**  $45^\circ$       **c**  $90^\circ$       **d**  $135^\circ$       **e**  $120^\circ$

**f**  $100^\circ$       **g**  $420^\circ$       **h**  $72^\circ$       **i**  $288^\circ$

**2** Express each angle in radians, correct to 4 decimal places where necessary.

**a**  $56^\circ$       **b**  $140^\circ$       **c**  $276^\circ$       **d**  $267.8^\circ$       **e**  $566.42^\circ$

Example  
**1**

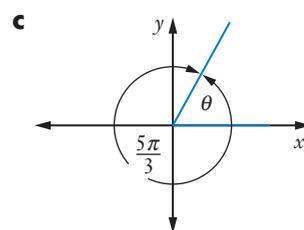
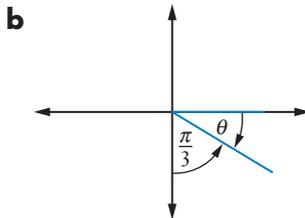
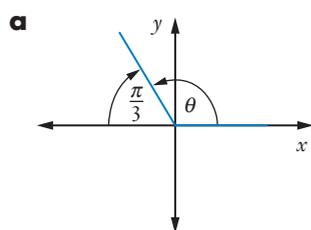
**3** Convert each angle to degrees, correct to 1 decimal place where necessary.

**a**  $\frac{2\pi}{3}$       **b**  $\frac{5\pi}{6}$       **c**  $\frac{7\pi}{4}$       **d**  $\frac{6\pi}{11}$       **e**  $\frac{11\pi}{5}$

**4** Convert each angle to degrees, correct to 1 decimal place where necessary.

**a** 1      **b** 2.2      **c** 0.8      **d** 3.3      **e** 2.06

**5** Find the value of  $\theta$  in each diagram.



Example  
**2**

**6** Find the exact value of:

**a**  $\cos\left(\frac{\pi}{3}\right) \times \sin\left(\frac{\pi}{4}\right)$       **b**  $\tan\left(\frac{\pi}{3}\right) + \sin\left(\frac{5\pi}{6}\right)$       **c**  $\sin^2\left(\frac{11\pi}{4}\right) + \cos^2\left(\frac{11\pi}{4}\right)$

**d**  $\frac{3 \cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{6}\right)}$       **e**  $\frac{\cos\left(\frac{\pi}{4}\right) + 1}{\sin\left(\frac{\pi}{4}\right) - 1}$       **f**  $\sin^2\left(\frac{2\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) + 1$

Example  
**3**

**7** Calculate correct to two decimal places:

**a**  $\cos\left(\frac{8\pi}{7}\right)$       **b**  $\sin\left(\frac{4\pi}{9}\right)$       **c**  $\cos\left(\frac{2\pi}{5}\right)$       **d**  $\tan\left(\frac{\pi}{13}\right)$

**8** Calculate correct to two decimal places:

**a**  $\cos(1.31)$       **b**  $\sin(0.41)$       **c**  $\tan(2.04)$       **d**  $\sin(1.63)$       **e**  $\cos(0.18)$

Problem solving

**9** Show that  $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right) \sin\left(\frac{\pi}{4}\right)$ .

**10** Show that  $\sin^2\left(\frac{\pi}{6}\right) + \cos^2\left(\frac{\pi}{6}\right) = 1$ .

## 8.02 Arc length and sector area

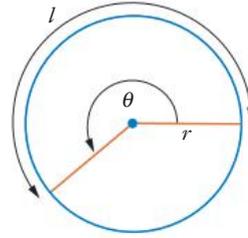
Suppose you have an **arc**  $l$  that subtends an angle  $\theta$  at the centre of a circle of radius  $r$ .

$$C = 2\pi r,$$

$$\text{so } \frac{l}{C} = \frac{\theta}{2\pi}$$

$$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

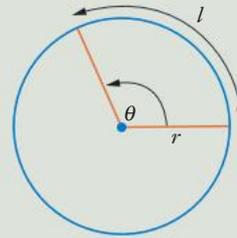
$$\text{so } l = r\theta$$



### Arc length

The **arc length** subtended by an angle  $\theta$  at the centre of a circle is given by

$$l = r\theta$$



### EXAMPLE 4

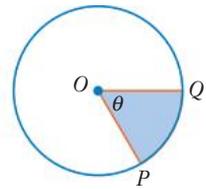
- a** Find the exact length of an arc that subtends an angle of  $\frac{5\pi}{6}$  in a circle of radius 10 cm.
- b** A circle has a radius of 12 cm. An arc measuring 25 cm subtends the central angle  $\theta$ . Calculate the angle in radians, correct to 2 decimal places.

### Solution

- a** Write the formula.  $l = r\theta$
- Substitute known values.  $= 10 \times \frac{5\pi}{6}$
- Evaluate.  $= \frac{25\pi}{3}$
- State the result. The length of the arc is  $\frac{25\pi}{3}$  cm.
- b** Write the formula.  $l = r\theta$
- Substitute known values.  $25 = 12 \times \theta$
- Isolate  $\theta$ .  $\theta = \frac{25}{12}$
- Evaluate.  $= 2.0833\dots$
- State the result. The central angle is approximately 2.08 radians.

Consider the diagram on the right showing **sector**  $POQ$  shaded.

The area of the sector is proportional to the angle it subtends at the centre of the circle.

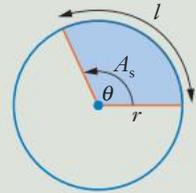


$$\begin{aligned} \text{So } \frac{\text{Area of sector}}{\text{Area of circle}} &= \frac{\theta}{2\pi} \\ \frac{A}{\pi r^2} &= \frac{\theta}{2\pi} \\ A_s &= \frac{1}{2} r^2 \theta \end{aligned}$$

### Area of a sector

For a sector with radius  $r$  and central angle  $\theta$  (measured in radians):

$$A_s = \frac{1}{2} r^2 \theta$$



### EXAMPLE 5

Find the area of a sector of a circle if the arc length of the sector is 15 cm and the radius is 10 cm.

#### Solution

Write the rule for the central angle.	$\theta = \frac{l}{r}$
Substitute known values.	$= \frac{15}{10}$
Evaluate.	$= 1.5 \text{ radians}$
Write the rule for area of a sector.	$A_s = \frac{1}{2} r^2 \theta$
Substitute known values.	$= \frac{1}{2} \times 10^2 \times 1.5$
	$= 75$
Write the result.	The area of the sector is $75 \text{ cm}^2$ .

## EXAMPLE 6

The area of the sector of a circle formed by the central angle of  $\frac{2\pi}{5}$  is  $\frac{33\pi}{4}$  cm<sup>2</sup>. Find the radius of the circle.

### Solution

Write the rule for the area of a sector.

$$A_s = \frac{1}{2}r^2\theta$$

Substitute known values.

$$\frac{33\pi}{4} = \frac{1}{2} \times r^2 \times \frac{2\pi}{5}$$

Isolate  $r^2$ .

$$r^2 = \frac{165}{4}$$

Evaluate.

$$= 41.25$$

Calculate  $r$ .

$$r = 6.4226\dots$$

Round off and state the result.

The radius of the circle is about 6.4 cm.

## Exercise 8.02 Arc length and sector area

- 1 Calculate the length of each arc in terms of  $\pi$  (correct to 2 decimal places for part **f**) that subtends central angle  $\theta$  in a circle with radius  $r$ .

**a**  $r = 5$  cm,  $\theta = \frac{\pi}{3}$

**b**  $r = 17$  mm,  $\theta = \frac{5\pi}{6}$

**c**  $r = 1.8$  m,  $\theta = \frac{\pi}{2}$

**d**  $r = 12$  cm,  $\theta = \frac{2\pi}{3}$

**e**  $r = 36$  mm,  $\theta = 30^\circ$

**f**  $r = 10.9$  cm,  $\theta = 125^\circ$

- 2 Calculate the length of the radius in terms of  $\pi$  (correct to 2 decimal places for part **f**) for a circle with arc length,  $l$ , and central angle,  $\theta$ , as shown.

**a**  $l = 30$  mm,  $\theta = \frac{\pi}{5}$

**b**  $l = 21$  cm,  $\theta = \frac{2\pi}{3}$

**c**  $l = 60$  cm,  $\theta = \frac{\pi}{4}$

**d**  $l = 8.4$  m,  $\theta = \frac{\pi}{7}$

**e**  $l = 43$  cm,  $\theta = \frac{6\pi}{13}$

**f**  $l = 1.43$  m,  $\theta = 0.85$

- 3 Calculate the central angle (correct to 2 decimal places for part **c**) subtended by an arc of length,  $l$ , in a circle of radius  $r$ .

**a**  $r = 7$  cm,  $l = 14$  cm

**b**  $r = \frac{16\pi}{3}$  mm,  $l = \frac{4\pi}{15}$  mm

**c**  $r = 2.4$  m,  $l = 4.7$  m

**d**  $r = \frac{5\pi}{6}$  m,  $l = \frac{5\pi}{12}$  m

**e**  $r = 28$  cm,  $l = 35$  cm

**f**  $r = \frac{7\pi}{3}$  cm,  $l = \frac{21\pi}{8}$  cm

- 4 Calculate correct to one decimal place the length of an arc that subtends an angle of  $65^\circ$  at the centre of a circle of radius 20 cm.

Example  
4

Example  
5

5 An arc, 25 cm long, subtends an angle of  $36^\circ$  at the centre of a circle. Calculate correct to one decimal place the radius of the circle.

Example  
6

6 Each of the following circles has its radius ( $r$ ) and central angle ( $\theta$ ) stated. In each case, calculate the area of the sector (correct to 2 decimal places if necessary).

**a**  $r = 30$  mm,  $\theta = 60^\circ$       **b**  $r = 8.4$  m,  $\theta = 120^\circ$       **c**  $r = 1.43$  m,  $\theta = 1.32$

7 Calculate the radius (correct to 2 decimal places) of each circle given the sector area,  $A_s$ , cut off by the central angle,  $\theta$ .

**a**  $A_s = 20$  cm<sup>2</sup>,  $\theta = 45^\circ$       **b**  $A_s = 8.4$  m<sup>2</sup>,  $\theta = 0.74$   
**c**  $A_s = 4.3$  cm<sup>2</sup>,  $\theta = 76^\circ$       **d**  $A_s = 1.43$  cm<sup>2</sup>,  $\theta = 1.84$

8 A circle has a central angle of  $\frac{7\pi}{6}$  that subtends an arc measuring 33.27 cm. Calculate the radius of the circle and the area of the sector subtended by the central angle.

9 The area of sector  $XOY$  of a circle with centre  $O$  and radius 18 cm is found to be  $210$  cm<sup>2</sup>. Calculate:

**a** the magnitude of  $\angle XOY$       **b** the length of arc  $XY$

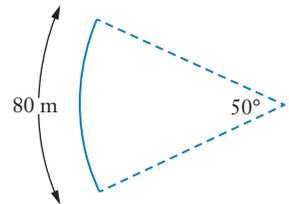
10 In a circle of radius 1.2 m, an arc subtends an angle of  $50^\circ$  at the centre. Calculate:

- a** the arc length  
**b** the length of the chord that joins the endpoints of the arc.

### Problem solving

11 The minute hand of a wall clock is 15 cm long. How far does the tip of the minute hand move in 18 minutes?

12 A curve on an athletics track is designed to be laid on the arc of a circle. If 80 m of track is required for a change of direction of  $50^\circ$ , what radius should be used?



13 A windscreen wiper is 48 cm long. The blade reaches from the endpoint of the wiper to a point that is 8 cm from the pivot point. Find the area (to the nearest cm<sup>2</sup>) of the windscreen swept by the blade when the wiper moves through an angle of  $140^\circ$ .



14 A screen printer creates printed designs by rotating a circular drum over the screen, which has been placed over the material on which the print is required. The drum has a diameter of 50 cm and the screen is 75 cm long. What percentage (correct to one decimal place) of the drum needs to be inked?

## 8.03 The unit circle and trigonometric ratios



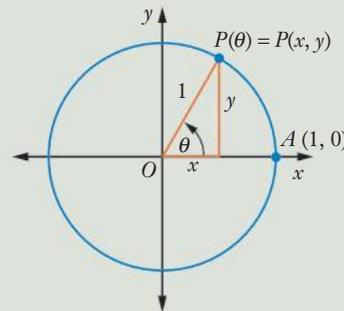
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The definitions of sin, cos and tan using triangles only work for angles up to  $90^\circ$ . You can use the **unit circle** to extend the definitions to angles of any size.

### Trigonometric ratios using the unit circle

If  $P(x, y)$  is a point on the unit circle found by rotating the point  $A(1, 0)$  anticlockwise through an angle of  $\theta$ , then you define:

$$\begin{aligned}\sin(\theta) &= \frac{y}{1} = y \\ \cos(\theta) &= \frac{x}{1} = x \\ \tan(\theta) &= \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}\end{aligned}$$



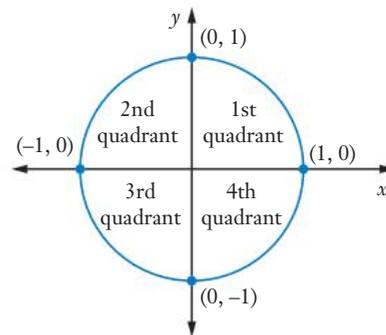
These definitions are sometimes extended to a circle of radius  $r$  as:

$$\sin(\theta) = \frac{y}{r} \quad \cos(\theta) = \frac{x}{r} \quad \tan(\theta) = \frac{y}{x} = \frac{\sin(\theta)}{\cos(\theta)}$$

$\sin(\theta) = y$ , so  $\sin(\theta)$  will be positive in the quadrants where  $y$  is positive, i.e. the first and second quadrants.

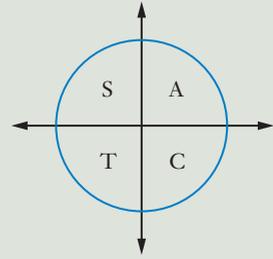
$\cos(\theta) = x$ , so  $\cos(\theta)$  will be positive in the quadrants where  $x$  is positive, i.e. the first and fourth quadrants.

$\tan(\theta) = \frac{y}{x}$ , so  $\tan(\theta)$  will be positive when  $x$  and  $y$  have the same sign, i.e. in the first and third quadrants.



## Signs of the trigonometric ratios

- 1st quadrant: All ratios are positive (**A**)  
 2nd quadrant: Sin only is positive (**S**)  
 3rd quadrant: Tan only is positive (**T**)  
 4th quadrant: Cos only is positive (**C**)



This is called the ‘ASTC’ rule and you can remember it by making up a phrase like ‘All Science Teachers are Curious’ or ‘All Stations To Central’.

### Second quadrant

$$\sin(\theta) = \sin(180^\circ - \beta) = \sin(\pi - \beta) = \sin(\beta)$$

$$\cos(\theta) = \cos(180^\circ - \beta) = \cos(\pi - \beta) = -\cos(\beta)$$

$$\tan(\theta) = \tan(180^\circ - \beta) = \tan(\pi - \beta) = -\tan(\beta)$$

### Third quadrant

$$\sin(\theta) = \sin(180^\circ + \beta) = \sin(\pi + \beta) = -\sin(\beta)$$

$$\cos(\theta) = \cos(180^\circ + \beta) = \cos(\pi + \beta) = -\cos(\beta)$$

$$\tan(\theta) = \tan(180^\circ + \beta) = \tan(\pi + \beta) = \tan(\beta)$$

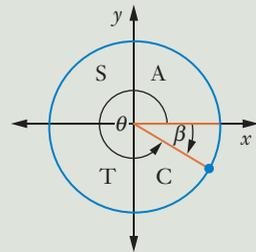
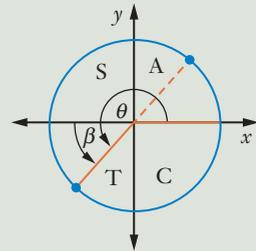
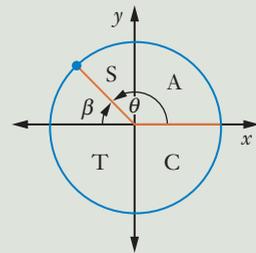
### Fourth quadrant

$$\sin(\theta) = \sin(360^\circ - \beta) = \sin(2\pi - \beta) = -\sin(\beta)$$

$$\cos(\theta) = \cos(360^\circ - \beta) = \cos(2\pi - \beta) = \cos(\beta)$$

$$\tan(\theta) = \tan(360^\circ - \beta) = \tan(2\pi - \beta) = -\tan(\beta)$$

The **reference angle**  $\beta$  is the acute angle between the given angle and  $0, \pi, 2\pi, -\pi$ , etc.



## EXAMPLE 7

Find the exact value of:

**a**  $\sin(225^\circ)$

**b**  $\cos\left(\frac{11\pi}{6}\right)$

**c**  $\tan\left(\frac{2\pi}{3}\right)$

### Solution

**a** Draw a sketch.

Calculate the reference angle and mark it on your sketch.

Using ASTC,  $\sin(225^\circ)$  is negative.

Write the trig ratio using the reference angle.

Use the standard triangle.

**b** Draw a sketch.

Calculate the reference angle and mark it on your sketch.

Using ASTC,  $\cos\left(\frac{11\pi}{6}\right)$  is positive.

Write the trig ratio using the reference angle.

Use the standard triangle.

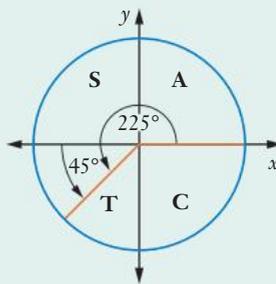
**c** Draw a sketch.

Calculate the reference angle and mark it on your sketch.

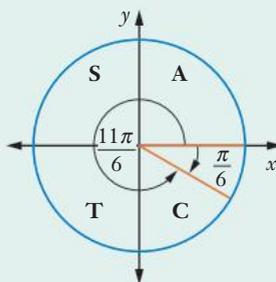
Using ASTC,  $\tan\left(\frac{2\pi}{3}\right)$  is positive.

Write the trig ratio using the reference angle.

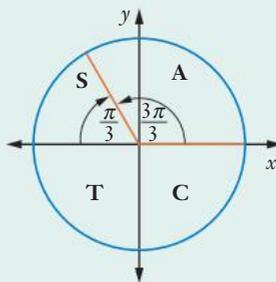
Use the standard triangle.



$$\begin{aligned}\sin(225^\circ) &= -\sin(45^\circ) \\ &= -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}\end{aligned}$$



$$\begin{aligned}\cos\left(\frac{11\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$



$$\begin{aligned}\tan\left(\frac{2\pi}{3}\right) &= -\tan\left(\frac{\pi}{3}\right) \\ &= -\sqrt{3}\end{aligned}$$

You need more than one revolution for trigonometric ratios of angles greater than  $360^\circ$ .

### EXAMPLE 8

- a** In which quadrant is  $850^\circ$ ?  
**b** Find the exact value of  $\sin\left(\frac{10\pi}{3}\right)$ .

#### Solution

- a** Find the number of revolutions that are needed to reach  $850^\circ$ .

Calculate the angle.

State the result.

$360^\circ$  is one revolution.

$720^\circ$  is two revolutions.

$$850^\circ - 720^\circ = 130^\circ$$

$130^\circ$  is in the 2nd quadrant, so  $850^\circ$  is in the 2nd quadrant.

- b** Find the number of revolutions that are needed to reach  $600^\circ$ .

Calculate the angle.

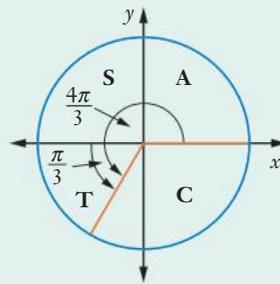
Draw a sketch.

Calculate the reference angle.

Using ASTC,  $\sin\left(\frac{4\pi}{3}\right)$  is negative.

$360^\circ$  is one revolution.

$$\frac{10\pi}{3} - 2\pi = \frac{4\pi}{3}$$



Write the trig ratio using the reference angle.  $\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$

Use the standard triangle.

$$= -\frac{\sqrt{3}}{2}$$

State the result.

$$\sin\left(\frac{10\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

You can change a negative angle to the positive equivalent to work out the trigonometric ratios.

### Exercise 8.03 The unit circle and trigonometric ratios

- 1** Use the general rules for trigonometric ratios to copy and complete each statement.

**a**  $\cos(\pi + a) =$

**b**  $\sin(360^\circ - x) =$

**c**  $\tan(180^\circ + \theta) =$

**d**  $\tan(360^\circ + \beta) =$

**e**  $\tan(2\pi - \alpha) =$

**f**  $\sin(\pi + \alpha) =$

Example  
7

**2** Find the exact value of:

- a**  $\cos(120^\circ)$                       **b**  $\sin\left(\frac{4\pi}{3}\right)$                       **c**  $\tan(135^\circ)$   
**d**  $\cos(300^\circ)$                       **e**  $\tan\left(\frac{7\pi}{6}\right)$                       **f**  $\cos(330^\circ)$

**3** Use the unit circle to find:

- a**  $\sin(90^\circ)$                       **b**  $\cos(180^\circ)$                       **c**  $\tan(2\pi)$   
**d**  $\cos(270^\circ)$                       **e**  $\tan(\pi)$                       **f**  $\sin(180^\circ)$

**4** Find the exact value of:

- a**  $\sin\left(\frac{13\pi}{6}\right)$                       **b**  $\cos(510^\circ)$                       **c**  $\tan(4\pi)$                       **d**  $\cos(450^\circ)$

Example  
**8**

### Problem solving

**5 a** Draw a right-angled triangle in the Cartesian plane where  $\theta$  is in quadrant 1 and

$$\sin(\theta) = \frac{4}{5}.$$

**b** Calculate the length of the other side of the triangle.

**c** Without actually calculating the value of  $\theta$ , find the exact values of the other two trig ratios using the information supplied.

**6** Without actually calculating the value of  $\theta$ , find the exact values of the other two trigonometric ratios using the information supplied.

**a**  $\cos(\theta) = \frac{5}{13}$  and  $\theta$  is in quadrant 1                      **b**  $\cos(\theta) = -\frac{7}{25}$  and  $\theta$  is in quadrant 2

**c**  $\sin(\theta) = -\frac{3}{5}$  and  $\theta$  is in quadrant 3                      **d**  $\sin(\theta) = -\frac{1}{\sqrt{2}}$  and  $\theta$  is in quadrant 4

**e**  $\cos(\theta) = \frac{1}{\sqrt{2}}$  and  $\theta$  is in quadrant 1                      **f**  $\cos(\theta) = -\frac{24}{25}$  and  $\theta$  is in quadrant 3

**g**  $\sin(\theta) = \frac{12}{13}$  and  $\theta$  is in quadrant 2

**7** Find the exact value of:

**a**  $\tan(135^\circ) - \cos(135^\circ)$                       **b**  $\sin(240^\circ) + \sin(150^\circ)$

**c**  $\cos(120^\circ) + \sin(240^\circ)$                       **d**  $\sin^2(330^\circ)$

**e**  $\sin^2(300^\circ) + \cos^2(300^\circ)$

**8** Simplify:

**a**  $\tan(\pi + x) + 3 \tan(x)$

**b**  $\cos(180^\circ - x) + \cos(360^\circ - x)$

**c**  $\sin(180^\circ + 5y) - 5 \sin(5y)$

**d**  $\cos(\pi + x) + 2 \cos(2\pi - x)$



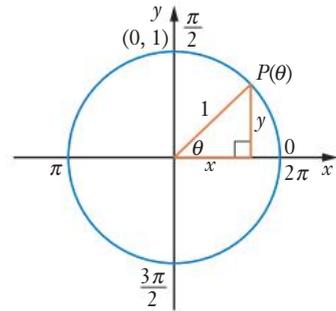
# 8.04 The graphs of $y = \sin (x)$ , $y = \cos (x)$ and $y = \tan (x)$

For the angle  $\theta$  in the unit circle, the point  $P(\theta)$  is the trigonometric point corresponding to  $\theta$ .

$$\sin (\theta) = \frac{y}{1} = y$$

$$\cos (\theta) = \frac{x}{1} = x$$

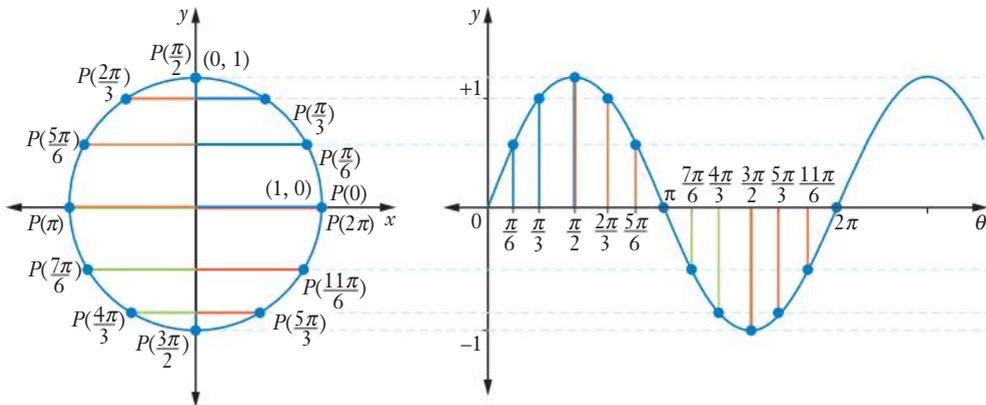
$$\tan (\theta) = \frac{y}{x}$$



You can use the unit circle to draw graphs of  $y = \sin (\theta)$ ,  $y = \cos (\theta)$  and  $y = \tan (\theta)$ .

Draw the unit circle and divide it into multiples of  $\frac{\pi}{6}$ , i.e.,  $\frac{\pi}{6}, \frac{2\pi}{6} = \frac{\pi}{3}, \frac{3\pi}{6} = \frac{\pi}{2}, \frac{4\pi}{6} = \frac{2\pi}{3}, \frac{5\pi}{6}$  and so on.

Because  $\sin (\theta) = y$ , you can transfer the  $y$  values of the points  $P\left(\frac{\pi}{6}\right), P\left(\frac{\pi}{3}\right), P\left(\frac{\pi}{2}\right), \dots, P(2\pi)$  to a set of axes on the Cartesian plane as shown below. Joining these points with a smooth curve gives the graph of  $y = \sin (\theta)$ .



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You can extend the graph in either direction of the  $x$ -axis using the symmetry of the circle.

You can see from the graph that the shape of the graph repeats. This is why the sin function is called a **periodic function**.

The basic unit that repeats itself is known as the **period** ( $T$ ).

The period is the length of one **cycle** of the function.

You can also see from the graph that value of  $\sin(\theta)$  is between 1 and  $-1$ . The function has a **mean value** of 0. The average distance between the maximum and minimum values is called the **amplitude** ( $A$ ) of the function.

### Key features of the graph of $y = \sin(x)$

Domain is all real numbers

Range is  $-1 \leq y \leq 1$ .

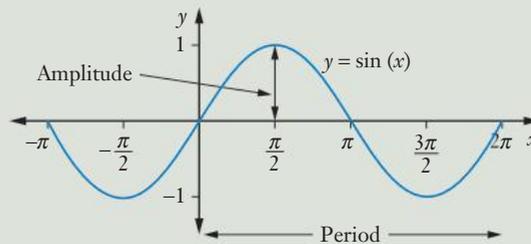
Period =  $2\pi$

Amplitude = 1

Zeros occur at  $\dots, -\pi, 0, \pi, 2\pi, \dots$

Maxima occur at  $\dots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \dots$

Minima occur at  $\dots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \dots$



The graph of  $y = \cos(\theta)$  can be drawn in a similar way to  $y = \sin(x)$  using the unit circle.

### Key features of the graph of $y = \cos(x)$

Domain is all real numbers

Range is  $-1 \leq y \leq 1$ .

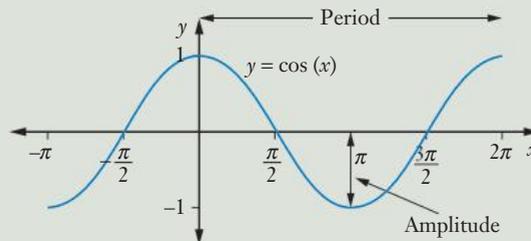
Period =  $2\pi$

Amplitude = 1

Zeros occur at  $\dots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

Maxima occur at  $\dots, -2\pi, 0, 2\pi, 4\pi, \dots$

Minima occur at  $\dots, -\pi, \pi, 3\pi, 5\pi, \dots$



You can draw the graph of  $y = \tan(\theta)$  using the tangent through the point  $A(1, 0)$ .

## Key features of the graph of $y = \tan(x)$

Domain is all real numbers except odd integral multiples of  $\frac{\pi}{2}$ .

Range is all real numbers.

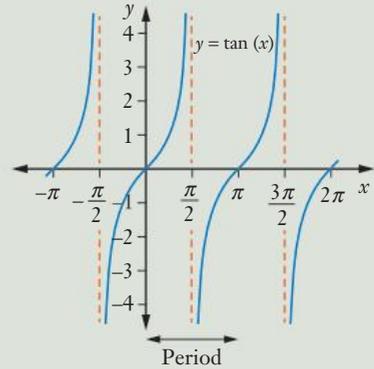
Period =  $\pi$

Zeros occur at  $\dots, -\pi, 0, \pi, 2\pi, \dots$

The graph has asymptotes

at  $\dots, x = -\frac{\pi}{2}, x = \frac{\pi}{2}, x = \frac{3\pi}{2}, x = \frac{5\pi}{2}, \dots$

At these values, the function is undefined, but near them, the values of  $\tan(x)$  become very large or very small ( $y \rightarrow \pm\infty$ ).



## EXAMPLE 9

Sketch the graph of  $y = \cos(x)$  in the domain  $-\pi \leq x \leq \pi$ .

### Solution

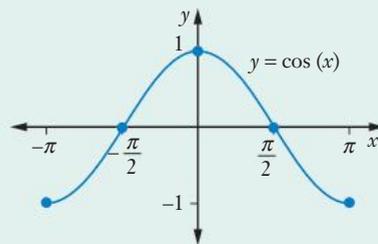
Construct a table of values for  $-\pi \leq x \leq \pi$ .

You can limit the number of points you choose because you already know the shape of the graph  $y = \cos(x)$ .

$y = \cos(x)$

$x$	$-\pi$	$-\frac{\pi}{2}$	$0$	$\frac{\pi}{2}$	$\pi$
$y$	$-1$	$0$	$1$	$0$	$-1$

Plot the points and join them with a smooth curve.



What happens to the shape of the graphs of trigonometric functions when they are translated vertically?

## EXAMPLE 10

Sketch the graph of  $y = \sin(x) - 2$  in the domain  $0 \leq x \leq 2\pi$ .

### Solution

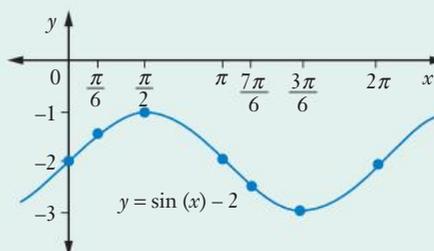
Construct a table of values for

$0 \leq x \leq 2\pi$ . You know that  $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ , so it's convenient to use values such as  $x = \frac{\pi}{6}$  and  $x = \frac{7\pi}{6}$ .

Plot the points and join them with a smooth curve.

$$y = \sin(x) - 2$$

$x$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\pi$	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$2\pi$
$y$	-2	-1.5	-1	-2	-2.5	-3	-2



### Translation of trigonometric functions

The graphs of  $y = \sin(x) + D$ ,  $y = \cos(x) + D$  and  $y = \tan(x) + D$  are **translated**  $D$  units vertically compared with the graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$  respectively.

If  $D > 0$ , the graph is translated up.

If  $D < 0$ , the graph is translated down.

### Reflection of trigonometric functions in the $x$ -axis

The graphs of  $y = -\sin(x)$ ,  $y = -\cos(x)$  and  $y = -\tan(x)$  are **reflections** in the  $x$ -axis of the graphs of  $y = \sin(x)$ ,  $y = \cos(x)$  and  $y = \tan(x)$  respectively.

## Exercise 8.04 The graphs of $y = \sin(x)$ , $y = \cos(x)$ and $y = \tan(x)$

1 Sketch the graph of each function in the domain  $-\frac{3\pi}{2} \leq x \leq \frac{\pi}{2}$ .

a  $f(x) = \sin(x)$

b  $f(x) = \cos(x)$

c  $f(x) = \tan(x)$

2 Sketch the graph of each function in the domain  $0 \leq x \leq 2\pi$ .

a  $f(x) = \cos(x) - 5$

b  $f(x) = \tan(x) + 3$

c  $f(x) = \sin(x) + 4$

Example  
9

Example  
10

- 3** Sketch the graph of each function in the domain  $-2\pi \leq x \leq \pi$ .
- a**  $y = -\sin(x)$                       **b**  $y = -\tan(x)$                       **c**  $y = -\cos(x)$
- 4** Sketch the graph of each function in the domain  $0 \leq x \leq 2\pi$ .
- a**  $y = 5 - \cos(x)$                       **b**  $y = 3 - \sin(x)$                       **c**  $y = 1 - \tan(x)$

### Problem solving

- 5** Draw the graph of the function  $f(t) = 3 - \cos(t)$  in the domain  $0 \leq t \leq 2\pi$ .
- a** Use the equation of the function to calculate the exact value of the function when  $t = \frac{5\pi}{6}$ .
- b** Use the graph to find the value(s) of  $t$  when  $f(t) = 2.5$ .
- c** Use the graph to find the approximate value of  $f(t)$  when  $t = 3$ .
- d** Use the graph to find the approximate value(s) of  $t$  when  $f(t) = 2.8$ .
- 6** The velocity (in  $\text{m s}^{-1}$ ) of an object is given by  $v = 5 + \sin(t)$  in the domain  $0 \leq t \leq 4\pi$ , where  $t$  is measured in seconds.
- a** Calculate the velocity after:
- i**  $\frac{5\pi}{6}$  seconds                      **ii**  $\pi$  seconds                      **iii**  $\frac{\pi}{2}$  seconds
- iv**  $\frac{\pi}{3}$  seconds                      **v**  $\frac{\pi}{4}$  seconds                      **vi**  $\frac{4\pi}{3}$  seconds
- b** Draw a graph of the velocity over the stated domain.
- c** Use the graph to find the time(s) when the velocity is  $5.7 \text{ m s}^{-1}$ .



Trigonometric graphs match-up



Sketching periodic functions: amplitude and period



Sketching periodic functions: phase and vertical shift

## 8.05 Transformations of trigonometric functions

You know that  $y = \sin(x) + D$  and  $y = \cos(x) + D$  are translated up or down compared to  $y = \sin(x)$  and  $y = \cos(x)$ . You also saw that  $y = -\sin(x)$  and  $y = -\cos(x)$  are reflections of the basic graphs in the  $x$ -axis. This is consistent with the work you did in Chapter 4.

Are the effects of the parameters  $A$ ,  $B$  and  $C$  on  $y = A \sin(Bx + C) + D$  and  $y = A \cos(Bx + C) + D$  also consistent with the transformations of graphs in Chapter 4?

## INVESTIGATION

### THE GRAPHS OF $y = A \sin (Bx + C)$ AND $y = A \cos (Bx + C)$

Use your CAS or graphics calculator for this investigation.

#### TI-84 Plus CE

Make sure that the angle measure is set to radians. Press

$\boxed{\text{window}}$  and set the domain and range values as

$$-3\pi \leq X \leq 3\pi \text{ and } -4 \leq Y \leq 4.$$

Press  $\boxed{y=}$  and enter the function  $Y_1 = \sin(x)$  into the function line using  $\boxed{X,T,\theta,n}$  to enter the variable.

Press  $\boxed{\text{enter}}$ .

Then press  $\boxed{\text{graph}}$  to display the graph of the function.

Now press  $\boxed{y=}$  and enter the function  $Y_2 = 3 \sin(x)$  into the function line.

Press  $\boxed{\text{enter}}$ .

Then press  $\boxed{\text{graph}}$  to display the graph of the function.

Now follow the same procedure to enter  $Y_3 = \sin(x)$  and display the graph.

You can use the  $\boxed{\text{zoom}}$  button or the  $\boxed{\text{window}}$  button to modify the domain and range values if you want to see more detail.

#### Casio fx-CG20AU

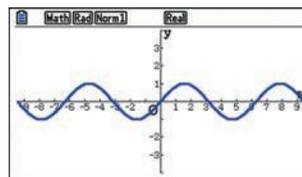
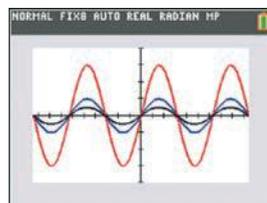
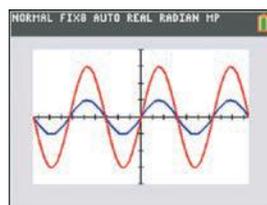
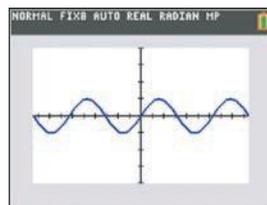
Press  $\boxed{\text{MENU}}$  and select the **Graph** page.

Press  $\boxed{\text{SHIFT}} \boxed{\text{F3}}$  (**V-Window**) and set the domain and range values as  $-3\pi \leq X \leq 3\pi$  and  $-4 \leq Y \leq 4$ .

Enter the function  $Y_1 = \sin(x)$  into the function line using  $\boxed{X,\theta,T}$  to enter the variable.

Press  $\boxed{\text{EXE}}$ .

Then press  $\boxed{\text{F6}}$  (**DRAW**) to display the graph of the function.



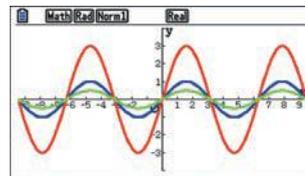
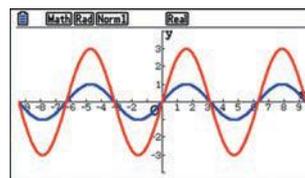
Now press **EXIT** and enter the function  $Y_2 = 3 \sin(x)$  into the function line.

Press **EXE**.

Then press **F6** (DRAW) to display the graph of the function.

Now follow the same procedure to enter  $Y_3 = \sin(x)$  and display the graph.

You can press **SHIFT F2** (Zoom) or press **SHIFT F3** (V-Window) to modify the domain and range values if you want to see more detail.



### Both calculators:

Now use your calculator to examine the effects of inserting the parameters  $B$  and  $C$  on the graphs of  $y = A \sin(Bx + C)$  and  $y = A \cos(Bx + C)$ .

Compare the following graphs to  $y = \sin(x)$  and  $y = \cos(x)$ :

$$y = \sin(2x), y = \sin(-2x), y = \sin(0.5x), y = \cos(2x),$$

$$y = \cos(-2x) \text{ and } y = \cos(0.5x)$$

Compare the following graphs to  $y = \sin(x)$  and  $y = \cos(x)$ :

$$y = \sin(-x) \text{ and } y = \cos(-x);$$

$$y = \sin\left(x + \frac{\pi}{6}\right), y = \sin\left(x - \frac{\pi}{6}\right), y = \sin(x - 1), y = \cos\left(x + \frac{\pi}{6}\right),$$

$$y = \cos\left(x - \frac{\pi}{6}\right) \text{ and } y = \cos(x + 1).$$

Finally do a few combined graphs, such as

$$y = -2 \sin\left(2x + \frac{\pi}{4}\right) + 3 \text{ and } y = 0.45 \cos\left(3x - \frac{\pi}{4}\right) + 3.$$

- Write statements that summarise the effects that changing the values of  $A$ ,  $B$  and  $C$  have on the shape of the graphs of  $y = A \sin(Bx + C)$  and  $y = A \cos(Bx + C)$  compared with the graphs of  $y = \sin(x)$  and  $y = \cos(x)$ .

You can measure the amplitude of a function directly from its graph.

## Features of sine and cosine functions

For the functions  $y = A \sin (Bx + C) + D$  and  $y = A \cos (Bx + C) + D$

- the amplitude is the magnitude of  $A$
- the period is  $\frac{2\pi}{B}$
- the **phase shift** is  $\frac{C}{B}$  and it is the horizontal translation
- the average (mean) value is  $D$ .

## Features of the tangent function

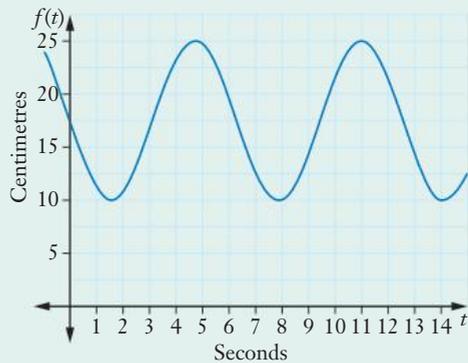
For the function  $y = A \tan (Bx + C) + D$

- the period is  $\frac{\pi}{B}$
- the phase shift is  $\frac{C}{B}$  and it is the horizontal translation

### EXAMPLE 11

Use the graph of the periodic function to calculate:

- its amplitude
- its period.



### Solution

- Measure the distance between the maximum and minimum values.

$$y_{\max} - y_{\min} = 25 - 10 = 15$$

Calculate the amplitude.

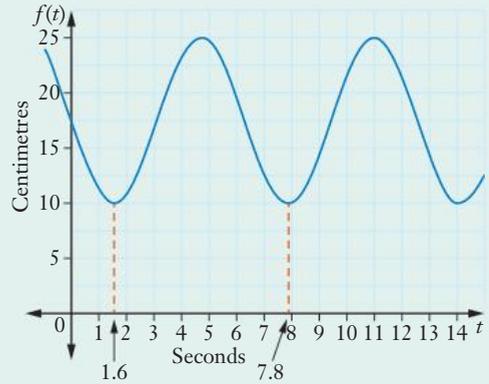
$$a = \frac{15}{2} = 7.5$$

State the result.

The amplitude is 7.5 cm.

- b** Select points at the beginning and end of a period.

Use the graph to find the values of  $t$  at each point.



Calculate the period.

$$\begin{aligned}\text{Period} &= 7.8 - 1.6 \\ &= 6.2\end{aligned}$$

State the result.

The period is 6.2 s.



Amplitude and period

### EXAMPLE 12

State the amplitude and period of  $y = 4 \cos(3x)$  and draw its graph in the domain  $0 \leq x \leq 2\pi$ .

#### Solution

Compare  $y = 4 \cos(3x)$  with  $y = a \cos(bx)$ .

$$a = 4$$

The period is  $\frac{2\pi}{b}$ .

$$\frac{2\pi}{b} = \frac{2\pi}{3}$$

State the result.

The amplitude is 4 and the period is  $\frac{2\pi}{3}$ .

Calculate the number of periods that  $y = 4 \cos(3x)$  has in the domain  $0 \leq x \leq 2\pi$ .

$$2\pi \div \frac{2\pi}{3} = 2\pi \times \frac{3}{2\pi} = 3$$

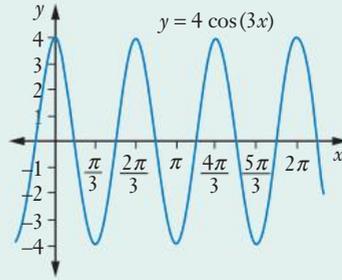
The periods or cycles are evenly distributed between 0 and  $2\pi$ .

There are 3 cycles for  $0 \leq x \leq 2\pi$ .

$$0 \leq x \leq \frac{2\pi}{3}, \quad \frac{2\pi}{3} \leq x \leq \frac{4\pi}{3} \text{ and}$$

$$\frac{4\pi}{3} \leq x \leq 2\pi.$$

Use this information to draw the graph.



### EXAMPLE 13

Sketch the graph of  $y = 3 \cos\left(x - \frac{\pi}{4}\right) + 2$  in the domain  $0 \leq x \leq 2\pi$ .

#### Solution

The graph of  $y = 3 \cos\left(x - \frac{\pi}{4}\right) + 2$  has the same basic shape as the graph of  $y = \cos(x)$ . List how the graph of  $y = \cos(x)$  has been transformed.

The graph of  $y = 3 \cos\left(x - \frac{\pi}{4}\right) + 2$  has:

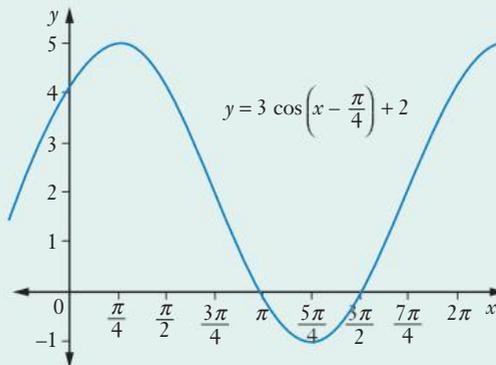
Period =  $2\pi$

Amplitude = 3

Phase change =  $-\frac{\pi}{4}$ , i.e.,  $\frac{\pi}{4}$  to the right

Vertical translation = 2 units upwards

Draw the graph of  $y = 3 \cos\left(x - \frac{\pi}{4}\right) + 2$  by transforming the graph of  $y = \cos(x)$ .



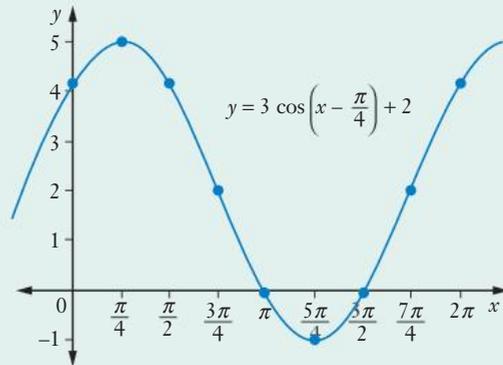
*Alternative method*

If you are unable to transform the graph of  $y = \cos(x)$ , draw up a table of values.

$$y = 3 \cos\left(x - \frac{\pi}{4}\right) + 2$$

$x$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\pi$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	$2\pi$
$y$	4.12	5	4.12	2	-0.12	-1	-0.12	2	4.12

Join the points with a smooth curve.



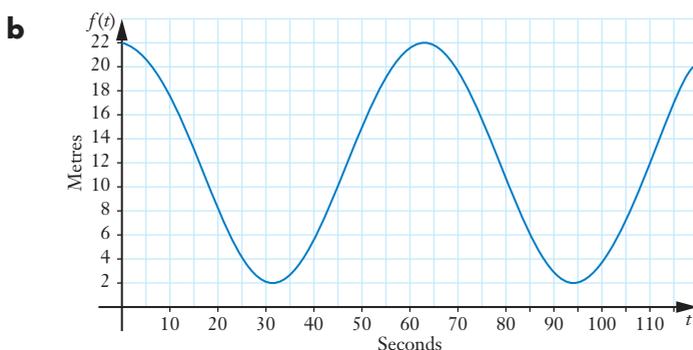
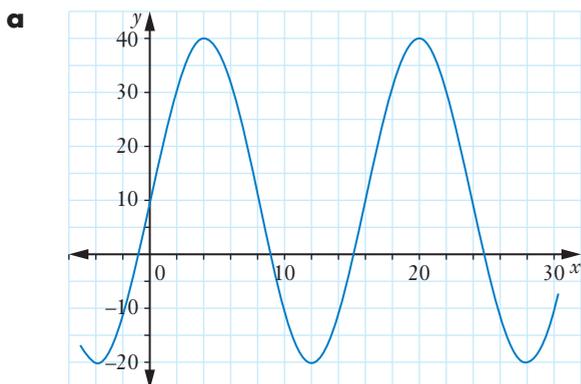
Shutterstock.com/Ferenc Szelepcsényi

A pure musical note makes the air vibrate in a sine wave pattern.

## Exercise 8.05 Transformations of functions

1 Use the graph of each periodic function to calculate its amplitude and the period.

Example  
11



2 Calculate the amplitude of each trigonometric function.

a  $y = 3 \cos(x)$

b  $y = 7 \sin(x)$

c  $y = -2 \sin(x)$

d  $y = 2.8 \cos(x)$

e  $y = 5 \cos(x) + 9$

f  $y = 8 \sin(x) - 6$

g  $f(t) = 12 - 10 \sin(t)$

h  $f(x) = -15 \cos(x) - 18$

Example  
12

3 Calculate the maximum and minimum values for each trigonometric function for  $0 \leq x \leq 2\pi$ .

a  $y = \sin(x)$

b  $y = 3 \sin(x)$

c  $y = 4 \cos(x)$

d  $y = \sin(x) - 4$

e  $y = \cos(x) + 3$

f  $y = 5 - 2 \sin(x)$

4 Find the period of each function.

a  $y = \cos(x)$

b  $y = \sin(2x)$

c  $y = \tan(0.5x)$

d  $y = -4 \tan(5x)$

e  $y = 3 \cos\left(\frac{x}{2}\right)$

f  $y = 7 \sin\left(\frac{x}{3}\right)$

5 Draw a graph of each function for  $0 \leq x \leq 2\pi$ .

a  $y = 0.5 \cos(x)$

b  $y = -4 \sin(x)$

6 Draw a graph of each function for  $0 \leq x \leq 2\pi$ .

a  $y = 5 + 2 \sin(x)$

b  $y = 3 \cos(x) - 2$

**7** Draw the graph of each function showing 1 complete cycle and state the amplitude and period.

**a**  $y = 2 \cos(3x)$       **b**  $y = \frac{1}{2} \sin(2x)$       **c**  $y = 4 \sin\left(\frac{x}{2}\right)$

**d**  $y = -3 \cos\left(\frac{x}{3}\right)$       **e**  $y = 5 \cos\left(\frac{x}{4}\right)$       **f**  $y = -\frac{1}{3} \sin(4x)$

**8** Sketch the graph of each function showing a full cycle. Check the graph with a graphics calculator.

**a**  $y = \sin\left(x - \frac{\pi}{2}\right)$       **b**  $y = \sin\left(x + \frac{\pi}{2}\right)$       **c**  $y = \tan\left(x - \frac{\pi}{4}\right)$

**d**  $y = \cos\left(x - \frac{\pi}{6}\right)$       **e**  $y = \cos\left(x + \frac{\pi}{6}\right)$       **f**  $y = \tan\left(x + \frac{\pi}{4}\right)$

Example  
13

**9** Sketch the graph of each function and check with a graphics calculator, showing a full cycle of each.

**a**  $y = 4 \sin\left[2\left(x - \frac{\pi}{3}\right)\right] + 2$       **b**  $y = 2 \cos\left[3\left(x - \frac{\pi}{6}\right)\right] - 3$

**c**  $y = 5 \sin\left[4\left(x + \frac{\pi}{8}\right)\right] - 2$       **d**  $y = 2 \tan\left[2\left(x - \frac{\pi}{4}\right)\right] + 3$

**e**  $y = 3 - 3 \sin\left[2\left(x - \frac{\pi}{4}\right)\right]$       **f**  $y = 5 - \tan\left[3\left(x + \frac{\pi}{6}\right)\right]$

**10** Sketch the graph of each function and check with a graphics calculator, showing a full cycle of each.

**a**  $y = 2 \sin\left(2x - \frac{\pi}{3}\right) + 3$       **b**  $y = 3 \cos(4x + \pi) - 2$

**c**  $y = 4 - 3 \sin\left(6x + \frac{3\pi}{2}\right)$       **d**  $y = 5 \cos\left(2x - \frac{3\pi}{4}\right)$

### Problem solving

**11** The tide height at a jetty is approximately periodic.

A high tide of 1.8 m occurs at 11 a.m. followed by a low tide of 0.2 m at 5 p.m. on the same day.

- Calculate the amplitude of the tide height.
- Calculate the period of the tide height.



Shutterstock.com/Mark Winfrey

- 12** The temperature,  $T$  in  $^{\circ}\text{C}$  measured by a weather station is modelled by  $T = 24 - 3 \cos\left(\frac{\pi t}{12}\right)$ , where  $t$  is time in hours and  $t = 0$  is midnight.

- Draw a graph of the temperature for the 24 hours after midnight.
- Use the graph to calculate when the maximum temperature occurs.
- When does the minimum temperature occur?
- At what time is the temperature  $26^{\circ}\text{C}$ ?



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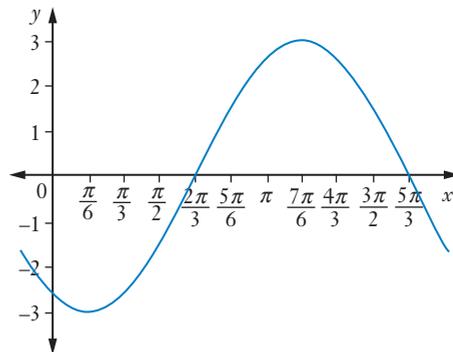
- 13** The height of the tide,  $h$ , in metres above sea level at a pier is modelled by

$$h = 4 \sin\left(\frac{\pi t}{6}\right) + 2, \text{ where } t \text{ is time in hours after midnight.}$$

- Draw a graph of the tide height for  $0 \leq t \leq 24$ .
- Use the graph to calculate the height of the high tide.
- What was the height of the tide at 11:30 a.m.?
- A ferry can only reach the pier when the tide is 3 m above sea level. When could the ferry reach the pier?

- 14** This graph is a function whose equation is in the form of  $y = a \cos(x + c)$ .

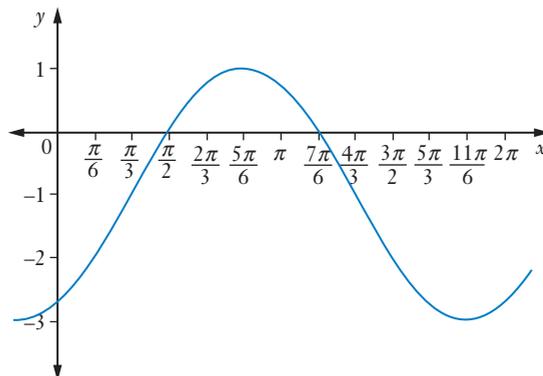
Determine the equation of the function.



- 15** This graph at right is a function whose equation is in the form of

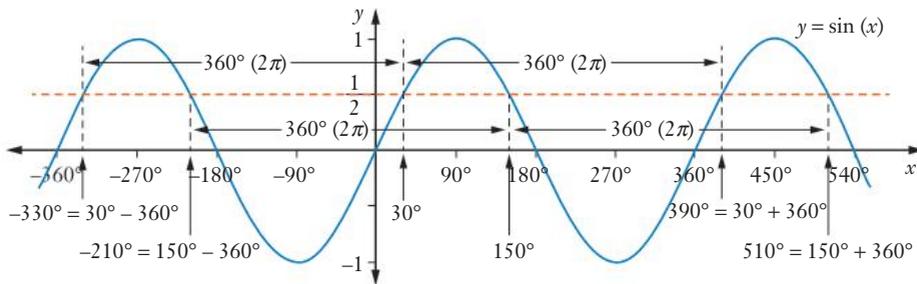
$$y = a \sin b(x + c) + d.$$

Find the values of  $a$ ,  $b$ ,  $c$  and  $d$  and hence write the equation of the function.



## 8.06 Trigonometric equations

Trigonometric equations have an infinite number of solutions, as shown on the graph below for  $\sin(x) = 0.5$ .



In this course, you will be given a domain so the number of solutions is restricted.

A scientific calculator only gives one answer for each value of  $\sin^{-1}$ ,  $\cos^{-1}$  or  $\tan^{-1}$ . You have to work out the other answers needed. Always give an exact answer if you can.

### EXAMPLE 14

Solve each equation.

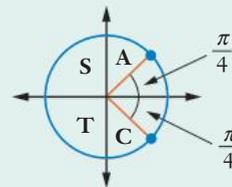
- a  $\cos(x) = \frac{1}{\sqrt{2}}$  for  $-\pi < x \leq \pi$
- b  $\sqrt{2} \cos(x) = 1$  for  $-180^\circ \leq x \leq 180^\circ$
- c  $3 \sin(x) + 1 = 0$  for  $0 \leq x \leq 2\pi$

### Solution

- a Use the exact value for the acute (reference) angle.

$$\cos = \frac{\pi}{4}$$

Draw a sketch. Use the reference angle and ASTC to mark the other solution.



Write the solutions in the restricted domain.

$$x = \frac{7\pi}{4}$$

**b** Write the equation.

Rearrange and simplify.

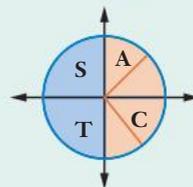
Ignore the sign to get the reference angle.

Draw a sketch.

$$\sqrt{2} \cos(x) = 1$$

$$\cos(x) = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}}$$



Use the reference angle and the domain.

$$x = -45^\circ, 45^\circ$$

**c** Write the equation.

Rearrange and simplify.

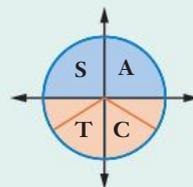
The question does not ask for an approximation, so you need an exact reference angle.

Draw a sketch.

$$3 \sin(x) + 1 = 0$$

$$\sin(x) = -\frac{1}{3}$$

$$0 \leq \sin^{-1}\left(\frac{1}{3}\right) \leq \frac{\pi}{2}$$



Use the reference angle and the domain  
( $0 \leq \theta \leq 2\pi$ ).

$$x = \pi + \sin^{-1}\left(\frac{1}{3}\right) \text{ or}$$

$$2\pi - \sin^{-1}\left(\frac{1}{3}\right).$$

In part **c** above,  $\sin^{-1}\left(\frac{1}{3}\right) = 0.3398\dots$  so the answers to 4 decimal places would be  $\pi + 0.3398$ ,  
 $2\pi - 0.3398 = 3.4814$  or  $5.9433$ .

You can use your graphics calculator to find approximate solutions for trigonometric equations.

## TECHNOLOGY

### Solution of trigonometric equations

Solve  $2 \sin(x) = \sqrt{3}$ ,  $0 \leq x \leq 2\pi$ , correct to 2 decimal places.

Rearrange the equation:  $\sin(x) = \frac{\sqrt{3}}{2}$

#### TI-84 Plus CE

Make sure that the angle measure is set to radians.

Press  $\boxed{Y=}$  and enter the functions

$$Y_1 = \sin(x)$$

$$Y_2 = \frac{\sqrt{3}}{2}$$

Then press  $\boxed{\text{graph}}$  to display the graphs of the functions.

Press  $\boxed{\text{window}}$  and change the window settings to:

$$0 \leq X \leq 2\pi \text{ and } -1 \leq Y \leq 1$$

to make the intersections clearer.

Press  $\boxed{2\text{nd}} \boxed{\text{trace}} \boxed{\text{calc}}$  and select **5: intersect**.

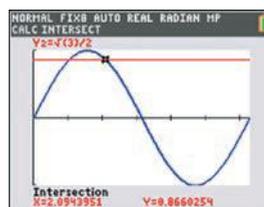
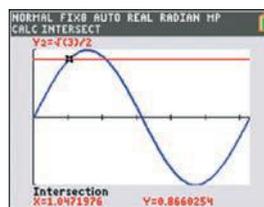
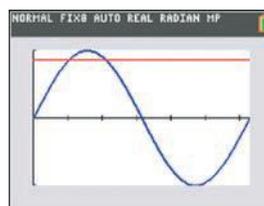
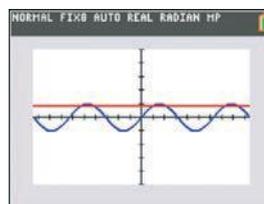
Move the cursor to position the cross on  $Y_1$  close to the first point of intersection of the graphs. Press  $\boxed{\text{enter}}$ .

Then move the cursor to position the cross on  $Y_2$  close to the point of intersection of the graphs. Press  $\boxed{\text{enter}}$ .

Finally move the cursor to position the cross close to your guess for the point of intersection of the graphs. Press  $\boxed{\text{enter}}$ .

Repeat the procedure to find the second intersection.

State the result.



$$x \approx 1.05, 2.09$$



TI-Nspire CX  
Chapter 8

## Casio fx-CG20AU

Press **MENU** and select the **Graph** page.

Enter the functions

$$Y_1 = \sin(x)$$

$$Y_2 = \frac{\sqrt{3}}{2}$$

Press **EXE**.

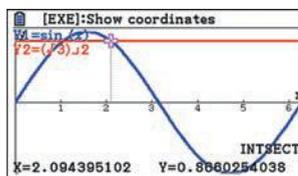
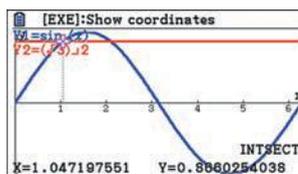
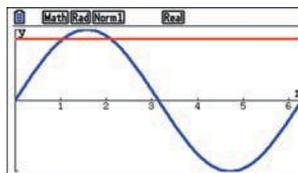
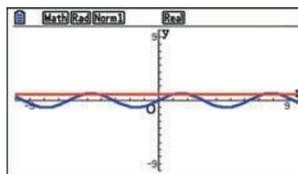
Then press **F6** (**DRAW**) to display the graph of the function.

Press **SHIFT** **F3** (**V-Window**) and set the domain and range values as  $0 \leq X \leq 2\pi$  and  $-1 \leq Y \leq 1$  to make the intersections clearer.

Press **SHIFT** **F5** (**G-Solv**) and select **F5** (**INTSECT**).

The first point of intersection is then displayed.

Press the right arrow button to display the second intersection.



State the result.

$$x \approx 1.05, 2.09$$

## Both calculators

Use the same procedure to find the solutions to:

- $y = \cos(x)$  and  $y = x - 3$  in the domain  $-\pi \leq x \leq \pi$
- $y = 2 \sin(x) + 3$  and  $y = 3x - 4$  in the domain  $0 \leq x \leq \pi$
- $y = \frac{1}{2} \tan\left(x + \frac{\pi}{4}\right)$  and  $y = 4 - 5x$  in the domain  $0 \leq x \leq \pi$ .

## Solving trigonometric equations

To solve equations of the form  $\sin(x) = a$ ,  $\cos(x) = a$  or  $\tan(x) = a$ :

- use the sign of the function to decide the quadrants
- find the reference angle using either exact values or inverse functions
- sketch the solutions in the unit circle or using a graph
- examine the given domain to determine the required values.

You sometimes need algebra to solve trigonometric equations.

### EXAMPLE 15

**a** Solve  $\sin(2x) = 1.5 \cos(2x)$  for  $0 \leq x \leq 2\pi$ , correct to 2 decimal places.

**b** Solve  $2 \sin^2(B) + 7 \sin(B) - 4 = 0$ , where  $\frac{\pi}{2} \leq B \leq \frac{3\pi}{2}$

### Solution

**a** Write the equation.

$$\sin(2x) = 1.5 \cos(2x)$$

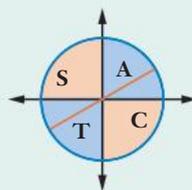
Divide both sides by  $\cos(2x)$ .

$$\tan(2x) = 1.5$$

Use ASTC.

$\tan > 0$  in 1st and 3rd quadrants.

Draw a sketch.



Use a calculator to find  $\tan^{-1}$ .

$$\tan^{-1}(1.5) = 0.9827\dots$$

Domain is  $0 \leq x \leq 2\pi$ , so you need to find values of  $2x$  in the domain  $0 \leq 2x \leq 4\pi$ .

$$2x = 0.9827\dots, \pi + 0.9827\dots,$$

$$2\pi + 0.9827\dots, 3\pi + 0.9827\dots$$

Evaluate.

$$\approx 0.9827\dots, 4.1243\dots, 7.2659\dots, 10.4075\dots$$

Solve for  $x$  and round off.

$$x \approx 0.49, 2.06, 3.63, 5.20$$

Compare with the domain for  $x$ .

The solutions are within the given domain.

**b** The equation  $2 \sin^2(B) + 7 \sin(B) - 4 = 0$  is a quadratic.

Replace  $\sin(B)$  with  $x$ .

$$2x^2 + 7x - 4 = 0$$

Factorise.

$$(2x - 1)(x + 4) = 0$$

Solve for  $x$ .

$$2x - 1 = 0 \quad \text{or} \quad x + 4 = 0$$

$$x = \frac{1}{2} \quad \quad \quad x = -4$$

Replace  $x$  with  $\sin(B)$ .

$$\sin(B) = \frac{1}{2} \quad \text{or} \quad \sin(B) = -4$$

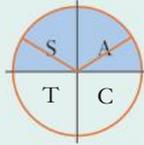
Discard the false solution.

$$\sin(B) \neq -4, \text{ so } \sin(B) = \frac{1}{2} \text{ only.}$$

Use ASTC.

$\sin > 0$  in 1st and 2nd quadrants.

Draw a sketch.



Write the reference angle.

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Use the reference angle and domain.

$$B = \frac{\pi}{6}, \frac{5\pi}{6}$$

Refer to the domain  $\frac{\pi}{2} \leq B \leq \frac{3\pi}{2}$ .

$$B = \frac{5\pi}{6}$$

## Exercise 8.06 Trigonometric equations

1 Solve each trigonometric equation for the domain given.

**a**  $2 \cos(x) = -2, -\pi \leq x \leq \pi$

**b**  $\sin(x) - 1 = 0, 0 \leq x \leq 2\pi$

**c**  $1 - \tan(x) = 0, -2\pi \leq x \leq 0$

**d**  $6 \cos(x) - 3 = 3, -\pi \leq x \leq 2\pi$

**e**  $10 \cos(x) + 5 = -5, -\pi \leq x \leq \pi$

**f**  $\cos(x) + 8 = 8, 0 \leq x \leq 3\pi$

**g**  $3 \tan(x) + 3 = 0, -\pi \leq x \leq \pi$

**h**  $2 \sin(x) - 1 = 0, -\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$

2 Solve each equation for  $0^\circ \leq \theta \leq 360^\circ$ .

**a**  $\tan(\theta) = 1$

**b**  $\sin(\theta) = \frac{1}{2}$

**c**  $\cos(\theta) = -\frac{\sqrt{3}}{2}$

**d**  $\sin(\theta) = -\frac{1}{\sqrt{2}}$

**e**  $\cos(\theta) = 0$

**f**  $\tan(\theta) = \sqrt{3}$

**g**  $\sin(\theta) = \sin(50^\circ)$

**h**  $\cos(\theta) = \cos(115^\circ)$

3 Solve each equation over the domain  $0 \leq x \leq \pi$ , correct to 4 decimal places.

**a**  $3 \sin(x) = 1$

**b**  $4 \cos(x) + 1 = 0$

**c**  $5 \tan(x) - 2 = 0$

**d**  $9 \cos(x) = 5$

**e**  $7 \sin(x) - 3 = 0$

**f**  $3 \tan(x) + 20 = 0$

4 Find all values of  $x$  in the domain  $0 \leq x \leq \pi$  for which:

**a**  $\sin(2x) = 0.5$

**b**  $\cos(2x) = -1$

**c**  $\sin(2x) = 1$

5 Solve each equation for  $0^\circ \leq \beta \leq 360^\circ$ .

**a**  $\cos^2(\beta) = \frac{1}{4}$

**b**  $\sin^2(\beta) = 1$

**c**  $6 \tan^2(\beta) = 2$

**d**  $2 \cos^2(\beta) = 1$

6 Solve each equation for the domain indicated.

**a**  $2 \sin^2(\theta) + \sin(\theta) = 0$ , where  $90^\circ \leq \theta \leq 270^\circ$

**b**  $\cos^2(\theta) - \frac{1}{2} \cos(\theta) = 0$ , where  $180^\circ \leq \theta \leq 360^\circ$

**c**  $\tan^2(\theta) = \tan(\theta)$ , where  $0 \leq \theta \leq 2\pi$

**d**  $4 \sin^2(\theta) + 2 \cos^2(\theta) - 3 = 0$ , where  $-\pi \leq \theta \leq \pi$

Example  
14

Example  
15

7 Solve each equation for  $0 \leq \alpha \leq 2\pi$ , giving approximate answers where appropriate.

a  $2 \sin^2(\alpha) + 3 \sin(\alpha) - 5 = 0$

b  $2 \cos^2(\alpha) + 5 \cos(\alpha) - 3 = 0$

c  $3 \cos^2(\alpha) - 4 \cos(\alpha) = 4$

d  $6 \sin^2(\alpha) + \sin(\alpha) - 1 = 0$

### Problem solving

8 A particle  $P$  bobs up and down on the end of an elastic string that is fixed at  $O$ . Its distance below  $O$  is given by the rule

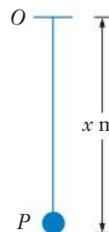
$$x = 0.34 + 0.04 \sin(4t)$$

where  $x$  is in metres and  $t$  is the time in seconds since observation began.

a How far below  $O$  is the particle initially?

b How far below  $O$  is the particle at  $t = 0.6$ ?

c What are the distance below  $O$  of the lowest point in the particle's path, and the value of  $t$  when the particle first reaches this point?



9 Find, correct to 2 decimal places, the two values of  $x$  closest to 0 for which

$$5 \sin\left(2x - \frac{\pi}{6}\right) - 2 = 0.$$



Trigonometric functions assignment



Trigonometric functions extension

## 8.07 Modelling with trigonometric functions

The sine and cosine functions can be used to model tides, circular motion and diurnal and annual temperature variation.

A **sinusoidal** model has the shape of a sine graph (or cosine graph). Cosine is also called sinusoidal as it has the same shape as sine.

You can use the graph of a periodic function to help solve a problem. Use whichever of sine or cosine is the more convenient for any particular problem.

### EXAMPLE 16

The table shows the average monthly temperature on the Sunshine Coast over a 12-month period.

$x$	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
$T(^{\circ}\text{C})$	24.1	21.5	18.2	14.5	11.9	11.2	12.1	14.7	17.8	21.8	23.8	25.2

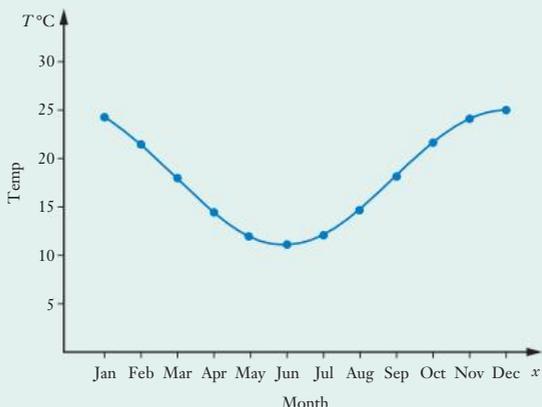
a Draw a graph to represent this data.

b Find the period.

c Find the amplitude.

## Solution

- a** Plot the points and join them with a smooth curve.



- b** Describe the shape of the graph.

The graph is approximately a cosine curve over a full period.

State the period.

Period = 12 months

- c** You need to find the maximum and minimum temperatures to calculate the amplitude.

Maximum temperature  $\approx 25.2^\circ\text{C}$

Minimum temperature  $\approx 11.1^\circ\text{C}$

Calculate the amplitude.

$$A \approx \frac{25.2 - 11.1}{2} = 7$$

State the result.

The amplitude is about  $7^\circ\text{C}$ .

You can use the equation of a periodic function to calculate results.

## EXAMPLE 17

In a harbour, high tides occur 12 h 20 min apart. A high tide occurs at 2 a.m. on 3 October. Low tides occur halfway between high tides. The depth of water at the entrance bar is 2.8 m at low tide and 8.4 m at high tide.

- Sketch a rough version of the tidal heights.
- Find a sinusoidal model for the depth of the water,  $d$ , as a function of the time  $t$  hours since midnight on 3 October.
- Find the depth of water at 4 p.m. on 5 October.
- Use a graphics calculator to find the times on 3 October at which a ship with a draught of 5 m can enter the harbour.
- Evaluate the usefulness of your model.

## Solution

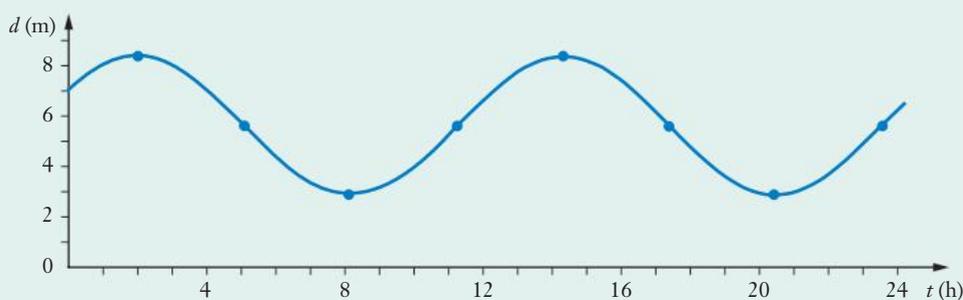
- a** Low tides occur halfway between high tides. High tides are 12 h 20 min apart, so the next low tide will be 6 h 10 min after 2 a.m., i.e., at 8:10 a.m.

The average depth will be halfway between the tides, so the first average depth will occur 3 h 5 min after 2 a.m., i.e., 5:05 a.m. The depth will be  $(8.4 + 2.8) \div 2 = 5.6$  m.

You can continue adding 3 h 5 min to the last times to get the following times and depths.

Time	2:00 a.m.	5:05 a.m.	8:10 a.m.	11:15 a.m.	2:20 p.m.	5:25 p.m.	8:30 p.m.	11:35 p.m.
$d$	8.4	5.6	2.8	5.6	8.4	5.6	2.8	5.6

Draw a sketch using the points found and knowledge of the shape of sine and cosine.



- b** Since the graph starts near a maximum, it is more convenient to use cosine than sine.

Write the cosine function.

$$y = A \cos \left[ B \left( x + \frac{C}{B} \right) \right] + D$$

Find the amplitude.

$$\begin{aligned} \text{Amplitude} &= (8.4 - 2.8) \div 2 \text{ m} \\ &= 2.8 \text{ m} \end{aligned}$$

Find the period.

$$\begin{aligned} \text{Period} &= 12 \text{ h } 20 \text{ min} \\ &= 12\frac{1}{3} \text{ h} \\ &= \frac{37}{3} \text{ h} \end{aligned}$$

The period =  $\frac{2\pi}{B}$ .

$$\frac{2\pi}{B} = \frac{37}{3}$$

Rearrange to find  $B$ .

$$B = \frac{6\pi}{37}$$

Find the phase shift.

Maximum is 2 hours after  $t = 0$ .

The graph has been shifted to the right, so the phase shift,  $\frac{C}{B} < 0$ .

$$\frac{C}{B} = -2 \text{ h}$$

Find the mean value.

Choose variables.

Substitute into  $y = A \cos \left[ B \left( x + \frac{C}{B} \right) \right] + D$ .

- c** At 4 p.m. on 5 October,  $t = 64$  h.

Evaluate and round off.

**d TI-84 Plus CE**

Make sure that the angle measure is set to radians. Press  $\boxed{Y=}$  and enter the functions

$$Y_1 = 2.8 \cos \left( \frac{6\pi x}{37} - \frac{12\pi}{37} \right) + 5.6$$

$$Y_2 = 5$$

Then press  $\boxed{\text{graph}}$  to display the graphs of the functions.

Press  $\boxed{\text{window}}$  and change the window settings to:

$$0 \leq X \leq 24 \text{ and } 0 \leq Y \leq 10$$

Press  $\boxed{2\text{nd}} \boxed{\text{trace}} \boxed{[\text{calc}]}$  and select **5:intersect**.

Follow the procedure as outlined previously to locate the first point of intersection of the graphs.

Repeat the procedure to find the other intersections.

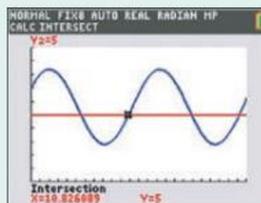
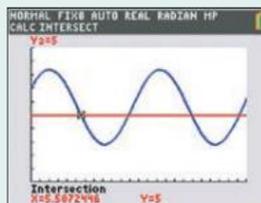
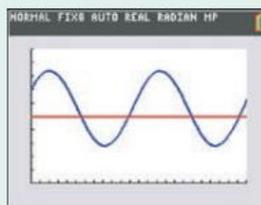
$$D = (8.4 + 2.8) \div 2 = 5.6 \text{ m}$$

Let  $t = x$  and  $d = y$ .

$$d = 2.8 \cos \left[ \frac{6\pi}{37} (t - 2) \right] + 5.6$$

$$= 2.8 \cos \left[ \frac{6\pi}{37} (64 - 2) \right] + 5.6$$

$$\approx 8.36 \text{ m}$$



$$x \approx 5.507, 10.83, 17.84, 23.16$$



TI-Nspire CX  
Chapter 8

### Casio fx-CG20AU

Press **[MENU]** and select the **Graph** page.

Enter the functions

$$Y_1 = 2.8 \cos\left(\frac{6\pi x}{37} - \frac{12\pi}{37}\right) + 5.6$$

$$Y_2 = 5$$

Then press **[F6]** (**DRAW**) to display the graph of the function.

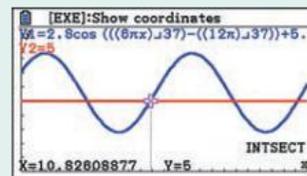
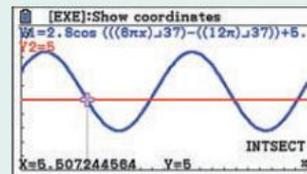
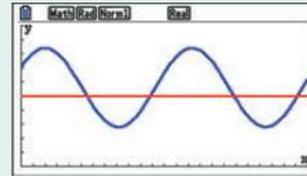
Press **[SHIFT]** **[F3]** (**V-Window**) and change the window settings to:

$$0 \leq X \leq 24 \text{ and } 0 \leq Y \leq 10$$

Press **[SHIFT]** **[F5]** (**G-Solv**) and select **[F5]** (**INTSECT**).

The first point of intersection is then displayed.

Press the right arrow button to display the next intersection and repeat until all intersections are found.



$$x \approx 5.507, 10.83, 17.84, 23.16$$

### Both calculators

Use the fact that  $t$  = hours after midnight to convert the  $x$  values into times.

$$5.507 \approx 5:30 \text{ a.m.}$$

$$10.83 \approx 10:50 \text{ a.m.}$$

$$17.84 \approx 5:50 \text{ p.m.}$$

$$23.15 \approx 11:10 \text{ p.m.}$$

Use the graph and the points of intersection to state the result.

The ship could enter from midnight to 5:30 a.m., from 10:50 a.m. to 5:50 p.m., and after 11:10 p.m.

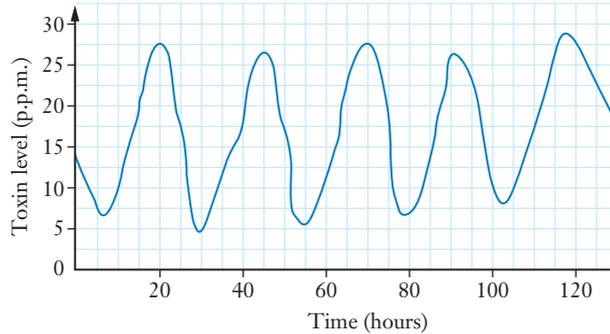
- e Consider the application of the model.

Tides change height with the phases of the moon, so the model could not be used for other dates.

## Exercise 8.07 Modelling with trigonometric functions

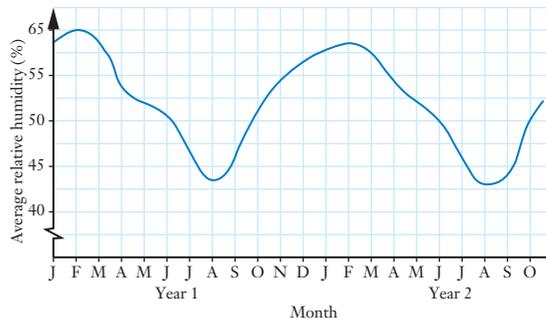
Example  
16

- 1 A factory discharges waste into a local stream and this is causing problems for wildlife in the area. The toxin levels in the stream were measured over a period of days. The toxin levels are recorded in parts per million (ppm). The first reading was taken at midnight and the graph of the readings is shown.



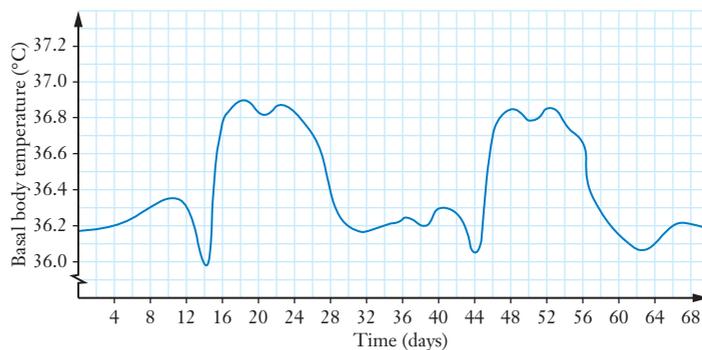
- Does the toxin level appear to be a periodic function over time?
- At about what time does the toxin level seem to peak?
- When does the toxin level appear to be at its lowest?
- What is the period?
- What is the amplitude?

- 2 This graph shows the average relative humidity (%) for Brisbane over two consecutive years.



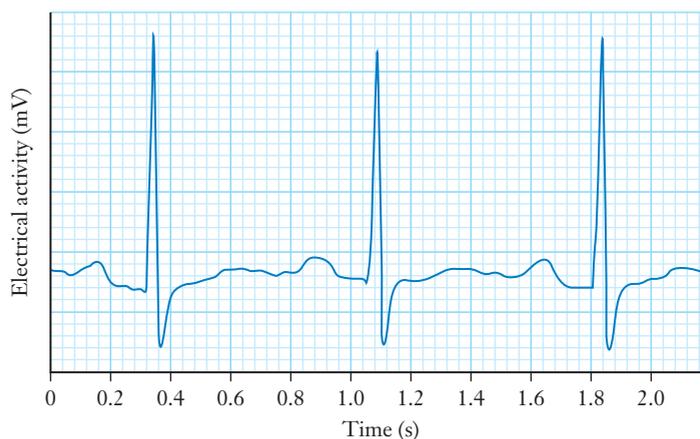
- What is Brisbane's most humid month?
- Calculate the period and the amplitude.

- 3 The following graph shows the average daily body temperature of an adult female.

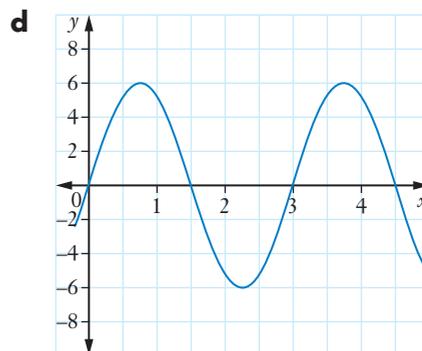
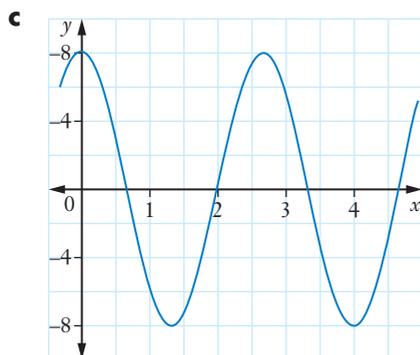
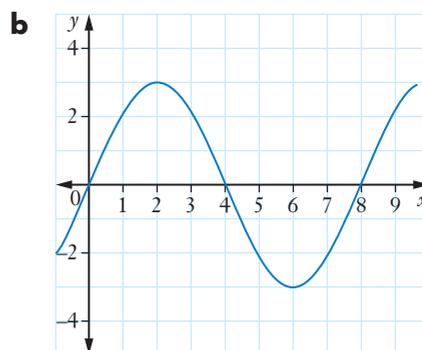
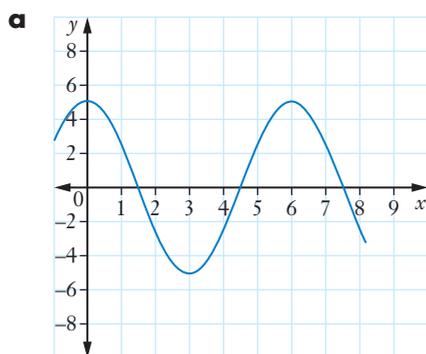


- Calculate the period.
- Calculate the amplitude.

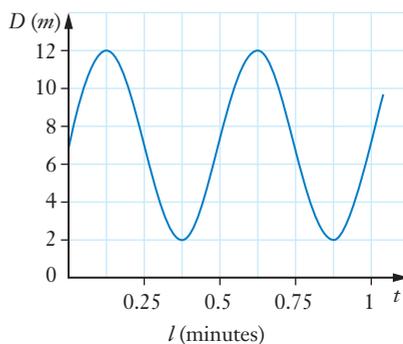
- 4** This graph shows the electrocardiogram (ECG) for an adult male. The peaks show when the ventricles of the heart contract.



- a** Calculate the period.  
**b** What does the period represent?  
**c** How long does it take for the heart to beat 10 000 times?
- 5** Find an equation for each of periodic function.



- 6 Juanita is lying on a tropical beach, enjoying the sound of the waves. She has just finished her exams and so has plenty of time to observe the movement of the waves. She notices that the waves appear to roll up the beach at regular time intervals, and she is able to estimate the distance of the wave front from her toes over time. Idly, she scratches Cartesian axes in the sand and sketches the distance of the wave front from her toes against time in minutes.



At this point she realises that the distance can be modelled by a sine curve

$$D = a \sin(kt) + d$$

where  $a$ ,  $k$  and  $d$  are positive constants.

- State the maximum and minimum distances of the waves from her feet.
- How many waves wash up on the beach each hour?
- Find the values of  $a$ ,  $k$  and  $d$ .

Her beach umbrella is stuck into the sand 4.5 m closer to the wave front than her toes and the waves are washing over its base.

- Calculate the percentage of time for which the base of the umbrella is in the water.
- Evaluate the usefulness of her model.

- 7 The sales,  $S$  in 100s of units, of a seasonal product are modelled by

$$S = 54.8 + 32.5 \cos\left(\frac{\pi t}{6}\right)$$

where  $t$  is the time in months ( $t = 1$  is January and  $t = 12$  is December).

- Draw a graph of the sales for a period of 12 months.
- Use the graph to determine the months for which sales exceed 6800 units.
- Is the model likely to be useful in the long term?

Example  
17

### Problem solving

- 8 In an unusual meteorological investigation, the temperature,  $T$  °C, in a town in central Queensland was found to fluctuate approximately according to the rule  $T = 25 + 6 \sin(0.1\pi t)$ , where  $t$  is the number of hours after 10:00 a.m.
- Sketch a graph of the temperature fluctuations for a sufficient number of hours to be able to determine the maximum and minimum temperatures for that day and the next night.
  - Use the graph to determine the maximum and minimum temperatures.
  - When did they occur?
  - At what time was the temperature:
    - 27°C?
    - 20°C?
  - Why was this unusual?

- 9** In Calcutta, the highest mean monthly temperature is  $29.45^{\circ}\text{C}$  in June and the lowest is  $18.3^{\circ}\text{C}$  in December. Find a model for the temperature throughout the year and graph it. Evaluate the usefulness of the model.
- 10** In Upernivik, Greenland, the average temperature varies between  $-23^{\circ}\text{C}$  in January and  $5^{\circ}\text{C}$  in July. Find and graph a model for the average temperature and evaluate your model.



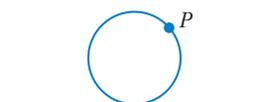
Alamy Stock Photo/Premium Stock Photography GmbH

- 11** A large hoop is rolling along the ground. The vertical distance above the ground of a point  $P$  on the rim of the hoop is given by:

$$y = 1.25 - 1.25 \cos(2\pi t)$$

where  $y$  is in metres and  $t$  is in seconds.

Find the first three times at which  $P$  is 2.00 m above the ground. How good is this model?



- 12** The time between successive high tides at a pier is 12 h 20 min. The average depth of water is 4 m, but at low tide it is 3.1 m. Write an equation to model this relationship and use it to find the lengths of time that a boat with a draught of 3.5 m and 4.5 m can use the pier.

Consider the usefulness of the model.

- 13** An oscilloscope tracing signals from a microphone shows a sinusoidal curve. The amplitude relates to the volume of the sound while the frequency gives the pitch.

A violin string vibrates at 200 Hz (1 Hz is 1 vibration per second). The oscilloscope trace amplitude is 12 V.

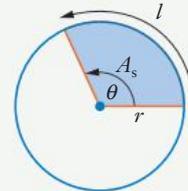
- a** Find the period of the sound wave.
- b** Write an equation for the sound wave.
- c** Evaluate the usefulness of the model.

# 8. CHAPTER SUMMARY

## Trigonometric functions

### Radian facts

- $\pi = 180^\circ$   
 $\frac{\pi}{6} = 30^\circ$        $\frac{\pi}{4} = 45^\circ$        $\frac{\pi}{3} = 60^\circ$        $\frac{\pi}{2} = 90^\circ$   
 $\frac{3\pi}{2} = 270^\circ$        $2\pi = 360^\circ$



- Given that  $l$  and  $r$  are in the same units and  $\theta$  is in radians:

**Arc length**  $= l = r\theta$

- For a circle with radius  $r$  and central angle  $\theta$

**Area of sector**  $= A_s = \frac{1}{2}r^2\theta$

- The graph of a **periodic function** repeats itself. The basic unit that repeats itself is known as the **period** ( $T$ ). The period is the smallest value of  $T$  such that  $f(x) = f(x + T)$ . The period is the length of one **cycle** of the function.
- The **amplitude** ( $A$ ) of a periodic function is the average distance between the maximum and minimum values.

### Trigonometric graphs

- $y = \sin(x)$

Period  $= 2\pi$

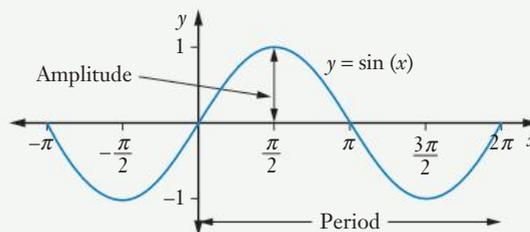
Amplitude  $= 1$

Range is  $-1 \leq y \leq 1$

Zeros occur at  $\dots, -\pi, 0, \pi, 2\pi, \dots$

Maxima occur at  $\dots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \dots$

Minima occur at  $\dots, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \dots$



- $y = \cos(x)$

Period  $= 2\pi$

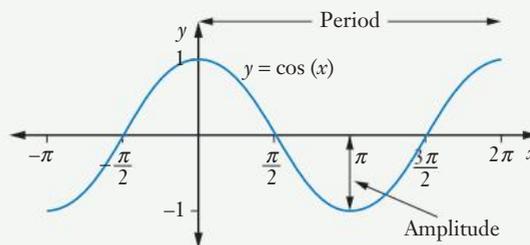
Amplitude  $= 1$

Range is  $-1 \leq y \leq 1$ .

Zeros occur at  $\dots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

Maxima occur at  $\dots, -2\pi, 0, 2\pi, \dots$

Minima occur at  $\dots, -\pi, \pi, 3\pi, \dots$



- $y = \tan(x)$

Range = real numbers.

Period =  $\pi$

Zeros occur at  $\dots, -\pi, 0, \pi, \dots$

Asymptotes at  $\dots, x = -\frac{\pi}{2}, x = \frac{\pi}{2}, \dots$

- For the functions  $y = A \sin(Bx + C) + D$  and  $y = A \cos(Bx + C) + D$

or  $y = A \sin\left[B\left(x + \frac{C}{B}\right)\right] + D$  and

$y = A \cos\left[B\left(x + \frac{C}{B}\right)\right] + D$

- the amplitude is the magnitude of  $A$

- the period is  $\frac{2\pi}{B}$

- the phase shift is  $\frac{C}{B}$  and it is the horizontal translation

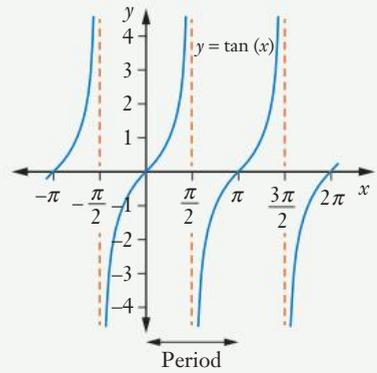
- the average (mean) value is  $D$ .

- For the function  $y = A \tan(Bx + C) + D$  or  $y = A \tan\left[B\left(x + \frac{C}{B}\right)\right] + D$

- the period is  $\frac{\pi}{B}$

- the phase shift is  $\frac{C}{B}$  and it is the horizontal translation

- the average (mean) value is  $D$ .



### Solving trigonometric equations

- To solve equations of the form  $\sin(x) = a$ ,  $\cos(x) = a$  or  $\tan(x) = a$ :

- 1 use the sign of the function to decide the quadrants

- 2 find the reference angle using either exact values or inverse functions

- 3 sketch the solutions in the unit circle or using a graph

- 4 examine the given domain to determine the required values

- Periodic phenomena can be modelled by the **sinusoidal** functions  $y = \sin(x)$  and  $y = \cos(x)$ . A sinusoidal model can be sine or cosine, whichever is more convenient.

# 8. CHAPTER REVIEW

## Trigonometric functions

- 1** Express each angle in radians, leaving your answers in terms of  $\pi$ .
- a**  $270^\circ$       **b**  $225^\circ$       **c**  $330^\circ$       **d**  $288^\circ$
- 2** Express in radians, correct to 4 decimal places.
- a**  $47^\circ$       **b**  $220.75^\circ$
- 3** Express in degrees, correct to 1 decimal place.
- a**  $\frac{3\pi}{4}$       **b**  $0.43$       **c**  $3.21$       **d**  $102^\circ 23'$
- 4** Find the exact value of:
- a**  $\cos\left(\frac{\pi}{3}\right) \times \sin\left(\frac{3\pi}{4}\right)$       **b**  $\sin\left(\frac{5\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)$
- 5** Find the exact value of:
- a**  $\cos\left(\frac{13\pi}{6}\right)$     **b**  $\sin\left(\frac{-2\pi}{3}\right)$     **c**  $\cos\left(\frac{7\pi}{3}\right)$     **d**  $\tan\left(\frac{-\pi}{4}\right)$     **e**  $\sin\left(\frac{22\pi}{3}\right)$
- 6** An arc 17 cm long subtends an angle of  $73^\circ$  at the centre of a circle. Find the radius of the circle.
- 7** An arc subtends an angle of  $\frac{3\pi}{5}$  at the centre of a circle with radius 12 cm. Calculate the area of a sector formed by the arc and the radii.
- 8** The area of a circle is  $\frac{49\pi}{16}$  cm<sup>2</sup>. An arc subtends an angle of  $\frac{4\pi}{7}$  at the centre of the circle. Calculate correct to two decimal places:
- a** the length of the arc      **b** the area of the sector
- 9** The sector of a circle with a radius of 6 cm has a curved length of 10.57 cm. Find the central angle that subtends the sector and the area of the sector.
- 10** Find the exact value of:
- a**  $\tan\left(\frac{\pi}{3}\right)$       **b**  $\sin\left(\frac{7\pi}{4}\right)$
- 11** Find the exact value of:
- a**  $\sin(210^\circ)$       **b**  $\cos(-120^\circ)$       **c**  $\tan(300^\circ)$
- d**  $\sin\left(-\frac{5\pi}{3}\right)$       **e**  $\cos\left(\frac{4\pi}{3}\right)$       **f**  $\tan\left(-\frac{5\pi}{4}\right)$

Example  
1

Example  
1

Example  
1

Example  
2

Example  
2

Example  
4

Example  
5

Example  
6

Example  
6

Example  
7

Example  
7

Example  
8

**12 a** In which quadrant is  $-460^\circ$ ? **b** Find the exact value of  $\cos(-480^\circ)$ .

Example  
8

**13** For each trigonometric ratio, without actually calculating the value of  $\theta$ , find the exact value of the other two trigonometric ratios using the information supplied.

**a**  $\sin(\theta) = \frac{5}{13}$  and  $\theta$  is in quadrant 2 **b**  $\cos(\theta) = \frac{3}{5}$  and  $\theta$  is in quadrant 4

Example  
8

**14** Find the exact value of:

**a**  $\tan(150^\circ)$  **b**  $\sin(-60^\circ)$  **c**  $\cos(210^\circ)$

Examples  
9, 10

**15** Sketch the graph of each trigonometric function.

**a**  $y = 3 \sin(x)$  for  $0 \leq x \leq 2\pi$  **b**  $y = -\cos(4x)$  for  $0 \leq x \leq \frac{\pi}{2}$

Examples  
11, 12

**16** Find the amplitude and range of the graph with the rule  $y = 2 \sin\left(\frac{x}{12}\right) + 2$ .

Example  
12

**17** What is the range of the function with the rule  $y = 3 \sin(x) + 2$ ?

Example  
12

**18** What is the period of the function given by  $y = 2 \cos\left(\frac{\pi t}{4}\right) - 5$ ?

Example  
12

**19** Find the amplitude, period and range of the function  $y = 2 \sin(3x + \pi) + 2$ .

Example  
12

**20** Find the amplitude, period and range of the function  $y = 3 \cos 2(x - \pi) + 1$ .

Example  
13

**21** Sketch the graph of each function, showing one full cycle.

**a**  $y = -\sin\left(x - \frac{\pi}{4}\right)$  **b**  $y = 5 - 3 \sin\left(2x + \frac{\pi}{6}\right)$   
**c**  $y = 6 \cos\left(3x - \frac{\pi}{4}\right) + 4$  **d**  $y = \tan\left(2x - \frac{\pi}{4}\right) - 6$

Example  
14

**22** Solve each equation for the domain indicated.

**a**  $2 \sin(\beta) = \sqrt{2}$ , where  $0 \leq \beta \leq 2\pi$   
**b**  $1 - \cos(\theta) - 2 \cos^2(\theta) = 0$ , where  $0 \leq \theta \leq \pi$   
**c**  $2 \cos^2(\alpha) + 3 \cos(\alpha) = 2$ , where  $-180^\circ \leq \alpha \leq 180^\circ$

Example  
15

**23** Find the values of  $x$  for which  $\cos(2x) = \frac{1}{\sqrt{2}}$  in the domain  $0 \leq x \leq \pi$ .

Example  
15

**24** Solve the equation  $2 \sin(4x) = 1$  for  $0 \leq x \leq 2\pi$ .

Example  
15

**25** Find all the values of  $x$  between 0 and  $2\pi$  for which  $\sin(2x) = \cos(2x)$ .

Example  
15

**26** Solve  $\sin(2x) = 3 \cos(2x)$  for the domain  $0^\circ \leq x \leq 90^\circ$ , to the nearest minute.

Example  
15

**27** A function has the equation  $f(x) = 5 + 2 \sin(2x)$ .

- a** State the period and range of the function.  
**b** Sketch the graph for  $0 \leq x \leq 2\pi$ .  
**c** Find the values of  $x$  in this domain for which  $f(x) < 4$ .

- 28** The table shows the monthly average overnight minimum temperature for Melbourne.

$m$	0	1	2	3	4	5	6	7	8	9	10	11
$T$	15	14.5	12.7	10.6	8.2	6.5	6	6.7	8.3	10.4	12.9	14.3

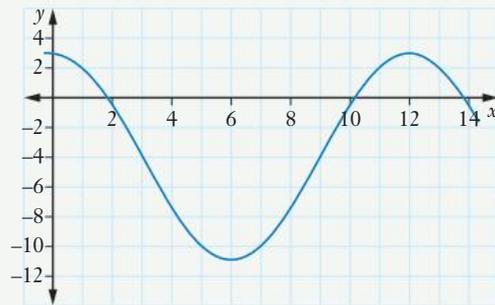
where  $T$  is the temperature in degrees Celsius and  $m$  is the number of months since January.

- a** Draw a graph to represent this data.    **b** Calculate the amplitude.  
**c** Calculate the period.  
**d** During what months is the average overnight minimum increasing?
- 29** A clock stands on a shelf. The centre of the clock, where the hands are attached, is 15 cm above the shelf, and the minute hand is 10 cm long. The distance of the tip of the hand from the shelf can be modelled by a sinusoidal function. Write an equation for the model in terms of the time  $t$ , where  $t$  is the number of minutes past midnight. Hence find the number of times during the day when the tip is 10 cm above the shelf.

### Problem solving

**30** Show that  $\cos\left(\pi + \frac{\pi}{3}\right) = \cos(\pi)\cos\left(\frac{\pi}{3}\right) - \sin(\pi)\sin\left(\frac{\pi}{3}\right)$

- 31** Work out a model for the periodic function shown here.



- 32** The voltage of an AC current that is nominally 240 volts is given by  $V = 240\sqrt{2}\sin(100\pi t)$ , where  $V$  is in volts and  $t$  in seconds.

- a** What is the period?  
**b** What is the maximum voltage?  
**c** What is the amplitude?
- 33** Find, correct to 2 decimal places, the values of  $x$  in the interval  $0 \leq x \leq 2\pi$  for which

$$3 \cos\left(2x + \frac{\pi}{6}\right) = 2.$$

- 34** The intensity of light from the moon can be modelled by the function with the rule

$$y = 0.5 \cos\left(\frac{\pi d}{14}\right) + 0.5$$

where  $d$  gives the number of days since the full moon.

- a** What is the maximum intensity?  
**b** When does this occur?  
**c** When is the intensity at 75% of the maximum?



Practice quiz

# 9.

## EXPONENTIAL FUNCTIONS AND LOGARITHMS

The presence of algae in a swimming pool makes the water appear green. The amount of algae in a pool can double every three or four days. It can seem that a neglected pool goes green overnight, but this is an illusion because of the way the algae grow. The growth of algae is an example of an **exponential function**, and many natural phenomena are like this.

- 9.01 Exponential functions
- 9.02 Exponential graphs
- 9.03 Translations of exponential graphs
- 9.04 Exponential equations
- 9.05 Modelling exponential functions
- 9.06 Common logarithms
- 9.07 General logarithms
- 9.08 The change of base theorem
- 9.09 Solving equations with logarithms
- Chapter summary
- Chapter review

## SYLLABUS SUBJECT MATTER

### Introduction to exponential functions

- recognise and determine the qualitative features of the graph of  $y = a^x$  ( $a > 0$ ) including asymptotes, and of its translations ( $y = a^x + b$  and  $y = a^{x+c}$ )
- recognise and determine the features of the graphs of  $y = b \cdot a^x$  and  $y = a^{kx}$  ( $k \neq 0$ )
- identify contexts suitable for modelling by exponential functions and use models to solve practical problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis
- solve equations involving exponential functions with and without technology

### Introduction to logarithms

- define logarithms as indices:  $a^x = b$  is equivalent to  $x = \log_a b$
- recognise the inverse relationship between logarithms and exponentials:  $y = a^x$  is equivalent to  $x = \log_a y$
- solve equations involving indices with and without technology

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Prior learning

# TERMINOLOGY

asymptote  
decay factor  
exponential  
general logarithm  
half-life  
indicial equations  
power  
vertical dilation

base  
decay function  
exponential equation  
growth factor  
horizontal dilation  
initial value  
reflection

common logarithm  
exponent  
exponential function  
growth function  
index (indices)  
logarithm  
translation



Exponential functions

## 9.01 Exponential functions

The geometric sequence with first term 1 and common ratio 1.189 doubles about every fourth term. The GP is 1, 1.189, 1.414, 1.681, 2.00, 2.376, 2.825, 3.359, 3.994, ...,  $(1.189)^{n-1}$ ,  $(1.189)^n$ , ...

If you make the common ratio  $\sqrt[4]{2}$ , it exactly doubles every fourth term. You can use this to model the growth of algae in a pool, but it only works for whole numbers of days.

Instead of the power  $n$ , you can use  $x$  to define the function  $f(x) = (\sqrt[4]{2})^x = (1.189)^x$ , which has the same values as the GP for  $x = 0, 1, 2, 3, 4, \dots$  but you can also find  $f(10.5) \approx 6.169$ , the growth halfway through the tenth day.

A general **exponential function** has the form  $y = b \cdot a^x$ , where  $a > 0$ .

The variable of an exponential function is in the **exponent** (power). This is different to power functions like  $y = x^3$ .

### EXAMPLE 1

$f(x) = 2 \times (3.2)^x$  and  $g(x) = 5 \times 0.4^x$ . Find the values of  $f(x)$  and  $g(x)$  at  $-4, 0, 1, 2.5$  and  $5$  correct to 3 decimal places and comment on the nature of the functions.

### Solution

Find the values of  $f(x)$ .

$$f(-4) \approx 0.019, f(0) = 2, f(1) = 6.4,$$

$$f(2.5) \approx 362.636, f(5) \approx 671.089$$

Find the values of  $g(x)$ .

$$g(-4) \approx 195.313, g(0) = 1, g(1) = 2,$$

$$g(2.5) \approx 0.506, g(5) \approx 0.051$$

Comment.

$f(x)$  is increasing and  $g(x)$  is decreasing.

From the previous example, you can see that an exponential function  $f(x) = b \cdot a^x$  with  $a > 1$  is always increasing and one with  $a < 1$  is always decreasing. Notice also that the value of  $f(x)$  at  $x = 0$  is always  $b$ .

- An exponential function  $y = b \cdot a^x$  where  $a > 1$  is called a **growth function**.
- An exponential function  $y = b \cdot a^x$  where  $0 < a < 1$  is called a **decay function**.
- $a$  is called the **growth** or **decay factor**. The **initial value** is  $y(0) = b$ .

## EXAMPLE 2

A bacterial colony has about 2000 bacteria. The number of bacteria triples every day. Assuming that growth continues in this way, how many bacteria are in the colony after:

- a** 60 hours?                      **b** 5 days?

### Solution

- a** Write the known data.  
‘Triples’ means  $= 300\% = 3$ .

Write the function.  
Change 60 hours to days.

Substitute into the function.

Round the answer appropriately.

- b** Find the required value.  
Write the answer.

The initial value is 2000 and the growth factor is 3.

$n(x) = 2000 \times 3^x$ , where  $x$  is in days.

$$60 \text{ h} = \frac{60}{24} = 2.5 \text{ days}$$

$$n(2.5) = 2000 \times 3^{2.5} = 31\,126.91\dots$$

There are about 31 000 bacteria after 60 hours.

$$n(5) = 2000 \times 3^5 = 486\,000$$

There are about 486 000 bacteria after 5 days.

### EXAMPLE 3

The amount of air left in a leaking spaceship is decreasing by 30% every minute. When the leak first occurred, the pressure was 105 kPa. What will the pressure be after 6 minutes?

#### Solution

Write the known data.

The initial value is 105 and the decay factor is 0.7.

Decrease by 30% =  $100\% - 30\%$   
=  $70\% = 0.7$ .

Write the function.

$$P(t) = 105 \times 0.7^t$$

Find the required value.

$$P(6) = 105 \times 0.7^6 = 12.353\dots$$

Round the answer appropriately.

After 6 minutes the pressure is about 12.4 kPa.

## Exercise 9.01 Exponential functions

Example  
1

1  $f(x) = 0.3 \times 4.1^x$  and  $g(x) = 8 \times 0.93^x$ . Find the values of  $f(x)$  and  $g(x)$  at  $-2$ ,  $0$ ,  $2$  and  $4$  correct to 3 decimal places and comment on the nature of the functions.

Example  
2

2 An algal bloom covers about  $3 \text{ m}^2$ . The area increases by about 5% each day. What area will it cover after 20 days?

Example  
3

3 The value of a car depreciates by about 15% per year. What is a car bought for \$24 900 worth after 5 years?

### Problem solving

- 4 The atmospheric pressure decreases by about 12% for every 1000 m increase in altitude. The pressure at sea level is approximately 101 kPa. People need a minimum pressure of about 35 kPa to breathe. What is the highest altitude at which you could breathe?
- 5 There were about 20 900 assaults reported to police in Queensland in 2016. The number of reported assaults has grown by about 1.2% a year over the last 20 years. How many assaults would you expect to be reported in 2030? Does the increase in assaults mean that our society is becoming more violent?



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## 9.02 Exponential graphs

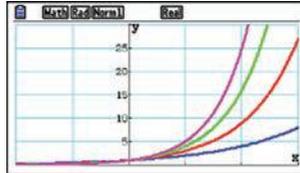
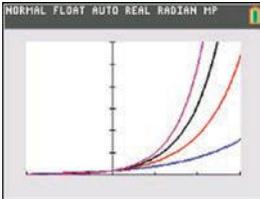
The graphs of exponential functions have a distinctive shape.

### INVESTIGATION

#### EXPONENTIAL GRAPHS

- 1 Draw the graphs of the following functions on your graphics calculator:

$$y_1 = 2^x, y_2 = 3^x, y_3 = 4^x, y_4 = 5^x \text{ for } -2 \leq x \leq 3 \text{ and } 0 \leq y \leq 30.$$

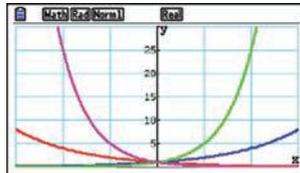
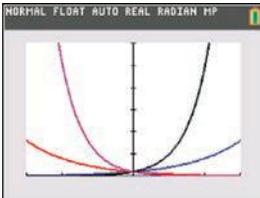


- 2 Use **Trace** or **Calc/G-Solv** to find the values of each function at  $x = -1$ ,  $x = 0$ ,  $x = 1$ ,  $x = 2$ .

- 3 What do you notice about the graphs?

- 4 Now draw the graphs of the following functions on your graphics calculator:

$$y_1 = 2^{-x}, y_2 = 0.5^x, y_3 = 5^{-x}, y_4 = 0.2^x \text{ for } -3 \leq x \leq 3 \text{ and } 0 \leq y \leq 30.$$



- 5 Use **Trace** or **Calc/G-Solv** to find the values of each function at  $x = -1$ ,  $x = 0$ ,  $x = 1$ ,  $x = 2$ .

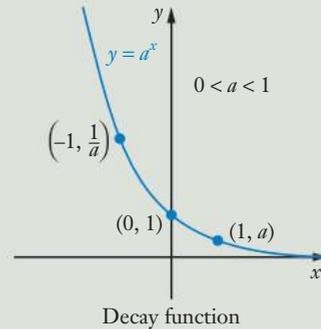
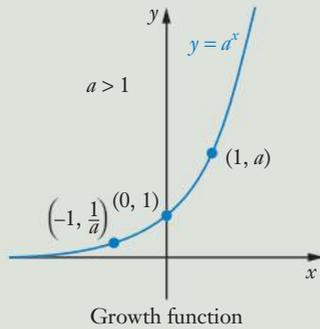
- 6 What do you notice about the graphs?

The graph of  $y = a^x$  passes through  $\left(-1, \frac{1}{a}\right)$ ,  $(0, 1)$  and  $(1, a)$ .

It has a horizontal **asymptote**,  $y = 0$ .

For  $a > 1$ , as  $x \rightarrow -\infty$ ,  $y \rightarrow 0$  and as  $x \rightarrow \infty$ ,  $y \rightarrow \infty$ .

For  $0 < a < 1$ , as  $x \rightarrow -\infty$ ,  $y \rightarrow \infty$  and as  $x \rightarrow \infty$ ,  $y \rightarrow 0$ .



#### EXAMPLE 4

Sketch the graph of  $f(x) = 4^x$ .

#### Solution

State the general shape.

$a = 4 > 0$ , so the function is increasing with asymptote  $y = 0$ .

Find important points.

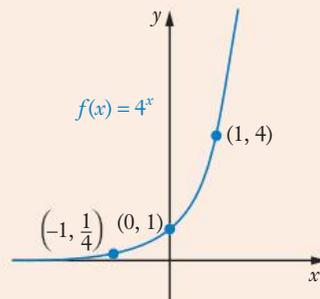
$$f(0) = 1, f(1) = 4$$

State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow 0$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow \infty$$

Sketch the graph.



The function  $y = 3 \times 4^x$  is 3 times the size of  $y = 4^x$  at any particular value of  $x$ . The graph is a vertical dilation of  $y = 4^x$  by the factor 3. The graph of  $y = -3 \times 4^x$  is negative for all values of  $x$  instead of positive, so it is the reflection of  $y = 3 \times 4^x$  in the  $x$ -axis.

The graph of  $y = b \cdot a^x$  is the **vertical dilation** of the graph of  $y = a^x$ . If  $b$  is negative, it will also be a reflection in the  $x$ -axis.

The graph will pass through  $\left(-1, \frac{b}{a}\right)$ ,  $(0, b)$  and  $(1, ab)$ . Label these points when you sketch the graph.

### EXAMPLE 5

Sketch the graph of:

**a**  $y = 3 \times 0.8^x$

**b**  $f(x) = -2 \times 3^x$

#### Solution

**a** State the general shape.

$a = 0.8 < 1$ , so the function is decreasing with asymptote  $y = 0$ .

Find important points.

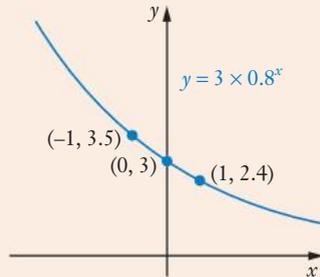
$f(0) = 3, f(1) = 2.4, f(-1) = 3.5$

State the behaviour as  $x \rightarrow \pm\infty$ .

As  $x \rightarrow -\infty, f(x) \rightarrow \infty$

As  $x \rightarrow \infty, f(x) \rightarrow 0$

Sketch the graph.



**b** State the general shape.

$b = -2 < 0$ , so the function is reflected below the  $x$ -axis and decreasing.

Find important points.

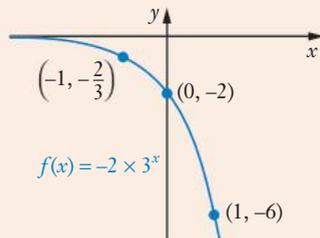
$f(0) = -2, f(1) = -6, f(-1) = -\frac{2}{3}$

State the behaviour as  $x \rightarrow \pm\infty$ .

As  $x \rightarrow -\infty, f(x) \rightarrow 0$

As  $x \rightarrow \infty, f(x) \rightarrow -\infty$

Sketch the graph.



Consider the graphs of  $7^x$  and  $\left(\frac{1}{7}\right)^x = (7^{-1})^x = 7^{-x}$ . The points (1, 7), (2, 49) and (3, 343) on the graph of  $7^x$  will correspond to the points (-1, 7), (-2, 49) and (-3, 343) on the graph of  $7^{-x}$ .

The graphs of  $7^x$  and  $\left(\frac{1}{7}\right)^x = 7^{-x}$  are reflections of each other in the  $y$ -axis.

A function of the form  $y = a^{kx}$  can be re-written as  $y = (a^k)^x$ , so it is of the same form as  $y = a^x$ , but it has the base  $a^k$  instead of  $a$ . In other words, it is a horizontal dilation of  $y = a^x$ .

- The graphs of  $y = a^x$  and  $y = a^{-x}$  are reflections of each other in the  $y$ -axis.
- The graph of  $y = a^{kx}$  is a **horizontal dilation** of  $y = a^x$ , passing through (0, 1) and  $\left(\frac{1}{k}, a\right)$ .
  - For  $k > 1$ , it is compressed by the factor  $k$ .
  - For  $0 < k < 1$ , it is stretched by the factor  $\frac{1}{k}$ .
  - For  $k < 0$ , it is also reflected in the  $y$ -axis.

### EXAMPLE 6

Sketch the graph of  $y = 3^{-2x}$ .

#### Solution

State the general shape.

The graph of  $y = 3^{-2x}$  is decreasing and is reflected in the  $y$ -axis from  $y = 3^{2x}$ . It is vertically compressed by the factor 2.

Find important points.

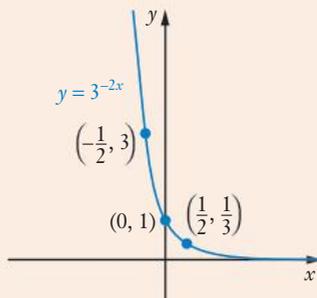
$$f(0) = 1, f(-0.5) = 3, f(0.5) = \frac{1}{3}$$

State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow \infty$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow 0$$

Sketch the graph.



## Exercise 9.02 Exponential graphs

Sketch the graph of each exponential function.

1  $f(x) = 6^x$

2  $y = 0.7^x$

3  $f(x) = 0.3^x$

4  $y = 2.5^x$

5  $f(x) = 2 \times 3^x$

6  $y = 0.5 \times 4^x$

7  $f(x) = 3 \times 0.6^x$

8  $y = 5 \times 1.5^x$

9  $f(x) = -3^x$

10  $y = -4 \times 0.8^x$

11  $f(x) = -3 \times 0.4^x$

12  $y = -0.5 \times 2.5^x$

13  $f(x) = 3^{0.4x}$

14  $y = 0.6^{-2x}$

15  $f(x) = 2^{-3x}$

16  $y = 0.2^{-0.5x}$

Example  
4

Example  
5

Example  
6

## 9.03 Translations of exponential graphs

You saw how a function  $f(x)$  was transformed to  $af(bx+c)+d$  in Chapter 4 and similar transformations of the trigonometric functions in Chapter 8. The transformations of exponential functions occur in the same way.

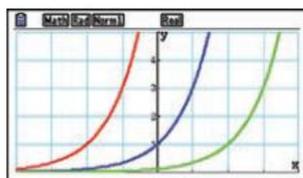
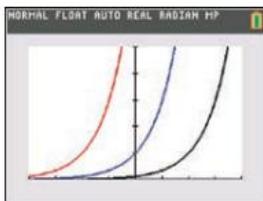


### INVESTIGATION

#### TRANSLATIONS OF EXPONENTIAL GRAPHS

- 1 Draw the graphs of the following functions on your graphics calculator.

$$f_1(x) = 3^x, f_2(x) = 3^{x+2}, f_3(x) = 3^{x-2} \text{ for } -4 \leq x \leq 4 \text{ and } 0 \leq y \leq 5.$$



- 2 Use **Trace** or **Calc/G-Solv** to find the values of:

a  $f_1(-1), f_2(-3)$  and  $f_3(1)$

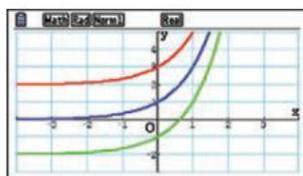
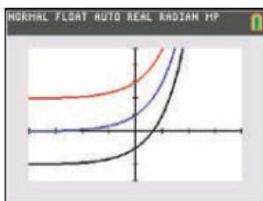
b  $f_1(0), f_2(-2)$  and  $f_3(2)$

c  $f_1(1), f_2(-1)$  and  $f_3(3)$

- 3 What can you say about these graphs?

- 4 Now draw the graphs of the following functions on your graphics calculator.

$$f_1(x) = 3^x, f_2(x) = 3^x + 2, f_3(x) = 3^x - 2 \text{ for } -4 \leq x \leq 4 \text{ and } -3 \leq y \leq 5.$$



5 Use **Trace** or **Calc/G-Solv** to find the values of:

a  $f1(-1), f2(-1)$  and  $f3(-1)$

b  $f1(0), f2(0)$  and  $f3(0)$

c  $f1(1), f2(1)$  and  $f3(1)$

6 What can you say about these graphs?

- Compared to the graph of  $y = a^x$ :
  - the graph of  $y = a^{x+b}$  is **translated** horizontally by  $b$ , left for  $b > 0$ , right for  $b < 0$
  - the graph of  $y = a^x + c$  is **translated** vertically by  $c$ , up for  $c > 0$ , down for  $c < 0$ .  
The asymptote is also moved up or down to  $y = c$ .
- Some important points on the graph of  $y = a^{x+b} + c$  are  $(-b - 1, \frac{1}{a} + c)$ ,  $(-b, 1 + c)$ ,  $(-b + 1, a + c)$  and the  $y$ -intercept  $(0, a^b + c)$ .

### EXAMPLE 7

Sketch the graph of:

a  $f(x) = 4^{x+3}$

b  $y = 0.3^{x-2}$

#### Solution

a State the general shape.

$a = 4 > 0$ : function is increasing.

Find important points.

$$f(-3) = 1, f(-4) = \frac{1}{4}, f(-2) = 4, f(0) = 64$$

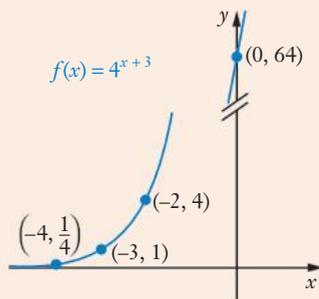
State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow 0$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow \infty$$

Sketch the graph.

In some cases you may have to show the  $y$ -intercept on a different scale. In this case, show a break in the axis by a zig-zag or two sloping lines through the axis.



**b** State the general shape.

$a = 0.3 < 1$ : function is decreasing.

Find important points.

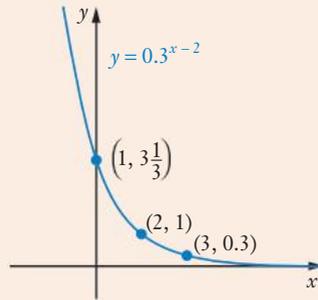
$$f(2) = 1, f(1) = 3\frac{1}{3}, f(3) = 0.3$$

State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow \infty$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow 0$$

Sketch the graph.



### EXAMPLE 8

Sketch the graphs of

**a**  $f(x) = 0.5^x - 3$

**b**  $y = 3^x + 2$

### Solution

**a** State the general shape.

$a = 0.5 < 1, c = -3$ : decreasing with asymptote  $y = -3$ .

Find important points.

$$f(0) = -2, f(1) = -2\frac{1}{2}, f(-1) = -1$$

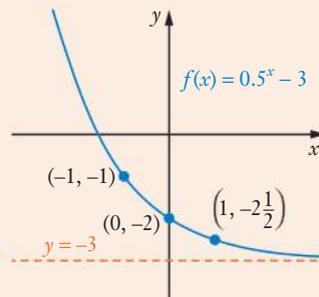
State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow \infty$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow -3$$

Sketch the graph.

Make sure you label the asymptote when it is not the x-axis.



**b** State the general shape.

$a = 3 > 1, c = 2$ : increasing with asymptote  $y = 2$ .

Find important points.

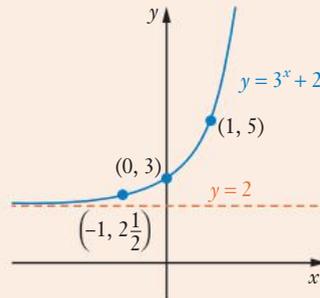
$$f(0) = 3, f(-1) = 2\frac{1}{3}, f(1) = 5$$

State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow 2$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow \infty$$

Sketch the graph.



Different transformations can be combined in one function.

### EXAMPLE 9

Sketch the graph of  $y = 5^{x-1} + 2$ .

#### Solution

State the general shape.

$a = 5 > 1, c = 2$ : increasing with asymptote  $y = 2$ .

Find important points.

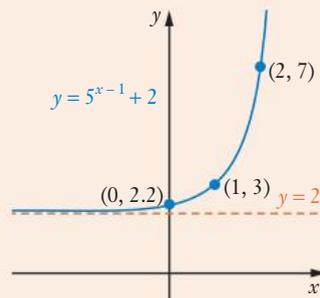
$$f(1) = 3, f(0) = 2.2, f(2) = 7$$

State the behaviour as  $x \rightarrow \pm\infty$ .

$$\text{As } x \rightarrow -\infty, f(x) \rightarrow 2$$

$$\text{As } x \rightarrow \infty, f(x) \rightarrow \infty$$

Sketch the graph.



## Exercise 9.03 Translations of exponential graphs

Sketch the graph of each exponential function.

**1**  $y = 0.2^{x-2}$

**2**  $f(x) = 3^{x+3}$

**3**  $y = 0.5^{x-1}$

**4**  $f(x) = 2^x + 3$

**5**  $y = 0.8^x - 2$

**6**  $f(x) = 5^x + 1$

**7**  $y = 0.7^{x-2} + 3$

**8**  $f(x) = 6^{x+2} - 1$

**9**  $y = 0.9^{x-1} - 3$

**10**  $f(x) = 4^{x+1} + 4$

Problem solving

**11**  $y = -0.6^{x-3} + 2$

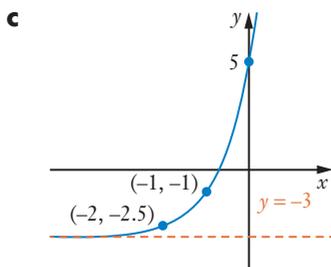
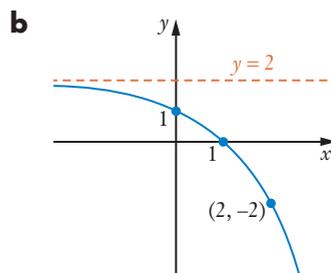
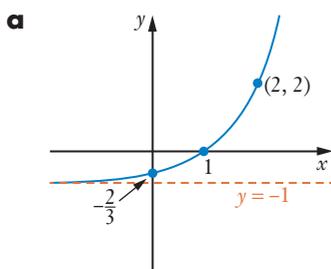
**12**  $f(x) = 2 \times 3^{x+1} + 4$

**13**  $y = -0.4 \times 0.7^{2x-4} - 1$

**14**  $f(x) = 0.5 \times 2^{-x+2} - 4$

**15**  $y = -3 \times 0.4^{0.5x-2} + 1$

**16** Find a possible exponential function for each graph.





## 9.04 Exponential equations

The variable in an **exponential equation** is part of the exponent (power, index).

An exponential equation is also called an **indicial equation**.

You may be able to solve some exponential equations by making the base the same on both sides and equating the indices. Always check your solutions in the original equation. You may have to discard false solutions.

### EXAMPLE 10

Solve each exponential equation.

**a**  $3^{2x-5} = 27$

**b**  $16^x - 5 \times 4^x = 24$

#### Solution

**a** Write in powers to the same base.

$$3^{2x-5} = 3^3$$

Equate indices and solve.

$$2x - 5 = 3$$

$$2x = 8$$

$$x = 4$$

Check in the original equation.

$$\text{LHS} = 3^{2 \times 4 - 5}$$

$$= 3^3$$

$$= 27$$

$$= \text{RHS} \quad \text{OK} \checkmark$$

Write the answer.

The solution is  $x = 4$ .

**b** Write in powers to the same base.

$$(4^2)^x - 5 \times 4^x - 24 = 0$$

$$4^{2x} - 5 \times 4^x - 24 = 0$$

$$(4^x)^2 - 5 \times 4^x - 24 = 0$$

Let  $y = 4^x$  to make a quadratic equation, then solve it.

$$y^2 - 5y - 24 = 0 \text{ for } y = 4^x$$

$$(y - 8)(y + 3) = 0$$

$$y = 8 \quad \text{or} \quad y = -3$$

Express in terms of  $4^x$ .

$$4^x = 8 \quad \text{or} \quad 4^x = -3$$

Discard the false solution.

$$4^x = -3 \text{ is not possible, so } 4^x = 8$$

Write in powers to the same base and equate indices to solve.

$$\begin{aligned}2^{2x} &= 2^3 \\ 2x &= 3 \\ x &= \frac{3}{2}\end{aligned}$$

Check in the original equation.

$$\begin{aligned}\text{LHS} &= 16^{\frac{3}{2}} - 5 \times 4^{\frac{3}{2}} \\ &= (\sqrt{16})^3 - 5 \times (\sqrt{4})^3 \\ &= 64 - 5 \times 8 \\ &= 24 \\ &= \text{RHS} \quad \text{OK} \checkmark\end{aligned}$$

Write the answer.

$$\text{The solution is } x = \frac{3}{2}.$$

In part **b** above, it would actually be more efficient to change to base 2 to start with, but that does make it more difficult to see the quadratic equation.

In most real applications, you must use technology to solve exponential equations. You can also use a graph to solve these equations.

### EXAMPLE 11

Solve each equation, correct to 3 decimal places, using a graphics calculator.

**a**  $3^x = 12$

**b**  $5 \times 3^x - 2 = 0$

#### Solution

**a** **TI-84 Plus CE**

Press  $\boxed{Y=}$  and enter  $Y_1 = 3^x$  and  $Y_2 = 12$ .

Set the  $\boxed{\text{window}}$  to  $0 \leq x \leq 5$  and  $0 \leq y \leq 15$  and draw the  $\boxed{\text{graph}}$ .

Use  $\boxed{\text{calc}}$ , **5: intersect** to find the required value.

**Casio fx-CG20AU**

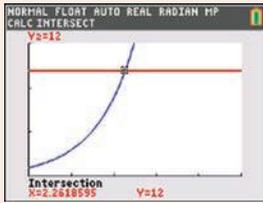
Choose **5 Graph** and enter  $Y1 = 3^x$  and  $Y2 = 12$ .

Set the **V-Window** to  $0 \leq x \leq 5$  and  $0 \leq y \leq 5$ ,  $\boxed{\text{EXIT}}$  and draw the graph ( $\boxed{\text{F6}}$ ).

Use **G-solv**, **INTSECT**( $\boxed{\text{F5}}$ ) to find the required value.



TI-Nspire CX  
Chapter 9



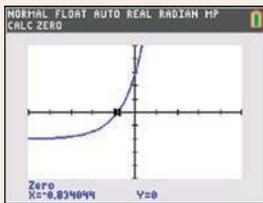
$$x \approx 2.262$$

### b TI-84 Plus CE

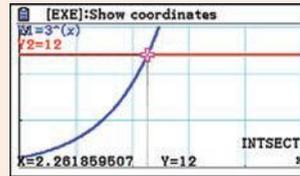
Press  $\boxed{Y=}$  and enter  $Y_1 = 5 \times 3^x - 2$ .

Set the  $\boxed{\text{window}}$  to  $-5 \leq x \leq 5$  and  $-5 \leq y \leq 5$  and draw the  $\boxed{\text{graph}}$ .

Use  $\boxed{\text{calc}}$ , **2: zero** to find the required value.



$$x \approx -0.834$$



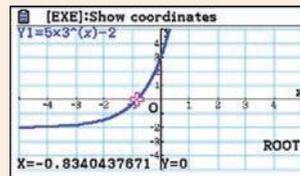
$$x \approx 2.262$$

### Casio fx-CG20AU

Choose **5 Graph** and enter  $Y1 = 5 \times 3^x - 2$ .

Set the **V-Window** to  $-5 \leq x \leq 5$  and  $-5 \leq y \leq 5$ ,  $\boxed{\text{EXIT}}$  and draw the graph.

Use **G-solv**, **ROOT** ( $\boxed{\text{F1}}$ ) to find the required value.



$$x \approx -0.834$$

You may prefer to change the equation  $3^x = 12$  to  $3^x - 12 = 0$  or  $5 \times 3^x - 2 = 0$  to  $5 \times 3^x = 2$  so you can use the same method for both equations.

## Exercise 9.04 Exponential equations

Example  
10

1 Solve each exponential equation without using technology.

**a**  $2^{x+1} = 32$

**b**  $9^{x-5} = 243$

**c**  $4^{3x+1} = 32$

**d**  $5^{3-2x} - 125 = 0$

**e**  $16^{x-2} - 8 = 0$

**f**  $49 - 343^{3x+3} = 0$

**g**  $64 - 16^{5-2x} = 0$

**h**  $25^{1-x} - 125 = 0$

**i**  $64^{3x+1} = 32^{5x-4}$

2 Solve each equation without using technology.

**a**  $9^x - 90 \times 3^x + 729 = 0$

**b**  $4^x + 32 = 12 \times 2^x$

**c**  $16^x + 1024 = 5 \times 4^{x+2}$

**d**  $5^{2x+1} + 25 = 5^{x+3} + 5^x$

**e**  $4^x + 2^{x+2} = 32$

**f**  $5^{2x+1} = 625 - 124 \times 5^{x+1}$

**g**  $9^x = 26 \times 3^x + 27$

**h**  $49^{x+1} = 350 \times 7^x - 49$

**i**  $4 \times 4^x = 3 \times 2^{x+3} + 64$

- 3 Solve each equation correct to 3 decimal places using a graphics calculator.
- a  $16^x = 5$                       b  $5^x = 16$                       c  $4^{2x} = 85.4$   
 d  $50^{2x-3} - 28 = 0$             e  $5^x - 4^x - 3 = 0$             f  $10^{3x} - 150 = 2^{4x}$

### Problem solving

- 4 Solve  $8^x + 7 \times 2^{x+1} = 7 \times 4^x + 8$  without using technology.  
 5 Solve  $3^{3x} - 12 \times 9^x + 63 \times 3^x = 4 \times 3^3$  without using technology.

## 9.05 Modelling exponential functions

You saw how exponential functions are used to model population growth, compound interest, appreciation and depreciation in the first part of this chapter. You can use exponential functions for other situations where the change in an amount depends on the amount itself.



A quantity  $A(t)$  such that  $\frac{A(t+1)}{A(t)} = k$ , is given by the exponential equation  $A(t) = A_0 k^t$ , where  $k$  is a constant and  $A_0$  is the initial value of  $A$ .

### INVESTIGATION

#### RADIOCARBON DATING

**Radiocarbon dating** uses the naturally occurring isotope carbon-14 ( $^{14}\text{C}$ ) to find the age of dead animals or plants discovered by archeologists. Carbon-14 is created by radiation striking the upper atmosphere. It makes up about one-trillionth of the carbon in the atmosphere. The proportion of carbon-14 is the same in all living plants and animals.

When the animal or plant dies, the carbon-14 gradually changes back to nitrogen. It takes 5730 years for half the carbon-14 present at death to decay. This is called the **half-life** of carbon-14. About 11.4% of the carbon-14 decays every 1000 years after death.

The amount of carbon-14 in the atmosphere is quite stable.



National Geographic Creative/LYNN JOHNSON

- 1 Write an equation for the percentage of carbon-14 left in a dead animal or plant after  $x$  thousand years.
- 2 Copy this table and use your calculator to complete it.

Proportion of original carbon 14 left	Age (time since death)
90%	
80%	
70%	
60%	
50%	
40%	
30%	
20%	
10%	
5%	
3%	
1%	

- 3 Find out if the amount of carbon-14 in the atmosphere has always been the same.
- 4 What are the limits of the usefulness of carbon-14 dating?

### EXAMPLE 12

The temperature  $T$  of a hot object as it cools is modelled by  $(T - T_E) = (T_0 - T_E) \times 0.9^t$ , where  $T_E$  is the temperature of the environment,  $T_0$  is the initial temperature and  $t$  is the time in minutes.

The initial temperature is  $88^\circ\text{C}$  and the temperature of the surroundings is  $26^\circ\text{C}$ .

- a Find the temperature after 10 minutes.
- b Find the time taken to cool to  $40^\circ\text{C}$ .
- c Verify and evaluate the model.

## Solution

- a** Substitute values into the equation.

$$T - 26 = (88 - 26) \times 0.9^t$$

Rearrange to make  $T$  the subject.

$$T = 26 + 62 \times 0.9^t$$

Substitute  $t = 10$ .

$$\begin{aligned} &= 26 + 62 \times 0.9^{10} \\ &= 47.618\dots \end{aligned}$$

Write the answer.

After 10 minutes, it is about  $48^\circ\text{C}$ .

- b** Try values.

$$T(11) = 26 + 62 \times 0.9^{11} = 45.456\dots$$

$$T(12) = 26 + 62 \times 0.9^{12} = 43.510\dots$$

$$T(13) = 26 + 62 \times 0.9^{13} = 41.759\dots$$

$$T(14) = 26 + 62 \times 0.9^{14} = 40.183\dots$$

$$T(15) = 26 + 62 \times 0.9^{15} = 38.765\dots$$

Write the answer.

It takes a bit over 13 minutes to cool to  $40^\circ\text{C}$ .

- c** Verify the model against the data.

$$T(0) = 26 + 62 \times 0.9^0 = 26 + 62 = 88^\circ\text{C}$$

$$\text{As } t \rightarrow \infty, 0.9^t \rightarrow 0, \text{ so } T(t) \rightarrow 26^\circ\text{C}$$

so the model fulfils the conditions.

Evaluate the model.

The model allows the calculation of temperatures for any time, so is a good predictor of future conditions.

You could try substituting 14.1, 14.2, and so on to narrow the time further, but temperature measurement is not generally accurate to less than  $1^\circ\text{C}$  anyway.

You can use your graphics calculator to find an exponential model for given data. In regressions, the  $r^2$  value shows how close the points are to a function. A value of 1 shows a perfect fit.

## EXAMPLE 13

The amplitude of a pendulum is found to change with time as shown below.

Try an exponential model for the amplitude and predict the amplitude after 20 minutes. Evaluate the usefulness of your model.

Time in minutes	1	2	4	6	10
Amplitude in cm	12	11.5	11	10	9

### Solution

#### TI-84 Plus CE

Press **[mode]** and make sure that **STAT DIAGNOSTICS** is on. Press **[stat]** **1: edit** and enter the data in L1 and L2.

L1	L2	L3	L4	L5	2
1	12				
2	11.5				
4	11				
6	10				
10	9				

Press **[stat]** again, move the cursor right to **CALC** and go down to 0: **ExpReg** and press **[enter]**. Leave XList as L1, YList as L2, move the cursor down to **Calculate** and press **[enter]** to find the function.

```

ExpReg
y=a*b^x
a=12.34020952
b=0.9683652595
r^2=0.9891858482
r=-0.9945782263
    
```

Write the function.

Find the value after 20 minutes.

Write the answer.

Evaluate the model.

#### Casio fx-CG20AU

Choose **2 Statistics** and enter the data in List 1 and List 2.

List 1	List 2	List 3	List 4
3	4	11	
4	6	10	
5	10	9	
6			

Use **CALC** (**[F2]**), **REG** (**[F3]**), **EXP** (**[F6]** **[F2]**) and  $a^x$  (**[F2]**) to find the function.

```

ExpReg(a*b^x)
a = 12.3402095
b = 0.96836525
r = -0.9945782
r^2 = 0.98918584
MSE = 1.928E-04
y = a * b^x
    
```

The function is  $A(t) \approx 12.34 \times (0.9684)^t$

$$A(20) \approx 12.34 \times (0.9684)^{20} \approx 6.492$$

The amplitude is given by  $12.34 \times (0.9684)^t$ , where  $t$  is in minutes. It is about 6.5 cm after 20 minutes.

The value of  $r^2$  is very close to 1, so the exponential model works well for the 10 minutes shown. It may not be as good a model over a longer time.



TI-Nspire CX  
Chapter 9

## Exercise 9.05 Modelling exponential functions

Example  
12

- 1 A full 30 000 L cylindrical rainwater tank has sprung a leak in the seam at the bottom of the tank. As the water leaks out, the pressure of water decreases. This means that the tank leaks more slowly over time. The level of water in the tank is given by  $L = 295 \times 0.97^t$  cm, where  $t$  is the time in hours from the start of the leak. The leak was discovered 3 hours after it started.

  - a What is the diameter of the tank?
  - b How much water leaked out before the leak was discovered?
  - c How long will it take for 5000 L to leak out?
- 2 a The concentration of paracetamol in an adult's blood is about  $25 \times (0.65)^t$   $\mu\text{g/mL}$   $t$  hours after taking a standard adult dose of 1 gram. What is the level after 2 hours?

b The therapeutic level is considered to be a minimum of 5  $\mu\text{g/mL}$ , but high levels are extremely toxic. How long after taking a dose does the effect wear off?
- 3 The population of rabbits in an area increases by an average of 50% every month. How long would it take two rabbits to overrun an area at a population of 3000 rabbits?
- 4 A sample of radioactive material loses 4% of its radioactivity per annum. Calculate the half-life of the material, the time it takes to decay to half its original amount of radioactivity.
- 5 A sample of carbon from an ancient midden has only 40% of the original amount of carbon-14 present. The carbon-14 is known to have decayed at a rate of 1.2% per century from the time the midden was used. How old is the midden?
- 6 The voltage across a capacitor as it discharged through a resistor was measured by students. The values are shown below.

Example  
13

Time (s)	4	8	12	20	30	40
Voltage (kV)	49	40	33	22	13	8

Use an exponential model to find the original voltage. The capacitor is considered to be fully discharged when the voltage drops below 20 V. How long does this take? If the original voltage was 3 times as high, how long would the capacitor take to drop to a voltage of 5 kV?

Evaluate your model and answers.

### Problem solving

- 7 The size of a bacteria colony on an agar plate in an incubator increases by 13.5% every hour. Construct a function to model the size and use it to find the size of a colony after a day, given that it begins with a size of  $0.1 \text{ mm}^2$ . Are there any limitations on your model?

- 8** The temperature in the core of a nuclear reactor is about  $64^{\circ}\text{C}$ . In a disaster such as that at Fukushima, the temperature can rise exponentially to cause a dangerous explosion. The table below shows temperatures recorded in a nuclear core after a problem was encountered.

Time (min)	0	5	10	20	30
Temperature ( $^{\circ}\text{C}$ )	64	69	73	89	101

If the temperature exceeds  $1500^{\circ}\text{C}$ , the core begins to melt down.

- Find an exponential equation for the temperature.
  - Verify the model and evaluate its limitations.
  - What will the temperature be after 3 hours?
  - How long have the operators got to avert a disaster?
- 9** Iodine-131 decays to xenon, so the mass of a sample decreases as the xenon escapes into the atmosphere. The mass of a sample of iodine-131 prepared for use in thyroid therapy was measured over a week with the following results.

Time (days)	0	1	2	3	4	5	6	7
Mass (mg)	87	77	71	67	59	56	51	46

- Find an exponential model for the amount of iodine-131 remaining.
  - Verify the model and evaluate its limitations.
  - Find the half-life of the iodine-131, the time it takes to decay to half its original mass.
  - How long would it take to decay to only 10 mg?
- 10** The difference in temperature between a hot pie and its surroundings decreases by 4% every minute. A hot pie bought at a Broncos–Raiders game starts at a temperature of  $72^{\circ}\text{C}$ , while the temperature at the ground is  $15^{\circ}\text{C}$ . It takes 3 minutes to walk back to the stand.
- If you don't start eating the pie until you get back to the stand, what is its temperature when you start eating?
  - Find a model for the temperature of the pie  $t$  minutes after it is bought.
  - What is the temperature of the pie after 10 minutes?
  - How long will it take for the temperature of the pie to drop below  $40^{\circ}\text{C}$ ?
  - Evaluate the usefulness of your model.

## 9.06 Common logarithms

Suppose you wanted to find out the **power** to which a **base** must be raised to obtain a given number? To solve the exponential equation  $4^m = 1024$ , you can use **logarithms**.

In particular, exponential equations involving base 10 such as  $10^m = 1$ ,  $10^m = 2$ ,  $10^m = 3$ ,  $10^m = 4$ ,  $10^m = 5$  require **common logarithms**.



Exponential and  
log graphs  
match-up

### Common logarithms

- The **common logarithm**,  $m$ , of the number  $x$  is the solution of the equation  $10^m = x$ .
- You write the common logarithm of the number  $x$  as  $m = \log_{10}(x)$  or just  $m = \log(x)$ .
- It is also called the **logarithm to the base 10** (or **log base 10**).

$$m = \log_{10}(x) \Leftrightarrow 10^m = x$$

### EXAMPLE 14

Find the value of:

**a**  $\log(100)$

**b**  $\log(\sqrt[5]{10})$

#### Solution

**a** Express as  $\log_{10}$ .

Use  $x = \log_{10}(a) \Leftrightarrow 10^x = a$ .

Write 100 as a power of 10.

Equate the indices.

State the result.

$$\log(100) = \log_{10}(100)$$

$$m = \log_{10}(100) \Leftrightarrow 10^m = 100$$

$$10^m = 10^2$$

$$m = 2$$

$$\log(100) = 2$$

**b** Express as  $\log_{10}$  and in index form.

Rewrite in index form.

Equate the indices.

State the result.

$$\log_{10}(\sqrt[5]{10}) = \log_{10}\left(10^{\frac{1}{5}}\right)$$

$$m = \log_{10}\left(10^{\frac{1}{5}}\right) \Leftrightarrow 10^m = \left(10^{\frac{1}{5}}\right)$$

$$m = \frac{1}{5}$$

$$\log(\sqrt[5]{10}) = \frac{1}{5}$$

So the common logarithm of a number is the power to which 10 must be raised to give that number. When you are asked to find the common logarithm of a number, think 'What power of 10 gives the number?'

### EXAMPLE 15

Find the value of  $\log(47)$  using a calculator and show that it is the solution of the exponential equation  $10^x = 47$ .

#### Solution

Use a calculator to calculate  $\log(47)$  and store the result.

$$\log(47) = 1.6720\dots$$

Rewrite in index form.

$$1.6720\dots = \log(47) \Leftrightarrow 10^{1.6720\dots} = 47$$

Use a calculator to check the equation.

$$10^{1.6720\dots} = 47$$

State the result.

So  $\log(47) = 1.6720\dots$  is the solution of the equation  $10^x = 47$ .

### TECHNOLOGY

#### Investigating the graphs of common logarithms

The graph of the common logarithm function  $y = \log(x)$  has only one basic shape, as shown on the right.

- 1 Use a graphics calculator to draw the graph of  $y = \log(x)$ . Set the domain values between 0 and 10 and the range values between  $-3$  and 4.

On the same set of axes, draw the graph of  $y = \log(x) + 2$  and  $y = \log(x) - 2$ .

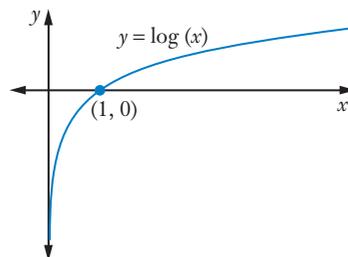
Make a labelled sketch of the graphs and describe the relationship between them.

- 2 Clear the graphs of  $y = \log(x) + 2$  and  $y = \log(x) - 2$  and change the window settings to  $-3 \leq x \leq 7$ .

On the same set of axes, draw the graphs of  $y = \log(x)$ ,  $y = \log(x + 2)$  and  $y = \log(x - 2)$ .

Describe the relationship between the graphs. Make a labelled sketch as a record of the graphs.

- 3 Comment on your findings.



The properties of common logarithms follow from the properties of indices. The most useful of these properties are shown below.

### Properties of common logarithms

$$\log(ab) = \log(a) + \log(b)$$

$$\log(1) = 0$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$

$$\log(10) = 1$$

$$\log(a^n) = n \log(a)$$

$$\log\left(\frac{1}{a}\right) = -\log(a)$$

### EXAMPLE 16

Show that  $\log(ab) = \log(a) + \log(b)$  for any positive numbers  $a$  and  $b$ .

#### Solution

Assign variables to  $\log(a)$  and  $\log(b)$ .

Let  $\log(a) = x$  and  $\log(b) = y$ .

Write  $a$  and  $b$  in index form.

$$a = 10^x \text{ and } b = 10^y$$

Write  $ab$  as a product of powers.

$$ab = 10^x \times 10^y$$

Simplify.

$$= 10^{x+y}$$

Use the definition of a common logarithm.

$$\log(ab) = x + y$$

Replace  $x$  and  $y$  with  $\log(a)$  and  $\log(b)$ .

$$= \log(a) + \log(b)$$

Write the result.

$$\log(ab) = \log(a) + \log(b)$$

### Exercise 9.06 Common logarithms

1 Find, without using a calculator:

**a**  $\log(10)$

**b**  $\log(1000)$

**c**  $\log(1)$

**d**  $\log(0.1)$

**e**  $\log(\sqrt{10})$

**f**  $\log(0.001)$

**g**  $\log(\sqrt[3]{10})$

**h**  $\log(10^{\frac{3}{2}})$

**i**  $\log(\sqrt{10^5})$

2 Use a calculator to find the value of each logarithm correct to 4 decimal places, and show that they are the solutions of exponential equations.

**a**  $\log(58)$

**b**  $\log(160)$

**c**  $\log(16)$

**d**  $\log(0.05)$

**e**  $\log(15)$

**f**  $\log(890)$

**g**  $\log(67)$

**h**  $\log(63)$

**i**  $\log(62.5)$

3 **a** Find correct to 3 decimal places the values of  $\log(3.5)$  and  $\log(350)$ .

**b** Find the relationship between  $\log(3.5)$ ,  $\log(350)$  and  $\log(100)$ .

Example  
14

Example  
15

- 4 a** Find the values of  $\log(18)$ ,  $\log(3)$  and  $\log(6)$ .  
**b** Find the relationship between  $\log(18)$ ,  $\log(3)$  and  $\log(6)$ .
- 5** Show that  $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$ .
- 6** Show that  $\log(a^x) = x \log(a)$ .
- 7** Show that  $\log(10) = 1$  and  $\log(1) = 0$ .
- 8** Show that  $\log\left(\frac{1}{a}\right) = -\log(a)$ .
- 9** Show that  $\log(a \times 10^n) = n + \log(a)$ .
- 10** Given  $\log(5.7) \approx 0.7559$ , find:  
**a**  $\log(57)$                       **b**  $\log(5700)$                       **c**  $\log(0.057)$
- 11** Given  $\log(2.84) \approx 0.4533$ , find:  
**a**  $\log(28.4)$                       **b**  $\log(2840)$                       **c**  $\log(0.284)$
- 12** Given  $\log(9.735) \approx 0.9883$ , find:  
**a**  $\log(97\,350)$                       **b**  $\log(0.9735)$                       **c**  $\log(0.000\,973\,5)$
- 13** Given  $\log(1.23) \approx 0.0899$ , find:  
**a**  $\log(123)$                       **b**  $\log(12\,300)$                       **c**  $\log(0.123)$

### Problem solving

- 14 a** Copy each table and use a calculator to complete it correct to 4 decimal places.

$x$	5	10	15	20	25	30	35	40	45	50
$\log(x)$										

$x$	55	60	65	70	75	80	85	90	95	100
$\log(x)$										

- b** Draw a graph of the function  $y = \log(x)$  for  $5 \leq x \leq 100$ .  
**c** Use the fact that  $y = \log_{10}(x) \Leftrightarrow 10^y = x$  and the values in the completed table to draw the graph of  $y = 10^x$  for  $5 \leq y \leq 100$ .  
**d** Comment on the shapes of the graphs of  $y = \log(x)$  and  $y = 10^x$ .
- 15 a** Use a calculator to complete the following table of values.

$x$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\log(x)$										

$x$	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$\log(x)$										

- b** Draw a graph of the function  $y = \log(x)$  for  $0.1 \leq x \leq 2$ .
- c** Use the fact that  $y = \log(10^x) \Leftrightarrow 10^y = x$  and the values in the completed table to draw the graph of  $y = 10^x$  for  $0.1 \leq y \leq 2$ .
- d** Comment on the shapes of the graphs of  $y = \log(x)$  and  $y = 10^x$ .

**16 a** Use a calculator to complete the following table of values.

$x$	0.001	0.005	0.01	0.05	0.1	0.5	1	5	10	50	100	500	1000
$\log(x)$													

- b** Draw a graph of the function  $y = \log(x)$  for  $0.001 \leq x \leq 1000$ .
  - c** Use the fact that  $y = \log(10^x) \Leftrightarrow 10^y = x$  and the values in the completed table to draw the graph of  $y = 10^x$  for  $0.001 \leq y \leq 1000$ .
  - d** Comment on the shapes of the graphs of  $y = \log(x)$  and  $y = 10^x$ .
- 17** Use a graphics calculator to draw the graphs of  $y = \log(x)$  and  $y = 10^x$  on the same axes, and comment on the shapes of the graphs.
- 18 a** What are the domain and range of  $y = 10^{3x}$ ?
- b** What are the domain and range of  $y = \log(x)$ ?

## 9.07 General logarithms

Logarithms that have a base other than 10 are called **general logarithms**.



### General logarithms

The logarithm of  $x$  to the base  $a$ , is the power  $m$  such that  $a^m = x$  (where  $a > 0$  and  $a \neq 1$ ).

$$m = \log_a(x) \text{ (where } a > 0, a \neq 1 \text{ and } x > 0\text{)}.$$

So

$$a^m = x \Leftrightarrow \log_a(x) = m$$

The index statement  $a^m = x$  means the same as the logarithm statement  $\log_a(x) = m$ , so the statements are said to be equivalent.

### EXAMPLE 17

**a** Write in index form:

**i**  $\log_3(243) = 5$       **ii**  $\log_4\left(\frac{1}{16}\right) = -2$

**b** Write in logarithmic form:

**i**  $5^3 = 125$       **ii**  $2^{-5} = \frac{1}{32}$

### Solution

**a i** Write the equivalence statement.

Substitute  $a = 3$ ,  $x = 243$  and  $m = 5$ .

State the result.

$$m = \log_a(x) \Leftrightarrow a^m = x$$

$$\log_3(243) = 5 \Leftrightarrow 3^5 = 243$$

The index form is  $3^5 = 243$ .

**ii** Write the equivalence statement.

Substitute  $a = 4$ ,  $x = \frac{1}{16}$  and  $m = -2$ .

State the result.

$$m = \log_a(x) \Leftrightarrow a^m = x$$

$$\log_4\left(\frac{1}{16}\right) = -2 \Leftrightarrow 4^{-2} = \frac{1}{16}$$

The index form is

$$4^{-2} = \frac{1}{16}$$

**b i** Write the equivalence statement.

Substitute  $a = 5$ ,  $x = 125$  and  $m = 3$ .

State the result.

$$a^m = x \Leftrightarrow \log_a(x) = m$$

$$5^3 = 125 \Leftrightarrow \log_5(125) = 3$$

The log form is  $\log_5(125) = 3$ .

**ii** Write the equivalence statement.

Substitute  $a = 2$ ,  $x = \frac{1}{32}$  and  $m = -5$ .

State the result.

$$a^m = x \Leftrightarrow \log_a(x) = m$$

$$2^{-5} = \frac{1}{32} \Leftrightarrow \log_2\left(\frac{1}{32}\right) = -5$$

The log form is  $\log_2\left(\frac{1}{32}\right) = -5$ .

When you are asked to find the logarithm of a number to a base, think ‘What power of the base gives the number?’.



Logarithms

### EXAMPLE 18

Evaluate each logarithm.

**a**  $\log_2(32)$

**b**  $\log_3\left(\frac{1}{27}\right)$

**c**  $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$

**d**  $\log_{15}(1)$

**e**  $\log_{23}(23)$

## Solution

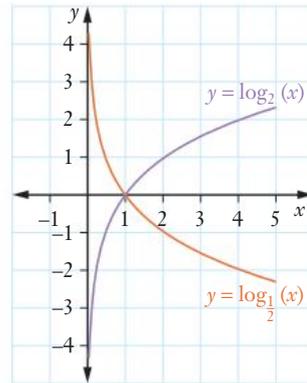
- a** Write the problem.  $\log_2(32)$   
Write the index form.  $2^x = 32$   
What power of 2 is 32?  $2^5 = 32$   
State the result.  $\log_2(32) = 5$
- b** Write the problem.  $\log_3\left(\frac{1}{27}\right)$   
Write the index form.  $3^x = \frac{1}{27}$   
The power of 3 that is  $\frac{1}{27}$  must be negative.  $3^{-3} = \frac{1}{27}$   
State the result.  $\log_3\left(\frac{1}{27}\right) = -3$
- c** Write the problem.  $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$   
Write the index form.  $\left(\frac{1}{2}\right)^x = \frac{1}{16}$   
Express the RHS as a power of 2.  $= \frac{1}{2^4}$   
Use the index laws.  $= \left(\frac{1}{2}\right)^4$   
State the result.  $\log_{\frac{1}{2}}\left(\frac{1}{16}\right) = 4$
- d** Write the problem.  $\log_{15}(1)$   
Write the index form.  $15^x = 1$   
What power of 15 is 1?  $15^0 = 1$   
State the result.  $\log_{15}(1) = 0$
- e** Write the problem.  $\log_{23}(23)$   
Write the index form.  $23^x = 23$   
What power of 23 is 23?  $23^1 = 23$   
State the result.  $\log_{23}(23) = 1$

## INVESTIGATION

### THE GRAPHS OF LOGARITHMIC FUNCTIONS

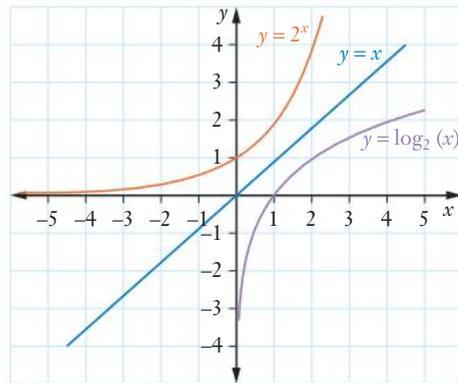
1 The graphs of  $y = \log_2(x)$  and  $y = \log_{\frac{1}{2}}(x)$  are shown on the right.

- Record your observations regarding the shape and general features of each graph.
- Is it possible to transform one of the graphs into the other using a single transformation?



2 The graphs of  $y = \log_2(x)$ ,  $y = 2^x$  and  $y = x$  are shown on the right.

- Compare the graphs of  $y = \log_2(x)$  and  $y = 2^x$ .
- Is it possible to transform one of the graphs into the other using a single transformation?
- How could you describe the relationship between  $y = \log_2(x)$  and  $y = 2^x$ ?



From the properties of indices you know that  $a^1 = a$ ,  $a^0 = 1$  and  $a^{-1} = \frac{1}{a}$ . These facts have equivalent expressions for logarithms.

#### Logarithm properties

For  $a > 0$  and  $a \neq 1$ :

$$\log_a(1) = 0$$

$$\log_a(a) = 1 \quad \log_a\left(\frac{1}{a}\right) = -1$$

$$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$$

$$a^{\log_a(x)} = x$$

If  $\log_a(x) = \log_a(y)$ , then  $x = y$ .

If  $\log_a(x) = \log_b(x)$ , then  $a = b$ .

Note:  $\log_a(0)$  is undefined.

Since logarithms are the inverses of exponents, they obey similar laws, as shown in the following.

Let  $x = a^m$  and  $y = a^n$ .

Then

$$x \times y = a^m \times a^n$$

Simplify.

$$xy = a^{m+n}$$

Apply the logarithm definition.

$$\log_a(xy) = m + n$$

Substitute, using the definition.

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

Other logarithm laws may be derived in a similar way and are summarised below.

## Laws of logarithms

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n \log_a(x)$$



Logarithm laws



Logarithm laws

### EXAMPLE 19

**a** Simplify:

**i**  $\log_3(\sqrt{10})$

**ii**  $\log_4(28) + \log_4(5) - \log_4(35)$

**iii**  $\frac{\log_7(8)}{\log_7(32)}$

**b** Write as a single logarithm:  $2 \log_3(x) + 3 \log_3(x) - 4 \log_3(y)$ .

**c** Expand using the properties and laws of logarithms:  $\log_5\left(\frac{x^4}{\sqrt{yz^5}}\right)$ .

### Solution

**a i** Write  $\sqrt{10}$  in index form.

$$\log_3(\sqrt{10}) = \log_3\left(10^{\frac{1}{2}}\right)$$

Use the logarithm laws.

$$= \frac{1}{2} \log_3(10)$$

**ii** Write the expression.

$$\log_4(28) + \log_4(5) - \log_4(35)$$

Use the logarithm laws.

$$= \log_4\left(\frac{28 \times 5}{35}\right)$$

$$= \log_4(4)$$

Use the logarithm properties.

$$= 1$$

iii Express the terms using powers.

$$\frac{\log_7(8)}{\log_7(32)} = \frac{\log_7(2^3)}{\log_7(2^5)}$$

Use the logarithm laws.

$$= \frac{3 \log_7(2)}{5 \log_7(2)}$$

$$= \frac{3}{5}$$

b Write the expression.

$$2 \log_3(x) + 3 \log_3(x) - 4 \log_3(y)$$

Use the logarithm laws.

$$= \log_3(x^2) + \log_3(x^3) - \log_3(y^4)$$

Use the logarithm laws.

$$= \log_3\left(\frac{x^2 \times x^3}{y^4}\right)$$

Simplify.

$$= \log_3\left(\frac{x^5}{y^4}\right)$$

c Write the expression.

$$\log_5\left(\frac{x^4}{\sqrt{yz^5}}\right)$$

Express all terms as powers.

$$= \log_5\left(\frac{x^4}{y^{\frac{1}{2}}z^{\frac{5}{2}}}\right)$$

Use the logarithm laws.

$$= \log_5(x^4) - \log_5\left(y^{\frac{1}{2}}\right) - \log_5(z^{\frac{5}{2}})$$

Use the logarithm laws.

$$= 4 \log_5(x) - \frac{1}{2} \log_5(y) - 5 \log_5(z)$$

## Exercise 9.07 General logarithms

1 Write each equation in index form.

a  $\log_5(125) = 3$

b  $\log_2(32) = 5$

c  $\log_4(64) = 3$

d  $\log_7(7) = 1$

e  $\log_3(2187) = 7$

f  $\log_9(1) = 0$

g  $\log_5(625) = 4$

h  $\log_{13}(169) = 2$

2 Write in index form.

a  $\log_{16}(4) = \frac{1}{2}$

b  $\log_8\left(\frac{1}{2}\right) = -\frac{1}{3}$

c  $\log_3\left(\frac{1}{9}\right) = -2$

d  $\log_4\left(\frac{1}{16}\right) = -2$

e  $\log_{\frac{1}{2}}(8) = -3$

f  $\log_{\frac{1}{4}}(2) = -\frac{1}{2}$

g  $\log_{32}(4) = \frac{2}{5}$

h  $\log_5\left(\frac{1}{\sqrt[3]{5}}\right) = -\frac{1}{3}$

3 Write in logarithmic form.

a  $8^2 = 64$

b  $3^4 = 81$

c  $2^6 = 64$

d  $12^0 = 1$

e  $15^1 = 15$

f  $6^3 = 216$

Example  
17

4 Write in logarithmic form.

a  $36^{\frac{1}{2}} = 6$

b  $27^{-\frac{1}{3}} = \frac{1}{3}$

c  $4^{-2} = \frac{1}{16}$

e  $\left(\frac{1}{2}\right)^{-4} = 16$

f  $\left(\frac{1}{9}\right)^{-\frac{1}{2}} = 3$

g  $9^{\frac{3}{2}} = 27$

5 Evaluate each logarithm.

a  $\log_2(16)$

b  $\log_3(81)$

c  $\log_5(125)$

d  $\log_6(1296)$

e  $\log_{17}(1)$

f  $\log_{13}(13)$

g  $\log_{10}(1000)$

h  $\log_4(4096)$

6 Write the value of:

a  $\log_3(1)$

b  $\log_5(5)$

c  $\log_4(1)$

d  $\log_7(1)$

e  $\log_{12}(12)$

f  $\log_{10}(1)$

g  $\log_{10}(10)$

h  $\log_9(9)$

7 Write the value of:

a  $\log_8(0)$

b  $\log_4(-4)$

c  $\log_a(0), a > 0$

d  $\log_b(-x), x > 0$

8 Evaluate each logarithm.

a  $\log_2\left(\frac{1}{32}\right)$

b  $\log_3\left(\frac{1}{9}\right)$

c  $\log_5\left(\frac{1}{625}\right)$

d  $\log_4\left(\frac{1}{64}\right)$

e  $\log_{\frac{1}{2}}\left(\frac{1}{8}\right)$

f  $\log_{\frac{1}{2}}(64)$

g  $\log_{\frac{1}{3}}\left(\frac{1}{81}\right)$

h  $\log_{\frac{1}{5}}(125)$

9 Write the value of:

a  $\log_3(27)$

b  $\log_5(625)$

c  $\log_8(2)$

d  $\log_2(16)$

e  $\log_8(4)$

f  $\log_{10}(100)$

g  $\log_{100}(10)$

h  $\log_4(0.25)$

i  $\log_7(1)$

j  $\log_2(\sqrt{8})$

k  $\log_8(\sqrt{2})$

l  $\log_4(0.5)$

10 If  $\log_z(x) = a$  and  $\log_z(y) = b$ , express each logarithm in terms of  $z$ ,  $a$  and  $b$ .

a  $\log_z(x^2)$

b  $\log_z(xy^3)$

c  $\log_z\left(\frac{x^3}{y^2}\right)$

d  $\log_z(x^y)$

e  $\log_z(z^x)$

f  $\log_z\left(\frac{y}{x^3}\right)$

g  $\log_z(y\sqrt{x})$

h  $\log_z(\sqrt[3]{x^2y})$

11 Simplify each logarithm.

a  $\log_7(\sqrt{7})$

b  $\log_5(10) + \log_5(4) - \log_5(5)$

c  $\frac{\log_6(16)}{\log_6(4)}$

d  $\log_2(16) + \log_2(8) - \log_2(4)$

e  $\log_3(27) - \log_3(9) + 3$

f  $\frac{\log_3(27)}{\log_3\left(\frac{1}{3}\right)}$

g  $\log_8\left(\frac{1}{4}\right) + \log_8(2) + 1$

h  $\frac{\log_2(0.25)}{\log_2(0.5)}$

Example  
18

Example  
19

**12** Simplify:

**a**  $\log_2(32)$

**b**  $\log_{10}(0.001)$

**c**  $\log_2\left(\frac{1}{256}\right)$

**d**  $\log_2(32) - \log_3(27)$

**e**  $\log_{10}(200) - \log_{10}(100)$

**f**  $\log_2(16) + \log_2(32) + \log_2(64)$

**g**  $8 \log_2(2) - 2 \log_2(8)$

**h**  $\log_3\left(\frac{1}{9}\right) + \log_3(81) - \log_3(27)$

**13** Write each expression as a single logarithm.

**a**  $3 \log_2(x) + \log_2(x) - 2 \log_2(x)$

**b**  $3 \log_5(x) + 4 \log_5(x) - 6 \log_5(x)$

**c**  $2 \log_4(x^2) + 4 \log_4(x) - 3 \log_4(x^3)$

**d**  $4 \log_{10}(x) - 5 \log_{10}(x) + \log_{10}(y)$

**e**  $\log_7(x+1)^2 + 3 \log_7(x+1)$

**f**  $4 \log_3(x) + 2 \log_3(y) - 3 \log_3(y)$

**g**  $4 \log_8(x) - \log_8(x) + \log_8(y^2)$

**h**  $2 \log_4(x-2) - \log_4[(x-2)^3]$

**14** Expand each logarithm using the properties and laws of logarithms.

**a**  $\log_2\left(\frac{xy}{z}\right)$

**b**  $\log_4\left(\frac{4x}{y}\right)$

**c**  $\log_7\left(\frac{6}{xy}\right)$

**d**  $\log_5\left(\frac{5x}{y^2z}\right)$

**e**  $\log_{10}\left(\frac{x^3\sqrt{y}}{z^5}\right)$

**f**  $\log_9\left(\frac{x^4\sqrt[3]{y}}{\sqrt{z}}\right)$

**g**  $\log_5\left(\sqrt{\frac{x^3y}{z^3}}\right)$

**h**  $\log_8\left(\sqrt[3]{\frac{x^4y^2}{z^2}}\right)$

**15** If  $\log_7(5) = 0.827$ , then calculate each logarithm correct to 3 decimal places.

**a**  $\log_7(\sqrt[3]{5})$

**b**  $\log_7(25)$

**c**  $\log_7(245)$

**Problem solving**

**16** Given that  $a^b = 36$ , find  $\log_6(a)$  in terms of  $b$ .

**17** If  $\log_a(5) - 2 \log_a(x) = 0$ , find  $x$ .

**18** Prove  $\log_p\left(\frac{a}{b}\right) = \log_p(a) - \log_p(b)$ .

**19** Prove  $\log_m(a^x) = x \log_m(a)$

## 9.08 The change of base theorem

You can see that  $\log_4(8) = \log_4(\sqrt{64}) = \log_4(\sqrt{4^3}) = \log_4\left(4^{\frac{3}{2}}\right) = \frac{3}{2}$ .

But it is also true that  $\log_4(8) = \frac{\log_2(8)}{\log_2(4)}$  because  $\log_2(8) = 3$  and  $\log_2(4) = 2$ .

This is an example of a general rule, where the logarithm to a particular base can be written as the ratio of logarithms to a different base.

### Change of base theorem

For  $a, b, x > 0$  and  $a, b \neq 0$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

You prove the theorem by taking logarithms of the equivalent index expression.

Rewrite in index form.

$$\text{Let } y = \log_a(x)$$

$$x = a^y$$

Take logarithms to base  $b$  of both sides.

$$\log_b(x) = \log_b(a^y)$$

Use the laws of logarithms.

$$\log_b(x) = y \log_b(a)$$

Rearrange to isolate  $y$ .

$$y = \frac{\log_b(x)}{\log_b(a)}$$

But  $y = \log_a(x)$ .

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

QED

### EXAMPLE 20

Find the value of  $\log_8(29)$ , correct to four decimal places.

#### Solution

Use the change of base theorem.

$$\log_8(29) = \frac{\log_b(29)}{\log_b(8)}$$

Use common logarithms (base 10) because we can evaluate them on a calculator.

$$= \frac{\log_{10}(29)}{\log_{10}(8)}$$

$$= 1.619326\dots$$

$$\approx 1.6193$$

You can use the change of base theorem to solve equations involving indices.

### EXAMPLE 21

Solve the equation  $7^x = 12$ , correct to 4 significant figures.

#### Solution

Write the equation.

$$7^x = 12$$

Rewrite in logarithmic form.

$$x = \log_7(12)$$

Use the change of base theorem.

$$= \frac{\log(12)}{\log(7)}$$

$$= 1.2769\dots$$

$$\approx 1.277$$

## EXAMPLE 22

Solve  $25^{2x-1} = 64$ , correct to 4 significant figures.

### Solution

Write the equation.

$$25^{2x-1} = 64$$

Rewrite in logarithmic form

$$2x - 1 = \log_{25}(64)$$

Use the change of base theorem.

$$= \frac{\log(64)}{\log(25)}$$
$$= 1.2920\dots$$

Solve for  $x$ .

$$2x = 2.2920\dots$$

$$x = 1.1460\dots$$

$$x \approx 1.1460$$

## Exercise 9.08 The change of base theorem

1 Use the change of base theorem to express each logarithm as a ratio of common logarithms.

**a**  $\log_5(12)$

**b**  $\log_4(20)$

**c**  $\log_2(50)$

**d**  $\log_9(300)$

**e**  $\log_{11}(0.5)$

**f**  $\log_7(15)$

2 Use the change of base theorem to evaluate each logarithm correct to 4 significant figures.

**a**  $\log_2(10)$

**b**  $\log_5(15)$

**c**  $\log_3(12)$

**d**  $\log_8(24)$

**e**  $\log_{17}(4.8)$

**f**  $\log_{21}(17)$

3 Solve each equation correct to 4 significant figures.

**a**  $3^x = 50$

**b**  $5^x = 10$

**c**  $8^x = 100$

**d**  $7^x = 3$

**e**  $0.5^x = 10$

**f**  $1.8^x = 20$

4 Find  $x$  correct to 4 decimal places.

**a**  $10^x = 20$

**b**  $20^x = 8$

**c**  $15^x = 10$

**d**  $0.1^x = 15$

**e**  $0.8^x = 21$

**f**  $0.05^x = 17$

5 Solve each equation correct to 4 significant figures.

**a**  $5^{x+1} = 40$

**b**  $9^{2x+3} = 72$

**c**  $7^{3x-1} = 59$

**d**  $8^{2-x} = 17$

**e**  $11^{3-2x} = 41$

**f**  $20^{5x-4} = 36$

Example  
20

Example  
21

Example  
22

## Problem solving

6 Show that:

a  $\log_{\frac{1}{3}}(8) = -\frac{\log(8)}{\log(3)}$

c  $\log_{25}(5) = \frac{1}{2}$

e  $\frac{1}{\log_2(5)} + \frac{1}{\log_2(5)} = \log_5(4)$

b  $\log_{16}(36) = \frac{1}{2} + \log_2(3)$

d  $\log_{81}(27) = \frac{3}{4}$

## 9.09 Solving equations with logarithms

You can use logarithms to solve equations with indices. You will also have to use basic algebra.

### EXAMPLE 23

Solve each equation, correct to 4 significant figures where necessary.

a  $\log_6(x-1) = 2$

b  $9^x = 3^{x+1} + 4$

### Solution

a Write the equation.

Rewrite in index form, and solve.

$$\log_6(x-1) = 2$$

$$6^2 = x-1$$

$$36 = x-1$$

$$x = 37$$

b Write the equation.

Write in base 3.

Apply the index laws.

Let  $a = 3^x$ .

Write as a quadratic.

Factorise.

Solve.

Substitute  $3^x$  for  $a$ .

But  $3^x > 0$ .

Write in logarithmic form and evaluate.

$$9^x = 3^{x+1} + 4$$

$$3^{2x} = 3^{x+1} + 4$$

$$(3^x)^2 = 3^x \times 3^1 + 4$$

$$a^2 = 3a + 4$$

$$a^2 - 3a - 4 = 0$$

$$(a-4)(a+1) = 0$$

$$a = -1 \text{ or } 4$$

$$3^x = -1 \text{ or } 3^x = 4$$

$$3^x = 4$$

$$x = \log_3(4)$$

$$= 1.261\ 859\dots$$

$$\approx 1.262$$



Logarithms - Solving equations 1



Logarithms - Solving equations 2



Exponential functions assignment



Exponential functions extension

You might have to use logarithm laws to solve an equation.

### EXAMPLE 24

Solve  $12^x = 4^{x-3}$ , correct to 4 significant figures.

#### Solution

Write the equation.

$$12^x = 4^{x-3}$$

Take logarithms of both sides.

$$\log(12^x) = \log(4^{x-3})$$

Use the laws of logarithms.

$$x \log(12) = (x-3) \log(4)$$

Expand.

$$x \log(12) = x \log(4) - 3 \log(4)$$

Group like terms.

$$x \log(12) - x \log(4) = -3 \log(4)$$

Factorise and use the laws of logarithms.

$$x [\log(12) - \log(4)] = -3 \log(4)$$

$$x \log\left(\frac{12}{4}\right) = -3 \log(4)$$

$$x \log(3) = -3 \log(4)$$

Isolate  $x$ .

$$x = \frac{-3 \log(4)}{\log(3)}$$

Evaluate.

$$x = -3.7855\dots$$
$$\approx -3.786$$

To solve equations with logarithms of variables you need a single logarithm.

### EXAMPLE 25

Solve each equation, correct to 4 significant figures where necessary.

**a**  $\log_3(7x+3) = \log_3(5x+9)$

**b**  $\log_4(x) + \log_4(x-12) = 3$

**c**  $[\log(x)]^2 + \log(x) = 4$

## Solution

**a** State the equation.

$$\log_3 (7x + 3) = \log_3 (5x + 9)$$

Group the logarithmic terms.

$$\log_3 (7x + 3) - \log_3 (5x + 9) = 0$$

Write as a single logarithm.

$$\log_3 \left( \frac{7x+3}{5x+9} \right) = 0$$

Rewrite in index form.

$$\frac{7x+3}{5x+9} = 3^0 = 1$$

Simplify.

$$7x + 3 = 1(5x + 9)$$

Expand.

$$7x + 3 = 5x + 9$$

Simplify.

$$2x = 6$$

Solve.

$$x = 3$$

**b** State the equation.

$$\log_4 (x) + \log_4 (x - 12) = 3$$

Write as a single logarithm.

$$\log_4 [x(x - 12)] = 3$$

Rewrite in index form.

$$x(x - 12) = 4^3$$

Expand.

$$x^2 - 12x - 64 = 0$$

Factorise.

$$(x + 4)(x - 16) = 0$$

Solve.

$$x = -4 \text{ or } 16$$

But  $x > 0$ .

$$x = 16$$

**c** Write the equation.

$$[\log (x)]^2 + \log (x) = 4$$

Rearrange.

$$[\log (x)]^2 + \log (x) - 4 = 0$$

Let  $z = \log (x)$ .

$$z^2 + z - 4 = 0$$

The quadratic doesn't factorise, so use the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute  $a = 1$ ,  $b = 1$  and  $c = -4$

$$= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-4)}}{2 \times 1}$$

Evaluate using a calculator.

$$z = -2.5615... \quad \text{or} \quad z = 1.5615...$$

Substitute  $z = \log (x)$ .

$$\log (x) = -2.5615... \quad \text{or} \quad \log (x) = 1.5615...$$

Rewrite in index form.

$$x = 10^{-2.5615...} \quad \text{or} \quad x = 10^{1.5615...}$$

Evaluate using a calculator.

$$x = 0.002\,744... \quad \text{or} \quad x = 36.437...$$

Check the solutions.

Both solutions work. ←

Round off and write the result.

$$x \approx 0.002\,744 \text{ or } x \approx 36.44.$$

It is a good idea to check solutions to equations involving logarithms to ensure that the results are defined.

## Exercise 9.09 Solving equations with logarithms

1 Solve each equation, correct to 3 decimal places where necessary.

<b>a</b> $2^x = 7$	<b>b</b> $5^x = 30$	<b>c</b> $3^x = 7$
<b>d</b> $3^{2x} = 5$	<b>e</b> $7^{3x} = 50$	<b>f</b> $4^x = 15$

2 Solve each equation, correct to 3 decimal places where necessary.

<b>a</b> $4^{x+1} = 3$	<b>b</b> $3^{x-3} = 5$	<b>c</b> $5^{3x-1} = 15$
<b>d</b> $7^{2x+1} = 11$	<b>e</b> $12^{3x-4} = 32$	<b>f</b> $64^{6x-7} = 20$

3 Solve each equation.

<b>a</b> $\log(x+1) = 2$	<b>b</b> $\log_2(5x+7) = 5$	<b>c</b> $\log_3(9x+2) = 4$
<b>d</b> $2 \log_4(x) = 5$	<b>e</b> $\log(3x+1) = 2$	<b>f</b> $\log_3(2x+1) = 2$

4 Solve each equation, correct to 4 significant figures where necessary.

<b>a</b> $4^{x+2} = 64$	<b>b</b> $3^{x-1} = 81$	<b>c</b> $4^x = 3^{x+1}$
<b>d</b> $6^x = 2^{x-3}$	<b>e</b> $9^x = 3^{x+3} - 5$	<b>f</b> $16^x = 4^{x+2} + 1$

5 Solve each equation correct to 4 significant figures.

<b>a</b> $4^x = 7^{x-1}$	<b>b</b> $8^x = 5^{x+1}$	<b>c</b> $7^x = 25^{x-2}$
<b>d</b> $15^{x+1} = 6^{2x-3}$	<b>e</b> $24^{x+1} = 9^{2x-5}$	<b>f</b> $13^{2x} = 7^{5x-1}$

6 Solve each equation, correct to 4 significant figures where necessary.

<b>a</b> $\log_{19}(-5x-6) = \log_{19}(2-3x)$	<b>b</b> $\log_7(x-2) + \log_7(x+3) = \log_7(14)$
<b>c</b> $\log_6(x+4) + \log_6(x-2) = \log_6(4x)$	<b>d</b> $\log_8(x) + \log_8(x+6) = \log_8(5x+12)$
<b>e</b> $\log(x-2) - \log(2x-3) = \log(2)$	<b>f</b> $\log_7(2x-1) + \log_7(3) = \log_7(5x+3)$

7 Solve each equation, correct to 4 significant figures where necessary.

<b>a</b> $\log_3(x-4) + \log_3(x+4) = 2$	<b>b</b> $\log_2(x) - \log_2(x+3) = 1$
<b>c</b> $\log_7(x) - \log_7(x-1) = 1$	<b>d</b> $\log_6(5) - \log_6(x-7) = 1$
<b>e</b> $\log_3(8) + \log_3(-5x) = 3$	<b>f</b> $\log_7(x-3) - \log_7(x) = 3$

8 Solve each equation, correct to 4 significant figures where necessary.

<b>a</b> $[\log(x)]^2 + \log(x) = 2$	<b>b</b> $[\log(x)]^2 + \log(x) - 5 = 0$
<b>c</b> $[\log_3(x)]^2 - \log_3(x) = 3$	<b>d</b> $[\log_8(x)]^2 + \log_8(x) = 2$
<b>e</b> $[\log_5(x)]^2 + 3 \log_5(x) - 4 = 0$	<b>f</b> $[\log_4(x)]^2 - 5 \log_4(x) = 1$

9 Solve each equation, correct to 4 significant figures where necessary.

<b>a</b> $7^{2x} - 3 \times 7^x + 1 = 0$	<b>b</b> $10^{x+3} = 6^{2x}$
<b>c</b> $6^x = 2^{x-3}$	<b>d</b> $5^{x+2} = 4$
<b>e</b> $4^{x-3} = \frac{1}{16}$	<b>f</b> $8^{x+4} = 4^{3x-1}$

Example  
23

Example  
24

Example  
25

**10** Solve each equation, correct to 4 significant figures where necessary.

**a**  $\log_7 (25 - x) = 3$

**b**  $\log_5 (4x + 11) = 2$

**c**  $\log_3 (x^2 - 6x) = 3$

**d**  $\log (4x - 5) = \log (2x - 1)$

**e**  $\log_{12} (x^2 + 35) = \log_{12} (-12x - 1)$

**f**  $\log_9 (-11x + 2) = \log_9 (x^2 + 30)$

**11** Solve each equation, correct to 4 significant figures where necessary.

**a**  $\log_6 (x) + \log_6 (x - 9) = 2$

**b**  $\log (x + 6) - \log (x) = \log (x + 2)$

**c**  $4 + 3 \log (2x) = 16$

**d**  $\log (x + 2) + \log (x - 1) = 1$

**e**  $\log (x^2 - 1) - \log (x + 1) = 3$

**f**  $\log_2 (x + 1) - \log_2 (x - 4) = 3$

**g**  $\log_4 (8) - \log_4 (x + 6) = 1$

**h**  $\log_6 (x + 1) - \log_6 (x) = \log_6 (29)$

### Problem solving

**12** If \$ $P$  is deposited in an account with an annual interest rate of  $i\%$ , the amount ( $A$ ) after  $n$  years in the account is given by:

$$A = P \left( 1 + \frac{i}{k} \right)^{kn}$$

where  $k$  is the number of times the interest is calculated per year.

**a** How long will it take for the amount to double if it is invested at 5% compounded monthly?

**b** If the money is invested at 8% compounded quarterly, how long will it take for the amount to be 5 times as big?

**13** In 1935 Charles Richter defined the magnitude of an earthquake to be

$$M = \log \left( \frac{I}{S} \right)$$

where  $I$  is the intensity of the earthquake and  $S$  is the intensity of a 'standard earthquake'.

In 1906 an earthquake in San Francisco registered 9.3 on the Richter scale. Years later, an earthquake in South America was recorded that was five times stronger.

What was the magnitude of the earthquake in South America?

**14** An earthquake in Chile in 2012 measured 7.2 on the Richter scale. How many times stronger was the San Francisco earthquake of 1906?



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# 9. CHAPTER SUMMARY

## Exponential functions and logarithms

### Exponential functions

- A general **exponential function** has the form  $y = b \cdot a^x$ , where  $a > 0$ .
  - For  $a > 1$ , it is called a **growth function**.
  - For  $0 < a < 1$ , it is called a **decay function**.
  - $a$  is called the **growth** or **decay factor**. The **initial value** is  $y(0) = b$ .
- The graph of a function of the form  $y = b \cdot a^x$  is a vertical dilation of the graph of  $y = a^x$ .  
If  $b$  is negative, it is also a reflection in the  $x$ -axis. The graph passes through  $\left(-1, \frac{b}{a}\right)$ ,  $(0, b)$  and  $(1, ab)$ .
- The graphs of  $a^x$  and  $\left(\frac{1}{a}\right)^x = a^{-x}$  are reflections of each other in the  $y$ -axis.
- The graph of  $y = a^{kx}$  is a horizontal dilation of  $y = a^x$ , passing through  $(0, 1)$  and  $\left(\frac{1}{k}, a\right)$ .
  - For  $k > 1$ , it is compressed by the factor  $k$ .
  - For  $0 < k < 1$ , it is stretched by the factor  $\frac{1}{k}$ .
  - For  $k < 0$ , it is also reflected in the  $y$ -axis.
- The variable in an **exponential equation** (or **indicial equation**) is part of the exponent (power, index).
- A quantity  $A(t)$  such that  $\frac{A(t+1)}{A(t)} = k$ , is given by the exponential equation  $A(t) = A_0 k^t$ , where  $k$  is a constant and  $A_0$  is the initial value of  $A$ .

### Logarithms

- The **logarithm** of  $x$  to the **base**  $a$ , is the **power**  $m$  such that  $a^m = x$ , where  $a > 0$  and  $a \neq 1$ .
- $a^m = x \Leftrightarrow \log_a(x) = m$  (where  $a > 0$ ,  $a \neq 1$  and  $x > 0$ ).
- A **common logarithm** is also called the **logarithm to the base 10** (or **log base 10**).  
 $m = \log_{10}(x) \Leftrightarrow 10^m = x$
- A logarithm to a base other than 10 is called a **general logarithm**.

- The following **properties of logarithms** hold for  $a > 0$  and  $a \neq 1$ :

$$\log_a(1) = 0$$

$$\log_a(a) = 1$$

$$\log_a\left(\frac{1}{a}\right) = -1$$

$$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$$

$$a^{\log_a(x)} = x$$

If  $\log_a(x) = \log_a(y)$ , then  $x = y$ .

If  $\log_a(x) = \log_b(x)$ , then  $a = b$ .

$\log_a(0)$  is undefined.

- Logarithms obey the following laws for  $a, x, y > 0$  and  $a \neq 1$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$\log_a(x^n) = n \log_a(x)$$

- The **change of base theorem** for logarithms states that for  $a, b, x > 0$  and  $a, b \neq 0$ ,

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

# 9. CHAPTER REVIEW

## Exponential functions and logarithms

Example  
**1**

**1** Find the values of  $f(x) = 6 \times 1.2^x$  for  $x = -1, 1, 2$  and  $3$  and comment on the nature of the function.

Example  
**2**

**2** The temperature of an electric heater increases by 10% a minute immediately after it is switched on. When first switched on, the temperature is  $12^\circ$ . What is its temperature after 5 minutes?

Example  
**3**

**3** The value of a car depreciates by about 17% per year. What is a car bought for \$23 790 worth after 4 years?

Examples  
**4-6**

**4** Sketch the graph of:

**a**  $y = 3 \times 4^x$

**b**  $f(x) = 0.4^{2x}$

Examples  
**7-9**

**5** Sketch the graph of  $y = 3^{x-2} + 1$

Example  
**10**

**6** Solve each equation without using technology.

**a**  $2^{3x+2} = 256$

**b**  $4^{x+1} + 32 = 24 \times 2^x$

Example  
**11**

**7** Solve  $3^{x-2} - 45 = 0$

Example  
**12**

**8** The cost of fresh oranges from 1990 until 2015 was about  $\$1.5 \times 1.046^y$  /kg, where  $y$  is the number of years after 1990.

**a** What was the cost in 2015?

**b** Using the same model, what is the price likely to be in 2025?

Example  
**13**

**9** The height to which a ball rebounds is found to be the following for successive bounces.

Bounce	1	2	3	4	5
Height (m)	4.02	2.62	2.01	1.27	0.88

**a** Find an exponential model for the bounce height.

**b** What is the bounce height likely to be after 10 bounces with the same ball?

Example  
**14**

**10** Evaluate each logarithm without using a calculator.

**a**  $\log(100)$

**b**  $\log(0.01)$

**c**  $\log(\sqrt{10})$

**d**  $\log\left(10^{\frac{4}{3}}\right)$

**e**  $\log(7\sqrt{10})$

**f**  $\log(\sqrt{10^3})$

Example  
**15**

**11** Use a calculator to find the approximate value of  $\log(35)$  and show that it is the solution of the exponential equation  $10^x = 35$ .

**12 a** Find the values of  $\log(6.5)$  and  $\log(6500)$ .

**b** Write a relationship between  $\log(6.5)$ ,  $\log(6500)$  and  $\log(1000)$ .

**13** Show that  $\log\left(\frac{p}{q}\right) = \log(p) - \log(q)$ .

**14** Write in index form.

**a**  $\log_7(343) = 3$

**b**  $\log_2(128) = 7$

**c**  $\log_2\left(\frac{1}{8}\right) = -3$

**d**  $\log_{\frac{1}{2}}(16) = -4$

**15** Write in logarithmic form.

**a**  $5^3 = 125$

**b**  $64^{\frac{1}{3}} = 4$

**c**  $\left(\frac{1}{3}\right)^{-5} = 243$

**d**  $64^{-\frac{1}{6}} = \frac{1}{2}$

**16** Evaluate each logarithm.

**a**  $\log_2(128)$

**b**  $\log_{12}(1)$

**c**  $\log_6(216)$

**d**  $\log_{17}(0)$

**e**  $\log_2\left(\frac{1}{64}\right)$

**f**  $\log_5\left(\frac{1}{125}\right)$

**g**  $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$

**h**  $\log_{\frac{1}{5}}(25)$

**17** If  $\log_m(p) = a$  and  $\log_m(q) = b$ , express each expression in terms of  $m$ ,  $a$  and  $b$ .

**a**  $\log_m(pq^4)$

**b**  $\log_m\left(\frac{p^2}{q^3}\right)$

**c**  $\log_m(p^3\sqrt{q})$

**d**  $\log_m(p^d)$

**18** Simplify each expression.

**a**  $\log_{11}(\sqrt{11})$

**b**  $\log_8(20) + \log_8(15) - \log_8(25)$

**c**  $\frac{\log_7(36)}{\log_7(6)}$

**d**  $\log_4\left(\frac{1}{16}\right) + \log_4(64) + 2$

**e**  $\log_3(81) - \log_3\left(\frac{1}{9}\right) + 1$

**f**  $4 \log_3(9) - 3 \log_3(27)$

**19** Write each expression as a single logarithm.

**a**  $2 \log_3(x) + \log_3(x^3) - 4 \log_3(x)$

**b**  $3 \log_7(x) - \log_7(xy) + \log_7(y^3)$

**20** Expand using the properties and laws of logarithms.

**a**  $\log_6\left(\frac{6x^2}{yz^3}\right)$

**b**  $\log_8\left(\frac{x\sqrt{y^3}}{z^2}\right)$

**21** Express  $\log_{12}(0.4)$  in terms of common logarithms.

**22** Use the change of base theorem to evaluate each logarithm correct to 4 significant figures.

**a**  $\log_3(15)$

**b**  $\log_{14}(27)$

Example  
**16**

Example  
**17**

Example  
**17**

Example  
**18**

Example  
**18**

Example  
**19**

Example  
**19**

Example  
**19**

Example  
**20**

Example  
**20**

Example  
21

**23** Solve each equation, correct to 4 decimal places.

**a**  $7^x = 30$

**b**  $25^x = 12$

**c**  $0.7^x = 32$

Example  
22

**24** Solve each equation, correct to 4 significant figures.

**a**  $8^{x+3} = 50$

**b**  $7^{2-3x} = 59$

**c**  $5^{2x-1} = 79$

Example  
23

**25** Solve each equation.

**a**  $\log_2(x+3) = 5$

**b**  $\log_3(2x+3) = 3$

**c**  $\log_4(5x-1) = 3$

**d**  $9^x = 3^{x-3}$

**e**  $4^x = 2^{x+3} - 2$

**f**  $36^x = 6^{x+2} + 2$

Example  
24

**26** Solve each equation, correct to 4 significant figures if necessary.

**a**  $2^x = 5^{x-2}$

**b**  $8^{x-1} = 4^{x+1}$

**c**  $9^x = 37^{x-3}$

Example  
25

**27** Solve each equation, correct to 4 significant figures if necessary.

**a**  $\log_9(x+6) - \log_9(x+1) = \log_9(2)$

**b**  $\log_3(x+4) - \log_3(x-3) = \log_3(10)$

**c**  $\log_7(x+1) - \log_7(x-4) = \log_7(40)$

**d**  $\log_5(x+2) - \log_5(x-5) = \log_5(45)$

**e**  $\log_2(x+5) - \log_2(2x-3) = 2$

**f**  $\log_6(x+5) - \log_6(x-2) = 3$

**g**  $\log_3(x+8) + \log_3(2x-1) = 1$

**h**  $\log_6(2x+1) - \log_6(x-6) = 2$

**i**  $[\log(x)]^2 + \log(x) = 7$

**j**  $[\log(x)]^2 + 2 \log(x) - 3 = 0$

**k**  $5^x = 3^{x-2}$

**l**  $7^{x+1} = 20$

### Problem solving

**28** Average wages have risen about 6% a year from about \$280/week in 1980.

**a** What was the average weekly wage in 2005?

**b** Predict the average weekly wages in 2025.

**c** How useful is this prediction?

**29** How long would it take an investment at a compound interest rate of 4% to double?

**30** Sketch the graph of  $f(x) = -0.6 \times 2^{-x+1} + 3$ .

**31** The numbers of instructions per second possible on computers has grown enormously, from about 500 000 in 1975 to about 300 000 000 000 in 2015.

**a** Construct an exponential model of the number of instructions.

**b** Find the likely number of instructions per second in 2020.

**c** Evaluate the model.

**32** The numbers of people who died in house fires is as follows.

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Number	52	46	47	56	39	44	58	40	38	41	30

- a** Try an exponential model of the numbers of deaths from house fires.
- b** Use the model to predict the number of deaths in house fires in 2025 and 2055.
- c** Verify and evaluate the model.

**33** Prove  $\log_m(ab) = \log_m(a) + \log_m(b)$ .

**34** Show that  $\log_{64}(32) = \frac{5}{6}$ .

**35** If  $\log_a(2) = 0.36$ ,  $\log_a(3) = 0.56$  and  $\log_a(5) = 0.83$ , use the properties of logarithms to evaluate:

**a**  $\log_a\left(\frac{5}{12}\right)$

**b**  $\log_a\left(\frac{1}{\sqrt{30}}\right)$

**36** The decibel scale is used to compare relative sound intensities using

$$S = 10 \log\left(\frac{I}{I_0}\right)$$

where  $S$  is the relative sound intensity measured in decibels (dB) and  $I_0 = 10^{-12}$  watts/m<sup>2</sup> is the smallest audible sound that has an intensity.

- a** What is the decibel level of a piece of heavy machinery that has a sound intensity of  $10^{-4}$  watts/m<sup>2</sup>?
- b** Calculate the sound intensity of a sound with decibel level 95.

**37** The intensity of one sound is 100 times larger than the intensity of another. What is the difference on the decibel level of the two sounds?

**38** The amount ( $A$ ) that a principal of  $\$P$  becomes when invested at an annual interest rate of  $i\%$ , after  $n$  years in the account is given by:

$$A = P\left(1 + \frac{i}{k}\right)^{kn}$$

where  $k$  is the number of times the interest is calculated per year.

At what rate of interest would an investment double in 7 years if interest is calculated daily?



Practice quiz

## Practice examination 3



**Time: 90 minutes**  
**Perusal time: 5 minutes**  
**Marks: 50**

### Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

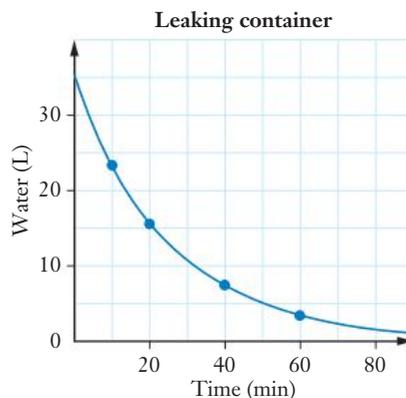


Year 11  
formulas

### Question 1 (3 marks)

The amount of water in a leaking container is shown in the graph.

- Calculate the average rate of loss from 20 min to 40 min.
- Describe how the rate changes.



### Question 2 (2 marks)

Examine the behaviour of  $\frac{3x^2 - 11x - 4}{x - 4}$  as  $x \rightarrow 4$  and determine if a limit exists.

**Question 3** (3 marks)

Find the derivative of  $7x + 5$  at  $x = -2$  from first principles.

**Question 4** (3 marks)

Find the derivative of  $f(x) = 3x^2 + 7x - 5$  from first principles.

**Question 5** (2 marks) (Technology-active)

The size of an oil slick is growing by 3% a day. When first measured, it covered an area of  $300 \text{ m}^2$ .

- a Write a function for the size  $a(t)$  of the oil slick (in square metres) after  $t$  days.
- b Find, correct to the nearest square metre, the size after 10 days.

**Question 6** (4 marks)

Sketch each graph, labelling important features.

- a  $f(x) = 3^x$
- b  $f(x) = 0.5^{x-3} + 1$ .

**Question 7** (2 marks)

- a Find the value of  $\log_4(64)$ .
- b Simplify  $\log_5(27) \times \log_3(5)$ .

**Question 8** (3 marks)

Solve each equation without using technology.

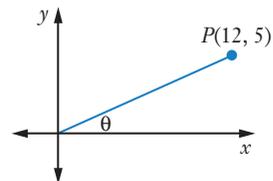
- a  $4^{2x+3} = 8^{x-2}$
- b  $\log_3(x) + \log_3(2x+3) = 2$

**Question 9** (1 mark) (Technology-active)

Calculate the area of a sector with central angle  $\frac{5\pi}{6}$  and radius 10 cm.

**Question 10** (2 marks)

For this diagram, find the exact values of  $\sin(\theta)$  and  $\cos(\theta)$ .

**Question 11** (3 marks)

State the amplitude, period and range of the function  $y = 3 \sin(2x) - 4$ .

**Question 12** (2 marks)

Solve the equation  $2 \cos(\alpha) = \sqrt{3}$ , where  $0 \leq \alpha \leq 2\pi$ , without using technology.

**Question 13** (3 marks)

Find the values of  $x$  for which the gradient of  $f(x) = 5x^2 - 7x - 4$  is greater than the gradient of  $g(x) = 2x^2 + 4x - 3$ .

**Question 14** (2 marks)

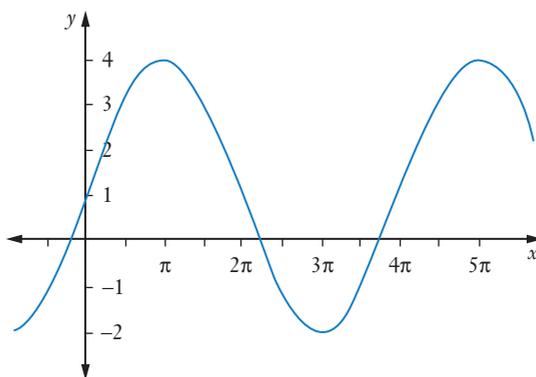
Solve  $25^x + 5^x = 6$  without using technology.

**Question 15** (2 marks) (Technology-active)

Solve  $12^{x-2} = 5^{2x+1}$ , correct to 3 decimal places.

**Question 16** (3 marks)

Determine the equation for this periodic function.



**Question 17** (3 marks)

The marginal cost of production is the cost of producing the last item. It is the rate of change of the cost of production. The cost of producing  $n$  widgets is

$$C(n) = 0.0005n^2 + 0.35n + 11, \quad \text{where } C(n) \text{ is in dollars.}$$

- a Find and use a rule for the cost of production of the 1001th widget.
- b Validate and evaluate your rule.

**Question 18** (4 marks) (Technology-active)

The population of sugar gliders released on an island near Tasmania is shown in this table.

Years after release	0	2	4	6	8
Population	8	12	19	30	47

- a Find and evaluate an exponential model for the population.
- b Predict the population in 20 years and comment on your prediction.

**Question 19** (3 marks) (Technology-active)

The water level reaches a high of 3.6 m in a tidal creek near Gladstone at midnight and has a low of 0.4 m at 6 a.m.

- a Model the water height as a sinusoidal function.
- b Use the model to predict the height at 8 a.m. six days later and comment on the validity of your prediction.

**END OF EXAMINATION**

# 10.

## PROPERTIES OF DERIVATIVES

Sir Isaac Newton (1643–1727) and Gottfried Leibniz (1646–1716) independently invented calculus. Each accused the other of stealing their work, but they used different notation anyway.

Calculus is a system that provides understanding about how situations alter when variables change. It is vital in modern physics, engineering and other disciplines.

- 10.01 The gradient function
- 10.02 The derivative of  $x^n$
- 10.03 The derivative of a linear combination
- 10.04 The derivative of a polynomial
- 10.05 Tangents
- 10.06 Applications of the derivative
- Chapter summary
- Chapter review



## SYLLABUS SUBJECT MATTER

### Properties and computation of derivatives

- examine examples of variable rates of change of non-linear functions
- establish the formula  $\frac{d}{dx}(x^n) = nx^{n-1}$  for positive integers
- understand the concept of the derivative as a function
- recognise and use properties of the derivative  $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
- calculate derivatives of power and polynomial functions

### Applications of derivatives

- determine instantaneous rates of change
- determine the gradient of a tangent and the equation of the tangent

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Prior learning

# TERMINOLOGY

average rate of change  
 derivative  
 increasing  
 linear combination  
 marginal revenue  
 stationary

binomial expansion  
 differentiate  
 instantaneous rate of change  
 marginal cost  
 polynomial  
 tangent

decreasing  
 gradient function  
 linear product  
 marginal profit  
 power rule



Gradient functions

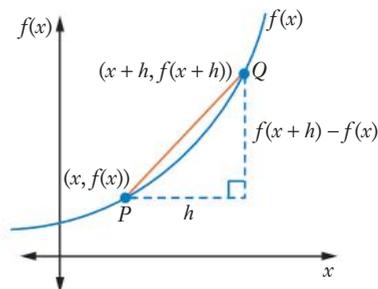
## 10.01 The gradient function

The diagram on the right shows a function,  $f(x)$ , that passes through  $P$  and  $Q$  whose coordinates are

$P(x, f(x))$  and  $Q(x+h, f(x+h))$ .

As  $Q$  gets closer to  $P$ ,  $h$  gets closer and closer to 0 and the gradient of  $PQ$  gets closer to the gradient of the curve at  $P$ .

From coordinate geometry you know the gradient of  $PQ$  is:



$$m_{PQ} = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

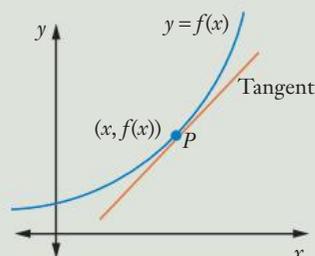
Using limit notation, you can write:

$$\lim_{Q \rightarrow P} m_{PQ} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

But  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ , so the derivative is the gradient of the curve.

### Derivative and gradient

The gradient of a function  $f(x)$  at any point  $P(x, f(x))$  is  $f'(x)$ .



You can use the gradient of a curve to see how a function changes.

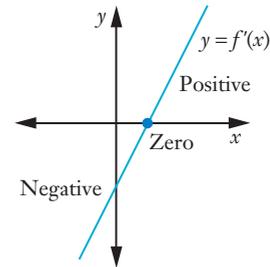
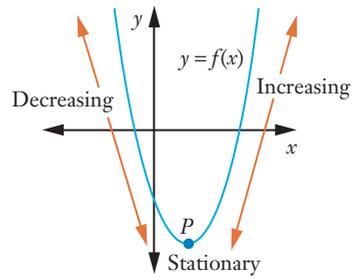
The gradient of the quadratic function on the right is negative to the left of  $P$ , zero at  $P$  and positive to the right of  $P$ .

Where the gradient of a function is zero, the function is called **stationary** and  $P$  is called a **stationary point**.

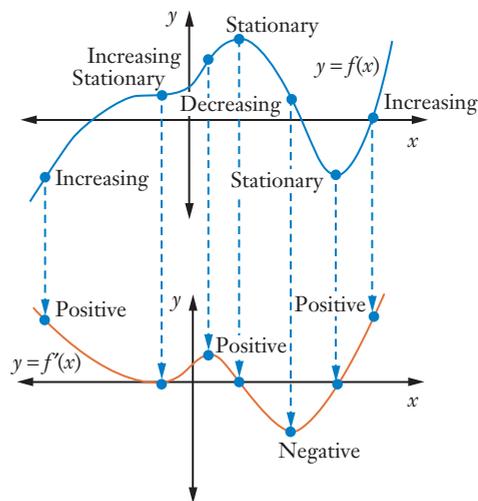
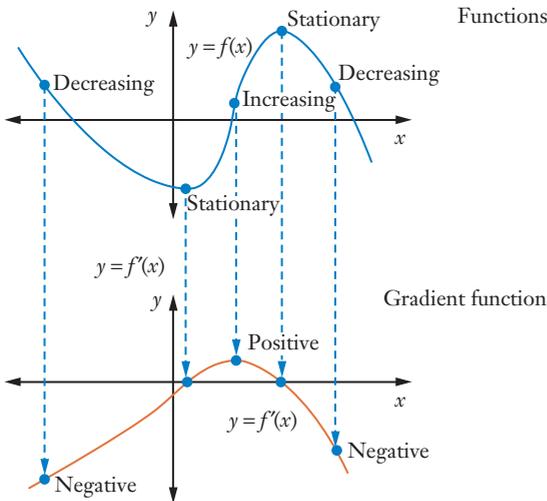
The graph of the **gradient function**,  $y = f'(x)$ , is shown on the right.

You can see how the graphs are related.

- Where the function is **decreasing**,  $f'(x) < 0$ .
- Where the function is **stationary**,  $f'(x) = 0$ .
- Where the function is **increasing**,  $f'(x) > 0$ .



The following diagrams show two functions with the corresponding gradient functions underneath.

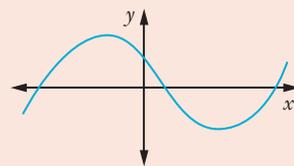


### Function and gradient function

- If a function is increasing, its gradient is positive. The gradient function is above the  $x$ -axis.
- If the function is decreasing, its gradient is negative. The gradient function is below the  $x$ -axis.
- If the function is stationary, its gradient is zero. The gradient function intersects the  $x$ -axis.

## EXAMPLE 1

Sketch a possible gradient function for the function shown on the right.



## Solution

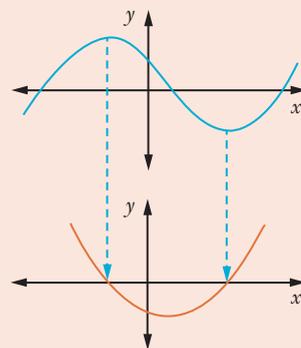
The function is at first increasing, so the gradient function is above the  $x$ -axis (positive).

There is a stationary point just before the  $y$ -axis, so the gradient function has an  $x$ -intercept.

Between the top and bottom of the curve, the function is decreasing, so the gradient function is below the  $x$ -axis (negative).

The gradient function again cuts the  $x$ -axis at the  $x$  value where the second stationary point occurs.

After the second stationary point, the function is increasing, so the gradient function is above the  $x$ -axis (positive).



## INVESTIGATION

### GRADIENT FUNCTIONS

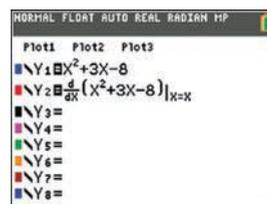
You can use a graphics calculator to examine functions and their gradient functions.



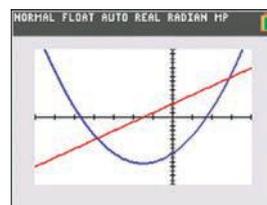
TI-Nspire CX  
Chapter 10

#### TI-84 Plus CE

- Enter the graphing mode by pressing the  $\boxed{y=}$  button.
- Enter the function  $Y_1 = x^2 + 3x - 8$ .
- Now enter the gradient function
- $Y_2 = \frac{d}{dx}(x^2 + 3x - 8)$  by pressing the  $\boxed{\text{math}}$  button then selecting **8: nDeriv**(. Then enter  $x^2 + 3x - 8$ . You then need to enter  $x = x$  to complete the operation.



Press the  $\boxed{\text{graph}}$  button to graph the function and its gradient function. Adjust the  $\boxed{\text{window}}$  settings as required.



### Casio fx-CG20AU

Select the **Graph** page.

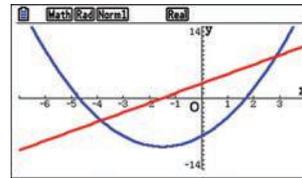
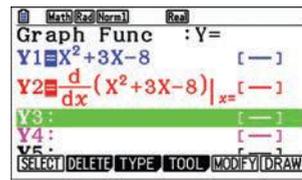
Enter the function  $Y1 = x^2 + 3x - 8$ .

Now enter the gradient function

$Y2 = \frac{d}{dx}(x^2 + 3x - 8)$  by pressing the **OPTN** button,

selecting **F2** (**CALC**) followed by **F1** ( $d/dx$ ) and then entering  $x^2 + 3x - 8$ .

Press **F6** (**DRAW**) to graph the function and its gradient function. Adjust the **V-Window** settings as required.



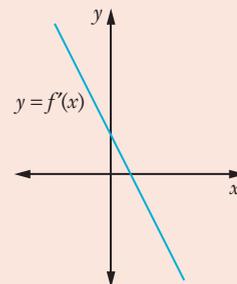
### Both calculators:

- What is the value of the gradient function at the stationary point on the function?
- Describe the behaviour of the function when the gradient function is:
  - a below the  $x$ -axis
  - b above the  $x$ -axis.
- Repeat the procedure using different functions including:
$$f(x) = 5x \quad f(x) = -5x \quad f(x) = x^3 \quad f(x) = x^3 - 2x^2 + 5x - 10$$
- Describe the graphs of  $f(x) = x^3$  and its gradient function.

You can go backwards from the gradient function to sketch a possible function.

### EXAMPLE 2

Draw a sketch of a possible function for this gradient function.



## Solution

Draw a set of axes under the axes of the gradient function. This is the plane where the function will be drawn.

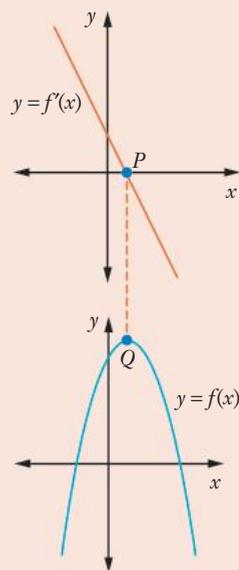
From left to right, the gradient function is at first positive (above the  $x$ -axis) so the function is increasing.

The gradient function is 0 at  $P$ , so the function has a stationary point at this  $x$  value.

Draw a vertical line down from  $P$  into the plane of the function. The function will have a stationary point at  $Q$  somewhere on this line.

After the stationary point, the gradient function is negative (below the  $x$ -axis) so the function is decreasing.

Draw the sketch so that  $y = f(x)$  is increasing to the left of  $Q$  and decreasing to the right of  $Q$ . The important points match with the gradient function.



The sketch drawn in Example 2 is that of only one possible function.

More information is required to locate the exact position of  $Q$ .

## EXAMPLE 3

Given that  $f(x) = 3 + 8x - x^2$ ,

- find the values of  $x$  for which  $f(x)$  is increasing
- determine if the function is increasing, decreasing or stationary at  $(5, 18)$ .

## Solution

- Write the rule for the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Substitute for  $f(x)$ .

$$= \lim_{h \rightarrow 0} \frac{3 + 8(x+h) - (x+h)^2 - (3 + 8x - x^2)}{h}$$

Expand the numerator.

$$= \lim_{h \rightarrow 0} \frac{3 + 8x + 8h - (x^2 + 2xh + h^2) - 3 - 8x + x^2}{h}$$

Simplify.

$$= \lim_{h \rightarrow 0} \frac{8h - 2xh - h^2}{h}$$

Factorise the numerator.

$$= \lim_{h \rightarrow 0} \frac{h(8 - 2x - h)}{h}$$

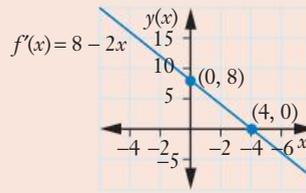
Cancel.

$$= \lim_{h \rightarrow 0} (8 - 2x - h)$$

Find the limit as  $h \rightarrow 0$ .

$$= 8 - 2x$$

Draw the graph of  $f'(x)$ .



Use the graph to find where the slope of the tangent is positive.

$$f'(x) > 0 \text{ when } x < 4.$$

State the result.

$$f(x) \text{ is increasing when } x < 4.$$

- b** Write the rule for the gradient function.

$$f'(x) = 8 - 2x$$

Substitute  $x = 5$ .

$$f'(5) = 8 - 2 \times 5$$

Evaluate.

$$= -2$$

$$< 0$$

If  $f'(x) < 0$ , the function is decreasing.

$$\text{The function is decreasing at } x = 5.$$

## INVESTIGATION

### TANGENTS ON A CURVE

It is possible to look at the tangents on simple curves to see how the gradient of a function changes. The spreadsheet 'Tangent tracer' can be used to investigate the way in which the tangent and its slope change.

- 1 Open the spreadsheet 'Tangent tracer' from NelsonNet.
- 2 Change the function to  $f(x) = x^2 - 4x + 1$  by making  $a_3 = 0$ ,  $a_2 = 1$ ,  $a_1 = -4$  and  $a_0 = 1$ .
- 3 Move the tangent along the curve.
  - Is the curve rising or falling before  $x = 2$ ?
  - What happens to the slope at  $x = 2$ ?
  - Is the curve rising or falling after  $x = 2$ ?
- 4 Change the function and observe how the tangent and slope of the curve change. Note that the vertical scale changes to accommodate the curve.

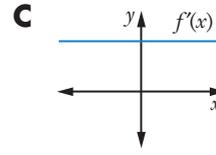
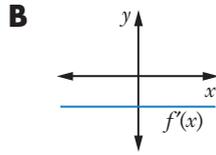
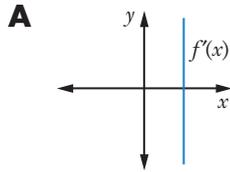
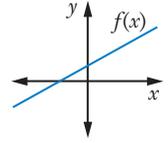


Tangent tracer

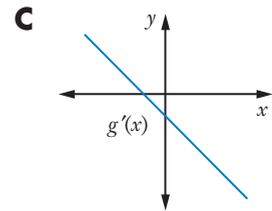
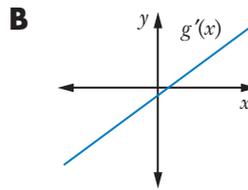
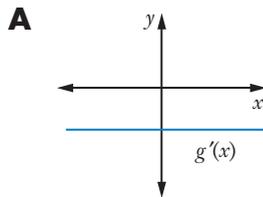
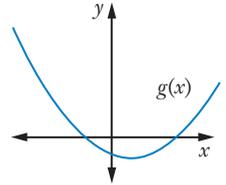
## Exercise 10.01 The gradient function

Example  
1

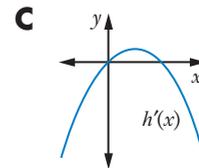
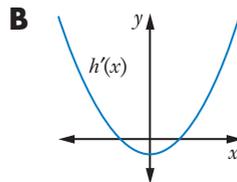
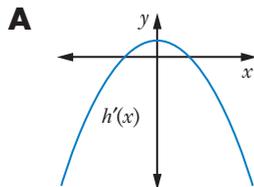
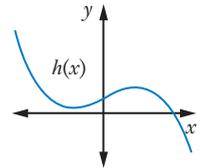
- 1 The graph of  $f(x)$  is shown on the right.  
Which of the following graphs shows a possible gradient function for  $f(x)$ ?



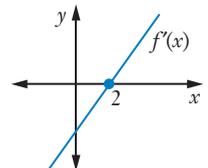
- 2 The graph of  $g(x)$  is shown on the right.  
Which graph below shows a possible gradient function for  $g(x)$ ?



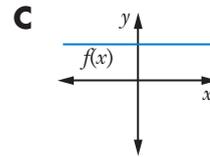
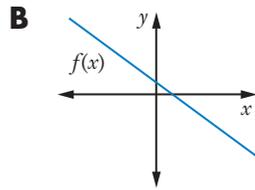
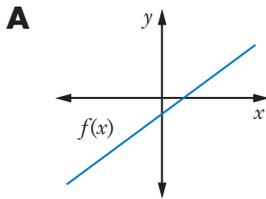
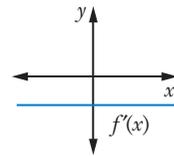
- 3 The graph of  $h(x)$  is shown on the right. Which of the graphs below shows a possible gradient function for  $h(x)$ ?



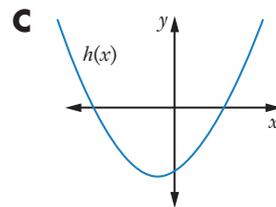
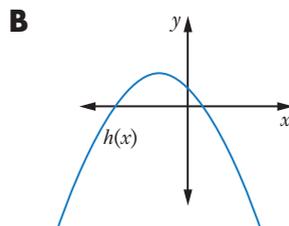
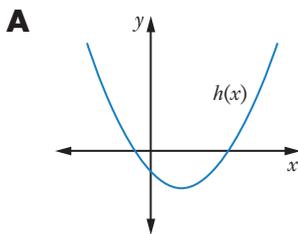
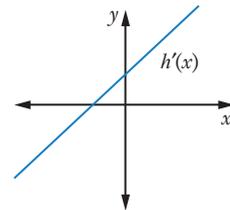
- 4 For the graph of  $f'(x)$  shown, draw a possible graph of  $f(x)$ .



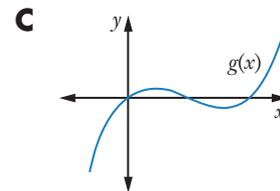
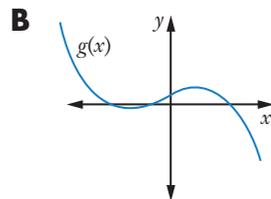
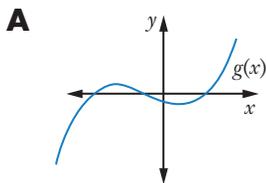
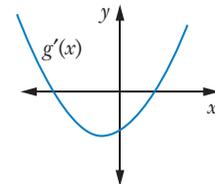
- 5** The graph of the gradient function  $f'(x)$  is shown on the right. Which graph below shows a possible graph of  $f(x)$ ?



- 6** The graph of the gradient function  $h'(x)$  is shown on the right. Which graph below shows a possible graph of  $h(x)$ ?



- 7** The graph of the gradient function  $g'(x)$  is shown on the right. Which graph below shows a possible graph of  $g(x)$ ?



- 8** Consider the function  $f(x) = 4x^2 + 3x - 1$ . Show that the function is:
- a** increasing when  $x = 8$
  - b** decreasing when  $x = -4$
  - c** stationary at  $x = -0.375$ .

**9** Consider the function  $f(t) = t^3 - 3t^2 + t - 5$ . Show that the function is:

- a** increasing when  $t = -1$
- b** decreasing when  $t = 1$
- c** increasing at  $t = 3$ .

**10** State whether  $y = x^2 - 4x$  is increasing, decreasing or stationary at:

- a**  $x = -4$                       **b**  $x = -3$                       **c**  $x = -2$                       **d**  $x = -1$
- e**  $x = 0$                           **f**  $x = 1$                           **g**  $x = 2$                           **h**  $x = 3$
- i**  $x = 4$                           **j**  $x = 5$

**11** If  $r(t) = 16 - 15t - 5t^2$ , state whether  $r(t)$  is increasing, decreasing or stationary at:

- a**  $t = -5$                       **b**  $t = -3.5$                       **c**  $t = -2$                       **d**  $t = -2.5$
- e**  $t = -1$                       **f**  $t = -1.5$                       **g**  $t = 0$                       **h**  $t = 1.5$
- i**  $t = 3$                           **j**  $t = 5$

**12** For  $w = m^3 - 12m + 9$ , state whether the function is increasing, decreasing or stationary at:

- a**  $m = -4$                       **b**  $m = -2$                       **c**  $m = -1$                       **d**  $m = 0$
- e**  $m = 1$                           **f**  $m = 2$                           **g**  $m = 4$                           **h**  $m = 6$

**13** For each function, sketch the graph of the gradient function and use the graph to state where  $f(x)$  is increasing and decreasing.

- a**  $f(x) = 3 + 2x - x^2$                       **b**  $f(x) = x^2 + 6x + 9$                       **c**  $f(x) = 3x - x^2 + 10$

**14** Find the values of  $x$  where each function has stationary points.

- a**  $f(x) = 2x^2 - 8x + 3$                       **b**  $f(x) = 6 + 4x - x^2$
- c**  $f(x) = x^3 - x^2 - 8x + 10$                       **d**  $f(x) = 2x^3 + 3x^2 - 12x$

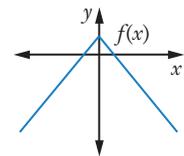
**15** Find any stationary points on each function.

- a**  $g(x) = -x^2 + 4x - 3$                       **b**  $f(x) = -x^2 - 2x + 4$
- c**  $y = x^3 - 12x$                           **d**  $g(x) = 3x - x^3$

### Problem solving

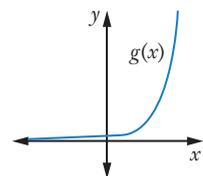
**16** The graph on the right represents a function,  $f(x)$ .

Draw a possible corresponding gradient function.

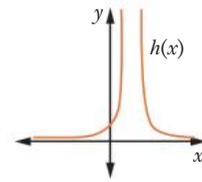


**17** The graph on the right represents a function,  $g(x)$ .

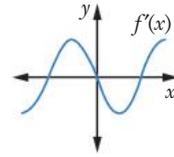
Draw a possible corresponding gradient function.



- 18** The graph on the right represents a function,  $h(x)$ .  
Draw a possible corresponding gradient function.



- 19** The graph on the right represents a gradient function,  $f'(x)$ .  
Draw a possible corresponding function.



## 10.02 The derivative of $x^n$

The coefficients of the binomial expansion of  $(x + y)^n$  are worked out as follows:  
 $(x + y)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1}y + {}^n C_2 x^{n-2}y^2 + {}^n C_3 x^{n-3}y^3 + \dots + {}^n C_1 xy^{n-1} + {}^n C_0 y^n$ .

### EXAMPLE 4

Find the derivative of  $f(x) = x^5$  from first principles.

#### Solution

Use the definition of a derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Substitute for  $f(x) = x^5$ .

$$= \lim_{h \rightarrow 0} \frac{(x+h)^5 - x^5}{h}$$

Write the expansion of  $(x+h)^5$ .

$$(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$$

Substitute into the formula for the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5 - x^5}{h}$$

Simplify.

$$= \lim_{h \rightarrow 0} \frac{5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5}{h}$$

Factorise.

$$= \lim_{h \rightarrow 0} \frac{h(5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)}{h}$$

Cancel.

$$= \lim_{h \rightarrow 0} (5x^4 + 10x^3h + 10x^2h^2 + 5xh^3 + h^4)$$

Find the limit using the fact that as  $h \rightarrow 0$ ,  $10x^3h \rightarrow 0$ , etc.

$$= 5x^4$$

State the result.

The derivative of  $f(x) = x^5$  is  $f'(x) = 5x^4$ .

You can also write the expansion of  $(x + y)^n$  as:

$$(x + y)^n = x^n + nx^{n-1}y + a_3x^{n-2}y^2 + a_4x^{n-3}y^3 + a_5x^{n-4}y^4 + \dots$$

where  $a_3, a_4, a_5, \dots$  are the coefficients of the 3rd, 4th, 5th, ... terms.

This expansion can be used to find the derivative of  $f(x) = x^n$  from first principles.

Write the function.  $f(x) = x^n$

Calculate  $f(x + h)$ .  $f(x + h) = (x + h)^n$   
 $= x^n + a_1x^{n-1}h + a_2x^{n-2}h^2 + a_3x^{n-3}h^3 + a_4x^{n-4}h^4 + \dots$

Use the definition of a derivative.  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$

Substitute for  $f(x) = x^n$ .  $= \lim_{h \rightarrow 0} \frac{(x + h)^n - x^n}{h}$

Use the expansion of  $(x + h)^n$ .  $= \lim_{h \rightarrow 0} \frac{(x^n + nx^{n-1}h + a_3x^{n-2}h^2 + a_4x^{n-3}h^3 + a_5x^{n-4}h^4 + \dots) - x^n}{h}$

Simplify.  $= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + a_3x^{n-2}h^2 + a_4x^{n-3}h^3 + a_5x^{n-4}h^4 + \dots}{h}$

Factorise.  $= \lim_{h \rightarrow 0} \frac{h(nx^{n-1} + a_3x^{n-2}h + a_4x^{n-3}h^2 + a_5x^{n-4}h^3 + \dots)}{h}$

Cancel.  $= \lim_{h \rightarrow 0} (nx^{n-1} + a_3x^{n-2}h + a_4x^{n-3}h^2 + a_5x^{n-4}h^3 + \dots)$

Find the limit using the fact that as  $h \rightarrow 0$ ,  $a_3x^{n-2}h \rightarrow 0$ , etc.  $= nx^{n-1}$

### Derivative of $x^n$

The derivative of  $f(x) = x^n$  is  $f'(x) = nx^{n-1}$  (where  $n = 0, 1, 2, \dots$ ).

This is called the **power rule** for derivatives.

Another word for finding the derivative of a function is **differentiate**. The process is called **differentiation**.

### EXAMPLE 5

Differentiate  $f(x) = x^8$ .

#### Solution

Write the function.

$$f(x) = x^8$$

Use the rule for the derivative.

$$f'(x) = nx^{n-1}$$

Substitute  $n = 8$ .

$$= 8x^{8-1}$$

Simplify.

$$= 8x^7$$

You can also use first principles to find the derivative of algebraic terms involving coefficients and powers such as  $8x^4$ .

### EXAMPLE 6

Use first principles to find the derivative of  $f(x) = 8x^4$ .

#### Solution

Write the function.

$$f(x) = 8x^4$$

Calculate  $f(x+h)$ .

$$f(x+h) = 8(x+h)^4$$

Expand the binomial.

$$\begin{aligned} &= 8(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) \\ &= 8x^4 + 32x^3h + 54x^2h^2 + 32xh^3 + 8h^4 \end{aligned}$$

Use the definition of a derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Substitute for  $f(x) = 8x^4$ .

$$= \lim_{h \rightarrow 0} \frac{8x^4 + 32x^3h + 54x^2h^2 + 32xh^3 + 8h^4 - 8x^4}{h}$$

Simplify.

$$= \lim_{h \rightarrow 0} \frac{32x^3h + 54x^2h^2 + 32xh^3 + 8h^4}{h}$$

Factorise.

$$= \lim_{h \rightarrow 0} \frac{8h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$$

Cancel.

$$= \lim_{h \rightarrow 0} 8(4x^3 + 6x^2h + 4xh^2 + h^3)$$

Find the limit using the fact that as  $h \rightarrow 0$ ,  $6x^2h \rightarrow 0$ , etc.

$$= 32x^3$$

State the result.

The derivative of  $f(x) = 8x^4$  is  $f'(x) = 32x^3$ .

## Exercise 10.02 The derivative of $x^n$

1 Expand each expression.

**a**  $(x + y)^5$

**b**  $(a + b)^3$

**c**  $(m + h)^4$

**d**  $(g + h)^6$

**e**  $(x + h)^7$

**f**  $(x + h)^8$

2 Expand each expression.

**a**  $(n + 3m)^4$

**b**  $(2c + d)^5$

**c**  $(4 + p)^3$

**d**  $(a + 1)^6$

Example  
**4**

3 Use first principles to find the derivative of:

**a**  $f(x) = x^4$

**b**  $f(x) = x^6$

**c**  $f(x) = x^7$

**d**  $f(x) = x^9$

Example  
**5**

4 Use the power rule for derivatives to differentiate each function.

**a**  $f(x) = x^2$

**b**  $y = x^3$

**c**  $f(x) = x^5$

**d**  $g(x) = x^8$

**e**  $y = x^{20}$

**f**  $f(x) = x^{17}$

**g**  $h(x) = x^{12}$

**h**  $y = x^{22}$

Example  
**6**

5 Use first principles to find the derivative of:

**a**  $f(x) = 5x^3$

**b**  $f(x) = 6x^4$

**c**  $f(x) = 3x^5$

**d**  $f(x) = 2x^6$

### Problem solving

6 Given that  $f(z) = z^n$ , use first principles to prove that  $f'(z) = nz^{n-1}$ .

7 Given that  $f(q) = bq^n$ , use first principles to prove that  $f'(q) = nbq^{n-1}$ .



Derivative of a sum of terms



Derivatives of linear products

## 10.03 The derivative of a linear combination

You can use first principles to find the derivative of the sum of two functions.

### EXAMPLE 7

If  $f(x) = x^3$  and  $g(x) = x^2$ :

**a** find  $\frac{d}{dx}[f(x) + g(x)]$  using first principles

**b** compare  $\frac{d}{dx}[f(x) + g(x)]$  with  $\frac{d}{dx}f(x)$  and  $\frac{d}{dx}g(x)$ .

### Solution

**a** Let  $[f(x) + g(x)]$  be a new function  $q(x)$ .

$$[f(x) + g(x)] = q(x) = x^3 + x^2$$

Find  $q(x + h)$ .

$$q(x + h) = (x + h)^3 + (x + h)^2$$

Expand. 
$$= (x^3 + 3x^2h + 3xh^2 + h^3) + (x^2 + 2xh + h^2)$$

Simplify. 
$$= x^3 + x^2 + 3x^2h + 2xh + 3xh^2 + h^2 + h^3$$

Find  $q(x+h) - q(x)$ . 
$$q(x+h) - q(x)$$
  

$$= x^3 + x^2 + 3x^2h + 2xh + 3xh^2 + h^2 + h^3 - (x^3 + x^2)$$

Simplify. 
$$= 3x^2h + 2xh + 3xh^2 + h^2 + h^3$$

Use the definition of the derivative. 
$$q'(x) = \lim_{h \rightarrow 0} \frac{q(x+h) - q(x)}{h}$$

Substitute for  $q(x+h) - q(x)$ . 
$$= \lim_{h \rightarrow 0} \frac{3x^2h + 2xh + 3xh^2 + h^2 + h^3}{h}$$

Factorise the numerator. 
$$= \lim_{h \rightarrow 0} \frac{h(3x^2 + 2x + 3xh + h + h^2)}{h}$$

Cancel. 
$$= \lim_{h \rightarrow 0} (3x^2 + 2x + 3xh + h + h^2)$$

Find the limit. 
$$= 3x^2 + 2x$$

State the result. 
$$\frac{d}{dx}[f(x) + g(x)] = 3x^2 + 2x$$

**b** Find  $f'(x)$  and  $g'(x)$  using the power rule. 
$$f(x) = x^3 \text{ so } f'(x) = 3x^{3-1} = 3x^2$$
  

$$g(x) = x^2 \text{ so } g'(x) = 2x^{2-1} = 2x$$

Compare  $q'(x)$  with  $f'(x)$  and  $g'(x)$ . 
$$q'(x) = f'(x) + g'(x).$$

Write the result. 
$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x).$$

Now you will use a similar approach to find the derivative of the difference of two functions.

### EXAMPLE 8

If  $p(x) = x^2$  and  $q(x) = x^4$

- a** find  $\frac{d}{dx}[p(x) - q(x)]$  from first principles
- b** compare  $\frac{d}{dx}[p(x) - q(x)]$  with  $\frac{d}{dx}p(x)$  and  $\frac{d}{dx}q(x)$ .

### Solution

- a** Let  $[p(x) - q(x)]$  be a new function  $f(x)$ .

$$[p(x) - q(x)] = f(x) = x^2 - x^4$$

Find  $f(x + h)$ .

$$f(x + h) = (x + h)^2 - (x + h)^4$$

Expand.

$$= (x^2 + 2xh + h^2) - (x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4)$$

Simplify.

$$= x^2 - x^4 + 2xh - 4x^3h - 6x^2h^2 - 4xh^3 + h^2 - h^4$$

Find  $f(x + h) - f(x)$ .

$$\begin{aligned} f(x + h) - f(x) &= x^2 - x^4 + 2xh - 4x^3h - 6x^2h^2 - 4xh^3 + h^2 - h^4 - (x^2 - x^4) \\ &= 2xh - 4x^3h - 6x^2h^2 - 4xh^3 + h^2 - h^4 \end{aligned}$$

Simplify.

$$= 2xh - 4x^3h - 6x^2h^2 - 4xh^3 + h^2 - h^4$$

Use the definition of the derivative.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Substitute for  $f(x + h) - f(x)$ .

$$= \lim_{h \rightarrow 0} \frac{2xh - 4x^3h - 6x^2h^2 - 4xh^3 + h^2 - h^4}{h}$$

Factorise the numerator.

$$= \lim_{h \rightarrow 0} \frac{h(2x - 4x^3 - 6x^2h - 4xh^2 + h - h^3)}{h}$$

Cancel.

$$= \lim_{h \rightarrow 0} (2x - 4x^3 - 6x^2h - 4xh^2 + h - h^3)$$

Find the limit.

$$= 2x - 4x^3$$

State the result.

$$\frac{d}{dx}[p(x) - q(x)] = 2x - 4x^3$$

- b** Find  $p'(x)$  and  $q'(x)$  using the power rule.

$$p(x) = x^2 \text{ so } p'(x) = 2x^{2-1} = 2x$$

$$q(x) = x^4 \text{ so } q'(x) = 4x^{4-1} = 4x^3$$

Compare  $f'(x)$  with  $p'(x)$  and  $q'(x)$ .

$$f'(x) = p'(x) - q'(x).$$

Write the result.

$$\frac{d}{dx}[p(x) - q(x)] = \frac{d}{dx}p(x) - \frac{d}{dx}q(x).$$

## Derivative of a sum or difference of functions

For any functions  $f(x)$  and  $g(x)$ :

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

and

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

You can prove the general rule for the derivative of a sum as shown below.

Write the derivative for  $[f(x) + g(x)]$ .

$$\frac{d}{dx}[f(x) + g(x)] = \lim_{h \rightarrow 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$

Rearrange the numerator.

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

Write as separate fractions.

$$= \lim_{h \rightarrow 0} \left[ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right]$$

Find limits of each fraction.

$$= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

Use the definition of a derivative.

$$= f'(x) + g'(x) \quad \text{QED}$$

You can show the rule for the derivative of a difference in a similar way.

The power rule is easily extended.

## Derivative of $kx^n$

$$\frac{d}{dx}[kf(x)] = kf'(x)$$

$$\frac{d}{dx}(kx^n) = knx^{n-1}.$$

This is called the **linear product rule** for derivatives.

The linear product rule is relatively easy to prove.

Use the definition of the derivative. 
$$\frac{d}{dx}[kf(x)] = \lim_{h \rightarrow 0} \frac{kf(x+h) - kf(x)}{h}$$

Factorise the numerator. 
$$= \lim_{h \rightarrow 0} \frac{k[f(x+h) - f(x)]}{h}$$

$k$  is constant. 
$$= k \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Use the definition of the derivative. 
$$= kf'(x) \quad \text{QED}$$

### Derivative of a constant

If  $f(x) = c$ , where  $c$  is a constant:

$$\frac{d}{dx}f(x) = \frac{d}{dx}(c) = 0$$

You can prove the derivative of a constant as an exercise.

A **linear combination** is an expression formed by adding or subtracting linear products. For example, a linear combination of  $x$  and  $y$  could be an expression like:

$$3x + 5y, \frac{1}{2}x - 8y, x + \sqrt{2}y.$$

A linear combination of the functions  $f(x)$  and  $g(x)$  could be:

$$2f(x) + g(x), 3f(x) - \frac{3}{5}g(x)$$

You can combine the rules for the derivative of a sum or difference of functions and the linear product rule to get the rule for the derivative of a linear combination of functions.

### Derivative of a linear combination of functions

For any functions  $f(x)$  and  $g(x)$  and constants  $a$  and  $b$ :

$$\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$$

## EXAMPLE 9

Differentiate  $7x^5 - 5x^8$ .

### Solution

Use the linear combination rule.

$$\frac{d}{dx}(7x^5 - 5x^8) = 7 \frac{d}{dx}(x^5) + (-5) \frac{d}{dx}(x^8)$$

Calculate the derivatives.

$$= 7 \times 5x^4 - 5 \times 8x^7$$

Simplify.

$$= 35x^4 - 40x^7$$

## Exercise 10.03 The derivative of a linear combination

- 1 If  $f(x) = x$  and  $g(x) = x^2$ , find  $\frac{d}{dx}[f(x) + g(x)]$  using first principles.
- 2 If  $p(x) = x^3$  and  $q(x) = x$ , find  $\frac{d}{dx}[p(x) + q(x)]$  using first principles.
- 3 If  $m(x) = x^4$  and  $n(x) = x^2$ , find  $\frac{d}{dx}[m(x) + n(x)]$  using first principles.
- 4 If  $f(x) = x^3$  and  $g(x) = x^5$ , find  $\frac{d}{dx}[f(x) + g(x)]$  using first principles.
- 5 If  $u(x) = x^2$  and  $v(x) = x^3$ , find  $\frac{d}{dx}[u(x) - v(x)]$  using first principles.
- 6 If  $f(x) = x$  and  $g(x) = x^6$ , find  $\frac{d}{dx}[f(x) - g(x)]$  using first principles.
- 7 If  $p(x) = x^5$  and  $q(x) = x^2$ , find  $\frac{d}{dx}[p(x) - q(x)]$  using first principles.
- 8 If  $f(x) = x^3$  and  $g(x) = x^4$ , find  $\frac{d}{dx}[f(x) - g(x)]$  using first principles.
- 9 Use the rules of differentiation to find the derivative of each expression.

**a**  $\frac{d}{dx}(x^6 + x^3)$

**b**  $\frac{d}{dx}(x^5 + x^8)$

**c**  $\frac{d}{dt}(t^7 - t^4)$

**d**  $\frac{d}{dx}(x^6 - x^{10})$

**e**  $\frac{d}{dr}(r^7 + r^4)$

**f**  $\frac{d}{dx}(x^3 - x^{10})$

**g**  $\frac{d}{du}(u^5 + u^9)$

**h**  $\frac{d}{dx}(x^3 - x^6)$

Example  
7

Example  
8

**10** Use the rules of differentiation to find the derivative of each expression.

**a**  $\frac{d}{dx} (x^3 + x^5 + x^7)$

**b**  $\frac{d}{dx} (x^2 - x^6 + x^8)$

**c**  $\frac{d}{du} (u^3 + u^4 - u^5)$

**d**  $\frac{d}{dx} (x^3 - x^7 - x^{10})$

**e**  $\frac{d}{dz} (z^2 + z^5 - z^6 + z^8)$

**f**  $\frac{d}{dr} (r - r^2 + r^5 - r^8)$

Example  
**9**

**11** Use the linear combination rule to find the derivative of each expression.

**a**  $2x^4 - 7x^5$

**b**  $18x^2 + 45x^8$

**c**  $16x - 12x^{11}$

**d**  $10x^7 + 4x^4$

**e**  $6x^6 + 11x^5$

**f**  $9x^7 + 4x^{10}$

**g**  $2x^3 - 10x^8$

**h**  $4x^5 - 7x^7$

### Problem solving

**12** If  $y = 4x^5 - 8x^3$ , calculate:

**a**  $3y'$

**b**  $\frac{3}{2}y'$

**c**  $-4y'$

**d**  $3\sqrt{5}y'$

**e**  $5.5y'$

**13** If  $g(x) = 3x^3 - 4x^7$ , calculate:

**a**  $5g'(x)$

**b**  $-4g'(x)$

**c**  $\frac{5}{3}g'(x)$

**d**  $7\sqrt{2}g'(x)$

**e**  $2.8g'(x)$

**14** If  $g'(x) = 3x^2 - 4x^3$  and  $h'(x) = 2x^4 + 3x^3$ , find the derivative of:

**a**  $2g(x) + 3h(x)$

**b**  $4g(x) - 2h(x)$

**c**  $-5g(x) + 4h(x)$

**d**  $-3g(x) - 4h(x)$

**15** If  $f(x) = 3x^3 - 4x^2 + 7$  and  $g(x) = 2x^4 - 3x^3 + x^2 - 12$ , find the derivative of:

**a**  $3f(x) + 2g(x)$

**b**  $5f(x) - 3g(x)$

**c**  $-2f(x) - 5g(x)$

## 10.04 The derivative of a polynomial

You should remember that a **polynomial** function is defined as:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0, \text{ where } a_i \in \mathbf{R}.$$

All of the following are polynomial functions:

$$f(x) = 5x + 2, y = \frac{x^3}{6} + \sqrt{5}x \text{ and } g(x) = 15 - x^2 + 7x^3$$

All of the following are *not* polynomial functions:

$$h(x) = 4x^{-2} + 7x, g(x) = \frac{8}{x} + 3x^2, y = 2x^2 - 3\sqrt{x} \text{ and } f(x) = 6xy + 2x$$

You use the linear combination rule to differentiate polynomials.

### EXAMPLE 10

If  $f(x) = 3x^3 + x^2 - 10x + 7$ , find:

**a**  $f'(x)$                       **b**  $f'(-2)$

#### Solution

**a** Write the function.

$$f(x) = 3x^3 + x^2 - 10x + 7$$

Use the linear combination rule.

$$f'(x) = 9x^2 + 2x - 10$$

**b** Substitute  $x = -2$ .

$$f'(-2) = 9 \times (-2)^2 + 2 \times (-2) - 10$$

Evaluate.

$$= 36 - 4 - 10$$

$$= 22$$

State the result.

$$f'(-2) = 22$$

In some cases you have to expand an expression before it can be differentiated.

### EXAMPLE 11

Find the derivative of  $g(x) = (2x - 3x^2)(6x - x^2 + x^3)$ .

#### Solution

Expand.

$$\begin{aligned}g(x) &= (2x - 3x^2)(6x - x^2 + x^3) \\ &= 2x(6x - x^2 + x^3) + (-3x^2)(6x - x^2 + x^3) \\ &= 12x^2 - 2x^3 + 2x^4 - 18x^3 + 3x^4 - 3x^5 \\ &= 12x^2 - 20x^3 + 5x^4 - 3x^5\end{aligned}$$

Differentiate.

$$g'(x) = 24x - 60x^2 + 20x^3 - 15x^4$$

In the previous section you saw that for any functions  $f(x)$  and  $g(x)$ :

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

So the derivative of a sum of two functions is the sum of their derivatives.

What about the product of two functions?

### EXAMPLE 12

Given that  $f(x) = 4x^2 - 3x$  and  $g(x) = x^3 + 2x^2 - 5$ :

- calculate  $\frac{d}{dx}[f(x) \times g(x)]$
- calculate  $\frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$
- compare  $\frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$  and  $\frac{d}{dx}[f(x) \times g(x)]$ .

#### Solution

- Find the product of the functions.  $f(x) \times g(x) = (4x^2 - 3x)(x^3 + 2x^2 - 5)$

Expand.

$$= 4x^5 + 8x^4 - 20x^2 - 3x^4 - 6x^3 + 15x$$

Simplify.

$$= 4x^5 + 5x^4 - 6x^3 - 20x^2 + 15x$$

Find the derivative.

$$\frac{d}{dx}[f(x) \times g(x)] = 20x^4 + 20x^3 - 18x^2 - 40x + 15$$

**b** Calculate  $\frac{d}{dx}f(x)$ .

$$\frac{d}{dx}f(x) = \frac{d}{dx}(4x^2 - 3x)$$

Differentiate each term.

$$= 8x - 3$$

Calculate  $\frac{d}{dx}g(x)$ .

$$\frac{d}{dx}g(x) = \frac{d}{dx}(x^3 + 2x^2 - 5)$$

Differentiate each term.

$$= 3x^2 + 4x$$

Find the product of the derivatives.

$$\frac{d}{dx}f(x) \times \frac{d}{dx}g(x) = (8x - 3)(3x^2 + 4x)$$

Expand.

$$= 24x^3 + 32x^2 - 9x^2 - 12x$$

Simplify.

$$= 24x^3 + 23x^2 - 12x$$

State the result.

$$\frac{d}{dx}f(x) \times \frac{d}{dx}g(x) = 24x^3 + 23x^2 - 12x$$

**c** Compare the results from **a** and **b**.

$$\frac{d}{dx}[f(x) \times g(x)] \neq \frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$$

Example 12 shows that the derivative of the product of two functions is *not* the same as the product of their derivatives.

In general, for the functions  $f(x)$  and  $g(x)$ :

$$\frac{d}{dx}[f(x) \times g(x)] \neq f'(x) \times g'(x)$$

### Exercise 10.04 The derivative of a polynomial

**1** Find the derivative of each polynomial.

**a**  $6x^4 - 5x^3 + 2x^2 + 8x - 9$

**b**  $9x^3 + 10x^4 - 3x^5 + 18$

**c**  $x^3 + x^2 + x + 1$

**d**  $4x^9 - 8x^{10} + 20$

**e**  $15x^{12} + 12x^{15} - 36x^5$

**f**  $5 - x^7 + 7x^9 + x^{10}$

**g**  $5x^8 + 12x^9 - 6x^{10} + x^{12}$

**h**  $11 - 3x + x^2 + 8x^3 - x^4$

**i**  $5x^4 - 17x + 28 + 16x^3$

**j**  $25x^7 - 8x^6 + 14x^5 + 2x - 2$

Example  
**10**

**2** Find each derivative.

**a**  $\frac{d}{dx}(3x^4 - 6x^2 + 18)$

**b**  $\frac{d}{du}(3u^4 - 6u^2 + 18)$

**c**  $\frac{d}{dm}(5m^6 + 8m^5 - 24m + 8)$

**d**  $\frac{d}{dy}(9y + 6y^5 - 4y^3 + 2y^7)$

**e**  $\frac{d}{dz}(3z^2 - 9z^3 + 38z - 5)$

**f**  $\frac{d}{dt}(5t^7 + 9t^4 - 16t)$

**g**  $\frac{d}{dr}(21 - 13r - 6r^2 + 9r^3)$

**h**  $\frac{d}{da}(-6a^4 + 2a^2 - 10)$

**i**  $\frac{d}{dp}(8p - 3p^4 - 6p^7 - 5p^9)$

**j**  $\frac{d}{dt}(8 + 4t^2 - 2t^3 + 11t^4)$

**3** If  $f(x) = 2x^4 - 5x^3 - 7x + 9$ , find:

**a**  $f'(1)$

**b**  $f'(-1)$

**c**  $f'(5)$

**d**  $f'(-3)$

**4** Find the derivative of each product.

**a**  $(7 + 2x)(3 + 5x)$

**b**  $(4 - 3x)(9 - 2x)$

**c**  $(8 + 3x)(6 - 7x)$

**d**  $(4x + 7)(8 - 3x)$

**e**  $(9x + 2)(2x - 11)$

**f**  $(8 - x)(7x - 2)$

**5** Find the derivative of each product.

**a**  $(7 + x)^2$

**b**  $(x - 4)^2$

**c**  $(3 + 2x)^2$

**d**  $(4 + 3x)^2$

**e**  $(4x - 5)^2$

**f**  $(8 + 3x)^2$

**6** Find the derivative of each product.

**a**  $(x - 3x^2)(1 + x^3)$

**b**  $(2x^2 - x^3)(5x + 3x^2)$

**c**  $(4x^2 + 2x^3)(5x^3 - 3x^4)$

**d**  $(x - 7x^2)(2 + 3x^2 + x^3)$

**e**  $(2x^2 + 3x^3)(3 - 4x^3 - 6x^4)$

**f**  $(8x + 3x^2 - 4x^3)(2x^2 + 7x^3)$

**7** Find the derivative of each quotient.

**a**  $\frac{7x^5 - 3x^2}{x}, x \neq 0$

**b**  $\frac{12x^7 + 8x^5 - 4x^3}{x^2}, x \neq 0$

**c**  $\frac{3x^2 + 2x - 5}{x - 1}, x \neq 1$

**d**  $\frac{8x^2 - 2x - 21}{2x + 3}, x \neq -\frac{1}{2}$

**8** Given that  $f(x) = 3x - 7$  and  $g(x) = 4x^2 + 5x^2 - 9$ , demonstrate that:

$$\frac{d}{dx}[f(x) \times g(x)] \neq \frac{d}{dx}f(x) \times \frac{d}{dx}g(x)$$

**9** Given that  $y = 2x^2 - 3x$  and  $z = 3 - 7x^2 + 2x^3$ :

**a** calculate  $y' \times z'$

**b** calculate  $(y \times z)'$

**c** compare the results of **a** and **b**.

Example  
**11**

Example  
**12**

## Problem solving

**10 a** Differentiate each expression and factorise the result.

**i**  $(x + 1)^2$

**ii**  $(x + 1)^3$

**iii**  $(2x + 1)^2$

**iv**  $(2x + 1)^3$

**v**  $(3x + 1)^2$

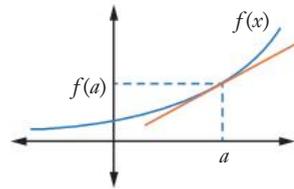
**vi**  $(3x + 1)^3$

**b** Using the results from **a**, write an expression for the derivative of  $(ax + 1)^n$ , where  $a$  and  $n$  are constants.

## 10.05 Tangents

Remember that the derivative is the gradient of a tangent to a function.

For the function shown here, the gradient of the tangent at  $x = a$  is  $f'(a)$ .



Slopes of curves



Tangents to a curve

### EXAMPLE 13

Find any values of  $x$  for which the slope of the tangent to the curve  $y = 3x^3 - 3x^2 + 4$  is equal to 3.

#### Solution

Find the derivative.

$$y' = 9x^2 - 6x$$

The derivative is the slope of the tangent so let  $y' = 3$ .

$$9x^2 - 6x = 3$$

Rearrange.

$$9x^2 - 6x - 3 = 0$$

Factorise.

$$3(3x^2 - 2x - 1) = 0$$

$$3(x - 1)(3x + 1) = 0$$

Solve for  $x$ .

$$x - 1 = 0 \text{ or } 3x + 1 = 0$$

$$x = 1 \text{ or } -\frac{1}{3}$$

Since the tangent is a straight line, you can find its equation using the slope and point.



Equation of  
a tangent

### EXAMPLE 14

The equation of a parabola is  $y = 5x - 3x^2 + 8$ .

- a What is the gradient at  $x = 2$ ?
- b What is the equation of the tangent at  $x = 2$ ?

#### Solution

- a Find the derivative.

$$f'(x) = 5 - 6x$$

Find its value at  $x = 2$ .

$$f'(2) = 5 - 12 = -7$$

Write the answer.

The gradient of the tangent at  $x = 2$  is  $m = -7$ .

- b Find  $y$  at  $x = 2$ .

$$y = 5 \times 2 - 3 \times 2^2 + 8 = 6$$

The tangent passes through  $(2, 6)$  and has a gradient of  $-7$ .

Use the point–gradient formula to find the equation of the tangent.

Write the values of  $m$  and  $(x_1, y_1)$ .

$$m = -7 \text{ and } (x_1, y_1) = (2, 6)$$

Write the gradient–point formula.

$$y - y_1 = m(x - x_1)$$

Substitute and simplify.

$$y - 6 = -7(x - 2)$$

Expand brackets.

$$y - 6 = -7x + 14$$

Write in standard form.

$$7x + y - 20 = 0$$

Write the answer.

The equation of the tangent is  $7x + y - 20 = 0$ .

### EXAMPLE 15

Tangent  $AB$  touches curve  $f(x) = 12 + 4x - x^2$  at point  $A$  and cuts the  $x$ -axis at point  $B$ .  $A$  has an  $x$ -coordinate of 3. Find:

- a the equation of the tangent
- b the length of  $AB$ , correct to 2 decimal places
- c the angle that  $AB$  makes with the  $x$ -axis.

## Solution

**a** Find  $A$ .

$f(3) = 15$ , so  $A$  is the point  $(3, 15)$ .

Find  $f'(x)$ .

$$f'(x) = 4 - 2x$$

Find the gradient of the tangent at  $A$ .

$$f'(3) = 4 - 6 = -2$$

State the result.

Gradient of tangent at  $(3, 15)$ ,  $m = -2$

Use the point–gradient formula.

$$y - y_1 = m(x - x_1)$$

Use  $(x_1, y_1) = (3, 15)$  and  $m = -2$

$$y - 15 = -2(x - 3)$$

Expand.

$$y - 15 = -2x + 6$$

Write in standard form.

$$2x + y - 21 = 0$$

**b**  $B$  lies on the  $x$ -axis and the tangent, so  $y = 0$ .

$$2x + 0 - 21 = 0$$

Solve.

$$x = 10.5$$

State the result.

$B$  is the point  $(10.5, 0)$ .

Write the formula for length.

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Use  $A = (3, 15)$  and  $B = (10.5, 0)$ .

$$= \sqrt{(10.5 - 3)^2 + (0 - 15)^2}$$

Evaluate and round off.

$$= 16.7705\dots$$

Round off and state the result.

$AB$  is about 16.77 units long.

**c** Identify the angle that  $AB$  makes with the  $x$ -axis.

Let  $\theta$  be the angle that  $AB$  makes with the  $x$ -axis.

Use the fact that  $m = \tan(\theta)$ .

$$\tan^{-1}(m) = \theta$$

Substitute.

$$\tan^{-1}(-2) = \theta$$

Evaluate.

$$\theta = -63.4349\dots^\circ \text{ or } 116.5650\dots^\circ$$

Round off and state the result.

$AB$  makes an angle of about  $116.57^\circ$  with the  $x$ -axis.

## Exercise 10.05 Tangents

- 1** Find the gradient of the tangent for each function at  $x = -1$ .
- a**  $y = x^2 - 6x + 5$       **b**  $f(x) = 7 + 4x - 5x^2$       **c**  $f(x) = 2x^2 - 9x - 14$   
**d**  $f(x) = 5x - 3x^2 + 12$       **e**  $y = x^3 + 2x^2 - 3x$       **f**  $f(x) = 2x^3 - 4x^2 + 7x - 12$
- 2** For each function, calculate the gradient of the tangent at the  $x$  value indicated.
- a**  $f(x) = x^2 - 5x - 10$  at  $x = 0$       **b**  $y = 3x^2 + 4x - 11$  at  $x = 1$   
**c**  $f(x) = 7 + 4x - 3x^2$  at  $x = -1$       **d**  $y = 2x^2 - 9x + 7$  at  $x = 2$   
**e**  $f(x) = x^3 - 6x^2 + 3x$  at  $x = 2$       **f**  $f(x) = 2x^3 - 3x^2 + 4x - 8$  at  $x = 1$
- 3** Find all points on each function where the gradient of the tangent is equal to 0.
- a**  $f(x) = 2x^2 - 8x + 7$       **b**  $y = \frac{3x^2}{2} + 12x + 17$       **c**  $f(x) = 5 - 3x - x^2$   
**d**  $y = 3x^2 - 12x - 15$       **e**  $f(x) = 2x^3 - 6x$       **f**  $f(x) = x^3 - 2x^2 + x - 2$
- 4** Given  $y = x^3 - 6x^2 + 11x - 6$ , find all points on the function where the gradient of the tangent is:
- a** 2      **b** -1      **c** 26
- 5** Find the equation of the tangent to each function at the  $x$  value indicated.
- a**  $f(x) = x^2 - 1$  at  $x = 2$       **b**  $y = x^2 + 4x + 1$  at  $x = -1$   
**c**  $f(x) = x^3 - x^2 + 2x - 2$  at  $x = -1$       **d**  $y = x^3 - 2x + 3$  at  $x = 1$   
**e**  $f(x) = x^3 - 2x^2$  at  $x = 1$       **f**  $f(x) = x^3 - 2x^2 + 4$  at  $x = 2$
- 6** The tangent to the curve  $y = 2x^2 + 4x - 1$  at  $A(0, -4)$  cuts the  $x$ -axis at point  $B$ . Find:
- a** the equation of  $AB$   
**b** the length  $AB$  (correct to 2 decimal places)  
**c** the acute angle that  $AB$  makes with the  $x$ -axis (correct to 1 decimal place).
- 7** The tangent to the curve  $y = 2x^2 + 4x - 1$  drawn at  $P$  where  $x = -2$ , cuts the  $y$ -axis at point  $Q$ . Find:
- a** the gradient of  $PQ$   
**b** the equation of the tangent at  $x = -2$   
**c** the length  $PQ$  (correct to 2 decimal places)  
**d** the angle that  $PQ$  makes with the  $x$ -axis (correct to 1 decimal place).

Example  
13

Example  
14

Example  
15

- 8** Given  $f(x) = 3x^2 - 5x + 2$ , find:
- the acute angle that the tangent at  $x = 1$  makes with the  $x$ -axis
  - the equation of the tangent at  $x = 1$
  - the acute angle that the tangent at  $x = -3$  makes with the  $x$ -axis
  - the length of the tangent at  $x = -3$  cut off by the axes
  - the length of the tangent drawn at  $x = 5$  that is cut off by the axes.
- 9** Given that  $y = x^2 - 2x^3 + 5x - 1$ , find:
- the equation of the tangent at  $x = -2$
  - the gradient of the tangent at  $x = 1$
  - the equation of the tangent at  $x = 1$
  - the angle that the tangent at  $x = 1$  makes with the  $x$ -axis.
- 10** Given  $f(x) = x^2 + 4x - 4x^3 - 4$ , find:
- a general expression for the gradient of the tangent to the function
  - the point(s) where the gradient of the tangent to the function is zero
  - the gradient of the tangent at  $x = 1$
  - the gradient of the tangent at  $x = 0$
  - the angle that the tangent in part **d** makes with the  $x$ -axis
  - the equation of the tangent at  $x = 3$
  - the equation of the tangent at  $x = -2$
  - the angle that the tangent in part **f** makes with the  $x$ -axis.
- 11** A curve has the equation  $y = x^3 - 3x^2 - 2x$ . Find the equation of the tangent to this curve with gradient 7.

### Problem solving

- 12** Find the equations of the tangent to the curve  $f(x) = 12 - x - x^2$  at the points where the curve crosses the  $x$ -axis.
- 13** The function  $f(x) = x^3 + ax^2 - 12x + b$  has two points where the gradient of the tangent is 0. If one point is  $(2, 0)$ , find:
- the values of  $a$  and  $b$
  - the other point.
- 14** Find the equation of the tangent to  $y = x^2 - 6x - 4$  that is parallel to the line  $4x - y + 7 = 0$ .
- 15** A function,  $y = ax^2 + bx + c$ , passes through the point  $(2, 15)$ . The slope of the tangent is 4 at  $x = 1$  and  $-8$  at  $x = -1$ . Find the equation of the function.



Instantaneous rates of change



Properties of derivatives assignment



Properties of derivatives problems

## 10.06 Applications of the derivative

### Instantaneous rate of change

The derivative of a function is the **instantaneous rate of change** of the function.

#### EXAMPLE 16

Find the rate of change of  $f(x) = -2x^3 - 10x + 8$  at  $x = 3$ .

#### Solution

Write down the function.

$$f(x) = -2x^3 - 10x + 8$$

Differentiate.

$$f'(x) = -6x^2 - 10$$

Find  $f'(3)$ .

$$f'(3) = -6 \times 3^2 - 10$$

Evaluate.

$$= -64$$

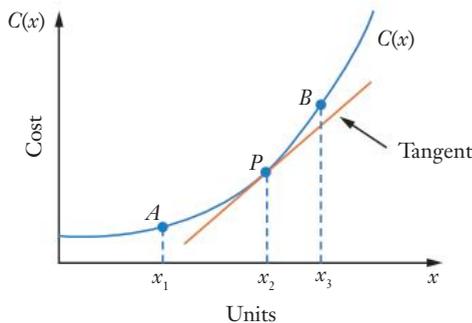
State the result.

The rate of change of  $f(x)$  at  $x = 3$  is  $-64$ .

Economics is often concerned with non-constant quantities like cost, revenue, profit, demand and so on. For example, the cost of producing initial items may be greater than those later on due to set-up costs. Eventually, items may cost more per item to produce because of scarcity of resources, overuse of equipment and so on.

Economists try to make functions (equations) to model quantities.

For example, the cost function for producing widgets could be like this:



The cost function shows that the cost of producing widgets increases as more are produced.

The cost of producing  $x_1$  units is  $C(x_1)$ .

The cost of producing  $x_3$  units is  $C(x_3)$ .

The average rate of change of cost between

$$x_1 \text{ and } x_3 = \frac{\Delta C(x)}{\Delta x} = \frac{C(x_3) - C(x_1)}{x_3 - x_1}$$

Economists (and producers) are interested in the **marginal cost** of production. This is the cost of producing the last item.

The marginal cost of producing  $x_2$  units is approximated by the slope of the tangent drawn to the curve at  $x_2$ .

So marginal cost  $\approx C'(x)$ .

Similarly:

**Marginal revenue**  $\approx R'(x)$ , where  $R(x)$  is the revenue function.

**Marginal profit**  $\approx P'(x)$ , where  $P(x)$  is the profit function.

### EXAMPLE 17

The total production cost per week for producing  $x$  widgets is given by:

$$C(x) = 50 + 5x + 0.2x^2 \text{ for } 0 \leq x \leq 200$$

where cost is in dollars and  $x$  is the number of units sold.

- a** What is the average rate of change in cost as  $x$  increases from 2 to 7 units?
- b** What is the marginal cost when  $x = 5$ ?
- c** What is the marginal cost when  $x = 15$ ?
- d** Each widget is sold for \$61. When is the cost to produce a widget greater than the revenue from selling it?

### Solution

**a** Calculate the cost at  $x = 2$ .

$$C(2) = 50 + 5 \times 2 + 0.2 \times 2^2 = 60.8$$

Calculate the cost at  $x = 7$ .

$$C(7) = 50 + 5 \times 7 + 0.2 \times 7^2 = 94.8$$

Calculate the average rate of change.

$$\frac{\Delta C(x)}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1}$$

Substitute.

$$= \frac{94.8 - 60.8}{7 - 2}$$

Evaluate.

$$= 6.8$$

State the result.

The average rate of change of cost from  $x = 2$  to  $x = 7$  is \$6.8/unit.

- b** Find the instantaneous rate of change.  $C'(x) = 5 + 0.4x$
- Let  $x = 5$ .  $C'(5) = 5 + 0.4 \times 5 = 7$
- State the result. The marginal cost at  $x = 5$  is \$7/unit.
- c** Let  $x = 15$ .  $C'(15) = 5 + 0.4 \times 15 = 30$
- State the result. The marginal cost at  $x = 15$  is \$30/unit.
- d**  $C'(x)$  is the cost of production per unit. The revenue from selling each unit is \$61.
- Write an expression for cost and revenue as the same.  $C'(x) = 61$
- Substitute for  $C'(x)$ .  $5 + 0.4x = 61$
- Solve for  $x$ .  $0.4x = 61 - 5$
- $$x = \frac{56}{0.4} = 140$$
- State the result. The cost to produce a widget is greater than the revenue from selling it when  $x > 140$ .

You can model the rates of change of physical quantities like area, volume, pressure and temperature using derivatives.

### EXAMPLE 18

The volume of liquid in an underground tank being gravity-filled from a tanker is given by

$$V = 1200 + 1020t - 17t^2$$

where the volume  $V$  is in litres and the time  $t$  is in minutes.

- a** What is the filling rate after 5 minutes?
- b** What is the flow rate in the pipe after 10 minutes?

### Solution

Both the filling rate and the flow rate are given by the instantaneous rate of change of the volume, so you use the derivative.

Calculate the derivative.

$$\frac{dV}{dt} = 1020 - 34t$$

- a** Find the value of the derivative at  $t = 5$ .

$$\left. \frac{dV}{dt} \right|_{t=5} = 1020 - 34 \times 5$$

Evaluate.

$$= 850$$

State the result.

The filling rate after 5 minutes is 850 L/min.

- b** Find the value of the derivative at  $t = 10$ .

$$\left. \frac{dV}{dt} \right|_{t=10} = 1020 - 34 \times 10$$

Evaluate.

$$= 680$$

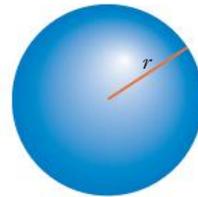
State the result.

The filling rate after 5 minutes is 680 L/min.

## INVESTIGATION

### ERRORS AND DERIVATIVES

The volume of a sphere is given by  $V = \frac{4\pi r^3}{3}$ .



- 1 Calculate the volume of a sphere of radius 8 m.
- 2 Calculate the volume of a sphere of radius 8.1 m.
- 3 What effect does an error of 0.1 m in the radius have on the volume of a sphere (from these calculations)?
- 4 Calculate the effect on the volume of an error of 1 m in the radius (e.g. from 7.5 to 8.5 m).
- 5 Calculate the value  $\left. \frac{dV}{dr} \right|_{r=8}$ . What unit does this value have?
- 6 Compare your answers to questions 4 and 5.
- 7 What would you expect the approximate error to be if there were a 0.4 m error in the radius (e.g. from 7.8 to 8.2 m)?
- 8 Repeat questions 4 to 6 for  $r = 50$  m (error of 1 m).
- 9 Use the derivative to calculate the approximate error in volume if a 1 m error is made in the radius and  $r \approx 700$  m.
- 10 What approximate error in the volume would occur if a mistake of 0.2 m were made in the radius at a radius of 700 m?

## Exercise 10.06 Applications of the derivative

Example

16

- 1 Find the rate of change of  $y = 7x^3 + 2x^2 - 12x + 5$  at:  
**a**  $x = 1$       **b**  $x = -2$       **c**  $x = 3$       **d**  $x = 0$       **e**  $x = 5$
- 2 Find the rate of change of  $P(x) = 2x^2 + 4x^3 - 18x$  at:  
**a**  $x = -2$       **b**  $x = 3$       **c**  $x = -1$       **d**  $x = 0$       **e**  $x = -4$
- 3  $g(t) = t^2 - 2t^3 + 5t - 6$   
**a** Find the average rate of change from  $t = 2$  to  $t = 8$ .  
**b** Find the instantaneous rate of change at:  
**i**  $t = 2$       **ii**  $t = 8$       **iii**  $t = 5$   
**c** Compare the average rate of change and the instantaneous rates of change.

Example

17

- 4 The cost function for the production of ovens is given by  
$$C(x) = 0.04x^3 - 10x^2 + 1200x + 2000$$
where the cost  $C(x)$  of producing  $x$  ovens is given in dollars.  
**a** Find the cost of producing 60 ovens.  
**b** Find the average cost of producing each oven if 60 ovens are made.  
**c** Use the derivative to find the marginal cost of producing an extra oven at 60 ovens.  
**d** Compare the average and marginal costs at:  
**i** 60 ovens      **ii** 100 ovens      **iii** 130 ovens.

Example

18

- 5 A gas-filled balloon has a small leak. Its volume is given by:  
$$V(t) = 400 - 5t - \frac{t^2}{50}$$
where  $V$  is in  $\text{cm}^3$  and  $t$  is in seconds.  
**a** Find the rate of change of volume after 20 seconds.  
**b** Find the rate of change of volume after 30 seconds.  
**c** When does the volume function cease to be valid?

### Problem solving

- 6 The profit function for the monthly sales of cars by a dealership is  
$$P(x) = 900x - 2x^2 - 100$$
where  $x$  is the number of cars sold.  
**a** Find the instantaneous rate of change of profit when  
**i** 150 cars are sold      **ii** 250 cars are sold  
**b** Explain the meaning of the results obtained in **a**.
- 7 The total revenue function for a particular cake is  
$$R(x) = 3x$$
where the revenue is in dollars and  $x$  is the number of cakes sold.

The total cost function is

$$C(x) = 300 + 2x - 0.2x^2$$

- a** Calculate the instantaneous rate of change of profit if:
- 3 cakes are produced and sold
  - 10 cakes are produced and sold
- b** What can you conclude from the answer to **a** above?
- 8** The total revenue function for the sale of irons is  $R(x) = 40x - 0.1x^2$ , where the revenue is in dollars and  $x$  is the number of irons sold. The total cost function is  $C(x) = 2340 + 5x$ .
- a** Calculate the instantaneous rate of change of profit if:
- 100 irons are produced and sold
  - 200 items are produced and sold
- b** Find the number of irons produced and sold when a profit ceases to be made.
- 9** A block of ice in the shape of a cube is melting uniformly.
- Find the rate of change of surface area with respect to side length,  $x$  cm.
  - At what rate is the block's surface area changing when the side length is 4 cm?
- 10** Refer to the investigation 'Errors and derivatives' on p.432 before this exercise.
- Use derivatives to find the approximate error made in the volume of a cylindrical water tank of radius 4 m and height 3 m if the radius has an error of 0.09 m.
  - What approximate error would be made in the volume of the water tank above if an error of 0.09 m were made in the height instead of the radius?
- 11** An iceberg approximates the shape of a rectangular prism and is about 800 m long, 400 m wide and 120 m thick. As it moves into warmer water, it melts evenly by 1 m per day along each measurement.
- Find the volume of the iceberg after  $t$  days.
  - Find the rate of change of the volume after  $t$  days.
  - How much ice melts on the 13th day?
  - How much melts on the 15th day?
- 12** An antlion makes a hole in the sand in the shape of an inverted cone and sits in the bottom of the hole. The slope of the sides is  $45^\circ$ . When an ant falls into the trap, the antlion gets its reward but has to re-dig the hole.
- Write an expression for the volume of the hole in terms of its depth.
  - What is the rate of change of volume at depth  $d$ ?
  - If the hole is 1.5 cm deep, how much sand must the antlion dig to make it 1.7 cm deep?



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# 10. CHAPTER SUMMARY

## Properties of derivatives

- For a function  $f(x)$ , the **gradient** of the function at any point  $P(x, f(x))$  is  $f'(x)$ .  
If  $f'(x) > 0$ , the function is increasing.  
If  $f'(x) < 0$ , the function is decreasing.  
If  $f'(x) = 0$ , the function is stationary.
- $\frac{d}{dx}(x^n) = nx^{n-1}$  (the **power rule** for derivatives, where  $n = 0, 1, 2, \dots$ ).
- $\frac{d}{dx}(c) = 0$ , where  $c$  is a constant.
- $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- $\frac{d}{dx}[kf(x)] = kf'(x)$
- $\frac{d}{dx}(kx^n) = knx^{n-1}$  (the **linear product rule** for derivatives)
- $\frac{d}{dx}[af(x) + bg(x)] = af'(x) + bg'(x)$ , where  $a$  and  $b$  are constants (the **linear combination rule** for derivatives)
- $\frac{d}{dx}[f(x) \times g(x)] \neq f'(x) \times g'(x)$
- A polynomial function is:  
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x^1 + a_0$$
, where  $n = 0, 1, 2, 3, \dots$
- The derivative of a function is the gradient of the tangent to the function. For  $f(x)$ , the gradient of the tangent at  $x = a$  is  $f'(a)$ .
- $f'(x)$  is the **instantaneous rate of change** of  $f(x)$ .
- If  $C(x)$  = the cost function,  $R(x)$  = the revenue function, and  $P(x)$  = the profit function

$$\text{Marginal cost} = \lim_{x \rightarrow 0} \frac{\Delta C(x)}{\Delta x} = C'(x)$$

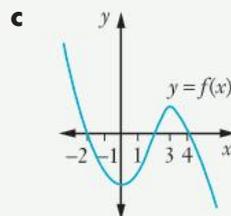
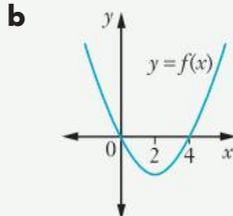
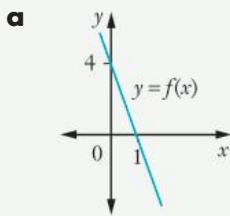
$$\text{Marginal revenue} = \lim_{x \rightarrow 0} \frac{\Delta R(x)}{\Delta x} = R'(x)$$

$$\text{Marginal profit} = \lim_{x \rightarrow 0} \frac{\Delta P(x)}{\Delta x} = P'(x)$$

# 10. CHAPTER REVIEW

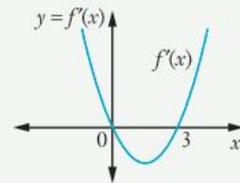
## Properties of derivatives

1 Sketch a possible graph of the derivative of each function.



Example  
**1**

2 The diagram on the right shows the graph of  $f'(x)$  for a particular function  $f(x)$ . Use this graph to draw a possible graph of  $f(x)$ .



Example  
**2**

3 Sketch the graph of the gradient function of  $f(x) = x^2 + 6x + 12$  and use the graph to state where  $f(x)$  is increasing and decreasing.

Example  
**3**

4 Given that  $f(x) = 2x^2 - x - 21$ , find the values of  $x$  for which  $f(x)$  is decreasing.

Example  
**3**

5 Expand each expression.

**a**  $(x + y)^7$

**b**  $(2a + b)^4$

Example  
**4**

6 Find the derivative of each function using first principles.

**a**  $f(x) = x^6$

**b**  $f(x) = 5x^4$

Examples  
**4-6**

7 Use the power rule to find the derivative of:

**a**  $f(x) = x^{10}$

**b**  $y = x^{14}$

Example  
**5**

8 If  $g(x) = x^3$  and  $h(x) = 5x^2$ , find each of the following using first principles.

**a**  $\frac{d}{dx}[g(x) + h(x)]$

**b**  $\frac{d}{dx}[h(x) - g(x)]$

Examples  
**7,8**

9 Use the linear combination rule to find the derivative of:

**a**  $5x^9 - 4x^6$

**b**  $3x^7 + 7x^3$

Example  
**9**

10 If  $p(x) = x^4 - 3x^2 + 5x - 10$ , find:

**a**  $p'(x)$

**b**  $p'(3)$

**c**  $p'(0)$

Example  
**10**

Example  
10

11 Find the derivative of each polynomial.

- a**  $5x^6 - 3x^2 + x + 10$       **b**  $4 - u - 3u^2 + 6u^3$       **c**  $q^7 + 4q^3 - q + 1$   
**d**  $15y^6 - y^4 - 28y$       **e**  $12 + w - 3w^2 + w^3$

Example  
11

12 Find the derivative of  $g(x) = (3x^3 - 5x^2 + 2x)(4x^2 + x^3)$ .

13 If  $h(x) = 4x^5 - 7x^2$  and  $g(x) = 2 + 6x^2 - x^4$ , show that:

$$\frac{d}{dx}[h(x) \times g(x)] \neq \frac{d}{dx}h(x) \times \frac{d}{dx}g(x)$$

Example  
12

Example  
13

14 Find any values of  $x$  for which the slope of the tangent to  $y = \frac{8}{3}x^3 + 5x^2 + 2x - 7$  is equal to 5.

Example  
14

15  $y = x^3 - 4x^2 + 3x + 8$ . Find the gradient of the tangent at:

- a**  $x = 2$       **b**  $x = -3$       **c**  $x = 1$       **d**  $x = 0$       **e**  $x = -5$

Example  
14

16  $P(v) = 5v^2 - 3v + 9$ . Find the gradient of the tangent at:

- a**  $v = -1$       **b**  $v = 3$       **c**  $v = 0$       **d**  $v = -3$       **e**  $v = 10$

Example  
14

17  $y = 3x^4 - 4x^3 + x - 3$ . Find the equation of the tangent at:

- a**  $x = 1$       **b**  $x = -1$       **c**  $x = 0$       **d**  $x = 2$       **e**  $x = -2$

Example  
15

18 Given  $f(x) = x^3 + 3x^2 - 6x - 8$ , find:

- a** the angle that the tangent at  $A$ , where  $x = -2$ , makes with the  $x$ -axis  
**b** the equation of the tangent at  $x = -2$   
**c** the point  $B$ , where the tangent at  $x = -2$  intersects with  $f(x)$   
**d** the length of  $AB$

Example  
16

19 Find the rate of change of  $f(x) = 6x^4 - 5x^3 + 8x$  at:

- a**  $x = 1$       **b**  $x = -2$       **c**  $x = -4$       **d**  $x = 5$       **e**  $x = 0$

Example  
16

20 Find the instantaneous rate of change of  $y = 3t^2 + 5t + 7$  at:

- a**  $t = 3$       **b**  $t = -5$       **c**  $t = -1$       **d**  $t = 0$       **e**  $t = 8$

Example  
17

21 A manufacturer produces items where the cost of producing  $n$  items is given by the function

$$C(n) = 0.0004n^2 + 0.27n + 5, \text{ where } C(n) \text{ is in dollars.}$$

- a** Find a function to work out the approximate marginal cost of producing the  $n$ th item.  
**b** Find the marginal cost of producing the 500th item.  
**c** Find the marginal cost of producing the 1000th item.  
**d** Find the marginal cost of producing the 1500th item.  
**e** If the items are sold for \$1.53 each, at what point does the cost of production outstrip the revenue from sales?

- 22** The volume of air in a balloon is given by

$$V(t) = t^3 - 6t^2 + 35$$

where  $V$  is in  $\text{cm}^3$  and  $t$  is in minutes.

- Calculate the rate of change of the volume when  $t = 4$ .
- What happens to the volume of the balloon when  $t > 4$ ?



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### Problem solving

- 23** The volume of a spherical cell is  $V = \frac{4}{3}\pi r^3$ , where the radius  $r$  is measured in micrometres ( $1 \mu\text{m} = 10^{-6} \text{m}$ ). Find the rate of change of the volume of the cell when the radius is  $5 \mu\text{m}$ .

- 24** The electrical resistance of a metal wire is given by

$$R(T) \approx R_0(1 + 0.004T + 0.000\,001T^2)$$

where  $R_0$  is the resistance at  $0^\circ\text{C}$  and  $T$  is the temperature in degrees Celsius.

A particular platinum wire has a resistance of  $12 \Omega$  (ohms) at  $0^\circ\text{C}$ .

- What is its resistance at  $400^\circ\text{C}$ ?
  - What is the rate of change of resistance when the temperature is at  $400^\circ\text{C}$ ?
  - What is the resistance at  $500^\circ\text{C}$ ?
  - What is the average rate of change of resistance from  $400^\circ\text{C}$  to  $500^\circ\text{C}$ ?
  - What is the resistance at  $-120^\circ\text{C}$ ?
  - What is the rate of change of resistance at  $-120^\circ\text{C}$ ?
  - Suggest a use for such a wire. Note that platinum has a melting point of  $1770^\circ\text{C}$ .
- 25** Use derivatives to find the approximate error in volume if a 2% error in radius occurred when cylinders of a four-cylinder car engine were bored. The correct ratio of radius to length was 1 : 1 and the engine capacity was supposed to be 2000 cc. What would be the capacity if the cylinders were over-bored?



Practice quiz

# 11.

## DISCRETE RANDOM VARIABLES

The Bureau of Meteorology's forecast for Townsville for one day appears next page.

The probability of any rain that day was 70%. The amount of rain that falls is a **continuous variable**, so, for example, the probability of getting exactly 3.48 mm of rain is effectively zero. This applies to any other particular amount. With a continuous variable, you can only talk about the probability of a *range* of values. For the Townsville forecast, if it rains, then the probability of getting at least 2 mm of rain is 50% and the probability of getting at least 10 mm is 25%.

On the other hand, the number of points scored by the North Queensland Cowboys in a game against the Brisbane Broncos is a **discrete variable** as it can only be whole numbers. You can estimate the probability of the discrete value of, say, 12 points.

- 11.01 Discrete random variables
- 11.02 Uniform and non-uniform discrete random variables
- 11.03 Expected value of a discrete random variable
- 11.04 Variance and standard deviation of a discrete random variable
- 11.05 Applications of discrete random variables
- Chapter summary
- Chapter review

## TOWNSVILLE WEATHER FORECAST



Min **27** Max **33**

**Shower or two.**

Possible rainfall: **2 to 10 mm**

Chance of any rain: **70%** ■■■■■■□□

Partly cloudy. High (70%) chance of showers.  
Light winds becoming northeasterly 15 to 20 km/h during the day.

Sun protection recommended from 8:10 am to 4:00 pm, UV index predicted to reach 15 [Extreme]

Source: From bom.gov.au, 10-Feb-2017

## SYLLABUS SUBJECT MATTER

### General discrete random variables

- understand the concepts of a discrete random variable and its associated probability function, and its use in modelling data
- use relative frequencies obtained from data to determine point estimates of probabilities associated with a discrete random variable
- recognise uniform discrete random variables and use them to model random phenomena with equally likely outcomes
- examine simple examples of non-uniform discrete random variables
- recognise the mean or expected value of a discrete random variable as a measurement of centre, and evaluate it in simple cases
- recognise the variance and standard deviation of a discrete random variable as measures of spread, and evaluate these in simple cases
- use discrete random variables and associated probabilities to solve practical problems



# TERMINOLOGY

at random  
 discrete variable  
 geometric probability distribution  
 point estimate  
 standard deviation

continuous variable  
 expected value  
 house percentage  
 probability function  
 uniform

discrete probability distribution  
 fair game  
 odds  
 random variable  
 variance

## 11.01 Discrete random variables

A **random variable** has a numerical value determined by the outcome of a chance experiment. It is **discrete** or **continuous**. A random variable is usually shown by a capital letter and specific values by the lower-case letter.

A **discrete probability distribution** shows probabilities of a discrete random variable. It is also called a **probability function**. You write the values of a probability function as  $P(X = x)$  or  $p(x)$ . The values are all non-negative and the sum of the values,  $\sum p(x) = 1$ .

### EXAMPLE 1

Determine whether each function could be a discrete probability distribution.

<b>a</b>	$x$	-2	0	2	4	6	8
	$f(x)$	0.3	0.1	0.1	0.1	0.2	0.3

**b**  $f(x) = \frac{x}{15}, x = 1, 2, 3, 4, 5$

**c**  $f(x) = \frac{x^2}{60}, x = -4, -3, -2, -1, 0, 1, 2, 3, 4$

### Solution

**a** Check the values. All the values are positive, either 0.1, 0.2 or 0.3.

Find the sum.

$$\begin{aligned} \sum f(x) &= 0.3 + 0.2 + 0.1 + 0.1 + 0.2 + 0.3 \\ &= 1.2 \end{aligned}$$

A discrete function with negative  $f(x)$  values or with a sum of values not equal to 1 cannot be a probability function.

Write the answer.  $f(x)$  is not a probability function because the sum is not 1.

**b** Check the values.

$x$	1	2	3	4	5
$f(x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

Find the sum.

$$\sum f(x) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = \frac{15}{15} = 1$$

Write the answer.

$f(x)$  could be a probability function because all the values are non-negative and the sum is 1.

**c** Check the values.

$x$	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	$\frac{16}{60}$	$\frac{9}{60}$	$\frac{4}{60}$	$\frac{1}{60}$	$\frac{0}{60}$	$\frac{1}{60}$	$\frac{4}{60}$	$\frac{9}{60}$	$\frac{16}{60}$

Find the sum.

$$\sum f(x) = \frac{16+9+4+1+0+1+4+9+16}{60} = \frac{60}{60} = 1$$

Write the answer.

$f(x)$  could be a probability function because all the values are non-negative and the sum is 1.

You can use the frequency of occurrence of different eye colours to estimate the probability of a person having blue eyes. This is called a **point estimate** because the probability is estimated for a single possibility.

## EXAMPLE 2

Estimate the probability of a Year 11 student having 4 siblings from the results for a Year 11 class shown below.

Number of siblings	0	1	2	3	4	5
Number of students	3	6	6	3	5	1

### Solution

Find the total frequency.

$$\sum f = 24$$

Estimate the desired probability.

$$P(4) = \frac{5}{24} = 0.2083\dots$$

Write the answer.

The probability of a Year 11 student having 4 siblings is about 0.2.

## INVESTIGATION

### YEAR 11 PROBABILITIES

Your teacher may want you to estimate probabilities of Year 11 traits such as number of siblings, mode of transport to school.

Consider how the data could be collected.

How accurate would the information collected be?

How typical of Year 11 students across Australia would your estimates be?

### Exercise 11.01 Discrete random variables

1 Classify each variable as discrete or continuous.

- a The mass of a bus.
- b The number of passengers in a bus.
- c The number of bus stops on a route.
- d The distance between bus stops.
- e The price of a washing machine.
- f The capacity of a washing machine.
- g The energy star rating of a washing machine.
- h The volume of water used by a washing machine.

Example  
1

2 State whether each function could be a probability distribution, giving reasons.

a

$x$	1	2	3
$f(x)$	0.3	0.4	0.2

b

$x$	-3	-1	1	3
$f(x)$	0.1	0.2	0.3	0.4

c

$x$	0	2	6	8
$f(x)$	0.4	-0.1	0.2	0.5

3 a State whether  $p_2(x) = \frac{5x^3}{12} - \frac{x^4}{24} - \frac{3}{4} - \frac{35x^2}{24} + \frac{25x}{12}$  for  $x = 1, 2, 3, 4$  could be a probability function, giving reasons.

b Comment on the nature of the function.

Example  
2

4 The numbers of As, Bs, Cs, Ds and Es awarded in a Maths Methods class at the end of semester 1 are shown.

Grade	A	B	C	D	E
Number of students	3	6	10	4	2

- a Estimate the probability that a student received an A.
- b Estimate the probability that a student in the class passed (C or better).

- 5 From 300 people seen at a fracture clinic, 57 were diagnosed with sprains, 102 with broken arms, 47 with fractures of leg bones, 48 with other fractures and the rest had no serious problem. Estimate the probability that a new patient:
- will have a broken arm
  - will not have a fracture.
- 6 The numbers of offences committed in Queensland in 2015 were as follows.

Type	Against person	Theft	Fraud	Drug	Public order	Other
Number	12 000	17 000	2 000	23 000	20 000	15 000

What is the probability that an offence:

- involves injury to another person?
- involves theft?
- is a public order offence?

### Problem solving

- 7 Construct a discrete probability distribution on the domain  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  such that the values decrease steadily from 0 to 9 and  $p(9) \neq 0$ .

## 11.02 Uniform and non-uniform continuous discrete variables

When you roll a normal die you can get outcomes of  $d = 1, 2, \dots, 6$ . Each has the same probability. As there are 6 possible values, all the probabilities are  $\frac{1}{6}$ . This is a natural example of a **uniform discrete random variable**  $D$ .

The probability function for a **uniform discrete random variable**  $X$  is a constant.

If  $X$  has  $n$  values, then  $P(X = x) = \frac{1}{n}$ .

If you assign values of 0 and 1 to the outcomes heads and tails, then flipping a coin produces a uniform discrete random variable. The probability function is  $p(x) = \frac{1}{2}$  for  $x = 0, 1$ .

### EXAMPLE 3

The variable  $N$  is equal to the number chosen at random from the numbers 1, 2, ..., 20.

- Find the probability function.
- State whether it is uniform.
- Find the probability that  $n$  is over 12.

## Solution

- a** Find the probability of each number. At random means equally likely, so  $p(x) = \frac{1}{20}$  for  $x = 1, 2, \dots, 20$ .
- b** State the answer. All the values are the same, so  $N$  is a uniform discrete random variable.
- c** Find the probability. There are 8 outcomes over 12, so  $P(n > 12) = \frac{8}{20} = 0.2$ .

In the card game Rummy, aces have a score of 1, number cards score their own number and king, jack and queen score 10 points. The variable  $R$  consisting of the score values is a discrete random variable. For random cards,  $p(4) = \frac{1}{13}$  but  $p(10) = \frac{3}{13}$ , so  $R$  is not uniform.

## EXAMPLE 4

One marble each from a cup containing 16 marbles with the numbers 1 to 16 on them is given at random to each of 7 people. The players receive  $m$  points, given by the remainder when the number is divided by 6.

- a** Find the probability function.
- b** State whether it is uniform.
- c** Find the probability that  $m$  is below 2.

## Solution

- a** Find the probability of each remainder.  $p(0) = \frac{2}{16}$  for 6 and 12  
 $p(1) = \frac{3}{16}$  for 1, 7 and 13  
 $p(2) = \frac{3}{16}$  for 2, 8 and 14  
 $p(3) = \frac{3}{16}$  for 3, 9 and 15  
 $p(4) = \frac{3}{16}$  for 4, 10 and 16  
 $p(5) = \frac{2}{16}$  for 5 and 11
- b** State the answer. Not all values are the same so  $M$  is not a uniform discrete random variable.
- c** Find the probability.  $P(n < 2) = \frac{2}{16} + \frac{3}{16} = \frac{5}{16}$

Sometimes you are interested in the probability of repeating an experiment, say,  $n$  times before a particular outcome occurs. For example, you might want to know the probability of rolling a normal die 10 times before you get a six. This is the same as the probability of failing to get a six 9 times in a row, and then getting a six. Because the individual events are independent, you can simply multiply the probabilities, so it is given by

$$P(\text{10 rolls to get a six}) = \left(\frac{5}{6}\right)^9 \times \frac{1}{6} = \frac{5^9}{6^{10}} \approx 0.0323.$$

This is a discrete probability situation, but it is not finite. There is always a probability of getting any number of rolls, even though it might be small. This is an example of an **infinite discrete random probability**.

A **geometric probability distribution** of a discrete random variable  $X$  is the distribution that arises as the probabilities of  $x$  failures before a success occurs in an experiment, with fixed independent probabilities of failure and success.

The probability distribution is given by  $P(X = x) = pq^x$ , where  $x$  is a positive integer.

Some mathematicians call the geometric probability distribution to mean the number of repetitions  $x$  to attain the first success (that is, including the successful trial). In this case,  $P(X = x) = pq^{x-1}$ . However, in this book, it means the number of failures before a success.

### EXAMPLE 5

- a State the probability function for the random variable  $N$  from tossing a normal coin  $n$  times in a row before the first head appears.
- b Use the distribution to find the probability that a coin is tossed 9 times before a head appears.

#### Solution

- a State the probabilities of success and failure.

$$p = q = \frac{1}{2}$$

State the probability function.

$$pP(n) = \left(\frac{1}{2}\right)^n \times \frac{1}{2} = \frac{1}{2^{n+1}}$$

- b Find the probability.

$$\begin{aligned} P(9) &= \frac{1}{2^{9+1}} \\ &= \frac{1}{1024} \end{aligned}$$

## Exercise 11.02 Uniform and non-uniform continuous random variables

Example  
3

- 1 A number  $p$  is chosen at random from the prime numbers from 1–20.
- Find the probability function.
  - State whether it is uniform.
  - Find the probability that  $p$  is below 6.
  - Find the probability that  $p$  is between 14 and 16 inclusive.

Example  
4

- 2 A number is chosen at random from 1 to 20 and divided by 5. The number  $n$  is obtained by adding the remainder to the quotient.
- Find the probability function of  $N$ .
  - State whether it is uniform.
  - Find the probability that  $n$  is less than 6.
  - Find the probability that  $n$  is between 3 and 6 inclusive.

Example  
5

- 3 A marble is drawn at random from a bag containing 8 green and 4 red marbles. The marble is replaced before another marble is drawn.
- State the distribution of  $N$ , the random variable for the number of marbles drawn before you get the first red one.
  - Find  $P(5)$ .
  - What is  $P(8)$ ?
- 4  $m$  is the remainder when a number from 1 to 24 is divided by 8.
- Find the probability function of  $M$ .
  - State whether it is uniform.
  - Find the probability that  $m$  is below 8.
  - Find the probability that  $m$  is between 4 and 6 inclusive.
- 5  $n$  is the remainder when a number from 1 to 24 is divided by 9.
- Find the probability function of  $N$ .
  - State whether it is uniform.
  - Find the probability that  $n$  is less than 8.
  - Find the probability that  $n$  is between 4 and 6 inclusive.
- 6 A card is dealt at random from a normal pack of cards. The card is replaced and the pack shuffled before another card is dealt.
- State the distribution of  $H$ , the random variable for the number of cards drawn before you get the first heart.
  - Find  $P(4)$ .
  - What is  $P(7)$ ?

## Problem solving

- 7** A player rolls a die. If a prime number occurs, the player wins that number of dollars; but if a non-prime number occurs, the player loses that number of dollars. If  $X$  is the number of dollars the player stands to win,
- construct the probability distribution for  $X$
  - draw a graph of the distribution.
- 8** A game consists of rolling 3 dice. If all 3 dice show the same number, you win \$20. If 2 numbers are the same, you win \$5. If all 3 are different, you lose \$5.  $X$  is the random variable representing the amount you win.
- Find the probability distribution of  $X$ .
  - What is the probability of winning?
- 9** The individual prices of lollies in a pick-and-pay display were as follows. All prices are in cents.



Alamy Stock Photo/Dragon Zivkovic

5   10   5   10   20   5   5   10   20   20   15   5   5   10   10  
 20   15   50   50   20   25   25   15   5   10   10   5   10   5   20

- Construct the probability distribution function for the prices  $P$  of lollies chosen at random.
  - What is the probability of getting a lolly worth less than 20 cents?
- 10** How many times must you roll a die for the probability of getting at least one 6 to exceed 0.9?



## 11.03 Expected value of a discrete random variable

You can construct a random variable  $X$  for the age of a student chosen at random from a group of 5 Year 11 students of ages 16, 17, 17, 18 and 18.

The probability distribution of  $X$  is shown below.

$x$	16	17	18
$P(X = x)$	0.2	0.4	0.4

You can work out the mean of their ages as  $\frac{16 + 17 + 17 + 18 + 18}{5} = 17.2$ .

You *expect* to get the average if you randomly choose a student from the group many times.

$$\begin{aligned} \text{Notice that } 17.2 &= \frac{16 \times 1 + 17 \times 2 + 18 \times 2}{5} \\ &= 16 \times \frac{1}{5} + 17 \times \frac{2}{5} + 18 \times \frac{2}{5} \\ &= \sum x \cdot p(x) \end{aligned}$$

Now think of the general case with  $k$  different scores  $x_i$  of frequencies  $f_i$  where the total frequency is  $\sum f_i = n$ . If you construct a discrete variable  $X$  by choosing at random, you get this probability distribution.

$x$	$x_1$	$x_2$	$x_3$	...	$x_i$	...	$x_k$
$P(X = x)$	$\frac{f_1}{n}$	$\frac{f_2}{n}$	$\frac{f_3}{n}$	...	$\frac{f_i}{n}$	...	$\frac{f_k}{n}$

The mean of the data is given by  $\sum \frac{fx}{n} = \sum x \cdot \frac{f}{n} = \sum x \cdot f(x)$ .

This rule is used for any discrete random variable, even if it doesn't come from data.

The **expected value** of a discrete random variable  $X$  is defined by

$$\mu = E(X) = \sum x \cdot p(x)$$

### EXAMPLE 6

Find the expected value of a discrete random variable with values 1–8 and probabilities  $\frac{1}{8}$ .

#### Solution

Use the formula.

$$\begin{aligned}\mu = E(x) &= \sum x \cdot p(x) \\ &= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + \dots + 8 \times \frac{1}{8} \\ &= 4.5\end{aligned}$$

State the answer.

The expected value is 4.5.

Calculating the expected value for a uniform distribution is a little easier than for a non-uniform distribution.

### EXAMPLE 7

A ‘Lucky’ lottery has first, second and third prizes of \$100 000, \$10 000 and \$5000, 2 of \$1000, 2 of \$500, 10 of \$200, 20 of \$100, 100 of \$50, 600 of \$25, 570 of \$15 and 2480 of \$10. There are 270 000 tickets costing \$2.20 each. What is the expected value for the prizes and the expected value for a buying a ticket?

#### Solution

Use the formula to find the expected value of the prizes.

$$\begin{aligned}E(x) &= \sum x \cdot p(x) \\ &= \$100\,000 \times \frac{1}{270\,000} + \$10\,000 \times \frac{1}{270\,000} + \dots \\ &\quad + \$10 \times \frac{2480}{270\,000} \\ &= \frac{1}{270\,000} \times (1 \times \$100\,000 + 1 \times \$10\,000 + \dots + 2480 \times \$10) \\ &= \frac{\$175\,350}{270\,000} \\ &= \$0.6494\dots\end{aligned}$$

Subtract the ticket cost. Expected value of ticket  $\approx$  \$0.65 – \$2.20

The Lucky lottery is not so lucky after all! Since lotteries are set up to make money for the promoters, this is not really surprising. You should always expect to lose.  $\rightarrow = -\$1.55$

State the answer.

The expected prize value is about 65 cents and the expected value of buying a ticket is about  $-\$1.55$ .

Notice in Example 7 that it is easier to add up the total value of the prizes to calculate the expected value than to directly follow the formula.

## TECHNOLOGY

### Expected value

You can use your graphics calculator to find the expected value of a random normal variable. The following instructions show how to do this for the lottery prize in Example 7.

#### TI-84 Plus CE

Press **[stat]** and **[enter]** and choose **1: edit**.

Move up to  $L_1$  and press **[clear]** and if you need to remove old data. Do the same for  $L_2$  if needed.

Put the values of 100 000, 10 000, ..., 10 in  $L_1$ .

Put the probabilities as  $1 \div 270\,000$ , ...,  $2480 \div 270\,000$  in  $L_2$ .

Press **quit** (**[2nd]** **[mode]**) and **list** (**[2nd]** **[stat]**) and move across to **MATH** and choose **5: sum**(.

Put in  $L_1 \times L_2$ ) to get the expected value.

Use **[2nd]** **1** for  $L_1$ .

$L_1$	$L_2$	$L_3$	$L_4$	$L_5$	$L_6$
100000	3.7E-6				
10000	3.7E-6				
5000	3.7E-6				
1000	7.4E-6				
500	7.4E-6				
200	3.7E-5				
100	7.4E-5				
50					
25					
15					
10					

$L_1(0)=100/270000$

Normal	Float	Auto	Real	Radian	MP
Sum(L1xL2)					
0.6494444444					

#### Casio fx-CG20AU

Choose **2 Statistics** and if you need to remove old data use **[F6]** **[F4]** and **[F1]** (**DEL-ALL**).

Do the same for List 2 if needed.

Put the values of 100 000, 10 000, ..., 10 in List 1.

Put the probabilities as  $1 \div 270\,000$ , ...,  $2480 \div 270\,000$  in List 2.

Press **[MENU]** and choose **1: Run-Matrix**.

Press **[OPTN]** **[F6]** **[F6]** and **[F1]** (**Sum**).

Put in (List 1  $\times$  List 2) to get the expected value.

Use **[OPTN]** **[F1]** **1** to get List 1.

	List 1	List 2	List 3	List 4
SUB				
5	500	7.4E-6		
8	200	3.7E-5		
7	100	7.4E-5		
8	50			

$100 \div 270000$

Math	Stat/Normal	Calc/Inv
Sum (List 1 x List 2)		
0.6494444444		



TI-Nspire CX  
Chapter 11

### Exercise 11.03 Expected value of a discrete random variable

Use your graphics calculator as directed by your teacher for this exercise.

- Find the expected value for choosing a number at random from 5 to 25.
- Find the expected value for the discrete random variable with this distribution.

Example  
6

$X$	2	3	5	7	11	13	17	19
$P(X = x)$	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

- Find the expected value of each distribution.

Example  
7

**a**

$x$	0	5	10	15
$P(X = x)$	0.5	0.1	0.2	0.2

**b**

$x$	2	3	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

**c**

$x$	-5	-4	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

**d**

$x$	1	3	4	5
$p(x)$	0.4	0.1	0.2	0.3

- The results of a Physics test out of 40 are shown below.

22    36    12    16    35    30    25    28    26    20  
18    15    32    27    21    13    8    30    17    27

Find the expected value of the variable  $S$  for the test results.

- Results in a History test, out of 100, are shown below.

32    45    68    90    45    63    55    17    50    60  
36    67    70    85    55    65    67    57    43    80

- Find the expected value of the variable  $F$  for the test results.
- The test results are rescaled to a mark  $M$  out of 30. Find the expected value of the variable  $M$ .

#### Problem solving

- What is the expected value from rolling a normal 6-sided die?
- Two normal dice are thrown.  $S$  is the equal or smaller number of the numbers on the dice. Find the expected value of  $S$ .
- Another 'Lucky' lottery has first, second and third prizes of \$200 000, \$20 000 and \$5000. There are 7 prizes of \$1000, 10 of \$500, 25 of \$100, 75 of \$75, 600 of \$40, 700 of \$20 and 2800 of \$12. Tickets cost \$5.50 and there are 200 000 tickets. Find your expected value from buying a ticket.



## 11.04 Variance and standard deviation of a discrete random variable

**Standard deviation** is a measure of the spread of statistical data and is given by

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$$

For the ages of a group of five Year 11 students of ages 16, 17, 17, 18 and 18 and the random variable  $X$  you get by choosing a student at random from the group, the mean,  $\bar{x} = \mu = E(X) = 17.2$ .

$$\begin{aligned} \text{Now } \frac{\sum (x - \bar{x})^2}{n} &= \frac{(16 - 17.2)^2 + (17 - 17.2)^2 + (17 - 17.2)^2 + (18 - 17.2)^2 + (18 - 17.2)^2}{5} \\ &= \frac{1.2^2 + 0.2^2 + 0.2^2 + 0.8^2 + 0.8^2}{5} \\ &= \frac{1.44 + 0.04 + 0.04 + 0.16 + 0.16}{5} \\ &= \frac{1.44 \times 1 + 0.04 \times 2 + 0.16 \times 2}{5} \\ &= (16 - 17.2)^2 \times \frac{1}{5} + (17 - 17.2)^2 \times \frac{2}{5} + (18 - 17.2)^2 \times \frac{2}{5} \\ &= \sum (x - \bar{x})^2 \cdot p(x) \end{aligned}$$

You can see that for the general case with scores  $x$  and frequencies  $f$ :

$$\begin{aligned} \frac{\sum (x - \bar{x})^2 f}{n} &= \sum (x - \bar{x})^2 \cdot \frac{f}{n} \\ &= \sum (x - \bar{x})^2 \cdot p(x) \end{aligned}$$

This is applied to any discrete random variable as follows.

The **variance** of a discrete random variable  $X$  is defined as

$$\text{Var}(X) = \sum (x - \mu)^2 \cdot p(x) = \sum x^2 \cdot p(x) - \mu^2$$

where  $\mu = E(X)$  is the expected value of  $X$ .

The **standard deviation** of a discrete random variable  $X$  is defined as

$$\sigma_X = SD(X) = \sqrt{\text{Var}(X)}$$

You can show that the two formulas for variance are the same as follows.

$$\begin{aligned}
 \sum (x - \mu)^2 \cdot p(x) &= \sum (x^2 - 2x\mu + \mu^2) \cdot p(x) \\
 &= \sum x^2 \cdot p(x) - 2\sum x\mu \cdot p(x) + \sum \mu^2 \cdot p(x) \\
 &= \sum x^2 \cdot p(x) - 2\mu \sum x \cdot p(x) + \mu^2 \sum p(x), \text{ taking out the constant} \\
 &\quad \text{factors} \\
 &= \sum x^2 \cdot p(x) - 2\mu \times \mu + \mu^2 \times 1, \text{ using } \mu = \sum x \cdot p(x) \text{ and } \sum p(x) = 1 \\
 &= \sum x^2 \cdot p(x) - \mu^2 \quad \text{QED}
 \end{aligned}$$

### EXAMPLE 8

Find the variance and standard deviation of a discrete random variable with values 1 to 8 and probabilities  $\frac{1}{8}$ .

#### Solution

Find the expected value.

$$\begin{aligned}
 \mu &= \sum x \cdot p(x) \\
 &= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + \dots + 8 \times \frac{1}{8} \\
 &= 4.5
 \end{aligned}$$

Use the formula for variance.

$$\begin{aligned}
 \text{Var}(X) &= \sum x^2 \cdot p(x) - \mu^2 \\
 &= 1^2 \times \frac{1}{8} + 2^2 \times \frac{1}{8} + \dots + 8^2 \times \frac{1}{8} - 4.5^2 \\
 &= \frac{1}{8} \times (1^2 + 2^2 + \dots + 8^2) - 20.25 \\
 &= 25.5 - 20.25 \\
 &= 5.25
 \end{aligned}$$

Find the standard deviation.

$$SD(X) = \sqrt{5.25} = 2.291\dots$$

State the answer.

The variance is 5.25 and the standard deviation is about 2.29.

You can create a formula for the variance of a uniform discrete variable, but it is probably not worthwhile.



Variance and standard deviation of a discrete random variable

### EXAMPLE 9

Find the variance and standard deviation of the random variable  $M$  with the probability distribution shown.

$x$	2	3	5	7	11
$p(x)$	0.1	0.2	0.3	0.2	0.2

### Solution

Find the expected value.

$$\begin{aligned}\mu &= \sum x \cdot p(x) \\ &= 2 \times 0.1 + 3 \times 0.2 + \dots + 11 \times 0.2 \\ &= 5.9\end{aligned}$$

Use the formula for variance.

$$\begin{aligned}\text{Var}(X) &= \sum x^2 \cdot p(x) - \mu^2 \\ &= 4 \times 0.1 + 9 \times 0.2 + \dots + 121 \times 0.2 - 5.9^2 \\ &= 43.7 - 34.81 \\ &= 8.89\end{aligned}$$

Find the standard deviation.

$$SD(X) = \sqrt{8.89} = 2.981\dots$$

State the answer.

The variance is 5.9 and the standard deviation is about 2.98.



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## TECHNOLOGY

### Variance and standard deviation

You can use your graphics calculator to find the variance and standard deviation of a random normal variable. The following instructions show how to do this for the random variable in Example 9.

#### TI-84 Plus CE

Put the values into  $L_1$  and  $L_2$  as shown on page 452.

**Quit**, find the expected value as shown on page 452 and store it in **E**.

Use  $\text{sum}(L_1^2 \times L_2) - E^2$  to find the variance.

Find the square root of the answer to get the standard deviation.

Store the values as shown if you need them for other calculations.

L1	L2	L3	L4	L5	2
2	0.1				
3	0.2				
4	0.3				
5	0.2				
11	0.2				

Normal Float Auto Real Radian HP
Sum(L1 * L2) → E
5.9
Sum(L1^2 * L2) - E^2 → V
8.89
√Ans → S
2.981610303

#### Casio fx-CG20AU

Put the values into List 1 and List 2 as shown on page 453.

Press **MENU** and choose **1: Run-Matrix**.

Find the expected value as shown on page 453 and store in **E**.

Use  $\text{sum}((\text{List } 1)^2 \times \text{List } 2) - E^2$  to find the variance.

You need the brackets around List 1.

Find the square root of the answer to get the standard deviation.

Press **F-D** to get a decimal answer.

Store the values as shown if you need them for other calculations.

Note that the rest of the first two calculations display is shown at the bottom.

Std Norm	d/c Run		
List 1	List 2	List 3	List 4
SUB			
3	5	0.3	
4	7	0.2	
5	11	0.2	

Math Std Norm	d/c Run
Sum (List 1 * List 2) → E	5.9
Sum ((List 1)^2 * List 2) - E^2	8.89
√ → S	2.981610303
JUMP DELET MAT MATH	

Math Std Norm	d/c Run
in (List 1 * List 2) → E	5.9
st 1)^2 * List 2) - E^2 → V	8.89



TI-Nspire CX  
Chapter 11

## Exercise 11.04 Variance and standard deviation of a discrete random variable

Example  
8

- 1 Find the variance and standard deviation of a discrete random variable with values 1 to 10 and probabilities  $\frac{1}{10}$ .
- 2 Find the variance and standard deviation of a discrete random variable with values 6 to 15 and probabilities  $\frac{1}{10}$ .
- 3 Find the variance and standard deviation of a discrete random variable with values 3, 6, 9, 11, 13, 15, 17, 19 and probabilities  $\frac{1}{8}$ .

Example  
9

- 4 Find the variance and standard deviation of the random variable with each probability distribution.

**a**

$x$	0	5	10	15
$P(X=x)$	0.5	0.1	0.2	0.2

**b**

$x$	2	3	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

**c**

$x$	-5	-4	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

**d**

$x$	1	3	4	5
$p(x)$	0.4	0.1	0.2	0.3

### Problem solving

- 5 What are the variance and standard deviation from rolling a normal 6-sided die?
- 6 Two normal dice are thrown.  $T$  is the equal or smaller number of the numbers on the dice. Find the variance and standard deviation of  $T$ .



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Discrete probability assignment



Discrete probability problems

## 11.05 Applications of discrete random variables

You have calculated the expected value of a few lotteries and games with entry fees. Commercial gambling games are set up so the expected value for players is negative. The average amount that players win is less than the entry fee. You might think this is unfair, particularly given that anyone who wins consistently is banned from playing in casinos.

A **fair game** is a game with an expected value of 0.

In the case of games played for money, the expected value includes any entry fee.

### EXAMPLE 10

A dice game is played with two dice. It costs \$2 to play.

If the total is 11 or 12, you are paid \$12 and if the total is 2, 3 or 4, you are paid \$6. You lose on other totals. Is the game fair?

#### Solution

Write the probabilities of each total.

$x$	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Choose a random variable. Let  $X$  be the amount won on a roll of the dice.

Write the probability of each result.

$$p(\$10) = \frac{2}{36} + \frac{1}{36} = \frac{3}{36}$$

$$p(\$4) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$$

$$p(-\$2) = \frac{4}{36} + \frac{5}{36} + \dots + \frac{3}{36} = \frac{27}{36}$$

Find the expected value.

$$\begin{aligned} E(X) &= \$10 \times \frac{3}{36} + \$4 \times \frac{6}{36} + \$-2 \times \frac{27}{36} \\ &= \$ \frac{30 + 24 - 54}{36} \\ &= \$0 \end{aligned}$$

Write your conclusion. Since the expected value is \$0, the game is fair.

In games such as the one above, the amounts won and lost are usually stated as **odds**.

The **odds** for a bet is the ratio of the amount won to the amount bet. If you win, you get back both the amount won and the amount bet. The odds for being paid back \$12 in the example are 5 : 1 because you win \$10 by betting \$2.

By changing the amounts paid for winning and losing, you can run a game so that you win in the long run. Casinos, lotteries, bookmakers and other betting agencies make sure they do this.

The **house percentage** is the proportion of money bet that casinos and other betting agencies win in the long run, expressed as a percentage.

### EXAMPLE 11

A game consists of a wheel divided into 12 equal sectors numbered 1 to 12. You may place a \$2 bet on any number. If your number comes up, you get \$18 back. That is, you win \$16 and get your \$2 back.

- What is the house percentage?
- How much should you win for the game to be fair?

### Solution

- Choose the random variable.

Let  $B$  be the amount won on a bet.

Write the probability distribution.

$b(\$)$	16	-2
$p(b)$	$\frac{1}{12}$	$\frac{11}{12}$

Find the expected value.

$$E(B) = 16 \times \frac{1}{12} + (-2) \times \frac{11}{12} \\ = -0.5$$

Express as a percentage.

$$\text{House percentage} = \frac{\$0.50}{\$2} \times 100\% = 25\%$$

- Choose a variable for the amount won.

Let  $\$a$  be the amount won on a bet.

Recalculate the expected value.

$$E(B) = a \times \frac{1}{12} + (-2) \times \frac{11}{12}$$

Solve  $E(B) = 0$ .

$$a \times \frac{1}{12} + (-2) \times \frac{11}{12} = 0$$

$$\frac{a - 22}{12} = 0$$

$$a = 22$$

Write the answer.

You would have to win \$22 for the game to be fair.

## Exercise 11.05 Applications of discrete random variables

- 1** A game is played with two normal dice. It costs \$5 to enter, and you receive \$18 (that is, win \$13) if the bigger number shown on the dice is 3 or less. Otherwise you lose.
- a** Is the game fair?
  - b** What would you need to win to make the game fair?
- 2** It costs \$3 to play a game with two dice. You throw the dice and win \$6 (that is, receive \$9) if you get a total of 2, 3 or 4. You lose for a total of 5, 6, 7, 8 or 9, otherwise you throw again until you get a total of 2 to 9.
- a** What is the house percentage?
  - b** What would you need to win to make the game fair?

Example  
**10**

Example  
**11**

### Problem solving

- 3** A charity raffle has 500 tickets for \$10 each and three prizes worth \$200, \$100 and \$50. Find the expected value of this raffle and the house percentage.
- 4** Life insurance companies effectively make a bet with their clients that they will live beyond a given age. For term insurance, the bet is that they will live for the next year. This means that if the person insured dies within a year of taking out the policy, the beneficiary (for example, his family) is paid \$1 000 000.

According to the Australian Bureau of Statistics, the probability of a 20-year-old dying in the next year is 0.000 27 for a woman and 0.000 65 for a man.

Term insurance for a 20-year-old non-smoker for \$1 000 000 cover costs about \$55 a month for women and \$75 a month for men.

- a** Calculate the expected value and house percentage for these life insurance policies.
- b** Explain the differences between the cost of insurance for women and men.
- c** Why are the ratios of the costs not the same as the ratios of the probabilities?

- 5** In the game of roulette, a ball travels round a wheel and lands on one of 38 coloured numbers. Half the numbers from 1 to 36 are red and the others from 1 to 36 are black. There is a green 0 and a green 00. You cannot bet on the green numbers.

The odds for some of the possible bets are shown below.

odd, even, red or black 1 : 1                      single numbers 35 : 1

two numbers 17 : 1                                  four numbers 8 : 1

column 2 : 1    row (street) 11 : 1

Calculate the house percentage for each bet and explain your answer. Remember that the house wins if the ball lands on 0 or 00.



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# 11. CHAPTER SUMMARY



## Discrete random variables

- A **random variable** has a numerical value determined by the outcome of a chance experiment. A random variable is usually shown by a capital letter and specific values by the lower-case letter.
- A **discrete probability distribution** shows probabilities of a discrete random variable. It is also called a **probability function**. The probabilities are shown as  $P(X = x)$  or  $p(x)$ .
- $p(X = x) \geq 0$  and  $\sum p(x) = 1$ .
- A **point estimate** is an estimate of the probability of a single outcome (of a discrete variable) occurring.
- The probability function of a **uniform discrete random variable**  $X$  with  $n$  values is given by  $P(X = x) = \frac{1}{n}$ .
- Outcomes chosen '**at random**' from a discrete set are equally likely.
- A **geometric probability distribution** of a discrete random variable  $X$  is the distribution that arises as the probabilities of  $x$  failures before a success occurs in an experiment, with fixed independent probabilities of failure and success.
- The **expected value** of a discrete random variable  $X$  is defined by  $\mu = E(X) = \sum x \cdot p(x)$ .
- The **variance** of a discrete random variable  $X$  is defined as  $Var(X) = \sum (x - \mu)^2 \cdot p(x) = \sum x^2 \cdot p(x) - \mu^2$ , where  $\mu = E(X)$  is the expected value of  $X$ .
- The **standard deviation** of a discrete random variable  $X$  is defined as  $\sigma_X = SD(X) = \sqrt{Var(X)}$ .
- A **fair game** is one with an expected value of 0.
- The **odds** for a bet is the ratio of the amount won to the amount bet.
- The **house percentage** is the proportion of money bet that casinos or other betting agencies win in the long run, expressed as a percentage.

# 11. CHAPTER REVIEW

## Discrete random variables

Example  
1

- 1 Determine whether each function could be a discrete probability distribution, giving reasons.

**a**

$x$	1	2	3	4	5
$p(x)$	0.1	0.3	0.1	0.1	0.3

**b**

$x$	-2	1	3	7	22
$p(x)$	0.2	0.1	0.3	0.25	0.15

**c**

$x$	2	3	4	5	6
$p(x)$	-0.1	0.2	0.3	0.2	0.4

Example  
2

- 2 Estimate the probability of a random student in the school tuckshop line being 15 years old, from the information below.

Age	12	13	14	15	16	17	18	19
Number of students	8	7	7	6	9	5	3	5

Example  
3

- 3 Find the probability function for the random variable  $X$ , a number chosen at random from the odd numbers from 7 to 23 inclusive and comment on the nature of the variable.

Example  
4

- 4 A number is chosen at random between 25 and 45 inclusive and the digits are added together to give  $q$ .

- a** Find the probability distribution of the random variable  $Q$ .  
**b** Find  $P(q < 9)$ .

Example  
5

- 5 A counter is taken at random from a cloth bag containing 4 black and 5 white counters.

- a** State the probability distribution of the probability of drawing  $c$  counters before you get a black one.  
**b** Find  $P(C = 5)$ , correct to 4 decimal places.

Example  
6

- 6 Find the expected value of choosing a number at random from the numbers between 5 and 20 inclusive, that is divisible by 2 or 3 or both.

Example  
7

- 7 Find the expected value of the following distribution.

$x$	1	2	3	4	5	6
$p(x)$	0.1	0.2	0.25	0.2	0.15	0.1

- 8** Find the variance and standard deviation of the value of a coin chosen at random from the Australian coins: 5c, 10c, 20c, 50c, \$1 and \$2.
- 9** Find the variance and standard deviation of the probability distribution shown below.

$x$	1	2	3	5	8	13
$p(x)$	0.05	0.2	0.3	0.2	0.15	0.1

Example  
**8**

Example  
**9**

Example  
**10**

Example  
**11**

- 10** A game is played with one person throwing two normal dice. If the total is even, the person throwing the dice wins and if the total is odd, the other person wins. Is the game fair, and does it make any difference if they take it in turns?
- 11** It costs \$5 to play a game that uses many packs of cards with all except the cards numbered 2 to 6 removed. Two cards are dealt and if the total is 7, 8 or 9 the house wins, otherwise you win. What is the house percentage?

### Problem solving

- 12** A purse contains 4 twenty-cent coins, 2 ten-cent coins and two fifty-cent coins. Three coins are taken out at random without replacement. Construct the probability distribution for the number of twenty-cent coins.
- 13** A card is drawn at random from a normal pack. The card is replaced and the pack is shuffled before another card is drawn. What is the probability of drawing 5 cards before getting a king?
- 14** What is the expected value and variance of the total of choosing one number at random from 3 to 8 inclusive and another number at random from 2 to 6 inclusive?
- 15** A chocolate wheel is divided into 52 equal sections. 24 of the sections have heart symbols, 12 have club symbols, 8 have anchor symbols, 4 have smiley face symbols, 2 have pirate flag symbols and 2 have Australian flag symbols. Players may bet on any symbol before the wheel is spun. After the wheel comes to rest, bets are paid at the odds: hearts 1 : 1, clubs 3 : 1, anchors 5 : 1, smiley faces 11 : 1 and flags 23 : 1. What is the house percentage?



# 12.

## APPLICATIONS OF DERIVATIVES

You have already studied the average rate of change, the derivative of a function and how these are related to the graph of a function.

In this chapter you will see how calculus is used to help sketch the graphs of polynomials and look for optimal solutions to real world problems.

- 12.01 Instantaneous rates of change
- 12.02 Straight-line motion
- 12.03 The product and quotient rules
- 12.04 Stationary points
- 12.05 Sketching polynomials
- 12.06 The chain rule
- 12.07 Greatest and least values on an interval
- 12.08 Optimisation
- Chapter summary
- Chapter review

## SYLLABUS SUBJECT MATTER

### Applications of derivatives

- determine instantaneous rates of change
- determine the gradient of a tangent and the equation of the tangent
- construct and interpret displacement–time graphs, with velocity as the slope of the tangent
- sketch curves associated with polynomials up to and including degree 4; find stationary points, and local and global maxima and minima with and without technology, and algebraically where appropriate and examine behaviour as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$
- identify contexts suitable for modelling optimisation problems involving polynomials up to degree 4 on finite interval domains and use models to solve practical problems with and without technology; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis

### Differentiation rules

- understand and apply the product and quotient rules for power and polynomial functions
- understand the notion of composition of power and polynomial functions and use the chain rule for determining the derivatives of composite functions
- select and apply the product, quotient and chain rule to differentiate power and polynomial functions; express derivative in simplest and factorised form

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Prior learning

## TERMINOLOGY

acceleration  
composite function  
flow rate  
global minimum  
least value  
optimisation  
quotient rule

average rate of change  
displacement  
gradient function  
greatest value  
local maximum  
point of horizontal inflection  
stationary point

chain rule  
dominant term  
global maximum  
instantaneous rate of change  
local minimum  
product rule  
velocity

## 12.01 Instantaneous rates of change

For  $y = f(x)$ , the **average rate of change** in the function from  $x_1$  to  $x_2$  is given by:

$$\text{Average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

and the **instantaneous rate of change** of  $y = f(x)$  is:

$$\text{Instantaneous rate of change} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = y' = f'(x).$$

### EXAMPLE 1

If  $f(x) = 3x^2 - 2x + 1$ , calculate:

- a the average rate of change from  $x = 1$  to  $x = 3$
- b the instantaneous rate of change at  $x = -1$
- c the value of  $x$  at which the rate of change is 0.

### Solution

- a Use the rate of change formula.  $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

$$= \frac{f(3) - f(1)}{3 - 1}$$

Substitute.

$$= \frac{22 - 2}{2}$$

$$= 10$$

State the result.

The average rate of change from  $x = 1$  to  $x = 3$  is 10.

- b Calculate the derivative.

$$f'(x) = 6x - 2$$

Calculate  $f'(x)$  when  $x = -1$ .

$$f'(-1) = 6 \times (-1) - 2 = -8$$

State the result.

The instantaneous rate of change at  $x = -1$  is  $-8$ .

c Let  $f'(x) = 0$ .

$$6x - 2 = 0$$

Solve.

$$x = \frac{1}{3}$$

State the result.

The rate of change is 0 when  $x = \frac{1}{3}$ .

The derivative of a function makes a *new* function called the **gradient function** of the original function. The gradient function of  $f(x) = 4x^3 - 2x + 5$  is  $f'(x) = 12x^2 - 2$ .

You can see that for polynomials the degree of the gradient function must be 1 less than the degree of the original function.

The gradient is zero at a **stationary point**. The gradient function must cross the  $x$ -axis at this point. The gradient is positive when the function is **increasing**, so the gradient function will be above the  $x$ -axis. The gradient is negative when the function is **decreasing**, so the gradient function will be below the  $x$ -axis.

## EXAMPLE 2

The function  $p(x)$  is shown. Sketch the graph of the gradient function.

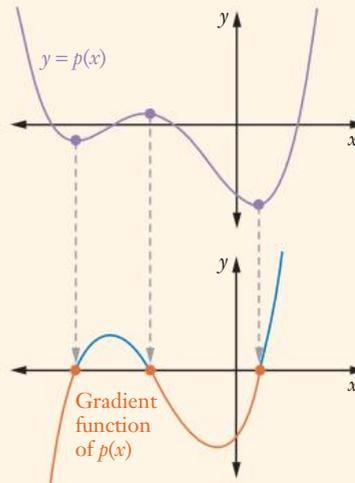
### Solution

Put zeros where the function has a stationary point.

The function is decreasing before the first stationary point, so make the gradient function negative.

Between the first and second stationary points the function is increasing, so make the gradient function positive.

Continue in this way to get the entire gradient function.



### EXAMPLE 3

The population of a group of koalas in a remote location is modelled by:

$$P(t) = \frac{t^3}{3} - 6t^2 + 32t + 40, 0 \leq t \leq 10$$

where  $t$  is in years. Calculate:

- the average rate of population growth for the first 3 years
- the instantaneous rate of population growth after 2 years
- the instantaneous rate of population growth after 6 years
- the time at which the population growth is 0.



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### Solution

**a** Use the rate of change formula. 
$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$= \frac{f(3) - f(0)}{3 - 0}$$

Substitute.

$$= \frac{91 - 40}{3} = 17$$

State the result.

The population growth for the first 3 years is 17 koalas/year.

- b** Calculate the derivative.

$$P'(t) = t^2 - 12t + 32$$

Calculate  $P'(t)$  when  $t = 2$ .

$$P'(2) = 2^2 - 12 \times 2 + 32 = 12$$

State the result.

The instantaneous population growth after 3 years is 12 koalas/year.

- c** Find  $P'(6)$ .

$$P'(6) = 6^2 - 12 \times 6 + 32$$

Evaluate.

$$= -4$$

State the result.

The instantaneous population growth after 6 years is  $-4$  koalas/year, i.e. the population is declining by 4 koalas/year.

- d** Let  $P'(t) = 0$ .

$$t^2 - 12t + 32 = 0$$

Factorise.

$$(t - 8)(t - 4) = 0$$

Solve.

$$t = 4 \text{ or } 8.$$

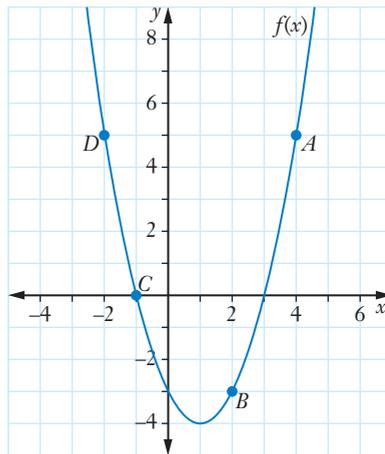
State the result.

The population growth is 0 at 4 years and 8 years.

## Exercise 12.01 Instantaneous rates of change

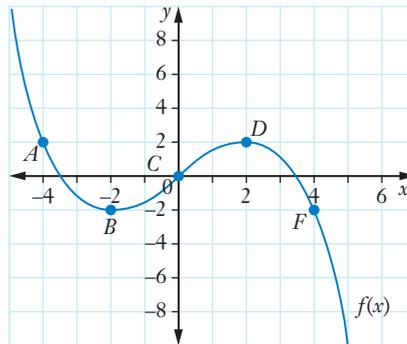
1 Use this graph for the following.

- a** Calculate the average rate of change from:
- i**  $B$  to  $A$
  - ii**  $D$  to  $B$
  - iii**  $D$  to  $A$
  - iv**  $C$  to  $A$
- b** Estimate the instantaneous rate of change at:
- i**  $A$
  - ii**  $B$
  - iii**  $D$
  - iv**  $C$



2 Use the graph of the function shown here to answer the following.

- a** Calculate the average rate of change from:
- i**  $A$  to  $B$
  - ii**  $B$  to  $D$
  - iii**  $A$  to  $D$
  - iv**  $C$  to  $F$



- b** Estimate the instantaneous rate of change at:
- i**  $B$
  - ii**  $D$
  - iii**  $C$

3 If  $y = 4x^3 - 4x^2 + 15$ ,

- a** find the average rate of change from:
- i**  $x = -1$  to  $x = 2$
  - ii**  $x = 1$  to  $x = 3$
  - iii**  $x = -2$  to  $x = 2$ .
- b** find the rate of change at:
- i**  $x = -1$
  - ii**  $x = 1$
  - iii**  $x = 2$ .
- c** find where the instantaneous rate of change is 0.

4 If  $y = 6x^2 + 7x - 2x^3 + 9$ ,

- a** find the average rate of change from:
- i**  $x = -2$  to  $x = 2$
  - ii**  $x = 1$  to  $x = 5$
  - iii**  $x = -1$  to  $x = 4$ .
- b** find the rate of change at:
- i**  $x = -2$
  - ii**  $x = 1$
  - iii**  $x = 3$ .

Example  
1

5 If  $y = x^4 + 3x - 5x^3 + 7$ ,

**a** find the average rate of change from:

**i**  $x = -2$  to  $x = 2$

**ii**  $x = 2$  to  $x = 4$

**iii**  $x = -2$  to  $x = 4$ .

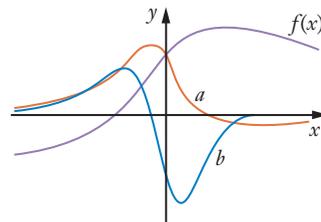
**b** find the rate of change at:

**i**  $x = -2$

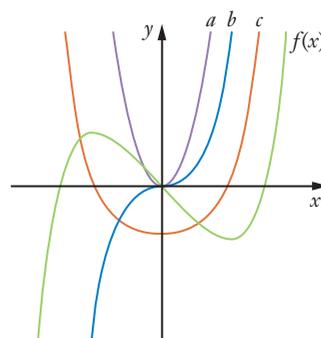
**ii**  $x = 2$

**iii**  $x = 4$ .

6 The graph of  $f(x)$  is shown on the right. Which of the graphs marked  $a$  and  $b$  represents the gradient function of  $f(x)$ ? Explain your choice.

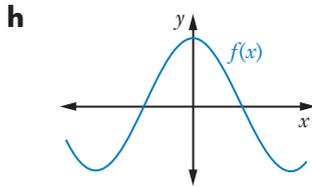
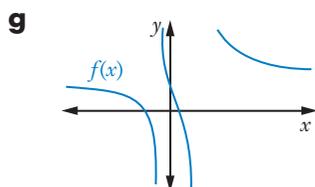
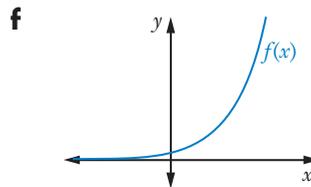
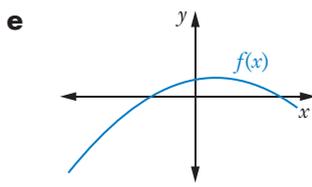
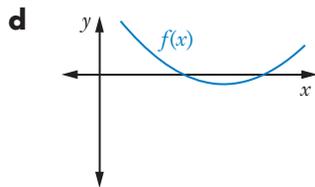
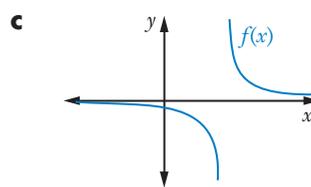
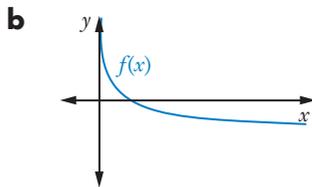
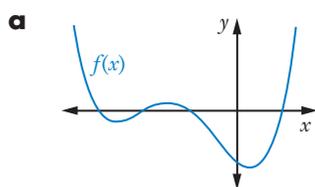


7 The graph of  $f(x)$  is shown on the right. Which of the graphs marked  $a$ ,  $b$  and  $c$  represents the gradient function of  $f(x)$ ? Explain your choice.



8 Sketch the gradient function for each function and comment on its domain and range.

Example  
2



## Problem solving

- 9 The cost (in dollars) of producing units of a certain commodity is

$$C(x) = 5000 - 10x + 0.5x^2$$

- a Find the average rate of change of cost when the production level is changed
- i from 100 to 105      ii from 100 to 101
- b Find the marginal cost when  $x = 100$ .
- 10 A stone is dropped into a lake, causing circular waves to be formed. After  $t$  seconds the radius of one of these waves is  $40t$  cm. Find in terms of  $\pi$  the rate of change with respect to  $t$  of the area of the circle formed by the wave at:
- a  $t = 1$       b  $t = 2$       c  $t = 3$

- 11 Tina is on her daily jog. Her pulse rate (in beats/min)  $t$  seconds after she begins is given by

$$P(t) = 56 + 2t^2 - t, \text{ where } 0 \leq t \leq 7.$$

Find the rate of change of her pulse rate when:

- a  $t = 2$       b  $t = 4$       c  $t = 6$
- 12 The volume of a spherical balloon (in  $\text{cm}^3$ )  $t$  hours after 1 p.m. is given by

$$V(t) = \frac{4}{3}\pi(9 - 2t)^2, \text{ where } 0 \leq t \leq 4$$

Find correct to 2 decimal places the rate of change at 4 p.m. for:

- a  $V(t)$       b the radius of the balloon      c the surface area of the balloon

## 12.02 Straight-line motion

You usually use  $x$  for the **position** of an object and  $s$  for its **displacement**; the change in position from the start to the end point without regard to the path travelled.

Displacement has a direction, and for straight-line motion, this is positive or negative.

The rate of change in displacement,  $\frac{\Delta s}{\Delta t}$ , is the average **velocity** ( $v$ ) of the object.

So, instantaneous velocity,  $v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$ .

When the object is stationary, you say that it is at rest, so  $v = 0$ .

The rate of change in velocity,  $\frac{\Delta v}{\Delta t}$ , is the average **acceleration** ( $a$ ) of the object.

So, instantaneous acceleration,  $a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$ .

For an object travelling at constant velocity,  $a = 0$ .

Example  
3



Straight-line  
motion 1

## Straight-line motion

Position =  $x$

Displacement = change in position =  $s$

Velocity,  $v = \frac{ds}{dt}$

Acceleration,  $a = \frac{dv}{dt}$

### EXAMPLE 4

The displacement of an object from a fixed point is given by

$$s = t^3 - 5t + 5$$

where  $t$  is in seconds and  $s$  is in metres.

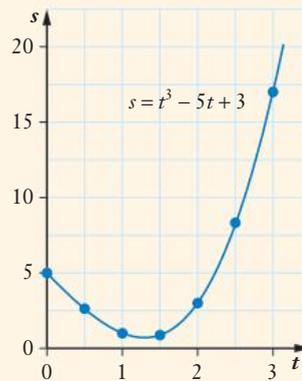
- Sketch a graph of the displacement.
- Describe the velocity.
- Find the velocity and acceleration of the object after 3 seconds.

### Solution

- There are no simple factors, so do a table of values.

$t$	0	0.5	1	1.5	2	2.5	3
$s$	5	2.63	1	0.88	3	8.33	17

Join points by a smooth line.



- Describe the changes in velocity.

The slope is negative, so the velocity is negative at the start. It changes direction near  $t = 1.5$  and then increases rapidly.

- c** Differentiate to find the velocity.  $v = \frac{ds}{dt} = 6t^2 - 5$
- Differentiate again to find the acceleration.  $a = \frac{dv}{dt} = 12t$
- Substitute the value of  $t$ .  $v(3) = 6 \times 3^2 - 5 = 49$   
 $a(3) = 12 \times 3 = 36$
- Write the answer, including units. The velocity and acceleration after 3 s are 49 m/s and 36 m/s<sup>2</sup> respectively.

You often need to find the time at which an object comes to rest during motion.

### EXAMPLE 5

The displacement  $s$  cm of an object over time  $t$  seconds is given by

$$s = 9 + 6t - t^2.$$

- Sketch a displacement–time graph.
- Describe the velocity.
- Find any times when the object is at rest.
- What is the object's displacement after 5 seconds?
- What is the total distance travelled in the first 5 seconds?

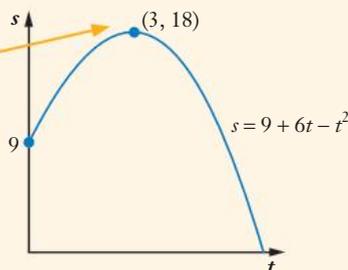
### Solution

- a** Complete the square.

$$\begin{aligned} s(t) &= 9 - (t^2 - 6t) \\ &= 9 - (t^2 - 6t + 9) + 9 \\ &= -(t - 3)^2 + 18 \end{aligned}$$

Sketch the graph.

The object comes to rest as it changes direction



- b** Describe the velocity.

The velocity is positive at  $t = 0$ , zero at  $t = 3$  and negative after that.

- c** Find an expression for velocity.

$$v = \frac{ds}{dt} = 6 - 2t$$

The object is at rest, so  $v = 0$ .

$$6 - 2t = 0$$

Solve.

$$t = 3$$

State the result.

The object is at rest after 3 seconds.

- d** Find  $s$  when  $t = 5$ .

$$s(5) = 9 + 6 \times 5 - 5^2 = 14$$

State the result.

The object is 14 cm from the origin after 5 seconds.

- e** For  $0 \leq t \leq 3$ , the object moves away from the origin; for  $3 < t \leq 5$ , it moves towards the origin.

Find  $s$  at  $t = 3$ .

$$s(3) = 9 + 6 \times 3 - 3^2 = 18$$

You can represent the path of the object using a diagram.



Find the *distance*.

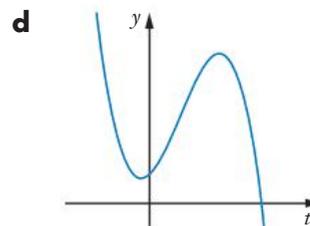
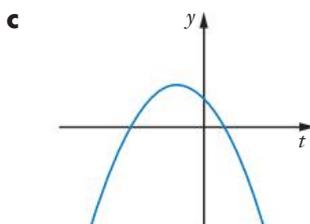
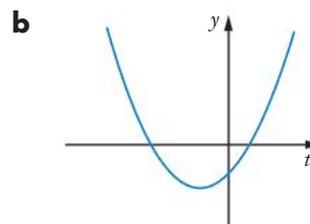
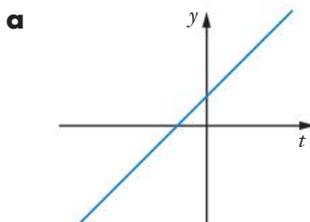
$$\text{Total distance} = 9 + 4 = 13$$

State the result.

The object travels a total of 13 cm in the first 5 seconds.

## Exercise 12.02 Straight-line motion

- 1** Each graph below shows the position function,  $y = s(t)$ , of an object. Draw a sketch of the velocity and the acceleration that correspond to each position function shown.



- 2** The displacement of an object from a fixed point is given by  $s(t) = 4t^2 + 3t - 17$ , where  $t$  is in seconds and  $s$  is in metres.
- Sketch a displacement–time graph.
  - Describe the velocity.
  - Calculate:
    - an equation for the velocity of the object at  $t$  seconds
    - an equation for the acceleration of the object at  $t$  seconds
    - the velocity and acceleration of the object after 4 seconds.
- 3** The position of a particle is given by  $s = \frac{t^3}{2} - 8t$ , where  $t$  is in seconds and  $x$  is in cm.
- Show that the particle is initially at the origin.
  - Find any other times that the particle is at the origin.
  - Sketch a displacement–time graph.
  - Describe the velocity.
  - Find the velocity of the particle after 3 seconds.
  - Find the acceleration of the particle after 4 seconds.
  - When is the velocity  $5\frac{1}{2}$  m/s?
- 4** The position of a body is given by  $x = 3t^2 - 14t + 8$ , where  $t$  is in seconds and  $x$  is in metres.
- Find the initial velocity of the body.
  - Show that the acceleration is constant.
  - What is the position of the body after 5 seconds?
  - When does the body come to rest?
  - What is the position of the body when it comes to rest?
- 5** The position of an object is given by  $s = 8t - t^2$ , where  $s$  is in cm and  $t$  is in seconds.
- When is the particle at the origin?
  - When is the object is at rest?
  - What is the object's displacement after 6 seconds?
  - Sketch a displacement–time graph.
  - Describe the velocity.
  - What is the total distance travelled in the first 6 seconds?



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- 6** The position function of a point,  $P$ , over time  $t$  seconds is given by  
 $s(t) = t^3 - 12t^2 + 36t - 20$ .
- When is the point at rest?
  - What is the total distance travelled in the first 8 seconds?
- 7** The position function of a particle is given by  $x = t^3 - 6t^2 + 9t$ , where  $x$  is in metres and  $t$  is in seconds.
- What is the velocity when  $t = 4$ ?
  - What is the velocity after 2 seconds?
  - When is the object at rest?
  - When is the particle moving forward (i.e. positive)?
  - When is the particle moving backward (i.e. negative)?
  - What is the total distance travelled in the first 7 seconds?
- 8** The height (in metres) of a ball thrown vertically into the air is given by  $h(t) = 17 + 5t - 5t^2$ , where  $t$  is in seconds.
- Find the initial displacement and velocity of the ball.
  - Calculate the ball's acceleration.
  - When will the ball be at the origin?
  - Calculate the ball's maximum displacement from the origin.
  - Sketch the graph showing the ball's motion up until it reaches the origin and describe the velocity.
- 9** An astronaut throws a piece of moon rock vertically from the surface of the moon. Its height (in metres) after  $t$  seconds is  $h = 12t - 0.82t^2$ , where  $t$  is in seconds.
- What is the velocity of the rock after 5 s?
  - When does the rock first come to rest?
  - When does the rock return to the surface of the moon?
  - Calculate the greatest height reached by the rock.
  - Sketch a graph showing the rock's height above the moon's surface and describe the velocity.
- 10** The position of an object is given by  $s(t) = 20t - 5t^3$ , where  $-1 < t < 3$ ,  $s$  is in metres and  $t$  is in seconds.
- When is the object's velocity zero?
  - When is the object's acceleration zero?
  - When is the velocity negative?
  - When is the acceleration positive?
  - Sketch the graphs of  $s(t)$ ,  $v(t)$  and  $a(t)$ .
- 11** A ball thrown vertically upwards some distance above the ground has the position function  $x(t) = 9t - 5t^2 + 18$ , where  $x$  is in metres and  $t$  is in seconds. Find:
- the greatest height reached by the ball
  - the velocity of the ball when it reaches the ground.

## Problem solving

- 12** The position of a particle moving along a horizontal line is  $s(t) = -t^3 + 18t^2 - 81t$  for  $0 < t < 12$ , where  $s$  is in metres and  $t$  is in seconds. Find the greatest velocity reached by the particle and the time at which this occurs.
- 13** In a video game, a point is programmed to move across the screen. The position of the point is given by  $x(t) = 9t - \frac{3t^3}{4}$ , where  $x$  is in cm and  $t$  is in seconds. When the point reaches the edge of the screen ( $x = 0$  or  $15$ ), time  $t$  is reset to 0 and the point begins moving again according to  $x(t)$ .
- Where and when does the point come to rest?
  - What is the acceleration when the spot comes to rest?
  - When does the point first reach the edge of the screen after  $t = 0$ ?
- 14** The position of ball  $A$ , falling vertically from rest from the top of a tower is given by  $h_A = 63 - 4.9t^2$ . At the same instant as ball  $A$  begins to fall, another ball,  $B$ , is projected vertically upwards from the bottom of the tower. The position of  $B$  is given by  $h_B = 21t - 4.9t^2$ . In both cases,  $h$  is in metres and  $t$  is in seconds. The balls collide. Find the distance of the point where the balls collide from the bottom of the tower.

## 12.03 The product and quotient rules

The derivative of  $(x + 2)$  is 1 and the derivative of  $(x - 4)$  is also 1.

The derivative of  $(x + 2) \times (x - 4) = x^2 - 2x - 8$  is  $2x - 2$ , which is not  $1 \times 1$ .

It is  $1 \times (x - 4) + 1 \times (x + 2)$ .

The rule for the derivative of a product is shown below.



Product rule



Quotient rule

### Product rule

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

or more simply as

$$(uv)' = uv' + vu' \text{ for functions } u(x) \text{ and } v(x).$$

You prove it as follows for the derivative of the product function  $F(x) = f(x)g(x)$ .

$$\begin{aligned} \frac{d}{dx} F(x) &= \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

This expression needs to be separated. You subtract and add  $f(x+h)g(x)$  in the numerator.

$$\begin{aligned}\frac{d}{dx}F(x) &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[ \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h} \right] \\ &= \lim_{h \rightarrow 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h \rightarrow 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h \rightarrow 0} f(x+h) \frac{g(x+h) - g(x)}{h} + \lim_{h \rightarrow 0} g(x) \frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)h'(x) \quad \text{QED}\end{aligned}$$

### EXAMPLE 6

Find the derivative of:

**a**  $y = 3x^4(5x + 6)$

**b**  $f(x) = (4x^3 - 3x^2 - 4)(4x^5 + 7x)$

#### Solution

**a** Let  $u = 3x^4$  and  $v = 5x + 6$ .

$$y = uv$$

Find  $\frac{du}{dx}$  and  $\frac{dv}{dx}$ .

$$u' = 12x^3 \quad v' = 5$$

Use the product rule.

$$y' = uv' + vu'$$

Substitute for  $u$ ,  $u'$ ,  $v'$  and  $v$ .

$$= (3x^4) \times 5 + (5x + 6) \times 12x^3$$

State the result.

$$y' = 15x^4 + 12x^3(5x + 6)$$

**b** Let  $u = 4x^3 - 3x^2 - 4$  and  $v = 4x^5 + 7x$ .

$$f = uv$$

Find  $\frac{du}{dx}$  and  $\frac{dv}{dx}$ .

$$u' = 12x^2 - 6x \quad v' = 20x^4 + 7$$

Use the product rule.

$$f' = uv' + vu'$$

Substitute for  $u$ ,  $u'$ ,  $v'$  and  $v$  and state the result.

$$\begin{aligned}f'(x) &= (4x^3 - 3x^2 - 4)(20x^4 + 7) \\ &\quad + (4x^5 + 7x)(12x^2 - 6x)\end{aligned}$$

For a function that is divided to make a quotient, you can change it to a product like

$$\begin{aligned}\frac{d}{dx}\left(\frac{3x-5}{x+4}\right) &= \frac{d}{dx}(3x-5)(x+4)^{-1} \\ &= 3 \times (x+4)^{-1} + (3x-5) \times (-1)(x+4)^{-2} \\ &= 3(x+4) \times (x+4)^{-2} - 1 \times (3x-5) \times (x+4)^{-2} \\ &= \frac{3(x+4) - 1 \times (3x-5)}{(x+4)^2}\end{aligned}$$

With  $f(x) = 3x - 5$  and  $g(x) = x + 4$ , you get the rule below.

### Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

or more simply as

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \text{ for functions } u(x) \text{ and } v(x).$$

### EXAMPLE 7

Find the derivative of  $y = \frac{3t-4}{2t+5}$ .

#### Solution

Write the equation.

$$y = \frac{3t-4}{2t+5}$$

Let  $u = 3t - 4$  and  $v = 2t + 5$ .

$$= \frac{u}{v}$$

Find  $u'$  and  $v'$ .

$$u' = 3 \text{ and } v' = 2$$

Use the quotient rule.

$$y' = \frac{vu' - uv'}{v^2}$$

Substitute for  $u'$ ,  $v'$ , etc.

$$= \frac{(2t+5) \times 3 - (3t-4) \times 2}{(2t+5)^2}$$

Expand brackets in the numerator.

$$= \frac{6t+15-6t+8}{(2t+5)^2}$$

Simplify and state the result.

$$\frac{dy}{dt} = \frac{23}{(2t+5)^2}$$

There are times when you may need to find the derivatives of functions involving negative or fractional powers.

### Negative and fractional powers

$$x^{-n} = \frac{1}{x^n} \qquad x^{\frac{1}{n}} = \sqrt[n]{x} \qquad x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

The power rule is true for any real number except 0 but the proof is beyond the scope of this course.

### Power rule (general case)

For any real number except 0,  $\frac{d}{dx}(x^n) = nx^{n-1}$ ,  $n \neq 0$

### EXAMPLE 8

Find the derivative of each of the following.

**a**  $\frac{5}{2x^3}$

**b**  $4\sqrt[3]{x^2}$

**c**  $\frac{\sqrt{x}}{1-x^2}$

#### Solution

**a** Write with a negative power.

$$\frac{5}{2x^3} = \frac{5}{2}x^{-3}$$

Use the power rule.

$$\frac{d}{dx}\left(\frac{5}{2}x^{-3}\right) = -3 \times \frac{5}{2}x^{-3-1}$$

Evaluate.

$$= -\frac{15}{2}x^{-4}$$

Write in the original form.

$$= -\frac{15}{2x^4}$$

**b** Write with a fractional power.

$$4\sqrt[3]{x^2} = 4x^{\frac{2}{3}}$$

Use the power rule.

$$\frac{d}{dx}4x^{\frac{2}{3}} = \frac{2}{3} \times 4x^{\frac{2}{3}-1}$$

Evaluate.

$$= \frac{8}{3}x^{-\frac{1}{3}}$$

Write in original form.

$$= \frac{8}{3\sqrt[3]{x}}$$

**c** Write the expression as  $\frac{u}{v}$ .

$$\frac{\sqrt{x}}{1-x^2} = \frac{u}{v}$$

Identify  $u$  and  $v$ .

$$u = \sqrt{x} = x^{\frac{1}{2}} \text{ and } v = 1 - x^2$$

Find  $u'$  and  $v'$ .

$$u' = \frac{1}{2}x^{-\frac{1}{2}} \text{ and } v' = -2x$$

Use the quotient rule.

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Substitute for known values.

$$= \frac{(1-x^2) \times \frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}} \times (-2x)}{(1-x^2)^2}$$

Evaluate.

$$= \frac{\frac{1}{2\sqrt{x}}(1-x^2) + 2x\sqrt{x}}{(1-x^2)^2}$$

Change the second term in the numerator to form the common factor  $\frac{1}{2\sqrt{x}}$ .

$$= \frac{\frac{1}{2\sqrt{x}}(1-x^2) + 2x \times \frac{2\sqrt{x}}{2\sqrt{x}}\sqrt{x}}{(1-x^2)^2}$$

Factorise the numerator.

$$= \frac{\frac{1}{2\sqrt{x}}[(1-x^2) + 2x \times 2\sqrt{x}\sqrt{x}]}{(1-x^2)^2}$$

Simplify.

$$= \frac{(1-x^2) + 4x^2}{2\sqrt{x}(1-x^2)^2}$$

$$= \frac{1+3x^2}{2\sqrt{x}(1-x^2)^2}$$

### Exercise 12.03 The product and quotient rules

**1** Use the product rule to differentiate each function.

**a**  $x^4(3x+2)$

**b**  $(5x-7)(6x+1)$

**c**  $2x^2(5+2x)$

**d**  $3x^3(2x^2-1)$

**e**  $6x(4x^3+3)$

**f**  $(4x-3)(2+5x)$

**2** Find the derivative of:

**a**  $(z^5 - z^3 + z - 1)(9 + 3z^2 - z^4)$

**b**  $(3p+4)(p^2-2p-4)$

**c**  $(5v^3 - 3v^2 + v - 9)(7v^4 - 2v^3 - v^2 + 8v + 6)$

**d**  $(3t^2 - 2t + 5)(t^3 + 4t^2 + t - 6)(5t^4 + t^3 - 7t + 8)$

**e**  $(4b^5 + 3b^2 - 6b + 2)(b^4 + 3b^2 - 6b^3 + 21)$

**f**  $(3u - 8u^2 - 6u^3 + 8)(7u^3 - 4u^2 + 9u - 6)$

Example

6

Example  
**7**

**3** Find the derivative of each function with respect to the given variable.

$$\begin{array}{lll} \mathbf{a} & \frac{5q+9}{3-5q} & \mathbf{b} \quad \frac{8m-7}{2m-5} & \mathbf{c} \quad \frac{3x+10}{2x-5} \\ \mathbf{d} & \frac{t+5}{t-5} & \mathbf{e} \quad \frac{3(2t-5)}{4(5t+8)} & \mathbf{f} \quad \frac{12-5u}{5(2u-9)} \end{array}$$

**4** Find the derivative of each function with respect to the given variable.

$$\begin{array}{lll} \mathbf{a} & \frac{1}{2x^2-3} & \mathbf{b} \quad \frac{5}{2x+3x^2} & \mathbf{c} \quad \frac{x^2}{x^3-1} \\ \mathbf{d} & \frac{3x^2}{4-x^2} & \mathbf{e} \quad \frac{x^2-2}{x^2+1} & \mathbf{f} \quad \frac{3x^2-x}{2x^3+3x} \end{array}$$

Example  
**8**

**5** Find the derivative of each function with respect to the given variable.

$$\begin{array}{lllll} \mathbf{a} & 5x^{-1} & \mathbf{b} \quad \frac{8}{x} & \mathbf{c} \quad 3x^{-4} & \mathbf{d} \quad \frac{3}{t^2} & \mathbf{e} \quad 7m^{-10} \\ \mathbf{f} & \frac{-3}{u^7} & \mathbf{g} \quad \frac{1}{y^{20}} & \mathbf{h} \quad \frac{4}{v^{15}} & \mathbf{i} \quad 16d^{-9} & \mathbf{j} \quad \frac{-8}{z^9} \end{array}$$

**6** Find the derivative of each function with respect to the given variable.

$$\begin{array}{lllll} \mathbf{a} & \sqrt{x} & \mathbf{b} \quad \sqrt[3]{m} & \mathbf{c} \quad 7x^{\frac{1}{2}} & \mathbf{d} \quad 5t^{\frac{1}{3}} & \mathbf{e} \quad 7p^{\frac{2}{3}} \\ \mathbf{f} & \frac{1}{\sqrt{x}} & \mathbf{g} \quad -9z\sqrt{z^5} & \mathbf{h} \quad \frac{-6u}{\sqrt[4]{u^3}} & \mathbf{i} \quad 4d^2\sqrt[6]{d} & \mathbf{j} \quad \frac{5m^2}{\sqrt{m}} \end{array}$$

**7** Differentiate each function.

$$\mathbf{a} \quad x^3(1-\sqrt{x}) \qquad \mathbf{b} \quad \left(\frac{3}{x}+x^4\right)(x^2-1) \qquad \mathbf{c} \quad \frac{x^2}{(x-2)(2-x^2)}$$

**8** Differentiate each function.

$$\begin{array}{l} \mathbf{a} \quad (3x^5-3x^2+8)(2x^2+5x+2) - (2x^4+5x^3)(x^5-6) \\ \mathbf{b} \quad \frac{3}{2f-5} + \frac{5}{6f+10} - \frac{4}{5f-11} \end{array}$$

Problem solving

**9** Find  $\frac{d}{dx} \left( \frac{x}{x-\frac{k}{x}} \right)$ .

**10** For the graph of the function  $y = \frac{x^2-1}{x+2}$ , find any points where the gradient of the tangent is  $\frac{11}{16}$ .

**11** For the graph of the function  $y = x^2(2x-5)$ , find any points where the gradient of the tangent is 4.

## 12.04 Stationary points



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Remember from Chapter 10 that for any function:

- if  $f'(x) > 0$ , the slope is positive and the function is increasing
- if  $f'(x) = 0$ , the slope is 0 and the function is stationary
- if  $f'(x) < 0$ , the slope is negative and the function is decreasing

### EXAMPLE 9

For the function  $f(x) = x^2 - 4x + 7$ , find:

- the point at which the function is stationary
- all  $x$  values for which the function is increasing
- all  $x$  values for which the function is decreasing

### Solution

- a** The function is stationary when  $f'(x) = 0$ .  $f'(x) = 2x - 4 = 0$

Solve.  $x = 2$

Find the value of the function at  $x = 2$ .  $f(2) = 2^2 - 4 \times 2 + 7$   
 $= 3$

State the result.

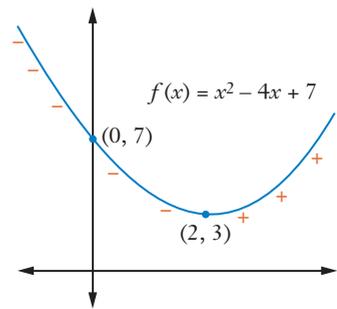
The function is stationary at  $(2, 3)$ .

- b** The function is increasing when  $f'(x) > 0$ . Let  $f'(x) > 0$ .  
 Substitute.  $2x - 4 > 0$   
 Solve.  $x > 2$   
 State the result. The function is increasing for  $x > 2$ .
- c** The function is decreasing when  $f'(x) < 0$ . Let  $f'(x) < 0$ .  
 Substitute.  $2x - 4 < 0$   
 Solve.  $x < 2$   
 State the result. The function is decreasing for  $x < 2$ .

In Example 9, there is a stationary point at  $(2, 3)$ , the function is increasing to the right of the stationary point and decreasing to the left of it.

There must be a minimum at  $(2, 3)$ .

It is easy to see that  $f(0) = 7$ , so the graph crosses the  $y$ -axis at  $(0, 7)$ .



You can use the derivative to identify a local **maximum**, local **minimum** or point of horizontal **inflection**. You check the *sign of the derivative* just before and just after a stationary point.

You use the term *local* to indicate that the function has a maximum or minimum value for  $x$  values *near the stationary point* and not necessarily for all values of  $x$  in the domain of the function.

	$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of point	Change in $f'(x)$ as $x$ increases
$f(x)$	+	0	-	Maximum	Decreasing
$f(x)$	-	0	+	Minimum	Increasing
$f(x)$	+	0	+	Inflection on a rising curve	Increasing to zero, then increasing again
$f(x)$	-	0	-	Inflection on a falling curve	Decreasing to zero, then decreasing again

**EXAMPLE 10**

Identify the stationary points of  $f(x) = x^3 + x^2 - 5x - 4$ .

**Solution**

Differentiate.

$$f'(x) = 3x^2 + 2x - 5$$

Let  $f'(x) = 0$ .

$$3x^2 + 2x - 5 = 0$$

Factorise the LHS.

$$(3x + 5)(x - 1) = 0$$

Solve.

$$3x + 5 = 0 \quad \text{or} \quad x - 1 = 0$$

$$x = -\frac{5}{3} \quad \text{or} \quad x = 1$$

Find  $f\left(-\frac{5}{3}\right)$ .

$$\begin{aligned} f\left(-\frac{5}{3}\right) &= \left(-\frac{5}{3}\right)^3 + \left(-\frac{5}{3}\right)^2 - 5\left(-\frac{5}{3}\right) - 4 \\ &= 2\frac{13}{27} \end{aligned}$$

Find  $f(1)$ .

$$\begin{aligned} f(1) &= (1)^3 + (1)^2 - 5 \times 1 - 4 \\ &= -7 \end{aligned}$$

State the result.

The stationary points are at  $\left(-1\frac{2}{3}, 2\frac{13}{27}\right)$  and  $(1, -7)$ .

$x$ -interval/value	$x < -1\frac{2}{3}$ say $x = -2$	$-1\frac{2}{3}$	$-1\frac{2}{3} < x < 1$ say $x = 0$	1	$x > 1$ say $x = 2$
$f'(x)$	3	0	-5	0	11
$f(x)$	Increasing	Stationary	Decreasing	Stationary	Increasing
Shape					

Use the values in the table to identify the stationary points.

There is a local maximum at  $\left(-1\frac{2}{3}, 2\frac{13}{27}\right)$  and a local minimum at  $(1, -7)$ .



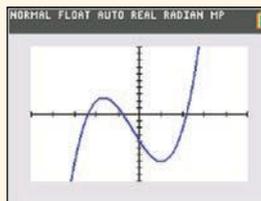
You can also use your graphics calculator to identify stationary points.

### TI-84 Plus CE

Press the  $\boxed{Y=}$  button and enter the function

$$Y_1 = x^3 + x^2 - 5x - 4.$$

You will need to set appropriate values of X and Y in the  $\boxed{\text{window}}$  settings to see the main features of the graph.



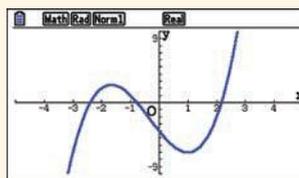
The shape of the graph tells you that the first stationary point is a local maximum and the second is a local minimum.

### Casio fx-CG20AU

Select the **GRAPH** screen and enter the function

$$Y_1 = x^3 + x^2 - 5x - 4.$$

Press  $\boxed{\text{SHIFT}} \boxed{\text{F3}}$  (**V-WINDOW**) and set appropriate values of X and Y in the settings to see the main features of the graph.



The shape of the graph tells you that the first stationary point is a local maximum and the second is a local minimum.

## EXAMPLE 11

Identify any stationary points of  $f(x) = x^3 + 6x^2 + 12x + 9$ .

### Solution

Differentiate.

$$f'(x) = 3x^2 + 12x + 12$$

Let  $f'(x) = 0$ .

$$3x^2 + 12x + 12 = 0$$

Factorise the LHS.

$$3(x^2 + 4x + 4) = 0$$

The brackets contain a perfect square.

$$3(x + 2)^2 = 0$$

Solve.

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

Find  $f(-2)$ .

$$\begin{aligned} f(-2) &= (-2)^3 + 6(-2)^2 + 12 \times (-2) + 9 \\ &= 1 \end{aligned}$$

State the result.

There is only one stationary point, at  $(-2, 1)$ .

$x$ -interval/value	$-\infty \rightarrow -2$	$-2$	$-2 \rightarrow \infty$
$f'(x)$	$f'(-3) = 3 > 0$	$f'(-2) = 0$	$f'(0) = 12 > 0$
$f(x)$	Increasing	Stationary	Increasing
Shape			

State the result.

$(-2, 1)$  is a point of horizontal inflection on a rising curve.

### Exercise 12.04 Stationary points

Example  
9

- For the function  $f(x) = 3x^2 - 12x + 5$ , find:
  - the point at which the function is stationary
  - all  $x$  values for which the function is increasing
  - all  $x$  values for which the function is decreasing
- Find the values of  $x$  for which each of the following functions is increasing and decreasing.
  - $f(x) = x^3 + 6x^2 + 8$
  - $f(x) = -x^3 - 2x^2 + 7x + 5$
  - $f(x) = x^4 - 3x^3 + x^2$
- Find any stationary points of each function.
  - $f(x) = 2x^2 - 8x + 3$
  - $f(x) = 6 + 4x - x^2$
  - $f(x) = x^3 - x^2 - 8x + 10$
  - $f(x) = 2x^3 + 3x^2 - 12x$
  - $f(x) = x^3 + 3x - 5$
  - $f(x) = x^3 + 3x^2 - 9x$
- Identify the stationary points of:
  - $f(x) = x^3 - 3x + 1$
  - $f(x) = x^3 - 9x^2 + 24x$
  - $f(x) = 2 + 3x - x^3$
- Identify any stationary points of:
  - $f(x) = 3x^4 - 4x^3$
  - $f(x) = x^3 + 6x^2 + 12x + 10$
  - $f(x) = 12x^2 - x^3 - 48x + 69$
- Identify any stationary points of each function.
  - $g(x) = -x^2 + 4x - 3$
  - $f(x) = -x^2 - 2x + 4$
  - $y = x^3 - 12x$
  - $g(x) = 3x - x^3$
  - $f(x) = 2x^3 - 12x^2 + 7$
  - $h(r) = 4\pi r^2 - \pi r$
  - $y = x^3 - x^2 - 8x + 2$
  - $y = x^3 - 6x + 1$
  - $y = x^4 - 4x + 5$
  - $g(x) = x^2(8 - x)$

Example  
10

Example  
11

## Problem solving

- 7  $f(x) = x^2 + kx + 7$  has a stationary point at  $x = 2$ .
- Find the value of  $k$ .
  - Identify the stationary point.
- 8  $f(x) = x^3 + ax^2 + bx + 2$  has stationary points at  $x = 1$  and  $x = 3$ .
- Find the values of  $a$  and  $b$ .
  - Identify the stationary points.
- 9 If  $f(x) = x^4 + ax^2 + b$  has a stationary point at  $(-1, 2)$ :
- find the values of  $a$  and  $b$ .
  - identify all the stationary points.



## 12.05 Sketching polynomials

You can use key features of a polynomial to sketch its graph.

### Procedure for sketching functions

- Zeros:** If possible, determine the  $x$ -intercepts—the values of  $x$  for which  $f(x) = 0$ .
- Sign:** Determine the sign of  $f(x)$  between zeros.
- Derivative:** Use the derivative to:
  - identify stationary points
  - determine the intervals in which the function is increasing or decreasing.
- $y$ -intercept:** Find the value of  $f(0)$ .
- Behaviour as  $x \rightarrow \pm\infty$ : a **dominant term** is of higher power than any other term. As  $x \rightarrow \pm\infty$ , the magnitude of the dominant term becomes much larger than the magnitudes of the other terms, so the function behaves like the dominant term. Alternatively, a function may have a **limit** as  $x \rightarrow \pm\infty$  and this makes a horizontal asymptote.

Then use your results to sketch the graph.

## EXAMPLE 12

Sketch  $f(x) = (x + 1)^2(x - 3)$ .

### Solution

Use headings.

State zeros, including type.

Find the sign of  $f(x)$  before, between and after the zeros.

Do it as a table.

Multiply out the brackets first.

Now find the derivative.

Find the zeros of  $f'(x)$ .

This will factorise.

Find signs of  $f'(x)$  before and after the stationary points.

Do it as a table.

Use diagrams of the slope.

Find the values at the points.

Write the stationary points.

### Zeros

There is a double zero at  $x = -1$  and there is a single zero at  $x = 3$ .

### Sign of function

$x$	-2	-1	0	3	4
$f(x)$	-5	0	-3	0	25
Sign	-		-		+

### Derivative

$$\begin{aligned}f(x) &= (x + 1)^2(x - 3) \\ &= x^3 - x^2 - 5x - 3\end{aligned}$$

$$f'(x) = 3x^2 - 2x - 5$$

### Stationary points

$$3x^2 - 2x - 5 = 0$$

$$(3x - 5)(x + 1) = 0$$

$$x = -1 \text{ or } x = \frac{5}{3}$$

There are stationary points at  $x = -1$  and  $x = \frac{5}{3}$ .

$x$	-2	-1	0	$\frac{5}{3}$	2
$f'(x)$	11	0	-5	0	3
Sign	+	0	-	0	+



$$f(-1) = 0 \text{ and } f\left(\frac{5}{3}\right) = -9\frac{13}{27}.$$

There is a local maximum at  $(-1, 0)$  and a local minimum at  $\left(1\frac{2}{3}, -9\frac{13}{27}\right)$ .

Use the table to state where  $f(x)$  is increasing and decreasing.

Find the value of  $f(0)$ .

Write the dominant term.

Place all information on the graph, marking known points.

Join up with smooth curves to finish the sketch.

### Increasing and decreasing intervals

$f(x)$  is increasing for  $x < -1$

$f(x)$  is decreasing for  $-1 < x < \frac{5}{3}$

$f(x)$  is increasing for  $\frac{5}{3} < x$

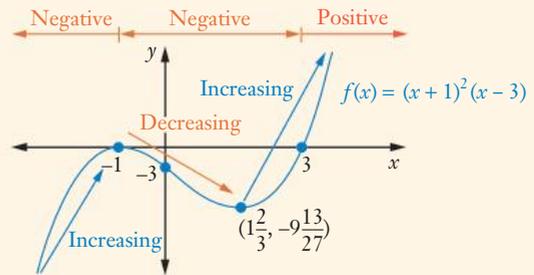
$f(0) = -3$

### Behaviour as $x \rightarrow \pm\infty$

$f(x) = x^3 - x^2 - 5x - 3$  has dominant term  $x^3$ .

As  $x \rightarrow \infty, f(x) \rightarrow \infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow -\infty$ .

### Sketch



If you cannot factorise the polynomial, use the other information or your graphics calculator.

### EXAMPLE 13

Sketch  $y = 2x^2 + 3x - x^3 - 5$ .

### Solution

Find the derivative.

Find the zeros of  $\frac{dy}{dx}$ .

Rearrange with  $x^2$  positive.

### Zeros and sign of function

The expression does not factorise easily.

### Derivative

$$\frac{dy}{dx} = 4x + 3 - 3x^2$$

### Stationary points

$$4x + 3 - 3x^2 = 0$$

$$3x^2 - 4x - 3 = 0$$

This will not factorise, so use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute values.

$$\begin{aligned} &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-3)}}{2 \times 3} \\ &= \frac{4 \pm \sqrt{52}}{6} \\ &\approx -0.54 \text{ or } 1.87 \end{aligned}$$

Keep the exact values on your calculator.

Stationary points are at  $x \approx -0.54$  and  $x \approx 1.87$ .

Find the signs of  $\frac{dy}{dx}$  just before and after the stationary points.

$x$	-1	-0.54	0	1.87	2
$\frac{dy}{dx}$	-4	0	3	0	-1
Sign	-	0	+	0	-

Use diagrams of the slope.



Find the values at the points.

$$f(-0.54) \approx -5.88 \text{ and } f(1.87) \approx 1.06$$

Write the stationary points.

There is a minimum at about  $(-0.54, -5.88)$  and a maximum at about  $(1.87, 1.06)$ .

Use the table.

### Increasing and decreasing intervals

$f(x)$  is decreasing before  $x \approx -0.54$ , increasing between  $x = -0.54$  and  $x \approx 1.87$ , and decreasing after  $x \approx 1.87$ .

Find  $y$  when  $x = 0$ .

### $y$ -intercept

$$y = -5$$

State the dominant term.

### Behaviour as $x \rightarrow \pm\infty$

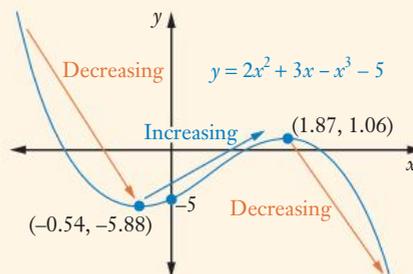
$$y = 2x^2 + 3x - x^3 - 5 \text{ has dominant term } -x^3.$$

As  $x \rightarrow \infty, f(x) \rightarrow -\infty$  and as  $x \rightarrow -\infty, f(x) \rightarrow \infty$ .

Place all information on the graph, marking known points.

Join up with smooth curves to finish the sketch.

### Sketch



Example  
12

## Exercise 12.05 Sketching polynomials

1 Sketch the graph of each function, showing all key features.

<b>a</b>	$g(x) = 10x - x^2 - 21$	<b>b</b>	$h(x) = x^2 - x - 8$	<b>c</b>	$h(x) = 2x^2 + x - 4$
<b>d</b>	$q(x) = x^3 - x^2 - x - 1$	<b>e</b>	$f(x) = 3x - x^3 - 18$	<b>f</b>	$f(x) = x^3 + 3x^2 - 24x$
<b>g</b>	$f(x) = 3x^5 - 5x^3 + 5$	<b>h</b>	$f(x) = 2x^6 - 6x^4$	<b>i</b>	$f(x) = 7 + 2x^2 - x^4$

Example  
13

2 Sketch the graph of each function, showing all key features.

<b>a</b>	$f(x) = x^3 - 3x^2 + 3x - 1$	<b>b</b>	$y = 2x^3 - 6x^2 + 6x - 5$	<b>c</b>	$g(x) = x^4 - 2x^3 + 6$
<b>d</b>	$y = 2 + 5x - x^3$	<b>e</b>	$f(x) = x^3 - 9x^2 + 20x$	<b>f</b>	$h(x) = (2x + 1)^2(x + 3)$

### Problem solving

- 3 **a** Use a graphics calculator to draw the graph of  $y = 10x^2 - 2x^3 + 14x - 33$  for  $-10 < x < 10$ , and hence estimate the coordinates of the stationary points.
- b** Estimate where the rate of change is positive.
- c** Estimate the rate of change at  $x = -4$ .
- d** Estimate the rate of change at  $x = 3$ .
- 4 **a** Use a graphics calculator to draw the graph of  $y = 2x^4 - 17x^3 + x^2 + 137x - 79$  for  $-10 < x < 10$ , and hence estimate the coordinates of the stationary points.
- b** Estimate where the rate of change is negative.
- c** Estimate the rate of change at  $x = -1$ .
- d** Estimate the rate of change at  $x = 2$ .
- 5 For a cubic polynomial function, e.g.,  $f(x) = ax^3 + bx^2 + cx + d$ , how many stationary points are possible?



Chain rule

## 12.06 The chain rule

You could write  $F(x) = x^{40}$  like  $F(x) = f[g(x)] = (x^8)^5$  with  $f(x) = x^8$  and  $g(x) = x^5$ .

The derivative of  $f(x)$  is  $8x^7$ , so the derivative of  $f(g)$  is  $8g^7$  and  $8g^7 = 8(x^5)^7 = 8x^{35}$ .

The derivative of  $g(x)$  is  $5x^4$ .

But the derivative of  $F(x)$  is  $40x^{39}$ .

Notice that  $8x^{35} \times 5x^4 = 40x^{39}$ , so  $F'(x) = f'(g(x)) \times g'(x)$ .

This is a general rule for functions you can write as a function of a function: a **composite function**.



Mixed differentiation problems

## The chain rule

If  $F(x) = f(g(x))$ , then  $F'(x) = f'(g(x)) \times g'(x)$ .

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \times g'(x)$$

If  $y = f(u)$  and  $u = g(x)$ , then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

### EXAMPLE 14

- a** Find the derivative of  $y = 3(6x - 5)^3$ .  
**b** Find the equation of the tangent to the curve  $y = 3(6x - 5)^3$  when  $x = 1$ .

### Solution

- a** Write down the function.

$$y = 3(6x - 5)^3$$

Let  $u = 6x - 5$ .

$$y = 3u^3$$

Calculate  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$y'(u) = 9u^2 \quad u'(x) = 6$$

Use the chain rule.

$$y'(x) = y'(u) \times u'(x)$$

Substitute for  $y'(u)$  and  $u'(x)$ .

$$y'(x) = 9u^2 \times 6$$

Substitute for  $u$  and simplify.

$$= 54(6x - 5)^2$$

State the result.

$$\frac{dy}{dx} [3(6x - 5)^3] = 54(6x - 5)^2$$

- b** Find the point on the curve where  $x = 1$ .

$$y = 3(6 \times 1 - 5)^3 = 3$$

State the point.

$(1, 3)$  lies on the curve.

Identify the gradient.

Gradient of tangent,  $m = y'(x)$

Find the gradient of the tangent at  $x = 1$ .

$$m = 54(6x - 5)^2 = 54(6 \times 1 - 5)^2 = 54$$

Write the equation of the tangent in point–gradient form.

$$y - y_1 = m(x - x_1)$$

Let  $(x_1, y_1) = (1, 3)$  and  $m = 54$ .

$$y - 3 = 54(x - 1)$$

Simplify.

$$y = 54x - 51$$

State the result in standard form.

The equation of the tangent to  $y = 3(6x - 5)^3$  at  $(1, 3)$  is  $54x - y - 51 = 0$ .

**EXAMPLE 15**

Differentiate each function.

**a**  $\frac{6}{(8x-3)^9}$

**b**  $7\sqrt{5x^3-4x}$

**Solution**

**a** Write as a function.

Let  $u = 8x - 3$ .

Calculate  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

Write the chain rule.

Substitute for known values.

Substitute for  $u$  and simplify.

Write in the original form.

**b** Write as a function.

Let  $u = 5x^3 - 4x$ .

Calculate  $\frac{dy}{du}$  and  $\frac{du}{dx}$ .

$$y = \frac{6}{(8x-3)^9}$$

$$= \frac{6}{u^9}$$

$$y = 6u^{-9}$$

$$\frac{dy}{du} = -54u^{-10} \qquad \frac{du}{dx} = 8$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -54u^{-10} \times 8$$

$$= -54(8x-3)^{-10} \times 8$$

$$= -432(8x-3)^{-10}$$

$$\frac{d}{dx} \left( \frac{6}{(8x-3)^9} \right) = -\frac{432}{(8x-3)^{10}}$$

$$y = 7\sqrt{5x^3-4x}$$

$$= 7\sqrt{u} = 7u^{\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{2} \times 7u^{-\frac{1}{2}} \qquad \frac{du}{dx} = 15x^2 - 4$$

Write the chain rule.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitute for known values.

$$= \frac{7}{2} u^{-\frac{1}{2}} \times (15x^2 - 4)$$

Substitute for  $u$ .

$$= \frac{7}{2} (5x^3 - 4x)^{-\frac{1}{2}} \times (15x^2 - 4)$$

Simplify

$$= \frac{7(15x^2 - 4)}{2(5x^3 - 4x)^{\frac{1}{2}}}$$

Write in the original form.

$$\frac{d}{dx} 7\sqrt{5x^3 - 4x} = \frac{7(15x^2 - 4)}{2\sqrt{5x^3 - 4x}}$$

$\frac{dy}{du}$  and  $\frac{du}{dx}$  are not fractions. They are symbols for the derivative of a function with respect to a variable.

Sometimes you need to use the chain rule with the product or quotient rules.

### EXAMPLE 16

Differentiate:

**a**  $y = \left(\frac{3x-4}{2x+5}\right)^5$

**b**  $y = 12(3x+4)^3(5x-3)^4$

### Solution

**a** Write the equation.

$$y = \left(\frac{3x-4}{2x+5}\right)^5$$

$$\text{Let } p = \frac{3x-4}{2x+5}.$$

$$y = p^5 \text{ and } \frac{dy}{dp} = 5p^4$$

$$\text{Let } u = 3x - 4 \text{ and } v = 2x + 5.$$

$$p = \frac{3x-4}{2x+5} = \frac{u}{v}$$

Find  $u'$  and  $v'$ .

$$u' = 3 \text{ and } v' = 2.$$

Calculate  $p'(x)$ .

$$p'(x) = \frac{vu' - uv'}{v^2}$$

Substitute for known values.

$$= \frac{(2x+5) \times 3 - (3x-4) \times 2}{(2x+5)^2}$$

Expand the numerator.

$$= \frac{6x+15-6x+8}{(2x+5)^2}$$

Simplify.

$$= \frac{23}{(2x+5)^2}$$

Calculate  $\frac{dy}{dx}$  using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

Substitute for known values.

$$= 5p^4 \times \frac{23}{(2x+5)^2}$$

Substitute for  $p$ .

$$= 5 \left( \frac{3x-4}{2x+5} \right)^4 \times \frac{23}{(2x+5)^2}$$

Simplify.

$$= \frac{115(3x-4)^4}{(2x+5)^6}$$

**b** Write the equation.

$$y = 12(3x+4)^3(5x-3)^4$$

Let  $u = 12(3x+4)^3$  and  $v = (5x-3)^4$ .

$$y = u \times v$$

Find  $u'$  using the chain rule.

$$u' = 3 \times 12(3x+4)^2 \times 3$$

$$= 108(3x+4)^2$$

Find  $v'$  using the chain rule.

$$v' = 4(5x-3)^3 \times 5$$

$$= 20(5x-3)^3$$

Now use the product rule.

$$y' = vu' + uv'$$

Substitute for  $u'$ ,  $v'$ , etc.

$$= (5x-3)^4 \times 108(3x+4)^2$$

$$+ 12(3x+4)^3 \times 20(5x-3)^3$$

Simplify.

$$= 108(5x-3)^4(3x+4)^2 + 240(3x+4)^3(5x-3)^3$$

Factorise.

$$= 12(3x+4)^2(5x-3)^3[9(5x-3) + 20(3x+4)]$$

Expand brackets.

$$= 12(3x+4)^2(5x-3)^3[45x-27+60x+80]$$

Simplify.

$$= 12(3x+4)^2(5x-3)^3(105x+53)$$

## Exercise 12.06 The chain rule

1 Differentiate each function.

**a**  $7(3t - 6)^9$

**b**  $(4m^2 + 1)^5$

**c**  $8(3v - 6)^3$

**d**  $12(3x^3 + 8)^5$

**e**  $(7 - 2u)^{11}$

**f**  $(5 - 4v)^4$

**g**  $7(3v - 9)^8$

**h**  $-9(x^2 + 4x + 6)^7$

**i**  $-2(3 - 5y)^3$

Example  
14

2 Find the derivative of each function.

**a**  $4(3x + 2)^4 + 5(3x + 2)^2 - 9$

**b**  $(6x - 8) - (6x - 8)^3 + 7(6x - 8)^4$

**c**  $2(5 - 3x)^3 + 5(5 - 3x)^2 - 7(5 - 3x) + 16$

**d**  $3(t + 4)^5 - 2(t + 4)^3 + 5(t + 4)^2 - 3(t + 4) + 9$

**e**  $6(3 - 4u)^4 - 5(3 - 4u)^2 + 3 - 4u$

**f**  $5(3p + 7)^6 + 4(3p + 7)^3 - 6p - 14$

**g**  $(3 - 2x)^3 - 5(3 - 2x)^2 + 5x - 8$

3 Find the derivative of each function.

**a**  $-2(3x - 7)^{-4}$

**b**  $(8p + 3)^{-6}$

**c**  $\frac{8}{(3g - 4)^5}$

**d**  $\frac{-5}{(2z + 8)^4}$

**e**  $\frac{7}{5t - 9}$

**f**  $\frac{3}{(4x - 5)^6}$

**g**  $\frac{1}{6(t + 4)^3}$

**h**  $\frac{8}{5(7v + 3)^4}$

**i**  $\frac{-7}{2(4q - 7)^8}$

Example  
15

4 Find the derivative of each function.

**a**  $\sqrt{x^2 - 1}$

**b**  $\sqrt{4x^3 + 3}$

**c**  $11\sqrt{1 - 2x^2}$

**d**  $3(6 - 5x^3)^{\frac{1}{2}}$

**e**  $8(4x^3 - 3)^{\frac{1}{2}}$

**f**  $5\sqrt[3]{3x^4 - 1}$

5 Differentiate:

**a**  $5(3x - 2)^4(x + 8)^3$

**b**  $(4t + 5)^3(2t - 7)^2$

**c**  $6(m - 6)(4m + 3)^8$

**d**  $(2k + 9)^4(3k - 5)^7$

**e**  $4(3g - 4)^5(5g - 4)^6$

**f**  $-6(v + 7)^5(v - 6)^4$

**g**  $-3(6 - 5r)^3(2r + 9)^{11}$

**h**  $-5(8 - 6h)^7(-4 - 3h)^6$

**i**  $(5t + 8)(2t - 3)$

**j**  $-(3p + 10)^7(8p + 1)^8$

Example  
16

6 Differentiate:

**a**  $(x + 3)(x + 5)^3$

**b**  $(3x - 1)^4(x - 5)^5$

**c**  $2x\sqrt{1 - x}$

**d**  $x^{\frac{1}{3}}(2 - x)^{\frac{2}{3}}$

**e**  $\frac{1}{\sqrt{2x^3 - 3}}$

**f**  $\frac{x}{\sqrt{3 - x^2}}$

7 Differentiate:

a  $5(2x - 3)^5 + 2(2x - 3)^2 + \frac{4}{(2x - 3)^2} - \frac{1}{3(2x - 3)^5}$

b  $\frac{(3v - 4)^2}{3(2v + 3)^3}$

c  $\frac{15c^7}{5(9 - 4c)^2}$

d  $\frac{1}{(t - 6)^3} - \frac{4}{(2t + 7)^2} + \frac{2(3t - 5)}{7(2t + 9)^4}$

e  $\frac{4p^5 - 5p^4 + 2p^3 - p^2 - 4p + 2}{(5p - 1)^6}$

8 If  $f(x) = \frac{2x^2 + 3x - 7}{(3x - 2)^3}$ , find  $f'(-2)$ .

Problem solving

9 Find any points on the curve  $y = 2(x^2 - 3)^4$  where the gradient of the tangent is equal to 0.

10 Find the point where the tangent to the curve  $y = (2x - 3)^3$  at the point where  $x = \frac{7}{4}$  intersects the  $y$ -axis.



Greatest and least values

## 12.07 Greatest and least values on an interval

The **greatest value** of a function over an interval includes values at the ends of the interval. The **least value** also includes the ends.

### EXAMPLE 17

Find the greatest and least values of  $f(x) = x^3 + \frac{3}{2}x^2 - 6x + 4$  for  $-3 \leq x \leq 3$ .

#### Solution

Find the derivative.

**Stationary points**  
 $f'(x) = 3x^2 + 3x - 6$

Find the zeros of  $f'(x)$ .

$$3x^2 + 3x - 6 = 0$$

$$3(x + 2)(x - 1) = 0$$

$$x = -2 \text{ or } x = 1$$

There are stationary points at  $x = -2$  and  $x = 1$ .

Find signs of  $f'(x)$  before and after the stationary points.

$x$	-3	-2	0	1	2
$f'(x)$	12	0	-6	0	12
<b>Sign</b>	+	0	-	0	+

Use diagrams of the slopes.



Find the values at the points.

$$f(-2) = 14 \text{ and } f(1) = \frac{1}{2}.$$

Write in words.

There is a maximum at  $(-2, 14)$  and a minimum at  $(1, \frac{1}{2})$ .

**y-intercept**

Find the value of  $f(0)$ .

$$f(0) = 4$$

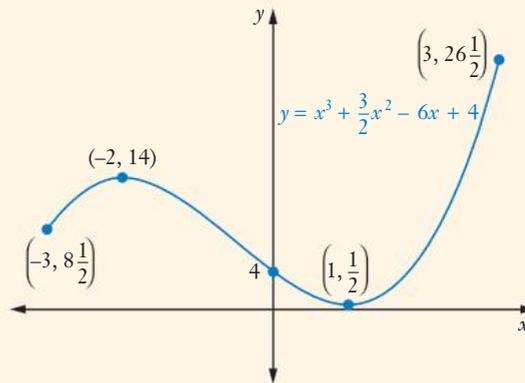
**Boundaries**

Find the values at the ends of the interval.

$$f(-3) = 8\frac{1}{2} \text{ and } f(3) = 26\frac{1}{2}$$

**Sketch**

Use the information to make a sketch over the required domain.



Write the answer.

The greatest value is  $26\frac{1}{2}$  and the least value is  $\frac{1}{2}$ .

A **global maximum**  $f(x_0)$  is a value for which  $f(x_0) \geq f(x)$  for all values of  $x$  in the domain of  $f$ . A **global minimum**  $f(x_0)$  is a value for which  $f(x_0) \leq f(x)$  for all values of  $x$  in the domain of  $f$ .

You can also use a graphics calculator to find the least or greatest value of a function for a given domain.

### EXAMPLE 18

Use a graphics calculator to find the greatest and least values of  $y = x^4 - 6x^2 + 5x - 3$  for  $-3 \leq x \leq 3$ .

#### Solution

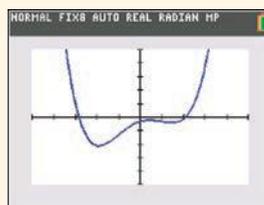
##### TI-84 Plus CE

Press  $\boxed{Y=}$  and enter the function into  $Y_1$ . Press  $\boxed{\text{graph}}$  to display the graph.

Set the  $\boxed{\text{window}}$  to  $-4 \leq X \leq 4$  and  $-50 \leq Y \leq 50$ .

The graph shows that the least value is the local minimum. To find the local minimum, press  $\boxed{2\text{nd}}$   $\boxed{\text{trace}}$  (calc) and then select 3:minimum. Set the left and right bounds and the estimate.

The graph shows that the greatest value occurs at  $x = 3$ . To calculate the value of  $y$ , press  $\boxed{2\text{nd}}$   $\boxed{\text{trace}}$  (calc) and then select 1:value. Enter 3 for the  $x$  value.

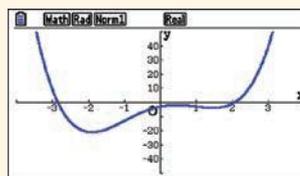


##### Casio fx-CG20 AU

Select the Graph page and enter the function into  $Y_1$ . Press  $\boxed{F6}$  (DRAW) to display the graph.

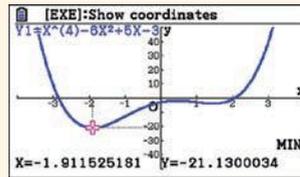
Press  $\boxed{\text{SHIFT}} \boxed{F3}$  (V-Window) and set

$-4 \leq X \leq 4$  and  $-50 \leq Y \leq 50$ .

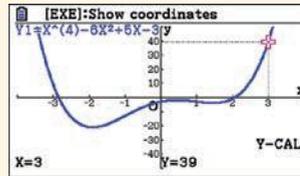


TI-Nspire CX  
Chapter 12

The graph shows that the least value is the local minimum. To find the local minimum, press **[SHIFT]** **[F5]** (G-Solv) and then press **[F3]** (MIN).



The graph shows that the greatest value occurs at  $x = 3$ . To calculate the value of  $y$ , press **[SHIFT]** **[F5]** (G-Solv), then **[F6]** and then **[F1]** (Y-CAL). Enter 3 as the  $x$  value, then press **[EXE]**.



### Both calculators

Write the answer.

For  $y = x^4 - 6x^2 + 5x - 3$  over the domain  $-3 \leq x \leq 3$ , the greatest value is 39 and the least value is about  $-21.13$ .

## Exercise 12.07 Greatest and least values on an interval

- 1 Sketch the function  $f(x) = x^2 - 2x + 3$  for  $-2 \leq x \leq 2$ , showing its major features. Find the greatest and least values of the function over the stated domain.
- 2 Sketch the function  $f(x) = 5x - 2x^2 + 10$  for  $-1 \leq x \leq 3$ , and find the greatest and least values of the function over the stated domain.
- 3 Find the global maximum of the function  $f(x) = x^2 + 6x + 5$  for  $-6 \leq x \leq 1$ .
- 4 Find the global minimum of the function  $f(x) = -3x^2 - 10x + 8$  for  $-5 \leq x \leq 2$ .
- 5 Find the greatest and least values of the function  $f(x) = x^3 - 5x^2 + 6x + 4$  for  $-3 \leq x \leq 1$ .
- 6 Find the global maximum and global minimum of the function  $f(x) = x^3 - 12x$  for  $-3 \leq x \leq 5$ .
- 7 Find the greatest and least values of the function  $f(x) = x^3 - 3x^2 + 1$  for  $\frac{1}{2} \leq x \leq 4$ .
- 8 Use a graphics calculator to find the greatest and least values of the function  $f(x) = x^4 + x^3 - 16x^2 - 4x + 48$  for  $-3 \leq x \leq 3$ .
- 9 Use a graphics calculator to find the global maximum and global minimum of the function  $f(x) = x^5 - 5x^3$  for  $-2 \leq x \leq 2$ .

Example  
17

Example  
18

## Problem solving

- 10** Find the points on the graph of the function  $f(x) = 1 - 2x + 3x^2 - x^3$  between  $x = 0$  and  $x = 3$  where the gradient is the greatest and the least.
- 11** Find the points on the graph of the function  $f(x) = -x^3 + \frac{15}{2}x^2 - 12x - 2$  for  $0 \leq x \leq 6$  where the gradient is the greatest and the least.



Application of  
optimisation



Applications  
of derivatives  
assignment



Applications  
of derivatives  
problems

## 12.08 Optimisation

**Optimisation** is the process of finding an optimal (best) solution to a problem, usually for particular conditions. An *optimal* value depends on the situation. For example, a manufacturer will probably want to minimise the amount of material required to produce a product, while a sales manager will want to maximise the profit for the sale of goods held and so on.

### Solving optimisation problems on an interval

- 1 Read the problem carefully—several times if needed. Think about the given facts, together with the unknown quantities that are to be found/optimised.
- 2 If possible, make a diagram and label it appropriately. Choose variables for the unknown quantities. Even when a diagram is not possible, identify variables and choose a symbol.
- 3 List all known facts and relationships. Use equations if possible.
- 4 Determine the quantity to be maximised or minimised. Express it as a function of the other variables. Use the other equations to express all variables in terms of a single variable. Express the quantity as a function of the same variable.
- 5 Find the interval over which the quantity is to be optimised by examining restrictions on the domain, whether explicit or implied.
- 6 Find the maximum or minimum and express your answer in terms of the question.
- 7 State any assumptions you made and check your solution. Look for possible extensions to other cases.

## EXAMPLE 19

An open box with a rectangular base is to be constructed from a rectangular piece of cardboard 16 cm wide and 21 cm long by cutting a square from each corner and then bending up the sides. Find the size of the corner square that will produce a box having the largest possible volume.



Shutterstock.com/Roman Samokhin

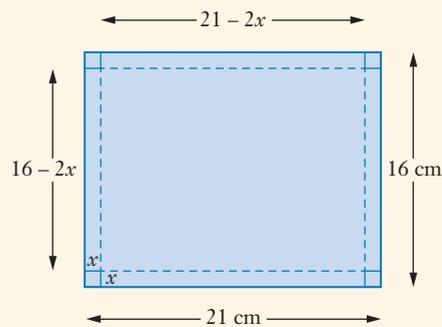


Maximum volume

## Solution

Draw a diagram of the cardboard and mark in the corners that are to be cut out.

Let the length of the corner be  $x$ .



Write an equation for the volume of the box.

$$V = \text{length} \times \text{width} \times \text{height}$$

Express as a function of  $x$ .

$$V(x) = x(16 - 2x)(21 - 2x)$$

Expand the brackets.

$$= x(336 - 74x + 4x^2)$$

Expand again.

$$= 336x - 74x^2 + 4x^3$$

Write restrictions on the domain of  $x$ .

$$x \geq 0 \text{ and } 2x < 16, \text{ i.e., } x < 8.$$

Express the volume as the greatest value over an interval.

You want the greatest value of  $V(x)$  over  $0 \leq x < 8$ .

Differentiate.

$$V'(x) = 336 - 148x + 12x^2$$

Factorise and rearrange.

$$= 4(3x^2 - 37x + 84)$$

Factorise the quadratic.

$$= 4(3x - 28)(x - 3)$$

Let  $V'(x) = 0$ .

$$0 = 4(3x - 28)(x - 3)$$

Solve.

$$x = 3 \text{ or } x = \frac{28}{3} = 9\frac{1}{3}$$

But  $x \neq 9\frac{1}{3}$ , since  $2x < 16$ . State the result.

$$x = 3$$

Identify the stationary point by examining the gradients on either side of  $x = 3$ .

When  $x < 3$ ,  $V'(x) > 0$

When  $x > 3$ ,  $V'(x) < 0$



State the result.

There is a maximum at  $x = 3$ .

Calculate the maximum volume.

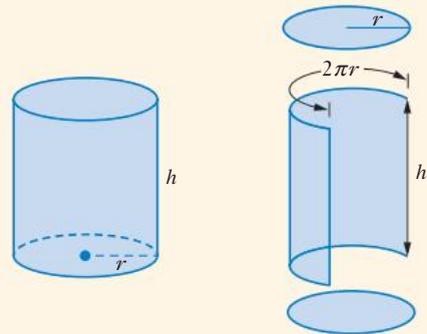
$$V(3) = 3(16 - 2 \times 3)(21 - 2 \times 3) = 450$$

Write the result.

The maximum volume of  $450 \text{ cm}^3$  is obtained from a corner square 3 cm long.

### EXAMPLE 20

A cylindrical container is to be constructed from a total of  $1000 \text{ cm}^2$  of metal sheet as shown below. Find, correct to 2 decimal places, the greatest volume the container can have. Consider the usefulness of your method.



### Solution

Assign the variables.

Let the radius =  $r$

Let the height =  $h$

Let the surface area =  $SA$

Let the volume =  $V$

Write the equation for surface area.

$$SA = 2\pi r^2 + 2\pi r h$$

Write the equation for volume.

$$V = \pi r^2 h$$

You need to differentiate the volume function to find any stationary points. You cannot do this at present because there are 2 variables,  $r$  and  $h$ . Use the  $SA$  to make an expression for volume with only one variable.

Let  $SA = 1000$ .

$$1000 = 2\pi r^2 + 2\pi r h$$

Rearrange to obtain an expression for  $h$ .

$$2\pi r h = 1000 - 2\pi r^2$$

$$h = \frac{1000 - 2\pi r^2}{2\pi r}$$

Factorise.

$$= \frac{1}{\pi r} (500 - \pi r^2)$$

Substitute in volume.

$$V = \frac{\pi r^2}{\pi r} (500 - \pi r^2)$$

Cancel and expand.

$$= 500r - \pi r^3$$

Differentiate.

$$V' = 500 - 3\pi r^2$$

Let  $V' = 0$  to get the stationary point.

$$0 = 500 - 3\pi r^2$$

Rearrange.

$$3\pi r^2 = 500$$

Solve.

$$r^2 = \frac{500}{3\pi}$$

$$r = \pm \sqrt{\frac{500}{3\pi}}$$

The radius can't be negative.

$$r = \sqrt{\frac{500}{3\pi}} \approx 7.28$$

Identify the stationary point by examining the gradients on either side of  $r = \sqrt{\frac{500}{3\pi}}$ .

$$\text{When } r < \sqrt{\frac{500}{3\pi}}, V' > 0$$

$$\text{When } r > \sqrt{\frac{500}{3\pi}}, V' < 0$$



State the result.

$$\text{The volume is a maximum when } r = \sqrt{\frac{500}{3\pi}}.$$

Calculate the volume.

$$\begin{aligned} V &= 500 \sqrt{\frac{500}{3\pi}} - \pi \left( \sqrt{\frac{500}{3\pi}} \right)^3 \\ &= 2427.8854\dots \end{aligned}$$

State the result

The maximum volume of about  $2427.89 \text{ cm}^3$  occurs when the radius is about  $7.3 \text{ cm}$ .

Consider your method.

In practice, the ends will be cut from squares and some overlap will be needed.

The true maximum volume in practice will be less than that given.

## Exercise 12.08 Optimisation

Example  
19

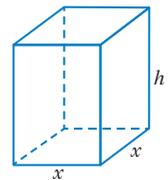
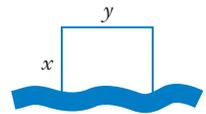
- 1 The sum of two positive numbers,  $x$  and  $y$ , is 50.
  - a Write an equation for the sum,  $S$ , of the numbers.
  - b Show that the product,  $P$ , of the numbers is given by  $P = 50x - x^2$ .
  - c Find the numbers,  $x$  and  $y$ , if their product is a maximum.
- 2 The sum of two positive numbers, say  $x$  and  $y$ , is 20.
  - a Write an equation for the sum of the numbers.
  - b Find the numbers if the sum of their squares is a minimum.

Example  
20

- 3 The perimeter of a piece of glass that will be used in a picture frame is 30 cm.
  - a Show that the area of the piece of glass is given by  $A = 15x - x^2$ .
  - b Find the greatest area of the piece of glass.

### Problem solving

- 4 A farmer has 2400 m of fencing and wants to fence a rectangular field. A straight river will form one of the boundaries of the field as shown.
  - a Write an equation for the length of the fencing using  $x$  and  $y$ .
  - b Write an equation for the area of the field,  $A$ .
  - c Find the dimensions of the field that has the greatest area.
- 5 An object is projected vertically upwards. The height in metres of the object at  $t$  seconds is given by  $h(t) = 6t - 5t^2$ .
  - a When will the object reach its maximum height?
  - b What is the maximum height reached by the object?
- 6 A metal container in the shape of a rectangular prism is constructed from a total of 400 cm<sup>2</sup> of metal sheeting. The base of the container is square, as shown here.
  - a Write an equation for the total surface area of the container.
  - b Show that the height,  $h = \frac{100}{x} - \frac{x}{2}$ .
  - c Write an equation for the volume in terms of  $x$ .
  - d Find the maximum volume of the container and assess the method.
- 7 Find two nonnegative numbers whose sum is 12 and such that the product of one of the numbers and the square of the other number is a maximum.



- 8** Find the greatest volume of a rectangular box with a square base if the sum of the height and the side of the base must not exceed 24 cm.
- 9** 1200 cm<sup>2</sup> of material is available to make a box with a square base and an open top. Find the largest possible volume of the box.
- 10** A rectangle is formed using a wire of fixed length. Show that of all the rectangles that can be formed with the piece of wire, the one with the greatest area is a square.
- 11** A farmer wants to build a rectangular enclosure with three parallel partitions using 100 m of fencing, as shown here.



What dimensions will maximise the area of the enclosure?

- 12** Find the point  $P(x, y)$  on the  $x$ -axis that minimises the sum of the squares of the distances from  $P$  to the origin and to  $Q(5, 2)$ .
- 13** In an experiment, it is found that the temperature ( $^{\circ}\text{C}$ ) of an object is approximated by the function

$$f(t) = 40 - 24t + 9t^2 - t^3$$

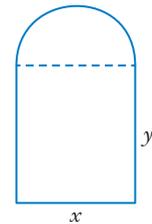
where  $t$  is time in minutes after the start of the experiment, given that  $1\frac{1}{2} \leq t \leq 6$ .

Find the greatest and least temperatures of the object during the given time interval.

- 14** An open cylinder has a surface area of  $3\pi \text{ m}^2$ . Find:
- the dimensions of the cylinder with the greatest volume
  - the greatest volume of the cylinder.
  - Would the surface area be sufficient information to make it from flat material?

- 15** A large window is to be constructed in the shape of a semicircle over a rectangle, as shown here.

If the distance around the outside of the window is 12 m, what dimensions will result in the rectangle having the largest possible area?



# 12. CHAPTER SUMMARY



## Applications of derivatives

- The **average rate of change** in a function  $f(x)$  from  $x_1$  to  $x_2$  is given by  $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ .
- The **instantaneous rate of change** of  $y = f(x)$  is the derivative,  $f'(x)$ .  
The derivative is a new function called the **gradient function**.

- For **straight-line motion**:

Displacement ( $s$ ) is the change in position ( $x$ )

$$\text{Velocity, } v = \frac{dx}{dt} \text{ or } \frac{ds}{dt}$$

$$\text{Acceleration, } a = \frac{dv}{dt}$$

- **Product rule**

$$\text{For the functions } f(x) \text{ and } g(x): \frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\text{For the functions } u \text{ and } v: \frac{d}{dx} (uv) = uv' + vu'$$

- **Quotient rule**

$$\text{For the functions } f(x) \text{ and } g(x): \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

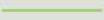
$$\text{For the functions } u \text{ and } v: \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$$

- **Power rule**

$$\text{If } n \text{ is any real number, then } \frac{d}{dx} (x^n) = nx^{n-1}$$

- Where a function has a **stationary point**, the gradient is zero. Where the function is **increasing**, the gradient is positive, and where the function is **decreasing**, the gradient is negative.

- A stationary point can be a local **maximum**, local **minimum** or point of horizontal **inflection**.

	$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	Type of point	Change in $f'(x)$ as $x$ increases
$f(x)$	+ 	0 	- 	Maximum 	Decreasing
$f(x)$	- 	0 	+ 	Minimum 	Increasing
$f(x)$	+ 	0 	+ 	Inflection on a rising curve 	Increasing to zero, then increasing again
$f(x)$	- 	0 	- 	Inflection on a falling curve 	Decreasing to zero, then decreasing again

- The **chain rule**

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x)$$

If  $y = f(u)$  and  $u = g(x)$ , then  $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

- The **greatest value** and **least value** over an interval include the values at the ends. If they exist, the **global maximum** and **global minimum** of a function are the maximum and minimum values across the whole domain.
- To sketch the graph of a function
  - 1 Calculate the zeros if possible.
  - 2 Determine the sign of *the function* between zeros.
  - 3 Identify stationary points and determine the intervals in which the function is increasing or decreasing.
  - 4 Find the  $y$ -intercept.
  - 5 Determine the behaviour of the function as  $x \rightarrow \pm \infty$ .
- To solve optimisation problems on an interval
  - 1 Read the problem carefully.
  - 2 Draw a diagram and introduce variables for the unknown quantities.
  - 3 List known facts and relationships using equations.
  - 4 Determine the quantity to be maximised or minimised and express it as a function of one of the variables.
  - 5 Find the maximum or minimum and express your answer in terms of the question.
  - 6 State any assumptions you have made and examine your solution for possible extensions to other cases.

# 12. CHAPTER REVIEW

## Applications of derivatives

Example

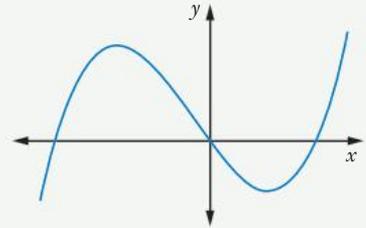
1

- 1 If  $f(x) = 2x^2 - 4x + 11$ , calculate:
- the average rate of change from  $x = 2$  to  $x = 5$
  - the instantaneous rate of change when  $x = -1$
  - the value of  $x$  at which the rate of change is 0.

Example

2

- 2 The diagram shows the graph of a function  $y = f(x)$ .  
Copy this diagram and sketch the gradient function on the same axes.



Example

3

- 3 The volume (in litres) of liquid in a container with a slow leak is given by  $V = 500 - 10t - \frac{1}{50}t^2$ , where  $t$  is in seconds. Find:
- the average rate of change from  $t = 3$  to  $t = 10$
  - the instantaneous rate of change when  $t = 5$ .

Example

4

- 4 The displacement in cm after time  $t$  s of a particle moving in a straight line is given by

$$x = 4 + 9t - \frac{1}{3}t^3 \text{ for } 0 \leq t \leq 7$$

- Find the initial displacement.
- Find the displacement after 5 s.
- How far does the particle move in the first 5 seconds?
- Find its velocity after 2 s.
- When is the particle at rest?

Example

5

- 5 A marble dropped into a barrel of water falls such that its height after  $t$  s is given by  $h(t) = 1.5 - 0.1t^2$ . Find the height, velocity and acceleration of the marble after 3 s.

Example

5

- 6 The height, in metres, of a stone thrown vertically upwards is given by  $h(t) = 8 + 30t - 5t^2$ , where  $t$  is in seconds.
- Find the initial displacement and velocity of the stone.
  - Calculate the stone's acceleration.
  - When will the stone be at the origin?
  - Calculate the stone's maximum displacement from the origin.
  - Sketch the graph showing the stone's motion up until it reaches the origin.

Example  
**6**

**7** Differentiate each function.

**a**  $x(2x^2 - 3)$       **b**  $(2x - 7)(3x^2 + x^3)$       **c**  $(3x^3 - 2x^2 + 1)(2x^2 - 3x + 2)$

Example  
**7**

**8** Differentiate each function.

**a**  $\frac{1}{3x+5}$       **b**  $\frac{1}{5x^2-2x}$       **c**  $\frac{7x+3}{2x-1}$

Example  
**8**

**9** Differentiate each function.

**a**  $\frac{3}{2x^4}$       **b**  $3\sqrt{x} + 4x^2$       **c**  $\frac{1}{x} - \frac{1}{\sqrt{x}}$       **d**  $\frac{x - \sqrt{x}}{x^2}$

Example  
**9**

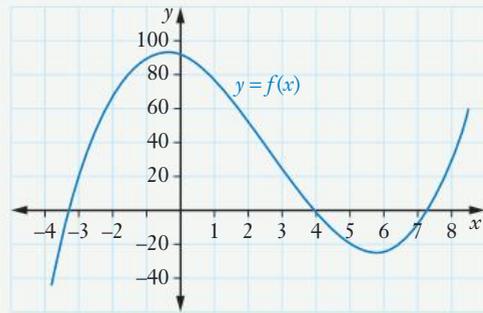
**10** For the function  $f(x) = x^2 - x - 12$ , find:

- a** the point at which the function is stationary
- b** all  $x$  values for which the function is increasing
- c** all  $x$  values for which the function is decreasing.

**11** Identify the stationary points of  $f(x) = x^3 - 5x^2 + 3x + 2$ .

**12** Use the graph of  $y = f(x)$  to estimate:

- a** the coordinates of stationary points
- b** the  $y$  value at  $x = 1$
- c** the intervals where the function is increasing
- d** the rate of change at  $x = 2$ .



Examples  
**10, 11**

**13** Sketch the graph of each function.

**a**  $f(x) = x^2 - 5x - 36$       **b**  $f(x) = 5 - (x - 5)^3$       **c**  $y = x^3 - 12x^2 + 48x - 20$

Example  
**12**

**14** Sketch the graph of  $h(x) = x^3 - x^2 - 5x + 9$ .

Example  
**12**

**15** Sketch the graph of  $y = 36x - 3x^2 - 2x^3 + 5$ .

Example  
**12**

**16** Sketch the graph of  $y = 5x - 3x^2 + 6 - x^3$ , identifying all key features.

Example  
**13**

**17** Find the derivative of each function.

**a**  $7(4x + 7)^8$       **b**  $6(2x - 5)^6$

Example  
**14**

**18** Differentiate:

**a**  $3(2x - 9)^{-5}$       **b**  $\frac{12}{5(7 - 4x)^3}$

Example  
**15**

Example  
16

19 Differentiate:

a  $7(12 - 5x)^9(4x - 11)^4$

b  $-3(3 - x)(4x + 15)^6$

Example  
17

20 Find the greatest and least values of the function  $f(x) = x^2 - 3x - 10$  over the domain  $-3 \leq x \leq 4$ .

Example  
18

21 Find the greatest and least values of the function  $f(x) = 9x^2 - x^3 - 24x + 20$  for  $1 \leq x \leq 6$ .

Example  
18

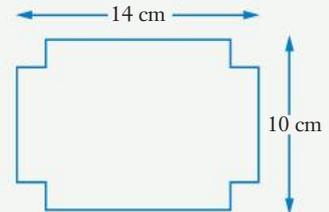
22 Use a graphics calculator to find the global maximum and global minimum of the function  $f(x) = 5x^2 - 2x^3 + 17x + 10$  for  $-1 \leq x \leq 4$ .

Example  
19

23 A 10 cm by 14 cm rectangular piece of cardboard has equal square corners cut out and is folded up to make an open box.

Calculate the maximum volume of the box.

Verify and evaluate your model.



### Problem solving

24 A woman of height 1.8 m is walking away from a street light at a speed of 1.2 m/s. The street light is 4.2 m above the road. You are interested in the rate of change of the length of the woman's shadow.

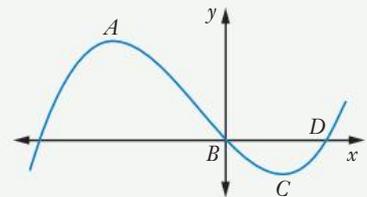


- a Find the relationship between the shadow length  $y$  and the woman's distance  $x$  from the street light.
- b State the relationship between the rate of increase of the shadow length and the woman's walking speed.
- c Find the rate of increase of the shadow length when she is:
  - i 3 m from the light
  - ii 6 m from the light.

25 At a theme park, the flume ride track follows the curve

$$y = \frac{x^3 + 20x^2 - 1500x}{720}$$

as shown in the diagram.



The highest point is at  $A$  and the lowest point is at  $C$ .  
The 'log' enters a tunnel at  $B$  and leaves the tunnel at  $D$ .

- a Find the greatest height.
- b Find the lowest point.
- c Find the drop from the highest point to the lowest.

- d** Find the average gradient of the track from  $A$  to  $C$ .
- e** Find the steepest point on the track between  $A$  and  $C$ .
- f** What is the gradient at the steepest point?

**26** Find the point on the line  $y = 3x + 4$  that is closest to  $(1, 2)$ .

**27** A closed cylindrical container is to be constructed from a maximum of  $280 \text{ cm}^2$  of aluminium sheet. The greatest radius allowable is  $4 \text{ cm}$ . Find, correct to 2 decimal places, the greatest volume the container can have. Verify and evaluate the usefulness of your method.

**28** Find the greatest and least values of the function  $y = 3x^4 + 6x^3 - 15x^2$  for  $-3 \leq x \leq 2$ .



## Practice examination 4

**Time: 90 minutes**  
**Perusal time: 5 minutes**  
**Marks: 50**

### Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

### Question 1 (2 marks)

- a** Find the gradient function of  $f(x) = 2x^2 - 5x + 7$ .
- b** Use the gradient function to determine the values of  $x$  where  $f(x)$  is decreasing.

### Question 2 (3 marks)

Find any values of  $x$  for which the gradient function of  $y = x^3 - 2x^2 - 4x + 7$  is equal to 3.

### Question 3 (2 marks)

Find the derivative of each function.

- a**  $f(x) = 10x^7$
- b**  $g(x) = 2x^3 + x^4 - 6x + 2$

### Question 4 (1 mark)

Find the rate of change of  $f(x) = 3x^4 - 2x^3 + 7x$  at  $x = 2$ .



Year 11  
formulas

**Question 5** (2 marks)

Find the equation of the tangent to the curve  $y = x^3 - 5x^2 + x + 7$  at  $x = -1$ .

**Question 6** (3 marks)

A random prime number between 1 and 20 inclusive is divided by 4.

$x$  is the remainder.

- Find the probability distribution of  $x$ .
- State the nature of the variable  $X$ .

**Question 7** (2 marks)

A random variable  $V$  has the following probability distribution.

$v$	2	3	5	7	10
$P(V=v)$	0.1	0.1	0.2	0.4	0.2

Find  $E(V)$  and  $Var(V)$ , correct to 2 decimal places.

**Question 8** (2 marks)

Find the instantaneous rate of change of  $f(x) = 3x^2 - 5x + 9$  at  $x = -1$ .

**Question 9** (2 marks)

An object is moving in a straight line so its distance from the origin is  $s(t) = t^2 + 3t$  metres at  $t$  seconds.

Find the velocity and acceleration of the object after 2 s.

**Question 10** (6 marks)

Find the derivative of each function.

- $(3x - 5)(2x^2 + x^3 - 5)$
- $\frac{3-x}{x^2 + 4x}$
- $(3x^2 - 4x + 8)^5$

**Question 11** (2 marks)

Find the derivative of  $(2x - 5)^3(x + 2)^4$  and express it in factorised form.

**Question 12** (3 marks) (Technology-active)

Find the greatest and least values of the function  $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$  over the domain  $-1 \leq x \leq 4$ .

**Question 13** (3 marks)

The volume of air in a balloon is given by

$$V(t) = \frac{1}{3}t^3 - 6t^2 + 32t \quad \text{for } 0 \leq t \leq 4$$

where  $V$  is in  $\text{cm}^3$  and  $t$  is in minutes.

Calculate the rate of change of the volume when  $t = 2$ .

**Question 14** (2 marks) (Technology-active)

The grades given to students in the Mathematical Methods classes for Semester 1 were distributed as shown below.

<b>Grade</b>	A	B	C	D	E
<b>Number</b>	7	11	29	15	17

Use  $A = 6$ ,  $B = 4$ ,  $C = 3$ ,  $D = 2$  and  $E = 0$  to calculate the expected value of the grade of a randomly-selected student.

**Question 15** (5 marks) (Technology-active)

Sketch the graph of  $h(x) = x^3 + 4x^2 - 3x + 2$ , identifying important features.

**Question 16** (3 marks)

The tangent to  $2x^3 + x^2 - 6x + 2$  at  $A(1, 7)$  crosses the  $x$ -axis at  $B$ .

Find the length of  $AB$ , correct to 2 decimal places.

**Question 17** (2 marks)

Suzanne and Daphne agree to play badminton until one of them wins 2 games in a row. From past experience, Suzanne has a probability of winning of 0.4 and Daphne 0.6, and on this occasion Daphne wins the first game. What is the probability that Suzanne wins eventually?

**Question 18** (5 marks) (Technology-active)

An open cylindrical tank is to have a volume of  $64 \text{ cm}^3$ .

- a Find the dimensions of the tank that minimise the surface area.
- b Estimate the area of sheet metal that will actually be used to make it.

**END OF EXAMINATION**

# ANSWERS

## Chapter 1

### Investigation: Fibonacci numbers

Leonardo of Pisa

Most of the numbers of petals are Fibonacci numbers.

Breeding (of rabbits or cows), Spirals on pine cones, sunflower seed heads, etc.

Binet's formula  $F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}$ , where  $\phi = \frac{1+\sqrt{5}}{2}$   
(the golden ratio)

### Exercise 1.01

- 1 a** 13, 15, 17, 19      **b** 25      **c**  $t_{14}$   
**2 a** 44      **b** -42      **c** 225      **d** 12 288  
**3 a**  $t_{23}$       **b**  $t_{11}$       **c**  $t_8$       **d**  $t_7$   
**4 a** 5, 8, 11, 14, 17, ...      **b**  $t_n = 3n + 2$   
**5 a** 12, 8, 4, 0, -4  
**b** 12, 6, 3, 1.5, 0.75  
**c** 4, 7, 13, 25, 49  
**d** 6, 5, 4.5, 4.25, 4.125  
**e**  $\frac{3}{16}, \frac{3}{4}, 3, 12, 48$   
**6 a**  $t_n = 3n + 4$   
**b**  $t_n = \frac{320}{4^n}$  or  $t_n = 80 \times 4^{1-n}$   
**c**  $t_n = 2.5n + 5.5$   
**d**  $t_n = n^3$   
**e**  $t_n = 4 + \left(\frac{1}{2}\right)^{n-1}$  or  $4 + 2^{1-n}$   
**7 a** \$400, \$440 and \$640      **b** 28 weeks  
**8 a**  $16\frac{1}{8}$  seconds      **b** 16 seconds

### Exercise 1.02

- 1 a** AP,  $d = -2$       **b** AP,  $d = 1 - 2x$   
**c** Not an AP      **d** Not an AP  
**e** AP,  $d = \frac{1}{12}$       **f** Not an AP  
**2 a**  $t_n = 4n - 1$       **b**  $t_n = 105 - 3n$   
**c**  $t_n = \frac{22-7n}{6}$       **d**  $t_n = 2.1n - 0.3$   
**e**  $t_n = 2n - 17$       **f**  $t_n = 2nx + (4 - n)y$   
**3 a**  $t_7 = -16, t_{20} = -55$       **b**  $t_{11} = 83, t_{16} = 123$   
**c**  $t_9 = 19x - 16, t_{32} = 65x - 62$   
**d**  $t_6 = 1\frac{1}{4}, t_{12} = 5\frac{3}{4}$       **e**  $t_9 = 2.\bar{2}, t_{15} = 3.\bar{5}$   
**f**  $t_{20} = -107, t_{30} = -167$   
**4**  $-8\frac{3}{4}$   
**5** 11  
**6**  $\frac{12y-7x}{5}$   
**7**  $t_{50}$   
**8** 112  
**9**  $22\frac{1}{2} \leq a < 22\frac{12}{13}$

### Investigation: Diagonals

Convex polygons

Number of sides	4	5	6	7	8
Number of diagonals	2	5	9	14	20

$$\text{Number of diagonals} = 2 + 3 + 4 + 5 + \dots = \frac{n(n-3)}{2}$$

Prove formula using the fact that each vertex has  $n - 3$  diagonals.

### Exercise 1.03

- 1 a** 365      **b** 1666      **c** 356  
**d** -20      **e**  $9x + 9$       **f** 943.5  
**2 a** 255      **b** -2124      **c**  $50x + 380$   
**d** 2409      **e** -2284      **f**  $3966\frac{2}{3}$   
**3** 20  
**4** 19  
**5** 200  
**6** -21  
**7** 42  
**8** 6875  
**9** 350  
**10** 7

### Exercise 1.04

- 1** 182  
**2 a** The odd numbers: 1, 3, 5, 7, ...  
**b**  $1 + 3 + 5 + \dots + (2n - 1) = n^2$   
**3**  $n$ th pentagonal number =  $\frac{n(3n-1)}{2}$   
**4**  $n$ th hexagonal number =  $n(2n - 1)$   
**5 a** \$8520 (7 years interest)  
**b** \$8160      **c** \$58 080  
**6 a** 800 L    **b** 5300 L    **c** 7:15 p.m.  
**7**  $V = \$138\,500 - \$8500n$ , 13 years  
**8** 2.55 km

### Chapter 1 review

- 1 a** 127      **b**  $t_{10}$   
**2 a** 1, 6, 16, 36, 76, ...    **b**  $t_n = 5 \times 2^{n-1} - 4$   
**3 a** Not an AP      **b** 4  
**c** Not an AP      **d** -2  
**4 a**  $t_n = \frac{121-23n}{7}$     **b**  $-15\frac{4}{7}$       **c**  $t_{15} = 32$   
**5** 62  
**6** 654  
**7**  $495p - 570q$   
**8** 11  
**9** 1680  
**10**  $S_n = \frac{n(3n-1)}{2}$

- 11**  $2\frac{6}{7}$   
**12** 451 000  
**13** -752  
**14** About 121 000

## Chapter 2

### Exercise 2.01

- 1 a**  $\frac{1}{a^2}$     **b**  $\frac{1}{4^3}$     **c**  $\frac{1}{g^7}$     **d**  $\frac{1}{5^4}$   
**e**  $\frac{1}{y^6}$     **f**  $\frac{1}{c^4d^8}$     **g**  $\frac{4}{h^4}$     **h**  $\frac{1}{256h^4}$   
**i**  $\frac{2}{k}$     **j**  $\frac{7}{p^4}$     **k**  $\frac{8}{q^6}$     **l**  $\frac{v^2}{4}$   
**2 a**  $y^{-4}$     **b**  $4^{-3}$     **c**  $h^{-9}$     **d**  $t^4k^{-6}$   
**e**  $y^2p^{-6}$     **f**  $4j^{8i^{-6}}$     **g**  $3y^{-2}m^{-5}$     **h**  $2m^{-1}z^{-4}$   
**i**  $x^8$     **j**  $2x^4y^4$     **k**  $3k^{-3}b$     **l**  $4r^{-9}w^6$   
**3 a**  $\sqrt{6}$     **b**  $\sqrt[4]{z}$   
**c**  $\sqrt[3]{216}$     **d**  $\sqrt[5]{y^3}$  or  $(\sqrt[5]{y})^3$   
**e**  $\sqrt{49^3}$  or  $(\sqrt{49})^3$     **f**  $\sqrt[3]{v^7}$  or  $(\sqrt[3]{v})^7$   
**g**  $\sqrt[3]{8^7}$  or  $(\sqrt[3]{8})^7$     **h**  $\sqrt[4]{16^5}$  or  $(\sqrt[4]{16})^5$   
**i**  $\sqrt{m^5}$  or  $(\sqrt{m})^5$   
**j**  $\sqrt[4]{u^{-1}}$  or  $(\sqrt[4]{u})^{-1}$  or  $\sqrt[4]{\frac{1}{u}}$  or  $\frac{1}{\sqrt[4]{u}}$   
**k**  $\sqrt{10}$   
**l**  $\sqrt[5]{p^{-3}}$  or  $(\sqrt[5]{p})^{-3}$   
**4 a**  $\frac{1}{k^2}$     **b**  $3^4$     **c**  $5^2$     **d**  $4^5$   
**e**  $\frac{1}{k^3}$     **f**  $\frac{3}{c^2}$     **g**  $\frac{4}{6^3}$     **h**  $\frac{3}{e^4}$   
**i**  $28^{\frac{4}{9}}$     **j**  $q^{\frac{6}{5}}$     **k**  $8^{\frac{5}{2}}$     **l**  $16^{\frac{7}{8}}$   
**5 a**  $\frac{1}{r^2}$     **b**  $\frac{12}{y^9}$     **c**  $3p^4$     **d**  $\frac{2}{j^6}$   
**e**  $\frac{1}{6m^{13}}$     **f**  $\frac{1}{4k^{14}}$     **g**  $\frac{15g}{h^2}$     **h**  $5k^{18}m^{14}$   
**i**  $16v^{11}$     **j**  $\frac{4q^{17}}{p^{13}}$     **k**  $\frac{4e^{12}}{3f^{17}}$     **l**  $\frac{c^3d^{19}}{2}$

6 a  $a^{\frac{4}{2}}$       b  $b^{\frac{3}{2}}$       c  $\frac{6}{n^{\frac{1}{2}}}$

d  $12g^{\frac{6}{3}}$       e  $12w^{\frac{3}{3}}p^{\frac{2}{2}}$       f  $10m^{\frac{3}{2}}n^{\frac{2}{3}}$

g  $\frac{21w^3}{s^{\frac{3}{2}}}$       h  $\frac{24p^{\frac{1}{2}}}{q^{\frac{5}{12}}}$

7 a 8      b 1      c 2      d 1

e 6      f 9      g 1      h 9

i 343      j 216      k 16      l 64

m  $\frac{1}{4}$       n  $\frac{16}{9}$       o 2      p 1

q  $\frac{64}{27}$       r  $\frac{1}{36}$       s 125      t  $\frac{2}{5}$

u  $\frac{27}{8}$       v 4      w 100 000      x  $\frac{1}{10000}$

8 a  $2^4$       b  $5^{k+1}$       c  $3 \times 2^3$

d  $2^m 3^{2n+3m}$       e  $3^{2i+2} \times 7^3$       f  $2^3$

### Exercise 2.02

1 a  $6.3 \times 10^3$       b  $1.42 \times 10^8$       c  $7.5 \times 10^{-3}$

d  $9.38 \times 10^{-7}$       e  $5 \times 10^5$       f  $6 \times 10^{-4}$

g  $2.1 \times 10^2$       h  $7.9 \times 10^{-1}$       i  $3.109 \times 10^7$

j  $4.062 \times 10^{-2}$       k  $9.623 \times 10^6$       l  $2 \times 10^{-9}$

m  $4 \times 10^1$       n  $5.6 \times 10^0$       o  $5 \times 10^{-1}$

p  $4 \times 10^3$       q  $7 \times 10^{-3}$       r  $3.02 \times 10^{11}$

s  $7.4 \times 10^{-10}$       t  $1.02 \times 10^5$       u  $5.3 \times 10^{-5}$

v  $2.71 \times 10^4$       w  $5.8 \times 10^1$       x  $9.5 \times 10^{-2}$

2 a 430 000      b 0.0021      c 9 840 000

d 0.319      e 0.000 000 006

f 20 000      g 42      h 0.058

i 1.6      j 0.000 095      k 464

l 2100      m 50 000      n 0.3

o 0.000 004 08      p 0.098 42      q 70 000

r 60 300 000      s 0.005 106      t 0.000 000 9

u 324 000 000      v 16 300      w 2950

x 0.83

3 a  $1.472 \times 10^2$       b  $7.314 148 \times 10^{10}$

c  $2.1996 \times 10^{19}$       d  $4.10 \times 10^{-6}$

e  $1.020 625 \times 10^{11}$       f  $4.1884 \times 10^9$

g  $8.629 804 \times 10^{-8}$       h  $7.226 \times 10^7$

4 a  $2.715 642 5 \times 10^{14}$       b  $6.541 430 8 \times 10^8$

c  $1.171 875 \times 10^{-9}$       d  $1.969 \times 10^{21}$

e  $3.111 251 8 \times 10^{-3}$       f  $2.309 994 6 \times 10^{-33}$

5 a 3      b 2      c 1      d 1

e 1      f 3      g 4      h 5

i 2      j 4      k 3      l 2

m 3      n 2      o 4      p 5

q 6      r 2      s 4      t 1

u 3      v 7      w 2      x 3

6 a 40      b 480      c 248

d 0.000 006 0      e 20 000      f 3.6

g 63 500      h 18 950 000 000      i 36.0

### Exercise 2.03

1 a  $15h + 30$       b  $-6mn + 12n$

c  $-10q^2 + 26q$       d  $-21g^2h + 49h^2$

e  $20t - 8tx + 28ty$       f  $6hj^2 - 15h^2j + 3hj$

g  $5y^2 - 16y + 18$

h  $-6y^4 + 2xy^4 + 4by^4 + 12y^4z^2$

i  $8bk + 28 - 6b^2k + 3b^2k^2$

j  $-10x^4yz^2 + 25y^2z^2 - 15xy^2z^2$

k  $2fg - 2f^2 - 3g^2 + 3f^3g$

l  $3p^2 - 3pq^3 - 2q^2 + 8p^4q$

2 a  $2x^2 + 4x + 5xy + 4y + 3y^2$

b  $6x^2 - 17xy + 10x + 12y^2 - 15y$

c  $6a^2 + 8ab + 9a - 4b - 6$

d  $6c^2 - cd - 3c - d^2 - d$

e  $-2v^2 + 2vw - 2cv + 4w^2 + 4cw$

f  $-3b^3 - 12ab^2 + 15b^2 - 9a^2b + 15ab$

3 a  $9r^3 - 15r^2 - 2r + 8$

b  $6t^3 - 5t^2g - 7tg^2 + 4g^3$

c  $m^3 + 7fm^2 - 5f^2m - 75f^2$

d  $25k^3 - 95k^2a + 64ka^2 - 12a^3$

e  $15v^3 - 85v^2 - 40v + 560$

f  $42b^4 - 231b^3 + 210b^2 + 168b$

g  $4q^2 + 5p^2 - 9pq + 24q - 28p + 32$

h  $8g^2 - 3h^2 + 2gh - 34g - 3h + 36$

i  $12y^2 - 9z^2 + 12yz - 4y + 6z - 1$

j  $2z^4 + 7z^3 - 7z^2 + 14z + 8$

k  $c^5 + 3c^4 + c^3 + 5c^2 + 23c - 7$

l  $45c^4d^2 - 36c^3d^3 + 30c^2d^2 - 30c^2d^5 - 24cd^3 + 24cd^6$

- 4 a**  $3a^2 - a - 4$       **b**  $15v^2 - 4v - 32$   
**c**  $28h^2 + 41h + 15$       **d**  $36g^2 - 55g - 14$   
**e**  $42m^2 + m - 56$       **f**  $5a^2 - 8ab + 3b^2$   
**g**  $10p^2 - 31gp + 15g^2$       **h**  $21t^2 + 16jt + 3j^2$   
**i**  $35u^2 - 9uw - 18w^2$       **j**  $64x^2 + 32xy - 45y^2$   
**k**  $25e^2 - 65ef + 42f^2$       **l**  $35z^2 - 29uz - 28u^2$
- 5 a**  $a^2 - 2ab + b^2$       **b**  $k^2 - 2km + m^2$   
**c**  $x^2 + 2xy + y^2$       **d**  $9y^2 - 24y + 16$   
**e**  $25f^2 + 40f + 16$       **f**  $4a^2 - 28a + 49$   
**g**  $16h^2 + 24hm + 9m^2$       **h**  $4p^2 - 20jp + 25j^2$   
**i**  $49a^2 + 70ab + 25b^2$       **j**  $4f^2 - 36f + 81$   
**k**  $64v^2 + 16ev + e^2$       **l**  $36f^2 - 120cf + 100c^2$
- 6 a**  $a^2 - b^2$       **b**  $x^2 - y^2$       **c**  $g^2 - 9h^2$   
**d**  $y^2 - 36$       **e**  $9q^2 - 16$       **f**  $16r^2 - 25s^2$   
**g**  $9p^2 - 49q^2$       **h**  $64i^2 - 9j^2$       **i**  $25 - x^2$   
**j**  $64 - 25k^2$
- 7 a**  $x^2 + 2xy + y^2 - 1$       **b**  $a^2 + 4ab + 4b^2 - 16$   
**c**  $9t^2 - s^2 - 4s - 4$       **d**  $16w^2 - z^2 - 10z - 25$
- 8 a**  $x^3 + 3ax^2 + 3a^2x + a^3$   
**b**  $x^3 - 15x^2 + 75x - 125$   
**c**  $27x^3 - 54x^2 + 36x - 8$   
**d**  $x^3 + 3x^2 + 3x + 1$   
**e**  $27v^3 - 108v^2w + 144vw^2 - 64w^3$   
**f**  $x^3 + 9x^2 + 27x + 27$   
**g**  $8m^3 + 36m^2n + 54mn^2 + 27n^3$   
**h**  $x^3 - 12x^2 + 48x - 64$   
**i**  $8x^3 + 12x^2 + 6x + 1$   
**j**  $x^3 + 21x^2 + 147x + 343$   
**k**  $x^3 + 6x^2y + 12xy^2 + 8y^3$   
**l**  $x^6 - 9x^4y^2 + 27x^2y^4 - 27y^6$
- 9 a**  $a^3 + b^3$       **b**  $m^3 + n^3$       **c**  $a^3 - b^3$   
**d**  $x^3 - y^3$       **e**  $z^3 + 1$       **f**  $d^3 - 27$   
**g**  $p^3 + 125$       **h**  $8 + 27k^3$       **i**  $27u^3 - 64v^3$   
**j**  $125e^3 - 8i^3$

### Exercise 2.04

- 1 a**  $2(3a + 4)$       **b**  $2(9m - 8)$       **c**  $-7(2x + y)$   
**d**  $x(x + 1)$       **e**  $m(m + 4)$       **f**  $2x(x - 2)$   
**g**  $ab(a + 1)$       **h**  $cd(d - c)$       **i**  $8m(3m + n)$   
**j**  $2ab(a - 2)$       **k**  $2ab(5a + 6b)$       **l**  $12ab(3b + 4a)$

- 2 a**  $(2 + x)(a + b)$       **b**  $(3a^2 - 12a)(x + y)$   
**c**  $-(3a + 5)(a + 2)$   
**d**  $4a(9abc - 6ac + 1)(3b - 2)$   
**e**  $(a + b)(c - 1)$       **f**  $(5i - 2j)(g - h)$   
**g**  $(5p + 3q)(3m - 2n)$       **h**  $(4d + 10)(2c - 3)$   
**i**  $(g - h)(i + j)$       **j**  $(k - 2n)(m^2 + h^2)$
- 3 a**  $(a + b)(c - 1)$       **b**  $(g - 2h)(3i - 2j)$   
**c**  $(3m - 2n)(5p + 3q)$       **d**  $2(2c - 3)(2d + 5)$   
**e**  $(x^2 + 1)(x - 1)$       **f**  $(4g - 3h)(2h^2 - 9)$   
**g**  $(2x - 3y)(3x^2 + y^2)$       **h**  $(3h + 2i)(h^2 - 3i)$   
**i**  $(g - h)(i + j)$       **j**  $(a^3 - 3b)(a + c)$
- 4 a**  $3(2a - 5b)$       **b**  $4c(3c - 5)$   
**c**  $4ac(3ac - b)$       **d**  $4ab(3a - 2b)$   
**e**  $6mn^2(2m^3 - 3mn + 5)$       **f**  $8x^2y(5x - 3y + 8)$
- 5 a**  $\frac{p-4q}{8}$       **b**  $\frac{2b-7}{8}$       **c**  $\frac{c+3d}{3d}$   
**d**  $\frac{3p}{4q}$       **e**  $\frac{8v-17}{6}$       **f**  $\frac{22q-13}{20}$   
**g**  $\frac{v^2+3v+3}{v(v+1)}$       **h**  $\frac{19f+4g}{(2f-g)(3f+g)}$   
**i**  $\frac{5w^2+w}{w^2-1}$       **j**  $\frac{2a^2+5a}{a+5}$
- 6 a**  $\frac{1}{2}$       **b**  $\frac{14x+56}{15x+30}$       **c**  $\frac{m}{5m+5}$       **d**  $\frac{7}{2}$   
**e**  $\frac{t^2}{6t-6}$       **f**  $\frac{5x-15}{2x-60}$       **g**  $\frac{3q}{10q-2}$       **h**  $\frac{1}{2p}$

### Exercise 2.05

- 1 a**  $(3k - 5)(k - 2)$       **b**  $(v - 2)(v + 8)$   
**c**  $(5m + n)(m + n)$       **d**  $(7p - 3)(p - 3)$   
**e**  $(7x - 6y)(x + y)$       **f**  $(2z - 3)(z - 4)$   
**g**  $(3m - 8y)(m + 4y)$       **h**  $(j - 3k)(j + 4k)$   
**i**  $(7v - 2w)(v + 3w)$       **j**  $(3c + 7)(c + 4)$   
**k**  $(3x - 5y)(x - y)$       **l**  $(5e + 4f)(e - f)$   
**m**  $(5z + 1)(z + 7)$       **n**  $(3c + 7d)(c - 7d)$   
**o**  $(2n - 7)(n + 5)$       **p**  $(3y - 5)(y - 6)$   
**q**  $(7g + 4h)(g - 8h)$       **r**  $(5m - 4n)(m - 8n)$   
**s**  $(3a - 7)(a - 8)$       **t**  $(7x - 2y)(x - y)$   
**u**  $(7m - 9p)(m - 3p)$       **v**  $(3z - 5y)(z + 3y)$   
**w**  $(c + 2d)(c - 6d)$       **x**  $(3r - 4s)(r + s)$
- 2 a**  $(3x - 5)(x - 2)$       **b**  $(x + 8)(x - 2)$   
**c**  $(7x - 6)(x + 1)$       **d**  $(2x - 3)(x - 4)$   
**e**  $(4x + 3)(x - 4)$       **f**  $(4x + 3)(2x + 3)$   
**g**  $(5x + 1)(x + 7)$       **h**  $(3x + 7)(x - 7)$

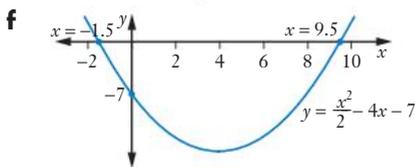
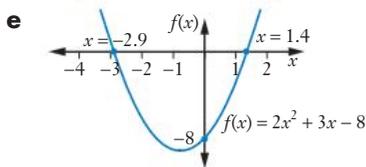
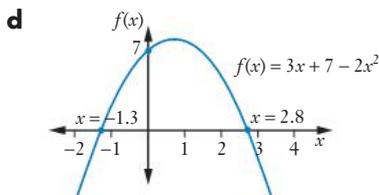
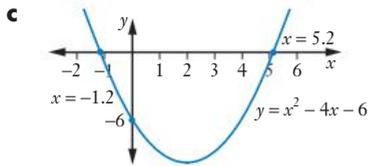
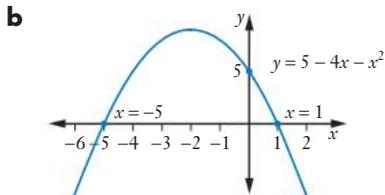
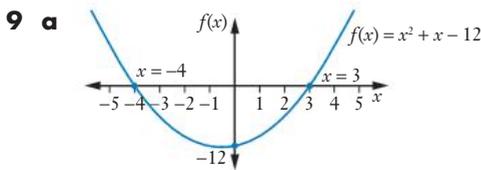
**i**  $(7x+4)(x-8)$       **j**  $(5x-4)(x-8)$   
**k**  $(5x+2)(4x-9)$       **l**  $(5x+2)(5x-8)$   
**3 a**  $(x+4y)^2$       **b**  $(2b-5c)^2$   
**c**  $(n-7)(n+7)$       **d**  $(5p+7q)^2$   
**e**  $(7b-4)^2$       **f**  $(9v-11w)(9v+11w)$   
**g**  $(6g-1)^2$       **h**  $(h+8)(h-8)$   
**i**  $(10+9g)(10-9g)$       **j**  $(12m+5e)^2$   
**k**  $(3y-8)^2$       **l**  $(9a-7b)^2$   
**m**  $(10p+13r)^2$       **n**  $(8g+11)(8g-11)$   
**4 a**  $(t-u)(t^2+tu+u^2)$   
**b**  $(4t+3u)(16t^2-12tu+9u^2)$   
**c**  $(2j-m)(4j^2+2jm+m)$   
**d**  $(f+2)(f^2-2f-4)$   
**e**  $(h-4)(h^2+4h+16)$   
**f**  $(2k+3)(4k^2+6k+9)$   
**g**  $(6j-5k)(36j^2+30jk+25k^2)$   
**h**  $(w+7)(w^2-7w+49)$   
**i**  $(4d+e)(16d^2-4de+e^2)$   
**j**  $(1-3m^2)(1+3m^2+9m^4)$

### Exercise 2.06

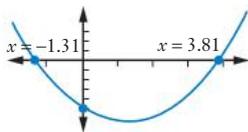
**1 a**  $x=0$  or  $x=12$       **b**  $x=5$  or  $x=-2$   
**c**  $x=0$  or  $x=8$       **d**  $x=0$  or  $x=-\frac{3}{2}$   
**e**  $x=-7$  or  $x=-2$       **f**  $x=5$  or  $x=3$   
**2 a**  $x=\frac{4}{3}$  or  $x=-3$       **b**  $x=1$  or  $x=11$   
**c**  $x=-1$  or  $x=-4$       **d**  $x=-1$  or  $x=8$   
**e**  $x=-5$  or  $x=-2$       **f**  $x=-1$  or  $x=3$   
**3 a**  $x=5$  or  $x=-1$       **b**  $x=3$  or  $x=-2$   
**c**  $x=0$  or  $x=13$       **d**  $x=3$   
**e**  $x=-\frac{1}{2}$  or  $x=-2$       **f**  $x=-\frac{1}{3}$  or  $x=-3$   
**4 a**  $a=\frac{2}{5}$  or  $a=-2$       **b**  $v=6$  or  $v=3\frac{1}{2}$   
**c**  $n=-8$  or  $n=8$       **d**  $c=3\frac{1}{2}$   
**e**  $e=-1\frac{1}{2}$  or  $e=-\frac{6}{7}$       **f**  $p=\frac{8}{9}$  or  $p=-2$   
**g**  $h=-\frac{4}{5}$  or  $h=7$       **h**  $x=6$  or  $x=-8$   
**i**  $u=-7$  or  $u=-\frac{5}{9}$       **j**  $r=-2\frac{1}{2}$  or  $r=\frac{1}{2}$   
**k**  $y=-2\frac{1}{2}$  or  $y=\frac{5}{8}$       **l**  $g=-\frac{15}{16}$  or  $g=1\frac{1}{2}$

**5 a**  $b=3\frac{1}{2}$  or  $b=2$       **b**  $d=-2$  or  $d=-\frac{5}{7}$   
**c**  $x=-7$  or  $x=7$       **d**  $a=4$   
**e**  $g=-5$  or  $g=-\frac{3}{4}$       **f**  $x=\frac{3}{8}$  or  $x=\frac{1}{2}$   
**g**  $f=-2$  or  $f=3$       **h**  $e=-1\frac{2}{3}$  or  $e=1\frac{3}{5}$   
**i**  $z=1\frac{4}{5}$  or  $z=6$       **j**  $t=1$  or  $t=1\frac{1}{2}$   
**k**  $m=-\frac{5}{6}$  or  $m=1$       **l**  $p=\frac{1}{3}$  or  $p=3$   
**6 a**  $x=1$  or  $x=7$       **b**  $x=-3$  or  $x=-6$   
**c**  $x=-2$  or  $x=\frac{4}{9}$       **d**  $x=-3$  or  $x=2$   
**e**  $x=1\frac{1}{8}$  or  $x=3$       **f**  $x=-2\frac{1}{3}$  or  $x=6$   
**g**  $x=7$  or  $x=-8$       **h**  $x=-\frac{1}{4}$  or  $x=9$   
**i**  $x=-1\frac{1}{2}$  or  $x=3\frac{1}{2}$       **j**  $x=-\frac{1}{2}$  or  $x=8$   
**k**  $x=-\frac{1}{2}$  or  $x=2\frac{1}{4}$       **l**  $x=-4$  or  $x=-5$   
**7 a**  $k=-5$  or  $k=1$       **b**  $v=-7$  or  $v=5$   
**c**  $j=1$  or  $j=8$       **d**  $x=-11$  or  $x=12$   
**e**  $m=-1\frac{1}{3}$  or  $m=\frac{2}{3}$   
**f**  $b=-5.16=-2-\sqrt{10}$  or  $b=1.16=-2+\sqrt{10}$   
**g**  $a=-1.62=\frac{-1+\sqrt{5}}{2}$  or  $a=0.62=\frac{-1-\sqrt{5}}{2}$   
**h**  $z=-0.85=\frac{3-\sqrt{41}}{4}$  or  $z=2.35=\frac{3+\sqrt{41}}{4}$   
**i**  $v=0.35=\frac{9+\sqrt{69}}{2}$  or  $v=8.65=\frac{9-\sqrt{69}}{2}$   
**j**  $x=3.47=\frac{15+\sqrt{65}}{2}$  or  $x=11.53=\frac{15-\sqrt{65}}{2}$   
**k**  $x=-2$  or  $x=1\frac{1}{2}$       **l**  $c=-2$  or  $c=1\frac{2}{3}$   
**m**  $m=0.18=\frac{3+\sqrt{7}}{2}$  or  $m=2.82=\frac{3-\sqrt{7}}{2}$   
**n**  $r=0.30=\frac{6+\sqrt{15}}{7}$  or  $r=1.41=\frac{6-\sqrt{15}}{7}$   
**o**  $h=-0.19=\frac{7+\sqrt{79}}{10}$  or  $h=1.59=\frac{7-\sqrt{79}}{10}$

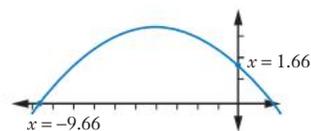
- 8 **a**  $x = -3.5$  and  $x = 0.7$     **b**  $x = -1.4$  and  $x = 4.8$   
**c**  $x = -1.5$  and  $x = 3.2$     **d**  $x = -3.6$  and  $x = 2.7$



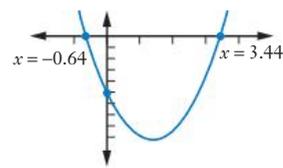
10 **a**  $f(x) = 2x^2 - 5x - 10$



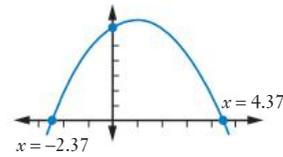
**b**  $f(x) = 8 - \frac{x^2}{2} - 4x$



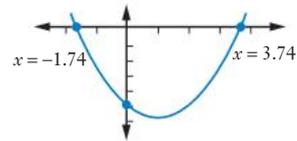
**c**  $y = 5x^2 - 14x - 11$



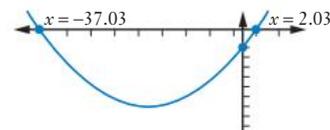
**d**  $y = 6x + 31 - 3x^2$



**e**  $y = 4x^2 - 8x - 26$



**f**  $y = \frac{x^2}{5} + 7x - 15$



11 Equations:  $l^2 - 0.9l = 16.2$  or  $w^2 + 0.9w = 16.2$   
 Dimensions: 3.6 m  $\times$  4.5 m

12 Equation:  $l^2 + 12l = 160$   
 Length of room: 8 m

13 Equations:  $6h^2 + 32h = 88$  or  $\frac{2l^2}{3} + \frac{32l}{3} = 88$   
 Dimensions: 2 cm  $\times$  4 cm  $\times$  6 cm

### Exercise 2.07

1 **a**  $x = -9$  or  $x = -5$

**b**  $x = -17$  or  $x = -3$

**c**  $x = -14$  or  $x = 21$

**d**  $x = -2 \pm \sqrt{11} \approx -5.32$  or  $1.32$

**e**  $x = 3 \pm \sqrt{30} \approx -2.48$  or  $8.48$

**f**  $x = -7$  or  $x = 20$

2 **a**  $f = -2\frac{1}{3}$  or  $f = 2$     **b**  $t = -1\frac{1}{5}$  or  $t = 3$

**c**  $x = -\frac{1}{6}$  or  $x = 1$

**d**  $x = -1.79$  or  $x = 2.79$

**e**  $p = -0.28$  or  $p = 1.95$     **f**  $d = -2.64$  or  $d = 1.14$

- 3 a**  $x = -7.36$  or  $x = 1.36$    **b**  $x = -1\frac{1}{3}$  or  $x = \frac{1}{2}$   
**c**  $x = -2.12$  or  $x = 6.12$    **d**  $x = 3.41$  or  $x = 2.73$   
**e**  $x = 2.54$  or  $x = 9.46$    **f**  $x = -2$  or  $x = -1\frac{1}{2}$   
**g**  $x = -28.79$  or  $x = -0.21$    **h**  $x = 0.11$  or  $x = 1.76$
- 4 a** 2 roots   **b** 1 root   **c** no roots  
**d** no roots   **e** 1 root   **f** 2 roots
- 5** Oscar 4 km/h and Hans 5 km/h  
**6** \$20  
**7** Large pipe 10 hours and small pipe 15 hours  
**8** 10 km/h  
**9** 40 ohms and 60 ohms

### Exercise 2.08

- 1 a**  $m = kV$    **b** 270 g  
**2 a** 38.4 m   **b** 30 pedal turns  
**3 a** 5625 kg   **b** 2 cm  
**4 a** About 1.4 seconds   **b** 392 cm  
**5 a**  $x = 3$    **b**  $x = 6$    **c**  $x = 2$   
**d**  $y = 4$    **e**  $y = 6$    **f**  $y = 12$   
**6 a**  $a = 5$    **b**  $a = 3$    **c**  $b = 3$   
**7 a** 6 minutes 24 seconds   **b** 8 cm  
**8 a** 5.12 km/L   **b** 1073 kg  
**9 a** 3.46 distance units   **b** 27 force units

### Exercise 2.09

- 1 a** no   **b** yes   **c** no   **d** yes  
**e** no   **f** yes   **g** no   **h** yes
- 2 a**  $-x^4$ , degree 4   **b**  $3x^7$ , degree 7  
**c**  $5x$ , degree 1   **d**  $t^9$ , degree 9  
**e** 5, degree 0   **f**  $-y^{10}$ , degree 10
- 3 a** 6   **b**  $\frac{1}{4}$   
**c** 7   **d** -2
- 4 a** 194   **b** 8   **c** -4   **d** 44  
**e** 212   **f** 548   **g** 19.66   **h** 23.80
- 6** 33 544
- 7 a**  $b = 6$    **b**  $c = 2\frac{2}{3}$    **c**  $a = 3$   
**d**  $a = -2$  and  $b = 1$    **e**  $d = -6$    **f**  $a = -1$
- 8 a**  $f(-2) = -2$    **b**  $x = 0$  or  $x = -1$   
**c**  $f(a+2) = a^3 + 8a^2 + 21a + 18$

**d**  $f(-2)f(x+h) = -2x^3 - 6x^2h - 6xh^2 - 2h^3 - 4x^2 - 8xh - 4h^2 - 2x - 2h$

### Exercise 2.10

- 1**  $Q(x) = 3x^2 + 3x + 4$ ,  $R(x) = 8x - 1$   
**2**  $Q(x) = 3x^3 - 2x^2 + 7x + 9$ ,  $R(x) = 25x - 3$   
**3**  $Q(x) = 4x^3 - 3x + 5$ ,  $R(x) = 8$   
**4**  $x^3 - 2x^2 - 16x + 32 = (x+4)(x-4)(x-2)$   
**5 a**  $(x+1)(x-2)(2x+3)$    **b**  $(x-1)^2(x+3)$   
**c**  $(x+1)(x-1)(x-2)$    **d**  $(x+2)(x-2)(2x-1)$   
**e**  $(x+2)(x+3)(3x+1)$    **f**  $(x-3)(x+2)(2x+5)$   
**g**  $(3x-1)(x+1)(x+3)$    **h**  $(3x+4)(x+2)(x-3)$   
**i**  $(x+2)(x+4)(4x-3)$

### Exercise 2.11

- 1 a**  $x = -1$  or  $x = 1\frac{1}{2}$   
**b**  $x = -2$  or  $x = -1\frac{1}{3}$  or  $x = 2$   
**c**  $x = -3$  or  $x = 1$  or  $x = 5$   
**d**  $x = -6$  or  $x = 2$   
**e**  $x = -3$  or  $x = -2$  or  $x = 1$  or  $x = 3$   
**f**  $x = -2$  or  $x = -\frac{1}{2}$  or  $x = 1$  or  $x = 2$   
**g**  $x = -4$  or  $x = -3$  or  $x = \frac{2}{3}$  or  $x = 2$   
**h**  $x = -1$  or  $x = -\frac{1}{2}$  or  $x = \frac{2}{3}$  or  $x = 3$
- 2 a**  $x = -4$  or  $x = -2$  or  $x = 3$   
**b**  $x = -2\frac{1}{2}$  or  $x = 1$   
**c**  $x = -3$  or  $x = -2$  or  $x = -1$   
**d**  $x = -1$  or  $x = 3$  or  $x = 4$

### Chapter 2 review

- 1 a**  $\frac{3p^{14}}{q^6}$    **b**  $\frac{1}{2}$    **c**  $\frac{1}{27}$   
**d** 4   **e** 81   **f**  $\frac{p^3}{4\sqrt{2}}$
- 2 a**  $2.064 \times 10^{-3}$    **b**  $3.6320 \times 10^4$   
**c**  $9.5 \times 10^1$    **d**  $2.2 \times 10^{-2}$
- 3 a** 68 700   **b** 32 100 000  
**c** 427   **d** 0.000 89
- 4 a** 6   **b** 2   **c** 4   **d** 1
- 5 a**  $3x^2 + 3xy$    **b**  $15h - 30$

- c**  $6h^2j^3 - 15h^2j^2 + 3h^2j$     **d**  $8p^3 + 27$   
**e**  $3p^4 - 3pq^3 - 2q^4 + 8p^4q$   
**f**  $20r^4 - 8t^3 + 28t^2$     **g**  $9x^2 - 30x + 25$   
**h**  $4a^2 - 28a + 49$     **i**  $2a^3 + 17a^2 + 20a - 75$   
**j**  $64v^2 + 32ev + 4e^2$   
**k**  $2k^2 - 2h^2 + 3hk + 13h - k - 15$   
**l**  $x^2 - y^2$     **m**  $y^2 - 36$   
**n**  $64 - 25k^2$     **o**  $25 - x^2$   
**p**  $a^3 + 3a^2b + 3ab^2 + b^3$   
**q**  $8x^3 - 48x^2y + 96xy^2 - 64y^3$   
**r**  $128d^4e + 768d^3e^2 + 1536d^2e^3 + 1024de^4$
- 6 a**  $5m^2n(m - mn - 1)$     **b**  $(3h + 2g)(2i - h)$   
**c**  $(3m + 2)(m - 6)$     **d**  $(4v + 3w)(2v + 5w)$   
**e**  $(r + 2s)(r - 2s)(3r^5s + 3r^2s^4)$
- 7 a**  $\frac{3a - 9b}{a + 3b}$     **b**  $\frac{2p^2 + 7p + 5}{4p^2 + 4p - 8}$   
**c**  $\frac{17r^2 + 30r - 2}{(5r - 2)(3r + 2)} \cdot 17r^2 + 15r - 14$
- 8 a**  $(7p - 3q)(p - 3q)$     **b**  $(3y - 5)(y + 6)$   
**c**  $(3a - 7)(a - 8)$     **d**  $(3x - 5y)(x - y)$   
**e**  $(5e + 4f)(e - f)$     **f**  $(8z - 7a)(2z + 9a)$
- 9 a**  $(x + 4y)^2$     **b**  $(h + 8)(h - 8)$   
**c**  $(5p + 7q)^2$     **d**  $(3y - 8)^2$   
**e**  $(4m + 3f^2)(4m - 3f^2)$     **f**  $(3p + 1)(9p^2 - 3p + 1)$   
**g**  $(4y - 3)^2$   
**h**  $(2 - 5q^2)(4 + 10q^2 + 25q^4)$   
**i**  $(p + 2)(p - 2)(p^4 + 4p^2 + 16)$   
**j**  $(3g + 5f)^2$   
**k**  $(f + 2)(f^2 - 2f + 4)$   
**l**  $(1 - 3m^2)(1 + 3m^2 + 9m^4)$
- 10 a**  $x = \frac{4}{3}$  or  $x = -2$   
**b**  $g = -2.5$  or  $g = -2$   
**c**  $m = 1.56$  or  $m = -1.16$   
**d**  $h = 2.39$  or  $h = -1.39$
- 11 a**  $x = 1$  or  $x = 5$     **b**  $x = -2$  or  $x = -7$   
**c**  $x = -4$  or  $x = -2$     **d**  $x = 6$  or  $x = 4$
- 12 a**  $x = 4 - \sqrt{14}, 4 + \sqrt{14}$   
**13 a**  $x = -4.854$  or  $x = 1.854$   
**b**  $x = -4.562$  or  $x = -0.438$   
**c**  $x = 0.382$  or  $x = 2.618$
- 14 a**  $x = -1.791$  or  $x = 2.791$   
**b**  $x = -0.285$  or  $x = 1.951$
- c**  $x = -0.243$  or  $x = 1.643$   
**d**  $x = -0.681$  or  $x = 0.881$
- 15 a** 2    **b** 0    **c** 1
- 16 a** Polynomial because all the powers are non-negative integers.  
**b** Not a polynomial as the power of the 2nd term is  $x$ .  
**c** Not a polynomial as the power of the 2nd term is  $1/2$ , a fraction.  
**d** Polynomial because all the powers are non-negative integers.  
**e** Not a polynomial as the power of the 1st term is negative.  
**f** Polynomial because all the powers are non-negative integers.
- 17 a**  $-6x^3$ , degree 3    **b**  $-x^5$ , degree 5  
**c** 4, degree 0    **d**  $t^9$ , degree 9
- 18 a** 20    **b** 55    **c**  $34 - 2\sqrt{6}$   
**d**  $-6859$     **e** 125    **f**  $-1$   
**g**  $5(4x - 3)^6 - 2(4x - 3)^3 + 4$   
**h** 8860471
- 19 a**  $Q(x) = x^2 + 4x + 21, R = 77$   
**b**  $Q(x) = 2x^5 + x^4 - x^3 + 4, R = 2$   
**c**  $Q(x) = 9x - 4, R = 12x^2 - 9x - 1$
- 20 a**  $(x + 1)(x - 1)(x - 2)$     **b**  $(x + 2)(x + 4)(4x - 3)$   
**c**  $(x + 2)(x + 3)(3x + 1)$
- 21 a**  $x = -1$  or  $x = -1\frac{2}{3}$  or  $x = 2$   
**b**  $x = -2$  or  $x = \frac{1}{2}$  or  $x = 3$   
**c**  $x = -3$  or  $x = -1\frac{1}{2}$  or  $x = 0$  or  $x = 2$   
**d**  $x = -2$  or  $x = 0$  or  $x = 1\frac{1}{2}$  or  $x = 2\frac{1}{2}$  or  $x = 3$   
**e**  $x = -2$  or  $x = -\frac{1}{3}$  or  $x = 3$   
**f**  $x = -\frac{1}{2}$  or  $x = 1\frac{2}{3}$  or  $x = 3$
- 22 a**  $k = 0.0008$  cm/N    **b**  $F = \frac{C}{0.0008}$  or 1250C  
**c** 4 cm    **d** 5625 N
- 23 a**  $k = 120$  km    **b**  $s = \frac{120}{t}$   
**c**  $t = 1.5$  hours    **d**  $s = 240$  km/h
- 24** Length = 12 units and Width = 9 units  
**25** Length = 9 units and Width = 4 units  
**26** 6 tourists in the group. Each paid \$12.  
**27** Height = 2 m, Width = 4 m and Length = 8 m

## Chapter 3

### Investigation: Power sets

- 1 8
- 2 256
- 3 2
- 4 2
- 5 4, 16 and 32
- 6 Power set of  $A$  has cardinality  $2^{|A|}$

### Exercise 3.01

- 1 **a**  $A = \{14, 16, 18, 20, 22, 24, 26, 28, 30, 32\}$   
**b**  $B = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$   
**c**  $C = \{9, 11, 13, 15, 17, 19, 21, 23, 25\}$   
**d**  $D = \{\emptyset, \{1\}, \{3\}, \{5\}, \{7\}, \{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}, \{1, 3, 5\}, \{1, 3, 7\}, \{1, 5, 7\}, \{3, 5, 7\}, \{1, 3, 5, 7\}\}$   
**e**  $F = \{\text{Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec}\}$
- 2 **a** 10    **b** 15    **c** 9    **d** 16    **e** 12
- 3 **a** 9    **b** 9    **c** 4    **d** 15    **e** 0
- 4 **a** {letters of the alphabet}    **b** {b}  
**c** {m, p, r, t}    **d** {a, e, i, o, u}  
**e** {a, b, c, m, p, r}    **f**  $\emptyset$     **g** 21  
**h**  $A$  and  $C$ ,  $A$  and  $D$ ,  $B$  and  $C$ ,  $B$  and  $D$   
**i**  $\emptyset, \{p\}, \{r\}, \{t\}, \{p, r\}, \{p, t\}, \{r, t\}, \{p, r, t\}$   
**j**  $\emptyset, \{m\}, \{p\}, \{r\}, \{m, p\}, \{m, r\}, \{p, r\}$
- 5 **a**  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$   
**b**  $\{2, 3, 4, 6, 7, 8, 9\}$   
**c**  $\{0, 9\}$   
**d**  $\{0, 1, 2, 3, 4, 6, 7, 8, 9\}$   
**e**  $\{0, 1, 2, 3, 4, 6, 7, 8, 9\}$   
**f**  $\{0, 9\}$   
**g**  $V = \{0, 3, 6, 9\}$     **h**  $M$  and  $P$   
**i**  $\emptyset, \{2\}, \{4\}, \{6\}, \{8\}, \{2, 4\}, \{2, 6\}, \{2, 8\}, \{4, 6\}, \{4, 8\}, \{6, 8\}, \{2, 4, 6\}, \{4, 6, 8\}, \{2, 6, 8\}, \{2, 4, 8\}, \{2, 4, 6, 8\}$   
**j**  $\emptyset, \{5\}, \{6\}, \{7\}, \{8\}, \{5, 6\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{6, 8\}, \{7, 8\}, \{5, 6, 7\}, \{6, 7, 8\}, \{5, 7, 8\}, \{5, 6, 8\}$
- 6 **a** {HH, HT, TH, TT}    **b**  $A = \{\text{HH, HT}\}$   
**c**  $B = \{\text{HH, TH}\}$     **d** {HH}  
**e** {HH, HT, TH}    **f**  $C = \{\text{HT, TH, TT}\}$   
**g**  $D = \{\text{HT, TH}\}$     **h**  $E = \{\text{TT}\}$   
**i**  $F = \{\text{HH}\}$     **j** {HH}

- 7 **a** {Blue, Red, Green, Pink, Orange}  
**b** {Blue, Red}  
**c** {Green, Pink, Orange}  
**d**  $\emptyset$   
**e** {Blue, Red, Green, Pink, Orange}  
**f** 2, 3, 0 and 5 respectively
- 8  $n(A) = 37, n(B) = 2, n(C) = 3, n(D) = 4, n(E) = 6, n(F) = 12, n(G) = 12, n(H) = 18$

### Exercise 3.02

- 1 **a**  $\frac{1}{12}$     **b**  $\frac{7}{24}$     **c**  $\frac{1}{2}$
- 2 **a** {R, R, R, R, R, G, G, G, G, G, G, G, G, B, B, B, B, B, B}  
**b** 0.35    **c** 0    **d** 0.4    **e** Green
- 3 **a**  $\frac{2}{5} = 0.4$     **b**  $\frac{3}{5} = 0.6$     **c** White    **d**  $\frac{1}{3}$   
**e**  $\frac{2}{3}$
- 4 **a** {V, V, V, V, V, V, H, H, H, H, H, H}  
**b**  $\frac{6}{11}$     **c**  $\frac{5}{11}$     **d**  $\frac{6}{11}$     **e** 1
- 5 **a**  $\frac{1}{21}, \frac{2}{21}, \frac{1}{7}, \frac{4}{21}, \frac{5}{21}, \frac{2}{7}$     **b**  $\frac{4}{7}$     **c**  $\frac{10}{21}$   
**d**  $\frac{3}{7}$
- 6 **a**  $\frac{184}{365}$  or  $\frac{92}{183}$  (leap year)  
**b**  $\frac{2}{7}$     **c** About  $\frac{1}{12}$   
**d**  $\frac{1}{6}$     **e**  $\frac{1}{7}$
- 7  $P(A) = 0.12, P(B) = 0.06, P(C) = 0.03$
- 8 **a**  $\frac{2}{7}$     **b**  $\frac{1}{21}$     **c**  $\frac{2}{21}$   
**d**  $\frac{2}{21}$     **e** 0    **f**  $\frac{2}{3}$

### Exercise 3.03

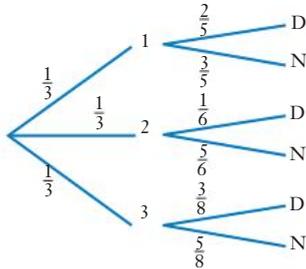
- 1 **a**  $\frac{8}{125}$     **b**  $\frac{44}{125}$
- 2  $\frac{5}{6}$
- 3 **a**  $\frac{1}{4}$     **b**  $\frac{3}{13}$     **c**  $\frac{3}{52}$     **d**  $\frac{11}{26}$

4  $\frac{1}{3}$

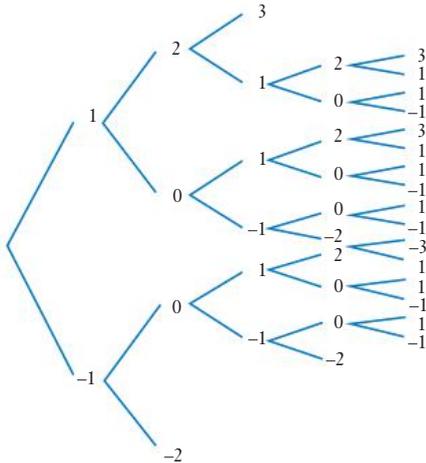
5  $\frac{8}{663}$

6 a  $\frac{1}{55}$       b  $\frac{41}{55}$

7  $\frac{113}{360}$



8 a

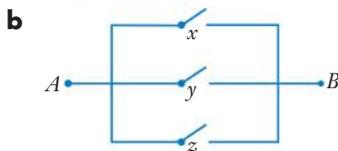


b  $\frac{1}{2}$

9 a  $\frac{4}{11}$       b  $\frac{1}{11}$

10  $\frac{3}{8}$

11 a i  $\frac{3}{8}$       ii  $\frac{5}{8}$



12  $\frac{5}{12}$

### Investigation: Flipping boxes

The probabilities are in the order of the areas of the sides of the boxes, but they are not exactly proportional to the areas.

### Exercise 3.04

1 a 0.075    b 0.025    c 0.275    d 138

2 a 0.06, 0.34, 0.26, 0.29, 0.02, 0.034

b  $\frac{43}{150}$     c 7    d 8

3 a 100

b

No.	0	1	2	3	4	5	6	7	8	9
<i>f</i>	7	14	32	23	11	5	6	1	0	1

c 0.32

4 a 61    b 51    c 69

5 a 48    b 22    c 0.458

d 0.104    e 23

6 a 0.0135    b 0.0016    c 0.0025

d 0.000 02    e 2

### Chapter 3 review

1 a  $A = \{7, 10, 13, 16, 19\}$ ,  
 $B = \{23, 29, 31, 37\}$

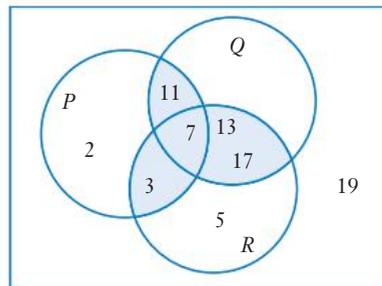
b  $|A| = 5, |B| = 4$

2 a  $\{5, 13, 17, 19\}$

b  $\{7, 13, 17\}$

c  $\{\}, \{7\}, \{11\}, \{13\}, \{17\}, \{7, 11\}, \{7, 13\}, \{7, 17\},$   
 $\{11, 13\}, \{11, 17\}, \{13, 17\}, \{7, 11, 13\},$   
 $\{7, 11, 17\}, \{7, 13, 17\}, \{11, 13, 17\}$

d  $\{3, 5, 7, 11, 13, 17\}$



3 a  $\{WB, WP, WY, WR, WR, BW, BP, BY, BR, BR,$   
 $PW, PB, PY, PR, PR, YW, YB, YP, YR, YR, RW,$   
 $RB, RP, RY, RR, RW, RB, RP, RY, RR\}$

b  $\{BW, BP, BY, BR, BR, WB, PB, YB, RB, RB\}$

- c** {WR, WR, BR, BR, PR, PR, YR, YR, RW, RB, RP, RY, RR, RW, RB, RP, RY, RR}  
**d** {WB, WY, BW, BP, BY, BR, BR, PB, PY, YW, YB, YP, YR, YR, RB, RY, RB, RY}  
**e** {BY, YB}  
**f** {}

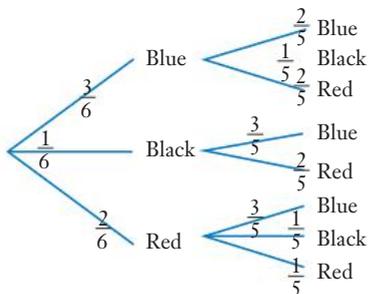
**4**  $\frac{2}{23}$

**5**  $\frac{1}{3}$

**6 a** 0.475

**b** 0.55

**7 a**



**b**  $\frac{1}{5}$

**c**  $\frac{11}{15}$

**8**  $\frac{2}{3}$

**9**  $\frac{4}{5}$

**10 a**  $\frac{3}{14}$

**b** About 57

**11 a** {Scarlet, Brad, Jason, Lisa, Merryl, Tony, Peter, Debbie}

**b** {Mary, Samantha, Paul, Ross, Zoe, Alan, David, Tammy, Wayne, Skye}

**c** {Ross, Zoe}

**d** {Jenny, Kathy, Andrew, William}

**e** Students are more likely to follow the same path as their parents.

**12** 25

**13**  $\frac{3}{16}$  (or 0.125)

**14**  $\frac{149}{198}$

### Practice examination 1

**1 a** 34, 41, 48

**b**  $t_{15} = -2.6$  (1 mark each)

**2 a**  $S_{20} = 780$

**b**  $n = 50$  (1 mark each)

**3 a**  $2^{17}$

**b**  $\frac{3}{2}$  (2, 3 marks)

**4 a**  $2ab^2 + 13a^4 - 5a^2b$

**b**  $9m^2 - 30mp + 25p^2$  (1 mark each)

**5 a**  $x = 1$  or  $x = \frac{3}{2}$

**b**  $w = \frac{3-\sqrt{5}}{4}$  or  $w = \frac{3+\sqrt{5}}{4}$

(2, 3 marks – simplification required)

**6**  $\frac{2x+1}{x+1}$ ;  $g(f(4)) = \frac{9}{5}$  (2 marks – simplification required)

**7**  $\Delta = 72$ , there are 2 different real roots (2 marks)

**8 a** 5      **b**  $P(-2) = 0$  (1 mark each)

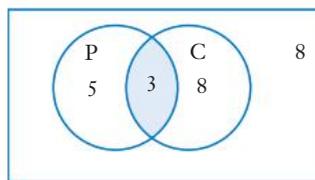
**9**  $A \cap B = \{3, 9, 15\}$

$A \cup B = \{1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 17, 18, 19\}$   
 (1 mark each)

**10**  $\frac{2}{7}$  (1 mark)

**11** 0.4 (1 mark)

**12**



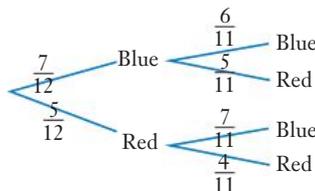
$P(\text{Physics and Chemistry}) = \frac{1}{8}$  (1 mark each part)

**13 a**  $\frac{2}{5}$

**b**  $\frac{3}{5}$  (1 mark each)

**14** 2 (1 mark)

**15**



$P(\text{different}) = \frac{35}{66}$  (1 mark each part)

**16**  $p = \frac{3-\sqrt{3}}{3}$  or  $p = \frac{3+\sqrt{3}}{3}$  (1 mark each main step)

**17**  $x = 0, 1$  or  $2$  (1 mark each main step)

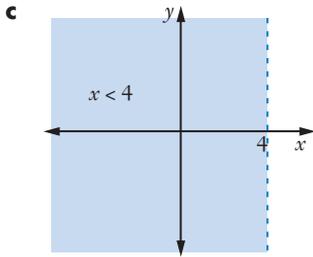
**18** 40 017 (1 correct interpretation, 1 answer)

- 19 8 (1 correct interpretation, 1 answer)  
 20 **b**  $27.15 \text{ m}^3$  when  $r = 1.63 \text{ m}$  (2 proof, 1 interpretation, 1 answer)  
 21 3.904% (1 correct interpretation, 1 answer)

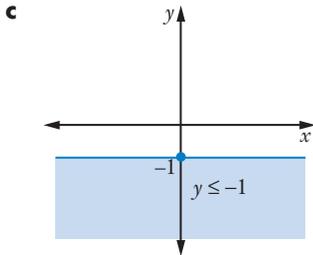
## Chapter 4

### Exercise 4.01

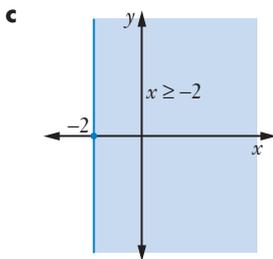
- 1 **a**  $x < 4$  **b**  $R$



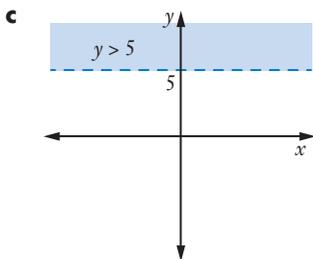
- 2 **a**  $R$  **b**  $y \leq -1$



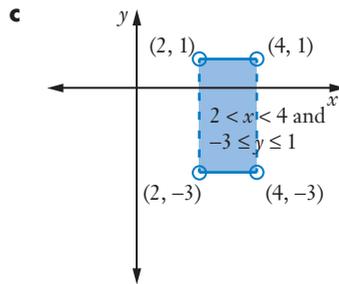
- 3 **a**  $x \geq -2$  **b**  $R$



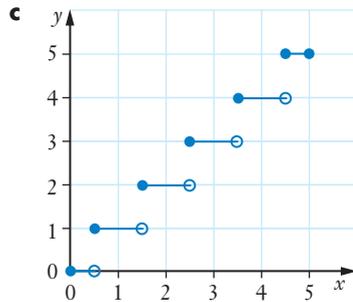
- 4 **a**  $R$  **b**  $y > 5$



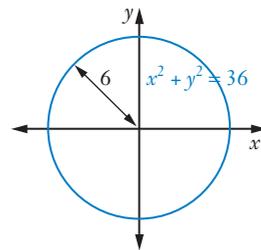
- 5 **a**  $2 < x < 4$  **b**  $-3 \leq y \leq 1$



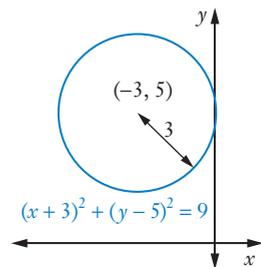
- 6 **a**  $x \geq 0$  **b**  $y \in \mathbb{Z}, y \geq 0$



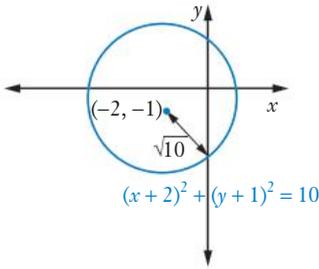
- 7 **a** Circle with centre  $(0, 0)$  and radius 6



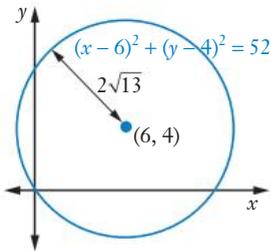
- b** Circle with centre  $(-3, 5)$  and radius 3



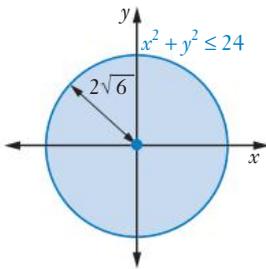
- c** Circle with centre  $(-2, -1)$  and radius  $\sqrt{10}$



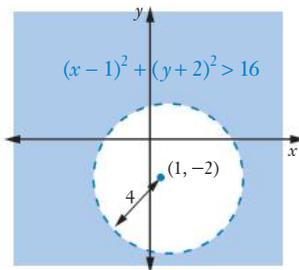
- d** Circle with centre  $(6, 4)$  and radius  $2\sqrt{13}$



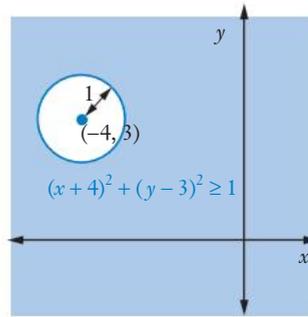
- 8 a** Circle and region inside, centre  $(0, 0)$ , radius  $2\sqrt{6}$



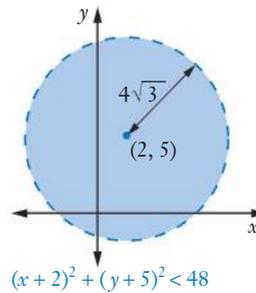
- b** Region outside circle, centre  $(1, -2)$  and radius 4



- c** Circle and region outside, centre  $(-4, 3)$  and radius 1

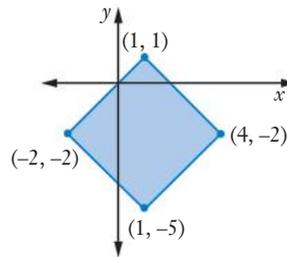


- d** Region inside circle, centre  $(2, 5)$  and radius  $4\sqrt{3}$



- 9 a**  $(3, -4), 7$                       **b**  $(1, 3), 6$   
**c**  $(2.5, -3.5), 5$               **d**  $(-2, -4), 3$   
**e**  $(-4, 2), 9$

- 10** Square and region inside, corners  $(4, -2), (1, -5), (-2, -2), (1, 1)$

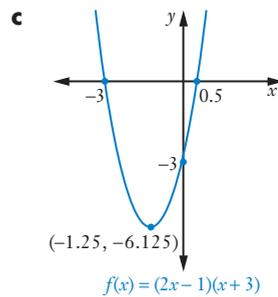
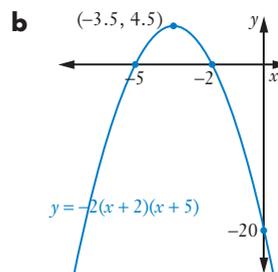
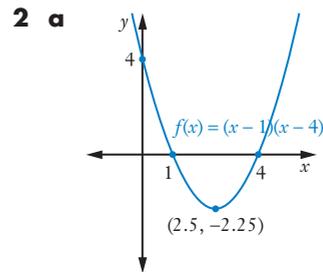
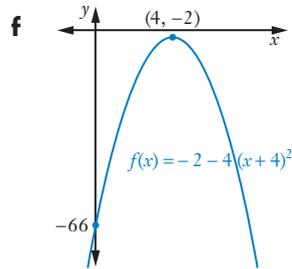
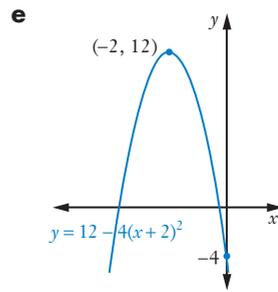
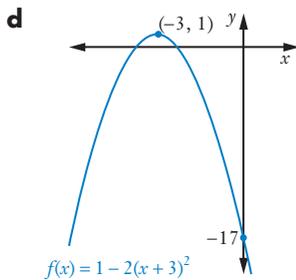
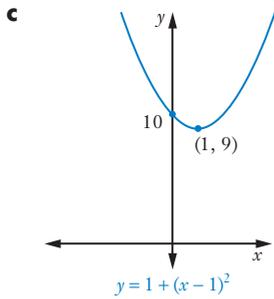
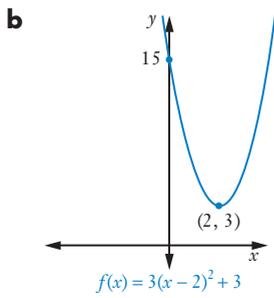
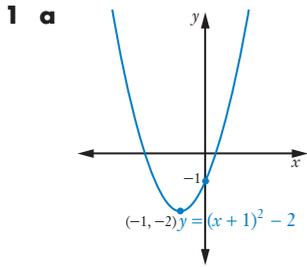


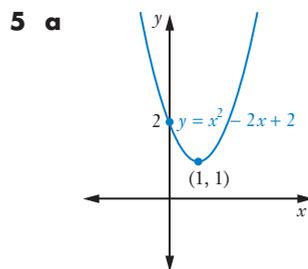
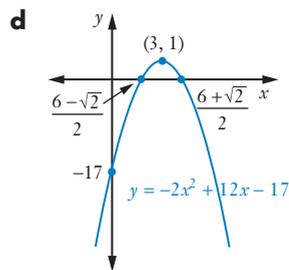
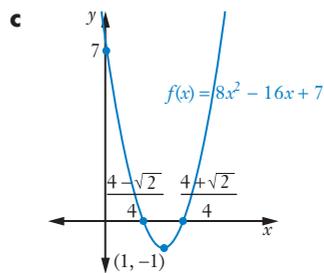
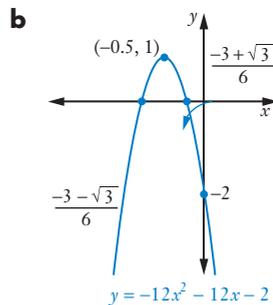
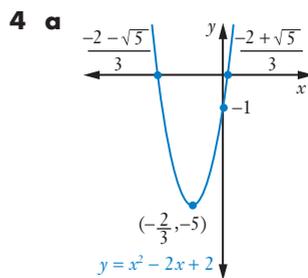
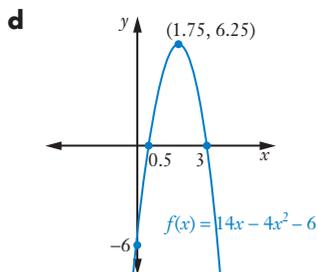
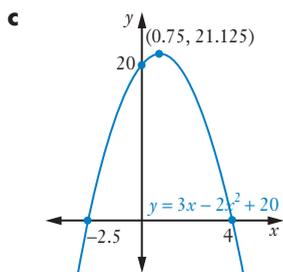
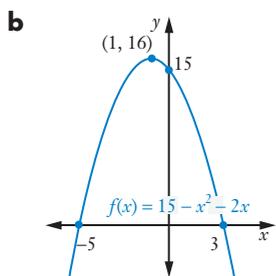
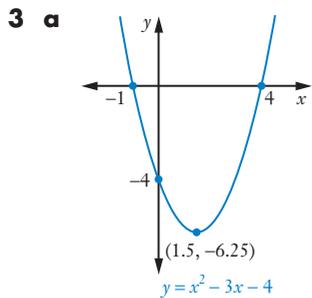
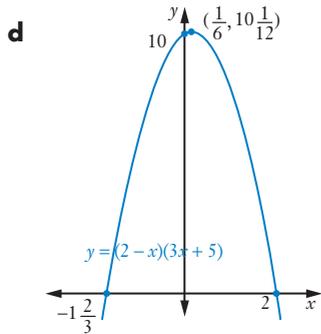
### Exercise 4.02

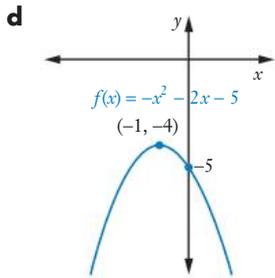
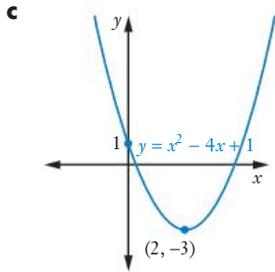
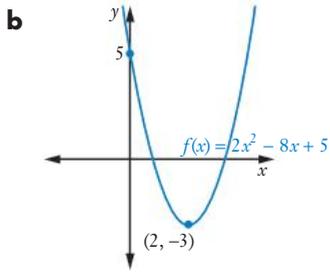
- |                         |                         |
|-------------------------|-------------------------|
| <b>1 a</b> Function     | <b>b</b> Function       |
| <b>c</b> Not a function | <b>d</b> Not a function |
| <b>e</b> Function       | <b>f</b> Function       |
| <b>g</b> Function       | <b>h</b> Not a function |
| <b>i</b> Function       | <b>j</b> Function       |

- 2 a** Function                      **b** Not a function  
**c** Function                        **d** Not a function  
**e** Not a function                **f** Function

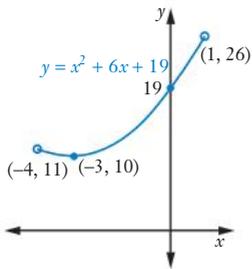
Exercise 4.03



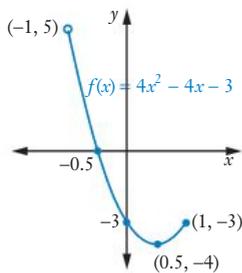




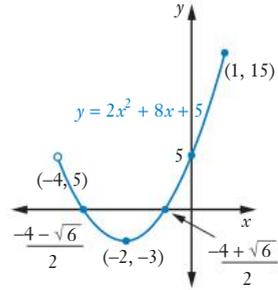
**6 a** Range:  $10 \leq y < 26$



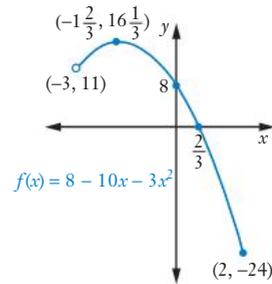
**b** Range:  $-4 \leq y < 5$



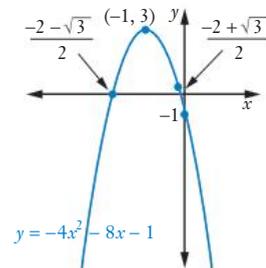
**c** Range:  $-3 \leq y \leq 15$



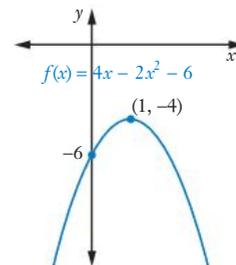
**d** Range:  $-24 \leq y \leq 16\frac{1}{3}$



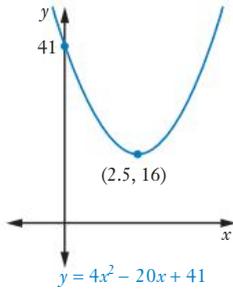
**e** Range:  $y \leq 3$



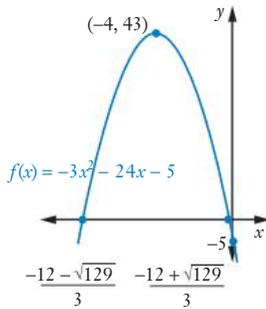
**f** Range:  $y \leq -4$



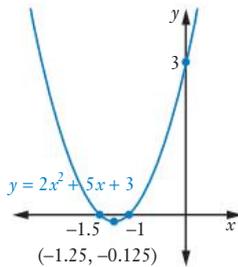
**g** Range:  $y \geq 16$



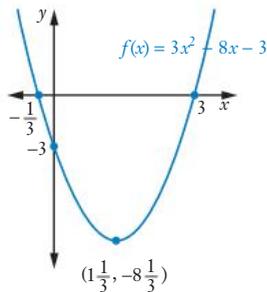
**h** Range:  $y \leq 43$



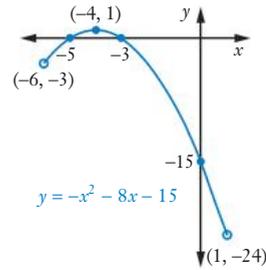
**i** Range:  $y \geq -0.125$



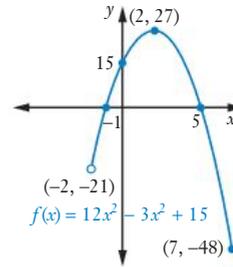
**j** Range:  $y \geq -8\frac{1}{3}$



**k** Range:  $-24 < y \leq 1$



**l** Range:  $-48 \leq y \leq 27$



**7 a**  $y$ -intercept: 2; zeros:  $-5.65, -0.35$ ; minimum,  $(-3, -7)$

**b**  $y$ -intercept: 9; zeros:  $-3.71, 1.21$ ; maximum,  $(-1.25, 12.125)$

**c**  $y$ -intercept:  $-1$ ; zeros:  $0.04, 5.71$ ; maximum,  $(2.875, 32.0625)$

**d**  $y$ -intercept: 7; zeros: none; minimum,  $(1.25, 3.875)$

**8 a**  $y = -x^2 + 4x - 6$       **b**  $y = 1.5x^2 - 1.5x - 3$

**c**  $y = -0.8x^2 + 0.8x + 4.8$

**9** 25

### Exercise 4.04

**1**  $5\sqrt{5}$

**2**  $\frac{8\sqrt{2}}{3}$

**3 a**  $\$22x^2 - 110x + 210$  per panel

**b** 2.5 m      **c**  $\$7250$

**4**  $f(x) = 0.25x^2 - 4.5x + 50$

**5**  $f(x) = \frac{85x - 5x^2 - 30}{6}$

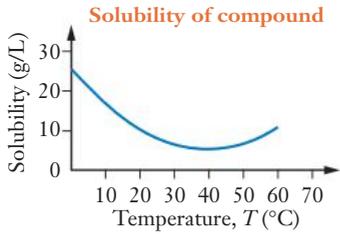
**6** 1 km horizontally and 75 m vertically up (with maximum gradient).

The function matches the data. It only really matters that the entry and exit points of the tunnel match the model, as long as the rest of the escarpment is above it.

7 12 000 000 chips, no maximum profit.

8  $R(T) = 80 + \frac{13T}{1200} - \frac{T^2}{360\,000}$ ,  $89\ \Omega$ . The data matches the model, but the model is unlikely to work when the wire gets close to melting. The resistance will certainly not become negative.

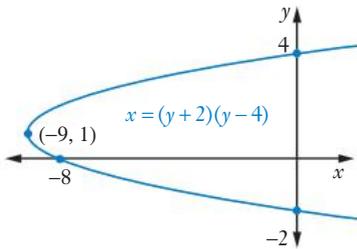
9  $40^\circ\text{C}$



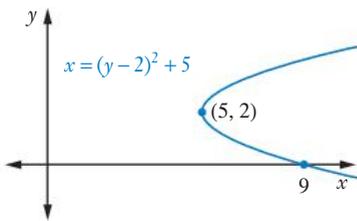
10 Maximum,  $(2.5, 33.25)$ ;  $h$ -intercept: 2;  $t$ -intercepts:  $-0.08, 5.08$ . The ball takes 2.5 s to reach maximum height and 5.08 s to reach the ground. The model matches the data given.

### Exercise 4.05

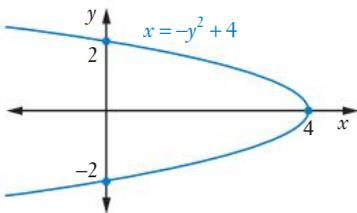
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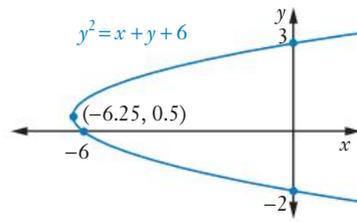
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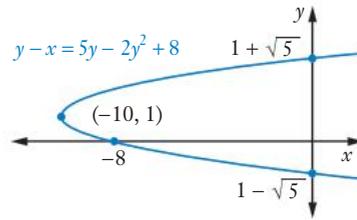
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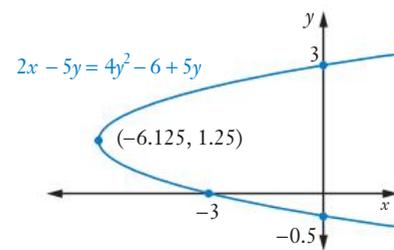
4



5



6

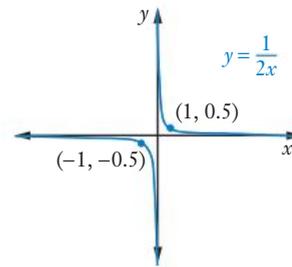


7  $x = 2y - y^2 - 8$

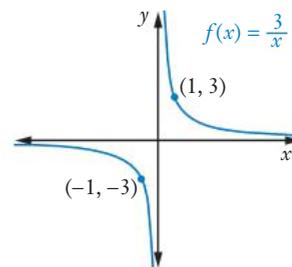
8  $x = 2y^2 - 16y + 24$

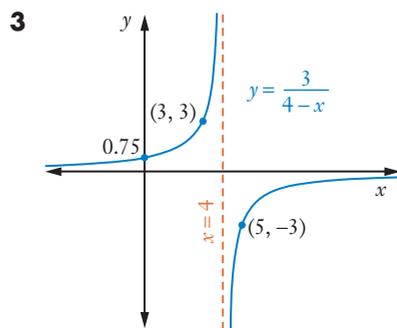
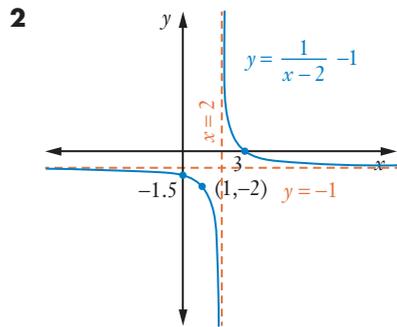
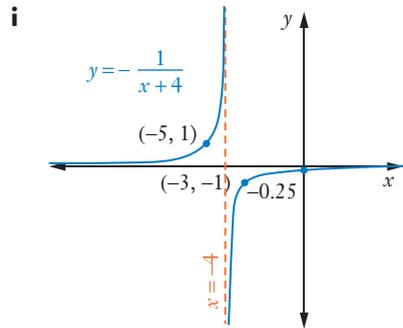
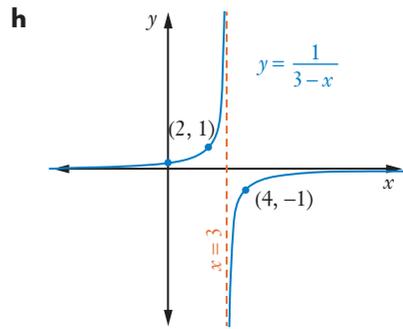
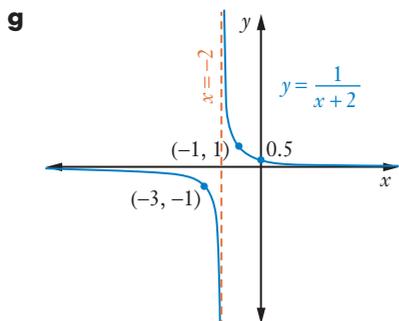
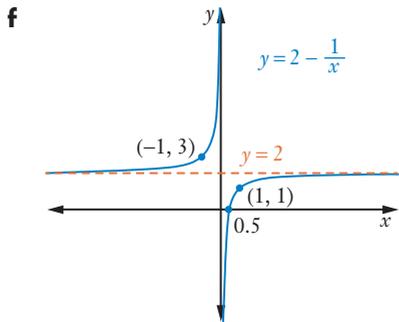
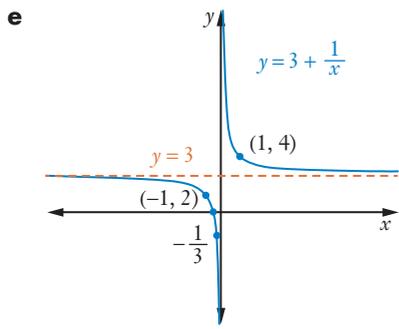
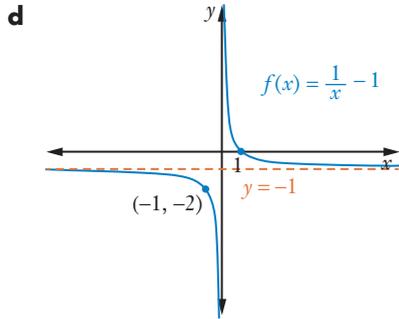
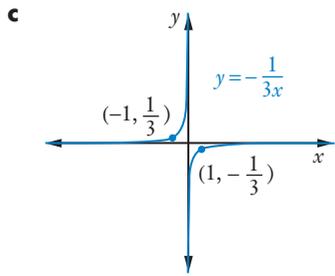
### Exercise 4.06

1 a

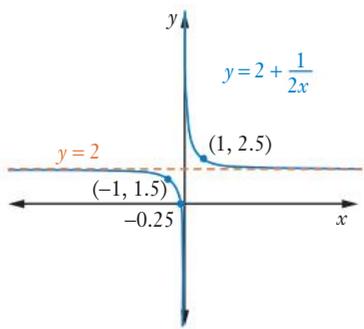


b

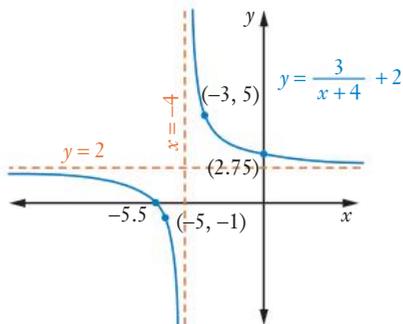




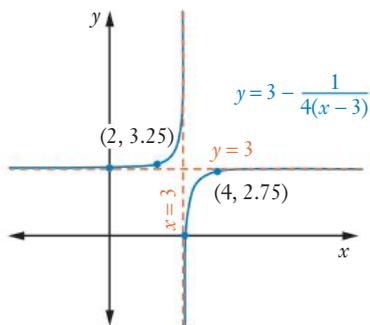
4



5



6



7  $f(x) = 2 + \frac{4}{x-3}$ , the equation matches the data

8  $f(x) = 4 - \frac{3}{x+2}$ , the equation matches the data

9  $f(x) = \frac{2}{x+1} + 3$ , the equation matches the data

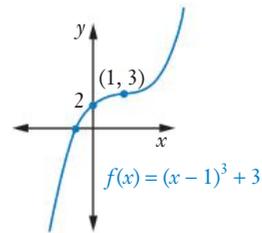
10  $f(x) = 4 - \frac{3}{x-1}$ , the equation matches the data

### Investigation $y = k(x-a)(x-b)(x-c)$

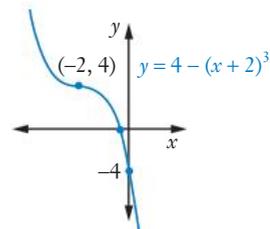
The graph is vertically stretched if  $k > 1$  and compressed if  $0 < k < 1$ . If  $k < 0$  it is inverted. It has zeros at  $a$ ,  $b$  and  $c$ .

### Exercise 4.07

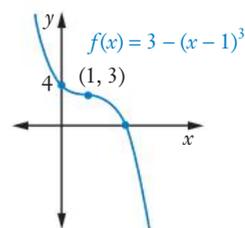
1 a



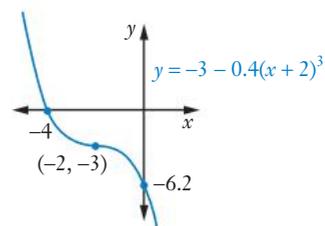
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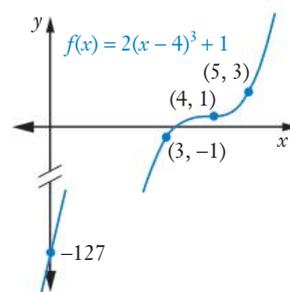
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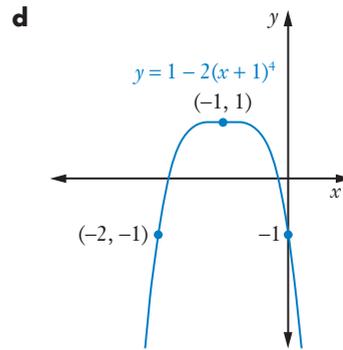
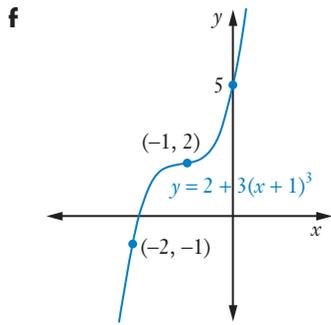


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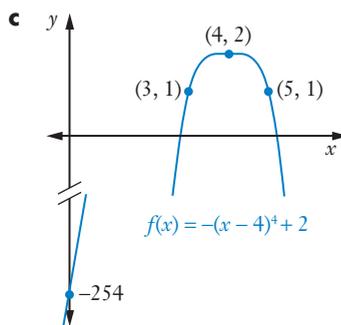
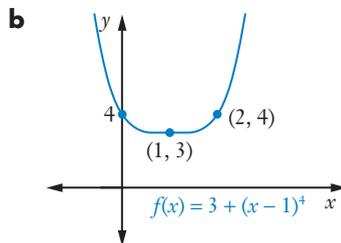
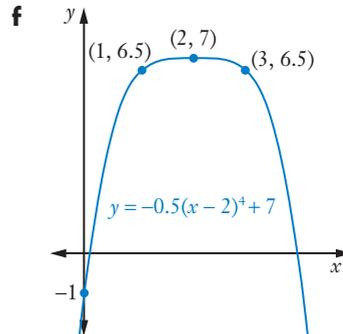
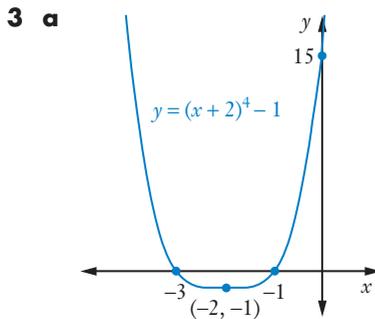
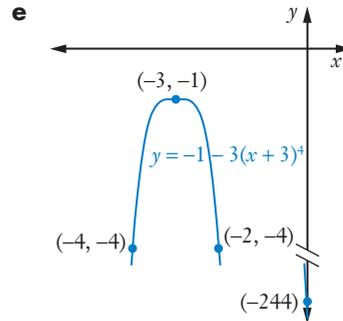


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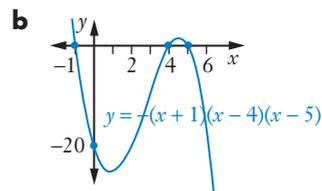
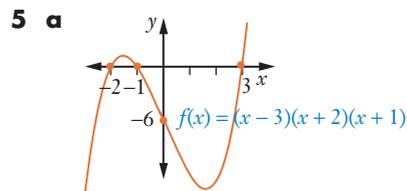


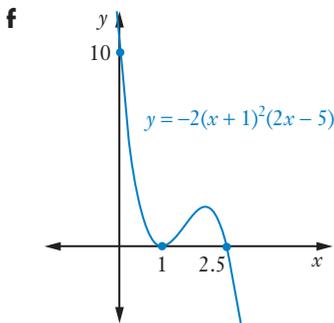
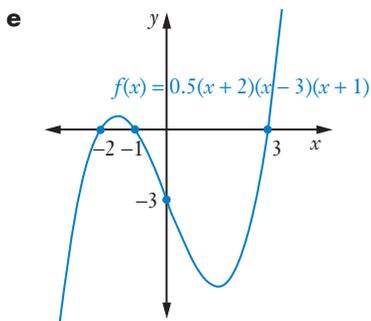
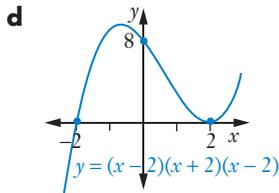
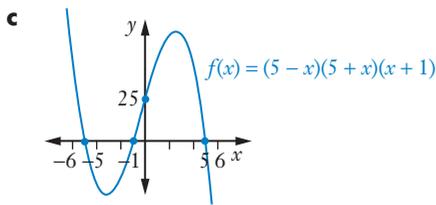


- 2 a** Range:  $-11 \leq y \leq 24$     **b** Range:  $-62 < y \leq 10$   
**c** Range:  $-67 < y \leq 24$     **d** Range:  $-124 \leq y \leq 28$   
**e** Range:  $-11.8 < y < 49$     **f** Range:  $-123 \leq y \leq 7$



- 4 a**  $-84 < y \leq -3$     **b**  $-252 < y \leq 4$   
**c**  $-6 \leq y \leq 15.6$     **d**  $-244 \leq y \leq -1$   
**e**  $-65 < y \leq -1$     **f**  $-3 \leq y < 29$





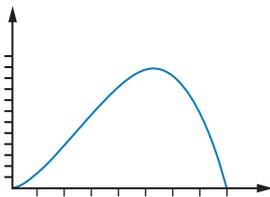
**6**  $f(x) = -x(x-3)(x+1)$

**7 a**  $f(x) = a(x-3)^3 + 2$

**b**  $f(x) = -0.5(x-3)^3 + 2$

**8**  $y = 5 - 0.25(x-2)^4$

**9 a**



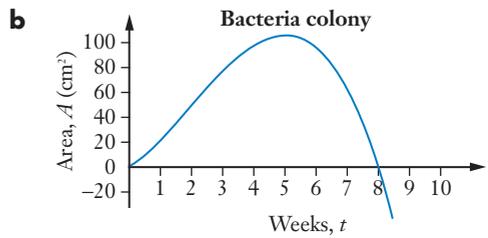
**b** -1.8366, 0, 40.8366

**c** 36.5, 91, 103.5 g/L

**d** 107.73 g/L. More nutrient, more algae.

**e** The mass cannot be negative, so the domain must be  $0 \leq D \leq 40.84$  or smaller.

**10 a**  $0.5 \text{ cm}^2$



**c** After 8 weeks

**d** 85.3%

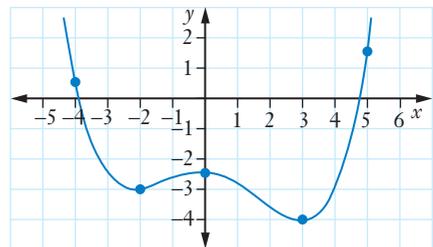
**e** 93.3%

### Investigation The effect of $b$ on $f(bx)$

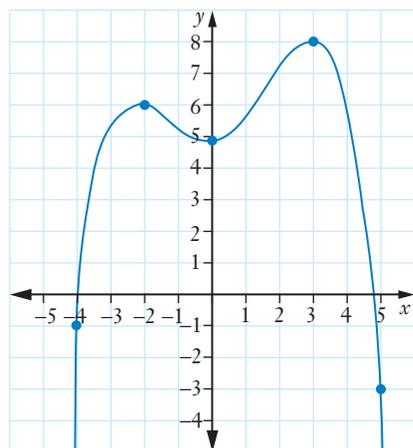
- 1** For  $b > 1$ ,  $f(bx)$  is horizontally compressed.
- 2** For  $0 < b < 1$ ,  $f(bx)$  is horizontally stretched.
- 3** For  $b < 0$ ,  $f(x)$  is inverted.

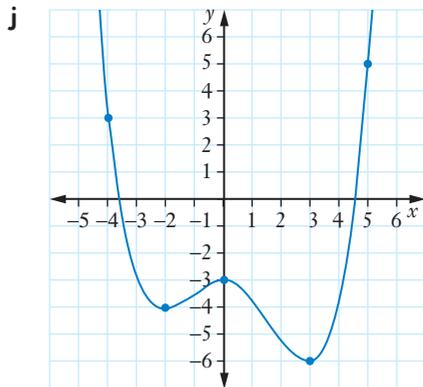
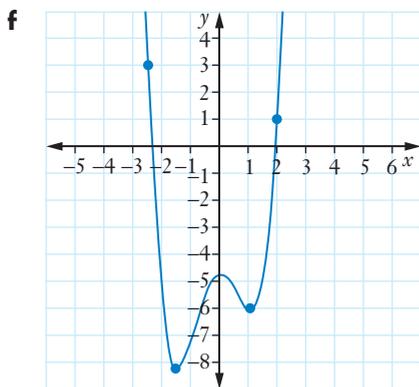
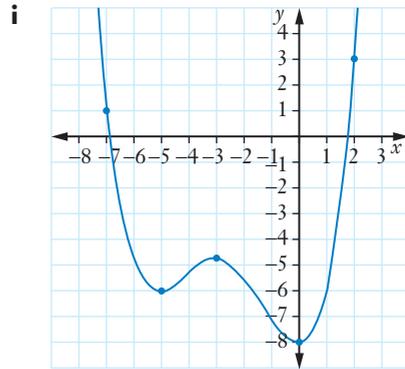
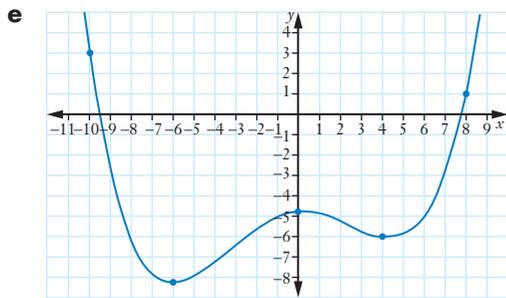
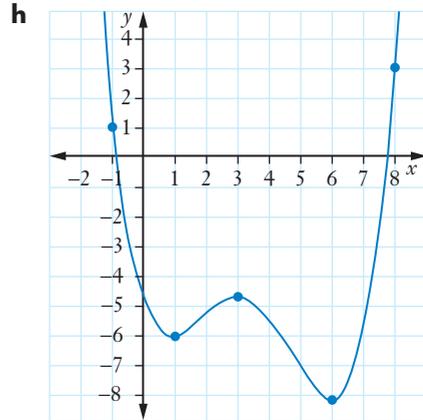
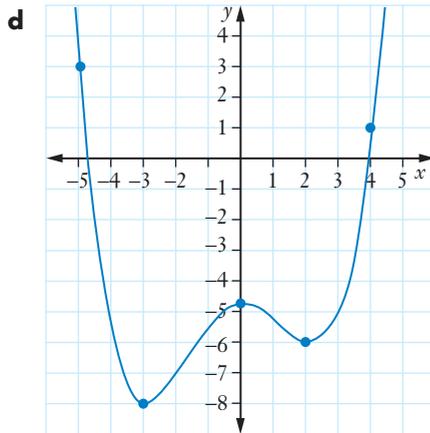
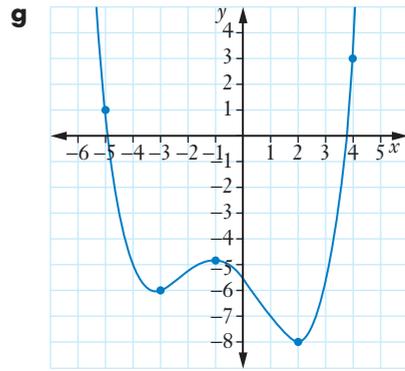
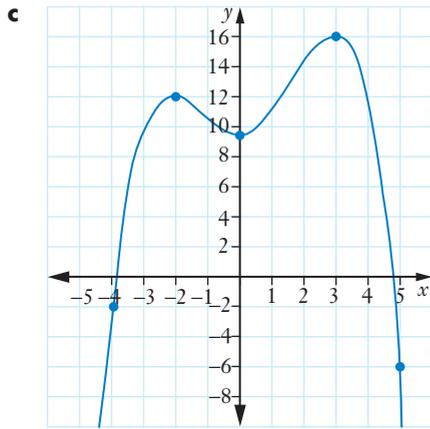
### Exercise 4.08

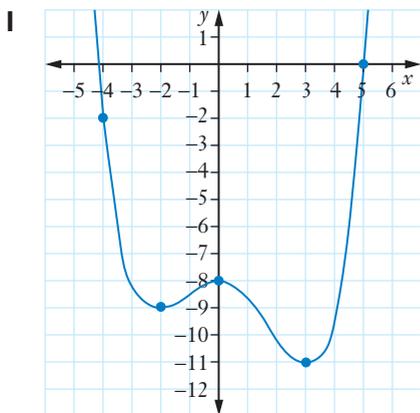
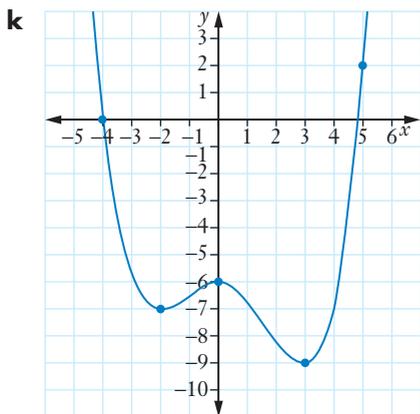
**1 a**



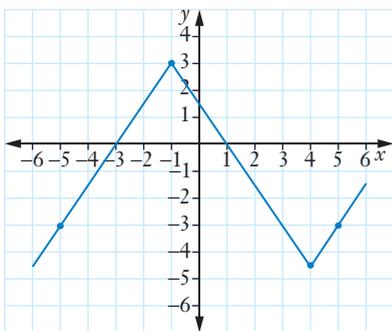
**b**



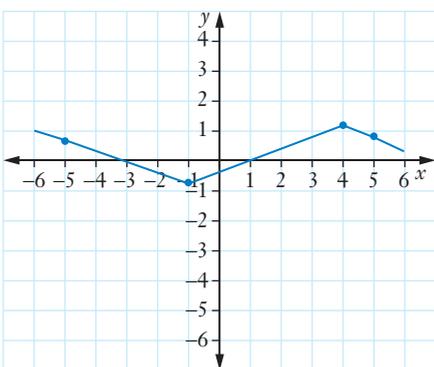




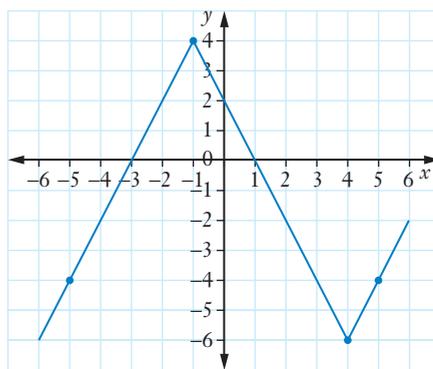
**2 a**



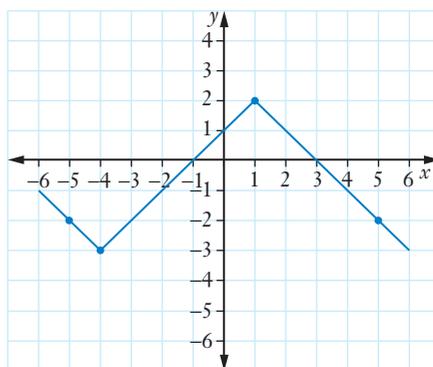
**b**



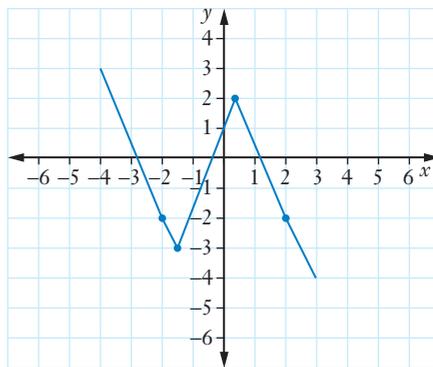
**c**



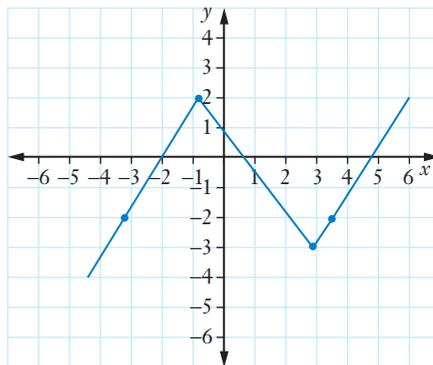
**d**

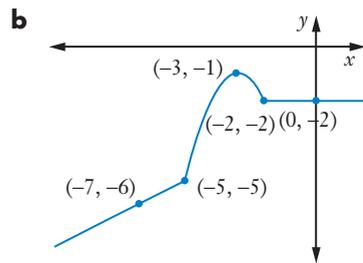
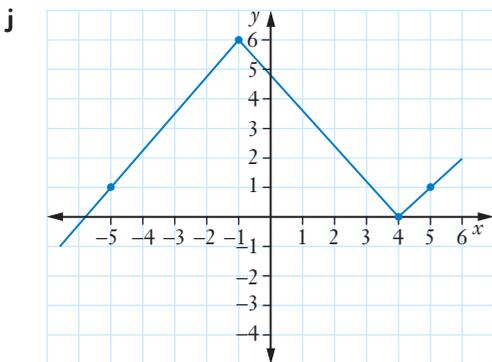
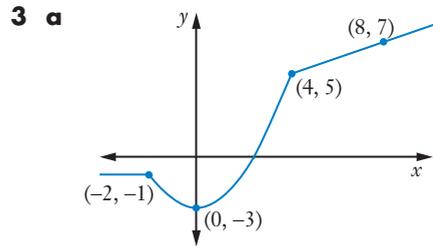
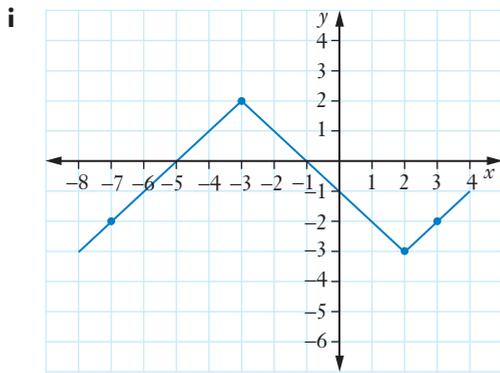
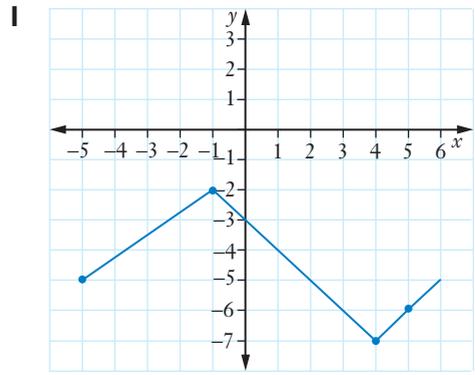
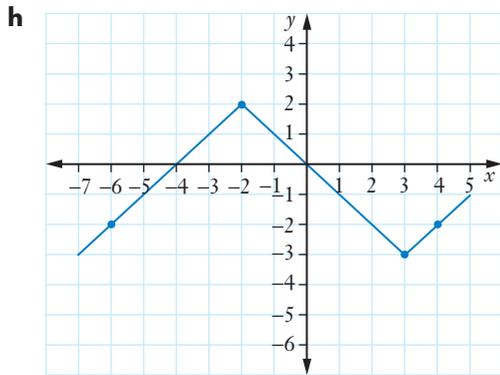
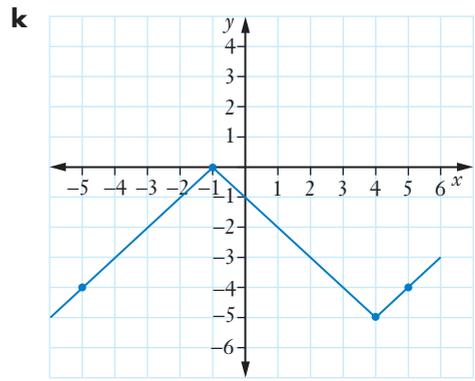
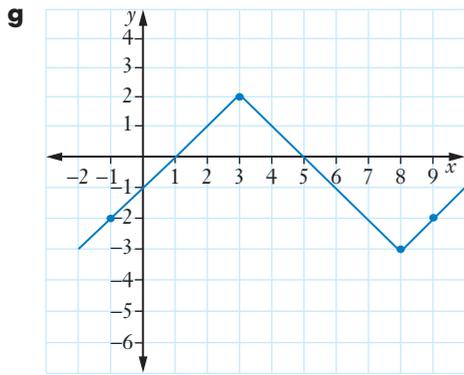


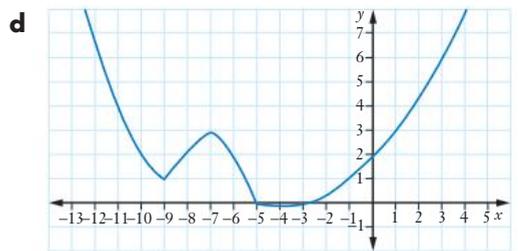
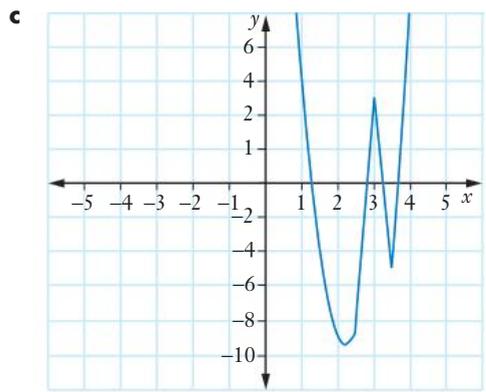
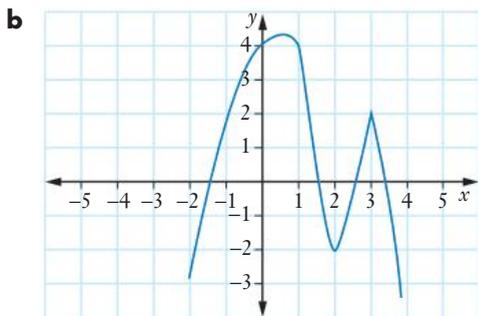
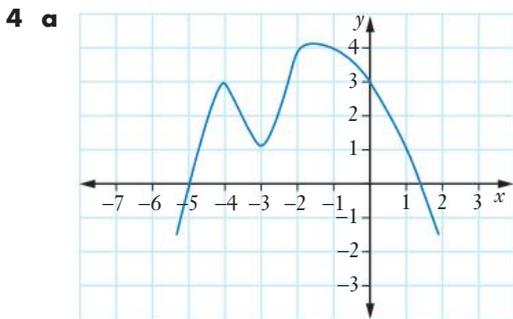
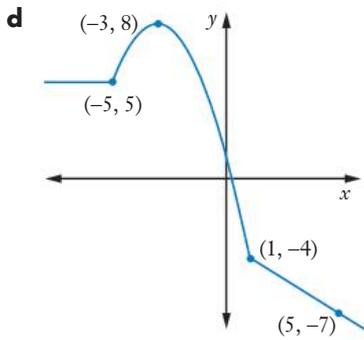
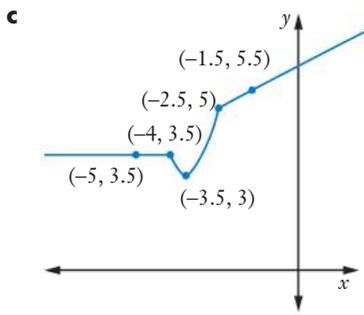
**e**



**f**







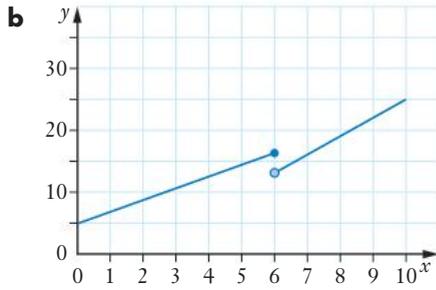
- 5**  $g(x) = -f(2x - 2) + 1$
- 6**  $q(x) = 1.5p(-x - 2) - 3$
- 7**  $f(x) = -g(0.5x + 4) + 1$
- 8**  $p(x) = 0.5q(-x - 2) + 1$

### Exercise 4.09

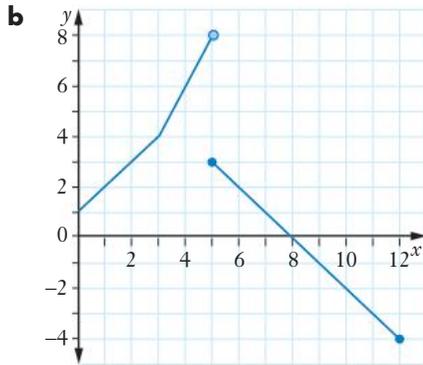
- 1**  $x \approx -2.05$
- 2**  $x \approx -2.39, x \approx 0.81$  or  $x \approx 2.58$
- 3**  $x \approx -2.66, x \approx -1.21$  or  $x \approx 1.87$
- 4**  $x \approx -1.78, x \approx 1.29$  or  $x \approx 3.49$
- 5**  $x \approx -1.46$  or  $x \approx 2.12$
- 6**  $x \approx -1.58$  or  $x \approx 2.34$
- 7**  $x \approx -2.90$  or  $x \approx 1.59$
- 8**  $x = 0, x = \frac{1 - \sqrt{17}}{2}$  or  $x = \frac{1 + \sqrt{17}}{2}$
- 9**  $x = -2, x = 1$  or  $x = 1.5$
- 10**  $x = 2, x = -2$  or  $x = -3$ .

### Exercise 4.10

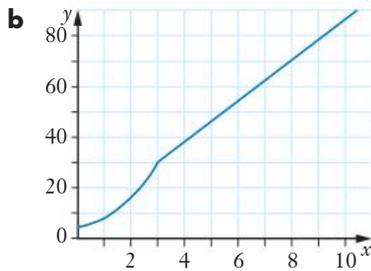
1 a  $f(5) = 14, f(6) = 16, f(7) = 16,$



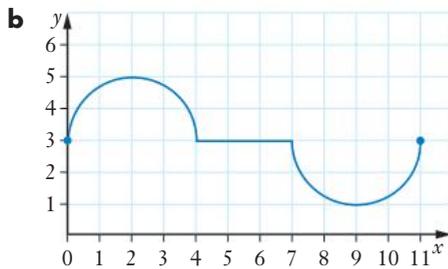
2 a  $f(2) = 3, f(3) = 4, f(4) = 6, f(5) = 3, f(7) = 1$



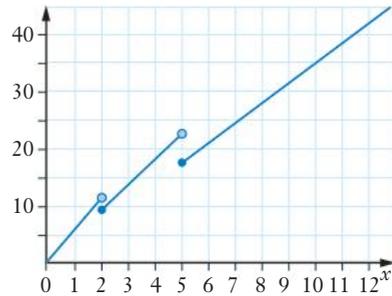
3 a  $f(1) = 7, f(3) = 31, f(5) = 47$



4 a  $f(2) = 5, f(6) = 3, f(8) = 3 - \sqrt{3} \approx 1.27$



5 
$$P(x) = \begin{cases} 5.5x, & x < 2 \\ 4.5x, & 2 \leq x < 5 \\ 3.5x, & x \geq 5 \end{cases}$$

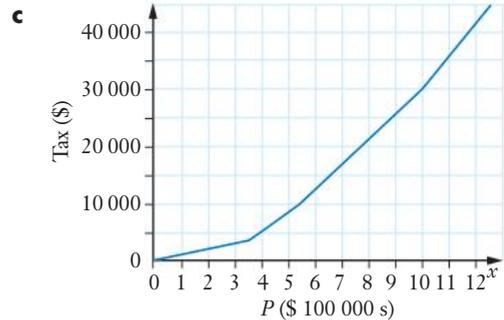


6 a

$$T(P) = \begin{cases} 0.01P, & 0 < P < \$350\,000 \\ 3500 + 0.035(P - 350\,000), & \$350\,000 \leq P < \$540\,000 \\ 10\,150 + 0.045(P - 540\,000), & \$540\,000 \leq P < \$1\,000\,000 \\ 30\,850 + 0.0575(P - 1\,000\,000), & \$1\,000\,000 \leq P \end{cases}$$

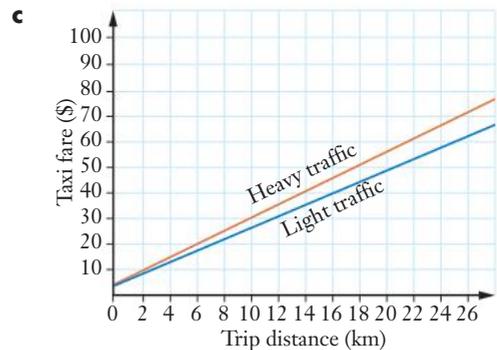
where the price  $P$  is rounded up to the nearest \$100

b \$3000, \$12 850, \$26 350



7 a  $C = 4.4 + 2.17d + \frac{60d}{45} \times 0.05 \times 0.81$

b  $C = 4.4 + 2.17d + \frac{60d}{30} \times 0.25 \times 0.81$

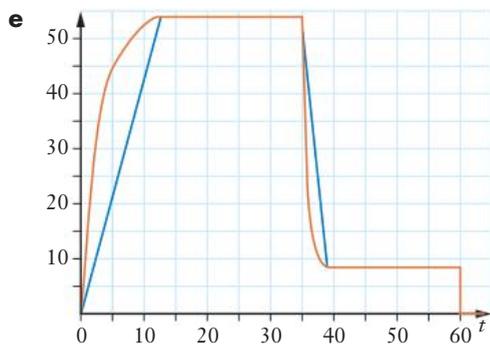


$$8 \text{ a } v = \begin{cases} 54 \times \frac{t}{12}, & t \leq 12 \\ 54, & 12 \leq t < 35 \\ 54 - 46 \left( \frac{t-35}{3.5} \right), & 35 \leq t < 38.5 \\ 8, & 38.5 \leq t < 60 \\ 0, & 60 \leq t \end{cases}$$

$$8 \text{ b } v = \begin{cases} 56(1-0.75^t), & t \leq 12 \\ 54, & 12 \leq t < 35 \\ 8 + 46 \times 0.25^{t-35}, & 35 \leq t < 38.5 \\ 8, & 38.5 \leq t < 60 \\ 0, & 60 \leq t \end{cases}$$

**c** First: 13.5 m/s, 22.5 m/s, 45 m/s  
 Second: 32.4 m/s, 42.7 m/s, 52.8 m/s  
 The second increases faster.

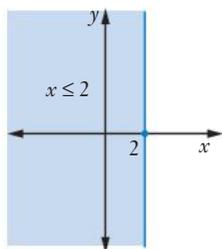
**d** First: 40.9 m/s, 27.7 m/s, 14.6 m/s, 8 m/s  
 Second: 19.5 m/s, 10.9 m/s, 8.7 m/s, 8 m/s  
 The second decreases faster.



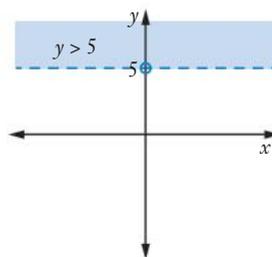
The first function curves up to 54 m/s after jumping and curves down to 8 m/s after the parachute is opened.

### Chapter 4 review

**1 a** Domain:  $x \leq 2$ , range:  $\mathbb{R}$

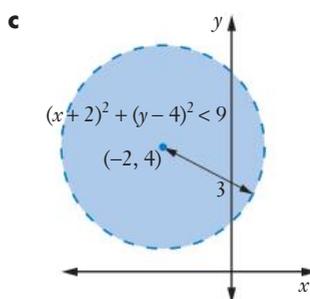


**b** Domain:  $\mathbb{R}$ , range:  $y > 5$



**2 a** It is the inside of a circle with centre  $(-2, 4)$  and radius 3.

**b** Domain:  $-5 < x < 1$ , range:  $1 < y < 7$



**3 a**  $(-1, 4)$ , 4

**b**  $(-3, -7)$ , 5

**4 a** Not a function

**b** Function

**c** Not a function

**d** Function

**e** Function

**f** Not a function

**g** Function

**h** Not a function

**i** Function

**j** Function

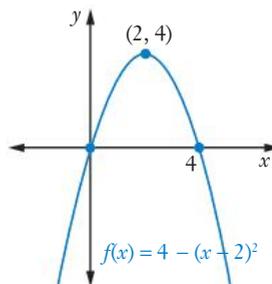
**5 a** Function

**b** Not a function

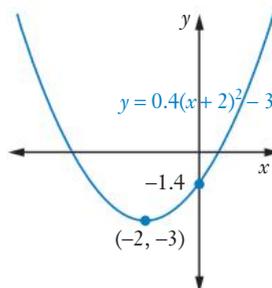
**c** Not a function

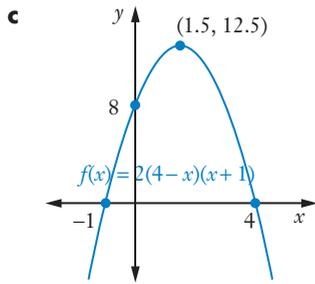
**d** Function

**6 a**

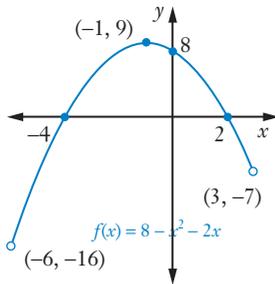


**b**

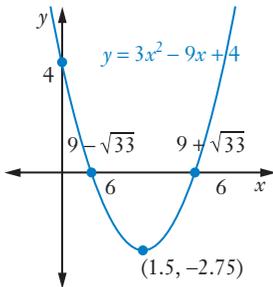




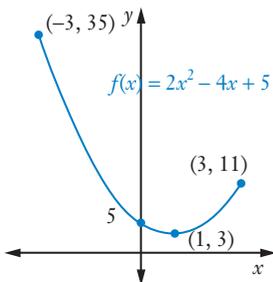
**7 a** Range:  $-16 < y \leq 9$



**b** Range:  $y \geq -2\frac{3}{4}$



**c** Range:  $3 \leq y \leq 35$

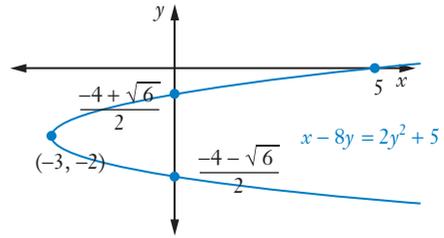


**8** 7.78

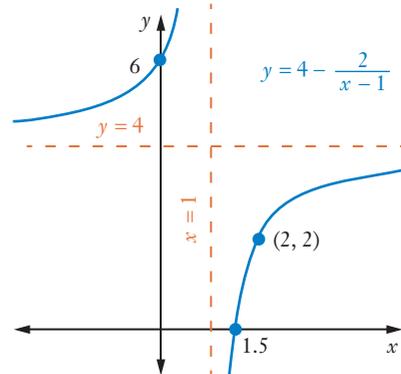
**9** 1012.5 m<sup>2</sup>

**10**  $y = 2x^2 + 3x + 1$

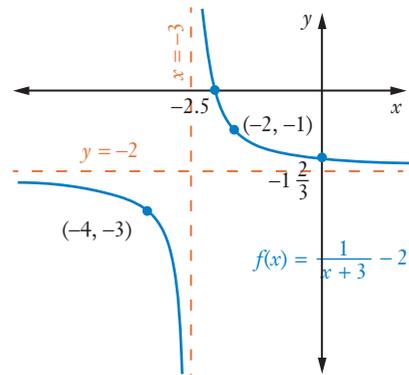
**11** Domain:  $x \geq -2$



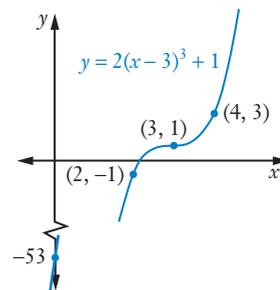
**12 a** Domain:  $\mathbb{R} \setminus 1$ , range:  $\mathbb{R} \setminus 4$

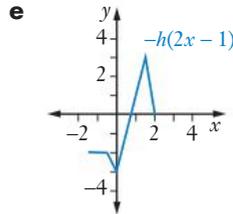
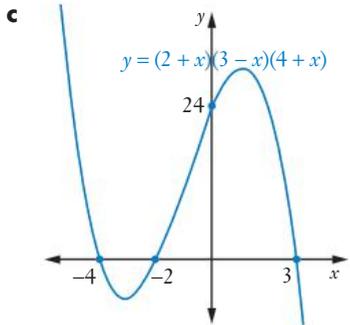
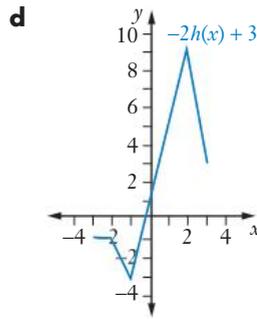
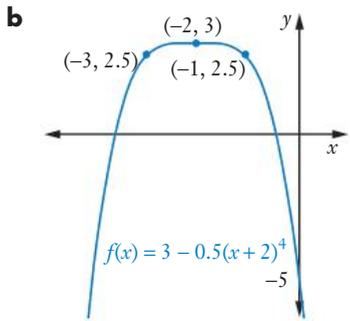


**b** Domain:  $\mathbb{R} \setminus -3$ , range:  $\mathbb{R} \setminus -2$



**13 a**





**16 a** -1.41      **b** -0.26 or 2.70

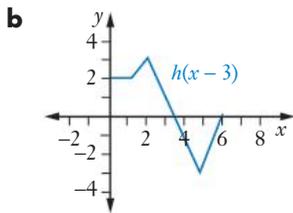
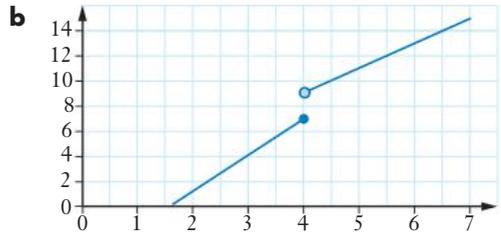
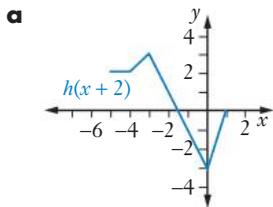
**c**  $-1 - \sqrt{7}$ ,  $-1 + \sqrt{7}$  or 2

**17 a** 1, 7, 13

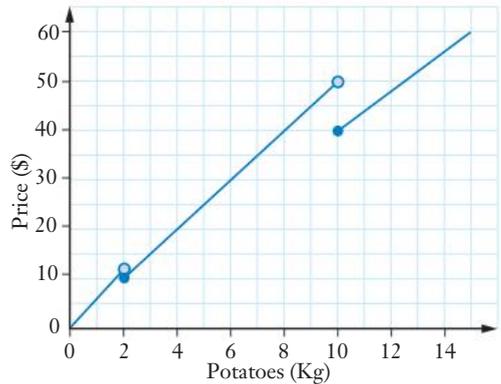
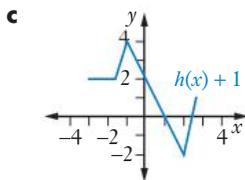
**14 a**  $R$

**b**  $y \geq -2$

**15**

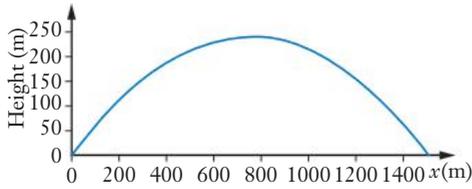


**18** 
$$P(x) = \begin{cases} 6x, & x < 2 \\ 5x, & 2 \leq x < 10 \\ 4x, & x \geq 10 \end{cases}$$



19  $f(x) = 6 - 0.5(x - 3)^2$

20 a



b  $h(x) = \frac{240}{750^2}x(1500 - x) = \frac{4x(1500 - x)}{9375}$

c 4.27 m

d The model fits the data given, but is unlikely to match elsewhere.

21 Maximum volume is about 95 000 cm<sup>3</sup> with dimensions 28.29 × 71.71 × 46.41 cm. Small changes reduce the volume, so the method seems to work well.

22  $x = 0.5(y + 2)^2 - 3$

23 a 51.5 m by 103 m      b 5304.5 m<sup>2</sup>

24  $y = 3 + \frac{2}{x+1}$

25  $g(x) = 3 - f(2x - 1)$

26

$T(P) =$

$$\begin{cases} 0, & 0 < P < \$5000 \\ 0.015(P - 5000), & \$5000 \leq P < \$75000 \\ 1050 + 0.035(P - 75000), & \$75000 \leq P < \$540000 \\ 17325 + 0.045(P - 540000), & \$540000 \leq P < \$1000000 \\ 30850 + 0.0575(P - 1000000), & \$1000000 \leq P \end{cases}$$



The model works correctly for the boundary conditions of each section. The function is an effective calculation method.

## Chapter 5

### Exercise 5.01

- 1
- |   |
|---|
| 1                                       |
| 1 1                                     |
| 1 2 1                                   |
| 1 3 3 1                                 |
| 1 4 6 4 1                               |
| 1 5 10 10 5 1                           |
| 1 6 15 20 15 6 1                        |
| 1 7 21 35 35 21 7 1                     |
| 1 8 28 56 70 56 28 8 1                  |
| 1 9 36 84 126 126 84 36 9 1             |
| 1 10 45 120 210 252 210 120 45 10 1     |
| 1 11 55 165 330 462 462 330 165 55 11 1 |
- 2 a 35      b 56      c 126      d 15  
e 210      f 84      g 8      h 38  
i 120      j 36
- 3 a 1287      b 1820      c 1330      d 5005  
e 3432      f 8568      g 7315      h 1716  
i 77 520      j 27 405
- 4 495
- 5 125 970
- 6 a 4      b 2      c 10      d 6  
e 12      f 16
- 7 133 784 560
- 8 10
- 9 3 838 380
- 10 924
- 11 5005
- 12 560
- 13 270 725
- 14 a 45      b 120
- 15 60

### Exercise 5.02

- 1  $\frac{1}{30}$   
2  $\frac{1}{204}$   
3  $\frac{5}{44}$   
4  $\frac{14}{33}$

$$5 \quad \mathbf{a} \quad \frac{63}{256} \approx 0.246 \quad \mathbf{b} \quad \frac{258\,048}{9765\,625} \approx 0.026$$

$$6 \quad \frac{6}{4165} \approx 0.0014$$

### Exercise 5.03

$$1 \quad \mathbf{a} \quad t^8 - 8t^7v + 28t^6v^2 - 56t^5v^3 + 70t^4v^4 - 56t^3v^5 + 28t^2v^6 - 8tv^7 + v^8$$

$$\mathbf{b} \quad n^{12} - 12n^{11}t + 66n^{10}t^2 - 220n^9t^3 + 495n^8t^4 - 792n^7t^5 + 924n^6t^6 - 792n^5t^7 + 495n^4t^8 - 220n^3t^9 + 66n^2t^{10} - 12nt^{11} + t^{12}$$

$$\mathbf{c} \quad a^{15} + 15a^{14}q + 105a^{13}q^2 + 455a^{12}q^3 + 1365a^{11}q^4 + 3003a^{10}q^5 + 5005a^9q^6 + 6435a^8q^7 + 6435a^7q^8 + 5005a^6q^9 + 3003a^5q^{10} + 1365a^4q^{11} + 455a^3q^{12} + 105a^2q^{13} + 15aq^{14} + q^{15}$$

$$\mathbf{d} \quad r^{14} - 14r^{13}u + 91r^{12}u^2 - 364r^{11}u^3 + 1001r^{10}u^4 - 2002r^9u^5 + 3003r^8u^6 - 3432r^7u^7 + 3003r^6u^8 - 2002r^5u^9 + 1001r^4u^{10} - 364r^3u^{11} + 91r^2u^{12} - 14ru^{13} + u^{14}$$

$$\mathbf{e} \quad q^{11} + 11q^{10}g + 55q^9g^2 + 165q^8g^3 + 330q^7g^4 + 462q^6g^5 + 462q^5g^6 + 330q^4g^7 + 165q^3g^8 + 55q^2g^9 + 11qg^{10} + g^{11}$$

$$2 \quad \mathbf{a} \quad n^5 + 20n^4c + 160n^3c^2 + 640n^2c^3 + 1280nc^4 + 1024c^5$$

$$\mathbf{b} \quad y^6 - 30y^5a + 375y^4a^2 - 2500y^3a^3 + 9375y^2a^4 - 18\,750ya^5 + 15\,625a^6$$

$$\mathbf{c} \quad p^7 - 21p^6q + 189p^5q^2 - 945p^4q^3 + 2835p^3q^4 - 5103p^2q^5 + 5103pq^6 - 2187q^7$$

$$\mathbf{d} \quad m^8 + 16m^7k + 112m^6k^2 + 448m^5k^3 + 1120m^4k^4 + 1792m^3k^5 + 1792m^2k^6 + 1024mk^7 + 256k^8$$

$$\mathbf{e} \quad k^9 - 27k^8e + 324k^7e^2 - 2268k^6e^3 + 10\,206k^5e^4 - 30\,618k^4e^5 + 61\,236k^3e^6 - 78\,732k^2e^7 + 59\,049ke^8 - 19\,683e^9$$

$$3 \quad \mathbf{a} \quad 81v^4 + 216v^3n + 216v^2n^2 + 96vn^3 + 16n^4$$

$$\mathbf{b} \quad 81n^4 - 432n^3m + 864n^2m^2 - 768nm^3 + 256m^4$$

$$\mathbf{c} \quad 243m^5 - 1620m^4g + 4320m^3g^2 - 5760m^2g^3 + 3840mg^4 - 1024g^5$$

$$\mathbf{d} \quad 128m^7 + 1344m^6c + 6048m^5c^2 + 15\,120m^4c^3 + 22\,680m^3c^4 + 20\,412m^2c^5 + 10\,206mc^6 + 2187c^7$$

$$\mathbf{e} \quad 2187x^7 - 10\,206x^6y + 20\,412x^5y^2 - 22\,680x^4y^3 + 15\,120x^3y^4 - 6048x^2y^5 + 1344xy^6 - 128y^7$$

$$4 \quad \mathbf{a} \quad q^9 - 9q^8x + 36q^7x^2$$

$$\mathbf{b} \quad n^{20} + 20n^{19}y + 190n^{18}y^2$$

$$\mathbf{c} \quad 4096g^{12} - 24\,576g^{11}v + 67\,584g^{10}v^2$$

$$\mathbf{d} \quad e^{20} - 60e^{19}m + 1710e^{18}m^2$$

$$\mathbf{e} \quad 16\,384m^7 + 143\,360m^6u + 537\,600m^5u^2$$

$$5 \quad 3^{12} \times 2^4 \times {}^{16}C_4 x^{12}y^4 = 15\,475\,561\,920x^{12}y^4$$

$$6 \quad 2^8 \times 5^4 \times {}^{12}C_4 m^8n^4 = 79\,200\,000m^8n^4$$

### Investigation: The space shuttle

At the temperature of the Challenger launch ( $-1^\circ\text{C}$ ) the probability of failure of the o-ring was about 0.13.

Postponement to a later time and a temperature of  $16^\circ\text{C}$  would have reduced the risk to at most 0.02.

The Columbia breakup on re-entry was caused by debris at take-off damaging the heat shielding tiles. This had occurred previously without causing a problem, so the risk had been ignored. The risk was managed in later flights by inspecting for damage in orbit, and launching only when another shuttle was available for a rescue mission if needed.

Although risks in travel by car are higher than most other risks we take, space travel is much riskier, whether assessed by trip numbers or by trip lengths.

### Exercise 5.04

$$1 \quad \mathbf{a} \quad \frac{5}{33} \quad \mathbf{b} \quad \frac{7}{22} \quad \mathbf{c} \quad \frac{35}{66} \quad \mathbf{d} \quad \frac{1}{22}$$

$$2 \quad \frac{58}{135}$$

$$3 \quad \mathbf{a} \quad \frac{1}{3} \quad \mathbf{b} \quad \frac{3}{11}$$

$$4 \quad P(H | O) = 0.517, P(H | \bar{O}) = 0.132, \text{ so for obese women the risk is nearly 4 times greater.}$$

$$5 \quad \mathbf{a} \quad \frac{1}{1024} \quad \mathbf{b} \quad \frac{15}{1024} \quad \mathbf{c} \quad \frac{45}{512} \quad \mathbf{d} \quad \frac{243}{1024}$$

$$6 \quad 0.025\%$$

$$7 \quad \frac{3}{11}$$

$$8 \quad \frac{9}{16}$$

### Exercise 5.05

$$1 \quad \mathbf{a} \quad \frac{1}{45} \quad \mathbf{b} \quad \frac{2}{45} \quad \mathbf{c} \quad \frac{14}{45}$$

$$2 \quad \mathbf{a} \quad \frac{4}{9} \quad \mathbf{b} \quad \frac{1}{9} \quad \mathbf{c} \quad \frac{4}{9} \quad \mathbf{d} \quad \frac{1}{2}$$

$$\mathbf{e} \quad \frac{5}{18} \quad \mathbf{f} \quad \frac{7}{18}$$

$$3 \quad \mathbf{a} \quad \frac{11}{40} \quad \mathbf{b} \quad \frac{3}{8}$$

4  $\frac{1}{16}$

5 a  $\frac{299}{600}$       b  $\frac{1}{1800}$

6 a  $\frac{1}{5}$       b  $\frac{4}{5}$       c  $\frac{1}{125}$       d  $\frac{64}{125}$

e  $\frac{16}{125}$       f  $\frac{12}{125}$       g  $\frac{48}{125}$

7 0.525

8 a  $\frac{1}{12}$       b  $\frac{1}{2}$       c  $\frac{1}{2}$

d  $\frac{1}{4}$       e  $\frac{1}{6}$

9 a 0.384      b 0.992      c 0.02      d 0.98

10 a 0.26      b  $\frac{4}{17}$       c No

11 a 8.5%      b 1.445%

12 0.569

### Exercise 5.06

1 300

2 a

		Height			Total
		Short (< 166 cm)	Average (166–175 cm)	Tall (> 175 cm)	
Number of computers	Few (0–2)	0.33	0.27	0.24	0.28
	Average (3–4)	0.19	0.32	0.24	0.25
	Many (> 4)	0.48	0.41	0.52	0.47

b No, considering the small numbers.

3 a 0.035      b 0.285      c 0.015      d 7 : 57

4 a 800

b The park might not be typical, birds might leave the park or come in, some birds may die after tagging, tagging may make the birds easier to catch.

5 a 197      b 430

c The high incidence of hooking suggest they may have a high mortality, they might leave the area, other sharks could come in, tagging might make them more vulnerable or easier to catch.

6 a

	Returning to work		Not returning
	Full-time	Part-time	
Professional	0.31	0.38	0.31
Administrative	0.12	0.24	0.65
Clerical	0.23	0.24	0.53
	0.21	0.26	0.53

b Professional women are more likely to return to work and those in administrative positions are less likely to return.

7 a

	30 minutes	45 minute	1 hour
Workshop	0.30	0.43	0.26
On-site	0.5	0.35	0.15
Office	0.22	0.30	0.48

b Those in the workshop prefer a 45 min break, those on-site a shorter break and those in the office a longer break.

8 a Male 0.43, female 0.33

b Male 0.37, female 0.44

c Men have greater support for a republic than women.

9 a 0.1982      b 0.0072

c That false positives are 27.5 times more likely than true positives.

10 a 0.095      b 0.0495

### Chapter 5 review

1 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1

2 210

3  $\frac{95}{253} \approx 0.3755$

4  $\frac{18}{143} \approx 0.1259$

5 a  $x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9$

b  $6561x^8 + 34\,992x^7y + 81\,648x^6y^2 + 108\,864x^5y^3 + 90\,720x^4y^4 + 48\,384x^3y^5 + 16\,128x^2y^6 + 3072xy^7 + 256y^8$

6  $729p^6 - 5832p^5q + 19\,440p^4q^2$

7  $256\,256t^4q^{10}$

8  $\frac{1}{15}$

- 9  $\frac{1}{64}$   
 10  $\frac{1}{16}$   
 11  $\frac{41}{96} \approx 0.427$   
 12 About 327  
 13 a

	Short	Average	Tall	Total
Small shoes	0.556	0.286	0.1	0.3
Average shoes	0.333	0.524	0.5	0.475
Large shoes	0.111	0.19	0.4	0.225
Total	1	1	1	1

- b Yes  
 14 a 0.097    b 0.027    c 0.009 409  
 15  $n = 8$   
 16  $\frac{1}{56}$   
 17 0.46  
 18 a  $\frac{1}{88}$     b  $\frac{1}{2728}$   
       c  $\frac{1}{8184}$     d  $\frac{1}{237\,336}$   
 19 a 0.046    b 0.609    c 5  
 20 About 0.493  
 21 About 0.259 and 0.227, 3% higher for smokers  
 22  $P(\text{HD} | \overline{\text{GF}}) \approx 0.4534$  and  
 $P(\text{HD} | \text{GF}) \approx 0.3208$ , so there is a about a 13%  
 higher risk for those with the genetic factor.  
 Genetic factors may be important predictors of  
 heart trouble, but obesity is definitely related.

## Chapter 6

### Exercise 6.01

- 1 a GP,  $r = 2$     b GP,  $r = 3$   
       c not GP    d GP,  $r = -\frac{1}{3}$   
       e not GP    f GP,  $r = \sqrt{2}x^{-2}$   
 2 a 3    b  $-\frac{2}{3}$     c -5  
       d 3    e  $\frac{3}{2}$     f  $-\frac{3}{2}$   
 3 a 2, -6, 18, -54, ...  
       b -1, -5, -25, -125, ...  
       c 2, 3, 4,  $\frac{1}{2}$ ,  $6\frac{3}{4}$ , ...

- d  $9x, -3x^2, x^3, -\frac{1}{3}x^4, \dots$   
 e  $50y^{-2}, 10y^{-1}, 2, \frac{2}{5}y, \dots$   
 f  $a, ar, ar^2, ar^3, \dots$   
 4 a -5, 10, -20, 40, ...    b  $-\frac{1}{4}, 1, -4, 16, \dots$   
       c  $-12\frac{1}{2}, 2\frac{1}{2}, -\frac{1}{2}, \frac{1}{10}, \dots$     d  $\frac{1}{2}x^{-2}, -x^{-1}, 2, -4x, \dots$   
       e  $20, -5, 1\frac{1}{4}, -\frac{5}{16}, \dots$     f 20, 30, 45, 67.5, ...  
 5 There is a common ratio of 1.065.  
 6 There is a common ratio of 1.06.  
 7 \$17 803.48  
 8 If the depreciation rate is  $x\%$ , there is a common  
 ratio of  $(1 - 0.01x)$ .

### Investigation: Limiting behaviour

- 1 a 157 464, about  $9 \times 10^9$ , about  $5 \times 10^{14}$   
       b 157 464, about  $-9 \times 10^9$ , about  $5 \times 10^{14}$   
       c 0.015 625, about  $1.5 \times 10^{-5}$ , about  $1.5 \times 10^{-8}$   
       d 0.015 625, about  $-1.5 \times 10^{-5}$ ,  
       about  $1.5 \times 10^{-8}$   
 2 For  $r > 1$ ,  $r < -1$  they get larger in magnitude, for  
 $-1 < r < 1$  they get closer to 0  
 3 For  $r > 1$ ,  $r < -1$ ,  $t_n \rightarrow \pm\infty$ , for  $-1 < r < 1$ ,  $t_n \rightarrow 0$   
 4  $t(n) = 10 \times 2^n$  steeply increases as  $n$  increases  
 $t(n) = 10 \times (0.2)^n$  decreases asymptotically to 0  
 5 As  $n \rightarrow \infty$ ,  $t(n) = 10 \times 2^n \rightarrow \infty$ ,  
 $t(n) = 10 \times (0.2)^n \rightarrow 0$   
 6 For  $r > 1$ ,  $r < -1$ ,  $t_n \rightarrow \pm\infty$ , for  $-1 < r < 1$ ,  $t_n \rightarrow 0$

### Exercise 6.02

- 1 a 53.7824    b  $\frac{3}{512}$     c  $-324\sqrt{3}$   
       d  $-\frac{5}{64}$     e  $\frac{x^6\sqrt{y}}{y^{11}}$     f 8418.89  
 2  $t_5 = 3280.5$   
 3  $t_{15} \approx 0.0078$   
 4  $t_6 = 972$   
 5 No  
 6 a 7    b 11    c 12    d 13    e 7  
 7 a 640    b  $\frac{1}{2}\sqrt{2}$  or  $-\frac{1}{2}\sqrt{2}$     c  $-45\sqrt[3]{3}$   
 8 a \$3651.52    b \$13 183.14    c \$2022.07  
       d \$775.71    e \$114 622.87    f \$33 715.46

- 9 9 years  
 10 5.5% with quarterly rests, but there is little difference.

### Exercise 6.03

- 1 a 213.75 b 1.998 c 17 476 d 29 524  
 2 a 1275 b 7651 c 6141 d 2294  
 3 9  
 4 9  
 5 1640  
 6 a 0.622 b 19.1  
 7 a 1215 b 2868  
 8 320, 620  
 9 a \$47 158.95 b \$318 748  
 10 a 3.355 m b 126.45 m  
 11  $2\frac{40}{61}, -645\frac{21}{61}$ ; no, the power of  $r$  is odd  
 12 a 5th week b 134.8 km  
 c The regime could cause an injury or take too much time.

### Exercise 6.04

- 1 a 27 b  $13\frac{1}{2}$  c 125  
 d  $83\frac{1}{3}$  e  $1\frac{3}{5}$   
 2 a  $\frac{5}{11}$  b  $\frac{4}{11}$  c  $\frac{169}{330}$  d  $\frac{41}{165}$  e  $\frac{241}{1998}$   
 3 About 1667 litres  
 4 120 m  
 5 The time taken also decreases. Achilles passes the tortoise at the 2 km mark.

### Exercise 6.05

- 1 \$8228.96  
 2 \$39 605.70  
 3 \$86 752.47  
 4 \$18 255.57  
 5 \$4078.26  
 6 \$3559.66  
 7 \$12 945.11  
 8 a 42c b 56c c 32c d 68c  
 9 Andrea's \$5000 was worth about \$5565 when David got his \$5500, so it was reasonably fair if you think they should have the same amount.

- 10 \$1581.65  
 11 The future values add up to \$8058.73, so it is very fair.  
 12 \$2.47  
 13 \$368.40

### Exercise 6.06

- 1 a \$21 701.27 b \$23 242.06  
 2 a 12.5779 b 9.7122  
 c 557.8304 d 113.5515  
 3 a 10.1945 b 99.9578 c 64.7957  
 4 a 13.1808 b 307.3859 c 1372.9211  
 5 a \$50 756.32 b \$536 083.05  
 c \$534 371.51  
 6 a \$14 233.27 b \$11 621.26 c \$11 609.63  
 7 \$215.40  
 8 a \$116 000 b \$104 000  
 c \$115 000 d \$157 000  
 9 a \$2511.12 b \$2130.55  
 c \$490.07 d \$2758.50  
 10 Plan A gives \$160 292.85, plan B \$174 403.60 and plan C \$174 728.39. Fortnightly and weekly payments actually invest \$62 400, while monthly payments invest only \$57 600, so this is why they give so much more. The smaller difference between weekly and fortnightly results from the extra interest from weekly payments, as part of the payment is made earlier.  
 11 a \$26 681.53 b She will get only \$21 345.23.  
 12 \$82.03  
 13 a \$2164.19  
 b Their payment goes up by \$471.84 (21.8%).  
 14 a \$1266.12 b \$339.13  
 15 Five years is a reasonable time to pay for a car, so assume a 5 year loan. The flat-rate would be \$310/month and the reducing interest \$291.81/month, so the reducing interest loan is better.

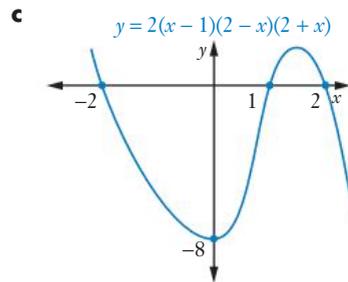
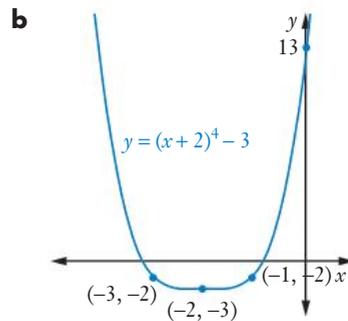
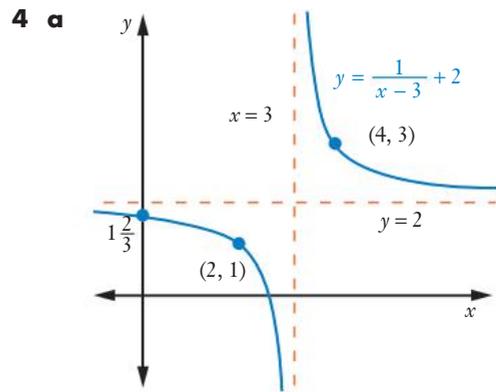
### Chapter 6 review

- 1 a  $r = \frac{1}{4}$  b  $r = 2\sqrt{3}$  c Not a GP  
 2 a 7, -21, 63, -189, ...  
 b 100,  $20\sqrt{5}$ , 20,  $4\sqrt{5}$ , ...  
 c  $m, n, \frac{n^2}{m}, \frac{n^3}{m^2}$

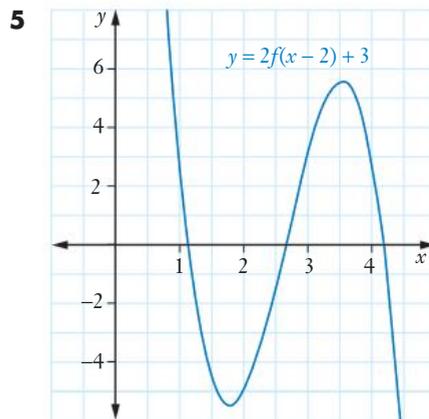
- 3** 160  
**4**  $t_6 = 3690.5625$   
**5**  $t_{10} = -\frac{9\sqrt{2}}{4}$  or  $t_{10} = \frac{9\sqrt{2}}{4}$   
**6** About \$75 605  
**7** 23.977  
**8** 9  
**9** 7 and 3584  
**10** 27  
**11**  $\frac{9}{55}$   
**12** \$8415.31  
**13** \$15 887.95  
**14** \$19.72  
**15** \$18 024.45  
**16** **a** 11.8594      **b** 11.4699  
**17** \$20 460.86  
**18** \$190 512.65  
**19** \$45 857.22  
**20** 8 years  
**21** 6.95 m  
**22** 1 m  
**23** Assuming 50% includes the car payments and the mortgage has monthly payments over 30 years, \$847 000. Current mortgages generally take a working lifetime to pay off, so a 30-year loan is reasonable. If interest rates go up or if one of them stops working or loses their job, they could have difficulties.

## Practice examination 2

- 1** Centre  $(3, 0)$ , radius  $2\sqrt{2}$ , domain  $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$ , range  $[-2\sqrt{2}, 2\sqrt{2}]$   
 (1 mark centre/radius, 1 mark domain/range)  
**2** A is a function and B is not, vertical line test  
 (1 mark)  
**3** **a** Minimum,  $(5, 2)$       **b**  $(3, 1)$   
 (1 mark each part)

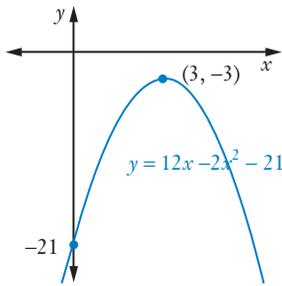


(2 marks each part)



(2 marks)

- 6 1820 (1 mark)
- 7 a 220    b 56 (1 mark each part)
- 8  $\frac{3}{11}$  (2 marks)
- 9  $x^{16} + 16x^{15}y + 120x^{14}y^2 + \dots$  (1 mark)
- 10  $\frac{12}{95}$  (2 marks)
- 11  $\frac{1}{1296}$  (1 mark)
- 12 a  $\frac{6}{31}$     b  $\frac{8}{27}$  (1 mark each part)
- 13 a 81, 243, 729    b  $-51.2$  (1 mark each part)
- 14 a  $80\frac{8}{9}$     b 81 (1 mark each part)
- 15 a \$9835.76    b \$7683.68 (1 mark each part)
- 16



(2 marks)

$$17 \quad C(x) = \begin{cases} 1.3x & 0 \leq x \leq 10 \\ 1.1x & 10 < x \leq 100 \\ 0.95x & 100 < x \end{cases} \quad (2 \text{ marks})$$

18  $\frac{25}{66}$  (2 marks)

19  $P(Q | \text{born Q}) \approx 0.614$ ,  $P(Q) \approx 0.524$ ,  
so  $P(Q | \text{born Q}) \neq P(Q)$

For this sample, support for Queensland is not independent of birthplace. (2 marks)

20 About \$263 000 (2 marks)

21 a  $y = 7.6 - \frac{8.2}{x+2}$ , maximum yield 7.6 t/ha

b 3.125 kg/ha gives a 6 t/ha yield (4 marks)

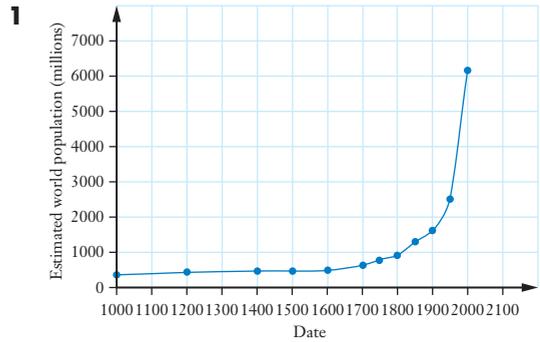
22 Assuming they are spread out,

$$[1 - (0.875)^2]^2 = \frac{225}{4096} \approx 0.055 \quad (4 \text{ marks})$$

23  $1\frac{7}{9}$  (2 marks)

## Chapter 7

### Investigation: Population predictions



2 Rate of increase in millions/year

Century	15th	16th	17th	18th	19th	20th
Rate of increase	0.1	0.15	1.4	2.7	7	45

3 Assuming a similar pattern, about 200 million/year

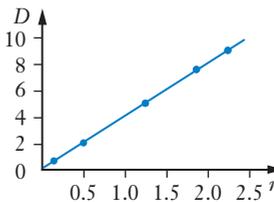
4  $6100 + 200 \times 50 = 16\,100$  million, assuming similar patterns of increase.

### Exercise 7.01

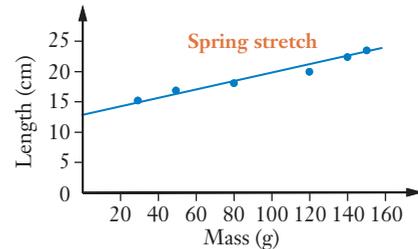
- 1 a \$2.45/kg    b 45 rev/min  
c 30 m/s    d  $6.25 \text{ m/s}^2/\text{N}$   
e AU\$1.25/US\$1 or AU\$1/US\$0.80
- 2 a 0.4 mL/L, 250 L    b 80 mL  
c 144 L    d 1429 seeds  
e 1908 bricks

3 Yes, rate = 0.15

4 Yes

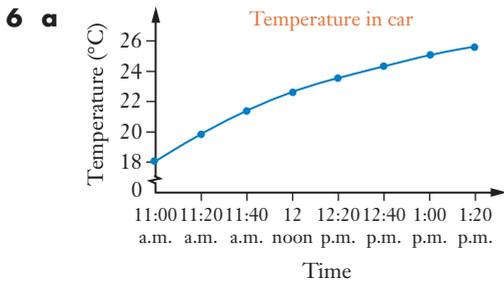


5 a



b The graph is a straight line, within experimental error.

- c About 0.07 cm/g.  
 d About 13 cm, the unstretched length.



- b The rate of change of temperature gradually decreases.  
 c  $3.2^\circ\text{C}/\text{h}$   
 d It will probably be lowering the temperature about  $1^\circ\text{C}/\text{h}$ , so it won't be much use.

e No

- 7 a** 88 149 km  
**b** 3673 km/h  
 c Because of the motion of the Earth around the Sun, it has to orbit about  $30^\circ$  further than a complete orbit to appear at the same place in the sky.

- 8 a** 13.3 km/h      **b** The first leg  
 c The second leg      **d** 8 km/h

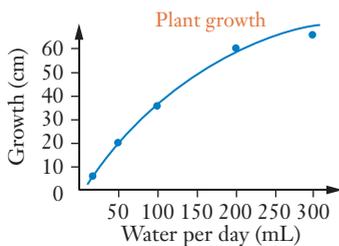
**9** 91 km/h

**10**  $4 \text{ m/s}^2$

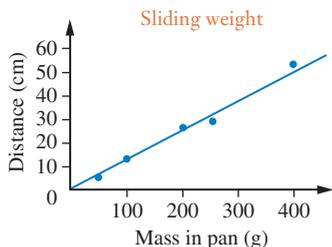
- 11 a** 0.5 m/s      **b** 1.5 m/s  
 c 2.5 m/s      **d** 1.5 m/s

- 12 a** 0.72      **b** 0.46      **c** 0.40      **d** 0.97

**13** No



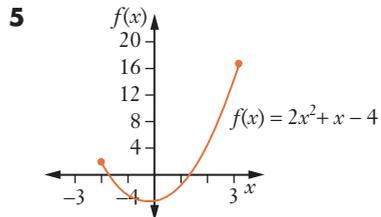
**14** Yes, within experimental error



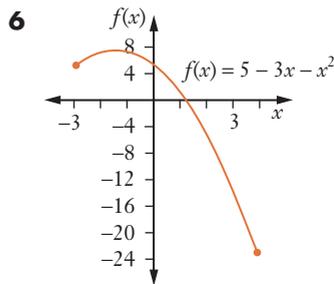
## Exercise 7.02

Answers in this section based on tangents may vary a little from the stated answer.

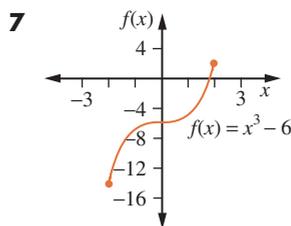
- 1**  $10 \text{ m/s}^2, -9.1 \text{ m/s}^2, -20 \text{ m/s}^2$   
**2**  $-10.7 \text{ mL/day}, 4 \text{ mL/day}$   
**3**  $112.5^\circ\text{C}/\text{min}, 0^\circ\text{C}/\text{min}, -100^\circ\text{C}/\text{min}$   
**4**  $-2, -2, -2$



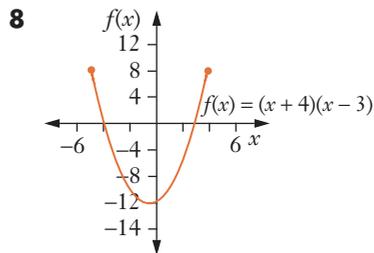
- a** -3      **b** 5      **c** 9



- a** 1      **b** -3      **c** -9



- a** 3      **b** 0      **c** 3



- a** -5      **b** -1      **c** 3

**9** Velocity at 2 s is 25 m/s. It takes 5 s to go 100 m. Its velocity at 100 m is 10 m/s.

**10** 6

### Exercise 7.03

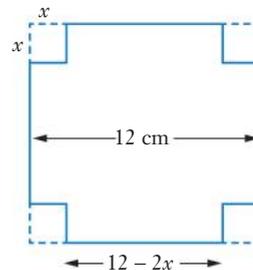
- 1 **a**  $4x + 4h + 1$     **b**  $4h$     **c**  $4$   
**2 a**  $7 - 3x - 3h$     **b**  $-3h$     **c**  $-3$   
**3 a**  $2$     **b**  $-5$     **c**  $6$   
**4 a**  $3x^2 + 6xh + 3h^2$     **b**  $6xh + 3h^2$   
**c**  $6x + 3h$   
**5 a**  $x^2 + 2xh + h^2 - 5x - 5h$     **b**  $2xh + h^2 - 5h$   
**c**  $2x + h - 5$   
**6 a**  $4x + 2h - 4$     **b**  $8x + 4h + 3$     **c**  $-x - \frac{3}{2}h$   
**7 a**  $\frac{1}{x+h}$     **b**  $\frac{-h}{x(x+h)}$     **c**  $\frac{-1}{x(x+h)}$   
**8 a**  $\frac{-1}{(x+h+1)(x+1)}$     **b**  $\frac{-3}{(x+h-2)(x-2)}$   
**c**  $\frac{2}{(3-2x-2h)(3-2x)}$   
**9**  $3x^2 + 3xh + h^2 - 1$   
**10**  $\frac{2}{(5-x-h)(5-x)}$   
**11**  $\frac{x^2 + xh - 64}{x(x+h)}$   
**12** *Hint:* Multiply the numerator and denominator of  $\frac{f(x+h) - f(x)}{h}$  by  $(\sqrt{(x+h)^2 - 1} + \sqrt{x^2 - 1})$  then simplify.

### Exercise 7.04

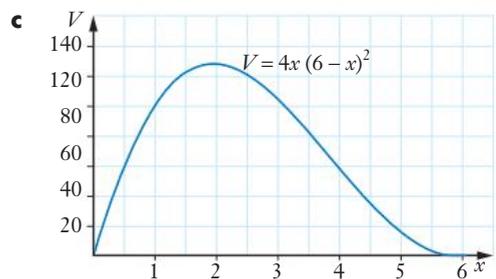
- 1 **a**  $9$     **b**  $-2$     **c**  $0.5$   
**d**  $0.0567$     **e**  $1$   
**2 a** Undefined    **b**  $\lim_{x \rightarrow 0} f(x) = 1$   
**3 a** Undefined    **b**  $\lim_{x \rightarrow 0} f(x) = 3$   
**4 a** Undefined    **b**  $\lim_{x \rightarrow 1} g(x) = 2$   
**5 a**  $7$     **b**  $-2$     **c**  $21$   
**d**  $4$     **e**  $0$     **f**  $2.5$   
**6 a**  $2$     **b**  $-8$     **c**  $-4$   
**d**  $1$     **e**  $0.289$     **f**  $1$   
**7 a**  $\lim_{x \rightarrow 0} f(x) = -2$   
**b** Limit does not exist because  $f(x)$  approaches different numbers as  $x \rightarrow -3$  from below and above.  
**c**  $\lim_{x \rightarrow -3} g(x) = -6$

- d** Limit does not exist because  $f(x)$  decreases without bound as  $x \rightarrow -5$ .  
**8 a** Limit does not exist because  $f(x)$  increases without bound as  $x \rightarrow -1$  from below and decreases without bound as  $x \rightarrow -1$  from above.  
**b**  $\lim_{x \rightarrow 1} f(x) = \frac{3}{2}$   
**c** Limit does not exist because  $g(x)$  approaches different numbers as  $x \rightarrow 0$  from below and above.  
**d**  $\lim_{x \rightarrow -1} h(x) = -1$   
**e** Limit does not exist because  $h(x)$  decreases without bound as  $x \rightarrow 3$ .  
**f** Limit does not exist because  $g(x)$  increases without bound as  $x \rightarrow 2$  from below and decreases without bound as  $x \rightarrow 2$  from above.  
**9 a**  $25$     **b** Limit does not exist.  
**c**  $4$     **d**  $10$   
**e** Limit does not exist.    **f**  $-3$   
**10 a**  $-5$     **b**  $-3$   
**c** Limit does not exist because  $f(x)$  increases without bound as  $x \rightarrow -5$  from below and decreases without bound as  $x \rightarrow -5$  from above.  
**d**  $-2$     **e**  $1$     **f**  $3$     **g**  $-1$     **h**  $\frac{1}{2}$

**11 a**



**b**  $V = 4x(6 - x)^2$



- d**  $x = 2$     **e**  $\lim_{x \rightarrow 2} (V) = 128$   
**12 a**  $A = \frac{1}{2}x\sqrt{18 - x^2}$     **b**  $x = 3$

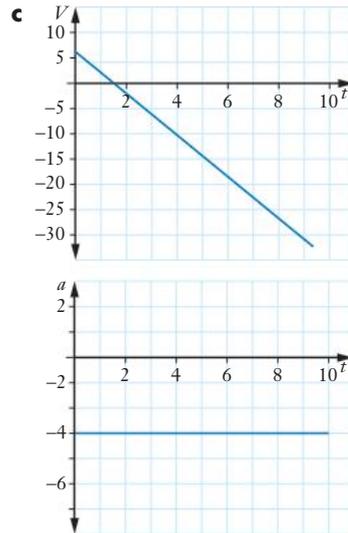
### Exercise 7.05

- 1 **a**  $f'(1) = 3$                       **b**  $f'(3) = -1$   
**c**  $g'(2) = 4$                       **d**  $h'(-1) = 3$
- 2  $f'(x) = 3x^2$
- 3 17
- 4 **a**  $f'(x) = 5$                       **b**  $g'(x) = -2$   
**c**  $h'(x) = 2x$                       **d**  $f'(x) = 6x - 4$
- 5  $f'(x) = 6x^2$
- 6  $g'(x) = 8x + 20$
- 7  $f'(1) = -4$
- 8  $f'(1) = 12$
- 9 **a** 11                      **b** 11                      **c** -8  
**d** -4                      **e** 12                      **f** -1228
- 10 **a** 9                      **b** 1                      **c**  $6x$   
**d** 0                      **e**  $20x + 3$                       **f**  $-5 - 2x$   
**g**  $3 - 8x$                       **h**  $21x^2$                       **i**  $4x^3$
- 11 Use the definition of the derivative. Remember that  $c$  is a constant.
- 12 Use the definition of the derivative. This time  $q$  is the variable instead of  $x$ .
- 13  $\frac{-1}{(x+2)^2}$
- 14  $f'(3) = 13$  and  $g'(3) = 11$ , so  $f(x)$  is steeper at  $x = 3$ .

### Exercise 7.06

- 1 **a**  $-\$5800/\text{year}$                       **b**  $-\$2117.50/\text{year}$   
**c**  $-\$2095/\text{year}$                       **d**  $-\$980/\text{year}$   
**e** The depreciation rate decreases over time.
- 2 **a** 0    **b** 2    **c** -3    **d** 5    **e** 2.5  
**f** 21
- 3 **a** 3    **b** -5    **c** 11    **d** -16    **e** -2
- 4 **a**  $14^\circ\text{C}$     **b**  $18^\circ\text{C}$     **c**  $2^\circ\text{C}/\text{min}$   
**d**  $1^\circ\text{C}/\text{min}$     **e**  $3.5^\circ\text{C}/\text{min}$
- 5 **a** About  $-1.57 \text{ L}/\text{min}$     **b**  $-1.6 \text{ L}/\text{min}$   
**c**  $-1.4 \text{ L}/\text{min}$
- 6 **a** 40 m above the ground  
**b** 25 m above the ground  
**c** 45 m                      **d**  $20 \text{ m}/\text{s}$   
**e**  $-10 \text{ m}/\text{s}$                       **f**  $-10 \text{ m}/\text{s}^2$

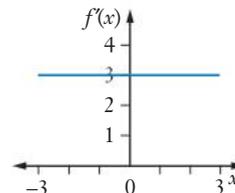
- 7 **a**  $-14 \text{ m}/\text{s}$                       **b**  $-4 \text{ m}/\text{s}^2$

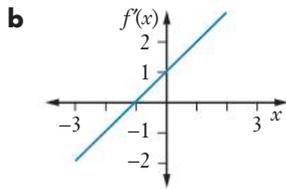


- 8 **a**  $v(t) = 6t + 5$                       **b**  $a(t) = 6$
- 9 **a** At  $t = -3$ ,  $v = 12$ ; at  $t = 4$ ,  $v = 12$   
**b** At  $t = 0$ ,  $v = 7$ ; at  $t = 5$ ,  $v = 47$   
**c** At  $t = 2$ ,  $v = -3$ ; at  $t = -4$ ,  $v = 33$   
**d** At  $t = -1$ ,  $v = 10$ ; at  $t = 3$ ,  $v = 10$
- 10 **a** At  $t = -2$ ,  $a = 51$ ; at  $t = 3$ ,  $a = 111$   
**b** At  $t = -3$ ,  $a = -59.75$ ; at  $t = 7$ ,  $a = 650.25$   
**c** At  $t = 1$ ,  $a = -0.1$ ; at  $t = 3$ ,  $a = -39.3$   
**d** At  $t = -2$ ,  $a = 12.8$ ; at  $t = 5$ ,  $a = 10$
- 11  $a = 12t$
- 12 **a** 64 m                      **b**  $64 \text{ m}/\text{s}$  downwards
- 13 **a**  $v(t) = \frac{-20}{(t+2)^2}$                       **b** About  $-0.408 \text{ m}/\text{min}$

### Exercise 7.07

- 1 -5
- 2 **a** C, E and G                      **b** A and C; E and G  
**c** C and E
- 3 **a**





**4 a** 0      **b** 1      **c**  $-\frac{1}{2}$       **d** -4

**5 a**  $2x + y + 4 = 0$       **b**  $x + y + 4 = 0$

**c**  $2x + y - 18 = 0$       **d**  $x - y - 6 = 0$

**e**  $10x + y + 17 = 0$       **f**  $6x - y - 4 = 0$

**6 a**  $x = 0$       **b**  $x = -1\frac{1}{2}$       **c**  $x = 2\frac{1}{2}$

**d**  $x = -1\frac{1}{4}$       **e**  $x = \frac{1}{3}$       **f**  $x = 0$

**7 a**  $f'(x) = 6x + 2$       **b**  $f'(1) = 8$

**c**  $8x - y - 2 = 0$

**8**  $3x - y - 4 = 0$

**9**  $x - 2y - 3 = 0$

**10 a**  $f'(x) = 4x - 9$       **b**  $x = 2\frac{1}{4}$

**c**  $x = 1$       **d**  $x = 3$

**11 a** 54, 22      **b** -45, -2805

**c**  $-1, -\frac{1}{36}$       **d**  $-\frac{6}{25}, -\frac{3}{2}$

**12**  $x = \pm\sqrt{\frac{2}{3}}$

**13**  $g(x)$  is steeper than  $f(x)$  at  $x = 5$ .

**14**  $f'(3) = 13$  and  $g'(3) = 11$ , so  $f(x)$  is steeper at  $x = 3$ .

**15**  $f(x)$  is steeper than  $g(x)$  when  $x > 6$

**16** (2, 3)

**17**  $\left(\frac{1}{3}, -9\frac{23}{27}\right)$  and (1, -10)

**18** Yes. For example, the functions  $f(x) = ax + b$  and  $g(x) = ax + c$  have the same gradient function as do the functions  $p(x) = ax^2 - bx + c$  and  $q(x) = ax^2 - bx + d$ .

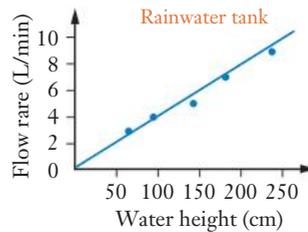
**19**  $\left(\frac{1}{3}, 2\frac{8}{9}\right)$

**20**  $\left(-\frac{1}{2}, -\frac{5}{8}\right)$  and  $\left(\frac{1}{2}, \frac{5}{8}\right)$

**21**  $f'(x) = 6x - 5$  and  $g'(x) = 6x + 1$ , so they are parallel so  $f'(x)$  is never greater than  $g'(x)$ .

## Chapter 7 review

**1** The results are consistent with a constant rate.



**2 a** Section F      **b** 10 km/h

**c** 2 h      **d** 20 km/h

**e**  $11\frac{3}{7}$  km/h      **f** 0 km/h

**3 a i** 33 000 bacteria/day

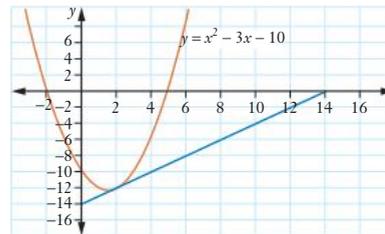
**ii** 26 000 bacteria/day

**iii** 1400 bacteria/day

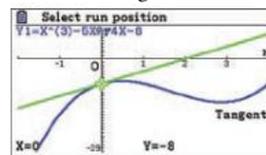
**b** The rate of increase in bacteria numbers decreases over time.

**4** About 2.3 cm/min

**5** Rate of change = 1.



**6** Rate of change = 4.



**7 a** 4      **b** -3      **c**  $10x + 5h + 7$

**d**  $3x^2 + 3xh + h^2$       **e**  $\frac{1}{(3-x-h)(3-x)}$

**8 a** Undefined      **b** -2

**9 a** Limit does not exist because  $f(x)$  increases without bound as  $x \rightarrow 0$  from below and increases without bound as  $x \rightarrow 0$  from above.

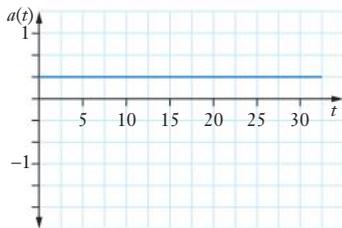
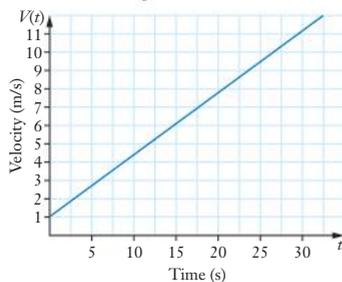
**b** -4

**c** -11

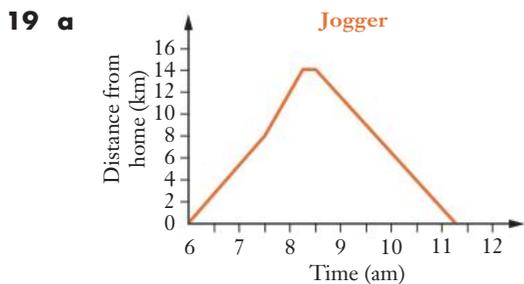
**d** Limit does not exist because  $f(x)$  decreases without bound as  $x \rightarrow 2$  from below and increases without bound as  $x \rightarrow 2$  from above.

**10 a**  $6x - 2$       **b** -14

- 11 a  $8x + 9$       b  $-15$   
 12 a 13 mL      b 58 mL  
     c 15 mL/s      d 13 mL/s  
     e 21 mL/s  
 13 a 128 m above the ground  
     b 77 m above the ground  
     c  $-4$  m      d 6 m/s (upwards)  
     e  $-22$  m/s (downwards)      f  $-10$  m/s<sup>2</sup>  
 14 a  $-9.8$  m/s      b  $-19.6$  m/s  
     c  $-29.4$  m/s      d  $-39.2$  m/s  
 15 a 6 m/s      b  $\frac{1}{3}$  m/s<sup>2</sup>



- 16 After 1 s, displacement = 2.5 m, velocity = 4 m/s;  
 after 2 s, displacement = 8 m, velocity = 7 m/s  
 17  $5x + y - 4 = 0$   
 18 a  $f'(x) = 4x - 7$       b  $x = 1\frac{3}{4}$       c  $x = 3$



- b From 7:30 a.m. to 8:15 a.m.  
 c On the way home  
 20 a  $-32$  cm/s      b 7 cm/s      c 14 cm/s      d  $-2$  cm/s  
 21 a i \$240      ii \$240.67  
     b \$0.67

- c i \$675      ii \$676.07  
 d \$1.07  
 e i \$1310      ii \$1311.47  
 f \$1.47      g  $C'(n) = 0.0008n + 0.27$   
 22  $x < 3$   
 23  $A'(r) = 2\pi h + 4\pi r$

## Chapter 8

### Exercise 8.01

- 1 a  $\frac{\pi}{3}$       b  $\frac{\pi}{4}$       c  $\frac{\pi}{2}$   
     d  $\frac{3\pi}{4}$       e  $\frac{2\pi}{3}$       f  $\frac{5\pi}{9}$   
     g  $\frac{7\pi}{3}$       h  $\frac{2\pi}{5}$       i  $\frac{8\pi}{5}$   
 2 a 0.9774      b 2.4435      c 4.8171  
     d 4.6740      e 9.8859  
 3 a 120°      b 150°      c 315°  
     d 98.2°      e 396°  
 4 a 57.3°      b 126.1°      c 45.8°  
     d 189.1°      e 118.0°  
 5 a  $\frac{2\pi}{3}$       b  $-\frac{\pi}{6}$       c  $\frac{\pi}{3}$   
 6 a  $\frac{\sqrt{2}}{4}$       b  $\frac{2\sqrt{3}+1}{2}$       c 1  
     d  $-3$       e  $-2\sqrt{2}-3$       f  $\frac{2\sqrt{3}+7}{4}$   
 7 a  $-0.90$       b 0.98      c 0.31      d 0.25  
 8 a 0.26      b 0.40      c  $-1.97$       d 1.00      e 0.98

### Exercise 8.02

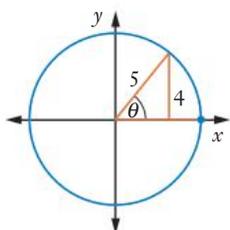
- 1 a  $\frac{5\pi}{3}$  cm      b  $\frac{85\pi}{6}$  mm      c  $\frac{9\pi}{10}$  m  
     d  $8\pi$  cm      e  $6\pi$  mm      f 23.78 cm  
 2 a  $\frac{150}{\pi}$  mm      b  $\frac{63}{2\pi}$  cm  
     c  $\frac{240}{\pi}$  cm      d  $\frac{58.8}{\pi}$  m  
     e  $\frac{559}{6\pi}$  cm      f 1.68 m  
 3 a 2      b 0.05      c 1.96  
     d 0.5      e 1.25      f 1.125

- 4** 22.7 cm  
**5** 39.8 cm  
**6 a**  $471.24 \text{ mm}^2$    **b**  $73.89 \text{ m}^2$    **c**  $1.35 \text{ m}^2$   
**7 a** 7.14 cm   **b** 4.76 m   **c** 2.55 cm  
**d** 1.25 cm  
**8** Radius = 9.08 cm, Area =  $151 \text{ cm}^2$   
**9 a** 1.30 (or  $74.3^\circ$ )   **b** 23.3 cm  
**10 a** 1.047 m   **b** 1.014 m  
**11** 28.3 cm  
**12** 91.7 m  
**13**  $2737 \text{ cm}^2$   
**14** 47.7%

### Exercise 8.03

- 1 a**  $-\cos(\alpha)$    **b**  $-\sin(x)$    **c**  $\tan(\theta)$   
**d**  $\tan(\beta)$    **e**  $-\tan(\alpha)$    **f**  $-\sin(\alpha)$   
**2 a**  $-\frac{1}{2}$    **b**  $-\frac{\sqrt{3}}{2}$    **c** -1  
**d**  $\frac{1}{2}$    **e**  $\frac{1}{\sqrt{3}}$    **f**  $\frac{\sqrt{3}}{2}$   
**3 a** 1   **b** -1   **c** 0  
**d** 0   **e** 0   **f** 0  
**4 a**  $\frac{1}{2}$    **b**  $-\frac{\sqrt{3}}{2}$    **c** 0   **d** 0

**5 a**



**b** 3   **c**  $\cos(\theta) = \frac{3}{5}, \tan(\theta) = \frac{4}{3}$

- 6 a**  $\sin(\theta) = \frac{12}{13}, \tan(\theta) = \frac{12}{5}$   
**b**  $\sin(\theta) = \frac{24}{25}, \tan(\theta) = -\frac{24}{7}$   
**c**  $\cos(\theta) = -\frac{4}{5}, \tan(\theta) = \frac{3}{4}$   
**d**  $\cos(\theta) = \frac{\sqrt{2}}{2}, \tan(\theta) = -1$   
**e**  $\sin(\theta) = \frac{\sqrt{2}}{2}, \tan(\theta) = 1$

**f**  $\sin(\theta) = -\frac{7}{25}, \tan(\theta) = -\frac{7}{24}$

**g**  $\cos(\theta) = -\frac{5}{13}, \tan(\theta) = -\frac{12}{5}$

**7 a**  $-1 - \frac{-\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - 1 = \frac{\sqrt{2}-2}{2}$

**b**  $\frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{1-\sqrt{3}}{2}$

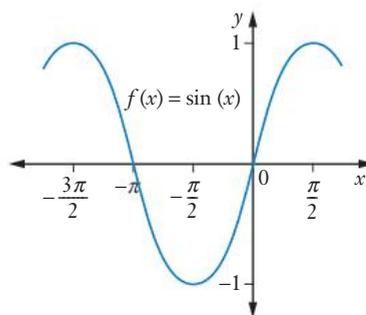
**c**  $-\frac{1}{2} + \frac{-\sqrt{3}}{2} = \frac{-1-\sqrt{3}}{2}$

**d**  $\frac{1}{4}$    **e** 1

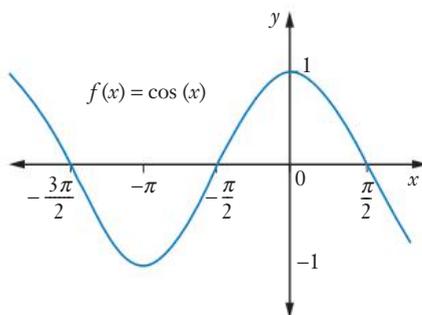
- 8 a**  $4 \tan(x)$    **b** 0   **c**  $-6 \sin(5y)$    **d**  $\cos(x)$

### Exercise 8.04

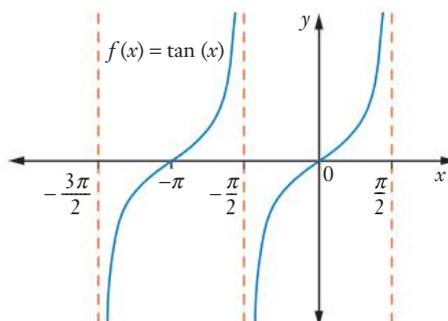
**1 a**

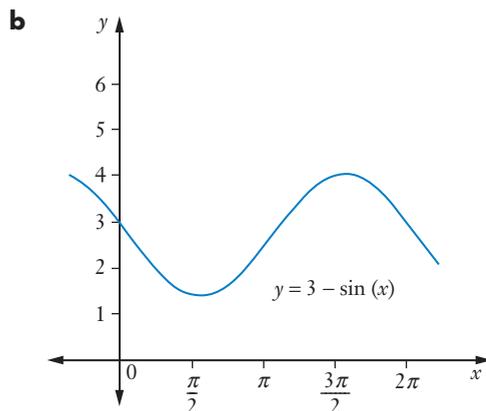
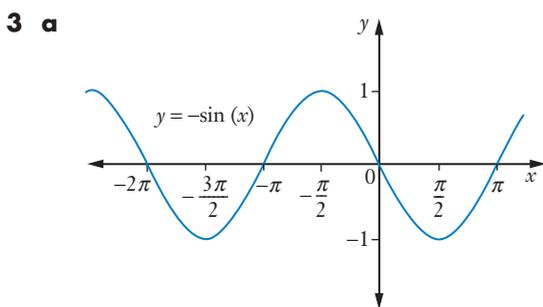
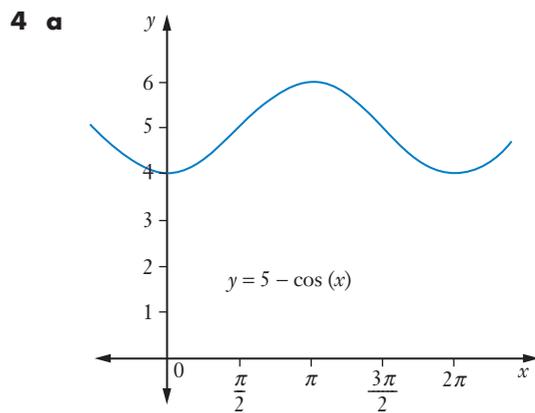
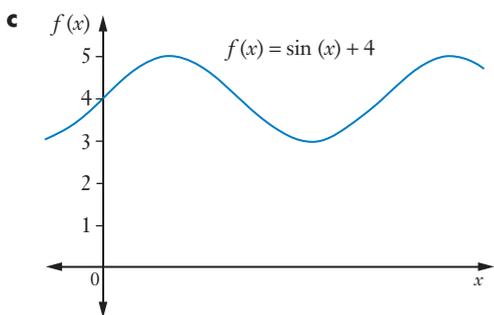
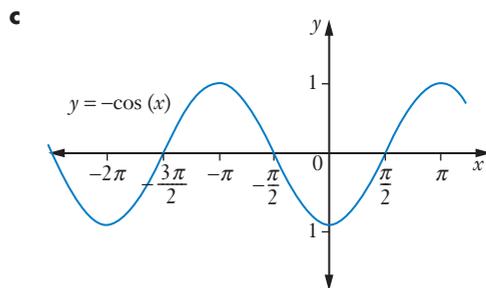
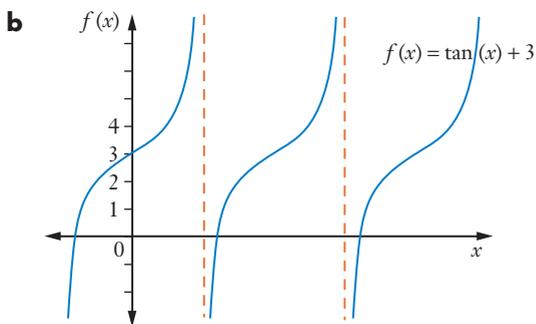
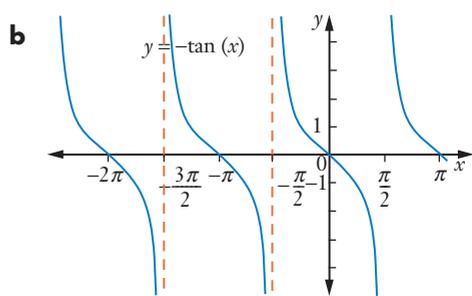
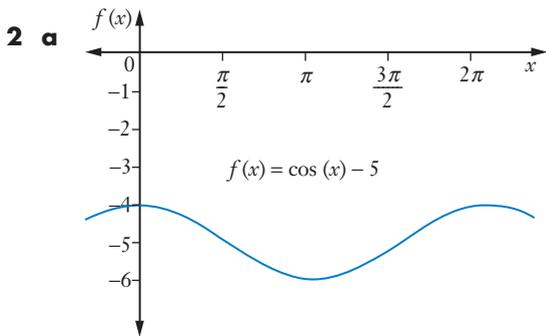


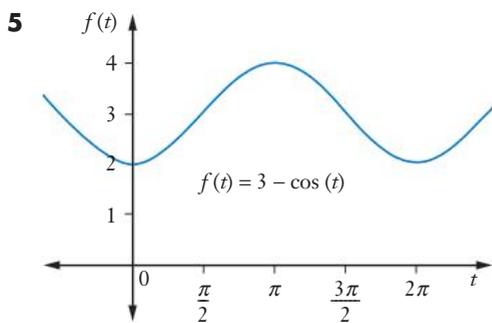
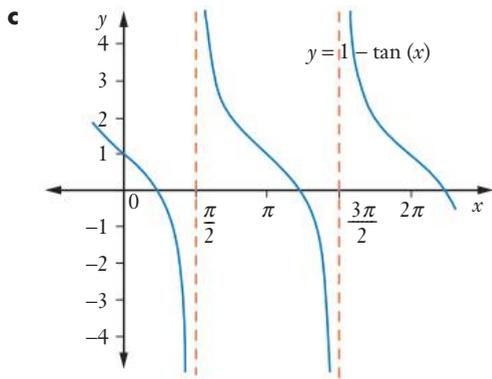
**b**



**c**

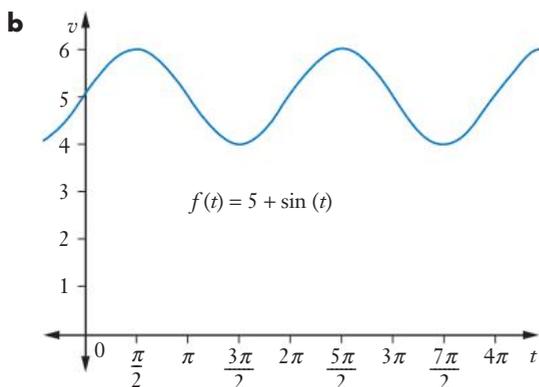






- a**  $\frac{6 + \sqrt{3}}{2}$       **b**  $\frac{\pi}{3}$  and  $\frac{5\pi}{3}$   
**c** 4      **d** 1.37 and 4.91

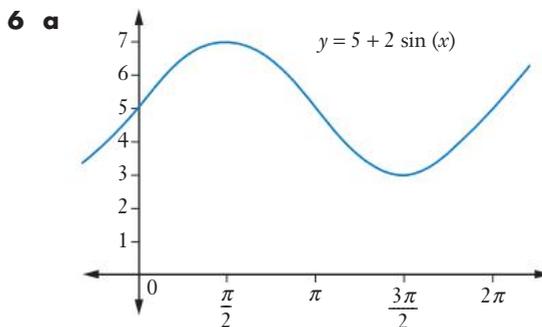
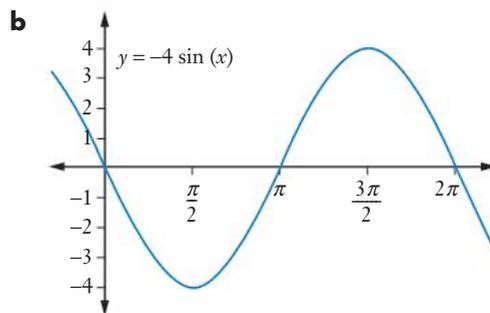
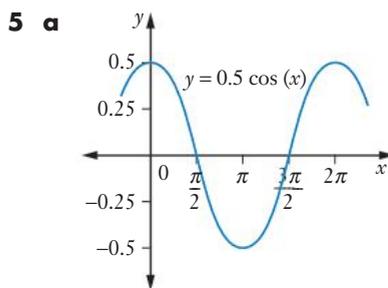
- 6 a i**  $5.5 \text{ m s}^{-1}$     **ii**  $5 \text{ m s}^{-1}$     **iii**  $6 \text{ m s}^{-1}$   
**iv**  $\frac{10 + \sqrt{3}}{2} \text{ m s}^{-1}$     **v**  $\frac{10 + \sqrt{2}}{2} \text{ m s}^{-1}$   
**vi**  $\frac{10 - \sqrt{3}}{2} \text{ m s}^{-1}$

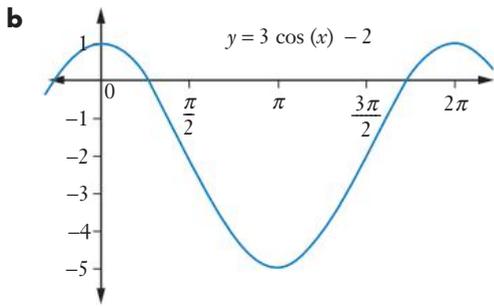


- c** 0.78 s, 2.36 s, 7.06 s, 8.65 s

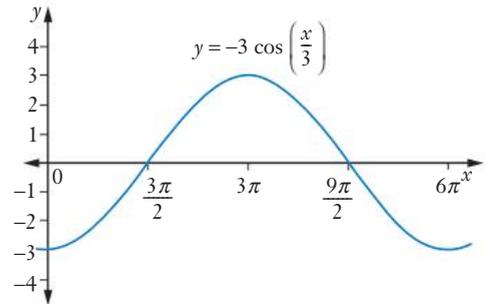
## Exercise 8.05

- 1 a** Amplitude = 30, period = 16  
**b** Amplitude = 10 m, period  $\approx 63$  s
- 2 a** 3      **b** 7      **c** 2  
**d** 2.8      **e** 5      **f** 8
- g** 10      **h** 15
- 3 a** Maximum = 1, minimum = -1  
**b** Maximum = 3, minimum = -3  
**c** Maximum = 4, minimum = -4  
**d** Maximum = -3, minimum = -5  
**e** Maximum = 4, minimum = 2  
**f** Maximum = 7, minimum = 3
- 4 a**  $2\pi$       **b**  $\pi$       **c**  $2\pi$   
**d**  $\frac{\pi}{5}$       **e**  $4\pi$       **f**  $6\pi$

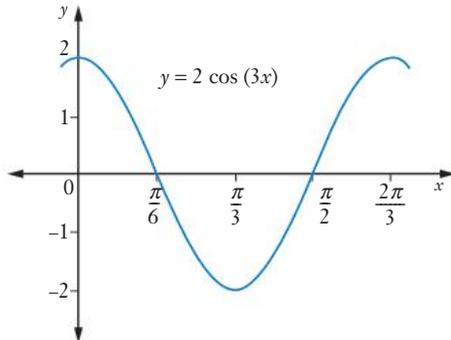




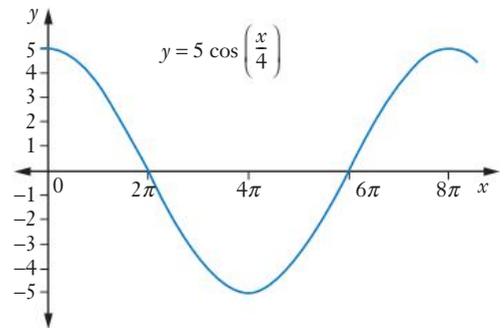
**d** Amplitude = 3, period =  $6\pi$



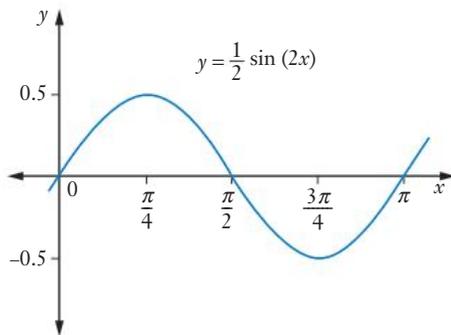
**7 a** Amplitude = 2, period =  $\frac{2\pi}{3}$



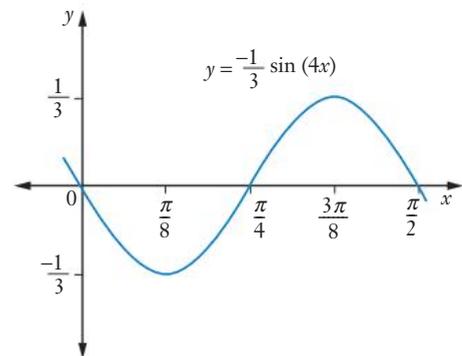
**e** Amplitude = 5, period =  $8\pi$



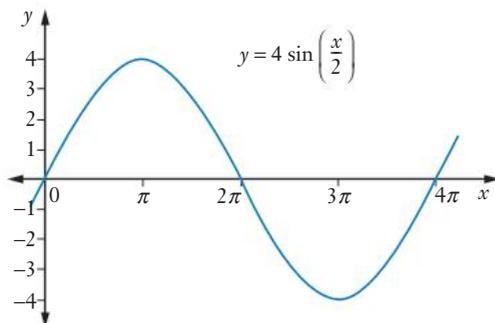
**b** Amplitude = 0.5, period =  $\pi$



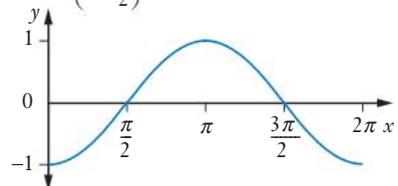
**f** Amplitude =  $\frac{1}{3}$ , period =  $\frac{\pi}{2}$

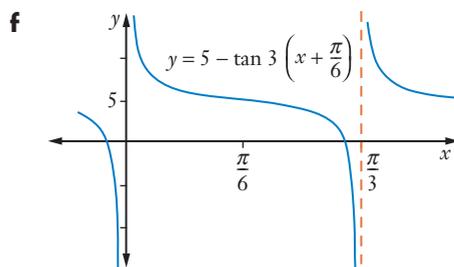
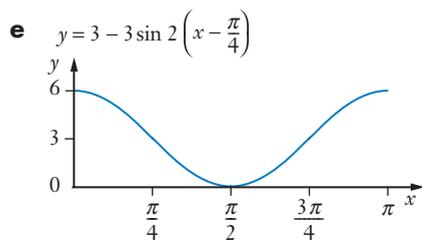
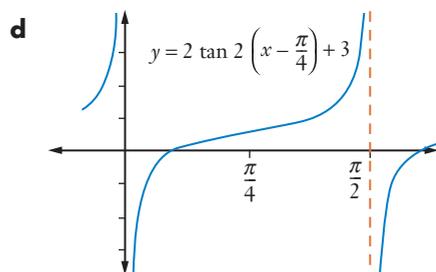
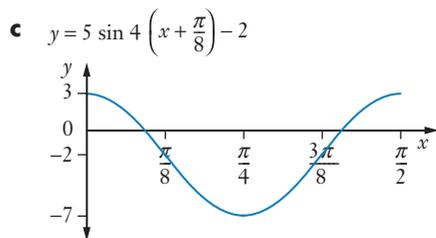
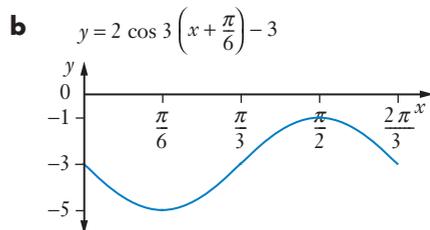
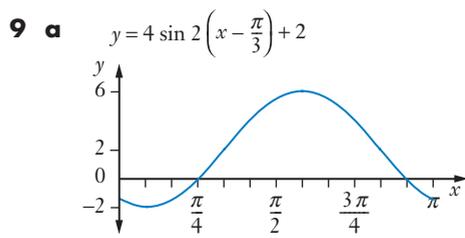
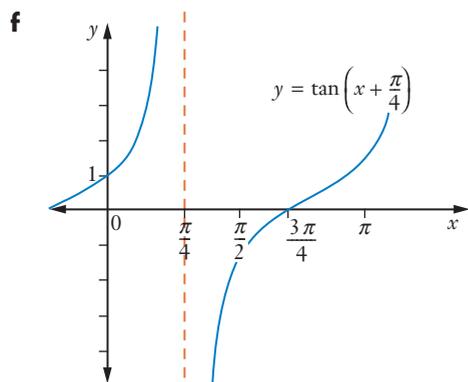
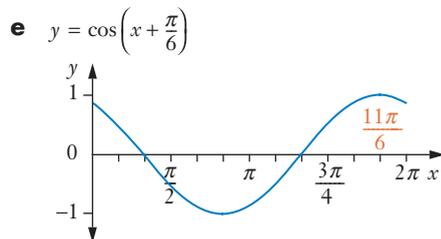
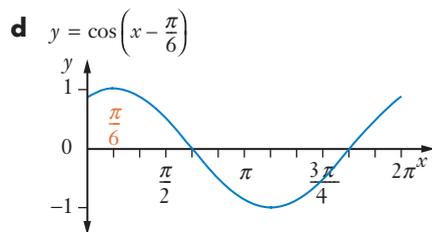
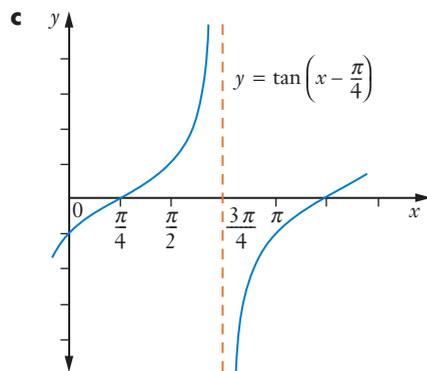
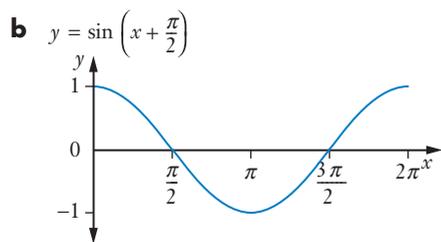


**c** Amplitude = 4, period =  $4\pi$

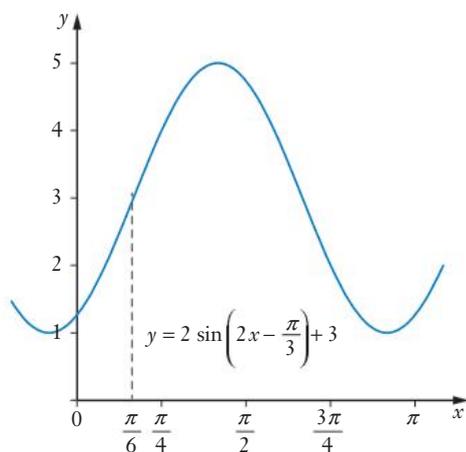


**8 a**  $y = \sin\left(x - \frac{\pi}{2}\right)$

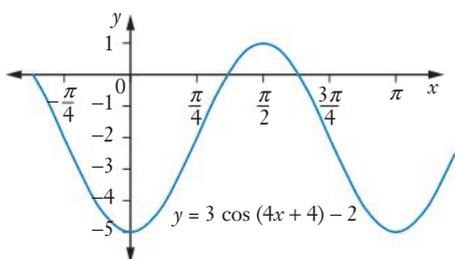




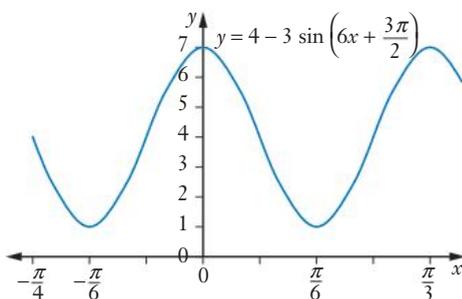
10 a



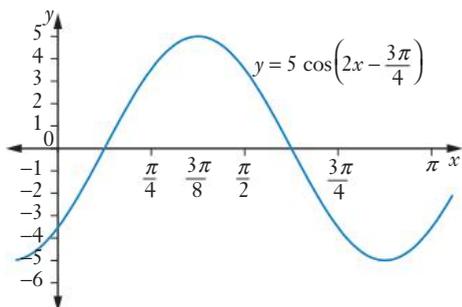
b



c

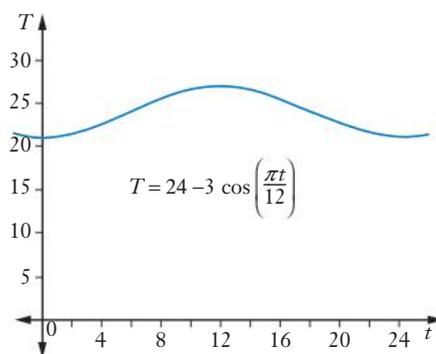


d



11 a Amplitude = 0.8 m    b Period = 12 h

12 a

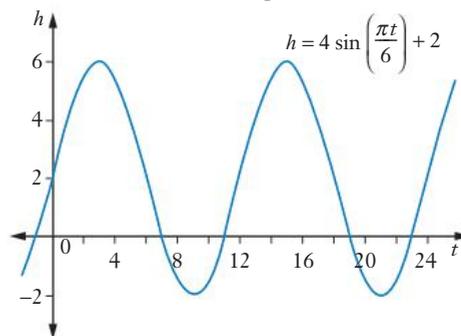


b 12 midday

c 12 midnight

d About 8:48 a.m. and 3:12 p.m.

13 a



b 6 m

c 0.96 m

d Between 12:29 a.m. and 5:31 a.m. and between 12:29 p.m. and 5:31 p.m.

14  $y = 3 \cos\left(x + \frac{5\pi}{6}\right)$

15  $a = 2, b = 1, c = -\frac{\pi}{3}$  and  $d = -1$ .

$y = 2 \sin\left(x - \frac{\pi}{3}\right) - 1$ .

### Exercise 8.06

1 a  $x = -\pi, \pi$

b  $x = \frac{\pi}{2}$

c  $x = -\frac{7\pi}{4}, -\frac{3\pi}{4}$

d  $x = 0, 2\pi$

e  $x = -\pi, \pi$

f  $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

g  $x = -\frac{\pi}{4}, \frac{3\pi}{4}$

h  $x = \frac{\pi}{6}, \frac{5\pi}{6}$

2 a  $\theta = 45^\circ, 225^\circ$

b  $\theta = 30^\circ, 150^\circ$

c  $\theta = 150^\circ, 210^\circ$

d  $\theta = 225^\circ, 315^\circ$

e  $\theta = 90^\circ, 270^\circ$

f  $\theta = 60^\circ, 240^\circ$

g  $\theta = 50^\circ, 130^\circ$

h  $\theta = 115^\circ, 245^\circ$

- 3 a**  $x = 0.3398, 2.8018$     **b**  $x = 1.8235$   
**c**  $x = 0.3805$     **d**  $x = 0.9818$   
**e**  $x = 0.4429, 2.6987$     **f**  $x = 1.7197$

**4 a**  $x = \frac{\pi}{12}, \frac{5\pi}{12}$

**b**  $x = \frac{\pi}{2}$

**c**  $x = \frac{\pi}{4}$

**5 a**  $\beta = 60^\circ, 120^\circ, 240^\circ, 300^\circ$

**b**  $\beta = 90^\circ, 270^\circ$

**c**  $\beta = 30^\circ, 150^\circ, 210^\circ, 330^\circ$

**d**  $\beta = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

**6 a**  $\theta = 180^\circ, 210^\circ$

**b**  $\theta = 270^\circ, 300^\circ$

**c**  $\theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

**d**  $\theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

**7 a**  $\alpha = \frac{\pi}{2}$

**b**  $\alpha = \frac{\pi}{3}, \frac{5\pi}{3}$

**c**  $\alpha = 2.3005, 3.9827$

**d**  $\alpha = 0.3398, 2.8018, 3.6652, 5.7596$

**8 a** 0.34 m

**b** 0.37 m

**c** 0.38 m when  $t = \frac{\pi}{8}$

**9**  $x = 0.47, -1.51$

### Exercise 8.07

**1 a** Yes

**c** About 6:00 a.m.

**e** About 10.5 ppm

**2 a** February

**b** Period = 12 months, amplitude = 9%

**3 a** 30 days

**b** About  $0.45^\circ\text{C}$

**4 a** 0.75 s

**b** Time between heart beats.

**c** 2 h 5 min

**5 a**  $y = 5 \cos\left(\frac{\pi x}{3}\right)$

**b**  $y = 3 \sin\left(\frac{\pi x}{4}\right)$

**c**  $y = 8 \cos\left(\frac{3\pi x}{4}\right)$

**d**  $y = 6 \sin\left(\frac{2\pi x}{3}\right)$

**6 a** Maximum = 12 m, minimum = 2 m

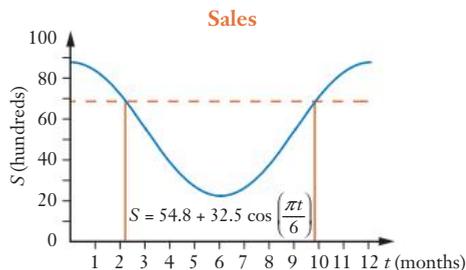
**b** 120

**c**  $a = 5, k = 4\pi, d = 7$

**d**  $33\frac{1}{3}\%$

**e** It will only work for a limited time.

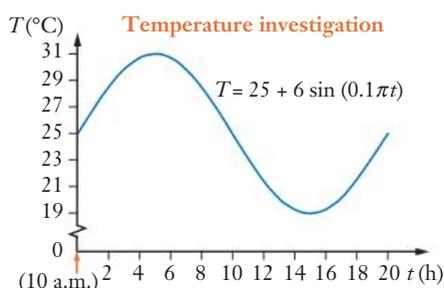
**7 a**



**b** Jan, Feb, Oct, Nov and Dec

**c** No, conditions change over time.

**8 a**



**b**  $31^\circ\text{C}, 19^\circ\text{C}$

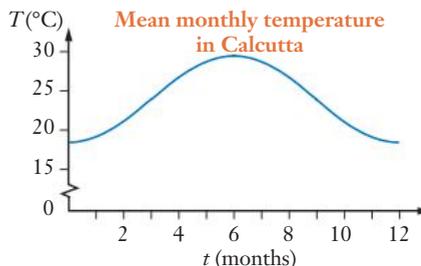
**c** 3 p.m., 1 a.m.

**d i** 11:05 a.m., 6:55 p.m., ...

**ii** 11:08 p.m., 2:52 a.m., ...

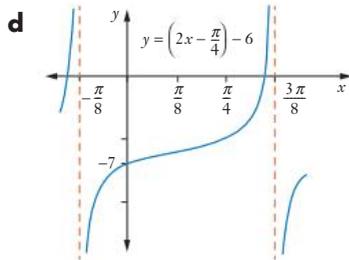
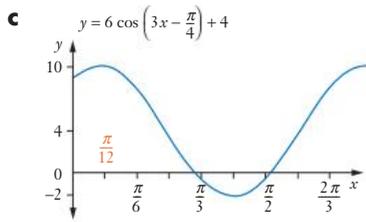
**e** Because the period was not 24 hours.

**9**  $T = 23.875 - 5.575 \cos\left(\frac{\pi t}{6}\right)$



In the absence of climate change, averages over a long period will be good predictors of monthly trends, but not daily weather. General warming of the Earth makes conditions more extreme because there is more energy in the atmospheric system.





**22** **a**  $\beta = \frac{\pi}{4}, \frac{3\pi}{4}$     **b**  $\theta = \frac{\pi}{3}, \pi$     **c**  $\alpha = \pm 60^\circ$

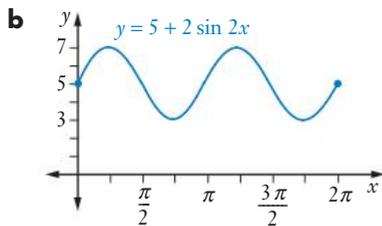
**23**  $x = \frac{\pi}{8}, \frac{7\pi}{8}$

**24**  $x = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{13\pi}{24}, \frac{17\pi}{24}, \frac{25\pi}{24}, \frac{29\pi}{24}, \frac{37\pi}{24}, \frac{41\pi}{24}$

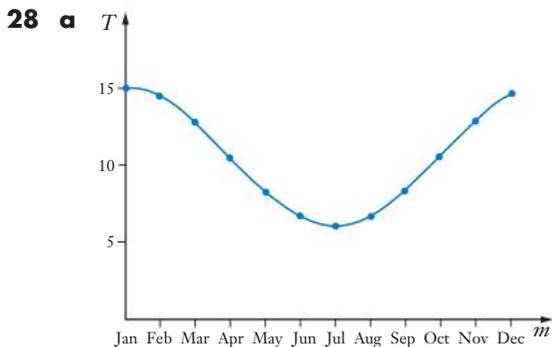
**25**  $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

**26**  $x = 35^\circ 47'$

**27** **a** Period  $\pi$ , range  $3 \leq y \leq 7$



**c**  $\frac{7\pi}{12} \leq x \leq \frac{11\pi}{12}$  and  $\frac{19\pi}{12} \leq x \leq \frac{23\pi}{12}$



**b**  $4.5^\circ\text{C}$                       **c** 12 months

**d** August to December

**29**  $d = 15 + 10 \cos\left(\frac{\pi t}{30}\right)$ . The tip is 10 cm above the shelf 48 times a day (20 past and 20 to the hour). If the clock mechanism gets worn the model will need to be altered.

**31**  $y = 7 \cos\left(\frac{\pi x}{6}\right) - 4$

**32** **a**  $\frac{1}{50}$  s                              **b**  $240\sqrt{2} \approx 340$  V

**c**  $240\sqrt{2}$  V

**33**  $x = 0.16, 2.46, 3.30, 5.60$

**34** **a** 1

**b** At  $t = 0$  and every 28 days thereafter

**c**  $4\frac{2}{3}$  and  $23\frac{1}{3}$  days after the full moon

## Chapter 9

### Exercise 9.01

**1**  $f(x)$  values: 0.018, 0.3, 5.043 and 84.773.

$g(x)$  values: 9.250, 8, 6.919, 5.984.

$f(x)$  is a growth function and  $g(x)$  is a decay function.

**2** About  $8 \text{ m}^2$

**3** About \$11 000

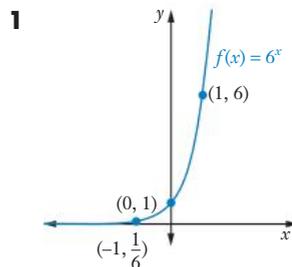
**4** About 8000 metres

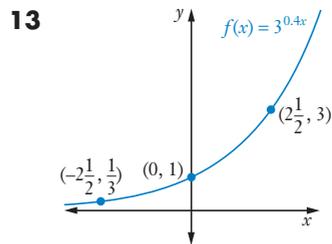
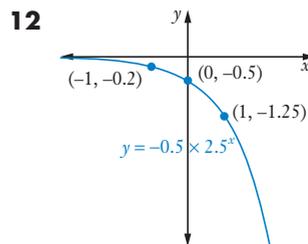
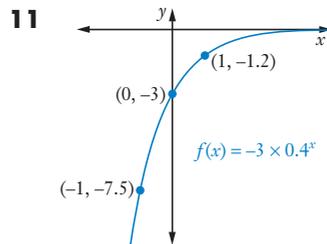
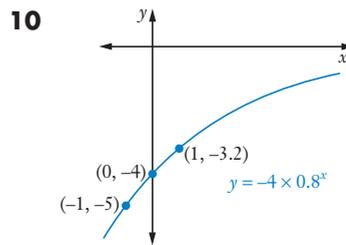
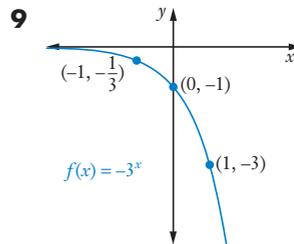
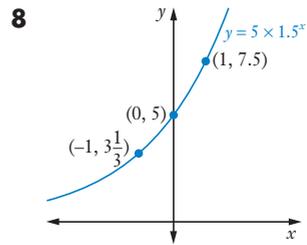
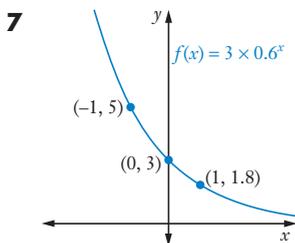
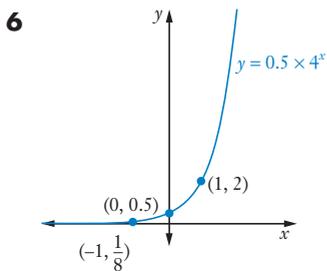
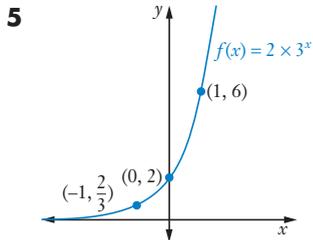
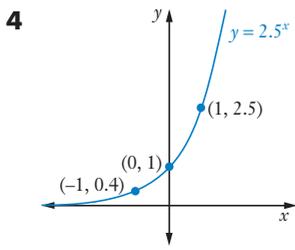
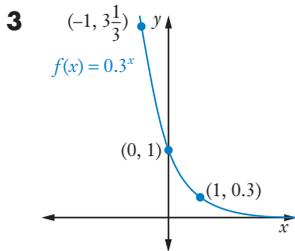
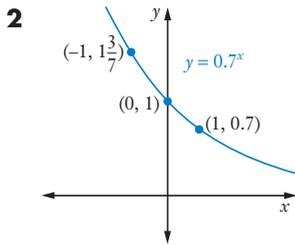
**5** About 24 700 assaults. It needs to be compared to population growth to find if our society is becoming more violent.

### Investigation: Exponential graphs

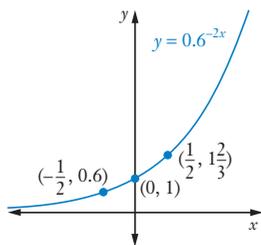
The graphs of exponential functions with bases larger than 1 increase as  $x$  increases. The graphs of exponential functions with bases smaller than 1 decrease as  $x$  increases.

### Exercise 9.02

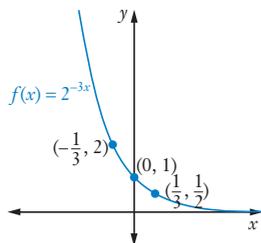




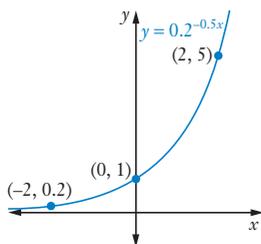
14



15

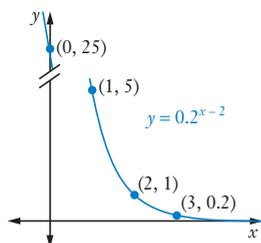


16

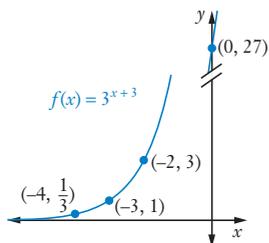


### Exercise 9.03

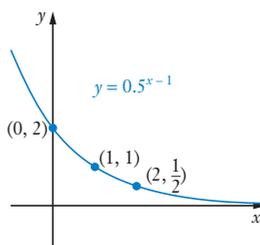
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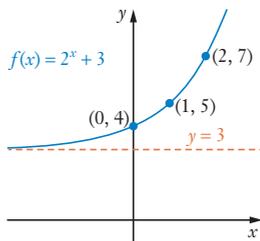
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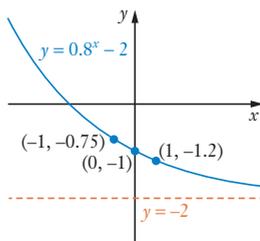
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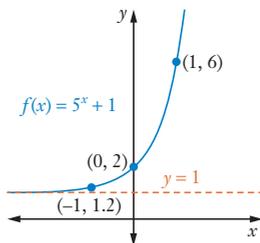
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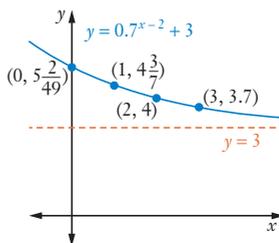
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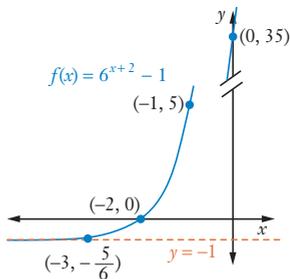
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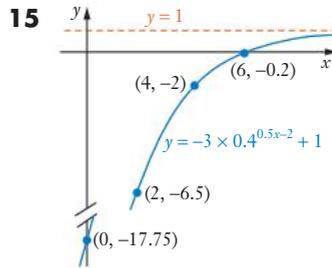
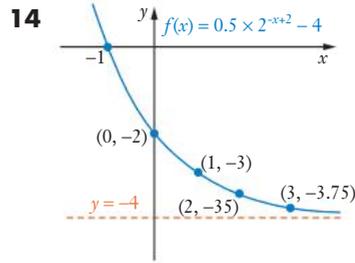
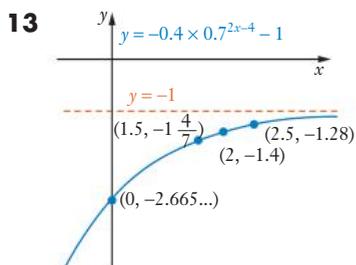
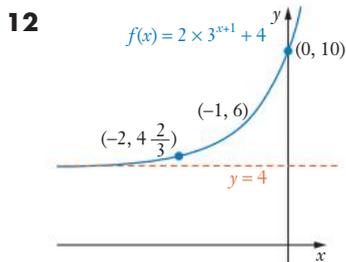
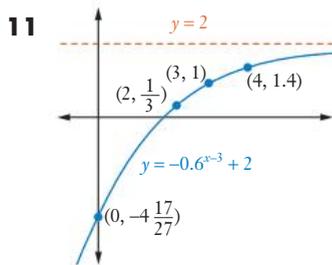
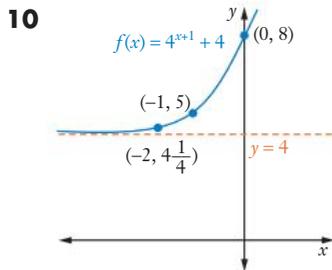
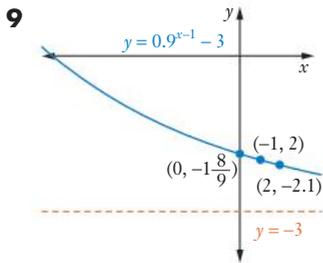


7



8





- 16**
- a**  $y = 3^{x-1} - 1$
- b**  $y = -2^x + 2$
- c**  $y = 2^{2x+3} - 3$

### Exercise 9.04

- 1**
- a**  $x = 4$       **b**  $x = 7\frac{1}{2}$       **c**  $x = \frac{1}{2}$
- d**  $x = 0$       **e**  $x = 2\frac{3}{4}$       **f**  $x = -\frac{7}{9}$
- g**  $x = 1\frac{3}{4}$       **h**  $x = -\frac{1}{2}$       **i**  $x = 3\frac{5}{7}$
- 2**
- a**  $x = 2$  or  $4$       **b**  $x = 2$  or  $3$       **c**  $x = 2$  or  $3$
- d**  $x = -1$  or  $2$       **e**  $x = 2$       **f**  $x = 0$
- g**  $x = 3$       **h**  $x = -1$  or  $1$       **i**  $x = 3$
- 3**
- a**  $x \approx 0.580$       **b**  $x \approx 1.723$       **c**  $x \approx 1.604$
- d**  $x \approx 1.926$       **e**  $x \approx 1.473$       **f**  $x \approx 0.733$
- 4**  $x = 0, 1$  or  $2$
- 5**  $x = 1$

### Exercise 9.05

- 1**
- a** About 3.6 m      **b** About 2619 L
- c** Almost 6 hours (5.986)
- 2**
- a** About 10.6  $\mu\text{g}/\text{m}$       **b** 3 h 44 m
- 3** 18.04 months
- 4** 16.98 years
- 5** 7590 years

- 6** 156 s, 70 s.  $r^2$  is very high, but the model is only for 40 seconds. The capacitor might not work with a higher voltage.
- 7** About  $2.1 \text{ mm}^2$ . The bacteria might become overcrowded or exhaust the nutrients on the plate.
- 8** **a**  $T \approx 63.7 \times 1.016^t \text{ }^\circ\text{C}$ , where  $t$  is in minutes.  
**b** The nature of the reaction could change, another nuclear reactor might not work the same way.  
**c** About  $1100^\circ\text{C}$ .  
**d** About 3 hours and 19 minutes.
- 9** **a**  $m = 85.6 \times 0.916^t \text{ mg}$ , where  $t$  is in days  
**b** The data is only over 1 week, the initial value seems to be inaccurate.  
**c** 7.7 days  
**d** About 24.5 days
- 10** **a** About  $65^\circ\text{C}$   
**b**  $T = 15 + 57 \times 0.96^t \text{ }^\circ\text{C}$ , where  $t$  is in minutes.  
**c** About  $53^\circ\text{C}$   
**d** About 20 min, assuming that the temperature changes in the same way, even while it is being eaten.  
**e** There are a lot of assumptions in the model, so it is really only a general indicator.

### Exercise 9.06

- 1** **a** 1                      **b** 3                      **c** 0  
**d** -1                      **e**  $\frac{1}{2}$                       **f** -3  
**g**  $\frac{1}{3}$                       **h**  $1\frac{1}{2}$                       **i**  $2\frac{1}{2}$
- 2** **a**  $1.7634, 10^{1.7634} \approx 58$   
**b**  $2.2041, 10^{2.2041} \approx 160$   
**c**  $1.2041, 10^{1.2041} \approx 16$   
**d**  $-1.3010, 10^{-1.3010} \approx 0.05$   
**e**  $1.1761, 10^{1.1761} \approx 15$   
**f**  $2.9494, 10^{2.9494} \approx 890$   
**g**  $1.8261, 10^{1.8261} \approx 67$   
**h**  $1.7993, 10^{1.7993} \approx 63$   
**i**  $1.7959, 10^{1.7959} \approx 62.5$

- 3** **a**  $\log(3.5) \approx 0.5441, \log(350) \approx 2.5441$   
**b**  $\log(350) = \log(3.5) + \log(100)$
- 4** **a**  $\log(18) \approx 1.2553, \log(3) \approx 0.4771,$   
 $\log(6) \approx 0.7782$   
**b**  $\log(18) = \log(3) + \log(6)$

**5–9** Teacher to check proofs.

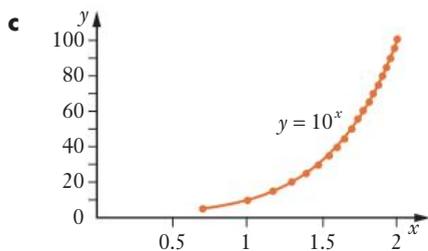
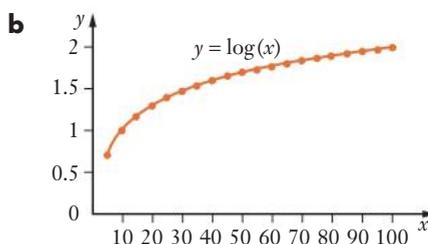
- 10** **a** 1.7559                      **b** 3.7559                      **c** -1.2441  
**11** **a** 1.4533                      **b** 3.4533                      **c** -0.5467  
**12** **a** 4.9883                      **b** 0.0117                      **c** -3.0117  
**13** **a** 2.0899                      **b** 4.0899                      **c** -0.9101  
**14** **a**

$x$	5	10	15	20	25
$\log(x)$	0.6990	1	1.1761	1.3010	1.3979

$x$	30	35	40	45	50
$\log(x)$	1.4771	1.5441	1.6021	1.6532	1.6990

$x$	55	60	65	70	75
$\log(x)$	1.7404	1.7782	1.8129	1.8451	1.8751

$x$	80	85	90	95	100
$\log(x)$	1.9031	1.9294	1.9542	1.9777	2



- d** At the same scale, the graphs are mirror images in the line  $y = x$ .

15 a

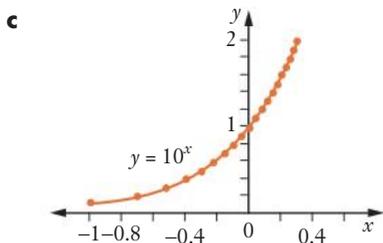
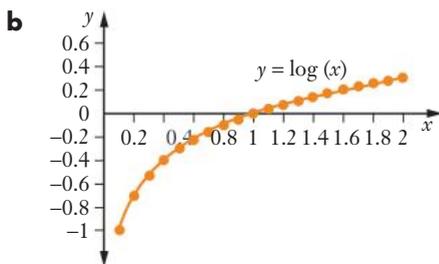
$x$	0.1	0.2	0.3	0.4
$\log(x)$	-1	-0.6990	-0.5229	-0.3979

$x$	0.5	0.6	0.7	0.8
$\log(x)$	-0.3010	-0.2218	-0.1549	-0.0969

$x$	0.9	1.0	1.1	1.2
$\log(x)$	-0.0458	0	0.0414	0.0792

$x$	1.3	1.4	1.5	1.6
$\log(x)$	0.1139	0.1461	0.1761	0.2041

$x$	1.7	1.8	1.9	2
$\log(x)$	0.2304	0.2553	0.2788	0.3010



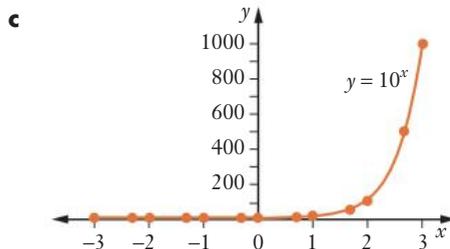
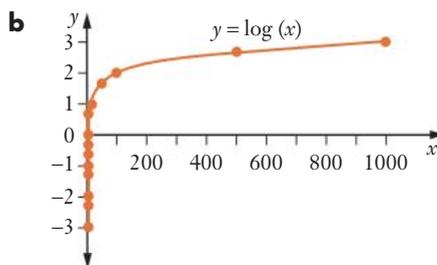
d At the same scale, the graphs are mirror images in the line  $y = x$ .

16 a

$x$	0.001	0.005	0.01	0.05	0.1
$\log(x)$	-3	-2.301	-2	-1.301	-1

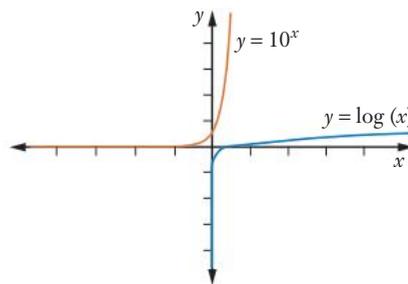
$x$	0.5	1	5	10	50
$\log(x)$	-0.301	0	0.699	1	1.699

$x$	100	500	1000
$\log(x)$	2	2.699	3



d At the same scale, the graphs are mirror images in the line  $y = x$ .

17 a



The graphs are mirror images in the line  $y = x$ .

18 a Domain = real numbers, range = positive numbers

b Domain = positive numbers, range = real numbers

### Exercise 9.07

1 a  $5^3 = 125$       b  $2^5 = 32$       c  $4^3 = 64$

d  $7^1 = 7$       e  $3^7 = 2187$       f  $9^0 = 1$

g  $5^4 = 625$       h  $13^2 = 169$

2 a  $16^{\frac{1}{2}} = 4$       b  $8^{-\frac{1}{3}} = \frac{1}{2}$       c  $3^{-2} = \frac{1}{9}$

d  $4^{-2} = \frac{1}{16}$       e  $\left(\frac{1}{2}\right)^{-3} = 8$       f  $\left(\frac{1}{4}\right)^{\frac{1}{2}} = 2$

g  $32^{\frac{2}{5}} = 4$       h  $5^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{5}}$

**3 a**  $\log_8(64) = 2$   
**c**  $\log_2(64) = 6$   
**e**  $\log_{15}(15) = 1$

**4 a**  $\log_{36}(6) = \frac{1}{2}$   
**c**  $\log_4\left(\frac{1}{16}\right) = -2$

**e**  $\log_{\frac{1}{9}}(3) = -\frac{1}{2}$

**5 a** 4      **b** 4      **c** 3      **d** 4  
**e** 0      **f** 1      **g** 3      **h** 6  
**6 a** 0      **b** 1      **c** 0      **d** 0  
**e** 1      **f** 0      **g** 1      **h** 1

**7** All undefined

**8 a** -5      **b** -2      **c** -4      **d** -3  
**e** 3      **f** -6      **g** 4      **h** -3

**9 a** 3      **b** 4      **c**  $\frac{1}{3}$       **d** 4  
**e**  $\frac{2}{3}$       **f** 2      **g**  $\frac{1}{2}$       **h** -1  
**i** 0      **j**  $\frac{3}{2}$       **k**  $\frac{1}{6}$       **l**  $-\frac{1}{2}$

**10 a**  $2a$       **b**  $a+3b$       **c**  $3a-2b$       **d**  $z^b \times a$   
**e**  $z^a$       **f**  $b-3a$       **g**  $b+\frac{a}{2}$       **h**  $\frac{2a+b}{3}$

**11 a**  $\frac{1}{2}$       **b**  $\log_5(8)$       **c** 2  
**d** 5      **e** 4      **f** -3  
**g**  $\frac{2}{3}$       **h** 2

**12 a** 5      **b** -3      **c** -8      **d** 2  
**e**  $\log_{10}(2)$       **f** 15      **g** 2      **h** -1

**13 a**  $\log_2(x^2)$       **b**  $\log_5(x)$       **c**  $\log_4\left(\frac{1}{x}\right)$   
**d**  $\log_{10}\left(\frac{y}{x}\right)$       **e**  $\log_7[(x+1)^5]$       **f**  $\log_3\left(\frac{x^4}{y}\right)$

**g**  $\log_8(x^3y^2)$       **h**  $-\log_4(x-2)$

**14 a**  $\log_2(x) + \log_2(y) - \log_2(z)$   
**b**  $1 + \log_4(x) - \log_4(y)$   
**c**  $\log_7(6) - \log_7(x) - \log_7(y)$   
**d**  $1 + \log_5(x) - 2 \log_5(y) - \log_5(z)$

**b**  $\log_3(81) = 4$

**d**  $\log_{12}(1) = 0$

**f**  $\log_6(216) = 3$

**b**  $\log_{27}\left(\frac{1}{3}\right) = -\frac{1}{3}$

**d**  $\log_{\frac{1}{2}}(16) = -4$

**f**  $\log_9(27) = \frac{3}{2}$

**e**  $3 \log_{10}(x) + \frac{1}{2} \log_{10}(y) - 5 \log_{10}(z)$

**d**  $4 \log_9(x) + \frac{3}{2} \log_9(y) - \frac{1}{2} \log_9(z)$

**g**  $\frac{1}{2}[3 \log_5(x) + \log_5(y) - 3 \log_5(z)]$

**h**  $\frac{1}{3}[4 \log_8(x) - 2 \log_8(y) - 2 \log_8(z)]$

**15 a** 0.276      **b** 1.654      **c** 2.827

**16**  $\frac{2}{b}$

**17**  $x = \sqrt{5}$

**18–19** Proofs

### Exercise 9.08

**1 a**  $\frac{\log(12)}{\log(5)}$       **b**  $\frac{\log(20)}{\log(4)}$       **c**  $\frac{\log(50)}{\log(2)}$

**d**  $\frac{\log(300)}{\log(9)}$       **e**  $\frac{\log(0.5)}{\log(11)}$       **f**  $\frac{\log(15)}{\log(7)}$

**2 a** 3.322      **b** 1.683      **c** 2.262

**d** 1.528      **e** 0.5537      **f** 0.9306

**3 a**  $x = 3.561$       **b**  $x = 1.431$       **c**  $x = 2.215$

**d**  $x = 0.5646$       **e**  $x = -3.322$       **f**  $x = 5.097$

**4 a**  $x = 1.3010$       **b**  $x = 0.6941$

**c**  $x = 0.8503$       **d**  $x = -1.1761$

**e**  $x = -13.6438$       **f**  $x = -0.9457$

**5 a**  $x = 1.292$       **b**  $x = -0.5268$       **c**  $x = 1.032$

**d**  $x = 0.6375$       **e**  $x = 0.7257$       **f**  $x = 1.039$

**6 a**  $\log_{\frac{1}{3}}(8) = \frac{\log(8)}{\log\left(\frac{1}{3}\right)} = \frac{\log(8)}{\log(1) - \log(3)} = \frac{\log(8)}{-\log(3)}$

$= -\frac{\log(8)}{\log(3)}$

**b**  $\frac{\log_2(36)}{\log_2(16)} = \frac{\log_2(4 \times 9)}{\log_2(2^4)} = \frac{2 + 2 \log_2(3)}{4}$

$= \frac{1}{2} + \frac{1}{2} \log_2(3)$

**c** Change to base 5 then simplify.

**d** Change to base 3 then simplify.

**e** Change to base 5 then simplify.

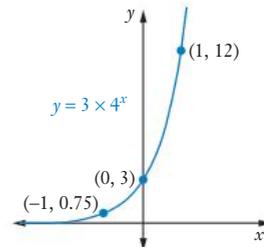
### Exercise 9.09

- 1 a** 2.807      **b** 2.113      **c** 1.771  
**d** 0.732      **e** 0.670      **f** 1.953  
**2 a** -0.208      **b** 4.465      **c** 0.894  
**d** 0.116      **e** 1.798      **f** 1.287  
**3 a** 99      **b** 5      **c**  $\frac{79}{9}$   
**d** 32      **e** 33      **f** 4  
**4 a** 1      **b** 5      **c** 3.819  
**d** -1.893      **e** -1.529 or 2.994  
**f** 2.003  
**5 a** 3.477      **b** 3.424      **c** 5.057  
**d** 9.233      **e** 11.644      **f** 0.4231  
**6 a** -4      **b** 4      **c** 4  
**d** 3      **e**  $\frac{4}{3}$       **f** 6  
**7 a** 5      **b** No solution  
**c**  $\frac{7}{6}$       **d**  $\frac{47}{6}$   
**e**  $-\frac{27}{40}$       **f** No solution  
**8 a**  $\frac{1}{100}$  or 10      **b** 0.001 617 or 61.84  
**c** 0.2390 or 12.55      **d**  $\frac{1}{64}$  or 8  
**e** 0.0016 or 5      **f** 0.7657 or 1337  
**9 a**  $\pm 0.4946$       **b** 5.393      **c** -1.893  
**d** -1.139      **e** 1      **f** 4.667  
**10 a** -318      **b**  $\frac{7}{2}$       **c** -3 or 9  
**d** 2      **e** -6      **f** -7 or -4  
**11 a** 12      **b** 2      **c** 5000  
**d** 3      **e** 1001      **f**  $\frac{33}{7}$   
**g** -4      **h**  $\frac{1}{28}$   
**12 a** About 13.9 years      **b** About 20.3 years  
**13** About 10  
**14** About 126 times more intense.

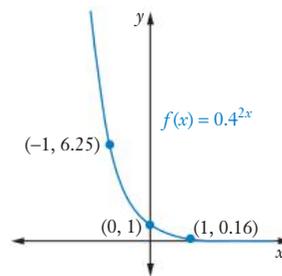
### Chapter 9 review

- 1** 5, 7.2, 8.64, 10.368;  $f(x)$  is increasing  
**2** About 19.3°C  
**3** About \$11 290

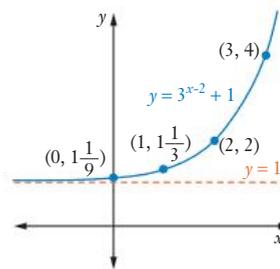
**4 a**



**b**

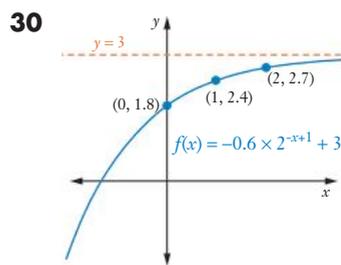


**5**



- 6 a**  $x = 2$       **b**  $x = 1$  or  $x = 2$   
**7**  $x \approx 5.465$   
**8 a** About \$4.62/kg      **b** About \$7.24/kg  
**9 a**  $h(b) \approx 5.82 \times 0.686^b$   
**b** About 0.13 m (13 cm)  
**10 a** 2      **b** -2      **c**  $\frac{1}{2}$   
**d**  $\frac{4}{3}$       **e**  $\log(7) + \frac{1}{2}$       **f**  $\frac{3}{2}$   
**11** 1.5441;  $10^{1.5441} \approx 35$   
**12 a**  $\log(6.5) \approx 0.8129$ ,  $\log(6500) \approx 3.8129$   
**b**  $\log(6500) = \log(6.5) + \log(1000)$   
**13** Demonstration

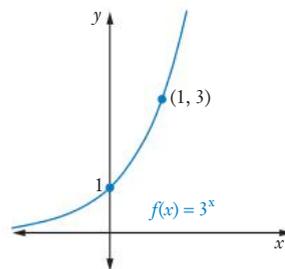
- 14 a  $7^3 = 343$       b  $2^7 = 128$   
 c  $2^{-3} = \frac{1}{8}$       d  $\left(\frac{1}{2}\right)^{-4} = 16$
- 15 a  $\log_5(125) = 3$       b  $\log_{64}(4) = \frac{1}{3}$   
 c  $\log_{\frac{1}{3}}(243) = -5$       d  $\log_{64}\left(\frac{1}{2}\right) = -\frac{1}{6}$
- 16 a 7      b 0      c 3  
 d Undefined      e -6      f -3  
 g 4      h -2
- 17 a  $a + 4b$       b  $2a - 3b$   
 c  $a + \frac{b}{3}$       d  $am^b$
- 18 a  $\frac{1}{2}$       b  $\log_8(12)$       c 2  
 d 3      e 7      f -1
- 19 a  $\log_3(x)$       b  $\log_7(x^2y^2)$
- 20 a  $2 \log_6(x) - \log_6(y) - 3 \log_6(z) + 1$   
 b  $\log_8(x) + \frac{3}{2} \log_8(y) - 2 \log_8(z)$
- 21  $\frac{\log(0.4)}{\log(12)}$
- 22 a 2.465      b 1.249
- 23 a  $x = 1.7479$       b  $x = 0.7720$       c  $x = -9.7168$
- 24 a  $x = -1.119$       b  $x = -0.03181$       c  $x = 1.857$
- 25 a  $x = 29$       b  $x = 12$       c  $x = 13$   
 d  $x = -3$       e  $x \approx 2.953$  or  $-1.953$   
 f  $x = 2$
- 26 a  $x = 3.513$       b  $x = 5$       c  $x = 7.663$
- 27 a  $x = 4$       b  $x = 3.778$       c  $x = 4.128$   
 d  $x = 5.159$       e  $x = 2.429$       f  $x = 2.033$   
 g  $x = 0.6730$       h  $x = 6.382$   
 i  $x = 0.000\ 6418$  or  $155.8$   
 j  $x = 0.001$  or  $10$   
 k  $x = -4.301$       l  $x = 0.5395$
- 28 a About \$1202  
 b Assuming the conditions stay the same, about \$3854.  
 c It is unlikely to be very accurate.
- 29 About 18 years

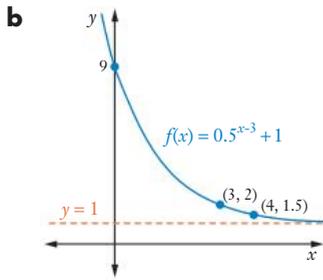


- 31 a  $I(y) \approx 5 \times 10^5 \times 1.395^y$  where  $y$  is the number of years since 1975.  
 b  $1.6 \times 10^{12}$   
 c It assumes that the same kind of advances in computer design will continue. You would expect some kind of limit eventually.
- 32 a  $d = 53 \times 0.963^y$ , where  $y$  is the number of years since 2005.  
 b 25, 8  
 c The  $r^2$  was only 0.42, so the model is not a good fit. 2055 is too far into the future to use the model, particularly with a rising population.
- 33 Proof
- 34 Demonstration.
- 35 a -0.45      b -0.875
- 36 a 80      b  $3.16 \times 10^{-3}$  watts/m<sup>2</sup>
- 37 20 dB
- 38 About 9.9%

### Practice examination 3

- 1 a 0.4 L/min      b 0.2 L/min  
 c The rate decreases (1 mark each part)
- 2  $\lim_{x \rightarrow 4} \left( \frac{3x^2 - 11x - 4}{x - 4} \right) = 13$  (2 marks)
- 3 7 (3 marks)
- 4  $6x + 7$  (3 marks)
- 5 a  $a(t) = 300 \times 1.03^d$  m<sup>2</sup>  
 b 403 (1 mark each part)
- 6 a





(2 marks each)

**7 a** 3                      **b** 3                      (1 mark each)

**8 a**  $x = -12$             **b**  $x = \frac{3}{2}$  only            (1, 2 marks)

**9**  $\frac{125\pi}{3} \text{ cm}^2$                       (1 mark)

**10**  $\sin(\theta) = \frac{5}{13}$ ,  $\cos(\theta) = \frac{12}{13}$                       (2 marks)

**11** Amplitude = 3, period =  $\pi$ , range =  $[-7, -1]$                       (1 mark each)

**12**  $\alpha = \frac{\pi}{6}, \frac{11\pi}{6}$                       (2 marks)

**13**  $x > 1\frac{5}{6}$                       (3 marks)

**14**  $\log_5(2)$  only                      (2 marks)

**15**  $-8.96$                       (2 marks)

**16**  $y = 3 \sin\left(\frac{x}{2}\right) + 1$                       (3 marks)

**17 a** Marginal cost =  $0.001n + 0.35$ , \$1.35                      (1 mark each)

**b**  $C(1001) - C(1000) = 862.351 - 861.0 = 1.351$ ,  
so the rule works exactly                      (1 mark)

**18 a** Using a graphics calculator,  
 $n(t) \approx 7.85 \times 1.25^t$  with  $r^2 = 0.9995$

**b** About 680, but this assumes that the  
population continues to grow in the same  
way.                      (2 marks each part)

**19 a**  $H = 1.6 \cos\left(\frac{\pi t}{6}\right) + 2$  m at  $t$  hours past  
midnight on that day.

**b** 1.2 m, but this is unlikely to be correct as the  
tides will have changed by that time.                      (1, 2 marks)

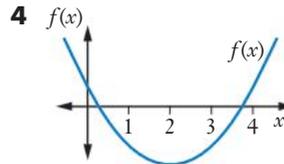
## Chapter 10

### Exercise 10.01

**1 C**

**2 B**

**3 A**



**5 B**

**6 C**

**7 A**

**8 a**  $f'(x) > 0$             **b**  $f'(x) < 0$             **c**  $f'(x) = 0$

**9 a**  $f'(t) > 0$             **b**  $f'(t) < 0$             **c**  $f'(t) > 0$

**10 a-f** Decreasing

**g** Stationary

**h-j** Increasing

**11 a-d** Increasing

**f** Stationary

**e, g-j** Decreasing

**12 a** Increasing

**b** Stationary

**c** Decreasing

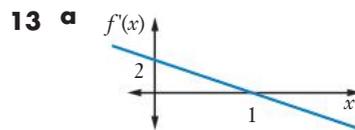
**d** Decreasing

**e** Decreasing

**f** Stationary

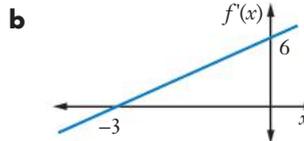
**g** Increasing

**h** Increasing



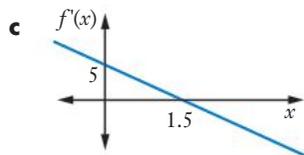
$f(x)$  is increasing when  $x < 1$

$f(x)$  is decreasing when  $x > 1$



$f(x)$  is increasing when  $x > -3$

$f(x)$  is decreasing when  $x < -3$



$f(x)$  is increasing when  $x < 1.5$

$f(x)$  is decreasing when  $x > 1.5$

**14 a**  $x = 2$

**b**  $x = 2$

c  $x = \frac{4}{3}, 2$

d  $x = 1, -2$

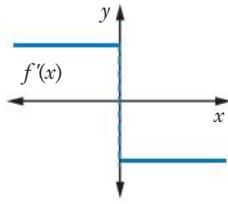
15 a (2, 1)

b (-1, 5)

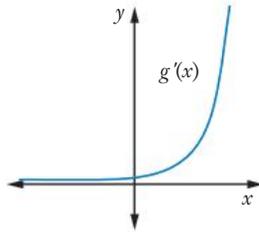
c (-2, 16) and (2, -16)

d (-1, -2) and (1, 2)

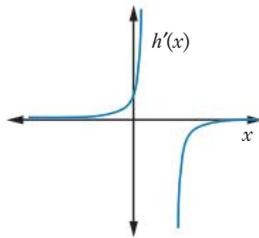
16



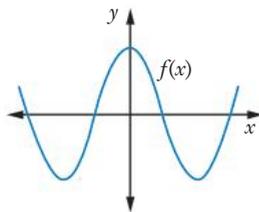
17



18



19



### Exercise 10.02

1 a  $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

b  $a^3 + 3a^2b + 3ab^2 + b^3$

c  $m^4 + 4m^3h + 6m^2h^2 + 4mh^3 + h^4$

d  $g^6 + 6g^5h + 15g^4h^2 + 20g^3h^3 + 15g^2h^4 + 6gh^5 + h^6$

e  $x^7 + 7x^6h + 21x^5h^2 + 35x^4h^3 + 35x^3h^4 + 21x^2h^5 + 7xh^6 + h^7$

f  $x^8 + 8x^7h + 28x^6h^2 + 56x^5h^3 + 70x^4h^4 + 56x^3h^5 + 28x^2h^6 + 8xh^7 + h^8$

2 a  $n^4 + 12n^3m + 54n^2m^2 + 108nm^3 + 81m^4$

b  $32c^5 + 80c^4d + 80c^3d^2 + 40c^2d^3 + 10cd^4 + d^5$

c  $64 + 48p + 12p^2 + p^3$

d  $a^6 + 6a^5 + 15a^4 + 20a^3 + 15a^2 + 6a + 1$

3 a  $4x^3$       b  $6x^5$       c  $7x^6$       d  $9x^8$

4 a  $2x$       b  $3x^2$       c  $5x^4$       d  $8x^7$

e  $20x^{19}$       f  $17x^{16}$       g  $12x^{11}$       h  $22x^{21}$

5 a  $15x^2$       b  $24x^3$       c  $15x^4$       d  $12x^5$

### 6-7 Proofs

### Exercise 10.03

1  $1 + 2x$

2  $3x^2 + 1$

3  $4x^3 + 2x$

4  $3x^2 + 5x^4$

5  $2x - 3x^2$

6  $1 - 6x^5$

7  $5x^4 - 2x$

8  $3x^2 - 4x^3$

9 a  $6x^5 + 3x^2$       b  $5x^4 + 8x^7$       c  $7t^6 - 4t^3$

d  $6x^5 - 10x^9$       e  $7r^6 + 4r^3$       f  $3x^2 - 10x^9$

g  $5u^4 + 9u^8$       h  $3x^2 - 6x^5$

10 a  $3x^2 + 5x^4 + 7x^6$

b  $2x - 6x^5 + 8x^7$

c  $3u^2 + 4u^3 - 5u^4$

d  $3x^2 - 7x^6 - 10x^9$

e  $2z + 5z^4 - 6z^5 + 8z^7$

f  $1 - 2r + 5r^4 - 8r^7$

11 a  $8x^3 - 35x^4$

b  $36x + 360x^7$

c  $16 - 132x^{10}$

d  $70x^6 + 16x^3$

e  $36x^5 + 55x^4$

f  $63x^6 + 40x^9$

g  $6x^2 - 80x^7$

h  $20x^4 - 49x^6$

12 a  $60x^4 - 72x^2$

b  $30x^4 - 36x^2$

c  $96x^2 - 80x^4$

d  $60\sqrt{5}x^4 - 72\sqrt{5}x^2$

e  $110x^4 - 132x^2$

13 a  $45x^2 - 140x^6$

b  $112x^6 - 36x^2$

c  $15x^2 - 46\frac{2}{3}x^6$

d  $63\sqrt{2}x^2 - 196\sqrt{2}x^6$

e  $25.2x^2 - 78.4x^6$

14 a  $6x^2 + x^3 + 6x^4$

b  $12x^2 - 22x^3 - 4x^4$

c  $-15x^2 + 32x^3 + 8x^4$

d  $-9x^2 - 8x^4$

15 a  $16x^3 + 9x^2 - 20x$

b  $-24x^3 + 72x^2 - 46x$

c  $-40x^3 + 27x^2 + 6x$

### Exercise 10.04

- 1 a**  $24x^3 - 15x^2 + 4x + 8$   
**b**  $27x^2 + 40x^3 - 15x^4$   
**c**  $3x^2 + 2x + 1$   
**d**  $36x^8 - 80x^9$   
**e**  $180x^{11} + 180x^{14} - 180x^4$   
**f**  $-7x^6 + 63x^8 + 10x^9$   
**g**  $40x^7 + 108x^8 - 60x^9 + 12x^{11}$   
**h**  $-3 + 2x + 24x^2 - 4x^3$   
**i**  $20x^3 - 17 + 48x^2$   
**j**  $175x^6 - 48x^5 + 70x^4 + 2$
- 2 a**  $12x^3 - 12x$   
**b**  $12u^3 - 12u$   
**c**  $30m^5 + 40m^4 - 24$   
**d**  $9 + 30y^4 - 12y^2 + 14y^6$   
**e**  $6z - 27z^2 + 38$   
**f**  $35t^6 + 36t^3 - 16$   
**g**  $27r^2 - 13 - 12r$   
**h**  $4a - 24a^3$   
**i**  $8 - 12p^3 - 42p^6 - 45p^8$   
**j**  $8t - 6t^2 + 44t^3$
- 3 a**  $-14$     **b**  $-30$     **c**  $618$     **d**  $-358$
- 4 a**  $20x + 41$     **b**  $12x - 35$   
**c**  $-42x - 38$     **d**  $-24x + 11$   
**e**  $36x - 95$     **f**  $-14x + 58$
- 5 a**  $2x + 14$     **b**  $2x - 8$     **c**  $8x + 12$   
**d**  $18x + 24$     **e**  $32x - 40$     **f**  $18x + 48$
- 6 a**  $-15x^4 + 4x^3 - 6x + 1$   
**b**  $-15x^4 + 4x^3 + 30x^2$   
**c**  $-42x^6 - 12x^5 + 100x^4$   
**d**  $-35x^4 - 80x^3 + 9x^2 - 28x + 2$   
**e**  $-126x^6 - 144x^5 - 40x^4 + 27x^2 + 12x$   
**f**  $-168x^5 + 65x^4 + 248x^3 + 48x^2$
- 7 a**  $28x^3 - 3$     **b**  $60x^4 + 24x^2 - 4$   
**c**  $3$     **d**  $4$
- 8** Demonstration using the process shown in Example 12.
- 9 a**  $24x^3 - 74x^2 + 42x$   
**b**  $20x^4 - 80x^3 + 63x^2 + 12x - 9$   
**c**  $y' \times z' \neq (y \times z)'$

- 10 a i**  $2(x + 1)$     **ii**  $3(x + 1)^2$   
**iii**  $4(2x + 1)$     **iv**  $6(2x + 1)^2$   
**v**  $6(3x + 1)$     **vi**  $9(3x + 1)^2$   
**b**  $\frac{d}{dx}(ax + 1)^n = an(ax + 1)^{n-1}$ .

### Exercise 10.05

- 1 a**  $-8$     **b**  $14$     **c**  $-13$   
**d**  $11$     **e**  $-4$     **f**  $21$
- 2 a**  $-5$     **b**  $10$     **c**  $10$   
**d**  $-1$     **e**  $-9$     **f**  $4$
- 3 a**  $(2, -1)$     **b**  $(-4, -7)$   
**c**  $\left(-1\frac{1}{2}, 7\frac{1}{4}\right)$     **d**  $(2, -27)$   
**e**  $(-1, 4), (1, -4)$     **f**  $\left(\frac{1}{3}, -1\frac{23}{27}\right), (1, -2)$
- 4 a**  $(1, 0), (3, 0)$     **b**  $(2, 0)$   
**c**  $(-1, -24), (5, 24)$
- 5 a**  $4x - y - 5 = 0$     **b**  $2x - y = 0$   
**c**  $7x - y + 1 = 0$     **d**  $x - y + 1 = 0$   
**e**  $x + y = 0$     **f**  $4x - y - 4 = 0$
- 6 a**  $4x - y - 1 = 0$  (or  $y = 4x - 1$ )  
**b**  $1.03$  units  
**c**  $77.0^\circ$
- 7 a**  $-4$   
**b**  $4x + y + 9 = 0$  ( $y = -4x - 9$ )  
**c**  $8.25$  units  
**d**  $104.0^\circ$  ( $-76.0^\circ$ )
- 8 a**  $45^\circ$   
**b**  $x - y - 1 = 0$  ( $y = x - 1$ )  
**c**  $92.5^\circ$  ( $-87.5^\circ$ )  
**d**  $25.02$   
**e**  $73.06$
- 9 a**  $y = -23x - 37$     **b**  $1$   
**c**  $y = x + 2$     **d**  $45^\circ$
- 10 a**  $m = 2x + 4 - 12x^2$   
**b**  $\left(-\frac{1}{2}, -5\frac{1}{4}\right)$  and  $\left(\frac{2}{3}, -2\frac{2}{27}\right)$   
**c**  $-6$     **d**  $4$     **e**  $76^\circ$   
**f**  $98x + y - 203 = 0$  ( $y = -98x - 203$ )  
**g**  $48x + y + 72 = 0$  ( $y = -48x - 72$ )  
**h**  $90.6^\circ$  ( $-89.4^\circ$ )

- 11  $7x - y + 5 = 0$  ( $y = 7x + 5$ ) and  
 $7x - y - 27 = 0$  ( $y = 7x - 27$ )
- 12  $7x - y + 28 = 0$  ( $y = 7x + 28$ ) and  
 $7x + y - 21 = 0$  ( $y = -7x + 21$ )
- 13 a  $a = 0, b = 16$       b  $(-2, 32)$
- 14  $4x - y - 29 = 0$  ( $y = 4x - 29$ )
- 15  $y = 3x^2 - 2x + 7$

### Investigation: Errors and derivatives

- 1  $2144.66 \text{ m}^3$
- 2  $2226.09 \text{ m}^3$
- 3  $81.43 \text{ m}^3$
- 4  $805.29 \text{ m}^3$
- 5  $804.25 \text{ m}^2$
- 6 They have almost the same value
- 7  $321.77 \text{ m}^3 \approx 0.4 \times 805.29 \text{ m}^3$
- 8  $31\,417 \text{ m}^3, 31\,416 \text{ m}^2$ , almost the same
- 9  $6\,158\,000 \text{ m}^3$
- 10 About  $1\,231\,000 \text{ m}^3$

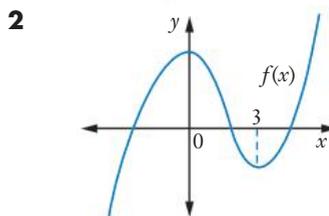
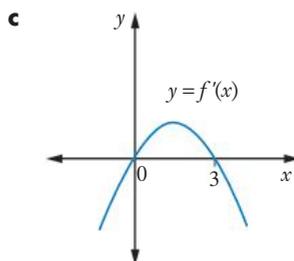
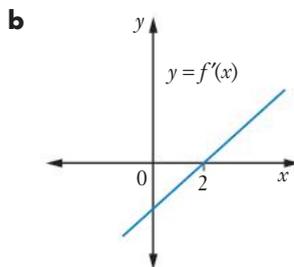
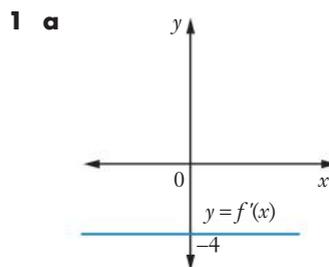
### Exercise 10.06

- 1 a 13      b 64      c 189  
d -12      e 533
- 2 a 22      b 102      c -10  
d -18      e 158
- 3 a -153  
b i -15    ii -363    iii -135  
c The average rate is not the average of the instantaneous rates.
- 4 a \$46 640    b \$777.33    c \$432  
d i The average cost (\$777.33) is higher than the marginal cost (\$432).  
ii The average cost (\$620) is higher than the marginal cost (\$400).  
iii The average cost (\$591.38) is lower than the marginal cost (\$628).
- 5 a -5.8    b -6.2  
c The function ceases to be valid when  $V < 0$ , i.e. after about 63.7 s.
- 6 a i \$300/car      ii -\$100/car  
b When 150 cars are sold a profit of \$300/car is made. When 250 cars are sold a loss of \$100/car is made.
- 7 a i \$2.20/cake      ii \$5/cake

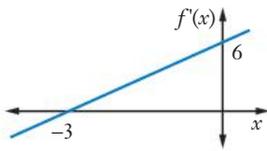
b The sale of cakes is more profitable as the number of cakes increases.

- 8 a i \$15/iron      ii -\$5/iron  
b 175 irons
- 9 a  $S'(x) = -12x$  (The negative sign indicates the surface area is decreasing)  
b Decreasing by  $48\text{cm}^2/\text{cm}$
- 10 a  $6.786 \text{ m}^3$       b  $4.524 \text{ m}^3$
- 11 a  $V(t) = 38\,400\,000 - 464\,000t + 1320t^2 - t^3$   
b  $V'(t) = -464\,000 + 2640t - 3t^2$   
c  $431\,469 \text{ m}^3$   
d  $426\,351 \text{ m}^3$
- 12 a  $V = \frac{\pi d^3}{3}$     b  $\pi d^2$       c  $1.61 \text{ cm}^3$

### Chapter 10 review



3



$F(x)$  is increasing when  $x > -3$  and  $f(x)$  is decreasing when  $x < -3$ .

4 When  $x < \frac{1}{4}$

5 a  $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

b  $16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$

6 (Expand first then use first principles.)

a  $6x^5$

b  $20x^3$

7 a  $10x^9$

b  $14x^{13}$

8 a  $3x^2 + 10x$

b  $10x - 3x^2$

9 a  $45x^8 - 24x^5$

b  $21x^6 + 21x^2$

10 a  $4x^3 - 6x + 5$  b 95

c 5

11 a  $30x^5 - 6x + 1$

b  $-1 - 6u + 18u^2$

c  $7q^6 + 12q^2 - 1$

d  $90y^5 - 4y^3 - 28$

e  $1 - 6w + 3w^2$

12  $18x^5 + 35x^4 - 72x^3 + 24x^2$

13 Show by demonstration.

14  $x = \frac{3}{2}$  or  $-\frac{1}{4}$

15 a -1

b 54

c -2

d 3

e 118

16 a -13

b 27

c -3

d -33

e 97

17 a  $x - y - 4 = 0$

b  $23x + y + 20 = 0$

c  $x - y - 3 = 0$

d  $49x - y - 83 = 0$

e  $143x + y + 211 = 0$

18 a  $-80.5^\circ$  or  $90.5^\circ$

b  $y + 6x + 4 = 0$

c  $B(1, -10)$

d 18.25 units

19 a 17

b -244

c -1768

d 2633

e 8

20 a 23

b -25

c -1

d 5

e 53

21 a  $C'(n) = 0.0008n + 0.27$

b \$0.67

c \$1.07

d \$1.47

e When  $n > 1575$

22 a 0

b When  $t > 4$ , the rate of change of volume is positive, so the volume of the balloon is increasing.

23  $100\pi \mu\text{m}^3/\mu\text{m}$  or about  $314.2 \mu\text{m}^3/\mu\text{m}$

24 a  $33.12 \Omega$  b  $0.0576 \Omega/^\circ\text{C}$

c  $39 \Omega$

d  $0.0588 \Omega/^\circ\text{C}$

e  $6.4128 \Omega$

f  $0.0451 \Omega/^\circ\text{C}$

g A high-temperature gauge: any reasonable answer that does not expose the wire to temperatures above  $\approx 1760^\circ\text{C}$  is acceptable.

25 Approximate error in volume would be 100 cc, so the capacity would be 2100 cc.

## Chapter 11

### Exercise 11.01

1 a Continuous

b Discrete

c Discrete

d Continuous

e Discrete

f Continuous

g Discrete

h Continuous

2 a No, the sum of the values is not 1.

b Yes, all values are non-negative and add up to 1.

c No, not all the values are positive.

3 a Yes all values are positive and add up to 1.

b All the values are  $\frac{1}{4}$ , so it would be easier to

write  $f(x) = \frac{1}{4}$ .

4 a 0.12

b 0.76

5 a 0.34

b 0.3433

6 a About 0.135

b About 0.191

c About 0.225

7  $f(x) = \frac{11-x}{66}$ ,  $f(x) = \frac{30-x}{275}$  or  $f(x) = \frac{22-2x}{132}$ , etc.

for  $0 < x < 10$

### Exercise 11.02

1 a  $p(p) = \frac{1}{20}$

b Uniform

c  $\frac{1}{4}$

d  $\frac{3}{20}$

2 a  $p(1) = \frac{1}{10}$ ,  $p(2) = \frac{3}{20}$ ,  $p(3) = \frac{1}{5}$ ,  $p(4) = \frac{1}{4}$ ,

$p(5) = \frac{3}{20}$ ,  $p(6) = \frac{1}{10}$ ,  $p(7) = \frac{1}{20}$

b Not uniform

c  $\frac{17}{20}$

d  $\frac{7}{10}$

3 a  $p(n) = \frac{2^n}{3^{n+1}}$       b  $\frac{2^5}{3^6} \approx 0.0439$

c  $\frac{2^8}{3^9} \approx 0.0130$

4 a  $p(m) = \frac{1}{8}, m \in \mathbb{Z}^+, 0 \leq m \leq 7$

b Uniform      c 1      d  $\frac{3}{8}$

5 a  $p(n) = \frac{1}{12}, x = 0, 7, 8; p(n) = \frac{1}{8}, n \in \mathbb{Z}^+, 1 \leq n \leq 6$

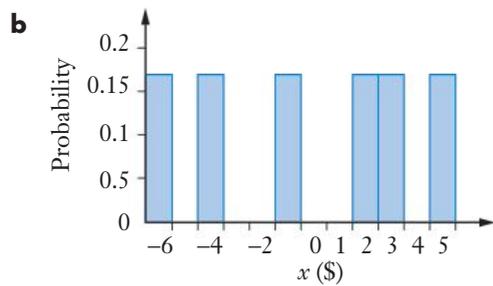
b Not uniform      c  $\frac{11}{12}$       d  $\frac{3}{8}$

6 a  $p(h) = \frac{3^h}{4^{h+1}}$       b  $\frac{3^4}{4^5} \approx 0.0791$

c  $\frac{3^7}{4^8} \approx 0.0334$

7 a

$x$	-6	-4	-1	2	3	5
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$



8 a

$x$	-5	5	20
$p(X=x)$	$\frac{5}{9}$	$\frac{5}{12}$	$\frac{1}{36}$

b  $\frac{4}{9}$

9 a

$P$	5	10	15	20	25	50
$P(P=p)$	$\frac{3}{10}$	$\frac{4}{15}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{15}$

b  $\frac{2}{3}$

10 13 times

### Exercise 11.03

- 1 15  
 2 9.625  
 3 a 5.5      b 4      c -1      d 3  
 4 22.9  
 5 a 57.5      b 17.25  
 6 3.5  
 7 2.53  
 8 \$-4

### Exercise 11.04

- 1 8.25, about 2.87  
 2 8.25, about 2.87  
 3 About 26.23, 5.12  
 4 a 37.25, about 6.10      b 10, about 3.16  
     c 8.25, about 2.87      d 3, about 1.73  
 5 About 2.92, 1.71  
 6 About 1.96, 1.4

### Exercise 11.05

- 1 a No,  $E(X) < 0$       b \$15  
 2 a 40%      b \$12  
 3 -\$9.30 and 93%  
 4 a Female -\$390, 59.1%; male -\$250, 27.8%  
     b Men are more likely to die than women (more than double the chances), so their insurance costs more.  
     c Some of the costs are fixed (e.g. administration), so these costs are the same for men and women.  
 5 About 5.26% for all bets because of the different odds.

### Chapter 11 review

- 1 a No,  $\sum p(x) \neq 1$   
 b Yes,  $\sum p(x) = 1$  and  $p(x) \geq 0$  for all values  
 c No,  $p(2) < 0$   
 2 0.12  
 3  $p(x) = \frac{1}{9}$ , a uniform discrete random variable  
 4 a

$q$	3	4	5	6	7	8	9	10	11	12
$P(q)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{1}{21}$

b  $\frac{13}{21}$

- 5 **a**  $p(c) = \frac{4 \times 5^c}{9^c + 1}$       **b** 0.0235
- 6 12.8
- 7 3.4
- 8 Variance is about 4720 cents and SD about 68.7 cents.
- 9 Variance about 11.53, SD about 3.40
- 10 The expected value is 0, so it is fair and it makes no difference if they take it in turns.
- 11 4%
- 12

$q$	0	1	2	3
$P(q)$	$\frac{1}{14}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{14}$

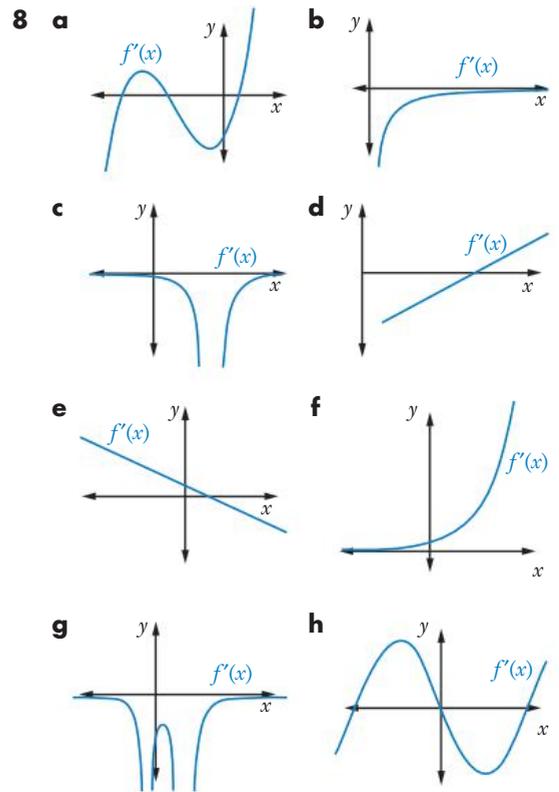
- 13 About 0.0516
- 14 9.5, 4.92
- 15 About 7.692%

## Chapter 12

### Exercise 12.01

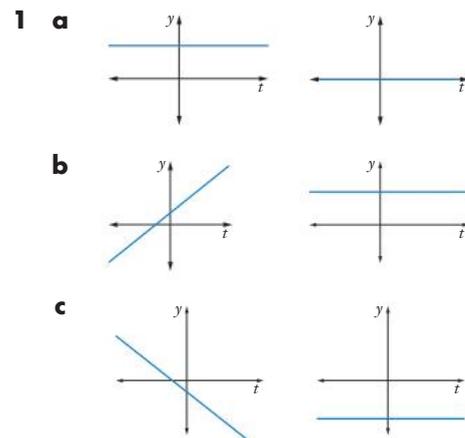
- 1 **a** **i** 4      **ii** -2      **iii** 0      **iv** 1  
**b** **i** 6      **ii** 2      **iii** -6      **iv** -3
- 2 **a** **i** -2      **ii** 1      **iii** 0      **iv** -0.5  
**b** **i** 0      **ii** 0      **iii** 1.5
- 3 **a** **i** 8      **ii** 36      **iii** 16  
**b** **i** 20      **ii** 4      **iii** 32  
**c**  $x = 0$  or  $\frac{2}{3}$
- 4 **a** **i** -1      **ii** -19      **iii** -1  
**b** **i** -41      **ii** 13      **iii** -11
- 5 **a** **i** -17      **ii** -17      **iii** -17  
**b** **i** -89      **ii** -25      **iii** 19

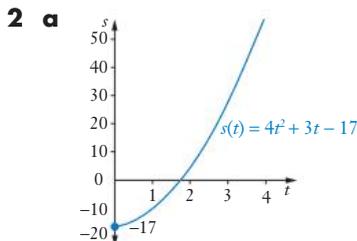
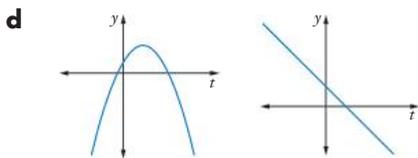
- 6 The graph marked *a* represents the gradient function because it crosses the  $x$ -axis at the same  $x$ -value where  $f(x)$  has a stationary point, it is above the  $x$ -axis when the gradient of  $f(x)$  is positive and is below the  $x$ -axis when the gradient of  $f(x)$  is negative.
- 7 The graph marked *c* represents the gradient function because it crosses the  $x$ -axis at the same  $x$ -values where  $f(x)$  has a stationary points, it is above the  $x$ -axis when the gradient of  $f(x)$  is positive and is below the  $x$ -axis when the gradient of  $f(x)$  is negative.



- 9 **a** **i** \$92.50/unit      **ii** \$90.50/unit      **b** \$90/unit
- 10 **a**  $3200\pi$  cm<sup>2</sup>/cm      **b**  $6400\pi$  cm<sup>2</sup>/cm  
**c**  $9600\pi$  cm<sup>2</sup>/cm
- 11 **a** 7 beats/min      **b** 15 beats/min      **c** 23 beats/min
- 12 **a**  $-50.27$  cm<sup>3</sup>/h      **b**  $-0.92$  cm/h      **c**  $-48.33$  cm<sup>2</sup>/h

### Exercise 12.02

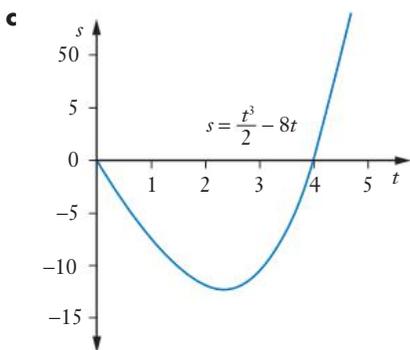




**b** The velocity is increasing at an increasing rate.

**c i**  $v = 8t + 3$  **ii**  $a = 8$  **iii**  $v = 35$  m/s,  $a = 8$  m/s<sup>2</sup>

**3 a**  $s(0) = 0$  **b**  $t = -4$  or  $4$  s



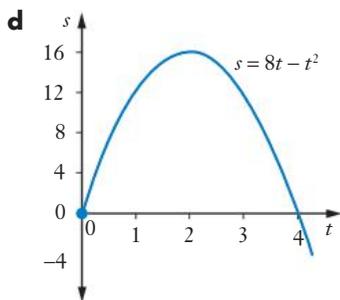
**d** The velocity is negative at the beginning. It decreases in size, becomes zero and then increases in the positive direction.

**e** 5.5 cm/s **f** 12 cm/s<sup>2</sup> **g**  $t = -3$  or  $3$  s

**4 a**  $v = -14$  m/s **b**  $a = 6$  m/s<sup>2</sup> **c** 13 m

**d**  $t = 2\frac{1}{3}$  s **e**  $-8\frac{1}{3}$  m

**5 a**  $t = 0$  or  $8$  s **b**  $t = 4$  s **c** 12 cm



**e** The velocity is positive and decreases until  $t = 4$  when it is zero. It then increases in the negative direction.

**f** 20 cm

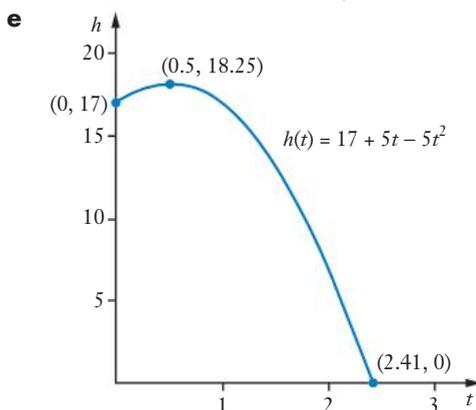
**6 a**  $t = 2$  or  $6$  s **b** 96 m

**7 a** 9 m/s **b**  $-3$  m/s **c**  $t = 1$  or  $3$  s

**d**  $x < 1$  and  $x > 3$  **e**  $1 < x < 3$  **f** 12 m

**8 a**  $s = 17$  m,  $v = 5$  m/s **b**  $a = -10$  m/s<sup>2</sup>

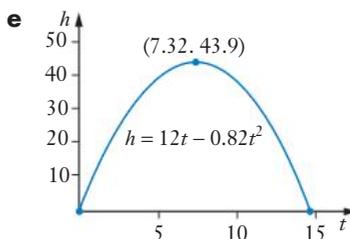
**c**  $t \approx 2.41$  s **d**  $18\frac{1}{4}$  m



The ball is thrown vertically into the air with an initial velocity of 5 m/s, after which it slows, coming to rest at its maximum height, before falling at an increasing negative velocity to the origin.

**9 a**  $v = 3.8$  m/s **b**  $t \approx 7.32$  s

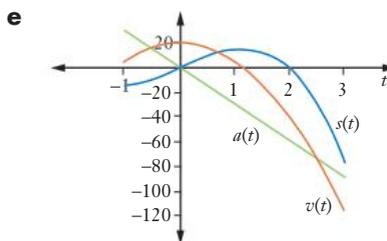
**c**  $t \approx 14.63$  s **d** About 43.9 m



The moon rock is thrown into space with an initial velocity of 12 m/s, after which it slows, coming to rest at its maximum height, before falling at an increasing negative velocity back to the moon.

**10 a**  $t \approx 1.15$  s **b**  $t = 0$  s

**c**  $t > 1.15$  s **d**  $t < 0$  s



**11 a** 22 m **b**  $-21$  m/s

**12** Greatest velocity = 27 m/s at  $t = 6$  s.

**13 a**  $t = 2$  s,  $x = 12$  cm **b**  $a = -9$  m/s<sup>2</sup>

**c**  $t \approx 3.46$  s

**14** 18.9 m

### Exercise 12.03

1 **a**  $x^3(15x + 8)$    **b**  $60x - 37$    **c**  $4x(3x + 5)$

**d**  $3x^2(10x^2 - 3)$    **e**  $96x^3 + 18$    **f**  $40x - 7$

2 **a**  $-9z^8 + 28z^6 + 25z^4 + 4z^3 - 18z^2 - 6z + 9$

**b**  $9p^2 - 4p - 20$

**c**  $245v^6 - 186v^5 + 40v^4 - 88v^3 + 69v^2 - 2v - 66$

**d**  $135t^8 + 424t^7 + 70t^6 - 126t^5 + 195t^4 - 212t^3 - 90t^2 - 238t + 346$

**e**  $36b^8 - 192b^7 + 84b^6 + 18b^5 + 300b^4 + 188b^3 - 90b^2 + 138b - 126$

**f**  $-252u^5 - 160u^4 - 4u^3 + 24u^2 + 86u + 54$

3 **a**  $\frac{60}{(3-5q)^2}$    **b**  $\frac{-26}{(2m-5)^2}$    **c**  $\frac{-35}{(2x-5)^2}$

**d**  $\frac{-10}{(t-5)^2}$    **e**  $\frac{123}{4(5t+8)^2}$    **f**  $\frac{21}{5(2u-9)^2}$

4 **a**  $-\frac{4x}{(2x^2-3)^2}$    **b**  $-\frac{10(3x+1)}{x^2(3x+2)^2}$

**c**  $-\frac{x(x^3+2)}{(x^3-1)^2}$    **d**  $\frac{24x}{(4-x^2)^2}$

**e**  $\frac{6x}{(x^2+1)^2}$    **f**  $-\frac{6x^2-4x-9}{(2x^2+3)^2}$

5 **a**  $-5x^{-2}$    **b**  $-\frac{8}{x^2}$    **c**  $-12x^{-5}$    **d**  $-\frac{6}{t^3}$

**e**  $-70m^{-11}$    **f**  $\frac{21}{u^8}$    **g**  $-\frac{20}{y^{21}}$    **h**  $-\frac{60}{v^{16}}$

**i**  $-144d^{-10}$    **j**  $\frac{72}{z^{10}}$

6 **a**  $\frac{1}{2\sqrt{x}}$    **b**  $\frac{1}{3\sqrt[3]{m^2}}$    **c**  $\frac{7x^{-\frac{1}{2}}}{2}$

**d**  $\frac{20\sqrt[3]{t}}{3}$    **e**  $\frac{-14p^{-\frac{5}{3}}}{3}$    **f**  $\frac{-x^{-\frac{3}{2}}}{2}$

**g**  $-\frac{63z^{\frac{5}{2}}}{2}$    **h**  $\frac{-3u^{-\frac{3}{4}}}{2}$    **i**  $\frac{26d^{\frac{7}{3}}}{3}$

**j**  $\frac{15\sqrt{m}}{2}$

7 **a**  $\frac{-x^2(7\sqrt{x}-6)}{2}$    **b**  $\frac{6x^7-4x^5+3x^2+3}{x^2}$

**c**  $\frac{x(x^3+2x-8)}{(x-2)^2(2-x^2)^2}$

8 **a**  $-18x^8 - 40x^7 + 42x^6 + 90x^5 + 30x^4 + 24x^3 + 45x^2 + 20x + 40$

**b**  $-\frac{6}{(2f-5)^2} - \frac{15}{2(3f+5)^2} + \frac{20}{(5f-11)^2}$

9  $\frac{-2kx}{(x^2-k)^2}$

10  $(-5.0984, -8.0666), (1.0984, 0.0666)$

11  $\left(\frac{-1}{3}, \frac{-13}{27}\right), (2, -4)$

### Exercise 12.04

1 **a**  $x = 2$    **b**  $x > 2$    **c**  $x < 2$

2 **a** Increasing:  $x < -4, x > 0$ ;  
decreasing:  $-4 < x < 0$

**b** Increasing:  $-2\frac{1}{3} < x < 1$ ;

decreasing:  $x < -2\frac{1}{3}, x > 1$

**c** Increasing:  $0 < x < 0.25, x > 2$ ;  
decreasing:  $x < 0, 0.25 < x < 2$

3 **a**  $(2, -5)$    **b**  $(2, 10)$

**c**  $(-\frac{4}{3}, 16\frac{14}{27}), (2, -2)$    **d**  $(1, -7), (-2, 20)$

**e** None   **f**  $(1, -5), (-3, 27)$

4 **a** Local maximum at  $(-1, 3)$ , local minimum at  $(1, -1)$

**b** Local maximum at  $(2, 20)$ , local minimum at  $(4, 16)$

**c** Local minimum at  $(-1, 0)$ , local maximum at  $(1, 4)$

5 **a** Point of horizontal inflection on a falling curve at  $(0, 0)$ , local minimum at  $(1, -1)$ .

**b** Point of horizontal inflection on a rising curve at  $(-2, 2)$ .

**c** Point of horizontal inflection on a falling curve at  $(4, 5)$ .

6 **a** Maximum at  $(2, 1)$

**b** Maximum at  $(-1, 5)$

**c** Maximum at  $(-2, 16)$ , minimum at  $(2, -16)$

**d** Minimum at  $(-1, -2)$ , maximum at  $(1, 2)$

**e** Maximum at  $(0, 7)$ , minimum at  $(4, -57)$

**f** Minimum at  $\left(\frac{1}{8}, -\frac{\pi}{16}\right)$

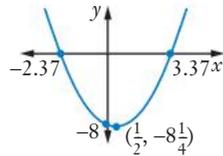
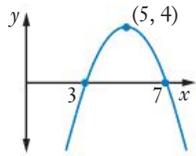
**g** Maximum at  $\left(-\frac{4}{3}, 8\frac{14}{27}\right)$ , minimum at  $(2, -10)$

**h** Maximum at  $(-\sqrt{2}, 1 + 4\sqrt{2})$ ,  
minimum at  $(\sqrt{2}, 1 - 4\sqrt{2})$

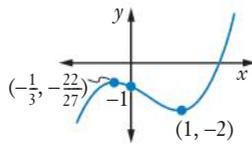
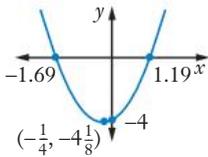
- i** Minimum at  $(1, 2)$
- j** Minimum at  $(0, 0)$ , maximum at  $(5\frac{1}{3}, 75\frac{23}{27})$
- 7 a**  $k = -4$     **b** Local maximum at  $(2, 3)$
- 8 a**  $a = -6, b = 9$
- b** Local maximum at  $(1, 6)$ , local minimum at  $(3, 2)$
- 9 a**  $a = -2, b = 3$
- b** Local minima at  $(-1, 2)$  and  $(1, 2)$ , local maximum at  $(0, 3)$

### Exercise 12.05

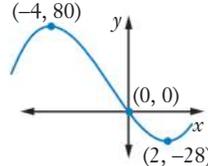
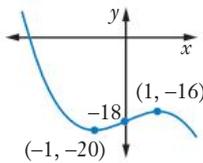
**1 a**  $g(x) = 10x - x^2 - 21$     **b**  $h(x) = x^2 - x - 8$



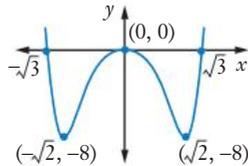
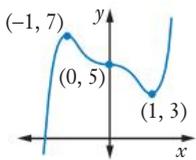
**c**  $h(x) = 2x^2 + x - 4$     **d**  $g(x) = x^3 - x^2 - x - 1$



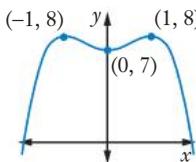
**e**  $f(x) = 3x - x^3 - 18$     **f**  $f(x) = x^3 + 3x^2 - 24x$



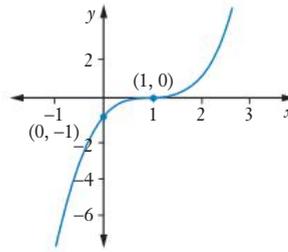
**g**  $f(x) = 3x^5 - 5x^3 + 5$     **h**  $f(x) = 2x^6 - 6x^4$



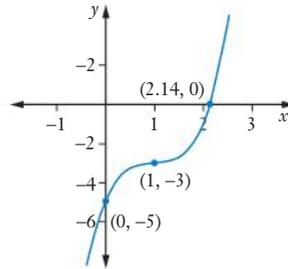
**i**  $f(x) = 7 + 2x^2 - x^4$



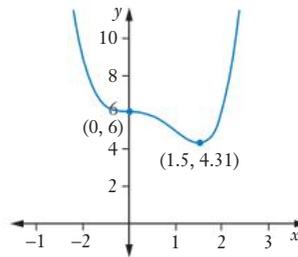
**2 a**  $f(x) = x^3 - 3x^2 + 3x - 1$



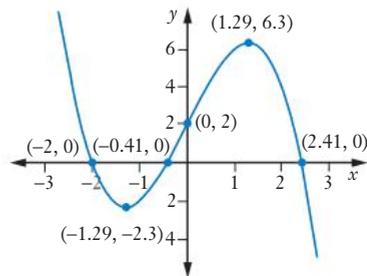
**b**  $y = 2x^3 - 6x^2 + 6x - 5$



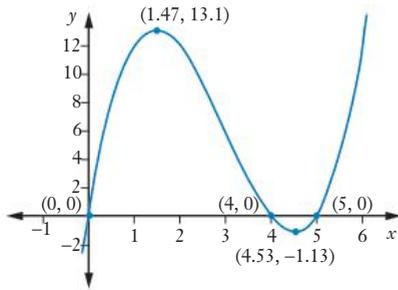
**c**  $g(x) = x^4 - 2x^3 + 6$



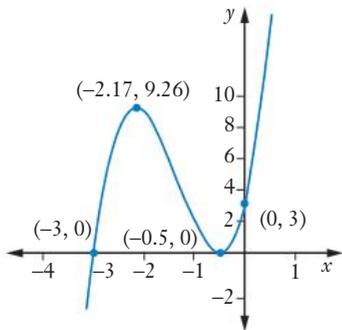
**d**  $y = 2 + 5x - x^3$



e  $f(x) = x^3 - 9x^2 + 20x$



f  $h(x) = (2x + 1)^2(x + 3)$



- 3 a  $(-0.59, -37.37)$  and  $(3.93, 55.07)$   
 b  $-0.59 < x < 3.93$   
 c  $-162$  d  $20$   
 4 a  $(-1.46, -214.9)$ ,  $(2.01, 95)$  and  $(5.83, -304.5)$   
 b  $x < -1.46$  or  $2.01 < x < 5.83$   
 c  $76$  d  $1$   
 5 If  $f(x) = ax^3 + bx^2 + cx + d$ ,  $f'(x) = 3ax^2 + 2bx + c$ . Stationary points exist when  $f'(x) = 0$ , i.e. when  $3ax^2 + 2bx + c = 0$ . This is a quadratic equation. Quadratic equations can have 0, 1 or 2 solutions. Hence, there can be 0, 1 or 2 stationary points.

### Exercise 12.06

- 1 a  $189(3t - 6)^8$  b  $40m(4m^2 + 1)^4$   
 c  $72(3v - 6)^2$  d  $540x^2(3x^3 + 8)^4$   
 e  $-22(7 - 2u)^{10}$  f  $-16(5 - 4v)^3$   
 g  $168(3v - 9)^7$  h  $-63(2x + 4)(x^2 + 4x + 6)^6$   
 i  $30(3 - 5y)^2$   
 2 a  $48(3x + 2)^3 + 30(3x + 2)$   
 b  $6 - 18(6x - 8)^2 + 168(6x - 8)^3$   
 c  $-18(5 - 3x)^2 - 30(5 - 3x) + 21$   
 d  $15(t + 4)^4 - 6(t + 4)^2 + 10(t + 4)^2 - 3$   
 e  $-96(3 - 4u)^3 + 40(3 - 4u) - 4$

f  $90(3p + 7)^5 + 36(3p + 7)^2 - 6$

g  $-6(3 - 2x)^2 + 20(3 - 2x) + 5$

3 a  $-24(3x - 7)^{-5}$  b  $-48(8p + 3)^{-7}$  c  $\frac{-120}{(3g - 4)^6}$

d  $\frac{40}{(2z + 8)^5}$  e  $\frac{-35}{(5t - 9)^2}$  f  $\frac{-72}{(4x - 5)^7}$

g  $\frac{-1}{2(t + 4)^4}$  h  $\frac{-224}{5(7v + 3)^5}$  i  $\frac{112}{(4q - 7)^9}$

4 a  $\frac{x}{\sqrt{x^2 - 1}}$  b  $\frac{6x^2}{\sqrt{4x^3 + 3}}$  c  $\frac{-22}{\sqrt{1 - 2x^2}}$

d  $\frac{-45x^2}{2\sqrt{6 - 5x^3}}$  e  $\frac{48x^2}{\sqrt{4x^3 - 3}}$  f  $\frac{20x^3}{\sqrt[3]{(3x^4 - 1)^2}}$

5 a  $15(3x - 2)^3(x + 8)^2(7x + 30)$

b  $8(4t + 5)^2(2t - 7)(5t - 8)$

c  $54(4m - 21)(4m + 3)^7$

d  $(2k + 9)^3(3k - 5)^6(66k + 149)$

e  $60(3g - 4)^4(5g - 4)^5(11g - 12)$

f  $-6(v + 7)^4(v - 6)^3(9v - 2)$

g  $3(6 - 5r)^2(2r + 9)^{10}(140r + 3)$

h  $30(8 - 6h)^6(-4 - 3h)^5(-4 - 39h)$

i  $20t + 1$

j  $-(3p + 10)^6(8p + 1)^7(360p + 661)$

6 a  $2(2x + 7)(x + 5)^2$

b  $(3x - 1)^3(x - 5)^4(27x - 65)$

c  $2\sqrt{1-x} - \frac{x}{\sqrt{1-x}}$

d  $\frac{-(2-x)^{\frac{2}{3}}}{3x^{\frac{2}{3}}} - \frac{1}{2x^{\frac{1}{3}}}$  e  $\frac{-3x^2}{\sqrt{(2x^3 - 3)^3}}$

f  $\frac{1}{\sqrt{3-x^2}} + \frac{x^2}{\sqrt{(3-x^2)^3}}$

7 a  $50(2x - 3)^4 + 8(2x - 3) + \frac{16}{(2x - 3)^3} + \frac{10}{3(2x - 3)^6}$

b  $\frac{2(3v - 4)(7 - v)}{(2v + 3)^4}$

c  $\frac{3c^6(63 - 20c)}{(9 - 4c)^3}$

d  $\frac{-3}{(t - 6)^4} + \frac{16}{(2t + 7)^3} + \frac{2(67 - 18t)}{7(2t + 9)^5}$

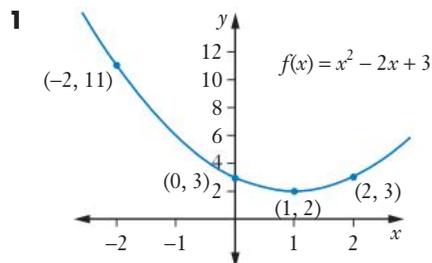
e  $\frac{-20p^5 + 30p^4 - 10p^3 + 14p^2 + 102p - 56}{(5p - 1)^7}$

8  $\frac{85}{4096}$

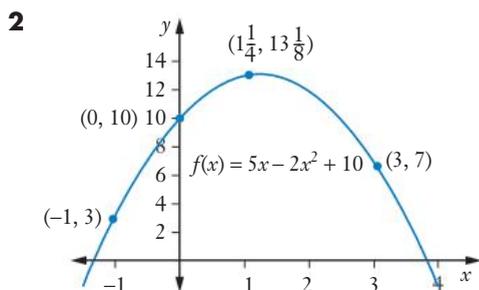
9  $(0, 162), (-\sqrt{3}, 0), (\sqrt{3}, 0)$

10  $(0, -2.5)$

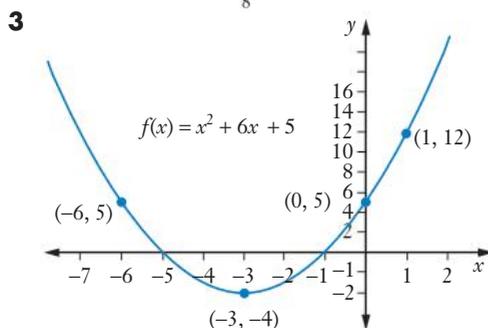
### Exercise 12.07



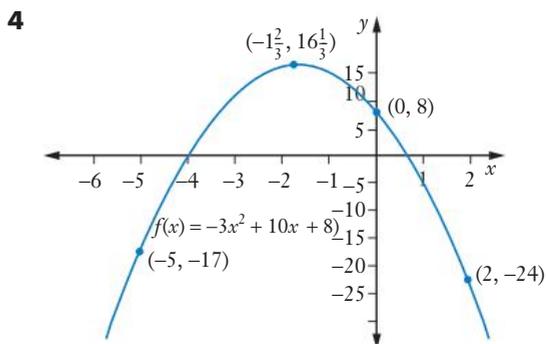
Greatest value = 11 and least value = 2



Greatest value =  $13\frac{1}{8}$  and least value = 3



Global maximum = 12



Global minimum = -24

- 5 About 6.113 and -86  
 6 65 and -16  
 7 17 and -3  
 8 About 48.248 and -30  
 9 About 10.39 and about -10.39  
 10  $(1, 1)$  and  $(3, -5)$  (gradients of 1 and -11 respectively)  
 11 Greatest gradient of  $6\frac{3}{4}$  at  $(2\frac{1}{2}, -\frac{3}{4})$ , least gradient of -30 at  $(6, -20)$

### Exercise 12.08

- 1 a  $50 = x + y$   
 b  $y = 50 - x$ .  $P = xy = x(50 - x)$   
 c  $x = y = 25$
- 2 a  $20 = x + y$                       b  $x = y = 10$
- 3 a  $30 = 2(x + y)$  so  $y = 15 - x$ .  $A = xy = x(15 - x)$ .  
 b  $56.25 \text{ cm}^2$
- 4 a  $2400 = 2x + y$                       b  $A = x(2400 - 2x)$   
 c 600 m by 1200 m
- 5 a  $t = 0.6 \text{ s}$                           b  $h = 1.8 \text{ m}$
- 6 a  $SA = 400 = 2x^2 + 4xh$   
 b  $400 = 2x^2 + 4xh$  so  $4xh = 400 - 2x^2$  and  
 $h = \frac{100}{x} - \frac{x}{2}$   
 c  $V = x^2h = x^2\left(\frac{100}{x} - \frac{x}{2}\right) = 100x - \frac{x^3}{2}$   
 d About  $544.33 \text{ cm}^3$ . The method is probably fairly accurate because the sides are flat.
- 7 4 and 8  
 8  $2048 \text{ cm}^3$   
 9  $4000 \text{ cm}^3$

**10** If the sides are  $x$  and  $y$ ,  $P = 2(x + y)$  so  $y = \frac{P - 2x}{2}$   
and  $A = \frac{P}{2}x - x^2$ .  $A$  has a minimum when  
 $x = y = \frac{P}{4}$  i.e. the rectangle is a square.

**11** 10 m by 25 m

**12**  $P\left(\frac{5}{2}, 0\right)$

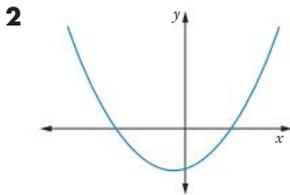
**13**  $24^\circ\text{C}$  and  $4^\circ\text{C}$

**14 a**  $r = h = 1$  m      **b**  $V = \pi \text{ m}^3$   
**c** No, because there will be a lot of scrap from cutting the circular base.

**15**  $x \approx 2.33$  m and  $y = 3$  m

### Chapter 12 review

**1 a** 10      **b** -8      **c**  $x = 1$



**3 a** About -10.26 L/s      **b** -10.2 L/s

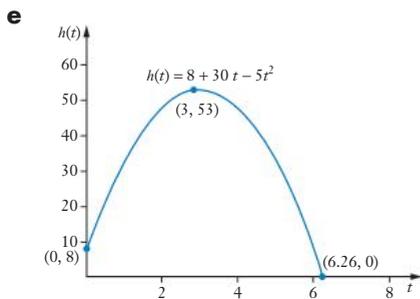
**4 a** 4 cm      **b**  $7\frac{1}{3}$  cm      **c**  $32\frac{2}{3}$  cm

**d** 5 cm/s      **e**  $t = 3$  s

**5** 0.6 m, -0.6 m/s, -0.2 m/s<sup>2</sup>

**6 a** 8 m, 30 m/s      **b** -10 m/s<sup>2</sup>

**c** About 6.26 s      **d** 53 m



**7 a**  $6x^2 - 3$       **b**  $x(8x^2 - 3x - 42)$

**c**  $30x^4 - 52x^3 + 36x^2 - 4x - 3$

**8 a**  $\frac{-3}{(3x+5)^2}$       **b**  $\frac{-2(5x-1)}{x^2(5x-2)^2}$       **c**  $\frac{-13}{(2x-1)^2}$

**9 a**  $\frac{-6}{x^5}$       **b**  $\frac{3}{2\sqrt{x}} + 8x$

**c**  $\frac{1}{2\sqrt{x^3}} - \frac{1}{x^2}$       **d**  $\frac{-(2\sqrt{x}-3)}{2\sqrt{x^5}}$

**10 a**  $\left(\frac{1}{2}, -12\frac{1}{4}\right)$       **b**  $x > \frac{1}{2}$       **c**  $x < \frac{1}{2}$

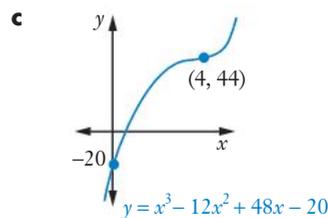
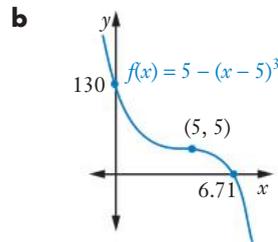
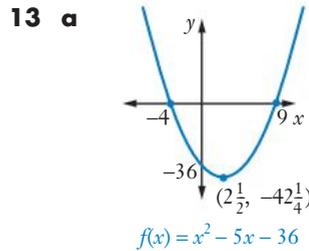
**11** Local maximum at  $\left(\frac{1}{3}, 2\frac{13}{27}\right)$

local minimum at  $(3, -7)$

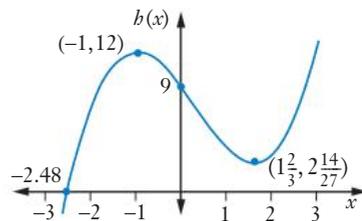
**12 a** About  $(-0.4, 93)$  and  $(5.8, -27)$

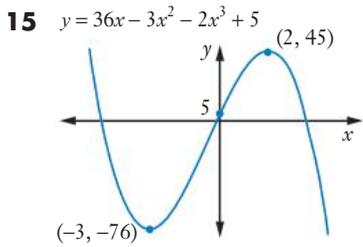
**b** About 77      **c**  $x < -0.4, x > 5.8$

**d** About -30

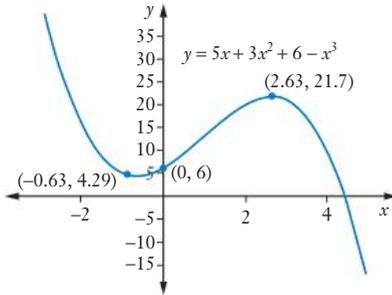


**14**  $h(x) = x^3 - x^2 - 5x + 9$





- 16** Minimum at about  $(-0.63, 4.29)$  and maximum at about  $(2.63, 21.7)$



**17 a**  $224(4x + 7)^7$       **b**  $72(2x - 5)^5$

**18 a**  $\frac{-30}{(2x - 9)^6}$       **b**  $\frac{144}{5(7 - 4x)^4}$

**19 a**  $7(12 - 5x)^8(4x - 11)^3(687 - 260x)$

**b**  $3(28x - 57)(4x + 15)^5$

**20** 8 and  $-12.25$

**21** 4 and  $-16$

**22** 53 and 0

**23** About  $120.16 \text{ cm}^3$ . The dimensions fit the model, and since the sides are flat, it will be accurate.

**24 a**  $y = 0.75x$

**b** Rate of increase of shadow length  
 $= 0.75 \times \text{walking speed}$

**c i** 0.9 m/s    **ii** 0.9 m/s

**25 a** 50 m      **b**  $-20.58 \text{ m}$     **c** 70.58 m

**d**  $-1.51$       **e**  $(-6\frac{2}{3}, 14.71)$     **f**  $-2.27$

**26**  $(-\frac{1}{2}, 2\frac{1}{2})$

**27** About  $359.72 \text{ cm}^3$ . The true volume will be less because of waste from cutting the circular parts.

**28** 36 and about  $-70.3$

## Practice examination 4

**1 a**  $y = 4x - 5$

**b** Decreasing when  $x < 1.25$  (1 mark each)

**2**  $x = -1$  or  $\frac{7}{3}$  (3 marks)

**3 a**  $70x^6$       **b**  $6x^2 + 4x^3 - 6$  (1 mark each)

**4** 79 (1 mark)

**5**  $y = 14x + 14$  (2 marks)

**6 a**

$x$	1	2	3
$p(x)$	$\frac{4}{9}$	$\frac{1}{9}$	$\frac{4}{9}$

**b** A discrete random variable (2, 1 marks)

**7**  $E(V) = 6.3, \text{Var}(V) = 6.21$  (1 mark each)

**8**  $-11$  (2 marks)

**9**  $v = 7 \text{ m/s}, a = 2 \text{ m/s}^2$  (1 mark each)

**10 a**  $12x^3 + 3x^2 - 20x - 15$     **b**  $\frac{x^2 - 6x - 12}{x^2(x + 4)^2}$

**c**  $10(3x - 2)(3x^2 - 4x + 8)^4$  (2 marks each)

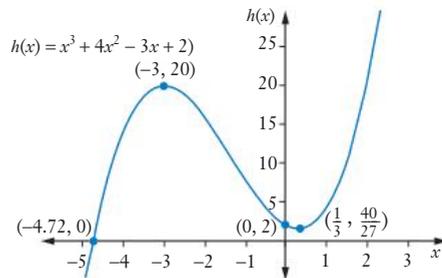
**11**  $2(7x - 4)(2x - 5)^2(x + 2)^3$  (2 marks)

**12** Greatest value  $14\frac{1}{6}$ , least value  $\frac{2}{3}$  (3 marks)

**13**  $12 \text{ cm}^3/\text{cm}$  (3 marks)

**14**  $E(X) \approx 2.57$  (2 marks)

**15**



(5 marks)

**16** 7.83 units (3 marks)

**17**  $\frac{4}{19}$  (2 marks)

**18 a**  $r = h = \frac{4}{\sqrt[3]{\pi}} \approx 2.73 \text{ cm}$

**b** About  $77 \text{ cm}^2$ , using a square and rectangle (3, 2 marks)

# GLOSSARY AND INDEX

**1 : 1 ('one-to-one')**: A function or relation with each  $x$ -value and each  $y$ -value in at most one pair  $(x, y)$ . (p. 157)

**acceleration**: The rate of change of velocity with respect to time. (p. 240, 272, 286, 467)

**amplitude**: Distance from the average position to the extreme values of a periodic function where  $a_0, a_1, a_2, \dots, a_n$  are real numbers and  $n$  is a positive integer. (p. 309, 337)

**and**: Mathematically,  $A$  and  $B$  means both  $A$  and  $B$  are true or both happen. (p. 80, 100)

**annuity**: A series of equal payments at regular intervals with compound interest. (p. 218, 226)

**annuity factor**: The value of an annuity with payments of \$1; shown as  $S_{\overline{n}|i}$  for the future value or  $a_{\overline{n}|i}$  for the present value of  $n$  payments at interest rate  $i$ . (p. 218, 226)

**annuity-due**: An annuity with payments made at the beginning of the interest period. (p. 218, 226)

**appreciation**: Increase in the value of an item over time. (p. 204, 205, 225)

**arc**: A portion of the circumference of a circle. (pp. 294, 299)

**arithmetic sequence, arithmetic progression (AP)**: A sequence given by  $t_{n+1} = t_n + d$ ; successive terms differ by a constant  $d$ . (p. 7, 18)

**arithmetic series**: A sequence of sums  $s_n = t_1 + t_2 + \dots + t_n$  of an arithmetic sequence  $t_n$ . (p. 11, 18)

**asymptote**: A function or relation characteristic: a horizontal line  $y = a$  such that as  $x \rightarrow \pm\infty, y \rightarrow a$ ; or a vertical line  $x = a$  such that as  $x \rightarrow a, y \rightarrow \pm\infty$ ; or a sloping line  $y = mx + c$  such that as  $x \rightarrow \pm\infty, y \rightarrow mx + c$ . (p. 348)

**at random**: having no pattern; outcomes equally likely to occur; determined only by chance in which one quantity changes with respect to another calculated over a period of time. (p. 457)

**average rate of change**: The rate of change a first variable quantity determined over an interval of a second variable quantity given by  $\frac{\text{change of first quantity}}{\text{change of second quantity}}$  often stated as first quantity with respect to second quantity. (p. 239, 285, 462, 504)

**average speed**: The total distance travelled by an object divided by the elapsed time to cover that distance. (p. 242)

**axis of symmetry**: A straight line such that a function or relation is symmetrical about the line. (p. 118, 157, 158)

**base**: The value  $a$  in a term of the form  $a^n$ . (p. 22, 365, 370, 384)

**binomial**: In algebra, an expression with two terms. (p. 30)

**binomial expansion**: Simplification of expansion of expressions of the type  $(x + y)^n$  to produce  $n + 1$  terms beginning with  $x^n$ . (p. 193)

**cardinality**: The number of elements in a set, shown as  $|A|$ . (p. 76, 100)

**certain**: An event that must occur, having a probability of 1. (p. 85, 101)

**chain rule**: The differential calculus rule for the derivative of a function of a function, or for changing the derivative variable. (p. 505)

**chord**: A straight line between two points on a curve. (p. 242)

**circular measure**: (p. 298) See **radian**.

**codomain**: The set of numbers allowed as values of  $y$  in the pairs  $(x, y)$  of a relation. (p. 110, 157)

**coefficient**: A constant placed before and multiplied by a variable. For example, the 3 in  $3x$ . (p. 58, 69)

**combination**: A selection of a group of objects without regard to order. (p. 167, 193)

- common difference:** The difference  $d = t_{n+1} - t_n$  between successive terms of an AP. (p. 7, 18)
- common logarithm:** A logarithm with base 10 written as  $\log(x)$  or  $\log_{10}(x)$  such that  $y = \log x \Leftrightarrow 10^y = x$ . (p. 365, 384)
- common ratio:** The ratio  $r$  of successive terms of a geometric progression. (p. 198, 225)
- complement:** For a set or event  $A$ , the set of all elements excluding those in  $A$ ; shown as  $\bar{A}$  or  $A'$ . (p. 78, 85, 100, 101)
- composite function:** The function  $f[g(x)]$  with values given by  $f(y)$  where  $y = g(x)$ . (p. 488)
- compound interest:** Interest added to a loan or investment and calculated periodically on its current value. (p. 204, 205, 225)
- concave downwards:** A curve with its inside below the curve. (p. 137, 159)
- concave upwards:** A curve with its inside above the curve. (p. 137, 159)
- conditional probability:** The probability of  $A$  given that  $B$  is true, shown as  $P(A|B)$ . (p. 193)
- constant:** A fixed number such as 2 or  $\pi$ . (p. 58)  
*See variable.*
- Consumer Price Index (CPI):** A measure of inflation calculated quarterly by the Australian Bureau of Census. (p. 216, 225)
- contingency table:** A table showing two characteristics of data across rows and down columns. (p. 187, 193)
- continuous variable:** A variable with values from an interval or the real numbers. (p. 434)
- converge:** A sequence with terms  $t_n$  converges to  $a$  if  $t_n \rightarrow a$  as  $n \rightarrow \infty$ . (p. 203, 225)
- cosine (cos):** For  $0^\circ \leq \theta \leq 90^\circ$ ; the ratio of the adjacent side to the hypotenuse of the right-angled triangle containing  $\theta$ ; For any angle; the  $x$ -coordinate of the trigonometric point  $P(x, y)$  with angle  $\theta$  on a unit circle. (p. 295)
- countable:** The elements of a countable set can be paired with counting numbers. (p. 77, 100)
- cubic:** A polynomial of degree 3. (p. 140, 159)
- cycle:** One complete repetition of the pattern of a periodic function. (p. 309, 337)
- decay factor:** the factor by which a quantity decreases in unit time. (p. 345, 384)
- decay function:** An exponential function  $y = b \cdot a^x$ , where  $0 < a < 1$ ;  $a$  is the decay factor. (p. 345, 384)
- decreasing function:** A function (or part of a function) that becomes less as  $x$  increases. (p. 397)
- deflation:** Increase of the value of money over time; general decrease of prices. (p. 216, 225)
- degree<sup>1</sup>:** The angle measure that is  $\frac{1}{360}$  of a revolution, shown as  $^\circ$ . (p. 294) *See radian.*
- degree<sup>2</sup>:** The highest power of the variable in a polynomial. (p. 58, 69)
- dependent events:** Events  $A, B$  that are not independent;  $P(A|B) \neq P(A)$ . (p. 181, 182, 193)
- dependent variable:** A variable whose value depends on that of another; the second variable of an ordered pair  $(x, y)$  of a function or relation. (p. 110, 157)  
*See independent variable.*
- depreciation:** Decrease in the value of an item over time. (p. 204, 205, 225)
- derivative:** Symbolised by  $f'(x)$  or  $\frac{dy}{dx}$ , the instantaneous rate of change of a function with respect to one of its variables. (p. 267, 285)
- difference quotient:**  $\frac{f(x+h) - f(x)}{h}$ ; calculates the slope of the secant through the points on the graph of  $f(x)$  with  $x$ -coordinates  $x$  and  $x+h$ . (p. 253, 285)
- differentiate:** To find the derivative. (p. 406)
- dilation:** A transformation that magnifies or reduces  $x$  or  $y$  values of a function or relation. (p. 118, 137, 158)
- direct proportion:** Two variables that increase or decrease in the same ratio. (p. 69)
- discontinuity:** A value at which a function is not defined. (p. 132, 158)
- discount rate:** The interest rate used in calculations of annuities, future and present value. (p. 214, 225)
- discrete probability distribution:** The probabilities of a discrete random variable. (p. 436, 457)
- discrete variable:** A variable with values from a finite set of numbers. (p. 434)
- discriminant:** Symbolised by  $\Delta$ , the value inside the radical in the quadratic formula:  $\Delta = b^2 - 4ac$ . Its value determines whether the quadratic equation has 1, 2 or no solutions. (p. 52)
- disjoint:** Sets or events with no elements in common. (p. 78, 100)
- displacement:** A change of position without reference to the route including both the size and direction of the change, measured from the starting point to the end. (p. 240, 272, 286, 467)
- distributive law:** The identity showing how to expand brackets:  $a(b \pm c) = ab \pm ac$ . (p. 30)
- diverge:** A sequence with terms  $t_n$  that increase in size;  $t_n \rightarrow \pm\infty$  as  $n \rightarrow \infty$ . (p. 203, 225)
- domain:** The set of first ( $x$ ) values of the ordered pairs  $(x, y)$  of a function or relation. (p. 110, 157)
- element:** A member of a set or event. (p. 76, 100)
- empty set:** The set with no elements. (p. 76, 100)

**event:** A collection of outcomes of a chance situation. (p. 80, 100)

**expected frequency:** The number of times an event is expected to occur in a chance situation. (p. 95, 101)

**expected value:** The mean value of a random variable; given by  $\mu = E(X) = \sum x \cdot p(x)$  for a discrete random variable  $X$ . (p. 444, 457)

**exponent:** The number at the top right of a power that shows the number of times the base is to be multiplied; the value  $n$  in a term of the form  $a^n$ . (pp. 22, 344.) See **power**.

**exponential equation:** An equation where the variable is part of an exponent (index). (p. 356, 384)

**exponential function:** A function of the general form  $y = b \cdot a^x$ , where  $a > 0$ . (p. 344, 384)

**factor theorem:** For a polynomial  $P(x)$ , if  $P(b) = 0$ , then  $(x - b)$  is a linear factor of  $P(x)$ . (p. 69)

**factorial  $n$ , ( $n!$ ):** The product  $n \times (n - 1) \times (n - 2) \times \dots \times 3 \times 2 \times 1$ ; with  $0! = 1! = 1$ . (p. 168, 193)

**fair game:** A game with an expected value of 0. (p. 453, 457)

**favourable outcome:** An outcome or event whose frequency is required. (p. 95, 101)

**finite:** Having a definite number of elements. (p. 77, 100)

**flow rate:** The rate of change of volume; the rate at which a liquid moves. (p. 240)

**function:** A  $1 : n$  relation; each  $x$ -value is in at most one pair  $(x, y)$ . (p. 114, 157)

**future value:** The value of an annuity or investment at compound interest at a future date. (p. 214, 218, 225, 226)

**general logarithm:** logarithm with a base other than  $e$  or 10. (p. 369, 384)

**general term:** A non-recursive rule giving the value of any term of a sequence. (p. 5, 8, 18, 202, 225)

**geometric probability distribution:** The distribution of probabilities of the number of failures before a success is achieved in multiple trials with the same individual probabilities. (p. 441, 457)

**geometric sequence, geometric progression (GP):** A sequence given by  $t_{n+1} = rt_n$ ; successive terms have a ratio of a constant  $r$ . (p. 198, 225)

**geometric series:** A sequence of sums  $S_n = t_1 + t_2 + \dots + t_n$  of a geometric sequence  $t_n$ . (p. 207, 225)

**global maximum:** The greatest value of a function over its entire domain. (p. 495, 505)

**global minimum:** The least value of a function over its entire domain. (p. 495, 505)

**gradient:** The slope of a line, given by  $m = \tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$  for a straight line with inclination  $\theta$  passing through  $(x_1, y_1)$  and  $(x_2, y_2)$ . (p. 124, 158, 430)

**gradient function:** The new function obtained by finding the gradient at every point on the graph of a function. (p. 397, 463, 504)

**greatest value:** The largest value of a function over a stated interval. (p. 494, 505)

**grid:** A table showing outcomes as a 2D combination of simpler or partial outcomes. (p. 90, 101)

**growth factor:** The factor by which a quantity increases in unit time; the factor  $(1 + i)$  for compound interest at the rate  $i$  with  $i$  negative for depreciation. (p. 205, 225)

**growth function:** An exponential function  $y = b \cdot a^x$ , where  $a > 1$ ;  $a$  is the growth factor. (p. 345, 384)

**highest common factor:** The largest term that divides into every term of an expression. (p. 35, 68)

**house percentage:** the proportion of money bet that casinos or other betting agencies win in the long run expressed as a percentage. (p. 454, 457)

**hybrid function:** A function with different rules for different parts of the domain. (p. 152, 159)

**hyperbola:** The shape of the function with the equation  $y = \frac{1}{x}$ . (p. 132, 158)

**identity:** An equation which shows the equivalence of different expressions involving the same variables such as  $(a + b)(a - b) \equiv a^2 - b^2$ . (p. 33)

**immediate annuity:** An annuity with payments made at the end of the interest period. (p. 218, 226)

**impossible:** An event that cannot occur, having a probability of 0. (p. 85, 101)

**inclination:** The angle made by a straight line with a horizontal line. (p. 124, 158)

**increasing function:** A function (or part of a function) that becomes greater as  $x$  increases. (p. 397)

**independent events:** Events  $A, B$  that do not affect each other's probabilities;  $P(A|B) = P(A)$ . (p. 182, 193)

**independent variable:** A variable whose variation does not depend on other variables; the first variable of an ordered pair. (p. 110, 157) See **dependent variable**.

**index (plural: indices):** The number at the top right of a power that shows the number of times the base is to be multiplied; the value  $n$  in a term of the form  $a^n$ . (p. 22) See **power**.

**indicial equation:** An equation where the variable is part of an exponent (index). (p. 356, 384)

**infinitely countable:** A countable set that is not finite. (p. 77, 100)

- inflation:** Decrease of the value of money over time; general increase of prices. (p. 216, 225)
- initial value:** The coefficient  $b$  of an exponential function; the value at  $x = 0$ . (p. 345, 384)
- instantaneous rate of change:** The rate of change calculated at a particular instant. (p. 247, 270, 285, 286, 424, 430, 462, 504)
- intersection:** The set or event  $A \cap B$  containing elements both  $A, B$ . (p. 78, 100)
- inverse proportion:** Two variables that increase or decrease in the same opposite ratio to each other. (p. 56, 69)
- leading term:** The term in a polynomial which contains the highest power. (p. 58)
- least value:** The smallest (least) value of a function over a stated interval. (p. 494, 505)
- limit of a function:** The value that a function 'approaches' as the independent variable (input) approaches some value. (p. 257)
- linear:** The relationship between variables is linear if the power of each is 1. (p. 58)
- linear combination:** An expression formed from a set of terms by multiplying each term by a constant and adding the results. (p. 412, 430)
- linear product:** Product that has no powers, roots, exponentials, etc. (p. 411)
- local maximum:** The greatest value of a function over an interval, particularly a short interval including the point with this value. (p. 480, 505)
- local minimum:** The smallest value of a function over an interval, particularly a short interval including the point with this value. (p. 480, 505)
- logarithm:** The exponent of a number expressed as a power to a particular base. (p. 365, 384)
- mantissa:** The non-leading zeros of a decimal or non-trailing zeros of a whole number; the part before the  $\times$  sign of a number in scientific notation. (p. 26, 68)
- marginal cost:** In economics, the cost of production of one more item. (p. 425, 430)
- marginal profit:** In economics, the profit made from the sale of one more item. (p. 425, 430)
- marginal revenue:** In economics, the revenue earned from the sale of one more item. (p. 425, 430)
- maximum:** The largest value. (p. 118, 157)
- minimum:** The smallest value. (p. 118, 137, 157, 159)
- monic:** A polynomial in which the coefficient of the leading term is 1. (p. 58, 69)
- mutually exclusive:** Sets or events with no elements in common. (p. 78, 100)
- $n$ th term:** The general term of a sequence, usually expressed in terms of the term number. (p. 4, 18, 202, 225)
- null factor law:** If the product of two terms  $x$  and  $y$  is zero then either one or both terms must be equal to zero. If  $xy = 0$  then  $x = 0$  or  $y = 0$  or  $x = y = 0$ . (p. 43, 69)
- null set:** The set with no elements. (p. 76, 100)
- odds:** The ratio of the amount to be won to the amount to be lost on a bet. (p. 453, 457)
- optimisation:** The process of finding an optimal (best) solution to a problem, usually for particular conditions. (p. 498)
- or:** Mathematically,  $A$  or  $B$  means at least one of  $A$  and  $B$  are true or happen. (p. 80, 100)
- order:** The leading power of a number; the power of 10 of a number in scientific notation. (p. 26, 68)
- ordered pair:** A pair of numbers in brackets;  $(x, y)$ . (p. 110, 157)
- ordinary annuity:** An annuity with payments made at the end of the interest period. (p. 218, 226)
- outcome:** An individual result of chance situation. (p. 80, 100)
- parabola:** The shape of the function  $y = x^2$ . (p. 47, 118, 157)
- Pascal's triangle:** Triangular arrangement in rows of numbers of combinations of  $n$  objects. (p. 167, 193)
- period:** The interval of the independent variable (often time) needed to complete one cycle of a periodic function. (p. 309, 337)
- periodic function:** A function that repeats the same values after a constant interval called the period. (p. 309, 337)
- phase shift:** The measure of how far a periodic function is horizontally to the right of the usual position. (p. 315)
- piecewise function:** A function with different rules for different parts of the domain. (p. 152, 159)
- point estimate:** An estimate of the probability of the value of a random variable being in a particular value. (p. 95, 437, 457)
- point of horizontal inflection:** A point on a curve function changes between concave up and concave down and has a horizontal tangent; point where the derivative is zero but there is no change of sign of the derivative. (p. 480, 505)
- point of inflection:** Point where a curve changes between concavity upwards and downwards. (p. 137, 159)
- polynomial:** An expression of the form  $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x^1 + a_0$ . (p. 58, 69, 415)

- power:** A repeated multiplication of the form  $a^n = a \times a \times a \times \dots$  to  $n$  terms, with base  $a$  and exponent (index)  $n$ . (p. 22, 365, 384)
- power rule:** The rule that states  $\frac{d}{dx}(ax^n) = anx^{n-1}$ . (p. 406, 430, 504)
- present value:** The value of an annuity at the present date; or investment needed now at compound to obtain an amount at some future date. (p. 214, 218, 225, 226)
- probability:** A number from 0 to 1 showing the mathematical chance that an event will occur. (p. 84, 101)
- probability function:** A function  $p(x)$  giving the probabilities of a discrete random variable. (p. 436, 457)
- product rule:** The differential calculus rule for the derivative of the product of two functions. (p. 504)
- proper subset:** A subset of  $A$  that excludes at least one element of  $A$ , shown as  $B \subset A$ . (p. 78, 100)
- quadratic:** A polynomial in which the leading term has a power of 2.  $P(x) = ax^2 + bx + c$ , where  $a$ ,  $b$  and  $c$  are constants and  $a \neq 0$ , is a quadratic. (p. 158)
- quadratic formula:** The formula for the solutions of a quadratic equation; given by  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  for  $ax^2 + bx + c = 0$ . (p. 50)
- quadratic function:** A function of the form  $f(x) = ax^2 + bx + c$ ; a polynomial of degree 2. (p. 118, 157)
- quadratic relation:** A relation of the form  $x = ay^2 + by + c$ . (p. 129, 158)
- quotient rule:** The differential calculus rule for the derivative of the quotient of two functions. (p. 504)
- radian:** The angle measure subtended at the centre of a circle by an arc equal in length to the radius of the circle; the circular angle measure that this produces. (p. 294) *See degree.*
- radical:** The mathematical operator  $\sqrt[n]{\phantom{x}}$  used to find the  $n$ th root of a number. For example,  $\sqrt[3]{8} = 2$ . (p. 23)
- random:** having no pattern; outcomes equally likely to occur; determined only by chance. (p. 84, 101)
- random variable:** A variable with numerical values determined by a chance experiment; usually shown by a capital letter. (p. 436, 457)
- range:** The set of second  $y$ -values of the ordered pairs  $(x, y)$  of a function or relation. (p. 110, 157)
- rate:** The quotient of two quantities, with units separated by / in the order of division. (p. 236)
- rate of change:** The value of the quotient of the change in one quantity and the change in the quantity the rate is 'with respect to'. (p. 236, 285)
- recursive rule:** An expression for a numbered rule in terms of the previous rule such as  $t_{n+1} = t_n + d$ . (p. 5, 18, 198)
- reducing interest loan:** A loan at compound interest with payments that exceed interest. (p. 221, 226)
- reference angle:** For a given angle in the unit circle, the acute angle that is the smallest angle between the terminal arm of the angle and the  $x$ -axis. (p. 304)
- reflection:** A transformation that reflects the graph of a given function across a line;  $-f(x)$  across the  $x$ -axis (vertical) or  $f(-x)$  across the  $y$ -axis (horizontal). (p. 146)
- relation:** A set of ordered pairs  $(x, y)$ ; shown by a list, rule, mapping or part of a Cartesian plane. (p. 110, 157)
- relative frequency:** The proportion of occurrence of a data item, given by  $\frac{\text{frequency}}{\text{total frequency}}$ . (p. 95, 101)
- remainder theorem:** When a polynomial  $P(x)$  is divided by  $(x - b)$  the remainder is  $P(b)$ . (p. 69)
- replacement:** Putting an object back before choosing another. (p. 88, 101)
- root:**  $b$  is a root of a function  $f(x)$  if  $f(b) = 0$ . (p. 45, 69)
- sample space:** The collection of all possible outcomes in a chance situation. (p. 80, 100)
- scientific notation:** A number written in the form  $b \times 10^n$  where  $1 \leq b < 10$  and  $n$  is an integer. For example,  $3.1 \times 10^{-2}$ . (p. 26, 68) *See mantissa, order.*
- secant:** A straight line that passes through a curve. (p. 242)
- sector:** Part of a circle that lies between two radii and the circumference. (p. 300)
- sequence:** An ordered list of numbers with a 1st number, 2nd number, ... and so on. (p. 4, 18)
- series:** A sequence made by adding terms of another sequence. (p. 11, 18, 207)
- set:** A well-defined collection of objects: you can tell whether or not an object is in the set. (p. 76, 100)
- significant figures:** The digits in a number that show how accurately the number has been written such as 38 000 has two significant figures. (p. 28)
- simplified tree diagram:** A tree diagram with similar branches combined. (p. 88, 101)
- simultaneous equations:** A set of equations in the same variables. (p. 149, 159)
- sine (sin):** For  $0^\circ \leq \theta \leq 90^\circ$ ; the ratio of the opposite side to the hypotenuse of the right-angled triangle containing  $\theta$ ; For all angles; the  $y$ -coordinate of the trigonometric point  $P(x, y)$  with angle  $\theta$  on a unit circle. (p. 295)

- sinusoidal:** A function that has the general shape of the sine or cosine function. (p. 328, 338)
- slope of a curve (at a point):** The slope of the tangent line drawn at that point. (p. 278)
- speed:** The rate of change of distance. (p. 240)  
See **velocity**.
- square numbers:** The numbers of dots in squares of dots with side lengths 1, 2, 3, ... (p. 16, 18)
- standard deviation:** A quantity that represents the spread of a random variable in the same units as the variable; given by  $\sigma_X = SD(X) = \sqrt{Var(X)}$ . (p. 448, 449, 457)
- standard triangle:** A right-angled triangle with a unit side, used to find exact trigonometric values of  $30^\circ$ ,  $45^\circ$  or  $60^\circ$ . (p. 300)
- stationary point:** A point where the gradient of a function is zero; a point where the derivative is zero. (p. 397, 463, 504)
- subset:** All the elements of a subset  $B$  of  $A$  are also in  $A$ , shown as  $B \subseteq A$ . (p. 78, 100)
- sum of a GP:** The sum of the terms  
 $S_n = t_1 + t_2 + \dots + t_n$  of a geometric sequence  $t_n$ . (p. 207, 225)
- sum of an AP:** The sum of the terms  
 $S_n = t_1 + t_2 + \dots + t_n$  of an arithmetic sequence  $t_n$ . (p. 11)
- sum to infinity:** The value that a convergent sum  $S_n$  approaches as  $n$  becomes large. (p. 212, 225)
- sum to  $n$  terms:** The sum of the first  $n$  terms of a sequence. (p. 11)
- table:** A table showing outcomes as a 2D combination of simpler or partial outcomes. (p. 90, 101)
- tangent<sup>1</sup> (tan):** For  $0^\circ \leq \theta \leq 90^\circ$ ; the ratio of the opposite side to the adjacent of the right-angled triangle containing  $\theta$ ; For all angles; the quotient of the  $y$ -coordinate and  $x$ -coordinate of the trigonometric point  $P(x, y)$  with angle  $\theta$  on a unit circle. (p. 295)
- tangent<sup>2</sup> (to a curve):** A straight line drawn so that it just touches a curve. (p. 247)
- term:** One of the numbers in a sequence. (p. 4, 18)
- translation:** A transformation that moves the graph of a given function. (p. 132)
- tree diagram:** A diagram showing successive stages of a compound event as branches. (p. 88, 101)
- trials:** The number of items in a data collection is called the number of trials. (p. 95, 101)
- triangular numbers:** The numbers of dots in triangles of dots with side lengths 1, 2, 3, ... (p. 16, 18)
- trigonometric function (circular function):**  
A function of an angle expressed as a ratio of two of the sides of a right-angled triangle containing the angle. (p. 295)
- turning point:** Point where a curve changes between rising and falling. (p. 118, 157)
- uniform:** Constant, particularly a random variable with a constant value. (p. 439)
- union:** The set or event  $A \cup B$  containing elements in at least one of  $A, B$ . (p. 78, 100)
- unit circle:** A circle of radius 1, particularly used for trigonometric ratios. (p. 303)
- universal set:** The set  $U$  or  $E$  containing all possible elements of sets in a particular situation. (p. 76, 100)
- variable:** A value in an equation or formula, represented by a letter which can vary. (p. 43)
- variance:** A quantity that represents the spread of a random variable; given by  $Var(X) = \sum (x - \mu)^2 \cdot p(x) = \sum x^2 \cdot p(x) - \mu^2$  for a discrete random variable  $X$ . (p. 449, 457)
- velocity:** The rate of change of displacement. (p. 240, 272, 286, 467) See **speed**.
- Venn diagram:** A diagram showing sets or events as circles in a rectangle representing the universal set or sample space. (p. 80, 101)
- vertex:** Turning point of a curve from a horizontal or vertical viewpoint. (p. 158)
- vertical dilation:** A dilation of the form  $af(x)$  that magnifies or reduces a graph vertically. (p. 132, 158)
- vertical line test:** A function is intersected in only one point by any vertical line. (p. 157)
- well-defined:** Theoretically able to distinguish from other values or members. (p. 76, 100)
- worth of money:** The interest rate used in calculations of annuities, future and present value. (p. 214, 225)
- $y$ -intercept:** Point(s) where the curve of a function or relation crosses the  $y$ -axis; the distance of this point from the origin. (p. 124, 158)
- zero:** Point(s) where the curve of a function or relation crosses the  $x$ -axis. (p. 119, 158)

## OVERVIEW

Nelson QMaths 11 & 12 has been rewritten and updated to cover the new Queensland senior courses: *Essential Mathematics, General Mathematics, Mathematical Methods and Specialist Mathematics.*

**Mathematical Methods** is a calculus-based course that can lead to working in STEM fields involving science, technology and engineering.



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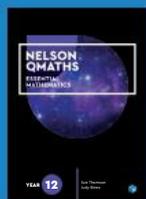
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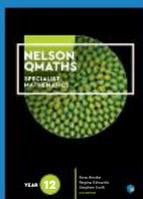
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## FEATURES

- Written by a successful Queensland author team of experienced teachers, including the leading authors of **Nelson Senior Maths for the Australian Curriculum**
- Each chapter begins with a chapter table of contents, syllabus subject matter and a **Terminology** word list
- Graded exercises are linked to worked examples and feature **Problem Solving** sections
- **Investigations** explore the syllabus in more detail through group work and modelling activities
- **Technology** sections promote ICT in the classroom, featuring Casio and Texas Instruments graphics calculator activities, spreadsheets and the Internet
- Each chapter ends with a **Chapter summary** and **Chapter review** exercise
- **Practice examinations** revise the skills and knowledge of previous chapters
- Syllabus grid, answers and glossary/index included

