MATHEMATICS METHODS

ATAR COURSE TEXT BOOK

YEAR 11 UNITS 1 & 2

O. T. LEE

ACADEMIC TASK FORCE

MATHEMATICS METHODS

YEAR 11 ATAR COURSE
UNITS 1 & 2

SECOND EDITION

A fully worked out Solution Manual for this book is available as a resource to Teachers at www.academicgroup.com.au

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About the Author
Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

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Preface

This book addresses the syllabus requirements of Units 1 and 2 of the Mathematics Methods course of Western Australia.

The use of CAS/graphic calculators is seamlessly integrated into the teaching and learning process. Questions have become more explicit in terms of the required methods and techniques. Knowledge of CAS/graphic calculator techniques empower students to appreciate the relative efficiencies (and accuracies) of *machine based* techniques against traditional pencil and paper techniques. However, the traditional pencil and paper techniques are the ones that convey the actual mathematical concepts and processes and form the backbone of this book. Machine based techniques are at best interpretative techniques.

The use of Hands-on-Tasks is continued in this book. These tasks allow students to conceptualise mathematical concepts on their own without being explicitly "taught". This promotes relational understanding rather than factual knowledge of mathematical concepts and ideas.

A fully worked out ***Solution Manual*** is available as a resource for teachers.

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01 Lines

1.1 Independent and Dependent Variables

- Let x and y be two variables.
- If x and y are related by an expression which describes y in terms of x, like:
  $y = x + 4$ or $y = x^2 + 2x + 5$ or $y = x^3 - 4$ or $y = 2^x$ or $y = 1/x$ …
  then, in general, x is said to be the *independent* variable and y is the *dependent* variable. The values taken by y depend on the values taken by x.
- If, however, we describe x in terms of y, like:
  $x = y - 4$, or $x = 4y^2$ or $x = 1/y$ or $x = 3y$ …
  then, y is the independent variable and x is the dependent variable.

1.2 Equation of Lines

1.2.1 Gradient-Intercept form: *y = mx + c*

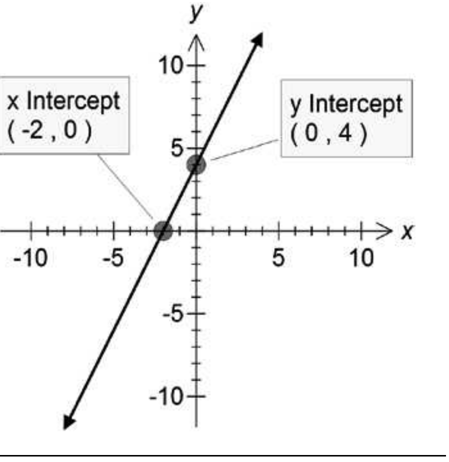
- Let x and y be two variables.
  Lines have equations of the form $y = mx + c$, where m and c are constants.
  - In sketching lines, the dependent variable is plotted horizontally and the independent variable is plotted vertically.
  - The constant m controls the steepness of the line and is referred to as the gradient of the line. The constant c denotes where the line crosses the vertical axis and is termed the vertical intercept of the line.
- Note the form of the equation of a line:
  Dependent variable = constant 1 × dependent variable + constant 2.

Example 1.1

Sketch the line with equation $y = 2x + 4$, indicating clearly the intercepts:

Solution:

Vertical intercept: (0, 4)
Horizontal intercept: $y = 0$
$\Rightarrow \quad 2x + 4 = 0 \quad \Rightarrow \quad x = -2$.
Hence $(-2, 0)$.



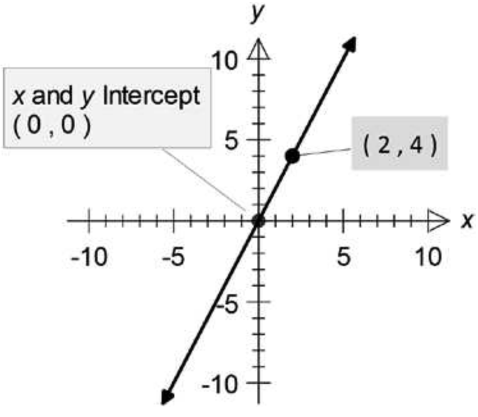
Note:
- *To sketch a line, only two points are required. Typically, the intercepts are used.*

Example 1.2

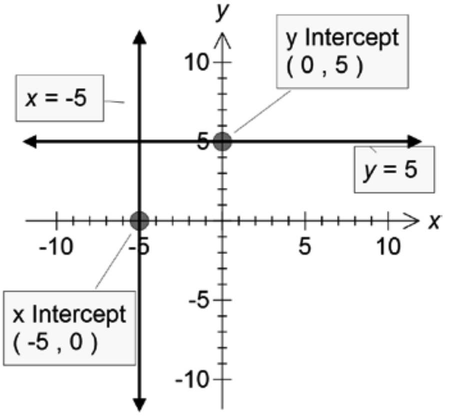
Sketch the lines with the following equations, indicating clearly the intercepts, if any:
(a) $y = 2x$ (b) $y = 5$ (c) $x = -5$

Solution:

(a) Two Points: (0, 0) and (2, 4)



(b) and (c)



Note:
- *In (a), the line passes through the origin, hence the intercepts coincide. To sketch this line, another point is required. In this case, a random convenient point of (2, 4) was chosen.*
- *In (b), the line y = 5 is a horizontal line.*
- *In (c), the line x = –5 is a vertical line.*

1.2.2 General form: $ax + by = c$

- Let x and y be two variables. Consider the expression $3x + 4y = 12$.
  This expression can be rewritten as $y = -\frac{3x}{4} + 3$.
- In other words, the expression $3x + 4y = 12$ is identical to $y = -\frac{3x}{4} + 3$.
  Since $y = -\frac{3x}{4} + 3$ represents the equation of a line, $3x + 4y = 12$ also represents the equation of a line.
- In general, the equation of a line may be written as $ax + by = c$, where a, b and c are constants. In this case, the identities of the independent and dependent variables are unclear and need to be considered in the context of the question.

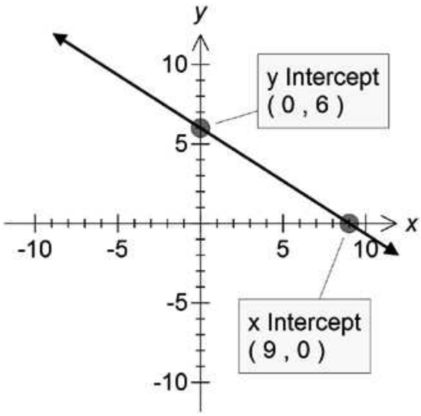
Example 1.3

Sketch the lines with the following equations, indicating clearly the intercepts, if any:

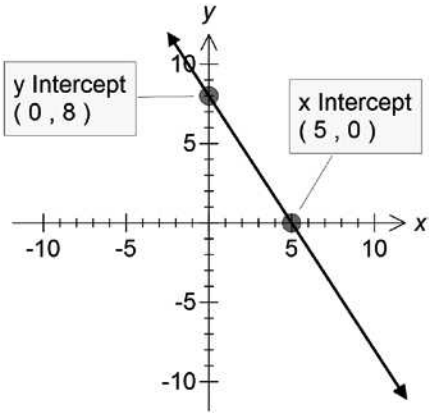
(a) $2x + 3y = 18$ (b) $\dfrac{x}{5}+\dfrac{y}{8}=1$

Solution:

(a) Vertical intercept: (0, 6)
Horizontal intercept: (9, 0)



(b) Vertical intercept: (0, 8)
Horizontal intercept: (5, 0)



Example 1.4

Find the gradient and coordinates of the vertical intercepts of each of the following lines:
(a) $y = 2x + 5$, independent variable is x (b) $x + 2y = 10$, independent variable is x
(c) $3p + 2q = 6$, independent variable is p (d) $3p + 2q = 6$, independent variable is q.

Solution:

(a) $y = 2x + 5$. Hence, the gradient is 2 and the vertical intercept has coordinates (0, 5) .

(b) As x is the independent variable, rewrite as: $2y = 10 - x \Rightarrow y = -\dfrac{x}{2} + 5$

Hence, the gradient is $-\dfrac{1}{2}$ and the vertical intercept has coordinates (0, 5)

(c) As the independent variable is p, rewrite as: $2q = 6 - 3p \Rightarrow q = -\dfrac{3p}{2} + 3$

Hence, the gradient is $-\dfrac{3}{2}$ and the vertical intercept has coordinates (0, 3) .

(d) As the independent variable is q, rewrite as: $3p = 6 - 2q \Rightarrow p = -\dfrac{2q}{3} + 2$

Hence, the gradient is $-\dfrac{2}{3}$ and the vertical intercept has coordinates (0, 2) .

Exercise 1.1

1. State the *exact* coordinates of the intercepts of each of the following:
   (a) $y = -2x + 3$
   (b) $2y = 4x + 16$
   (c) $x = y - 3$
   (d) $2x = 3y + 4$
   (e) $x + y = 7$
   (f) $2x + y = 14$
   (g) $y = 8$
   (h) $4x - 12 = 0$

2. Sketch the graphs of each of the following. Indicate clearly the *exact* coordinates of the intercepts, if any. [x is plotted on the horizontal axis]
   (a) $y = -x/2 + 4$
   (b) $4y = 3x + 12$
   (c) $3x = 5y + 15$
   (d) $3x + y = 15$
   (e) $2x + 5y - 20 = 0$
   (f) $3x - 24 - 8y = 0$
   (g) $6x + 12 = 0$
   (h) $4 - 2y = 0$
   (i) $\frac{x}{4} + \frac{y}{6} = 1$
   (j) $\frac{x}{2} - \frac{y}{3} = 1$

3. Find the gradient and coordinates of the vertical intercepts of each of the following lines, with the independent variable as indicated:
   (a) $y = -4x + 6$, x
   (b) $y = -4x + 6$, y
   (c) $y = \frac{(1-2x)}{3}$, x
   (d) $y = \frac{(1-2x)}{3}$, y
   (e) $2x + 5y = 10$, x
   (f) $2x + 5y = 10$, y
   (g) $\frac{u}{4} + \frac{v}{11} = 1$, u
   (h) $\frac{u}{4} + \frac{v}{11} = 1$, v

1.2.3 Gradient of a Line

- Through two given points, only one line can be drawn.
- Hence, to find the gradient or slope of a line, two convenient points (x_1, y_1) and (x_2, y_2) are chosen.
- The gradient is calculated through the formula:

$$\text{gradient } m = \frac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$$

$$= \frac{y_1 - y_2}{x_1 - x_2}.$$

Example 1.5

Find the gradient of the line passing through the following pairs of points:

(a) (5, 9) and (1, 5)　　(b) (5, 3) and (2, 7)

(c) (3, –5) and (5, –5)　　(d) (4, 8) and (4, 2)

Solution:

(a) gradient = $\frac{9-5}{5-1} = 1$　　(b) gradient = $\frac{3-7}{5-2} = \frac{-4}{3}$

(c) gradient = $\frac{-5-(-5)}{3-5} = 0$　　(d) gradient = $\frac{8-2}{4-4}$ = indeterminate

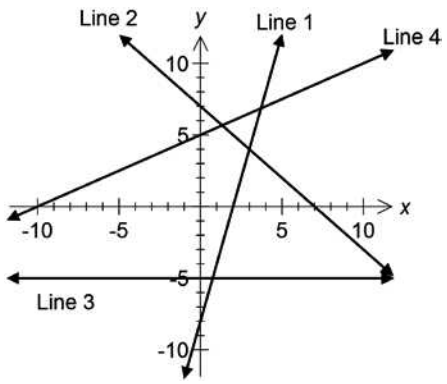
Note:

- *In part (d), the two points (4, 8) and (4, 2) lie on a line parallel to the y-axis. As such, the line on which they lie has an indeterminate gradient.*

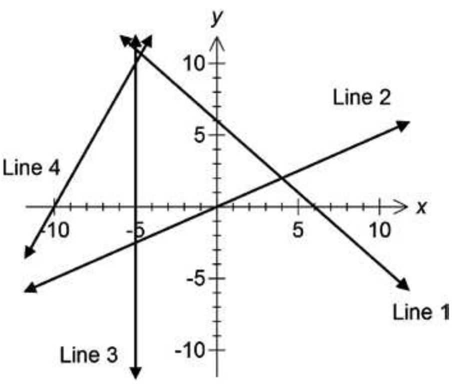
Exercise 1.2

1. Find the gradient of the following lines:

(a)



(b)



2. Find the gradient of the line joining each of the following pairs of points:
   (a) (1, 5) and (–2, 4)　　(b) (–2, 5) and (2, 1)　　(c) (4, –4) and (3, 3)
   (d) (2, 4) and (–4, –9)　　(e) (–3, –2) and (–4, –5)　　(f) (–8, –9) and (–2, –11)

3. The points A, B and C have coordinates (0, 0), (2, 2) and (–4, –4) respectively.
   (a) Find the gradients of the line segments AB and BC.
   (b) Hence, determine if the points A, B and C are *collinear* (i.e. lie on the same line).

4. Determine mathematically whether the following points are collinear.
   (a) (0, 3), (1, 5) and (4, 11)　　(b) (1, 4), (3, 5) and (5, 7)

5. Find the value(s) of *k* if the following points are collinear.
   (a) (1, 3), (5, 10) and (8, *k*)　　(b) (–2, –4), (*k*, 4) and (6, 10).

1.2.4 Equation of a line passing through a fixed point with a given gradient

- The equation of a straight line with gradient *m*, passing through a fixed point with coordinates (*h*, *k*) is given by:

$$y - k = m(x - h)$$

Example 1.6

Find the equation of each of the following lines:

(a) gradient = 4, passing through the point with coordinates (5, –1)

(b) passing through the points with coordinates (10, 6) and (4, 10).

Solution:

(a) Equation of line is $y - (-1) = 4(x - 5)$

$\Rightarrow\ y = 4x - 21$

(b) Gradient of line = $\dfrac{6-10}{10-4} = -\dfrac{2}{3}$.

Take (10, 6) as the point of passage.

Hence, $y - 6 = -\frac{2}{3}(x-10)$

$\Rightarrow\ y = -\frac{2}{3}x + \frac{38}{3}$

> ***Alternative Solution:***
> Since the gradient of the line is 4, the line has equation of the form $y = 4x + c$.
> This line passes through (5, –1).
> Hence, when $x = 5$, $y = -1$.
> Substitute into the equation:
> $-1 = 4(5) + c \Rightarrow c = -21$
> Therefore the equation of the line is
> $y = 4x - 21$.

Exercise 1.3

1. Find the equation of the lines with the given gradients and the coordinates of a given point on the line:
   (a) gradient = 2 ; (0, 4)
   (b) gradient = 2 ; (0, –4)
   (c) gradient = 1/5 ; (1, 5)
   (d) gradient = 1/5 ; (–1, –4)
   (e) gradient = –2/7 ; (6, –5)
   (f) gradient = 8/5 ; (12, 8)

2. Find the equation of the line passing through each of the following pairs of points:
   (a) (0, 0) and (2, 8)
   (b) (–5, 8) and (0, 0)
   (c) (1, 4) and (3, 2)
   (d) (5, 1) and (2, 7)
   (e) $(\frac{1}{2}, -1)$ and $(4, \frac{3}{2})$
   (f) $(4, \frac{3}{5})$ and $(\frac{1}{3}, 2)$

3. The straight line $y = mx + 7$ passes through the point (5, 4). Find the value of *m*.

4. The straight line $2x + ky = 20$ passes through the point (6, 9). Find the value of *k*.

5. The straight line $y = 2x - 9$ passes through the point (*k*, *k*). Find the value of *k*.

6. The straight line $3x + 7y = 21$ passes through the point (*k*, 2*k*). Find the value of *k*.

*7. Find the values of a, b and c, given that the straight line $ax + by = c$ passes through:
(a) (0, 7) and (2, 0) (b) (–1, 4) and (2, 5).

*8. Find the values of a and b, given that the straight line $\frac{x}{a}+\frac{y}{b}=1$ passes through:
(a) (5, 0) and (0, 7) (b) (0, –3) and (–4, 0)

9. Given the line with equation $\frac{x}{a}+\frac{y}{b}=1$, find the relationship between the values of a and b and the intercepts of the line.

1.3 Parallel and Perpendicular Lines

- Two lines are *parallel* if the gradients are identical.
  If two lines have identical gradients then the two lines are *parallel*.
- Two lines are *perpendicular* if the product of the gradients is *negative one*.
  If the product of the gradients of two lines is *negative one*, then the two lines are *perpendicular*.

Example 1.7

Determine if the following pairs of lines are parallel, perpendicular or otherwise:

(a) $y = 10x + 5$ and $y = 5 - 0.1x$ (b) $3x + 4y = 36$ and $4x + 3y = 12$

(c) $\frac{x}{2}+\frac{y}{4}=1$ and $y = -2x - 10$ (d) $y - 5 = 0$ and $x - 5 = 0$

Solution:

(a) The line $y = 10x + 5$ has a gradient of 10. The line $y = 5 - 0.1x$ has a gradient of -0.1 .
Product of the gradients = $10 \times -0.1 = -1$. Hence, the two lines are perpendicular.

(b) The lines $3x + 4y = 36$ and $4x + 3y = 12$ have gradients of $-\frac{3}{4}$ and $-\frac{4}{3}$ respectively.
Product of the gradients = $-\frac{3}{4}\times-\frac{4}{3} \neq -1$.
Hence, the two lines are neither parallel nor perpendicular.

(c) The line $\frac{x}{2}+\frac{y}{4}=1$ has a gradient of -2. The line $y = -2x - 10$ has a gradient of -2 .
The two lines have identical gradients. Hence, the two lines are parallel.

(d) The line $y - 5 = 0$ is a horizontal line. The line $x - 5 = 0$ is a vertical line.
Hence, the two lines are perpendicular.

Example 1.8

Find the equation of the line passing through the point (2, 1) and:

(a) parallel to the line $y = x + 2$ (b) perpendicular to the line $2x + 3y = 6$

Solution:

(a) The line $y = x + 2$ has gradient 1.
Since the required line is parallel to $y = x + 2$, the gradient of the required line is 1.
The required line passes through (2, 1).
Hence, the equation of the required line is $y - 1 = 1(x - 2) \Rightarrow y = x - 1$.

(b) The line $2x + 3y = 6$ has gradient $-\frac{2}{3}$.
Since the required line is perpendicular to $2x + 3y = 6$,
the gradient of the required line is $\frac{3}{2}$. The required line passes through (2, 1).
Hence, the equation of the required line is $y - 1 = \frac{3}{2}(x-2) \Rightarrow y = \frac{3}{2}x - 2$.

Example 1.9

Find the equation of the line passing through the point (2, 1) and the point of intersection between the lines $y = 2x + 1$ and $3x + y = 6$.

Solution:

For the point of intersection between $y = 2x + 1$ and $3x + y = 6$:
Substitute $y = 2x +1$ into $3x + y = 6$;

$$3x + (2x + 1) = 6$$
$$x = 1 \Rightarrow y = 3$$

That is, the point of intersection of the two given lines is (1, 3).

Therefore, the required line passes through (2, 1) and (1, 3).
The gradient of the required line is -2 .
Hence, the equation of the line is

$$y - 1 = -2(x - 2)$$
$$y = -2x + 5$$

Exercise 1.4

1. Determine if the following pairs of lines are parallel, perpendicular or otherwise:
   (a) $y = x + 3$ and $y = 4 + x$
   (b) $y = -x + 4$ and $y = x + 1$
   (c) $y = 5 + 2x$ and $y = -\frac{1}{5} + 2x$
   (d) $y = 3x + 5$ and $y = \frac{1-x}{3}$
   (e) $2x + 5y = 10$ and $2x + 5y = 12$
   (f) $3x + 8y = 24$ and $-8x + 3y = 24$
   (g) $2(x - 1) = 0$ and $y = 5$
   (h) $\frac{x}{2} + \frac{y}{5} = 1$ and $\frac{x}{5} + \frac{y}{2} = 1$

2. Find the equation of the line passing through the given point and parallel to the given line:
   (a) (0, 3); $y = 2x + 4$
   (b) (−1, 1); $2x + 3y = 6$
   (c) (4, −1); $\frac{x}{2} + \frac{y}{3} = 1$
   (d) (−2, 4); $x = 3$

3. Find the equation of the line passing through the given point and perpendicular to the given line:
   (a) (1, 4); $y = -x/2 + 1$
   (b) (−1, 3); $3x + 5y = 15$
   (c) (3, 1); $\frac{x}{5} + \frac{y}{6} = 1$
   (d) (4, 6); $y = 3$

4. The line $5x + 3y = 15$ meets the x-axis at P. Find the equation of the line through P perpendicular to $5x + 3y = 15$.

5. Find the equation of the line passing through (1, 3) and the point of intersection between $2x + y = 10$ and $4x + 3y = 12$.

6. Find the equation of the line passing through (2, 5) and the point of intersection between $5x + 4y = 19$ and $3x + 7y = 16$.

7. The lines $y = x + 4$ and $y = 2x + 3$ meet at A. The lines $0.01x + 0.03y = 0.14$ and $0.05x - 0.02y = 0.02$ meet at B. Find the equation of the line passing through A and B.

8. The lines $2x + y = 5$ and $x + 2y = 4$ meet at the point P. Find the equation of the line through P and: (a) parallel (b) perpendicular to the line $5x + 3y = 15$.

9. The lines $y = -2x + 5$ and $x + 2y = 4$ meet at the point Q. Find the equation of the line through Q and: (a) parallel (b) perpendicular to the line $\frac{x}{2} + \frac{y}{5} = 20$.

10. Determine if the following lines meet at the same point. Give the coordinates of this common point, where it exists.
    (a) $y = 2x + 1$, $y = 3x - 3$, $y = -2x + 17$
    (b) $y = -x + 4$, $2x + 3y = 10$, $y = x + 3$

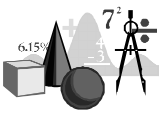
1.4 Mid-Points and Distances

- Consider the points A and B with coordinates (x_1, y_1) and (x_2, y_2) respectively.
- The distance between A and B is given by:

$$s = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- The mid-point of the line segment connecting A and B has coordinates:

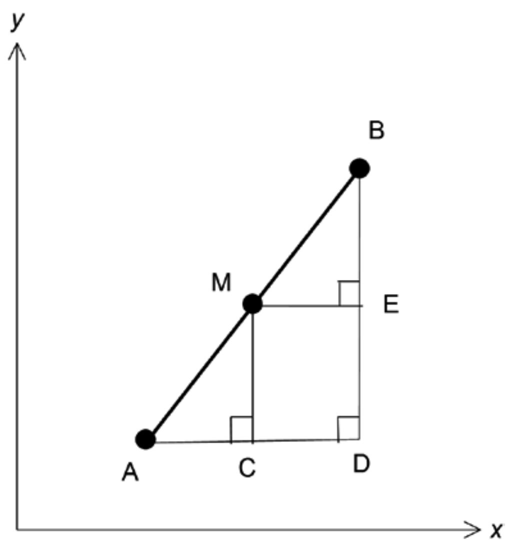
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Hands On Task 1.1

In this task we will examine the processes involved in arriving at the two results quoted above.

1. In the accompanying diagram, the points A and B have coordinates (x_1, y_1) and (x_2, y_2) respectively. ΔADB is a right triangle such that AD is parallel to the x-axis and DB is parallel to the y-axis.
   (a) Find the length of AD.
   (b) Find the length of DB.
   (c) Hence, use Pythagoras' Theorem to show that AB = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.



2. Further, the points M, C and E are respectively the mid-points of AB, AD and DB. Clearly, CM is parallel to DB and the y-axis. Also, ME is parallel to AD and the x-axis.
   (a) Find the lengths of AC and DE.
   (b) Show that the x-coordinate of C is $\dfrac{x_1 + x_2}{2}$.
   (c) Find the y-coordinate of E.
   (d) Hence, deduce that the coordinates of M, the mid-point of AB is $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

3. The two results quoted above are hence true for A and B in the first quadrant. Show that these two results are true for A and B located anywhere in the x-y plane.

Example 1.10

The line L passes through the points A and B with coordinates (–2, 4) and (6, 10) respectively. Find:

(a) the distance between A and B

(b) the coordinates of M, the mid-point of AB

(c) the coordinates of the point C, on line L, which is as far from B as A is to B

(d) the equation of the perpendicular bisector of AB (that is, the line perpendicular to AB passing through its mid-point M).

Solution:

(a) Distance AB = $\sqrt{(6--2)^2+(10-4)^2}$ = 10.

(b) M has coordinates $\left(\frac{-2+6}{2}, \frac{4+10}{2}\right) \equiv (2, 7)$.

(c) Let coordinates of C be (h, k). Since CB = AB, B must be the mid-point of AC.

Hence, B as the mid-point of AC has coordinates $\left(\frac{-2+h}{2}, \frac{4+k}{2}\right)$.

But B has coordinates (6, 10).

Therefore, $\frac{-2+h}{2} = 6 \Rightarrow h = 14$ and $\frac{4+k}{2} = 10 \Rightarrow k = 16$.

Hence, C has coordinates (14, 16).

(d) The line segment AB has gradient = $\frac{10-4}{6--2} = \frac{3}{4}$.

Hence, the line perpendicular to AB has gradient $-\frac{4}{3}$.

Equation of perpendicular bisector of AB: $y - 7 = -\frac{4}{3}(x-2)$

$$y = -\frac{4}{3}x + \frac{29}{3}.$$

Exercise 1.5

1. Given the points A and B, find the distance between A and B. Find also, the coordinates of the mid-point of the line joining A to B.
   (a) A (4, 8) & B (10, 16)
   (b) A (–10, 15) & B (5, –10)
   (c) A $(7k, -3k)$ & B $(2 + k, 3k - 2)$
   (d) A $(t + 2, 2t - 4)$ & B $(t + 12, t + 2)$.

2. M is the mid-point of the line segment joining A and B. Find the coordinates of B. Hence, or otherwise, find the distance between A and B.
   (a) A (6, 12) & M (–2, –8)
   (b) A (–7, –2) & M (–11, –17)
   (c) A (8, –20) & M (12, 14)
   (d) A $(3k + 4, -2k - 3)$ & M $(k + 4, -2)$

3. The line L passes through the points A (5, 13) and B (–3, 3). Find:
   (a) the distance between A and B
   (b) the coordinates of M, the mid-point of AB
   (c) the coordinates of the point C, on line L, which is as far from B as A is to B
   (d) the equation of the line through M that is parallel to the line $3x + 4y = 12$.

4. The line L passes through the points A (–16, 2) and B (4, –10). Find:
   (a) the distance between A and B
   (b) the coordinates of M, the mid-point of AB
   (c) the coordinates of the point C, on line L, which is as far from A as A is to B
   (d) the equation of the line through C that is perpendicular to the line $\frac{x}{2} + \frac{y}{3} = 1$.

5. Find the equation of the perpendicular bisector of the line joining the points A and B.
   (a) A(1, 4) & B (7, 2)　　(b) A (–1, 6) & B (7, –4)

6. The points P and Q have coordinates (4, 2) and (8, k) respectively. The perpendicular bisector of the line PQ has equation $x + 2y = 18$. Find the value(s) of k.

7. The points P and Q have coordinates (k, 10) and (4,–14) respectively. The perpendicular bisector of the line PQ has equation $x - 3y = 6$. Find the value(s) of k.

8. The points A (2, 4) and B lie on the line $x + 2y = 10$. The perpendicular bisector of AB has equation $2x - y = 5$. Find the coordinates of the point B.

9. The points A and B (–2, 2) lie on the line $-4x + y = 10$. The perpendicular bisector of AB has equation $x + 4y = -28$. Find the coordinates of the point A.

*10. The perpendicular bisector of the line joining the point S (2, 2) with the point T has equation $x + y = 8$. Find the coordinates of M, the mid-point of ST.
[Hint: Let coordinates of point T be (α, β).]

*11. The perpendicular bisector of the line joining the point S (9, 4) with the point T has equation $2x + y = 7$. Find the coordinates of M, the mid-point of ST.

1.5 Linear Relationships

- Let x and y be two variables.
- If the relationship between x and y is expressed in the form $ax + by = c$ where a, b and c are constants, then:
  - we say that x and y are linearly related
  - the graph of y against x (or x against y) will be a straight line.

1.5.1 Direct Proportion

- Let x and y be two variables.
- Consider the following table of values for x and y.

x	0	1	2	3	4	5	6
y	0	5	10	15	20	25	30

- Clearly, the value of y increases by 5 units for each unit increase in the value of x. The rule for y in terms of x is $y = 5x$.
- y is said to be directly proportional to x. This is expressed symbolically as $y \propto x$.
- In general, if y is directly proportional to x:
  - $y \propto x \quad \Leftrightarrow \quad y = kx$ where the constant or proportionality $k > 0$.
  - y is linearly related to x.
  - the graph of y against x (or x against y) will be a straight line.
- Note that if y is linearly related to x, y need not necessarily be in direct proportion to x. y is in direct proportion to x if only if y is linearly related to x in the form $y = kx$ where $k > 0$.

Example 1.11

Consider the following table of values.

x	0	1	2	3	4	5	6
y	5	10	15	20	25	30	35

Determine with reasons if y is directly proportional to x.

Solution:

The algebraic relationship between x and y is clearly $y = 5x + 5$.
Hence, the linear relationship between x and y is not of the form $y = kx$.
Therefore, y is not directly proportional to x.

Note:

- *y is not directly proportional to x, but y is directly proportional to $(x + 1)$*

Example 1.12

Consider the following table of values.

x^2	0	1	2	3	4	5	6
y	0	2	4	6	8	10	12

Determine with reasons if y is directly proportional to x^2.

Solution:

The algebraic relationship between x and y is clearly $y = 2x^2$.
Hence, the linear relationship between x^2 and y is the form $y = kx^2$.
Therefore, y is directly proportional to x^2.

Note:

- *y is not directly proportional to x. But y is directly proportional to x^2.*

1.5.2 Applications involving linear relationships

Example 1.13

Kate, a plumber, charges her customers \$20 per 15 minutes or part thereof. In addition, she charges every customer a flat rate call out fee of \$50. Find:
(a) the charge for a job that took 2 hours to complete.
(b) the charge for a job that took 100 minutes to complete.
(c) how long Kate has to work on a single job to earn at least \$500.

Solution:

(a) When $t = 2$ hours, $C = 50 + 20 \times (2 \times 4)$
$= \$210$

(b) When $t = 100$ minutes, $t = 1$ hour and 40 minutes rounded to 1 hour and 45 minutes.
Hence, $C = 50 + 20 \times (7)$
$= \$190$

(c) $50 + 80t \geq 500$
$t \geq 5.625$ rounded to 5.75 hours
Hence, Kate has to work for at least 5 hours 45 minutes.

Exercise 1.6

1. Consider the following table of values:
Determine with reasons if y is directly proportional to: (a) x (b) $x + 1$:

x	0	1	2	3	4	5	6
y	2	4	6	8	10	12	14

2. Consider the following table of values.
Determine with reasons if y is directly proportional to: (a) x (b) $x - 2$.

x	0	1	2	3	4	5	6
y	4	2	0	−2	−4	−6	−8

3. Consider the following table of values.
Determine with reasons if y is directly proportional to: (a) $\sqrt{x}$ (b) $\sqrt{x} - 1$.

$\sqrt{x}$	0	1	2	3	4	5	6
y	1	3	5	7	9	11	13

4. Consider the following table of values. Determine with reasons if:

$\frac{1}{x}$	1	2	3	4	5	6	7
y	4	8	12	16	20	24	28

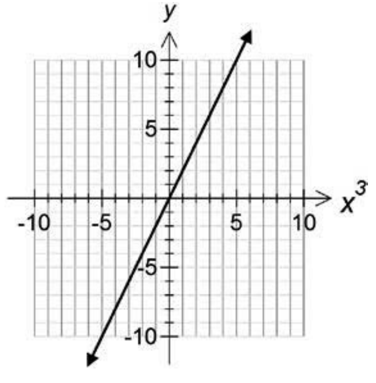
(a) y is directly proportional to $\frac{1}{x}$ (b) x is directly proportional to $\frac{1}{x}$.

5. Consider the following table of values.
Determine with reasons if y is directly proportional to: (a) x (b) x^2.

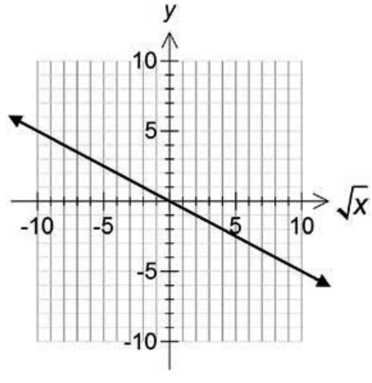
x	0	1	2	3	4	5	6
y	0	−1	−4	−9	−16	−25	−36

6. In each of the following graphs, find the algebraic relationship between the variables:

(a)



(b)



7. In an experiment to find the relationship between "the time between the sighting of a lightning flash and the hearing of the thunder clap" the following data was collected.

Distance from observer to lightning flash (d km)	2	5	10
Time (t seconds)	6	15	30

(a) Plot the graph of d against t.
(b) Find an algebraic relationship between d and t.
(c) Estimate the value of t when $d = 90$ km and discuss the validity of your answer.

8. The table below shows some data on a, the age of a child (in months) and v, the number of words in the child's vocabulary.

Age (a months)	18	24	30	36
Number of words (v)	180	540	900	1260

(a) Find an algebraic relationship between v and a
(b) Use this model to determine the age when a child's vocabulary first starts to build up. Discuss the validity of your answer.
(c) Determine if it would be valid to use this model for a child who is 12 months old. Justify your answer.
(d) How old would a child have to be to have a vocabulary of at least 10 000 words?
(e) Use this model to determine the number of words a 50 year old person would have in his vocabulary. Comment on your answer.

9. Under controlled conditions an entomologist models N, the number of cricket chirps (per minute) against temperature C (in degrees Celsius) using the formula:

$$C = 2.4N + 4$$

(a) Rewrite this formula with C as the independent variable.
(b) On average, what is the change in the number of chirps per minute for every twelve degrees Celsius rise in temperature?
(c) Comment on the validity of the model when $C = 0$.
(d) Find N when $C = 100$ and comment on your answer.
(e) Under a different set of conditions, the entomologist uses the model $C = 1.8N + 6$.
    (i) Under which set of conditions are the crickets more responsive to changes in temperature? Justify your answer.
    (ii) At which temperature would the number of chirps per minute be the same?

10. Granny, a recreational parachutist, jumps out of a light aircraft at a given height. Her downward vertical speed (v ms^{-1}) t seconds after she jumps out of the plane is given by: $v = 0.5 + 0.91t$.
(a) With what vertical speed did Granny jump out of the plane?
(b) What is the change in Granny's vertical speed over a five second interval?
(c) How long will it take Granny to reach a vertical speed of 60 kmh^{-1} ?
(d) Use this model to find Granny's vertical speed (in kmh^{-1}) after 2 minutes. Comment on the validity of your answer.

11. The cost of printing a children's storybook is made up of a fixed cost (cost due to rental, insurance, etc) and a variable cost which is dependent on the number of books printed. The cost C (\$) is given by: $C = 5\,000 + 7.50n$ where n is the number of books printed.
    (a) What is the fixed cost of printing the books?
    (b) What is the cost of printing 10 000 copies of the book?
    (c) What is the average cost per book if 10 000 copies are printed?
    (d) What is the average cost per book if n copies are printed?

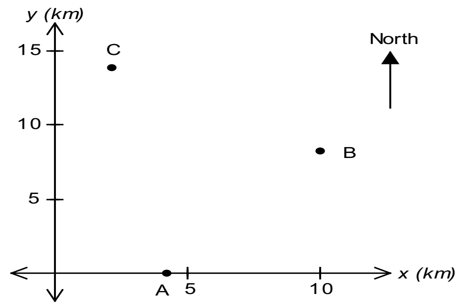
12. Debbie runs a home based "hand-painted" greetings card business. The cost C (\$) of producing n cards is given by $C = 1.50n + 500$.
    (a) If 200 cards are produced and the selling price of each card is \$8.00, find an expression for the profit P (\$) if (i) 150 copies and (ii) k copies were sold .
    (b) If 1 000 cards are produced and the cards are sold for \$5.00 each, find the number of cards that need to be sold for Debbie to:
    (i) break-even (ii) make a profit of at least \$500
    (c) Find an expression for the average cost A (\$) of producing n cards. What is the average cost per card for very large production runs?

13. WestByte produces high density DVD disks. The disks cost \$0.80 each with total fixed costs of \$20 000. Disks are sold in packs of 10 at \$15 per pack.
    (a) Given that ($n \times 10$) disks were produced and sold, find an expression, in terms of n, for (i) revenue R (\$) and (ii) profit P (\$) .
    (b) Given that ($n \times 10$) disks were produced and 80% of these are expected to be sold.
    (i) Find an expression, in terms of n, for the expected profit P (\$) .
    (ii) Find the number of disks that need to be produced to break-even.
    (iii) Find the number of disks that need to be produced to make a profit of \$10 000.

14. Ron, an electrician, charges a call-out fee of \$80. He charges his customers \$60 per hour, in lots of half-hours or part thereof.
    (a) Find the charge C (\$) for a job that took 3 hours to complete.
    (b) What is the average cost per hour if a job takes 3 hours to complete?
    (c) Sam, who is also an electrician, charges \$100 per hour, in lots of half-hours or part thereof. For what jobs (determined by the length of time to complete the jobs) would it be more economical to call Sam? Justify your answer mathematically.

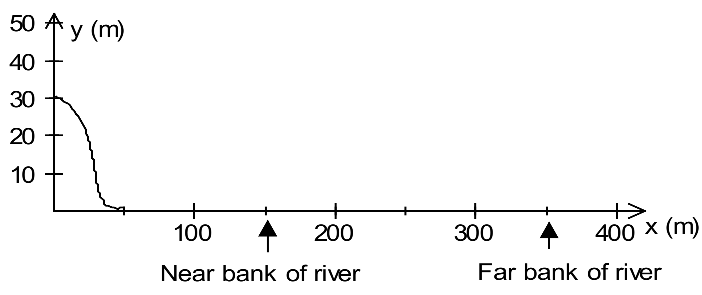
15. Claire is a computer repair specialist and charges a \$100 repair-assessment fee, where she assesses the amount of time needed (and the parts needed) to repair a faulty computer. For performing the required repairs she charges a labour fee of \$80 per hour, charged in lots of 15 minutes or part thereof. "Parts" are charged separately.
    (a) Claire takes 4.5 hours to complete the repairs on Richard's computer system. The cost of replacement parts amounted to \$650.00. What was the total charge?
    (b) The cost K (\$) to Claire for a job that takes t hours to complete is given by $K = 20 + 1.20t$. On average, Claire makes a profit of 15% on the cost of parts supplied and installed. Calculate the profit made by Claire on the repairs to Richard's computer.

16. The accompanying diagram shows the location of landing beacons that guide aeroplanes onto the main runway. Beacons A, B and C are located at (4, 0), (10, 8) and (2, 14) respectively. Aeroplanes which land at the airport are required to fly along the path AB and then turn into the path BC which leads to the main runway.



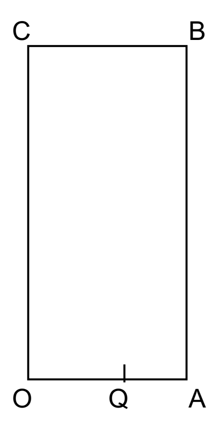
(a) Find the distance between the beacons A and B, and B and C.
(b) Determine if a school located at (5.5, 2) is directly beneath the flight path of planes attempting to land at this airport. Show your method clearly.
(c) Show that planes attempting to land have to make a 90° turn from AB to BC.

17. Louise owns a riverside property that overlooks a vacant block and a river. The accompanying diagram shows a cross-section view of the location of the property. From Louise' patio at (0, 30) she has a clear view of the near and far banks of the river, with coordinates (150, 0) and (350, 0) respectively.



(a) Find the equation of the line of sight from the patio to:
 (i) the near bank of the river (ii) the far bank of the river .
(b) A tree (which grows to a maximum height of 30 m) is planted at (50, 0) and grows at a rate of 1 m per year. How long (to the nearest month) will it be before the tree obstructs the view of the near bank?
(c) An apartment block with the highest point at (100, 20) is built on the vacant block.
 (i) Verify mathematically that the line of sight to the near bank will be obstructed.
 (ii) What would be the nearest point on the river that would be unobstructed by the apartment block?

18. The 4 corners O, A, B and C of a billiard table are located at (0, 0), (5, 0), (5, 10) and (0, 10) respectively. A ball is struck from point Q with coordinates (3,0) and moves along the path $y = x - 3$. The ball hits the side AB at R and is "reflected" across the table to point S on another side of the table, and from S the ball is "reflected" to T on yet another side of the table. Assume that the table surface and the sides are smooth and the ball is "reflected" away from each side with the same angle it strikes the side.



(a) Find the coordinates of R.
(b) Find the equation of the path RS.
(c) Find the coordinates of S.
(d) A second ball is located at (1, 8). Determine mathematically if the first ball will strike the second ball.

02 Quadratics

2.1 Quadratics

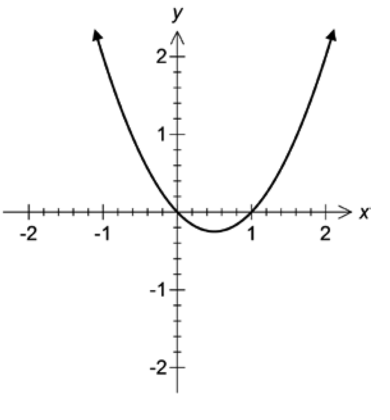
2.1.1 Quadratic Relationships

- Let x and y be two variables.
- If the relationship between x and y is expressed in the form $y = ax^2 + bx + c$ where a, b and c are constants, then:
  - we say that x and y have a *quadratic* relationship
  - the graph of y against x will be *parabolic* in shape.

2.1.2 Features of quadratic curves

- Quadratics curves have equations of the form:
  - $y = 2x^2 + 2x + 3$
  - $y = 2(x + 3)(3 - 5x)$
  - $y = 2(x - 1)^2 + 4$
- In general, quadratics have equations of the form:
  - $y = ax^2 + bx + c$ general form
  - $y = k(x - a)(x - b)$ factored form
  - $y = k(x - p)^2 + q$ completed square form
- All quadratic curves have the following features:
  - a turning point which can either be a maximum or a minimum turning point
  - a line of symmetry or an axis of symmetry [the turning point lies on the line of symmetry]
  - a vertical intercept.

  However, *not all quadratics curves have horizontal intercepts*.
- A quadratic curve has a minimum turning point if the x^2 term has a positive coefficient and a maximum turning point otherwise.
- For example, the accompanying diagram shows the graph of the curve $y = x^2 - x$.
  The curve has:
  - a minimum turning point at (0.5, –0.25)
  - an axis of symmetry with equation $x = 0.5$ (the parabola is symmetrical about this line and the turning point lies on this line)
  - a vertical intercept at (0,0)
  - horizontal intercepts at (0,0) and (1,0).



- For the quadratic curve with equation $y = ax^2 + bx + c$:
  - the turning point occurs at $x = -\dfrac{b}{2a}$
  - the y-intercept is at $(0, c)$.
- For the quadratic curve with equation $y = k(x - p)(x - q)$:
  - the x-intercepts are located at $(p, 0)$ and $(q, 0)$
  - the turning point occurs at the x-value corresponding to the midpoint of the x-intercepts.
- For the quadratic curve with equation $y = k(x - p)^2 + q$:
  - the turning point has coordinates (p, q).

Example 2.1

Use an algebraic method to find the coordinates of the turning point, and state the nature of this turning point for: (a) $y = 2x^2 + 8x - 5$ (b) $y = (x - 2)(4 - x)$ (c) $y = 1 - (x + 5)^2$

Solution:

(a) The coefficient of the x^2 term is positive.

Hence, the curve has a minimum turning point at $x = -\dfrac{8}{2(2)} = -2$.

Substitute into the equation: $y = 2(-2)^2 + 8(-2) - 5 = -13$

Hence, the curve has a minimum turning point at $(-2, -13)$.

(b) The coefficient of the x^2 term is negative.

x-intercepts are at $x = 2$ and $x = 4$.

Hence, the curve has a maximum turning point at $x = \dfrac{2+4}{2} = 3$.

Substitute into the equation: $y = (3 - 2)(4 - 3) = 1$

Hence, the curve has a maximum turning point at $(3,1)$.

(c) The coefficient of the x^2 term is negative.

Hence, the curve has a maximum turning point at $(-5, 1)$.

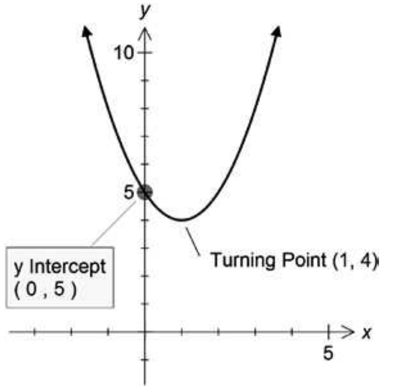
Example 2.2

Without using a calculator, sketch each of the following, indicating the coordinates of the turning point and intercepts:

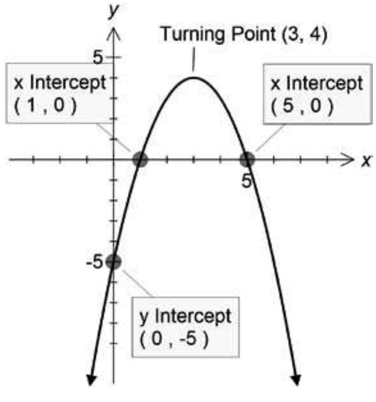
(a) $y = (x-1)^2 + 4$ (b) $y = (x-5)(1-x)$ (c) $y = 2x^2 - 4x - 6$

Solution:

(a) Curve has a minimum turning point at (1, 4).
When $x = 0$, $y = 5$ $\Rightarrow$ the y-intercept is at (0, 5).
When $y = 0$, $(x-1)^2 = -4$.
This equation has no real solutions.
Hence, the curve does not intersect the x-axis.

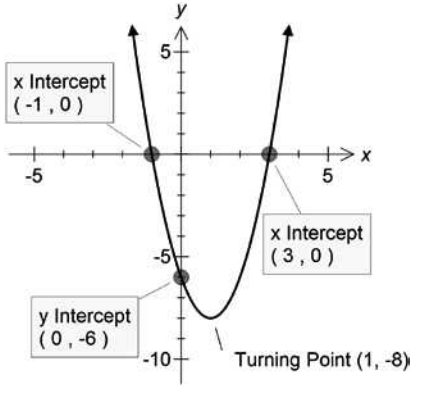


(b) Curve has horizontal intercepts at $x = 1$ and $x = 5$.
The turning point occurs when $x = \frac{1+5}{2} = 3$.
Hence, the turning point is at (3, 4).
When $x = 0$, $y = -5$.
Hence, the y-intercept is at (0, –5)



(c) This curve has a y-intercept at (0, –6).
The turning point occurs at $x = -\frac{(-4)}{2(2)} = 1$.
Hence, the turning point is at (1, –8).

$2x^2 - 4x - 6 = 0 \Rightarrow (2x-6)(x+1) = 0$
x-intercepts are at $x = 3$ and $x = -1$.



Note:

- *The essential features required in sketching a quadratic curve are its turning point and the y-intercept. The x-intercepts need only be given when they are easily obtained.*

Exercise 2.1

1. Find the exact coordinates and describe the nature of the turning points for:
   (a) $y = x^2 + 6x + 8$ (b) $y = 5x^2 + 35x - 10$
   (c) $y = 6 + 4x - 3x^2$ (d) $y = 1.5 - 0.2x + 0.01x^2$

2. Find the exact coordinates and describe the nature of the turning points for:
   (a) $y = (x + 4)(x - 6)$ (b) $y = (2x + 5)(x - 3)$
   (c) $y = 2(x + 6)(5 - x)$ (d) $y = 3x(5 - 2x)$

3. Determine the exact coordinates and describe the nature of the turning points for:
   (a) $y = (x - 2)^2 + 4$ (b) $y = (x + 3)^2 - 5$
   (c) $y = 3 - 2(x - 1)^2$ (d) $y = (2x - 5)^2$

4. Determine the exact coordinates and describe the nature of the turning points for:
   (a) $y = 200x^2 + 5000x - 1500$ (b) $y = \dfrac{x^2 + 3x - 10}{2}$
   (c) $y = 2 + \dfrac{(1 - x)^2}{2}$ (d) $y = \dfrac{-(2 - 3x)(4 + x)}{3}$

5. Without using a calculator, make a quick sketch of each of the following, indicating the coordinates of the turning point and the intercepts.
   (a) $y = -x^2$ (b) $y = (x - 1)^2$
   (c) $y = -(x + 1)^2 + 2$ (d) $y = 2(x - 1)^2 + 1$

6. Without using a calculator, sketch each of the following, indicating the coordinates of the turning point and the intercepts.
   (a) $y = (x + 1)(x - 3)$ (b) $y = (1 - x)(2 - x)$
   (c) $y = \dfrac{(x - 1)(x + 2)}{2}$ (d) $y = \dfrac{(2x + 1)(2 - x)}{2}$

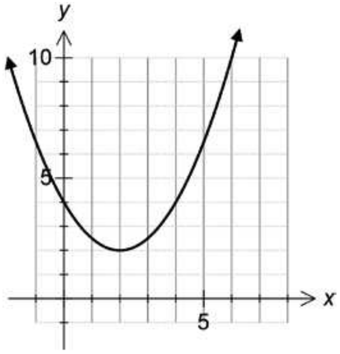
7. Without using a calculator, sketch each of the following, indicating the coordinates of the turning point and the intercepts.
   (a) $y = x^2 + 5x$ (b) $y = 9 - x^2$
   (c) $y = 4[1 - (x + 2)^2]$ (d) $y = \dfrac{2 + (x + 3)^2}{5}$

8. Find the value of k if $y = 4 - k(x + 3)^2$ has a vertical intercept at $(0, -14)$.

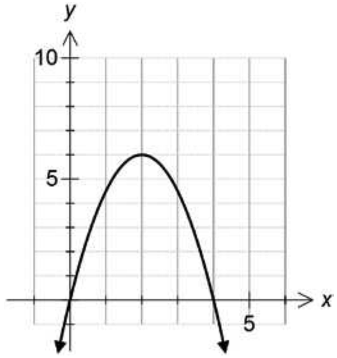
9. Find the value of k if the given quadratic curve has the stated axis of symmetry:
   (a) $y = 3x^2 + kx + 10$; $x = 2$ (b) $y = kx^2 + 12x - 5$; $x = \dfrac{3}{2}$
   (c) $y = (x + 3)(x - k)$; $x = 1$ (d) $y = k(x - 1)^2 + 2$; $x = 1$.

10. Find the equation of the quadratic curve whose graph is shown below.

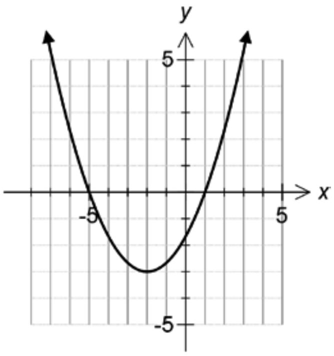
(a)



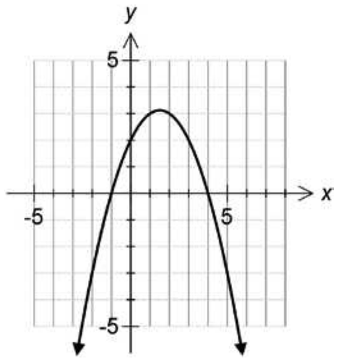
(b)



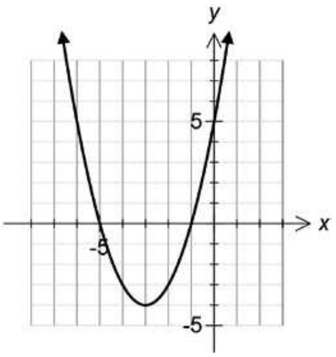
(c)



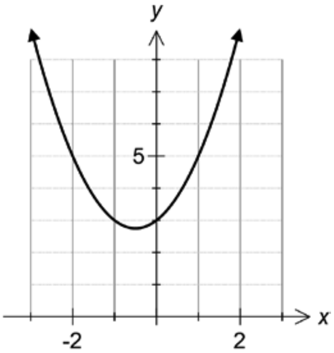
(d)



(e)



*(f)



11. Find the equation of the quadratic curve that:
    (a) has a turning point at (2, 3) and a vertical intercept at (0, 7)
    (b) has a turning point at (–1, –4) and a vertical intercept at (0, –2)
    (c) has a turning point at (2, 0) and a vertical intercept at (0, 2)
    (d) has a turning point at (2, 8) and a horizontal intercept at (4, 0) .

12. Find the equation of the quadratic curve that:
    (a) has intercepts at (2, 0), (–8, 0) and (0, 48)
    (b) has intercepts at (–5, 0), (4, 0) and (0, 10)
    (c) has exactly one *x*-intercept at (3, 0) and a vertical intercept at (0, 18)
    (d) has exactly one *x*-intercept at (–5, 0) and a vertical intercept at (0,–5).

2.2 Completion of Squares

- Consider the quadratic curve with equation in completed squares form
  $y = k(x - p)^2 + q$.
- The turning point has coordinates (p, q).
- Hence, to convert $y = ax^2 + bx + c$ into the completed squares form, we need to find the coordinates of the turning point.

Example 2.3

Rewrite $2x^2 + 8x - 9$ in the completed squares form.

Solution:

Let $y = 2x^2 + 8x - 9$. $\Rightarrow$ The turning point occurs at $x = -\dfrac{8}{2(2)} = -2$.

When $x = -2$, $y = -17$. That is, the turning point is at $(-2,-17)$.

Hence, $y = k(x + 2)^2 - 17$.

Clearly $k = 2$ as the x^2 coefficient is 2. $\Rightarrow$ $y = 2(x + 2)^2 - 17$.

Alternative Solution:

A more traditional approach to completion of squares is shown below:

$$\begin{aligned} 2x^2 + 8x - 9 &= 2[x^2 + 4x - \tfrac{9}{2}] \\ &= 2[(x + 2)^2 - 4 - \tfrac{9}{2}] \\ &= 2(x + 2)^2 - 17 \end{aligned}$$

Exercise 2.2

1. Complete the squares for each of the following:

   (a) $x^2 + 8x + 16$ (b) $x^2 + 8x + 17$ (c) $x^2 + 8x + 14$
   (d) $x^2 + 8x - 10$ (e) $x^2 + 5x - 10$ (f) $x^2 + 3x + 20$
   (g) $x^2 - 7x + 12$ (h) $x^2 - 9x - 1$

2. Complete the squares for each of the following:

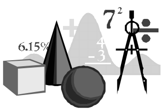
   (a) $-x^2 + 4x + 5$ (b) $-x^2 + 6x - 2$ (c) $1 + 4x - x^2$
   (d) $2 - 3x - x^2$ (e) $2x^2 + 10x + 16$ (f) $3x^2 + 12x - 7$
   (g) $-5x^2 - 10x - 7$ (h) $4 - 3x - 2x^2$

2.3 Solving Quadratic Equations

2.3.1 Algebraic techniques for solving quadratic equations

- Consider the quadratic equation in factored form $k(ax - p)(bx - q) = 0$.
  The **roots** or **solutions** or **zeros** are simply $x = \dfrac{p}{a}$ and $\dfrac{q}{b}$.

- For a quadratic equation in general form $ax^2 + bx + c = 0$,
  the following algebraic techniques may be used to determine the roots.
  - Rewrite the equation in factored form (this is not possible in many cases).
  - Use the quadratic root formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.



Hands On Task 2.1

In this task we will derive the quadratic root formula.

Consider the quadratic equation $ax^2 + bx + c = 0$.

(a) Rewrite this equation as $a[x^2 + \dfrac{b}{a}x + \dfrac{c}{a}] = 0$.

(b) Show that this can then be rewritten in the completed square form as

$$a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right\} = 0 .$$

(c) Show that this can then be simplified as $\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}$.

(d) Hence show that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, stating any assumptions made.

Example 2.4

Complete the squares for $2x^2 + 8x - 9$. Hence, find the exact solutions to $2x^2 + 8x - 9 = 0$.

Solution:

From Example 2.2: $\quad 2x^2 + 8x - 9 = 2(x + 2)^2 - 17$

Hence, $\quad 2x^2 + 8x - 9 = 0$ is identical to $2(x + 2)^2 - 17 = 0$

$$(x + 2)^2 = \frac{17}{2} \quad \Rightarrow \quad x + 2 = \pm\sqrt{\frac{17}{2}}$$

$$x = -2 \pm\sqrt{\frac{17}{2}}$$

Exercise 2.3

1. Without the use of a calculator, solve for real values of x:
   (a) $x(x + 4) = 0$ (b) $(x - 1)(x + 5) = 0$
   (c) $(2x - 0.01)(x + 0.05) = 0$ (d) $(0.01x + 10)(5 - 0.02x) = 0$
   (e) $x^2 + 10x = 0$ (f) $x^2 = 4x$
   (g) $(x + 1)^2 = 4$ (h) $(2x + 1)^2 = 9$

2. Without the use of a calculator, find the real zeros for each of the following equations:
   (a) $2(1 - x)^2 = 18$ (b) $5(3x + 1)^2 = 405$ (c) $x^2 + 7x - 8 = 0$
   (d) $x(x + 1) - 2 = 0$ (e) $x^2 + 4x - 1 = 0$ (f) $5 + 3x - x^2 = 0$

3. Without the use of a calculator, solve for real values of x:
   (a) $2x^2 + 10x - 5 = 0$ (b) $4x - 3x^2 + 1 = 0$ (c) $\frac{x^2}{2} + 2x - 1 = 0$
   (d) $\frac{1-3x}{2} + x^2 = 0$ (e) $2x^2 + 3x - 4 = x^2 - x + 3$ (f) $2x^2 + 3x + 1 = (x + 1)(x - 3)$

4. Without the use of a calculator, find the real zeros for each of the following:.
   (a) $\frac{x+1}{x} = \frac{2}{x}$ (b) $\frac{x+1}{4} - \frac{(x+1)^2}{2} = 0$ (c) $\frac{4}{x^2} + \frac{1}{x} - 1 = 0$
   (d) $\frac{(x+1)^2}{x} - \frac{3x}{2} = 0$ (e) $\frac{x+1}{2} + \frac{x^2-1}{3} = 1$ (f) $\frac{\sqrt{x}}{2} = 3x - 1$

5. Redo questions 3 and 4 using a CAS calculator.

2.3.2 Existence of solutions

- Consider the equation $ax^2 + bx + c = 0$.

  The solutions are $x = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

- The expression under the radical symbol, $b^2 - 4ac$, is termed the discriminant, denoted Δ.

- Depending on the value of Δ, one of the following three cases is possible.
  - If $\Delta = 0$, then the equation has exactly one real solution, $x = -\dfrac{b}{2a}$.
  - If $\Delta > 0$, then the equation has exactly two distinct real solutions,

    $$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$
  - If $\Delta < 0$, then the equation has no real solutions.
    This is because the square root of a negative number is not a real number.

Example 2.5

Find the value(s) for k for which $x^2 - 4x + k = 0$ has:
(a) exactly one real solution (b) no real solution.

Solution:

The discriminant is $(-4)^2 - 4k = 16 - 4k$.

(a) For exactly one solution, $16 - 4k = 0 \;\Rightarrow\; k = 4$.

(b) For no real solutions $16 - 4k < 0 \;\Rightarrow\; k > 4$

Note:

- *For the equation to have real solutions,* $16 - 4k \geq 0 \;\Rightarrow\; k \leq 4$.

Exercise 2.4

1. Find the value(s) for k for which $x^2 - 10x + k = 0$ has exactly one solution.

2. Find the value(s) for k for which $2x^2 + 5x - k = 0$ has real solutions.

3. Find the value(s) for k for which $kx^2 + 6x - 2 = 0$ has no real solutions.

4. Find the value(s) for k for which $3x^2 + kx + 1 = 0$ has exactly one solution.

*5. Find the value(s) for k for which $(x + 1)^2 + (k - 1) = 0$ has real solutions.

*6. Find the value(s) for k for which $2(x - 1)^2 + 2k = 10$ has:
(a) exactly one real solution (b) no real solutions.

7. The curve $y = x^2 + 4x - 2k$ has two distinct real zeros. Find the value(s) of k.

*8. The curve $y = k(x + 4)^2 + 2$ intersects the x-axis twice. Find the value(s) of k.

*9. The curve $y = (x - 4)(2x + k)$ intersects the x-axis once. Find the value(s) of k.

*10. Find the value(s) of k for which $y = (x - k)^2 + 10x$ does not intersect the x-axis.

*11. Find the value of k for which $x^2 + kx + 4 = 0$ has real roots.

*12. Find the value of k for which $v^2 + kv + 9k = 0$ has no real roots.

*13. Find the value of k for which $ku^2 + ku + 10 = 0$ has two distinct real roots.

2.3.3 Working with quadratic inequalities

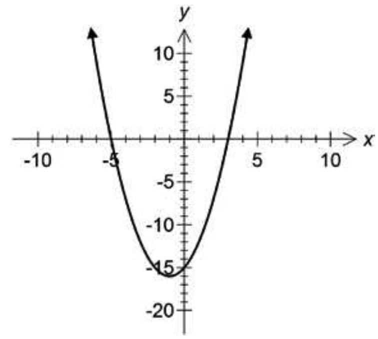
Example 2.6

Without the use of a calculator, make a quick sketch of $y = (x - 3)(x + 5)$.
Hence, find the value(s) of x for which $(x - 3)(x + 5) < 0$.

Solution:

The horizontal intercepts are at $x = -5$ and $x = 3$.
The vertical intercept is at $y = -15$.
The sketch of $y = (x - 3)(x + 5)$ is drawn as shown.

From the sketch, $(x - 3)(x + 5) < 0$ for $-5 < x < 3$.



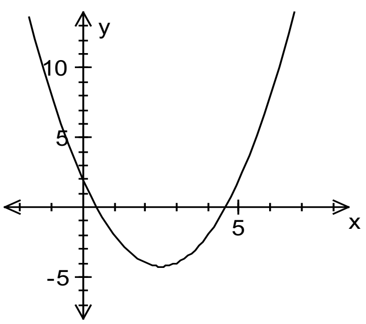
Note:

- *The sketch requires the following features:*
  - *the horizontal intercepts (this is a must),*
  - *the shape of the curve* $\cup$ *or* $\cap$.
- *The expression* $(x - 3)(x + 5)$ *is represented graphically as the "height" of the curve* $y = (x - 3)(x + 5)$. *Hence, when* $(x - 3)(x + 5) < 0$, *the height* $y < 0$.

Exercise 2.5

1. Without the use of a calculator, solve for real values of x:
   (a) $(x + 1)(2x - 3) \geq 0$ (b) $x^2 + 4x + 4 > 0$
   (c) $16 - x^2 \leq 0$ (d) $2(x - 3)^2 + 4 \geq 0$
   (e) $x^2 + 5x \leq 6$ *(f) $1 - 2x^2 > 4x$
   (g) $(x + 1)(x - 5) \geq 2x - 10$ (h) $(x + 2)(2x - 1) \leq (x - 1)(2 + x)$
   (i) $(x - 5)^2 + 8 \leq 2(x + 1)^2 - 1$ (j) $(2x - 1)^2 - 1 > 4 + (x - 1)^2$

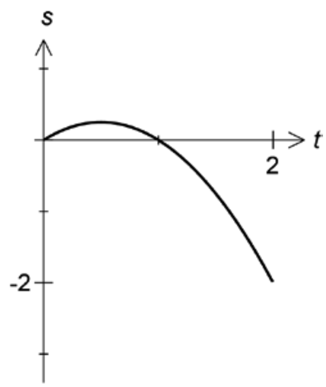
2. Given the graph of $y = x^2 - 5x + 2$, ***describe*** how you would use the graph to solve:
   (a) $x^2 - 5x + 2 = 0$
   (b) $x^2 - 5x + 2 \geq 0$
   (c) $x^2 - 5x + 2 = 2$
   (d) $x^2 - 5x + 2 \leq 5$
   (e) $x^2 - 5x + 2 > x + 5$
   (f) $x^2 - 5x + 2 < (x - 1)(5 - x)$



3. Redo question 1 using a CAS calculator.

2.4 Applications Using Quadratic Models

Example 2.7



A particle P travels in a straight line. Its displacement s (m), t seconds after it passes a fixed point O, is given by $s = -t(t-1)$ for $0 \le t \le 2$. The sketch of $s = -t(t-1)$ for $0 \le t \le 2$ is given in the accompanying diagram. Find:

(a) displacement of P after 1 second and after 2 seconds.
(b) the furthest displacement for P for $0 \le t \le 1$ seconds
(c) the distance travelled within the first 2 seconds.

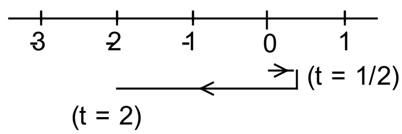
Solution:

(a) When $t = 1$, $s = -1(1-1) = 0$ m
When $t = 2$, $s = -2(2-1) = -2$ m

(b) s is maximum when $t = \dfrac{-1}{2(-1)} = 0.5$.

Hence, furthest displacement = $-0.5(0.5-1) = 0.25$ m

(c) The particle changes direction when $t = 0.5$ seconds.
The "trail" for P is given below:

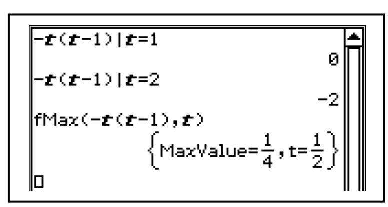


Hence, distance travelled ($0 \le t \le 2$) = 0.25 + 0.25 + 2 = 2.5 m

Note:

- *In part (c), when t = 0.5, P is 0.25 m to the right of O.*
  At t = 1, P is back at O.
  At t = 2, P is 2 m to the left of O.
  Hence, distance travelled = 0.25 + 0.25 + 2 = 2.5 m .

Alternative Solution for parts (a) and (b):



(a) When $t = 1$, $s = 0$ m
When $t = 2$, $s = -2$ m

(b) Maximum point on curve has coordinates (0.5, 0.25).
Hence, furthest displacement = 0.25 m.

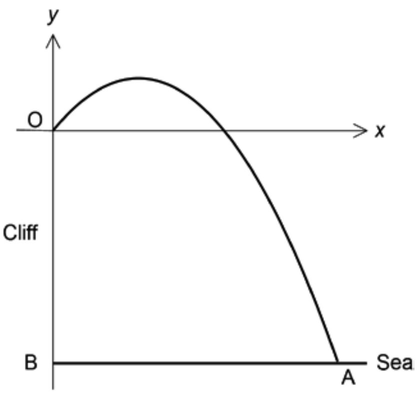
Example 2.8

A ball is thrown off the top of a cliff, 100 m above sea level. Taking the point of projection O as the origin of the coordinate axes, the path taken by the ball is given as $y = 0.1x(30 - x)$. The ball hits the surface of the sea at A.

(a) Find the height above sea level for the highest point reached by the ball.

(b) Find the distance from A to B, the base of the cliff.

(c) Find the horizontal distance from O when the ball is 110 m above sea level.



Solution:

(a) $y = 0.1x(30 - x) \;\Rightarrow\;$ roots are $x = 0, 30$

Line of symmetry: $x = \dfrac{0 + 30}{2} = 15$.

When $x = 15$, $y = 22.5$. That is, maximum point on curve is at (15, 22.5).

Hence, highest point reached by the ball is 100 + 22.5 = 122.5 m above sea level.

(b) At A, $y = -100$. $\Rightarrow\; -0.1x^2 + 3x = -100$.

Rewrite as $x^2 - 30x - 1000 = 0$.

$(x - 50)(x + 20) = 0$

Roots are $x = 50$ and $x = -20$ (reject this) .

Hence, A is 50 m from B.

(c) When the ball is 110 m above the sea level, $y = 10$.

Hence, $-0.1x^2 + 3x = 10$.

Rewrite as $0.1x^2 - 3x + 10 = 0$.

Roots are $x = 3.8$ and $x = 26.2$ [From calculator].

Hence, the ball is 100 m above sea level when its horizontal distance from O is 3.8 m and 26.2 m .

Alternative Solution:

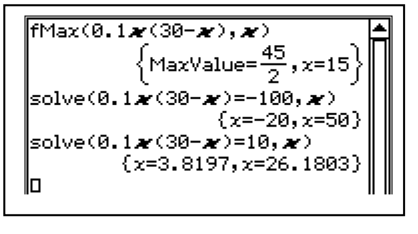
(a) From CAS calculator, maximum point on curve has coordinates (22.5, 15).

Hence, highest point reached by the ball is 100 + 22.5 = 122.5 m above sea level.

(b) At A, $y = -100$. $\Rightarrow\; -0.1x^2 + 3x = -100$.

From CAS calculator, $x = 50$ and $x = -20$ (reject this) .

Hence, A is 50 m from B.



(c) When the ball is 110 m above the sea level, $y = 10$.

Hence, $-0.1x^2 + 3x = 10$

From CAS calculator, $x = 3.8$ and $x = 26.2$.

Hence, the ball is 100 m above sea level when its horizontal distance from O is 3.8 m and 26.2 m .

Exercise 2.6

1. A particle P moves along a straight line. Its displacement s (m) from a fixed point O, t seconds after it passes O, is given by $s = 8t - 4t^2$.
   (a) Sketch the graph of s against t for $0 \le t \le 3$ seconds.
   (b) Find the displacement after (i) 1 second and (ii) 3 seconds.
   (c) Find the time when P returns to O.
   (d) Find the distance travelled by P in the (i) first second and (ii) first three seconds.

2. A particle P moves in a straight line. Its displacement s (m) t seconds after it passes a fixed point O, is given by $s = -15t + 5t^2$.
   (a) Sketch the graph of s against t for $0 \le t \le 4$ seconds.
   (b) Find the time when P returns to O.
   (c) Find the furthest distance P is from O for (i) $0 \le t \le 3$ (ii) $0 \le t \le 4$ seconds.
   *(d) Find the times when P is 10 m from O.

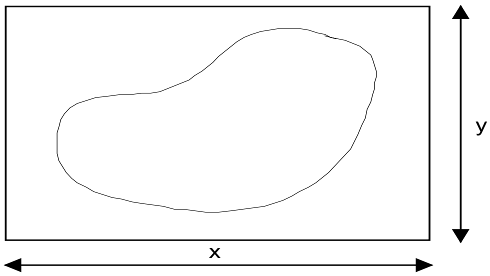
3. A particle P moves in a straight line. Its displacement s (m) t minutes after it passes a fixed point O, is given by $s = 3t + 9t^2$.
   (a) Sketch the graph of s against t for $0 \le t \le 3$ minutes.
   (b) Verify that P never returns to O.
   (c) Find the distance travelled in the third second.
   (d) Find the average speed of the particle in the third second.

4. A particle P moves in a straight line. Its displacement s (m) from a fixed point O is related to its velocity v (ms^{-1}) by the equation $s = -10 + 0.1v^2$, where $s \ge 0$ and $v \ge 0$.
   (a) Find u, the velocity of P when it is at O.
   (b) Find its displacement when $v = 2u$.
   (c) Find the velocity when $s = 90$ m.
   *(d) Show that the velocity of P can never be less than u.

5. Jane runs a home cottage industry where she bottles “wildflower honey”. The cost C (\$) of producing x jars of “wildflower honey” is given by $C = 20 + 0.005x^2$, for $0 < x \le 200$.
   (a) What is the fixed cost component for Jane’s cottage industry.
   (b) Find the cost of producing 100 jars of “wildflower honey” and find the average cost per jar when 100 jars are produced.
   (c) Find the average cost per jar, A (\$), when x jars are produced.
   *(d) Graph A against x and comment briefly on the relationship between A and x.

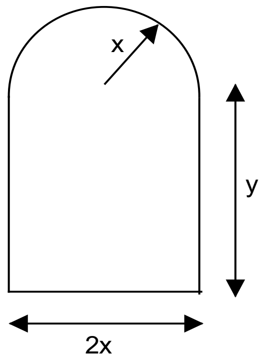
6. The selling price, S (\$), for each of Jade’s hand painted cards is given by $S = 10 - 0.01x^2$.
   The sales demand , D, for Jade’s cards is given by $D = -x^2 + 20x + 50$.
   (a) Find the value of x when the cards are sold at \$6.00 each.
   Hence, find the sales demand when the cards are sold at \$6.00 each.
   (b) Find the selling price per card when there is a demand for 150 cards.
   (c) Find the selling price that will give her maximum sales demand.

7. Mr Sumich needs to fence in his new swimming pool. The fenced area that encloses his pool is to be rectangular in shape and Mr Sumich would like this to be at least 18 m long and 10 m wide. Mr Sumich has 70 m of pool fencing that he can use. Let the fenced rectangular area be x m long and y m wide.



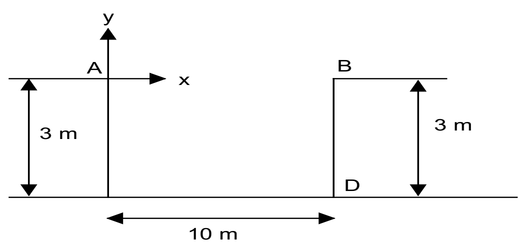
   (a) Show that $y = 35 - x$.
   (b) Show that the area of the region to be fenced is given by $A = 35x - x^2$.
   (c) Find the maximum area that can be enclosed with the 70 m of fencing available.
   (d) Determine if the dimensions that give the maximum area enclosed fits Mr Sumich's plan for the enclosure. Justify your answer.
   (e) Using all the available fencing, what is the largest possible area that will satisfy Mr Sumich's plan?
   (f) Find the difference in area between the answer in (c) and (d).

8. The accompanying diagram is a schematic diagram of a garden bed made up of a rectangle and a semicircle placed at one end of the rectangle. The radius of the semi-circular part is x. All measurements are in metres. Sydney plans to line the edges of the garden bed with kerbing, and he has 20 m of kerbing.



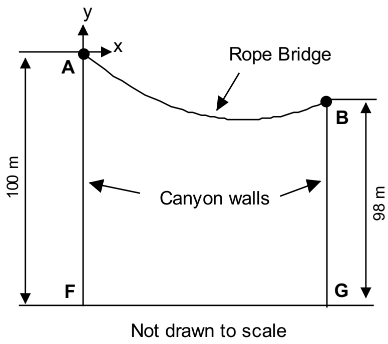
   (a) Show that $y = 10 - x - \dfrac{\pi x}{2}$.
   *(b) Show that the area of the garden bed is given by $A = 20x - (2 + \dfrac{\pi}{2})x^2$.
   *(c) Show that the maximum value of A occurs when $x = \dfrac{20}{4+\pi}$.
   *(d) Find the maximum value of A.

9. Points A, B and D lie on the same vertical plane. A and B are on the edges of the tops of two pergolas. The tops of the two pergolas are both parallel to the ground. A and B are both 3 m above the ground level and are 10 m apart (horizontally). A boy sitting at A throws a ball towards B.



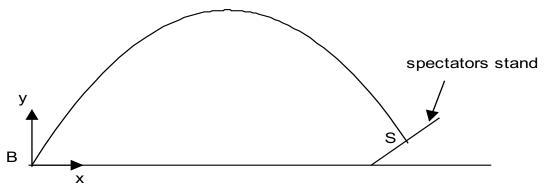
   Taking A as the origin of the x-y axis, the path of the ball is given by $y = 0.1x(k - x)$. x and y are measured in metres.
   (a) Find the value of k for the ball to hit B.
   (b) For $k = 12$, what is the highest point above the ground reached by the ball?
   (c) For $k = 12$, how far to the right of point B does the ball hit the top of the pergola?
   (d) For $k = 8$, the ball will hit the supporting post BD. Find how far up the post BD will the ball strike the post.

10. A rope suspension bridge connects the edges A and B of two canyon cliffs as shown in the accompanying diagram. Point A is 100 m above the canyon floor while point B is 98 m above canyon floor. F is the foot of the cliff vertically below A while G is the foot of the cliff vertically below B, (A, B, F and G are in the same vertical plane). Assume that the cliff walls AF and BG are parallel. Using A as the origin of the coordinate axes, the platform part of the bridge is given by the equation $y = 0.005x(x - 50)$.



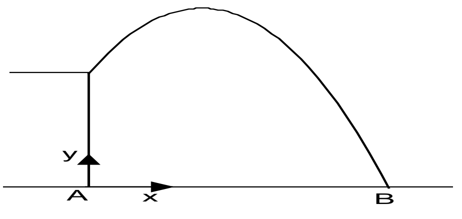
(a) Find the height of the lowest point of the bridge platform.
(b) Find the width of the canyon.
(c) Find the horizontal distance between the two points on the bridge platform that are 97 m above the canyon floor.

11. In a cricket match, a batsman at B hits a ball into the spectators stand. Taking B as the origin of the coordinate axes, the path of the ball can be modelled by the equation $y = 0.002x(115 - x)$. The profile of the spectator stand can be modelled by the equation $y = 0.1x - 10$ for $100 \le x \le 115$. x and y are in metres.



(a) Find the highest point reached by the cricket ball.
(b) Find the x-coordinate of the points where the cricket ball is 5 m above the ground.
(c) A sight screen 4 m high is located at (90, 0) . Find how high above the screen the cricket ball was as it flew over the sight screen.
(d) The cricket ball hits the spectator stand at S. Find the coordinates of S.

12. A ball hits the edge of a table and bounces onto the ground. The ball hits the floor at B. It then bounces off the ground and hits the ground again at C. Assume that A, B and C are in the same vertical plane. Taking A, the foot of the table, as the origin of the coordinate axes, the path of the ball from the table to B, may be modelled by the equation $y = 1 + x - \dfrac{4x^2}{9}$. x and y are measured in metres



(a) Find the height of the table.
(b) Find the height of the highest point reached by the ball.
(c) Find the distance from A where the ball first hits the floor.
*(d) Assume the path from B to C is parabolic in shape. The greatest height reached by the ball between B and C is 1 m and the point C has coordinates (5, 0). Find the equation of the path of the ball between B and C.
*(e) For $0 \le x \le 5$, find the value(s) of x when the ball is 0.5 m above the ground.

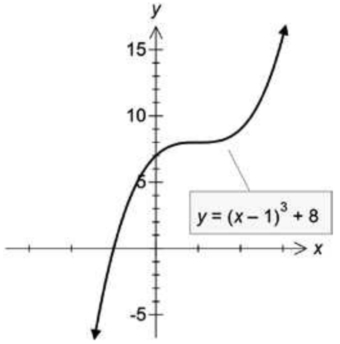
03 Cubics

3.1 Cubics

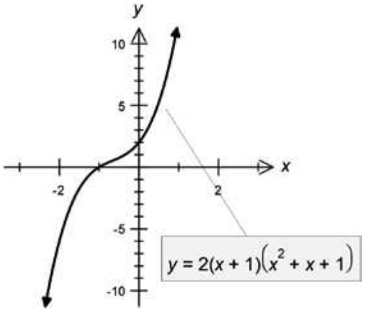
- Cubics have equations of the form:
  - $y = 2x^3 + x^2 + 3x - 1$
  - $y = 2(x + 3)^2(3 - 5x)$
  - $y = 2(x - 1)^3 + 4$
  - $y = -3(x + 3)(x - 1)(x + 2)$

3.2 Features of Cubics (cubic curves)

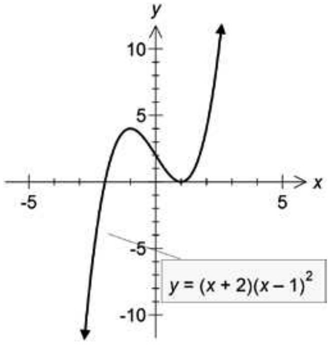
- The general cubic expression $ax^3 + bx^2 + cx + d$, depending on the values of a, b, c and d, may be rewritten as:
  - I an exact cube add a constant, e.g. $(x - 1)^3 + 8$.
  - II a product of a linear factor and a quadratic factor, e.g. $2(x + 1)(x^2 + x + 1)$.
  - III a product of two distinct factors, one of which is repeated, e.g. $(x + 2)(x - 1)^2$.
  - IV a product of three distinct factors, e.g. $(x + 1)(x - 2)(x + 3)$.
- Depending on the number of distinct linear factors in its factored form, cubic curves have:
  - one root (form I)



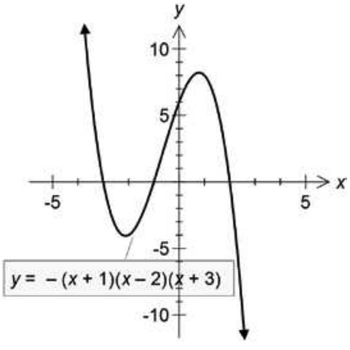
  - one root (form II)



  - two roots (form III)

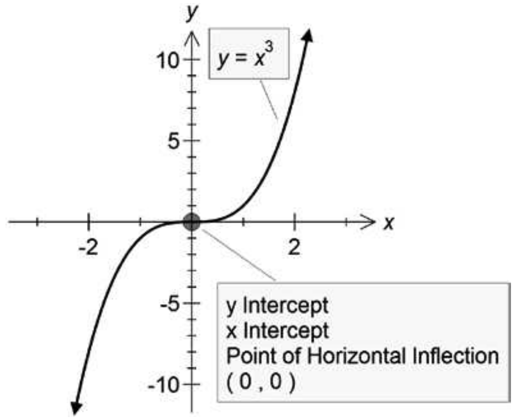


  - three roots (form IV)

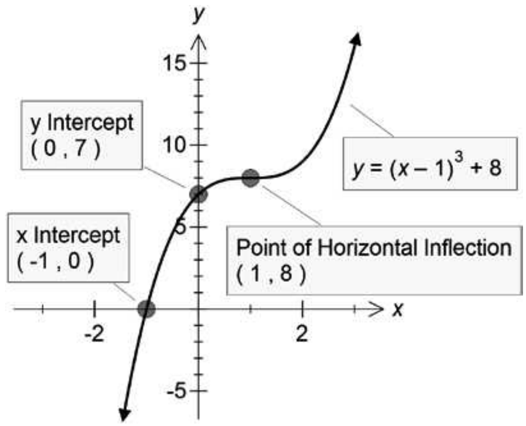


- All cubic curves have exactly one vertical intercept.

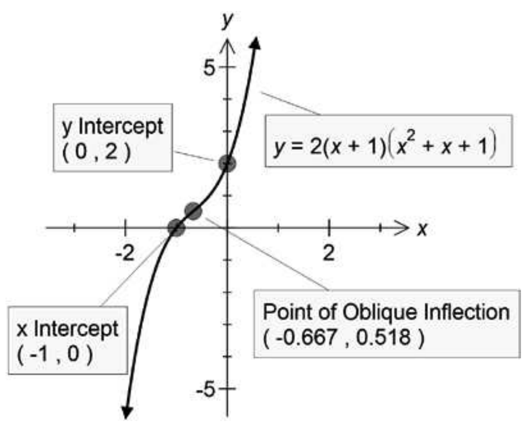
3.2.1 Cubics with one root



- The accompanying diagram shows the graph of the basic cubic curve $y = x^3$.
  - The curve has one root at $x = 0$ and one vertical intercept at (0, 0).
  - The curve has a "saddle point" called the point of *horizontal inflection* at (0, 0).



- Consider now the graph of the cubic $y = (x - 1)^3 + 8$.
  - The curve has one root at $x = -1$ and a vertical intercept at (0, 7).
  - Notice that the point of *horizontal inflection* is now at (1, 8).



- Consider now the graph of the cubic $y = 2(x + 1)(x^2 + x + 1)$.
  - It can be shown that $2(x + 1)(x^2 + x + 1)$ cannot be rewritten in the form $k(x - a)^3 + b$.
  - The curve has a root at $x = -1$ and a vertical intercept at (0, 2).
  - The curve has a point of *oblique inflection* at (–0.667, 0.518).

- In general, a cubic curve with equation $y = k(x - a)^3 + b$ will have:
  - one root,
  - a vertical intercept at $(0, -ka^3 + b)$,
  - a point of *horizontal inflection* at (a, b).

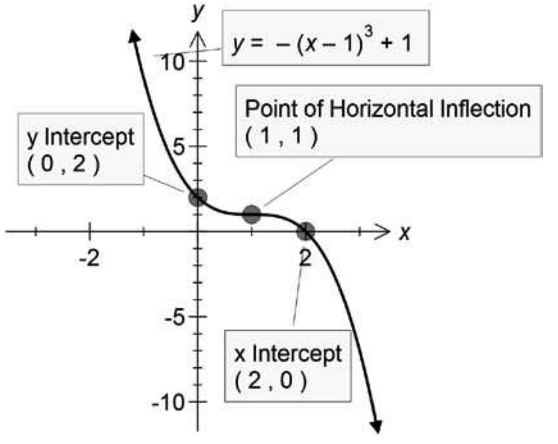
- In general, a cubic curve with equation $y = k(x - a)(x^2 + bx + c)$ where $k(x - a)(x^2 + bx + c)$ cannot be rewritten in the form $k(x - p)^3 + q$, will have:
  - one root at $x = a$,
  - a vertical intercept at $(0, -kac)$
  - a point of *oblique inflection*.

Example 3.1

Without the use of calculator, sketch $y = 1 - (x - 1)^3$.
Indicate all essential features of this curve.

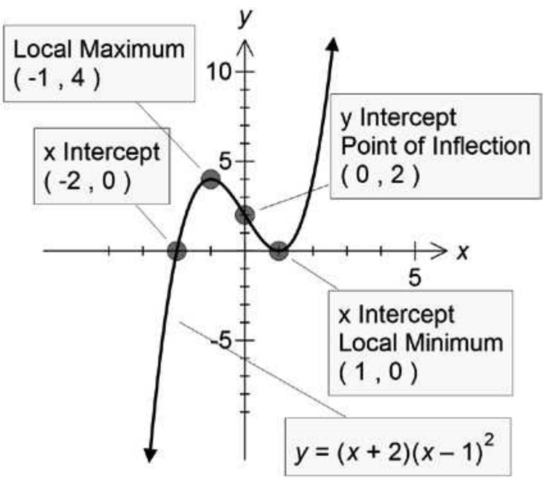
Solution:

Rewrite as $y = -(x - 1)^3 + 1$.
Vertical Intercept (0, 2).
Root: $-(x - 1)^3 + 1 = 0$
$x = 2$



3.2.2 Cubics with two roots

- The accompanying diagram shows the graph of the curve $y = (x + 2)(x - 1)^2$.



  - The curve has one root at $x = -2$ corresponding to the factor $(x + 2)$ and a "bouncing root" at $x = 1$ corresponding to the repeated factor $(x - 1)$.
  - The curve has a vertical intercept at (0, 2).
  - The curve has a maximum turning point as well as a minimum turning point and an oblique inflection point.

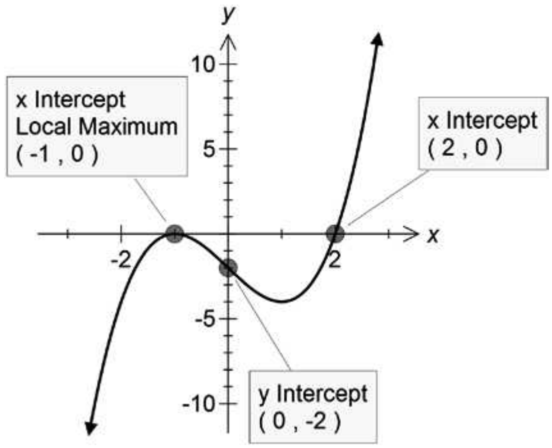
- In general, a cubic curve with equation $y = k(x - a)(x - b)^2$ where $a \neq b$, will have:
  - a root at $x = a$ and a root at $x = b$ which is simultaneously a turning point
  - a vertical intercept at $(0, -kab^2)$
  - a minimum turning point and a maximum turning point
  - an oblique inflection point.

Example 3.2

Without the use of calculator, sketch $y = -(2 - x)(x + 1)^2$.
Indicate all intercepts of this curve.

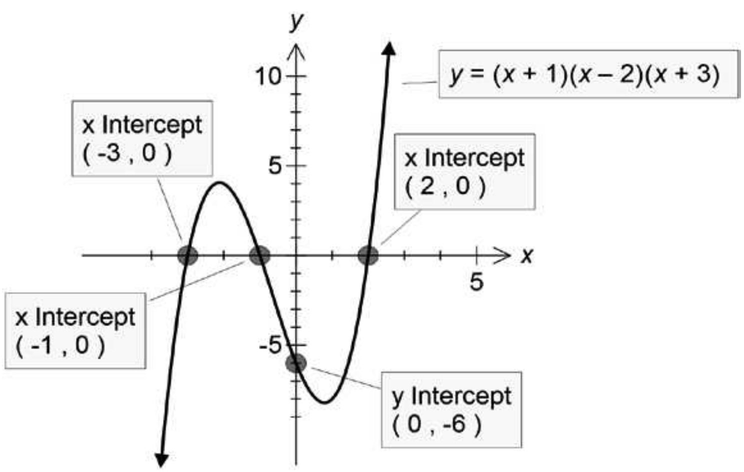
Solution:

Vertical Intercept $(0, -2)$.
Roots: $x = 2$
$x = -1$ (repeated root)
Maximum turning point at $(-1, 0)$.



3.2.3 Cubics with three roots

- The accompanying diagram shows the graph of the curve $y = (x + 1)(x - 2)(x + 3)$.



  - The curve has roots at $x = -3$, -1 and 2 corresponding to the factors $(x + 3)$, $(x + 1)$ and $(x - 2)$ respectively
  - The curve has a vertical intercept at $(0, -6)$.
  - The curve has a maximum turning point as well as a minimum turning point and an oblique inflection point.

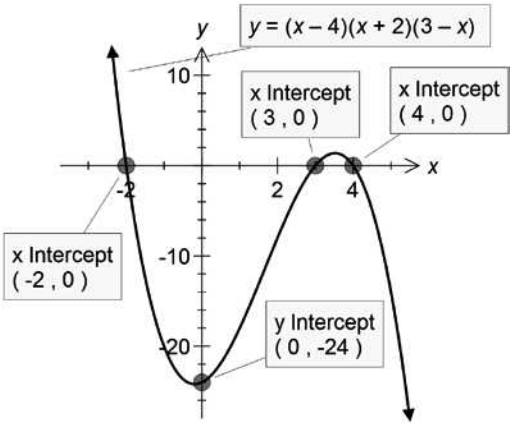
- In general, a cubic curve with equation $y = k(x - a)(x - b)(x - c)$ where $a \neq b \neq c$, will have:
  - roots at $x = a$, $x = b$ and $x = c$
  - a vertical intercept at $(0, -kabc)$
  - a minimum turning point and a maximum turning point
  - an oblique inflection point.

Example 3.3

Without the use of calculator, sketch $y = (x - 4)(x + 2)(3 - x)$.
Indicate all intercepts of this curve.

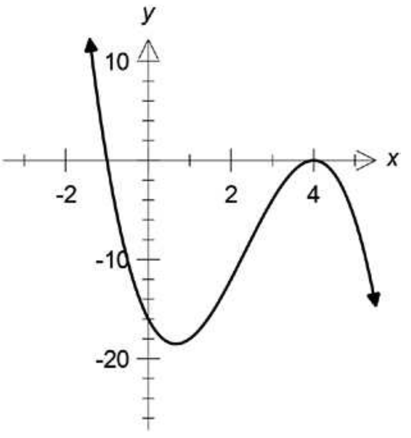
Solution:

Vertical Intercept (0, –24).
Roots: $x = -2$
$x = 3$
$x = 4$



Example 3.4

Determine the equation of the given curve.



Solution:

Roots are: $x = -1$ and $x = 4$ (repeated).
Hence, $y = k(x + 1)(x - 4)^2$.
y-intercept is (0, –16) $\Rightarrow$ when $x = 0$, $y = -16$.
Therefore, $-16 = k\,(1)(16)$
$k = -1$
Hence, equation of curve is $y = -(x + 1)(x - 4)^2$.

Exercise 3.1

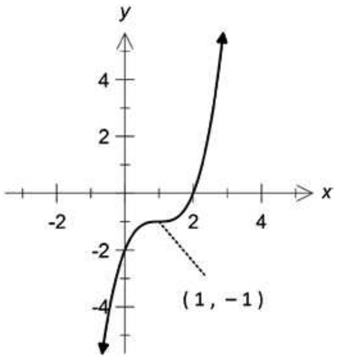
1. Without the use of calculator, sketch each of the following curves.
   Indicate all essential features of this curve.
   (a) $y = x^3 + 1$ (b) $y = (x + 2)^3$ (c) $y = (x - 2)^3 + 1$
   (d) $y = 2(x + 1)^3 - 16$ (e) $y = -2(x - 1)^3 + 12$ (f) $y = 8 - (x + 2)^3$

2. Without the use of calculator, sketch each of the following curves.
   Indicate all intercepts of this curve.
   (a) $y = x^2 (x + 1)$ (b) $y = x(x + 1)^2$ (c) $y = (x + 2)(x + 1)^2$
   (d) $y = -(x - 3)(1 - x)^2$ (e) $y = (2x + 3)(x - 2)^2$ (f) $y = 0.5\,(x - 2)(x + 4)^2$

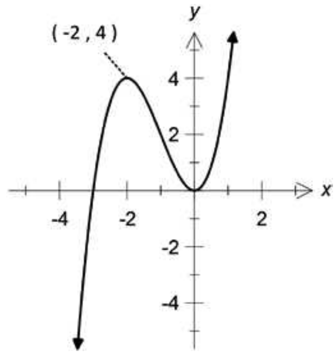
3. Without the use of calculator, sketch each of the following curves.
   Indicate all intercepts of this curve.
   (a) $y = x(x - 2)(x + 3)$ (b) $y = (x + 4)(x + 2)(5 + x)$ (c) $y = (x - 1)(x - 2)(x - 3)$
   (d) $y = (x^2 - 4)(1 - x)$ (e) $y = x^3 - x^2$ (f) $y = x^3 - 2x^2 + x$

4. Find the equation of the cubic curve whose graph is shown below.

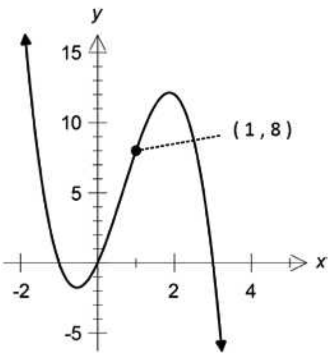
(a)



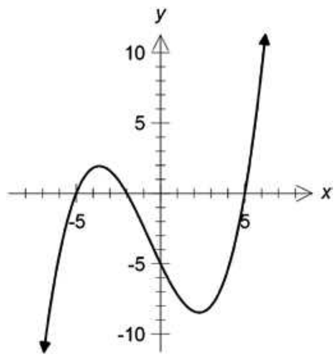
(b)



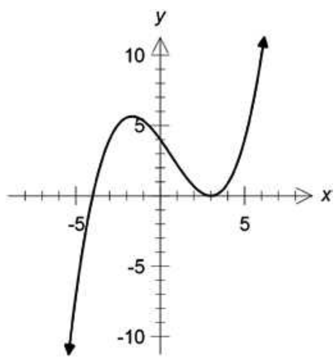
(c)



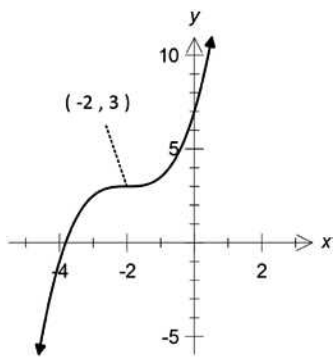
(d)



(e)



(f)



5. Find the equation of the cubic curve that:
   (a) has a point of horizontal inflection at (2, 3) and a vertical intercept at (0, –5)
   (b) has a point of horizontal inflection at (–3, –4) and a vertical intercept at (0, 5)
   (c) has a point of horizontal inflection at (1, –4) and a horizontal intercept at (2, 0)
   (d) has a point of horizontal inflection at (–1, 9) and a horizontal intercept at (2, 0).

6. Find the equation(s) of the cubic curve(s) with:
   (a) intercepts at (1, 0), (3, 0), (5, 0) and (0, –5)
   (b) intercepts at (–2, 0), (–3, 0), (7, 0) and (0, 7)
   (c) intercepts at (–1, 0), (1, 0) and (0, 3)
   (d) intercepts at (2, 0), (–3, 0) and (0, 3).

3.3 Solving cubic equations

- Without the use of a calculator, to solve the cubic equation $ax^3 + bx^2 + cx + d = 0$, we first need to factorise the cubic expression.
- As mentioned in Section 3.2, a cubic may be factorised into:
  - I a product of a linear factor and a quadratic factor, e.g. $2(x + 1)(x^2 + x + 1)$
  - II a product of two distinct factors, one of which is repeated, e.g. $(x + 2)(x - 1)^2$
  - III a product of three distinct factors, e.g. $(x + 1)(x - 2)(x + 3)$.
- Hence, a cubic has at least one linear factor corresponding to a root of the equivalent cubic curve. To find this root, we simply need to find a value of x that makes the cubic expression zero; this value is called a *zero* of the expression. This can be done by simple inspection involving a process of trial and error.
- To a fully factorise a cubic expression, the following procedure may be used.
  - Find the first linear factor (find a zero of the cubic expression).
  - By inspection, the cubic can then be rewritten as the product of a linear factor and a quadratic factor as in I.
  - The quadratic factor may then be further factorised (if possible).

Example 3.5

Without the use of a calculator, solve each of the following equations.

(a) $x^3 + 3x^2 + 7x + 5 = 0$ (b) $x^3 + 3x^2 - 10x - 24 = 0$

Solution:

(a) A root of the equation will be a factor of the constant "5" in the cubic expression.

Try: $x = -1 \Rightarrow$ Expression $= -1 + 3 - 7 + 5 = 0$.

Hence, $x = -1$ is a zero of the expression and $(x + 1)$ is a factor.

Therefore: $x^3 + 3x^2 + 7x + 5 \equiv (x + 1)(ax^2 + bx + c)$.

By inspection: $a = 1$ *[As the coefficient of the x^3 term is 1]*

$c = 5$ *[As the constant term is 5]*

Hence, $x^3 + 3x^2 + 7x + 5 \equiv (x + 1)(x^2 + bx + 5)$

By further inspection: $b = 2$

Therefore: $x^3 + 3x^2 + 7x + 5 = 0 \Rightarrow (x + 1)(x^2 + 2x + 5) = 0$

The quadratic factor $(x^2 + 2x + 5)$ has no real roots. $\Rightarrow x = -1$.

(b) Try: $x = 2$ Expression $= 8 + 12 - 20 - 24 \neq 0$

Try: $x = -2$ Expression $= -8 + 12 + 20 - 24 = 0$

Hence, $(x + 2)$ is a linear factor.

By inspection: $x^3 + 3x^2 - 10x - 24 \equiv (x + 2)(x^2 + x - 12)$

$\equiv (x + 2)(x + 4)(x - 3)$.

Hence, $x^3 + 3x^2 - 10x - 24 = 0 \Rightarrow (x + 2)(x + 4)(x - 3) = 0$

$x = -4, -2, 3$

Example 3.6

Without the use of a calculator, sketch $y = -2x^3 + x^2 - 2x + 1$. Indicate all essential features.

Solution:

Rewrite $-2x^3 + x^2 - 2x + 1$ as $-2\left(\frac{x^3}{2} - \frac{x^2}{2} + x - \frac{1}{2}\right)$.

A zero will be a factor of $\frac{1}{2}$.

Try: $x = \frac{1}{2}$ Expression $= \frac{1}{4} - \frac{1}{4} + 1 - 1 = 0$.

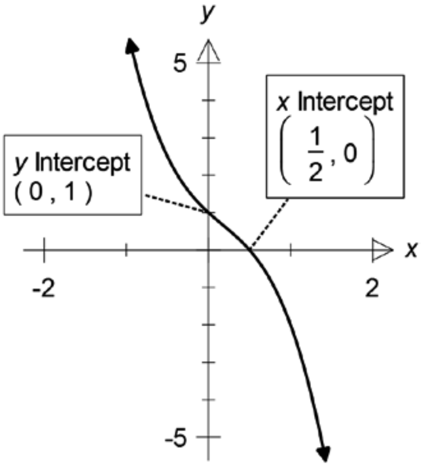
Hence, $(2x - 1)$ is a factor.

By inspection: $-2x^3 + x^2 - 2x + 1 \equiv -(2x - 1)(x^2 + 1)$.

Hence:

$y = -2x^3 + x^2 - 2x + 1 \Rightarrow y = -(2x - 1)(x^2 + 1)$

The sketch of $y = -(2x - 1)(x^2 + 1)$ is as shown.



Notes:

- *The dominant term in the cubic expression is* $-2x^3$.
- *For very large positive values of x, the value of this term is far larger than the sum of the values of the other terms. Hence, this term determines the approximate value of the cubic.*
  That is: as $x \to \infty$, $x^3 \to \infty$,, $-2x^3 \to -\infty$, *and* $y \to -\infty$,.
  Hence, the graph of the cubic tends towards the bottom right of the x-y plane.
- *Similarly, as* $x \to -\infty$,, $x^3 \to -\infty$, *and* $-2x^3 \to \infty$,.
  Hence, the graph of the cubic tends towards the top left of the x-y plane.

Example 3.7

A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 1)(t - 3)(t - 4)$ for $0 \le t \le 4$.

(a) Find the initial displacement of P.

(b) Find when P is O.

(c) Find when P is 2 m from O (accurate to 1 decimal place).

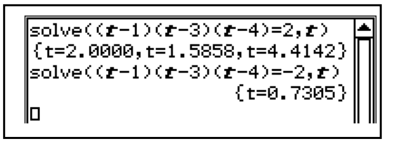
Solution:

(a) When $t = 0$, $s = -12$ metres.

(b) When P is at O, $s = 0$. $\Rightarrow$ $t = 1, 3, 4$ seconds.

(c) When P is 2 m from O, $(t - 1)(t - 3)(t - 4) = 2$ or $(t - 1)(t - 3)(t - 4) = -2$

$t = 2, 1.6, 4.4$ or 0.7 seconds.



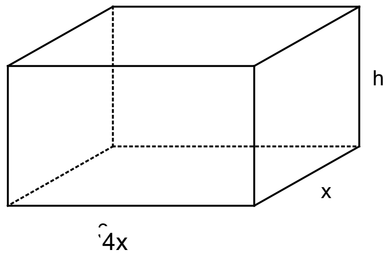
Exercise 3.2

1. Without the use of a calculator, sketch the following curves. Indicate all intercepts.
   (a) $y = x^3 - 2x^2 - x + 2$ (b) $y = x^3 + 3x^2 - 4x - 12$ (c) $y = 2x^3 + x^2 - 2x - 1$
   (d) $y = x^3 + 7x^2 + 16x + 12$ (e) $y = x^3 + 6x^2 + 12x + 8$ (f) $y = 2x^3 - 14x - 12$
   (g) $y = x^3 + x^2 + 3x + 3$ (h) $y = x^3 + 2x^2 + 4x + 3$ (i) $y = 2x^3 + 2x^2 + 8x + 8$

2. Without the use of a calculator, solve each of the following:
   (a) $x^3 - 16x = 0$ (b) $3x^3 - 15x = 0$ (c) $x^3 + 5x^2 + 3x - 9 = 0$
   (d) $x^3 + x^2 - 16x - 16 = 0$ (e) $x^3 - 3x^2 - 4x + 6 = 0$ (f) $x^3 - 11x^2 + 35x - 25 = 0$
   (g) $x^3 - 3x^2 - 10x + 24 = 0$ (h) $2x^3 - 5x^2 - 2x + 5 = 0$ (i) $2x^3 + 3x^2 - 10x + 4 = 0$

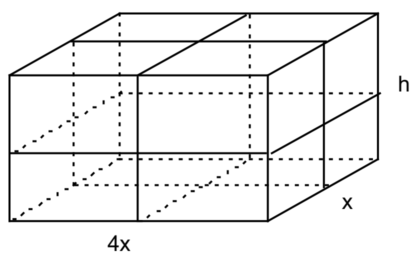
3. Use your calculator to solve for x, to two decimal places:
   (a) $x^3 + 4x^2 - 6x + 11 = 0$ (b) $63x^3 - 33x^2 + x + 1 = 0$ (c) $8x^3 + 4x^2 - 10x + 2 = 0$

4. A piece of wire, 600 cm long is used to make the 12 edges of a rectangular cage. The length of the rectangular cage is 4 times the width of the cage, x cm.



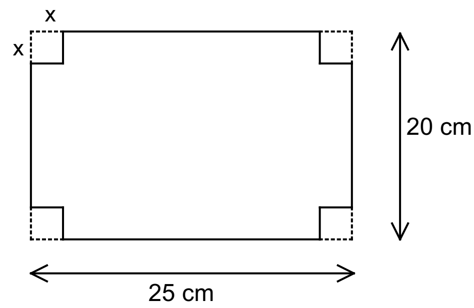
   (a) Show that the height, h, of the rectangular cage is given by, $h = 150 - 5x$.
   (b) Show that the volume, V, of the box is given by $V = 600x^2 - 20x^3$.
   (c) Explain why it is not possible to construct a cage with width greater than 30 cm.
   (d) Find, to the nearest mm, the width of the cage when its volume is 50 000 cm^3.
   (e) Find, to the nearest mm, the height of the cage when its volume is 80 000 cm^3.

5. A piece of wire, 10 m long, is used to make the 24 edges of the frame of a rectangular cage as shown in the accompanying diagram. The length of the cage is 4 times the width of the cage, x cm.



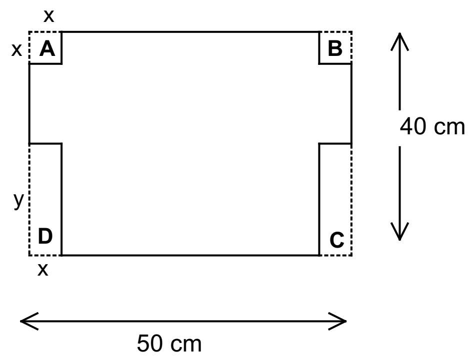
   (a) Show that the height, h, of the cage is given by, $h = 125 - 5x$.
   (b) Show that the volume, V, of the cage is given by $V = -20x^3 + 500x^2$.
   (c) What are the possible values for x? Explain how you obtained your answer.
   (d) Find, to the nearest mm, the width of the cage when its volume is 40 000 cm^3.
   (e) By referring to the graph of V against x, explain why it is not possible to construct a cage with a volume of 50 000 cm^3.

6. A rectangular sheet of cardboard, 25 cm by 20 cm, is to be made into an open rectangular box. Four squares, each of side, x cm, are removed from each corner of the cardboard.



   (a) Show that the volume, V, of the box is given by $V = 2x(25 - 2x)(10 - x)$.
   (b) Why is it impossible to construct a box with *height* 13 cm?
   (c) What are the possible values for V ? Explain how you obtained your answer.
   (d) Find, to the nearest mm, the height of the box when its volume is 500 cm^3.

7. A rectangular sheet of cardboard, 40 cm by 50 cm, is to be made into a closed rectangular box. Squares, each of side x cm, are removed from the corners A and B and rectangles each of dimension x cm by y cm, are removed from the corners C and D of the cardboard as shown.



   (a) Show that the volume, V, of the box is given by $V = x(20 - x)(50 - 2x)$.
   (b) What are the possible values for x and V?
   (c) Find, to the nearest mm, the length of the box when its volume is 2000 cm^3.

8. A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 2)(4 - t)(t - 6)$ for $0 \le t \le 6$.
   (a) Find when the particle is at O.
   (b) Find when P reverses direction
   (c) Find when P is 20 m from O.

9. A particle P moves along a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = (t - 2)^2(t - 7)$ for $0 \le t \le 8$.
   (a) Find when P returns to O. (b) Find when P is 10 m from O.
   (c) Find the distance travelled in the first 8 seconds.
   (d) Find the average speed in the first 8 seconds.

10. An object P is travelling in a straight line. Its displacement, s metres, from a fixed point O, at time t seconds, is given by, $s = 5 - 10t + \frac{1}{2}t^3$ for $0 \le t \le 6$.
   (a) Find the initial displacement of P.
   (b) Find when P is at O.
   (c) Find the farthest P is from O.
   (d) Find when P is 15 m from O.

04 Rectangular Hyperbolae

4.1 Inverse Proportion

- Let x and y be two variables.
- Consider the following table of values for x and y.

x	1	2	3	4	5	6
y	120	60	40	30	24	20

- Notice that for each pair (x, y), the product $xy = 120$. Hence, $y = \dfrac{120}{x}$.
- When the relationship between 2 variables x and y are in the form
  $xy = k$ or $y = \dfrac{k}{x}$, where $k > 0$,
  the two variables are in an inverse proportion relationship.
  That is, y is inversely proportional to x and x is inversely proportional to y.
- Hence, in general, if y is inversely proportional to x:
  - $y \propto \dfrac{1}{x} \quad \Leftrightarrow \quad y = \dfrac{k}{x}$ or $xy = k$ where $k > 0$.
  - the graph of y against x (or x against y) is called a rectangular hyperbola.
- Note that if y is inversely proportional to x, then y is directly proportional to $\dfrac{1}{x}$.

Example 4.1

Find the algebraic relationship between x and y, given that:
(a) y is inversely proportional to $(x + 1)$, and when $x = 5$, $y = 10$
(b) y is inversely proportional to x^2, and when $x = 5$, $y = 10$.

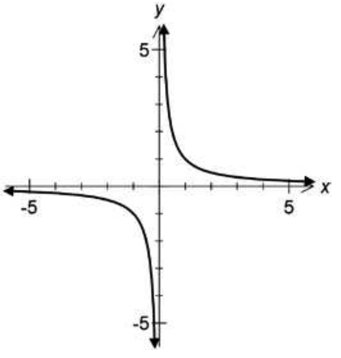
Solution:

(a) y is inversely proportional to $(x + 1)$. $\Rightarrow$ $y \times (x + 1) = k$
When $x = 5$, $y = 10$ $\Rightarrow$ $10 \times (5 + 1) = k$ $\Rightarrow$ $k = 60$
Hence, $(x + 1)\, y = 60$ or $y = \dfrac{60}{x+1}$.

(b) y is inversely proportional to x^2. $\Rightarrow$ $y \times x^2 = k$
When $x = 5$, $y = 10$ $\Rightarrow$ $10 \times 5^2 = k$ $\Rightarrow$ $k = 250$
Hence, $x^2 y = 250$ or $y = \dfrac{250}{x^2}$.

4.2 Rectangular Hyperbolae

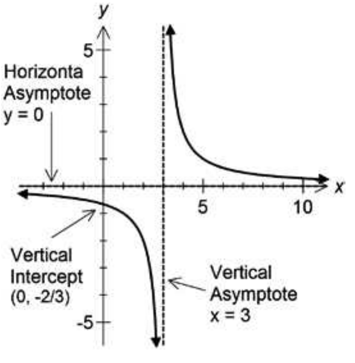
- The graph of the curve with equation $y = \dfrac{k}{x}$ is called a rectangular hyperbola.
- The accompanying diagram shows the graph of the rectangular hyperbola $y = \dfrac{1}{x}$.



    - Notice that there are no intercepts.
    - As x assumes a very large positive value, y has a very small positive value.
      As x assumes a very large negative value, y has a very small negative value.
      That is, $x \to \pm\infty$, $y \to 0$.
      The line y = 0 is termed the *horizontal asymptote*.
    - As x assumes a very small positive value, y has a very large positive value.
      As x assumes a very small negative value, y has a very large negative value.
      That is, $x \to \pm 0$, $y \to \pm\infty$.
      The line x = 0 is termed the *vertical asymptote*.

- In general, the rectangular hyperbola $y = \dfrac{k}{x}$ has:
    - no intercepts
    - a horizontal asymptote with equation y = 0
    - a vertical asymptote with equation x = 0.

- The accompanying diagram shows the graph of the rectangular hyperbola $y = \dfrac{2}{x-3}$. The curve has:



    - a vertical intercept at $\left(0, -\dfrac{2}{3}\right)$
    - no horizontal intercept
    - a horizontal asymptote with equation y = 0
    - a vertical asymptote with equation x = 3.

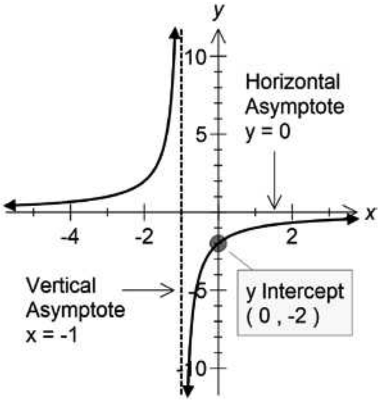
- In general, the rectangular hyperbola $y = \dfrac{k}{x-a}$ has:
    - a vertical intercept at $\left(0, -\dfrac{k}{a}\right)$ but no horizontal intercept.
    - a horizontal asymptote with equation y = 0.
    - a vertical asymptote with equation $x = a$.
- The curve of the rectangular hyperbola is also known as the reciprocal curve.

Example 4.2

Without the use of calculator, sketch $y = \dfrac{-2}{x+1}$. Indicate all essential features of this curve.

Solution:

Vertical Intercept (0, −2).
Vertical Asymptote $x = -1$.
Horizontal Asymptote $y = 0$.



Note:

- *In this example, y is not inversely proportional to x as the constant k is negative.*
- *The curve is part of the rectangular hyperbola family.*

Example 4.3

Find the algebraic relationship between the variables w, x and t given that w is inversely proportional to x and is directly proportional to $\sqrt{t}$ and $w = 500$ when $x = 5$ and $t = 25$.

Solution:

$$w = \frac{k\sqrt{t}}{x}$$

$w = 500$ when $x = 5$ and $t = 25$: $\quad k = \dfrac{500\times 5}{\sqrt{25}} = 500.$

Hence $\quad w = \dfrac{500\sqrt{t}}{x}$

Example 4.4

A project can be completed by 200 workers in 40 weeks. Assume that the time t to complete the project is inversely proportional to the number of workers available w.
(a) How many workers would be required to complete the project in half the time?
(b) How long would one fifth as many workers take to complete the project?

Solution:

(a) $\quad w \times t = k$.

When $w = 200$, $t = 40$: $\quad k = 200 \times 40 = 8000$

Therefore: $\quad w \times t = 8000$.

For $t = 20$ weeks: $\quad w = \dfrac{8000}{20} = 400.$

That is, 400 workers would be required to complete the project in half the time.

(b) For $w = \dfrac{1}{5} \times 200 = 40$: $\quad t = \dfrac{8000}{40} = 200.$

That is, one fifth as many workers would take 200 weeks to complete the project.

Exercise 4.1

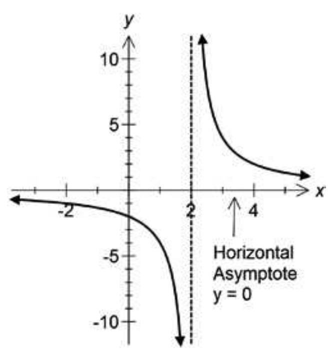
1. Find the algebraic relationship between x and y, given that:
   (a) y is inversely proportional to x, and when $x = -20$, $y = -2$
   (b) x is inversely proportional to y, and when $x = -4$, $y = -15$
   (c) y is inversely proportional to $(x - 1)$, and when $x = 4$, $y = 25$
   (d) y is inversely proportional to $(x + 5)$, and when $x = -3$, $y = 20$
   (e) y is inversely proportional to x^2, and when $x = -2$, $y = 40$
   (f) y is inversely proportional to x^3, and when $x = 5$, $y = 10$
   (g) y is inversely proportional to $\sqrt{x}$, and when $x = 25$, $y = 10$
   (h) x is inversely proportional to y^2, and when $x = 4$, $y = 4$.

2. Without the use of calculator, sketch each of the following curves.
   Indicate all essential features of this curve.
   (a) $y = \dfrac{-2}{x}$ (b) $xy = 4$ (c) $y = \dfrac{1}{x+4}$
   (d) $y = \dfrac{-4}{1-x}$ (e) $y = \dfrac{-1}{2(x+1)}$ (f) $y = \dfrac{1}{2x+4}$
   (g) $(x + 2)\, y = 10$ (h) $(3 - x)\, y = 6$

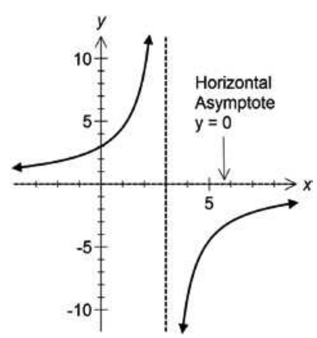
3. Find the equation of the rectangular hyperbola with horizontal asymptote $y = 0$ and :
   (a) vertical asymptote $x = 2$ and vertical intercept (0, 4)
   (b) vertical asymptote $x = -3$ and vertical intercept (0, -9)
   (c) vertical asymptote $x = 4$ and vertical intercept (0, 2)
   (d) vertical asymptote $x = 0.5$ and vertical intercept (0, -2).

4. Find the equation of each of the following hyperbolas.

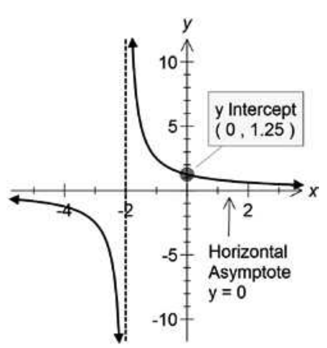
(a)



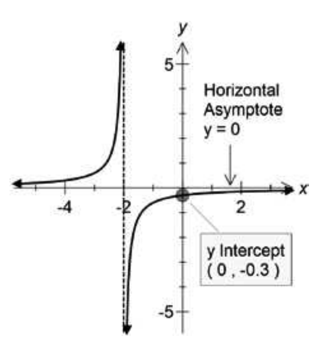
(b)



(c)



(d)



5. The variable w is inversely proportional to the variable x and is directly proportional to the variable y. Find the algebraic relationship between w, x and y given that:
   (a) $w = 500$ when $x = 5$ and $y = 10$ (b) $w = 1600$ when $x = 4$ and $y = 5$

6. The variable w is inversely proportional to the variable x^2 and is directly proportional to the variable y. Find the algebraic relationship between w, x and y given that:
   (a) $w = 2000$ when $x = 10$ and $y = 5$ (b) $w = 3600$ when $x = 3$ and $y = 20$

7. A project can be completed by 100 workers in 45 weeks. Assume that the time to complete the project is inversely proportional to the number of workers available.
   (a) How many workers would be required to complete the project in a third the time?
   (b) How long would one fifth as many workers take to complete the project?

8. The amount of current (I amps) flowing through an electrical circuit is inversely proportional to the total resistance of the circuit (R ohms). The circuit has a device that can alter its resistance. If the current flow is 2 amps when the total resistance is 12 ohms, find:
   (a) the current flow when the resistance is 18 ohms
   (b) the total resistance if the current flow is 5 amps.

9. A road trip can be completed in 5 hours travelling at an average speed of 80 kmh^{-1}.
   (a) What should the average speed be if the journey is to be completed in 6 hours?
   (b) How long would the road trip take if the average speed was 90 kmh^{-1}?

10. The gravitational attraction (F Newtons) between two objects varies inversely as the square of the distance between them (r metres). Given that $F = 6674$ when $r = 10$, find:
    (a) the gravitational attraction when $r = 50$ m
    (b) the distance between the two objects when the gravitational attraction is 1000 N.

11. The pressure (P pascals) exerted by a gas is directly proportional to its temperature (T degrees Kelvin) and inversely proportional to the volume of the gas (V m^3).
    At a fixed temperature, 20 m^3 of a gas exerts a pressure of 50 pascals
    (a) Find: (i) the pressure exerted by 100 m^3 of gas
    (ii) the volume of gas required to exert 100 pascals of pressure.
    (b) If the temperature was doubled, find the pressure exerted by 20 m^3 of a gas.
    (c) If the temperature was halved, what volume of gas would exert a pressure of 50 pascals?

*12. The electrical resistance of a cable (R ohms) is proportional to the length of the cable (L m) and inversely proportional to the square of its diameter (D cm). A cable 1 km long with diameter 0.5 cm has resistance 25 ohms.
    (a) Find the percentage change in resistance of the cable corresponding to a 20% decrease in its diameter (the length remaining unchanged).
    (b) By how much should the diameter of the cable change if its resistance is to be reduced by 40% (the length remaining unchanged)?

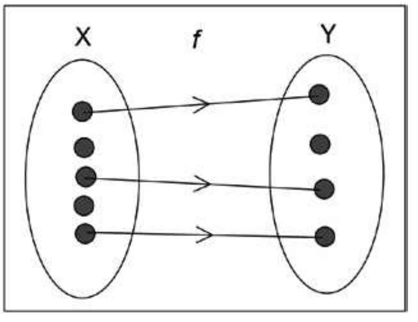
05 Functions & Relations

5.1 Definition of a Relation

- A relation r between sets X and Y is a rule that associates (maps) elements in set X with elements in set Y.

5.2 Definition of a Function

- A function f between sets X and Y is a rule that associates *each* element in set X with a *unique* element in set Y.
- X is called the *domain* of the function f, while Y is the *codomain* of f.
- The set of all images of f in Y is called the *range* of f. The range is a subset of the codomain.



- A function f is said to be an *onto function* if its range is identical to its codomain.
- A function is a relation with the added condition that *every* member in set X must be associated with *exactly one* element in set Y.
  - Where two different elements in X is mapped to the same Y element the function is called a *many-to-one* function.
  - Where no two elements in X is mapped to the same Y element the function is called a *one-to-one* function.
- All functions are relations but not all relations are functions.
- Mathematically, a function is written symbolically as:

$$f : \mathrm{X} \to \mathrm{Y}$$
$$f : x \to 2x + 1$$

  - The first line describes the sets involved in the mapping.
  - The second line is an algebraic rule that describes the actual mapping between the elements in the domain and the codomain and is often written as $f(x) = 2x + 1$. This is referred to as the *function notation*.
- Note that the sets involved as well as the actual mapping rule should be specified.
- Where the domain and codomain of a function is not specified, the *natural domain* is assumed: this is the largest set that qualifies the mapping rule as a function rule. The codomain will then be the *natural range* of the function. This is the set of all images corresponding to the elements in the natural domain.

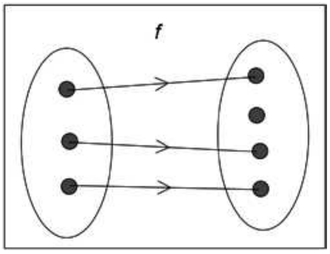
Hands On Task 5.1

In this task, we will explore in greater detail the idea of functions.

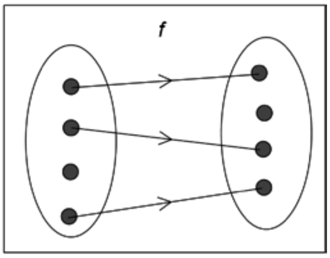
Study the definition of a function carefully and answer the following questions to obtain a better appreciation of what a function is.

1. Determine if the following mappings are functions or otherwise. Justify each answer.

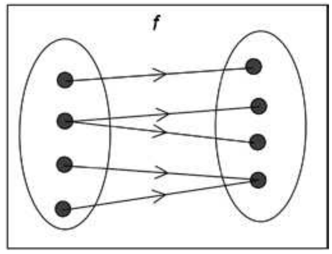
(a)



(b)

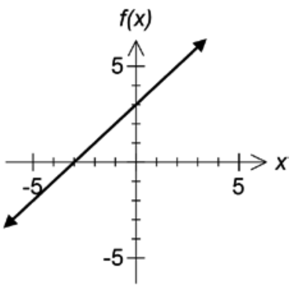


(c)

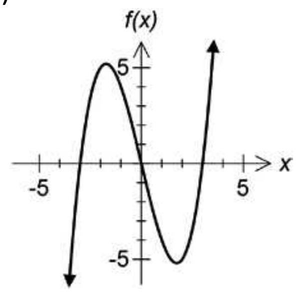


2. Shown below are the graphs of some mapping rules. Determine which of these are rules for functions. Justify your answer in each case.

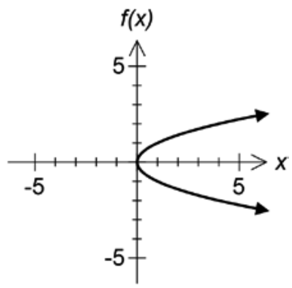
(a)



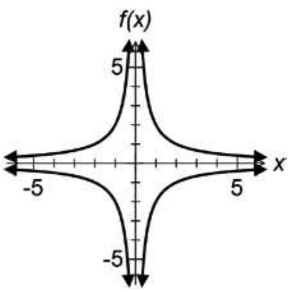
(b)



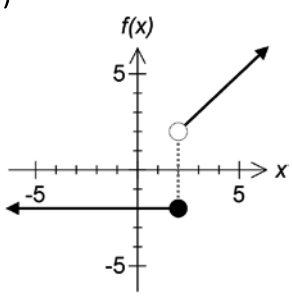
(c)



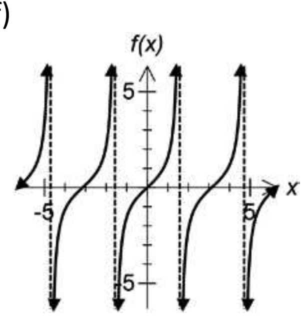
(d)



(e)



(f)



3. Draw a vertical line through each of the graphs shown in Question 2. Determine if a test can be formed using this vertical line to identify the graphs of functions.

4. The following mappings are defined over the set of all real numbers to the set of all real numbers. That is $f: \mathbb{R} \to \mathbb{R}$. Determine if the mapping rules are rules for functions. Justify each answer.

(a) $f(x) = 2x + 1$ (b) $f(x) = 2^x$ (c) $f(x) = \sqrt{(x - 1)}$

(d) $f(x) = 1 - x^2$ (e) $f(x) = 1/x$ (f) $f(x) = x^3 - 2x$

Summary

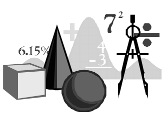
- When a vertical line is drawn through the graph of a function, at most there should only be one point of intersection. This is known as the *vertical line test.*

5.3 Polynomials

- A polynomial of *degree n* is an algebraic expression of the form:

$$a_1x^n + a_2x^{n-1} + a_3x^{n-2} + \ldots + a_nx + a_{n+1}$$

  where the coefficients a_1, a_2, a_3, …, a_{n+1} are all constants.
  - The powers are *all* positive integers.
  - The *degree* of the polynomial refers to the highest power in the expression.
- The linear expression $ax + b$ is a polynomial of degree 1.
  The quadratic expression $ax^2 + bx + c$ is a polynomial of degree 2.
  The cubic expression $ax^3 + bx^2 + cx + d$ is a polynomial of degree 3.
- The graphs of lines, quadratics and cubics all pass the vertical line test.
  In general, graphs of polynomials pass the vertical line test.
  Hence, *all polynomials are functions*.
- The coefficient of the term with the highest power is called the leading coefficient.
  - The leading coefficient of the polynomial $f(x) = -2x^3 + 4x + 10x + 20$ is -2.



Hands On Task 5.2

In this task, we will explore polynomial functions of the form $f(x) = x^n$ where n is a positive integer.

1. On your CAS/graphic calculator graph $y = x^n$ for n = 2, 4, 6, 8, …. .
   Describe clearly the common features of these graphs.

2. On your CAS/graphic calculator graph $y = x^n$ for n = 3, 5, 7, 9, …. .
   Describe clearly the common features of these graphs.

3. Hence, summarize your investigations on the graphs of $y = x^n$ where n is a positive integer.

Summary

- For n as a positive even integer, the graph of $y = x^n$:
  - has a turning point at (0, 0)
  - has a "parabolic" shape.
- For n as a positive odd integer, where $n > 1$, the graph of $y = x^n$:
  - has an inflection point at (0, 0)
  - has a "cubic" shape.

5.4 Domain and Range

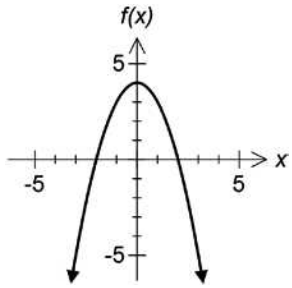
- The natural domain of all polynomials is $\mathbb{R}$, the set of all real numbers.
  - The natural range of a quadratic function is determined by its turning point.
  - The natural range of a cubic function is $\mathbb{R}$, the set of all real numbers.
- The natural domain of a rectangular hyperbola is determined by its vertical asymptote while its natural range is determined by its horizontal asymptote.

Example 5.1

State the natural domain and range for each function as drawn below.

Solution:

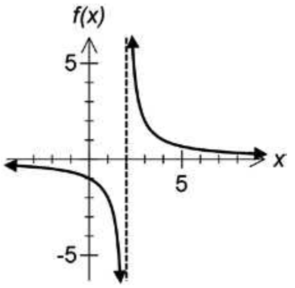
(a)



The natural domain is the set of all real numbers $\mathbb{R}$.

The range of the function is$\{y: y \le 4, y \in \mathbb{R}\}$.

(b)



The natural domain of the function is the set of all real numbers except 2. The domain is written as:
$\{x: x \ne 2, x \in \mathbb{R}\}$.

The natural range is the set of all real numbers except 0. This is written as:
$\{y: y \ne 0, y \in \mathbb{R}\}$.

Example 5.2

State the natural domain and range for each of the following functions:

(a) $f(x) = x^2 + 4x + 5$ (b) $g(x) = \dfrac{-2}{x+4}$

Solution:

(a) $f(x) = x^2 + 4x + 5 \equiv (x + 2)^2 + 1$.

Domain: $\mathbb{R}$ Range: $\{y: y \geq 1, y \in \mathbb{R}\}$.

Notes:

- *$f(x) = (x + 2)^2 + 1 \equiv$ A square number add one.*
- *A square number must be positive. This square number can be zero.*
- *Since, $f(x)$ is a square number add one, $f(x)$ must be at least one. Hence, $f(x) \geq 1$.*
- *Alternatively, the minimum turning point has coordinates $(-2, 1)$.*
  Hence, minimum value for $f(x)$ is 1. That is $f(x) \geq 1$.

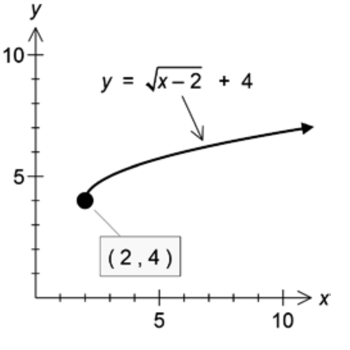
(b) Domain: $\{x: x \neq -4, x \in \mathbb{R}\}$ Range: $\{y: y \neq 0, y \in \mathbb{R}\}$.

Notes:

- *The vertical asymptote is $x = -4$. Hence, x cannot assume the value of -4.*
- *The horizontal asymptote is $y = 0$. Hence, y cannot assume the value of 0.*

5.4.1 Square Root Functions

- Consider the function $f(x) = \sqrt{x-a} + b$.
- $(x - a)$ must not be negative, otherwise, we would find ourselves dealing with the square root of a negative number (which is not a real number).
  - Hence, $x - a \geq 0$. That is $x \geq a$.
  - Therefore, the domain of $f(x)$ is $\{x: x \geq a, x \in \mathbb{R}\}$.
- The expression $\sqrt{x-a}$ is by definition non-negative; that is, $\sqrt{x-a} \geq 0$.
  - Hence, $f(x) \equiv$ A non-negative number add b.
  - Therefore, $f(x) \geq b$. The range for $f(x)$ is $\{y: y \geq b, y \in \mathbb{R}\}$.
- The accompanying diagram shows the graph of $y = \sqrt{x-2} + 4$.
  - The domain is $\{x: x \geq 2, x \in \mathbb{R}\}$.
  - The range is $\{y: y \geq 4, y \in \mathbb{R}\}$.
  - As $x \to \infty$, $y \to \infty$.



Example 5.3

Without the use of a calculator, state the natural domain and range for each of the following functions. Hence, sketch these functions.

(a) $f(x) = \sqrt{x+3} - 5$ (b) $g(x) = 3 - \sqrt{x-2}$

Solution:

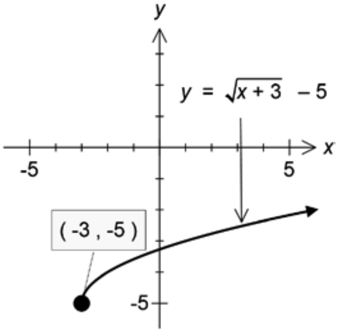
(a) $f(x) = \sqrt{x+3} - 5$

$x + 3 \ge 0 \Rightarrow x \ge -3$.

Hence, domain is $\{x: x \ge -3, x \in \mathbb{R}\}$.

$f(x) = \sqrt{x+3} - 5 \equiv$ A non-negative number less 5.

Therefore, range: $\{y: y \ge -5, y \in \mathbb{R}\}$.



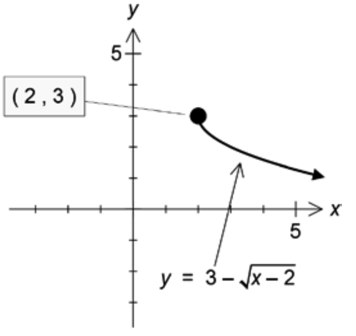
(b) $g(x) = 3 - \sqrt{x-2}$

$x - 2 \ge 0 \Rightarrow x \ge 2$.

Hence, domain is $\{x: x \ge 2, x \in \mathbb{R}\}$.

$g(x) = 3 - \sqrt{x-2} \equiv$ Three less a non-negative number.

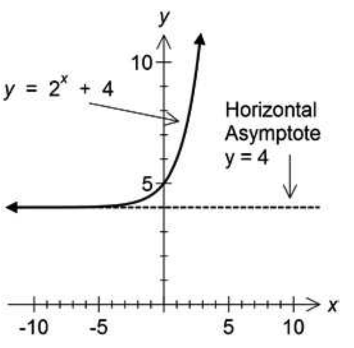
Therefore, range: $\{y: y \le 3, y \in \mathbb{R}\}$.



5.4.2 Exponential Functions

- Consider the function $f(x) = a^x + b$, where $a > 0$.
- The domain of $f(x)$ is $\mathbb{R}$ as all real values of x will yield a value of $f(x)$.
- The expression a^x is always positive. That is $a^x > 0$.
  - Hence, $f(x) \equiv$ A positive number add b.
  - Therefore, $f(x) > b$. The range for $f(x)$ is $\{y: y > b, y \in \mathbb{R}\}$.
- The accompanying diagram shows the graph of $y = 2^x + 4$.
  - The domain is **R**.
  - The range is $\{y: y > 4, y \in \mathbf{R}\}$.
  - As $x \to \infty$, $y \to \infty$.
  - As $x \to -\infty$, $y \to 4$.

  Hence, $y = 4$ is a horizontal asymptote.



Example 5.4

Without the use of a calculator, state the natural domain and range for each of the following functions. Hence, sketch these functions.

(a) $f(x) = 3^x - 2$ (b) $g(x) = 2 - 0.5^x$

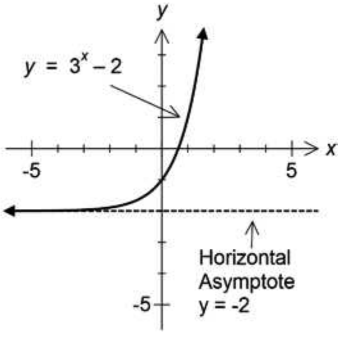
Solution:

(a) $f(x) = 3^x - 2$

Domain is $\mathbb{R}$.

$f(x) = 3^x - 2 \equiv$ A positive number less 2.

Therefore, range: $\{y: y > -2, y \in \mathbb{R}\}$.

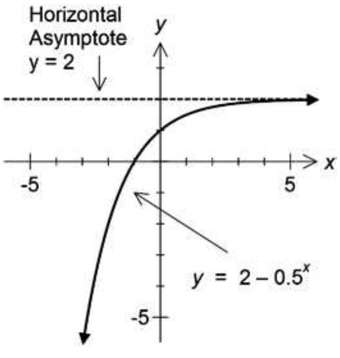


(b) $g(x) = 2 - 0.5^x$

Domain is $\mathbb{R}$.

$f(x) = 2 - 0.5^x \equiv$ 2 less a positive number.

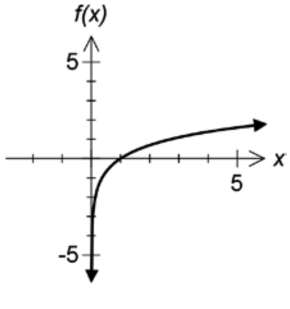
Therefore, range: $\{y: y < 2, y \in \mathbb{R}\}$.



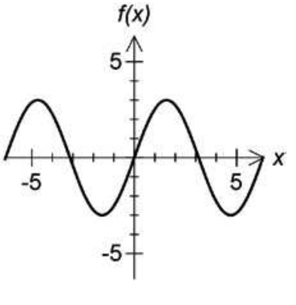
Exercise 5.1

1. Shown below are the graphs of some mapping rules. Determine with reasons which of these are rules of functions. Give the natural domain and range for those that are function rules.

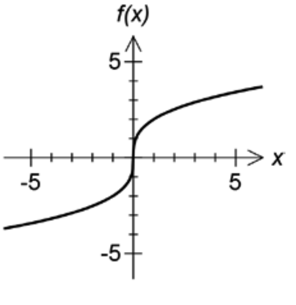
(a)



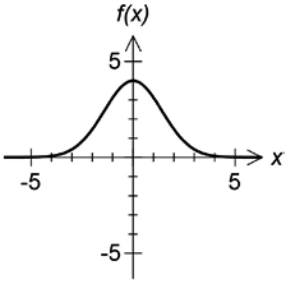
(b)



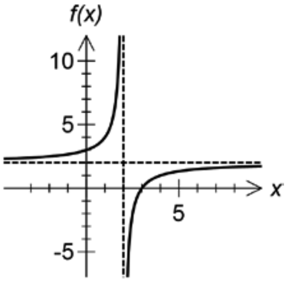
(c)



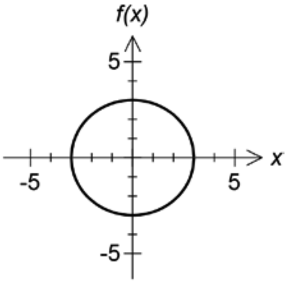
(d)



(e)



(f)



2. Without the use of a calculator, find the natural domain and range for the following functions. Sketch the graphs of each of these functions.
   (a) $f(x) = 1 - (x+1)^2$ (b) $f(x) = (x-2)^2 - 1$ (c) $f(x) = x^2 - 4x + 6$
   (d) $f(x) = \dfrac{1}{x+1}$ (e) $f(x) = \dfrac{1}{4-x}$ (f) $f(x) = \dfrac{2}{3x+1}$
   (g) $f(x) = \sqrt{x+2}$ (h) $f(x) = \sqrt{4-x}$ (i) $f(x) = 3 - \sqrt{x-2}$
   (j) $f(x) = 5^x + 3$ (k) $f(x) = 4 - 3^x$ (l) $f(x) = 0.2^x - 1$

3. Given the following function rules and the domains, find their corresponding ranges.
   (a) $f(x) = 2x - 3, -3 \le x \le 5$ (b) $f(x) = x^2 + 3, -3 \le x \le 5$
   (c) $f(x) = (x-1)^3 + 3, -3 \le x \le 5$ (d) $f(x) = \sqrt{(x+1)}, -1 \le x \le 10$
   (e) $f(x) = \dfrac{1}{4-x}, x > 4$ (f) $f(x) = 1 - 2^x, -2 < x < 2$

4. Given that $f(x) = x^2 + 2x - 2$, find:
   (a) $f(-a)$ (b) $f(2a)$ (c) $f\left(\dfrac{a}{2}\right)$ (d) $f(a-2)$ (e) k such that $f(2k) = f(-k)$.

5. Given that $f(x) = \dfrac{2}{x-1}$, find:
   (a) $f(-a)$ (b) $f\left(\dfrac{a}{2}\right)$ (c) $f(2a+1)$ (d) $f\left(\dfrac{1}{a}\right)$ (e) k such that $f(k) = f\left(\dfrac{1}{k}\right)$.

6. Given that $f(x) = \sqrt{5-x}$, find:
   (a) $f(-a)$ (b) $f\left(\dfrac{a}{2}\right)$ (c) $3 - 2f(a)$ (d) $f(a-5)$ (e) k such that $f(k) = 3 - 2f(k)$.

7. Given that $f(x) = 10^x$, find:
   (a) $f(a)$ (b) $f(-a)$ (c) $1 - 2f(a)$ (d) $f(1-2a)$ (e) k such that $f(k) = f(1-2k)$.

8. Given that $f(x) = 2x + 1$ and $g(x) = x^2 + 5$, find:
   (a) $f(x-1)$ (b) $g(x-3)$ (c) $[f(x)]^2$ (d) $f(x) \times g(x+1)$ (e) k such that $f(k-1) = g(k-3)$

9. Let $f(x) = x - 4$. Find the natural domain and range of g given that $g(x) = f(x^2)$.

*10. Given that $f(x) = \sqrt{x}$, find the natural domain and range of g given that $g(x) = f(x^2)$.

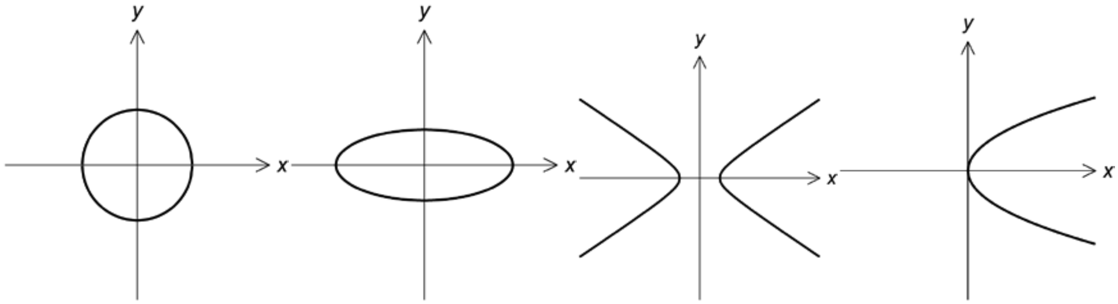
*11. Given that $f(x) = x^2$, find the natural domain and range of g given that $g(x) = f(\sqrt{x})$.

*12. Given that $f(x) = x + 4$ and $g(x) = f\left(\dfrac{1}{x+1}\right)$ find the natural domain and range of g.

*13. Let $f(x) = \sqrt{9-x}$. Find the natural domain and range of g given that $g(x) = f(3^x)$.

5.5 Graphs of some relations

- Recall that functions are relations that "pass the vertical line test.
- The diagrams below show the graphs of some relations that are not functions.



Notice that each of these graphs fail the "vertical line" test.

5.5.1 Circles

- Circles are relations with equations that can be written in the form

$$(x - a)^2 + (y - b)^2 = r^2 .$$

  - The centre of the circle has coordinates (a, b).
  - The radius of the circle has length r.

Example 5.5

Find the coordinates of the centre and the radius of the circle with equation:

(a) $(x + 2)^2 + (y - 4)^2 = 25$ (b) $x^2 + y^2 - 2x + 6y = 6$.

Solution:

(a) Centre of circle has coordinates $(-2, 4)$ and radius of circle = $\sqrt{25} = 5$.

(b) Rewrite as $x^2 - 2x + y^2 + 6y = 6$.
Complete the squares for the x and y terms:

$$(x - 1)^2 - 1 + (y + 3)^2 - 9 = 6$$
$$(x - 1)^2 + (y + 3)^2 = 16.$$

Hence, centre of circle has coordinates $(1, -3)$ and radius of circle = $\sqrt{16} = 4$.

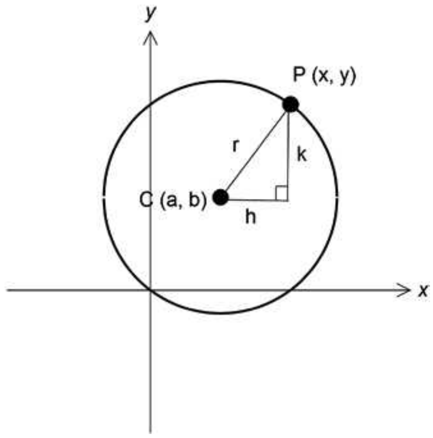
Notes:

- *The general equation of a circle is of the form $ax^2 + ay^2 + bx + cy + d = 0$.*
  Note that the coefficients of the x^2 and y^2 term must be the same and there must be no xy term in the equation.

Hands On Task 5.3

In this task, we will explore how the equation of a circle is constructed.

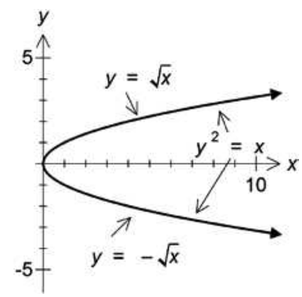
1. A circle is the path traced by a moving point that is always at constant distance from a fixed point. The fixed point is the centre of the circle.



2. The accompanying diagram shows a moving point P (x, y) on a circle with centre at C (a, b) and radius r. Clearly CP = r.

3. Find h and k in terms of a, b, x and y. Hence, show that $r^2 = (x - a)^2 + (y - b)^2$.

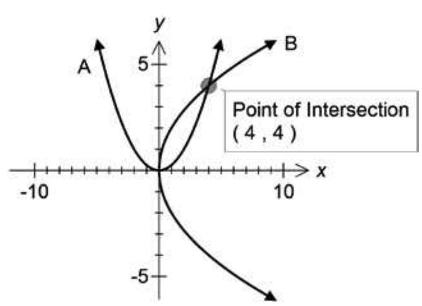
5.5.2 The relation $y^2 = kx$



- Consider the equation $y^2 = x$.
- $y^2 = x$ is the equation of a relation because for a single value of x, *two* values of y are possible. For example, if $x = 4$, $y = -2$ or 2.
- Clearly, the domain of the relation is $\{x: x \geq 0, x \in \mathbb{R}\}$. The range is $\mathbb{R}$.
- The graph of $y^2 = x$ consists of the graphs of $y = \sqrt{x}$ and $y = -\sqrt{x}$ as shown in the accompanying diagram.
- The graph is parabolic in shape and is symmetrical about the x-axis.

Example 5.6

Find the equation of the parabolae A and B as shown in the accompanying diagram.



Solution:

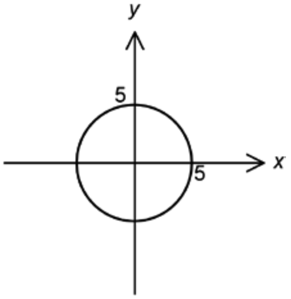
Parabola A has equation $y = ax^2$.
For $x = 4$, $y = 4$: $\quad 4 = 16a \;\Rightarrow\; a = 0.25$
Hence, parabola A has equation $y = 0.25x^2$.

Parabola B has equation $y^2 = bx$.
For $x = 4$, $y = 4$: $\quad 16 = 4a \;\Rightarrow\; b = 4$
Hence, parabola B has equation $y^2 = 4x$.

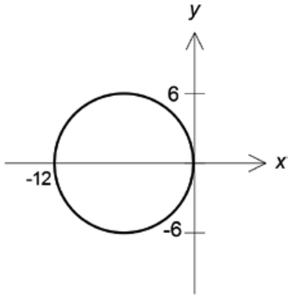
Exercise 5.2

1. Find the equations of the following circles.

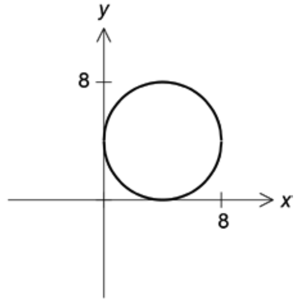
(a)



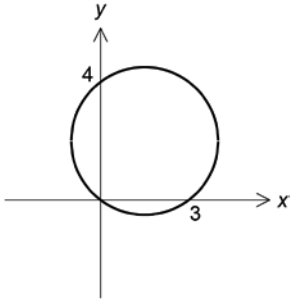
(b)



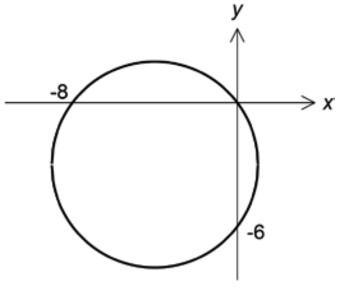
(c)



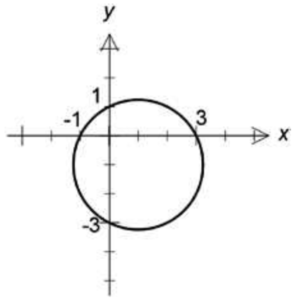
(d)



(e)



(f)



2. Find the equation of the circle in the form $(x - a)^2 + (y - b)^2 = r^2$:
   (a) with centre at $(-2, 3)$ and radius 5 (b) with centre at $(3, 2)$ and diameter 6.

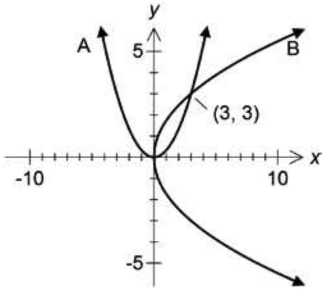
3. Find the equation in the form $x^2 + y^2 + ax + by + c = 0$ of the circle:
   (a) with centre at $(5, -2)$ and radius 8 (b) with centre at $(4, 5)$ and diameter 2.

4. Find the coordinates of the centre and the radius of the following circles.
   (a) $(x - 1)^2 + (y + 2)^2 = 9$ (b) $(x + 10)^2 + (y + 12)^2 = 400$.
   In each case, find the exact coordinates of all the intercepts (if any).

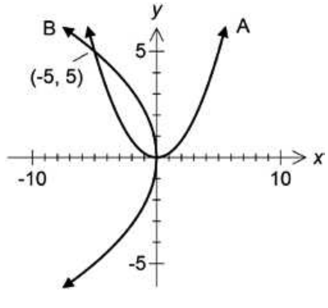
5. Find the coordinates of the centre and the radius of the following circles.
   (a) $x^2 + y^2 + 6x - 14y = 6$ (b) $x^2 + y^2 - 4x + 16y - 32 = 0$.
   In each case, find the exact coordinates of all the intercepts (if any).

6. Find the equation of the following parabolas.

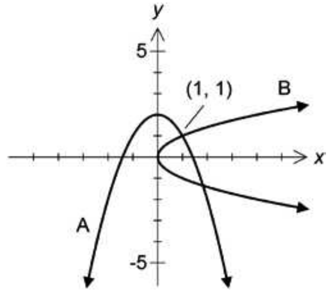
(a)



(b)

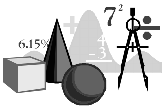


(c)



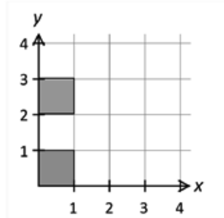
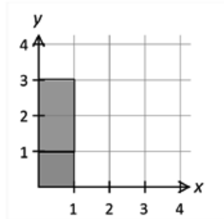
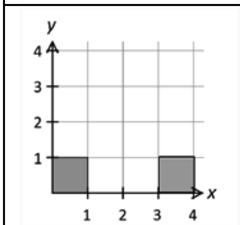
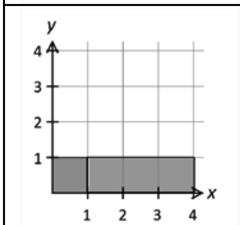
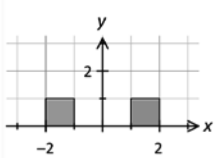
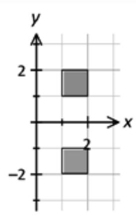
5.6 Transformations on functions

- In this section we will examine the effects certain transformations have on the equation and graph of functions.

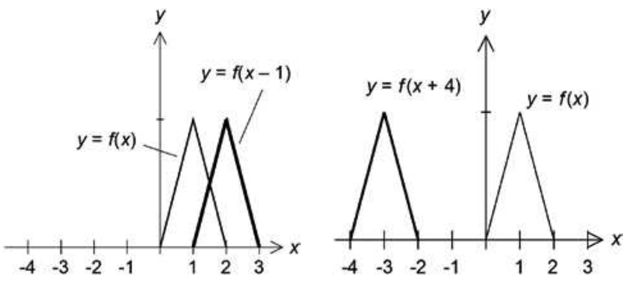


Hands On Task 5.4

Listed below are the transformations and their corresponding geometrical effects on the unit square.

Sketch	Transformation	Geometrical Effects
	Vertical Translation 2 units upwards. (Translation parallel to the y-axis +2 units)	Each point on the body is shifted 2 units upwards parallel to the y-axis.
	Vertical Dilation of factor 3.	The distance of each point to the x-axis (y-coordinate) is multiplied by 3
	Horizontal Translation 3 units to the right (Translation parallel to the x-axis +3 units)	Each point is shifted 3 units to the right parallel to the x-axis.
	Horizontal Dilation of factor 4.	The distance of each point to the y-axis (x-coordinate) is multiplied by 4
	Reflection about the y-axis.	Each point has the sign of its x-coordinate reversed.
	Reflection about the x-axis.	Each point has the sign of its y-coordinate reversed.

1. The accompanying diagrams show the relative positions of the graphs of $y = f(x)$ and $y = f(x-1)$ and $y = f(x)$ and $y = f(x+4)$ on the same set of axes.

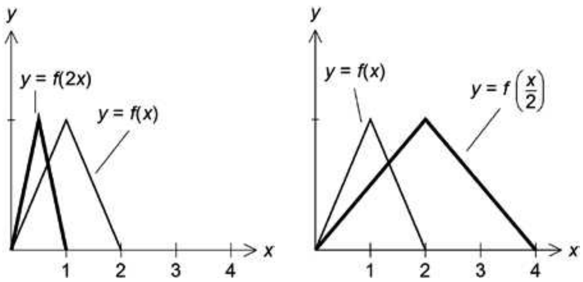


(a) Study the two diagrams carefully and using the table of transformations provided, describe the transformation used to transform $y = f(x)$ to:
(i) $y = f(x-1)$ (ii) $y = f(x+4)$.

(b) On your CAS/graphic calculator graph $y = x^3$.
(i) Now graph $y = (x-2)^3$.
Describe the transformation used to transform $y = x^3$ to $y = (x-2)^3$.
(ii) Now graph $y = (x+1)^3$.
Describe the transformation used to transform $y = x^3$ to $y = (x+1)^3$.

(c) Given the curve with equation $y = \sqrt{x}$, describe the transformations required to transform $y = \sqrt{x}$ into each of the following: (i) $y = \sqrt{x+5}$ (ii) $y = \sqrt{x-4}$.

2. The accompanying diagrams show the relative positions of the graphs of $y = f(x)$ and $y = f(2x)$ and $y = f(x)$ and $y = f\left(\frac{x}{2}\right)$ on the same set of axes.

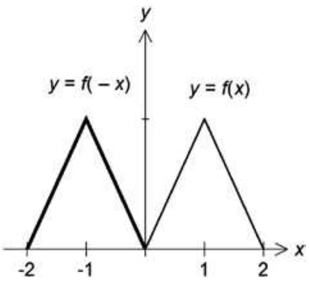


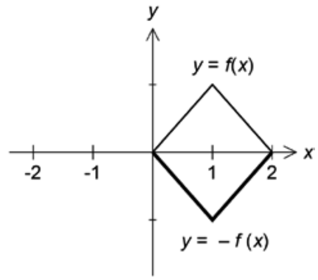
(a) Study the two diagrams carefully and using the table of transformations provided, describe the transformation used to transform $y = f(x)$ to:
(i) $y = f(2x)$ (ii) $y = f\left(\frac{x}{2}\right)$.

(b) On your CAS/graphic calculator graph $y = 16 - x^2$, $y = 16 - (2x)^2$ and $y = 16 - \left(\frac{x}{2}\right)^2$.
(i) Describe the transformation used to transform $y = 16 - x^2$ to $y = 16 - (2x)^2$.
(ii) Describe the transformation used to transform $y = 16 - x^2$ to $y = 16 - \left(\frac{x}{2}\right)^2$.

(c) Given the curve with equation $y = 5^x$, describe the transformations required to transform $y = 5^x$ into each of the following.
(i) $y = 5^{2x}$ (ii) $y = 5^{0.5x}$

3. The accompanying diagrams show the relative positions of the graphs of $y = f(x)$ and $y = f(-x)$ and $y = f(x)$ and $y = -f(x)$ on the same set of axes.



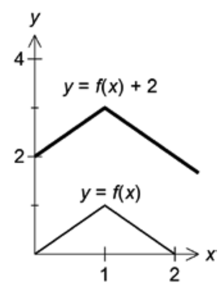


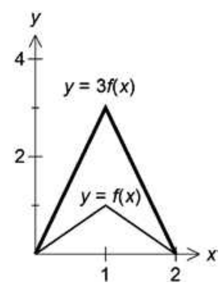
   (a) Study the two diagrams carefully and using the table of transformations provided, describe the transformation used to transform $y = f(x)$ to (i) $y = f(-x)$ (ii) $y = -f(x)$.

   (b) On CAS/your graphic calculator graph $y = x^2 - 4x$, $y = (-x)^2 - 4(-x)$ and $y = -(x^2 - 4x)$. Describe the transformation used to transform:
   (i) $y = x^2 - 4x$ to $y = (-x)^2 - 4(-x)$ (ii) $y = x^2 - 4x$ to $y = -(x^2 - 4x)$.

   (c) Given the curve with equation $y = \sqrt{x+1}$, describe the transformations required to transform $y = \sqrt{x+1}$ into each of the following: (i) $y = \sqrt{1-x}$ (ii) $y = -\sqrt{x+1}$.

4. The accompanying diagrams show the relative positions of the graphs of $y = f(x)$ and $y = f(x) + 2$ and $y = f(x)$ and $y = 3f(x)$ on the same set of axes.





   (a) Study the two diagrams carefully and using the table of transformations provided, describe the transformation used to transform $y = f(x)$ to:
   (i) $y = f(x) + 2$ (ii) $y = 3f(x)$.

   (b) On your CAS/graphic calculator graph
   $y = \dfrac{1}{x+2}$, $y = \dfrac{1}{x+2} + 2$ and $y = \dfrac{3}{x+2}$. Describe the transformation used to transform: (i) $y = \dfrac{1}{x+2}$ to $y = \dfrac{1}{x+2} + 2$ (ii) $y = \dfrac{1}{x+2}$ to $y = \dfrac{3}{x+2}$.

   (c) Given the curve with equation $y = 5^x$, describe the transformations required to transform $y = 5^x$ into each of the following: (i) $y = 5^x - 5$ (ii) $y = 2(5^x)$.

5. (a) Form a tentative summary on the effect of a Horizontal Translation of magnitude k units (where $k > 0$) on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

   (b) Form a tentative summary on the effect of a Horizontal Dilation of factor k units (where $k > 0$) on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

   (c) Form a tentative summary on the effect of a Reflection about the y-axis on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

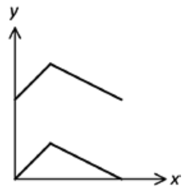
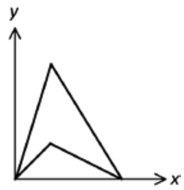
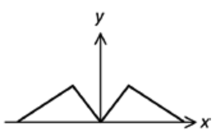
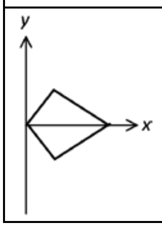
   (d) Form a tentative summary on the effect of a Reflection about the x-axis on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

   (e) Form a tentative summary on the effect of a Vertical Translation of magnitude k units where $k > 0$ on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

   (f) Form a tentative summary on the effect of a Vertical Dilation of factor k (where $k > 0$) on (i) the graph of $y = f(x)$ (ii) the equation $y = f(x)$.

Summary

The table below summarises the effects various transformations have on the points of the curve $y = f(x)$ and on its equation.

Sketch	Transformation	Effect on point (x, y) on $y = f(x)$	Effect on equation $y = f(x)$
	Vertical Translation k units upwards. (Translation parallel to the y-axis $+k$ units)	$(x, y) \to (x, y + k)$	$y = f(x) + k$ Add k to the function part of the equation.
	Vertical Dilation of factor k.	$(x, y) \to (x, ky)$	$y = k\,f(x)$ Multiply the function part of the equation with k.
	Horizontal Translation k units to the right (Translation parallel to the x-axis $+k$ units)	$(x, y) \to (x + k, y)$	$y = f(x - k)$ Replace every "x" term with "$x - k$".
	Horizontal Dilation of factor k.	$(x, y) \to (k\,x, y)$	$y = f\left(\dfrac{x}{k}\right)$ Replace every "x" term with "$\dfrac{x}{k}$"
	Reflection about the y-axis.	$(x, y) \to (-x, y)$	$y = f(-x)$ Replace every "x" term with "$-x$"
	Reflection about the x-axis.	$(x, y) \to (x, -y)$	$y = -f(x)$ Multiply the function part of the equation with "-1".

- The transformations, horizontal dilation, horizontal translation and reflection about the y -axis have a "*term replacement*" effect on the equation of the original curve. For example, a horizontal translation of 1 unit in the direction of the positive x-axis:
  - increases the x-coordinate of each point by 1
  - causes every "x" term in the equation $y = f(x)$ to be replaced by "$x - 1$".
  - Hence, the point (a, b) becomes $(a + 1, b)$ and the equation of the curve $y = f(x)$ becomes $y = f(x - 1)$.

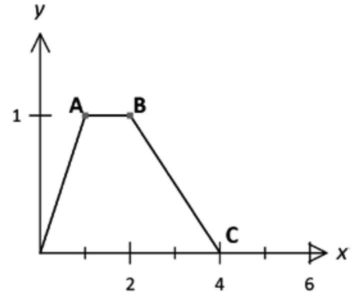
- The transformations, vertical dilation, vertical translation and reflection about the x-axis affect as a single entity, the function part $f(x)$ of the equation.
  For example, a vertical dilation of factor 2:
    - multiplies the y-coordinate of each point of the curve by a factor of 2
    - causes the $f(x)$ part of the equation $y = f(x)$ to be multiplied by 2.
    - Hence, the point (a, b) becomes $(a, 2b)$ and the equation of the curve $y = f(x)$ becomes $y = 2f(x)$.
- A reflection may be considered as a dilation with scale factor of -1.

Example 5.7

The accompanying diagram shows the sketch of $y = f(x)$.

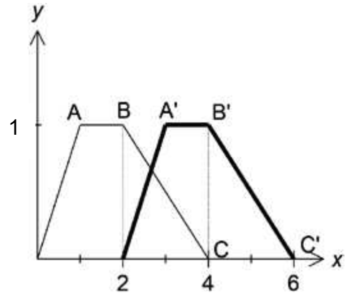
(a) Sketch the curve $y = f(x - 2)$. Indicate the images of the points A, B and C.

(b) Sketch the curve of $y = f(x/2)$. Indicate the images of the points A, B and C.



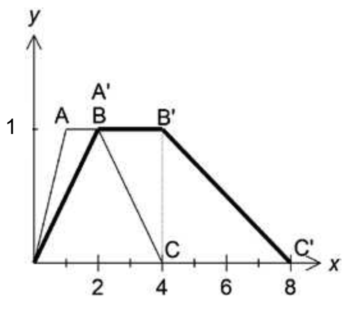
Solution:

(a)



The transformation that maps $y = f(x)$ into $y = f(x - 2)$ is a horizontal translation of magnitude 2 units (to the right). This transformation has the effect of adding 2 units to the x-coordinate of each point of the curve. Hence, the coordinates of A′, B′ and C′ are respectively (3, 1), (4, 1) and (6, 0).

(b)



The transformation that maps $y = f(x)$ into $y = f(x/2)$ is a horizontal dilation of factor 2 units. This transformation has the effect of multiplying the x-coordinate of each point on the curve by 2. Hence, the coordinates of A′, B′ and C′ are respectively (2 ,1), (4, 1) and (8, 0).

Example 5.8

Find the equation of the resulting curve when a *horizontal translation* of magnitude 2 units in the direction of the negative x-axis is applied to the following curves:

(a) $y = x^3 - x^2 - 2x + 4$ (b) $y = 3^{2x+1}$ (c) $y = \dfrac{1}{1-3x}$

***Solution**:*

(a) *A horizontal translation 2 units in the direction of the negative x-axis has a "term replacement" effect on the equation of the curve. Each "x" term in the original curve is replaced by "x + 2".*

Hence $y = x^3 - x^2 - 2x + 4$ becomes:

$$\begin{aligned} y &= (x + 2)^3 - (x + 2)^2 - 2(x + 2) + 4 \\ &= x^3 + 6x^2 + 12x + 8 - (x^2 + 4x + 4) - 2(x + 2) + 4 \\ &= x^3 + 6x^2 + 12x + 8 - x^2 - 4x - 4 - 2x - 4 + 4 \\ &= x^3 + 5x^2 + 6x + 4 \end{aligned}$$

(b) Similarly $y = 3^{2x+1}$ becomes $y = 3^{2(x+2)+1}$ $\Rightarrow$ $y = 3^{2x+5}$.

(c) $y = \dfrac{1}{1-3x}$ becomes $y = \dfrac{1}{1-3(x+2)}$. $\Rightarrow$ $y = \dfrac{-1}{3x+5}$

Example 5.9

Find the equation of the resulting curve when the curve with equation $y = \sqrt{2x+1}$ is subjected to each of the following transformations:

(a) a reflection about the y-axis (b) a horizontal dilation of factor $\dfrac{1}{2}$.

(c) a vertical translation 3 units in the direction of the positive y-axis.

Solution:

(a) *A reflection about the y-axis has a "term replacement" effect on the equation of the original curve. Each "x" term is replaced with a "–x" term.*

Hence, $y = \sqrt{2x+1}$ becomes $y = \sqrt{2(-x)+1}$. $\Rightarrow$ $y = \sqrt{1-2x}$.

(b) *A horizontal dilation of factor of 1/2 has a "term replacement" effect on the equation of the original curve. Each "x" term is replaced with a "x/(1/2) ≡ 2x" term.*

Hence, $y = \sqrt{2x+1}$ becomes $y = \sqrt{2(2x)+1}$. $\Rightarrow$ $y = \sqrt{4x+1}$.

(c) *A vertical translation of 3 units in the direction of the positive y-axis adds 3 to the function part of the equation.*

Hence, $y = \sqrt{2x+1}$ becomes $y = 3 + \sqrt{2x+1}$.

Exercise 5.3

1. The curve $y = f(x)$ has intercepts at (1, 0), (3, 0) and (0, 2). Find the equation of the resulting curve in the form $y = a + b\,f(cx + d)$ and the coordinates of the images of these intercepts when the following transformations are applied to $y = f(x)$:
   (a) a reflection about the x-axis
   (b) a reflection about the y-axis
   (c) a horizontal translation of 3 units in the direction of the positive x-axis
   (d) a horizontal dilation of factor 3.

2. The curve $y = f(x)$ has a maximum point at (2, 5) and a minimum point at (–2, –5). Determine the equation of the resulting curve in the form $y = a + b\,f(cx + d)$ and the coordinates of the maximum and minimum points when the following transformations are applied to $y = f(x)$.
   (a) a vertical translation of 3 units in the direction of the negative y-axis
   (b) a horizontal translation of 2 units in the direction of the positive x-axis
   (c) a horizontal dilation of factor 0.5
   (d) a vertical dilation of factor 4
   (e) a reflection about the x-axis.

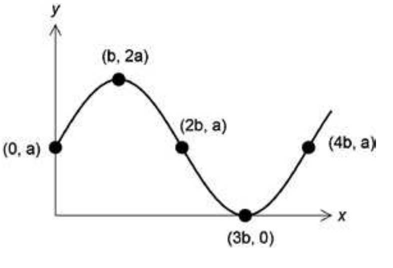
3. Describe the transformation required to map $y = \sqrt{x}$ to each of the following:
   (a) $y = 3\sqrt{x}$
   (b) $y = \sqrt{3x}$
   (c) $y = \sqrt{0.25x}$
   (d) $y = 3 + \sqrt{x}$
   (e) $y = \sqrt{x} - 3$
   (f) $y = \sqrt{x+3}$
   (g) $y = \sqrt{x-3}$
   (h) $y = -\sqrt{x}$
   (i) $y = \sqrt{-x}$.

4. Describe the transformation required to map each of the following curves back to the curve with equation $y = 2^x$:
   (a) $y = 5\,(2^x)$
   (b) $y = 2^{5x}$
   (c) $y = 2^{0.2x}$
   (d) $y = 5 + 2^x$
   (e) $y = 2^x - 5$
   (f) $y = 2^{x+5}$
   (g) $y = 2^{x-5}$
   (h) $y = -2^x$
   (i) $y = 2^{-x}$

5. The curve $y = f(x)$ has intercepts at (2, 0) and (0, –4). Find the intercepts of the curves with the following equations:
   (a) $y = 2\,f(x)$
   (b) $y = f(0.5x)$
   (c) $y = f(2x)$
   (d) $y = f(-x)$
   (e) $y = -f(x)$
   (f) $y = |f(x)|$

6. The curve $y = f(x)$ has a minimum point at (4, 1) and a maximum point at (–3, 6). Find the coordinates of the minimum and maximum points of the curves with the equations:
   (a) $y = -1 + f(x)$
   (b) $y = f(x + 2)$
   (c) $y = f(-2 + x)$
   (d) $y = f(0.25x)$
   (e) $y = f(-x)$
   (f) $y = -f(x)$.

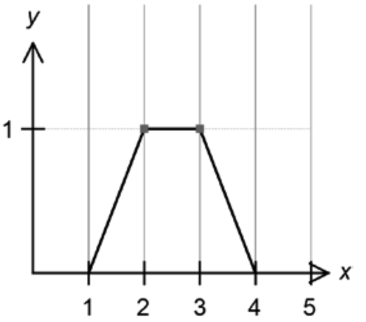
7. The accompanying diagram shows the sketch of $y = f(x)$. Find, in terms of a and b the coordinates of:
   (a) the minimum point of $y = f(x - 2)$,
   (b) the turning points and its nature of $y = -f(x)$,
   (c) the x-intercept(s) for $y = f(x) - a$.



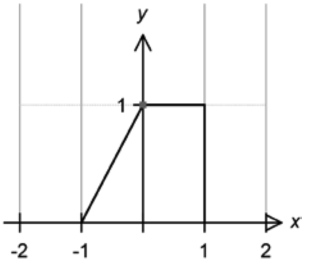
8. Find the equation of the resulting curve when the curve with equation $y = -x^2 - 2x + 1$ is subjected to each of the following transformations:
   (a) a reflection about the x-axis
   (b) a reflection about the y-axis
   (c) a horizontal dilation of factor 4
   (d) a horizontal dilation of factor 0.25
   (e) a vertical translation 2 units in the direction of the negative y-axis
   (f) a horizontal translation 2 units in the direction of the positive x-axis
   (g) a vertical dilation of factor 3.

9. Describe the transformation that maps:
   (a) $y = 10^x$ to $y = 10^{x+1}$
   (b) $y = \sqrt{2x}$ to $y = 2 + \sqrt{2x}$
   (c) $y = \dfrac{1}{x-3}$ to $y = \dfrac{1}{3-x}$
   (d) $y = 4^{x+1}$ to $y = 4^{x-2}$
   (e) $y = \sqrt{x+1}$ to $y = \sqrt{4x+1}$
   (f) $y = \dfrac{1}{x+2} + 5$ to $y = \dfrac{1}{x+2} - 5$

10. Given the sketch of the curve $y = f(x)$, sketch the curve obtained under each of the following transformations:
    (a) a reflection about the y-axis
    (b) a reflection about the x-axis
    (c) a horizontal translation of 1 unit in the direction of the negative x-axis
    (d) a vertical translation of 1 unit in the direction of the positive y-axis
    (e) a horizontal dilation of factor 2
    (f) a vertical dilation of factor 1/2.



11. Given the sketch of the curve $y = f(x)$ below, sketch the curve with equation:
    (a) $y = f(x + 2)$
    (b) $y = f(x) + 2$
    (c) $y = 2f(x)$
    (d) $y = f(0.5x)$
    (e) $y = -f(x)$
    (f) $y = f(-x)$



5.7 Compound Transformations

- In this section we will examine the effects of successive applications of transformations on a function and its graph.
- Let the curve $y = f(x)$ be mapped to $y = -a f(-bx + c) + d$ where a, b, c and d are positive constants. To determine the order in which the various transformations are applied to achieve the desired result it is important to remember that:
  - horizontal translations, horizontal dilations and reflections about the y-axis have term replacement effects on the equation of the curve
  - vertical translations, vertical dilations and reflections about the x-axis affect the $f(x)$ part of the equation as a whole.
- There are numerous ways of achieving the desired result.
  One possible sequence would be:
  1. Horizontal translation of c units in the direction of the negative x-axis
  2. Horizontal dilation of factor $\frac{1}{b}$
  3. Reflection about the y-axis
  4. Vertical dilation of factor a (or reflection about the x-axis)
  5. Reflection about the x-axis (or vertical dilation of factor a)
  6. Vertical translation of d units in the direction of the positive y-axis
- This is schematically represented below:

$f(x) \to f(x + c)$	every "x" term is replaced by "$x + c$"
$\to f(bx + c)$	every "x" term is replaced by "bx"
$\to f(-bx + c)$	every "x" term is replaced by "$-x$"
$\to a\,f(-bx + c)$	$f(-bx + c)$ is multiplied by "a"
$\to -a\,f(-bx + \text{c})$	$a\,f(-bx + c)$ is multiplied by "-1"
$\to -a\,f(-bx + c) + d$	"d" is added to $-a\,f(-bx + c)$.

- Note that a dilation with a negative scale factor k, consists of a dilation of scale factor $|k|$ followed by a reflection about the appropriate axis or vice-versa.
- For the rest of this section, we will limit ourselves to two successive applications of the transformations discussed.

Example 5.10

The curve with equation $y = f(x)$ has a maximum point at (2, 6). Find the coordinates of the maximum point of the resulting curve when the curve $y = f(x)$ is translated 2 units along the positive x-axis, then horizontally dilated by a factor of 2.

Solution:

The horizontal translation of 2 units along the positive x-axis maps (2, 6) to (4, 6).
The horizontal dilation of factor 2 maps (4, 6) to (8, 6).
Hence, the coordinates of the required point is (8, 6)

Example 5.11

Describe the sequence of transformations that maps the curve with equation $y = f(x)$ to $y = f(2x - 1)$. Hence determine the image of the point (5, 2) under this sequence of transformations.

Solution:

Transformation	Equation
Horizontal Translation 1 unit right	$y = f(x) \rightarrow y = f(x-1)$ *Every "x" term is replaced by "x – 1".*
Horizontal Dilation factor $\frac{1}{2}$	$y = f(x-1) \rightarrow y = f(2x-1)$ *Every "x" term is replaced by "2x".*

Hence, $(5, 2) \xrightarrow{\text{Right 1 unit}} (6, 2) \xrightarrow{\text{Dilate horizontally factor 1/2}} (3, 2)$.

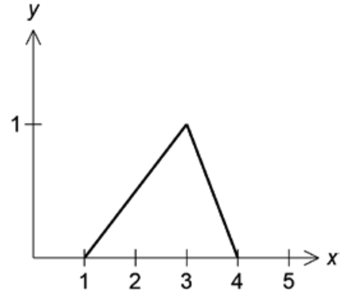
Alternative Solution:

Transformation	Equation
Horizontal Dilation factor $\frac{1}{2}$	$y = f(x) \rightarrow y = f(2x)$ *Every "x" term is replaced by "2x".*
Horizontal Translation $\frac{1}{2}$ unit right	$y = f(2x) \rightarrow y = f\left(2(x - \frac{1}{2})\right) = f(2x-1)$ *Every "x" term is replaced by "x – $\frac{1}{2}$"*

Hence, $(5, 2) \xrightarrow{\text{Dilate horizontally factor 1/2}} (\frac{5}{2}, 2) \xrightarrow{\text{Right 1/2 unit}} (3, 2)$.

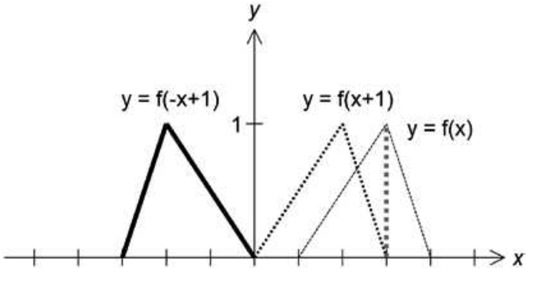
Example 5.12

Given the sketch of $y = f(x)$, sketch the curve with equation $y = f(1 - x)$.



Solution:

Transformation	Equation
Horizontal Translation 1 unit left	$y = f(x) \rightarrow y = f(x+1)$ *Each "x" term is replaced by "x + 1".*
Reflect about the y-axis	$y = f(x+1) \rightarrow y = f(-x+1)$ *Each "x" term is replaced by "–x".*



Exercise 5.4

1. The curve $y = f(x)$ has intercepts at $(-1, 0)$ and $(0, -2)$. Determine the equation of the resulting curve (using the function notation) and the coordinates of the images of the given intercepts when the following transformations are applied on $y = f(x)$ in succession:
   (a) a reflection about the x-axis followed by a reflection about the y-axis
   (b) a horizontal translation of 1 unit in the direction of the positive x-axis followed by a a horizontal dilation of factor 2.

2. The curve $y = f(x)$ has a maximum point at $(-1, 6)$ and a minimum point at $(3, -6)$. Determine the equation of the resulting curve (using the function notation) and the coordinates of the maximum and minimum points of the resulting curve when the following transformations are applied on $y = f(x)$ in succession:
   (a) a vertical translation of 1 unit in the direction of the positive y-axis followed by a horizontal translation of 2 units in the direction of the negative x-axis
   (b) a vertical dilation of factor 1/3 followed by a horizontal dilation of factor 4.

3. Describe the transformations required to map $y = \dfrac{1}{x}$ to each of the following:

   (a) $y = \dfrac{1}{2x+2}$ (b) $y = \dfrac{1}{2-x}$ (c) $y = \dfrac{1}{0.5x-2}$

   (d) $y = 2 + \dfrac{2}{x}$ (e) $y = -\dfrac{1}{x} - 2$ (f) $y = \dfrac{1}{2(x-1)}$

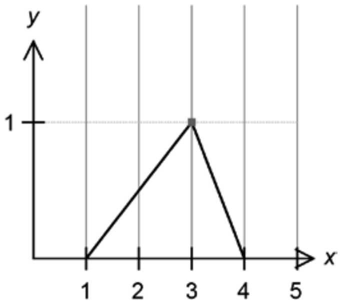
4. Describe the transformations required to map $y = \sqrt{x}$ to each of the following:
   (a) $y = \sqrt{1+2x}$ (b) $y = \sqrt{0.5x-1}$ (c) $y = \sqrt{1-x}$
   (d) $y = 5 + 2\sqrt{x}$ (e) $y = -\sqrt{x} - 5$ (f) $y = -2\sqrt{x+1}$

5. The curve $y = f(x)$ has a minimum point at $(4, 1)$ and a maximum point at $(-3, 6)$. Find the coordinates of the minimum and maximum points of the curves with the equations:
   (a) $y = -1 + f(0.5x)$ (b) $y = f(2x + 3)$ (c) $y = -f(-2 + x)$
   (d) $y = -1 + f(x - 1)$ (e) $y = 0.5f(4x)$ (f) $y = 1 - 4f(x)$

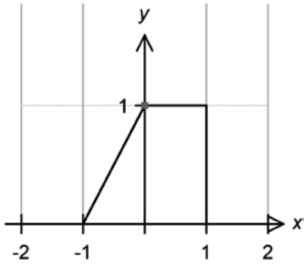
6. Find the equation of the resulting curve when the curve with equation $y = -x^2 - 2x + 1$ is subjected to the following transformations in succession:
   - a horizontal dilation of factor 4
   - a horizontal translation 1 unit in the direction of the negative x-axis

7. The curve $y = (x + 1)^2$ is transformed to the curve $y = (x - 2)^2 - 4$ by a series of transformations. Determine the transformations used and the order in which they were applied.

8. Given the sketch of the curve $y = f(x)$ below, sketch the curve obtained when the following transformations are applied in succession:
   - a reflection about the x-axis
   - a horizontal translation of 1 unit left.

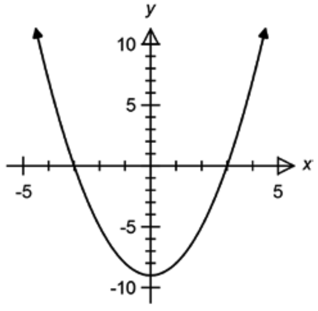


9. The sketch of $y = f(x)$ is given in the accompanying diagram.
   (a) Sketch the graph of $y = -2f(x) - 2$.
   (b) Sketch the graph of $y = f(-x + 2)$.

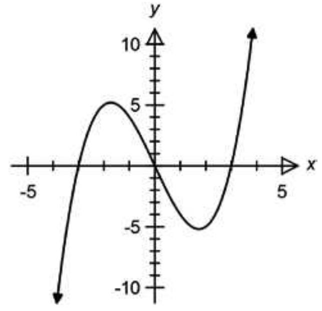


10. Given the graph of $y = f(x)$, sketch the graphs of the following:

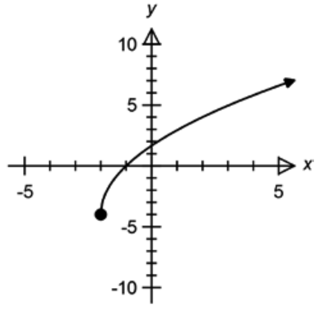
(a) $y = f(1 + 2x)$



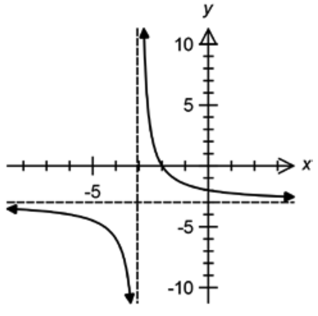
(b) $y = f(2x) + 1$



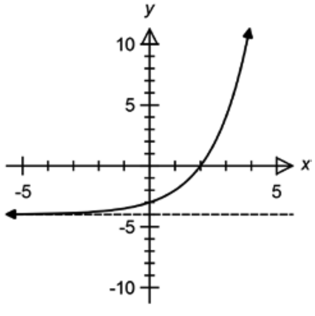
(c) $y = f[(x + 2)/2]$



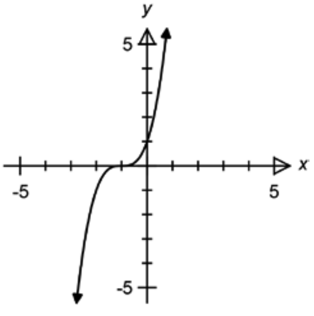
(d) $y = f(x/2 - 2)$



(e) $y = f(1 - 2x)$



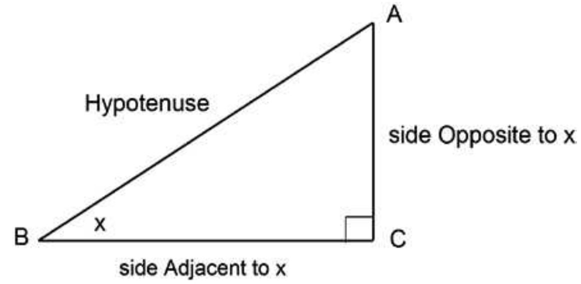
(f) $y = -f(-x)$



06 Right Triangle Trigonometry

6.1 Trigonometric Ratios

- The accompanying diagram shows a right angled triangle ABC.
- Let $\angle ABC = x$.



- The sine of x, $\sin x = \dfrac{\text{Length of the side opposite to } x}{\text{Length of the hypotenuse}}$.

  This is abbreviated to $\sin x = \dfrac{\text{Opposite}}{\text{Hypotenuse}}$ $\boxed{S = \dfrac{O}{H}}$

- The cosine of x, $\cos x = \dfrac{\text{Length of the side adjacent to } x}{\text{Length of the hypotenuse}}$.

  This is abbreviated to $\cos x = \dfrac{\text{Adjacent}}{\text{Hypotenuse}}$ $\boxed{C = \dfrac{A}{H}}$

- The tangent of x, $\tan x = \dfrac{\text{Length of the side opposite to } x}{\text{Length of the side adjacent to } x}$.

  This is abbreviated to $\tan x = \dfrac{\text{Opposite}}{\text{Adjacent}}$ $\boxed{T = \dfrac{O}{A}}$

- The labels "*opposite*" and "*adjacent*" are not fixed and change with respect to the angle of interest.
- These ratios are used to find unknown sides and angles in a right angled triangle.
- Note that in ΔABC, $\sin x = \dfrac{AC}{AB}$. However, $\cos(90^\circ - x) = \dfrac{AC}{AB}$.

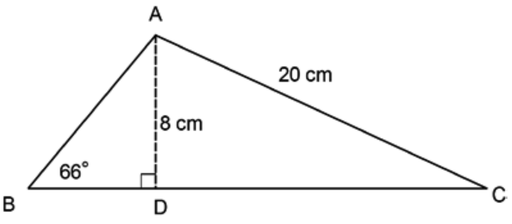
  Also, $\cos x = \dfrac{BC}{AB}$ and $\sin(90^\circ - x) = \dfrac{BC}{AB}$.

  Hence, we conclude that:

> - $\sin x = \cos(90^\circ - x)$
> - $\cos x = \sin(90^\circ - x)$

Example 6.1

In triangle ABC, $\angle ABD = 66^{\circ}$, AD = 8 cm and AC = 20 cm. AD is perpendicular to BC.
(a) Find BD (2 decimal places).
(b) Find $\angle DAC$ (1 decimal place).
(c) Find BC (2 decimal places).



Solution:

(a) In triangle ABD:

$$\tan 66^{\circ} = \frac{8}{BD} \quad \Rightarrow BD = \frac{8}{\tan 66^{\circ}}$$
$$= 3.5618$$
$$= 3.56 \text{ (2 decimal places)}$$

(b) In triangle ADC:

$$\cos \angle DAC = \frac{8}{20} \quad \Rightarrow \angle DAC = \cos^{-1}\left(\frac{8}{20}\right)$$
$$= 66.4^{\circ}$$

(c) In triangle ADC, using Pythagoras' Theorem:

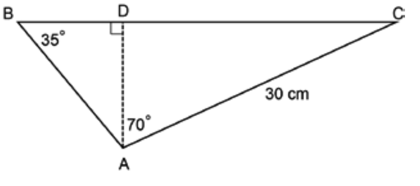
$DC^2 + 8^2 = 20^2 \quad \Rightarrow \quad DC = \sqrt{336} = 18.3303$

In triangle ABC, $\quad BC = BD + DC$.

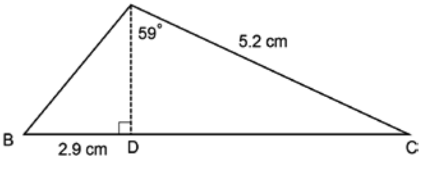
$\Rightarrow \quad BC = 3.5618 + 18.3303 = 21.89$ (2 decimal places)

Exercise 6.1

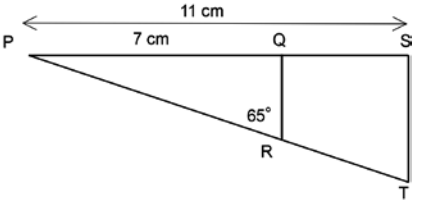
1. In ΔABC, AD is perpendicular to BC and AC = 30 cm. $\angle ABD = 35^{\circ}$ and $\angle DAC = 70^{\circ}$. Find BC (2 decimal places).



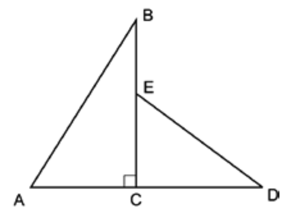
2. In ΔABC, BD = 2.9 cm and AC = 5.2 cm. $\angle BDA = 90^{\circ}$ and $\angle DAC = 59^{\circ}$. Find $\angle CAB$ (nearest degree).



3. Triangles PQR and PST are right triangles as shown in the accompanying diagram. PQ = 7 cm and PS = 11 cm. $\angle PRQ = 65^{\circ}$. Find RT (2 decimal places).



4. ΔABC is a right triangle with AB = 40 cm and $\angle BAC = 72^\circ$. CD = 31 cm and $\angle CDE = 47^\circ$. Find the area of ΔABE (1 decimal place).



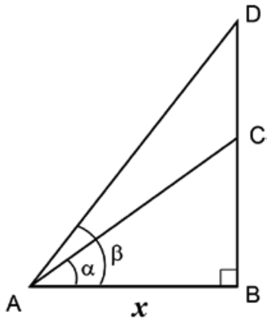
5. The Principal of a College steps onto the school oval at A and walks to B, 28 metres due N30°W. The Principal then turns and walks a further 45 metres in the direction α° bearing ($0 < \alpha < 90^\circ$) to meet a group of students at E. The Principal is now 53 metres from where he started. Show that $\angle ABE = 90^\circ$. Hence, find the bearing of this group of students from the point where stepped onto the school oval.

6. From T, the top of a 50 m high observation tower, Greg spots a kangaroo hop from A to B in 20 seconds. The angle of depression of the kangaroo changed by 10° as it hopped from A to B, with B further away from the tower than A. Greg estimates that the distance between A and K the foot of the observation tower is 60 m. Assume that the kangaroo hopped in a straight line and the points A, B and K are collinear. Find the speed of the kangaroo. Justify your answer.

7. A traffic drone hovering stationary 100 m vertically above a highway, measures that the angle of depression of a truck moving along the highway undergoes a change of 2° in 1 second. If the initial distance (direct) between the drone and the truck is 200 m and is increasing with time, determine if the truck is speeding. Assume that the highway runs in a straight line. The speed limit on the highway is 60 km per hour. Justify your answer.

*8. A lamp is located at L the top of a 15 metre vertical tower LT. A vertical pole of height h is located at P due West of the tower. Light from the lamp casts a shadow of length 3 m due West of P. A second vertical pole of height $1.5h$ is located at X due South of the tower. Light from the lamp casts a shadow of length 5 m due South of X. Find h if the distance between P and X is 20 metres. [Hint: Draw two separate diagrams, to find PT and XT in terms of h.]

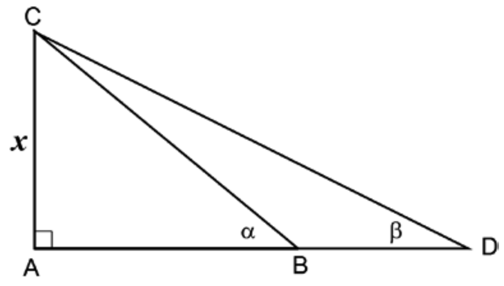
9. Triangles ABC and ABD are right triangles. $\angle CAB = \alpha$, $\angle DAB = \beta$ and AB = x. Prove that CD = $x(\tan\beta - \tan\alpha)$.



10. Triangles ABC and ADC are right triangles.
AC = x, $\angle CBA = \alpha$ and $\angle ADC = \beta$.
Prove that the area of ΔBDC is given by

$$\text{Area} = \frac{x^2(\tan\alpha - \tan\beta)}{2\tan\alpha\tan\beta}.$$

[You may need to use the "simplify" command on your CAS calculator for this question.]



11. *[You may need to use the "simplify" command on your CAS calculator for this question.]*

ΔABC, ΔADC, ΔADE and ΔDFC are right triangles.
AB = 1 unit and $\angle BAC = \theta$ and $\angle DAC = \phi$.

(a) Find AC in terms of θ.

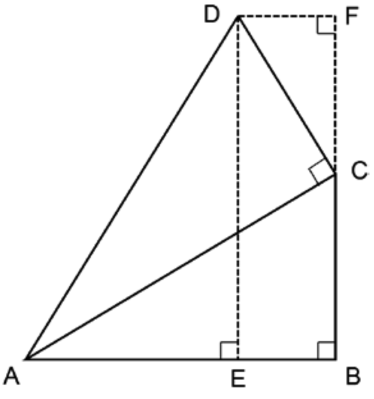
(b) Show that AD = $\dfrac{1}{\cos\theta\cos\phi}$ and DC = $\dfrac{\sin\phi}{\cos\theta\cos\phi}$.

(c) In ΔDFC, explain clearly why $\angle DCF = \theta$.

(d) Show that DF = $\dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}$.

(e) Show that AE = $\dfrac{\cos\theta\cos\phi - \sin\theta\sin\phi}{\cos\theta\cos\phi}$.

(f) Hence, prove that $\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$.



*12. Referring to the diagram in Question 11 and using some of the results in Question 11 where appropriate:

(a) Show that CB = $\dfrac{\sin\theta}{\cos\theta}$ and CF = $\dfrac{\sin\phi}{\cos\phi}$.

(b) Show that DE = $\dfrac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi}$.

(c) Hence, prove that $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

*13. Referring to the diagram in Question 11 and using some of the results in Question 11 where appropriate, prove that $\tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$.

6.2 Exact Ratios

- Often in Mathematics and Engineering, exact values are required. The table below lists the exact values for the trigonometric ratios of some acute angles.

Angle θ^o	$\sin\theta$	$\cos\theta$	$\tan\theta$
0^o	0	1	0
30^o	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45^o	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60^o	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90^o	1	0	∞

Example 6.2

Without using your calculator, given that $\tan 45^o = 1$, use an appropriate right triangle and Pythagoras Theorem to show that $\sin 45^o = \cos 45^o = \dfrac{\sqrt{2}}{2}$.

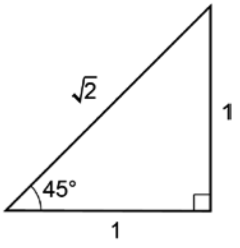
Solution:

Consider an isosceles right triangle with sides 1 unit each.

Clearly, the two base angles must be 45^o each.

Using Pythagoras' Theorem, the hypotenuse x is given by:

$$x^2 = \sqrt{1^2 + 1^2} \quad \Rightarrow \quad x = \sqrt{2}$$



Hence, $\sin 45^o = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. Similarly, $\cos 45^o = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

Notes:

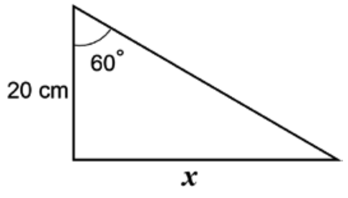
- *If $\tan 45^o = 1$, then we can draw a right triangle with an angle of 45^o and with the adjacent and opposite sides each of length 1 unit.*
- *Clearly using Pythagoras Theorem, x, the length of the hypotenuse can be found.*

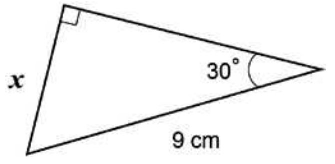
Exercise 6.2 This Exercise is to be completed without the use of a calculator.

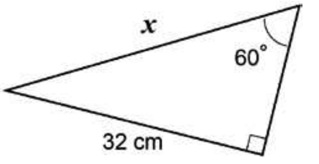
1. Given that $\sin 30^\circ = \frac{1}{2}$, use an appropriate right triangle to show that:

   (a) $\cos 60^\circ = \frac{1}{2}$ (b) $\sin 60^\circ = \frac{\sqrt{3}}{2}$ (c) $\tan 30^\circ = \frac{\sqrt{3}}{3}$ (d) $\tan 60^\circ = \sqrt{3}$

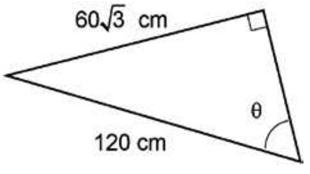
2. Find x in simplified exact form in each of the following diagrams:

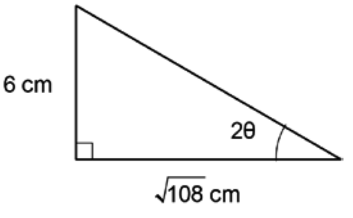
   (a)


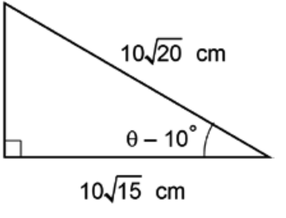
   (b)


   (c)


3. Find θ in each of the following diagrams.

   (a)


   (b)


   (c)


07 Trigonometric Ratios of non-acute angles

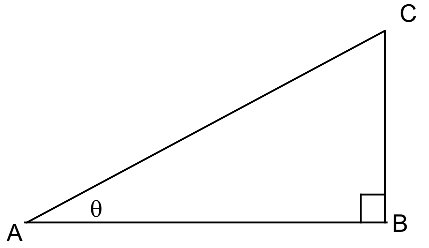
7.1 Review of Trigonometric Ratios of acute angles

- In the previous chapter, the trigonometric ratios of sine, cosine and tangent were defined for acute angles within right-angled triangles.
- In the right-angled triangle ABC:

$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{BC}{AB}$$



- These definitions do not apply when $\theta > 90^{\circ}$, that is when θ is not acute.
  The definitions of sine, cosine and tangent need to be enlarged to include angles greater than 90°.

7.2 The Four Quadrants

- First, we need to review how angles are referenced in a more formal manner.
- The horizontal *positive x-axis* is taken as the base line.
  Positive angles are measured in an anti-clockwise manner.
  The *x* and *y* axes divide the *x*-*y* plane into four quadrants. See Figure 1.

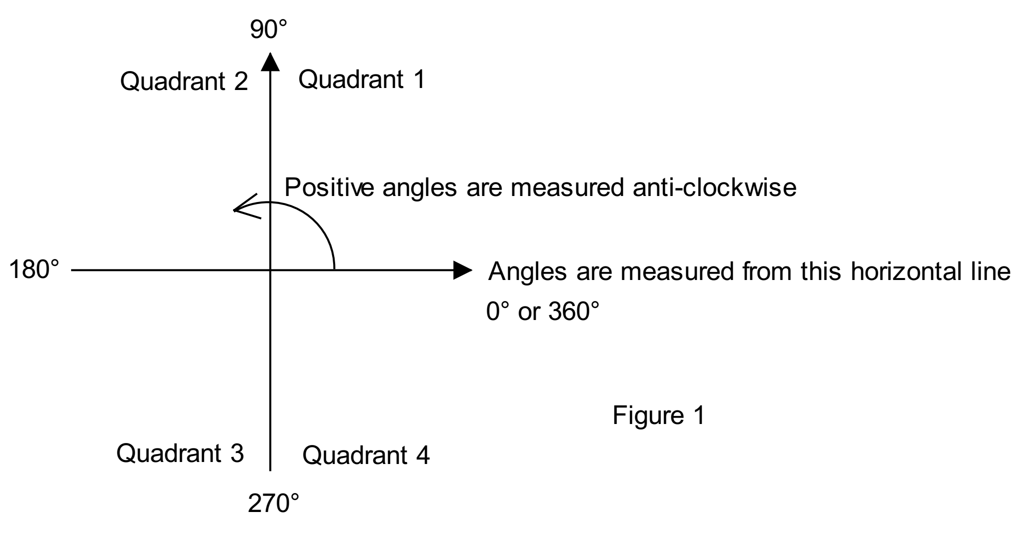


Figure 1

- Angles in the first quadrant are termed acute, angles in the second quadrant are termed obtuse and angles in the third or fourth quadrants are called reflex angles.

Example 7.1

Identify the quadrant location of each of the following angles: (a) 130° (b) 320° (c) 390°

Solution:

(a) 130° is greater than 90° but less than 180°. Hence, it is in Quadrant 2.

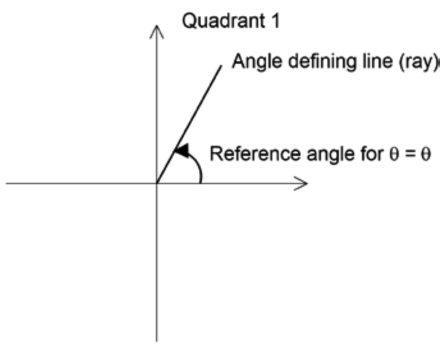
(b) 320° is greater than 270° but less than 360°. Hence, it is in Quadrant 4.

(c) 390° is an angle consisting of a full revolution of 360° + an angle of 30°.
Hence, it is equivalent to an angle of 30°. Hence, it is in Quadrant 1.

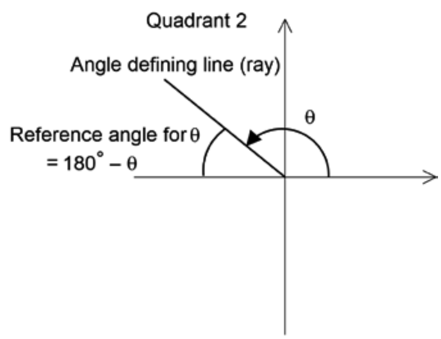
7.3 Reference Angles

- The reference angle associated with a given angle θ, is the acute angle, which the defining line (ray) makes with either the positive or the negative *x*-axis.

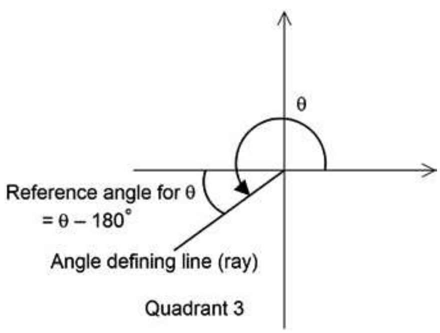
- If θ is in Quadrant 1:
  Reference angle for $\theta = \theta$



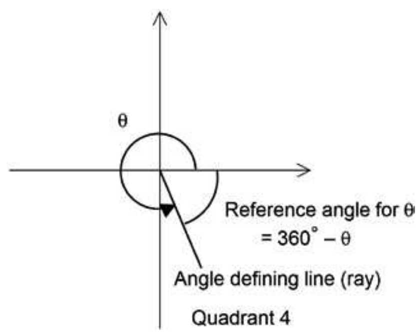
- If θ is in Quadrant 2:
  Reference angle for $\theta = 180^{\circ} - \theta$



- If θ is in Quadrant 3:
  Reference angle for $\theta = \theta - 180^{\circ}$.



- If θ is in Quadrant 4:
  Reference angle for $\theta = 360^{\circ} - \theta$.



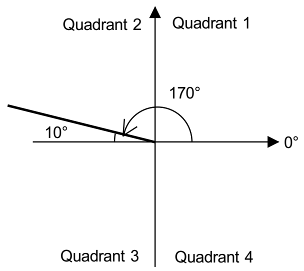
Example 7.2

Find α, the reference angle associated with each of the following angles.
(a) 50° (b) 170° (c) 200° (d) 290° (e) 500°.

Solution:

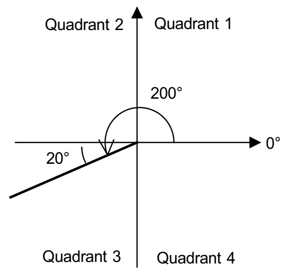
(a) 50° is an acute angle. Hence, $\alpha = 50^\circ$.

(b)



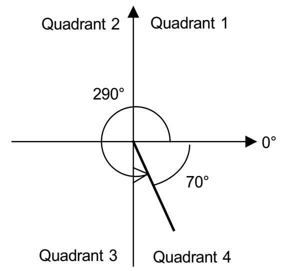
170° is in Quadrant 2.
$\Rightarrow \alpha = 180^\circ - 170^\circ = 10^\circ$.

(c)



200° is in Quadrant 3.
$\Rightarrow \alpha = 200^\circ - 180^\circ = 20^\circ$.

(d)



290° is in Quadrant 4.
$\Rightarrow \alpha = 360^\circ - 290^\circ = 70^\circ$

(e) 500° is an angle consisting of one full revolution of 360° + an angle of 140°.
500° is an angle equivalent to 140° in Quadrant 2. Hence, $\alpha = 180^\circ - 140^\circ = 40^\circ$.

Example 7.3

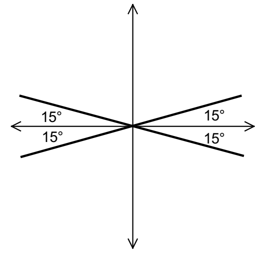
Find x where $0^\circ \le x \le 360^\circ$, with a reference angle of 15°.

Solution:

The reference angle is on either side of the x-axis as shown in the diagram below.

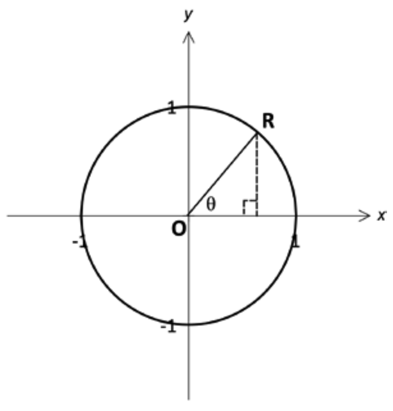
Hence,

$$x = 15^\circ, 180^\circ - 15^\circ, 180^\circ + 15^\circ, 360^\circ - 15^\circ$$
$$= 15^\circ, 165^\circ, 195^\circ, 345^\circ.$$



7.4 Circular Functions

- We shall now consider a more formal definition for sine, cosine and tangent.
  In formal mathematics, the sine, cosine and tangent functions are defined over the unit circle (a circle centred at the origin of radius 1) and are referred to as *Circular Functions*.
- Consider the unit circle (a circle of radius 1 with centre at the origin).
  (The equation of the circle is $x^2 + y^2 = 1$.)
- Let OR be the ray that defines angle θ. R is on the circumference of the unit circle.



- The Sine function is defined as:
  sin (θ) = *y-coordinate of R*
- The Cosine function is defined as:
  cos (θ) = *x-coordinate of R*
- The Tangent function is defined as:
  $$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
- Notice that in these definitions, the angle *x* need not be acute.
- As sin (θ) and cos (θ) are defined by the unit circle, clearly:
  - $-1 \le \sin(\theta) \le 1$
  - $-1 \le \cos(\theta) \le 1$

Example 7.4

The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates (0.8, –0.6).
Without the use of a calculator, find: (a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$.

Solution:

(a) $\sin\theta$ = *y*-coordinate of R. Hence, $\sin\theta = -0.6$.

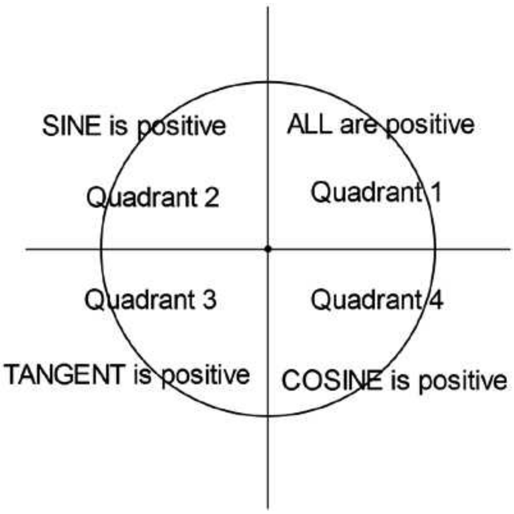
(b) $\cos\theta$ = *x*-coordinate of R. Hence, $\cos\theta = 0.8$.

(c) $\tan\theta = \dfrac{\sin(\theta)}{\cos(\theta)} = \dfrac{-0.6}{0.8} = -0.75$

Important Note:

- By noting the position of point R and the numerical sign of the coordinates of R, the following results may be drawn.

Quadrant Location of Angle	Function with Positive Value	Function with Negative Value
1	Sine, Cosine, Tangent	None
2	Sine	Cosine, Tangent
3	Tangent	Sine, Cosine
4	Cosine	Sine, Tangent



Example 7.5

Without the use of a calculator, for each value of x , state the sign of sin (x), cos (x) and tan (x): (a) 25° (b) 200° (c) 300° (d) 600°

Solution:

(a) 25° is in Quadrant 1. Hence, sin (25°), cos (25°) and tan (25°) all have positive signs.

(b) 200° is in Quadrant 3. Hence, only tan (200°) has a positive sign.
sin (200°) and cos (200°) both have negative signs.

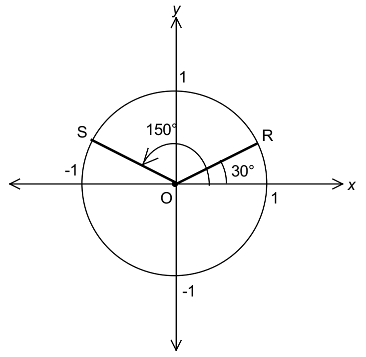
(c) 300° is in Quadrant 4. Hence, only cos (300°) has a positive sign.
sin (300°) and tan (200°) both have negative signs.

(d) $600^{\circ} = 360^{\circ} + 240^{\circ}$. Hence, 600° is in Quadrant 3.
Hence, only tan (600°) has a positive sign.
sin (600°) and cos (600°) both have negative signs.

7.5 Trigonometric ratios of non-acute angles

- We now need to establish the relationship between the trigonometric ratios of non-acute angles with the corresponding ratios for their reference angles. This is essential if exact values are required in calculations.

- *Case I*



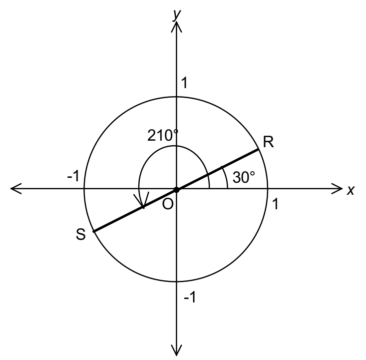
  - Consider the angle with size 150^{o} defined by the ray OS on the unit circle.
    The reference angle $\alpha = 180^{o} - 150^{o} = 30^{o}$
  - Consider now the angle with size 30^{o} defined by the ray OR on the unit circle.
  - R and S are symmetrically located about the *y*-axis. Hence, the *x*-coordinates for R and S share the same numerical value and differ only by the sign. The *y*-coordinates of R and S are identical. That is:

    x-coordinate of S = – (*x*-coordinate of R)
    y-coordinate of S = *y*-coordinate of R
  - Using the Circular function definition of Sine and Cosine:

$$\cos(150^{o}) = -\cos(30^{o})$$
$$\sin(150^{o}) = \sin(30^{o}) .$$

Hence, $$\tan(150^{o}) = -\tan(30^{o})$$

- *Case II*



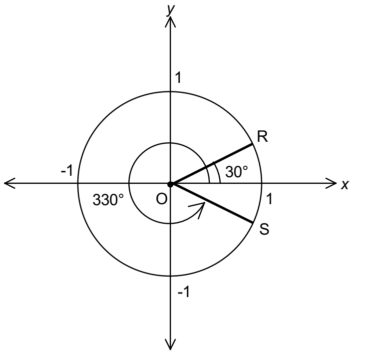
  - Consider the angle with size 210^{o} defined by the ray OS on the unit circle.
    The reference angle $\alpha = 210^{o} - 180^{o} = 30^{o}$.
  - Consider now the angle with size 30^{o} defined by the ray OR on the unit circle.
  - R and S are symmetrically located about the origin. Hence, the *x*-coordinates for R and S share the same numerical value and differ only by the sign. Also, the *y*-coordinates for R and S share the same numerical value and differ only by the sign. That is:

    x-coordinate of S = – (*x*-coordinate of R)
    y-coordinate of S = – (*y*-coordinate of R)
  - Using the Circular function definition of Sine and Cosine:

$$\cos(210^{o}) = -\cos(30^{o})$$
$$\sin(210^{o}) = -\sin(30^{o}) .$$

Hence, $$\tan(210^{o}) = \tan(30^{o})$$

- *Case III*
  - Consider the angle with size 330° defined by the ray OS on the unit circle.
    The reference angle $\alpha = 360^{\circ} - 330^{\circ} = 30^{\circ}$
  - Consider now the angle with size 30° defined by the ray OR on the unit circle.



  - R and S are symmetrically located about the horizontal *x*-axis. Hence, the *x*-coordinates for R and S are identical. The *y*-coordinates for R and S share the same numerical value and differ only by the sign. That is:
    x-coordinate of S = *x*-coordinate of R
    y-coordinate of S = – (*y*-coordinate of R)
  - Using the Circular function definition of Sine and Cosine:

$$\cos(330^{\circ}) = \cos(30^{\circ})$$
$$\sin(330^{\circ}) = -\sin(30^{\circ}).$$

Hence, $\tan(330^{\circ}) = -\tan(30^{\circ})$

- From these three cases, we can conclude that:

If α is the reference angle for *x*,

$$\sin(x) = \begin{cases} \sin(\alpha) & \text{if } x \text{ is in Q1\&Q2} \\ -\sin(\alpha) & \text{if } x \text{ is in Q3\&Q4} \end{cases}$$

$$\cos(x) = \begin{cases} \cos(\alpha) & \text{if } x \text{ is in Q1\&Q4} \\ -\cos(\alpha) & \text{if } x \text{ is in Q2\&Q3} \end{cases}$$

$$\tan(x) = \begin{cases} \tan(\alpha) & \text{if } x \text{ is in Q1\&Q3} \\ -\tan(\alpha) & \text{if } x \text{ is in Q2\&Q4} \end{cases}$$

Example 7.6

Express each of the following in terms of the appropriate reference angle.

(a) $\sin(200^{\circ})$ (b) $\tan(340^{\circ})$ (c) $\cos(420^{\circ})$

Solution:

(a) 200° is in Quadrant 3. Reference angle is $200^{\circ} - 180^{\circ} = 20^{\circ}$.
Sine is negative in this quadrant. Hence, $\sin(200^{\circ}) = -\sin(20^{\circ})$

(b) 340° is in Quadrant 4. Reference angle is $360^{\circ} - 340^{\circ} = 20^{\circ}$.
Tangent is negative in this quadrant. Hence, $\tan(340^{\circ}) = -\tan(20^{\circ})$

(c) 420° is in Quadrant 1. Reference angle is $420^{\circ} - 360^{\circ} = 60^{\circ}$.
Cosine is positive in this quadrant. Hence, $\cos(420^{\circ}) = \cos(60^{\circ})$

Example 7.7

If tan (x) = tan (50°), find x for $0^\circ \le x \le 360^\circ$.

Solution:

Reference angle for x is 50°. The tangent function is positive in Quadrant 1 and 3.
Hence, $x = 50^\circ$ or $180^\circ + 50^\circ = 50^\circ$ or 230° .

Example 7.8

If cos (25°) = – cos (x), find x for $0^\circ \le x \le 360^\circ$.

Solution:

Rewrite expression as cos (x) = – cos (25°).

Reference angle for x is 25°. The cosine function is negative in Quadrant 2 and 3.
Hence, $x = 180^\circ - 25^\circ$ or $180^\circ + 25^\circ = 155^\circ$ or 205° .

Example 7.9

Without the use of a calculator, find the exact values of each of the following.

(a) sin 135° (b) tan 330° (c) cos 660°

Solution:

(a) $\sin 135^\circ = +\sin 45^\circ = \dfrac{\sqrt{2}}{2}$

(b) $\tan 330^\circ = -\tan 30^\circ = -\dfrac{\sqrt{3}}{3}$

(c) $\cos 600^\circ = -\cos 60^\circ = -\dfrac{1}{2}$

Example 7.10

Given that tan (x) = $-\dfrac{1}{\sqrt{3}}$ and sin (x) = $-\dfrac{1}{2}$, find x for $0^\circ \le x \le 360^\circ$.

Solution:

Both the tangent and sine ratios are negative. Hence, x is in Quadrant 4. Reference angle for x is 30°.
Hence, $x = 360^\circ - 30^\circ = 330^\circ$.

Exercise 7.1

1. Identify the quadrant location of each of the following angles:
   (a) 100° (b) 210° (c) 260° (d) 290° (e) 335°
   (f) 350° (g) 390° (h) 750° (i) 910° (j) 550°

2. Determine the reference angle for each of the following angles.
   (a) 35° (b) 119° (c) 195° (d) 225° (e) 332°
   (f) 349° (g) 442° (h) 850° (i) 645° (j) 1000°

3. Find x where $0^{\circ} \le x \le 360^{\circ}$, with a reference angle of:
   (a) 65° (b) 31° (c) 42° (d) 6°

4. Find x where $360^{\circ} \le x \le 720^{\circ}$, with a reference angle of:
   (a) 13° (b) 72° (c) 45° (d) 30°

5. The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates $(-0.8, -0.6)$. Without the use of a calculator, find: (a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$.

6. The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates $(-\frac{1}{2}, k)$. Without the use of a calculator, find the two possible values of k and hence find (a) $\cos\theta$ (b) $\sin\theta$ (c) $\tan\theta$.

7. The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates $(k, \frac{3}{4})$. Without the use of a calculator, find the two possible values of k and hence find (a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$.

8. The angle θ is defined by the ray OR where O is the centre of the unit circle and R is a point on the unit circle with coordinates (k, k). Without the use of a calculator, find the two possible values of k and hence find (a) $\cos\theta$ (b) $\sin\theta$ (c) $\tan\theta$.

9. Without the use of a calculator, for each value of x, state the sign of $\sin(x)$, $\cos(x)$ and $\tan(x)$: (a) 55° (b) 204° (c) 505° (d) 650°

10. Given that $\sin(x) < 0$ and $\tan(x) < 0$, state the quadrant location of x.

11. Given that $\tan(x) > 0$ and $\cos(x) < 0$, state the quadrant location of x.

12. If $\cos(x) = -\cos(24^{\circ})$, find x for $0^{\circ} \le x \le 360^{\circ}$.

13. If $\tan(x) = \tan(56^{\circ})$, find x for $180^{\circ} \le x \le 360^{\circ}$.

14. If $\sin(x) = -\sin(79^{\circ})$, find x for $360^{\circ} \le x \le 720^{\circ}$.

15. If $\sin(35^{\circ}) = -\cos(x)$, find x for $0^{\circ} \le x \le 360^{\circ}$.

16. Without the use of a calculator, find the exact values of each of the following.
(a) $\sin 135^{\circ}$ (b) $\tan 330^{\circ}$ (c) $\cos 420^{\circ}$ (d) $\sin 450^{\circ}$
(e) $\cos 240^{\circ}$ (f) $\tan 540^{\circ}$ (g) $\sin 495^{\circ}$ (h) $\cos 840^{\circ}$

17. Given that $\cos(x) = \dfrac{\sqrt{3}}{2}$, find x for $0^{\circ} \le x \le 360^{\circ}$.

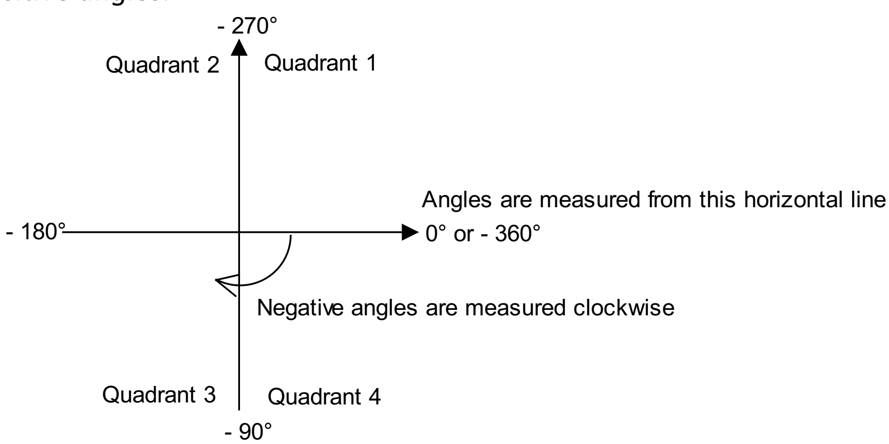
18. Given that $\cos(x) = \dfrac{\sqrt{3}}{2}$ and $\sin(x) = -\dfrac{1}{2}$, find x for $0^{\circ} \le x \le 360^{\circ}$.

19. Given that $\tan(x) = \sqrt{3}$ and $\sin(x) = -\dfrac{\sqrt{3}}{2}$, find x for $0^{\circ} \le x \le 360^{\circ}$.

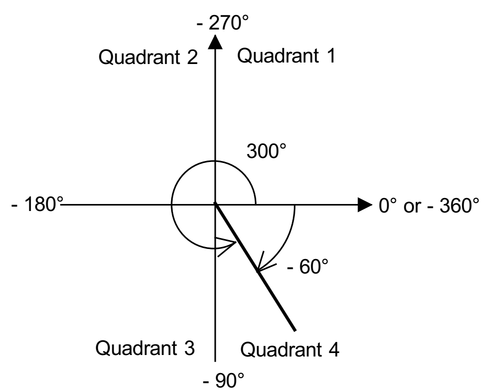
20. Given that $\tan(x) = -1$ and $\cos(x) = \dfrac{\sqrt{2}}{2}$, find x for $0^{\circ} \le x \le 720^{\circ}$.

7.5.1 Negative angles

- As mentioned in Section 7.2, positive angles are measured in an anti-clockwise direction from the positive x-axis. Hence, negative angles are measured in a clockwise direction from the positive x-axis.
- *The labels attached to the four quadrants remain unchanged from that used for positive angles.*



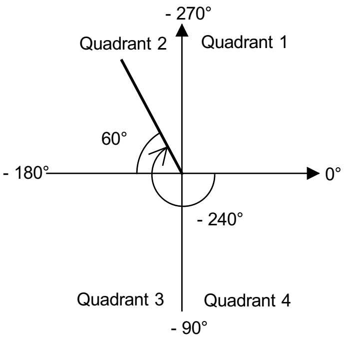
- Consider the negative angle -60°. This angle is located in the Quadrant 4. This angle can be described in positive terms as 300°.
- Using reference angles it can be shown that:



- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos(\theta)$
- $\tan(-\theta) = -\tan\theta$

Example 7.11

Find the reference angle associated with -240°.



Solution:

The reference angle for any angle θ is the acute angle between the angle defining line and the positive/negative x-axis.

From the given diagram, reference angle = $240^{\circ} - 180^{\circ}$
$= 60^{\circ}$

Example 7.12

If $\sin(22^{\circ}) = -\sin(x)$, find x for $-180^{\circ} \le x \le 180^{\circ}$.

Solution:

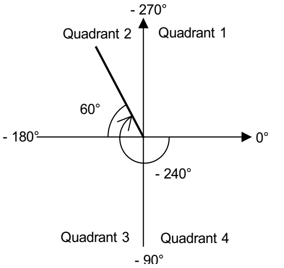
Rewrite expression as $\sin(x) = -\sin(22^{\circ})$. Reference angle for x is 22°.
The sine function is negative in Quadrant 3 and 4.

Hence, $x = -22^{\circ}, -(180^{\circ} - 22^{\circ}) = -22^{\circ}$ or -158°.

Example 7.13

Without the use of calculator, find the exact value of (a) $\sin(-240^{\circ})$ (b) $\cos(-300^{\circ})$

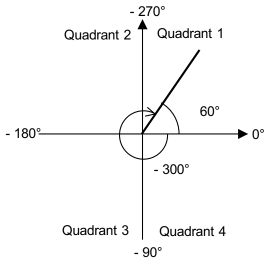
Solution:



(a) *-240° is in Quadrant 2 and sine is positive in Quadrant 2.*

The reference angle is = 60°.

Hence, $\sin(-240^{\circ}) = \sin(60^{\circ}) = \dfrac{\sqrt{3}}{2}$



(b) *-300° is in Quadrant 1 and cosine is positive in Quadrant 1.*

The reference angle is $360^{\circ} - 300^{\circ} = 60^{\circ}$.

Hence, $\cos(-300^{\circ}) = \cos(60^{\circ}) = \dfrac{1}{2}$

Example 7.14

Given that $\cos(x) = -\dfrac{\sqrt{2}}{2}$ and $\sin(x) = \dfrac{\sqrt{2}}{2}$, without the use of a calculator,

find x for $-360^{\circ} \le x \le 360^{\circ}$.

Solution:

The cosine ratio is negative and the sine ratio is positive in Quadrant 2. Reference angle for x is 45°.

Hence, $x = 180^{\circ} - 45^{\circ}$ or $-(180^{\circ} + 45^{\circ}) = 135^{\circ}$ or -225°

Exercise 7.2

1. Identify the quadrant location of each of the following angles:
   (a) -98° (b) -190° (c) -290° (d) -440° (e) -600°

2. Determine the reference angle for each of the following angles.
   (a) -45° (b) -155° (c) -235° (d) -295° (e) -336°

3. Find x where $-360^{\circ} \le x \le 0^{\circ}$, with a reference angle of:
   (a) 22° (b) 40° (c) 50° (d) 86°

4. Find x where $-180^{\circ} \le x \le 180^{\circ}$, with a reference angle of:
   (a) 5° (b) 33° (c) 52° (d) 75°

5. Without the use of a calculator, for each value of x , state the sign of sin (x), cos (x) and tan (x): (a) -27° (b) -210° (c) -475° (d) -780°

6. If cos (x) = – cos (26°), find x for $-360^{\circ} \le x \le 0^{\circ}$.

7. If tan (x) = tan (34°), find x for $-180^{\circ} \le x \le 180^{\circ}$.

8. If sin (x) = – sin (57°), find x for $-180^{\circ} \le x \le 180^{\circ}$.

9. If cos (41°) = sin (x), find x for $-360^{\circ} \le x \le 360^{\circ}$.

10. Without the use of a calculator, find the exact values of each of the following.
    (a) sin (-135°) (b) tan (-300°) (c) cos (-390°) (d) sin (-480°)

11. Given that cos (x) = $-\dfrac{\sqrt{3}}{2}$, find x for $-360^{\circ} \le x \le 0^{\circ}$.

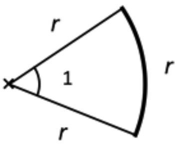
12. Given that cos (x) = $-\dfrac{\sqrt{3}}{2}$ and sin (x) = $\dfrac{1}{2}$, find x for $-180^{\circ} \le x \le 180^{\circ}$.

13. Given that tan (x) = $\sqrt{3}$ and sin (x) = $-\dfrac{\sqrt{3}}{2}$, find x for $-360^{\circ} \le x \le 360^{\circ}$.

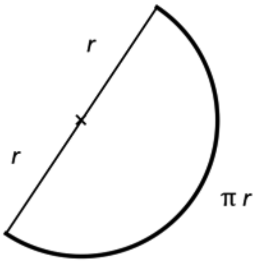
14. Given that tan (x) = -1 and sin (x) = $\dfrac{\sqrt{2}}{2}$, find x for $-360^{\circ} \le x \le 360^{\circ}$.

7.6 Using Radians

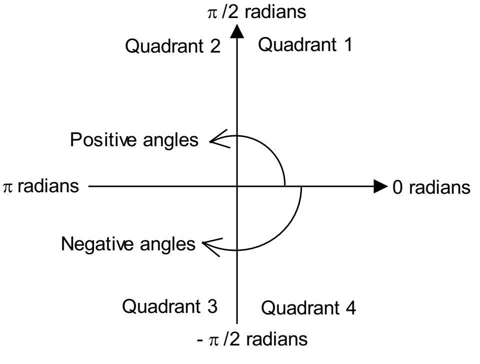
- Radians is another unit of measurement for the size of an angle.
- Consider a sector of a circle of radius r. If the length of the arc is equal to the radius of the circle, then the angle subtended at the centre of the sector is said to measure one radian. Note that this is independent of the value of r.



- When the sector corresponds to a semi-circle, the length of the arc is $\frac{1}{2} \times 2\pi r = \pi r$.
  Since, an arc of length r subtends an angle of 1 radian; a semi-circle of length πr will subtend an angle of $\frac{\pi r}{r} = \pi$ radians.



- But a semi-circle subtends an angle of 180°. Hence π radians = 180°. Therefore, one radian is approximately 57.3°.
- In advanced Mathematics, the default measure for angles is the radian measure. It is usual to use either $-\pi < x \le \pi$ or $0 \le x < 2\pi$. As in the previous case, positive angles are measured anti-clockwise from the positive x-axis.

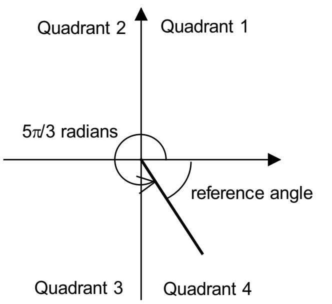


- A table of exact values using degrees and radians is reproduced below.

θ°	θ radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	∞

Example 7.15

Find the reference angle in exact form for an angle of size $\frac{5\pi}{3}$ radians.



Solution:

The reference angle for any angle θ is the acute angle between the angle defining line and the positive/negative x-axis.

From the given diagram,

reference angle = $2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$ radians

Example 7.16

Without the use of a calculator, find the exact values of (a) $\cos\left(\frac{7\pi}{6}\right)$ (b) $\tan\left(\frac{11\pi}{4}\right)$

Solution:

(a) $\frac{7\pi}{6}$ is in Quadrant 3 and cosine is negative in Quadrant 3.

The reference angle is $\frac{7\pi}{6} - \pi = \frac{\pi}{6}$.

Hence, $\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

(b) $\frac{11\pi}{4} = 2\pi + \frac{3\pi}{4}$ is in Quadrant 2.

The reference angle is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

Hence, $\tan\left(\frac{11\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$

Example 7.17

Given that $\cos(x) = \frac{1}{2}$ and $\sin(x) = -\frac{\sqrt{3}}{2}$,

without the use of a calculator, find x for $-\pi < x \le \pi$.

Solution:

The cosine ratio is positive and the sine ratio is negative in Quadrant 4.

Reference angle for x is $\frac{\pi}{3}$. Hence, $x = -\frac{\pi}{3}$.

Exercise 7.3

1. Determine the reference angle for each of the following angles.

 (a) $\frac{5\pi}{6}$ (b) $\frac{13\pi}{6}$ (c) $-\frac{7\pi}{4}$ (d) $-\frac{2\pi}{3}$

2. Find x where $0 \le x < 2\pi$, with a reference angle of:

 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

3. Find x where $-\pi < x \le \pi$, with a reference angle of:

 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

4. Without the use of a calculator, for each value of x , state the sign of sin (x), cos (x) and tan (x): (a) $\frac{\pi}{7}$ (b) $\frac{17\pi}{6}$ (c) $-\frac{15\pi}{8}$ (d) $-\frac{4\pi}{7}$

5. If cos (x) = – cos $\left(\frac{5\pi}{12}\right)$, find x for $-\pi < x \le \pi$.

6. If tan (x) = tan $\left(\frac{\pi}{5}\right)$, find x for $0 \le x < 2\pi$.

7. If sin (x) = – sin $\left(\frac{7\pi}{15}\right)$, find x for $-\pi < x \le \pi$.

8. Without the use of a calculator, find the exact values of each of the following.

 (a) $\sin\left(\frac{3\pi}{4}\right)$ (b) $-\tan\left(\frac{7\pi}{4}\right)$ (c) $\cos\left(\frac{13\pi}{3}\right)$ (d) $\sin\left(-\frac{5\pi}{4}\right)$

9. Given that cos (x) = $-\frac{\sqrt{3}}{2}$, without the use of a calculator, find x for $-\pi < x \le \pi$.

10. Given that cos (x) = $-\frac{\sqrt{3}}{2}$ and sin (x) = $\frac{1}{2}$, without the use of a calculator, find x for $-2\pi < x \le 2\pi$.

11. Given that tan (x) = $\sqrt{3}$ and cos (x) = $-\frac{1}{2}$, without the use of a calculator, find x for $-2\pi \le x \le 0$.

12. Given that tan (x) = -1 and cos (x) = $\frac{\sqrt{2}}{2}$, without the use of a calculator, find x for $-\pi \le x \le \pi$.

7.7 Inverse Trigonometric Functions

- Consider sin (x) = 0.5. There are many angles that satisfy this expression.
  Some possible angles are 30^o, 150^o, 390^o, …. . The sine of all these angles is 0.5.
- Hence, x is an angle whose sine is 0.5.
  This is written in a more formal manner as $x = \sin^{-1}(0.5)$
  [$\sin^{-1}$ is the inverse sine function and is commonly read as "the arc sine of".]
- All scientific and CAS/graphic calculators are equipped with "buttons" that provide quick retrievals of the values of the basic trigonometric ratios of sine, cosine and tangent of required angles.
- To retrieve the angle(s) corresponding to a given ratio is an involved process. As noted earlier, there are numerous angles which satisfy a given expression like cos (x) = 0.1234. The angles have to be matched with the domain for x. Scientific and CAS/graphic calculators do not provide ready answers and need to be used intelligently to obtain the correct answer.

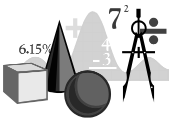
7.7.1 Principal Values

- The $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ "buttons" on most scientific and CAS/graphic calculators allow immediate retrieval of the *principal values* for the required angle.
- Consider the equation sin (x) = –0.5. $\Rightarrow$ $x = \sin^{-1}(-0.5)$
  Using the $\sin^{-1}$ "button" on a scientific/CAS/graphics calculator ,
  $$x = \sin^{-1}(-0.5) = -30^o$$
  The value of -30^o returned by the calculator is called the *principal value* of $\sin^{-1}(-0.5)$.
- Consider now the equation cos (x) = –0.5. $\Rightarrow$ $x = \cos^{-1}(-0.5)$
  Using the $\cos^{-1}$ "button" on a scientific/CAS/graphics calculator ,
  $$x = \cos^{-1}(-0.5) = 120^o$$
  The value of 120^o returned by the calculator is called the *principal value* of $\cos^{-1}(-0.5)$.
- The concept of principal value involves a deeper understanding of the concepts of functions, inverse functions and domains and is beyond the scope of this course. It is sufficient at this stage to note the following about the principal values of $\cos^{-1} k$, $\sin^{-1} k$ and $\tan^{-1} k$:

$0 \le \cos^{-1} k \le \pi$	or	$0^o \le \cos^{-1} k \le 180^o$
$-\frac{\pi}{2} \le \sin^{-1} k \le \frac{\pi}{2}$	or	$-90^o \le \sin^{-1} k \le 90^o$
$-\frac{\pi}{2} < \tan^{-1} k < \frac{\pi}{2}$	or	$-90^o < \tan^{-1} k < 90^o$

08 Non-Right Triangle Trigonometry

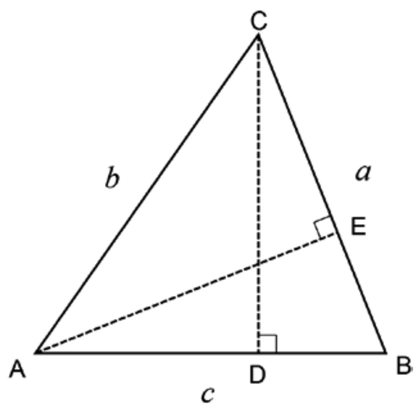
8.1 The Sine Rule



Hands On Task 8.1

In this task, we will prove a rule called the sine rule for an acute triangle (none of the angles are obtuse).

Consider ΔABC. Let D be the foot of the perpendicular from C to AB. Let E be the foot of the perpendicular from A to BC



1. In ΔACD, show that CD = $b \sin A$.
2. In ΔCBD, find CD in terms of $\angle B$ and a.
3. Hence, prove that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$.
4. In ΔAEC, find AE in terms of $\angle C$ and b.
5. In ΔAEB, find AE in terms of $\angle B$ and c.
6. Hence, prove that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
7. By rearranging the results in 3 and 6, show that $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Summary

The Sine Rule

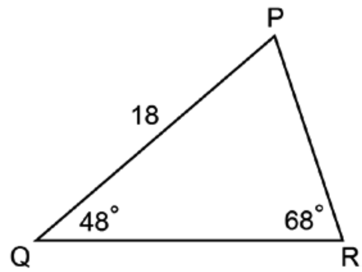
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
- $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

Note:
The Sine Rule establishes a relationship between two pairs of Angle-Opposite Side in the form:

$$\frac{\textit{length of side 1}}{\sin(\textit{angle opposite side 1})} = \frac{\textit{length of side 2}}{\sin(\textit{angle opposite side 2})}.$$

Example 8.1

In ΔPQR, find PR correct to 2 decimal places.



Solution:

Using the Sine Rule:

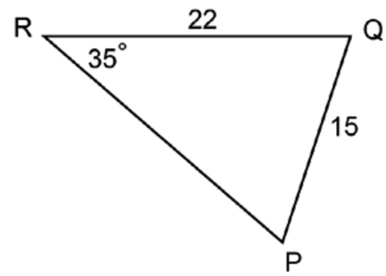
$$\frac{PR}{\sin P\hat{Q}R} = \frac{PQ}{\sin P\hat{R}Q}$$

$$\frac{PR}{\sin 48^{o}} = \frac{18}{\sin 68^{o}}$$

$$\Rightarrow \quad PR = \frac{18 \sin 48^{o}}{\sin 68^{o}} = 14.43^{o}$$

Example 8.2

In ΔPQR, find $\angle$RPQ correct to 1 decimal place.



Solution:

Using the Sine Rule:

$$\frac{\sin R\hat{P}Q}{RQ} = \frac{\sin P\hat{R}Q}{PQ}$$

$$\frac{\sin R\hat{P}Q}{22} = \frac{\sin 35^{o}}{15}$$

$$\Rightarrow \quad \sin R\hat{P}Q = \frac{22 \sin 35^{o}}{15} = 0.8412$$

Reference angle for $\angle RPQ = \sin^{-1} 0.8412 = 57.27^{o}$

$\Rightarrow \quad \angle RPQ = 57.27^{o}$ or $180 - 57.27^{o}$

Possible values for $\angle RPQ = 57.3^{o}$ or 122.7^{o}

Check: If $\angle RPQ = 122.7^{o}$, $\angle RPQ + \angle PRQ = 157.7^{o} < 180^{o}$.

Hence, $\angle RPQ = 57.3^{o}$ or 122.7^{o}.

> *Since sine is positive in Q1 and Q2, two solutions must be considered.*

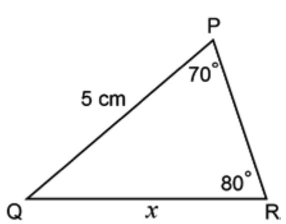
Notes:

- *In Example 8.2, for convenience, the second form of the Sine Rule was used.*
- *In using the Sine Rule to determine the size of an unknown angle, care must be taken to determine if the obtuse solution is valid. If the sum of the obtuse solution and the known angle exceeds 180^{o}, then the obtuse solution must be rejected.*

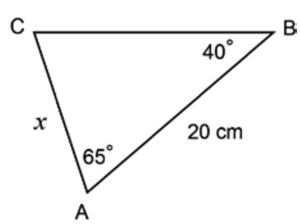
Exercise 8.1

1. Find the value of *x* in each of the following triangles correct to 2 decimal places.

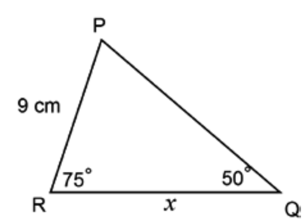
(a)



(b)

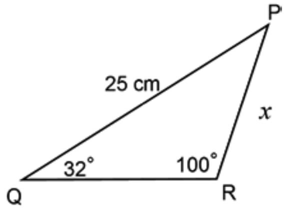


(c)

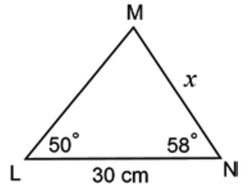


2. Find the value of x in each of the following triangles correct to 2 decimal places.

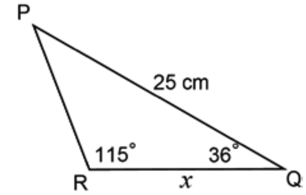
(a)



(b)

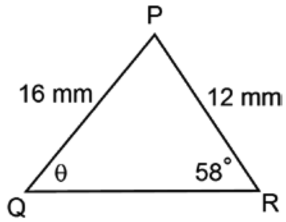


(c)

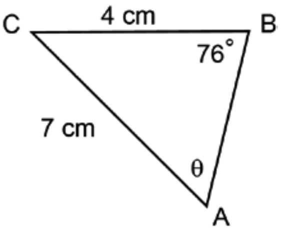


3. Find the value of θ° in each of the following triangles correct to 1 decimal.

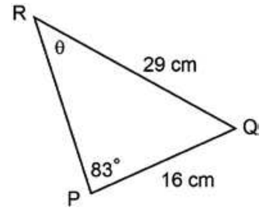
(a)



(b)

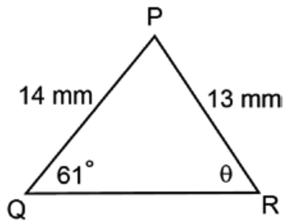


(c)

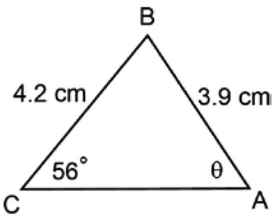


4. Find the value of θ° in each of the following triangles correct to 1 decimal place.

(a)



(b)



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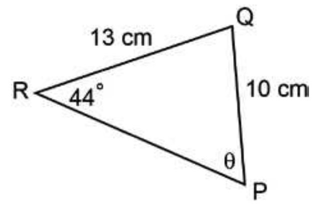
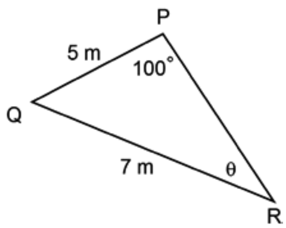
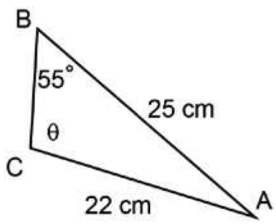


5. Find the value of θ° in each of the following triangles correct to 1 decimal place.

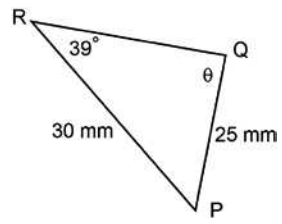
(a)



(b)

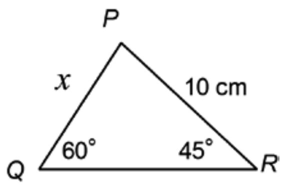


(c)

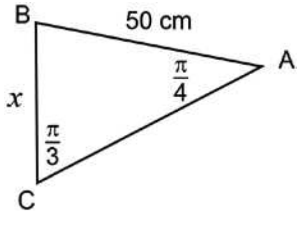


6. Without the use of a calculator, find the value of x in each of the following triangles.

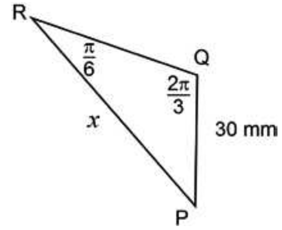
(a)



(b)

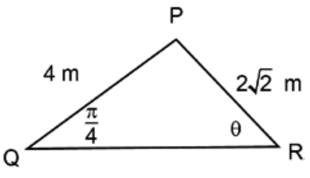


(c)

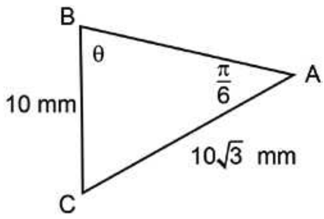


7. Without the use of a calculator, find the value of θ in each of the following triangles.

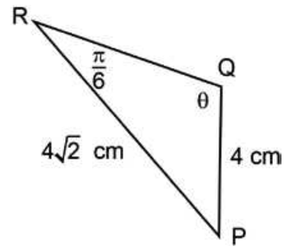
(a)



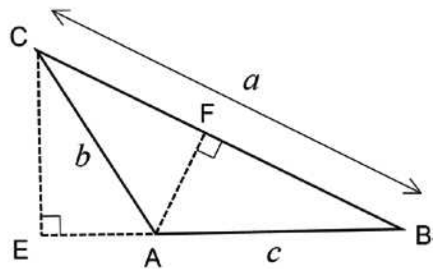
(b)



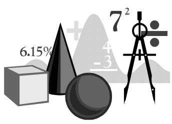
(c)



8. ΔABC is an obtuse triangle. F is the foot of the perpendicular from A to BC. E is the foot of the perpendicular to BA extended.
   (a) In ΔAFC, show that AF = b sin C.
   (b) In ΔAFB, find AF in terms of c and $\angle B$.
   (c) Hence show that $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.
   (d) In ΔCEB, find CE in terms of a and $\angle B$.
   *(e) In ΔCEA, show that CE = b sin A.
   (f) Hence, show that for a obtuse triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.



8.2 Cosine Rule

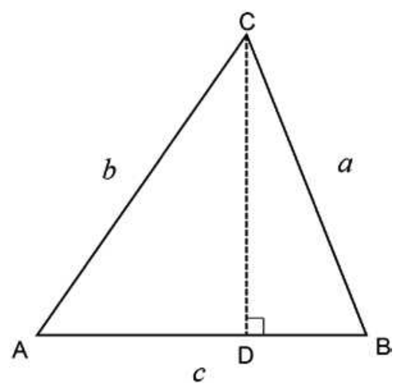


Hands On Task 8.2

In this task, we will prove a rule called the Cosine Rule for an acute triangle (none of the angles are obtuse).

Consider ΔABC. Let D be the foot of the perpendicular from C to AB.

1. In ΔACD, find show that AD = b cos A.
2. In ΔACD, use Pythagoras' Theorem to find CD in terms of b and $\angle A$.
3. Show that DB = $c - b$ cos A.
4. In ΔCDB, use Pythagoras' Theorem to show that $a^2 = b^2 + c^2 - 2bc \cos A$.
5. Use the result in 4 to show that cos A = $\dfrac{b^2+c^2-a^2}{2bc}$.



Summary

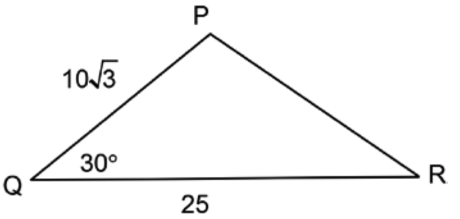
The Cosine Rule

- $a^2 = b^2 + c^2 - 2bc \cos A$
- cos A = $\dfrac{b^2+c^2-a^2}{2bc}$

Note:
(side opposite given angle)^2
= (included side 1)^2 + (included side 2)^2 − 2 × (included side 1) × (included side 2) × cos (given angle)

Example 8.3

In ΔPQR, find PR without the use of a calculator.



Solution:

Using the Cosine Rule:

$PR^2 = QP^2 + QR^2 - 2 \times QP \times QR \times \cos \angle PQR$

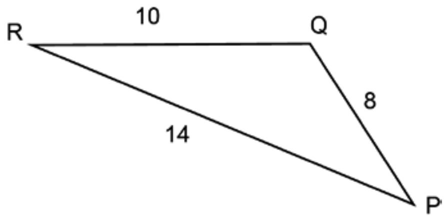
$= (10\sqrt{3})^2 + 25^2 - 2 \times 10\sqrt{3} \times 25 \times \cos 30$

$= 300 + 625 - 500\sqrt{3} \times \dfrac{\sqrt{3}}{2} = 175$

$PR = \sqrt{175} = 5\sqrt{7}$

Example 8.4

In ΔPQR, find $\angle$RQP correct to 1 decimal place.



Solution:

Using the Cosine Rule:

$\cos \angle RQP = \dfrac{QR^2 + QP^2 - PR^2}{2 \times QR \times QP}$

$= \dfrac{10^2 + 8^2 - 14^2}{2 \times 10 \times 8} = -0.2$

Reference angle for $\angle RQP = \cos^{-1}(0.2) = 78.5^\circ$

$\Rightarrow \quad \angle RQP = 180^\circ - 78.5^\circ = 101.5^\circ$

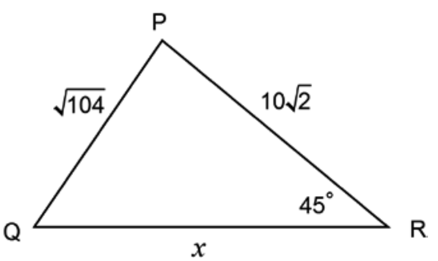
Cosine is negative in Q2 and Q3. Ignore Q3 as angles in a triangle < 180°.

Note:

- *If the concept of principal values is used, then $\angle PQP = \cos^{-1}(-0.2) = 101.5^\circ$.*

Example 8.5

In ΔPQR, find all possible values of *x*.



Solution:

Using the Cosine Rule:

$PQ^2 = RP^2 + RQ^2 - 2 \times RP \times RQ \times \cos \angle PRQ$

$(\sqrt{104})^2 = (10\sqrt{2})^2 + x^2 - 2 \times 10\sqrt{2} \times x \times \cos 45^\circ$

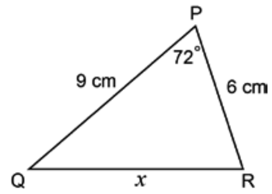
$104 = 200 + x^2 - 20x$

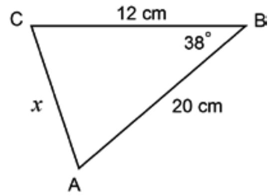
$\Rightarrow \; x^2 - 20x + 96 = 0$

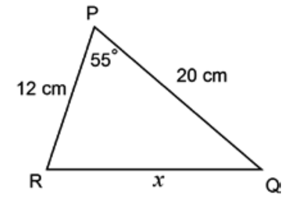
$x = 8$ or 12

Exercise 8.2

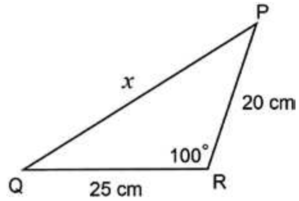
1. Find the value of *x* in each of the following triangles correct to 2 decimal places.

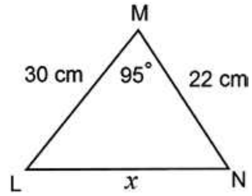
(a)


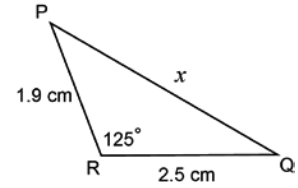
(b)


(c)


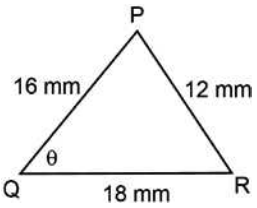
2. Find the value of *x* in each of the following triangles correct to 2 decimal places.

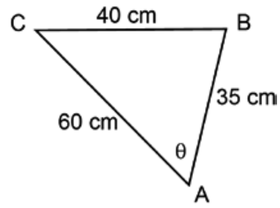
(a)


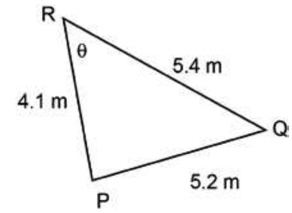
(b)


(c)


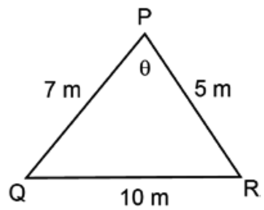
3. Find the value of θ^{o} in each of the following triangles correct to 1 decimal place.

(a)


(b)


(c)


4. Find the value of θ^{o} in each of the following triangles correct to 1 decimal place.

(a)


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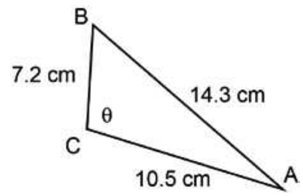

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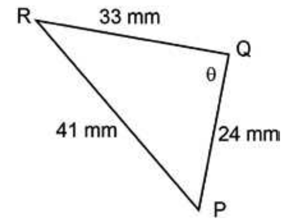
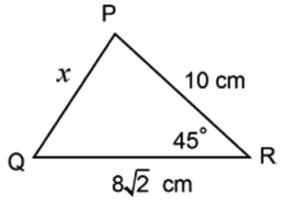
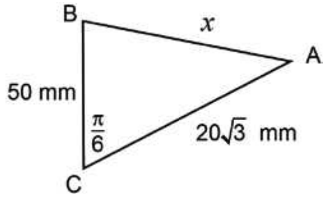
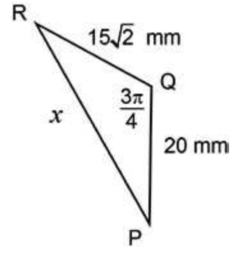


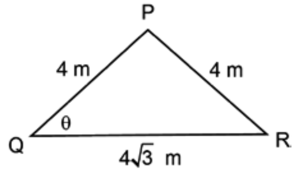
5. Without the use of a calculator, find the value of *x* in each of the following triangles.

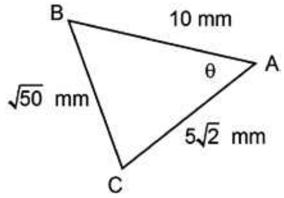
(a)


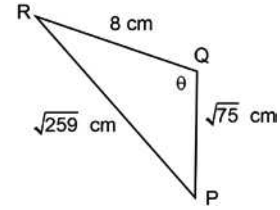
(b)


(c)


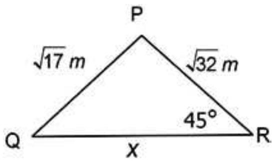
6. Without the use of a calculator, find the value of θ (radians) in each of the following triangles.

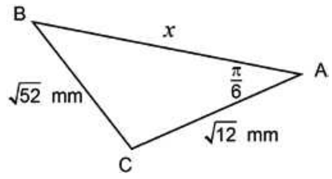
(a)


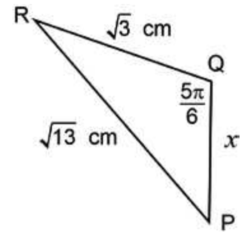
(b)


(c)


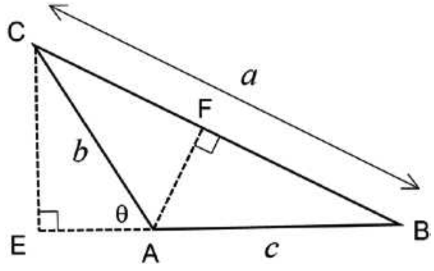
7. Find all possible values of x in each of the following triangles.

(a) 

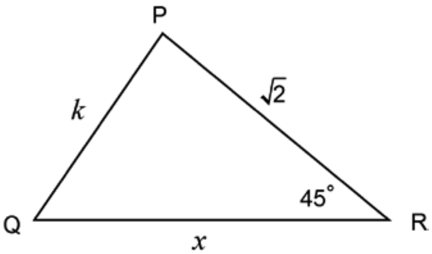
(b) 

(c) 

*8. ΔABC is an obtuse triangle. F is the foot of the perpendicular from A to BC. E is the foot of the perpendicular to BA extended. Let $\angle$CAE = θ.
(a) Explain clearly why cos A = – cos θ.
(b) In AEC, show that AE = $-b$ cos A.
(c) In ΔBEC, show that BE = $c - b$ cos A.
(d) Hence, in CEB, show that
$a^2 = b^2 + c^2 - 2bc \cos A$.



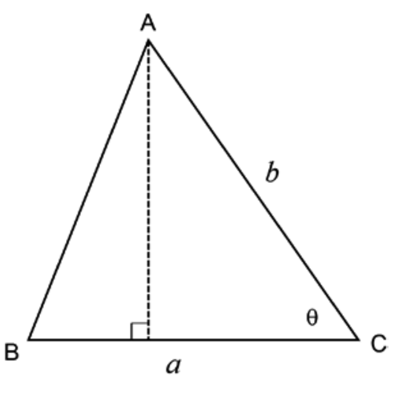
*9. In ΔPQR, find k if there is only one possible value for x.



8.3 Area of a non-right triangle

- Consider a non-right triangle ABC, where θ is the included angle between sides of lengths a and b.
- The height of the triangle, $h = b \sin \theta$.
- Hence, the area of Δ ABC

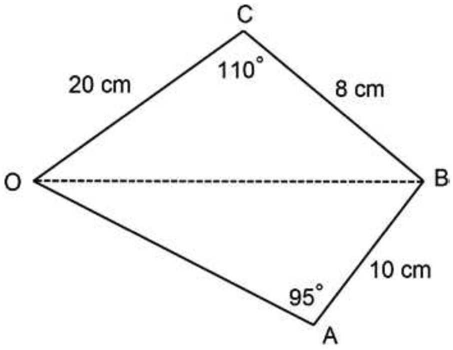
$$= \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times a \times b \sin \theta$$



> *Note the format:*
> *Area of Δ = $\frac{1}{2}$ × side 1 × side 2 × sine of included angle*

Example 8.6

Find the area of the quadrilateral OABC correct to 2 decimal places.



Solution:

Area of ΔOCB = $\frac{1}{2}\times 20\times 8\times \sin 110^{o}$ = 75.1754

In ΔOCB, using the cosine rule:

$OB^2 = 20^2 + 8^2 - 2 \times 20 \times 8 \times \cos 110^{o}$

$= 573.4464$

In ΔOAB, let OA = x, using the cosine rule:

$OB^2 = x^2 + 10^2 - 2 \times 10 \times x \times \cos 95^{o}$

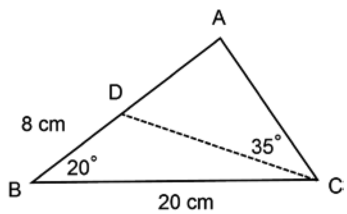
$x^2 - 20 \cos 95^{o}\, x - 473.4464 = 0 \quad \Rightarrow \quad x = 20.9047$ (reject -22.6478).

Hence, area of ΔOAB = $\frac{1}{2}\times 10\times 20.9047\times \sin 95^{o}$ = 104.1258 cm^2.

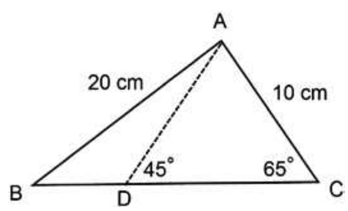
Therefore, area of OABC = 75.1754 + 104.1258 = 179.3011 = 179.3 cm^2.

Exercise 8.3

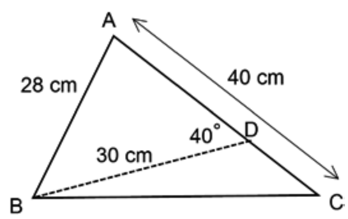
1. In the accompanying diagram:
   (a) Find $\angle$DCB correct to 1 decimal place.
   (b) Find AC correct to 2 decimal places.
   (c) Find the area of ΔABC correct to 2 decimal places.



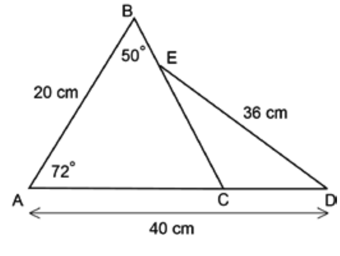
2. In the accompanying diagram, find:
   (a) AD (2 decimal places)
   (b) $\angle$ABD (1 decimal place)
   (c) BC (2 decimal places)
   (d) the area of ΔABC (2 decimal places.



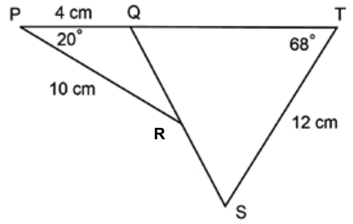
3. In the accompanying diagram, find:
   (a) DC (2 decimal places)
   (b) $\angle$ABC (1 decimal place)
   (c) the area of ΔABC (2 decimal places).



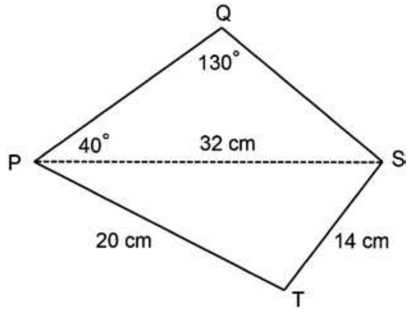
4. In the accompanying diagram, find correct to 2 decimal places:
   (a) AC
   (b) EC
   (c) the area of ΔCED.



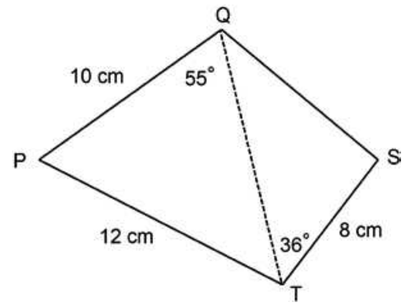
5. In the accompanying diagram, find:
   (a) $\angle PQR$ (1 decimal place)
   (b) RS (2 decimal places)
   (c) the area of ΔRST (2 decimal places).



6. In the accompanying diagram, find:
   (a) $\angle PTS$ (1 decimal place)
   (b) PQ (2 decimal places)
   (c) the area of quadrilateral PQST (2 decimal places).



7. In the accompanying diagram find correct to 2 decimal places:
   (a) the area of ΔPST
   (b) the area of quadrilateral PQST.



Exercise 8.4 Applications

1. Don sails his yacht 5 km from P to Q. From Q he sails his yacht to R which is 8 km from P. The bearings of P and R from Q are S30°W and S45°E respectively.
   (a) Find the distance between Q and R. (b) Find the bearing of P from R.

*2. Mike leaves the rose bush he was examining and walks 35m in the direction S20°W towards a pond. From there he walks 70 m towards a rotunda. Mike is now 100 m from the rose bush. Find the bearing of rotunda from the pond.

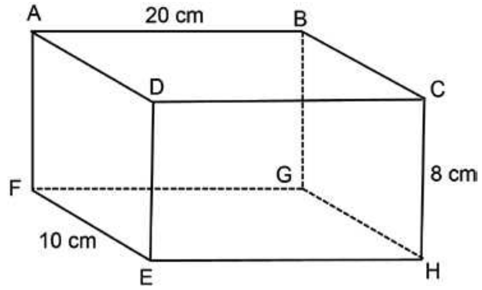
3. As Mahendra walks towards the base of a building, the angle of elevation of the top of the building changes from 25° to 30° over a distance of 50 m. How tall is this building?

*4. From P, the top of an observation tower of height h metres, Greg observes a pedestrian walk away in a straight line from S, the base of the tower. The angle of depression of the pedestrian changed by 20° over a period of 5 minutes. Greg estimates that the distance between him and the pedestrian was 60 m at the start, and 100 m 5 minutes later. Find h.

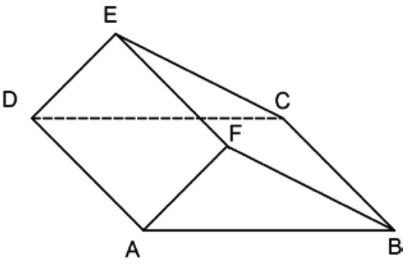
5. A police officer in a police helicopter hovering vertically above a freeway, measures that the angle of depression of a truck moving along the freeway at 25 ms^{-1} (away from the helicopter) undergoes a change of 1° in 1 second. If the helicopter was initially 1000 m from the truck, what was the height of the helicopter?

6. As Gwen walks towards a street lamp, she passes a point where the angle of elevation of the top of the street lamp is 25°. At this point Gwen estimates that she is 30 m from the top of the street lamp. She walks a further 50 m and is now on the other side of the street lamp. Assuming that she walked in a straight line (just avoiding the street lamp), what is the angle of elevation of the top of the street lamp at this stage?

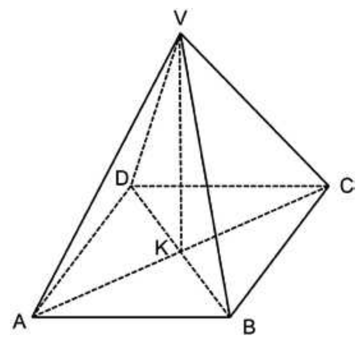
7. In the rectangular prism ABCDEFGH, AB = 20 cm, CH = 8 cm and FE = 10 cm.
   (a) Find $\angle DBH$.
   (b) Find the area of ΔBDH.
   (c) Find $\angle AKB$ where K is the point of intersection between AH and BE.



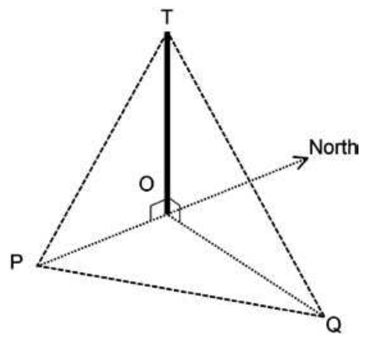
8. ABCDEF is a triangular wedge of uniform cross-section with AB = $5\sqrt{3}$ cm, AF = $\sqrt{19}$ cm, AD = 20 cm and $\angle ABF = 30^\circ$. Without the use of a calculator, find:
   (a) BF given that BF > 7 cm.
   (b) the surface area of the wedge.
   (c) the volume of the wedge.



9. VABCD is a right pyramid with a square base of length 12 cm. K is the foot of the perpendicular from V to the base ABCD with VK = 15 cm
   (a) Find the exact length of AC and VA.
   (b) Find $\angle AVC$.
   (c) Find $\angle BVC$.
   (d) Find the surface area of the pyramid.



10. The points O, P and Q lie on a horizontal plane with P due South of O. The bearing of Q from O is 100°. A tower of height h metres is erected at O. The angle of elevation of T, the top of the tower, from P and Q are 50° and 20° respectively.
    (a) Find OP and OQ in terms of h.
    (b) Find h if the distance between P and Q is 100 m.

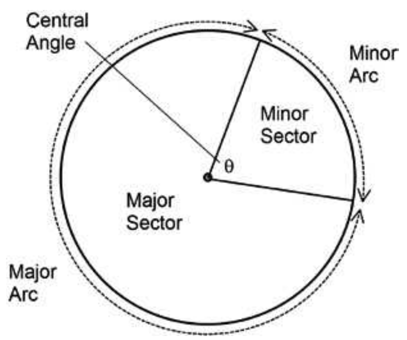
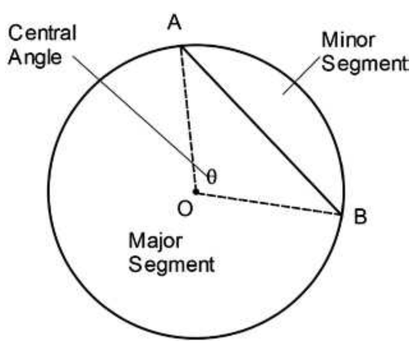


09 Arc, Sectors and Segments

9.1 Arcs, Sectors and Segments

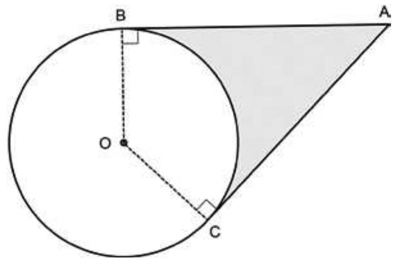
- The table below shows the formulae (formulas) for calculating the length of an arc, the area of a sector and the area of a segment of a circle corresponding to a central angle θ measured in degrees and radians.

- Note the simplicity of these results when the radian measure is used.

	θ in degrees	θ in radians
	• Length of minor arc: $L = \dfrac{\theta^{o}}{360^{o}} \times 2\pi\, r$ • Area of minor sector: $A = \dfrac{\theta^{o}}{360^{o}} \times \pi\, r^{2}$	• Length of minor arc $L = \dfrac{\theta}{2\pi} \times 2\pi\, r = r\,\theta$ • Area of minor sector $A = \dfrac{\theta}{2\pi} \times \pi\, r^{2} = \dfrac{1}{2} r^{2}\theta$
	• Area of minor segment = Area of minor sector $-$ Area of ΔOAB $= \dfrac{\theta^{o}}{360^{o}} \times \pi\, r^{2}$ $- \dfrac{1}{2} \times r^{2} \sin\theta$	• Area of minor segment = Area of minor sector $-$ Area of ΔOAB $= \dfrac{1}{2} r^{2}\theta - \dfrac{1}{2} \times r^{2} \sin\theta$ $= \dfrac{1}{2} r^{2}[\theta - \sin\theta]$

Example 9.1

The lines AB and AC are tangential at B and C respectively to a circle of radius 10 cm with centre O, $\angle BAC = 40^{\circ}$. Find the perimeter and area of the shaded region.

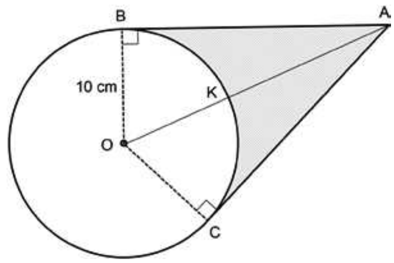


Solution:

In ΔOAB, $\angle BAO = 40^{\circ} \div 2 = 20^{\circ}$.

$\Rightarrow$ $AB = 10 \times \tan 70^{\circ} = 27.4748$ cm.

Let K be the point where OA intersects the circle.



Length of arc BK $= \dfrac{70^{\circ}}{360^{\circ}} \times 2\pi \times 10 = 12.2173$ cm

Hence, perimeter of shaded region

$= 2 \times (27.4748 + 12.2173) = 79.4$ cm.

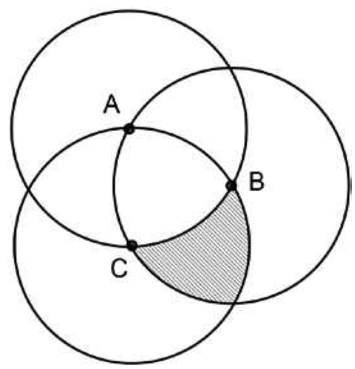
Area of $\Delta OBA = \dfrac{1}{2} \times 10 \times 27.4748 = 137.374\ \text{cm}^2$

Area of sector BAK $= \dfrac{70^{\circ}}{360^{\circ}} \times \pi \times 10^2 = 61.0865\ \text{cm}^2$

Hence, area of shaded region $= 2 \times (137.374 - 61.0865) = 152.6\ \text{cm}^2$.

Example 9.2

The accompanying diagram shows three circles each of radius 20 cm with centres at A, B and C respectively. Find the area of the shaded region.



Solution:

Let D be the point of intersection between the circles with centre B and C.

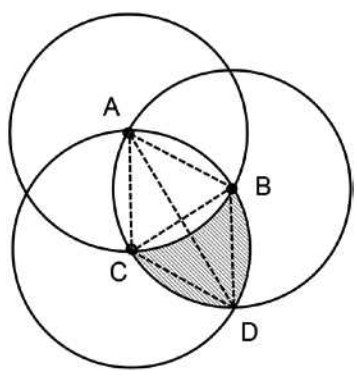
In ΔDBC, $DB = BC = CD = 20$ cm.

Hence, ΔDBC is an equilateral triangle. $\Rightarrow \angle BDC = \dfrac{\pi}{3}$.

Area of ΔDBC, $A_1 = \dfrac{1}{2} \times 20^2 \times \sin \dfrac{\pi}{3} = 173.2051\text{cm}^2$

Area of segment formed by chord BD, centre C:

$A_2 = \dfrac{1}{2} \times 20^2 \times [\dfrac{\pi}{3} - \sin \dfrac{\pi}{3}] = 36.2344\ \text{cm}^2$



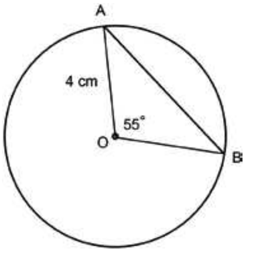
Hence, Area of shaded region $= A_1 + A_2$

$= 173.2051 + 36.2344$

$= 209.4\ \text{cm}^2$.

Exercise 9.1

1. In the accompanying diagram, find:
   (a) the length of the minor arc AB
   (b) the perimeter of the major sector OAB
   (c) the area of ΔOAB
   (d) the area of the minor segment formed by chord AB.



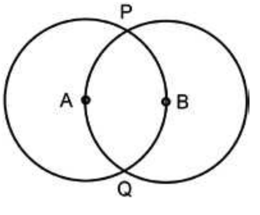
2. A circle of radius 20 cm has its centre at O. AB is a chord of the circle and AB = 20 cm.
   Find: (a) $\angle$AOB (b) the length of the minor arc AB
   (c) the area of the major segment formed by the chord AB.

3. A circle of radius 10 cm has its centre at O. AB is an arc of the circle of length 40 cm.
   Find: (a) $\angle$AOB (in radians) (b) the area of the minor sector AOB.

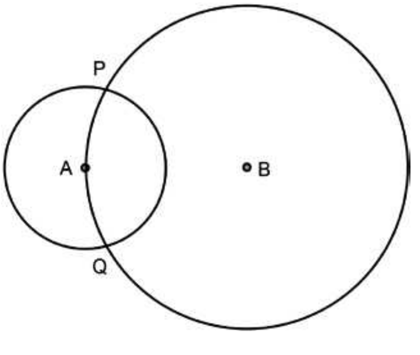
4. A circle has its centre at O. An arc of length 12 cm subtends an angle of 20° at O. Find:
   (a) the radius of the circle (b) the area of the minor segment.

5. A circle has its centre at O. OAB is a sector of the circle of area 50 cm^2 and subtends an angle of 35° at O. Find:
   (a) the radius of the circle (b) the perimeter of the major sector formed by AB.

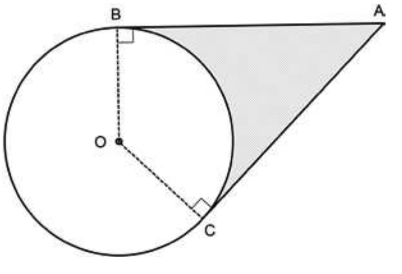
6. The circles with centres at A and B each have radius 10 cm. The circles intersect at P and Q.
   (a) Explain clearly why $\angle PAQ = 120^{\circ}$.
   (b) Find the area of the region trapped between the two circles.



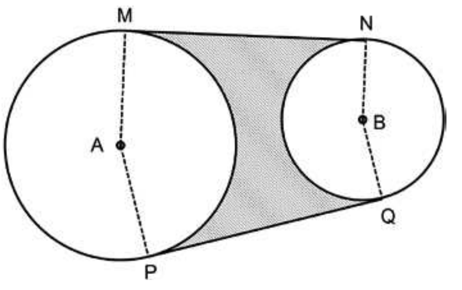
7. The circle with centre A has radius 10 cm. The circle with centre at B has radius twice that of the circle with centre at A. A is on the circumference of the larger circle. The circles intersect at P and Q. Find:
   (a) $\angle$PAB (in radians).
   (b) Find the area of the region trapped between the two circles.



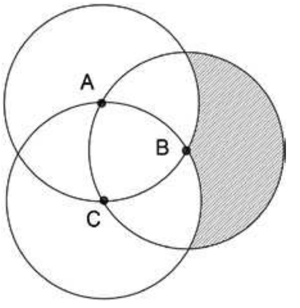
8. The lines AB and AC are tangents to a circle with centre O and radius 10 cm, at B and C respectively. AB = AC = 30 cm. Find:
   (a) $\angle$BOC (in degrees)
   (b) the perimeter of the shaded region
   (c) the area of the shaded region.



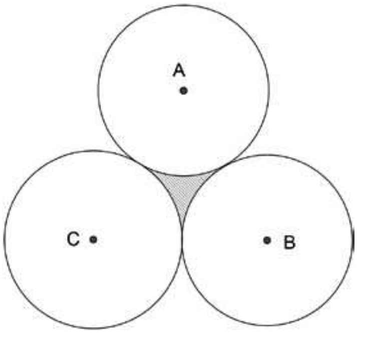
9. The points M and P are on a circle with centre A and radius 10 cm and the points N and Q are on the circle with centre B and radius 6 cm. The lines MN and PQ are tangential to both circles. Given that AB = 22 cm, find:
   (a) MN
   (b) $\angle$MAP (in radians)
   (c) the perimeter of the shaded region.



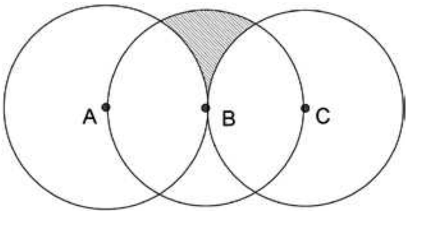
10. The accompanying diagram shows three circles each of radius 16 cm with centres at A, B and C respectively. Find:
    (a) the perimeter of the shaded region
    (b) the area of the shaded region.



11. The accompanying diagram shows three circles each of radius 2π cm with centres labelled A, B and C. The circles are each in contact with each other as shown. Find in exact form:
    (a) the area of ΔABC
    (b) the area of the shaded region
    (c) the perimeter of the shaded region.



12. The accompanying diagram shows three circles each of radius 3π cm. The circle with centre at B passes through the centres of the two other circles as shown.
    (a) Find the perimeter of the shaded region
    (b) Find the area of the shaded region.



10 Trigonometric Equations

10.1 Basic Trigonometric Equations

- The focus on this section is on the use of non-graphical algebraic techniques using scientific and CAS/graphic calculators.
- The use of the "SOLVE" command in CAS calculators will be discussed in Section 10.2.
- In solving basic trigonometric equations like sin (x) = -0.1304, we may choose to use either a method involving the use of reference angles or a method involving the use of principal values.

Example 10.1

Use reference angles/principal values to find x, where $0^\circ \leq x < 360^\circ$, if sin ($x$) = -0.1304.

Solution:

Method involving reference angles

In the given equation, the sine ratio is negative.
Hence, x must be in Quadrant 3 or Quadrant 4.

Consider now sin (x) = 0.1304,

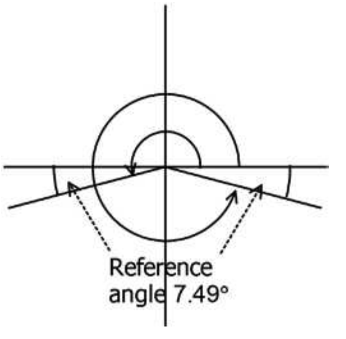
(*__Ignoring__ the negative sign in –0.1304)*

The reference angle = $\sin^{-1}(0.1304) = 7.49^\circ$.

Hence, $x = 180^\circ + 7.49^\circ$ *[In Q3]*

or $360^\circ - 7.49^\circ$ *[In Q4]*

$= 187.49^\circ$ or 352.51°.



Method involving principal values

In the given equation, the sine ratio is negative.
Hence, x must be in Quadrant 3 or Quadrant 4.

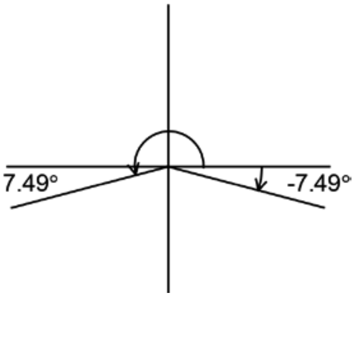
Consider sin (x) = -0.1304

$x = \sin^{-1}(-0.1304)$

Principal value for $\sin^{-1}(x) = -7.49^\circ$

Hence, $x = -7.49^\circ \equiv 360^\circ - 7.49^\circ = 352.51^\circ$ *[In Q3]*

or $(180^\circ + 7.49^\circ) = 187.49^\circ$. *[In Q4]*



Notes:

- To solve for $f(x) = a$ where $f(x)$ is a trigonometric function.

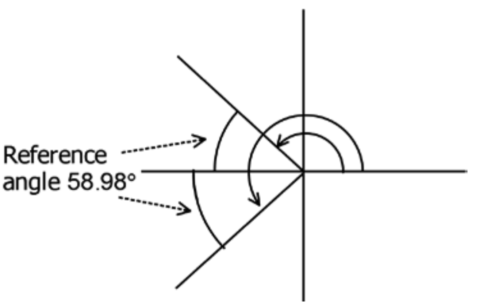
Method involving reference angles	Method involving principal values
• First locate the quadrant locations of x. • Work out the reference angle from $f^{-1}(\lvert a \rvert)$. • The values of x are then obtained by combining the quadrant location and the reference angle for x.	• First locate the quadrant locations of x. • Work out the principal angle from $f^{-1}(a)$. • The values of x are then obtained by combining the quadrant location and the properties of $f(x)$ within the context of the unit circle.

Example 10.2

Use reference angles/principal values to find x, where $0^\circ \le x < 360^\circ$ < if $\cos(x) = -0.5153$.

Solution:

In the given equation, the cosine ratio is negative.
Hence, x must be in Quadrant 2 or Quadrant 3.



Consider $\quad \cos(x) = -0.5153$.

The reference angle = $\cos^{-1}(0.5153) = 58.98^\circ$.

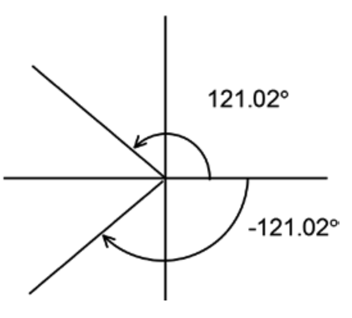
Hence $\quad x = 180^\circ - 58.98^\circ$ *[In Q2]*

or $180^\circ + 58.98^\circ$ *[In Q3]*

$= 121.02^\circ$ or 238.98°.

Or

In the given equation, the cosine ratio is negative.
Hence, x must be in Quadrant 2 or Quadrant 3.
Cos (x) is symmetrical about the x-axis.



Consider $\cos(x) = -0.5153$

$x = \cos^{-1}(-0.5153)$

Principal value for $\cos^{-1}(x) = 121.02^\circ$

Hence, $\quad x = 121.02^\circ$ *[In Q2]*

or -121.02° *[In Q3]*

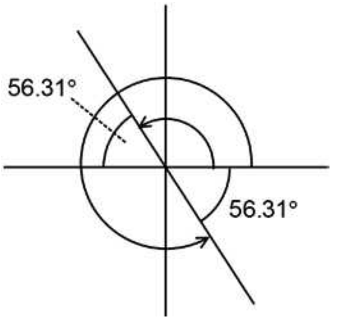
$= 121.02^\circ$ or 238.98°.

Example 10.3

Given that $\tan(2\theta) = -1.5$, use reference angles/principal values to find θ correct to 1 decimal place, where $0^{\circ} \le \theta < 360^{\circ}$.

Solution:

Initially, think of 2θ as a single angle.
The domain for θ is $0^{\circ} \le \theta < 360^{\circ}$.
Hence, the domain for 2θ must be adjusted to $0^{\circ} \le 2\theta < 720^{\circ}$.
In the given equation, the tangent ratio is negative.
Hence, 2θ must be in Quadrant 2 or Quadrant 4.



Consider $\tan(2\theta) = -1.5$.

The reference angle for $2\theta = \tan^{-1}(1.5) = 56.31^{\circ}$.

Hence, $2\theta = 180^{\circ} - 56.31^{\circ}$ *[In Q2]*

or $360^{\circ} - 56.31^{\circ}$ *[In Q4]*

or $(180^{\circ} - 56.31^{\circ}) + 360^{\circ}$ *[One complete revolution back to Q2]*

or $(360^{\circ} - 56.31^{\circ}) + 360^{\circ}$ *[One complete revolution back to Q4]*

$= 123.69^{\circ}$ or 303.69° or 483.69° or 663.69°

Therefore, $\theta = 61.8^{\circ}$ or 151.8° or 241.8° or 331.8°

[Divide each of the values for 2θ by 2 to obtain values for θ.]

Notes:

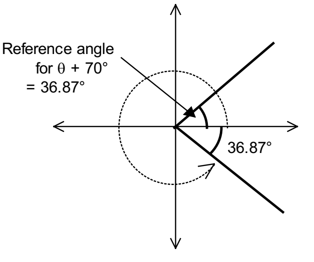
- *The third and fourth solutions involve complete revolutions back into the appropriate quadrants.*

Example 10.4

Use reference angles/principal values to find θ, where $0^{\circ} \le \theta < 360^{\circ}$, if $\cos(\theta + 70^{\circ}) = 0.8$.

Solution:

Initially, think of $(\theta + 70^{\circ})$ as a single angle.
The domain for θ is $0^{\circ} \le \theta < 360^{\circ}$. Hence, the domain for $(\theta + 70^{\circ})$ must be adjusted to $70^{\circ} \le \theta + 70^{\circ} < 420^{\circ}$.
In the given equation, the cosine ratio is positive.
Hence, $\theta + 70^{\circ}$ must be in Quadrant 1 or Quadrant 4.



Consider $\cos(\theta + 70^{\circ}) = 0.8$.

The reference angle for $\theta + 70^{\circ} = \cos^{-1}(0.8) = 36.87^{\circ}$.

Hence, $\theta + 70^{\circ} = 36.87^{\circ}$ *[In Q1. Reject as it is outside the adjusted domain.]*

or $360^{\circ} - 36.87^{\circ}$ *[In Q4]*

or $360^{\circ} + 36.87^{\circ}$ *[One complete revolution back to Q1]*

$= 323.13^{\circ}$ or 396.87°

Therefore, $\theta = 253.1^{\circ}$ or 326.9°

Example 10.5

Without the use of a calculator, find θ, where $0 \le \theta \le 2\pi$, if $\sin(\theta + \frac{\pi}{4}) = 0.5$.

Solution:

The reference angle for $(\theta + \frac{\pi}{4}) = \sin^{-1}(0.5) = \frac{\pi}{6}$.

Hence, $\theta + \frac{\pi}{4} = \frac{\pi}{6}$

or $\pi - \frac{\pi}{6}$ *[In Q2.]*

$= \frac{\pi}{6}$ or $\frac{5\pi}{6}$

Therefore, $\theta = -\frac{\pi}{12}, \frac{7\pi}{12}$

$= \frac{23\pi}{12}, \frac{7\pi}{12}$

Exercise 10.1

Use reference angles/principal values to solve each of the following questions.

1. Solve for θ in each of the following equations within the specified domain.
   (a) $\sin\theta = 0.2354$ $\quad 0^\circ \le \theta \le 360^\circ$ (b) $\cos\theta = -0.5319$ $\quad 0^\circ \le \theta \le 360^\circ$
   (c) $\tan\theta = 1.4943$ $\quad 0^\circ \le \theta \le 360^\circ$ (d) $\cos\theta = 0.5319$ $\quad -180^\circ < \theta \le 180^\circ$
   (e) $\sin\theta = -0.7872$ $\quad 0 \le \theta \le 2\pi$ (f) $\cos\theta = 0.1105$ $\quad 0 \le \theta \le 2\pi$
   (g) $\tan\theta = -5.2304$ $\quad -\pi < \theta \le \pi$ (h) $\sin\theta = 0.2579$ $\quad -\pi < \theta \le \pi$

2. Solve for θ in each of the following equations within the specified domain.
   (a) $\sin 2\theta = 0.6482$ $\quad 0^\circ \le \theta \le 360^\circ$ (b) $\cos\frac{\theta}{2} = -0.6491$ $\quad 0^\circ \le \theta \le 360^\circ$
   (c) $\tan(\theta + 25^\circ) = -2.5$ $\quad 0^\circ \le \theta \le 360^\circ$ (d) $\cos(\theta - 10^\circ) = 0.5319$ $\quad -180^\circ < \theta \le 180^\circ$
   (e) $\sin\frac{\theta}{3} = -0.8145$ $\quad 0 \le \theta \le 2\pi$ (f) $\cos 3\theta = 0.9056$ $\quad 0 \le \theta \le 2\pi$
   (g) $\tan(\pi - \theta) = -5.2304$ $\quad -\pi < \theta \le \pi$ (h) $\sin(\pi + \theta) = 0.7814$ $\quad -\pi < \theta \le \pi$

3. Solve for θ in each of the following equations within the specified domain.
   (a) $\sin(2\theta + 70^\circ) = 0.1111$ $\quad 0^\circ \le \theta \le 360^\circ$
   (b) $\cos 2(\theta + 15^\circ) = 0.6743$ $\quad -180^\circ < \theta \le 180^\circ$
   (c) $\sin(\theta + 1) = 0.1451$ $\quad 0 \le \theta \le 2\pi$
   (d) $\tan(1 - \theta) = -0.6990$ $\quad 0 \le \theta \le 2\pi$

4. Without the use of a calculator, solve for θ within the specified domain.

(a) $\sin\theta = 0.5$ $\quad 0 \le \theta \le 2\pi$

(b) $\cos\theta = \dfrac{1}{\sqrt{2}}$ $\quad 0 \le \theta \le 2\pi$

(c) $\tan\theta = 1$ $\quad -\pi \le \theta \le \pi$

(d) $2\cos\theta = \sqrt{3}$ $\quad -\pi < \theta \le \pi$

(e) $\sin 2\theta = -\dfrac{\sqrt{2}}{2}$ $\quad 0 \le \theta \le 2\pi$

(f) $\cos\dfrac{\theta}{3} = -\dfrac{1}{2}$ $\quad 0 \le \theta \le 2\pi$

(g) $\sqrt{3}\tan(\theta + \dfrac{\pi}{5}) = -1$ $\quad -\pi < \theta \le \pi$

(h) $\sin 2(\theta + \dfrac{\pi}{3}) = -1$ $\quad -\pi < \theta \le \pi$

5. Without the use of a calculator, solve for θ within the specified domain.

(a) $2\sin\left(\theta + \dfrac{\pi}{4}\right) = -1$ $\quad 0 \le \theta \le 2\pi$

(b) $\cos\left(\dfrac{\pi}{4} - \theta\right) = \dfrac{\sqrt{3}}{2}$ $\quad 0 \le \theta \le 2\pi$

(c) $\tan(\pi - \theta) = 1$ $\quad -\pi < \theta \le \pi$

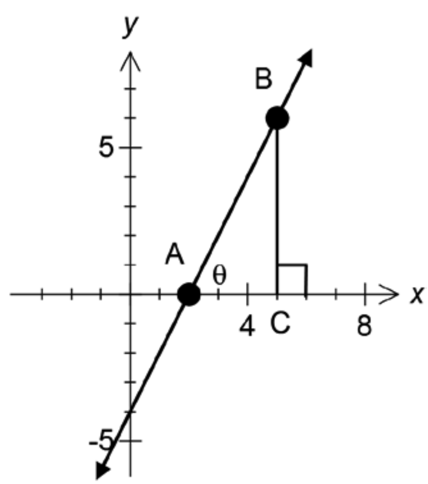
(d) $-2\sin\left(\dfrac{\pi}{3} - \theta\right) = \sqrt{3}$ $\quad -\pi < \theta \le \pi$

10.2 Gradient of a Line

Hands On Task 10.1

In this task, we will explore the relationship between the gradient of a line and the angle this line makes with the positive *x*-axis.

The accompanying diagram shows the line with equation $y = 2x - 4$. The line intersects the *x*-axis at A and the line makes an angle of size θ with the positive *x*-axis. The point B (5, 6) lies on this line and BC is perpendicular to the *x*-axis.



1. In ΔABC, find the length of AC and BC.

2. Find $\tan\theta$ and hence make a conjecture regarding the relationship between $\tan\theta$ and the gradient of this line.

3. Investigate if the conjecture is true for other lines.

Summary

- If the line $y = mx + c$ makes an angle of θ with the positive x-axis, then $m = \tan\theta$.

Example 10.6

Without the use of a calculator, find the angle the line with equation $x + \sqrt{3}y = 10$ makes with the positive x-axis.

Solution:

Gradient of line $m = -\dfrac{1}{\sqrt{3}}$.

Hence, $\tan\theta = -\dfrac{1}{\sqrt{3}} \Rightarrow \theta = \dfrac{5\pi}{6}$ or 150°.

Note:

- *The gradient is negative, hence, θ must be in the second quadrant.*

Exercise 10.2

1. Without the use of a calculator, find the equation of the line passing through the point A and inclined at an angle of θ with the positive x-axis as stated.

   (a) $\theta = 30^{\circ}$ A $(\sqrt{3}, 3)$ (b) $\theta = 120^{\circ}$ A $(\sqrt{3}, 6)$

   (c) $\theta = \dfrac{\pi}{3}$ A $(\sqrt{3}, 4)$ (d) $\theta = \dfrac{3\pi}{4}$ A $(-2, -2)$.

2. Without the use of a calculator, find the angle each of the given lines make with the positive x-axis.

   (a) $y = -x + 5$ (b) $y = 1 - \sqrt{3}x$ (c) $2x - 2y = 5$

   (d) $\sqrt{3}x + y = -3$ (e) $3x - \sqrt{3}y = 6$ (f) $-\sqrt{3}x + 3y = 9$

3. A line passes through the points A and B as given. Without the use of a calculator, find the angle this line makes with the positive x-axis.

   (a) A (5, 5) B (7, 7) (b) A (5, 5) B (8, 2)

   (c) A (1, 1) B $(2, 1 + \sqrt{3})$ (d) A $(4, -2)$ B $(4 - \sqrt{3}, 1)$

*4. Without the use of a calculator, find the acute angle between each of these two lines:

   (a) $x - y = 10$ and $x + y = 10$ (b) $y = x$ and $y = \dfrac{x}{\sqrt{3}}$

   (c) $y = -x - 1$ and $y = \sqrt{3}x + \sqrt{3}$ (d) $y = \sqrt{3} + x$ and $y = -\dfrac{\sqrt{3}x}{3} - 1$

10.3 Use of CAS Calculators

10.3.1 Extension: General Solutions

- The use of the "SOLVE" command in the context of trigonometric equations in most CAS calculators lead to the concept of general solutions which is beyond the scope of this course.
- For example, the general solution to the equation $\cos x = 0.5$ is

  $x = \pm 60^\circ + 360n^\circ$ or $x = \pm\frac{\pi}{3} + 2n\pi$ where n is an integer (positive or negative).

  The general solution provides a "formula" for determining the solutions to the given equation over any required domain.

Example 10.7

Given that the general solution to the equation $\tan x = 1$ is $x = 45^\circ + 180n^\circ$, find the solutions to this equation if: (a) $0 \le x \le 360^\circ$ (b) $-180^\circ \le x \le 180^\circ$.

Solution:

(a) $\tan x = 1 \Rightarrow x = 45^\circ + 180n^\circ$

When $n = 0$, $x = 45^\circ$

$n = 1$, $x = 45^\circ + 180^\circ = 225^\circ$

Hence, $x = 45^\circ$ or 225°.

(b) $\tan x = 1 \Rightarrow x = 45^\circ + 180n^\circ$

When $n = 0$, $x = 45^\circ$

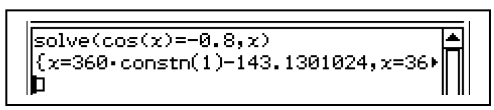
$n = -1$, $x = 45^\circ - 180^\circ = -135^\circ$

Hence, $x = 45^\circ$ or -135°.

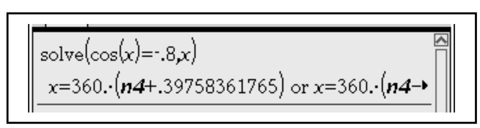
Example 10.8

Use your CAS calculator to find the general solution to $\cos x^\circ = -0.8$.
Hence, find x correct to one decimal place where $-360^\circ \le x \le 360^\circ$.

Solution:



$\cos x^\circ = -0.8 \Rightarrow x = 360n^\circ \pm 143.1^\circ$.



When $n = 0$, $x = -143.1^\circ$ or 143.1°

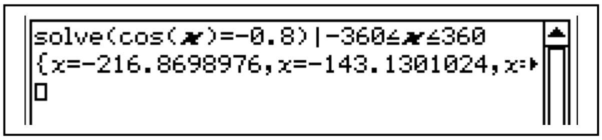
$n = 1$, $x = 216.9^\circ$ or 503.1°

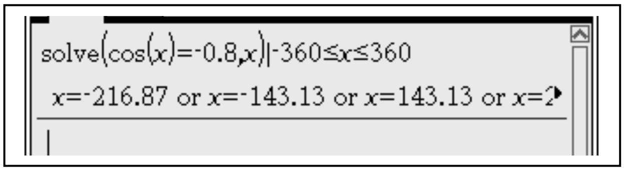
$n = -1$, $x = -216.9^\circ$ or -503.1°

Hence, $x = -216.9^\circ, -143.1^\circ, 143.1^\circ, 216.9^\circ$.

10.3.2 Using the "Conditional Solve" CAS command

- As observed in the previous section, when the "SOLVE" command is used in the context of trigonometric equations, most CAS calculators return the general solution.
- To retrieve numerical solutions within a given domain the "Conditional SOLVE" command should be used.





Exercise 10.3

Rework each question in Exercise 10.1 using the "SOLVE" and/or "Conditional SOLVE" command on your CAS calculator.

10.4 Basic Trigonometric Ratios within polynomials

- In this section we will work with trigonometric ratios embedded within polynomials.

Example 10.9

Without the use of calculator, solve exactly for θ in $(2 \sin \theta - 1)(\cos \theta + 1) = 0$ where $0 \le \theta \le 2\pi$.

Solution:

$(2 \sin \theta - 1)(\cos \theta + 1) = 0 \;\Rightarrow\; \sin \theta = \dfrac{1}{2}$ or $\cos \theta = -1$

For $\sin \theta = \dfrac{1}{2}$, reference angle for $\theta = \dfrac{\pi}{6} \quad \Rightarrow \quad \theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

For $\cos \theta = -1$, $\theta = \pi$.

Therefore, $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ or π.

Example 10.10

Without the use of a calculator, solve for θ in $2\cos^2\theta - 3\cos\theta - 2 = 0$ where $0^\circ \le \theta \le 360^\circ$. Note that $\cos^2\theta = (\cos\theta)^2$.

Solution:

$2\cos^2\theta - 3\cos\theta - 2 = 0$ *[This is in the form of the quadratic equation $2x^2 - 3x - 2 = 0$]*

Factorising the Left side of the equation:

$(2\cos\theta + 1)(\cos\theta - 2) = 0$

$\Rightarrow \cos\theta = -\frac{1}{2}$ or $\cos\theta = 2$

For $\cos\theta = -\frac{1}{2}$, reference angle for $\theta = 60^\circ$ $\Rightarrow$ $\theta = 120^\circ$ or 240°.

For $\cos\theta = 2$, there is no solution. *[Note: $-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$]*

Therefore, $\theta = 120^\circ$ or 240°.

Example 10.11

Use your calculator to solve for θ (1 decimal place) in $\tan^2\theta + \tan\theta - 1 = 0$ where $0^\circ \le \theta \le 360^\circ$. Note that $\tan^2\theta = (\tan\theta)^2$.

Solution:

From CAS calculator:

$\theta = 31.7^\circ, 121.7^\circ, 211.7^\circ$ or 301.7°

```
solve((tan(θ))^2+tan(θ)-1=0,θ)|0≤θ≤360
        {θ=121.7,θ=211.7,θ=301.7,θ=31.7}
```

Exercise 10.4

1. Without the use of a calculator, solve for θ where $0 \le \theta \le 360^\circ$.
   (a) $(\sin\theta + 1)(\cos\theta - 2) = 0$
   (b) $\cos\theta\,(2\cos\theta - 1) = 0$
   (c) $2\sin^2\theta + \sin\theta = 0$
   (d) $\tan^2\theta = \tan\theta$

2. Without the use of a calculator, solve for θ, where $0 \le \theta \le 2\pi$.
   (a) $\cos^2\theta + \cos\theta = 0$
   (b) $2\sin^2\theta - \sqrt{3}\sin\theta = 0$
   (c) $\sin\theta + \cos\theta = 0$
   (d) $\tan^2\theta - 3 = 0$

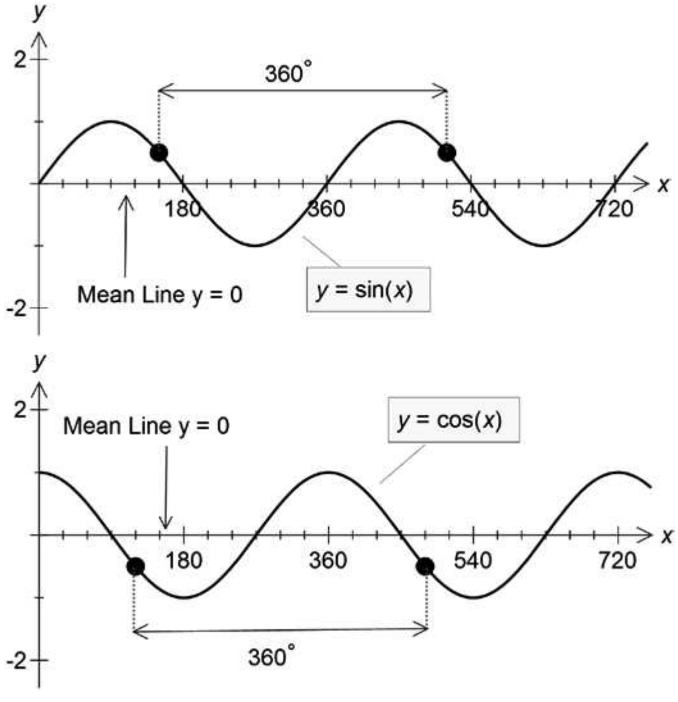
3. Solve for x where $0 \le x \le 360^\circ$.
   (a) $\tan^2 x - 4\tan x + 3 = 0$
   (b) $2\sin^2 x = 3\sin x + 2$
   (c) $2\cos^2 x + 5\cos x = 1$
   (d) $\cos^2 x + \cos x - 3 = 0$

4. Without the use of a calculator solve for t in the domain $0 \le t \le 2\pi$.
   (a) $(\sin t - 1)(\sin 2t + 1) = 0$
   (b) $\cos^2 2t - 2\cos 2t = -1$
   (c) $3\sin^2 t - \cos^2 t = 0$
   (d) $2\sin^2 2t - 1 = 0$

11 Trigonometric Graphs

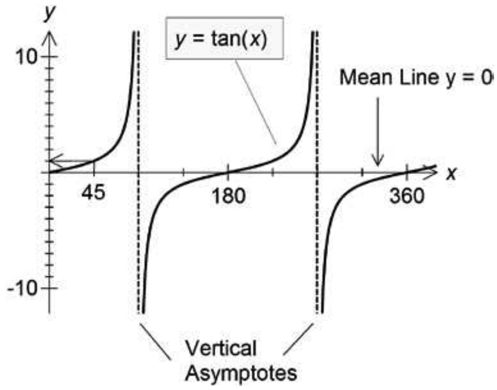
11.1 Graphs of basic trigonometric functions

- In this section we will examine trigonometric functions from a graphical perspective.
- Consider the graphs of $y = \sin x$ and $y = \cos x$ for $0 \leq x \leq 360^{\circ}$.



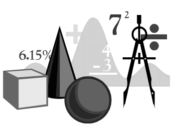
- The graphs of $y = \sin x$ and $y = \cos x$ consist of a repeated regular waveform. Such functions are referred to as *periodic functions*.
- In each case, the pattern covers a horizontal distance of 360° before the pattern repeats itself. The *period* of the sin x and cos x function is 360°.
- The height of the curve after every 360° is identical (see points A and B).
  That is: $\sin(x + 360^{\circ}) = \sin(x)$ and $\cos(x + 360^{\circ}) = \cos x$.
  In general, $\sin(x + 360^{\circ}n) = \sin(x)$ and $\cos(x + 360^{\circ}n) = \cos(x)$, where n is an integer.
- The mean line or equilibrium line cuts through the "middle" of the graph. The equation of the mean line is $y = 0$.
- The *amplitude* is the distance from the mean line to the maximum point or minimum point of the graph. The amplitude of the sin x and cos x function is 1.
- Hence for $y = \sin x$ and $y = \cos x$:
  - Equation of mean line is $y = 0$.
  - Period = 360° or 2π radians
  - Amplitude = 1 unit
  - Maximum value = 1
  - Minimum value -1.

- Notice that the $y = \cos x$ curve can be obtained by translating the $y = \sin x$ curve 90° to the left. Hence: $\sin (x + 90^{\circ}) = \cos x$.
- Similarly the $y = \sin x$ curve can be obtained by translating the $y = \cos x$ curve 90° to the right. Hence: $\cos (x - 90^{\circ}) = \sin x$.
- Note that $\cos (90^{\circ} - x) = \sin x$ and $\sin (90^{\circ} - x) = \cos x$.
- Consider now the graph of $y = \tan x$ for $0 \le x \le 360^{\circ}$.



  - The period of the $\tan x$ function is 180°.
    Hence, $\tan (x + 180^{\circ} n) = \tan (x)$, where n is an integer.
  - The equation of the mean line is $y = 0$.
  - The function does not have any maximum or minimum points.
    Hence, the concept of amplitude is not applicable.
  - The graph has vertical asymptotes at $x = 90^{\circ}$ and $x = 270^{\circ}$.
  - One point of interest is $(45^{\circ}, 1)$.

11.2 Graphs of Trigonometric functions of the form $y = a\,f\,(bx + c) + d$



Hands On Task 11.1

Recall that for $y = \sin x$ and $y = \cos x$:

- Equation of mean line is $y = 0$.
- Period = 360° or 2π radians
- Amplitude = 1 unit

1. By applying the appropriate translation to the curve of $y = \sin x$ or $y = \cos x$ determine the equation of the mean line for the curve with equation:

   (a) $y = \sin x + 5$ (b) $y = \cos 2x - 3$ (c) $y = 2 - \sin (x + 30^{\circ})$

2. By applying the appropriate dilation to the curve of $y = \sin x$ or $y = \cos x$ determine the amplitude for the curve with equation:

   (a) $y = 2 \sin x$ (b) $y = 3 \cos 2x + 3$ (c) $y = -4 \sin (x - \frac{\pi}{6})$

3. By applying the appropriate dilation to the curve of $y = \sin x$ or $y = \cos x$ determine the period for the curve with equation:

   (a) $y = \sin 2x$ (b) $y = \cos \frac{x}{2} + 3$ (c) $y = 1 - 2 \sin (x + 60^\circ)$

4. The graph of $y = \sin (x)$ passes through the point (0, 0). By applying the appropriate translation (and dilation) to the curve of $y = \sin x$ determine the image of this point for:

   (a) $y = \sin (x - 50^\circ)$ (b) $y = \sin (x + \frac{\pi}{3})$ (c) $y = \sin 2x$.

5. The graph of $y = \cos (x)$ passes through the point (0, 1). By applying the appropriate translation (and dilation) to the curve of $y = \cos x$ determine the image of this point for:

   (a) $y = \cos (x + 50^\circ)$ (b) $y = \cos (x - \frac{\pi}{3})$ (c) $y = \cos 2x$.

6. By applying the appropriate translation and dilation to the curve of $y = \sin x$ or $y = \cos x$ determine the minimum and maximum value for y for:

   (a) $y = -2 \sin (x + 10^\circ)$ (b) $y = 1 + 2 \cos 2x$ (c) $y = 3 - 4 \sin \frac{x}{2}$.

7. By applying the appropriate dilation to the curve of $y = \tan x$ determine the period for the curve with equation:

   (a) $y = \tan 2x$ (b) $y = -\tan \frac{x}{2}$ (c) $y = 1 + 2 \tan (x + 60^\circ)$

8. The graph of $y = \tan (x)$ passes through the point (0, 0). By applying the appropriate translation (and dilation) to the curve of $y = \tan x$ determine the image of this point for:

   (a) $y = \tan (x + 20^\circ)$ (b) $y = \tan (x - \frac{\pi}{4})$ (c) $y = \tan 2x$.

9. Use the observations made above to determine (i) the equation of the mean line (ii) the amplitude (iii) the period (iv) the minimum and maximum values for the curves with equations $y = a \sin (bx + c) + d$ and $y = a \cos (bx + c) + d$.

10. Use the observations made above to determine (i) the equation of the mean line (ii) the period for the curve with equation $y = a \tan (bx + c) + d$.

Summary

	Equation of Mean Line	Amplitude	Minimum/ Maximum y-value	Period	Phase shift
$y = \sin x$ $y = \cos x$	$y = 0$	1	Min: -1 Max: 1	360° or 2π	0
$y = a\sin(bx + c) + d$ $y = a\cos(bx + c) + d$	$y = d$	$\lvert a\rvert$	Min: $d - \lvert a\rvert$ Max: $d + \lvert a\rvert$	$\frac{360^\circ}{b}$ or $\frac{2\pi}{b}$	*Shifted* $\frac{c}{b}$ *degrees/ radians to the left*
$y = \tan x$	$y = 0$			180° or π	0
$y = a\tan(bx + c) + d$	$y = d$			$\frac{180^\circ}{b}$ or $\frac{\pi}{b}$	*Shifted* $\frac{c}{b}$ *degrees/ radians to the left*

Note:

- *The phase shift refers to the magnitude of the horizontal translation applied to the $y = \sin x$, $y = \cos x$ or $y = \tan x$ curve.*

Example 11.1

Without the use of a calculator, find the smallest positive value of x given that:

(a) $\cos 60^\circ = \sin x$ (b) $\sin\frac{7\pi}{6} = \cos x$.

Solution:

(a) Since $\cos(90^\circ - x) = \sin x$ $\Rightarrow 90^\circ - x = 60^\circ$

$\Rightarrow x = 30^\circ$

(b) Since $\sin(x + \frac{\pi}{2}) = \cos x$ $\Rightarrow x + \frac{\pi}{2} = \frac{7\pi}{6}$

$\Rightarrow x = \frac{2\pi}{3}$.

Example 11.2

For each of the given curves, (a) $y = 5\sin(x + 60^\circ)$ (b) $y = 10 - 5\cos(x - \frac{\pi}{6})$, state:

(i) the equation of the mean line, (ii) the amplitude, (iii) the period (iv) the phase shift (v) the maximum and minimum y-values.

Solution:

(a) (i) equation of mean line is $y = 0$
(ii) amplitude = 5
(iii) period = 360°
(iv) phase shift 60° to the left
(v) minimum y-value: -5 maximum y-value: 5

(b) (i) equation of mean line is $y = 10$
(ii) amplitude = $|-5| = 5$
(iii) period = 2π radians
(iv) phase shift is $-\frac{\pi}{6}$ to the left or $\frac{\pi}{6}$ to the right.
(v) minimum y-value: $10 - |-5| = 5$ maximum y-value: $10 + |-5| = 15$

Example 11.3

A tangent curve $y = f(x)$ has period 180° with mean line $y = -4$ and phase shift 10° to the right. Given that the curve passes through the point $(55^\circ, 0)$, find $f(x)$.

Solution:

Equation of mean line is $y = -4$.
Period is 180°.
Phase shift is 10° to the right.

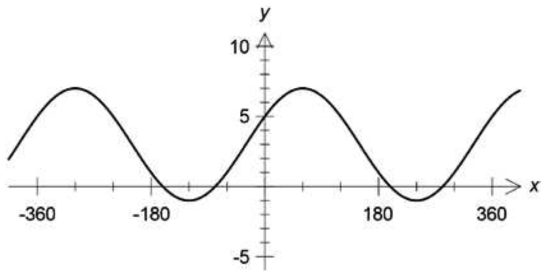
Hence, $y = a\tan(x - 10^\circ) - 4$.

When $x = 55$, $y = 0$ $\Rightarrow$ $0 = a \times \tan(55 - 10) - 4$
$0 = a - 4$ $\Rightarrow$ $a = 4$

Therefore, equation of curve is $y = 4\tan(x - 10^\circ) - 4$.

Example 11.4

The sketch of the curve $y = a \sin(x + b) + c$ is given in the accompanying diagram. Given that $a > 0$, find the values for a, b and c.



Solution:

Minimum y-value = -1
Maximum y-value = 7

Hence, mean y-value = $\dfrac{-1+7}{2}$ = 3.

Therefore, the equation of the mean line is $y = 3$. $\Rightarrow$ $c = 3$

Difference between maximum y-value and mean y-value = $7 - 3 = 4$
$\Rightarrow$ amplitude = 4 $\Rightarrow$ $|a| = 4$. But, $a > 0$, $\Rightarrow$ $a = 4$.

Hence $y = 4 \sin(x + b) + 3$.

In sketch above, when $x = 0$, $y = 5$ $\Rightarrow$ $5 = 4 \sin(b) + 3$
$\sin(b) = 0.5$ $\Rightarrow$ $b = 30^\circ$.

Therefore, equation of given curve is $y = 4 \sin(x + 30^\circ) + 3$.

Example 11.5

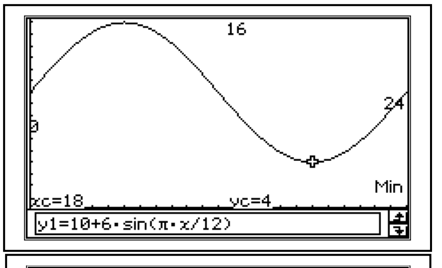
The temperature θ C inside a warehouse t hours after 6.00 am is modelled by the equation $\theta = 10 + 6 \sin \dfrac{\pi t}{12}$. Use a graphical method to determine the:

(a) minimum temperature inside the warehouse and the first time after 6 am when this occurs

(b) number of hours in a day when the temperature inside the warehouse is above 12°C.

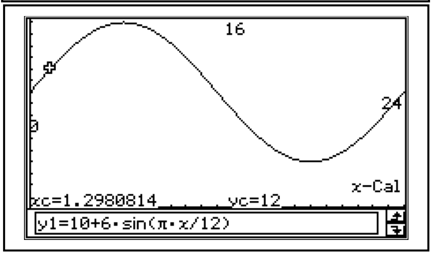
Solution:

(a) Use the "Min" command: .
The minimum temperature of 4 C occurs 18 hours after 6 am i.e. at 12 midnight.



(b) Use the "x-cal" command.
$\theta = 12$ for $t = 1.2981$ and 10.7019

From the graph drawn,
$\theta \geq 12$ for $1.2981 \leq t \leq 10.7019$
that is for 9.4038 hours a day.
Hence, 9 hours and 24 min.



Exercise 11.1

1. Without the use of a calculator, find the smallest positive value of x given that:
   (a) $\cos 20^\circ = \sin x$ (b) $\sin \frac{5\pi}{6} = \cos x$ (c) $\cos 100^\circ = \sin x$

2. For each of the given curves, state, (i) the equation of the mean line, (ii) the amplitude, (iii) the period (iv) the phase shift (v) the maximum and minimum y-values.
   (a) $y = -3 \sin x$ (b) $y = 0.5 \cos (2x)$ (c) $y = 2 \cos (x + 10^\circ)$
   (d) $y = 6 - 3 \cos (\pi x)$ (e) $y = 2 + 3 \cos (x + \frac{\pi}{6})$ (f) $y = 2\pi \cos (x - \frac{\pi}{12})$

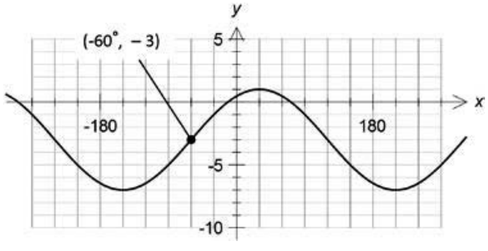
3. For each of the given curves, state, (i) the equation of the mean line, (ii) the period (iii) the phase shift (iv) the equation of the asymptotes.
   (a) $y = \tan 2x$ (b) $y = 1 + 0.5 \tan x$ (c) $y = \tan (x - 30^\circ)$
   (d) $y = 5 + 2 \tan (x)$ (e) $y = 3 - 3 \tan (x - \frac{\pi}{6})$

4. A sine curve $y = f(x)$ has period 360° with mean line $y = 2$ and no phase shift. Given that y has a maximum value of 5, find a possible equation of this curve.

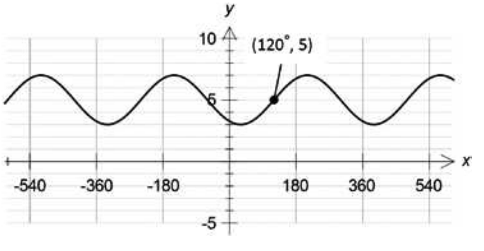
5. A cosine curve $y = f(x)$ has period 360° with mean line $y = 5$ and a phase shift of -30°. Given that y has a minimum value of 1, find a possible equation of this curve.

6. A tangent curve $y = f(x)$ has period 180° with mean line $y = 2$ and phase shift of 45°. Given that the curve passes through the point $(90^\circ, 7)$, find $f(x)$.

7. The sketch of the curve $y = a \cos (x + b) + c$ is given in the accompanying diagram. Find the values for a, b and c given that:
   (a) $a > 0$ and $b < 0$

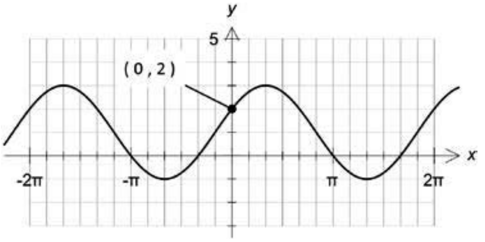


   (b) $a < 0$ and $b < 0$

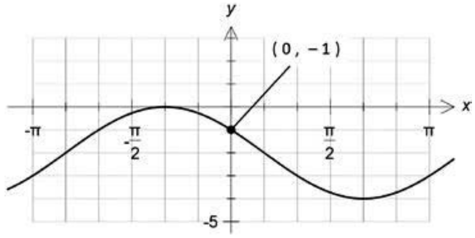


8. The sketch of the curve $y = a \sin (x + b) + c$ is given in the accompanying diagram. Find the values for a, b and c given that:

(a) $a > 0$ and $b > 0$

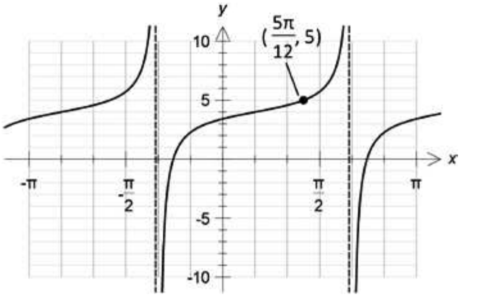


(b) $a < 0$ and $b < 0$

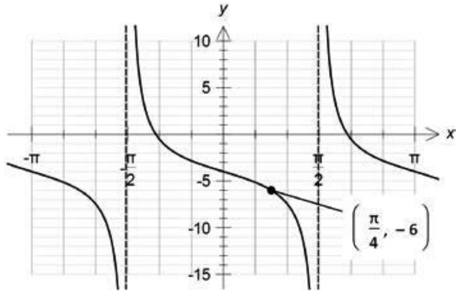


9. The sketch of the curve $y = a \tan (x + b) + c$ is given in the accompanying diagram. Find the values for a, b and c given that:

(a) $a > 0$ and $b < 0$



(b) $a < 0$



10. The temperature θ°C inside a garden shed t hours after 12 midnight is modelled by the equation $\theta = 25 - 15 \cos \dfrac{\pi t}{12}$. Determine the:
   (a) temperature inside the shed at 6 am
   (b) maximum temperature inside the shed and the first time after 12 am when this occurs
   (c) number of hours in a day when the temperature inside the shed is above 35°C.

11. The water depth, h meters, measured from the bottom of a lake, t hours after 6 am is modelled by the equation $h = 8 + 3 \sin \left(\dfrac{\pi t}{6} - \dfrac{\pi}{12}\right)$. Determine the:
   (a) water depth at 12 noon
   (b) minimum water depth and the first time after 6 am when this occurs
   (c) number of hours in one cycle when the water depth is below 9 metres.

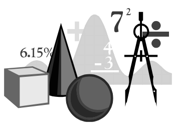
12 Trigonometric Identities

12.1 Compound Angle Formulae

- The following formulae were introduced in Exercise 6.1, for A and B as acute angles:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

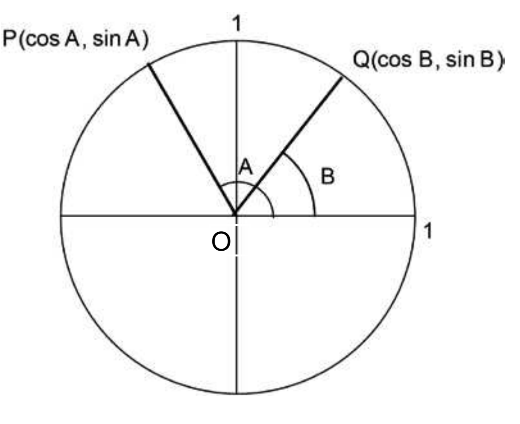
- These formulae are part of a set of formulae collectively referred to as the compound angle formulae and are true for all values of A and B.



Hands On Task 12.1

In this task, we will develop the proofs for the compound angle formulae as being true for all values of A and B.

1. Consider the points P and Q on the unit circle with centre O. OP, OQ are inclined with the positive x-axis with angles A and B respectively. Hence P and Q have coordinates ($\cos A$, $\sin A$) and ($\cos B$, $\sin B$) respectively.



2. Use Pythagoras' Theorem to find the length of the chord PQ in terms of A and B.

3. In ΔOPQ, clearly $\angle$POQ = $A - B$.

4. For ΔOPQ, use the expression for PQ in (1) and the cosine rule, $PQ^2 = OP^2 + OQ^2 - 2 \times OP \times OQ \times \cos P\hat{O}Q$, to prove that:
$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

5. By replacing B with $-B$ in the formula for $\cos(A - B)$ and using the results $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$, prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

6. Use the result $\sin\theta = \cos(90^\circ - \theta)$, and the formula for $\cos(A \pm B)$ to prove that $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$.

7. Use the relationship $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and the formulae for $\sin(A \pm B)$ and $\cos(A \pm B)$ to prove that $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$.

Summary

The complete set of compound angles formulae are:

- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ $\quad (A \neq \dfrac{\pi}{2}, B \neq \dfrac{\pi}{2})$

Example 12.1

Use an appropriate formula to find the exact values for (a) $\sin 15^\circ$ (b) $\cos \dfrac{7\pi}{12}$.

Solution:

(a) $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1).$$

Note: $\sin 15^\circ = \sin(60^\circ - 45^\circ)$ would yield exactly the same answer.

(b) $\cos \dfrac{7\pi}{12} = \cos\left(\dfrac{\pi}{3} + \dfrac{\pi}{4}\right)$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4}(1 - \sqrt{3}).$$

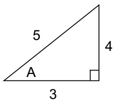
Example 12.2

Given that cos $A = \frac{3}{5}$ and sin $B = \frac{1}{3}$ where A and B are both acute, without the use of a calculator, find: (a) sin $(A - B)$ (b) tan $(A + B)$.

Solution:

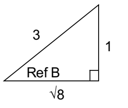
(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$$= \sin A \times \cos B - \frac{3}{5} \times \frac{1}{3}\ .$$



Since $\cos A = \frac{3}{5}$, from the reference triangle, $\sin A = \frac{4}{5}$.

Since $\sin B = \frac{1}{3}$, from the reference triangle, $\cos B = \frac{\sqrt{8}}{3}$



Hence, $\sin(A - B) = \frac{4}{5} \times \frac{\sqrt{8}}{3} - \frac{3}{5} \times \frac{1}{3} = \frac{1}{15}(8\sqrt{2} - 3)$.

(b) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

From the reference triangles in part (a), $\tan A = \frac{4}{3}$, $\tan B = \frac{1}{\sqrt{8}} = \frac{\sqrt{2}}{4}$.

Hence, $\tan(A + B) = \dfrac{\frac{4}{3} + \frac{\sqrt{2}}{4}}{1 - \frac{4}{3} \times \frac{\sqrt{2}}{4}} = \dfrac{25\sqrt{2}}{28} + \dfrac{27}{14}$

Example 12.3

Use an appropriate trigonometric identity to solve $\sin(\theta + \frac{\pi}{6}) = 2\cos(\theta - \frac{\pi}{3})$, giving exact answers for $0 \le \theta \le 2\pi$.

Solution:

$$\sin(\theta + \frac{\pi}{6}) = 2\cos(\theta - \frac{\pi}{3})$$

$$\Rightarrow \sin\theta \cos\frac{\pi}{6} + \cos\theta \sin\frac{\pi}{6} = 2\cos\theta\cos\frac{\pi}{3} + 2\sin\theta\sin\frac{\pi}{3}$$

$$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = \cos\theta + \sqrt{3}\sin\theta$$

$$-\frac{\sqrt{3}}{2}\sin\theta = \frac{1}{2}\cos\theta$$

$$\Rightarrow \tan\theta = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \frac{5\pi}{6}, \frac{11\pi}{6}$$

Exercise 12.1

1. Use an appropriate formulae to find the exact values for:
   (a) $\cos 15^\circ$ (b) $\tan 75^\circ$ (c) $\sin 105^\circ$ (d) $\cos 285^\circ$
   (e) $\sin \frac{5\pi}{12}$ (f) $\tan \frac{\pi}{12}$ (g) $\cos \frac{13\pi}{12}$ (h) $\sin (-\frac{11\pi}{12})$

2. Given that A and B are both acute angles and $\sin A = \frac{3}{5}$ and $\sin B = \frac{5}{13}$,
   use the appropriate compound angle formula to find in exact form:
   (a) $\sin (A + B)$ (b) $\cos (A - B)$ (c) $\tan (A + B)$.

3. Given that $\cos A = \frac{7}{25}$ and $\sin B = \frac{8}{17}$ where A is acute and B is obtuse,
   use the appropriate compound angle formula to find in exact form:
   (a) $\sin (A - B)$ (b) $\cos (A + B)$ (c) $\tan (A - B)$.

4. Given that $\sin A = \frac{1}{4}$ and $\tan B = \frac{4}{3}$ where A is acute and $180^\circ < B < 270^\circ$,
   use the appropriate compound angle formula to find in exact form:
   (a) $\sin (A + B)$ (b) $\cos (A - B)$ (c) $\tan (A + B)$.

5. Use an appropriate trigonometric identity to solve each of the following, giving answers in exact form for $0 \le \theta \le 2\pi$.
   (a) $\sin (\theta + \frac{\pi}{3}) = \sin \theta$ (b) $\sin (\theta + \frac{5\pi}{4}) = -\sqrt{2} \cos \theta$
   (c) $\cos (\theta - \frac{\pi}{6}) = 2 \sin (\theta + \frac{\pi}{3})$ (d) $\cos (\theta - \frac{\pi}{4}) = \sqrt{2} \sin (\theta + \frac{\pi}{3})$

*6. Use appropriate trigonometric identities to solve $\tan (x - \frac{\pi}{4}) = -\tan (x + \frac{\pi}{2})$,
   giving your answers in exact form for $0 \le x \le \pi$.

7. Prove each of the following, where n is an integer:
   (a) $\sin (x + 2n\pi) = \sin x$ (b) $\cos (x + 2n\pi) = \cos x$ (c) $\tan (x + n\pi) = \tan x$

13 Sets

13.1 Set Notation

- A set is a collection of items/objects each having a common defining property or attribute.
- Consider the set A of all *even* numbers between 1 and 100 inclusive.
  - The *members* or *elements* of set A are the numbers 2, 4, 6, ... 96, 98, 100.
  - The set A is written symbolically (using the *set builder notation*) as:
    A = {x : x is an even number between 1 and 100 inclusive}
  - Listing the elements in set A: A = {2, 4, 6, ..., 96, 98, 100}.
  - The number 2 is a member or element of set A.
    This is denoted symbolically as: $2 \in A$.
  - The number 3 is not divisible by 2 and hence is not a member of set A.
    This is denoted symbolically as: $3 \notin A$.
- The number of elements in set P is denoted $n(P)$ or $|P|$.
- The set of all elements within the context considered is called the *universal set* or *base set*.
- If all the elements in set P are also found in Q, then P is a *subset* of Q.
  This is denoted symbolically as: $P \subseteq Q$.
  If all the elements in P are found in Q but not all the elements in Q are found in P, then P is termed a *proper subset* of Q, denoted $P \subset Q$.
- If at least one of the elements in set P is not found in Q, then P is not a subset of Q. This is denoted symbolically as: $P \not\subset Q$.
- Consider the sets P and Q based on the universal set U.
  - The *intersection* between the sets P and Q is the set of all elements *common to both P and Q* and is denoted $P \cap Q$.
  - P and Q is said to be *disjoint* if there are no common elements between them; that is $P \cap Q = \phi$. ϕ is the symbol used to denote the empty set, the set with no elements.
  - The *union* between the sets P and Q is the set of all elements *found in either P or Q* and is denoted $P \cup Q$.
  - The *complement* of set P is the set of all elements in the universal set U that are not found in P, denoted $\overline{P}$ or P'. The complementation of a set must be done in the context of the universal set.
- For the sets A and B: $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
- The set of real numbers is conventionally denoted $\mathbb{R}$ or **R**.
  If A = {x: $x \geq 5$, $x \in \mathbb{R}$}, then set A consists of all real numbers greater than or equal to 5.

Example 13.1

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 8\}$, $C = \{3, 6, 9\}$ and $D = \{9\}$. Find:

(a) $A \cap B$ (b) $B \cap D$ (c) $B \cup C$ (d) $(A \cap B) \cap C$

(e) $(A \cap B) \cup D$ (f) $\overline{A}$ (g) $\overline{A \cap B}$ (h) $\overline{A} \cup \overline{B}$

Solution:

(a) $A \cap B = \{2, 4, 6, 8, 10\} \cap \{4, 8\} = \{4, 8\}$

(b) $B \cap D = \{4, 8\} \cap \{9\} = \phi$

(c) $B \cup C = \{4, 8\} \cup \{3, 6, 9\} = \{3, 4, 6, 8, 9\}$

(d) $(A \cap B) \cap C = \{2, 4, 6, 8, 10\} \cap \{4, 8\} \cap \{3, 6, 9\} = \phi$

(e) $(A \cap B) \cup D = \{2, 4, 6, 8, 10\} \cap (4, 8\} \cup \{9\} = \{4, 8, 9\}$

(f) $\overline{A} = \{1, 3, 5, 7, 9\}$

(g) $\overline{A \cap B} = \overline{\{4, 8\}} = \{1, 2, 3, 5, 6, 7, 9, 10\}$

(h) $\overline{A} \cup \overline{B} = \{1, 3, 5, 7, 9\} \cup \{1, 2, 3, 5, 6, 7, 9, 10\}$
$= \{1, 2, 3, 5, 6, 7, 9, 10\}$

Example 13.2

Let U: the set of all year 11 students at Swan College (SC)
G: the set of all year 11 girls at SC
B: the set of all year 11 boys at SC
A: set of all year 11 Mathematics Methods students at SC
C: set of all year 11 Mathematics Specialist students at SC
D: set of all year 11 Mathematics Essentials students at SC.

Describe in words, the following sets or statements:

(a) $A \cap G$ (b) $\overline{A}$ (c) $C \cup D$ (d) $C \cap D = \phi$

Solution:

(a) $A \cap G$: Set of year 11 girls at SC doing Mathematics Methods.

(b) $\overline{A}$ = Set of year 11 students at SC who do not do Mathematics Methods.

(c) $C \cup D$ = Set of year 11 Mathematics Specialist or Mathematics Essential students at SC

(d) $C \cap D = \phi \Rightarrow$ There are no students at SC that do both Mathematics Specialist and Mathematics Essentials.

Exercise 13.1

1. Let U = {x: $0 \le x \le 10, x \in \mathbb{R}$}, B = {$x$: $x \ge 4, x \in \mathbb{R}$}, C = {$x$: $x \le 6, x \in \mathbb{R}$} and D = {$x$: $x > 5, x \in \mathbb{R}$}. Find:
   (a) $B \cap C$ (b) $C \cup D$ (c) $B \cap D$
   (d) $(D \cap B) \cap C$ (e) $\overline{C \cap D}$ (f) $\overline{C} \cap \overline{D}$

2. Let U = {x: $90 \le x \le 110, x \in \mathbb{R}$}, P = {$x$: $90 \le x \le 100, x \in \mathbb{R}$}, Q = {$x$: $x \ge 105, x \in \mathbb{R}$} and R = {$x$: $x < 95, x \in \mathbb{R}$}. Find:
   (a) $P \cap Q$ (b) $Q \cap R$ (c) $Q \cup R$
   (d) $\overline{P} \cap R$ (e) $\overline{Q} \cup \overline{R}$ (f) $(P \cap R) \cap Q$

3. Let U = {x: $x \ge 50$}, A = (x: $60 < x < 70$}, B = {x: $x < 60$}, C = {60} and D = {70}.
   (a) Find $A \cap B$. (b) Find $A \cap C$.
   (c) Find $A \cup C$. (d) Find $A \cup D$.

4. Given that $A \subset B$, and $n(A) = 60$ and $n(B) = 100$, find (a) $n(A \cap B)$ (b) $n(A \cup B)$.

5. Given that A and B are disjoint and $n(A) = 50$ and $n(B) = 20$, find:
   (a) $n(A \cap B)$ (b) $n(A \cup B)$

6. Given that: A = {(x, y) | $0 \le x \le 2$ and x is an integer; $0 \le y \le 1$ and y is an integer} and B = {(x, y) | $0 \le x \le 1$ and x is an integer; $0 \le y \le 2$ and y is an integer}. Find:
   (a) $A \cap B$ (b) $A \cup B$.

7. Given that: A = {(x, y) | $2 \le x \le 5$ and x is an integer; $1 \le y \le 1$ and y is an integer} and B = {(x, y) | $4 \le x \le 5$ and x is an integer; $0 \le y \le 2$ and y is an integer}. Find:
   (a) $n(A \cap B)$ (b) $|A \cup B|$.

8. Let U: the set of all year 11 students at Elizabeth College (EC)
   C: set of all year 11 Chemistry students at EC
   P: set of all year 11 Physics students at EC
   D: set of all year 11 Dance students at EC.
   R: set of all year 11 Drama students at EC.
   Assume that none of these subsets are empty.
   Describe in words, the following sets or statements:
   (a) $P \subseteq C$ (b) $\overline{C \cap P}$ (c) $D \cup R = D$ (d) $C \cap D \cap R = \phi$

9. Let U: the set of all year 11 students at Bateman College (EC)
   C: set of all year 11 students who have visited China
   L: set of all year 11 students who have visited London
   N: set of all year 11 students who have visited New Zealand
   S: set of all year 11 students who have visited Singapore
   Assume that none of these subsets are empty.
   Describe in words, the following sets or statements:
   (a) $L \not\subset S$ (b) $\overline{C} \cap \overline{S}$ (c) $L \cap S = N$ (d) $C \cap N \cap S = \phi$

14 Combinations

14.1 The Combinatorial Notation

- Consider the product of all integers from 1 to n:
  $n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$
  - This product may be abbreviated as $n!$, read as n factorial.
  - For example: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

- Consider the expression $\dfrac{n!}{r!\,(n-r)!}$;
  where n is a whole number and r is a whole number less than or equal to n.
  - This expression is represented by the notation ${}^nC_r$,
    read as *n combinatorial r* or more simply as *n "C" r*.
  - It is also often represented by a more generalised notation $\dbinom{n}{r}$,
    read as *n "down" r*.
  - Hence:

$$ {}^nC_r \equiv \binom{n}{r} \equiv \frac{n!}{r!\,(n-r)!} = \frac{\overbrace{n\times(n-1)\times(n-2)\times\ldots\ldots\times(n-r+1)}^{r\ terms}}{r!}. $$

- An important feature of the combinatorial symbol is that:

$$ {}^nC_r = {}^nC_{n-r} \quad \text{or} \quad \binom{n}{r} = \binom{n}{n-r}. $$

- Note that $\dbinom{n}{0} = \dbinom{n}{n} = 1$ and $\dbinom{n}{1} = \dbinom{n}{n-1} = n$.

Example 14.1

Without the use of calculator, evaluate each of the following: (a) ${}^5C_2$ (b) $\dbinom{10}{4}$.

Solution:

(a) $${}^5C_2 = \frac{5\times 4}{2\times 1} = 10$$

(b) $$\binom{10}{4} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$$

Exercise 14.1

1. Without the use of calculator calculate each of the following:

   (a) $\begin{pmatrix} 9 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 10 \\ 7 \end{pmatrix}$ (c) $\begin{pmatrix} 9 \\ 4 \end{pmatrix} \times 4!$ (d) $\begin{pmatrix} 7 \\ 2 \end{pmatrix} \times \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

   (e) $^{50}C_2$ (f) $^{50}C_{47}$ (g) $^{100}C_{98}$ (h) $^{200}C_{199}$

2. Without the use of a calculator, find:

   (a) $^2C_0 + {}^2C_1 + {}^2C_2$

   (b) $^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$.

3. Verify that (a) $\begin{pmatrix} n \\ r \end{pmatrix} = \dfrac{\overbrace{n \times (n-1) \times (n-2) \times ...... \times (n-r+1)}^{r \text{ terms}}}{r!}$ (b) $\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}$.

4. Find the value(s) of r given that:

   (a) $^{20}C_r = {}^{20}C_{r+2}$ (b) $^{15}C_r = {}^{15}C_{r+3}$ (c) $^{12}C_{2r} = {}^{12}C_{r-3}$

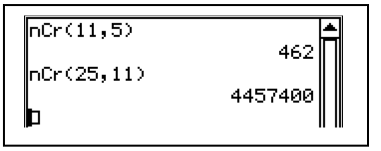
14.2 Combinations or Selections

- *r objects can be chosen from n unlike objects without replacement* in $^nC_r \equiv \begin{pmatrix} n \\ r \end{pmatrix}$ ways, where $r \le n$, r and n are positive whole numbers.
  - For example, there will be $^4C_2$ = 6 ways of choosing 2 elements from the set $\{a, b, c, d\}$. Namely:
    $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$.
- The total possible number of combinations of *n unlike objects taken r at a time* is given by $^nC_r \equiv \begin{pmatrix} n \\ r \end{pmatrix}$, where $r \le n$, r and n are positive whole numbers.
  - For example, in the set $\{a, b, c, d\}$, there will be $^4C_2$ = 6 combinations of two members of this set. Namely:
    $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$ and $\{c, d\}$.
- Clearly, given a set with n elements, there are ^nC_r subsets of r elements from n.
- Note that in counting the different combinations, the order in which the objects appear or the order in which the objects are chosen is not important.

Example 14.2

(a) How many different ways are there of choosing 5 students from 11 students?
(b) Find the number of different teams of 11 players each that can be formed from a shortlist of 25 players.

Solution:



(a) The number of ways of choosing 5 students from 11 is given by $\begin{pmatrix} 11 \\ 5 \end{pmatrix} = 462$.

(b) The total number of different teams possible is given by $\begin{pmatrix} 25 \\ 11 \end{pmatrix} = 4\,457\,400$

[If it takes 1 minute to write down the names of the players of any one team, then it will take at least 3 095 days non-stop, to list down all the possible teams!]

14.2.1 The Multiplication and Addition Rule

- Two tasks are *mutually exclusive* or *disjoint* if the set of outcomes of the first task and the set of outcomes of the second task have no common element.
- If the tasks A, B, C, D, ... , each *mutually exclusive* or *disjoint* from the other, can each respectively be completed in *a*, *b*, *c*, *d*, ... ways, then:
  - tasks A *and* B *and* C *and* D *and* ..... can be completed in
    $a \times b \times c \times d \times$ ..... ways
  - tasks A *or* B *or* C *or* D *or* ..... can be completed in
    $a + b + c + d +$ ..... ways.
- The first result is known as the *multiplication rule* for counting techniques while the second result is known as the *addition rule* for counting techniques.

Example 14.3

A committee consisting of three year 11 students and four year 12 students is to be formed from eleven year 11 students and twelve year 12 students. How many different committees can be formed?

Solution:

The three year 11 students can be selected from 11 in $\begin{pmatrix} 11 \\ 3 \end{pmatrix} = 165$ ways.

The four year 12 students can be selected from 12 in $\begin{pmatrix} 12 \\ 4 \end{pmatrix} = 495$ ways.

Using the multiplication rule, the three year 11 students *and* the four year 12 students can be selected in $165 \times 495 = 81\,675$ ways.

Example 14.4

A committee of 4 is to be formed from a nominated list of 5 men and 6 women. How many different committees with more women than men can be formed?

Solution:

For the committees to have more women than men, the committees must either have 3 women and 1 man ***or*** *4 women and no men.*

3 women can be chosen from 6 *and* 1 man can chosen from 5 in $\binom{6}{3} \times \binom{5}{1}$ = 100 ways.

4 women can be chosen from 6 in $\binom{6}{4}$ = 15 ways.

Hence, using the addition rule, committees of 3 women and 1 man *or* committees of 4 women only can be formed in 100 + 15 = 115 ways.

Notes:

- *The universal set of "committees with more women than men" is partitioned into 2 distinct disjoint sets: "committees with 3 women and 1 man" and "committees with 4 women".*
  An important skill in counting techniques is to recognise when a universal set needs to be partitioned into disjoint sets to satisfy the condition that all tasks must be disjoint each from another.

Exercise 14.2

1. Without the use of a calculator, find the number of ways of choosing:
   (a) 3 items from a total of 10 distinct items
   (b) 7 items from a total of 10 distinct items
   (c) 2 items from a total of 20 distinct items
   (d) 98 items from a total of 100 distinct items.

2. Find the number of ways the starting 5 players for a basketball game can be chosen from a group of 10 players if:
   (a) no other restrictions apply (b) the captain must always be in the starting 5

3. Find the number of ways a pair of tennis players can be selected from 5 male tennis players and 6 female tennis players if:
   (a) no other restrictions apply (b) the pair are of mixed sexes.

4. A student council comprising 10 students is to be set up. There were 20 year 11 nominees and 35 year 12 nominees for the student council. Determine the total number of possible student councils if:
   (a) the council is to comprise equal numbers of year 11s and 12s
   (b) the council is to comprise 4 year 11s and 6 year 12s
   (c) the council must have more year 12s than year 11s and at least 3 year 11s
   (d) the School Captain and Deputy School Captain (both of whom are in the list of nominees) must be in the council

5. An examination is divided into 2 sections, A and B, with 5 and 7 questions respectively. In how many ways can the examination be answered if students are required to answer:
   (a) all the questions in each section
   (b) all the questions in section A and any 2 questions from section B
   (c) a total of 5 questions with at least one question from each section
   (d) a total of 5 questions with not more than 3 questions from each section.

6. A multiple choice test is made up of 10 questions, each of which is provided with 5 answers of which only one is correct .In how many ways can a student:
   (a) correctly answer all 10 questions (b) incorrectly answer all 10 questions
   (c) correctly answer the first 6 questions and incorrectly answer the last 4 questions

7. In the game of Lotto, a player is required to pick 6 numbers from the numbers 1 through to and including the number 45. Each combination of 6 numbers is called a game. It costs 40 cents to play a game.
   (a) How many possible games are there if the numbers 7 and 13 must be included?
   (b) How many possible games are there if the numbers 8 and 37 must be excluded?
   (c) How many possible games are there if the numbers 7 and 13 must be included and the numbers 8 and 37 must be excluded?
   (d) How much will it cost a person to play all the possible games in Lotto?

8. Find the total number of subsets of 2 members from a set of 6 members.

9. Find the total number of subsets of 14 members from a set of 15 members.

10. Determine the total number of possible subsets in a set with 5 unlike objects.

11. Determine the total number of possible combinations of 8 objects if any number can be taken at a time.

12. Set A has 5 distinct elements and Set B has 6 distinct elements all different from those in A. Set C consisting of 4 elements is formed by choosing at least one element from each of the two sets. Find the total possible number of such sets.

13. Set A has 2 distinct elements and Set B has 3 distinct elements all different from those in A. Set C is formed by choosing elements from each of the two sets such that there are always more elements from set B than set A. Find the total number of such sets.

14.3 Binomial Expansions

- A binomial is an expression consisting of two terms: $(a + b)$, $(1 + x)$, $(1 - x^2)$.
- $(a + b)^5$ and $(1 + x^2)^5$ are powers of binomials.
- In this section, we will develop algebraic techniques for expanding the powers of binomials.

Hands On Task 14.1

In this task we will find a systematic way of writing out the expansion of a power of the binomial $(x + y)^n$, for n as a positive whole number.

Consider the following expansions:

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

1. Determine the number of terms in the expansions of $(x + y)^n$ for:
   (a) $n = 7$ (b) $n = 10$ (c) $n = 15$ (d) $n = k$.

2. Consider the expansion for $(x + y)^2 \equiv (x + y)(x + y) \equiv x^2 + 2xy + y^2$.
   The terms of the expansion consist of a "x^2" term, a "y^2" term and a "xy" term. Examine the expansions given above and then list all the terms (without the coefficients) for the following expansions:
   (a) $(x + y)^8$ (b) $(x + y)^{10}$ (c) $(x + y)^{12}$

3. Consider the expansion for $(x + y)^5 \equiv (x + y)(x + y)(x + y)(x + y)(x + y)$
   $\equiv x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.
   - The "x^5" term is the product of each "x" term from all five brackets of $(x + y)$. Therefore, there can only be one "x^5" term and the coefficient of the "x^5" term is 1.
   - Similarly the coefficient of the "y^5" term must be one.
   - The "x^3y^2" term is the product of three "x" terms and two "y" terms.
     - The three "x" terms must come from three different $(x + y)$ brackets.
     - These three brackets can be chosen from the five brackets in $\binom{5}{3} = 10$ ways.
     - The two "y" terms must come from the remaining two brackets.
     - There is only 1 way to obtain these "y" terms after first choosing the three "x" terms.
     - Hence, the "x^3y^2" terms can be formed in $10 \times 1 = 10$ ways.
     - Therefore the coefficient of the "x^3y^2" term is 10.

Use the above method to find the coefficient of the following terms:

(a) the x^2y term in the expansion of $(x + y)^3$

(b) the xy^3 term in the expansion of $(x + y)^4$

(c) the x^2y^3 term in the expansion of $(x + y)^5$

(d) the x^4y^3 term in the expansion of $(x + y)^7$

4. Use the ideas developed in Questions 1 through to 3, to expand the following:

(a) $(1 + x)^4$ (b) $(1 + x)^5$ (c) $(1 - x)^6$ (d) $(1 + 2x)^6$

Summary

The expansion for $(x + y)^n$, in *ascending* powers of x, where n is a whole number, is given by (the Binomial Theorem):

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 +$$

$$\ldots + \binom{n}{k}x^{n-k}y^k + \ldots$$

$$\ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- The expansion has $(n + 1)$ terms.
- The coefficients in the expansion are $\binom{n}{k}$ for $k = 0, 1, 2, 3, \ldots, n$.

  The coefficients $\binom{n}{k}$ are referred to as the binomial coefficients.
- The kth term is $\binom{n}{k-1}x^{n-k+1}y^{k-1}$.

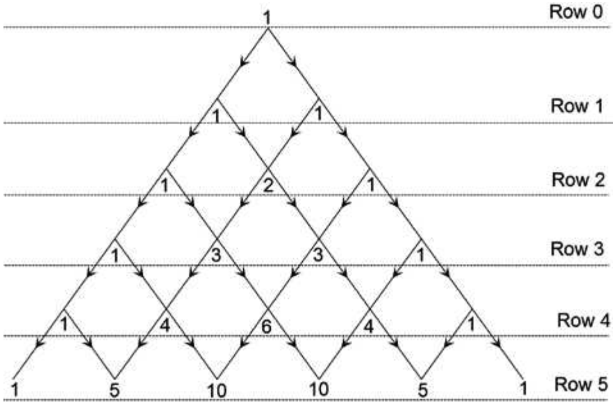
  Note that the sum of the powers of the x and y terms is always n.
- The term containing x^k is $\binom{n}{n-k}x^ky^{n-k} \equiv \binom{n}{k}x^ky^{n-k}$,

  since $\binom{n}{n-k} = \binom{n}{k}$.

Hands On Task 14.2

In this task we will explore some of the properties of Pascal's triangle.

The diagram below displays the first five rows of Pascal's triangle and the expansions for $(x + y)^n$ for n = 0, 1, 2, 3, 4 and 5.



$(x + y)^0 = 1$

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

1. Extend Pascal's triangle to the sixth row.

2. Compare the terms in Pascal's triangle with the coefficients of the expansions listed. Hence, state the expansion for $(x + y)^6$.

3. Express the terms in row 5 of Pascal's triangle in terms of the binomial coefficients $\binom{5}{k}$.

   Use this observation to write row 10 of Pascal's triangle.

4. (a) Add all the terms in row 5 of Pascal's triangle.
   (b) Substitute $x = 1$ and $y = 1$ into the expansion for $(x + y)^5$.
   Compare the value of $(x + y)^5$ with the sum of all terms in row 5 of Pascal's triangle.
   (c) Find the sum of terms in row 11 of Pascal's triangle.

5. Express each of the following as a sum of the binomial coefficients $\binom{n}{k}$ for $k = 0, 1, 2, ..., n$: (a) 2^8 (b) 2^{10} (c) 2^m where m is a positive integer.

Summary

- The terms in row n of Pascal's Triangle are identical to the binomial coefficients $\binom{n}{k}$ for $k = 0, 1, 2, \ldots, n$.

- When positive integer n is small the coefficients of the expansion for $(x + y)^n$ may be more easily derived from Pascal's Triangle than from the Binomial Theorem.

- The sum of the terms in row n of Pascal's Triangle = 2^n .

- Hence: $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{k} + \ldots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n}$.

  Therefore, the total number of possible subsets from a set of n distinct objects is 2^n. (This includes the empty set.)

Example 14.5

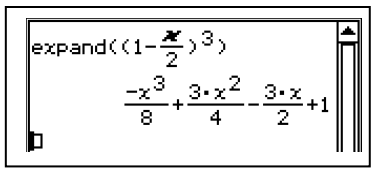
Without the use of a calculator, expand the following in ascending powers of x:

(a) $\left(1-\dfrac{x}{2}\right)^3$ (b) $(2 - x)^5$

Solution:

(a) $\left(1-\dfrac{x}{2}\right)^3 = (1)^3 + 3(1)^2\left(-\dfrac{x}{2}\right) + 3(1)\left(-\dfrac{x}{2}\right)^2 + \left(-\dfrac{x}{2}\right)^3$

$= 1 - \dfrac{3x}{2} + \dfrac{3x^2}{4} - \dfrac{x^3}{8}$.



(b) $(2 - x)^5 = 2^5 + 5(2)^4(-x)^1 + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)^1(-x)^4 + (-x)^5$

$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$

Example 14.6

Without the use of a calculator, in the expansion of $\left(4x-\dfrac{1}{x}\right)^4$ in descending powers of x, determine the term independent of x (that is, the constant term).

Solution:

$\left(4x-\dfrac{1}{x}\right)^4 = (4x)^4 + 4(4x)^3\left(-\dfrac{1}{x}\right) + 6(4x)^2\left(-\dfrac{1}{x}\right)^2 + 4(4x)\left(-\dfrac{1}{x}\right)^3 + \left(-\dfrac{1}{x}\right)^4$

Hence, term independent of x is: $6(4x)^2\left(-\dfrac{1}{x}\right)^2 \equiv 6 \times 16x^2 \times \dfrac{1}{x^2} \equiv 96$.

Alternative method

General term of expansion is $\binom{4}{k}(4x)^k\left(-\frac{1}{x}\right)^{4-k}$.

For this term to be independent of x, the powers of x must vanish.

Hence: $k = 4 - k \Rightarrow k = 2$.

Therefore, required term is $\binom{4}{2}(4x)^2\left(-\frac{1}{x}\right)^2 \equiv 6 \times 16x^2 \times \frac{1}{x^2} \equiv 96$.

Exercise 14.3

1. Without the use of a calculator, expand in ascending powers of x:
   (a) $(1 + x)^4$ (b) $(1 + 2x)^4$ (c) $(1 - x)^4$
   (d) $(1 + x)^5$ (e) $[1 + (x/2)]^5$ (f) $(2 + x)^5$

2. Expand in ascending powers of x up to and including the term in x^3:
   (a) $(1 + x)^{10}$ (b) $(1 - 2x)^{12}$ (c) $(2 - x)^8$ (d) $(2 + 3x)^9$

3. Expand in ascending powers of x up to and including the 4th term:
   (a) $[x + (1/x)]^7$ (b) $[x^2 + (1/x)]^9$ (c) $[x - (1/x^2)]^{11}$

4. Expand in descending powers of x up to and including the 4th term:
   (a) $[x + (2/x)]^8$ (b) $[x^2 - (1/x^2)]^{10}$ (c) $[(1/x) - x]^8$

5. Write in the simplest form:
   (a) the third term in the expansion of $(x + 5)^{10}$ in ascending powers of x
   (b) the fifth term in the expansion of $[x - (1/x)]^7$ in ascending powers of x
   (c) the sixth term in the expansion of $(2x - 1/2)^9$ in ascending powers of x
   (d) the fourth term in the expansion of $[x^2 + (1/x)]^5$ in descending powers of x.

6. Find the coefficient of the x^4 term in the expansion of: (a) $(x - 1)^{10}$ (b) $(2x + 5)^6$

7. Find the term independent of x in the expansion of: (a) $[2x + (1/x)]^{10}$ (b) $[x + (1/x^2)]^{12}$

8. Find the term containing:
   (a) x^3 in the expansion of $(x + 3)^8$ (b) x^{-3} in the expansion of $[(1/x) + 1]^5$
   (c) $1/x^3$ in the expansion of $[x + (1/x)]^5$ (d) x^7 in the expansion of $(x^2 + x)^5$

9. Express as the sum of binomial coefficients: (a) 2^7 (b) 4^3 (c) 256

10. Find the number of subsets in a set of: (a) 9 (b) $2n$ distinct elements.

15 Probability

15.1 Sample Spaces

- A statistical *experiment* consists of a sequence of actions with several resulting outcomes (for example: toss a coin, roll a die ....).
- The set of all possible outcomes of the experiment is known as its *sample space*.
- An *event* refers to a particular outcome or combination of outcomes of an experiment. In set language, an event is represented as a subset of the sample space.

Example 15.1

Australia and England play a series of 3 cricket test matches. Assume that the matches cannot be drawn. Describe the sample space for the series and the event that Australia wins the series. Determine the number of elements in the sample space.

Solution:

Let A: the outcome that Australia wins a match
E: the outcome that England wins a match
AEA: Australia wins the first and third match and England wins the second match

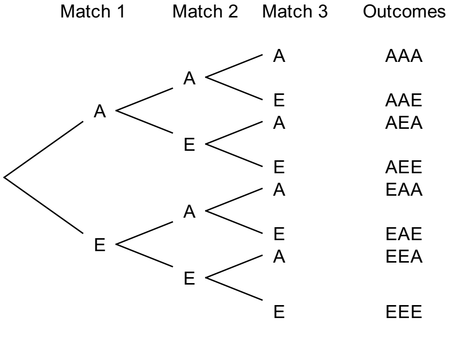
The sample space is: {AAA, AAE, AEA, EAA, AEE, EAE, EEA, EEE}.

The event that Australia wins the series is represented by the subset
{AAA, AAE, AEA, EAA}

The number of elements in the sample space is 8

Notes:

- *A systematic method for listing the sample space is to make use of a* tree diagram*, shown in the accompanying diagram.*
- *Clearly, the use of a tree diagram is only efficient when the total number of outcomes is small.*
- *The outcomes of the three matches are indicated by three levels of branches, one branch corresponding to one match.*



15.2 Theoretical (A Priori) Probability

- Consider a statistical experiment with sample space S.
  Let $n(\mathrm{S})$ denote the number of elements in S.
  Consider an event A associated with this experiment.
  Let $n(\mathrm{A})$ be the number of elements in the set describing the event A.
- Assuming that all the outcomes in S are *equally likely*, the *probability* of event A occurring, denoted P(A), is given by $\mathrm{P(A)} = \dfrac{n(\mathrm{A})}{n(\mathrm{S})}$.
- Since $0 \le n(\mathrm{A}) \le n(\mathrm{S})$, $0 \le \mathrm{P(A)} \le 1$.
- This definition of probability is applicable only when we are certain that the *outcomes of the experiment are equally likely*; that is, each outcome is equally likely to occur.
-

It is also known as the "a priori definition of probability" because the imputation of the outcomes being equally likely is done before (prior to) any scientific investigation or observation.

Example 15.2

Two fair die are rolled. Determine the probability of obtaining:
(a) exactly two sixes (b) two even numbers (c) a sum greater than six

Solution:

- *The sample space can be constructed using a tree diagram. However, the tree diagram would have 36 branches, which clearly is very unwieldy.*
- *A more efficient way would be to use a grid, as shown in the accompanying diagram.*
- *The numbers in the shaded part of the grid represents the sum of the numbers shown on the faces of the first and second die.*

		Die 1					
	+	1	2	3	4	5	6
Die 2	1	**2**	**3**	**4**	**5**	**6**	**7**
	2	**3**	**4**	**5**	**6**	**7**	**8**
	3	**4**	**5**	**6**	**7**	**8**	**9**
	4	**5**	**6**	**7**	**8**	**9**	**10**
	5	**6**	**7**	**8**	**9**	**10**	**11**
	6	**7**	**8**	**9**	**10**	**11**	**12**

The sample space S = {(1, 1), (1, 2), (1, 3), ..., (5, 6), (6, 6)}.
From the above grid, the number of elements in the sample space $n(\mathrm{S}) = 36$.

(a) The set representing event A: exactly 2 sixes are obtained = {(6,6)}. Hence, $n(\mathrm{A}) = 1$.
Since, the die is fair, all outcomes are equally likely.

Therefore, $\mathrm{P(A)} = \dfrac{1}{36}$

(b) Let B: 2 even numbers are obtained. From the grid above $n(\mathrm{B}) = 9$.

Therefore, $\mathrm{P(B)} = \dfrac{9}{36}$

(c) Let C: a sum greater than six is obtained. From the grid above, $n(\mathrm{C}) = 21$.

Hence, $\mathrm{P(C)} = \dfrac{21}{36}$

Example 15.3

A delegation of six students is to be selected from a group of 10 girls and 8 boys. Determine the probability that:
(a) equal numbers of girls and boys are chosen
(b) more girls than boys are chosen.

Solution:

Clearly, to list the entire sample space is quite a monumental task. In fact, we will dispense with listing the sample space altogether. Using the methods developed in Chapter 14 on Counting Techniques, we can easily "count" the number of elements in the sample space as well as the number of elements in the sets describing the relevant events.

(a) 6 students can be chosen from a total of 18 students in ${}^{18}C_6$ = 18 564.

The 3 girls and 3 boys can be chosen in ${}^{10}C_3 \times {}^{8}C_3$ = 6720

Hence, required probability = $\dfrac{6720}{18564} \approx 0.3620$.

(b) 4 girls and 2 boys can be chosen in ${}^{10}C_4 \times {}^{8}C_2$ = 5880

5 girls and 2 boys can be chosen in ${}^{10}C_5 \times {}^{8}C_1$ = 2016

6 girls and 0 boys can be chosen in ${}^{10}C_6 \times {}^{8}C_0$ = 210

Hence, number of delegations with more girls than boys = 8106

Therefore, required probability = $\dfrac{8106}{18564} \approx 0.4367$.

Summary

- Let n(S) denote the number of elements in the sample space S of an experiment. Let n(A) be the number of elements in the set describing an event A associated with the experiment.
- Assuming that all the outcomes in S are *equally likely*, the *probability* of event A occurring, denoted P(A) is given by:

$$P(A) = \frac{n(A)}{n(S)}, \text{ where } \quad 0 \le P(A) \le 1.$$

- The values of n(S) and n(A) can be obtained by using:
  - *a simple list (for small sample spaces)*
  - *a tree diagram (for larger sample spaces)*
  - *a grid (for yet larger sample spaces)*
  - *counting techniques (for very large sample spaces)* .

Exercise 15.1

1. A fair die is tossed. Determine the sample space. Calculate the probability of obtaining:
   (a) an even number (b) a prime number
   (c) a score greater than 2 (d) a score between 2 and 5 inclusive.

2. A fair eight sided die, labelled, A, B, C, D, E, F, G, H is tossed.
   Calculate the probability of obtaining:
   (a) a vowel (b) a letter from the word FACE (c) a letter from the word CARD.

3. A disc is drawn at random from a box containing a red disk, a white disk, a blue disk, a green disk, a black disk, a pink disc and a brown disk. Determine the probability that the disk drawn: (a) is a black disc (b) is a black or white disc
   (c) is a disc with one of the colours of the rainbow.

4. Jacques and Paige choose their date for a March wedding randomly. Given that March 1st is a Friday, calculate the probability that the chosen wedding date falls:
   (a) before the 15th of March (b) between the 6th and 28th of March
   (c) on a Saturday (d) on a Saturday before the 15th of March.

5. Three fair coins are tossed. Draw a tree diagram to illustrate the possible outcomes.
   Calculate the probability of obtaining:
   (a) exactly 2 heads (b) at least 2 heads (c) no more than 2 heads.

6. A fair die and a fair coin are tossed simultaneously. Draw a tree diagram to illustrate the possible outcomes. Calculate the probability of obtaining:
   (a) a six and a head (b) an even number and a head (c) a prime number.

7. A box contains one red, one black and one white ball. A ball is drawn randomly from the box. The colour of the ball is noted and the ball returned to the box. A second ball is randomly drawn from the box and its colour noted. Draw a tree diagram to illustrate all the possible outcomes. Calculate the probability of obtaining:
   (a) 2 white balls (b) 2 red balls
   (c) a red and a white ball (d) different coloured balls.

8. A box contains one red, one black and one white ball. A ball is drawn randomly from the box. The ball drawn is *not* placed back in the box. A second ball is randomly drawn from the box and its colour noted. Draw a tree diagram to illustrate all the possible outcomes. Given that all the outcomes are equally likely, calculate the probability of obtaining: (a) 2 white balls (b) 2 red balls
   (c) a red and a white ball (d) different coloured balls.

9. Tony goes to school either by bus, on foot or by bicycle. Tony never walks to school on two consecutive days. Draw a tree diagram to illustrate the different ways Tony travels on two consecutive days. Given that all the outcomes are equally likely, calculate the probability that Tony goes to school:
   (a) by bus on both days (b) by bus and by bicycle (c) by different modes.

10. Jamie has cereal, toast or porridge for breakfast. Jamie never has porridge for three consecutive days. On the day, Jamie has toast, the next day, Jamie must have cereal. Use a tree diagram to list all the possible breakfast combinations for 3 consecutive days. Given that all outcomes are equally likely, find the probability that Jamie has:
(a) cereal on 3 consecutive days (b) cereal on exactly 2 consecutive days
(c) the same type of breakfast for 3 consecutive days
(d) a different type of breakfast for each of the 3 days.

11. A fair octagonal die, labelled A, B, C, D, E, F, G and H is rolled twice.
Calculate the probability of obtaining:
(a) vowels on both rolls (b) a vowel and a consonant (c) at least one vowel.

12. Two fair special die are rolled simultaneously. Each of these die has the numeral six printed on two faces, and the numerals two, three, four and five printed on the remaining faces. Determine the probability of obtaining:
(a) two sixes (b) at least one six
(c) a total of 12 (d) a total between 6 and 12 inclusive
(e) a difference of 3 between the scores on the two die.

13. In an genetic experiment, a gene is chosen at random from a pool of 5 dominant genes A, B, C, D, E and is combined with a gene chosen at random from a pool of 5 recessive genes a, b, c, d and e. The combination Aa and aA being similar. Calculate the probability that the resulting combination: (a) is Aa
(b) has genes with the same letters of the alphabet
(c) has genes with consecutive letters of the alphabet.

14. Two flower seeds are selected at random from a bag containing 5 seeds for red flowers and 8 seeds for white flowers.
(a) Find the probability that the two seeds chosen are for white flowers.
(b) Find the probability that the two seeds chosen are for different coloured flowers.

15. A sample of 5 individuals is to be selected at random from a group of 10 smokers and 12 non-smokers. Find the probability that the sample chosen has:
(a) exactly 4 smokers (b) no smokers (c) more smokers than non-smokers.

16. A committee of 6 persons is to be selected at random from a group of 8 businessmen, 5 academics and 10 politicians. Find the probability that the committee formed has:
(a) equal numbers of businessmen, academics and politicians (b) all 5 academics
(c) more politicians than businessmen and academics combined.

17. The starting 5 players for a basketball team are to be chosen at random from 4 forwards, 2 centres and 4 guards. Andrew is a forward, James is a centre and Ricky is a guard. Find the probability that: (a) Andrew, James and Ricky are in the starting 5
(b) Andrew is not in the starting 5
(c) Andrew is not in the starting 5 and Ricky is in the starting 5
(d) there is 1 centre, 2 forwards and 2 guards in the starting 5.

15.3 Empirical (A Posteriori) Probability

- In n trials of an experiment, if event A occurs a times, then the relative frequency of event A occurring is $r_A = \frac{a}{n}$.
- The probability of A occurring, P(A), is defined as the value of r_A as n becomes very large (approaches infinity). That is: $P(A) \approx \frac{a}{n}$ for large values of n.
- This is termed the empirical (or a posteriori) definition of probability.
  The probability of an event A occurring is *assigned* after the experiments are concluded.
- Consider a community of 2 000 people where 890 are supporters of the Green Party. The probability that a person randomly selected from this community is a supporter of the Green Party is then *assigned* as $\frac{890}{2000}$ = 0.445.
  Note that the probability assigned to the stated event is based on *empirical data* and not based on logical considerations.

15.4 Combining Events & Conditional Probability I

Let A and B be two events .

- If event A precedes event B, then the probability of events A *and* B occurring, denoted P(A ∩ B) or P(AB), is given by:

  P(A ∩ ≡ × |

  | A) is the probability of B occurring *given* that A has occurred,
  or the probability of B occurring *conditional* upon A.
- If event B precedes event A, then the probability of A *and* B occurring is given by:

  P(A ∩ ≡ × |

  | B) is the probability of A occurring *given* that B has occurred,
  or the probability of A occurring *conditional* upon B.
  These two rules are known as the *multiplication rules of probability.*
- The probability of either events A ***or*** B occurring, denoted P(A ∪ B) is given by:

  P(A ∪ − ∩

  addition rule of probability.
  Note the similarity between this rule and the counting rule for sets.

Example 15.4

A blue box contains 2 white and 4 black balls while a green box contains 5 white and 2 black balls. Julie tosses a fair die. If an odd number appears, she selects a ball at random from the green box. Otherwise, she selects a ball at random from the blue box. Find the probability:
(a) that she selects a ball from the blue box
(b) that she selects a white ball *given* that an even number appeared on the die
(c) that an even number was obtained with the die *and* she selects a white ball
(d) that an odd number was obtained with the die *and* she selects a white ball
(e) that she selects a white ball.

Solution:

(a) A ball is selected from the blue box when an even number appears on the die tossed.

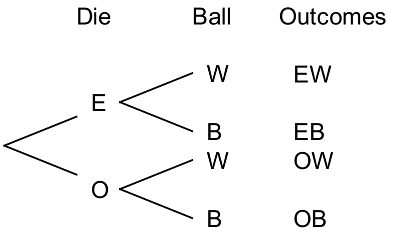
Since the die is fair, the probability of obtaining an even number is $\frac{3}{6}$.

Hence, the probability that a ball is selected from the blue box is $\frac{3}{6} = \frac{1}{2}$.

(b) Since an even number was obtained, the white ball must come from the blue box.

P(white ball chosen | even number) = P(white ball from the blue box) = $\frac{2}{6} = \frac{1}{3}$

(c) *The outcomes are best illustrated on a tree diagram:*
Define: E: die shows an even number
O: die shows a odd number
W: ball drawn is white
B: ball drawn is black
*The outcome of obtaining an even number and a white ball is described by EW. However, the outcomes {EW, EB, OW, OB} are **not equally likely** because the boxes have different numbers and compositions of balls.*



Using the multiplication rule: $P(EW) = P(E \cap W) = P(E) \times P(W | E) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

(d) Using the multiplication rule: $P(OW) = P(O \cap W) = P(O) \times P(W | O)$

$= \frac{1}{2} \times$ P(white ball from green box)

$= \frac{1}{2} \times \frac{5}{7} = \frac{5}{14}$

(e) *There are two ways of obtaining a white ball:*
- *An even number was obtained and a white ball was drawn from the blue box* ***or***
- *An odd number was obtained and a white ball was drawn from the green box*

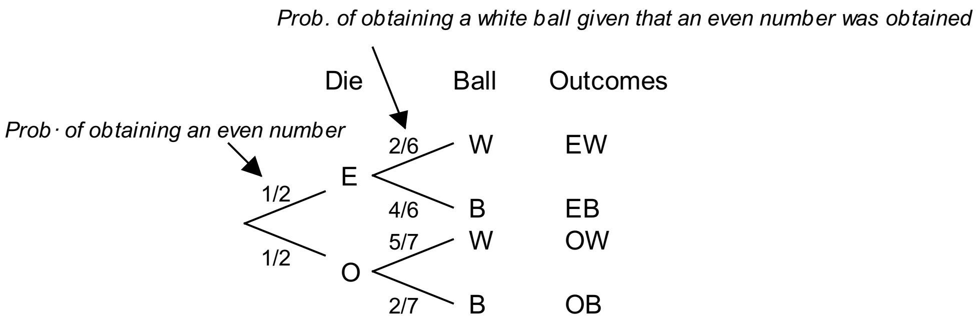
Hence using the addition rule: $P(W) = P(EW \cup OW) = P(EW) + P(OW) - P[(EW \cap OW)$

$= \frac{1}{6} + \frac{5}{14} - 0 = \frac{11}{21}$

*It is **impossible** to obtain a situation where an even number is obtained **together with** an odd number in a **single throw of a die**. Hence, P(EW ∩ OW) = 0.*

Special Notes:

- It is common to write down the relevant probabilities on the branches of a tree as illustrated below. Observe that the probabilities on the second level branches are *conditional probabilities.*



- All the possible outcomes at each stage of the tree must be listed, either individually or as a group as is the case in this example.
  - The outcomes “2”, “4” and “6” from the throw of the die have been grouped as “even”.
  - The outcomes “1”, “3” and “5” have been grouped as “odd”.
  - No other outcomes are possible with the throw of the die.
    As such we say that the events “even” and “odd” are *exhaustive*.
    They describe in total all the possible outcomes of this stage of the experiment. Observe that P(E) + P(O) = 1.
  - *That is, the sum of all the individual probabilities for a set of exhaustive outcomes is exactly one.*

- The outcomes that “spread out” from a point (for example, E and O from the start point) are such that both cannot occur at the same time. These outcomes are said to be *mutually exclusive*.
  - Since E and O cannot occur at the same time from a single throw of a die, $P(E \cap O)$ is assigned as 0.
  - Hence, if two events A and B are mutually exclusive, **P(A ∩**
    The addition rule becomes **P(A ∪** **P(A) + P(B)**.

- In summary, the outcomes that fork out from any point on a tree diagram are exhaustive (the sum of the probabilities is one) and mutually exclusive.

- The combined outcomes in the last column are such that none of them can ever occur at the same time as any of the others and are thus mutually exclusive.
  Hence, $P(EW \cap OW) = 0$, $P(EW \cap EB) = 0$ etc. .
  These outcomes are also exhaustive. That is, the sum of all the probabilities of the combined outcomes is one. Hence, P(EW) + P(EB) + P(OW) + P(OB) = 1.

Example 15.5

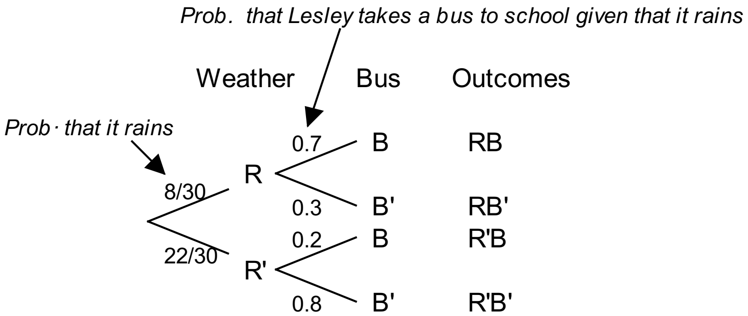
On average it rains on 8 days in April. If it rains, the probability that Lesley takes a bus to school is 70%. If it does not rain, the probability that Lesley takes a bus to school is 20%. Find the probability that on any April day:

(a) it rains and Lesley takes the bus to school

(b) Lesley takes the bus to school.

Solution:

Let R: event that it rains and B: the event that Lesley takes a bus to school



(a) P(rains and Lesley takes a bus to school) = P(RB)

$$= P(R) \times P(B \mid R)$$

$$= \frac{8}{30} \times \frac{7}{10}$$

$$= \frac{56}{300}$$

(b) P(Lesley takes a bus to school) = $P(RB \cup \bar{R}B)$

$$= P(RB) + P(\bar{R}B)$$

$$= \frac{56}{300} + P(\bar{R}) \times P(B \mid \bar{R})$$

$$= \frac{56}{300} + \frac{22}{30} \times \frac{2}{10}$$

$$= \frac{100}{300}.$$

Notes:

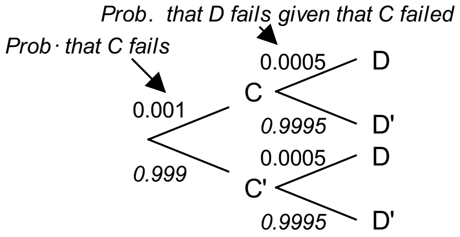
- *The probability that it rains on an April day is assigned as 8/30, as there are on average 8 rainy days in April.*
- *In part (b), $P(RB \cup \bar{R}B) = P(RB) + P(\bar{R}B)$ because RB and $\bar{R}B$ are mutually exclusive.*

Example 15.6

An electronic component has a capacitor (C) and a diode (D) that works independently of each other. The probability that C will fail on any given occasion is 0.001 while the probability that D will fail on any given occasion is 0.0005. Find the probability that on a given occasion: (a) both C and D will fail (b) at least one of the two components will fail.

Solution:

Let C: the event that C fails and D: the event that D fails.



(a) Since the components work independently of each other, the performance of C does not affect the performance of D. Hence, $P(D \text{ fails} \mid C \text{ failed}) = P(D \text{ fails}) = 0.0005$.

Therefore, $P(\text{both fail}) = P(C \cap D) = P(C) \times P(D \mid C)$

$$= 0.001 \times 0.0005 = 5 \times 10^{-7}$$

(b) $P(\text{at least one will fail}) = 1 - P(\text{none will fail})$

$$= 1 - P(\overline{C} \cap \overline{D})$$

$$= 1 - 0.999 \times 0.9995 = 0.001\,499\,5$$

Notes:

- *The tree diagram could have been drawn with D on the first level.*

Special Notes:

- Two events are said to be statistically *independent* if the outcome of one event has no impact on the outcome of the other.

. • For two events A and B that are *independent*:

  $P(A|$ $P(A|B) = P(A)$ and **$P(B|$ $|A) = P(B)$**

  The probability that A occurs is the same regardless of whether B has occurred or not.

- The event "at least one will fail" is *complementary* to the event "none will fail". These two complementary events are exhaustive. Hence:

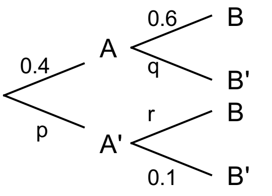
  $$P(\text{at least one will fail}) + P(\text{none will fail}) = 1$$

  From which we obtain the result: $P(\text{at least one will fail}) = 1 - P(\text{none will fail})$.

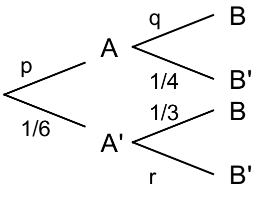
- For A and $\overline{A}$ (or A') as *complementary* events: $\mathbf{P(A) + P(\overline{A}) = 1}$.

Exercise 15.2

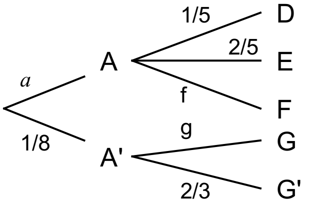
1. In the accompanying tree diagram find:
   (a) the values of *p*, *q* and *r*
   (b) $P(B \mid A)$
   (c) $P(\overline{B} \mid \overline{A})$
   (d) $P(AB \cup \overline{A}B)$



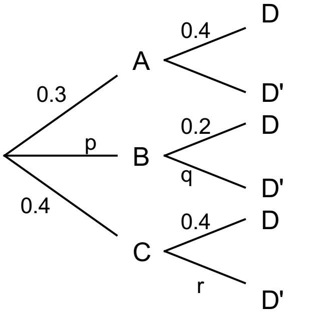
2. n the accompanying tree diagram find:
   (a) the values of *p*, *q* and *r*
   (b) $P(\overline{B} \mid A)$
   (c) $P(B \mid \overline{A})$
   (d) $P(\overline{B})$



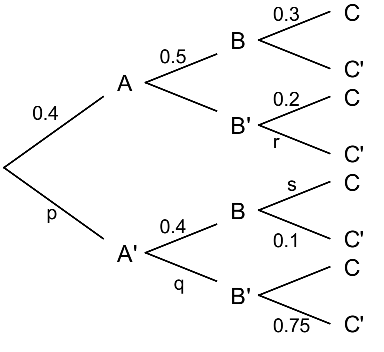
3. In the given tree diagram, the events that follow A are D, E and F *only*. Find:
   (a) the values of *a*, *f* and *g*
   (b) $P(E \mid A)$
   (c) $P(A \cap E)$
   (d) $P(AE \cup \overline{A}G)$



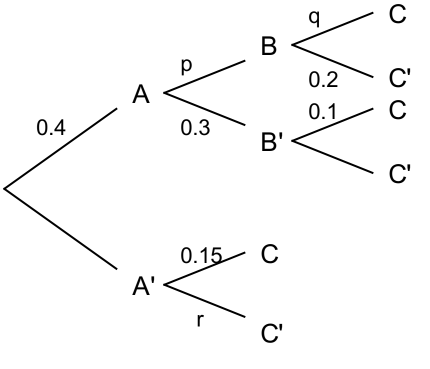
4. In the given tree diagram, the outcomes in the first level are A, B and C *only*. Find:
   (a) the values of *p*, *q* and *r*
   (b) $P(\overline{D} \mid A)$
   (c) $P(B \cap D)$
   (d) $P(BD \cup CD)$
   (e) $P(D)$



5. In the given tree diagram, find:
   (a) the values of *p*, *q*, *r* and *s*
   (b) $P(\overline{B} \mid A)$
   (c) $P(\overline{C} \mid AB)$
   (d) $P(C \mid A'B')$
   (e) $P(A \cap B \cap C)$
   (f) $P(C)$
   (g) $P(B)$
   (h) $P(BC)$
   (i) $P(\overline{A}\,\overline{C})$



6. In the accompanying tree diagram, find:
   (a) the values of *p*, *q* and *r*
   (b) $P(A')$
   (c) $P(C' \mid AB')$
   (d) $P(A \cap B \cap C)$
   (e) $P(A \cap C)$
   (f) $P(C)$



7. A box has 3 red balls and 4 blue balls. Damien picks a ball at random from the box, notes the colour of the ball and *returns* it to the box. He then picks a second ball at random from the box. Find the probability that Damien picked:
   (a) a red ball in the first draw
   (b) a red ball in the second draw given that the first ball was red
   (c) a red ball in the second draw (d) two balls of different colours.

8. Repeat Question 7, this time with Damien *not returning* the first ball into the box.

9. A bag holds 4 green marbles and 1 red marble. Two marbles are drawn randomly in succession *without replacement* from the bag. Calculate the probability that:
   (a) the second marble is red given that the first marble is red
   (b) the two marbles are of different colours.

10. A box has 4 white and 6 green balls. Three balls are drawn randomly in succession *without replacement* from the box. Find the probability that:
    (a) the second ball is green given that the first ball is white
    (b) the third ball is white given that the first two balls are both green
    (c) the three balls are all white (d) the three balls are all of the same colour
    (e) at least two of the balls are white (f) the second ball is white.

11. Repeat Question 10, for the balls drawn randomly *with replacement*.

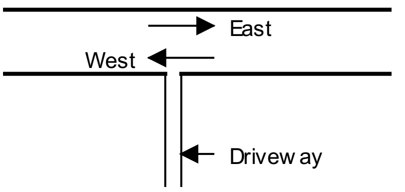
12. There are on average 17 wet mornings in the month of June (in Perth). If it rains, Shaun gets a lift to school from Dad. If it does not rain, the probability that Shaun rides to school is 0.9; otherwise he gets a lift to school from mum. Find the probability that:
    (a) it rains on a given June morning
    (b) it rains on a given June morning and Shaun rides to school
    (c) Shaun rides to school on a given June morning.

13. Larry is 5 years old and often throws a tantrum to get what he wants. The probability that Larry throws a tantrum over any encounter with mum is 70%. Given that he throws a tantrum, the probability that mum gives in to his demands is 20%. On occasions when he does not throw a tantrum, he has a probability of 80% of getting what he wants. Find the probability that on any given encounter Larry:
    (a) throws a tantrum and gets what he wants (b) gets what he wants.

14. In a certain tribal population, 40% of its members have detached ear lobes. Of these, 20% have curly hair. Of those without detached ear lobes, 70% have curly hair. A member of this tribe is chosen at random. Find the probability that this person has:
    (a) detached ear lobes and hair that is not curly
    (b) either detached ear lobes or curly hair but not both (c) curly hair.

15. Sam is scheduled to swim two races within an hour. The probability that she wins her first race is 0.95. The probability that she wins her second race given that she wins her first race is 0.75. Given that she fails to win the first race, the probability that she will win the second race is 0.99. Find the probability that Sam: (a) wins both races
    (b) wins either race 1 or race 2 but not both (c) wins at least one race.

16. Matt lives on a busy road and has to back out of his driveway onto the road. He starts off from his driveway each work day at 7.00 am. The main road has no median island facing his driveway. East bound traffic moves independently of westbound traffic. On any given workday, as he drives out from his drive way, the probability that the east bound lane has sufficient clearance for him to immediately reverse into it is 0.05. The corresponding probability for him to immediately reverse into the west bound lane is 0.01. Find the probability that on a given work day:



(a) Matt is unable to reverse immediately into the west bound lane
(b) Matt is able to reverse immediately into the east bound lane

17. Applicants for the Air Force in the island republic of Probland are required to undergo a medical test, a psychological test and a mathematics test. 70% and 40% of applicants are expected to pass the medical and psychological tests respectively. 10% of applicants pass the mathematics test and 1% who pass the mathematics test achieve a perfect score. Recruits are taken from applicants who pass all three tests or those who achieve a perfect score in the mathematics test. An applicant is selected at random. Assume that an applicant's performance in these tests are independent of each other. Find the probability that the applicant:
(a) will pass all three tests (b) will pass the medical and mathematics test
(c) will pass either the medical or the psychological test
(d) will pass either the psychological or medical test but not both
(e) pass at least one of the three tests (f) will be successful in joining the Air Force.

18. In a survey conducted in a particular suburb, it was found that 65% of the residents were university graduates. 40% of the university graduates earned in excess of \$100 000 a year and of these 80% were self-employed. 30% of university graduates earning less than \$100 000 a year were self-employed. Of those residents who were non-university graduates, 20% earned in excess of \$100 000 a year and of these 98% were self-employed. 40% of non-university graduates earning less than \$100 000 a year were self-employed. A resident was chosen at random. Find the probability that the resident:
(a) earned in excess of \$100 000 per year and was not self-employed
(b) was a university graduate or self-employed
(c) earned in excess of \$100 000 per year.

19. A machine contains 3 components A, B and C, each of which works independently from the others and may fail with probability 0.05, 0.01 and 0.02 respectively. Calculate the probability that:
(a) all three components fail (b) components A and C fail but not B
(c) A or B or C fail (d) at least one component fails

15.5 Conditional Probability II

- For any two events, A and B, $P(A \cap B) = P(B) \times P(A \mid B)$.

  Rearranging, $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

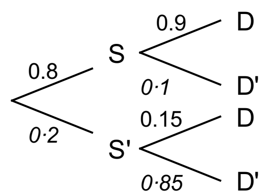
  This is known as the *Conditional rule of Probability*.

Example 15.7

At a reception dinner, 80% of guests had 6 or more oysters and of these 90% had diarrhoea later that evening. Of the guests that had less than 6 oysters, 15% had diarrhoea later that evening. Find the probability that a randomly selected guest:
(a) ate 6 or more oysters and had diarrhoea later that evening
(b) had diarrhoea later in the night
(c) who had diarrhoea had eaten 6 or more oysters at the reception dinner.

Solution:

Let S: guest eats 6 or more oysters
and D: guest has diarrhoea later.
The accompanying tree diagram illustrates the described scenario.



(a) Using the information on the tree diagram:

$$P(S \cap D) = 0.8 \times 0.9 = 0.72$$

(b) Using the addition rule:

$$P(D) = P(SD) + P(S'D)$$
$$= 0.72 + 0.2 \times 0.15 = 0.75$$

(c) Using the conditional rule:

$$P(S \mid D) = \frac{P(S \cap D)}{P(D)}$$
$$= \frac{0.72}{0.75}$$
$$= 0.96$$

Note:

- *$P(D/S) = 0.9$ is a conditional probability that can be located on the tree diagram. However, $P(S/D)$ is a conditional probability which in the context of the tree diagram drawn, cannot be located on the tree diagram. Hence, the need to use the conditional rule.*

Example 15.8

60% of the eligible voters in Probland are supporters of the Probland Labour Party while the remaining 40% are supporters of the Probland Liberal Party. 70% of eligible voters in Probland believe that Probland needs a new flag. 50% of eligible voters are both Labour supporters and supporters of a new flag. An eligible voter in Probland is chosen at random. Find the probability that the voter is:

(a) a Liberal supporter and a supporter of a new flag
(b) a supporter of a new flag given that the voter is a Liberal supporter
(c) a Liberal supporter given that the voter is a supporter of a new flag.

Solution:

*A table (called a **Two-way Table**) describing the situation is drawn below. The entries in each cell describe the percentage of Probland voters for the different categories. Entries in **bold** are obtained directly from information given in the Question while entries in italics are calculated from information given in the Question.*

	Labour	Liberal	Total
Supports new flag	**50**	*20*	**70**
Does not support new flag	*10*	*20*	*30*
Total	**60**	**40**	**100**

(a) $\text{P(Liberal supporter} \cap \text{supports new flag)} = \dfrac{20}{100} = 0.2$

(b) Using the Conditional rule:

$$\text{P(Supporter of new flag} \mid \text{Liberal supporter)} = \frac{\text{P(new flag supporter} \cap \text{Liberal)}}{\text{P(Liberal)}}$$

$$= \frac{0.2}{(40/100)} = 0.5$$

Alternative Method for (b):

Since, the voter is a Liberal supporter, we can *restrict the sample space* to consider only the Liberal supporters of which there are 40.
Of the 40 Liberal supporters 20 are also supporters of a new flag.
Hence, using the Theoretical definition of probability:

$$\text{P(Supporter of new flag} \mid \text{Liberal supporter)} = \frac{20}{40} = 0.5$$

(c) *Restricting the sample space* to only those who support the new flag, there are 70 of them. Of these 20 are also supporters of the Liberal party. Hence:

$$\text{P(Liberal} \mid \text{Supporter of new flag)} = \frac{20}{70} = 0.2857$$

Notes:

- *A tree diagram could also have been used. A third alternative would be to use a Venn Diagram. But in this case a Two-way table has distinct advantages over a tree diagram and a Venn diagram.*
- *In this example, it is clearly more efficient to calculate conditional probabilities by **restricting the sample space**.*

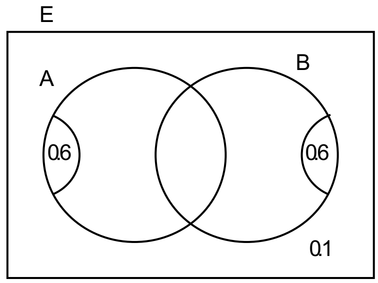
Exercise 15.3

1.

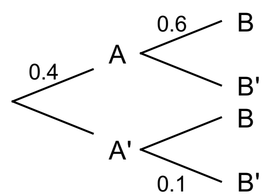
	A	A′	Total
B	**0.1**		**0.6**
B′		**0.4**	
Total	**0.1**		**1**

Use the Two-way Table above to find:
(a) $P(A' \cap B)$ (b) $P(B')$ (c) $P(B|A)$ (d) $P(A \cup B)$

2. The accompanying Venn diagram displays events A and B within the sample space E. An element is chosen at random from E. Find:
(a) $P(A \cup B)$ (b) $P(A \cap B)$
(c) $P(A' \cap B)$ (d) $P(A|B)$
(e) $P(B|A)$



3. Use the accompanying tree diagram to find:
(a) $P(A \cap B)$
(b) $P(B)$
(c) $P(B'|A)$
(d) $P(A|B)$

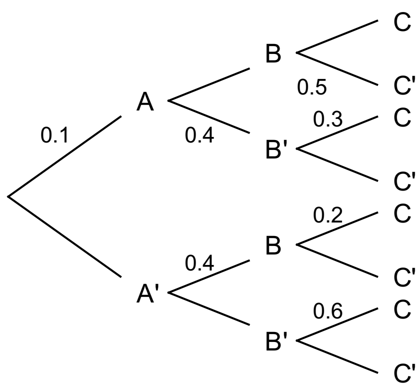


4. Given that $P(B|A) = 0.4$, $P(B|A') = 0.3$ and $P(B) = 0.32$, with the aid of a tree diagram, find: (a) $P(B'|A)$ (b) $P(A)$ (c) $P(A|B')$.

5. Use the contingency table (two way table) below to find:
(a) $P(A \cap D)$
(b) $P(C|E)$
(c) $P(F|B)$
(d) $P(A \cup F)$
(e) $P(D \cup C)$

	A	B	C	Total
D		**60**	**70**	**150**
E	**50**	**95**		**250**
F	**20**		**5**	
Total	**90**		**180**	**500**

6. Use the accompanying tree diagram to find:
(a) $P(A \cap B)$
(b) $P(B)$
(c) $P(A|B)$
(d) $P(A \cap C)$
(e) $P(C|A)$
(f) $P(A|C)$



7. The probability that components A and B of a combustion engine fail at the same time is 0.0009. The probability that A and B each fail individually are respectively 0.002 and 0.004. Find the probability that:
   (a) A fails given that B has failed
   (b) B fails given that A has failed
   (c) A or B fails.

8. A basketball team is scheduled to play two away back to back games (games on consecutive days). From past records, the probability of winning the first game is 0.9 and the probability of winning the next game after winning the first game is 0.85. The probability of winning the second game after losing the first game is 0.95.
   (a) Find the probability that the team wins at least one game.
   (b) Given that the team wins the second game, find the probability that it won the first game.

9. The proportions of a certain breed of cat having blue eyes, green eyes and mixed (blue and green) eyes are 60%, 30% and 10% respectively. 1% of the blue-eyed cats of this breed have long fur and the corresponding proportions for green-eyed cats and mixed-eyed cats of this breed are 5% and 40% respectively.
   (a) Find the probability that a randomly chosen cat of this breed has long fur.
   (b) Find the probability that a long-furred cat of this breed has blue eyes.

10. The proportion of year 11 students at Fremantle College enrolled in Mathematics Methods, Mathematics Applications and Mathematics Essential are 40%, 40% and 20% respectively. All year 11 students at Fremantle College are enrolled in exactly one of these subjects. 24% of year 11 students at Fremantle College are girls and are enrolled in Mathematics Methods. 50% of students in Mathematics Applications are girls. 12% of year 11 students in the College are boys and enrolled in Mathematics Essential.
    (a) Find the probability that a year 11 student chosen at random from the College is a boy who is enrolled in Mathematics Methods.
    (b) Find the probability that a year 11 student of Mathematics Essential chosen at random from the College is a girl.
    (c) Find the probability that a year 11 male student chosen at random from the College is enrolled in Mathematics Methods.

11. A river flows past a town. 60% of the town's taxpayers live north of the river while 40% of the town's taxpayers live south of the river. Of the taxpayers that live north of the river, 30% are employed by the state, 50% are employed in the private sector while 20% are self-employed. Of the taxpayers living south of the river, the corresponding proportions are 10%, 60% and 30%.
    (a) Calculate the probability that a taxpayer randomly selected from this town is self-employed
    (b) Calculate the probability that a taxpayer chosen randomly from those who are employed by the state lives south of the river.
    (c) Calculate the probability that a taxpayer chosen randomly from those who are self-employed or employed by the state lives south of the river.

12. A survey of 500 residents was conducted. 65% were born in Western Australia (WA), 25% were born in the Eastern States of Australia and 10% were born overseas. 80% of those born in WA had a WA born spouse, 90% of those born in the Eastern States had a WA born spouse and 60% of those born overseas had a WA born spouse. A resident was randomly chosen from those surveyed.
    (a) Find the probability that this resident has a spouse who was not WA born given that this resident was WA born.
    (b) Find the probability that the resident has a spouse that is WA born.
    (c) Find the probability that the resident was WA born given that the resident has a spouse that is WA born.

13. Of a group of 100 tourists visiting the natural bridge near Albany (400 km south of Perth), 40 had video cameras, 80 had cameras and 20 had no video cameras or cameras.
    (a) Find the probability that a tourist randomly chosen from this group has both a video camera and camera.
    (b) Calculate the probability that a tourist with a camera randomly chosen from this group also has a video camera.
    (c) Calculate the probability that 2 tourists randomly chosen from this group, both have video cameras.

14. In a survey of 140 students who had recently migrated with their families to Western Australia, students were asked to indicate the birth places of both their parents. The results are displayed in the accompanying table.

		Mother's birth place			
		UK	Europe	Asia	Total
Father's birth place	UK	45	16	4	65
	Europe	7	12	1	20
	Asia	8	2	45	55
	Total	60	30	50	140

    (a) Find the probability that a randomly selected student from this group has both parents born in the UK.
    (b) Find the probability that a student randomly selected from those with mothers born in the UK, has a father born in Asia.
    (c) Given that a student has a father born in Asia, find the probability that the student's mother is also born in Asia.
    (d) Find the probability that a student randomly chosen from this group have a parent born in either UK or Europe.

15. The accompanying table shows the number of students enrolled in the different subject combinations. Students are enrolled in only one of the given subject combinations.

	Physics	Chemistry	Biology
Geography	24	65	94
Economics	80	82	32
History	70	74	48

(a) Find the probability that a student who is enrolled in Chemistry is also enrolled in Geography.
(b) Find the probability that a student is enrolled in Physics and Economics.
(c) Five students were chosen randomly from this group.
Find the probability that exactly three are enrolled in Biology and Geography.
(d) Of five students randomly chosen from this group, three are enrolled in Biology and Geography. Find the probability that the remaining two students are enrolled in Chemistry and Economics.
(e) Nine students were randomly chosen from this group. Find the probability that all nine students are enrolled in different subject combinations.

16. A Prime Minister's Cricket XI (consisting of 11 players) is to be chosen from a list of 23 players to play against the visiting South African cricket team in a Twenty-20 cricket charity match. The list consists of 6 players from New South Wales (NSW), 7 from Victoria, 3 each from Western Australia (WA) and South Australia (SA) and 2 each from Tasmania and Queensland. All players stand an equal chance of being chosen.
(a) Calculate the probability that all the WA players will be selected.
(b) Find the probability that a team that has 3 WA players will also have all 2 players from Tasmania.
(c) Find the probability that the team will have at least 6 players from Victoria given that all 3 WA players are chosen.

17. A Senate Inquiry Committee of 6 senators is to be selected randomly from 4 Labor senators, 4 Liberal senators, 2 Green senators and 2 Independent senators. Mr. Falcon, a Labor senator refuses to be on the same committee as Mr. Turkey, a Liberal senator.
(a) Find the probability that Mr. Falcon is on the committee.
(b) Find the probability that both Mr. Falcon and Mr.Turkey are not on the committee.
(c) Find the probability that in all committees without Mr. Falcon, Mr. Turkey is a member of the committee.
(d) Find the probability that Mr. Falcon is on the committee given that there is 1 Green senator and 1 Independent senator.

18. The points A, B, C, D and E (in this order) lie on line L1 and the points F, G and H (in this order) lie on line L2 which is parallel to L1. Triangles are formed using the points on these two lines.
(a) Find the probability that a randomly chosen triangle drawn using the above procedure has exactly one vertex on L1.
(b) Given that a triangle has a vertex at F, find the probability that the second vertex is on L2.
(c) Given that a triangle has a vertex at A, find the probability that the second and third vertices are on L2.
(d) Quadrilaterals are now drawn using points on L1 and L2 as vertices. Find the probability that a randomly chosen quadrilateral:
(i) has vertices at B and G (ii) has BG as one of its sides.

15.6 Rules of Probability

- Listed below is a summary of the various probability rules established in the earlier sections.
  - Complement Rule $P(\overline{A}) = 1 - P(A)$
  - Conditional Rule $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
  - Condition for independence of two events:
    $$P(A) = P(A \mid B) = P(A \mid \overline{B})$$
  - Condition for two events to be mutually exclusive:
    $$P(A \cap B) = 0$$
  - Multiplication Rule $P(A \cap B) = P(A) \times P(B \mid A)$ or $P(B) \times P(A \mid B)$
  - Multiplication Rule for independent events: $P(A \cap B) = P(A) \times P(B)$
  - Addition Rule $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - Addition Rule for mutually exclusive events:
    $$P(A \cup B) = P(A) + P(B)$$

Example 15.9

Given that $P(A \cup B) = 0.8$, $P(A) = 0.4$ and $P(\overline{B}) = 0.3$, determine if A and B are mutually exclusive.

Solution:

Since, $P(\overline{B}) = 0.3$, using the complement rule, $P(B) = 1 - 0.3 = 0.7$

Using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.8 = 0.4 + 0.7 - P(A \cap B)$$

Hence, $P(A \cap B) = 0.3$

Since, $P(A \cap B) \neq 0$, A and B are *not* mutually exclusive.

Note:

- *To test if A and B are mutually exclusive, find P(A $\cap$ B):*
  - ***If P(A*** $\cap$
  - $\cap$ $\neq$

Given that $P(A \cup B) = 0.9$, $P(A \cap B) = 0.4$ and $P(A \mid B) = 0.5$, determine if A and B are independent.

Solution:

Using the multiplication rule:

$$P(A \cap B) = P(B) \times P(A \mid B)$$
$$0.4 = 0.5 \times P(B)$$

Hence, $P(B) = 0.8$

Using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.9 = P(A) + 0.8 - 0.4$$
$$P(A) = 0.5$$

Since $P(A) = P(A \mid B) = 0.5$, A and B are independent.

Note:

To determine if A and B are independent:

- *Show that* ***P(A) = P(A*** / /

  Alternatively, show that ***P(A*** $\cap$ $\times$

Example 15.11

Given that P(B) = 0.6 and P(A $\cup$ B) = 0.72, find P(A):
(a) if A and B are independent (b) if A and B are mutually exclusive.

Solution:

Let P(A) = p.

(a) Since A and B are independent:

$$P(A \cap B) = P(A) \times P(B) = 0.6p$$

Using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.72 = p + 0.6 - 0.6p$$
$$0.4p = 0.12$$
$$p = 0.3$$

(b) Since A and B are mutually exclusive, P(A $\cap$ B) = 0.

Using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.72 = p + 0.6 - 0$$
$$p = 0.12$$

Exercise 15.4 (Miscellaneous)

1. Given P(A $\cup$ B) = 0.8, P($\overline{A}$) = 0.4 and P($\overline{B}$) = 0.3, determine if A and B are mutually exclusive. Justify your answer.

2. Given that P($\overline{A} \cap \overline{B}$) = 0.6, P(A) = 0.4 and P(B) = 0.2, determine if A and B are mutually exclusive. Justify your answer.

3. Given that P(A) = 0.2, P(B) = 0.5 and P($\overline{A} \cap$ B) = 0.4, determine if A and B are mutually exclusive. Justify your answer.

4. Given that P(A $\cap \overline{B}$) = 0.1 and P(A) = 0.1, determine if A and B are mutually exclusive.

5. Given that P(A $\cap$ B) = 0.45, P(A) = 0.5 and P(B) = 0.9, determine if A and B are independent. Justify your answer.

6. Given that P(A $\cup$ B) = 0.6, P(A) = 0.3 and P(B) = 0.6, determine if A and B are independent. Justify your answer.

7. Given that P(A $\cup$ B) = 0.8, P(A $\cap$ B) = 0.2 and P(B $\mid$ A) = 0.5, determine if A and B are independent. Justify your answer.

8. Given that P(A) = 0.5 and P(B) = 0.4, find P(A $\cup$ B), given that A and B are:
(a) mutually exclusive (b) independent

9. Given that $P(\overline{A}) = 0.3$ and $P(B) = 0.2$, find $P(\overline{A} \cap \overline{B})$ given that A and B are:
   (a) mutually exclusive (b) independent.

10. If $P(A) = 0.3$ and $P(B) = 0.4$, find the minimum and maximum values for $P(A \cup B)$.

11. Two organic based pesticides A and B are sprayed onto each of 1 500 tomato plants to help stem the incidence of a particular plant disease and monitored through the season. The accompanying table shows the results of the study. Events A and B are defined as those plants that had been sprayed with A and B respectively and had subsequently picked up the disease. Events A' and B' are defined as those sprayed with A and B respectively that did not pick up the disease. Determine:

	A	A′	Total
B	0	504	504
B′	703	293	996
Total	703	797	1 500

   (a) $P(A \cup B)$ (b) $P(A \mid B')$
   (c) with reasons if A and B are mutually exclusive events
   (d) with reasons if A and B are statistically independent events.

12. The accompanying table shows the results of a study of 2 000 incidences of a particular kind of subatomic collision. Event A is defined as the particles released having an electric charge and event B is defined as the particles released having spins. Determine:

	A	A′	Total
B	1 120	280	1 400
B′	480	120	600
Total	1 600	4 00	2 000

   (a) $P(A \cup B)$ (b) $P(A' \mid B')$
   (c) with reasons if A and B are mutually exclusive, independent or neither.

13. A fair die is thrown twice. Define A: score obtained in the first throw is 3 or less, B: the two scores obtained differ by not more than one and C: the sum of the two scores is at least 10.
   (a) Find P(A), P(B) and P(C).
   (b) Show that only two of these events are mutually exclusive.
   (c) Show that only two of these events are independent.

14. Three events A, B and C are such that A and C are independent.
   Also, $P(A) = 0.2$, $P(B) = 0.1$, $P(A \cup C) = 0.5$ and $P(B \cup C) = 0.4$.
   (a) Find P(C).
   (b) Determine if B and C are independent, mutually exclusive or neither.

16 Indices

16.1 Review of the Laws of Indices

- For $a \neq 0$:
  - $a^x \times a^y = a^{x+y}$
  - $\dfrac{a^x}{a^y} = a^{x-y}$
  - $(a^x)^y = a^{xy}$
  - $a^0 = 1$

Example 16.1

Simplify each of the following: (a) $2x^5 \times 5x^2$ (b) $\dfrac{10x^{10}}{2x^2}$ (c) $(3x^3)^2$

Solution:

(a) $2x^5 \times 5x^2 = 10x^7$ (b) $\dfrac{10x^{10}}{2x^2} = 5x^8$ (c) $(3x^3)^2 = 9x^6$

Example 16.2

Simplify each of the following: (a) $3x^2y \times 4xy^3$ (b) $\dfrac{12x^4y^3}{4x^2y}$ (c) $(5x^2y^3)^2$

Solution:

(a) $3x^2y \times 4xy^3 = 12x^3y^4$ (b) $\dfrac{12x^4y^3}{4x^2y} = 3x^2y^2$ (c) $(5x^2y^3)^2 = 25x^4y^6$

Exercise 16.1 Calculator Free

1. Simplify each of the following.
   (a) $3x^2 \times 2x^3$ (b) $7x^4 \times 4x^7$ (c) $-2x^3 \times 5x^2$ (d) $\frac{1}{3}x^3 \times 6x^2$

2. Simplify (do not expand) each of the following.
   (a) $(1+x)^2(1+x)^3$ (b) $2(1-x)^2 \times 3(1-x)^3$
   (c) $\frac{1}{2}(1+x)^2 \times 4(1+x)$ (d) $\frac{-3}{2}(x-1)^2 \times \frac{4}{6}(x-1)^3$

3. Simplify each of the following.

(a) $\dfrac{4x^4}{2x^2}$ (b) $\dfrac{15x^{15}}{3x^3}$ (c) $\dfrac{7x^{14}}{14x^7}$ (d) $\dfrac{-8x^{24}}{24x^8}$

4. Simplify (do not expand) each of the following.

(a) $\dfrac{(1+x)^4}{(1+x)^2}$ (b) $\dfrac{10(1-x)^{10}}{5(1-x)^5}$ (c) $\dfrac{-3(x^2+1)^6}{6(x^2+1)^3}$ (d) $4(2+x)^2 \times \dfrac{1}{8(2+x)}$

5. Simplify each of the following.

(a) $(2x^3)^3$ (b) $(-2x^4)^2$ (c) $(-3x^4)^3$ (d) $\left(\dfrac{x^2}{2}\right)^3$

(e) $(x^2y)^3$ (f) $(3xy^3)^2$ (g) $(-4x^2y)^3$ (h) $\left(\dfrac{-xy^2}{4}\right)^2$

6. Simplify (do not expand) each of the following.

(a) $[(1+x)^2]^3$ (b) $[2(x-2)^2]^3$ (c) $[-(x+5)^3]^4$ (d) $[-2(x+2)^3]^3$

7. Simplify each of the following.

(a) $xy \times x^2y$ (b) $2x^2y \times 3x^2y^4$ (c) $-4x^2y \times -3x^3y^2$ (d) $-6x^2y \times \frac{1}{2}xy$

(e) $\dfrac{x^4y^5}{x^2y^3}$ (f) $\dfrac{4x^2y}{2xy}$ (g) $\dfrac{8x^3y^4}{24xy^3}$ (h) $\dfrac{-4x^4y^3}{6x^4y}$

8. Simplify (do not expand) each of the following.

(a) $x(1+x)^2 \times x^2(1+x)^3$ (b) $2x^2(1-x)^2 \times 3x(1-x)$

(c) $-2(1+x)^2(1+3x)^2 \times -4(1+x)^3(1+3x)^2$ (d) $\frac{1}{6}(3+x)^2(2-x)^3 \times -12(3+x)(2-x)$

9. Simplify (do not expand) each of the following.

(a) $\dfrac{x^3(1+x)^3}{x(1+x)^2}$ (b) $\dfrac{-4x^4(1+2x)^3}{2x^3(1+2x)^2}$

(c) $\dfrac{8(1+x)^2(1+4x)^3}{16(1+x)^3(1+4x)^2}$ (d) $\dfrac{-14(x+3)^3(x+5)^5}{21(x+3)^2(x+5)^4}$

10. Simplify (do not expand) each of the following.

(a) $[x^2(1+x)^4]^3$ (b) $[-2x(1+x)^2]^3$

(c) $[(1+2x)^2(x+3)]^2$ (d) $\left[\dfrac{(x+1)^2(2x-1)^3}{4}\right]^2$

16.2 Review of Negative Indices

- For $a \neq 0$: $\dfrac{1}{a^n} = a^{-n}$

Example 16.3

Simplify, leaving your answer in the numerator. (a) $2x^4 \times 5x^{-10}$ (b) $\dfrac{10x^3}{5x^4}$

Solution:

(a) $2x^4 \times 5x^{-10} = 10x^{-6}$

(b) $\dfrac{10x^3}{5x^4} = 2x^{-1}$

Example 16.4

Simplify. Leave your answers with positive indices. (a) $3x^2 \times 4x^{-5}$ (b) $\dfrac{2x^4}{4x^5}$

Solution:

(a) $3x^2 \times 4x^{-5} = 12x^{-3} = \dfrac{12}{x^3}$

(b) $\dfrac{2x^4}{4x^5} = \dfrac{1}{2x}$

Exercise 16.2 Calculator Free

1. Rewrite with the x term in the numerator.

   (a) $\dfrac{1}{x^4}$ (b) $\dfrac{2}{x^5}$ (c) $\dfrac{-1}{x^{-4}}$ (d) $\dfrac{1}{2x^3}$

2. Simplify each of the following, leaving the x and y terms in the numerator.

   (a) $x^3 \times -2x^4$ (b) $4x^{-2} \times 3x^{-2}$ (c) $5xy \times -2x^2y^{-2}$ (d) $10x^2y^{-3} \times \dfrac{x^{-3}y^2}{20}$

3. Simplify. Leave your answers with positive indices.

   (a) $\dfrac{x^4}{2x^5}$ (b) $\dfrac{16t^{-3}}{8t^{-4}}$ (c) $\dfrac{6x^2y^3}{3x^{-2}y^4}$ (d) $\dfrac{10p^{-3}q^{-2}}{15p^4q^{-3}}$

4. Simplify. Leave your answers with the x term in the numerator.

   (a) $\left(\dfrac{2}{x^2}\right)^3$ (b) $\left(\dfrac{1}{3x^2}\right)^3$ (c) $\left(\dfrac{1}{x^2}\right)^{-2}$ (d) $\left(\dfrac{1}{3x^2}\right)^{-2}$

5. Simplify. Leave your answers with the *x* and *y* terms with positive indices.

(a) $\left(\dfrac{2x^2}{y^3}\right)^{-2}$ (b) $\left(\dfrac{x^{-2}}{2y}\right)^{2}$ (c) $\left(\dfrac{2}{3x^{-2}y}\right)^{2}$ (d) $\left(\dfrac{4x^{-2}}{y^3}\right)^{-1}$

6. Simplify. Leave your answers with the variables with positive indices.

(a) $3p^2q^3 \times \dfrac{1}{4p^4q}$ (b) $\left(\dfrac{st}{2}\right)^2 \times \left(\dfrac{4s}{t^5}\right)$ (c) $6x^{-2}y \times \left(\dfrac{-x}{y^2}\right)^3$ (d) $\dfrac{5m^2n}{2} \times \left(\dfrac{m^2}{n}\right)^{-2}$

7. Simplify. Leave your answers in the numerator.

(a) $\dfrac{(1+x^2)^3}{(1+x^2)^4}$ *(b) $\dfrac{1-x^2}{(1+x)^3}$

*8. Simplify. Leave your answers with positive indices.

(a) $\dfrac{(4+x)^2(x+3)^2}{(4+x)^5(x+3)^3}$ (b) $\dfrac{(x+2)(x-2)^3}{(x^2-4)^2}$

16.3 Review of Fractional Indices

- For $a > 0$:
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (a^{\frac{1}{n}})^m = \left(\sqrt[n]{a}\right)^m$

Example 16.5

Rewrite each of the following without the radical sign $\sqrt{\ }$: (a) $\sqrt[3]{x}$ (b) $\dfrac{1}{\sqrt[4]{x}}$

Solution:

(a) $\sqrt[3]{x} = x^{\frac{1}{3}}$ (b) $\dfrac{1}{\sqrt[4]{x}} = \dfrac{1}{x^{\frac{1}{4}}}$ (or = $x^{-\frac{1}{4}}$)

Example 16.6

Without using a calculator, evaluate: (a) $8^{\frac{2}{3}}$ (b) $\dfrac{1}{\sqrt[3]{27}}$

Solution:

(a) $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4$ (b) $\dfrac{1}{\sqrt[3]{27}} = \dfrac{1}{3}$

Exercise 16.3 **Calculator Free**

1. Rewrite each of the following without the radical sign, leaving your answers in the numerator.

(a) $\sqrt[4]{x}$ (b) $\sqrt[4]{x^2}$ (c) $\sqrt[3]{x^2y^3}$ (d) $\sqrt{\left(\dfrac{x^4}{y^3}\right)}$

2. Simplify. Leave your answers in the numerator without the radical sign.

(a) $x^2 \times \sqrt{x}$ (b) $\sqrt{xy} \times \dfrac{2x}{y}$ (c) $\dfrac{9\left(\sqrt[3]{p}\right)^2}{3pq}$ (d) $\dfrac{4\sqrt{x^2y}}{y} \times \dfrac{y}{2\sqrt{x^3}}$

3. Simplify. Leave answers with positive indices.

(a) $\dfrac{4(1+x)^{\frac{1}{2}}}{6(1+x)}$ *(b) $\dfrac{(1-x^2)^{\frac{1}{2}}}{(1+x)(1-x)}$

4. Evaluate each of the following.

(a) $81^{\frac{1}{4}}$ (b) $27^{\frac{-1}{3}}$ (c) $\dfrac{1}{16^{\frac{1}{4}}}$ (d) $\left(\dfrac{125}{8}\right)^{\frac{1}{3}}$

(e) $8^{\frac{5}{3}}$ (f) $\dfrac{1}{27^{\frac{4}{3}}}$ (g) $\dfrac{1}{125^{-\frac{2}{3}}}$ (h) $\left(\dfrac{25}{16}\right)^{-\frac{3}{2}}$

5. Simplify. Leave your answers with the variables with positive indices.

(a) $\left(4x^3\right)^{\frac{3}{2}}$ (b) $\left(\dfrac{x^5}{9}\right)^{-\frac{3}{2}}$ (c) $\left(\dfrac{x^2}{16y^4}\right)^{\frac{1}{4}}$ (d) $\left(\dfrac{121x^2}{y^4}\right)^{-\frac{1}{2}}$

16.4 Review: Factorisation

Example 16.7

Without using a calculator, factorise each of the following:

(a) $8x^7 + 6x^5$ (b) $3(1 + x)^4 + 6(1 + x)^3$ (c) $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

Solution:

(a) $8x^7 + 6x^5 = 2x^5(4x^2 + 3)$

(b) $3(1 + x)^4 + 6(1 + x)^3 = 3(1 + x)^3[(1 + x) + 2] = 3(1 + x)^3(3 + x)$

(c) $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = x^{-\frac{1}{2}}(x+1)$

Example 16.8

Without using a calculator, factorise the numerator and/or the denominator and simplify each of the following. Leave answers with positive indices.

(a) $\dfrac{x^5+x^4}{x^3+2x^2}$ (b) $\dfrac{(1+x)(2+x)^2+(1+x)^2(2+x)}{(1+x)^2}$

Solution:

(a) $\dfrac{x^5+x^4}{x^3+2x^2} = \dfrac{x^4(x+1)}{x^2(x+2)} = \dfrac{x^2(x+1)}{(x+2)}$

(b) $\dfrac{(1+x)(2+x)^2+(1+x)^2(2+x)}{(1+x)^2} = \dfrac{(1+x)(2+x)[(2+x)+(1+x)]}{(1+x)^2}$

$= \dfrac{(2+x)(3+2x)}{(1+x)}$.

Exercise 16.4 Calculator Free

1. Factorise completely:

(a) $x^4 + x^2$ (b) $3x^2 + 4x^3$ (c) $x^2 - x^4$ (d) $x^{\frac{1}{2}} + x$

(e) $x^2y + xy^2$ (f) $3x^2y^3 + 6x^2y$ *(g) $4xy^{1/2} + 6x^{1/2}y$ *(h) $x^{-1/2}y + xy^{-1/2}$

2. Factorise completely:

(a) $x^{\frac{1}{2}} - x^{-\frac{1}{2}}$ (b) $(1 + x)^2 + (1 + x)^3$

(c) $8(1 - x)^2 + 12(1 - x)^3$ (d) $(1 + x)^2 + (1 + x)^{1/2}$

(e) $x^2(1 + x) + x(1 + x)^2$ (f) $3(x - 2)^2(x + 3)^3 + 6(x - 2)^2(x + 3)^2$

(g) $4(2 - x)(5 - x)^{1/2} + 6(2 - x)^{1/2}(5 - x)$ (h) $(1 + x)(1 - x)^{-1/2} + (1 - x)(1 + x)^{-1/2}$

3. Simplify. Leave all terms with positive indices.

(a) $\dfrac{x^5+x^7}{x^5}$ (b) $\dfrac{x^3}{x^3+x^2}$ (c) $\dfrac{x^3+x^2}{x^5+2x^4}$ (d) $\dfrac{2x^7+4x^5}{6x^5+8x^3}$

(e) $\dfrac{x^2(1-x^2)}{x^3(1+x)}$ (f) $\dfrac{x(9-x^2)}{2x(3-x)}$ (g) $\dfrac{x^{-1}+x}{x^2}$ (h) $\dfrac{x^{-\frac{1}{2}}+x^{\frac{1}{2}}}{x}$

*4. Simplify. Leave all terms with positive indices.

(a) $\dfrac{(1+x)^2+(1+x)^3}{(1+x)}$ (b) $\dfrac{(2+x)^3-(2+x)^2}{(2+x)^3}$

(c) $\dfrac{(1-2x)^2+4(1+x)(1-2x)}{(1-2x)^4}$ (d) $\dfrac{2(1-x)^{\frac{1}{2}}+(1-x)^{-\frac{1}{2}}(1+x)}{(1-x)}$

16.5 Review: Scientific Notation

.
- In expressing a number N in scientific/standard notation:

  $$N = m \times 10^n \quad \text{where } 1 \le m < 10.$$

  m is known as the mantissa.

- Very large and very small numbers are more efficiently expressed using scientific or standard notation. For example:
  - $0.000\,005\,286\,521 \equiv 5.\,286\,521 \times 10^{-6}$
  - $6\,235\,583\,011 \equiv 6.235\,583\,011 \times 10^{9}$

Example 16.9

Express each of the following numbers accurate to 3 significant figures in (i) decimal form (ii) scientific form. (a) 24.56^5 (b) 0.091^4

Solution:

(a) $24.56^5 = 8\,935\,972.269 \approx 8\,940\,000$ (3 SF)

$\approx 8.94 \times 10^6$ (scientific form 3SF)

(b) $0.091^4 = 0.000\,068\,574\,96 \approx 0.000\,068\,6$ (3 SF)

$\approx 6.86 \times 10^{-5}$ (scientific form 3SF)

Example 16.10

Without the use of a calculator evaluate each of the following, giving answers in scientific form accurate to 2 significant figures:

(a) $\dfrac{\left(1.60\times10^{-5}\right)\times\left(2.50\times10^{10}\right)}{8.00\times10^{16}}$ (b) $5.12 \times 10^4 + 6.39 \times 10^5$

Solution:

(a) $$\frac{\left(1.60\times10^{-5}\right)\times\left(2.50\times10^{10}\right)}{8.00\times10^{16}} = \frac{1.60\times2.50}{8.00}\times10^{-11}$$

$$= 0.2\times2.50\times10^{-11}$$

$$= 0.5\times10^{-11} = 5.0\times10^{-12} \quad \text{(2 SF)}$$

(b) $5.12 \times 10^4 + 6.39 \times 10^5 = 5.12 \times 10^4 + 63.9 \times 10^4$

$= 69.02 \times 10^4 \approx 6.9\times10^5$ (2 SF)

Exercise 16.5

1. Express each of the following numbers accurate to 3 significant figures in (i) decimal form and (ii) scientific form. (a) 3.1415^5 (b) 2.7218^{-2} (c) $\sqrt{180.5789}$ (d) 2^{3^3}

2. Without the use of a calculator evaluate each of the following, giving answers in scientific form accurate to 2 significant figures (SF):

   (a) $(2.5\times10^{-4})^2$ (b) $\dfrac{1}{3\times10^{15}}$ (c) $\dfrac{(5.00\times10^4)\times(1.44\times10^{20})}{6.00\times10^{12}}$

   (d) $3.75\times10^8+4.62\times10^7$ (e) $6.65\times10^{-5}-8.93\times10^{-6}$

3. A grain of rice weighs between 20 mg and 30 mg. Find in scientific form correct to 4 SF, the number of rice grains in 1 kg of rice.

4. The rest mass of an electron is approximately $9.109\,382\,2\times10^{-31}$ kg.
   (a) How many electrons (5 SF) are there in 1 kg of electrons?
   (b) The earth's mass is approximately $5.973\,6\times10^{24}$ kg. What number of electrons (4 SF) would be required to match the mass of the earth?

5. The speed of light in vacuum is exactly 299 792 458 ms^{-1}.
   (a) Express the speed of light in kmh^{-1} in scientific form correct to 3 SF.
   (b) Find the time taken by light to travel 1 km in vacuum, correct to 3 SF.
   (c) Find the distance travelled by light in one year (365 days) in vacuum. Give your answer in scientific form correct to 5 SF.

6. A mole of helium gas weighs 4.003 grams and has $6.022\,141\,5\times10^{23}$ helium atoms.
   (a) Find in scientific form, correct to 3 SF, the weight of one helium atom.
   (b) Find in scientific form, correct to 4 SF, the number of helium atoms in 1 kg of Helium gas.

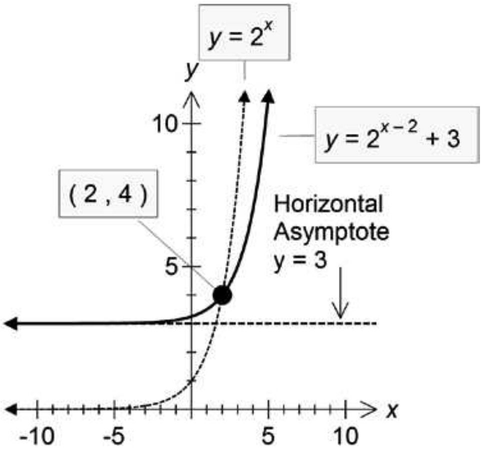
7. One nanometre is defined as equal to 1×10^{-9} metres and equal to 10 angstroms.
   (a) The thickness of a cell membrane is 6.2×10^{-7} metres. The diameter of a carbon nanotube measures 1 nanometre. How many carbon nanotubes placed side by side will be required to cover the width of this membrane?
   (b) A rectangular integrated chip measures 60 angstroms by 40 angstroms. Find the area of this plate in square metres.

8. The energy-mass equivalence principle states that the energy E joules possessed by a body of mass m kg is given by $E = mc^2$ where c = 299 792 458 ms^{-1}.
   (a) Find the energy equivalent of Ja'mie who weighs 60 kg correct to 3 SF.
   (b) The energy possessed by one gram of matter is equivalent to the energy released by the combustion of approximately 2 150 114 Litres of petrol. If petrol cost $1.50 per Litre, what would be the cost put on Ja'mie in terms of her energy equivalence? Give your answer in terms of billions of dollars, correct to 4 SF.

17 Exponential Functions

17.1 Exponential Functions

- Exponential functions were introduced briefly in Section 5.4.2.
- Exponential functions have equations of the form:
  $y = 2^x$, $y = 3^{x-1}$, $y = 4^x + 3$, $y = -5 \times 3^x$, $y = 3^{0.05x}$, …
  The independent variable appears as an *exponent* or *index* or *power*.
- $y = (-1.1)^x$ is not an exponential function despite the fact that the independent variable x is an exponent.
- Exponential functions take the basic form $y = a^x$, where $a > 0$.
  - The graph of the function has a y-intercept at (0, 1) and a horizontal asymptote with equation $y = 0$.
  - As discussed in Section 5.4.2,
    - the domain of the function is $\mathbb{R}$
    - and the range is $\{y: y > 0, y \in \mathbb{R}\}$.
- Consider the exponential function $y = a^{x-c} + b$.
  - Using the concepts introduced in Section 5.6, the graph of this function is obtained from the graph of $y = a^x$ by:
    - a horizontal translation of c units to the right
    - and a vertical translation of b units upwards.
  - Hence:
    - the horizontal asymptote now has equation $y = b$
    - the range of the function is now $\{y: y > b, y \in \mathbb{R}\}$
    - the domain of the function remains $\mathbb{R}$.
- The accompanying diagram shows the graph of $y = 2^{x-2} + 3$.
  - The graph of $y = 2^x$ has been translated two units to the right and three units upwards.
  - Hence, the y-intercept of $y = 2^x$, which is the point (0, 1) has been shifted to (2, 4) on the graph of $y = 2^{x-2} + 3$.
  - The horizontal asymptote of $y = 2^{x-2} + 3$ is obtained by shifting the horizontal asymptote of $y = 2^x$, 3 units upwards. Hence, $y = 3$.



Hands On Task 17.1

In this task, we will explore features of the graphs of $y = a^{kx}$ for varying values of $a > 0$ and k.

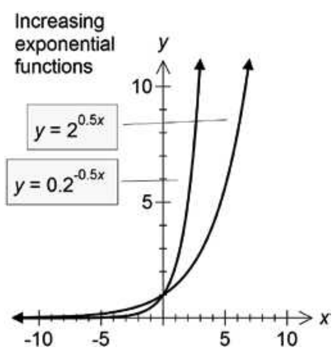
1. Graph on your CAS/graphic calculator the set of curves with equations, $y = 2^{kx}$, for:
   (a) $k = 1$ (b) $k = 0.05$ (c) $k = -1$ (d) $k = -0.05$.
   For what values of k does y increase as x increases (y is an increasing function)?

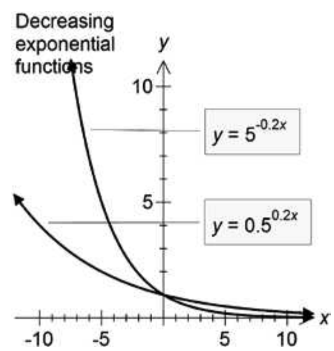
2. Graph on your CAS/graphic calculator the set of curves with equations, $y = a^{x}$, for:
   (a) $a = 2$ (b) $a = 3$ (c) $a = 0.8$ (d) $a = 0.2$.
   For what values of a does y decrease as x increases (y is a decreasing function)?

3. Graph on your CAS/graphic calculator, the set of curves with equations, $y = a^{kx}$, for:
   (a) a value of $a > 1$ and a value of $k > 0$ (b) a value of $a > 1$ and a value of $k < 0$
   (c) a value of $0 < a < 1$ and a value of $k > 0$ (c) a value of $0 < a < 1$ and a value of $k < 0$.
   Summarize your observations using the terms "increasing function" and "decreasing function".

Summary

- Consider the exponential curve $y = a^{kx}$, where $a > 0$.



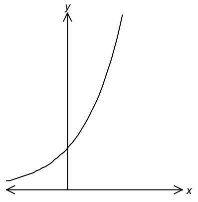


- $y = a^{kx}$, is an *increasing function* if:
  - $a > 1$ and $k > 0$
  - $0 < a < 1$ and $k < 0$
- $y = a^{kx}$, is a *decreasing function* if:
  - $a > 1$ and $k < 0$
  - $0 < a < 1$ and $k > 0$.

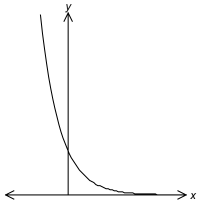
Exercise 17.1

1. Describe each of the following as either an increasing or a decreasing function.
   (a) $y = 10^{x}$ (b) $y = 10^{-x}$ (c) $y = 0.1^{x}$ (d) $y = 0.1^{-x}$

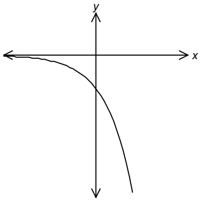
2. Match each of the following graphs with an equation from the given list.



Graph A



Graph B



Graph C

Equation I : $y = x^{2}$ Equation II : $y = -5^{x}$ Equation III : $y = 0.5^{-x}$
Equation IV : $y = -0.4^{x}$ Equation V : $y = 5^{-x}$ Equation VI : $y = -5^{-x}$

3. Sketch each of the following. Indicate at least two obvious points.
   (a) $y = 3^{x}$ (b) $y = 2^{-x}$ (c) $y = 0.25^{x}$
   (d) $y = 0.5^{-x}$ (e) $y = 100(2^{x})$ (f) $y = 100(5^{-x})$

4. Rewrite each of the following in the form $y = a^{x}$.
   (a) $y = 2^{2x}$ (b) $y = 3^{-2x}$ (c) $y = 4^{0.01x}$ (d) $y = 1.05^{2x}$

5. Given that $y = 0.3^{x}$ can be written as $y = a^{-x}$, find the value of a.

6. Given that $y = 1.25^{-x}$ can be written as $y = a^{x}$, find the value of a.

7. Given that $y = 5^{x+2}$ can be written as $y = k(5^{x})$, find the value of k.

8. Given that $y = 3^{x-1}$ can be written as $y = k(3)^{x}$, find the value of k.

17.2 Exponential Expressions and Equations

Example 17.1

Simplify, leaving your answer in the numerator: (a) $\dfrac{2^{x+1}}{2^{2x}}$ (b) $\dfrac{2^{x}+2^{x+1}}{3}$.

Solution:

(a) $\dfrac{2^{x+1}}{2^{2x}} = 2^{x+1-2x} = 2^{-x+1}$

(b) $\dfrac{2^{x}+2^{x+1}}{3} = \dfrac{2^{x}+2^{x}(2^{1})}{3}$

$= \dfrac{2^{x}(1+2)}{3} = 2^{x}$

Example 17.2

Without the use of calculator, solve: (a) $3^x = \dfrac{1}{81}$ (b) $(20)2^x = 5(2^{2x+1})$

Solution:

(a) $3^x = \dfrac{1}{81}$

$3^x = 3^{-4}$

$\Rightarrow\ x = -4$

(b) $(20)2^x = 5(2^{2x+1})$

$\dfrac{2^{2x+1}}{2^x} = \dfrac{20}{5}$

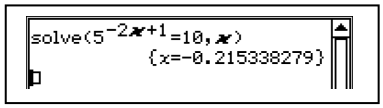
$2^{2x+1-x} = 2^2 \quad \Rightarrow\ x = 1.$

Example 17.3

Solve $5^{-2x+1} = 10$ giving your answer in scientific form correct to 4 significant figures.

Solution:

From CAS calculator, $x \approx -0.215338 \approx -2.153 \times 10^{-1}$



Exercise 17.2

1. Without the use of a calculator, simplify each of the following.

(a) $\dfrac{2^x}{2^2}$ (b) $\dfrac{3^x}{3^{-2}}$ (c) $\dfrac{2^y}{2^{y+1}}$ (d) $\dfrac{(5^{n+1})^2}{5^{1-n}}$

(e) $\dfrac{4^x}{2^{1+x}}$ (f) $\dfrac{3^{2x-1}}{9^x}$ (g) $\dfrac{25^{2t}}{125^t}$ (h) $\dfrac{8^{1-t}}{32^t}$

2. Without the use of a calculator, factorize and simplify each of the following.

(a) $2^{x+1} - 2^x$ (b) $3^{x+2} + 3^x$ (c) $5^x - 5^{x-1}$ (d) $(\tfrac{1}{2})^t + (\tfrac{1}{2})^{t-1}$

(e) $\dfrac{2^{x+2} + 2^x}{5}$ (f) $\dfrac{3^{x+3} - 3^{x+1}}{2^2}$ (g) $\dfrac{2^t - 2^{t-1}}{2}$ (h) $\dfrac{5^{n+1} + 5^n}{5^{n+2} + 5^{n+1}}$

3. Without the use of a calculator, solve each of the following:

(a) $2^x = 8$ (b) $2^x = 1/32$ (c) $3^{-x} = 1/27$ (d) $1/5^x = 125$

(e) $2^{2x} = 32$ (f) $3^{2x+1} = 81$ (g) $4^x = 8$ (h) $27^x = 1/81$

(i) $\dfrac{2^{x+1}}{2^{2x}} = 16$ (j) $\dfrac{3^{2x}}{3^{1-x}} = \dfrac{1}{243}$ (k) $\dfrac{4^x}{2^{x+1}} = 256$ (l) $\dfrac{121^{x+1}}{11^{3x}} = 11$

(m) $\dfrac{2^x}{2} = 2^{1-x}$ (n) $\dfrac{3^{x+1}}{3} = 3^{1-2x}$ (o) $2(5^{x+1}) = 50(5^{-x})$ (p) $6(2^{1+2x}) = 48(4^{1-x})$

4. By substituting $y = 2^x$, solve: (a) $2^{2x} - 8(2^x) + 16 = 0$ (b) $2^{2x} - 3(2^x) - 4 = 0$

5. Solve each of the following. Give your answer in correct to 4 SF.

(a) $3^{0.5x} = 5$ (b) $0.7^x = 0.1$ (c) $100 \times 2^{-0.5x} = 60$ (d) $5^x = 3^{2x+1}$

Example 17.4

The number of bacteria (N *hundreds*) in a laboratory culture is related to time t (hours) by the formula $N = 20 \times 1.15^t$.

(a) How many bacteria were there at: (i) the start (ii) after 24 hours?

(b) How long (nearest minute) will it take for the number of bacteria to exceed 10 000?

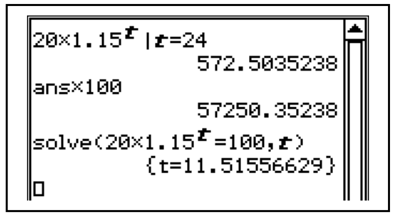
Solution:

(a) (i) When $t = 0$, $N = 20$. Hence, 2 000 bacteria.

(ii) When $t = 24$, $N = 20 \times 1.15^{24} \approx 572.5$

Hence, 57 250 bacteria.

(b) $N = 100$, $t = 11.5156$ hours $\approx$ 11 hours 31 minutes



Example 17.5

The mass (M g) of a radioactive substance at time t days is given by $M = 200 \times 0.82^t$.

(a) What mass of this substance has decayed after 10 days?

(b) How long (nearest hour) will it take for half the mass to decay?

Solution:

(a) When $t = 10$, $M = 27.4896$.

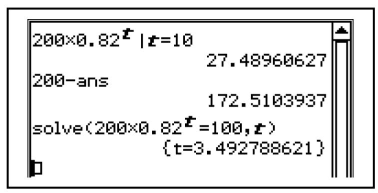
Hence, the mass that has decayed $= 200 - 27.4896$

≈ 172.5 g.

(b) When half the mass has decayed, $M = 100$ g.

$\Rightarrow \quad 200 \times 0.82^t = 100$

$t \approx 3.4928 \approx$ 3 days 12 hours.



Exercise 17.3

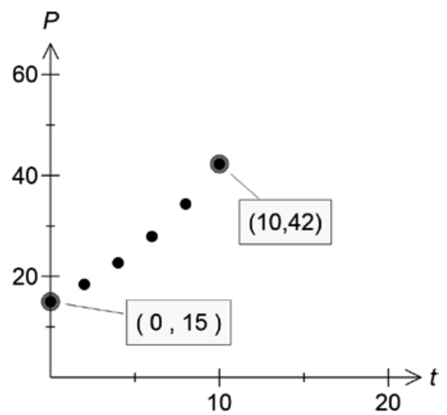
1. The number of bacteria in a laboratory culture is related to time t (hours) by the formula $N = 100 \times 1.23^t$.
   (a) How many bacteria were there at: (i) the start (ii) after 24 hours?
   (b) What is the *doubling time* (to the nearest minute) of the bacteria?
   [The time it takes for the bacteria to double in number.]

2. The number of organisms (N thousands) in a pond is related to time t (hours) by the formula $N = 3.56 \times 1.31^t$.
   (a) What was the increase in the number of organisms in the first 24 hours?
   (b) What is the average rate of increase in the first 24 hours?
   (c) The pond is no longer safe for humans when the number of organisms exceed a hundred thousand. Find to the nearest minute when this occurs.

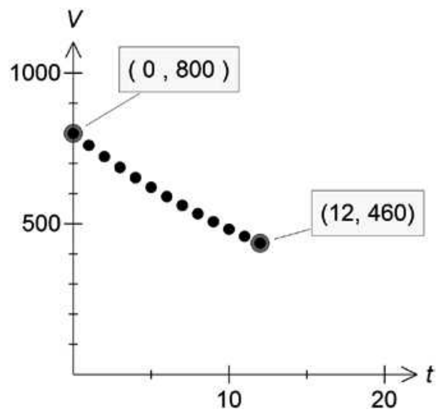
3. The mass (M mg) of a radioactive substance at time t hours is given by $M = 800 \times 0.95^{t}$.
   (a) What is the *half-life* of this substance?
   [The time taken for half the mass of the substance to decay.]
   (b) What mass of this substance has decayed:
   (i) in the first 24 hours, (ii) in the second day (iii) in the third day.
   (c) Comment on your answers in (b).

4. The mass (M g) of a radioactive substance at time t years is given by $M = 450 \times 2^{-0.05t}$.
   (a) What mass of this substance is left after 100 years?
   (b) What is the average rate of decay in the first 100 years?
   (c) What is the *half-life* of this substance?
   [The time taken for half the mass of the substance to decay.]

5. The accompanying diagram shows the plot of the population P of a colony of feral pigs in a forest reserve, every two months for a period of ten months. It is assumed that the population P follows the following model: $P = A \times 10^{kt}$, where t is the number of months into the study.
   (a) Use the information provided in the graph to find A and k.
   (b) Use your answers in (a) to estimate the feral pig population (nearest 10 pigs) after 20 months.



6. The accompanying diagram shows the plot of the number of viewers (V thousands) of a prime time news programme on a commercial television station each week for a period of 12 weeks. Assume that V follows the model $V = A \times 10^{kt}$, where t is the number of weeks into the survey.
   (a) Use the information provided in the graph to find A and k.
   (b) Use your answers in (a) to estimate when the number of viewers first falls below 400 000.



7. The number of dolphins, D, in a river is modelled by the equation $D = 100 \times 10^{0.08t}$, where t is the number of years after the start of 2010.
   (a) Estimate the number of dolphins (to the nearest 10) at the end of 2020.
   (b) When, to the nearest year, will the number of dolphins first exceed 500?
   (c) A toxic spill occurred at the start of 2015. The population of dolphins is now modelled by the equation $P = A \times 10^{-0.1t}$, where A is a constant and t is the number of years after the start of 2015.
   (i) Find A. (ii) Find the dolphin population at the end of 2020.

8. The number of quokkas (N) suffering from a virus infection is modelled by $N = 8500 \times 10^{-0.075t}$, where t is the number of years after the start of 2000.
   (a) Find the number of quokkas at the end of 2010.
   (b) A cure for the infection is discovered and a vaccine created and administered to as many of the surviving quokkas as was possible at the start of 2011. The number of quokkas is now modelled by the equation $P = A \times 10^{0.055t}$, where A is a constant and t is the number of years after the start of 2011.
   (i) Find A. (ii) Find the quokka population at the end of 2020.

9. Mammals A and B live in the same habitat. The number of mammals A and B, t years after the start of 2010 are modelled by the equations $N_A = 850 \times 10^{0.06t}$ and $N_B = 250 \times 10^{0.09t}$ respectively.
   (a) Which of these two mammals has a larger population at the start of 2010?
   (b) Which of these two mammals has a faster growing rate? Why?
   (c) Find to the nearest year, when the population of the mammal with a lower population at the start of 2010 first exceeds the population of the other mammal.

10. Organisms A and B live in the same habitat. The number of organisms A and B, t years after the start of 2012 is modelled by the equations $N_A = 5\,000 \times 10^{-0.09t}$ and $N_B = 2\,000 \times 10^{0.08t}$ respectively.
   (a) Which of these two organisms has a growing population? Why?
   (b) Find the average rate of growth or decline for organisms A and B.
   (c) When (to the nearest year) will the population of the two organisms be the same?

18 Sequences & Series

18.1 Recursive Rules

- Consider the sequence:

  3, 5, 7, 9, 11, …. .

  - The terms of this sequence may be referenced as:

    T_1, T_2, T_3, T_4, T_5, …… in subscript notation

    or T(1), T(2), T(3), T(4), T(5), …… in function notation.

  - In this instance, the first term of the sequence is denoted T_1 or T(1).
  - However, in many instances, it may be more appropriate to denote the first of the sequence as T_0 or T(0). In this case, the terms of the sequence are referenced as:

    T_0, T_1, T_2, T_3, T_4, …… in subscript notation

    or T(0), T(1), T(2), T(3), T(4), …… in function notation.

- A recursive rule allows the terms of any sequence to be generated.
  It consists of two parts:
  - a set of instructions detailing what needs to be done to obtain the next term of a sequence, given the current term(s).
  - the initial term(s) of the sequence.

- For example: $T_{n+1} = 2(T_n) + 1,\ T_1 = 5$
  - The recursive rule indicates that:

    Next Term = Current Term × 2 then Add 1, First Term = 5
  - Hence, the first five terms of this sequence are: 5, 11, 23, 47, 95, ….

- Consider the sequence:

  3, 5, 7, 9, 11, ….
  - If the first term of the sequence is denoted T_1 or T(1),
    then the recursive rule in subscript notation may be written as:

    $T_{n+1} = T_n + 2,\ T_1 = 3$ where n = 1, 2,3, … .
  - If the first term of the sequence is denoted T_0 or T(0),
    then the recursive rule in subscript notation may be written as:
    - $T_{n+1} = T_n + 2,\ T_0 = 3$ where n = 0, 1, 2, 3, …. .
    - $T_n = T_{n-1} + 2,\ T_0 = 3$ where n = 1, 2, 3, …. .

18.2 Arithmetic Sequences

- Consider the following sequences:

  - 1   3   5   7   9   11

    +2   +2   +2   +2   +2

    To get to the next term, add 2 to the current term.

  - 100   95   90   85   80   75

    − 5   − 5   − 5   − 5   − 5

    To get to the next term, subtract 5 from the current term.

- Each of the sequences listed above have the property:

  Next Term = Current Term + Constant   I

  or   Difference between two consecutive terms = Constant

- In general, these sequences follow the pattern:

  $a \qquad a + r \qquad a + 2r \qquad a + 3r \qquad a + 4r$ II

  - Sequences with the above pattern are called *arithmetic sequences*.
  - From I, the recursive rule for an arithmetic sequence is:

    $T_{n+1} = T_n + d\ ,\ \ T_1 = a$ where n = 1, 2,3, … .

    or $T_n = T_{n-1} + d,\ T_0 = a$ where n = 1, 1, 2,3, … .   .
  - From the pattern found in II, the nth term is given by:

    $T_n = a + (n-1)d$ where n = 1, 2,3, … .

    or $T_n = (a-d) + nd$ where n = 1, 2,3, … .

    This is also known as the explicit rule of the sequence.
  - The constant d is called the *common difference* of the sequence.

Example 18.1

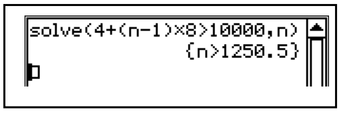
The first 3 terms of an arithmetic sequence are 4, 12 and 20 respectively. Find:
(a) the recursive rule of the sequence   (b) the 10th term of the sequence
(c) the term that first exceeds 10 000.

Solution:

(a) Common difference $d = 12 - 4 = 8$.
Hence, recursive rule is: $T_{n+1} = T_n + 8\ ,\ T_1 = 4$ or $T_n = T_{n-1} + 8\ ,\ T_0 = 4$.

(b) The 10th term of the sequence, $T_{10} = 4 + 9 \times 8 = 76$

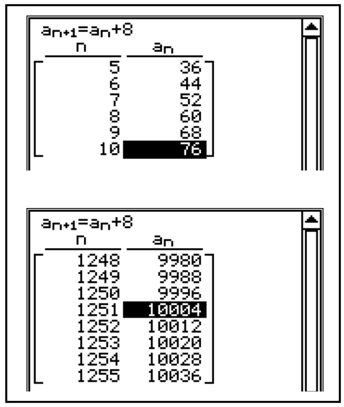
(c) $4 + 8(n-1) > 10\,000 \ \Rightarrow\ n = 1251$.
That is, the one thousand two hundred and fifty first term.



Note:

- *Clearly, part (c) is more efficiently done using the nth term formula.*

Alternative Solution using a CAS "Sequence Aplet":



(a) Common difference $d = 12 - 4 = 8$.
Hence, recursive rule is: $T_{n+1} = T_n + 8$, $T_1 = 4$.

(b) From Table generated:
The 10th term of the sequence, $T_{10} = 76$.

(c) From Table generated:
$T_n > 10\,000 \;\Rightarrow\; n = 1251$. That is, T_{1251} .

Example 18.2

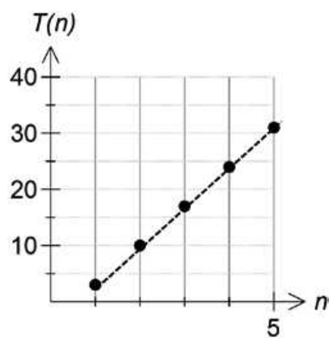
The third term and seventh term of an arithmetic sequence are respectively 17 and 45.
(a) Find the recursive rule and the general term of the sequence.
(b) Plot the first five terms of the sequence on a set of axes and comment on the plot.

Solution:

(a) Recognizing the main property of an arithmetic sequence:
Term 7 = Term 3 + (4 × common difference)
Hence, $17 + 4d = 45 \quad \Rightarrow \quad d = 7$
The first term = Term 3 − (2 × common difference)
= $17 - 2 \times 7 = 3$
Therefore, recursive rule is $T_{n+1} = T_n + 7$, $T_1 = 3$ or $T_n = T_{n-1} + 7$, $T_0 = 3$.
General Term: $T_n = 3 + (n-1) \times 7 \quad \Rightarrow \quad T_n = 7n - 4$.

(b) The first five terms are: 3, 10, 17, 24 and 31.
The plot is shown in the accompanying diagram.

The points representing the terms of the sequence are *collinear* , that is, they lie on the same line.



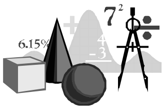
Notes:

- *In part (a), Term 7 = Term 3 + 4 × common difference. This is best illustrated as*

Term 3	*Term 4*	*Term 5*	*Term 6*	*Term 7*
17	*17 + d*	*17 + 2d*	*17 + 3d*	*17 + 4d = 45*

- *In general,* $T_m = T_n$ ***+ (m*** −
- *The general term is* $T_n = 7n - 4$. *Clearly the relationship between n and* T_n *is linear.*

18.2.1 Sum of an arithmetic sequence



Hands On Task 18.1

In this task, we will develop a formula for the sum of the first *n* terms of an arithmetic sequence.

1. Consider the arithmetic sequence:

   3 5 7 9 11 13 15 17

   - Pair the terms of the sequence as shown below:

     3 5 7 9
17 15 13 11

   - Notice that the sum of each pair is the same.
   - The sum of terms of this sequence = No. of pairs × sum of each pair
= 4 × 20
= 80.

2. Use the method in (1) to find the sum of the following sequences:
   (a) 8 13 18 23 28 33 38 43 48 53
   (b) 100 97 94 91 88 85 82 79 76 73 70 67

3. Use the method in (1) to find:
   (a) the sum of the first twenty terms in the sequence $T_{n+1} = T_n + 6$, $T_1 = 2$
   (b) the sum of the first forty terms of the arithmetic sequence with first term 1 and common difference 10.

4. Consider the arithmetic sequence:

   3 5 7 9 11 13 15 17 19

   - Pair the terms of the sequence as shown below:

     3 5 7 9 19
17 15 13 11

   - Notice that there are four pairs with the same sum and one term on its own.
   - The sum of terms of this sequence = No. of pairs × sum of each pair + single term
= 4 × 20 + 19 = 99.

5. Use the method in (4) to find the sum of the following sequences:
   (a) 8 13 18 23 28 33 38 43 48 53 58
   (b) 100 97 94 91 88 85 82 79 76 73 70 67 64

6. Use the method in (4) to find:
   (a) the sum of the first twenty first terms in the sequence $T_{n+1} = T_n + 6$, $T_1 = 2$
   (b) the sum of the first forty first terms of the arithmetic sequence with first term 1 and common difference 10.

7. Consider the arithmetic sequence consisting of n terms, with first term a and common difference d.
   (a) Find in terms of a and d, the last term l of this sequence (i.e. the T_n).
   (b) Given that n is an even number:
      (i) find the number of pairs with the same sum
      (ii) use the method in (1) to show that the sum of the first n terms of this sequence is given by $S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[a+l]$.
   (c) Given that n is an odd number, the sequence may be reformed into $\frac{(n-1)}{2}$ pairs with the same sum and a single term.
      (i) Show that the sum of these pairs is $\frac{(n-1)}{2}[2a+(n-2)d]$.
      (ii) Hence, show that the sum of the first n terms of this sequence is given by $S_n = \frac{n}{2}[2a+(n-1)d]$.

Summary

- The sum of the first n terms of an arithmetic sequence with first term a and common difference d is given by $S_n = \frac{n}{2}[2a+(n-1)d]$.
- The sum of the first n terms of an arithmetic sequence with first term a and nth term l is given by $S_n = \frac{n}{2}[a+l]$.

Example 18.3

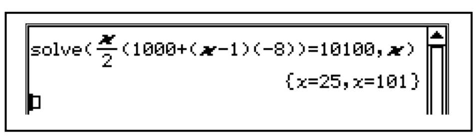
Consider an arithmetic sequence with first term 500 and common difference −8.
(a) Find the sum of the first 15 terms of the sequence.
(b) Find n if the sum of the first n terms is 10 100.

Solution:

(a) $S_{15} = \frac{15}{2}[2\times 500+(15-1)(-8)] = 6\,660$

(b) $\frac{n}{2}[2\times 500+(n-1)(-8)] = 10\,100$

$n = 25, 101$



Example 18.4

Evaluate the arithmetic series 6 + 10 + 14 + 18 + … + 122.

Solution:

Clearly first term a = 6 and common difference d = 4.

$T_n = 122 \quad \Rightarrow \quad 6 + 4(n-1) = 122$

$n = 30$

Hence, last term in series is the 30th term.

Therefore: $\quad S_{30} = \dfrac{30}{2}[6+122] = 1\,920.$

Notes:

- *When the terms of sequence of numbers are added together, the expression obtained is referred to as a series.*
- *6 10 14 18 …. 122 is an arithmetic sequence.*
  6 + 10 + 14 + 18 + … + 122 is an arithmetic series.
- *The term progression can refer to either a sequence or a series.*

Example 18.5

The sum of the first n terms of an arithmetic sequence is given by $S_n = 7n - n^2$.
Find the first term and common difference of this arithmetic progression.

Solution:

When $n = 1$: $\quad S_1 = 7 - 1 = 6 \quad \Rightarrow$ First term $T_1 = 6$.

When $n = 2$: $\quad S_2 = 14 - 4 = 10 \quad \Rightarrow \quad T_1 + T_2 = 10$.

But $T_1 = 6$. $\quad 6 + T_2 = 10 \quad \Rightarrow \quad T_2 = 4$

Therefore, common difference = $T_2 - T_1 = 4 - 6 = -2$.

Exercise 18.1

1. For each of the sequences given below, state the recursive rule and T_n in terms of n.
   (a) 5, 25, 45, 65, …. (b) 2.5, 2.8, 3.1, 3.4, ….
   (c) –10, –15, –20, –25, …. (d) 4, 1, –2, –5, ….

2. For each of the given recursive rules, state the first five terms of the sequence it represents and T_n in terms of n.
   (a) $T_{n+1} = T_n + 2.5$, $T_1 = 7.5$ (b) $T_{n+1} - T_n = 4$, $T_1 = 20$
   (c) $T_n = T_{n-1} - 11$, $T_0 = 33$ (d) $T_n = T_{n-1} - \dfrac{1}{3}$, $T_0 = 1$

3. The first 3 terms of an arithmetic sequence are 16, 21 and 26 respectively. Find:
   (a) the 15th term of the sequence (b) the term that first exceeds 5 000.

4. The first 3 terms of an arithmetic sequence are 1000, 991 and 982 respectively. Find:
   (a) the 50th term of the sequence (b) the number of terms that exceed 500.

5. The fifth term and tenth term of an arithmetic sequence are respectively 77 and 142. :
   (a) Find the recursive rule and the general term of the sequence.
   (b) Find the number of terms between 500 and 1000.

6. The sixth term and thirtieth term of an arithmetic sequence are respectively 480 and 384.
   (a) Find the recursive rule and the general term of the sequence.
   (b) How many positive terms are there in this sequence?

7. Consider an arithmetic series with first term 3.5 and common difference 1.2.
   (a) Find the sum of the first 10 terms of the series.
   (b) Find n if the sum of the first n terms is 200.

8. Consider an arithmetic series with first term 16 and common difference 7.
   (a) Find the sum of the second 10 terms of the series.
   (b) Find the least value of n for the sum of the first n terms to exceed 1 000.

9. Consider an arithmetic series with fourth term 31 and eighth term 67.
   (a) Find the sum of the first 8 terms of the series.
   (b) Find the sum of the 10 consecutive terms after the 8th term.
   (c) Find the least value of n for the sum of the first n terms to exceed 1 000.

10. An arithmetic series has first term 600 and common difference –30.
    (a) Find the sum of all the positive terms.
    (b) Find the least value of n for the sum of the first n terms to be zero.

11. Evaluate the following arithmetic series:
    (a) $3 + 9 + 15 + 21 + \ldots + 111$ (b) $-20 - 5 + 10 + 25 + \ldots + 400$
    (c) $60 + 58.2 + 56.4 + 54.6 + \ldots + 24$ (d) $15 + 9 + 3 - 3 + \ldots - 165$

12. Find the first term and common difference of an arithmetic sequence where the sum of the first n terms of an arithmetic sequence is given by:
    (a) $S_n = 4n + n^2$ (b) $S_n = 6n - n^2$.

13. The sum of the first n terms of an arithmetic sequence is given by $S_n = 8n + 2n^2$.
    (a) Find S_{20} and S_{21}. (b) Hence, or otherwise, find the twenty-first term.

14. The sum of the first n terms of an arithmetic sequence is given by $S_n = -13n + 3n^2$.
    (a) Find S_{10} S_{11} and S_{12}. (b) Hence, or otherwise, find the thirteenth term.

15. Find the recursive rule and general term of an arithmetic series where:
    (a) $S_5 = 250$ and $S_{20} = 1\,300$ (b) $S_5 = -110$ and $S_{15} = -705$.

18.3 Geometric Sequences

- Consider the following sequences:

  - 1 3 9 27 81 243

    × 3 × 3 × 3 × 3 × 3

    To get to the next term, multiply the current term by 3

  - 100 50 25 12.5 6.25 3.125

    × 0.5 × 0.5 × 0.5 × 0.5 × 0.5

    To get to the next term, multiply the current term by 0.5

- Each of the sequences listed above have the property:

  Next Term = Current Term × Constant I

  or Ratio between consecutive terms = Constant

- In general, these sequences follow the pattern:

  $a \quad ar \quad ar^2 \quad ar^3 \quad ar^4 \quad ar^5$ II

  - Sequences with the above pattern are called *geometric sequences*.
  - From I, the recursive rule for a geometric sequence is:

    $T_{n+1} = T_n \times r \ , \quad T_1 = a$

    or $T_n = T_{n-1} \times r \ , \ T_0 = a$
  - From the pattern found in II, the *n*th term or the explicit rule is given by

    $T_n = ar^{n-1}$ for integer $n \geq 1$

    or $T_n = ar^n$ for integer $n \geq 0$
  - The constant r is called the *common ratio* of the sequence.

Example 18.6

The first 3 terms of a geometric sequence are 4, 20 and 100 respectively. Find:

(a) the recursive rule of the sequence (b) the 10th term of the sequence

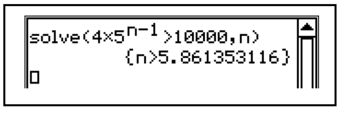
(c) the term that first exceeds 10 000.

Solution:

(a) Common ratio $r = \dfrac{20}{4} = 5$.

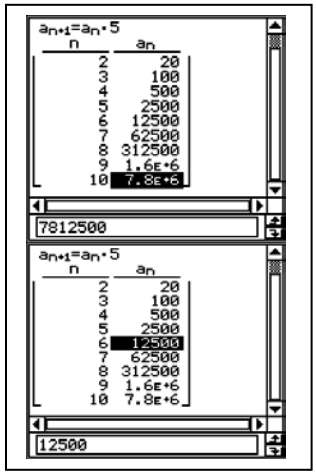
Hence, recursive rule is: $T_{n+1} = T_n \times 5 \ , \ T_1 = 4$ or $T_n = T_{n-1} \times 5 \ , \ T_0 = 4$.

(b) The 10th term of the sequence, $T_{10} = 4 \times 5^9 = 7\ 812\ 500$



(c) $4 \times 5^{n-1} > 10\ 000 \ \Rightarrow \ n = 6$. That is, the sixth term.

Alternative Solution using the CAS "Sequence Aplet":



(a) Common ratio $r = \dfrac{20}{4} = 5$.

Hence, recursive rule is: $T_{n+1} = T_n \times 5$, $T_1 = 4$

(b) The 10th term of the sequence, $T_{10} = 7\,812\,500$

(c) $T_n > 10\,000 \Rightarrow n = 6$. That is, the sixth term.

Example 18.7

The third and sixth terms of a geometric sequence are 18 and 486 respectively.
(a) Find the *n*th term of the sequence.
(b) Plot the first five terms of the sequence and comment on the nature of the plot.

Solution:

(a) Recognizing the main attribute of a geometric sequence:

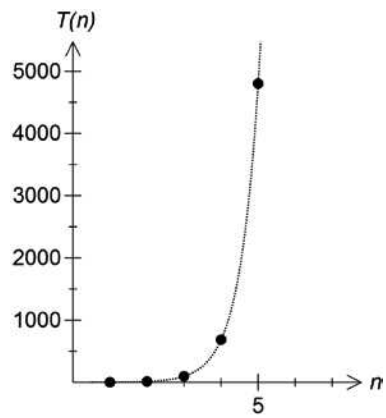
$$T_6 = T_3 \times r^3 \quad \Rightarrow \quad 486 = 18\,r^3$$
$$r = 3$$

Also, $T_3 = a \times 3^2 \quad \Rightarrow \quad 918 = 9a$
$$a = 2$$

Hence:

*n*th term $T_n = 2 \times 3^{n-1}$ for integer $n \geq 1$

or $T_n = 2 \times 3^n$ for integer $n \geq 0$.



(b) First 5 terms are: 2, 6, 18, 54, 162.
Clearly, the plot of T_n against n is exponential in shape.

Notes:

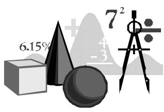
- *In part (a), Term 6 = Term 3 × (common ratio)^3. This is best illustrated as:*

Term 3	*Term 4*	*Term 5*	*Term 6*
18	*18 × r*	*18 × r × r*	*18 × r × r × r*

- *In general,* $T_m = T_n \times r^{m-}$ *for m > n.*
- *The general term is* $T_n = 2 \times 3^n$ *for integer n ≥ 0.*

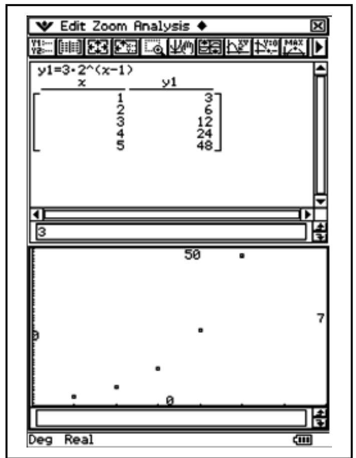
Clearly the relationship between n and T_n *is exponential.*

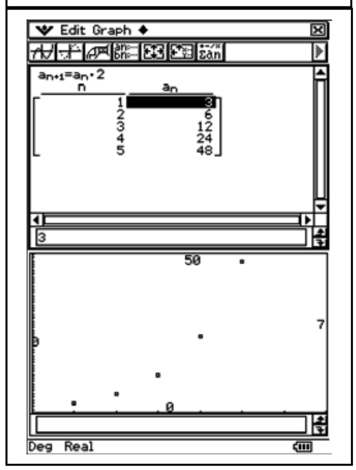
18.3.1 Types of Geometric Sequences



Hands On Task 18.2

In this task, we will explore the several geometric sequences and their *graphical plots*.





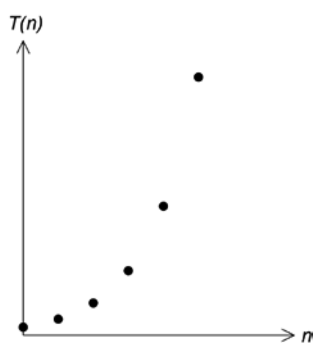
1. Consider the geometric sequence defined by:
$$T_{n+1} = T_n \times 2\ ,\ T_1 = 3 \ \text{ or } \ T_n = 3 \times 2^{n-1}\ .$$
   (a) Use an appropriate wizard/routine of your CAS calculator to generate the first five terms of the geometric sequence.
   (b) Use the point-plot routine to graph T_n against *n*.
   (c) Comment on the shape of the graph using the terms "increasing", "decreasing" and "oscillating".

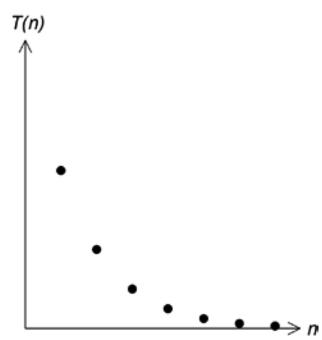
2. Repeat Question 1 for the following sequences:
   (a) $T_{n+1} = T_n \times 0.5\ ,\ T_1 = 3 \ \text{ or } \ T_n = 3 \times 0.5^{n-1}$
   (b) $T_{n+1} = T_n \times (-0.5)\ ,\ T_1 = 3 \ \text{ or } \ T_n = 3 \times (-0.5)^{n-1}$
   (c) $T_{n+1} = T_n \times (-2)\ ,\ T_1 = 3 \ \text{ or } \ T_n = 3 \times (-2)^{n-1}$

3. Summarize your observations.

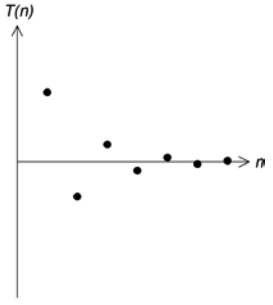
Summary

- For the geometric sequence represented by:
$$T_{n+1} = T_n \times r\ ,\ T_1 = a \quad \text{or} \quad T_n = a\,r^{n-1} \quad \text{for } a > 0 :$$
  - the sequence is a *growing sequence* if $r > 1$
  - the sequence is a *decaying sequence* if $0 < r < 1$

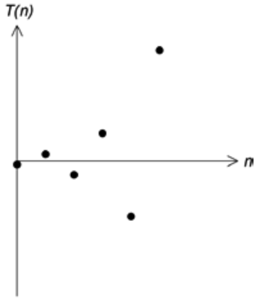




- the sequence is a *decaying oscillating sequence* if $-1 < r < 0$



- the sequence is an *increasing oscillating sequence* if $r < -1$



Exercise 18.2

1. For each of the sequences given below, state the recursive rule and T_n in terms of n.
   (a) 1, 4, 16, 64, … (b) 5, 10, 20, 40, … (c) −3, −6, −12, −24, ……
   (d) 4, 1, 1/4, 1/16, … (e) 4, −12, 36, −108… (f) 3, −1, 1/3, −1/9, 1/27, ……

2. For each of the given rules, state the first 5 terms of the sequence and T_n in terms of n.
   (a) $T_{n+1} = T_n \times 2.5$, $T_1 = 2$ (b) $T(n + 1) = 4\,T(n)$, $T(1) = 200$
   (c) $T_n = T_{n-1} \times -5$, $T_1 = 3$ (d) $T(n) = T(n - 1) \times \frac{1}{3}$, $T_1 = -2700$

3. For each of the given rules find the first 5 terms of the sequence and the recursive rule:
   (a) $T_n = -4\,(2.1)^{n-1}$ for $n = 1, 2, 3, ….$ (b) $T(n) = 5(-2)^{n-1}$ for $n = 1, 2, 3, ….$
   (c) $T_n = 3^n$ for $n = 1, 2, 3, ….$ (d) $T(n) = (1/4)^n$ for $n = 1, 2, 3, ….$

4. The 1st term and common ratio of a geometric sequence are 5 and 3 respectively. Find:
   (a) the 7th term (b) which term corresponds to 71 744 535
   (c) the term that first exceeds 1 000 000.

5. The first term and common ratio of a geometric sequence are 10 and 0.8 respectively.
   Find: (a) the 5th term (b) which term corresponds to 1.0737 (to 4 decimal places)
   (c) the first term that is less than 1.00.

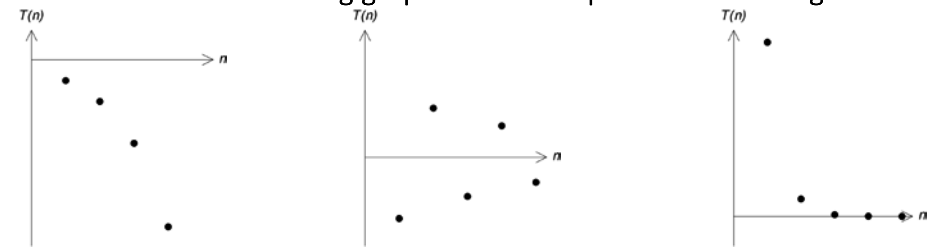
6. The 1st and 5th term of a geometric sequence are 9 and 2 304 respectively.
   Given that all the terms are positive, find:
   (a) the common ratio (b) the 10th term (c) the 10th term after the 5th term.

7. The 1st and 6th term of a geometric sequence are −16 and −1/2 respectively. Find:
   (a) the common ratio (b) the 10th term (c) the 10th term after −1/2.

8. The 5th and 8th term of a geometric sequence are 192 and 1 536 respectively. Find: (a) the common ratio (b) the 15th term (c) the 15th term after 1 536.

9. The 3rd and 5th terms of a geometric sequence are 252 and 9072 respectively. Find:
   (a) the 8th term (b) the 9th term after 252 (c) the 1st term that exceeds 5 000 000.

10. Match each of the following graphs with an equation from the given list.



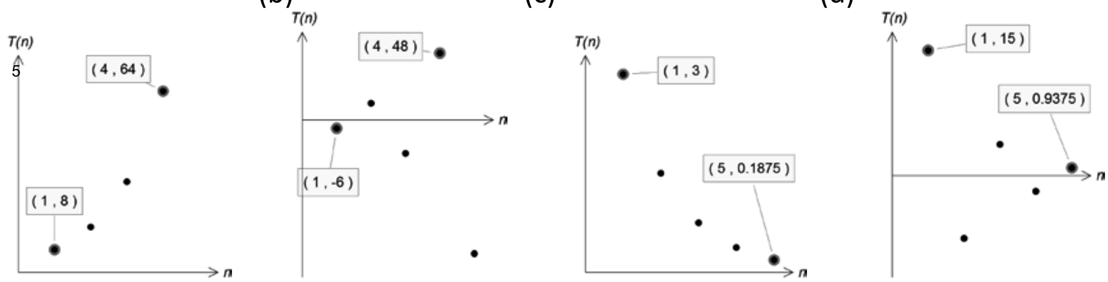
Graph A Graph B Graph C

Equation I : $T_n = -5(0.1)^n$ Equation II : $T(n) = 10(0.1)^n$

Equation III : $T(n) = 10(-0.8)^n$ Equation IV : $T_n = -3(2)^n$

11. The point plots of several geometric sequences are given below. Find the formula for the nth term of these sequences.

(a) (b) (c) (d)



18.3.2 Geometric Sum

- The sum of the first n terms of a geometric sequence with first term a and common ratio r is given by $S_n = \dfrac{a(1-r^n)}{1-r}$ where $r \neq 1$.

Hands On Task 18.3

In this task, you will be required to prove the formula $S_n = \dfrac{a(1-r^n)}{1-r}$ where $r \neq 1$.

Let $S_n = a + ar + ar^2 + ar^3 + \ldots.. + ar^{n-1}$, where $r \neq 1$.

1. Find an expression for $r \times S_n$.
2. Find an expression for $S_n - (r \times S_n)$ and hence an expression for $S_n(1-r)$.
3. Hence, show that $S_n = \dfrac{a(1-r^n)}{1-r}$ where $r \neq 1$.

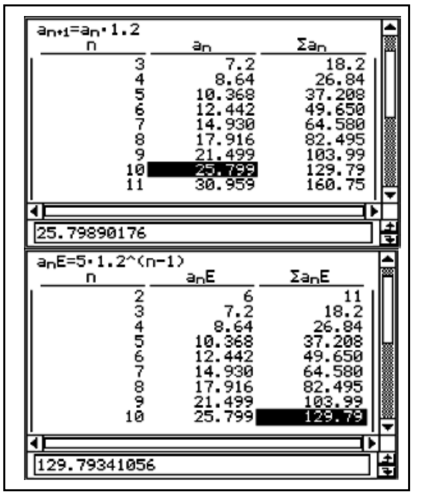
Example 18.8

The first term and common ratio of a geometric series are 5 and 1.2 respectively. Find:
(a) the 10th term (b) the sum of the first 10 terms.

Solution:

(a) $T_{10} = 5 \times 1.2^9 = 25.7989$ (4 D.P.)

(b) $S_{10} = \dfrac{5(1-1.2^{10})}{1-1.2} = 129.7934$ (4 D.P.)



Alternative Method using CAS "Sequence":

Use Recursive rule $T_{n+1} = T_n \times 1.2$, $T_1 = 5$

or Explicit Rule (General Term) is $T_n = 5 \times 1.2^{n-1}$

(a) Hence, $T_{10} = 25.7989$ (4 D.P.)

(b) $S_{10} = 129.7934$ (4 D.P.)

Example 18.9

The sum of the first *n* terms of a geometric series is given as $S_n = 5(1.5)^n - 5$. Find the recursive rule of this sequence.

Solution:

When $n = 1$, $S_1 = 5(1.5) - 5 = 2.5$ $\Rightarrow$ First term = 2.5.

When $n = 2$, , $S_2 = 5(1.5^2) - 5 = 6.25$

But $S_2 = T_1 + T_2$ $\Rightarrow$ $6.25 = 2.5 + T_2$

Hence $\Rightarrow$ $T_2 = 3.75$.

Therefore the common ratio $r = \dfrac{3.75}{2.5} = 1.5$.

Hence, recursive rule is $T_{n+1} = T_n \times 1.5$, $T_1 = 2.5$.

Example 18.10

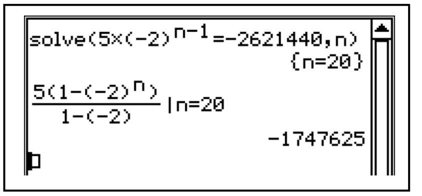
Find $5 - 10 + 20 - 40 + \ldots - 2\,621\,440$.

Solution:

General term $T_n = 5 \times (-2)^{n-1}$.

Consider $5 \times (-2)^{n-1} = -2\,621\,440 \Rightarrow n = 20$.

Hence: $S_{20} = \dfrac{5(1-(-2)^{20})}{1-(-2)} = -1\,747\,625$.



Exercise 18.3

1. The first term a and common ratio r of several geometric series are given below. Find in each case, T(n) and S(n) for the indicated value of $n \geq 1$.
   (a) $a = 1, r = 2.1, n = 10$ (b) $a = 4, r = 0.6, n = 12$
   (c) $a = 1, r = -3.2, n = 8$ (d) $a = 2, r = -0.4, n = 6$

2. For each of the given geometric sequences, find S_n for the indicated value of n.
   (a) 1, 1/2, 1/4, 1/8, … . $n = 50$ (b) 5, 10, 20, 40, … . $n = 20$
   (c) –1, –2, –4, –8, … . $n = 20$ (d) 3, –9, 27, –81, … . $n = 30$

3. For each of the given geometric sequences, find S_n for the indicated value of $n \geq 1$.
   (a) $T_n = 1.2(1.25)^{n-1}$, $n = 10$ (b) $T_n = 8000(0.75)^n$, $n = 100$
   (c) $T_n = 1000(-0.95)^{n-1}$, $n = 100$ (d) $T_n = -10(1.12)^n$, $n = 10$.

4. Two indicated terms of several geometric series are given below. In each case, find the sum of the first 10 terms.
   (a) T(4) = 19.4481, T(5) = 40.84101 (b) $T_3 = 20.25$, $T_5 = 0.164\,025$
   (c) $T_5 = 1\,024$, $T_{10} = 1\,048\,576$ (d) T(4) = –0.0256, T(8) = –0.000 655 36

5. The sum of the first n terms of several geometric series is given below. In each case, find the sum of the first 20 terms and the recursive rule.
   (a) $1 - 1.4^n$ (b) $1.5^{n+1} - 1.5$ (c) $0.25 - 0.5^{n+2}$ (d) $5 + (-0.2)^{n-1}$

*6. The first term of a geometric series is 3. The sum of the first 11 terms and the sum of the first 12 terms are 55.5936 and 64.1529 respectively. Find the common ratio and the sum of the first 5 terms of this sequence.

*7. The first term of a geometric series is 1 000. The sum of the first 10 terms and the sum of the first 11 terms are 9 561.792 5 and 10 466.174 6 respectively. Given that all the terms are positive, find the common ratio and the sum of the first 5 terms of this sequence.

8. The first term and common ratio of a geometric series are 500 and 2 respectively. Find the value of n so that the sum of the first n terms is exactly 32 767 500.

9. The first term and common ratio of a geometric series are 1 000 and 1.25 respectively. Find the smallest value of n such that the sum of the first n terms exceeds 67 000.

10. The first term and common ratio of a geometric series are 10 000 and 0.5 respectively. Find the largest value of n such that the sum of the first n terms is less than 19 999.995.

18.3.3 Sum to Infinity

- Consider S_n the sum of the first n terms of a geometric sequence with first term a and common ratio r. Rewrite $S_n = \dfrac{a(1-r^n)}{1-r}$ as $S_n = \dfrac{a}{1-r} - \dfrac{a \times r^n}{1-r}$.
  - Consider the case where $-1 < r < 1$.
    - As n becomes very large, because $-1 < r < 1$, the term $r^n \to 0$.
    - Hence, the second term in the expression for S_n disappears and $S_n \to \dfrac{a}{1-r}$.
    - $\dfrac{a}{1-r}$ is referred to as the sum to infinity (S_∞) of the geometric sequence.
    - Note that S_∞ exists only if the geometric sequence has $-1 < r < 1$.
- Hence, for a geometric sequence with first term a and common ratio $-1 < r < 1$,

$$S_\infty = \frac{a}{1-r}.$$

Example 18.11

Find the sum to infinity (if it exists) for the following:

(a) $5, -1, \frac{1}{5}, \frac{-1}{25}, \frac{1}{125}, \ldots$ (b) $5, -5, 5, -5, 5, \ldots$

Solution:

(a) The common ratio $r = -\frac{1}{5}$ and $-1 < -\frac{1}{5} < 1$. Hence S_∞ exists and $S_\infty = \dfrac{5}{1-\frac{-1}{5}} = \dfrac{25}{6}$.

(b) The common ratio $r = -1$. Hence, S_∞ does not exist.

Example 18.12

The sum of the first n terms for several geometric series is given below. In each case determine if the sum to infinity exists and find this sum where it exists.

(a) $S_n = 100(1 - 0.2^n)$ (b) $S_n = 100(2^n - 1)$

Solution:

(a) $S_n = 100(1 - 0.2^n) = 100 - 100 \times 0.2^n$.

As n becomes very large, 0.2^n tends to zero. That is, as $n \to \infty$, $0.2^n \to 0$.
Hence, as $n \to \infty$, $S_n \to 100 - 0$. Therefore, $S_\infty = 100$.

(b) $S_n = 100(2^n - 1) = 100 \times 2^n - 100$.

As n becomes very large, 2^n becomes very large and therefore S_n becomes very large. Therefore, as $n \to \infty$, $S_n \to \infty$ and S_∞ does not exist.

Exercise 18.4

1. Find the sum to infinity (if it exists) for the following sequences:
   (a) $-3, 3, -3, 3, -3, \ldots$ (b) $\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \ldots$
   (c) $10, \frac{10}{3}, \frac{10}{9}, \frac{10}{27}, \ldots$ (d) 50, 6.25, 0.781 25, 0.097 656 25

2. Find the sum to infinity (if it exists) for the following sequences:
   (a) $T(n) = 0.8^n$ (b) $T_n = 600(-0.2)^{n-1}$
   (c) $T(n + 1) = T(n) \times 1.1$, $T(1) = 2$ (d) $T(n) = T(n - 1) \times -1.1$, $T(1) = 0.4$

3. The sum of the first n terms of several geometric series is given below.
   In each case, find the sum to infinity, if it exists.
   (a) $S(n) = 50(1 - 0.1^n)$ (b) $S(n) = 100(1 + 0.1^n)$
   (c) $S_n = 1\,000(1 - 1.1^n)$ (d) $S_n = 50(1 - (-1)^n)$

4. Find the common ratio of the geometric series with first term a and sum to infinity S_∞.
   (a) $a = 10$, $S_\infty = 15$ (b) $a = 50$, $S_\infty = 45$

5. Find the first term of the geometric series with common ration r and sum to infinity S_∞.
   (a) $r = 0.45$, $S_\infty = 200$ (b) $r = -0.85$, $S_\infty = 160$.

18.4 Applications

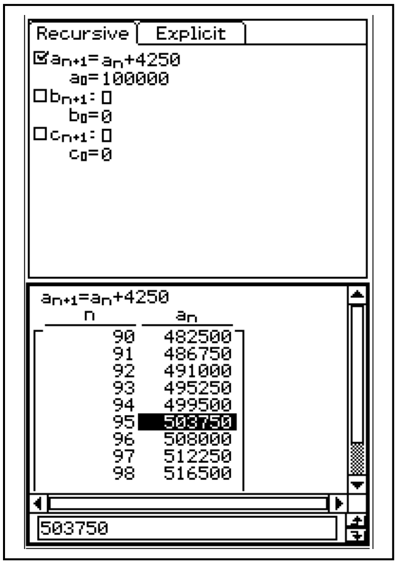
Example 18.13

A sum of \$100 000 is invested in an account that pays simple interest at a rate of 4.25% per annum. Let B(n) be the account balance at the end of n years.
(a) Find the recursive rule for the account balance at the end of n years.
(b) Find when the account balance first exceeds \$500 000.

Solution:

(a) Simple interest per year = 100 000 × 0.0425 = \$4 250.
The yearly account balances form a sequence:
100 000, (100 000 + 4250), (100 000 + 2×4250), ….
Hence: B(n) = B(n – 1) + 4250, B(0) = 100 000.

(b) Use CAS "Sequence":
B(94) = \$499 500   B(95) = \$503 750
Hence, at the end of 95 years!



Alternative Solution to part (b):

B(n) = 100 000 + 4 250 × n
B(n) > 500 000 ⇒ 100 000 + 4 250n > 500 000
n > 94.1
Hence, at the end of 95 years.

Example 18.14

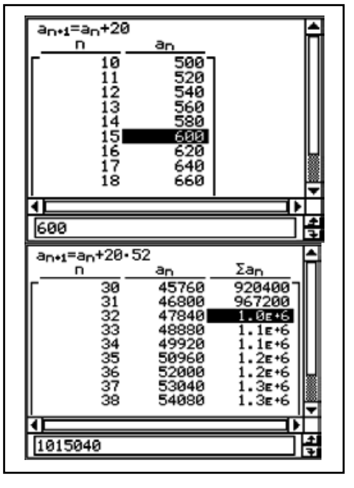
David rents his house out at \$300 per week. Each year, he increases the weekly rent by \$20.
Let R(n) be the weekly rent charged after n years.
(a) Find a recursive rule to describe the weekly rent charged after n years.
(b) How long would it take for the weekly rental to double?
(c) How long would it take for the total rent collected to exceed \$1 000 000?

Solution:

(a) R(n) = R(n – 1) + 20, R(0) = 300.

(b) Using CAS "Sequence":
R(15) = 600. Hence, after 15 years.

(c) Let Y(n): Total rent collected after n years.
Y(n + 1) = Y(n) + 20 × 52, Y(1) = 300 × 52.
Using CAS "Sequence":
Y(31) = 967 200   Y(32) = 1 015 040
Hence, by the end of 32 years.



Notes:

- *In writing recursive definitions, extreme care must be taken when assigning the initial terms.*
- *In part (a), by virtue of the word definition of R(n), the initial term is R(0).*
- *Whereas, in part (b), by virtue of the word definition of Y(n), the initial term is Y(1).*
  In this instance, Y(0) would be meaningless.

Alternative Solution:

(b) R(n) = 300 + 20 × n

R(n) = 600 $\Rightarrow$ 300 + 20n = 600 $\Rightarrow$ n = 15

Hence, after 15 years.

(c) Let Y(n): Total rent collected after n years.

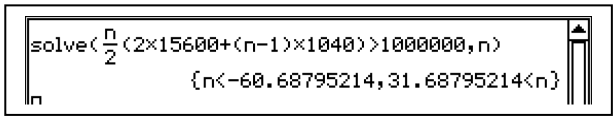
Y(n) represents an arithmetic sequence

with a = 300 × 52 = 15 600 and d = 20 × 52 = 1 040

S(n) > 1 000 000 $\Rightarrow$ $\frac{n}{2}\left(2\times15600+(n-1)\times1040\right)$ > 1 000 000

n > 31.69

Hence, by the end of 32 years.



Example 18.15

A sum of \$100 000 is invested in an account that pays interest at a rate of 4.25% per annum compounded annually. Let B(n) be the account balance at the end of n years.

(a) Find the recursive rule for the account balance at the end of n years.

(b) Find when the account balance first exceeds \$500 000.

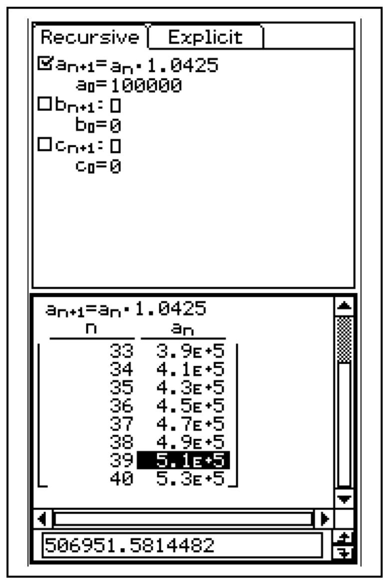
Solution:

(a) The yearly account balances form a sequence:

100 000, (100 000 × 1.0425), (100 000 × 1.0425^2), ….

Hence, recursive rule is:

B(n) = B(n – 1) × 1.0425, B(0) = 100 000.



(b) Using CAS "Sequence":

B(38) = \$486 284 B(39) = \$506 952

Hence, after 39 years.

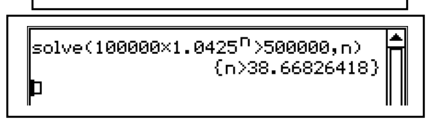
Alternative Solution to part (b):

(b) B(n) = 100 000 × 1.0425^n

B(n) > 500 000 $\Rightarrow$ 100 000 × 1.0425^n > 500 000

n > 38.7

Hence, at the end of 39 years.



Note:

- *Compare this answer with the answer in Example 18.13, where it took 95 years for an account paid only simple interest.*
- *Note also that in the case of a simple interest bearing account, the general term is linear in form. Whereas in the case of a compound interest bearing account, the general term is exponential in form.*

Example 18.16

A ball is dropped vertically from a height of 1 metre. It hits the floor and rebounds to a maximum height of 0.8 m. The ball then hits the floor again and rebounds to a maximum height that is 80% of the previous maximum height reached. This is repeated until the ball comes to rest on the floor. Find:

(a) a recursive rule to describe the successive maximum heights reached by the ball
(b) a rule to describe the maximum height (nearest mm) reached by the ball after it hits the floor for the *n*th time
(c) the distance (nearest cm) travelled by the ball before it hits the floor for the 10th time
(d) the total distance (nearest cm) travelled by the ball before it comes to rest on the floor.

Solution:

(a) The maximum heights reached by the ball is described by the geometric sequence:

$$0.8,\ 0.8 \times 0.8,\ 0.8 \times 0.8 \times 0.8,\ 0.8^4,\ 0.8^5,\ \dots\ .$$

Let H(n): Maximum height reached after hitting the floor the nth time
Hence, recursive rule is $H(n + 1) = H(n) \times 0.8$, $H(1) = 0.8$.

(b) Maximum height reached after hitting the floor the nth time, $H(n) = 0.8^n$.

(c) Distance travelled by the ball $= 1 + \underbrace{(0.8 \times 2)}_{\text{1st Bounce}} + \underbrace{(0.8^2 \times 2)}_{\text{2nd Bounce}} + \dots + \underbrace{(0.8^9 \times 2)}_{\text{9th Bounce}}$

$$= 1 + 2 \times [0.8 + 0.8^2 + \dots + 0.8^9]$$
$$= 7.9263 \text{ m} = 793 \text{ cm}$$

(d) Total distance travelled by ball before coming to rest

$$= 1 + 2 \times [0.8 + 0.8^2 + 0.8^3 + \dots]$$
$$= 1 + 2 \times \left(\frac{0.8}{1-0.8}\right) = 9 \text{ m} = 900 \text{ cm}$$

Note:

- *The maximum heights reached after subsequent bounces form a decaying geometric sequence. The total distance is described theoretically as* S_∞*, which means that the ball bounces an infinite number of times. However, in practice, this does not happen. The ball comes to rest after a finite number of bounces.*

Exercise 18.5

1. A relief agency has 30 tonnes of rice at its warehouse. Each day, 400 kg of rice is released to a nearby refugee camp.
   (a) Write a recursive rule to describe the amount of rice left in the warehouse after day *n*.
   (b) A critical level is reached when the amount of rice in the ware house drops below 5 tonnes. How long would this take?

2. A sum of $50 000 is invested in an account that pays simple interest at a rate of 5.15% per annum. Let B(*n*) be the account balance at the end of *n* years.
   (a) Find the recursive rule for the account balance at the end of *n* years.
   (b) Find when the account balance first exceeds $100 000.

3. A particle starts with a speed of 2 ms^{-1}. Each second its speed increases by 0.1 ms^{-1}.
   (a) Write a recursive rule for the speed of the particle after *t* seconds.
   (b) How long would it take the particle to reach a speed of 10 ms^{-1}.
   (c) After the particle reaches a speed of 10 ms^{-1}, its speed decreases by 0.7 ms^{-1} each second. How long would it take for the particle to come to rest?

4. In a business model, it is assumed that the number of sales made each day increases by 6 per a day from an initial 4 sales per day.
   (a) Find a recursive rule to describe the number of sales per day, *n* days after the shop first opens for business.
   (b) It is projected that the shop would be able to cope with at most 200 sales per day before the owner needs to employ more workers. How long would this take?
   (c) How many sales would have been made before more workers are required?

5. Mark bought his apartment for $350 000 and rents his apartment out at $400 per week. Each year, he increases the weekly rent by $50.
   (a) Find a recursive rule to describe the weekly rent charged after *n* years.
   (b) How long would it take for the weekly rental to exceed $800?
   (c) How long would it take for the total rent collected to exceed $350 000?

6. Kylie bought equipment worth $400 000 for her business. Kylie uses a prime cost depreciation method to depreciate the value of her equipment to a paper value of $0 in 10 years. In the prime cost depreciation method, the value of the equipment is reduced by the same amount each year.
   (a) Write a recursive rule to describe the value of her equipment after *n* years.
   (b) Determine the value of the equipment after 5 years.
   (c) As the equipment is a business asset, the annual decrease in its value can be claimed as a tax deduction. Calculate the total amount of tax deduction that can be claimed over 5 years using the prime cost method.

7. A farm produces 350 tonnes of wheat per year. Each year, because of soil degradation, the production reduces by 50 tonnes per year. It is no longer viable to run the farm if its production drops below 120 tonnes per year.
   (a) How long would it take for the farm to become unviable to run?
   (b) How much wheat would have been produced become the farm becomes unviable?

8. Logs are stacked to form a pyramid, with one log at the very top, two logs below it, three logs further below, four logs yet further below and so on. There were 40 logs at the very bottom (ground level).
   (a) Write a general rule to describe the number of logs in the *n*th row above the ground.
   (b) Find the total number of logs in the first *n* rows (from the ground level up).
   (c) How many logs were used to form this pyramid?

9. A colony of rabbits on an offshore island is subjected to a controlled release of a deadly virus. The population of rabbits on the island t months after the introduction of the virus is modelled by the pattern established in the following table.

t (months)	0	1	2	3
rabbit population	10 000	9 000	8 100	7 290

   (a) Find the rabbit population after 1 year.
   (b) Find when the number of rabbits first drops below 100.
   *(c) Find when the rabbit colony on this island is wiped out. Justify your answer.

10. A sum of $500 000 is invested in an account that pays interest at a rate of 3.6% per annum compounded (a) yearly (b) monthly. Let B(n) be the account balance at the end of n years. (i) Find the recursive rule for the account balance at the end of n years. (ii) Find long it will take for the initial amount to be doubled..

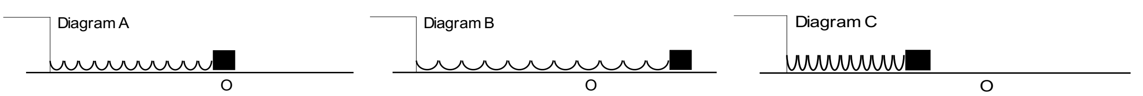
11. The "A series" international paper size, starts with size A0, which is a sheet of paper of area of approximately 1 m^2 with dimensions 841 mm × 1 189 mm. The next paper size, after A0, is A1 (594 mm × 841 mm), which has an area half that of the preceding size A0. This is followed by the A2 size (420 mm × 594 mm), which has an area half that of the preceding size A1. The paper sizes are thus labelled up to A10.
    (a) Find the area of a sheet of paper that has a labelled size of A4.
    (b) A sheet of paper of size A10 has length of 26 mm. Find its width (nearest mm).
    (c) Find the paper size of a sheet of paper in this series with dimensions 74 mm × 105 mm.

12. A little girl stands her doll exactly midway between 2 similar sized mirrors that are placed facing each other. The doll forms an image in each of the mirrors and these images in turn are reflected to form images in the mirrors, which in turn forms another set of images and hence a series of nested images is formed in each mirror. The size of each image is 70% of the size of the previous image. The height of the doll is measured at 30 cm. and the first image measures 21 cm in height.
    (a) Find the height of the 5th image (nearest 0.1 cm).
    (b) Which image has a height of 0.59 cm?
    (c) How many reflections are required before the image of the doll is less than 0.2 cm.

13. According to a Persian legend, the game of chess was invented by the servant of a bored Persian King to amuse the King. The King was very pleased with this game and the servant was asked what he would like as a reward. The servant requested that he be rewarded with grains of wheat determined in the following manner. Each square of the 64 square chess board was to be filled with a specific number of grains. Each square was to have twice the number of grains in the previous square, starting from a single grain in the first square. He was to have all the grains of wheat from each square of the chess board.
    (a) Find the number of grains of wheat in the 10th square.
    (b) Find the first square with at least 10 000 000 grains of wheat?
    (c) Find the total number of grains in the first 10 squares.
    (d) One tonne (metric) of wheat contains approximately 110 000 000 grains of wheat. Find the total tonnage of wheat requested by the King's servant.

14. A ball is dropped vertically from a height of 2 m and it rebounds to a maximum height of 1.8 m. It then hits the ground again and rebounds to a height that is 90% of its previous maximum height. This is repeated until the ball comes to rest on the ground.
    (a) Find the maximum height (nearest cm) reached by the ball after the 4th rebound.
    (b) Find the rebound that corresponds to a maximum rebound height of 0.775 m.
    (c) Find the total distance travelled by the ball immediately before the 10th rebound.
    (d) Find the total distance travelled before the ball comes to rest on the ground.

15.



To test the damping properties of a spring, a weight is attached to the spring and the system is set up on a smooth horizontal surface (diagram A). The weight is pulled to the right, extending the spring (diagram B), and released. The weight then "shoots" to the left compressing the spring (diagram C). At maximum compression, the spring pushes the weight to the right, extending the spring. The weight then moves back and forth until it comes to rest again at its equilibrium position at O. The initial displacement before release is 40 cm. The subsequent maximum displacements of the weight follow the sequence: $-30, 22.5, -16.875, \ldots$. All displacements are measured in cm and displacements to the right of O are denoted as positive displacements.
    (a) Find the maximum displacement of the weight after it passes O for the 7th time.
    (b) Find the total distance travelled just after the weight passes O for the 10th time,
    (c) Find after how many times would the weight have to pass O before its maximum displacement is less than 0.1 mm.

16. A spherical balloon is being inflated by a small air pump. The balloon is completely deflated before the air pump is started. The increase in the volume of the balloon during the t^{th} second (after the pump is started) is modelled by the table below.

t	1	2	3	4
Increase in volume (cm^3)	40 000	20 000	10 000	5 000

(a) Find the increase in volume of the balloon during the 10th second.
(b) Find the volume of air inside the balloon after 10 seconds
(c) Find the maximum volume and hence the maximum radius of the balloon.

17. An observation balloon is released and allowed to float vertically upwards from the ground into the atmosphere. The height increase during the first minute is 30 m. Its height increase during each subsequent minute is 95% of the height increase during the previous minute.
(a) Find the height increase during the 6th minute.
(b) Find the height of the balloon after 6 minutes
(c) Find when the height of the balloon first exceeds 200 m.
(d) Find if the balloon ever reaches the optimum observation altitude of 650 m. Justify your answer.

18. The machinery in a factory is valued at $60 000 when new. Over the first year, the value is depreciated by $24 000. Over each of the subsequent years, its value is depreciated by 40% of the previous year's depreciation.
(a) Find the depreciation for the 6th year.
(b) Find the value of the machinery after 6 years.
(c) Find when the value of the machinery first drops below $20 010.
(d) Will the value of the machinery ever drop below $20 000? Justify your answer.

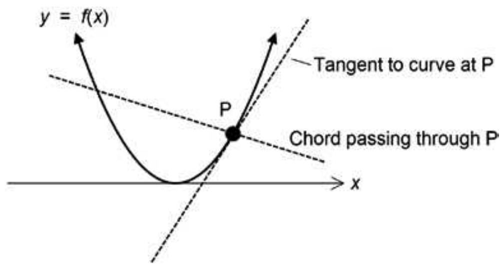
19. Lucy is to ride her bicycle from Perth to Albany (a distance of 400 km) as part of a fund raising campaign for a children's hospital. The distance she is to ride each day is to be 75% of the distance she rode the previous day.
(a) If Lucy rides 90 km on the first day, will she reach Albany in exactly 14 days. Justify your answer.
(b) Calculate the distance Lucy is to ride on the first day if she is to reach Albany at the end of the 10th day. How far does she have to ride on the 10th day?

20. Kerry has just started working. She hopes to save $50 000 to start her own business consultancy. Each year, she plans to save 5% more than what she saved the previous year.
(a) How much should Kerry save in the first year to achieve her target at the end of 10 years? How much must she save in the 10th year?
(b) Kerry saves $9 000 in the first year. Will she achieve her target at the end of the fifth year? Justify your answer.

19 Differentiation

19.1 Tangents and Chords

- The tangent to the curve $y = f(x)$ at the point P is a straight line that “touches” the curve at P.
- A chord of the curve $y = f(x)$ passing through the point P is a straight line that passes through P and at least one other point on the curve.



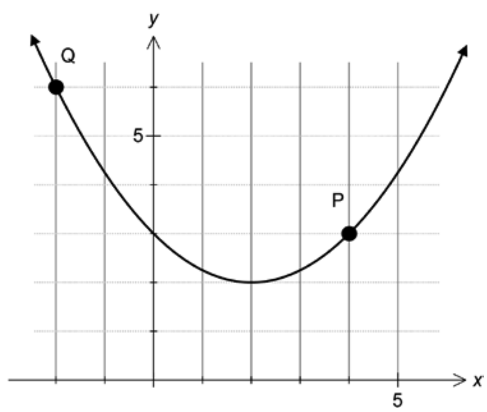
Hands On Task 19.1

In this task, we will explore the relationship between chords and tangents.

1. In the accompanying diagram, use a ruler and pencil, to draw in the tangent to the curve at P. Estimate the gradient of this tangent.

2. Draw in the chord PQ. Estimate the gradient of PQ.

3. Move the point Q along the curve, closer to P. Draw in this new chord and estimate its gradient.



4. Move the point Q progressively closer to P. Each time you do this; estimate the gradient of the chord PQ.

5. With reference to the tangent at P, what happens to the chord PQ as Q is moved progressively closer and closer to P?

6. What would happen to the chord PQ when Q is actually coincident with P?

7. Use your observations in Question 6 to verify the following statement:

$$\lim_{Q \to P}\left(\text{gradient of chord PQ}\right) = \text{Gradient of tangent at P}$$

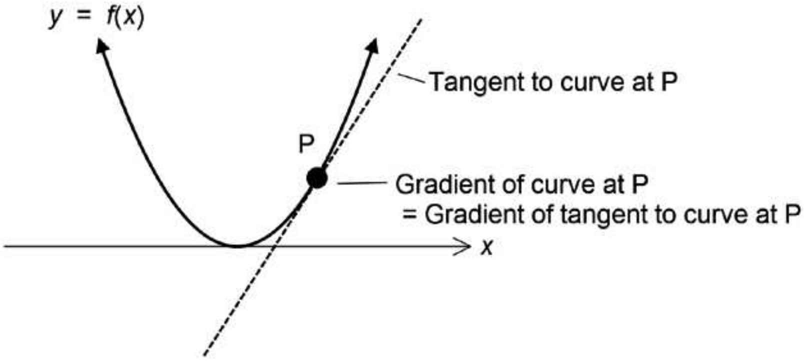
Summary

- Consider the chord PQ on the curve $y = f(x)$.
  As $Q \to P$, the chord becomes the tangent to the curve at P.
  Hence:

$$\lim_{Q \to P}\left(\text{gradient of chord PQ}\right) = \text{gradient of tangent at P}.$$

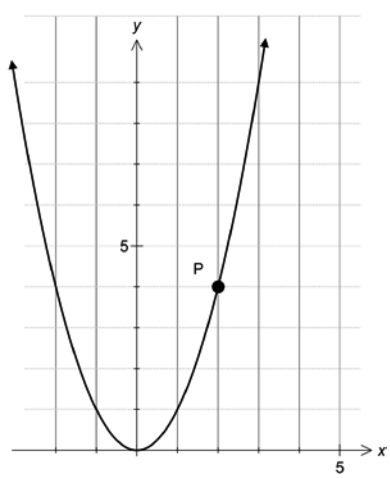
19.2 Gradient of a Curve at a Point

- The gradient of the curve $y = f(x)$ at the point P, is defined as the gradient of the tangent to the curve at the point P.



Hands On Task 19.2

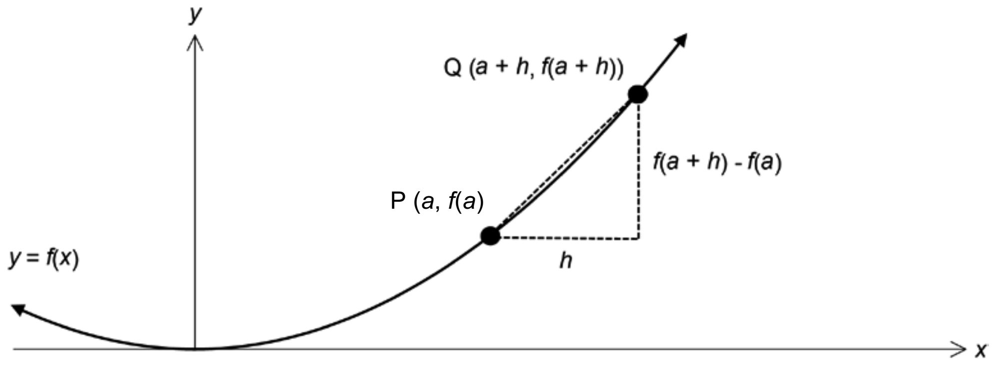
In this task, we will explore a graphical technique to estimate the gradient of a curve at a given point.



- The gradient of the curve $y = f(x)$ at the point P, is defined as the gradient of the tangent to the curve at the point P.

1. To find the gradient of the curve $y = x^2$ at P(2, 4):
   (a) Draw as accurately as possible the graph of $y = x^2$.
   (b) Locate the point P(2, 4) on the graph and draw the tangent to the curve $y = x^2$ at P(2, 4).
   (c) Hence, estimate the gradient of this tangent at P(2, 4).
2. Estimate the gradient of each of the following curves at the point P where $x = 1$:
   (a) $y = x^3$ (b) $y = \sqrt{x}$ (c) $y = \dfrac{1}{x}$

19.2.1 Analytical Method for determining Gradient of a Curve at a Point



- Consider the curve with equation $y = f(x)$.
  Let P be a point on this curve.
- Let the x-coordinate of P be a.
  Using the function notation, the y-coordinate of P is $f(a)$.
- Let Q (to the "right" of P) be a point such that the horizontal distance from Q to P is h. Hence, Q has coordinates $(a + h, f(a + h))$.
- Clearly, the chord PQ has gradient $m_{PQ} = \dfrac{f(a+h)-f(a)}{h}$.
- The gradient of the tangent at P = $\lim\limits_{Q \to P} (\text{gradient of chord PQ})$

  $$= \lim_{Q \to P}\left[\frac{f(a+h)-f(a)}{h}\right]$$

  But as Q $\to$ P, $h \to 0$. Hence:
  The gradient of the tangent at P = $\lim\limits_{Q \to P} (\text{gradient of chord PQ})$

  $$= \lim_{h \to 0}\left[\frac{f(a+h)-f(a)}{h}\right].$$

- The gradient of the curve $y = f(x)$ at the point P = Gradient of the tangent at P.
  Hence:

  Gradient of the curve $y = f(x)$ at the point P $(a, f(a))$ = $\lim\limits_{h \to 0}\left[\dfrac{f(a+h)-f(a)}{h}\right]$.

Summary

- The gradient of the curve $y = f(x)$ at the point where $x = a$ is given by:

$$m = \lim_{h \to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

Example 19.1

Use an analytical method to find the gradient of the curve $y = x^2$ at the point where $x = 2$.

Solution:

Gradient of curve at $x = 2$ is given by $m = \lim_{h \to 0}\left[\frac{f(2+h)-f(2)}{h}\right]$.

But $f(2 + h) = (2 + h)^2 = 4 + 4h + h^2$

and $f(2) = 2^2 = 4$.

Hence:

$$\begin{aligned} m &= \lim_{h \to 0}\left[\frac{f(2+h)-f(2)}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{(4+4h+h^2)-4}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{4h+h^2}{h}\right] \\ &= \lim_{h \to 0}\left[\frac{h(4+h)}{h}\right] \\ &= \lim_{h \to 0}\left[4+h\right] \\ &= 4. \end{aligned}$$

19.2.2 The Gradient Function

- Consider the curve with equation $y = f(x)$.
  - In the previous section it was established that the gradient of the curve at the point where $x = a$ is given by $m = \lim_{h \to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$.
  - In general, the gradient of the curve at any point on the curve is given by $\lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$.
  - The expression $\lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ represents the gradient of the curve $y = f(x)$ at any point on the curve. This expression is denoted $f'(x)$ and is called the gradient function of the curve.
- In summary, the gradient function of the curve $y = f(x)$ is given by:

  $$f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right].$$

  The gradient function allows us to determine the gradient of the curve at any point on the curve.

Hands On Task 19.3

In this task, we will develop a technique for obtaining $f'(x)$ for $f(x) = x^n$.

1. Consider $f(x) = x^2$.
   (a) Find an expression for $f(x + h)$ and verify that $f(x + h) - f(x) = 2hx + h^2$
   (b) Hence, verify that $f'(x) = 2x$.

2. Consider $f(x) = x^3$.
   (a) Find an expression for $f(x + h) - f(x)$. (b) Hence, verify that $f'(x) = 3x^2$.

3. Consider $f(x) = \dfrac{1}{x} = x^{-1}$.
   Verify that $f(x + h) - f(x) = \dfrac{-h}{x(x+h)}$ and hence verify that $f'(x) = \dfrac{-1}{x^2} = -x^{-2}$.

4. Consider $f(x) = \dfrac{1}{x^2} = x^{-2}$.
   Verify that $f(x + h) - f(x) = \dfrac{-2hx-h^2}{x^2(x+h)^2}$ and hence verify that $f'(x) = \dfrac{-2}{x^3} = -2x^{-3}$.

5. The following table lists $f'(x)$ of several expressions $f(x)$.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x	1	x^4	$4x^3$
x^2	$2x$	x^{-1}	$-x^{-2}$
x^3	$3x^2$	x^{-2}	$-2x^{-3}$

   (a) Determine expressions for $f'(x)$ for $f(x) = x^6$ and $f(x) = x^{-5}$.
   (b) Suggest an expression for $f'(x)$ for $f(x) = x^n$ where n is a non-zero integer.
   (c) Suggest an expression for $f'(x)$ for $f(x) = x^0$.

6. The following table lists the derivatives of several expressions.

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$x^{5/2}$	$\frac{5}{2}x^{3/2}$	$x^{-1/2}$	$\frac{-1}{2}x^{-3/2}$
$x^{3/2}$	$\frac{3}{2}x^{1/2}$	$x^{-3/2}$	$\frac{-3}{2}x^{-5/2}$
$x^{1/2}$	$\frac{1}{2}x^{-1/2}$	$x^{1/3}$	$\frac{1}{3}x^{-2/3}$

   Determine expressions for $f'(x)$ for $f(x) = x^{7/2}$ and $f(x) = x^{-3/5}$.

7. Given that $f(x) = x^n$, where n is any real number, suggest a rule for $f'(x)$.

Summary

- The process of obtaining $f'(x)$, the gradient function of $y = f(x)$, using the definition $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$ is called *differentiation using first principles*. $f'(x)$ is referred to as the derivative of $f(x)$ with respect to x.
- As suggested in this Task, there are simple rules for determining the derivative of a polynomial.

19.3 Differentiation of Polynomials

- Consider $y = f(x)$.
  The derivative of $f(x)$ with respect to x is defined as $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$.
- The process of obtaining the derivative is called *differentiation*.
- The following notations are often used to represent a derivative.
  If $y = f(x)$, then the derivative of y with respect to x is denoted $\frac{dy}{dx}$ or y'.
  The derivative of $f(x)$ with respect to x, is denoted $f'(x)$ of $\frac{d}{dx}f(x)$.
- The following results are quoted without proof.
  Given that a and b are constants and m and n are real numbers:
  - $\frac{d}{dx}(x^n) = nx^{n-1}$
  - $\frac{d}{dx}(a) = 0$
  - $\frac{d}{dx}(ax^n) = a(nx^{n-1})$
  - $\frac{d}{dx}(ax^n \pm bx^m) = a(nx^{n-1}) \pm b(mx^{m-1})$

Notes:

- *The first result was alluded to in Hands On Task 19.3.*
- *The second result indicates that the derivative of a constant is always zero.*
- *The third result indicates that the coefficient of x^n remains "undisturbed".*
- *The fourth result indicates that the derivative of a sum (or a difference) is the sum (or difference) of the derivatives of the separate terms.*

Example 19.2

Differentiate with respect to *x*.

(a) x^{15} (b) $4x^3$ (c) $\dfrac{1}{2x^4}$ (d) $2\sqrt{x}$

Solution:

(a) $\dfrac{d}{dx}(x^{15}) = 15x^{14}$

(b) $\dfrac{d}{dx}(4x^3) = 4(3x^2) = 12x^2$

(c) $\dfrac{d}{dx}\left(\dfrac{1}{2x^4}\right) = \dfrac{d}{dx}\left(\dfrac{1}{2}x^{-4}\right) = \dfrac{1}{2}(-4x^{-5}) = -2x^{-5}$

(d) $\dfrac{d}{dx}(2\sqrt{x}) = \dfrac{d}{dx}(2x^{\frac{1}{2}}) = 2\left(\dfrac{1}{2}x^{-\frac{1}{2}}\right) = x^{-\frac{1}{2}}$

Example 19.3

Differentiate the following with respect to *x*.

(a) $y = 3x^2 + 4$ (b) $y = 4x + \dfrac{1}{x}$ (c) $y = (x + 1)(x - 2)$ (d) $y = \dfrac{x^2+2}{x}$

Solution:

(a) $y = 3x^2 + 4$

$\dfrac{dy}{dx} = 3(2x) = 6x$

(b) $y = 4x + \dfrac{1}{x} = 4x + x^{-1}$

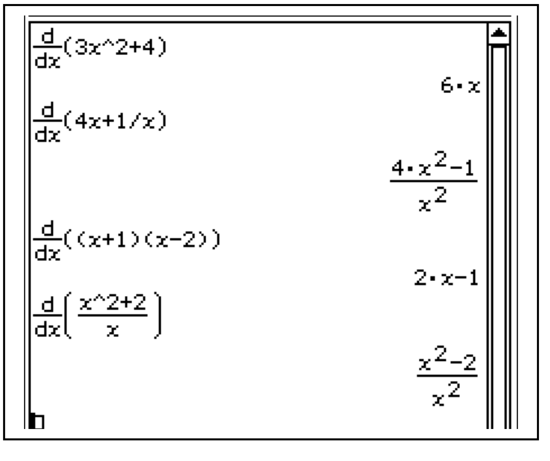
$\dfrac{dy}{dx} = 4 + (-1)\,x^{-2} = 4 - \dfrac{1}{x^2}$

(c) $y = (x + 1)(x - 2) = x^2 - x - 2$

$\dfrac{dy}{dx} = 2x - 1$

(d) $y = \dfrac{x^2+2}{x} = \dfrac{x^2}{x} + \dfrac{2}{x} = x + 2x^{-1}$

$\dfrac{dy}{dx} = 1 + 2[(-1)\,x^{-2}] = 1 - \dfrac{2}{x^2}$



Example 19.4

Let $f(x) = x^n$ where n is a non-negative integer.

(a) Use the Binomial Theorem to expand $(x + h)^n$ and hence factorise $(x + h)^n - x^n$.

(b) Use your answer in (a) to prove that $f'(x) = nx^{n-1}$.

Solution:

(a) $(x + h)^n = x^n + nx^{n-1}h + \dfrac{n(n+1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n$.

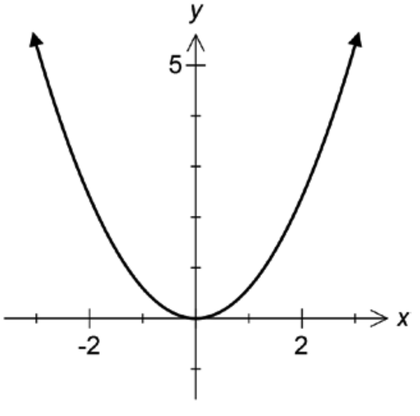
$$(x + h)^n - x^n = (x^n + nx^{n-1}h + \frac{n(n+1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n) - x^n$$
$$= nx^{n-1}h + \frac{n(n+1)}{2}x^{n-2}h^2 + ... + nxh^{n-1} + h^n$$
$$= h\,(nx^{n-1} + \frac{n(n+1)}{2}x^{n-2}h^1 + ... + nxh^{n-2} + h^{n-1})$$

(b) $$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{h\,(nx^{n-1} + \frac{n(n+1)}{2}x^{n-2}h^1 + \ \dots\ + nxh^{n-2} + h^{n-1})}{h}\right]$$
$$= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n+1)}{2}x^{n-2}h^1 + \ \dots\ + nxh^{n-2} + h^{n-1}\right]$$
$$= nx^{n-1}.$$

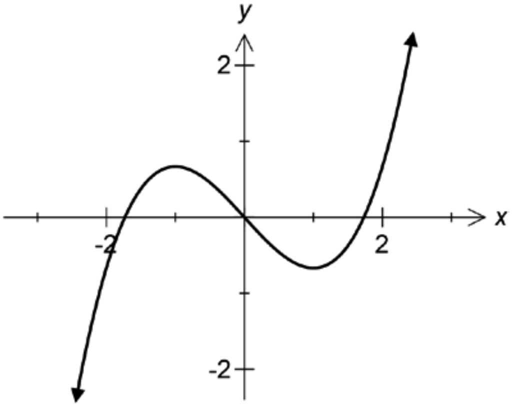
Exercise 19.1

1. Use the sketch of $y = f(x)$ to determine $f'(a)$ for the indicated values of a.

   (a) $a = -2$, $a = 0$, $a = 1$

   (b) $a = -1$, $a = 0$, $a = 2$

2. Without the use of a calculator, use first principles to differentiate the following:

   (a) 10 (b) $2x$ (c) x^3 (d) $x^2 + x$

3. Differentiate the following with respect to x.

(a) x^{2} (b) x^{12} (c) $x^{3/2}$ (d) $x^{1/2}$

(e) x^{-3} (f) x^{-7} (g) $x^{-1/2}$ (h) $x^{-5/2}$

(i) $\dfrac{1}{x^4}$ (j) $\dfrac{1}{x^6}$ (k) $\dfrac{1}{\sqrt[3]{x}}$ (l) $\left(\dfrac{1}{x}\right)^{7/2}$

4. Differentiate the following with respect to x.

(a) $4x$ (b) $-4x^{3}$ (c) 4^{3} (d) $\dfrac{x^3}{2}$

(e) $-5x^{-2}$ (f) $\dfrac{x^{-4}}{3}$ (g) $\dfrac{1}{4x}$ (h) $\dfrac{-3}{x^2}$

(i) $4\sqrt{x}$ (j) $5x^{5/2}$ (k) $\dfrac{x^{\frac{7}{2}}}{7}$ (l) $-(2x)^{4}$

5. Differentiate the following with respect to t.

(a) $\dfrac{4}{t}$ (b) $\left(\dfrac{2}{t}\right)^2$ (c) $\dfrac{-2}{\sqrt{t}}$ (d) $\left(\dfrac{8}{t}\right)^{\frac{1}{3}}$ (e) $\dfrac{2}{3t^{\frac{3}{2}}}$

(f) $-\dfrac{5}{2t^{\frac{2}{5}}}$ (g) $\left(\dfrac{25}{9t}\right)^{\frac{1}{2}}$ (h) $\dfrac{4}{27^{\frac{1}{3}}}$ (i) $-\dfrac{6}{t^{-3}}$ (j) $\dfrac{1}{8t^{-2}}$

6. Find:

(a) $\dfrac{d}{dx}(4x^3)$ (b) $\dfrac{d}{dt}(3t^2)$ (c) $\dfrac{d}{dy}(-3y^2)$ (d) $\dfrac{d}{dp}(5p^{-2})$

(e) $\dfrac{d}{dv}(\dfrac{1}{4}v^{-4})$ (f) $\dfrac{d}{dr}(14r^{\frac{1}{2}})$ (g) $\dfrac{d}{ds}\left(\dfrac{3}{4s^{\frac{3}{2}}}\right)$ (h) $\dfrac{d}{dk}\left(-\dfrac{3}{5k^{\frac{1}{2}}}\right)$

7. Find $\dfrac{dy}{dx}$ given:

(a) $y = x^{2} + 10$ (b) $y = -x^{3} + 2$ (c) $y = -3x^{2} + 4$

(d) $y = -3 + x^{5}$ (e) $y = \sqrt{x} - 3x$ (f) $y = x^{2} + \dfrac{1}{x}$

(g) $y = 5x - \dfrac{2}{x^2}$ (h) $y = \dfrac{1}{x} + \dfrac{1}{4x^2}$ (i) $y = \dfrac{1}{\sqrt{x}} - 4x$

8. Find $f'(x)$:

(a) $f(x) = x(x + 2)$ (b) $f(x) = -x(2x - 4)$ (c) $f(x) = 3(x - 3)$

(d) $f(x) = \dfrac{1}{2}(x + 4)$ (e) $f(x) = \dfrac{x}{2}(x + 4)$ (f) $f(x) = x(\sqrt{x} + 3)$

(g) $f(x) = x(1 - x^{3/2})$ (h) $f(x) = (x^{5/2} - 3)x^{1/2}$ (i) $f(x) = x^{-1}(x + 2)$

9. Find the derivative with respect to x for each of the following.
   (a) $(x + 1)(x - 6)$ (b) $(x - 1)(x + 2)$ (c) $(x + 1)(\sqrt{x} - 1)$
   (d) $(x^2 - 1)(1 - x)$ (e) $\dfrac{x+1}{x}$ (f) $\dfrac{x^2+x-2}{x}$
   (g) $\dfrac{x^2-3x}{x^2}$ (h) $\dfrac{4x-5x^2}{x^2}$ (i) $\dfrac{6-\sqrt{x}}{x^{\frac{3}{2}}}$

10. Given that $f(t) = \dfrac{(t+1)(t-1)}{t}$, find $f'(t)$. Hence, find $f'(1)$.

11. Given that $g(t) = \dfrac{(t-1)(t-2)}{t^2}$, find $g'(t)$. Hence, find $g'(1)$.

12. Given that $h(t) = t^2 + 2t - 3$, find $h'(t)$. Hence, find the value(s) of t such that $h'(t) = 0$.

13. Given that $v(t) = t^3 - 2t^2 + 3$, find $v'(t)$. Hence, find the value(s) of t such that $v'(t) = 0$.

14. For $P(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 4$, find $P'(t)$. Hence, find the value(s) of t such that $P'(t) = 0$.

15. For each of the following limits, identify the function being differentiated.
    Hence, evaluate each of these limits.
    (a) $\lim\limits_{h\to 0}\left[\dfrac{(x+h)^5 - x^5}{h}\right]$ (b) $\lim\limits_{h\to 0}\left[\dfrac{2(y+h)^2 - 2y^2}{h}\right]$ (c) $\lim\limits_{h\to 0}\left[\dfrac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}\right]$
    (d) $\lim\limits_{h\to 0}\left[\dfrac{\sqrt{t+h} - \sqrt{t}}{h}\right]$ (e) $\lim\limits_{h\to 0}\left[\dfrac{(1+x+h)^2 - (1+x)^2}{h}\right]$ (f) $\lim\limits_{h\to 0}\left[\dfrac{\frac{1}{\sqrt{(x+h)}} - \frac{1}{\sqrt{x}}}{h}\right]$

16. Use derivatives to find the exact values of each of the following:
    (a) $\lim\limits_{h\to 0}\left[\dfrac{(2+h)^2 - 2^2}{h}\right]$ (b) $\lim\limits_{h\to 0}\left[\dfrac{(-1+h)^2 - (-1)^2}{h}\right]$
    (c) $\lim\limits_{h\to 0}\left[\dfrac{(h-3)^3 + 27}{h}\right]$ (d) $\lim\limits_{h\to 0}\left[\dfrac{\frac{1}{(1+h)} - 1}{h}\right]$
    (e) $\lim\limits_{h\to 0}\left[\dfrac{\frac{2}{1+2h} - 2}{h}\right]$ (f) $\lim\limits_{h\to 0}\left[\dfrac{\sqrt{4+h} - 2}{h}\right]$
    (g) $\lim\limits_{h\to 0}\left[\dfrac{\frac{1}{\sqrt{(5+h)}} - \frac{\sqrt{5}}{5}}{h}\right]$ (h) $\lim\limits_{h\to 0}\left[\dfrac{\frac{-1}{\sqrt{(1+h)}} + 1}{h}\right]$

20 Gradient Function

20.1 Equation of tangent to a Curve

- As mentioned in the previous chapter, if $y = f(x)$ represents the equation of a curve, then, $f'(x)$ or $\dfrac{dy}{dx}$ or y' represents the *gradient function* of the curve.
- The numerical value of the gradient of the curve at any point is obtained by substituting the coordinates (usually, just the x-coordinate) into the equation for the gradient function.
- The gradient of the tangent to the curve at a point P on the curve is equal to the gradient of the curve at the point P. From this, the equation of the tangent may be easily obtained.

Example 20.1

Without the use of a CAS calculator, find the gradient of the curve $y = x^3 + 4x^2 + x - 1$ at the point (1, 5). Hence, find the equation of the tangent to the curve at this point.

Solution:

$y = x^3 + 4x^2 + x - 1 \Rightarrow$ Gradient Function $\dfrac{dy}{dx} = 3x^2 + 8x + 1$

At (1, 5), $x = 1$, $\dfrac{dy}{dx} = 3(1)^2 + 8(1) + 1 = 12$.

Hence, the gradient of the curve at (1, 5) is 12.

The gradient of the tangent to the curve at (1, 5) is 12.

The tangent passes through (1, 5).

Hence, the equation of the tangent is $y - 5 = 12(x - 1) \Rightarrow y = 12x - 7$

OR

Equation of the tangent is of the form $y = 12x + c$.

Since, it passes through (1, 5), when $x = 1$, $y = 5$.

Substitute into the equation of the tangent, $5 = 12(1) + c \Rightarrow c = -7$.

Hence, the equation of the tangent is $y = 12x - 7$.

Notes:

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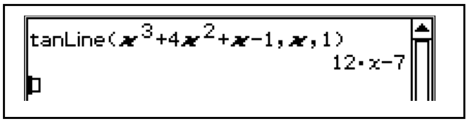


Exercise 20.1

1. Find the gradient function for the following curves:
   (a) $y = -x^2 + 4x + 3$ (b) $y = \frac{-1}{x^2}$ (c) $y = x^3 - 4x^2 + 5x - 1$ *(d) $y = (x - 1)^3$

2. Find the gradient of the curve with equation $y = -x^2 + 2x + 4$ at the point (2, 4).
   Hence, find the equation of the tangent to the curve at (2, 4).

3. Find the gradient of the curve with equation $y = -4/x$ at the point (−1, 4).
   Hence, find the equation of the tangent to the curve at (−1, 4).

4. Find the gradient of the curve with equation $y = \frac{16}{x^2}$ at the point where $x = 2$.
   Hence, find the equation of the tangent to the curve at the point where $x = 2$.

5. Find the equation of the tangent to the curve $y = -x^3 + x + 1$ at the point where $x = 0$.

6. Find the equation of the tangent to the curve $y = -(x + 1)^2 - 2$ at the point where $x = 1$.
   Hence, find the point(s) where this tangent intersects the curve.

7. Without the use of a calculator, find the coordinates of the point(s), where it exists, on the curve $y = f(x)$, where the gradient is zero:
   (a) $y = x^2 + 4x - 1$ (b) $y = x^3 + 4x^2 - 3x - 1$ (c) $y = (x + 3)^2 + 2$ (d) $y = x^2(1 - x)$

8. Without the use of a calculator, find the coordinates of the point(s), where they exist, on the curve $y = f(x)$ where the gradient is as indicated:
   (a) $y = 2x^2 + 3$, gradient = −1 (b) $y = (2 - x)^2$, gradient = −4
   (c) $y = \sqrt{x}$, gradient = $\frac{1}{4}$ (d) $y = \frac{-2}{x^2}$, gradient = $\frac{-1}{2}$

9. Find the equation of the tangent(s) to the curve $y = -x^2 + 4x - 1$ at the point(s) where the gradient of the curve is −2.

10. Find the equation of the tangent(s) to the curve $y = 2x^3 - 3x^2 - 4x$ at the point(s) where the gradient of the curve is −4.

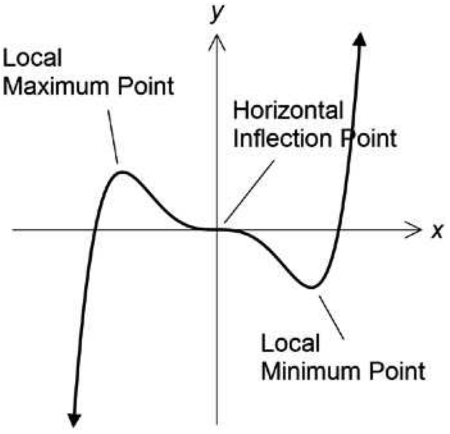
11. Find the points on the curve $y = f(x)$ where the tangent to the curve at these points is parallel to the indicated lines:
    (a) $y = -x^2 + 3x - 1$, $y = x + 2$ (b) $y = -x^3 + 2x^2 - x - 1$, x-axis
    (c) $y = \frac{-1}{x}$, $9x - y = 4$ (d) $y = \sqrt{x}$, $y = 1 + \frac{x}{2}$

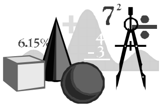
12. $y = x + 1$ is a tangent to the curve $y = ax^2 + bx$, at the point (1, 2). Find a and b.

13. $y = x + 2$ is a tangent to the curve $y = ax^2 + bx + c$ at the point (0, 2) and (1, 0) is another point on this curve. Find a, b and c.

20.2 Graphs of Gradient Functions and Stationary Points

- The accompanying diagram show the terms used to describe the stationary points of a curve. Stationary points consist of:
  - minimum turning points
  - maximum turning points
  - horizontal inflection points.
- Taken within the context of a defined domain for the function drawn, the turning points are referred to as local minimum or local maximum points. This will be discussed in greater detail in the next chapter.





Hands On Task 20.1

In this task, we will explore the relationship between the graph of a function and the graph of its gradient function.

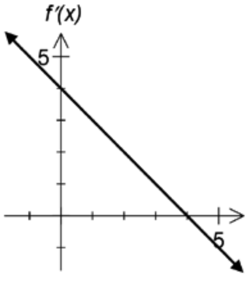
1. Graph on your CAS/graphic calculator, $y = f(x)$ and $y = f'(x)$; using the "derivative graphing" facility of your calculator. Complete the following table.

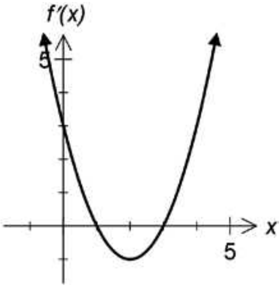
$y = f(x)$	Coordinates of stationary point(s) on $y = f(x)$	Coordinates of root(s) on $y = f'(x)$
$y = x^2 + 1$		
$y = (x-1)^2 + 1$		
$y = (x-1)(x-2)(x-3)$		
$y = x^3(x^2 - 4)$		

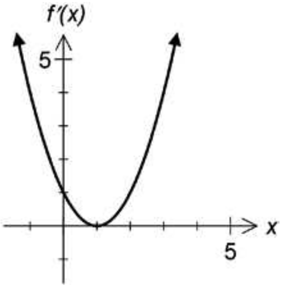
Compare the coordinates of the stationary points of $y = f(x)$ and the roots of $y = f'(x)$. Comment on your observations and suggest an explanation for your observations.

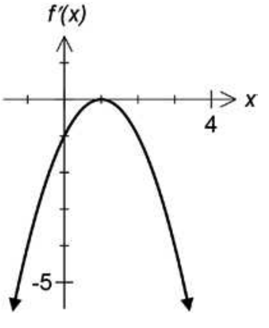
2. Given the sketch of $y = f'(x)$, describe how you would find the x-coordinates, where they exist, of the stationary point(s) of the curve of $y = f(x)$.

(a) (b) (c) (d)







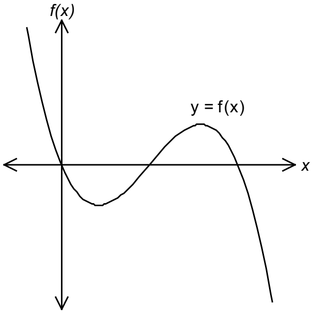


Summary

- Stationary points on $y = f(x)$ correspond to the roots on $y = f'(x)$.
- A root of $y = f'(x)$ corresponds to a either a turning point or a horizontal inflection point on $y = f(x)$.
- A root of $y = f'(x)$ which also happens to be a turning point corresponds to a point of horizontal inflection on $y = f(x)$.

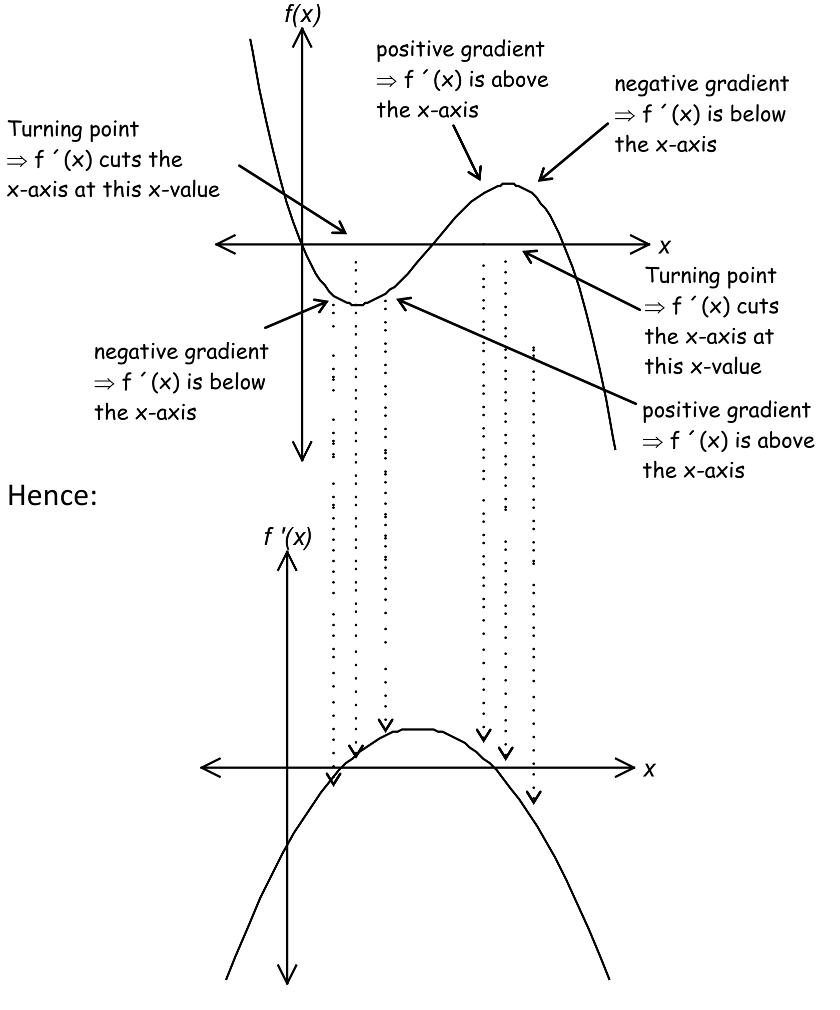
Example 20.2

Given the sketch of $y = f(x)$, sketch a possible graph of $y = f'(x)$.



Solution:

The turning points on $y = f(x)$ correspond to the roots of $y = f'(x)$.

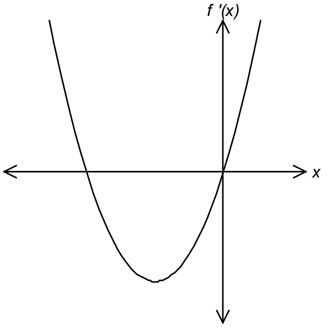


Note:

- *This is just a possible sketch of y = f ´(x). The x-intercepts of y = f ´(x) are fixed.*
  The x-intercept of the turning point of y = f ´(x) is fixed but its y-coordinate is unknown.

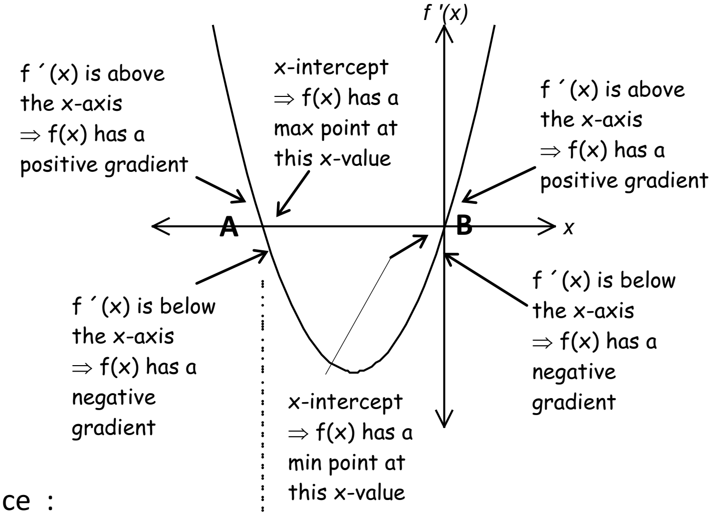
Example 20.3

Given the sketch of $y = f'(x)$, sketch a possible graph of $y = f(x)$.

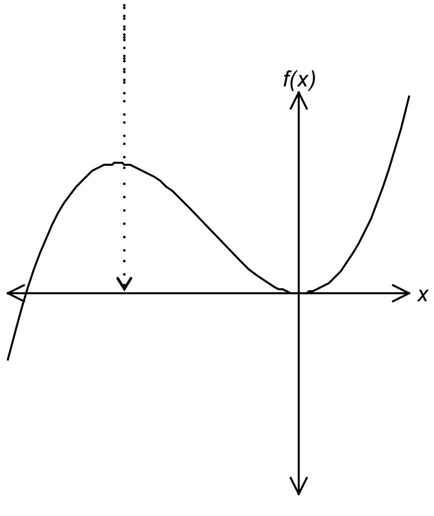


Solution:

Roots of $y = f'(x)$ correspond to turning points on $y = f(x)$.



Hence :



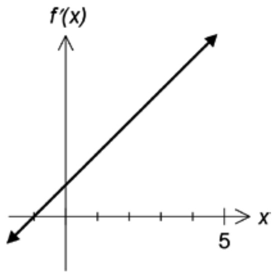
Note:

- *In the "neighbourhood" of A, the gradient of f(x) changes from a positive value, to zero and then to a negative value. Hence, A corresponds to a maximum point on y = f(x).*
- *In the "neighbourhood" of B, the gradient of f(x) changes from a negative value, to zero and then to a positive value. Hence, B corresponds to a minimum point on y = f(x).*

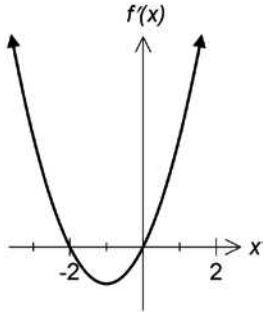
Exercise 20.2

1. Given the sketch of $y = f'(x)$, determine the x-coordinates of the stationary point(s) of $y = f(x)$, where they exist. State the nature of the stationary points, where they exist.

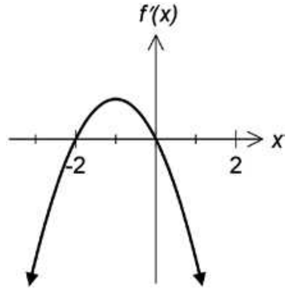
(a)



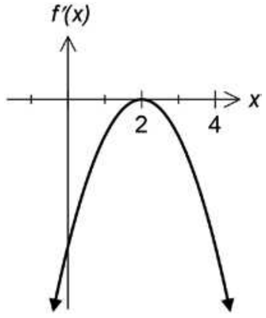
(b)



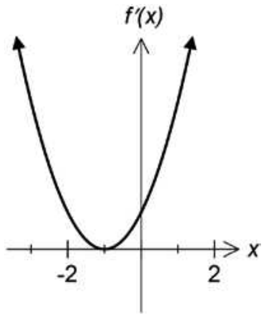
(c)



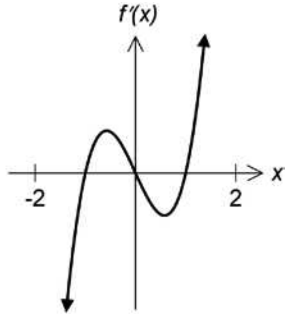
(d)



(e)

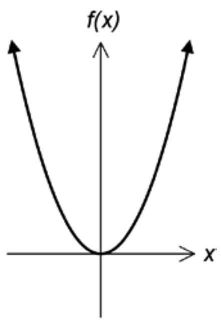


(f)

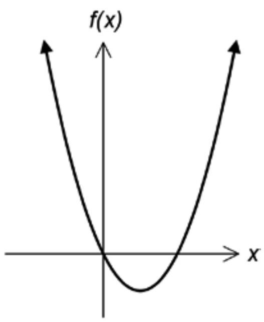


2. Given the sketch of $y = f(x)$, sketch a possible graph of $y = f'(x)$.

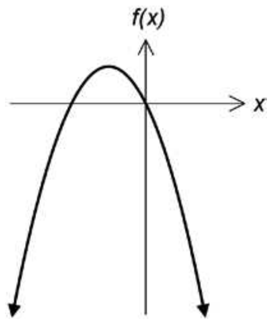
(a)



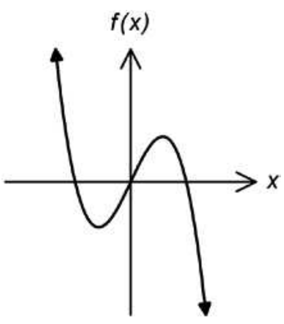
(b)



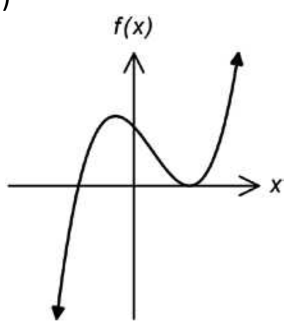
(c)



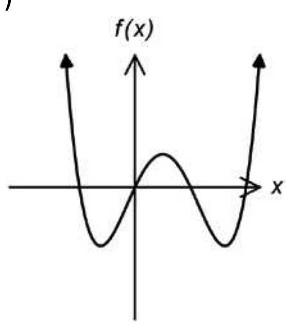
(d)



(e)

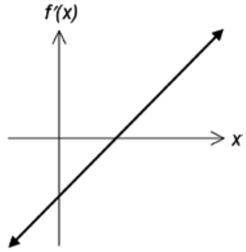


(f)

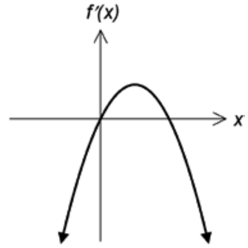


3. Given the sketch of $y = f'(x)$, give a possible sketch of $y = f(x)$.

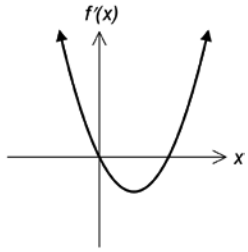
(a)



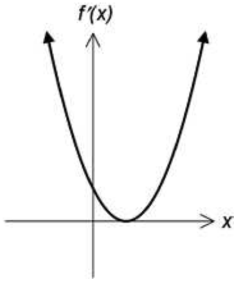
(b)



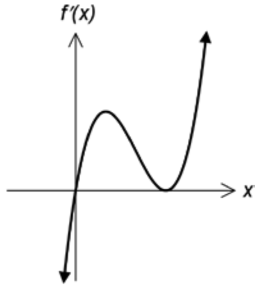
(c)



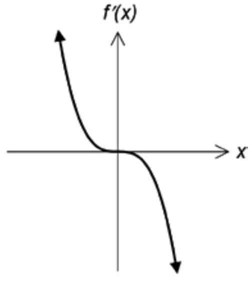
(d)



(e)



(f)



20.3 Stationary Points

- As noted in the previous section, stationary points on $y = f(x)$ correspond to the roots on $y = f'(x)$.
- Hence, the necessary condition for a stationary point on $y = f(x)$ is $f'(x) = 0$.
- That is, when $f'(a) = 0$, then the curve $y = f(x)$ has a stationary point at $x = a$.
- To determine the nature of the stationary point, the "sign" test may be used.
  - The stationary point at $x = a$, is a maximum turning point if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for $\frac{dy}{dx}$ or $f'(x)$	+	0	–

  - The stationary point at $x = a$, is a minimum turning point if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for $\frac{dy}{dx}$ or $f'(x)$	–	0	+

  - The stationary point at $x = a$, is a *horizontal inflection point* if :

x	$x = a^-$	$x = a$	$x = a^+$
sign for $\frac{dy}{dx}$ or $f'(x)$	– or +	0 0	– +

a^- is a value of x slightly less than a and a^+ is a value of x slightly greater than a.

Example 20.4

Without the use of a CAS calculator, find the coordinates of the turning point(s) for each of the following. State the nature of the turning point(s).

(a) $y = x^3 + x^2 - x + 1$ (b) $y = x^3 - 1$

Solution:

(a) $y = x^3 + x^2 - x + 1 \Rightarrow \dfrac{dy}{dx} = 3x^2 + 2x - 1$

For stationary points $\dfrac{dy}{dx} = 0.$

$\Rightarrow 3x^2 + 2x - 1 = 0$

$x = \frac{1}{3}$ or -1

When $x = \frac{1}{3}$, $y = \frac{22}{27}$.

Using the sign test:

x	$x = \frac{1}{3}^-$	$x = \frac{1}{3}$	$x = \frac{1}{3}^+$
sign for $\dfrac{dy}{dx}$	$-$ ╲	0 —	**+** ╱

Hence, $(\frac{1}{3}, \frac{22}{27})$ is a minimum turning point.

When $x = -1$, $y = 2$.

Using the sign test:

x	$x = -1^-$	$x = -1$	$x = -1^+$
sign for $\dfrac{dy}{dx}$	+ ╱	0 —	$-$ ╲

Hence, $(-1, 2)$ is a maximum turning point.

(b) $y = x^3 - 1 \Rightarrow \dfrac{dy}{dx} = 3x^2$

For stationary points, $\dfrac{dy}{dx} = 0.$

$\Rightarrow 3x^2 = 0$

$x = 0$

When $x = 0$, $y = -1$.

Using the sign test:

x	$x = 0^-$	$x = 0$	$x = 0^+$
sign for $\dfrac{dy}{dx}$	+ ╱	0 —	+ ╱

Hence, $(0, -1)$ is a horizontal inflection point.

Exercise 20.3

1. Without the use of a calculator, use calculus techniques to find the coordinates of the stationary point(s), for each of the following curves. Identify the nature of these points.

(a) $y = x^2 - 2x + 3$ (b) $y = -2x^2 - 4x - 3$ (c) $y = \frac{1}{2}x^2 + \frac{4}{3}x - 1$

(d) $y = \frac{1}{3}(x - 1)(2x + 3)$ (e) $y = x^3 - x^2$ (f) $y = x^3 - 2x^2 + x$

(g) $y = -x^3 + 3x$ (h) $y = x^4 - x^3$ (i) $y = x^4 + \frac{2}{3}x^3 - x^2 - 1$

2. The curve $y = ax^3 + bx^2 - 3x + 2$ has stationary points at $x = -1$ and $x = 1$. Find a and b.

3. The curve $y = ax^3 + bx^2 - 2x + 3$ has stationary points at $x = 2$ and $x = 4$. Find a and b.

4. The curve $y = ax^2 + \dfrac{b}{x^2}$ has stationary points at $x = -1$ and $x = 1$. Find a and b.

5. The curve $y = ax + \dfrac{b}{x}$ has a stationary point at (1, 1). Find a and b.

20.4 Curve Sketching

Example 20.5

Use derivatives to find the stationary point(s), where they exist, of the curve $y = x^2(x - 3)$. Sketch this curve. Indicate clearly the stationary point(s) and intercept(s), where they exist.

Solution:

$y = x^2(x - 3) = x^3 - 3x^2 \quad \Rightarrow \quad \dfrac{dy}{dx} = 3x^2 - 6x$

For stationary points $\dfrac{dy}{dx} = 0. \Rightarrow 3x^2 - 6x = 0 \quad \Rightarrow x = 0$ or 2

When $x = 0$, $y = 0$. When $x = 2$, $y = -4$

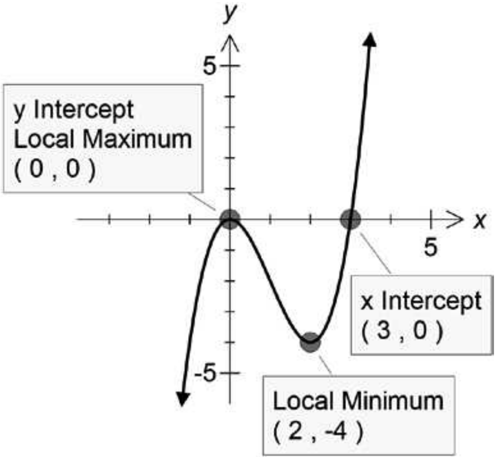
Hence, stationary points are (0, 0) and (2, –4).

Horizontal intercept(s) occur when $y = 0$.

$\Rightarrow x^2(x - 3) = 0 \Rightarrow x = 0, 3$

Vertical intercept: when $x = 0 \Rightarrow y = 0$.

The sketch of $y = x^2(x - 3)$ is given in the accompanying diagram.



Note:

- *In this instance, it is not necessary to verify the nature of the stationary points. The locations of the stationary points and the intercepts automatically define the nature of these points.*

Exercise 20.4

1. Use derivatives to find the stationary point(s), for each of the following curves. Hence, sketch these curves. Indicate clearly, the stationary point(s) and intercept(s).
   (a) $y = x^2(x - 1)$ (b) $y = -x^2(x + 1)$ (c) $y = x(x - 3)^2$
   (d) $y = -x(x + 4)^2$ (e) $y = x^4 - 2x^3$ (f) $y = \frac{1}{10}x^5 + 2x^2$

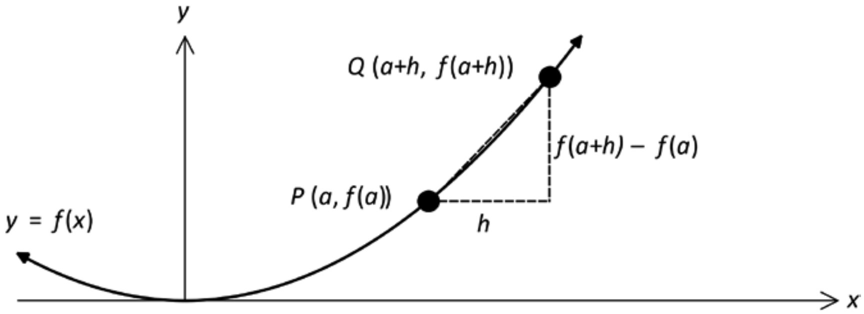
2. In each case, give a possible sketch for $y = f(x)$.
   (a) The curve $y = f(x)$ has exactly one turning point which is located at $(-2, 1)$ and intercepts at $(0, 0)$ and $(-4, 0)$ only.
   (b) The curve $y = f(x)$ has exactly one turning point which is located at $(2, 3)$ and an intercept at $(0, 5)$. For $x < 2$, the curve has a negative gradient and for $x > 2$, the curve has a positive gradient.
   (c) The curve $y = f(x)$, where $f(x)$ is a polynomial in x, has turning points at $(-2, 3)$ and $(1, \frac{-3}{2})$. For $x < -2$, $y' > 0$ and for $-2 < x < 1$, $y' < 0$ and for $x > 1$, $y' > 0$.
   (d) The curve $y = f(x)$ has turning points at $(-2, 2)$ and $(2, \frac{-26}{3})$.
   For $x < -2$ and $x > 2$, $y' > 0$ and for $-2 < x < 2$, $y' < 0$.

3. In each case, give a possible sketch for $y = f(x)$.
   (a) The curve $y = f(x)$ has stationary points at $(1, 0.92)$ and $(3, 2.25)$ and intercepts at $(0, 0)$ and $(3.83, 0)$. For $x < 3$, $y' \geq 0$ and for $x > 3$, $y' < 0$.
   (b) The curve $y = f(x)$ has stationary points at $(-2, 5)$, $(0, 1)$ and $(2, 5)$. It has intercepts at $(-2.91, 0)$ and $(2.91, 0)$. For $x < -2$, $0 < x < 2$, $y' > 0$. For $-2 < x < 0$ and $x > 2$, $y' < 0$.
   (c) The curve $y = f(x)$ has stationary points at $(1, -3.7)$ and $(3, -9)$. For $x < 3$, $y' \leq 0$. For $x > 3$, $y' > 0$.
   (d) The curve $y = f(x)$ has intercepts at $(0, 1)$ and $(1, 0)$ and no other intercepts. It has no stationary points. The curve has asymptotes with equations $x = -1$, $x = 2$ and $y = 0$. For $-1 < x < 2$, $x < -1$ and $x > 2$, $y' < 0$.

4. In each case, give a possible sketch for $y = f(x)$.
   (a) The curve $y = f(x)$ has no stationary points and has asymptotes with equations $x = 2$ and $y = 3$. For $x > 2$, $y' > 0$ and for $x < 2$, $y' > 0$.
   (b) The curve $y = f(x)$ has exactly one intercept which is located at $x = 2$ and exactly one asymptote with equation $x = 1$. It has no stationary points. For $x > 1$, $y' > 0$.
   (c) The curve $y = f(x)$ has a stationary point at $(2, 1)$ and an intercept at $(0, 0)$. For $x < 2$, $y' > 0$ and for $x > 2$, $y' < 0$. It has an asymptote with equation $y = 0$.
   (d) The curve $y = f(x)$ has asymptotes with equations $x = -1$, $x = 1$ and $y = 0$. It has an inflection point at $(0,-1)$. For $x < -1$ and $-1 < x < 1$ and $x > 1$, $y' < 0$.

21 Rates of Change

21.1 Instantaneous Rate of Change



- Consider the function $y = f(x)$.
  - When x changes from $x = a$ to $x = a + h$,
    the value of y changes from $f(a)$ to $f(a + h)$.
  - The average rate of change of y with respect to x
    as x changes from $x = a$ to $x = a + h$ is given by:

$$\text{Average Rate of Change} = \frac{\text{Change in } y - \text{value}}{\text{Change in } x - \text{value}}$$

$$= \frac{f(a+h)-f(a)}{(x+h)-x}$$

$$= \frac{f(a+h)-f(a)}{h}.$$

  - Notice that $\frac{f(a+h)-f(a)}{(x+h)-x}$ is the gradient of the chord PQ in the diagram above, where P is the point $(a, f(a))$ and Q is the point $(a + h, f(a + h))$.
  - As $h \to 0$, the gradient of the chord PQ $\to$ Gradient of tangent at P.
  - Hence, as $h \to 0$, the Average Rate of Change $\to$ Rate of Change at P.
  - But $\frac{f(a+h)-f(a)}{(x+h)-x} = f'(a)$.
  - Therefore, $f'(a)$ may be interpreted as the rate of change at P.
    $f'(a)$ is technically referred to as the instantaneous rate of change at P.
    That is, $f'(a)$ is the rate of change of y with respect to x at $x = a$.

- The gradient function can then be interpreted as:
  the instantaneous rate of change of y with respect to x.

- For example:
  If $P = f(t)$, represents the population of a town, at time t years,
  $\Rightarrow \frac{dP}{dt}$ is the instantaneous population growth rate.

Example 21.1

The temperature, θ° Celsius, of a body at time t minutes, is modelled by:
$\theta = 20 + 2t - 0.05t^2$, for $0 \le t \le 50$. Without the use of a calculator find:
(a) $\delta\theta$, the change in θ corresponding to a change in t from $t = 0$ to $t = 1$.
(b) the average rate of change of θ for $0 \le t \le 1$ minute. Interpret you answer.
(c) $\dfrac{d\theta}{dt}$.
(d) the instantaneous rate of temperature change when $t = 1$. Interpret your answer.

Solution:

(a) $\delta\theta = \theta(1) - \theta(0)$
$= 21.95 - 20 = 1.95$

(b) Average Rate of Change for $\theta = \dfrac{1.95}{1} = 1.95^{\circ}$ Celsius per minute.

That is, within the time interval $0 \le t \le 1$, θ increased with an average rate of 1.95° Celsius per minute.

(c) $\dfrac{d\theta}{dt} = 2 - 0.1t$.

(d) $\left.\dfrac{d\theta}{dt}\right|_{t=1} = 2 - 0.1(1) = 1.9^{\circ}$ Celsius per minute.

At the instant $t = 1$ minute, θ is increasing at a rate of 1.9° Celsius per minute.

Note:

- *The average rate of change is defined over an interval of time whereas the instantaneous rate of change is defined at a particular instant in time.*

21.1.1 The Leibniz Notation

- The change in y can be denoted δy while the change in x can be denoted δx.
- Hence, the average rate of change $= \dfrac{\text{Change in } y - \text{value}}{\text{Change in } x - \text{value}}$

$$= \frac{\delta y}{\delta x}$$

- From the previous section,

  the instantaneous rate of change of y with respect to $x = \lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right)$.

- The instantaneous rate of change of y with respect to x $\lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right)$ is denoted $\dfrac{dy}{dx}$.

  That is $\lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right) = \dfrac{dy}{dx}$. This is the origin of the expression $\dfrac{dy}{dx}$, used to represent the gradient function.

Example 21.2

The population, N, of a colony of alpacas being bred by a farmer, is modelled by
$N = 50 + 45t - 7t^2 + \frac{1}{3}t^3$, where t is time in years.

(a) Find the average rate of population growth for $3 \le t \le 9$. Interpret your answer.
(b) Find the instantaneous population growth when $t = 3$ and $t = 9$. Interpret your answers.
(c) Find when the instantaneous population growth is zero.

Solution:

(a) $N = 50 + 45t - 7t^2 + \frac{1}{3}t^3$

When $t = 3$, $N = 131$. When $t = 9$, $N = 131$.

$$\text{Average growth rate} = \frac{\text{change in population}}{\text{time period}}$$

$$\Rightarrow \text{Average growth rate} = \frac{131 - 131}{6} = 0$$

For $3 \le t \le 9$, the alpaca population had an average growth rate of zero.

(b) Instantaneous population growth, $N' = 45 - 14t + t^2$

$N'(3) = 12$.

$\Rightarrow$ At $t = 3$, the alpaca population is growing at a rate of 12 animals per year.

$N'(9) = 0$. $\Rightarrow$ At $t = 9$, the alpaca population is stable.

(c) $N' = 0 \Rightarrow 45 - 14t + t^2 = 0$

$t = 5, 9$.

Hence, instantaneous population growth is zero at $t = 5$ and $t = 9$ years.

Exercise 21.1

1. Given that, $A = 3t^2 + 4t - 5$, without the use of a calculator find:
   (a) an expression for the instantaneous rate of change of A with respect to t.
   (b) the instantaneous rate of change of A with respect to t when $t = 0$
   (c) the value of t when the instantaneous rate of change of A with respect to t is 5.

2. Given that, $A = -(t - 4)^2 + 10$ where $t > 0$, without the use of a calculator find:
   (a) an expression for the instantaneous rate of change of A with respect to t
   (b) the instantaneous rate of change of A with respect to t when $t = 4$
   (c) the value of t when the instantaneous rate of change of A with respect to t is -4.

3. Given that, $P = \frac{2}{3}t^3 + t^2 - 40t - 5$ where t is time, without the use of a calculator, find:
   (a) an expression for the instantaneous rate of change of P with respect to t
   (b) the instantaneous rate of change of P with respect to t when $t = 0$
   (c) the value of t when the instantaneous rate of change of P with respect to t is 0.

4. Given that, $V = 2(t-3)^3 + 4$, without the use of a calculator find:
   (a) the instantaneous rate of change of V with respect to t when $t = 3$ and $t = 4$
   (b) the average rate of change of V with respect to t for $3 \le t \le 4$.

5. The temperature, θ degrees Celsius, of a body, at time t minutes, is modelled by $\theta = 36 + 1.2t - 0.04t^2$. Without the use of a calculator find:
   (a) the initial temperature of the body (b) the maximum temperature of the body
   (c) the instantaneous rate of temperature change with respect to time when $t = 5$
   (d) the value of t when the instantaneous temperature change is 1° Celsius per minute.

6. The temperature of a body at time t hours, is modelled by $\theta = 0.1(t-10)^2 + 15$ Celsius.
   (a) Find the temperature when $t = 6$ and $t = 14$.
   (b) Find the average rate of temperature change for $6 \le t \le 14$. Interpret your answers.
   (c) Find the instantaneous rate of temperature change with respect to time when $t = 6$ and 14. Interpret your answers.
   (d) Find the minimum temperature of the body and the corresponding value of t.

7. The price, P cents, of a KHP share, on the Stock Exchange, is modelled by $P = \frac{1}{3}t^3 - 8t^2 + 55t + 56$, for $0 \le t \le 15$, where t is time in years after being first listed on the Stock Exchange. [P is the average price for the year.]
   (a) Find the price of the share when it was first listed.
   (b) Find the instantaneous rate of change of price with respect to time, for $t = 4$. Interpret your answer.
   (c) Find the average rate of change of price for $4 \le t \le 8$. Interpret your answer.
   (d) Find when the instantaneous rate of change of price is zero.

8. The price per kg, P cents, for an agricultural commodity at time t years is modelled by $P = -\frac{1}{3}t^3 + 13t^2 - 144t + 820$ for $0 \le t \le 20$. [P is the average price per kg for the year.]
   (a) Find the price of the commodity at $t = 20$.
   (b) Find the average rate of price change in the first 20 years. Comment on your answer.
   (c) Find the instantaneous rate of price change at $t = 20$. Comment on your answer.
   (d) Find during which years, the instantaneous rate of price change is zero.

9. Given that, $N = 2\sqrt{t} - t$, for $0 < t \le 6$, without the use of a calculator find::
   (a) the instantaneous rate of change of N with respect to t when $t = 4$
   (b) the average rate of change of N with respect to t for $1 \le t \le 4$.
   (c) the interval for which the instantaneous rate of change is negative.

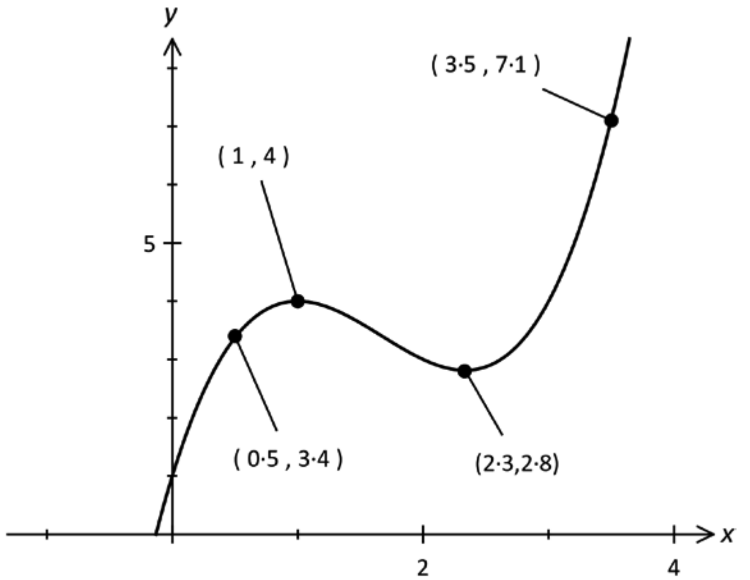
10. Given that, $N = \frac{1}{t} + 4t^2$ for $t > 0$, without the use of a calculator find::
   (a) the instantaneous rate of change of N with respect to t when $t = 1$.
   (b) the value(s) of t when the instantaneous rate of change of N with respect to t is negative.

21.2 Optimisation

- Optimisation deals with finding the maximum and minimum values of a given function within a given interval.

21.2.1 Global Points and Turning Points

- For a given function $f(x)$, within a given interval:
  - the point that corresponds to the highest value for $f(x)$ is called the global maximum point
  - the point that corresponds to the lowest value of $f(x)$ is called the global minimum point
  - the minimum and maximum turning points are referred to as local minimum and local maximum points respectively
- The diagram below shows the graph of a function $y = f(x)$.



- Consider $y = f(x)$ in the domain $0.5 \le x \le 3.5$.
  - $y = f(x)$ has a minimum turning point at (2.3, 2.8) and a maximum turning point at (1, 4).
  - The end points are at (0.5, 3.4) and (3.5, 7.1).
  - Hence, for $y = f(x)$ above, defined over the domain $0.5 \le x \le 3.5$:
    - (1, 4) is a *local maximum point*
    - (3.5, 7.1) is a *global maximum point*
    - (2.3, 2.8) is a *global minimum point* and a *local minimum point*.
- The local maximum and minimum points need not necessarily be the global maximum and minimum points respectively.

Example 21.3

Consider the curve with equation $y = x^3 - 3x - 1$ for $-3 \le x \le 2$.

(a) Use calculus to find and verify the local minimum and maximum points on this curve.

(b) Find the maximum and minimum values of y for $-3 \le x \le 2$.

Solution:

(a) $y = x^3 - 3x - 1 \;\Rightarrow\; \dfrac{dy}{dx} = 3x^2 - 3$

For turning points, $\dfrac{dy}{dx} = 0 \quad \Rightarrow 3x^2 - 3 = 0 \;\Rightarrow\; x = \pm 1$

When $x = 1$, $y = -3$.
Using the sign test:

x	$x = 1^-$	$x = 1$	$x = 1^+$
sign for $\dfrac{dy}{dx}$	$-$	0	+

Hence, (1, –3) is a local minimum point.

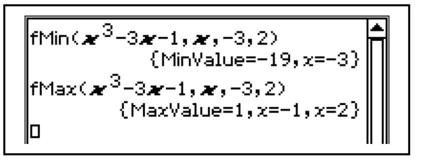
When $x = -1$, $y = 1$.
Using the sign test:

x	$x = -1^-$	$x = -1$	$x = -1^+$
sign for $\dfrac{dy}{dx}$	+	0	$-$

Hence, (–1, 1) is a local maximum point.

(b) At the end points: $x = -3$, $y = -19$ and $x = 2$, $y = 1$.

Comparing the values of y at the end points and at the turning points:
the maximum value of y is 1 and this occurs when $x = -1$ and when $x = 2$,
the minimum value of y is –19 and this occurs when $x = -3$.



Summary

- To find the maximum and minimum values of $y = f(x)$ over the domain $a \le x \le b$ using an analytical method (calculus):
  - find the maximum and minimum turning points on the curve $y = f(x)$
  - find the values of $f(x)$ at the end points
  - compare the values of $f(x)$ at the turning points and at the end points.

  [f(x) is assumed to be continuous (does not have any "breaks") for $a \le x \le b$.]
- CAS/graphic calculators have built-in routines, (*fMin* and *fMax* routines), for determining the global minimum and global maximum values of a function within a given interval.

Example 21.4

The number of litres, V (kL), of unleaded petrol, sold at an outlet each day, is modelled by $V = -\frac{t^3}{3} + \frac{15t^2}{2} - 44t + 100$, for $0 \le t \le 14$, where t is the number of days into a given fortnight. Use an analytical method to find:

(a) how many litres of unleaded petrol were sold at the start of the fortnight
(b) the minimum amount of unleaded petrol sold and the corresponding t value
(c) the maximum amount of unleaded petrol sold and the corresponding t value.

Solution:

(a) At the start of the fortnight, $t = 0$. $\Rightarrow$ $V = 100$ kL.

(b) $V = -\frac{t^3}{3} + \frac{15t^2}{2} - 44t + 100$

$\Rightarrow \frac{dV}{dt} = -t^2 + 15t - 44$

For Max/Min value of V, $\frac{dV}{dt} = 0$

$\Rightarrow$ $t = 4, 11$.

When $t = 4$, $V = 22.67$ kL.
When $t = 11$, $V = 79.83$ kL.
At the end point corresponding to $t = 14$, $V = 39.33$ kL.
Comparing values of V at the end points with that at the turning points, the minimum amount of unleaded petrol sold is 22.67 kL when $t = 4$.

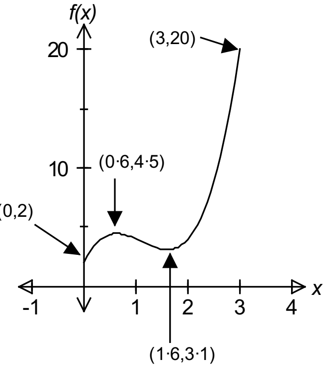
(c) Comparing values of V at the end points with that at the turning points, the maximum amount of unleaded petrol sold is 100 kL when $t = 0$.

Note:

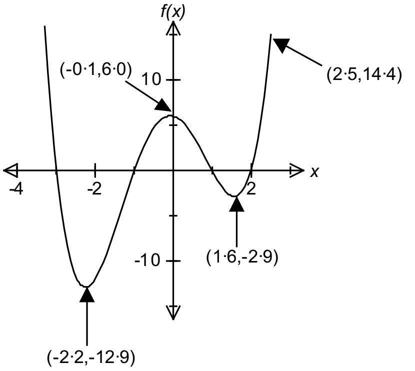
- *There is no actual need to confirm the nature of the turning points unless directed to.*

Exercise 21.2

1. The graph of $y = f(x)$ is given below for $0 \le x \le 3$.
   Find the maximum and minimum value for $f(x)$ and the value of x at which they occur.



2. The graph of $y = f(x)$ is given below.
   Find the maximum and minimum value for $f(x)$ and the value of x at which they occur within each of the following intervals:
   (a) $-3 \le x \le 2.5$
   (b) $-1 \le x \le 2.5$
   (c) $-3 \le x \le 2$
   (d) $-3 \le x \le 1$



3. Use a non-graphical method to find the global maximum and global minimum value for $y = x^3 + x^2 - x - 1$, and the value of x at which they occur within each of the following intervals: (a) $-2 \le x \le 2$ (b) $0 \le x \le 2$ (c) $-2 \le x \le 0$

4. Repeat Question 3 for $y = -x^3 + 4x^2 - 5x + 2$ for the following intervals:
   (a) $-2 \le x \le 3$ (b) $0 \le x \le 2$ (c) $1 \le x \le 3$

5. Without the use of a calculator, find the global maximum and global minimum value and the corresponding x-values for $y = \frac{x}{2} + \frac{1}{\sqrt{x}}$ for $\frac{1}{4} \le x \le 4$.

6. Without the use of a calculator, find the global maximum and global minimum value and the corresponding t-values for $v = 2t + \frac{8}{t}$ for $1 \le t \le 4$.

7. The number of litres, V (kL), of unleaded petrol, sold at an outlet each day, is modelled by $V = \frac{t^3}{3} - 9t^2 + 65t + 50$, for $0 \le t \le 14$, where t is the number of days into a given fortnight. Use an analytical method to find:
   (a) how many litres of unleaded petrol was sold at the start of the fortnight
   (b) the minimum amount of unleaded petrol sold and the corresponding t value
   (c) the maximum amount of unleaded petrol sold and the corresponding t value.

8. The amount, A (mg), of a drug in the bloodstream, t hours after it is introduced into a patient's system is given by $A = \frac{t^4}{2000} - \frac{3t^3}{250} - \frac{t^2}{20} + \frac{6t}{5}$. Use an analytical method to find:
   (a) the amount of the drug present after (i) 4 hours (ii) 8 hours.
   (b) the maximum amount of the drug present and when this occurs.

9. The number of customers per 15 minutes, N, through the checkout lane of a small delicatessen, on a particular afternoon, is modelled by $N = 0.5t(t-2)^2(1.5t-7)^2 + 5$, where t is the time in hours after 12 noon. Use an appropriate method to find the "slow" times between 12 noon and 6 pm and corresponding number of customers.

21.3 More Optimisation

- In this section, we confront problem solving situations where the function to be maximised has to be constructed from the contextual information given.

Example 21.5

Given that $x + y = A$ and $xy = 4$ where $x > 0$, find the minimum value of A and the value of x and y at which this occurs.

Solution:

Function to be optimised: $A = x + y$
Choose independent variable as: x
Rewriting A in terms of the chosen independent variable x:

$$xy = 4$$
$$\Rightarrow \quad y = \frac{4}{x}$$

Hence: $A = x + \dfrac{4}{x}$

Therefore: $\dfrac{dA}{dx} = 1 - \dfrac{4}{x^2}$

For max/min values: $\dfrac{dA}{dx} = 0 \qquad 1 - \dfrac{4}{x^2} = 0 \quad \Rightarrow \quad x = \pm 2$

When $x = 2$, $A = 4$:
Using the sign test:

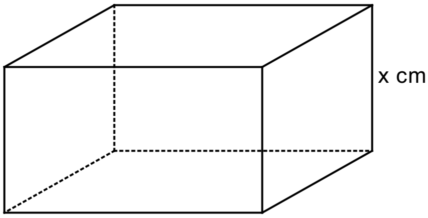
x	$x = 2^-$	$x = 2$	$x = 2^+$
sign for $\dfrac{dA}{dx}$	$-$		$+$

Hence, (2, 4) is a local minimum point.

For values of x close to 0, A becomes very large.
For very large values of x, A becomes very large.
Hence, the minimum value for A is 4 when $x = 2$ and $y = 2$.

Notes:
- *Initially the function to be maximised, A, is expressed in terms of two variables x and y.*
- *Hence, before A can be optimised, A needs to be expressed in exactly one variable.*
- *From information gleaned from the question, an algebraic relationship between the two variables x and y is found. This allows us to express one variable in terms of the other.*
- *The function to be optimised can now be expressed in terms of exactly one variable.*
- *We can now proceed to optimise A.*

Example 21.6



A piece of wire, 2 metres long, is used to make the 12 edges of the frame of a rectangular box. The height, x cm, of the rectangular frame is half that of the length of the frame.

(a) Show that the volume, V, of the rectangular box is given by, $V = 2x^2(50 - 3x)$.

(b) State for what values of x would the expression for V be valid.

(c) Use calculus to find the dimensions of the frame that will maximise the V.

Solution:

(a) Function to be maximised: Volume of rectangular box

Given variables: Height of frame, x and length of frame, $2x$.

Define variable: Width of frame, y.

Express V in terms of variables given and defined:

$$V = x \times 2x \times y$$

Choose independent variable as: x

Total length of wire used = 200 cm

$$4x + 4(2x) + 4y = 200$$

$$\Rightarrow \quad y = 50 - 3x$$

Hence: $V = 2x^2(50 - 3x)$

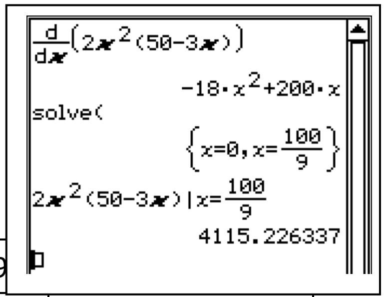
(b) From the expression for V, the expression is valid for $0 < x < \frac{50}{3}$.

(c) $V = 100x^2 - 6x^3 \qquad \frac{dV}{dx} = 200x - 18x^2$

For max/min values, $\frac{dV}{dx} = 0$.

$$200x - 18x^2 = 0 \quad \Rightarrow \quad x = 0 \text{ or } \frac{100}{9}$$

When $x = \frac{100}{9}$, $V = 4115.2$ cm^3 .



Using the sign test:

x	$x = (100/9)^-$	$x = 100/9$	
sign for $\frac{dV}{dx}$	+	0	–

Hence, $(\frac{100}{9}$, 4115.2) is a local maximum point.

Close to the end points, $V = 0$.

Hence, the maximum volume is 4115.2 cm^3

when the frame has a height of $\frac{100}{9}$ cm, a length of $\frac{200}{9}$ cm and a width of $\frac{50}{3}$ cm.

Exercise 21.3

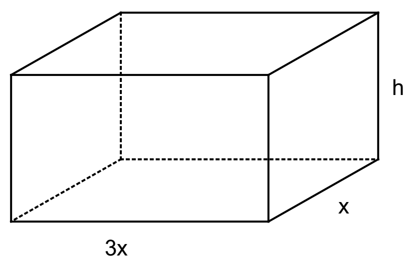
1. Given that $A = 2x + y$ and $xy = 8$, where $x > 0$, use an analytical method to find the minimum value of A and the corresponding values of x and y.

2. Given that $A = y - 4x^2$ and $xy = -1$, where $x > 0$, use an analytical method to find the maximum value of A and the corresponding values of x and y.

3. Given that $A = x^2 y$ and $x + y = 10$, where $x > 0$, use an analytical method to find the maximum value of A and the corresponding values of x and y.

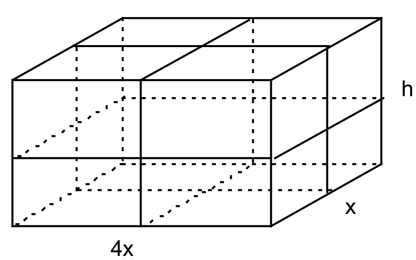
4. Given that $A = x^2 y$ and $x - y = 4$, where $0 \le x \le 4$, use a non-graphical method to find the minimum value of A and the corresponding values of x and y.

5. A piece of wire, 300 cm long is used to make the 12 edges of the frame of a rectangular box. The length of the rectangular frame is 3 times that of the width of the frame, x cm.



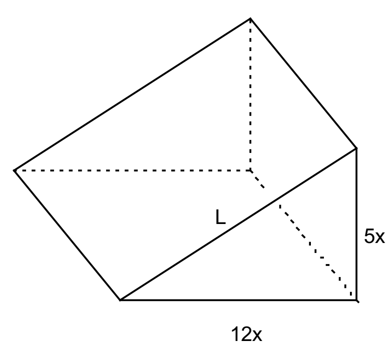
   (a) Show that the height, h, of the rectangular box is given by, $h = 75 - 4x$.
   (b) Show that the volume, V, of the box is given by $V = 225x^2 - 12x^3$.
   (c) Use an analytical method to find the dimensions of the frame that will maximise the volume of the box.

6. A piece of wire, 500 cm long is used to make the 24 edges of the frame of a closed rectangular cage. The length of the cage is 4 times that of the width of the cage, x cm.



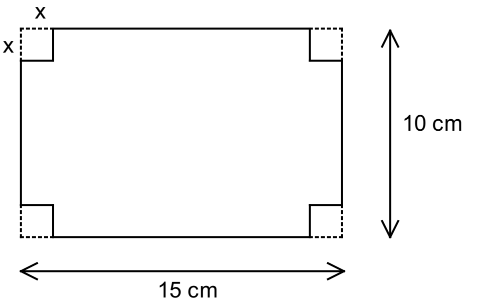
   (a) Show that the height, h, of the cage is given by, $h = 62.5 - 5x$.
   (b) Show that the volume, V, of the cage is given by $V = -20x^3 + 250x^2$.
   (c) Use an analytical method to find the dimensions of the frame that will maximise the volume of the cage.

7. A piece of wire, 300 cm long is used to make the 9 edges of the frame of a rectangular wedge. The height and length of the wedge are $5x$ and $12x$ respectively.

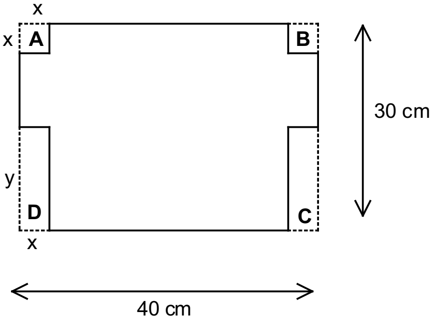


   (a) Given that L, is the length of the hypotenuse of its cross-section, show that $L = 13x$.
   (b) Show that the width of the wedge is given by $w = 100 - 20x$.
   (c) Show that the volume, V, of the wedge is given by $V = -600x^3 + 3\,000x^2$.
   (d) Use an analytical method to find the dimensions of the frame that will maximise the volume of the wedge.

8. A rectangular sheet of cardboard, 10 cm by 15 cm, is to be made into an open rectangular box. A square of side x cm, is removed from each of the four corners of the cardboard.
   (a) Show that the length of the box is given by $l = 15 - 2x$.
   (b) Show that the volume, V, of the box is given by $V = x(15 - 2x)(10 - 2x)$.
   (c) Use an analytical method to find the dimensions of the box that will maximise its volume.

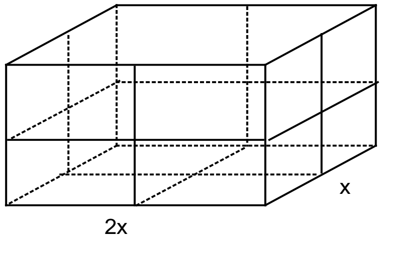


9. A rectangular sheet of cardboard, 30cm by 40 cm is to be made into a closed rectangular box. A square of side, x cm, is removed from each of the corners A and B of the cardboard. A rectangle x cm by y cm, is removed from each of the corners C and D of the cardboard.
   (a) Show that the length of the box is given by $L = 40 - 2x$.
   (b) Show that the volume, V, of the box is given by $V = x(15 - x)(40 - 2x)$.
   (c) Use calculus to find the dimensions of the box that will maximise its volume.

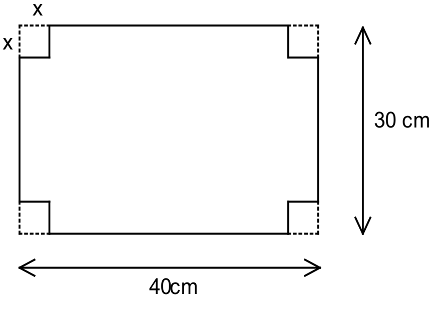


10. A piece of wire 240 cm long is used to make the 12 edges of the frame of a rectangular box. The length of the frame is 5 times its width, x cm. Use derivatives to find the dimensions of the box that will maximise its volume, and give this maximum volume.

11. A piece of wire 600 cm long is used to make the edges of an open rectangular cage. The length of the cage is twice that of its width. Use derivatives to find the maximum volume of the cage



12. A rectangular sheet of cardboard, 30 cm by 40 cm is to be made into an open rectangular box. A square of side x cm is removed from each of the four corners of the cardboard. Use derivatives to find the maximum volume of the box.



13. The total surface area of a closed rectangular box is 1 200 cm^2. The height of the box is four times its width. Find the dimensions of the box that will maximise its volume.

14. The total surface area of a closed cylindrical container is 1 800 cm^2. Find the dimensions of the container that will maximise its volume and give this maximum volume.

15. A closed rectangular box, has a volume of 10 000 cm^3. The height of the box is twice its width. Find the dimensions of the box that will minimise its surface area.

16. A rectangular box, open at one of its length $\times$ width end, has a volume of 15 000 cm^3. The length of the box is three times its width. Find the minimum surface area of the box.

17. A cylindrical canister, closed at both ends, has a volume of 16 000 cm^3. Find the minimum surface area of the canister

18. A cylindrical canister, open at one end, has a volume of 24 000 cm^3. Find the minimum surface area of the canister.

22 Anti-Differentiation

22.1 Anti-Differentiation of Polynomials

•

- Consider $\frac{d}{dx}\left(x^3\right) = 3x^2$.

  An anti-derivative of $3x^2 = x^3$. I

- Consider now $\frac{d}{dx}\left(x^3 + 10\right) = 3x^2$.

  An anti-derivative of $3x^2 = x^3 + 10$. II

- Consider now $\frac{d}{dx}\left(x^3 - 50\right) = 3x^2$.

  An anti-derivative of $3x^2 = x^3 - 50$. III

- Notice, that the anti-derivative of $3x^2$ is not unique (see I, II and III).
  Essentially, the variuos answers for the anti-derivative of $3x^2$ differ in terms of the "number" added.
  - Hence, in general, the anti-derivative of $3x^2 = x^3$ + Constant.
- The process of determining the anti-derivative of a function may sometimes be referred to as finding the integral of the function.
  - In this instance, the statement: the anti-derivative of $3x^2 = x^3$ + Constant is written symbolically as $\int 3x^2\, dx = x^3$ + Constant.
  - This is read as "the integral of $3x^2 = x^3$ + Constant".
    The constant is called the constant of integration.

22.1.1 Rules for Anti-Differentiation

- Listed below, are the rules for anti-differentiating x^n for integer $n \neq -1$ and $m \neq -1$.

• $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$	where C is a constant.
• $\int ax^n\, dx = a\int x^n\, dx$	where a and C are constants.
• $\int ax^m + bx^n\ dx = \int ax^m\, dx + \int bx^n\, dx$	where a and b are constants.

- In this unit, we will only consider the anti-derivatives of positive integral powers of x.

Example 22.1

Without the use of a calculator, find the anti-derivative with respect to *x*, for:

(a) x^5 (b) $\dfrac{3x^2}{4}$ (c) $-4x+3$ (d) $\dfrac{3x^4+2x^3}{x^2}$.

Solution:

(a) The anti-derivative of $x^5 = \dfrac{x^6}{6} + C$.

(b)
$$\int \frac{3x^2}{4}\,dx = \frac{3}{4}\int x^2\,dx = \frac{3}{4}\left(\frac{x^3}{3}\right) + C = \frac{x^3}{4} + C.$$

(c)
$$\int -4x+3\,dx = -4\left(\frac{x^2}{2}\right) + 3x + C = -2x^2 + 3x + C.$$

(d) Rewrite $\dfrac{3x^4+2x^3}{x^2} = \dfrac{3x^4}{x^2} + \dfrac{2x^3}{x^2} = 3x^2 + 2x.$

Hence:
$$\int \frac{3x^4+2x^3}{x^2}\,dx = \int 3x^2 + 2x\,dx = 3\left(\frac{x^3}{3}\right) + 2\left(\frac{x^2}{2}\right) + C = x^3 + x^2 + C.$$

Example 22.2

Given that $f'(x) = 2x+3$, without the use of a calculator, find $f(x)$, given that $f(1) = 1$.

Solution:

$$f'(x) = 2x+3 \Rightarrow f(x) = 2\left(\frac{x^2}{2}\right) + 3x + C = x^2 + 3x + C.$$

Substitute $x = 1$ into $f(x)$: $f(1) = 1 + 3 + C$

But $f(1) = 1$: $\Rightarrow 1 = 4 + C \quad \Rightarrow C = -3.$

Therefore, $f(x) = x^2 + 3x - 3.$

Note:

- *In this example, the constant of integration is determined by the condition that f (1) = 1.*

Exercise 22.1 **Calculator Free**

1. Find the anti-derivatives with respect to x, for each of the following:

   (a) $4x$ (b) 10 (c) $-3x^2$ (d) $4x^7$

   (e) $\dfrac{x^3}{2}$ (f) 5^4 (g) $\dfrac{4x^3}{5}$ (h) $-\dfrac{3x^2}{2}$

   (i) $(3x)^3$ (j) $\left(\dfrac{x^2}{2}\right)^3$ (k) $\sqrt{25x^6}$ (l) $\dfrac{x^7}{3x^4}$

2. Find:

   (a) $\int x^3 + 4\, dx$ (b) $\int 2x^2 + 4\, dx$ (c) $\int -x^3 + 3x\, dx$ (d) $\int 4x - x^4\, dx$

   (e) $\int t(t+2)\, dt$ (f) $\int 3t(t^2 + 4t)\, dt$ (g) $\int (u+1)^2\, du$ (h) $\int (2u-1)^2\, du$

   (i) $\int \dfrac{x^3 - x^2}{x^2}\, dx$ (j) $\int \dfrac{2x^5 - x^7}{4x^3}\, dx$ (k) $\int \dfrac{x^4 - x^3 + 2x}{5x}\, dx$ (l) $\int \dfrac{u^4 - 1}{u^2 + 1}\, du$.

3. Given that $f'(x) = x^2 + 2$, find $f(x)$ if $f(0) = 1$.

4. Given that $f'(t) = (t-1)^2$, find $f(t)$ if $f(3) = 0$.

5. Given that $\dfrac{dy}{dx} = \dfrac{-4x^6 + 3x^5}{x^3}$, find y if $x = 1$, $y = 2$.

6. Given that $y' = a$, where a is a constant, find y if $y(0) = 2$ and $y(1) = 1$.

7. Given that $\dfrac{dV}{dt} = at - 5$, where a is a constant, find V if $V(0) = 2$ and $V(2) = -10$.

8. Given that $f'(x) = ax + b$, where a and b are constants, find $f(x)$ if $f(0) = -5$, $f(1) = 2$ and $f(-1) = -6$.

22.2 Gradient Function and Integration

- If $y = f(x)$ represents the equation of a curve, then $f'(x)$ represents the gradient function of the curve.
- Hence, if the gradient function of a curve is $f'(x)$, the equation of the curve can be obtained using $y = \int f'(x)\, dx$.

Example 22.3

Find the equation of the curve with gradient function $f'(x) = 4x + 3$, given that the curve passes through the point (0, 3).

Solution:

$f'(x) = 4x + 3 \quad \Rightarrow \quad y = \int 4x + 3\, dx = 2x^2 + 3x + C.$

When $x = 0$, $y = 3$, $\quad \Rightarrow \quad 3 = 2(0)^2 + 3(0) + C \;\Rightarrow\; C = 3$

Hence, the equation of the curve is $y = 2x^2 + 3x + 3$.

Note: • *$y = \int f'(x)\, dx$ gives a family of curves.*
Particular curves are obtained only when a point of passage is known.

Exercise 22.2 Calculator Free

1. Find the equation of the curve with gradient function $f'(x) = 4$, given that the curve passes through the point (0, –1).
2. Find the equation of the curve with gradient function $f'(x) = -2x$, given that the curve passes through the point (0, 4).
3. Find the equation of the curve with gradient function $\frac{dy}{dx} = \frac{2x}{3} - 1$, given that the curve passes through the point (3, 2).
4. Find the equation of the curve with gradient function $y' = (2x - 1)^2$, given that the curve passes through the point (0, 1).
5. Find the equation of the curve with gradient function $\frac{dy}{dx} = \frac{-2x + 3x^2}{x}$, given that the curve passes through the point (2, 0).
6. Find the equation of the curve with gradient function $f'(x) = k^2$ where k is a constant, given that the curve passes through the points (0, 1) and (1, 2).
7. Find the equation of the curve with gradient function $f'(x) = x + a$ where a is a constant, given that the curve passes through the point (–1, 2) and (1, 3).
8. Find the equation of the curve with gradient function $y' = ax + 2$ where a is a constant, given that the curve passes through the point (0, 1) and has a turning point at $x = -1$.
9. Find the equation of the curve with gradient function $f'(x) = ax + b$ where a and b are constants, given that the $f(1) = 2$ and the curve has a turning point at (–1, 0).
10. Find the equation of the curve with gradient function $f'(x) = ax^2 + bx - 2$, given that the curve passes through the origin and has turning points at $x = -1$ and $x = 2$.

23 Rectilinear Motion

23.1 Rectilinear Motion and Differentiation

- In this section, attention is focused on objects that move along straight lines. Such objects are said to undergo rectilinear motion.

- Consider a body P moving along a straight line.
  Let its displacement from a fixed point O at time t be s.

  - Then $\frac{ds}{dt}$ represents the instantaneous rate of change of displacement with respect to time t. This is termed the *instantaneous velocity* of P, denoted v.

  - The speed of the body at time $t = |v|$.

  - The *average speed* of a body over a given time interval is given by:

$$\text{average speed} = \frac{\text{distance travelled}}{\text{time interval}}$$

Example 23.1

A particle P moves along a straight line. Its displacement s metres, t seconds after passing a fixed point O, is given by, $s = t\,(t-2)(t-5) \equiv t^3 - 7t^2 + 10t$, for $t \geq 0$.
(a) Find when the particle returns to O.
(b) Find the velocity of P when it returns to O.
(c) Find when the body is instantaneously at rest.

Solution:

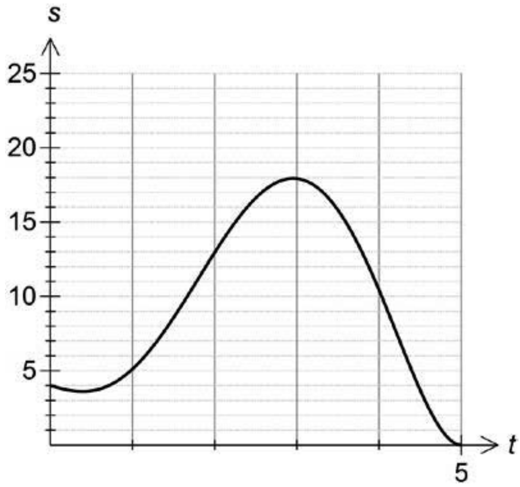
(a) Particle returns to O when $s = 0$.
Hence, $t\,(t-2)(t-5) = 0 \;\Rightarrow\; t = 0, 2, 5$
That is, P returns to O when $t = 2$ seconds and when $t = 5$ seconds.

(b) Velocity $v = \frac{ds}{dt} = 3t^2 - 14t + 10$
When $t = 2$, $v = -6$. When $t = 5$, $v = 15$
Hence, velocity of P when it returns to O is -6 ms^{-1} and 15 ms^{-1}.

(c) Body is at rest when $v = 0 \;\Rightarrow\; 3t^2 - 14t + 10 = 0 \;\Rightarrow\; t = 0.88, 3.79$
That is, P is instantaneously at rest when $t = 0.88$ s and 3.79 s.

Example 23.2

The accompanying diagram shows the displacement (metres) time graph of an object P moving in a straight line from a fixed point O for $0 \le t \le 5$ seconds. Use this diagram to estimate:
(a) the initial displacement of P
(b) the farthest distance P is from O
(c) the velocity of P at $t = 4$ seconds
(d) when the velocity of P is 5 ms^{-1}.



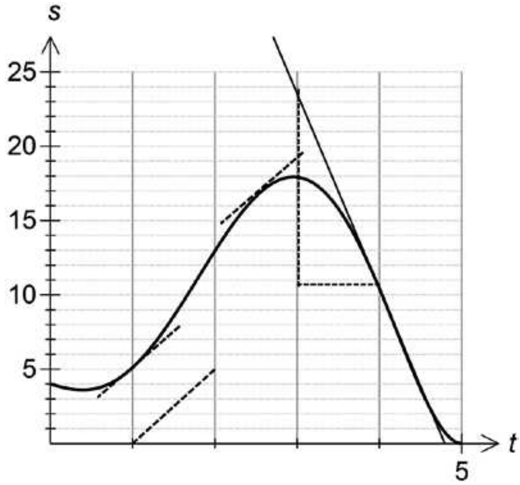
Solution:

(a) When $t = 0$, $s = 4$ metres.

(b) Farthest distance P is from O = 18 metres.

(c) Draw tangent to graph at $t = 4$ seconds.
From tangent drawn:
gradient of tangent ≈ -13.
Hence, gradient of curve at $t = 4 \approx -13$.
Therefore, velocity at $t = 4 \approx -13$ ms^{-1}.

(d) Use the grid provided to draw a line with gradient 5. "Float" this line until it just touches the curve.
Curve has gradient 5 at $t \approx 1$ and $t \approx 2.5$.
Hence, velocity of P is 5 ms^{-1} at $t \approx 1$ and $t \approx 2.5$ seconds.



Note:

- *For a displacement-time (s-t) graph, the gradient of the graph is* $\frac{ds}{dt}$; *which represents the velocity of the particle.*

Exercise 23. 1

1. A particle P moves along a straight line. Its displacement s metres, from a fixed point O, at time t seconds, is given by, $s = (t-1)(t-4)$ for $0 \le t \le 7$. Without the use of a calculator, find: (a) the initial displacement of P.
(b) when the particle is at O and its velocity when it is at O.
(c) when the body is instantaneously at rest.
(d) how long was P to the left of O?

2. A particle P moves along a straight line. Its displacement s metres, from a fixed point O, at time t seconds, is given by, $s = (t-1)(3-t)(t-4) \equiv -t^3 + 8t^2 - 19t + 12$ for $0 \le t \le 10$. Find:
(a) when the particle is at O. (b) exactly when P is instantaneously at rest
(c) when P was to the right of O?

3. A particle P moves along a straight line. Its displacement s metres, from a fixed point O, at time t seconds, is given by, $s = (t-1)^2(t-4) \equiv t^3 - 6t^2 + 9t - 4$ for $0 \le t \le 10$. Find:
   (a) when P returns to O and the corresponding velocities
   (b) the average velocity in the first 2 seconds.

4. A particle P moves along a straight line. Its displacement from a fixed point O, s metres, at time t seconds, is given by, $s = \sqrt{t} + \dfrac{1}{\sqrt{t}}$ for $t > 0$. Find:
   (a) the nearest distance P gets to O. *(b) when the speed of P is $\frac{3}{16}$ ms^{-1}.

5. A projectile P is launched vertically upwards. Its height h metres, at time t seconds, is given by, $h = 30t - 4.9t^2$ for $t \ge 0$.
   (a) Find the height after 5 seconds. (b) Find the velocity of P as it hits the ground.
   (c) Find when its velocity is 20 ms^{-1}. *(d) Find when its speed is 20 ms^{-1}.

6. An object P is thrown vertically upwards. Its height h metres, at time t seconds, is given by, $h = 10t - 4.9t^2$ for $t \ge 0$. Find:
   (a) the initial velocity of P. (b) the highest point reached by P.
   (c) the velocity of P as it hits the ground (d) the height of P when its speed is 5 ms^{-1}.

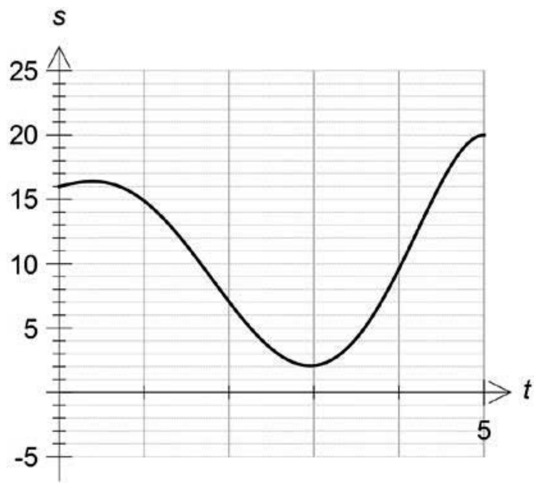
7. A ball is projected vertically upwards from a height of 2 m. Its height, h metres, from the point of projection, at time t seconds, is given by, $h = 5t - 4.9t^2$ for $t \ge 0$.
   (a) Find the time it takes for the ball to hit the ground.
   (b) Find the total distance travelled by the ball before hitting the ground.
   (c) Find the velocity of the ball when it is 3 m above the ground.
   (d) Find the speed with which the ball hits the ground.

8. A car P moves in a straight line. The distance travelled by the P, s metres, t seconds after it starts to slow down and before the P comes to a stop, is given by, $s = 12t - t^2$.
   (a) Find the time taken by P to come to a complete stop.
   (b) A stationary car Q is 30 m directly ahead of P when P starts to slow down.
   Will P hit the stationary car Q? Justify your answer.

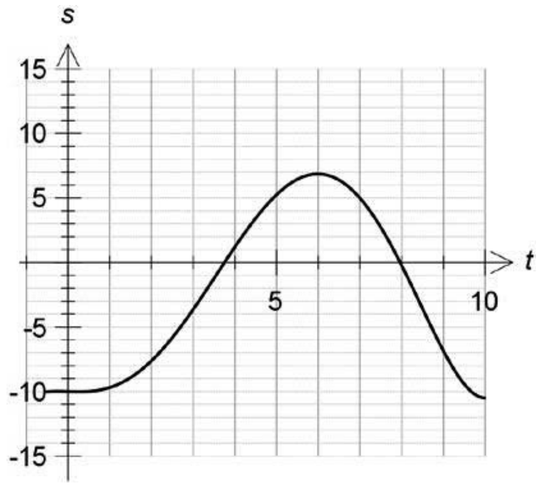
9. A car is travelling in a straight line. Its distance s metres, from a fixed point O, at time t seconds, is given by, $s = 10 + 14t + \frac{1}{3}t^3$ for $t \ge 0$.
   (a) Find the initial velocity of the car. (b) Find when the velocity of the car is 18 ms^{-1}.
   (c) Find the average speed of the car in the first 4 seconds.

10. A car passes a set of traffic lights (on the green cycle). Its distance s metres, t seconds after passing the set of lights, before it comes to a stop, is given by, $s = 10t + 2t^2 - \frac{1}{3}t^3$.
   (a) Find the speed of the car as it passed the lights.
   (b) Find the maximum speed achieved by the car after it passed the lights.
   (c) Find the distance travelled by the car before it comes to a stop.

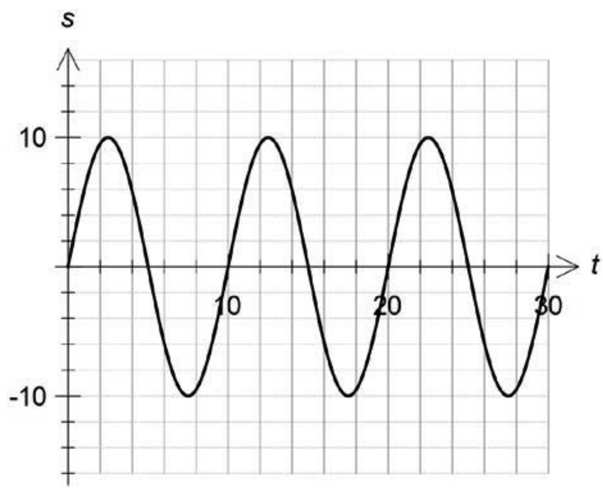
11. The accompanying diagram shows the displacement (metres) time graph of an object P moving in a straight line from a fixed point O for $0 \le t \le 5$ seconds. Use this diagram to estimate:
    (a) the initial displacement of P
    (b) the closest and farthest distance P is from O
    (c) the velocity of P at $t = 4$ seconds
    (d) when the velocity of P is -5 ms^{-1}.



12. The accompanying diagram shows the displacement (metres) time graph of an object P moving in a straight line from a fixed point O for $0 \le t \le 10$ seconds.
    (a) Estimate when P is instantaneously at rest.
    (b) Estimate the maximum speed of P, stating when this occurs.
    (c) Estimate the average speed of P in the first 8 seconds.



13. The accompanying diagram shows the displacement (metres) time graph of an object P moving in a straight line from a fixed point O for $0 \le t \le 30$ minutes.
    (a) Describe the motion of P.
    (b) Estimate the maximum speed of P.
    (c) Estimate the average speed of P in the first 30 seconds.



23.2 Rectilinear Motion and Integration

- Consider a body P moving along a straight line.
  Let its displacement from a fixed point at time t seconds be s metres.
- The instantaneous velocity $v = \dfrac{ds}{dt}$. Hence, $s = \int v\, dt$.
- The relationship between displacement s and velocity v, can be represented schematically as shown below.

$$\text{displacement} \underset{\text{integrate}}{\overset{\text{differentiate}}{\rightleftarrows}} \text{velocity}$$

Example 23.3

A particle P moves along the x-axis. Its velocity, v ms^{-1} at time t seconds is given by $v = -t^2 + 2t + 3$ for $0 \le t \le 5$ seconds.

(a) Find the maximum velocity of P.
(b) Find the change in displacement in the first 3 seconds.
(c) Find its displacement after 3 seconds given that P starts moving from a point 5 metres in the positive direction from the origin.

Solution:

(a) For maximum velocity, $\dfrac{dv}{dt} = 0$:

$$\Rightarrow \quad -2t + 2 = 0$$
$$t = 1$$

When $t = 1$: $\quad v = -1 + 2 + 3 = 4$ ms^{-1}.

Since v is a negative quadratic, maximum for $v = 4$ ms^{-1}.

(b) Displacement: $\quad s = \int -t^2 + 2t + 3\, dt$

$$= -\frac{t^3}{3} + t^2 + 3t + C$$

Change in displacement = $s(3) - s(0)$
= $(9 + C) - C$
= 9 metres.

(c) From (b): $\quad s = -\dfrac{t^3}{3} + t^2 + 3t + C$

But $s(0) = 5$. $\quad \Rightarrow \quad C = 5$

Hence: $\quad s = -\dfrac{t^3}{3} + t^2 + 3t + 5$

Therefore: $\quad s(3) = 14$ metres.

Example 23.4

A particle P travels in a straight line. Its velocity, v ms^{-1}, t seconds after passing a fixed point O, is given by $v = -t^2 + 1$, for $0 \le t \le 2$.
(a) Find when and where P changes direction.
(b) Find the total distance travelled in the first 2 seconds.

Solution:

(a) When P changes direction, $v = 0$.

$\Rightarrow$ $t = 1$ second.

Displacement: $s = \int -t^2 + 1 \, dt$

$= -\dfrac{t^3}{3} + t + C$.

But $s(0) = 0$. $\Rightarrow$ $C = 0$

Hence: $s = -\dfrac{t^3}{3} + t$

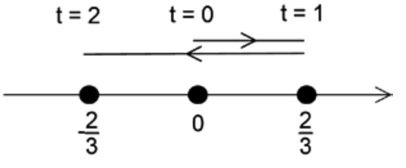
Therefore: $s(1) = \dfrac{2}{3}$ m.

That is, P changes direction at $t = 1$ second
at a point $\dfrac{2}{3}$ metres in the positive direction from O.

(b) $s(0) = 0$, $s(1) = \dfrac{2}{3}$ and $s(2) = -\dfrac{2}{3}$.

Hence, total distance travelled in first 2 seconds

$= \dfrac{2}{3} \times 3 = 2$ metres.



Note:

- *To work out the total distance travelled, we need to track the particle; taking note of the times and positions when the particle changes direction.*

Exercise 23.2

1. A particle P travels in a straight line. Its velocity, v ms^{-1}, t seconds after passing a fixed point O, is given by $v = t^2 + 2t + 3$.
   (a) Find the displacement of P after 2 seconds.
   (b) Find the distance travelled in the first two seconds.
   (c) Find the average speed in the first two seconds.

2. A particle P travels in a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by $v = -t^2 + 5t - 4$. P is initially 1 m from a fixed point O in the positive direction.
   (a) Find when and where P changes direction.
   (b) Find the distance travelled in the first 3 seconds.
   (c) Find the average speed in the first 3 seconds.

3. An object P is thrown vertically upwards from ground level. Its velocity, at time t seconds is given by $v = 19.6 - 9.8t$ for $t \geq 0$. Find:
(a) the total distance travelled in the first 3 seconds
(b) the change in displacement in the first 4 seconds
(c) the velocity of P when it is 14.7 metres from the ground.

4. An object P is thrown vertically upwards from a height of 2 metres above ground level. Its velocity, at time t seconds is given by $v = 25 - 10t$ for $t \geq 0$. Find:
(a) its displacement (measured from ground level) after 2 seconds
(b) the velocity of P when it is 20 metres above the point of projection
(c) the total distance travelled in the first 4 seconds.

5. A particle P is thrown vertically downwards from a cliff 50 metres above ground-level. Its velocity at time t seconds is given by $v = 2 + 9.8t$ for $t \geq 0$.
(a) Find how far P is above the ground-level after 2 seconds.
(b) Find the speed with which P hits the ground.
(c) Find the average speed of P.

6. A particle P is projected vertically upwards from a building 100 metres above ground-level. Its velocity at time t seconds is given by $v = 14 - 9.8t$ for $t \geq 0$.
(a) Find the speed of projection.
(b) Find how far P is above the ground-level after 3 seconds.
(c) Find the maximum height reached by P.
(d) Find when the speed of P is 7 ms^{-1}.

*7. A particle P moves along a straight line. Its velocity t seconds after passing a fixed point O, is given by $\frac{ds}{dt} = kt$ ms^{-1}, where k is a constant. The change in displacement in the first 2 seconds is 6 m.
(a) Find the value of k.
(b) Find the velocity of P when its displacement is 6 m from O in the positive direction.

*8. An object P is projected vertically upwards from a height of 10 metres above ground level. Its velocity, at time t seconds is given by $v = a - 10t$ for $t \geq 0$, where a is a constant. The change in displacement between $t = 2$ and $t = 3$ is 2 metres.
(a) Find the speed of projection.
(b) Find its displacement after 3 seconds.
(c) Find the height of P when it is travelling at 7 ms^{-1}.

*9. A particle P moves along a straight line. Its velocity, v ms^{-1}, t seconds after passing a fixed point O, is given by $v = kt + C$, where k and C are constants. The change in displacement in the first 4 seconds is 8 m. P reverses direction at $t = 3$ seconds.
(a) Find the values of k and C.
(b) Find the average speed for the first 4 seconds.

23.3 Acceleration

- Let v be the velocity of a particle P at time t.
- The acceleration experienced by P is defined as the rate of change of v with respect to time t.
- The instantaneous acceleration of P is given by $a(t) = \dfrac{dv}{dt}$.
- The average acceleration of P for $m \le t \le n$ is given by $\dfrac{v(n)-v(m)}{m-n}$.
- Conversely if a is the acceleration of P at time t, its velocity at time t is given by
$$v(t) = \int a \ dt .$$

Example 23.5

The velocity v of a particle P travelling in a straight line at time t seconds is given by $v(t) = 4 + 0.02t^2$ ms^{-1}. Find: (a) the acceleration of P at $t = 10$ seconds
(b) the average acceleration in the first 10 seconds.

Solution:

(a) Acceleration $a = \dfrac{dv}{dt} = 0.04t$.

Hence, $a(10) = 0.4$ ms^{-2}.

(b) Average acceleration in the first 10 seconds = $\dfrac{v(10)-v(0)}{10-0}$

$$= \frac{6-4}{10} = 0.2 \text{ ms}^{-2}.$$

Example 23.6

A particle P travels along a straight line. It's acceleration t seconds after passing a fixed point O is given by $a(t) = -10$ ms^{-2}. Given that the initial velocity of P is 8 ms^{-1}, find:
(a) the velocity of P (b) the displacement of P.

Solution:

(a) $v(t) = \int a \ dt = -10t + C$

$v(0) = 8 \Rightarrow C = 8.$

Hence, $v(t) = 8 - 10t$ ms^{-1}.

(b) Displacement $s(t) = \int v \ dt = 8t - 5t^2 + K$

$s(0) = 0 \Rightarrow K = 0.$

Hence, $s(t) = 8t - 5t^2$ m.

Exercise 23.3

1. A particle P moves along a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by $v = 2 - 0.01t^2$ for $t \geq 0$.
   (a) Find the initial acceleration for P.
   (b) Find the average acceleration for P in the first second.
   (c) Find the acceleration of P when the velocity is 1 ms^{-1}.
   (d) Find the velocity of P when the acceleration is -0.5 ms^{-2}.

2. A particle P moves along a straight line. Its velocity, v cms^{-1}, at time t seconds, is given by $v = 1 + \sqrt{t}$ for $t \geq 4$.
   (a) Find the acceleration for P at $t = 4$ seconds.
   (b) Find the average acceleration for P in the fifth second.
   (c) Find the acceleration of P when the velocity is 3 ms^{-1}.
   (d) Find the velocity of P when the acceleration is 0.125 ms^{-2}.

3. A particle P moves along a straight line. Its displacement, s, at time t seconds, is given by $s = 1 - \frac{1}{t}$ cm for $t \geq 2$.
   (a) Find the velocity of P at $t = 2$ seconds.
   (b) Find the acceleration of P at $t = 2$ seconds.
   (c) Find the average velocity of P in the third second.
   (d) Find the average acceleration of P in the third second.

*4. A particle P travels in a straight line. Its acceleration at time t seconds, is given by $a = 2 + 0.06t$ ms^{-2}. P has an initial velocity of 1 ms^{-1} and is initially 4 m from a fixed point O.
   (a) Find the velocity of P at time $t = 5$ seconds.
   (b) Find the displacement of P, from O after 5 seconds.
   (c) Find the acceleration of P when its displacement from O is 124 m.

*5. A particle P travels in a straight line. Its velocity as it passes a fixed point O, is 3 ms^{-1}. Its acceleration, a ms^{-2}, t seconds after passing O is given by $a = 6t - 6$.
   (a) Find the velocity of P after 2 seconds.
   (b) Find the maximum displacement of P for $0 \leq t \leq 2$.

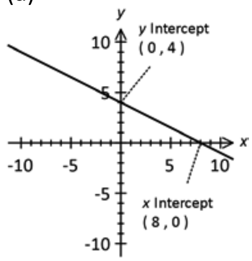
*6. A particle P travels in a straight line. Its velocity as it passes a fixed point O, is 3 ms^{-1}. Its acceleration, a ms^{-2}, t seconds after passing O is given by $a = -6t + 8$.
   (a) Find the change in displacement in the first 4 seconds.
   (b) Find when P changes direction.
   (c) Find the total distance travelled in the first 4 seconds.
   (d) Find the average speed in the first 4 seconds.

Answers

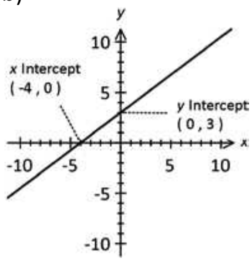
Exercise 1.1

1. (a) (0, 3), (3/2, 0) (b) (0, 8), (–4, 0)
   (c) (0, 3), (–3, 0) (d) (0, –4/3), (2, 0)
   (e) (0, 7), (7, 0) (f) (0, 14), (7, 0)
   (g) (0, 8) (h) (3, 0)

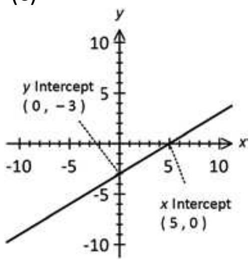
2. (a)



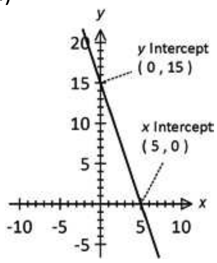
(b)



(c)



(d)



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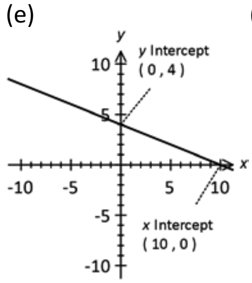

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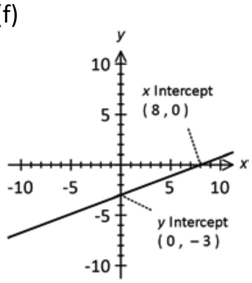
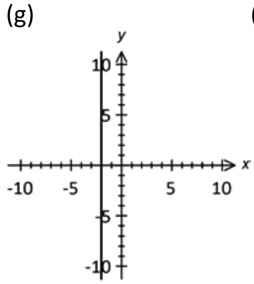
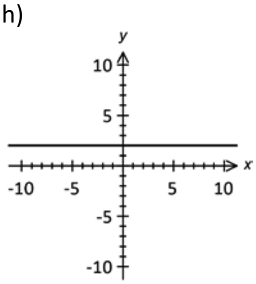


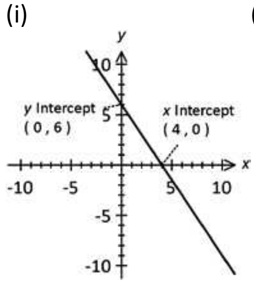
(g)



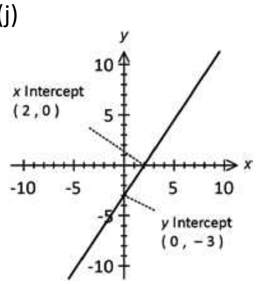
(h)



(i)



(j)



3. (a) $m = -4$; (0, 6) (b) $m = -1/4$; (0, 3/2)
   (c) $m = -2/3$; (0, 1/3) (d) $m = -3/2$; (0, 1/2)
   (e) $m = -2/5$; (0, 2) (f) $m = -5/2$; (0, 5)
   (g) $m = -11/4$; (0, 11) (h) $m = -4/11$; (0, 4)

Exercise 1.2

1. (a) line 1: $m = 4$ line 2: $m = -1$
   line 3: $m = 0$ line 4: $m = 1/2$
   (b) line 1: $m = -1$ line 2: $m = 1/2$
   line 3: $m \to \infty$ line 4: $m = 2$
2. (a) $m = 1/3$ (b) $m = -1$ (c) $m = -7$
   (d) $m = 13/6$ (e) $m = 3$ (f) $m = -1/3$
3. (a) $m_{AB} = 1$, $m_{BC} = 1$ (b) collinear
4. (a) collinear (b) not collinear
5. (a) $k = 61/4$ (b) $k = 18/7$

Exercise 1.3

1. (a) $y = 2x + 4$ (b) $y = 2x - 4$
   (c) $y = x/5 + 24/5$ (d) $y = x/5 - 19/5$
   (e) $y = -2x/7 - 23/7$ (f) $y = 8x/5 - 56/5$
2. (a) $y = 4x$ (b) $y = -8x/5$
   (c) $y = -x + 5$ (d) $y = -2x + 11$
   (e) $y = 5x/7 - 19/14$ (f) $y = -21x/55 + 117/55$
3. $m = -3/5$
4. $k = 8/9$
5. $k = 9$
6. $k = 21/17$
7. (a) $a = 7$, $b = 2$, $c = 14$
   (b) $a = 1$, $b = -3$, $c = -13$
8. (a) $a = 5$, $b = 7$ (b) $a = -4$, $b = -3$
9. Vertical intercept: $(0, b)$
   Horizontal intercept: $(a, 0)$.

Exercise 1.4

1. (a) parallel (b) perpendicular
   (c) parallel (d) perpendicular
   (e) parallel (f) perpendicular
   (g) perpendicular (h) neither
2. (a) $y = 2x + 3$ (b) $y = -2x/3 + 1/3$
   (c) $y = -3x/2 + 5$ (d) $x = -2$
3. (a) $y = 2x + 2$ (b) $y = 5x/3 + 14/3$
   (c) $y = 5x/6 - 3/2$ (d) $x = 4$
4. $y = 3x/5 - 9/5$
5. $y = -11x/8 + 35/8$
6. $y = -4x + 13$
7. $y = -x + 6$
8. (a) $y = -5x/3 + 13/3$ (b) $y = 3x/5 - 1/5$
9. (a) $y = -5x/2 + 6$ (b) $y = 2x/5 + 1/5$
10. (a) Lines meet at (4, 9).
    (b) Lines do not meet at a common point.

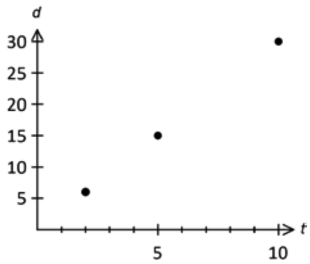
Exercise 1.5

1. (a) 10; (7, 12) (b) $5\sqrt{34}$, (–5/2, 5/2)
   (c) $2\sqrt{2}\,|3k-1|$, $(4k + 1, -1)$
   (d) $\sqrt{(t^2 - 12t + 136)}$, $(t + 7, 3t/2 - 1)$

2. (a) $(-10, -28)$; $8\sqrt{29}$ (b) $(-15, -32)$; $2\sqrt{241}$
   (c) $(16, 48)$; $4\sqrt{293}$
   (d) $(-k + 4, 2k - 1)$; $2\sqrt{(8k^2 + 4k + 1)}$
3. (a) $2\sqrt{41}$ (b) $(1, 8)$
   (c) $(-11, -7)$ (d) $y = -3x/4 + 35/4$
4. (a) $4\sqrt{34}$ (b) $(-6, -4)$
   (c) $(-36, 14)$ (d) $y = 2x/3 + 38$
5. (a) $y = 3x - 9$ (b) $y = 4x/5 - 7/5$
6. $k = 10$
7. $k = -4$
8. $(6, 2)$
9. $(-6, -14)$
10. T(6, 6), M(4, 4)
11. T(−3, −2), M(3, 1)

Exercise 1.6

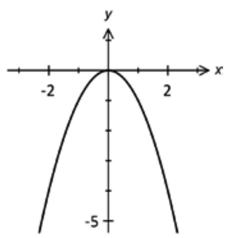
1. (a) No (b) Yes
2. (a) No (b) No
3. (a) No (b) No
4. (a) Yes (b) Yes
5. (a) No (b) No
6. (a) $y = 2x^3$ (b) $y = -(\sqrt{x})/2$
7. (a)



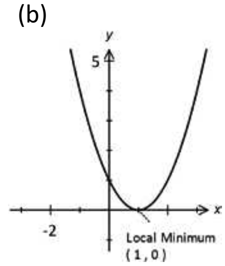
   (b) $d = 3t$
   (c) $t = 30$ seconds; not valid – distance is beyond normal auditory range.
8. (a) $v = 60a - 900$
   (b) After 15 months; not valid – young children do develop vocabulary before 12 months.
   (c) $v = -180$; not valid as $v > 0$.
   (d) At least 15 years 2 months.
   (e) $v = 35\,100$; far too large.
9. (a) $N = 5C/12 - 5/3$ (b) 5
   (c) Not valid as $N < 0$.
   (d) 40; not valid; cricket would be dead!
   (e) (i) $C = 1.8N + 6$; gradient is steeper
   (ii) 12
10. (a) 0.5 ms^{-1} (b) 4.55 ms^{-1}
   (c) 17.8 seconds
   (d) 394.9 kmh; highly unlikely to be valid - terminal velocity is about 320 kmh.
11. (a) \$5 000 (b) \$80 000
   (c) \$8 (d) \$(5000/n + 7.50)
12. (a) (i) \$400 (ii) \$(8k – 800)
   (b) (i) 400 (ii) 500
   (c) A = 1.5 + 500/n; \$1.50
13. (a) (i) $R = 15n$ (ii) $P = 7n - 20\,000$
   (b) (i) $P = 4n - 20\,000$
   (ii) 5 000 packs of 10 disks each
   (iii) 7 500 packs of 10 disks each
14. (a) \$260 (b) \$86.67
   (c) Cheaper to call Sam for jobs < 2 hours
15. (a) \$1 110 (b) \$532.10
16. (a) AB = 10 km; BC = 10 km
   (b) Yes
17. (a) (i) $y = -x/5 + 30$ (ii) $y = -3x/35 + 30$
   (b) 20 years (c) (ii) (300, 0)
18. (a) R(5, 2) (b) $y = -x + 7$
   (c) S(0, 7) (d) Yes

Exercise 2.1

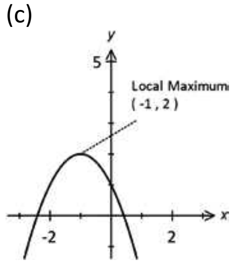
1. (a) Min (−3, −1) (b) Min (−7/2, −285/4)
   (c) Max (2/3, 22/3) (d) Min (10, 1/2)
2. (a) Min (1, −25) (b) Min (1/4, −121/8)
   (c) Max (−1/2, 121/2) (d) Max (5/4, 75/8)
3. (a) Min (2, 4) (b) Min (−3, −5)
   (c) Max (1, 3) (d) Min (5/2, 0)
4. (a) Min (−25/2, −32 750)
   (b) Min (−3/2, −49/8)
   (c) Min (1, 2) (d) Min (−5/3, −49/9)
5. (a)



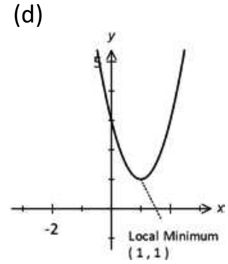
   (b)



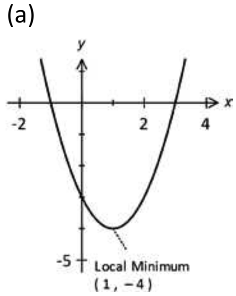
   (c)



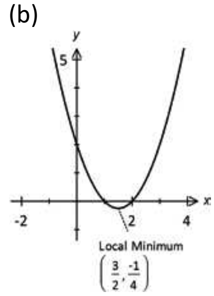
   (d)



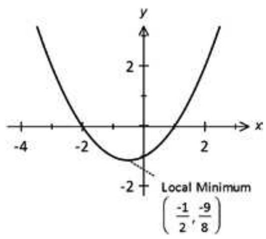
6. (a)



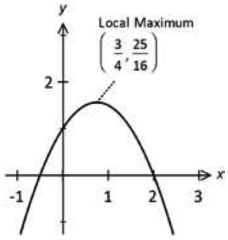
   (b)



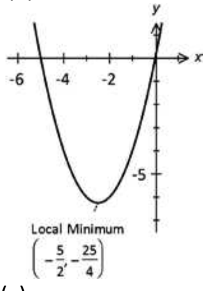
6. (c)



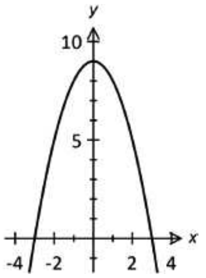
(d)



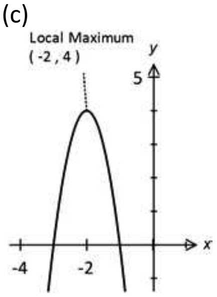
7. (a)



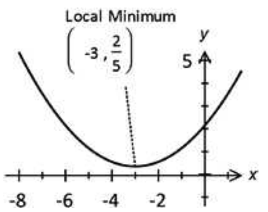
(b)



(c)



(d)



8. $k = 2$
9. (a) -12 (b) -4 (c) 5
   (d) Any real number except 0.
10. (a) $y = \frac{1}{2}(x-2)^2 + 2$ (b) $y = \frac{-3}{2}(x-2)^2 + 6$
    (c) $y = \frac{1}{3}(x+2)^2 - 3$ (d) $y = \frac{-1}{2}(x+1)(x-4)$
    (e) $y = (x + 1)(x + 5)$ (f) $y = x^2 + x + 3$
11. (a) $y = (x-2)^2 + 3$ (b) $y = 2(x+1)^2 - 4$
    (c) $y = \frac{1}{2}(x-2)^2$ (d) $y = -2(x-2)^2 + 8$
12. (a) $y = -3(x-2)(x+8)$
    (b) $y = \frac{-1}{2}(x+5)(x-4)$
    (c) $y = 2(x-3)^2$ (d) $y = \frac{-1}{5}(x+5)^2$

Exercise 2.2

1. (a) $(x+4)^2$ (b) $(x+4)^2 + 1$
   (c) $(x+4)^2 - 2$ (d) $(x+4)^2 - 26$
   (e) $(x+5/2)^2 - 65/4$ (f) $(x+3/2)^2 + 71/4$
   (g) $(x-7/2)^2 - 1/4$ (h) $(x-9/2)^2 - 85/4$
2. (a) $-(x-2)^2 + 9$ (b) $-(x-3)^2 + 7$
   (c) $-(x-2)^2 + 5$ (d) $-(x+3/2)^2 + 17/4$
   (e) $2(x+5/2)^2 + 7/2$ (f) $3(x+2)^2 - 19$
   (g) $-5(x+1)^2 - 2$ (h) $-2(x+3/4)^2 + 41/8$

Exercise 2.3

1. (a) 0, -4 (b) -5, 1
   (c) -0.05, 0.005 (d) -1000, 250
   (e) 0, 10 (f) 0, 4
   (g) -3, 1 (h) -2, 1
2. (a) 4, -2 (b) $-10/3$, $8/3$
   (c) -8, 1 (d) -2, 1
   (e) $-2 \pm \sqrt{5}$ (f) $(3/2) \pm (\sqrt{29})/2$
3. (a) $(-5/2) \pm (\sqrt{35})/2$ (b) $(2/3) \pm (\sqrt{7})/3$
   (c) $-2 \pm \sqrt{6}$ (d) $1/2$, 1
   (e) $-2 \pm \sqrt{11}$ (f) -4, -1
4. (a) 1 (b) -1, $-1/2$
   (c) $(1/2) \pm (\sqrt{17})/2$ (d) $2 \pm \sqrt{6}$
   (e) $-5/2$, 1 (f) $4/9$

Exercise 2.4

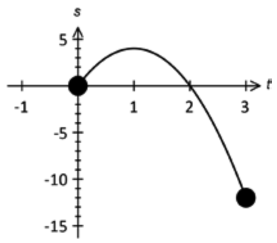
1. $k = 25$
2. $k \geq -25/8$
3. $k < -9/2$
4. $k = \pm 2\sqrt{3}$
5. $k \leq 1$
6. (a) $k = 5$ (b) $k > 5$
7. $k \geq -2$
8. $k < 0$
9. $k = -8$
10. $k > 5/2$
11. $k \geq 4$, $k \leq -4$
12. $0 < k < 36$
13. $k < 0$, $k > 40$

Exercise 2.5

1. (a) $x \leq -1$, $x \geq 3/2$ (b) $x \neq -2$
   (c) $x \leq -4$, $x \geq 4$ (d) All x.
   (e) $-6 \leq x \leq 1$
   (f) $-1 - \frac{\sqrt{6}}{2} < x < -1 + \frac{\sqrt{6}}{2}$
   (g) $x \leq 1$, $x \geq 5$ (h) $-2 \leq x \leq 0$
   (i) $x \leq -16$, $x \geq 2$ (j) $x < -1$, $x > 5/3$
2. (a) Solutions are the roots of the curve.
   (b) Solutions are values of x which correspond to the curve being on or above the x-axis.
   (c) Solutions are the x-coordinates of the points of intersection between the given curve and the line $y = 2$.
   (d) Solutions are values of x which correspond to the curve being on or below the line $y = 5$.
   (e) Solutions are values of x which correspond to the curve being above the line $y = x + 5$.
   (f) Solutions are values of x which correspond to the curve being below the curve $y = (x - 1)(5 - x)$.

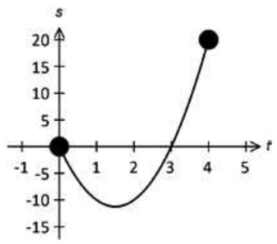
Exercise 2.6

1. (a)



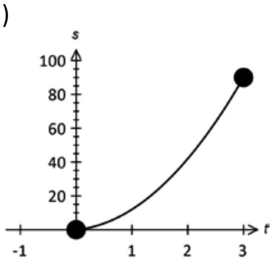
   (b) (i) 4 (ii) -12 (c) 2
   (d) (i) 4 m (ii) 20 m

2. (a)



   (b) 3 (c) (i) 45/4 m (ii) 20 m
   (d) 1, 2, $(3/2) + (\sqrt{17})/2$

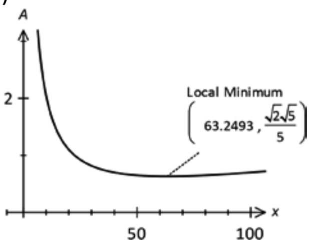
3. (a)



   (c) 48 m (d) 48 ms^{-1}

4. (a) 10 ms^{-1} (b) 30 m
   (c) $10\sqrt{10}$ ms^{-1}

5. (a) \$20 (b) \$70, \$0.70
   (c) $A(x) = 0.005x + 20/x$
   (d)



   A decreases as x increases until $x \approx 63$ when A starts to increase as x increases.

6. (a) $x = 20$; 50 (b) \$9
   (c) \$9

7. (c) 306.25 m^2
   (d) No. Dimensions for max. area is 17.5 m by 17.5 m which is less than the 18 m length required.
   (e) 306 m^2 (f) 0.25 m^2

8. (d) $200/(4 + \pi)$

9. (a) 10 (b) 6.6 m
   (c) 2 m (d) 1 m above ground.

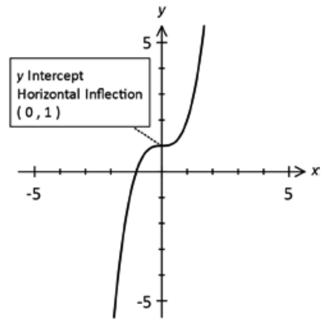
10. (a) 96.875 m (b) 40 m
    (c) 10 m

11. (a) 6.6125 m (b) 29.1 m, 85.9 m
    (c) 0.5 m above the top of sight screen
    (d) (110.32, 1.032)

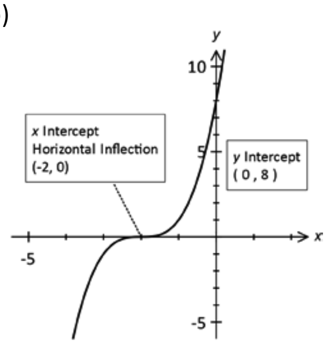
12. (a) 1 m (b) 1.5625 m
    (c) 3 m (d) $y = -(x - 3)(x - 5)$
    (e) 2.7m, 3.3 m, 4.7 m

Exercise 3.1

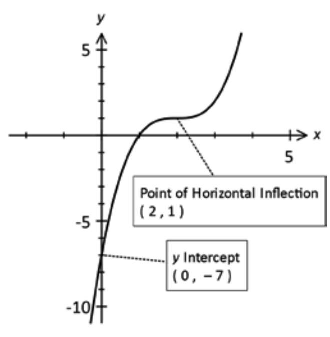
1. (a)



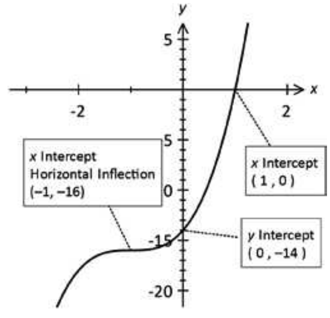
   (b)



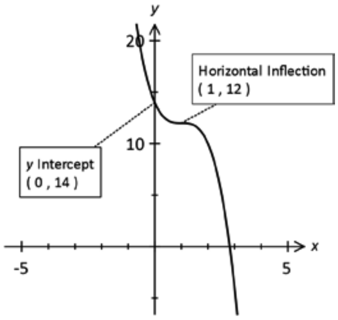
   (c)



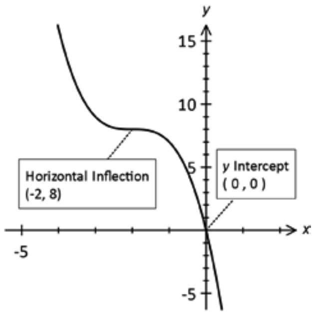
   (d)



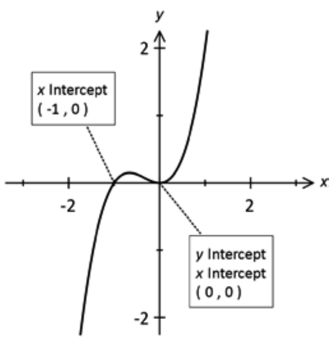
1. (e)



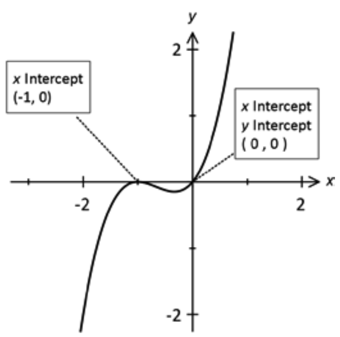
(f)



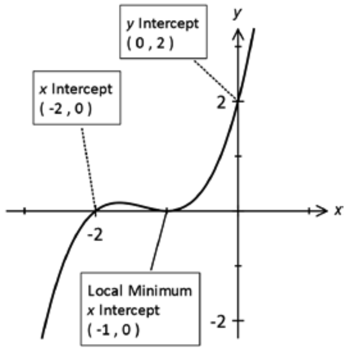
2. (a)



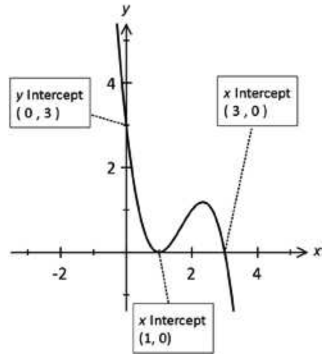
(b)



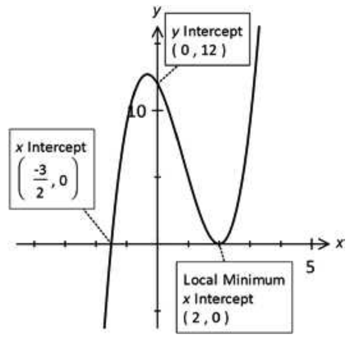
(c)



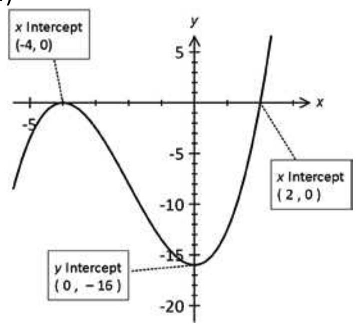
2. (d)



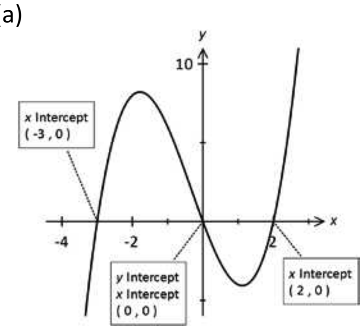
(e)



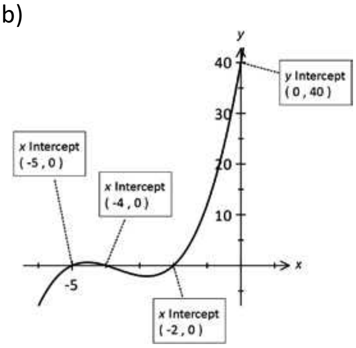
(f)



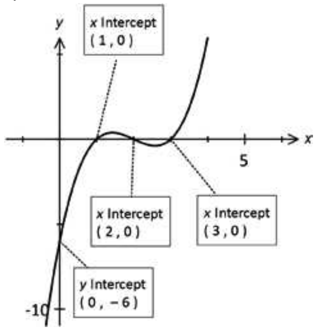
3. (a)



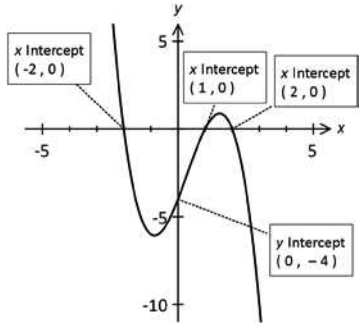
(b)



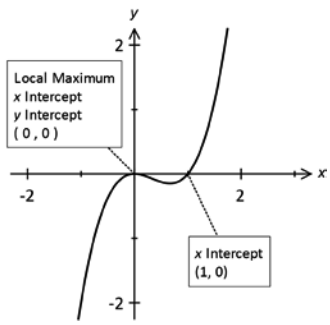
3. (c)



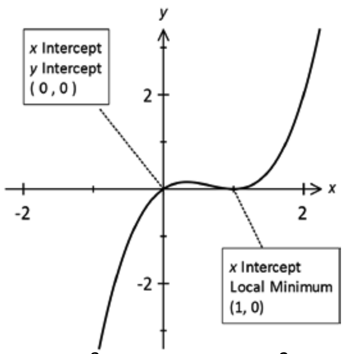
(d)



(e)



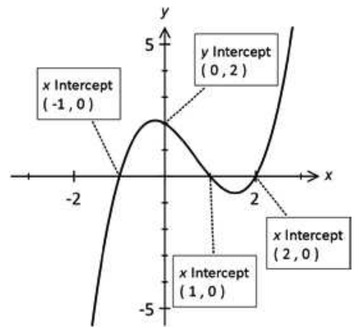
(f)



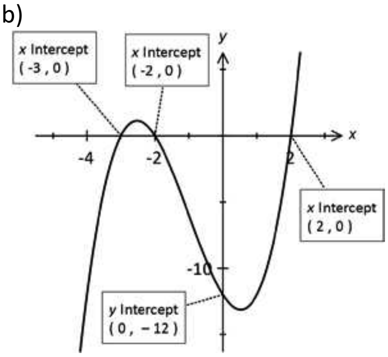
4. (a) $y = (x - 1)^3 - 1$ (b) $y = x^2(x + 3)$
   (c) $y = -2x(x + 1)(x - 3)$
   (d) $y = 0.1(x + 5)(x + 2)(x - 5)$
   (e) $y = [(x + 4)(x - 3)^2]/9$
   (f) $y = 0.5(x + 2)^3 + 3$
5. (a) $y = (x - 2)^3 + 3$
   (b) $y = -4 + [(x + 3)^3]/3$
   (c) $y = 4(x - 1)^3 - 4$
   (d) $y = -[(x + 1)^3]/3 + 9$
6. (a) $y = [(x - 1)(x - 3)(x - 5)]/3$
   (b) $y = -[(x + 2)(x + 3)(x - 7)]/6$
   (c) $y = 3(x + 1)(x - 1)^2$
   or $y = -3(x - 1)(x + 1)^2$
   (d) $y = -[(x - 2)(x + 3)^2]/6$
   or $y = [(x + 3)(x - 2)^2]/4$

Exercise 3.2

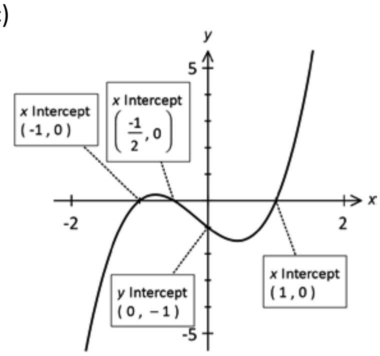
1. (a)



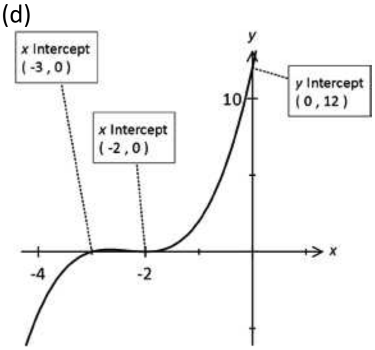
(b)



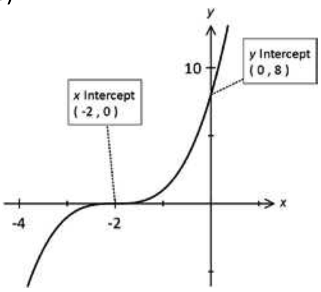
(c)



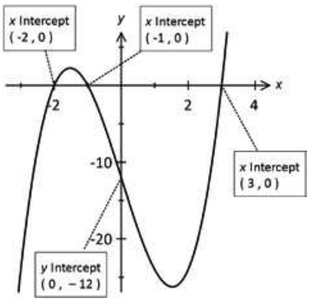
(d)



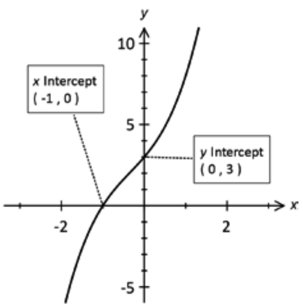
1. (e)



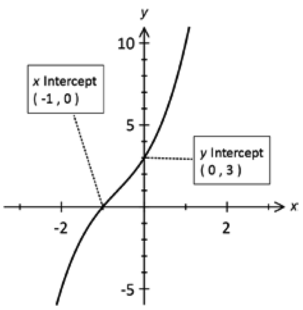
(f)



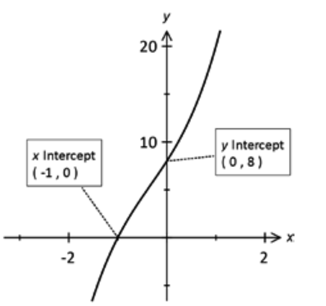
(g)



(h)



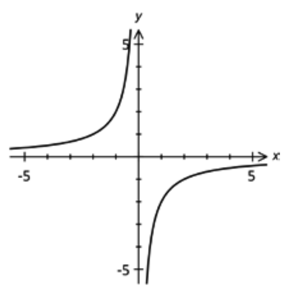
(i)



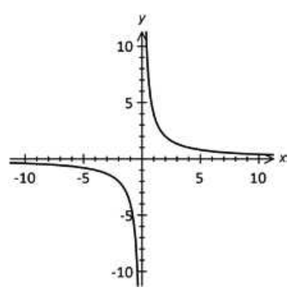
2. (a) $x = 0, \pm 4$ (b) $x = 0, \pm\sqrt{5}$
   (c) $x = -3, 1$ (d) $x = -1, \pm 4$
   (e) $x = 1, 1 \pm \sqrt{7}$ (f) $x = 1, 5$
   (g) $x = -3, 2, 4$ (h) $x = -1, 1, 5/2$
   (i) $x = 1/2, -1 \pm \sqrt{5}$
3. (a) $x = -5.47$ (b) $x = -0.14, 0.33$
   (c) $x = -1.47, 0.23, 0.74$
4. (c) If $x > 30$, then $h < 0$ which is impossible.
   (d) $x = 117$ mm or 264mm (e) $h = 500$ mm
5. (c) $h > 0 \Rightarrow 0 < x < 25$ cm
   (d) $x = 128, 200$ mm
   (e) Maximum volume is 46 296 cm^3.
6. (b) 0 < height < 10 mm
   (c) 0 < Volume ≤ 820.53 cm^3
   (d) height $x = 13, 68$ mm
7. (b) $0 < x < 20$ cm; 0 < Volume ≤ 3282.11 cm^3
   (c) length = 229, 449 mm
8. (a) $t = 2, 4, 6$ seconds
   (b) $t = 2.8, 5.2$ seconds (c) $t = 0.8$ seconds
9. (a) $t = 2, 7$ seconds
   (b) $t = 0.7, 3.8, 6.5, 7.3$ seconds
   (c) 101 m (d) 12.6 ms^{-1}
10. (a) 5 m (b) $t = 0.5, 4.2$ seconds
    (c) 53 m (d) 4.9 seconds

Exercise 4.1

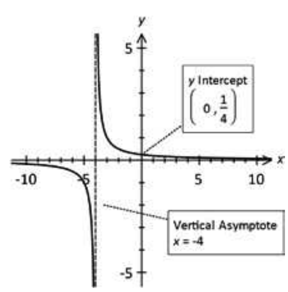
1. (a) $xy = 40$ (b) $xy = 60$
   (c) $(x - 1)\, y = 75$ (d) $(x + 5)\, y = 40$
   (e) $x^2 y = 160$ (f) $x^3 y = 1250$
   (g) $y\sqrt{x} = 50$ (h) $xy^2 = 64$
2. (a)



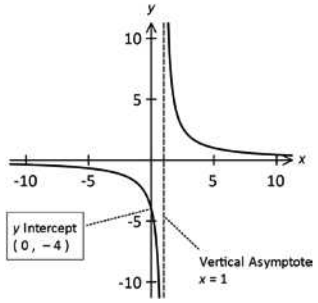
(b)



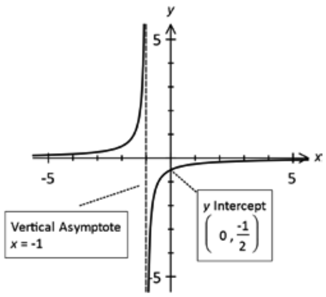
(c)



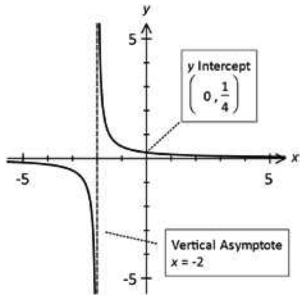
2. (d)



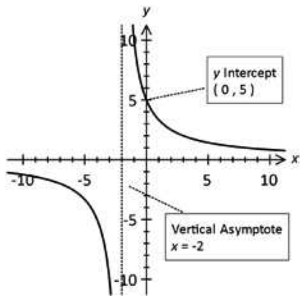
(e)



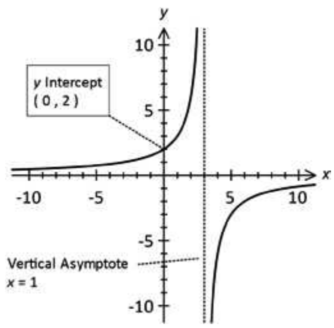
(f)



(g)



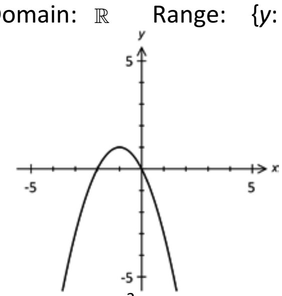
(h)



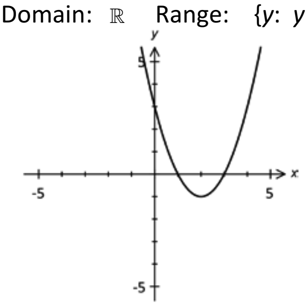
3. (a) $y = -8/(x-2)$ (b) $y = -27/(x+3)$
   (c) $y = -8/(x-4)$ (d) $y = 2/(2x-1)$
4. (a) $y = 4/(x-2)$ (b) $y = -9/(x-3)$
   (c) $y = 5/(2x+4)$ (d) $y = -3/(5x+10)$
5. (a) $w = 250y/x$ (b) $w = 1280y/x$
6. (a) $w = 40\,000y/x^2$ (b) $w = 1620y/x^2$
7. (a) 300 workers (b) 225 weeks
8. (a) 4/3 A (b) 24/5 Ohms
9. (a) 200/3 km/h (b) 4 hours 27 minutes
10. (a) 266.96 N (b) 25.83 m
11. (a) (i) 10 pascals (ii) 10 m^3
    (b) 100 pascals (c) 10 m^3
12. (a) 56.25% increase (b) Increase by 29.1%

Exercise 5.1

1. (a) $f(x)$ is a function as it passes the vertical line test.
   Domain: $\{x: x > 0, x \in \mathbb{R}\}$ Range: $\mathbb{R}$
   (b) $f(x)$ is a function as it passes the vertical line test.
   Domain: $\mathbb{R}$
   Range: $\{y: -3 \le y \le 3, y \in \mathbb{R}\}$
   (c) $f(x)$ is a function as it passes the vertical line test.
   Domain: $\mathbb{R}$ Range: $\mathbb{R}$
   (d) $f(x)$ is a function as it passes the vertical line test.
   Domain: $\mathbb{R}$
   Range: $\{y: 0 < y \le 4, y \in \mathbb{R}\}$
   (e) $f(x)$ is a function as it passes the vertical line test.
   Domain: $\{x: x \ne 2, x \in \mathbb{R}\}$
   Range: $\{y: y \ne 2, y \in \mathbb{R}\}$
   (f) $f(x)$ is a not a function as it fails the vertical line test.
2. (a) $f(x) = 1 - (x+1)^2$
   Domain: $\mathbb{R}$ Range: $\{y: y \le 1, y \in R\}$



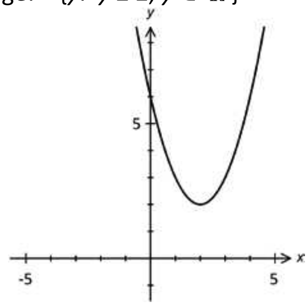
   (b) $f(x) = (x-2)^2 - 1$
   Domain: $\mathbb{R}$ Range: $\{y: y \ge -1, y \in \mathbb{R}\}$



2. (c) $f(x) = x^2 - 4x + 6 = (x-2)^2 + 2$

Domain: $\mathbb{R}$

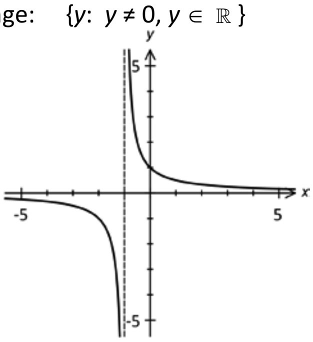
Range: $\{y\colon\ y \geq 2,\ y \in \mathbb{R}\}$



(d) $f(x) = \dfrac{1}{x+1}$

Domain: $\{x\colon\ x \neq -1,\ x \in \mathbb{R}\}$

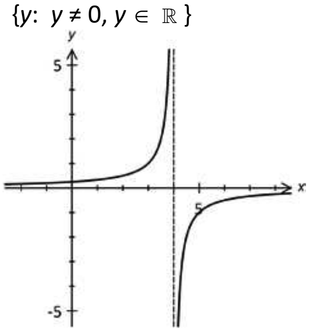
Range: $\{y\colon\ y \neq 0,\ y \in \mathbb{R}\}$



(e) $f(x) = \dfrac{1}{4-x}$

Domain: $\{x\colon\ x \neq 4,\ x \in \mathbb{R}\}$

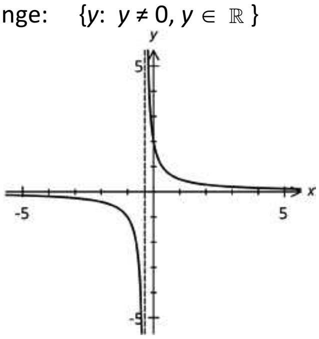
Range: $\{y\colon\ y \neq 0,\ y \in \mathbb{R}\}$



(f) $f(x) = \dfrac{2}{3x+1}$

Domain: $\{x\colon\ x \neq -\dfrac{1}{3},\ x \in \mathbb{R}\}$

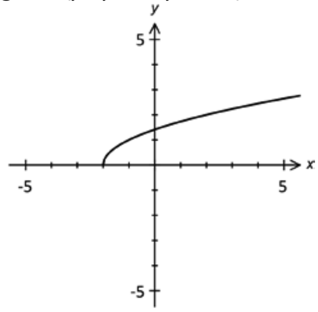
Range: $\{y\colon\ y \neq 0,\ y \in \mathbb{R}\}$



2. (g) $f(x) = \sqrt{x+2}$

Domain: $\{x\colon\ x \geq -2,\ x \in \mathbb{R}\}$

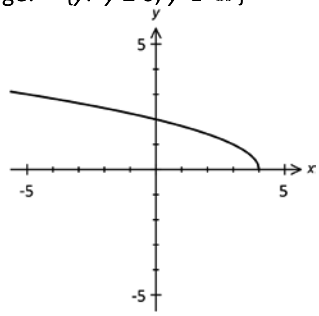
Range: $\{y\colon\ y \geq 0,\ y \in \mathbb{R}\}$



(h) $f(x) = \sqrt{4-x}$

Domain: $\{x\colon\ x \leq 4,\ x \in \mathbb{R}\}$

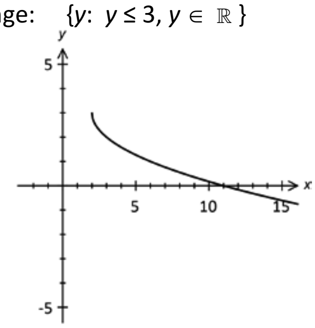
Range: $\{y\colon\ y \geq 0,\ y \in \mathbb{R}\}$



(i) $f(x) = 3 - \sqrt{x-2}$

Domain: $\{x\colon\ x \geq 2,\ x \in \mathbb{R}\}$

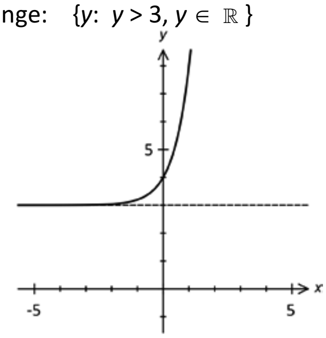
Range: $\{y\colon\ y \leq 3,\ y \in \mathbb{R}\}$



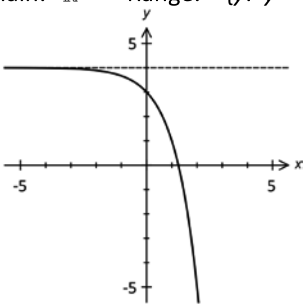
(j) $f(x) = 5^x + 3$

Domain: $\mathbb{R}$

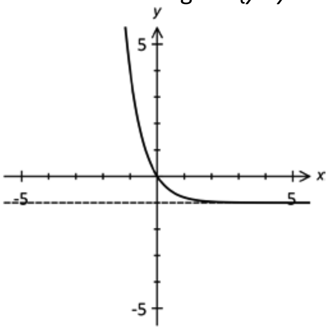
Range: $\{y\colon\ y > 3,\ y \in \mathbb{R}\}$



2. (k) $f(x) = 4 - 3^x$
 Domain: $\mathbb{R}$ Range: $\{y: y < 4, y \in \mathbb{R}\}$



 (l) $f(x) = 0.2^x - 1$
 Domain: $\mathbb{R}$ Range: $\{y: y > -1, y \in \mathbb{R}\}$



3. (a) $\{y: -9 \le y \le 7, y \in \mathbb{R}\}$
 (b) $\{y: 3 \le y \le 28, y \in \mathbb{R}\}$
 (c) $\{y: -61 \le y \le 67, y \in \mathbb{R}\}$
 (d) $\{y: 0 \le y \le \sqrt{11}, y \in \mathbb{R}\}$
 (e) $\{y: y < 0, y \in \mathbb{R}\}$
 (f) $\{y: -3 < y < 3/4, y \in \mathbb{R}\}$
4. (a) $a^2 - 2a - 2$ (b) $4a^2 + 4a - 2$
 (c) $a^2/4 + a - 2$ (d) $a^2 - 2a - 2$
 (e) $k = -2, 0$
5. (a) $-2/(a + 1)$ (b) $4/(a - 2)$
 (c) $1/a$ (d) $2a/(1 - a)$ (e) $k = -1$
6. (a) $\sqrt{(5 + a)}$ (b) $\sqrt{[(10 - a)/2]}$
 (c) $3 - 2\sqrt{(5 - a)}$ (d) $\sqrt{(10 - a)}$
 (e) $k = 4$
7. (a) 10^a (b) 10^{-a}
 (c) $1 - 2(10^a)$ (d) 10^{1-2a}
 (e) $k = 1/3$
8. (a) $2x - 1$ (b) $(x - 3)^2 + 5$
 (c) $(2x + 1)^2$ (d) $(2x + 1)(x^2 + 2x + 6)$
 (e) $x = 3, 5$
9. Domain: $\mathbb{R}$ Range: $\{y: y \ge -4, y \in \mathbb{R}\}$
10. Domain: $\mathbb{R}$ Range: $\{y: y \ge 0, y \in \mathbb{R}\}$
11. Domain: $\{x: x \ge 0, y \in \mathbb{R}\}$
 Range: $\{y: y \ge 0, y \in \mathbb{R}\}$
12. Domain: $\{x: x \ne -1, x \in \mathbb{R}\}$
 Range: $\{y: y \ne 4, y \in \mathbb{R}\}$
13. Domain: $\{x: x \le 2, x \in \mathbb{R}\}$
 Range: $\{y: 0 \le y < 3, y \in \mathbb{R}\}$

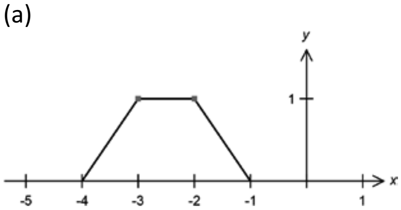
Exercise 5.2

1. (a) $x^2 + y^2 = 25$ (b) $(x + 6)^2 + y^2 = 36$
 (c) $(x - 4)^2 + (y - 4)^2 = 16$
 (d) $(x - 1.5)^2 + (y - 2)^2 = 6.25$
 (e) $(x + 4)^2 + (y + 3)^2 = 25$
 (f) $(x - 1)^2 + (y + 1)^2 = 5$
2. (a) $(x + 2)^2 + (y - 3)^2 = 25$
 (b) $(x - 3)^2 + (y - 2)^2 = 9$
3. (a) $x^2 + y^2 - 10x + 4y - 35 = 0$
 (b) $x^2 + y^2 - 8x - 10y + 40 = 0$
4. (a) Centre $(1, -2)$; radius = 3
 $(1 \pm \sqrt{5}, 0)$ & $(0, -2 \pm \sqrt{2})$
 (b) Centre $(-10, -12)$; radius = 20
 $(6, 0)$, $(-26, 0)$ & $(0, -12 \pm 10\sqrt{3})$
5. (a) Centre $(-3, 7)$; radius = 8
 $(-3 \pm \sqrt{15}, 0)$ & $(0, 7 \pm \sqrt{55})$
 (b) Centre $(2, -8)$; radius = 10
 $(-4, 0)$, $(8, 0)$ & $(0, -8 \pm 4\sqrt{6})$
6. (a) A: $y = x^2/3$ B: $y^2 = 3x$
 (b) A: $y = x^2/5$ B: $y^2 = -5x$
 (c) A: $y = -x^2 + 2$ B: $y^2 = x$

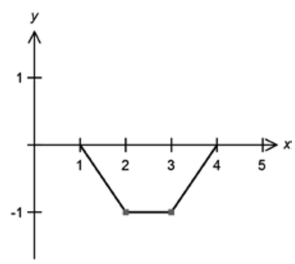
Exercise 5.3

1. (a) $y = -f(x)$; $(1, 0)$, $(3, 0)$ & $(0, -2)$
 (b) $y = f(-x)$; $(-1, 0)$, $(-3, 0)$ & $(0, 2)$
 (c) $y = f(x - 3)$; $(4, 0)$, $(6, 0)$ & $(3, 2)$
 (d) $y = f(x/3)$; $(3, 0)$, $(9, 0)$ & $(0, 2)$
2. (a) $y = f(x) - 3$; Max. $(2, 2)$, Min $(-2, -8)$
 (b) $y = f(x - 2)$; Max. $(4, 5)$, Min $(0, -5)$
 (c) $y = f(2x)$; Max. $(1, 5)$, Min $(-1, -5)$
 (d) $y = 4f(x)$; Max. $(2, 20)$, Min $(-2, -20)$
 (e) $y = -f(x)$; Max. $(-2, 5)$, Min $(2, -5)$
3. (a) Vertical dilation factor 3
 (b) Horizontal dilation factor 1/3
 (c) Horizontal dilation factor 4
 (d) Vertical translation 3 units upwards
 (e) Vertical translation 3 units downwards
 (f) Horizontal translation 3 units left
 (g) Horizontal translation 3 units right
 (h) Reflection about the x-axis
 (i) Reflection about the y-axis
4. (a) Vertical dilation factor 1/5
 (b) Horizontal dilation factor 5
 (c) Horizontal dilation factor 1/5
 (d) Vertical translation 5 units downwards
 (e) Vertical translation 5 units upwards
 (f) Horizontal translation 5 units right
 (g) Horizontal translation 5 units left
 (h) Reflection about the x-axis
 (i) Reflection about the y-axis

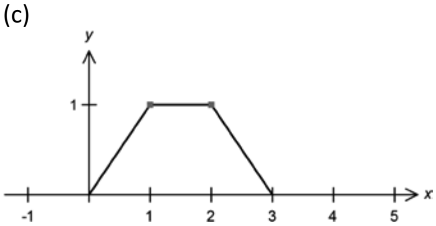
5. (a) (2, 0) & (0, –8) (b) (4, 0) & (0, –4)
   (c) (1, 0) & (0, –4) (d) (–2, 0) & (0, –4)
   (e) (2, 0) & (0, 4) (f) (2, 0) & (0, 4)
6. (a) Min. (4, 0); Max. (–3, 5)
   (b) Min. (2, 1); Max. (–5, 6)
   (c) Min. (6, 1); Max. (–1, 6)
   (d) Min. (16, 1); Max. (–12, 6)
   (e) Min. (–4, 1); Max. (3, 6)
   (f) Min. (–3, –6); Max. (4, –1)
7. (a) $(3b + 2, 0)$
   (b) Min. $(b, -2a)$; Max. $(3b, 0)$
   (c) $(0, 0)$, $(2b, 0)$ & $(4b, 0)$.
8. (a) $y = x^2 + 2x - 1$ (b) $y = -x^2 + 2x + 1$
   (c) $y = -x^2/16 - x/2 + 1$
   (d) $y = -16x^2 - 8x + 1$
   (e) $y = -x^2 - 2x - 1$ (f) $y = -x^2 + 2x - 1$
   (g) $y = 3(-x^2 - 2x + 1)$
9. (a) Horizontal translation 1 unit left
   (b) Vertical translation 2 units upwards
   (c) Reflection about the x-axis
   (d) Horizontal translation 3 units right
   (e) Horizontal dilation factor 1/4
   (f) Vertical translation 10 units downwards
10. (a)



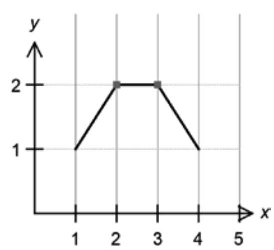
(b)



(c)

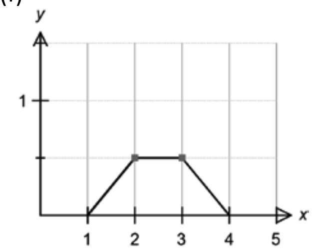


(d)

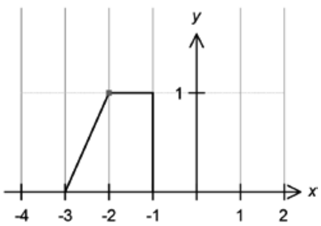


10. (e)

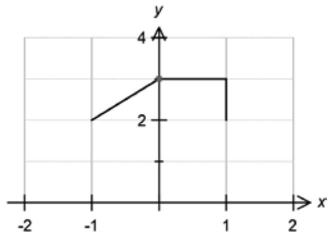
(f)



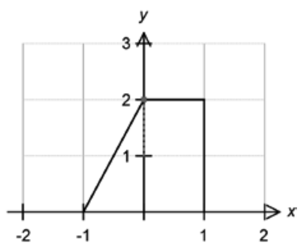
11. (a)



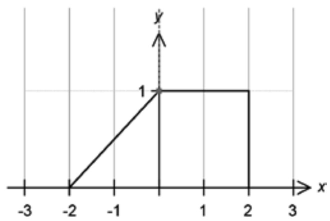
(b)



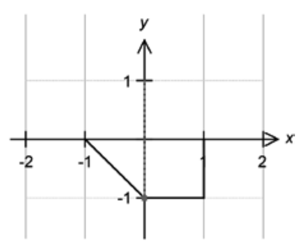
(c)



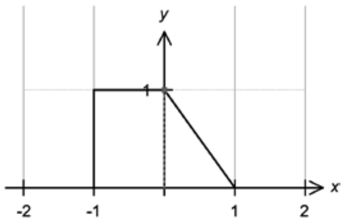
(d)



(e)

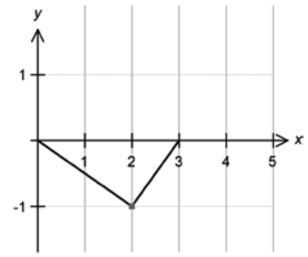


11. (f)

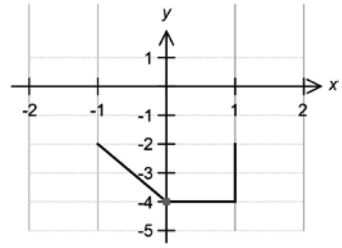


Exercise 5.4

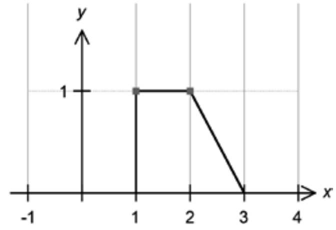
1. (a) $y = -f(-x)$; (1, 0) & (0, 2)
   (b) $y = f(x/2 - 1)$; (0, 0) & (2, –2)
2. (a) $y = f(x + 2) + 1$;
   Max. (–3, 7), Min (1, –5)
   (b) $y = (1/3)f(x/4)$;
   Max. (–4, 2), Min (12, –2)
3. *Several possible answers.*
   (a) Horizontal translation 2 units left
   Horizontal dilation factor 1/2
   (b) Horizontal translation 2 units left
   Reflection about the *y*-axis
   (c) Horizontal translation 2 units right
   Horizontal dilation factor 2
   (d) Vertical dilation factor 2
   Vertical translation 2 units upwards
   (e) Vertical translation 2 units upwards
   Reflection about the *x*-axis
   (f) Vertical dilation factor 1/2
   Horizontal translation 1 unit right
4. *Several possible answers.*
   (a) Horizontal translation 1 unit left
   Horizontal dilation factor 1/2
   (b) Horizontal translation 1 unit right
   Horizontal dilation factor 2
   (c) Horizontal translation 1 unit left
   Reflection about the *y*-axis
   (d) Vertical dilation factor 2
   Vertical translation 5 units upwards
   (e) Vertical translation 5 units upwards
   Reflection about the *x*-axis
   (f) Horizontal translation 1 unit left
   Vertical dilation factor –2
   OR Horizontal translation 1 unit left
   Vertical dilation factor 2
   Reflection about the *x*-axis.
5. (a) Min. (8, 0); Max. (–6, 5)
   (b) Min. (1/2, 1); Max. (–3, 6)
   (c) Min. (–1, –6); Max. (6, –1)
   (d) Min. (5, 0); Max. (–2, 5)
   (e) Min. (1, 1/2); Max. (–3/4, 3)
   (f) Min. (–3, –23); Max. (4, –3)
6. $y = -(x^2 + 10x - 7)/16$
7. Horizontal translation 3 units right
   Vertical translation 4 units downwards
8.



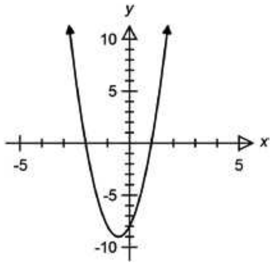
9. (a)



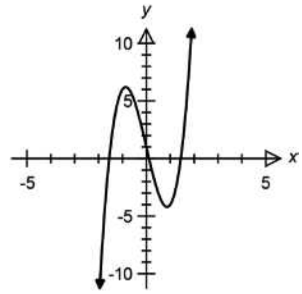
(b)



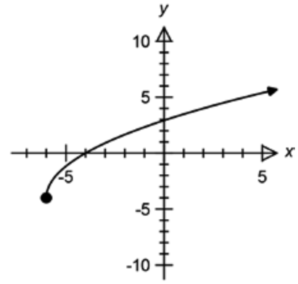
10. (a)



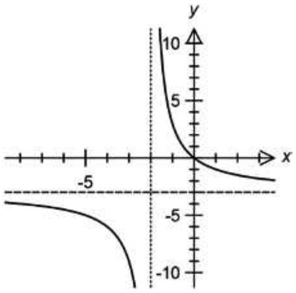
(b)



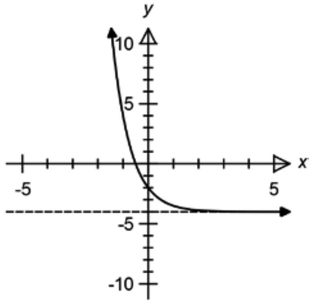
(c)



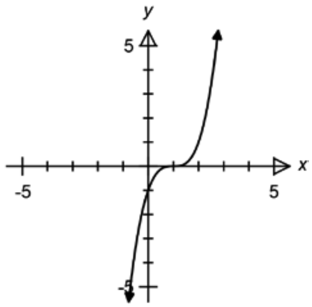
10. (d)



(e)



(f)



Exercise 6.1

1. 42.84 cm
2. 106°
3. 4.41 cm
4. 29.7 cm^2
5. 028.1°
6. 1.36 ms^{-1}
7. 53.5 km/h < 60 km/h. Hence, not speeding.
8. 2.62 m

Exercise 6.2

2. (a) $20\sqrt{3}$ (b) 9/2 (c) $(64\sqrt{3})/3$
3. (a) 60° (b) 15° (c) 40°

Exercise 7.1

1. (a) Q2 (b) Q3 (c) Q3 (d) Q4 (e) Q4
   (f) Q4 (g) Q1 (h) Q1 (i) Q3 (j) Q3
2. (a) 35° (b) 61° (c) 15° (d) 45° (e) 28°
   (f) 11° (g) 82° (h) 50° (i) 75° (j) 80°
3. (a) 65°, 115°, 245°, 295°
   (b) 31°, 149°, 211°, 329°
   (c) 42°, 138°, 222°, 318°
   (d) 6°, 174°, 186°, 354°
4. (a) 373°, 527°, 553°, 707°
   (b) 432°, 468°, 612°, 648°
   (c) 405°, 495°, 585°, 675°
   (d) 390°, 510°, 570°, 690°
5. (a) -0.6 (b) -0.8 (c) 0.75
6. $k = \pm(\sqrt{3})/2$ (a) $-1/2$ (b) $\pm(\sqrt{3})/2$ (c) $\pm\sqrt{3}$
7. $k = \pm(\sqrt{7})/4$ (a) 3/4 (b) $\pm(\sqrt{7})/4$
   (c) $\pm(3\sqrt{7})/7$
8. $k = \pm(\sqrt{2})/2$ (a) $\pm(\sqrt{2})/2$ (b) $\pm(\sqrt{2})/2$
   (c) 1
9. (a) > 0, > 0, > 0 (b) < 0, < 0, > 0
   (c) > 0, < 0, < 0 (d) < 0, > 0, < 0
10. Q4 11. Q3 12. 156°, 204° 13. 236°
14. 619°, 641° 15. 125°, 235°
16. (a) $(\sqrt{2})/2$ (b) $-(\sqrt{3})/3$ (c) 1/2 (d) 1
    (e) $-1/2$ (f) 0 (g) $(\sqrt{2})/2$ (h) $-1/2$
17. 30°, 330° 18. 330°
19. 240° 20. 315°, 675°

Exercise 7.2

1. (a) Q3 (b) Q2 (c) Q1 (d) Q4 (e) Q2
2. (a) 45° (b) 25° (c) 55° (d) 65° (e) 24°
3. (a) -22°, -158°, -202°, -338°
   (b) -40°, -140°, -220°, -320°
   (c) -50°, -130°, -230°, -310°
   (d) -86°, -94°, -266°, -274°
4. (a) $\pm 5^\circ$, $\pm 175^\circ$ (b) $\pm 33^\circ$, $\pm 147^\circ$
   (c) $\pm 52^\circ$, $\pm 128^\circ$ (b) $\pm 75^\circ$, $\pm 105^\circ$
5. (a) < 0, > 0, < 0 (b) > 0, < 0, < 0
   (c) < 0, < 0, > 0 (d) < 0, > 0, < 0
6. -154°, -206° 7. 34°, -146°
8. -57°, -123° 9. 49°, 131°, -311°, -229°
10. (a) $(-\sqrt{2})/2$ (b) $\sqrt{3}$ (c) $(\sqrt{3})/2$ (d) $-(\sqrt{3})/2$
11. -150°, -210° 12. 150°
13. 240°, -120° 14. 135°, -225°

Exercise 7.3

1. (a) $\pi/6$ (b) $\pi/6$ (c) $\pi/4$ (d) $\pi/3$
2. (a) $\pi/6$, $5\pi/6$, $7\pi/6$, $11\pi/6$
   (b) $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$
   (c) $\pi/3$, $2\pi/3$, $4\pi/3$, $5\pi/3$ (d) $\pi/2$, $3\pi/2$
3. (a) $\pm\pi/6$, $\pm 5\pi/6$ (b) $\pm\pi/4$, $\pm 3\pi/4$
   (c) $\pm\pi/3$, $\pm 2\pi/3$ (d) $\pm\pi/2$
4. (a) > 0, > 0, > 0 (b) > 0, < 0, < 0
   (c) > 0, > 0, > 0 (d) < 0, < 0, > 0
5. $\pm 7\pi/12$ 6. $\pi/5$, $6\pi/5$

7. $-7\pi/15, -8\pi/15$
8. (a) $(\sqrt{2})/2$ (b) 1 (c) 1/2 (d) $(\sqrt{2})/2$
9. $5\pi/6, -5\pi/6$ 10. $5\pi/6, -7\pi/6$
11. $-2\pi/3$ 12. $-\pi/4$

Exercise 8.1

1. (a) 4.77 (b) 13.31 (c) 9.62
2. (a) 13.45 (b) 24.16 (c) 13.37
3. (a) 39.5° (b) 33.7° (c) 33.2°
4. (a) $70.4^\circ, 109.6^\circ$ (b) $63.2^\circ, 116.8^\circ$ (c) $64.6^\circ, 115.4^\circ$
5. (a) 44.7° (b) $68.6^\circ, 111.4^\circ$ (c) $49.0^\circ, 131.0^\circ$
6. (a) $(10\sqrt{6})/3$ (b) $(50\sqrt{6})/3$ (c) $30\sqrt{3}$
7. (a) $\pi/2$ (b) $\pi/3, 2\pi/3$ (c) $\pi/4, 3\pi/4$

Exercise 8.2

1. (a) 9.14 (b) 12.87 (c) 16.39
2. (a) 34.62 (b) 38.72 (c) 3.91
3. (a) 40.8° (b) 39.8° (c) 64.7°
4. (a) 111.8° (b) 106.3° (c) 90.6°
5. (a) $2\sqrt{17}$ (b) $10\sqrt{7}$ (c) $5\sqrt{58}$
6. (a) $\pi/6$ (b) $\pi/4$ (c) $5\pi/6$
7. (a) 3 cm, 5 cm (b) 10 mm (c) 2 cm
9. $k = 1$ or $k > \sqrt{2}$

Exercise 8.3

1. (a) 12.4° (b) 7.41 cm (c) 54.52 cm^2
2. (a) 12.82 cm (b) 26.9° (c) 22.05 cm (d) 99.94 cm^2
3. (a) 37.32 cm (b) 25.8° (c) 385.67 cm^2
4. (a) 18.07 cm (b) 19.20 cm (c) 178.56 cm^2
5. (a) 147.6° (b) 14.40 cm (c) 84.97 cm^2
6. (a) 139.84° (b) 7.25 cm (c) 164.89 cm^2
7. (a) 47.13 cm^2 (b) 93.51 cm^2

Exercise 8.4

1. (a) 7. 7 km (b) N82.1°W
2. 162.3° or 237.7° 3. 121.2 m
4. 42.6 m 5. 710.5 m 6. 29.1°
7. (a) 69.6° (b) 134.2 cm^2 (c) 114.7°
8. (a) 8 m (b) $110\sqrt{3} + 20\sqrt{19} + 160$ cm^2 (c) $200\sqrt{3}$ cm^3
9. (a) $12\sqrt{2}, 3\sqrt{33}$ cm (b) 59.0° (c) 40.7° (d) 531.7 cm^2
10. (a) OP = h tan 40, OQ = h tan 70 (b) 36.6 m

Exercise 9.1

1. (a) 3.84 cm (b) 29.29 cm (c) 6.55 cm (d) 1.13 cm^2
2. (a) 60° (b) 20.94 cm (c) 1220.40 cm^2
3. (a) 4 rad. (b) 200 cm^2
4. (a) 34.38 cm (b) 4.16 cm^2
5. (a) 12.79 cm (b) 98.16 cm
6. (b) 122.84 cm^2
7. (a) 1.3181 rad (b) 140.31 cm^2
8. (a) 143.13° (b) 84.98 cm (c) 175.10 cm^2
9. (a) $6\sqrt{13}$ cm (b) 2.78 rad (c) 92.07 cm
10. (a) 83.78 cm (b) 355.74 cm^2
11. (a) $4\pi^2\sqrt{3}$ cm^2 (b) $4\pi^2\sqrt{3} - 2\pi^3$ cm^2 (c) $2\pi^2$ cm
12. (a) $3\pi^2$ cm (b) 30.42 cm^2

Exercise 10.1

1. (a) $13.62^\circ, 166.38^\circ$ (b) $122.13^\circ, 237.87^\circ$ (c) $56.21^\circ, 236.21^\circ$ (d) $57.87^\circ, -57.87^\circ$ (e) 4.0478, 5.3769 (f) 1.4601, 4.8231 (g) 1.7597, −1.3819 (h) 0.2608, 2.8807
2. (a) $20.20^\circ, 69.80^\circ, 200.20^\circ, 249.80^\circ$ (b) 260.95° (c) $86.80^\circ, 266.80^\circ$ (d) $67.87^\circ, -47.87^\circ$ (e) No soln. (f) 0.146, 1.948, 2.241, 4.043, 4.335, 6.137 (g) −1.7597, 1.3819 (h) −2.2447, −0.8969
3. (a) $51.81^\circ, 148.19^\circ, 231.81^\circ, 328.19^\circ$ (b) $-171.20^\circ, -38.80^\circ, 8.80^\circ, 141.20^\circ$ (c) 5.4288, 1.9960 (d) 4.7517, 1.6101
4. (a) $\pi/6, 5\pi/6$ (b) $\pi/4, 7\pi/4$ (c) $-3\pi/4, \pi/4$ (d) $-\pi/6, \pi/6$ (e) $5\pi/8, 7\pi/8, 13\pi/8, 15\pi/8$ (f) 2π (g) $19\pi/30, -11\pi/30$ (h) $5\pi/12, -7\pi/12$
5. (a) $11\pi/12, 19\pi/12$ (b) $\pi/12, 5\pi/12$ (c) $-\pi/4, 3\pi/4$ (d) $2\pi/3, \pi$

Exercise 10.2

1. (a) $y = (\sqrt{3})x/3 + 2$ (b) $y = -(\sqrt{3})x + 9$ (c) $y = (\sqrt{3})x + 1$ (d) $y = -x - 4$
2. (a) $3\pi/4$ (b) $2\pi/3$ (c) $\pi/4$ (d) $2\pi/3$ (e) $\pi/3$ (f) $\pi/6$
3. (a) $\pi/4$ (b) $3\pi/4$ (c) $\pi/3$ (d) $2\pi/3$
4. (a) $\pi/2$ (b) $\pi/12$ (c) $5\pi/12$ (e) $5\pi/12$

Exercise 10.3

See answers for Exercise 10.1.

Exercise 10.4

1. (a) 270° (b) $90^\circ, 270^\circ; 60^\circ, 300^\circ$
   (c) $0^\circ, 180^\circ, 360^\circ; 210^\circ, 330^\circ$
   (d) $0^\circ, 180^\circ, 360^\circ; 45^\circ, 225^\circ$
2. (a) $\pi/2, 3\pi/2; \pi$ (b) $0, \pi, 2\pi; \pi/3, 2\pi/3$
   (c) $3\pi/4, 7\pi/4$ (d) $\pi/3, 4\pi/3; 2\pi/3, 5\pi/3$
3. (a) $45^\circ, 225^\circ; 71.6^\circ, 251.6^\circ$ (b) $210^\circ, 330^\circ$
   (c) $79.3^\circ, 280.7^\circ$ (d) No soln.
4. (a) $\pi/2; 3\pi/4, 7\pi/4$ (b) $0, \pi, 2\pi$
   (c) $5\pi/6, 11\pi/6; \pi/6, 7\pi/6$
   (d) $5\pi/8, 7\pi/8, 13\pi/8, 15\pi/8;$
   $\pi/8, 3\pi/8, 9\pi/8, 11\pi/8$

Exercise 11.1

1. (a) 70° (b) $\pi/3$ (c) 190°
2. (a) $y = 0, 3, 360^\circ, 0^\circ, 3, -3$
   (b) $y = 0, 0.5, 180^\circ, 0^\circ, 0.5, -0.5$
   (c) $y = 0, 2, 360^\circ, -10^\circ, 2, -2$
   (d) $y = 6, 3, 2, 0, 9, 3$
   (e) $y = 2, 3, 2\pi, -\pi/6, 5, -1$
   (f) $y = 0, 2\pi, 2\pi, \pi/12, 2\pi, -2\pi$
3. (a) $y = 0, 90^\circ, 0^\circ, x = \pm(45^\circ + 90^\circ n)$
   (b) $y = 1, 180^\circ, 0^\circ, x = \pm(90^\circ + 180^\circ n)$
   (c) $y = 0, 180^\circ, 30^\circ,$
   $x = -60^\circ - 180^\circ n, x = 120^\circ + 180^\circ n$
   (d) $y = 5, 180^\circ, 0, x = \pm(90^\circ + 180^\circ n)$
   (e) $y = 3, \pi, \pi/6,$
   $x = 2\pi/3 + n\pi, x = -\pi/3 - n\pi$
4. $y = \pm 3 \sin x + 2$ 5. $y = \pm 4 \cos(x + 30^\circ) + 5$
6. $y = 5 \tan(x - 45^\circ) + 2$
7. (a) $y = 4 \cos(x - 30^\circ) - 3$
   (b) $y = -2 \cos(x - 30^\circ) + 5$
8. (a) $y = 2 \sin(x + \pi/6) + 1$
   (b) $y = -2 \sin(x - \pi/6) - 2$
9. (a) $y = \tan(x - \pi/6) + 4$ (b) $y = -2 \tan x - 4$
10. (a) 25 C (b) 40 C, 12 noon
    (c) 6 hours 26 minutes
11. (a) 8.78 m (b) 5m, 3.30 pm (c) 7 hrs 18 mins

Exercise 12.1

1. (a) $(1/4)(\sqrt{6} + \sqrt{2})$ (b) $2 + \sqrt{3}$
   (c) $(1/4)(\sqrt{6} + \sqrt{2})$ (d) $(1/4)(\sqrt{6} - \sqrt{2})$
   (e) $(\sqrt{2})(1 + \sqrt{3})/4$ (f) $2 - \sqrt{3})$
   (g) $(-\sqrt{2})(1 + \sqrt{3})/4$ (h) $(\sqrt{2})(1 - \sqrt{3})/4$
2. (a) 56/65 (b) 63/65 (c) 56/33
3. (a) $-416/425$ (b) $-297/425$ (c) $-416/87$
4. (a) $(-1/20)(3 + 4\sqrt{15})$ (b) $(-1/20)(4 + 3\sqrt{15})$
   (c) $(1/119)(192 + 25\sqrt{15})$
5. (a) $\pi/3, 4\pi/3$ (b) $\pi/4, 5\pi/4$
   (c) $2\pi/3, 5\pi/3$ (d) $\pi/2, 3\pi/2$
6. $3\pi/8, 7\pi/8$

Exercise 13.1

1. (a) $\{x: 4 \le x \le 6\}$ (b) U
   (c) $\{x: 5 < x \le 10\}$ (d) $\{x: 5 < x \le 6\}$
   (e) $\{x: 0 \le x \le 5, 6 < x \le 10\}$ (f) ϕ
2. (a) ϕ (b) ϕ
   (c) $\{x: 90 \le x < 95 \text{ or } 105 \le x \le 110\}$ }
   (d) ϕ (e) U (f) ϕ
3. (a) ϕ (b) ϕ (c) $\{x: 60 \le x < 70\}$
   (d) $\{x: 60 < x \le 70\}$
4. (a) 60 (b) 100 5. (a) 0 (b) 70
6. (a) {(0, 0), (0, 1), (1, 0), (1, 1)}
   (b) {(0, 0), (0, 1), (1, 0), (1, 1),
   (2, 0), (2, 1), (0, 2), (1, 2)}
7. (a) 4 (b) 14
8. (a) At EC, all year 11 Physics students are also year 11 Chemistry students.
   (b) The set of year 11 students at EC who do not do both Chemistry and Physics.
   (c) At EC, all year 11 Drama students are also year 11 Dance students.
   (d) There are no year 11 students at EC who do Chemistry, Dance and Drama.
9. (a) At BC, not all students who have visited London have visited Singapore.
   (b) The set of all students at BC who have not visited China nor Singapore.
   (c) At BC, all students who have visited both London and Singapore have also visited New Zealand *and vice-versa*.
   (d) At BC, no student has visited China and New Zealand and Singapore.

Exercise 14.1

1. (a) 36 (b) 120 (c) 3024
   (d) 756 (e) 1225 (f) 19 600
   (g) 4950 (h) 200
2. (a) 4 (b) 32 4. (a) 9 (b) 6 (c) 5

Exercise 14.2

1. (a) 120 (b) 120 (c) 190 (d) 4950
2. (a) 252 (b) 126 3. (a) 55 (b) 30
4. (a) 5 033 094 528 (b) 7 864 210 200
   (c) 15 530 163 000 (d) 886 322 710
5. (a) 1 (b) 21 (c) 770 (d) 560
6. (a) 1 (b) 1 048 576 (c) 256
7. (a) 123 410 (b) 6 096 454
   (c) 101270 (d) \$3 258 024
8. 15 9. 15 10. 32 11. 255
12. 310 13. 16

Exercise 14.3

1. (a) $1 + 4x + 6x^2 + 4x^3 + x^4$
   (b) $1 + 8x + 24x^2 + 32x^3 + 16x^4$
   (c) $1 - 4x + 6x^2 - 4x^3 + x^4$
   (d) $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$
   (e) $1 + 5x/2 + 5x^2/2 + 5x^3/4 + 5x^4/16 + x^5/32$
   (f) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
2. (a) $1 + 10x + 45x^2 + 120x^3 + \ldots$
   (b) $1 - 24x + 264x^2 - 1760x^3 + \ldots$
   (c) $256 - 1024x + 1792x^2 - 1792x^3 + \ldots$
   (d) $512 + 6912x + 41472x^2 + 145152x^3 + \ldots$
3. (a) $1/x^7 + 7/x^5 + 21/x^3 + 35/x$
   (b) $1/x^9 + 9/x^6 + 36/x^3 + 84$
   (c) $-1/x^{22} + 11/x^{19} - 55/x^{16} + 165/x^{13}$
4. (a) $x^8 + 16x^6 + 112x^4 + 448x^2 + \ldots$
   (b) $x^{20} - 10x^{16} + 45x^{12} - 120x^8 + \ldots$
   (c) $x^8 - 8x^6 + 28x^4 - 56x^2 + \ldots$
5. (a) $17\,578\,125x^2$ (b) $-35x$
   (c) $252x^5$ (d) $10x$
6. (a) 210 (b) 6 000
7. (a) 8 064 (b) 495
8. (a) $13\,608x^3$ (b) $10/x^3$
   (c) $5/x^3$ (d) $10x^7$
9. (a) $\sum_{r=0}^{7}\binom{7}{r}$ (b) $\sum_{r=0}^{6}\binom{6}{r}$ (c) $\sum_{r=0}^{8}\binom{8}{r}$
10. (a) 512 (b) 2^{2n}

Exercise 15.1

1. {1,2,3,4,5,6}
   (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{2}{3}$
2. (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{8}$
3. (a) $\frac{1}{7}$ (b) $\frac{2}{7}$ (c) $\frac{3}{7}$
4. (a) $\frac{14}{31}$ (b) $\frac{21}{31}$ (c) $\frac{5}{31}$ (d) $\frac{2}{31}$
5. (a) $\frac{3}{8}$ (b) $\frac{1}{2}$ (c) $\frac{7}{8}$
6. (a) $\frac{1}{12}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$
7. (a) $\frac{1}{9}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{2}{3}$
8. (a) 0 (b) 0 (c) $\frac{1}{3}$ (d) 1
9. (a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{3}{4}$
10. (a) $\frac{1}{16}$ (b) $\frac{1}{4}$ (c) $\frac{1}{16}$ (d) $\frac{1}{4}$
11. (a) $\frac{1}{16}$ (b) $\frac{3}{8}$ (c) $\frac{7}{16}$
12. (a) $\frac{1}{9}$ (b) $\frac{5}{9}$ (c) $\frac{1}{9}$ (d) $\frac{11}{12}$ (e) $\frac{1}{6}$
13. (a) $\frac{1}{25}$ (b) $\frac{1}{5}$ (c) $\frac{8}{25}$
14. (a) $\frac{14}{39}$ (b) $\frac{20}{39}$
15. (a) 0.095 69 (b) 0.030 075 (c) 0.4060
16. (a) 0.1248 (b) 0.000 1783 (c) 0.1968
17. (a) $\frac{1}{12}$ (b) $\frac{1}{2}$ (c) $\frac{5}{18}$ (d) $\frac{2}{7}$

Exercise 15.2

1. (a) $p = 0.6$, $q = 0.4$, $r = 0.9$
   (b) 0.6 (c) 0.1 (d) 0.78
2. (a) $p = \frac{5}{6}$, $q = \frac{3}{4}$, $r = \frac{2}{3}$
   (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{23}{72}$
3. (a) $a = \frac{7}{8}$, $f = \frac{2}{5}$, $g = \frac{1}{3}$
   (b) $\frac{2}{5}$ (c) $\frac{7}{20}$ (d) $\frac{47}{120}$
4. (a) $p = 0.3$, $q = 0.8$, $r = 0.6$
   (b) 0.6 (c) 0.06 (d) 0.22
   (e) 0.34
5. (a) $p = 0.6$, $q = 0.6$, $r = 0.8$, $s = 0.9$
   (b) 0.5 (c) 0.7 (d) 0.25
   (e) 0.06 (f) 0.406 (g) 0.44
   (h) 0.276 (i) 0.294
6. (a) $p = 0.7$, $q = 0.8$, $r = 0.85$
   (b) 0.6 (c) 0.9 (d) 0.224
   (e) 0.236 (f) 0.326
7. (a) $\frac{3}{7}$ (b) $\frac{3}{7}$ (c) $\frac{3}{7}$ (d) $\frac{24}{49}$
8. (a) $\frac{3}{7}$ (b) $\frac{1}{3}$ (c) $\frac{3}{7}$ (d) $\frac{4}{7}$
9. (a) 0 (b) $\frac{2}{5}$
10. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{30}$
    (d) $\frac{1}{5}$ (e) $\frac{1}{3}$ (f) $\frac{2}{5}$
11. (a) $\frac{3}{5}$ (b) $\frac{2}{5}$ (c) $\frac{8}{125}$
    (d) $\frac{7}{25}$ (e) $\frac{44}{125}$ (f) $\frac{2}{5}$
12. (a) $\frac{17}{30}$ (b) 0 (c) 0.39
13. (a) 0.14 (b) 0.38
14. (a) 0.32 (b) 0.74 (c) 0.5
15. (a) 0.7125 (b) 0.287 (c) 0.9995
16. (a) 0.99 (b) 0.0005
17. (a) 0.028 (b) 0.07 (c) 0.82
    (d) 0.54 (e) 0.838 (f) 0.02872
18. (a) 0.0534 (b) 0.8306 (c) 0.33
19. (a) 0.000 01 (b) 0.000 99 (c) 0.078 31
    (d) 0.078 31

Exercise 15.3

1. (a) 0.5 (b) 0.4 (c) 1 (d) 0.6

2. (a) 0.9 (b) 0.3 (c) 0.3
   (d) 0.5 (e) 0.5
3. (a) 0.24 (b) 0.78 (c) 0.4 (d) 0.3077
4. (a) 0.6 (b) 0.2 (c) $\frac{3}{17}$
5. (a) $\frac{1}{25}$ (b) $\frac{21}{50}$ (c) $\frac{15}{46}$
   (d) $\frac{17}{50}$ (e) $\frac{13}{25}$
6. (a) 0.06 (b) 0.42 (c) 1/7
   (d) 0.042 (e) 0.42 (f) 0.09589
7. (a) 0.225 (b) 0.45 (c) 0.0051
8. (a) 0.995 (b) 0.8895
9. (a) 0.061 (b) 0.098 36
10. (a) 4/25 (b) 2/5 (c) 1/3
11. (a) 6/25 (b) 2/11 (c) 8/23
12. (a) 0.2 (b) 0.805 (c) 0.6460
13. (a) 2/5 (b) 1/2 (c) 26/165
14. (a) 9/28 (b) 2/15 (c) 9/11 (d) 19/28
15. (a) 5/17 (b) 80/569 (c) 0.030 9
    (d) 0.029 50 (e) 0.000 473
16. (a) 0.09317 (b) 0.1474 (c) 0.004 438
17. (a) 3/11 (b) 5/22 (c) 6/11
    (d) 2/7
18. (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$
    (d) (i) $\frac{4}{15}$ (ii) $\frac{2}{15}$

Exercise 15.4

1. A and B are not mutually exclusive because $P(A \cap B) = 0.5 \neq 0$.
2. A and B are not mutually exclusive because $P(A \cap B) = 0.2 \neq 0$.
3. A and B are not mutually exclusive because $P(A \cap B) = 0.1 \neq 0$.
4. A and B are mutually exclusive because $P(A \cap B) = 0$.
5. A and B are independent because $P(A) \times P(B) = P(A \cap B) = 0.45$.
6. A and B are not independent because $P(A) \times P(B)\ 0.18 \neq P(A \cap B) = 0.3$.
7. A and B are not independent because $P(A) \times P(B)\ 0.24 \neq P(A \cap B) = 0.2$.
8. (a) 0.9 (b) 0.7
9. (a) 0.1 (b) 0.24
10. $0.4 \leq P(A \cup B) \leq 0.7$
11. (a) 0.8047 (b) 0.7058
    (c) A and B are mutually exclusive $[P(A \cap B) = 0]$.
    (d) A and B are not independent $[P(A) \neq P(A \mid B')]$
12. (a) 0.94 (b) 0.2
    (c) A and B are not mutually exclusive and are independent
13. (a) $P(A) = \frac{1}{2}$, $P(B) = \frac{4}{9}$, $P(C) = \frac{1}{6}$
    (b) Only A and C are mutually exclusive.
    (c) Only A and B are independent.
14. (a) 3/8
    (b) B and C are not mutually exclusive and are not independent

Exercise 16.1

1. (a) $6x^5$ (b) $28x^{11}$ (c) $-10x^5$ (d) $2x^5$
2. (a) $(1+x)^5$ (b) $6(1-x)^5$
   (c) $2(1+x)^3$ (d) $-(x-1)^5$
3. (a) $2x^2$ (b) $5x^{12}$ (c) $x^7/2$ (d) $-x^{16}/3$
4. (a) $(1+x)^2$ (b) $2(1-x)^5$
   (c) $-(x^2+1)^3/2$ (d) $(2+x)/2$
5. (a) $8x^9$ (b) $4x^8$ (c) $-27x^{12}$ (d) $x^6/8$
   (e) x^6y^3 (f) $9x^2y^6$
   (g) $-64x^6y^3$ (h) $x^2y^4/16$
6. (a) $(1+x)^6$ (b) $8(x-2)^6$
   (c) $(x+5)^{12}$ (d) $-8(x+2)^9$
7. (a) x^3y^2 (b) $6x^4y^5$
   (c) $12x^5y^3$ (d) $-3x^3y^2$
   (e) x^2y^2 (f) $2x$
   (g) $x^2y/3$ (h) $-2y^2/3$
8. (a) $x^3(1+x)^5$ (b) $6x^3(1-x)^3$
   (c) $8(1+x)^5(1+3x)^4$ (d) $-2(3+x)^3(2-x)^4$
9. (a) $x^2(1+x)$ (b) $-2x(1+2x)$
   (c) $\frac{(1+4x)}{2(1+x)}$ (d) $-2(x+3)(x+5)/3$
10. (a) $x^6(1+x)^{12}$ (b) $-8x^3(1+x)^6$
    (c) $(1+2x)^4(x+3)^2$ (d) $(x+1)^4(2x-1)^6/16$

Exercise 16.2

1. (a) x^{-4} (b) $2x^{-5}$ (c) $-x^4$ (d) $(x^{-3})/2$
2. (a) $-2x^7$ (b) $12x^{-4}$
   (c) $-10x^3y^{-1}$ (d) $(x^{-1}y^{-1})/2$
3. (a) $1/(2x)$ (b) $2t$
   (c) $2x^4/y$ (d) $2q/(3p^7)$
4. (a) $8x^{-6}$ (b) $x^{-6}/27$
   (c) x^4 (d) $9x^4$
5. (a) $y^6/(4x^4)$ (b) $1/(4x^4y^2)$
   (c) $(4x^4)/(9y^2)$ (d) $x^2y^3/4$

6. (a) $(3q^2)/(4p^2)$ (b) s^3/t^3
   (c) $-6x/y^5$ (d) $5n^3/(2m^2)$
7. (a) $(1 + x^2)^{-1}$ (b) $(1 - x)(1 + x)^{-2}$
8. (a) $\dfrac{1}{(4+x)^3(x+3)}$ (b) $(x - 2)/(x + 2)$

Exercise 16.3

1. (a) $x^{1/4}$ (b) $x^{1/2}$
   (c) $x^{2/3}y$ (d) $x^2y^{-3/2}$
2. (a) $x^{5/2}$ (b) $2x^{3/2}y^{-1/2}$
   (c) $3p^{-1/3}q^{-1}$ (d) $2x^{-1/2}y^{1/2}$
3. (a) $\dfrac{2}{3(1+x)^{1/2}}$ (b) $1/(1 - x^2)^{1/2}$
4. (a) 3 (b) 1/3 (c) 1/2 (d) 5/2
   (e) 32 (f) 1/81 (g) 25 (h) 64/125
5. (a) $8x^{9/2}$ (b) $27/x^{15/2}$
   (c) $x^{1/2}/(2y)$ (d) $y^2/(11x)$

Exercise 16.4

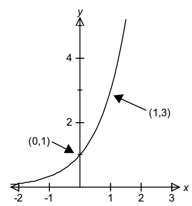
1. (a) $x^2(x^2 + 1)$ (b) $x^2(3 + 4x)$
   (c) $x^2(1 - x)(1 + x)$ (d) $x^{1/2}(1 + x^{1/2})$
   (e) $xy(x + y)$ (f) $3x^2y(y^2 + 2)$
   (g) $2x^{1/2}y^{1/2}[2x^{1/2} + 3y^{1/2}]$
   (h) $x^{-1/2}y^{-1/2}[y^{3/2} + x^{3/2}]$
2. (a) $x^{-1/2}(x - 1)$ (b) $(1 + x)^2(2 + x)$
   (c) $4(1 - x)^2(5 - 3x)$
   (d) $(1 + x)^{1/2}[(1 + x)^{3/2} + 1]$
   (e) $x(1 + x)(1 + 2x)$
   (f) $3(x - 2)^2(x + 3)^2(x + 5)$
   (g) $2(2 - x)^{1/2}(5 - x)^{1/2}[2(2 - x)^{1/2} + 3(5 - x)^{1/2}]$
   (h) $(1 + x)^{-1/2}(1 - x)^{-1/2}[(1 + x)^{3/2} + (1 - x)^{3/2}]$
3. (a) $(1 + x^2)$ (b) $x/(x + 1)$
   (c) $\dfrac{(x+1)}{x^2(x+2)}$ (d) $\dfrac{x^2(x^2+2)}{3x^2+4}$
   (e) $(1 - x)/x$ (f) $(3 + x)/2$
   (g) $(1 + x^2)/x^3$ (h) $(1 + x)/x^{3/2}$
4. (a) $(1 + x)(2 + x)$ (b) $(1 + x)/(2 + x)$
   (c) $(5 + 2x)/(1 - 2x)^3$ (d) $(3 - x)/(1 - x)^{3/2}$

Exercise 16.5

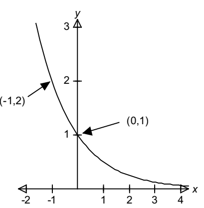
1. (a) (i) 306 (ii) 3.06×10^2
   (b) (i) 0.135 (ii) 1.35×10^{-1}
   (c) (i) 13.4 (ii) 1.34×10^1
   (d) (i) 134 000 000 (ii) 1.34×10^8
2. (a) 6.3×10^{-8} (b) 3.3×10^{-16}
   (c) 1.2×10^{12} (d) 4.2×10^8
   (e) 5.8×10^{-5}
3. $3.333 \times 10^4 < N < 5.000 \times 10^4$
4. (a) 1.0978×10^{30} (b) 6.558×10^{54}
5. (a) 1.08×10^9 km/h (b) 3.34×10^{-6} seconds
   (c) 9.4543×10^{12} km
6. (a) 6.65×10^{-24} g (b) 1.54×10^{26} atoms
7. (a) 6.2×10^2 (b) 2.4×10^{-17} m^2
8. (a) 5.39×10^{18} J (b) AUD$193.5 billion

Exercise 17.1

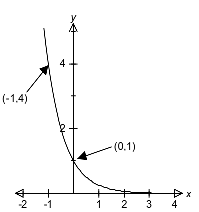
1. (a) Increasing function
   (b) decreasing function
   (c) decreasing function
   (d) increasing function
2. Graph A : Equation III
   Graph B : Equation V
   Graph C : Equation II
3. (a) $y = 3^x$



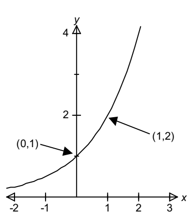
(b) $y = 2^{-x}$



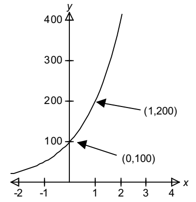
(c) $y = 0.25^x$



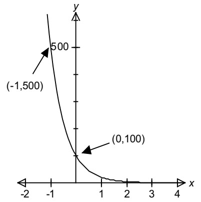
(d) $y = 0.5^{-x}$



3. (e) $y = 100(2^x)$



(f) $y = 100(5^{-x})$



4. (a) $y = 4^x$ (b) $y = (1/9)^x$
   (c) $y = (1.01396)^x$ (d) $y = (1.1025)^x$
5. 10/3
6. 4/5
7. 25
8. 1/3

Exercise 17.2

1. (a) 2^{x-2} (b) 3^{x+2}
   (c) 2^{-1} (d) 5^{3n+1}
   (e) 2^{x-1} (f) 3^{-1}
   (g) 5^t (h) 2^{3-8t}
2. (a) 2^x (b) $10(3^x)$
   (c) $4(5^{x-1})$ (d) $3(1/2)^t$
   (e) 2^x (f) $6(3^x)$ or $2(3^{x+1})$
   (g) $(2^t)/4$ or 2^{t-2} (h) 1/5
3. (a) 3 (b) −5 (c) 3 (d) −3
   (e) 5/2 (f) 3/2 (g) 3/2 (h) −4/3
   (i) −3 (j) −4/3 (k) 9 (l) 1
   (m) 1 (n) 1/3 (o) 1/2 (p) 1
4. (a) 2 (b) 2
5. (a) 2.930 (b) 6.456
   (c) 1.474 (d) −1.869

Exercise 17.3

1. (a) (i) 100 (ii) 14 379 (b) 3 h 21 min
2. (a) 2 318 969 (b) 96 624 per hour
   (c) 12 hours 21 minutes
3. (a) 13 hours 31 minutes
   (b) (i) 566 (ii) 166 (iii) 48
   (c) The amount decaying each day is decreasing.
4. (a) 14.1 g (b) 4.4 g/year (c) 20 years
5. (a) $A = 15$; $k = 0.044\ 72$ (b) 118
6. (a) $A = 800$ thousand; $k = -0.0200$
   (b) During the 16th week.
7. (a) 760 (b) During 2018.
   (c) (i) 251 (ii) 63
8. (a) 1272 (b) (i) 1272 (ii) 4 513
9. (a) A has a larger population at the start.
   (b) B has a faster growing rate.
   Exponent for 10 in equation for B is greater than the exponent for 10 in the equation for A.
   (c) During 2027.
10. (a) B has a growing population.
    Exponent for 10 in equation for B is positive whereas exponent for 10 in the equation for A is negative.
    (b) A: average decay rate of 18.7%
    B: average growth rate of 20.2%
    (c) During 2014.

Exercise 18.1

1. (a) $T_{n+1} = T_n + 20$, $T_1 = 5$, $n = 1, 2, 3, \ldots$
   $T_n = -15 + 20n$, $n = 1, 2, 3, \ldots$
   (b) $T_{n+1} = T_n + 0.3$, $T_1 = 2.5$, $n = 1, 2, 3, \ldots$
   $T_n = 2.2 + 0.3n$, $n = 1, 2, 3, \ldots$
   (c) $T_{n+1} = T_n - 5$, $T_1 = -10$, $n = 1, 2, 3, \ldots$
   $T_n = -5 - 5n$, $n = 1, 2, 3, \ldots$
   (d) $T_{n+1} = T_n - 3$, $T_1 = 4$, $n = 1, 2, 3, \ldots$
   $T_n = 7 - 3n$, $n = 1, 2, 3, \ldots$
2. (a) 7.5, 10, 12.5, 15, 17.5
   $T_n = 5 + 2.5n$, $n = 1, 2, 3, \ldots$
   (b) 20, 24, 28, 32, 36
   $T_n = 16 + 4n$, $n = 1, 2, 3, \ldots$
   (c) 33, 22, 11, 0, −11
   $T_n = 44 - 11n$, $n = 1, 2, 3, \ldots$
   (d) 1, 2/3, 1/3, 0, −1/3
   $T_n = 4/3 - n/3$, $n = 1, 2, 3, \ldots$
3. (a) 86 (b) 998th term
4. (a) 559 (b) 56 terms
5. (a) $T_{n+1} = T_n + 13$, $T_1 = 25$, $n = 1, 2, 3, \ldots$
   $T_n = 12 + 13n$, $n = 1, 2, 3, \ldots$
   (b) 38 terms
6. (a) $T_{n+1} = T_n - 4$, $T_1 = 500$, $n = 1, 2, 3, \ldots$
   $T_n = 504 - 4n$, $n = 1, 2, 3, \ldots$
   (b) 125 terms
7. (a) 89 (b) 16 terms
8. (a) 1175 (b) 16 terms
9. (a) 284 (b) 1165 (c) 15 terms
10. (a) 6300 (b) 41 terms

11. (a) 1083 (b) 5510 (c) 882
(d) −2325
12. (a) $a = 5, d = 2$ (b) $a = 5, d = -2$
13. (a) 960, 1050 (b) 90
14. (a) 170, 220, 276 (b) 62
15. (a) $T_{n+1} = T_n + 2, T_1 = 46, n = 1, 2, 3, \ldots$
$T_n = 44 + 2n, n = 1, 2, 3, \ldots$
(b) $T_{n+1} = T_n - 5, T_1 = -12, n = 1, 2, 3, \ldots$
$T_n = -7 - 5n, n = 1, 2, 3, \ldots$

Exercise 18.2

1. (a) $T_{n+1} = T_n \times 4, T_1 = 1, n = 1, 2, 3, \ldots$
$T_n = 4^{n-1}, n = 1, 2, 3, \ldots$
(b) $T_{n+1} = T_n \times 2, T_1 = 5, n = 1, 2, 3, \ldots$
$T_n = 5 \times 2^{n-1}, n = 1, 2, 3, \ldots$
(c) $T_{n+1} = T_n \times 2, T_1 = -3, n = 1, 2, 3, \ldots$
$T_n = (-3) \times 2^{n-1}, n = 1, 2, 3, \ldots$
(d) $T_{n+1} = T_n \times (1/4), T_1 = 4, n = 1, 2, 3, \ldots$
$T_n = 4(1/4)^{n-1}, n = 1, 2, 3, \ldots$
(e) $T_{n+1} = T_n \times (-3), T_1 = 4, n = 1, 2, 3, \ldots$
$T_n = 4(-3)^{n-1}, n = 1, 2, 3, \ldots$
(f) $T_{n+1} = T_n \times (-1/3), T_1 = 3, n = 1, 2, 3, \ldots$
$T_n = 3(-1/3)^{n-1}, n = 1, 2, 3, \ldots$
2. (a) 2, 5, 12.5, 31.25, 78.125
$T_n = 2(2.5)^{n-1}, n = 1, 2, 3, \ldots$
(b) 200, 800, 3200, 12 800, 51 200
$T_n = 200(4)^{n-1}, n = 1, 2, 3, \ldots$
(c) 3, −15, 75, −375, 1875
$T_n = 3(-5)^{n-1}, n = 1, 2, 3, \ldots$
(d) −2700, −900, −300, −100, −100/3
$T_n = -2700(1/3)^{n-1}, n = 1, 2, 3, \ldots$
3. (a) −4, −8.4, −17.64, −37.044, −77.7924
$T_{n+1} = T_n \times (2.1), T_1 = -4, n = 1, 2, 3, \ldots$
(b) 5, −10, 20, −40, 80
$T_{n+1} = T_n \times (-2), T_1 = 5, n = 1, 2, 3, \ldots$
(c) 3, 9, 27, 81, 243
$T_{n+1} = T_n \times (3), T_1 = 3, n = 1, 2, 3, \ldots$
(d) 1/4, 1/16, 1/64, 1/256, 1/1024
$T_{n+1} = T_n \times (1/4), T_1 = 1/4, n = 1, 2, 3, \ldots$
4. (a) 3645 (b) 16th term (c) 13th term
5. (a) 4.096 (b) 11th term (c) 12th term
6. (a) 4 (b) 2 359 296 (c) 2 415 919 104
7. (a) 1/2 (b) −1/32 (c) −1/2048
8. (a) 2 (b) 196 608 (c) 50 331 648
9. (a) ± 1 959 552 (b) ± 2 539 579 392
(c) 9th term
10. Graph A: Equation IV $T_n = -3(2)^n$
Graph B: Equation III $T_n = 10(-0.8)^n$
Graph C: Equation II $T_n = 10(0.1)^n$
11. (a) $T_n = 8(2)^{n-1}, n = 1, 2, 3, \ldots$
(b) $T_n = -6(-2)^{n-1}, n = 1, 2, 3, \ldots$
(c) $T_n = 3(1/2)^{n-1}, n = 1, 2, 3, \ldots$
(d) $T_n = 15(-1/2)^{n-1}, n = 1, 2, 3, \ldots$

Exercise 18.3

1. (a) $T_{10} = 794.2800$ $S_{10} = 1515.4437$
(b) $T_{12} = 0.01451$ $S_{12} = 9.9782$
(c) $T_8 = -3435.9738$ $S_8 = -2617.6467$
(d) $T_6 = -0.02048$ $S_6 = 1.4227$
2. (a) 2 (b) 5 242 875
(c) −1 048 575 (d) -1.544×10^{14}
3. (a) 39.9035 (b) 24 000
(c) 509.7843 (d) −196.5458
4. (a) 3182.4318
(b) If $r = 0.09$, $S_{10} = 2747.2527$
If $r = -0.09$, $S_{10} = 2293.5780$
(c) 1 398 100
(d) If $r = 0.4$, $S_{10} = -0.6666$
If $r = -0.4$, $S_{10} = 0.2857$
5. (a) $S_{20} = -835.6826$,
$T_{n+1} = 1.4T_n, T_1 = -0.4, n = 1, 2, 3, \ldots$
(b) $S_{20} = 4983.3851$,
$T_{n+1} = 1.5T_n, T_1 = 0.75, n = 1, 2, 3, \ldots$
(c) $S_{20} = 0.2500$,
$T_{n+1} = 0.5T_n, T_1 = 0.125, n = 1, 2, 3, \ldots$
(d) $S_{20} = 5$,
$T_{n+1} = -0.2T_n, T_1 = 6, n = 1, 2, 3, \ldots$
6. $r = 1.1$, $S_5 = 18.3153$
7. $r = 0.99$, $S_5 = 4900.99501$
8. $n = 16$ 9. $n = 13$ 10. $n = 21$

Exercise 18.4

1. (a) S_∞ does not exist (b) S_∞ does not exist
(c) $S_\infty = 15$ (d) $S_\infty = 400/7$
2. (a) 4 (b) 500
(c) S_∞ does not exist (d) S_∞ does not exist
3. (a) 50 (b) 100
(c) S_∞ does not exist (d) S_∞ does not exist
4. (a) 1/3 (b) −1/9 5. (a) 110 (b) 296

Exercise 18.5

1. (a) $T_{n+1} = T_n - 400$, $T_0 = 30 \times 1000$
(b) After 63 days.
2. (a) $B(n + 1) = B(n) + 2575$, $B(0) = 50\ 000$
(b) After 20 years.
3. (a) $v(t + 1) = v(t) + 0.1$, $v(0) = 2$ m/s
(b) After 80 seconds (c) After 95 seconds.
4. (a) $a(n + 1) = a(n) + 6$, $a(1) = 4$
(b) Need more workers for the 34th day.
(c) S(33) = 3300
5. (a) $w(n + 1) = w(n) + 50$, $w(0) = 400$
(b) After 9 years. (c) After 10 years.
6. (a) $v(n + 1) = v(n) - 40\ 000$, $v(0) = 400\ 000$
(b) \$200 000 (c) \$200 000
7. (a) During 6th year. (b) S(5) = 1250 tonnes
8. (a) $a(n) = 41 - n$, $n = 1, 2, 3, \ldots$
(b) $n(81 - n)/2$ (c) 820 logs.
9. (a) 2824 (b) After 44 months.
(c) After 88 months.
10. (a) (i) $B(n + 1) = 1.036B(n)$, $B(0) = 500\ 000$
(ii) At the end of 20 years.
(b) (i) $B(n + 1) = 1.003B(n)$, $B(0) = 500\ 000$
(ii) At the end of 19 years 4 months.
11. (a) 0.0625 m^2 (b) 38 mm (c) A7
12. (a) 5.0 cm (b) 11th image
(c) 15 reflections
13. (a) 512 (b) The 25th square.
(c) 1 023 grains (d) 1.6770×10^{11} tonnes
14. (a) 131 cm (b) 9th rebound
(c) 24.1 m (d) 38 m
15. (a) –5.3394 cm (b) 262.0 cm
(c) 29 times
16. (a) 78.1 cm^3 (b) 79921.9 cm^3
(c) 80 000 cm^3; 26.7 cm
17. (a) 23.2 m (b) 158.9 m
(c) After 8 minutes.
(d) No. Since $S_\infty = 600$ m < 650 m.
18. (a) \$245.76 (b) \$20 163.84
(c) After 10 years.
(d) No! Since S_∞ = \$20 000.
19. (a) No. Since $S_{14} = 353.6$ km < 400 km
(b) 105.967 km, 7.96 km
20. (a) \$3975.23, \$6166.89
(b) No. Since S_5 = \$49 730.68 < \$50 000

Exercise 19.1

1. (a) ≈ 2.4; 0; ≈ 1.2 (b) 0; ≈ –1; ≈ 3
2. (a) 0 (b) 2 (c) $3x^2$ (d) $2x+1$
3. (a) $2x$ (b) $12x^{11}$
(c) $(3/2)x^{1/2}$ (d) $(½)x^{-1/2}$
(e) $-3x^{-4}$ (f) $-7x^{-8}$
(g) $(-1/2)x^{-3/2}$ (h) $(-5/2)x^{-7/2}$
(i) $-4x^{-5}$ (j) $-6x^{-7}$
(k) $(-1/3)x^{-4/3}$ (l) $(-7/2)x^{-9/2}$
4. (a) 4 (b) $-12x^2$
(c) 0 (d) $(3/2)x^2$
(e) $10x^{-3}$ (f) $(-4/3)x^{-5}$
(g) $(-1/4)x^{-2}$ (h) $6x^{-3}$
(i) $2x^{-1/2}$ (j) $(25/2)x^{3/2}$
(k) $(½)x^{5/2}$ (l) $-64x^3$
5. (a) $-4t^{-2}$ (b) $-8t^{-3}$
(c) $t^{-3/2}$ (d) $(-2/3)t^{-4/3}$
(e) $-t^{-5/2}$ (f) $t^{-7/5}$
(g) $(-5/6)t^{-3/2}$ (h) 0
(i) $-18t^2$ (j) $t/4$
6. (a) $12x^2$ (b) $6t$
(c) $-6y$ (d) $-10p^{-3}$
(e) $-v^{-5}$ (f) $7r^{-1/2}$
(g) $(-9/8)x^{-5/2}$ (h) $(3/10)k^{-3/2}$
7. (a) $2x$ (b) $-3x^2$
(c) $-6x$ (d) $5x^4$
(e) $(1/2)x^{-1/2} - 3$ (f) $2x - (1/x^2)$
(g) $5 + (4/x^3)$ (h) $(-1/x^2) - (1/2x^3)$
(i) $(-1/2)x^{-3/2} - 4$
8. (a) $2x + 2$ (b) $-4x + 4$
(c) 3 (d) 1/2
(e) $x + 2$ (f) $(3/2)x^{1/2} + 3$
(g) $1 - (5/2)x^{3/2}$ (h) $3x^2 - 3x^{-1/2}/2$
(i) $-2x^{-2}$
9. (a) $2x - 5$ (b) $2x + 1$
(c) $(3/2)x^{1/2} - 1 + (1/2)x^{-1/2}$
(d) $2x - 3x^2 + 1$ (e) $-1/x^2$
(f) $1 + 2/x^2$ (g) $3/x^2$
(h) $-4/x^2$ (i) $-9x^{-5/2} + x^{-2}$
10. $f\,'(t) = 1 + 1/t^2$, $f\,'(1) = 2$
11. $g'(t) = 3/t^2 - 4/t^3$, $g'(1) = -1$
12. $h'(t) = 2t + 2$, $t = -1$
13. $v'(t) = 3t^2 - 4t$, $t = 0, 4/3$
14. $P'(t) = t^2 + t - 2$, $t = -2, 1$
15. (a) x^5, $5x^4$ (b) $2y^2$, $4y$
(c) $1/x^3$, $-3/x^4$ (d) $\sqrt{t}$, $1/(2\sqrt{t})$

15. (e) $(1 + x)^2$, $2 + 2x$ (f) $1/\sqrt{x}$, $-1/(2x^{3/2})$

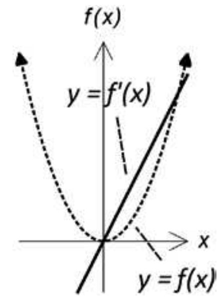
16. (a) 4 (b) -2 (c) 27
(d) -1 (e) -4 (f) $1/4$
(g) $-\sqrt{5}/50$ (h) $1/2$

Exercise 20.1

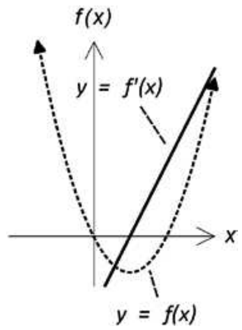
1. (a) $y' = -2x + 4$ (b) $y' = 2/x^3$
(c) $y' = 3x^2 - 8x + 5$ (d) $y' = 3x^2 - 6x + 3$
2. -2; $y = -2x + 8$ 3. 4; $y = 4x + 8$
4. -4, $y = -4x + 12$ 5. $y = x + 1$
6. $y = -4x - 2$, $(1, -6)$
7. (a) $(-2, -5)$ (b) $(-3, 17)$, $(1/3, -41/27)$
(c) $(-3, 2)$ (d) $(0, 0)$, $(2/3, 4/27)$
8. (a) $(-1/4, 25/8)$ (b) $(0, 4)$
(c) $(4, 2)$ (d) $(-2, -1/2)$
9. $y = -2x + 8$ 10. $y = -4x$, $y = -4x - 1$
11. (a) $(1, 1)$ (b) $(1/3, -31/27)$, $(1, -1)$
(c) $(1/3, -3)$, $(-1/3, 3)$
(d) $(1, 1)$
12. $a = -1$, $b = 3$ 13. $a = -3$, $b = 1$, $c = 2$

Exercise 20.2

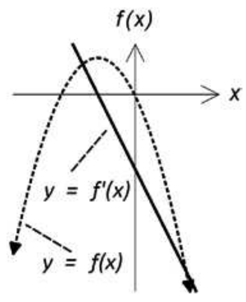
1. (a) Min. TP at $x = -1$
(b) Max. TP at $x = -2$; Min. TP at $x = 0$
(c) Min. TP at $x = -2$; Max. TP at $x = 0$
(d) Horizontal inflection point at $x = 2$
(e) Horizontal inflection point at $x = -1$
(f) Min. TP at $x = -1, 1$; Max. TP at $x = 0$
2. (a)



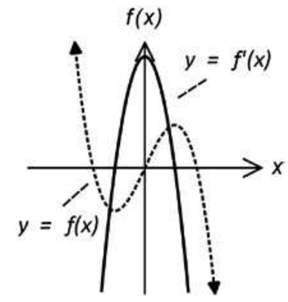
(b)



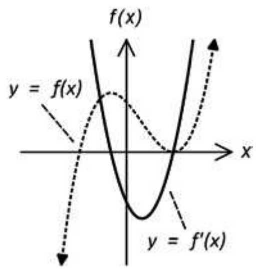
2. (c)



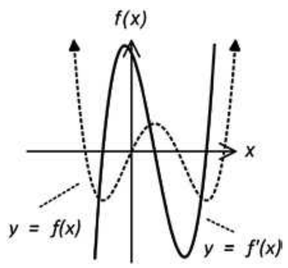
(d)



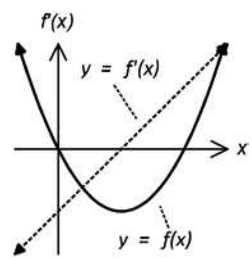
(e)



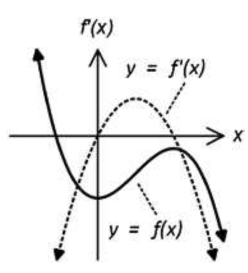
(f)



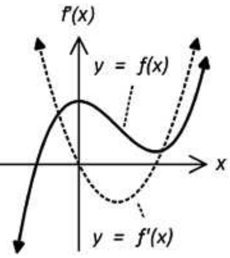
3. (a)



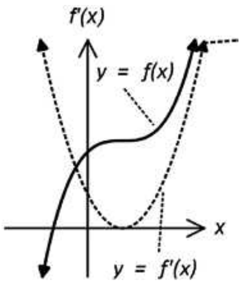
(b)



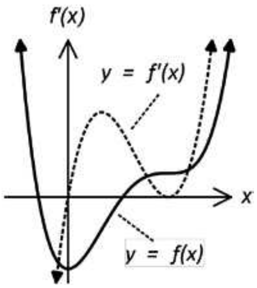
3. (c)



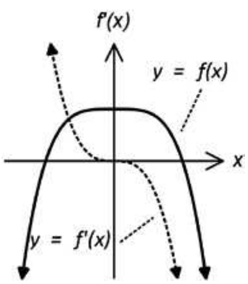
(d)



(e)



(f)

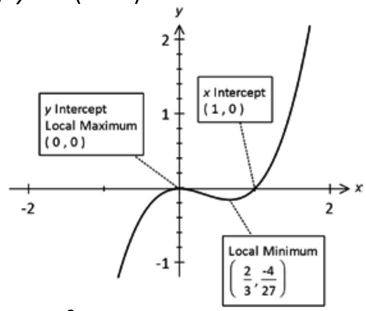


Exercise 20.3

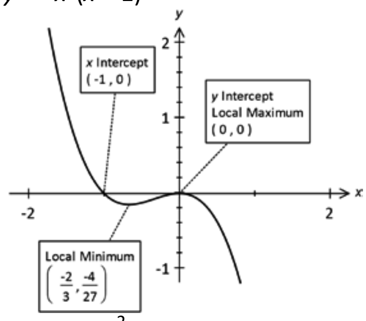
1. (a) Min at (1, 2) (b) Max at (–1, –1)
   (c) Min at (–4/3, –17/9)
   (d) Min at (–1/4, –25/24)
   (e) Min at (2/3, –4/27), Max at (0, 0)
   (f) Min at (1, 0), Max at (1/3, 4/27)
   (g) Max at (1, 2), Min at (–1, –2)
   (h) Min at (3/4, –27/256),
   Horizontal inflection at (0, 0)
   (i) Max at (0, –1), Min at (–1, –5/3),
   Min at (1/2, –53/48)
2. a = 1, b = 0 3. a = –1/12, b = 3/4
4. a = b = any real number
5. a = 1/2, b = 1/2

Exercise 20.4

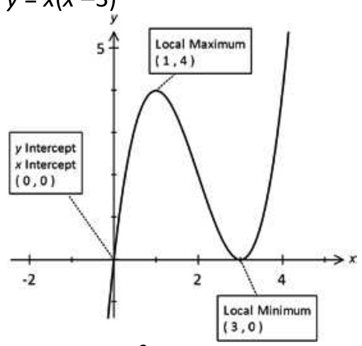
1. (a) $y = x^2(x - 1)$



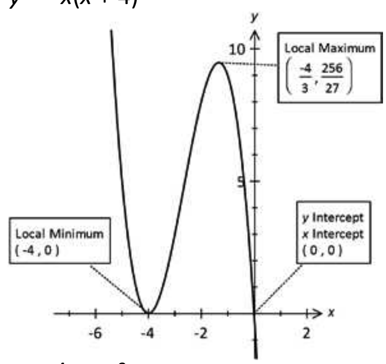
(b) $y = -x^2(x + 1)$



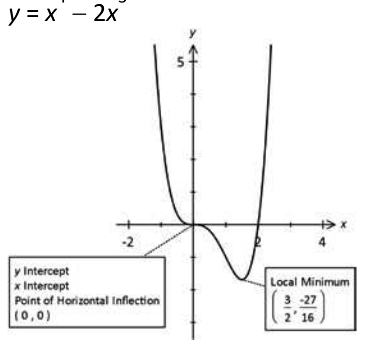
(c) $y = x(x - 3)^2$



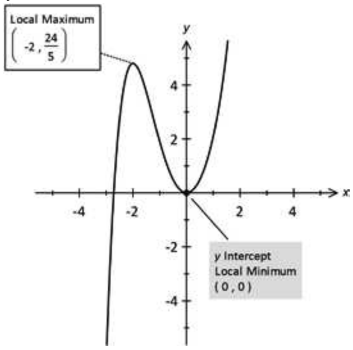
(d) $y = -x(x + 4)^2$



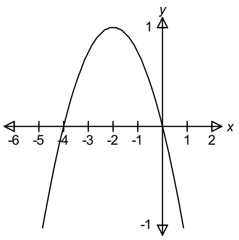
(e) $y = x^4 - 2x^3$



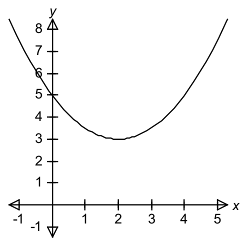
1. (f) $y = x^5/10 + 2x^2$



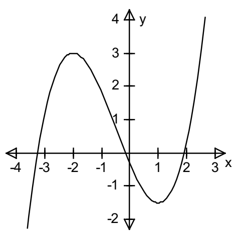
2. (a)



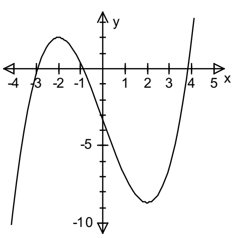
(b)



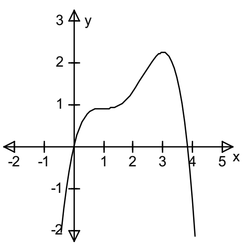
(c)



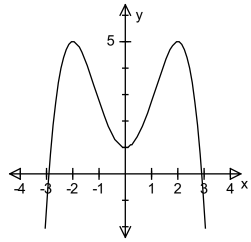
(d)



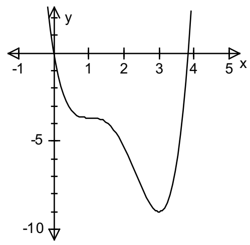
3. (a)



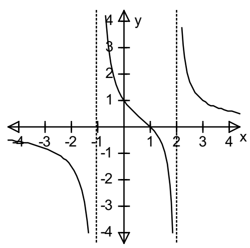
3. (b)



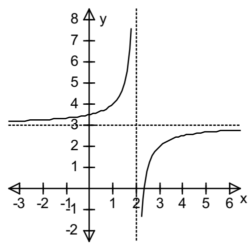
(c)



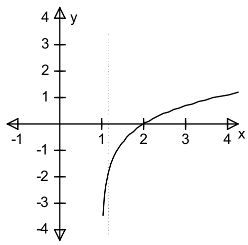
(d)



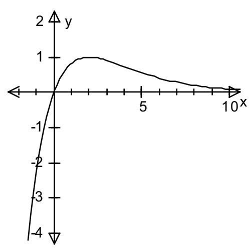
4. (a)



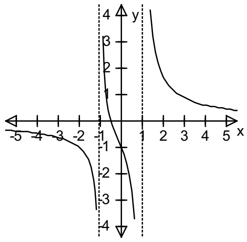
(b)



(c)



4. (d)



Exercise 21.1

1. (a) $dA/dt = 6t + 4$ (b) 4 (c) 1/6
2. (a) $dA/dt = -2(t - 4)$ (b) 0 (c) 6
3. (a) $dP/dt = 2t^2 + 2t - 40$
   (b) -40 (c) 4
4. (a) 0, 6 (b) 2
5. (a) 36 C (b) 45 C
   (c) 0.8 C /min (d) $t = 2.5$ min
6. (a) 16.6 C, 16.6 C
   (b) 0 ; For $6 \le t \le 14$, the average temp. change is 0 C per hour.
   (c) -0.8 C/hour, 0.8 C/hour
   At $t = 6$, the temp. is decreasing at 0.8 C per hour and at $t = 14$, the temp is increasing at a rate of 0.8 C per hour
   (d) Min temp = 15 C when $t = 10$ hours.
7. (a) 56 cents
   (b) 7 cents/year. At $t = 4$, the share price is increasing at a rate of 7 cents per year.
   (c) $-11/3$ cents/year. For $4 \le t \le 8$, the share price dropped by an average of 11/3 cents per year.
   (d) $t = 5$ or 11 years
8. (a) 473.33 cents
   (b) -17.33 cents/year. In the first 20 years, the price per kg dropped at a rate of 17.33 cents per year
   (c) -24 cents/year. At $t = 20$, the price per kg is dropping at a rate of 24 cents per year.
   (d) $t = 8$ or 18 years
9. (a) $-1/2$ (b) $-1/3$ (c) $t > 1$
10. (a) 7 (b) $0 < t < 1/2$

Exercise 21.2

1. Max. 20 when $x = 3$ Min 2 when $x = 0$
2. (a) Max 14.4 when $x = 2.5$
   Min -12.9, when $x = -2.2$
   (b) Max 14.4 when $x = 2.5$
   Min -2.9 when x = 1.6
   (c) Max 6.0 when $x = -0.1$
   Min -12.9 when $x = -2.2$
2. (d) Max 6.0 when $x = -0.1$
   Min -12.9 when $x = -2.2$
3. (a) Global max = 9 when $x = 2$
   Global min = -3 when $x = -2$
   (b) Global max = 9 when $x = 2$
   Global min = $-32/27$ when $x = 1/3$
   (c) Global max = 0 when $x = -1$
   Global min = -3 when $x = -2$
4. (a) Global max = 36 when $x = -2$
   Global min = -4 when $x = 3$
   (b) Global max = 2 when $x = 0$
   Global min = 0 when $x = 1$ and 2
   (c) Global max = 4/27 when $x = 5/3$
   Global min = -4 when $x = 3$
5. Global max = 5/2 when $x = 4$
   Global min = 3/2 when $x = 1$
6. Global max = 10 when $x = 1$ and $x = 4$
   Global min = 8 when $x = 2$
7. (a) 50 kL (b) 50 kL when $t = 0$
   (c) 191.66 kL when $t = 5$ days
8. (a) (i) 3.36 mg (ii) 2.31 mg
   (b) Max. 3.57 mg at $t = 5.16$ hours.
9. Min. 5 customers at 12 noon, 2 pm & 4.40 pm.

Exercise 21.3

1. Min for A = 8 when $x = 2$, $y = 4$
2. Max for A = -3 when $x = 1/2$, $y = -2$
3. Max for A = 4000/27 when $x = 20/3$, $y = 10/3$
4. Min for A = $-256/27$ when $x = 8/3$, $y = -4/3$
5. (c) 12.5 cm × 37.5 cm × 25.0 cm
6. (c) 25/3 cm × 100/3 cm × 125/6 cm
7. (d) Length = 40, H = 50/3 cm, W = 100/3 cm
8. (c) 11.08 cm × 6.08 cm × 1.96 cm
9. (c) 9.34 cm × 28.68 cm × 5.66 cm
10. Max volume = 40 000/9 cm^3
    with 20/3 cm × 100/3 cm × 20 cm
11. 18 140.59 cm^3 12. 3 032.3 cm^3
13. 7.07 cm × 28.28 cm × 11.32 cm
14. Max volume = 5 863.2 cm^3
    with radius = 9. 77 cm , height = 19.54 cm
15. 15.54 cm × 31.08 cm × 20.70 cm
16. 3 188.0 cm^2 17. 3 515.0 cm^2
18. 3 655.8 cm^2 .

Exercise 22.1

1. (a) $2x^2 + C$ (b) $10x + C$
   (c) $-x^3 + C$ (d) $x^8/2 + C$

1. (e) $x^4/8 + C$ (f) $625x + C$
   (g) $x^4/5 + C$ (h) $-x^3/2 + C$
   (i) $27x^4/4 + C$ (j) $x^7/56 + C$
   (k) $5x^4/4 + C$ (l) $x^4/12 + C$
2. (a) $x^4/4 + 4x + C$ (b) $2x^3/3 + 4x + C$
   (c) $-x^4/4 + 3x^2/2 + C$ (d) $2x^2 - x^5/5 + C$
   (e) $t^3/3 + t^2 + C$ (f) $3t^4/4 + 4t^3 + C$
   (g) $u^3/3 + u^2 + u + C$ (h) $4u^3/3 - 2u^2 + u + C$
   (i) $x^2/2 - x + C$ (j) $x^3/6 - x^5/20 + C$
   (k) $x^4/20 - x^3/15 + 2x/5 + C$
   (l) $u^3/3 - u + C$
3. $f(x) = x^3/3 + 2x + 1$
4. $f(t) = t^3/3 - t^2 + t - 3$
5. $y = -x^4 + x^3 + 2$
6. $y = -x + 2$
7. $v = -t^2/2 - 5t + 2$
8. $f(x) = 3x^2 + 4x - 5$

Exercise 22.2

1. $y = 4x - 1$
2. $y = -x^2 + 4$
3. $y = x^2/3 - x + 2$
4. $y = 4x^3/3 - 2x^2 + x + 1$
5. $y = 3x^2/2 - 2x - 2$
6. $y = x + 1$
7. $y = x^2/2 + x/2 + 2$
8. $y = x^2 + 2x + 1$
9. $y = x^2/2 + x + 1/2$
10. $y = x^3/3 - x^2/2 - 2x$

Exercise 23.1

1. (a) 4 m
   (b) $t = 1$, $v = -3$ ms^{-1}; $t = 4$, $v = 3$ ms^{-1}
   (c) 2.5 s (d) 3 seconds
2. (a) $t = 1, 3, 4$ seconds
   (b) $t = 8/3 \pm (\sqrt{7})/3$ seconds
   (c) $0 < t < 1$ and $3 < t < 4$ seconds
3. (a) $t = 1, 4$ seconds, $v(1) = 0$, $v(4) = 9$ ms^{-1}
   (b) 1 ms^{-1}
4. (a) 2 m (b) 0.75, 2.36, 4 s
5. (a) 27.5 m (b) –30.0 ms^{-1}
   (c) 1.02 s (d) 1.02 s , 5.10 s
6. (a) 10 ms^{-1} (b) 5.10 m
   (c) –10 ms^{-1} (d) 3.83 m
7. (a) 1.33 s (b) 4.55 m
   (c) ±2.32 ms^{-1} (d) 8.01 ms^{-1}
8. (a) 6 s
   (b) Yes. Stopping distance is 36 m.
9. (a) 14 ms^{-1} (b) 2 s
   (c) 58/3 ms^{-1}
10. (a) 10 ms^{-1} (b) 14 ms^{-1}
    (c) 60.26 m
11. (a) 16 m
    (b) Closest 2 m; Farthest 20 m
    (c) ≈ 13.1 ms^{-1} (d) ≈ 1, ≈ 2.5 s
12. (a) 6, 10 s (b) ≈ 7 ms^{-1}, t ≈ 8.5 s
    (c) 3 ms^{-1}
13. (a) P oscillates between –10 m and 10 m.
    (b) ≈ 6.5 ms^{-1} (c) 4 ms^{-1}

Exercise 23.2

1. (a) 38/3 m (b) 38/3 m
   (c) 19/3 ms^{-1}
2. (a) $s(1) = -5/6$, $s(4) = 11/3$
   (b) 31/6 m (c) 31/18 ms^{-1}
3. (a) 49/2 m (b) 0
   (c) ± 49/5 ms^{-1}
4. (a) 32 m (b) ± 15 ms^{-1}
   (c) 85/2 m
5. (a) 132/5 m (b) 31.4 ms^{-1}
   (c) 16.7 ms^{-1}
6. (a) 14 ms^{-1} (b) 97.9 m
   (c) 110 m (d) 5/7, 15/7 s
7. (a) $k = 3$ (b) 6 ms^{-1}
8. (a) 27 ms^{-1} (b) 46 m
   (c) 44 m
9. (a) $k = -2$, $C = 6$ (b) 5/2 ms^{-1}

Exercise 23.3

1. (a) 0 ms^{-2} (b) 1.99 ms^{-2}
   (c) –0.2 ms^{-2} (d) –4.25 ms^{-1}
2. (a) 1/4 cms^{-2} (b) 0.24 cms^{-2}
   (c) 1/4 cms^{-2} (d) 5 ms^{-1}
3. (a) 1/4 cms^{-1} (b) –1/4 cms^{-2}
   (c) 1/6 cms^{-1} (d) –5/36 cms^{-1}
4. (a) 11.75 ms^{-1} (b) 35.25 m
   (c) 2.6 ms^{-2}
5. (a) 3 ms^{-1} (b) 2 m
6. (a) 12 m (b) 3 seconds
   (c) 24 m (d) 6 ms^{-1}

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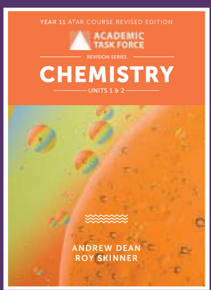

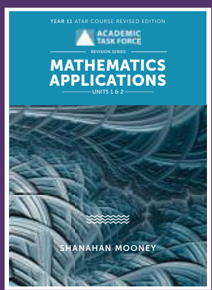

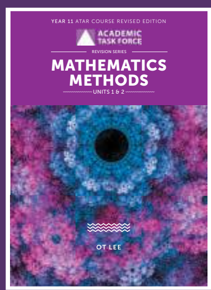

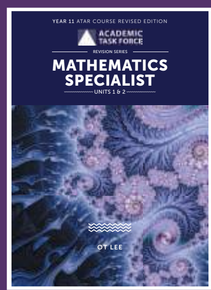

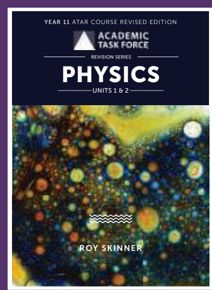

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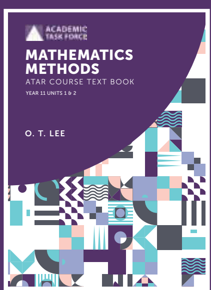

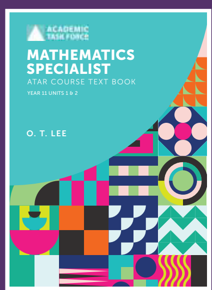

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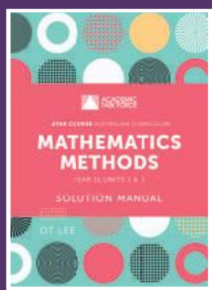

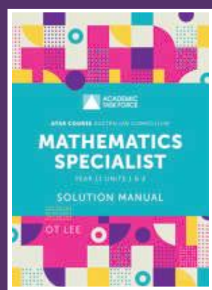

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