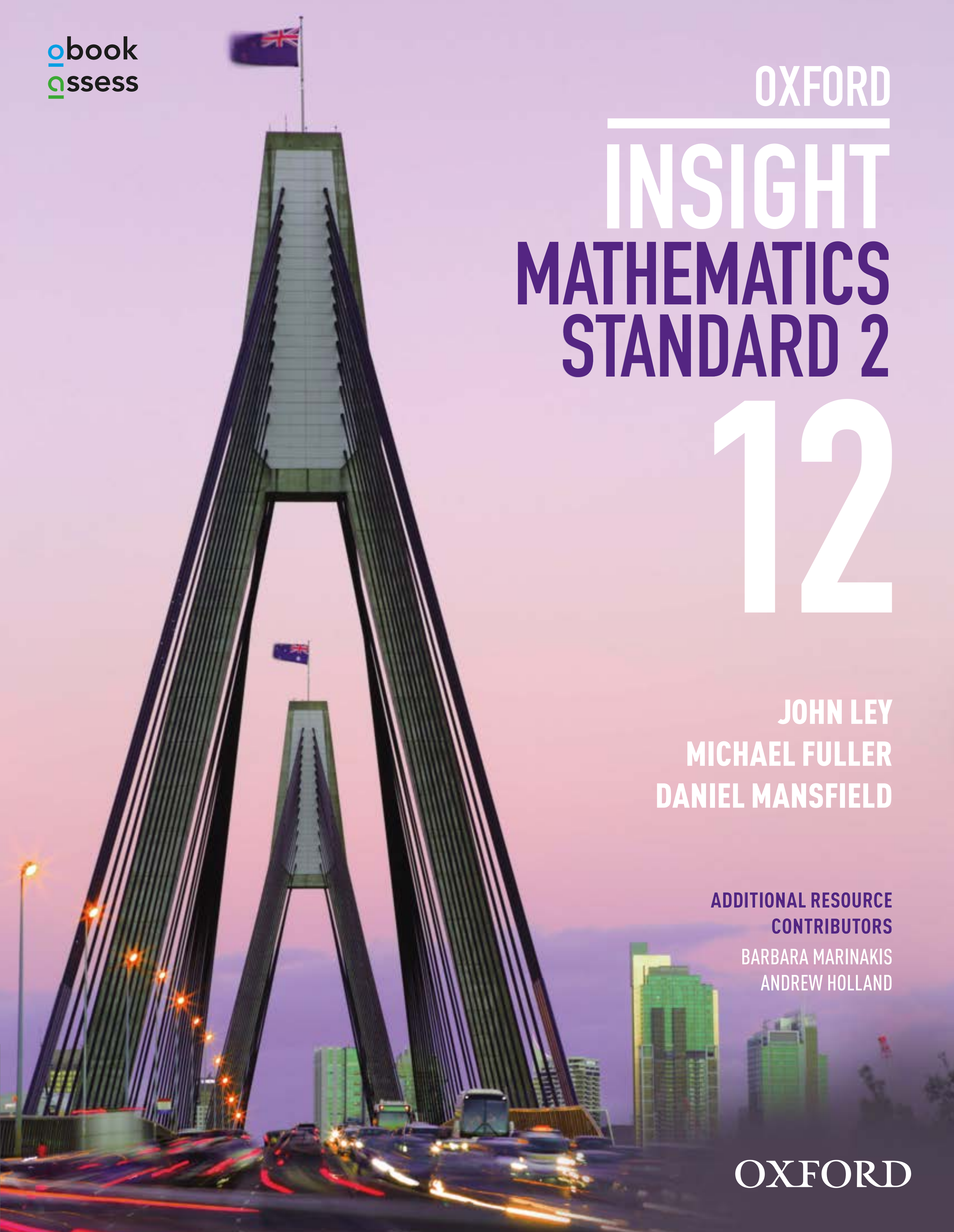

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# INSIGHT MATHEMATICS STANDARD 2 12


JOHN LEY
MICHAEL FULLER
DANIEL MANSFIELD

ADDITIONAL RESOURCE CONTRIBUTORS
BARBARA MARINAKIS
ANDREW HOLLAND


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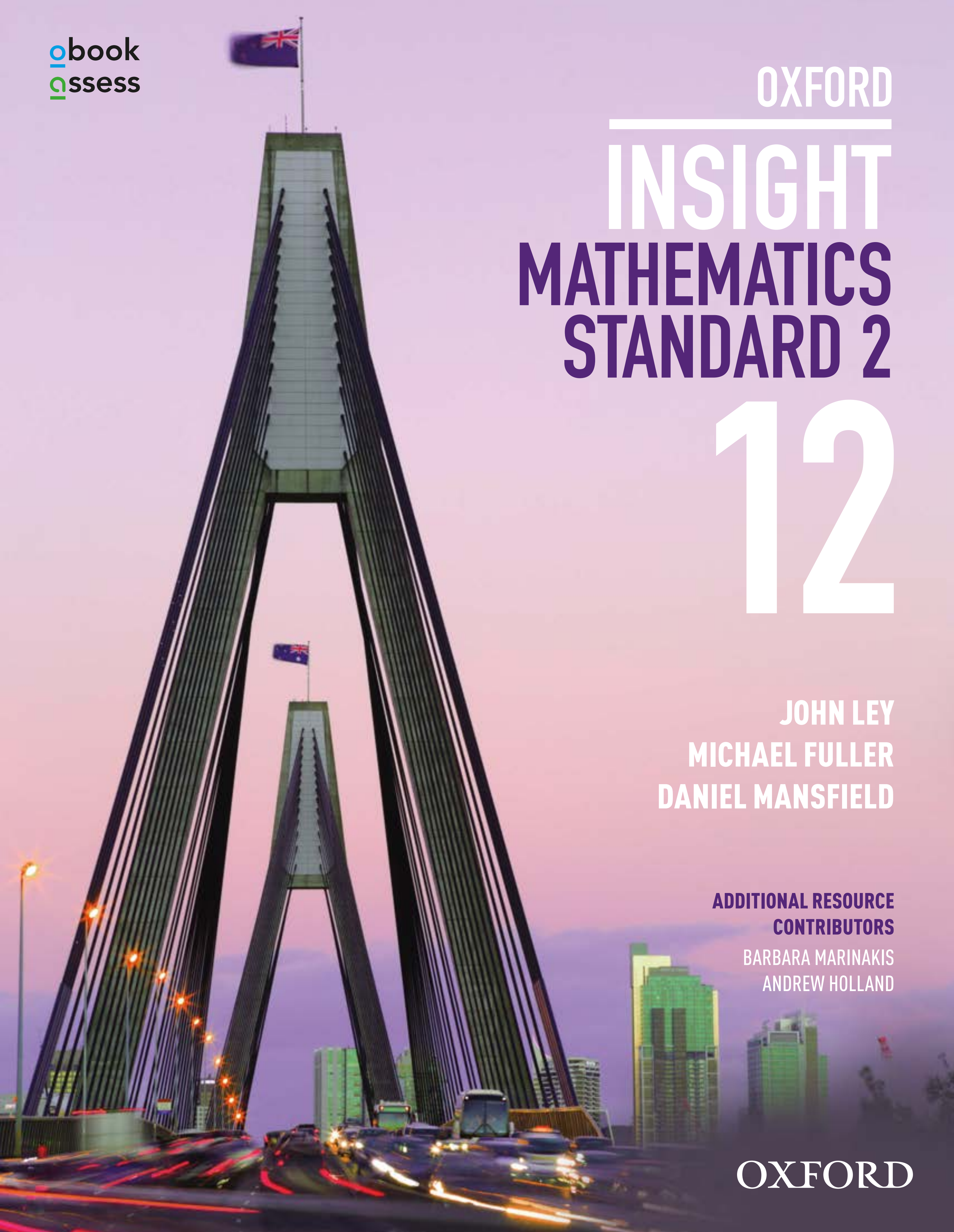

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Published in Australia by
Oxford University Press
Level 8, 737 Bourke Street, Docklands, Victoria 3008, Australia



First published 2019
Second Edition



NATIONAL LIBRARY OF AUSTRALIA

A catalogue record for this book is available from the National Library of Australia

ISBN 978 0 19 031214 5



Edited by Marta Veroni
Typeset by Newgen KnowledgeWorks Pvt. Ltd., Chennai, India
Proofread by Maja Vatric
Indexed by Neil Daly
Printed in Australia by Ligare Book Printers Pty Ltd

# CONTENTS

CONTENTS

## JOHN LEY

is a passionate and innovative mathematics educator. He has completed his PhD on mathematics education while lecturing and tutoring at the University of Western Sydney. John has held an array of teaching positions including Head of Mathematics, Assistant Principal, and Acting Principal. An experienced senior marker for the HSC, John was a member of the 2012–2014 assessment committees, setting the HSC calculus course examinations. John is the lead author of the Oxford Insight Mathematics series for NSW.

## MICHAEL FULLER

was involved in Mathematics in NSW for many years, and was a key author on the Oxford Insight Mathematics series. He held the position of Head of Mathematics at Killara High School in Sydney for 24 years.

## DR DANIEL MANSFIELD

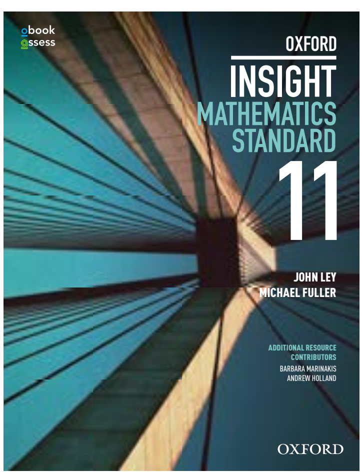
is an award-winning Lecturer in the School of Mathematics and Statistics at the University of New South Wales (UNSW). In 2017, his research into ancient Babylonian trigonometry made headlines around the world. Locally, Daniel is known for supporting secondary school mathematics teachers and their students. His passion for mathematics is further endorsed by his students at UNSW, who voted him the 'Most Inspiring Lecturer in First Year'.

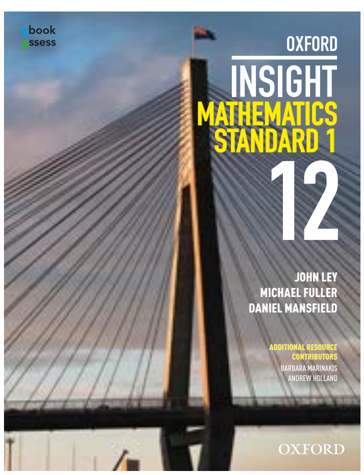
### ANDREW HOLLAND

has 18 years' experience teaching Mathematics to Secondary School students of varied levels of ability. He previously taught at St Andrew's Cathedral School and Shore School. He is now Head of Mathematics at St Joseph's College, Hunters Hill. Andrew previously authored a book on past HSC examination questions for General Mathematics.

### BARBARA MARINAKIS

has taught Mathematics to Secondary School and Tertiary Education students for 17 years and has widespread experience with students of all levels of ability. She has held teaching positions at Sydney Girls High School, Cranbrook School and is now teaching at Ascham School. Barbara has lectured the Year 12 HSC preparation lectures for The School for Excellence and has lectured and tutored at Australian Catholic University. She holds a Masters of Education from the University of NSW.



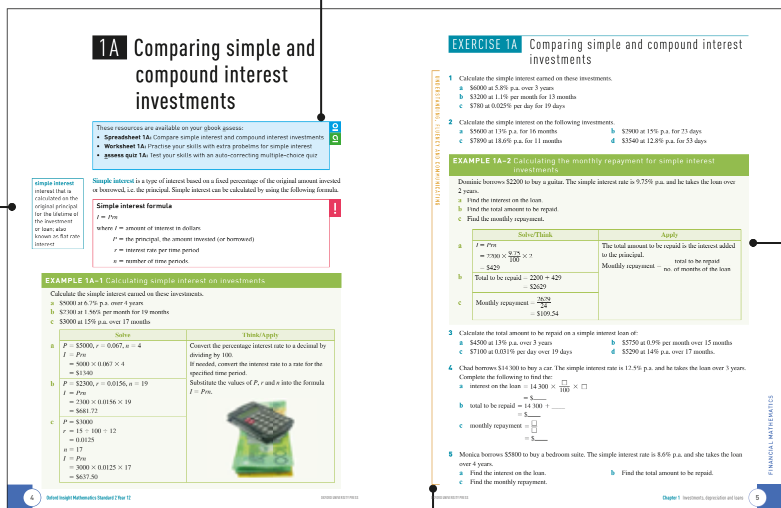



# Using Oxford Insight Mathematics Standard 2 Year 12

New South Wales' most trusted Mathematics series has been updated for the new Mathematics Standard Stage 6 syllabus. The new edition includes comprehensive exercise sets, carefully graded exercises and worked examples embedded where students need them. Enhanced opportunities for support and extension, as well as consolidation and practice are offered in reviews, cumulative reviews, exam-style questions and integrated technology.

helpful resources are outlined at the beginning of each unit

visual mathematics glossary boosts understanding of key concepts




worked examples and visuals located next to the relevant exercise

'Working Mathematically' syllabus components clearly signposted

## Student obook assess

*Oxford Insight Mathematics Standard 2 (Year 12)* is supported by a range of engaging and relevant digital resources via obook assess.

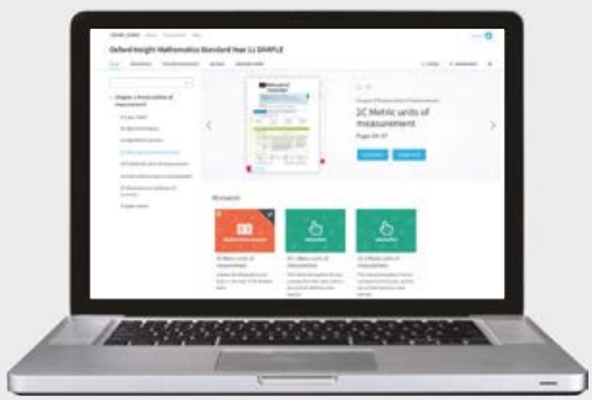
Students receive:

> a complete digital version of the Student book with notetaking and bookmarking functionality

> targeted instructional videos by a team of Australia's most experienced Mathematics Standard teachers designed to help students prepare for assessment tasks and exams

> interactive auto-correcting multiple-choice quizzes

> access to teacher-assigned work including readings, homework, tests and assignments.



## Teacher obook assess

In addition to the student resources, teachers also receive:

> detailed planning resources

> printable (and editable) class tests with exam-style questions and answers

> the ability to set up classes, set assignments, monitor progress and graph results, and to view all available content and resources in one place.

# Top tips for study success

## Tip 1 – read key documents

The first step to success is to gather all key documents and read them carefully.

> Your most important tool is the syllabus. It sets out all of the information about the course, including what you are expected to learn and how you will be assessed. You can download a copy from the NESA website.

> Keep all documents from your teacher relating to assessment tasks and copies of any assessment advice (e.g. marking criteria or assessment rubrics). Understanding exactly what is required in an assessment task is crucial.

## Tip 2 – study regularly

If you're going to perform at your best, you need to allocate time for regular periods of study and revision. Studying regularly will help you to continually reinforce new concepts and avoid the stress of last-minute cramming. During your study you might:

> summarise theory and key examples in your own words

> focus on topics you find difficult and work through the relevant examples and questions

> test your understanding with revision questions, practice papers and past exams.

## Tip 3 – manage your study time

When studying, it helps to put some practical strategies in place to stay on track. Try the following time management strategies.

> **Create a study timetable** to set up periods of regular study and revision around your school and personal schedule.

> **Use a diary, wall planner or calendar** to record the dates of upcoming assessment tasks, tests or exams and allow you to adequately prepare.

> **Make lists** of daily, weekly or monthly goals. It helps to keep the bigger picture in mind and breaks big tasks down into smaller, more manageable tasks, so that you gain a sense of achievement.

## Tip 4 – take care of yourself

Looking after yourself during for HSC is important:

> eat a balanced diet and stay hydrated – try to avoid too much caffeine and junk food

> get enough sleep and regular exercise

> make time for breaks from study – a walk to get some fresh air will help you reset before the next study session.

## Tip 5 – know the structure of exams

It's important for you to become familiar with the format of the exam and the types of questions that typically appear. In an exam you should also:

> show your working when answering a question – even if a question is incorrect or left unfinished, you might still get some marks for your working

> keep an eye on the clock to make sure you have enough time to answer every question

> re-read questions so you know that you have provided a complete and accurate answer.

## Tip 6 – understand key terms

Assessment tasks will likely include key terms. These range in level of difficulty. Some, such as *solve* or *find*, are simple to understand and master. Others, such as *justify*, are more challenging and will take practice to master. Below is a list of common key terms and an explanation of what they mean.

| TERM | DEFINITION |
|---|---|
| **analyse** | examine something complex by breaking it down into smaller parts and show how they relate to one another |
| **calculate** | work out an answer mathematically |
| **classify** | categorise into groups |
| **convert** | change to a different form without changing the value |
| **describe** | give a detailed account of the features |
| **evaluate** | determine the value |
| **explain** | make something clear by describing the relationships between different aspects and giving reasons |
| **express** | represent an answer as a number, figure, formula or symbol |
| **find** | determine the value or answer to a problem. |
| **identify** | determine and state clearly |
| **justify** | present an argument providing evidence |
| **solve** | work out the solution to a question |

# 1

# Investments, depreciation and loans

The main mathematical ideas investigated are:

- making compound interest calculations using the formula
- making compound interest calculations using a compounded value table
- comparing different investment strategies
- calculating the price of goods following inflation
- calculating new salaries after increases in line with inflation
- calculating the appreciated value of items
- the mathematics of shares
- calculating the salvage value of an item using the declining-balance method of depreciation
- calculating declining-balance loan repayments, including the use of tables
- calculating payments, charges and balances on credit cards.

FINANCIAL MATHEMATICS
MS-F4 Investments and Loans

# ARE YOU READY?

1A **1** What is the result of $2000 \times 0.05 \times 3$?

**A** 1200 **B** 30 000
**C** 2003.05 **D** 300

1A **2** What is 9.5% expressed as a decimal?

**A** 0.95 **B** 0.095
**C** 9.5 **D** 950

1A **3** Given that $a = 3$, $b = 4$ and $c = 8$, what is the value of $abc$?

**A** 348 **B** 15
**C** 96 **D** 3.48

1A **4** Given that $a = 2$, $b = 5$ and $c = 9$, what is the value of $a(b + c)$?

**A** 214 **B** 54 **C** 28 **D** 16

1A **5** What is 20% of 970?

**A** 19 400 **B** 194
**C** 1940 **D** 19.4

1A **6** What is 7.5% of \$11 300?

**A** \$84 750 **B** \$847.50
**C** \$1506.67 **D** \$1.51

1A **7** How much interest is earned if \$1000 is put into a simple interest account paying 5% p.a. for 1 year?

**A** \$5000 **B** \$50
**C** \$500 **D** \$1050

1A **8** \$2000 is put into a simple interest account paying 7% p.a. How much is in the account after 1 year?

**A** \$2070 **B** \$14 000
**C** \$140 **D** \$2140

1A **9** The options below show relationships between $x$ and $y$, where $k$ and $a$ are constants. Which options show a general equation for a linear relationship?

**A** $y = kx$ **B** $y = kx^2$
**C** $y = \frac{k}{x}$ **D** $y = ka^x$

1A **10** The options below show relationships between $x$ and $y$, where $k$ and $a$ are constants. Which option show a general equation for an exponential relationship?

**A** $y = kx$ **B** $y = kx^2$
**C** $y = \frac{k}{x}$ **D** $y = ka^x$

1B **11** How many days are there in 3 years (excluding leap years)?

**A** 1095 **B** 36
**C** 1098 **D** 156

1B **12** Given that $x = 150$ and $y = 1.1$, what is the value of $xy^2$?

**A** 181.5 **B** 165
**C** 123.97 **D** 151.1

1D **13** What is the value of $\$35 \times 1.05^3$?

**A** \$36.75 **B** \$36.86
**C** \$110.25 **D** \$40.52

1F **14** A washing machine purchased for \$1800 is depreciated by \$220 per year. What is the salvage value of the washing machine after 5 years?

**A** \$700 **B** \$2900
**C** \$1580 **D** \$920

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1** **Support sheet 1A.1 Multiplying and dividing decimal numbers**
**Q2** **Support sheet 1A.2 Converting percentages, fractions and decimals**
**Q3–4** **Support sheet 1A.3 Substituting for pronumerals**
**Q5–6** **Support sheet 1A.4 Percentage of a quantity**
**Q7–8** **Support sheet 1A.5 Understanding the simple interest formula**
**Q9–10** **Support sheet 1A.6 Linear and non-linear relationships**
**Q11** **Support sheet 1B.1 Converting units of time**
**Q12** **Support sheet 1B.2 Evaluating algebraic expressions involving powers**
**Q13** **Support sheet 1D.1 Finding cubes of numbers**
**Q14** **Support sheet 1F.1 Straight-line depreciation**

# 1A Comparing simple and compound interest investments

These resources are available on your obook assess:
- **Spreadsheet 1A:** Compare simple interest and compound interest investments
- **Worksheet 1A:** Practise your skills with extra problems for simple interest
- **assess quiz 1A:** Test your skills with an auto-correcting multiple-choice quiz

**simple interest** interest that is calculated on the original principal for the lifetime of the investment or loan; also known as flat rate interest

**Simple interest** is a type of interest based on a fixed percentage of the original amount invested or borrowed, i.e. the principal. Simple interest can be calculated by using the following formula.

**Simple interest formula**

$I = Prn$

where $I$ = amount of interest in dollars

$P$ = the principal, the amount invested (or borrowed)

$r$ = interest rate per time period

$n$ = number of time periods.

## EXAMPLE 1A–1 Calculating simple interest on investments

Calculate the simple interest earned on these investments.

**a** \$5000 at 6.7% p.a. over 4 years

**b** \$2300 at 1.56% per month for 19 months

**c** \$3000 at 15% p.a. over 17 months

| | Solve | Think/Apply |
|---|---|---|
| **a** | $P = \$5000,\ r = 0.067,\ n = 4$<br>$I = Prn$<br>$= 5000 \times 0.067 \times 4$<br>$= \$1340$ | Convert the percentage interest rate to a decimal by dividing by 100.<br>If needed, convert the interest rate to a rate for the specified time period.<br>Substitute the values of $P$, $r$ and $n$ into the formula $I = Prn$. |
| **b** | $P = \$2300,\ r = 0.0156,\ n = 19$<br>$I = Prn$<br>$= 2300 \times 0.0156 \times 19$<br>$= \$681.72$ | |
| **c** | $P = \$3000$<br>$r = 15 \div 100 \div 12$<br>$= 0.0125$<br>$n = 17$<br>$I = Prn$<br>$= 3000 \times 0.0125 \times 17$<br>$= \$637.50$ | |

# EXERCISE 1A Comparing simple and compound interest investments

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Calculate the simple interest earned on these investments.

**a** \$6000 at 5.8% p.a. over 3 years

**b** \$3200 at 1.1% per month for 13 months

**c** \$780 at 0.025% per day for 19 days

**2** Calculate the simple interest on the following investments.

**a** \$5600 at 13% p.a. for 16 months

**b** \$2900 at 15% p.a. for 23 days

**c** \$7890 at 18.6% p.a. for 11 months

**d** \$3540 at 12.8% p.a. for 53 days

## EXAMPLE 1A–2 Calculating the monthly repayment for simple interest investments

Dominic borrows \$2200 to buy a guitar. The simple interest rate is 9.75% p.a. and he takes the loan over 2 years.

**a** Find the interest on the loan.

**b** Find the total amount to be repaid.

**c** Find the monthly repayment.

| | Solve/Think | Apply |
|---|---|---|
| **a** | $I = Prn$<br>$= 2200 \times \frac{9.75}{100} \times 2$<br>$= \$429$ | The total amount to be repaid is the interest added to the principal.<br>Monthly repayment $= \frac{\text{total to be repaid}}{\text{no. of months of the loan}}$ |
| **b** | Total to be repaid $= 2200 + 429$<br>$= \$2629$ | |
| **c** | Monthly repayment $= \frac{2629}{24}$<br>$= \$109.54$ | |

**3** Calculate the total amount to be repaid on a simple interest loan of:

**a** \$4500 at 13% p.a. over 3 years

**b** \$5750 at 0.9% per month over 15 months

**c** \$7100 at 0.031% per day over 19 days

**d** \$5290 at 14% p.a. over 17 months.

**4** Chad borrows \$14 300 to buy a car. The simple interest rate is 12.5% p.a. and he takes the loan over 3 years. Complete the following to find the:

**a** interest on the loan $= 14\,300 \times \frac{\square}{100} \times \square$

$= \$\_\_\_\_$

**b** total to be repaid $= 14\,300 + \_\_\_\_$

$= \$\_\_\_\_$

**c** monthly repayment $= \frac{\square}{\square}$

$= \$\_\_\_\_$

**5** Monica borrows \$5800 to buy a bedroom suite. The simple interest rate is 8.6% p.a. and she takes the loan over 4 years.

**a** Find the interest on the loan.

**b** Find the total amount to be repaid.

**c** Find the monthly repayment.

**compound interest** interest that is calculated on the current balance of an investment, including the interest from the previous time period

For a **compound interest** investment, the interest earned at the end of each time period is added to the principal. This increases the principal that is used to calculate the interest for the next time period. Therefore, with compound interest you are earning interest on the interest you have previously earned.

## EXAMPLE 1A–3 Calculating the total value of an investment and compound interest earned

$2000 is invested for 3 years at 7% p.a. interest compounded annually.

**a** Find the amount the $2000 will grow to after 3 years.

**b** Find the amount of interest earned.

**Solve**

**a**

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | $2000 | $\frac{7}{100} \times 2000 = \$140$ | $2000 + 140 = \$2140$ |
| 2 | $2140 | $\frac{7}{100} \times 2140 = \$149.80$ | $2140 + 149.80 = \$2289.80$ |
| 3 | $2289.80 | $\frac{7}{100} \times 2289.80 = \$160.29$ | $2289.80 + 160.29 = \$2450.09$ |

The amount the $2000 will grow to after 3 years is $2450.09.

**b** The amount of interest earned $= 2450.09 - 2000 = \$450.09$

| | Think | Apply |
|---|---|---|
| **a** | Use $I = Prn$ with $P = 2000$, $r = 0.07$ and $n = 1$ to find the interest for the first year of $140. Add $140 to $2000 to get a new principal of $2140, then calculate the interest on $2140. | Use $I = Prn$ with $n = 1$ to calculate the interest each year. The principal each year is the previous principal plus the interest for that year. The interest earned is the total balance less the original principal. |
| **b** | Subtract $2000 from the total balance. | |

**6** **a** Complete the table to determine the final value of $2800 invested at 7% p.a. compound interest for 3 years.

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | $2800 | $196 | $2996 |
| 2 | $2996 | $209.72 | |
| 3 | | | |

**b** Calculate the total interest earned.

Interest = ____ − 2800 = $____

7 Toby invested \$6500 for 4 years at 6.5% p.a. interest compounded annually.

a Using a table, find the value of Toby's investment after 4 years.

b Find the amount of interest earned by Toby in the 4 years.

8 a Complete the table to determine the final value of \$980 invested at 3% p.a. compound interest for 4 years.

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

b Calculate the total interest earned.

c Use digital technology to produce a graph of the value of the investment over 4 years.

9 Adele decided to invest her savings of \$10 350 for 5 years at 7.7% p.a. compound interest.

a Complete the table.

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |

b If Adele intends to buy a car that is expected to be valued at \$14 495 when her investment matures, will she have enough to buy the car? Explain.

c By how much is the investment over or under the value of the car?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**10** June receives a gift of $5000 from her grandparents for her 21st birthday. She looks at different investment options, and wants to compare simple and compound interest investments at 5% p.a.

**a** Complete the following table to determine the final values of the two investments after 5 years.

| Time | Simple interest at 5% p.a. | Compound interest at 5% p.a. |
|---|---|---|
| Start of first year | $5000 | $5000 |
| Start of second year | $5250 | $5250 |
| Start of third year | | |
| Start of fourth year | | |
| Start of fifth year | | |
| Start of sixth year | | |

**b** What is the difference in the value of the investments after the 5 years?

**11** Leo wants to compare the potential value of two different investment opportunities. Bank A offers a simple interest rate of 6.2% p.a. and bank B offers a compound interest rate of 6.0% p.a., with the interest compounded monthly.

**a** If $10 000 is invested with each bank at the start of the year, which investment will have the higher balance at the end of the first year, and by how much?

**b** To receive the offered interest rates, the investments have to be made for a minimum of 2 years. Which investment will have the higher balance at the end of the second year, and by how much?

**c** Explain why the answers to parts **a** and **b** are different.

SPREADSHEET APPLICATION

We can use a spreadsheet to generate tables for compound interest investments. Open a new spreadsheet and type in the column headings from the example below into cells A1 to D1. Then follow these instructions.

Enter in the amount of the principal (in dollars) into cell B2.

Type the formula $=a$*B2 into cell C2, where $a$ represents the interest rate per time period expressed as a decimal. **Fill down** to C5 and beyond.

Type the formula =B2+C2 into cell D2 and **fill down** to D5 and beyond.

Type the formula =D2 into cell B3 and **fill down** to B5 and beyond.

| | A | B | C | D |
|---|---|---|---|---|
| **1** | **Year** | **Balance at start of year ($)** | **Interest ($)** | **Balance at end of year ($)** |
| **2** | 1 | | | |
| **3** | 2 | | | |
| **4** | 3 | | | |
| **5** | … | | | |

**12** **a** $1000 is invested at 7% p.a. interest compounding annually. Use a spreadsheet to calculate the value of the investment at the end of each year for 10 years.

**b** Use digital technology to produce a graph showing the value of the investment over a period of 10 years.

**c** On the same set of axes as part **b**, draw a simple interest graph for the same time period, but with a rate of 9% p.a.

**d** Use the graph drawn in part **c** to determine the number of years it takes for the two investments to be the same total amount.

SPREADSHEET APPLICATION

**13** **a** \$4000 is invested at 7.4% p.a. interest compounding annually. Use a spreadsheet to create a table showing the value of the investment at the end of each year for 10 years.

**b** Use digital technology to produce a graph showing the value of the investment over a period of 10 years.

**c** Find the time for the investment to be worth \$6500.

**d** On the same set of axes as part **b**, draw a straight line joining the point representing the initial value of the investment and the point representing the investment value after 10 years.

**e** Calculate the gradient of the straight line drawn for part **d** to help you determine the equivalent simple interest rate.

**14** An amount of \$70 000 is to be invested for 9 years.

**a** Use a spreadsheet to determine the total interest earned on the investment if the initial amount is invested at:

**i** 9.2% p.a. simple interest

**ii** 9.2% p.a. compound interest with interest compounded annually.

**b** Explain why the final value is different for the two investment options.

**c** On the one set of axes, draw graphs to show how the value of the investment changes for each option over the 9 years.

**d** The graph for simple interest shows a linear relationship. Explain why.

**e** The graph for compound interest shows a non-linear relationship. From the shape, what type of relationship can be seen? (Hint: consider whether it has the shape of a quadratic, cubic, exponential or reciprocal relationship.) Explain your choice.

**f** The graph for a simple interest investment has the equation $A = Prn + P$, where $A$ is the final value of the investment, $P$ is the principal, $r$ is the interest rate and $n$ is the number of time periods. How does this relate to the general equation of $y = mx + c$?

**g** The graph for compound interest has the equation $A = P(1 + r)^n$. How does this relate to the general equation of $y = ka^x$? Does this confirm your choice of non-linear relationship in part **e**?

**15** An amount of \$100 000 is to be invested for 10 years.

**a** Use a spreadsheet to determine the final value of the investment if the initial amount is invested at:

**i** 8.5% p.a. simple interest

**ii** 8.5% p.a. compound interest with interest compounded annually.

**b** Explain why the final value is different for the two investment options.

**c** On the one set of axes, use digital technology to draw graphs that show how the value of the investment changes for each option over the 10 years.

**d** What type of relationship does the graph for simple interest show? Explain.

**e** What type of relationship does the graph for compound interest show? Explain.

**f** Using the fact that simple interest follows a linear model and compound interest follows an exponential model, explain why investors prefer to invest their money so that it earns compound interest rather than simple interest.

# 1B The compound interest formula

These resources are available on your obook assess:
- **Video tutorial 1B:** Watch and listen to an explanation of Example 1B–1
- **Spreadsheet 1B:** Use the compound interest formula
- **Worksheet 1B:** Practise your skills with extra problems for compound interest
- **Investigation 1B:** Investigate another use of the compound interest formula
- **assess quiz 1B:** Test your skills with an auto-correcting multiple-choice quiz

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In the financial world, the principal, or initial amount, is known as the present value of the investment. The amount to which the principal grows is known as the future value of the investment.

Compound interest can be calculated by using the following formula.

**Compound interest formula**

$FV = PV(1 + r)^n$

where $FV$ = the future value

$PV$ = the present value

$r$ = interest rate per compounding period

$n$ = number of compounding periods

!

## EXAMPLE 1B–1 Finding the interest earned for a compound interest investment

**a** Use the compound interest formula to calculate the future value of a fixed term investment of $5000 over 5 years at 6.5% p.a. interest compounding yearly.

**b** Find the total interest earned.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $FV = PV(1 + r)^n$<br>$= 5000 \times (1 + 0.065)^5$<br>$= 5000 \times (1.065)^5$<br>$= \$6850.43$ | $PV = 5000$<br>$r = 0.065$, as the compounding period is annual, the interest rate is the annual rate.<br>$n = 5$, as the compounding period is annual, the number of time periods is the same as the number of years. | The interest rate and the time period must correspond. Substitute into the formula. The interest is calculated by subtracting the original investment amount (the present value) from the future value. |
| **b** | Interest $= 6850.43 - 5000$<br>$= \$1850.43$ | Subtract the present value of $5000 from the answer to part **a**. | |

# EXERCISE 1B The compound interest formula

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** **a** Using the compound interest formula, complete the following to calculate the future value when \$6500 is invested for 7 years at 4.2% p.a. interest compounding annually.

$PV =$ ____ $r = 4.2 \div$ ____ $= 0.$____ $n =$ ____

$FV = PV(1 + r)^n$

$= $ ____$(1 +$ ____$)^7$

$= 6500($____$)^{\square}$

$=$ ____

**b** Complete the following to find the total interest earned.

Interest $=$ ____ $- 6500 =$ ____

**2** **a** Use the compound interest formula to calculate the future value of a fixed-term investment of \$4000 over 6 years at 7.5% p.a. interest compounding yearly.

**b** Find the total interest earned.

**3** **a** Use the compound interest formula to calculate the future value of a fixed-term investment of \$6453 over 3 years at 4.95% p.a. interest compounding yearly.

**b** Find the total interest earned.

## EXAMPLE 1B–2 Calculating the future value of a compound interest investment

Use the compound interest formula to calculate the future value of a fixed-term investment of \$3500 over 7 years at 6.2% p.a. interest compounding quarterly.

| Solve | Think | Apply |
|---|---|---|
| $FV = PV(1 + r)^n$<br>$= 3500 \times (1 + 0.0155)^{28}$<br>$= 5000 \times (1.0155)^{28}$<br>$= \$5384.01$ | There are 4 quarters in a year, so $n = 7 \times 4 = 28$ time periods.<br>Quarterly interest rate is the annual rate divided by 4:<br>$r = 0.062 \div 4 = 0.0155$<br>Present value is \$3500. | Calculate to find the number of time periods, then find the interest rate for the required time period. Substitute into the compound interest formula. Usually, the time period is multiplied and the interest rate is divided by the same number. |

**4** Using the compound interest formula, complete the following to calculate the future value of a fixed-term investment of \$1200 over 5 years at 8.4% p.a. interest compounding quarterly.

$PV =$ ____ $n =$ ____ $\times 4 =$ ____ $r = 0.084 \div$ ____ $=$ ____

$FV = PV(1 + r)^n$

$= 1200(1 +$ ____$)^{\square}$

$= 1200 \times ($____$)^{\square}$

$=$ ____

**5** Use the compound interest formula to calculate the future value of a fixed-term investment of \$950 over 3 years at 4.1% p.a. interest compounding quarterly.

FINANCIAL MATHEMATICS

**6** Using the values in this table, calculate the future value of each fixed-term investment.

| | Principal | Rate p.a. | Years | Compounding period |
|---|---|---|---|---|
| **a** | \$700 | 6% | 3 | Quarterly |
| **b** | \$450 | 8% | 5 | Monthly |
| **c** | \$12 000 | 5% | 2 | Six-monthly |
| **d** | \$650 | 4% | 8 | Monthly |
| **e** | \$8540 | 6.5% | 6 | Quarterly |
| **f** | \$2205 | 3.8% | 4 | Monthly |

**7** Calculate the future value of the following investments to determine which option gives the best return over a year on \$5000 invested. How much better is the return on this investment than on the other two investments?

**a** 6% p.a. compounding yearly

**b** 5.9% p.a. compounded quarterly

**c** 5.85% p.a. compounding monthly

**!** The compound interest formula can be rearranged to determine the amount that needs to be invested if the required future value is known.

$$PV = \frac{FV}{(1 + r)^n}$$

### EXAMPLE 1B–3 Calculating the present value of a compound interest investment

Calculate the amount that must be invested at 6% p.a. interest compounding annually to have \$5000 at the end of 4 years.

| Solve | Think | Apply |
|---|---|---|
| $PV = \frac{FV}{(1 + r)^n}$ <br> $= \frac{5000}{(1 + 0.06)^4}$ <br> $= \frac{5000}{(1.06)^4}$ <br> $= \$3960.47$ <br> \$3960.47 must be invested. | $FV = 5000$, $r = 0.06$, $n = 4$ <br> Substitute into the rearranged formula to find $PV$. | Find the required values and substitute into the rearranged formula. Solve the equation to determine the required value. |

**8** Complete the following to calculate the amount that must be invested at 7% p.a. interest compounding annually to have \$6000 at the end of 5 years.

$FV = 6000$ $\qquad r =$ ____ $\qquad n =$ ____

$$PV = \frac{FV}{(1 + r)^n}$$

$$= \frac{6000}{(1 + 0.07)^{\square}}$$

$$= \frac{6000}{\phantom{000}}$$

$= \$$____

**9** Calculate the amount that must be invested at 5% p.a. interest compounding annually to have \$1600 at the end of 7 years.

**10** Calculate the amount that must be invested at 11.2% p.a. interest compounding annually to have \$10 000 at the end of 6 years.

## EXAMPLE 1B–4 Calculating the present value for a non-annual compounding period

Calculate the amount that must be invested at 7.5% p.a. interest compounding quarterly to have \$1700 at the end of 3 years.

| Solve | Think | Apply |
|---|---|---|
| $PV = \frac{1700}{(1+r)^n}$<br>$= \frac{1700}{(1+0.01\,875)^{12}}$<br>$= \frac{1700}{(1.01\,875)^{12}}$<br>$= \$1360.31$<br>\$1360.31 must be invested. | Quarterly interest rate<br>$= 0.075 \div 4 = 0.01\,875$<br>Number of quarters<br>$= 3 \times 4 = 12$<br>$FV = 1700$, $r = 0.01\,875$,<br>$n = 12$<br>Substitute into the formula and solve. | Calculate to find the number of time periods, then divide to find the interest rate for the required time period.<br>Substitute and solve. |

**11** Complete the following to calculate the amount that must be invested at 8.5% p.a. interest compounding quarterly to have \$2300 at the end of 7 years.

Quarterly interest rate $= 0.085 \div$ ____ $=$ ____

Number of quarters $=$ ____ $\times 4$ $\qquad n =$ ____

$$PV = \frac{FV}{(1+r)^n}$$
$$= \frac{2300}{(1 + \_\_\_)^{\square}}$$
$$= \underline{2300}$$
$$= \$\_\_\_\_$$

**12** Calculate the amount that must be invested at:

**a** 4% p.a. interest compounding quarterly to have \$1540 at the end of 8 years

**b** 10.2% p.a. interest compounding quarterly to have \$10 000 at the end of 4 years

**c** 9% p.a. interest compounding monthly to have \$3000 at the end of 3 years

**d** 4.5% p.a. interest compounding monthly to have \$950 at the end of 8 years

**e** 7.2% p.a. interest compounding six-monthly to have \$2000 at the end of 10 years

**f** 6.3% p.a. interest compounding six-monthly to have \$7500 at the end of 5.5 years.

**13** How much does Paul need to invest at 4.95% to have \$2500 in 3 years' time, if the interest compounds monthly?

**14** A company will need \$20 000 to replace its computer system in 4 years' time. How much needs to be invested at 4.95% p.a. interest compounding quarterly to have this amount available?

## EXAMPLE 1B–5 Choosing the best investment strategy

Justine wants to invest \$3000 for 3 years, but is struggling to decide where to invest her money. She narrows her choices down to three options.

Option A: A simple interest account at 9% p.a.

Option B: A compound interest account at 8.5% p.a., with interest compounded annually

Option C: A compound interest account at 8.3% p.a., with interest compounded daily

Determine which option will give Justine the highest future value on her investment.

| Solve | Think | Apply |
|---|---|---|
| Option A:<br>$I = Prn$<br>$= 3000 \times \frac{9}{100} \times 3$<br>$= \$810$<br>Future value $= 3000 + 810$<br>$= \$3810$ | Option A:<br>Substitute $P = 3000$, $r = \frac{9}{100}$ and $n = 3$ into $I = Prn$.<br>Add the interest to the principal to determine the future value. | Calculate the final value of each investment option by using the simple interest and compound interest formulas. |
| Option B:<br>$FV = PV(1 + r)^n$<br>$= 3000 \times (1 + 0.085)^3$<br>$= 3000 \times (1.085)^3$<br>$= \$3831.87$ | Option B:<br>Substitute $PV = 3000$, $r = 0.085$ and $n = 3$ into $FV = PV(1 + r)^n$. | |
| Option C:<br>$\text{FV} = PV(1 + r)^n$<br>$= 3000 \times \left(1 + \frac{0.083}{365}\right)^{1095}$<br>$= 3000 \times (1.000\,227...)^{1095}$<br>$= \$3848.12$ | Option C:<br>Substitute $PV = 3000$, $r = \frac{0.083}{365}$ and $n = 365 \times 3 = 1095$ into $FV = PV(1 + r)^n$. | |
| Option C will give Justine the highest future value on her investment. | | |

**15** Sienna is looking to invest \$5500 for 3 years, but would like some help in picking the best investment option. Determine which option will give Sienna the highest future value on her investment.

Option A: A simple interest account at 6.6% p.a.

Option B: A compound interest account at 6.2% p.a., with interest compounded annually

Option C: A compound interest account at 6% p.a., with interest compounded monthly

**16** Georgio has $12 000 to invest and wants to invest it for 4 years. Which investment option should he choose?

Option A: Compound interest at 4.7% p.a., with interest compounded annually

Option B: Compound interest at 4.45% p.a., with interest compounded monthly

Option C: Compound interest at 4.35% p.a., with interest compounded weekly

**17** Romeo is given a $6000 bonus at his job and decides to invest the full amount. He wants to investigate the effect that different compound interest rates will have on the future value of his investment.

Use the compound interest formula to determine the amount of interest Romeo will receive for investing his $6000 for 3 years at:

**a** 5% p.a. **b** 5.5% p.a. **c** 6% p.a.

**18** Juliet is saving for a deposit for a home. She has $24 500 in her savings account, and is looking to invest the full amount in a high-interest investment opportunity. To gain the special interest rate of 9.9% p.a. (compounding annually) Juliet must keep the full balance in the investment until the end of the agreed-upon term.

What will the future value of Juliet's investment be if she decides to invest her savings for:

**a** 2 years? **b** 3 years? **c** 4 years?

**19** Sam invested $3000 for three years in a compound interest account paying 8.1% p.a. with interest compounded monthly. Sam's brother, Lee, also invested $3000 for three years, but in a simple interest account paying 8.1% p.a. How much more interest did Sam earn over the period of the investment?

**20** Dijana's financial advisor suggested that Dijana put her $20 000 inheritance for 5 years into a compound interest investment paying 7.8% p.a. with interest compounded weekly.

**a** Calculate the future value of Dijana's investment after 5 years.

**b** What equivalent simple interest rate p.a. will Dijana need to find to achieve the same return? Give your answer to two decimal places.

**21** A sum of $5000 was invested for 4 years in a compound interest account paying 5.7% p.a. with interest compounded quarterly. How much would need to be invested to achieve the same amount of interest if the interest rate was 4.2% p.a. compounded monthly? Give your answer to the nearest dollar.

**22** Xavier's trust account matures on his 18th birthday. On that day it will be worth $12 500. His mother persuades him to invest the full amount for at least 3 years and together they find three different investment opportunities.

Option A: Compound interest at 7.55% p.a., with interest compounded monthly

Option B: Simple interest at 7.4% p.a., with a bonus payment of $1000 paid after 3 years

Option C: Compound interest at 4.5% p.a., with interest compounded annually, plus a bonus 5% of the future value added after 3 years.

**a** Calculate the potential future value of each of the investment options.

**b** What is the difference between the options with the lowest and highest future values?

**23** Julio is helping his father out with his finances and they're looking for a place to invest his savings. They've narrowed down their choices to the following options.

Option A: A compound interest account paying 5.6% p.a. compound interest, with interest compounded daily

Option B: Investing in the stock market, with an expected long-term return of 9.8% p.a.

Option C: Investing in the property market, with an expected long-term return of 9.5% p.a.

**a** Julio's father is willing to invest $50 000. If he invests his money for 5 years, determine the expected future value for each option.

**b** Why might Julio advise his father to not invest in the option that has the highest expected future value?

**24** Samantha inherited $40 000 from her grandparents, and decided to invest the full amount for at least 6 years. She narrows her investment options down to the following 3 three choices.

Option A: Compound interest at 6.5% p.a. for the first 3 years, and 7.5% p.a. for each subsequent year, with interest compounded monthly

Option B: Simple interest at 8.2% p.a. with a bonus of 2% of the future value at the end of the investment

Option C: Compound interest at 7.1% p.a. with interest compounded daily

**a** Which option should Samantha choose if she goes for a 6-year investment? How much does this option return?

**b** Which option should Samantha choose if she goes for a 10-year investment? How much does this option return?

**c** How much interest does Samantha earn in the 10th year under each investment option?

**25** Use trial and error to determine the compound interest rate that will give a future value of $5165.22 if $4000 is invested for 4 years with interest compounded annually.

**26** If the interest rate on a compound interest investment is doubled, will the interest earned also be doubled? Use an example to help you illustrate your answer.

SPREADSHEET APPLICATION

We can enter the compound interest formula into a spreadsheet to determine the future value of an investment given the present value, the interest rate per annum, the number of compounding periods per annum, and the term of the investment.

Type in the headings in column A, and enter in the formula into cell B1, as shown.

| | A | B |
|---|---|---|
| 1 | Future value ($) | =B2*(1+(B3/(100*B4)))^(B4*B5) |
| 2 | Present value ($) | |
| 3 | Interest rate (% p.a.) | |
| 4 | Compounding periods per annum | |
| 5 | Term of the investment (years) | |

To help you, you might like to use the prepared spreadsheet file (Spreadsheet 1B) provided on your obook assess.

**27** **a** Use a spreadsheet to calculate the future value of a $5000 investment, invested for 3 years with the interest compounded annually, at the following compound interest rates.

**i** 4% **ii** 6% **iii** 8% **iv** 10%

**b** Discuss how changing the interest rate of a compound interest investment affects the future value of the investment.

**28** **a** Use a spreadsheet to calculate the future value of a $5000 investment, invested for 2 years at 5% p.a. at the following compounding periods.

**i** Annually **ii** Quarterly **iii** Monthly **iv** Daily

**b** Discuss how changing the compounding period of a compound interest investment affects the future value of the investment.

**29** **a** Use a spreadsheet to calculate the future value of a $10 000 investment, invested at 6% p.a. with interest compounded monthly and the following term periods.

**i** 1 year **ii** 2 years **iii** 4 years **iv** 8 years

**b** Does doubling the term of the investment double the amount of interest earned on the investment? Discuss why or why not.

**30** **a** Research a number of different banks to find the interest rates for term deposits. Use a term deposit time period of 3 months and find the rates offered by three or four different banks.

**b** Find the value of an investment of $10 000 at the end of a 3-month term deposit at each bank.

**c** Use a spreadsheet to calculate the value of the $10 000 if it is reinvested (rolled over) at the same 3-month term deposit interest rate for 3 years. Graph the results using a spreadsheet.

**d** Find the value of $10 000 if it is reinvested in a 6-month term deposit, rolled over for 3 years. Graph the result.

**e** For the same bank, determine which of the two strategies results in a larger amount at the end.

**f** Repeat with other term deposits over different time periods rolled over for 3 years. Is there a pattern? Explain.

# 1C Using a compounded value table

These resources are available on your obook assess:

- **assess quiz 1C:** Test your skills with an auto-correcting multiple-choice quiz

A compounded value interest table can be used to quickly and efficiently calculate the future value of a compound interest investment, given the number of time periods and the interest rate per time period.

| Time period | Interest rate per time period | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1% | 2% | 3% | 4% | 5% | 6% | 8% | 9% | 10% | 15% |
| 1 | 1.010 | 1.020 | 1.030 | 1.040 | 1.050 | 1.060 | 1.080 | 1.090 | 1.100 | 1.150 |
| 2 | 1.020 | 1.040 | 1.061 | 1.082 | 1.103 | 1.124 | 1.166 | 1.188 | 1.210 | 1.323 |
| 3 | 1.030 | 1.061 | 1.093 | 1.125 | 1.158 | 1.191 | 1.260 | 1.295 | 1.331 | 1.521 |
| 4 | 1.041 | 1.082 | 1.126 | 1.170 | 1.216 | 1.262 | 1.360 | 1.412 | 1.461 | 1.750 |
| 5 | 1.051 | 1.104 | 1.159 | 1.217 | 1.276 | 1.338 | 1.469 | 1.539 | 1.611 | 2.011 |
| 6 | 1.062 | 1.126 | 1.194 | 1.265 | 1.340 | 1.419 | 1.587 | 1.671 | 1.772 | 2.313 |
| 7 | 1.072 | 1.149 | 1.230 | 1.316 | 1.407 | 1.504 | 1.714 | 1.828 | 1.949 | 2.660 |
| 8 | 1.083 | 1.172 | 1.267 | 1.369 | 1.477 | 1.594 | 1.851 | 1.993 | 2.144 | 3.059 |

### EXAMPLE 1C–1 Using the table of compounded values to find the present value

Taryn needs to have $5000 in 6 years' time. She can invest money at 4% p.a. compounded annually. What amount needs to be invested now? Use the compounded values in the table to help you answer the question.

| Solve | Think | Apply |
|---|---|---|
| Amount to be invested<br>$= 5000 \div 1.265$<br>$= \$3952.57$<br>Taryn needs to invest $3952.57 now to have $5000 in 6 years' time. | Find the value at the intersection of the 6th row and 4% column. This is 1.265. Divide 5000 by 1.265 to find the present value. | Find the value in the table at the intersection of the time and rate. Divide the future value given by this amount to find the present value. |

## EXERCISE 1C Using a compounded value table

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Use the interest table on the previous page to answer the following questions.

**a** Complete the following to find the value of \$8000 invested at 5% compounded annually for 3 years. Amount = 8000 × ____ = ____

**b** Find the value of \$5000 invested at 10% compounded annually for 7 years.

**c** Find the value of \$12 000 invested at 10% compounded annually for 6 years.

**d** Complete the following to find the value of \$1120 invested at 1% per quarter compounded quarterly for 6 quarters. Amount = 1120 × ____ = ____

**e** Find the value of \$10 500 invested at 5% per quarter compounded quarterly for 2 quarters.

**f** Find the value of \$1140 invested at 10% per quarter compounded quarterly for 5 quarters.

**g** Find the value of \$1550 invested at 1% per month compounded monthly for 7 months.

**h** Find the value of \$78 420 invested at 1% per month compounded monthly for 4 months.

**2** Use the interest table on the previous page to answer the following questions. Find the future value of these investments.

**a** \$5000 invested at 9% compounded annually for 4 years

**b** \$3500 invested at 6% compounded annually for 8 years

**c** \$7500 invested at 6% compounded annually for 7 years

**d** \$1300 invested at 3% per quarter compounded quarterly for 7 quarters

**e** \$6800 invested at 3% per quarter compounded quarterly for 3 quarters

**f** \$2350 invested at 6% per quarter compounded quarterly for 2 years

**g** \$890 invested at 12% per annum compounded monthly for 3 months

**h** \$8431 invested at 9% per month compounded monthly for 5 months

**3** Use the interest table on the previous page to answer the following questions.

**a** Complete the following to find the amount that needs to be invested at 4% compounded annually to have \$6000 in 5 years' time. Amount to be invested = 6000 ÷ ____ = ____

**b** What amount needs to be invested at 6% compounded annually to have \$3500 in 7 years' time?

**c** What amount needs to be invested at 8% compounded annually to have \$3800 in 4 years' time?

**d** Complete the following to find the amount that needs to be invested at 8% per quarter compounded quarterly for 8 quarters to have \$1500. Amount to be invested = ____ ÷ 1.851 = ____

**e** What amount needs to be invested at 4% per quarter compounded quarterly for 5 quarters to have \$2900?

**f** What amount needs to be invested at 2% per quarter compounded quarterly for 2 years to have \$2200?

**g** What amount needs to be invested at 2% per month compounded monthly for 5 months to have \$1150?

**h** What amount needs to be invested at 6% per month compounded monthly for 7 months to have \$5280?

**4** Janet invested \$3300 in a compound interest account paying 8% p.a., with interest compounded half-yearly. Use the interest table on the previous page to determine the future value of her investment after 3.5 years.

**5** A high-interest short-term loan charges 24% p.a., with interest compounded monthly. If \$1500 was borrowed for 7 months, use the interest table on the previous page to determine how much would need to be paid back.

**6** Stig wants to buy a new sound system that costs \$2200. Use the interest table on the previous page to determine how much he would need to invest now, in a compound interest account paying 8% p.a. with interest compounding quarterly, if he wants to purchase it in 18 months' time?

**7** Use the interest table on the previous page to find out how much more interest will be earned if \$5000 is invested for 6 years at 8% p.a. compounded annually compared to at 6% p.a. compounded annually.

# 1D Inflation and appreciated value

These resources are available on your obook assess:

- **Video tutorial 1D:** Watch and listen to an explanation of Example 1D–1
- **assess quiz 1D:** Test your skills with an auto-correcting multiple-choice quiz

**inflation**
an increase in the price of goods and services over a period of time

**appreciate**
a rise in the value or price of an item over a period of time

**Inflation** is the rise in the cost of consumer goods and services. It is usually expressed as a percentage rate. The inflation rate reflects the increase in the cost of items.

Some items, such as antiques, also **appreciate** in value, meaning that they gain in value each year.

We can calculate appreciated values on a year-by-year basis, or, if the rate of inflation is constant over the time period, we can use an adjusted version of the compound interest formula, $FV = PV(1 + r)^n$.

$$\text{Price after } n \text{ years} = \text{original price} \times \left(1 + \frac{\text{inflation rate}}{100}\right)^n$$

## EXAMPLE 1D–1 Calculating the price of an item after inflation

The price of petrol increases at the same rate as inflation. The inflation rate over 3 years is 4% p.a. If petrol is 139.50 c/L now, how much will it be in 3 years' time?

| Solve | Think | Apply |
|---|---|---|
| Price after 1 year $= 139.5 \times 1.04$<br>$= 145.08$ c/L<br>Price after 2 years $= 145.08 \times 1.04$<br>$= 150.8832$ c/L<br>Price after 3 years $= 150.8832 \times 1.04$<br>$= 156.918\,528$<br>$= 156.92$ c/L (2 decimal places)<br>*Using the formula:*<br>Price after $n$ years $= \text{original price} \times \left(1 + \frac{\text{inflation rate}}{100}\right)^n$<br>$= 139.50 \times \left(1 + \frac{4}{100}\right)^3$<br>$= 156.92$ c/L (2 decimal places) | Add 4% to 139.50 c by multiplying by 1.04. Repeat the multiplication three times. | If the rate of increase is constant (the same each year) apply the compound interest formula; otherwise, calculate the increase year by year. |

## EXERCISE 1D Inflation and appreciated value

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** A pizza costs \$12.95 today. The price increases by 3.8% p.a. Complete the following to find the expected price of the pizza after 4 years.

$$\text{Price after } n \text{ years} = \text{original price} \times \left(1 + \frac{\text{inflation rate}}{100}\right)^n$$

$$\text{Price after } \_\_\_\_ \text{ years} = \_\_\_\_ \times \left(1 + \frac{\square}{100}\right)^4 = \_\_\_\_$$

**2** A loaf of bread costs \$2.85 today. The price increases at the same rate as inflation. The rate of inflation is 5%. Find the expected price of a loaf of bread after:

**a** 1 year **b** 2 years **c** 3 years
**d** 4 years **e** 5 years.

**3** What price would you expect to pay for each of the following items after the number of years shown? Assume the rate of inflation remains constant.

| | Item | Present price | Inflation rate (%) | Number of years |
|---|---|---|---|---|
| **a** | Microwave oven | \$499 | 3.2 | 3 |
| **b** | Digital SLR camera | \$850 | 1.8 | 4 |
| **c** | Smart phone | \$553 | 2.8 | 2 |
| **d** | Office chair | \$140 | 3.5 | 3 |
| **e** | Blu-Ray player | \$385 | 2.3 | 4 |

**4** In 3 successive years the annual rate of inflation was 6.4%, 5.6% and 4.8%. How much would you expect to pay at the end of these 3 years for a dozen eggs that cost \$3.20 at the beginning of this period?

**5** In 3 successive years the annual rate of inflation was 2.8%, 3.5% and 4.1%. How much would you expect to pay at the end of these 3 years for a theatre ticket that cost \$29.99 at the beginning of this period?

**6** In 4 successive years the annual rate of inflation was 4.6%, 6.5%, 4.9% and 3.8%. How much would you expect to pay at the end of these 4 years for a dozen roses that cost \$30 at the beginning of this period?

**7** When Kate started her new job, she was told that her salary would increase in line with inflation. If the annual rate of inflation for three years was 2.1%, 3.0% and 3.7%, and her starting salary was \$58 000, how much was she earning at the end of her third year? Give your answer to the nearest dollar.

**8** François had a great year selling insurance. In his annual review meeting he is told that his salary will increase in line with inflation (4.1%), and on top of that he will receive a \$7000 pay rise. If his previous salary was \$61 000, determine his new salary.

### EXAMPLE 1D–2 Calculating the appreciated value of an item

Caitlin purchased a copy of a rare comic book for \$145. Over the next 8 years, the value of the comic book increased by an average of 9% p.a. How much is the comic book now worth?


| Solve | |
|---|---|
| $\text{Value after } n \text{ years} = \text{original value} \times \left(1 + \frac{\text{annual \% increase}}{100}\right)^n$ <br> $= 145 \times \left(1 + \frac{9}{100}\right)^8$ <br> $\approx \$288.90$ (to the nearest 5 cents) | |
| **Think** | **Apply** |
| $r = 9,\ n = 8$ <br> Original value $= \$145$ <br> Substitute these values and calculate. | Use the compound interest formula. |

**9** A stamp collection is purchased for \$2500. It increases in value by an average of 4.5% p.a. Find the appreciated value, to the nearest 5 cents, of the stamp collection after:

**a** 10 years **b** 20 years **c** 50 years.

**10** A signed Don Bradman cricket bat was purchased in 2012 for \$3500. It is expected to increase in value by 3% p.a. Find the appreciated value of the cricket bat, to the nearest 5 cents, after:

**a** 5 years **b** 10 years **c** 20 years.

**11** Lindsey's salary at the start of 2014 was \$52 000. She received a pay rise of 3.3% for each of the next 4 years. What was her salary at the start of 2018 correct to the nearest dollar?

**12** A Samurai sword is purchased for \$3800. It is expected to increase in value by 6% p.a. After 10 years it is sold for \$8600. By how much did this exceed the expected value?

**13** A signed 2011 World Cup Rugby jumper was purchased for \$2000. It was expected to increase in value by 8% p.a. After 5 years it was sold for \$2500. By how much is this over or under the expected value?

## EXAMPLE 1D–3 Calculating the annual rate of inflation

Three years ago a packet of crumpets cost $2.12. It now costs $2.36. What is the average annual rate of inflation of the cost of crumpets over the 3 years? Give your answer to one decimal place.

| Solve | Think | Apply |
|---|---|---|
| $FV = PV\left(1 + \frac{r}{100}\right)^n$<br>$2.36 = 2.12\left(1 + \frac{r}{100}\right)^3$<br>$\frac{2.36}{2.12} = \left(1 + \frac{r}{100}\right)^3$<br>$1.113\,207\,547 = \left(1 + \frac{r}{100}\right)^3$<br>$\sqrt[3]{1.113\,207\,547} = 1 + \frac{r}{100}$<br>$1.036\,395\,171 = 1 + \frac{r}{100}$<br>$0.036\,395\,171 = \frac{r}{100}$, so $r \approx 3.6\%$<br>The average annual inflation rate is 3.6%. | It is possible to use the compound interest formula, as the rate of inflation is constant. Use $FV = 2.36$ as the future value, $PV = 2.12$ as the present value, and $n = 3$. | As there is a constant increase in each time period, the compound interest formula is used. |

**14** Three years ago the price of chuck steak was $8.99/kg. It now costs $9.69/kg. Calculate the average yearly inflation rate, correct to two decimal places, that would produce this rise.

**15** In 2013, the cost of a standard university lecture was $152.13. In 2017 the price was $176.42. What was the annual rate of inflation (assuming it is constant) correct to two decimal places?

**16** **a** For the Samurai sword in question **12**, calculate the actual annual rate of increase (assuming it is constant) correct to two decimal places.

**b** For the rugby jumper in question **13**, calculate the actual annual rate of increase (assuming it is constant) correct to two decimal places.

**17** **a** Use the internet to research how to calculate the average rate of inflation over a number of years. Present a report containing your findings to your class.

**b** If the rate of inflation was 3.0% one year and 4.0% the following year, would the average rate of inflation over the two years be 3.5%? Explain your answer.

# 1E Shares

These resources are available on your obook assess:

- **assess quiz 1E:** Test your skills with an auto-correcting multiple-choice quiz
- **Website:** Visit the ASX website to play the trading game
- **Website:** Visit the MSN Money site for a share price summary

**share**
a proportion of a company given a monetary value, which can be sold or exchanged

**face** (or **par**) **value**
the value of a share when it is first issued

**market price**
the current price of a share

**dividend**
a payment made regularly from a company to its shareholders

**dividend yield**
the percentage of a current share's market price that is paid as a dividend

**brokerage**
a fee charged by stockbrokers to carry out trades on behalf of an investor

A **share** is a part ownership in a company. One method a company may use to raise capital is to sell shares to investors. The shares in the company when first issued have a **face** (or **par**) **value**. The shares are then listed on the stockmarket and the price fluctuates. The price of the share being traded on the stock exchange is called the **market price**.

Any profits made by the company are divided among the shareholders. This return to the shareholders is called a **dividend**. The dividend may be quoted as an amount per share or as a percentage of the market price of the share.

The theoretical return on the money invested is called the **dividend yield**. It is calculated according to the following formula.

$$\text{Dividend yield} = \frac{\text{amount of dividend}}{\text{market price}} \times 100\%$$

A stockbroker carries out the buying and selling of shares on behalf of an investor. The fee charged by stockbrokers is called **brokerage** and is usually a percentage of the value of the transaction or a fixed fee according to the amount of the transaction.


**EXAMPLE 1E–1** Calculating the total cost of purchasing shares

Calculate the total cost of purchasing 700 shares with a market price of \$1.49. Brokerage is 2.5%.

| Solve | Think | Apply |
|---|---|---|
| Cost of shares $= 700 \times 1.49$<br>$= \$1043$<br>Brokerage $= 2.5\%$ of $1043$<br>$= 2.5 \div 100 \times 1043$<br>$= \$26.08$ (2 decimal places)<br>Total cost $= 1043 + 26.08$<br>$= \$1069.08$ | The cost of the shares is the market price, so use \$1.49. The brokerage cost is 2.5% of \$1043. Add this value to the cost of the shares for the total cost. | Calculate the cost of the shares by multiplying the number of shares by the market price. Add the brokerage cost. |

# EXERCISE 1E Shares

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Complete the following to calculate the total cost of purchasing 600 shares with a market price of \$7.97. Brokerage is 2.5%.

Cost of shares = 600 × ____ = ____

Brokerage = ____% of 4782

= ____ ÷ 100 × ____ = ____

Total cost = ____ + ____ = ____

**2** Calculate the total cost of purchasing the following parcels of shares. The brokerage in each instance is 2.5%.

**a** 600 shares with a market price of \$4.65

**b** 300 shares with a market price of \$15.05

**c** 450 shares with a market price of \$2.85

**d** 300 shares with a market price of \$9.70

## EXAMPLE 1E–2 Calculating the amount obtained when shares are sold

Calculate the amount obtained when 1200 shares are sold at \$1.36 each. Brokerage is \$29.

| Solve | Think | Apply |
|---|---|---|
| Selling income = 1200 × 1.36<br>= \$1632<br>Amount obtained = 1632 − 29<br>= \$1603 | Multiply the selling price by the number of shares.<br>Subtract brokerage of \$29, as it is a cost. | The income from a sale of shares is the number of shares multiplied by the cost per share. Subtract the brokerage fee to determine the amount obtained. |

**3** Complete the following to calculate the amount obtained when 800 shares are sold at \$4.85. Brokerage is \$29.

Selling income = ____ × 4.85 = ____

Amount obtained = ____ − ____ = ____

**4** Calculate the amount obtained when:

**a** 600 shares are sold at \$2.46. Brokerage is \$29.

**b** 1000 shares are sold at \$1.70. Brokerage is \$27.50.

**c** 400 shares are sold at \$4.40. Brokerage is \$32.

**d** 300 shares are sold at \$11.54. Brokerage is \$45.

**5** Calculate the amount obtained when these shares are sold with brokerage of 2.2%.

| | Share | Number | Selling price |
|---|---|---|---|
| **a** | Share A | 800 | \$9.44 |
| **b** | Share B | 3000 | \$0.24 |
| **c** | Share C | 1500 | \$2.53 |
| **d** | Share D | 5000 | \$0.66 |
| **e** | Share E | 500 | \$5.28 |

## EXAMPLE 1E–3 Calculating the dividend yield for shares

A dividend of 34 cents per share is paid on shares with a market value of \$8.02. Find the dividend yield.

| Solve | Think | Apply |
|---|---|---|
| Dividend yield $= \dfrac{\text{amount of dividend}}{\text{market price}} \times 100\%$ <br> $= \dfrac{34}{802} \times 100$ <br> $\approx 4.24\%$ | Change \$8.02 to cents, so that the dividend and the price are both in cents. | The dividend and market value must both be in cents or dollars. Divide and find the percentage by multiplying by 100. |

**6** Complete the following to find the dividend yield on a dividend of 6.32 cents per share paid on shares with a market value of \$0.73.

$$\text{Dividend yield} = \frac{\text{amount of dividend}}{\text{market price}} \times 100\%$$
$$= 6.32 \div \_\_\_\_ \times 100$$
$$= \_\_\_\_\%$$

**7** A dividend of 25 cents per share is paid on shares with a market value of \$3.30. Find the dividend yield.

**8** A dividend of 4.65 cents per share is paid on shares with a market value of \$0.91. Find the dividend yield.

**9** A dividend of 10 cents per share is paid on shares with a market value of \$3.10. Find the dividend yield.

## EXAMPLE 1E–4 Calculating the total dividend for shares

Calculate the total dividend on 1500 shares with a market price of 50 cents if the dividend yield is 13%.

| Solve | Think | Apply |
|---|---|---|
| Dividend per share = 13% of 50 <br> = 6.5 cents <br> Total dividend = $1500 \times 6.5$ <br> = 9750 cents <br> = \$97.50 | Find 13% of 50 cents. Multiply 6.5 cents by 1500 shares. | Use the dividend yield to calculate the dividend per share. Multiply by the number of shares to find the total dividend. |

**10** Complete the following to calculate the total dividend on 1200 shares with a face value of 50 cents if the dividend is 11%.

$$\text{Dividend per share} = \_\_\_\_\% \text{ of } 50$$
$$= \frac{\square}{100} \times 50 = \_\_\_\_ \text{ cents}$$
$$\text{Total dividend} = \_\_\_\_ \times 5.5 = \_\_\_\_ \text{ cents}$$
$$= \$\_\_\_\_$$

**11** Calculate the total dividend on the following shares.

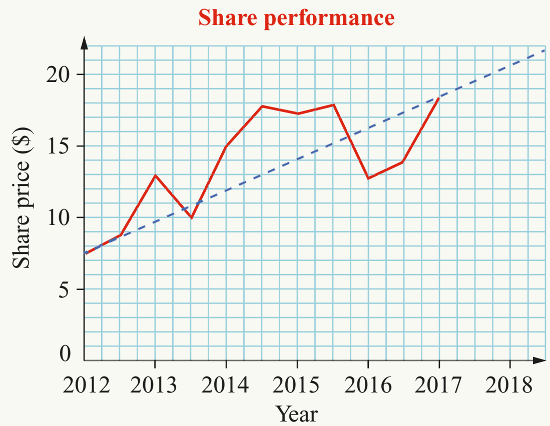
**a** 2000 shares with a market price of 50 cents if the dividend yield is 5%

**b** 600 shares with a market price of \$1.00 if the dividend yield is 7%

**c** 3500 shares with a market price of \$1.00 if the dividend yield is 4.6%

## EXAMPLE 1E–5 Estimating the future value of a share by using a share graph

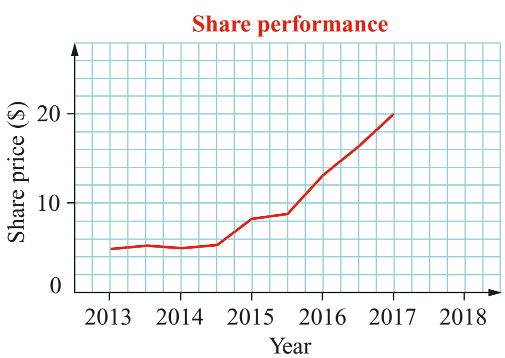
The graph shows the performance of a share over a period of time. Use this to estimate the value of the share in the year 2018, assuming the trend continues.



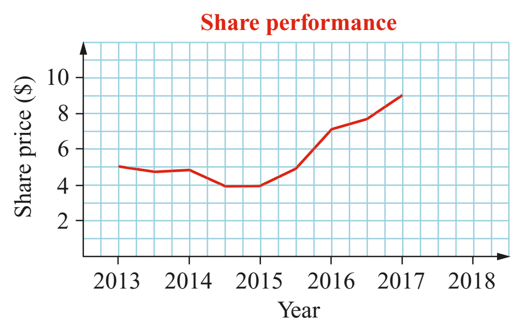

| Solve | Think/Apply |
|---|---|
| From the graph an estimate of the value in 2018 is about $21. | Draw a line that follows the trend of the graph and extend the line to 2018. This line gives an estimated value of $21. |

**12** The following graphs show the performance of some individual shares over a period of time. Use this to estimate the value of each share in 2018, assuming the trend continues.

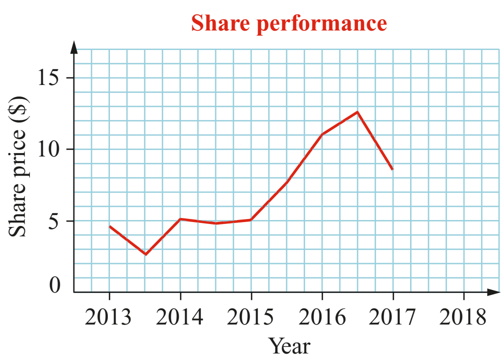
**a**



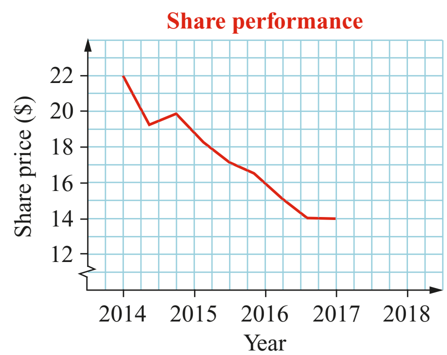

**b**




**c**




**d**



**13** An investor has 2500 shares. The current market price is $1.85 per share. The investor is paid a dividend of $0.25 per share.

**a** What is the current value of the investor's shares?

**b** Calculate the total dividend.

**c** Calculate the dividend yield.

**14** An investor has 3300 shares. The current market price is $2.22 per share. The investor is paid a dividend of $0.33 per share.

**a** What is the current value of the investor's shares?

**b** Calculate the total dividend.

**c** Calculate the dividend yield.

**15** **a** Find the total cost of buying 500 shares at $19.45 each if brokerage is 2.5%.

**b** The shares are sold 12 months later for $26.95 each and 2.5% brokerage. How much is obtained from the sale?

**c** Determine the gain from the investment.

**d** Find the percentage gain made on the original investment.

**16** Complete the following table.

| | Face value | Market price | Dividend per share in cents | Dividend yield |
|---|---|---|---|---|
| **a** | $1.00 | $1.80 | 9 | |
| **b** | $1.00 | $1.50 | | 12% |
| **c** | $0.50 | $1.24 | 4 | |
| **d** | $0.50 | $0.86 | | 14% |
| **e** | $1.00 | | 10 | 12.5% |

**17** Stephen owns 1500 shares with each share having a market value of $3.46. He is paid a dividend of $0.42 per share. Stephen's sister, Abigail, owns 2300 shares with each share having a market value of $2.59. She is paid a dividend of $0.30 per share.

**a** Whose shares have the higher current value?

**b** Who received the higher total dividend?

**c** Whose dividend yield was highest?

**18** An investor buys the following portfolio of shares from his broker. The end-of-year dividends are included.

| Shares | Face value | Market price | Dividend per share (in cents) |
|---|---|---|---|
| 100 A | $1.00 | $9.80 | 22 |
| 50 B | $1.00 | $19.90 | 18 |
| 300 C | $1.00 | $3.60 | 12 |
| 500 D | $0.50 | $1.10 | 12 |
| 2000 E | $0.50 | $0.28 | 8 |

**a** Find the market value of this portfolio of shares.

**b** Calculate the total cost of purchasing the shares if brokerage is 2.2%.

**c** Calculate the amount of the total dividend for the year.

**d** Using your answers from parts **a** and **c**, determine the dividend yield for this portfolio.

**e** The shares are then sold at the market price listed in the table below. Calculate the net profit or loss. Brokerage is 2.8%.

| Shares | Market price |
|---|---|
| 100 A | $10.20 |
| 50 B | $21.30 |
| 300 C | $2.90 |
| 500 D | $1.25 |
| 2000 E | $0.36 |

**19** Lotoya has recorded the value of a share each month for 6 months. The results are shown in the table.

| Month | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Share price | $1.25 | $1.30 | $1.20 | $1.50 | $1.80 | $1.85 |

**a** Draw a graph representing this information.

**b** Lotoya hopes the shares will be worth $3.00 at the end of 12 months. Do you agree? Explain by extending the graph.

**20** The price of Supersafe Financial's shares over a year have been recorded in the following table.

| Month | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Share price | $3.23 | $3.45 | $3.61 | $3.25 | $3.03 | $3.50 | $3.98 | $4.10 | $4.25 | $4.47 | $4.33 | $4.08 |

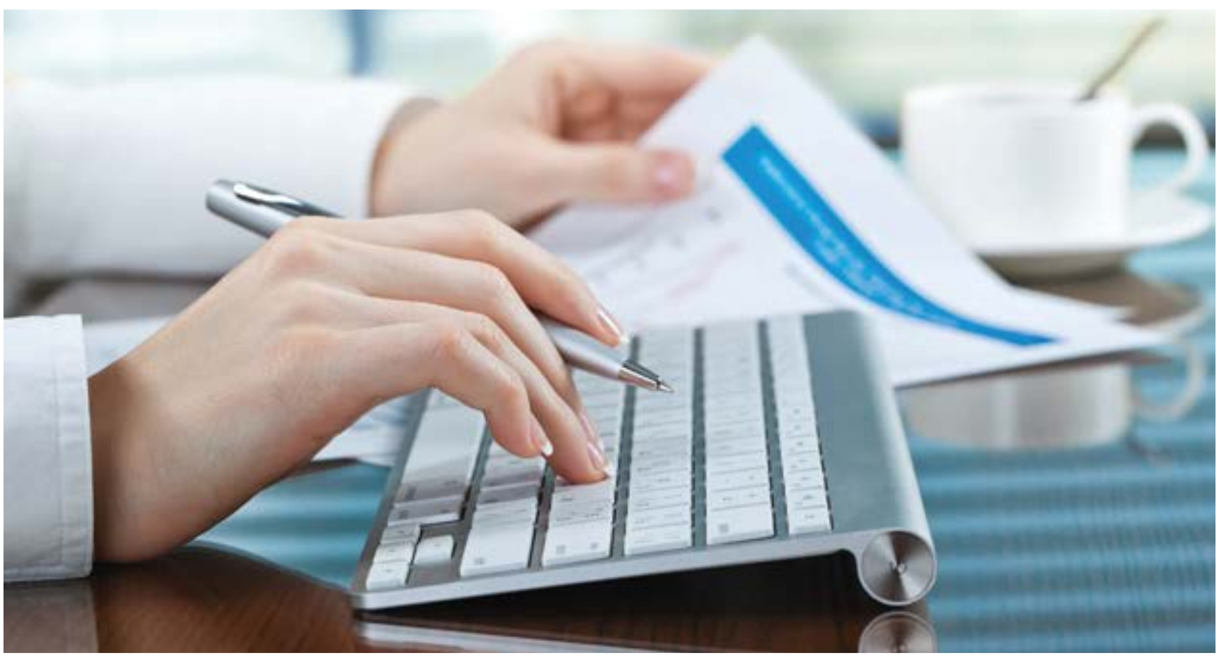
**a** Draw a graph representing this information.

**b** Did the share price drop below $3.00 at any point during the year? Explain your answer.



**21** Visit the ASX website and play the trading game. A link is provided on your obook assess.

**22** Choose your own portfolio of shares from five companies. Check the share prices and calculate the profit or loss each month. Don't forget the cost of buying and selling. The Money site will give a summary of share prices each day. A link is provided on your obook assess.



# 1F Declining-balance method of depreciation

These resources are available on your obook assess:
- **Video tutorial 1F:** Watch and listen to an explanation of Example 1F–1
- **assess quiz 1F:** Test your skills with an auto-correcting multiple-choice quiz

**depreciation**
a loss in the value of an item over time

**salvage value**
the reduced value of an item after depreciation

The **depreciation** of an item is its loss in value due to age and usage. The value of an item after depreciation is called its **salvage value**, book value, scrap value or written-down value. Some items can be depreciated by businesses or workers to help reduce the tax they have to pay. The value by which an item has been depreciated can be used as a tax deduction against taxable income.

In the Year 11 Mathematics Standard course we calculated straight-line depreciation, which depreciates the value of an item by a constant amount each year. In this course we will look at the declining-balance method of depreciation, which depreciates the value of an item by a constant percentage each year.

## EXAMPLE 1F–1 Constructing a depreciation table

Construct a table to calculate the value of a \$20 000 car after 3 years if its rate of depreciation is 20% p.a.

**Solve**

| Year | Value (\$) | Depreciation (\$) | Depreciated value (\$) |
|---|---|---|---|
| 1 | 20 000 | 4000 | 16 000 |
| 2 | 16 000 | 3200 | 12 800 |
| 3 | 12 800 | 2560 | 10 240 |

**Think**

Use $20\% = 0.2$.
Depreciation year 1
$= 0.2 \times 20\,000$
$= \$4000$
Value end of year 1
$= 20\,000 - 4000$
$= \$16\,000$
Depreciation year 2
$= 0.2 \times 16\,000$
$= \$3200$
Value end of year 2
$= 16\,000 - 3200$
$= \$12\,800$
Depreciation year 3
$= 0.2 \times 12\,800$
$= \$2560$
Value end of year 3
$= 12\,800 - 2560$
$= \$10\,240$

**Apply**

Amount of depreciation each year = rate of depreciation × the value of the car at the beginning of the year.
Subtract the amount of depreciation from the value at the beginning of the year.

## EXERCISE 1F Declining-balance method of depreciation

**1** Complete the table to calculate the value of a \$24 900 car after 3 years if its rate of depreciation is 22% p.a.

| Year | Value (\$) | Depreciation (\$) | Depreciated value (\$) |
|---|---|---|---|
| 1 | 24 900 | $0.22 \times 24\,900 = 5478$ | 19 422 |
| 2 | 19 422 | $0.22 \times$ _____ = ____ | |
| 3 | | | |

**2** Complete the table to calculate the value after 4 years of office furniture costing \$34 800 if its rate of depreciation is 18% p.a.

| Year | Value (\$) | Depreciation (\$) | Depreciated value (\$) |
|---|---|---|---|
| 1 | 34 800 | _____ $\times$ 34 800 = | |
| 2 | | | |
| 3 | | | |
| 4 | | | |

**!**

The formula for the declining-balance method of depreciation

$S = V_0(1 - r)^n$

where $S$ = the salvage value of the asset

$V_0$ = the purchase price of the asset

$r$ = the percentage depreciation rate per time period, expressed as a decimal

$n$ = the total number of time periods.

You will notice that this formula is very similar to the compound interest formula, but with a minus sign before the $r$, as the value steadily decreases over time.

### EXAMPLE 1F–2 Calculating the salvage value and amount of depreciation

A new car is purchased for \$32 000. It depreciates in value at a rate of 22% per year.

**a** Calculate the salvage value of the car after 3 years.

**b** By what amount has the car depreciated in value after 3 years?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $S = V_0(1 - r)^n$<br>$= 32\,000(1 - 0.22)^3$<br>$= \$15\,186$ (to nearest \$) | Substitute $V_0 = 32\,000$, $r = 22\% = 0.22$ and $n = 3$ into the formula. | Substitute the values of $V_0$, $r$ and $n$ into the formula $S = V_0(1 - r)^n$. |
| **b** | Depreciation $= 32\,000 - 15\,186$<br>$= \$16\,814$ | Subtract the salvage value (\$15 186) from the original price (\$32 000). | The amount of depreciation is the change in the value of the car. |

**3** A new car is purchased for \$35 000. It depreciates in value at a rate of 24% per year. Complete the following to find:

**a** the salvage value of the car after 3 years

$S = V_0(1 - r)^n$

$= \_\_\_\_ (1 - 0.24)^{\square}$

$= \$\_\_\_\_$ (to the nearest \$)

**b** the amount the car has depreciated in value after 3 years.

Change in value $= 35\,000 - \_\_\_\_$

$= \$\_\_\_\_$

**4** New office carpets are purchased for \$19 990. They depreciate in value at a rate of 28% per year.

**a** Calculate the salvage value of the carpets after 5 years.

**b** By what amount have the carpets depreciated in value after 5 years?

**5** Some manufacturing machinery is purchased for \$56 000. It depreciates in value at a rate of 35% per year.

**a** Calculate the salvage value of the machinery after 4 years.

**b** By what amount has the machinery depreciated in value after 4 years?

## EXAMPLE 1F–3 Calculating the annual percentage rate of depreciation

An electronic security system depreciates in value from \$35 000 to \$22 000 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

| Solve | Think | Apply |
|---|---|---|
| $S = V_0(1 - r)^n$<br>$22\,000 = 35\,000(1 - r)^2$<br>$\frac{22\,000}{35\,000} = (1 - r)^2$<br>$\sqrt{\frac{22\,000}{35\,000}} = 1 - r$<br>$0.7928 = 1 - r$<br>$r = 1 - 0.7928\ldots$<br>$= 0.2071\ldots$<br>$\approx 0.21$<br>$\therefore$ Rate of depreciation $\approx 21\%$ | Substitute $S = 22\,000$, $V_0 = 35\,000$ and $n = 2$ into the formula:<br>$22\,000 = 35\,000(1 - r)^2$<br>Divide both sides by 35 000.<br>Take the square root of both sides.<br>Add $r$ to both sides.<br>Subtract 0.7928 from both sides. | Substitute the values of $V_0$, $r$ and $n$ into the formula $S = V_0(1 - r)^n$ and solve the resulting equation. |

**6** A computer server depreciates in value from \$29 000 to \$20 462 in 2 years. Complete the following to calculate the annual percentage rate of depreciation using the declining-balance formula: $S = V_0(1 - r)^n$.

$20\,462 = \_\_\_ (1 - r)$

$\frac{20\,462}{\square} = (1 - r)$

$\sqrt{\frac{20\,462}{\square}} = (1 - r)$

$\therefore \; = (1 - r)$

$r = 1 - \_\_\_ \approx \_\_\_$ $\quad\therefore$ Rate of depreciation $\approx \_\_\_\%$

**7** An air conditioning system depreciates in value from \$36 000 to \$19 000 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

**8** Some office furniture depreciates in value from \$44 900 to \$32 440 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

**9** A plumbing fitting depreciates in value from \$15 000 to \$4500 in 3 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

**10** A network of office printers depreciates in value from \$68 000 to \$31 000 in 3 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**11** Peter purchases a computer for \$2300 which he uses only for work purposes. He is able to depreciate the value of the computer by 30% each year, and can use the depreciation amount as a tax deduction.

**a** What is the salvage value of Peter's computer after 1 year of use?

**b** What amount can Peter reduce his taxable income by in the first year that he depreciates his computer?

**c** What amount can Peter reduce his taxable income by in the second year that he depreciates his computer?

**12** How much will the company that depreciated their manufacturing machinery in question **5** be able to reduce their taxable income by in the first year of this depreciation?

**13** The percentage rate of depreciation of some items is not always consistent from year to year. Georgina is able to depreciate the value of her car by 40% in the first year, 18% in the second year and 12% in the third year.

**a** If Georgina's car cost \$38 000, what is the salvage value of the car after three years?

**b** What would be the equivalent annual percentage rate of depreciation? Give your answer to the nearest whole number.

**14** The graphs below show the depreciation of a car using:

**i** the straight-line method

**ii** the declining-balance method.

**a** What was the purchase price?

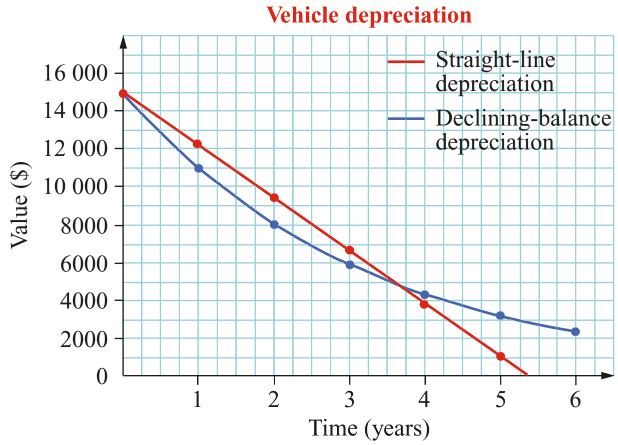
**b** What is the value of the car after 1 year using each method?

**c** When is the car worth \$8000, for each method?

**d** When is the salvage value the same for each type of depreciation? What is it?

**e** When is the greatest difference in book values? How much is it?

**f** For the straight-line method, what is the annual amount of depreciation?




FINANCIAL MATHEMATICS

**15** The graphs below show the depreciation of a scooter using:

- **i** the straight-line method
- **ii** the declining-balance method.

**a** What was its purchase price?

**b** What is the value of the scooter after 2 years using each method?

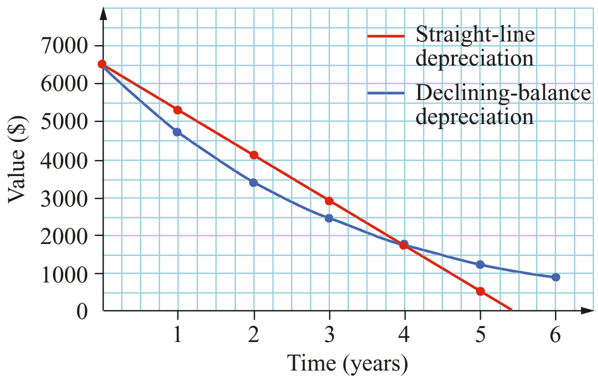
**c** When is it worth $4000 for each method?

**d** When is the salvage value the same for each type of depreciation? What is it?

**e** The scooter is scrapped after 5 years. What is its final written-down value for each method?

**f** For the straight-line method, what is the annual amount of depreciation?

**Vehicle depreciation**




**16** The graphs below show the depreciation of a car using:

- **i** the straight-line method
- **ii** the declining-balance method.

**a** What was the purchase price of the car?

**b** What is the car's value after 2 years for each method?

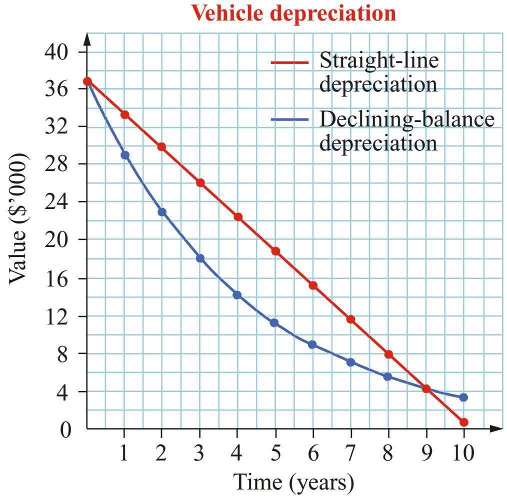
**c** When is the car worth half its original value for each method?

**d** When is the salvage value the same for each type of depreciation? What is it?

**e** For the straight-line method, what is the annual depreciation?

**f** What is the annual rate of depreciation for the declining-balance method?

**Vehicle depreciation**



PROBLEM SOLVING, REASONING AND JUSTIFICATION

**17** A hot water system is purchased for \$19 900. The straight-line depreciation amount is \$3700 and the declining-balance percentage is 30%.

**a** Complete the depreciation table.

**b** Draw a graph of the depreciation of the hot water system for each method, on the same set of axes.

**c** From the table, when $n = 2$, $S =$ ____ for the straight-line method and $S =$ ____ for the declining-balance method.

**d** From the graphs, when $n = 3.5$, $S \approx$ ____ for the straight-line method and $S \approx$ ____ for the declining-balance method.

**e** From the graph, the straight line intersects the curve when $n \approx$ ____.
The values are the same after approximately ____ years, or ____ years and ____ months.

| Year | Straight-line value (\$) | Declining-balance value (\$) |
|---|---|---|
| 0 | 19 900 | 19 900 |
| 1 | 16 200 | 13 930 |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |

**18** A motorcycle used for courier work is purchased for \$11 350. The depreciation can be calculated as either \$2100 per year using the straight-line method, or 32% per year using the declining-balance method.

**a** Complete a depreciation table showing the depreciated value each year for 5 years using each method.

**b** Draw a graph of the depreciation of the motorcycle for each method, on the same set of axes.

**c** Using the straight-line method, when is the salvage value less than that for the declining-balance method?

**d** What is the salvage value of the motorcycle after $3\frac{1}{2}$ years, using each method?

**e** When is the motorcycle worth half its original value, under each method?

**19** Use the internet to research the amount of depreciation of motor vehicles of different ages. Does the percentage of depreciation stay the same for the lifetime of the vehicle? Write a report on your findings.

# 1G Reducing-balance loans

These resources are available on your obook assess:

- **Spreadsheet 1G:** Create a reducing-balance loan table
- **Investigation 1G:** Investigate comparison interest rates
- **assess quiz 1G:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**reducing-balance loan**
a loan for which the interest is calculated on the balance owing, which gradually reduces over time

A home loan is an example of a **reducing-balance loan**. In this type of loan the interest is calculated on the balance owing at the start of each repayment period. For home loans the repayment period is often monthly, but may be fortnightly or weekly.

In a reducing-balance loan the repayment amount remains consistent throughout the lifetime of the loan, while the interest for each repayment period gradually reduces as the balance reduces. This means that early repayments on a reducing-balance loan pay a higher proportion of interest compared to latter repayments.

This topic examines various methods of calculating the balance owing and the monthly payments required. Usually, the information most frequently needed by borrowers concerns the amount that can be borrowed and the repayment. Most lending institutions have online calculators that can be used to find the monthly repayment, the amount of interest paid and the effects of changing the interest rate and making extra payments.

## EXAMPLE 1G–1 Looking up the monthly repayment on a loan

This table of home loan repayments was generated using a spreadsheet. The annual interest rate is 8.5% and the amount shown is the monthly repayment.

| Years | $160 000 | $200 000 | $240 000 | $280 000 | $320 000 | $360 000 | $400 000 |
|---|---|---|---|---|---|---|---|
| 5 | $3282.64 | $4103.30 | $4923.96 | $5744.62 | $6565.30 | $7385.96 | $8206.60 |
| 10 | $1983.78 | $2479.72 | $2975.66 | $3471.60 | $3967.54 | $4463.48 | $4959.44 |
| 15 | $1575.58 | $1969.48 | $2363.38 | $2757.28 | $3151.16 | $3545.06 | $3938.96 |
| 20 | $1388.52 | $1735.64 | $2082.78 | $2429.90 | $2777.04 | $3124.16 | $3471.28 |
| 25 | $1288.36 | $1630.46 | $1932.64 | $2254.64 | $2576.72 | $2898.82 | $3260.92 |
| 30 | $1230.26 | $1537.82 | $1845.40 | $2152.96 | $2460.52 | $2768.08 | $3075.64 |

Use the table to find the monthly repayment on a loan of $240 000 over:

**a** 30 years **b** 25 years **c** 10 years.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $1845.40 | Look down the $240 000 column and along the row showing the number of years. | Determine the appropriate column and row in the table. |
| **b** | $1932.64 | | |
| **c** | $2975.66 | | |

## EXERCISE 1G Reducing-balance loans

Use the table in Example 1G–1 to answer questions **1** to **4**.

**1** Find the monthly repayment on a loan of $320 000 over:

**a** 10 years **b** 15 years
**c** 25 years **d** 30 years.

**2** **a** Find the monthly repayment on $400 000 over 20 years.
**b** Find the monthly repayment on $360 000 over 20 years.
**c** How much larger is the monthly repayment on $400 000 than that on $360 000 over 20 years?

**3** From the amounts shown in the table, what is the maximum loan that can be taken if you can afford these monthly repayments?

**a** $2000 **b** $2300
**c** $2480 **d** $2600

**4** How must more would someone be able to borrow on a 20-year loan if they were able to afford a $3150 monthly repayment as opposed to a $2100 monthly repayment?

**5** Go to the website of a bank and use their online calculator to check the current results for questions **1** to **4**.

### EXAMPLE 1G–2 Calculating the amount still owing on a loan

A home loan of $250 000 is taken out at an interest rate of 7.75% p.a., reducible monthly, with a monthly repayment of $1888. Use the table to calculate the amount still owing after 6 months.

| Month | Principal at the start of the month | Monthly interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | $250 000 | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

**Solve**

| Month | Principal at the start of the month | Monthly interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | $250 000 | $1614.58 | $1888 | $249 726.58 |
| 2 | $249 726.58 | $1612.82 | $1888 | $249 451.40 |
| 3 | $249 451.40 | $1611.04 | $1888 | $249 174.44 |
| 4 | $249 174.44 | $1609.25 | $1888 | $248 895.69 |
| 5 | $248 895.69 | $1607.45 | $1888 | $248 615.14 |
| 6 | $248 615.14 | $1605.64 | $1888 | $248 332.78 |

| Think | Apply |
| --- | --- |
| Monthly interest rate $= \frac{7.75}{12}$<br>$= 0.645\ 83\%$<br>Interest for 1st month $I = \frac{250\ 000 \times 0.645\ 83 \times 1}{100}$<br>$= \$1614.58$<br>Hence:<br>Balance at end of 1st month<br>$= 250\,000 + 1614.58 - 1888$<br>$= \$249\,726.58$<br>The first row can now be completed. The balance at the end of the first month becomes the principal for the second month.<br>Hence for the second month:<br>$I = \frac{249\ 726.58 \times 0.645\ 83 \times 1}{100}$<br>$= \$1612.81$<br>Balance at end of 2nd month<br>$= 249\,726.58 + 1612.81 - 1888$<br>$= \$249\,451.40$<br>The balance at the end of the second month becomes the principal for the third month, etc. | Calculate the balance at end of the month by adding the interest to the principal and then subtracting the repayment. This amount then becomes the principal for the next month. |

*Note:* In Example 1G–2 the amount of interest being paid each month is reducing because the balance is reducing each month. The total amount repaid over 6 months $= 6 \times 1888 = \$11\,328$.

The total interest paid $= 1614.58 + 1612.82 + \ldots = \$9660.78$ and the principal has been reduced by $11\,328 - 9660.78 = \$1667.27$.

If this was a simple interest loan, the total amount of interest paid would be $6 \times 1614.58 = \$9687.48$.

**6** A home loan of \$420 000 is taken out at an interest rate of 7.2% p.a. with a monthly repayment of \$3306.83. Complete the table to calculate the amount still owing after 6 months.

| Month | Principal at start of the month | Monthly interest | Monthly repayment | Balance at end of month |
| --- | --- | --- | --- | --- |
| 1 | \$420 000 | \$2520.00 | \$3306.83 | \$419 213.17 |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

UNDERSTANDING, FLUENCY AND COMMUNICATING

**7** A home loan of $350 000 is taken out at an interest rate of 7.6% p.a. with a monthly repayment of $2471.26. Complete the table to calculate the amount still owing after 6 months.

| Month | Principal at start of the month | Monthly interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | $350 000 | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |

## EXAMPLE 1G–3 Calculating the monthly repayment on a loan

This table shows the repayment per $1000 borrowed on a monthly reducible loan. Find the monthly repayment on a loan of $210 000 over:

**a** 20 years at 7.75% p.a. **b** 25 years at 8.25% p.a.

| Term in years | 7% | 7.25% | 7.5% | 7.75% | 8% | 8.25% | 8.5% |
|---|---|---|---|---|---|---|---|
| 5 | $19.8012 | $19.9194 | $20.0379 | $20.1570 | $20.2765 | $20.3963 | $20.5164 |
| 10 | $11.6108 | $11.7401 | $11.8702 | $12.0011 | $12.1328 | $12.2653 | $12.3985 |
| 15 | $8.9883 | $9.1286 | $9.2701 | $9.4128 | $9.5566 | $9.7014 | $9.8474 |
| 20 | $7.7530 | $7.9036 | $8.0559 | $8.2095 | $8.3644 | $8.5207 | $8.6782 |
| 25 | $7.0678 | $7.2281 | $7.3899 | $7.5533 | $7.7182 | $7.8875 | $8.0522 |
| 30 | $6.6530 | $6.8218 | $6.9921 | $7.1641 | $7.3377 | $7.5127 | $7.6891 |

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Repayment = 8.2095 × 210<br>= $1724.00 | From the table, the repayment on $1000 at 7.75% over 20 years is $8.2095.<br>Thousands borrowed<br>$= \frac{210\,000}{1000} = 210$ | Determine the repayment on $1000 from the appropriate column and row in the table. Multiply this amount by the number of thousands being borrowed. |
| **b** | Repayment = 7.8875 × 210<br>= $1656.38 | Repayment on $1000 over 25 years at 8.25% is $7.8875. | |

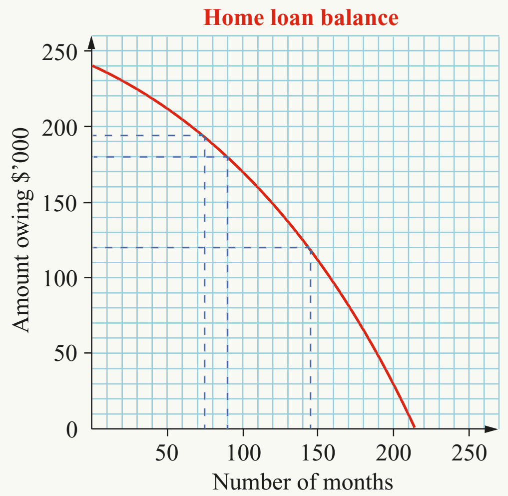
**8** Use the table in Example 1G–3 to find the monthly repayment on each of these loans.

| | Amount ($) | Term (years) | Interest rate (%) |
|---|---|---|---|
| **a** | 260 000 | 25 | 7.25 |
| **b** | 350 000 | 30 | 7 |
| **c** | 415 000 | 10 | 8.25 |
| **d** | 396 000 | 20 | 7.75 |
| **e** | 527 500 | 15 | 8 |
| **f** | 292 600 | 5 | 8.5 |

## EXAMPLE 1G–4 Interpreting information from a loan graph

The graph shows the amount outstanding after $n$ months for a home loan of \$240 000 at 8.4% p.a. with a monthly repayment of \$2160.

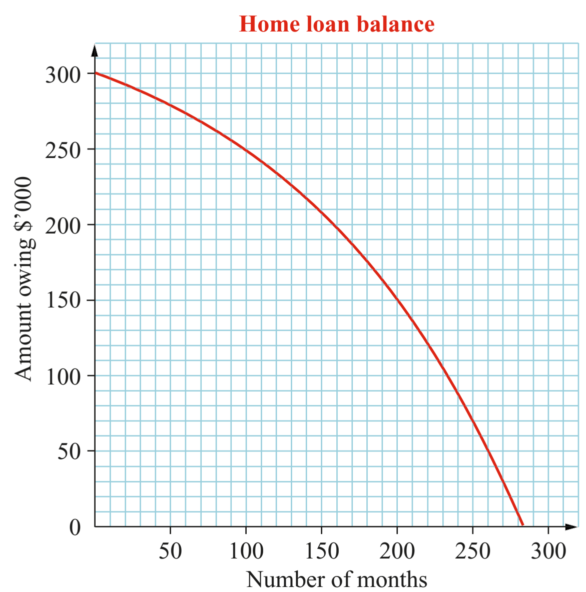
**a** How much is still owing after 75 months?
**b** When is the amount owing equal to \$180 000?
**c** When is the loan half paid?
**d** How long does it take to repay the loan in full?




| | Solve | Think | Apply |
|---|---|---|---|
| **a** | About \$195 000 | From the graph, when: $n = 75$ months, $A \approx \$195\,000$ | Locate the appropriate point on the graph. |
| **b** | About 90 months | From the graph, when: $A = \$180\,000$, $n \approx 90$ months | |
| **c** | About 145 months | From the graph, when: $A = \frac{1}{2} \times 240\,000 = \$120\,000$, $n \approx 145$ | |
| **d** | About 215 months | From the graph, when: $A = \$0$, $n \approx 215$ months | |

**9** Use the graph from Example 1G–4 to answer these questions.
- **a** How much is owing after 100 months?
- **b** How much is owing after 125 months?
- **c** When is the amount owing \$100 000?
- **d** When is the amount owing \$70 000?

**10** Use this graph of the amount owing on a home loan on the right to answer the following questions.
- **a** How much was borrowed?
- **b** How much is still owing after 100 months?
- **c** How much is still owing after 200 months?
- **d** How much is still owing after $12\frac{1}{2}$ years?
- **e** When is the amount owing \$200 000?
- **f** When is the amount owing \$50 000?
- **g** When is the loan half paid?
- **h** How long does it take to repay the loan in full?



**11** The amount $\$A_n$ owing on a home loan after $n$ months is shown in the table below.

| $n$ | 0 | 60 | 120 | 180 | 240 | 300 |
|---|---|---|---|---|---|---|
| $A_n$ | 450 000 | 404 720 | 346 920 | 261 260 | 150 135 | 0 |

Plot the values in the table and draw a smooth curve through them to draw a graph of amount owing versus time for this loan. Use the graph to answer these questions.

**a** How much was borrowed?
**b** How much is still owing after 100 months?
**c** How much is still owing after 18 years?
**d** When is the amount owing $300 000?
**e** When is the loan half paid?
**f** How long does it take to repay the loan in full?

**12** Here is a personal loan table showing the monthly repayments for amounts from $5000 to $15 000, over 1 to 3 years. The loan is monthly reducible with a rate of 10.45% p.a.

| Years | Amount borrowed | | | | | |
|---|---|---|---|---|---|---|
| | $5000 | $7000 | $9000 | $11 000 | $13 000 | $15 000 |
| 1 | $440.63 | $616.88 | $793.13 | $969.38 | $1145.63 | $1321.88 |
| 2 | $231.76 | $324.47 | $417.18 | $509.88 | $602.59 | $695.29 |
| 3 | $162.39 | $227.35 | $292.31 | $357.27 | $422.23 | $487.18 |

Find the monthly repayment on these personal loans.

**a** $7000 over 3 years
**b** $11 000 over 2 years
**c** $13 000 over 1 year
**d** $15 000 over 3 years

**13** The financial institution from Question **12** also has a higher interest rate for loans of more than 3 years. The table below shows the monthly repayments for personal loans for amounts from $5000 to $15 000, over 4 to 7 years. The loan interest is calculated on the monthly balance. The rate is 10.95% p.a. for loans of 4 to 5 years and 11.45% p.a. for loans of 6 to 7 years.

| Years | Amount borrowed | | | | | |
|---|---|---|---|---|---|---|
| | $5000 | $7000 | $9000 | $11 000 | $13 000 | $15 000 |
| 4 | $129.11 | $180.75 | $232.39 | $284.03 | $335.68 | $387.32 |
| 4.5 | $117.68 | $164.76 | $211.83 | $258.91 | $305.98 | $353.05 |
| 5 | $108.59 | $152.02 | $195.46 | $238.89 | $282.33 | $325.76 |
| 6 | $96.33 | $134.86 | $173.39 | $211.92 | $250.45 | $288.98 |
| 6.5 | $91.18 | $127.65 | $164.12 | $200.60 | $237.07 | $273.54 |
| 7 | $86.80 | $121.52 | $156.24 | $190.96 | $225.68 | $260.40 |

Find the monthly repayment on these loans.

**a** $7000 over 4.5 years
**b** $11 000 over 7 years
**c** $13 000 over 5 years
**d** $15 000 over 6.5 years

**14** The table below shows the personal loan monthly repayments for loans over \$15 000. The interest rate increases as the time period increases. It is 9.95% p.a. for 1 to 3 years, 10.45% p.a. for 4 to 5 years and 10.95% p.a. for 6 to 7 years.

| Years | Amount borrowed | | | | | |
|---|---|---|---|---|---|---|
| | \$15 000 | \$16 000 | \$17 000 | \$18 000 | \$19 000 | \$20 000 |
| 1 | \$1318.39 | \$1406.28 | \$1494.17 | \$1582.07 | \$1669.96 | \$1757.85 |
| 1.5 | \$900.51 | \$960.54 | \$1020.58 | \$1080.61 | \$1140.65 | \$1200.68 |
| 2 | \$691.83 | \$737.95 | \$784.07 | \$830.19 | \$876.32 | \$922.44 |
| 2.5 | \$566.82 | \$604.61 | \$642.40 | \$680.19 | \$717.98 | \$755.76 |
| 3 | \$483.66 | \$515.90 | \$548.14 | \$580.39 | \$612.63 | \$644.87 |
| 4 | \$383.69 | \$409.27 | \$434.85 | \$460.43 | \$486.01 | \$511.58 |
| 4.5 | \$349.38 | \$372.67 | \$395.96 | \$419.25 | \$442.54 | \$465.83 |
| 5 | \$322.04 | \$343.51 | \$364.98 | \$386.44 | \$407.91 | \$429.38 |
| 6 | \$285.13 | \$304.14 | \$323.14 | \$342.15 | \$361.16 | \$380.17 |
| 6.5 | \$269.64 | \$287.61 | \$305.59 | \$323.56 | \$341.54 | \$359.51 |
| 7 | \$256.44 | \$273.54 | \$290.63 | \$307.73 | \$324.83 | \$341.92 |

**a** Find the monthly repayment on these loans.

**i** \$17 000 over 4.5 years **ii** \$19 000 over 3 years

**iii** \$20 000 over 5 years **iv** \$16 000 over 2.5 years

**v** \$18 000 over 6.5 years

**b** \$17 000 is borrowed over 4 years.

**i** Find the total amount repaid over the term of the loan.

**ii** Find the amount of interest paid over the 4 years.

**iii** Find the equivalent simple interest rate over the 4 years.

**15** Carla and Ian take out a personal loan of \$15 000 to furnish their home. Interest is 14.4% p.a. monthly reducible. There is a loan establishment fee of \$390 and a monthly account-keeping fee of \$10 per month. Their monthly repayment is \$413.

Complete the following table to calculate how much they still owe after 6 months.

| Month | Principal at start of the month | Interest | Account-keeping fee | Monthly repayment | Balance at end of month |
|---|---|---|---|---|---|
| 1 | 15 000 + 390 = \$15 390 | \$184.68 | \$10 | \$413 | \$15 171.68 |
| 2 | \$15 171.68 | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |

**16** Harry takes out a personal loan of $18 000 to buy a car. Interest is 15.6% p.a. monthly reducible. There is a loan establishment fee of $260 and a monthly account-keeping fee of $8 per month. His monthly repayment is $434. Complete the following table to calculate how much he still owes after 6 months.

| Month | Principal at start of the month | Interest | Account-keeping fee | Monthly repayment | Balance at end of month |
|---|---|---|---|---|---|
| 1 | 18 000 + ___ = ___ | | | | |
| 2 | | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |
| 6 | | | | | |

SPREADSHEET APPLICATION

We can use a spreadsheet to model reducing-balance loans. Here we will create a spreadsheet in which you can modify the loan amount, the interest rate (p.a.) and the monthly repayment amount for a reducing-balance loan.

Type in the headings in column A and row 5, and enter in the formulas into cells B6, C6, D6, E6 and B7, as shown.

These formulas can be filled down into the subsequent cells as required.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| **1** | Loan amount ($) | | | | |
| **2** | Interest rate (%p.a.) | | | | |
| **3** | Monthly repayment ($) | | | | |
| **4** | | | | | |
| **5** | Month | Principal at start of the month ($) | Monthly interest ($) | Monthly repayment ($) | Balance ($) |
| **6** | 1 | =B1 | =B6*(B$2/1200) | =B$3 | =B6+C6-D6 |
| **7** | 2 | =E6 | | | |

To help you, you might like to use the prepared spreadsheet (Spreadsheet 1G) provided on your obook assess.

*Note*: The loan has been repaid in full when the balance in column E is either zero or negative.

**17 a** Tanya takes out an $8000 reducing-balance travel loan to pay for her dream holiday. The interest rate on the loan is 7.5% p.a. and she pays back the minimum monthly repayment of $300. What is the term of the loan?

**b** If the interest rate was 8% instead of 7.5%, how much longer would it take to pay back the loan?

**18 a** Patrick takes out a $25 000 reducing-balance loan to help pay for an extension to his home. The interest rate on the loan is 6.3% and the minimum monthly repayment amount is $490. If Patrick only paid the minimum monthly repayment, how long would it take him to repay the loan?

**b** Patrick wants to pay back the loan within 3 years. What is the minimum monthly amount he must pay back to achieve his goal? Round your answer up to the nearest $10.

**c** Patrick decides to pay back the amount calculated in part **b**, but before he starts repaying the loan the interest rate rises to 6.6%. Does this affect the time it will take him to pay back the loan?

# 1H Credit cards

These resources are available on your obook assess:
- **Spreadsheet 1H:** Create a credit card statement
- **assess quiz 1H:** Test your skills with an auto-correcting multiple-choice quiz

**credit card**
plastic card that allows the holder to purchase goods and services on credit

**Credit cards** are issued by banks and financial institutions as a convenient way for consumers to purchase goods and services from vendors. The vendor is paid by the bank and the bank recovers the money from the cardholder. Naturally, various fees and interest charges are applied by the banks for the use of this credit facility. The cardholder receives a monthly statement and must make a minimum payment of 2–5% of the balance owing. If the cardholder does not pay the account in full by the due date (for example, only the minimum payment is made), any amount outstanding will be carried over and interest charges will apply.

Credit cards usually have a higher rate of interest than most other consumer loans. Different rates of interest apply to purchases and cash advances for most cards. There is often an annual fee for the use of the card, and fees may be charged for exceeding the credit limit or making late payments. Many cards offer interest-free days under certain conditions.

When calculating the charges on a credit card payment, both the purchase date and the date of repayment must be included in the calculation.

Consider the monthly statements on the next page for a credit card that has up to 55 interest-free days. The minimum payment is the greater of $10 or 3% of the closing balance. The statement period is from the first to the last day of the month. The due date is 55 days from the first day of the statement period. The cardholder will not be charged any interest on retail purchases made during a statement period if the closing balance is paid in full by the due date and the closing balance of the previous statement in full by the due date.

*Note:*

- As the closing balance for each month is paid in full by the due date, there are no interest charges for March and April.
- The minimum payment in April is $10 since this is greater than 3% of $223 ($6.69).
- The due date for purchases made in March is 24 April, 55 days from 1 March (the start of the statement period).
- The number of interest-free days is up to 55. For example, the number of interest-free days for the purchase of furniture in March is 48 days (from 8 March to 24 April inclusive) and for the purchase of hardware is 39 days (from 17 March to 24 April inclusive).
- The interest-free period does not apply to cash advances (such as ATM withdrawals). These transactions attract interest from the day they appear on the statement.

| March statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| Mar 1 | Opening balance | 0 |
| Mar 8 | Furniture | 1680 |
| Mar 17 | Hardware | 67 |
| Opening balance: $0 | | |
| Closing balance: $1747 | | |
| Minimum payment due: $52.41 | | |
| Due date: 24 April | | |

| April statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| April 11 | Jeans | 88 |
| April 24 | Payment | −1747 |
| April 30 | Sunglasses | 135 |
| Opening balance: $1747 | | |
| Closing balance: $223 | | |
| Minimum payment due: $10 | | |
| Due date: 25 May | | |

| May statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| May 9 | Groceries | 48 |
| May 17 | Electrical goods | 136 |
| May 25 | Payment | −223 |
| Opening balance: $223 | | |
| Closing balance: $184 | | |
| Minimum payment due: $10 | | |
| Due date: 24 June | | |

## EXERCISE 1H Credit cards

Questions **1–3** refer to the credit card shown in the statements above.

**1** The minimum payment for this credit card is the greater of $10 or 3% of the closing balance. Calculate the minimum payment due on these closing balances.

**a** $96 **b** $390 **c** $1245 **d** $320

**2** For this credit card, what would be the due date for purchases made in:

**a** June? **b** December? **c** February (not a leap year)?

**3** How many interest-free days are available for the purchase of:

**a** jeans? **b** sunglasses? **c** groceries? **d** electrical goods?

**4** Complete these statements, given that the statement period is from the first to the last day of the month, the minimum payment is the greater of $10 or 4% of the closing balance, there are up to 55 interest-free days and the closing balance is paid in full on the due date.


| September statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Sep | Opening balance | 0 |
| 9 Sep | Clothes | 80 |
| 18 Sep | Make-up | 54 |
| Opening balance: | $0 | |
| Closing balance: | $___ | |
| Minimum payment due: | $___ | |
| Due date: | 25 October | |

| October statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 5 Oct | Shoes | 180 |
| 25 Oct | Payment | –___ |
| 29 Oct | Television set | 967 |
| Opening balance: | $___ | |
| Closing balance: | $___ | |
| Minimum payment due: | $___ | |
| Due date: | _____ | |

| November statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 10 Nov | Groceries | 48 |
| 16 Nov | DVDs | 66 |
| ___ Nov | Payment | –___ |
| Opening balance: | $___ | |
| Closing balance: | $___ | |
| Minimum payment due: | $___ | |
| Due date: | _____ | |

> **!** The interest charged on a credit card purchase is usually compounded on a daily basis. As such, a credit card is an example of a reducing-balance loan.

## EXAMPLE 1H–1 Calculating the balance on a credit card

Calculate the total amount due on an ATM cash withdrawal of $400 using a credit card if the full amount is repaid after 15 days. The annual percentage rate (APR) for cash for the card is 21.5% and there is a fee of 1.5% of the cash advance amount.

| Solve | Think | Apply |
| --- | --- | --- |
| Daily interest rate<br>$= \text{APR} \div 365$<br>$= 0.0589\ldots\%$<br>$= 0.000\,589\ldots$<br>Balance $= 400(1 + 0.000\,589\ldots)^{15}$<br>$= \$403.55$<br>Cash advance fee<br>$= \frac{1.5}{100} \times 400$<br>$= \$6$<br>Total amount to be repaid<br>$= 403.55 + 6$<br>$= \$409.55$ | Calculate the balance using $FV = PV(1 + r)^n$.<br>Calculate the cash advance fee. | Interest on a cash withdrawal is calculated daily from the date of the transaction, whether or not the card has an interest-free period.<br>Daily interest rate = APR ÷ 365 days<br>Balance<br>$= \text{withdrawal} \times (1 + \text{daily interest rate})^{\text{number of days}}$<br>Total amount due = balance + cash advance fee. |

**5** Complete the following to calculate the total amount due on an ATM cash withdrawal of $500 using a credit card if the full amount is repaid after 23 days. The annual percentage rate for cash for the card is 22.9% and there is a fee of 1.5% of the cash advance amount.

Daily interest rate $=$ ___% $\div 365 =$ ___% $=$ ___

Balance $= \square \times (1 + \square)^{\square} = \$$___

Cash advance fee $= \frac{\square}{100} \times 500 = \$$___

Total amount due $=$ ___ $+$ ___ $= \$$___

**6** Calculate the total amount due on an ATM cash withdrawal of $450 using a credit card if the amount is repaid after 17 days. The annual percentage rate for cash is 20.9% and there is a fee of 1.5% of the cash advance amount.

**7** Calculate the total amount due on an over-the-counter cash withdrawal of $150 using a credit card if the full amount is repaid after 21 days. The annual percentage rate for cash is 19.8% and the cash advance fee is the greater of $2.50 or 1.5% of the cash advance amount.

## EXAMPLE 1H–2 Calculating a credit card balance after multiple purchases

Calculate the balance on this credit card at the end of April given that the annual percentage rate is 19.9%. Assume that there is no interest-free period on these purchases.

| April statement | | |
| --- | --- | --- |
| **Date** | **Details** | **Amount ($)** |
| 1 Apr | Opening balance | 37 |
| 7 Apr | Purchase | 53 |
| 16 Apr | Purchase | 29 |
| 23 Apr | Payment | −80 |

| Solve | Think |
|---|---|
| $r = 19.9\% \div 365$<br>$= 0.0545...\%$<br>$= 0.000\,545...$<br>1 to 6 April (6 days):<br>$PV = 37, n = 6$<br>$FV = 37(1 + 0.000\,545...)^6$<br>$= \$37.12$<br>7 to 15 April (9 days):<br>$PV = 37.12 + 53 = 90.12, n = 9$<br>$FV = 90.12(1 + 0.000\,545...)^9$<br>$= \$90.56$<br>16 to 22 April (7 days):<br>$PV = 90.56 + 29 = 119.26, n = 7$<br>$FV = 119.56(1 + 0.000\,545...)^7$<br>$= \$120.02$<br>23 to 30 April (8 days):<br>$PV = 120.02 - 80 = 40.02, n = 8$<br>$FV = 40.02(1 + 0.000\,545...)^8$<br>$= \$40.19$<br>The balance at the end of April is \$40.19. | Split the month into time periods depending on when transactions are made.<br>Calculate the balance at the end of each time period using, $FV = PV(1 + r)^n$, where $FV$ is the balance at the end of the time period, $PV$ is the balance at the start of the time period, $n$ is the number of days in the time period, and $r$ is the daily rate expressed as a decimal. |
| **Apply** | |
| Future balance = previous balance × (1 + daily interest rate)$^{\text{number of days}}$ | |

**8** Calculate the balance at the end of May, given that the annual percentage rate is 18.6%. Assume that there is no interest-free period on these purchases.

| May statement | | |
|---|---|---|
| **Date** | **Details** | **Amount (\$)** |
| 1 May | Opening balance | 147 |
| 8 May | Purchase | 88 |
| 20 May | Purchase | 133 |
| 25 May | Payment | −100 |

**9** Calculate the balance due at the end of September, given that the annual percentage rate is 18.8% and there is no interest-free period on these purchases.

| September statement | | |
|---|---|---|
| **Date** | **Details** | **Amount (\$)** |
| 1 Sep | Opening balance | 135 |
| 10 Sep | Purchase | 59 |
| 19 Sep | Purchase | 136 |
| 24 Sep | Payment | −100 |

**!** If you have a credit card with an interest-free period, then these purchases do not appear on the balance, when calculating the interest on the balance, until the end of the interest-free period.

## EXAMPLE 1H–3 Calculating a credit card balance including interest-free period

Calculate the balance on this credit card at the end of August, given that the annual percentage rate is 17.9%. There is an interest-free period of 55 days for purchases and a 1.5% fee for cash advances (added on at the end of the month). Assume there is no interest-free period remaining on the opening balance.

| August statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Aug | Opening balance | 105 |
| 5 Aug | Purchase | 99 |
| 11 Aug | Cash advance | 100 |
| 21 Aug | Payment | −150 |

| Solve | Think |
|---|---|
| $r = 17.9\% \div 365$<br>$= 0.0490...\%$<br>$= 0.000\,490...$<br>1 to 4 August (4 days):<br>$PV = 105, n = 4$<br>$FV = 105\,(1 + 0.000\,490...)^4$<br>$= \$105.21$<br>5 to 10 August (6 days):<br>$PV = 105.21, n = 6$<br>$FV = 105.21\,(1 + 0.000\,490...)^6$<br>$= \$105.52$<br>11 to 20 August (10 days):<br>$PV = 105.52 + 100 = 205.52, n = 10$<br>$FV = 205.52\,(1 + 0.000\,490...)^{10}$<br>$= \$206.53$<br>21 to 31 August (11 days):<br>$PV = 206.53 - 150 = 56.53, n = 11$<br>$FV = 56.53\,(1 + 0.000\,490...)^{11}$<br>$= \$56.84$<br>Cash advance fee $= \frac{1.5}{100} \times 100$<br>$= \$1.50$<br>Final balance $= 56.84 + 99 + 1.50$<br>$= \$157.34$<br>The balance at the end of August is \$157.34. | Split the month into time periods depending on when transactions are made.<br>Calculate the balance at the end of each time period using $FV = PV(1 + r)^n$, where $FV$ is the balance at the end of the time period, $PV$ is the balance at the start of the time period (excluding interest-free purchases), $n$ is the number of days in the time period and $r$ is the daily rate expressed as a decimal.<br>Add the cost of the interest-free purchases and the cash advance fee to the balance at the end of the month. |

**Apply**

Future balance $=$ previous balance $\times$ $(1 + \text{daily interest rate})^{\text{number of days}}$

**10** Calculate the balance at the end of December, given that the annual percentage rate is 21.6%. Note: there is an interest-free period of 55 days for purchases (the council rates charge counts as a purchase). Assume there is no interest-free period remaining on the opening balance.

| December statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Dec | Opening balance | 87 |
| 7 Dec | Council rates | 488 |
| 14 Dec | Payment | −100 |
| 19 Dec | Purchase | 43 |
| 24 Dec | Payment | −100 |

**11** Calculate the balance on this credit card at the end of July. The interest rate on the credit is 22.9%, and there is an interest-free period of 55 days for purchases and a 1.5% fee for cash advances (added on at the end of the month). Assume there is no interest-free period remaining on the opening balance.

| July statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 July | Opening balance | 637 |
| 4 July | Purchase | 126 |
| 10 July | Cash advance | 200 |
| 14 July | Payment | −100 |
| 23 July | Purchase | 93 |
| 26 July | Payment | −150 |

**12** **a** Create a table to detail the following transactions for the month of November.
Opening balance = $517
Purchases on 6 November of $29, 13 November of $134, 19 November of $97 and 24 November of $166
Cash advances on 12 November of $200 and 25 November of $150
Payments on 7 November of $300 and 20 November of $100

**b** Calculate the closing balance for November, given that the annual percentage rate is 19.4% p.a. for both purchases and cash withdrawals, and there is no interest-free period for purchases and a 1.5% fee for cash advances (added on at the end of the month).

**c** What is the total interest and fees for November?

**d** Determine the minimum payment due if it is the greater of $25 or 4% of the closing balance.

**13** Use an internet calculator to find the time it would take to repay various credit card balances if you only make the minimum repayment each month. How much interest would be charged? How much can be saved by increasing your monthly repayment by $5, $10, etc.?

**14** Use the internet to compare credit card interest rates with interest rates for other loan types. Why do you think the interest rates for credit cards are typically higher than for other loans? Write a report on your findings.

**15** Zoe was comparing two different credit cards.

- Card A: $120 annual fee with an annual percentage rate of 15% p.a. for all purchases
- Card B: $75 annual fee with an annual percentage rate of 17.9% p.a. for all purchases

Zoe's credit card transactions for November were as shown in the table.
If there was no interest-free period on either card, which credit card would be more cost-efficient for Zoe?
*Note:* calculate the proportion of the annual card fee for one month.

| November statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Nov | Opening balance | 0 |
| 2 Nov | Purchase | 115 |
| 5 Nov | Purchase | 90 |
| 11 Nov | Payment | −65 |
| 12 Nov | Purchase | 93 |
| 20 Nov | Payment | −100 |
| 25 Nov | Purchase | 229 |

**16** Gil was trying to choose between two different credit cards with the following costs.
Card A: $60 annual fee plus 24.9% p.a. for all purchases
Card B: $150 annual fee plus 16.9% p.a. for all purchases
For both cards there is no interest-free period for purchases and a 1.5% fee for cash advances (added on at the end of the month).

Gil's credit card activity for March was as follows.

| March statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 March | Opening balance | 350 |
| 4 March | Cash advance | 100 |
| 13 March | Purchase | 49 |
| 17 March | Payment | −65 |
| 25 March | Purchase | 180 |

Taking into account the proportion of the annual fee for March, which credit card saves Gil more money, and by how much?

**17** **a** Anita has a credit card balance of $1650 at the end of April. The minimum payment due is the greater of $50 or 3% of the closing balance. How much is the minimum payment?

**b** Anita only makes the minimum payment at the end of April. If the annual percentage rate is 21.9%, how much interest is charged on April's remaining balance in May?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

SPREADSHEET APPLICATION

We can use a spreadsheet to create a credit card statement. Here we will create a spreadsheet in which you can add different purchases, the number of days since the purchase and the interest rate (p.a.) for a credit card.

- Type in the headings in rows 1 and 2, and enter in the formulas into cells D3 and D4, as shown.
- The formula in cell D4 can be filled down into the subsequent cells as required.
- Enter in the interest rate into cell B1, the name of the items into column A (starting at A4), the amount of the items into column B (starting at B3) and the days into the month for that transaction into column C (starting at C4).
- Finish each statement with an 'End of month' item. For this item, in the 'Days into month' cell enter the days in the month +1.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | Interest rate (% p.a.) | | | |
| 2 | Item | Amount ($) | Days into month | Balance |
| 3 | Opening balance | | 1 | =B3 |
| 4 | | | | =D3*(1+(B$1/36500)^(C4−C3)+B4 |
| 5 | | | | |

*Note:* this spreadsheet assumes that there is no interest-free period for the credit card.

To help you, you might like to use the prepared spreadsheet (Spreadsheet 1H) provided on your obook assess.

**18** Jacinta's purchases for May are shown in the table. Use a spreadsheet to calculate the difference in Jacinta's balance at the end of March if she makes these purchases on a credit card with an interest rate of 19.9% p.a. rather than using one with an interest rate of 27.9% p.a. Assume that there is no interest-free period on either card.

| Date | Item | Cost |
|---|---|---|
| 1 March | Groceries | $122 |
| 4 March | School books | $249 |
| 8 March | Groceries | $135 |
| 13 March | Coffee machine | $399 |
| 15 March | Groceries | $148 |
| 19 March | Concert tickets | $198 |
| 22 March | Groceries | $110 |
| 29 March | Groceries | $132 |

# CHAPTER 1 REVIEW INVESTMENTS, DEPRECIATION AND LOANS

You should be able to:

- ✔ calculate the future value of a compound interest investment using both tables and the compound interest formula, including for time periods other than years
- ✔ calculate the present value of a compound interest investment given the future value, the interest rate and the number of compounding periods
- ✔ use a compounded value table to calculate the future or present value of a compound interest investment
- ✔ determine the best investment strategy given a range of options
- ✔ calculate the effect of inflation on the cost of goods
- ✔ calculate the appreciated value of items
- ✔ complete calculations associated with the purchase and sale of shares
- ✔ calculate the total dividend and dividend yield
- ✔ analyse share performance and extrapolate to estimate future share values
- ✔ calculate the depreciation of an asset using the declining-balance method
- ✔ calculate values in a table of loan repayments
- ✔ calculate the balance at a given time for a reducing-balance loan
- ✔ calculate credit card payments, interest charges and balances.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

1A ▶ **1** What is the future value when \$3550 is invested at 5.8% p.a. compounded annually for 4 years?

**A** \$4373.60 **B** \$823.60 **C** \$898.06 **D** \$4448.06

1A ▶ **2** What is the amount of interest earned when \$680 is invested for 5 years at 4.9% p.a. interest compounding annually?

**A** \$863.75 **B** \$183.75 **C** \$166.60 **D** \$846.60

1B ▶ **3** What is the future value when \$1630 is invested at 4.3% p.a. compounded quarterly for 7 quarters?

**A** \$2188.66 **B** \$558.66 **C** \$1756.68 **D** \$126.68

1B ▶ **4** What is the amount that must be invested at 4.8% p.a. interest compounding monthly to have \$4500 at the end of 3 years?

**A** \$3897.61 **B** \$3909.57 **C** \$832.16 **D** \$3899.84

Use the table on the right to answer questions **5** and **6**.

| Periods (years) | Interest rate per period | | | |
|---|---|---|---|---|
| | **1%** | **5%** | **10%** | **15%** |
| 4 | 1.041 | 1.216 | 1.461 | 1.750 |
| 5 | 1.051 | 1.276 | 1.611 | 2.011 |
| 6 | 1.062 | 1.340 | 1.772 | 2.313 |
| 7 | 1.072 | 1.407 | 1.949 | 2.660 |

1C **5** What is the future value of \$3500 invested at 5% p.a. interest compounding annually for 5 years?

**A** \$4256 **B** \$4466 **C** \$4690 **D** \$4924.50

1C **6** What is the future value of \$1300 invested at 20% p.a. interest compounding quarterly for 6 quarters?

**A** \$1380.60 **B** \$1742 **C** \$3006.90 **D** \$2303.60

1D **7** A box of cereal costs \$3.78 today. The price increases at the same rate as inflation. The rate of inflation for the next 3 years is predicted to be constant at 6% p.a. What is the expected cost of the cereal in 3 years' time?

**A** \$4.07 **B** \$4.25 **C** \$4.50 **D** \$4.77

1D **8** A signed Australia–Pakistan cricket bat used in Australia's win in the 2008 series was purchased for \$600. It increased in value by 3% p.a. What was the appreciated value in 2018, correct to the nearest dollar?

**A** \$806 **B** \$780 **C** \$618 **D** \$3600

1E **9** What is the total cost of purchasing 400 shares with a market price of \$15.05 each and brokerage of 2.5%?

**A** \$5869.50 **B** \$6020 **C** \$6170.50 **D** \$6020.25

1E **10** A dividend of 24 cents per share is paid on shares with a market value of \$3.25 each and a face value of \$1.00. What is the dividend yield?

**A** 7.38% **B** 13.54% **C** 24% **D** 3.25%

1F **11** A car is bought for \$18 500. It depreciates in value by 22% per year. What is the salvage value of the car after 4 years?

**A** \$2220 **B** \$4070 **C** \$6848 **D** \$9.53

1F **12** A car depreciates in value from \$36 800 to \$23 550 in 2 years. Using the declining-balance method, what is the annual rate of depreciation?

**A** 18% **B** 20% **C** 36% **D** 56%

1G **13** The table shows the monthly payment per \$1000 on a monthly reducible loan.

What is the monthly repayment on a loan of \$15 600 at 8.25% over 10 years?

**A** \$154.25 **B** \$183.98

**C** \$195.76 **D** \$191.34

| Term in years | 7.75% | 8% | 8.25% | 8.5% |
|---|---|---|---|---|
| 5 | 20.1570 | 20.2765 | 20.3963 | 20.5164 |
| 10 | 12.0011 | 12.1328 | 12.2653 | 12.3985 |

Use the credit card statement on the right to answer questions **14** and **15**.

1H **14** If the annual percentage rate is 19.6% and there is no interest-free period on purchases, what would the balance at the end of August be?

**A** \$291.00 **B** \$293.63

**C** \$297.75 **D** \$300.72

1H **15** If there was an interest-free period of 40 days on purchases, what would the balance at the end of August be? Assume there is no interest-free period remaining on the opening balance.

**A** \$291.00 **B** \$293.66

**C** \$297.75 **D** \$300.72

| August statement | | |
|---|---|---|
| **Date** | **Details** | **Amount (\$)** |
| 1 Aug | Opening balance | 246 |
| 10 Aug | Water rates | 395 |
| 16 Aug | Purchases | 83 |
| 21 Aug | Payment | −500 |
| 27 Aug | Purchases | 67 |

# REVIEW SET 1

1 a Use the compound interest formula to calculate the future value of a fixed-term investment of $6000 over 5 years at 4.7% p.a., with interest compounding yearly.
b Find the total interest earned.

2 What amount must be invested at 7.5% p.a. interest compounding annually to have $5000 at the end of 6 years?

3 Calculate the future value when $3220 is invested at 2% per month compounded monthly for 7 months.

4 In three successive years the annual rate of inflation was 3.3%, 4.1% and 2.9%. How much would you expect to pay after these 3 years for a dozen eggs if they cost $5.20 at the beginning of this period?

5 A new tractor is purchased for $63 000. It depreciates in value at a rate of 18% per year.
a Calculate the salvage value of the tractor after 4 years.
b By what amount has the tractor depreciated in value over the 4 years?

6 Calculate the total cost of purchasing 400 shares with a market price of $10.35 and brokerage of 2.5%.

7 Calculate the amount of the dividend on 800 shares with a face value of $1.10 if the dividend is 6.5%.

8 Use the table in Example 1G–1 on page 36 to find the maximum loan that can be taken if you can afford the following monthly repayments.
a $1450 b $1900 c $2500 d $2700

9 Use the table in Example 1G–3 on page 39, which shows the payments per $1000 on a monthly reducible loan, to find the monthly payment on a loan of $70 000 over:
a 20 years at 7.75% b 25 years at 8.25%

10 The minimum payment on a credit card is the greater of $10 or 4% of the closing balance. Calculate the minimum payment on these closing balances.
a $520 b $240

11 A credit card offers 55 days interest free from the start of the statement period. The statement period is from the first to the last day of the month. Assuming that all conditions necessary for the interest-free days have been satisfied, determine the due date for purchases made in these months.
a June b October

12 A business owner must repay a $160 000 interest-free loan in full in 5 years' time. What is the single amount that he should invest now at 5.4% p.a. interest compounded monthly in order to be able to pay the debt?

# REVIEW SET 2

1 Use the compound interest formula to calculate the future value of a fixed-term investment of $2700 over 8 years at 4.99% p.a. interest compounding quarterly.

2 Calculate the amount that must be invested at 3.8% p.a. interest compounding quarterly to have $1350 at the end of 3 years.

3 Calculate the future value when $7000 is invested at 1% per quarter compounding quarterly for 5 quarters.

**4** In three successive years the annual rate of inflation was 2.8%, 3.1% and 3.8%. How much would you expect to pay after these 3 years for a box of chocolates that cost $24.95 at the beginning of this period?

**5** A dividend of 5.95 cents per share is paid on shares with a market value of $1.64. Find the dividend yield.

**6** Calculate the amount of dividend on 4000 shares with a face value of 60 cents if the dividend is 6.0%.

**7** A new car is purchased for $29 000. It depreciates in value at a rate of 21% per year.

**a** Calculate the salvage value of the car after 3 years.

**b** By what amount has the car depreciated in value over the 3 years?

**8** A car depreciates in value from $33 000 to $19 000 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

**9** **a** Use the table in Example 1G–1 on page 36 to find the monthly repayment on $280 000 over 20 years.

**b** Use the table in Example 1G–1 on page 36 to find the monthly repayment on $320 000 over 20 years.

**c** How much larger is the monthly repayment on $320 000 than on $280 000 over 20 years?

**10** The graph in Example 1G–4 on page 40 shows the amount outstanding after $n$ months for a home loan of $240 000 at 8.4% p.a. with a monthly repayment of $2160. Use this graph to answer the following questions.

**a** How much is owing after 140 months?

**b** When is the amount owing $150 000?

**11** The following credit card statement is for the month of March.

| March statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Mar | Opening balance | 721 |
| 9 Mar | Purchase | 356 |
| 19 Mar | Purchase | 49 |
| 24 Mar | Payment | −600 |
| 27 Mar | Purchase | 73 |

If the annual percentage rate is 21.6% and there is no interest-free period on purchases, calculate the balance at the end of March.

## REVIEW SET 3

**1** **a** Use the compound interest formula to calculate the future value of a fixed-term investment of $2458 over 2 years at 6.95% p.a. interest compounding yearly.

**b** Find the total interest earned.

**2** Calculate the amount when $5200 is invested at 2% per quarter compounded quarterly for 7 quarters.

**3** In three successive years the annual rate of inflation was 7.4%, 4.6% and 5.8%. How much would you expect to pay after these 3 years for a block of cheese that cost $4.70 at the beginning of this period?

**4** Some office furniture is purchased for $15 000. It depreciates in value at a rate of 13% per year.

**a** Calculate the salvage value of the office furniture after 3 years.

**b** By what amount has the office furniture depreciated in value over the 3 years?

5 A car depreciates in value from \$78 000 to \$41 000 in 3 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

6 Calculate the amount obtained when 3000 shares are sold at \$1.90 if brokerage is \$27.50.

7 Calculate the amount of dividend on 1800 shares with a face value of 50 cents if the dividend is 13.5%.

8 Use the table in Example 1G–1 on page 36 to find the monthly repayment on a loan of \$400 000 over:

**a** 10 years **b** 30 years.

9 The graph in Exercise 1G question **10** on page 40 shows the amount owing on a home loan of \$300 000 against repayment periods. Use the graph to answer the following questions.

**a** How much is owing after 15 years?

**b** When is the amount owing \$190 000?

10 The minimum payment on a credit card is the greater of \$10 or 3% of the closing balance. Calculate the minimum payment on these closing balances.

**a** \$460 **b** \$310

11 Calculate the future value of \$50 000 invested for 6 years at 4.9% p.a. interest compounded monthly.

12 What single amount must be invested now at 6.2% p.a. interest compounded quarterly in order to have \$30 000 in 5 years' time?

## REVIEW SET 4

1 Use the compound interest formula to calculate the future value of a fixed-term investment of \$2950 over 4 years at 5.1% p.a. interest compounding quarterly.

2 Calculate the amount of dividend on 2500 shares with a face value of \$1.00 if the dividend is 7.6%.

3 Calculate the amount that must be invested at 6% p.a. interest compounding annually to have \$1900 at the end of 5 years.

4 Calculate the amount that must be invested at 3.2% p.a. interest compounding quarterly to have \$10 000 at the end of 7 years.

5 In three successive years the annual rate of inflation was 5.8%, 4.5% and 3.1%. How much would you expect to pay after these 3 years for a hamburger that cost \$3.20 at the beginning of this period?

6 A computer depreciates in value from \$2200 to \$1200 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

7 Calculate the total cost of purchasing 3000 mining shares with a market price of \$5.85 if brokerage is 2.5%.

8 A dividend of 28 cents per share is paid on shares with a market value of \$3.96. Find the dividend yield.

9 Use the table in Example 1G–1 on page 36, which shows home loan monthly repayments, to find the monthly repayment on a loan of \$320 000 over:

**a** 30 years **b** 25 years.

**10** The graph in Exercise 1G question **10** on page 40 shows the amount owing on a home loan of $300 000 against repayment periods. Use the graph to answer the following questions.

**a** How much is owing on the home loan after 90 months?

**b** When is the amount owing $120 000?

**11** The credit card statement on the right is for the month of June. If the annual percentage rate is 22.1%, calculate the balance at the end of June. Assume that there is no interest-free period on these purchases.

| June statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Jun | Opening balance | 589 |
| 8 Jun | Purchase | 135 |
| 15 Jun | Purchase | 68 |
| 21 Jun | Payment | −450 |
| 24 Jun | Purchase | 33 |

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** **i** Convert 8% p.a. to a rate per quarter as a decimal. (1 mark)

**ii** How many quarters in 5 years? (1 mark)

**iii** Calculate the amount $6250 grows to when invested at 8% p.a. compounding quarterly for 5 years. (2 marks)

**iv** How much interest is earned? (1 mark)

**b** Kelly purchases 300 bank shares with a market value of $37.90 each. Brokerage is 2.3%.

**i** Determine the total cost of purchasing the shares. (2 marks)

**ii** Kelly receives a dividend cheque for $246. Find the amount of dividend per share. (1 mark)

**iii** Calculate the dividend yield. (1 mark)

**c** Calculate the future value of $1000 that is invested at 6% p.a. for 7 years. (1 mark)

**d** Tournya needs $6200 in 5 years' time. What amount does he need to invest now at 6% p.a. compounding annually? (1 mark)

**e** Megan signed a contract to rent a house for $1200 per month at the beginning of 2017. Her monthly rent increases each year by the amount of inflation for that year. If the inflation rate for 2017 was 4.5% p.a. and the inflation rate for 2018 was 5.1% p.a, find her rent at the start of 2019. (2 marks)

**f** A home loan of $320 000 is taken out over 30 years at an interest rate of 8.5% p.a. monthly reducible. The monthly repayment is $2460.52. Complete the next line of the table below to find the amount owing on the loan at the end of the second month. (2 marks)

| Month | Principal at start of the month | Monthly interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | $320 000 | $2266.67 | $2460.52 | $319 806.15 |
| 2 | | | | |

**TOTAL: 15 marks**

FINANCIAL MATHEMATICS

# 2

# Non-right-angled trigonometry

The main mathematical ideas investigated are:

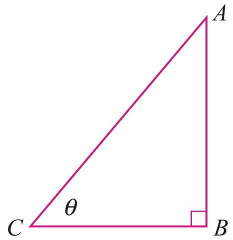
- finding unknown side lengths and angles in one or more right-angled triangles
- solving problems involving angles of elevation and depression
- solving problems involving compass and true bearings
- understanding different navigational methods
- determining the area of any triangle
- using the sine rule for non-right-angled triangles
- using the cosine rule for non-right-angled triangles
- constructing, interpreting and problem solving with compass radial surveys.

MEASUREMENT

MS-M6 Non-right-angled Trigonometry

# ARE YOU READY?

Use triangle $ABC$ shown below to answer questions **1–3**.




2A **1** In triangle $ABC$, which side is the hypotenuse?

**A** $AB$ **B** $AC$
**C** $BC$ **D** $CB$

2A **2** In triangle $ABC$, which side is the opposite side?

**A** $AB$ **B** $AC$
**C** $BC$ **D** $CB$

2A **3** In triangle $ABC$, which ratio represents the cosine of angle $\theta$?

**A** $\frac{AB}{AC}$ **B** $\frac{AB}{BC}$
**C** $\frac{BC}{AB}$ **D** $\frac{BC}{AC}$

2A **4** Using a calculator, what is the value of sin 42°, correct to four decimal places?

**A** 0.4200 **B** 0.6691
**C** 0.7431 **D** 0.9004

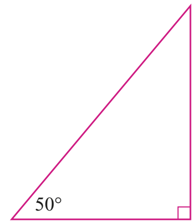
2A **5** Using a calculator, what is the size of the angle $\theta$ when $\tan \theta = 0.3561$, correct to nearest degree?

**A** 0° **B** 19° **C** 20° **D** 21°

2A **6** Which steps can you follow to solve $\frac{8}{x} = 0.75$ for $x$?

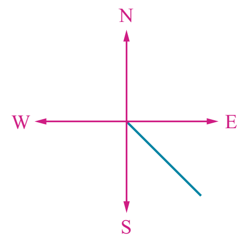
**A** Divide both sides by 8.
**B** Multiply both sides by 8 and then divide both sides by $x$.
**C** Divide both sides by $x$ and then multiply both sides by 0.75.
**D** Multiply both sides by $x$ and then divide both sides by 0.75.

2A **7** What is the size of the unknown angle in this triangle?




**A** 30° **B** 40°
**C** 50° **D** 90°

Use this diagram of a compass rose to answer questions **8** and **9**.



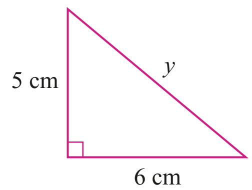

2C **8** What is the compass bearing shown in the diagram?

**A** SE **B** NE **C** ES **D** SW

2C **9** What is the true bearing (or three-figure bearing) shown in the diagram?

**A** 045° **B** 090°
**C** 135° **D** 225°

Use this triangle to answer questions **10** and **11**.




2D **10** What is the area of the triangle?

**A** 11 cm² **B** 15 cm²
**C** 30 cm² **D** 60 cm²

2F **11** Using Pythagoras' theorem, what is the length shown as $y$ in the triangle, correct to one decimal place?

**A** 7.0 cm **B** 7.8 cm
**C** 11 cm **D** 61 cm

2H **12** For a ratio scale of 1 : 100, what is the length on a scale drawing that represents an actual length of 2.4 m?

**A** 0.24 mm **B** 2.4 mm
**C** 2.4 cm **D** 24 cm

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1–3** **Support sheet 2A.1 Understanding trigonometric ratios**
**Q4–5** **Support sheet 2A.2 Using a calculator to find trigonometric and inverse trigonometric values**
**Q6** **Support sheet 2A.3 Solving linear equations where the unknown is the denominator of a fraction**
**Q7** **Support sheet 2A.4 Finding an unknown angle in a triangle**
**Q8–9** **Support sheet 2C.1 Understanding bearings**
**Q10** **Support sheet 2D.1 Area of a triangle**
**Q11** **Support sheet 2F.1 Using Pythagoras' theorem**
**Q12** **Support sheet 2H.1 Understanding ratio scales**

# 2A Review of right-angled trigonometry

These resources are available on your obook assess:

- **Interactive 2A.1:** Explore key ideas for right-angled trigonometry
- **Interactive 2A.2:** Explore key ideas for using trigonometry to find lengths
- **Interactive 2A.3:** Explore key ideas for using trigonometry to find angles
- **Worksheet 2A:** Practise your skills for finding unknown lengths and angles
- **assess quiz 2A:** Test your skills with an auto-correcting multiple-choice quiz

OI

GI

**trigonometry**
study of relationships between angles and side lengths of a triangle

**hypotenuse**
side that is opposite the right angle in a right-angled triangle

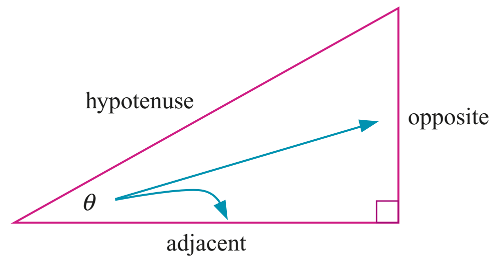
**opposite side**
side that is opposite the reference angle in a right-angled triangle

**adjacent side**
side that is adjacent or next to the reference angle in a right-angled triangle (but is not the hypotenuse)

In **trigonometry**, the sides of a right-angled triangle are given names in relation to the angles of the triangle.

The side opposite the right angle is called the **hypotenuse**.

The other sides are named relative to a reference angle, marked $\theta$ (theta) in this case.




The side that is directly opposite the angle $\theta$ is called the **opposite side**.

The side that is adjacent or next to the angle $\theta$ is called the **adjacent side**.

In previous years, we considered three trigonometric ratios: sine, cosine and tangent.

In a right-angled triangle, the sine of an angle $\theta$ is the ratio of the length of the side opposite $\theta$ to the length of the hypotenuse. This may be written as:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\text{O}}{\text{H}}$$

The cosine of an angle $\theta$ is the ratio of the length of the side adjacent to $\theta$ to the length of the hypotenuse. This may be written as:

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{A}}{\text{H}}$$

The tangent of an angle $\theta$ is the ratio of the length of the side opposite $\theta$ to the length of the side adjacent to $\theta$. This may be written as:

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{O}}{\text{A}}$$

To help remember the above definitions, try using this mnemonic: SOH CAH TOA.

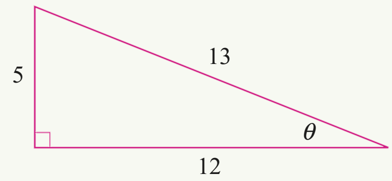
## EXAMPLE 2A–1 Writing trigonometric ratios

For this right-angled triangle, write these trigonometric ratios:

**a** $\sin\theta$

**b** $\cos\theta$

**c** $\tan\theta$



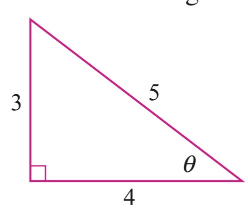

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\sin\theta = \dfrac{5}{13}$ | The length of the hypotenuse is 13 units, the length of the opposite side is 5 units and the length of the adjacent side is 12 units. | Use the trigonometric ratios: $\sin\theta = \dfrac{O}{H}$ $\cos\theta = \dfrac{A}{H}$ $\tan\theta = \dfrac{O}{A}$ |
| **b** | $\cos\theta = \dfrac{12}{13}$ | | |
| **c** | $\tan\theta = \dfrac{5}{12}$ | | |

# EXERCISE 2A Review of right-angled trigonometry

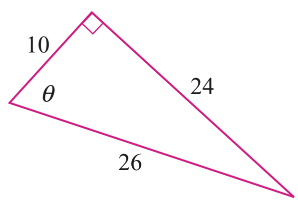


**1** For each of these right-angled triangles, write the trigonometric ratios $\sin\theta$, $\cos\theta$ and $\tan\theta$.

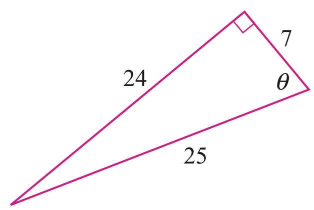
**a**




**b**




**c**




## EXAMPLE 2A–2 Finding trigonometric values for angles in degrees and minutes

Use your calculator to find these values, correct to four decimal places.

**a** $\sin 54°$ **b** $\tan 68.3°$ **c** $\cos 67°13'$ **d** $\sin 11°52'$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\sin 54° = 0.809\,0170$ $\approx 0.8090$ | One degree of angular measure can be divided into 60 smaller units called minutes: 1 degree = 60 minutes, written $1° = 60'$ 1 minute = 60 seconds, written $1' = 60''$ To find $\cos 67°13'$ on a CASIO calculator, press cos 67 °′″ 13 °′″ •. | Use the appropriate keys on your calculator to enter degrees and minutes. |
| **b** | $\tan 68.3° \approx 2.5129$ | | |
| **c** | $\cos 67°13' \approx 0.3872$ | | |
| **d** | $\sin 11°52' \approx 0.2056$ | | |

**2** Use your calculator to find these values, correct to four decimal places.

| | | | |
|---|---|---|---|
| **a** $\sin 47°$ | **b** $\cos 52°$ | **c** $\tan 66°$ | **d** $\sin 77.3°$ |
| **e** $\tan 13.7°$ | **f** $\cos 77.9°$ | **g** $\tan 74.9°$ | **h** $\sin 37.28°$ |
| **i** $\sin 43°45'$ | **j** $\cos 64°3'$ | **k** $\tan 34°26'$ | **l** $\cos 60°55'$ |
| **m** $\tan 71°41'$ | **n** $\sin 60°27'$ | **o** $\tan 55°21'$ | **p** $\sin 79°34'$ |

## EXAMPLE 2A–3 Finding the angle for a given trigonometric ratio (decimal value)

Use your calculator to find the size of angle $\theta$, correct to the nearest minute, when:

**a** $\sin\theta = 0.7511$ **b** $\cos\theta = 0.6725$ **c** $\tan\theta = 0.3685$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\sin\theta = 0.7511$<br>$\theta \approx 48°41'$ | These steps are appropriate for a CASIO calculator.<br>Press SHIFT sin 0.7511 = 48.6857...<br>The answer is in degrees. To change to degrees and minutes, press °′″.<br>The display shows 48°41°8.71, which is 48°41′8.71″.<br>If the number of seconds is < 30 round down, if the number of seconds is ≥ 30 round up.<br>Rounding gives $\theta \approx 48°41'$. | Find the angle for a given trigonometric value using the inverse key on a calculator. Change the angle in degrees to degrees, minutes and seconds, and then round to the nearest minute. |
| **b** | $\cos\theta = 0.6725$<br>$\theta \approx 47°44'$ | SHIFT cos 0.6725 = 47.7396...<br>To change to degrees and minutes, press °′″.<br>The display shows 47°44°22.88.<br>Rounding gives $\theta \approx 47°44'$. | |
| **c** | $\tan\theta = 0.3685$<br>$\theta \approx 20°14'$ | SHIFT tan 0.3685 = 20.2288...<br>To change to degrees and minutes, press °′″.<br>The display shows 20°13°43.83.<br>Rounding gives $\theta \approx 20°14'$. | |

**3** Use your calculator to find the size of angle $\theta$, correct to the nearest minute, when:

**a** $\sin\theta = 0.5789$ **b** $\cos\theta = 0.1427$ **c** $\tan\theta = 0.5499$
**d** $\tan\theta = 0.4952$ **e** $\sin\theta = 0.3367$ **f** $\cos\theta = 0.5247$
**g** $\cos\theta = 0.6723$ **h** $\tan\theta = 0.4459$ **i** $\sin\theta = 0.0496$

## EXAMPLE 2A–4 Finding the angle for a given trigonometric ratio (fraction value)

Use your calculator to find the size of angle $\theta$, correct to the nearest minute, when:

**a** $\cos\theta = \frac{8}{13}$ **b** $\tan\theta = \frac{15.3}{11.2}$ **c** $\sin\theta = \frac{6}{19}$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\cos\theta = \frac{8}{13}$<br>$\theta \approx 52°1'$ | SHIFT cos 8 a b/c 13 = 52.0201...°<br>°′″ gives 52°1′12.46″, which rounds to 52°1′. | Use the fraction key on a calculator to enter the ratio. Change angle in degrees to degrees, minutes and seconds, and then round to the nearest minute. |
| **b** | $\tan\theta = \frac{15.3}{11.2}$<br>$\theta \approx 53°48'$ | SHIFT tan 15.3 a b/c 11.2 = 53.7948...°<br>°′″ gives 53°47′41.63″, which rounds to 53°48′. | |
| **c** | $\sin\theta = \frac{6}{19}$<br>$\theta \approx 18°25'$ | SHIFT sin 6 a b/c 19 = 18.4084...°<br>°′″ gives 18°24′30.53″, which rounds to 18°25′. | |

**4** Use your calculator to find the size of angle $\theta$, correct to the nearest minute, when:

**a** $\sin\theta = \frac{12}{19}$ **b** $\cos\theta = \frac{15}{22}$ **c** $\tan\theta = \frac{3}{8}$

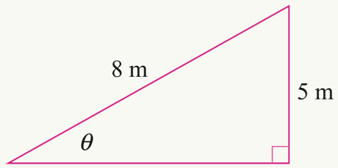
**d** $\tan\theta = \frac{9}{7}$ **e** $\sin\theta = \frac{7}{8}$ **f** $\cos\theta = \frac{5}{11}$

**g** $\cos\theta = \frac{27.6}{49.8}$ **h** $\tan\theta = \frac{18.1}{12.5}$ **i** $\sin\theta = \frac{6.2}{9.8}$

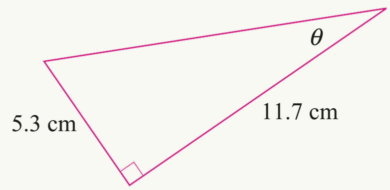
## EXAMPLE 2A–5 Finding the size of an unknown angle

Find the size of angle $\theta$ in each of these triangles, to the nearest minute.

**a**



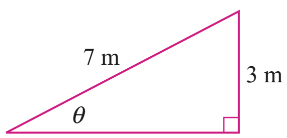

**b**



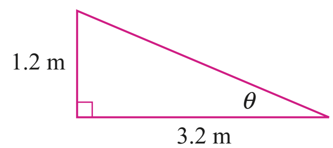

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\sin\theta = \frac{5}{8}$ $= 0.625$ $\theta = 38.682...^\circ$ $\approx 38^\circ 41'$ | The known side lengths are on the opposite side and the hypotenuse. The trigonometric ratio connecting these sides is the sine ratio. | Identify the sides with the given side lengths and use the appropriate trigonometric ratio. Change angle in degrees to degrees, minutes and seconds, and then round to the nearest minute. |
| **b** | $\tan\theta = \frac{5.3}{11.7}$ $= 0.45299...$ $\theta = 24.370...^\circ$ $\approx 24^\circ 22'$ | The known side lengths are on the opposite side and the adjacent side. The trigonometric ratio connecting these sides is the tangent ratio. | |

**5** Find the size of the angle $\theta$ in each of these triangles, to the nearest minute.

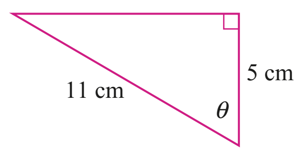
**a**



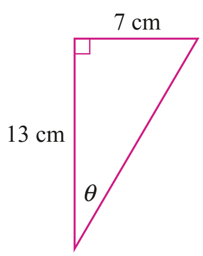

**b**



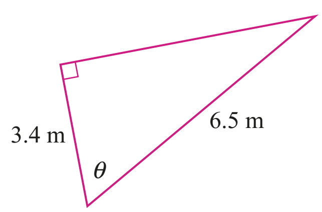

**c**



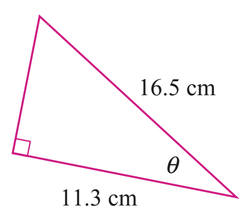

**d**



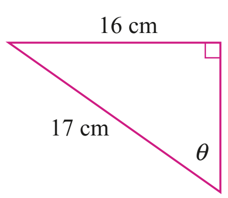

**e**



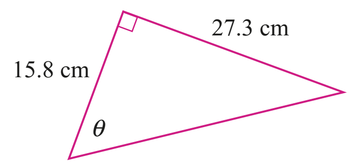

**f**



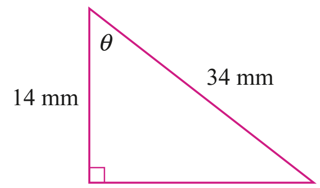

**g**




**h**



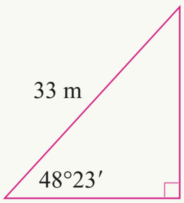

**i**



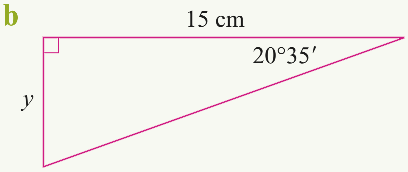
## EXAMPLE 2A–6 Finding an unknown side length (numerator)

Find the unknown side length labelled with a pronumeral in each of these triangles, correct to one decimal place.

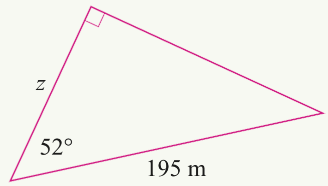
**a**




**b**



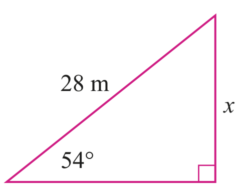

**c**



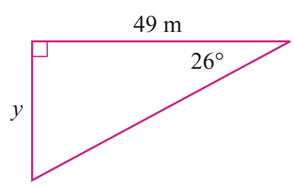

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\frac{x}{33} = \sin 48°23'$<br>$x = 33 \times \sin 48°23'$<br>$= 33 \times 0.7476...$<br>$= 24.67...$<br>$\approx 24.7\text{ m}$ | Relative to the given angle, $x$ is the length of the opposite side and 33 m is the length of the hypotenuse. The trigonometric ratio that connects these two sides is the sine ratio.<br>$\frac{x}{33} = \frac{\text{opposite}}{\text{hypotenuse}} = \sin 48°23'$ | Identify the sides with the unknown side length and the known side length and use the trigonometric ratio that connects them. Solve the resulting equation to find the unknown side length. |
| **b** | $\frac{y}{15} = \tan 20°35'$<br>$y = 15 \times \tan 20°35'$<br>$= 15 \times 0.3755...$<br>$= 5.633...$<br>$\approx 5.6\text{ cm}$ | Relative to the given angle, $y$ is the length of the opposite side and 15 cm is the length of the adjacent side. The trigonometric ratio that connects these two sides is the tangent ratio.<br>$\frac{y}{15} = \frac{\text{opposite}}{\text{adjacent}} = \tan 28°35'$ | |
| **c** | $\frac{z}{195} = \cos 52°$<br>$z = 195 \times \cos 52°$<br>$= 195 \times 0.6156...$<br>$= 120.053...$<br>$\approx 120.1\text{ m}$ | Relative to the given angle, $z$ is the length of the adjacent side and 195 m is the length of the hypotenuse. The trigonometric ratio that connects these two sides is the cosine ratio.<br>$\frac{z}{195} = \frac{\text{adjacent}}{\text{hypotenuse}} = \cos 52°$ | |

**6** Find the unknown side length labelled with a pronumeral in each of these triangles, correct to one decimal place.

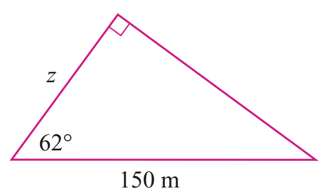
**a**



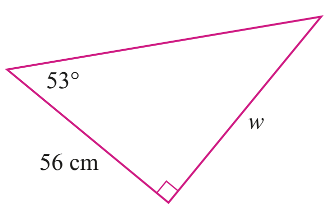

**b**



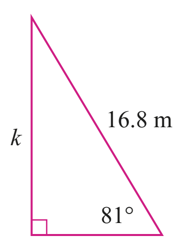

**c**



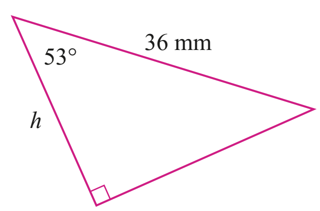

**d**



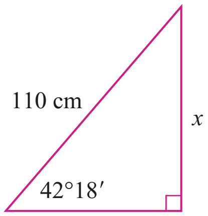

**e**



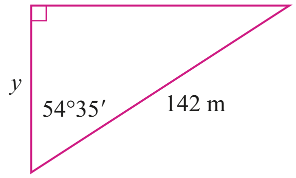

**f**



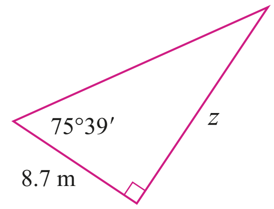
**g**




**h**



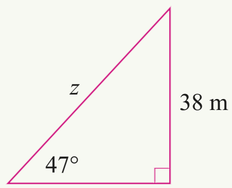

**i**



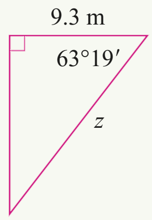

## EXAMPLE 2A–7 Finding the length of an unknown side (denominator)

Find the unknown side length labelled with a pronumeral in each of these triangles, correct to one decimal place.

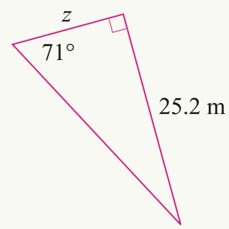
**a**




**b**




**c**

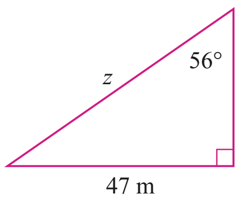



| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\frac{38}{z} = \sin 47°$<br>$38 = z \times \sin 47°$<br>$\frac{38}{\sin 47°} = z$<br>$z = 51.958...$<br>$\approx 52.0\,\text{m}$ | Relative to the 47° angle, 38 m is the length of the opposite side and $z$ is the length of the hypotenuse, hence use the sine ratio.<br>$\frac{\text{opposite}}{\text{hypotenuse}} = \frac{38}{z} = \sin 47°$<br>$38 = z \times \sin 47°$ (multiply both sides by $z$)<br>$z = \frac{38}{\sin 47°}$ (divide both sides by $\sin 47°$) | When the unknown side length is the denominator of a trigonometric equation, multiply both sides by the unknown side length and solve the resulting equation. |
| **b** | $\frac{9.3}{z} = \cos 63°19′$<br>$9.3 = z \times \cos 63°19′$<br>$\frac{9.3}{\cos 61°19′} = z$<br>$z = 20.709...$<br>$\approx 20.7\,\text{cm}$ | Relative to the 63°19′ angle, 9.3 m is the length of the adjacent side and $z$ is the length of the hypotenuse, hence use the cosine ratio.<br>$\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{9.3}{z} = \cos 63°19′$<br>$9.3 = z \times \cos 63°19′$ (multiply sides by $z$)<br>$z = \frac{9.3}{\cos 63°19′}$ (divide sides by $\cos 63°19′$) | |
| **c** | Third angle in the triangle<br>$= 180° - 90° - 71°$<br>$= 19°$<br>$\frac{z}{25.2} = \tan 19°$<br>$z = 25.2 \times \tan 19°$<br>$= 8.6770...$<br>$\approx 8.7\,\text{m}$ | Relative to the 71° angle, $z$ is the length of the adjacent side and 25.2 m is the length of the opposite side, hence use the tangent ratio.<br>$\frac{\text{opposite}}{\text{adjacent}} = \frac{25.2}{z} = \tan 71°$<br>We can use this equation to find $z$, but it is easier to use the third angle, 19°, with the tangent ratio. | When using the tangent ratio, determine which angle to use so that the unknown side length is in the numerator. |

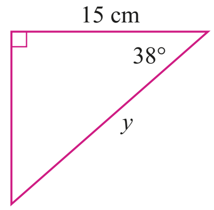
UNDERSTANDING, FLUENCY AND COMMUNICATING

**7** Find the unknown side length labelled with a pronumeral in each of these triangles, correct to one decimal place.

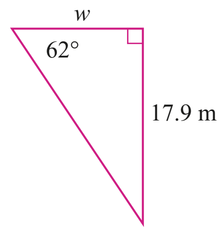
**a**



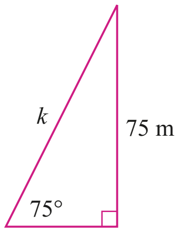

**b**



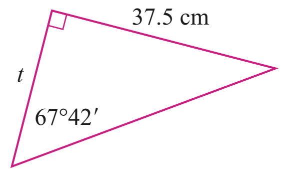

**c**



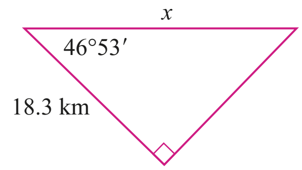

**d**



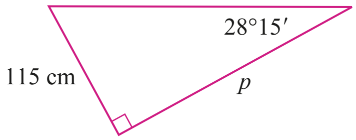

**e**



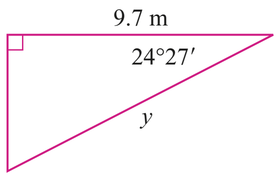

**f**



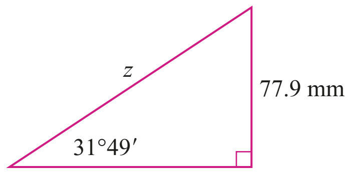

**g**




**h**




**i**



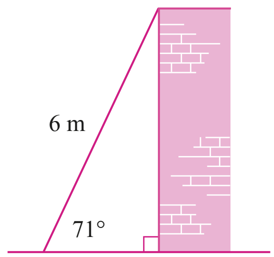

PROBLEM SOLVING, REASONING AND JUSTIFICATION

Unless stated otherwise, find all lengths correct to one decimal place and all angles to the nearest minute.

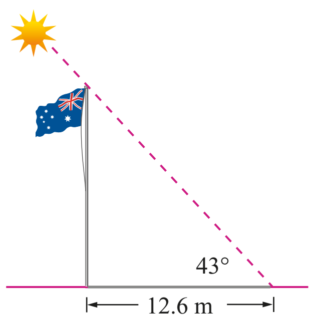
**8** A 6 m ladder makes an angle of 71° with the ground.

**a** How far does the ladder reach up the wall?

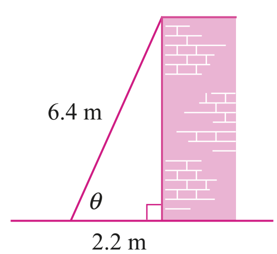
**b** How far is the foot of the ladder from the base of the wall?




**9** A flagpole casts a shadow 12.6 m long when the sun's rays make an angle of 43° with the ground. Find the height of the flagpole.

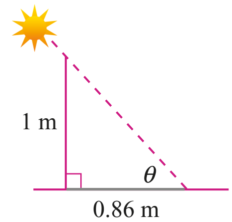



**10** The foot of a ladder 6.4 m long is placed 2.2 m from the base of a wall. Calculate the angle the ladder makes with the ground.

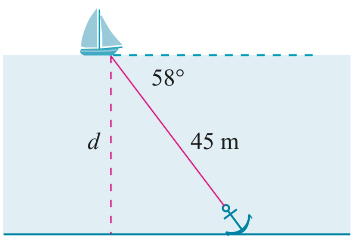


PROBLEM SOLVING, REASONING AND JUSTIFICATION

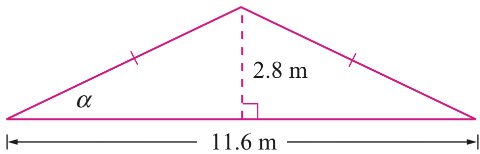
**11** A metre rule casts a shadow 0.86 m long. What is the angle of the sun's rays with the ground at this time?



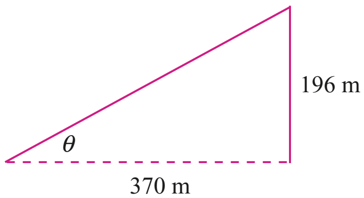

**12** The anchor rope of a boat is 45 m long. When it is let out fully, it makes an angle of 58° with the surface of the water. Calculate the depth of the water at this point.



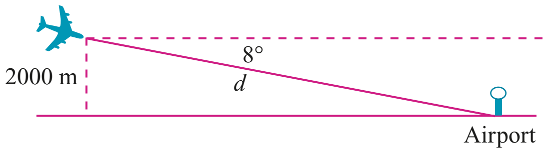

**13** The diagram shows the dimensions of the gable roof at the end of a house. Calculate the angle of pitch, α, of the roof.



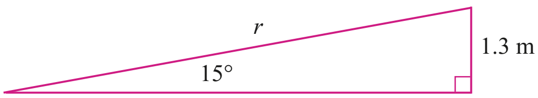

**14** A ski slope falls 196 m over a run of 370 m. Find the angle the slope makes with the horizontal.




**15** The pilot of a plane flying at an altitude of 2000 m notes that the angle from the horizontal line of sight to the base of the airport tower is 8°. How far, in a straight line to the nearest metre, is the plane from the tower?

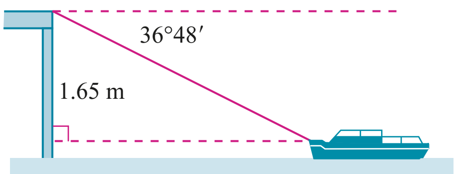



**16** A ramp, inclined at 15° to the ground, rises 1.3 m vertically to an entrance. How long is the ramp?




CHALLENGE

**17** A boat is tied to a wharf that is 1.65 m above the boat, as shown in the diagram. The rope makes an angle of 36°48′ with the horizontal. How long is the rope, to the nearest centimetre?




MEASUREMENT

# 2B Angles of elevation and depression

**angle of elevation**
angle between a horizontal line and the line of sight to an object above the horizontal

**angle of depression**
angle between a horizontal line and the line of sight to an object below the horizontal

These resources are available on your obook assess:
- **Interactive 2B:** Explore key ideas for angles of elevation and depression
- **Investigation 2B:** Apply your skills to finding the height of an object
- **assess quiz 2B:** Test your skills with an auto-correcting multiple-choice quiz

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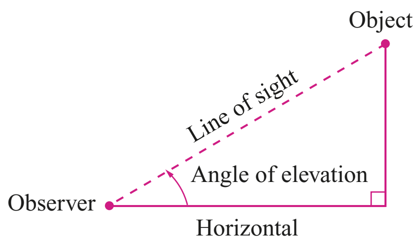
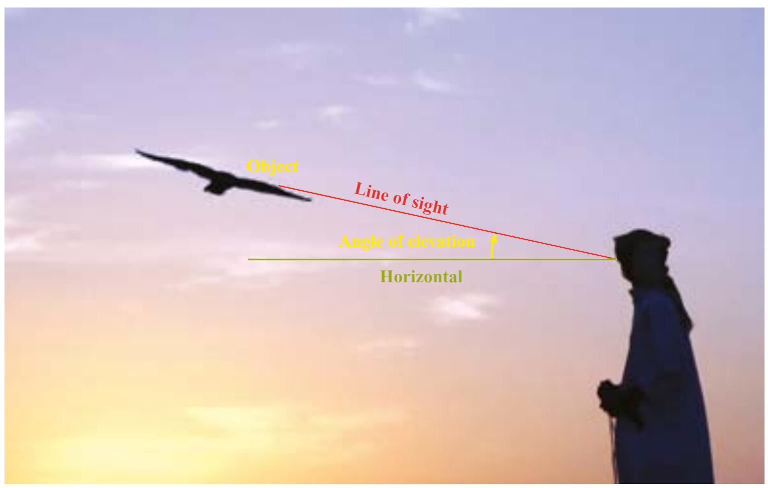
Right-angled triangle trigonometry is also used for problems involving an **angle of elevation** or an **angle of depression**.

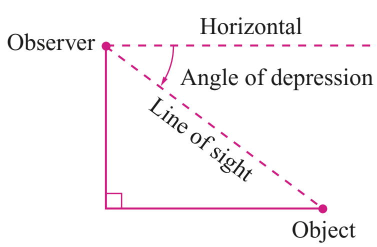
The angle of elevation of an object from an observer is the angle between the horizontal and the line of sight *up* to the object.

The angle of depression of an object from an observer is the angle between the horizontal and the line of sight *down* to the object.

*Note:* it is assumed that towers, buildings, trees or cliff faces are perpendicular to the horizontal ground or water surface described in problems involving angles of elevation or depression.

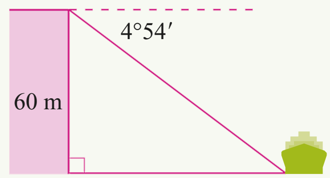


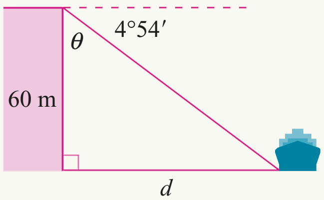




## EXAMPLE 2B–1 Calculating distance using an angle of elevation or depression

The angle of depression of a ship at sea from the top of a 60 m vertical cliff is 4°54′. Calculate the distance of the ship from the base of the cliff, to the nearest metre.



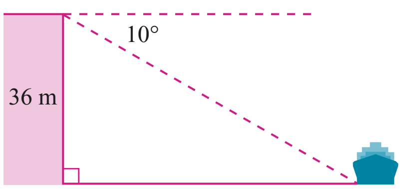

| Solve | Think | Apply |
| --- | --- | --- |
|  <br>$\theta = 90° - 4°54'$<br>$= 85°6'$<br>$\frac{d}{60} = \tan 85°6'$<br>$d = 60 \times \tan 85°6'$<br>$= 699.869...$<br>$\approx 700\,\text{m}$ | On the diagram, label the unknown distance as $d$ and the angle between the vertical cliff and the line of sight as $\theta$.<br>The angle between the vertical cliff and the horizontal through the top of the cliff is 90°.<br>Hence, $\theta = 90° - 4°54'$.<br>For the 85°6′ angle in the right-angled triangle, $d$ is the length of the opposite side and 60 m is the length of the adjacent side.<br>$\tan 85°6' = \frac{\text{opposite}}{\text{adjacent}}$<br>$= \frac{d}{60}$ | The angle of depression of an object from an observer is the angle between the horizontal and the line of sight down to the object. |

# EXERCISE 2B Angles of elevation and depression

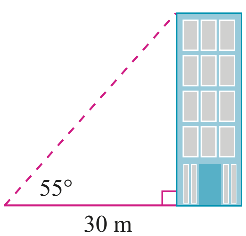
PROBLEM SOLVING, REASONING AND JUSTIFICATION

Unless stated otherwise, find all lengths correct to one decimal place and all angles to the nearest minute.

**1** The angle of depression of a boat at sea from the top of a vertical cliff 36 m high is 10°. Calculate the distance of the boat from the base of the cliff, to the nearest metre.



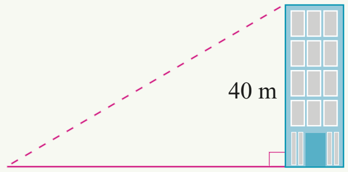

**2** The angle of elevation of the top of a building from a point 30 m from its base is 55°. Calculate the height of the building.

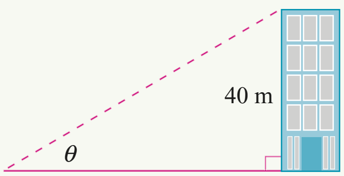



MEASUREMENT

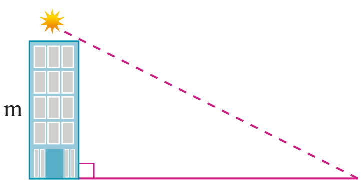
## EXAMPLE 2B–2 Using trigonometry to calculate the angle of elevation or depression

A building is 40 m high. Calculate the angle of elevation of the top of the building from a point that is 36 m from its base, correct to the nearest minute.

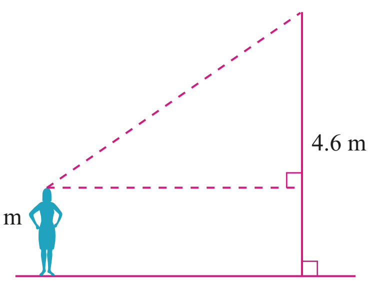



| Solve | Think | Apply |
|---|---|---|
|  <br>$\tan\theta = \frac{40}{36}$<br>$\theta \approx 48°1'$ | On the diagram, label the unknown angle as $\theta$. For $\theta$ in the right-angled triangle, 40 m is the length of the opposite side and 36 m is the length of the adjacent side.<br>Use $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. | The angle of elevation of an object from an observer is the angle between the horizontal and the line of sight up to the object. |

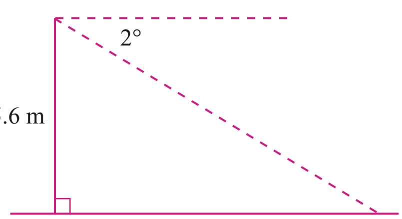
**3** A building 12.8 m high casts a shadow 7.2 m long. What is the angle of elevation of the Sun at this time?




**4** A person 1.5 m tall is standing 10 m from the base of a statue that is 4.6 m high. Calculate the angle of elevation from the person to the top of the statue.

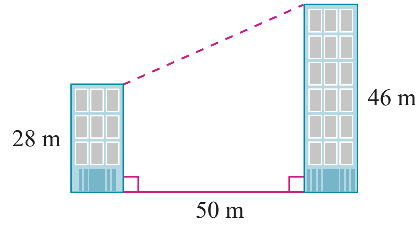



**5** From the top of a 15.6 m high observation tower, smoke is seen at an angle of depression of 2°. Assuming the land is flat, how far away is the fire?

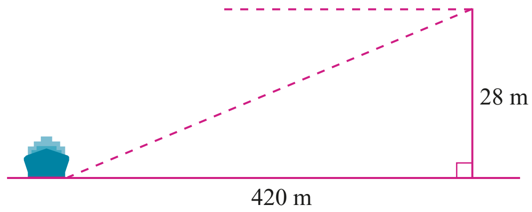


PROBLEM SOLVING, REASONING AND JUSTIFICATION

**6** Two buildings, 28 m and 46 m high respectively, are 50 m apart. Calculate the angle of elevation from the top of the shorter building to the top of the other.



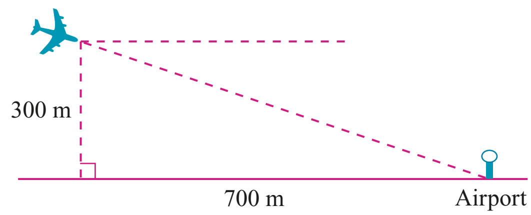

**7** A ship is 420 m from the base of a 28 m high cliff. Calculate the angle of depression of the ship from the top of the cliff.




**8** The pilot of a plane flying at an altitude of 300 m observes an airport runway that is 700 m horizontally from the plane.

**a** What is the angle of depression from the plane to the runway?

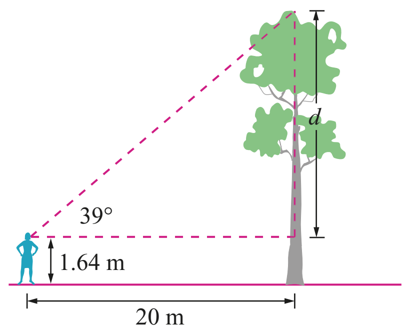
**b** What is the actual distance that the plane has to fly to reach the runway?



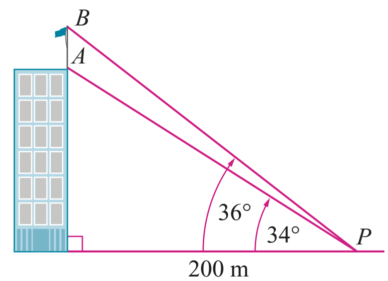

**9** From a point 20 m from the base of a tree, Sophia measures the angle of elevation of the top of the tree to be 39°. Her eye level is 1.64 m above the ground.

**a** Calculate the distance $d$ in the diagram.

**b** Hence find the height of the tree.




**10** From a point $P$, 200 m from the base of a building, the angles of elevation of the bottom and top of a flagpole on top of the building are 34° and 36° respectively. Calculate the height, $AB$, of the flagpole.




PRACTICAL TASKS

**11** Using the method of question **9**, find the height of buildings, trees or the flagpole at your school.

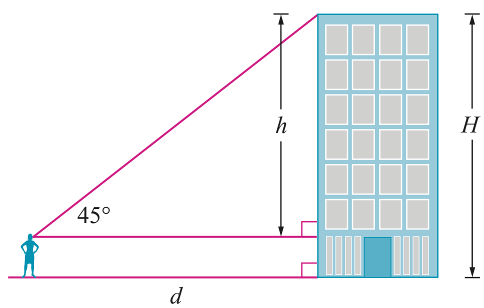
**a** Choose one of your school buildings that is on level ground.

**b** Walk away from the base of the building and, by trial and error, find the point on the ground from which the angle of elevation of the top of the building is 45°.

**c** Measure this distance, shown as $d$ on the diagram.

**d** Hence find the height $h$ on the diagram.

**e** Measure your height to eye level and calculate the height $H$ of the building.




MEASUREMENT

# 2C Bearings and navigational methods

These resources are available on your obook assess:
- **Interactive 2C:** Explore key ideas for compass bearings and true bearings
- **Worksheet 2C:** Practise your skills with extra problems for compass bearings and true bearings
- **Investigation 2C:** Apply your skills with bearings to mapping your school grounds
- **assess quiz 2C:** Test your skills with an auto-correcting multiple-choice quiz

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**bearing**
direction from one position to another, using the angles around a magnetic compass

**compass bearing**
indicates direction as an angle measured from north or south towards east or west; e.g. N40°W represents a bearing of 40° from north towards west and S75°E represents a bearing of 75° from south towards east

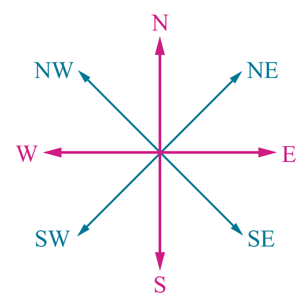
**true bearing**
indicates direction as an angle measured from north in a clockwise direction and written with three digits; e.g. east is represented by 090°T and south-west is represented by 225°T

Navigation involves determining position and direction. In the Year 11 Mathematics Standard course, you looked at describing position on the Earth's surface with latitude and longitude. Modern navigational methods rely on receiving electronic information from satellites via receivers. The Global Positioning System (GPS) is currently the world's most used satellite navigation system.

Navigating a course for a ship or an aircraft also requires accurate directions. These directions are usually given in the form of a **bearing**. There are two types of bearings: a **compass bearing** and a **true bearing**.

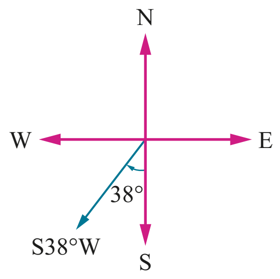
## Compass bearings

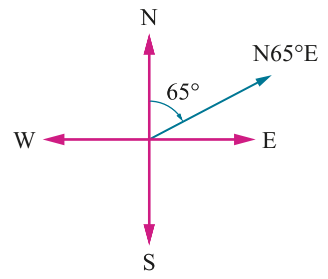
Compass bearings are stated using the compass rose. The compass rose has four major (or cardinal) points: north, east, south and west (N, E, S, W). The next four are in between these and are north-east, south-east, south-west and north-west (NE, SE, SW, NW). Note that north and south are written first and east and west are second. This compass rose shows these eight compass points.




For directions between these eight compass points, we start with N or S, whichever is closer, and end with E or W. Examples are S38° W or N65°E.

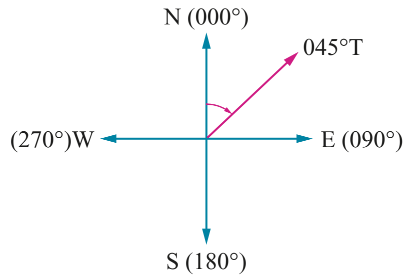
S38° W means face south, then turn 38° to west. N65°E means face north, then turn 65° to east.



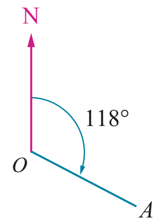
## True bearings

True bearings are measured as an angle of turn. The agreed convention is that the direction of travel is measured by a clockwise rotation from north and written using three digits.




For example, the diagram on the right shows us that the bearing of the direction north is 000°T, north-east is 045°T, east is 090°T, south is 180°T and west is 270°T. (*Note:* the letter T indicates it is a true bearing.)

In the diagram on the right, the bearing of *A* from *O* is the measure of the angle between the line *OA* and the line through *O* in the north direction. The bearing of *A* from *O* is written as 118°T.



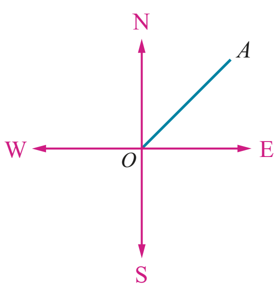

# EXERCISE 2C Bearings and navigational methods



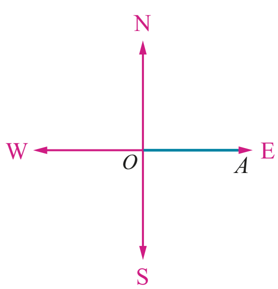
Unless stated otherwise, find all distances correct to one decimal place and all angles to the nearest degree.

**1** Use one of the eight points to write the compass bearing of *A* from *O* for each diagram.

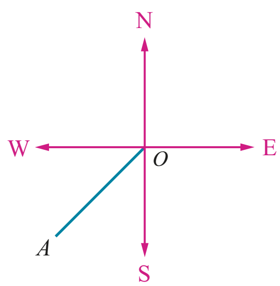
**a**




**b**

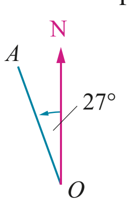



**c**

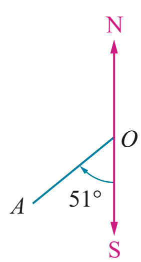



**2** Write the compass bearing of *A* from *O* for each diagram.

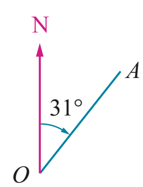
**a**




**b**

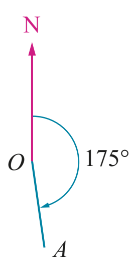



**c**

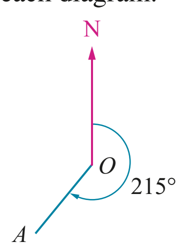



**3** Write the true bearing of *A* from *O* for each diagram.

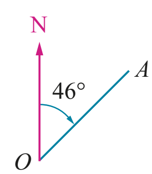
**a**




**b**




**c**

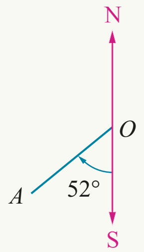


## EXAMPLE 2C–1 Writing compass bearings and true bearings

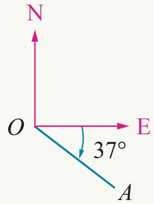
Write the bearing of $A$ from $O$ in the following diagrams as:

**i** a compass bearing **ii** a true bearing.

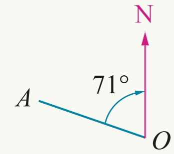
**a**

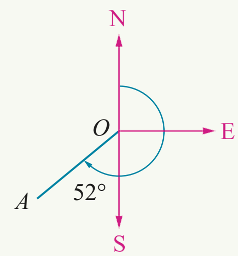
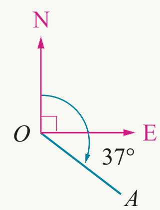
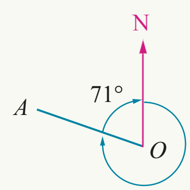



**b**




**c**

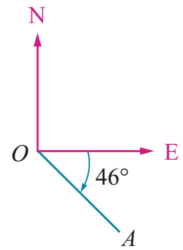


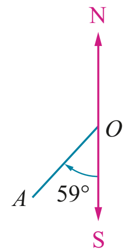

| | Solve | Think |
|---|---|---|
| **a i** | Compass bearing of $A$ from $O$ is S52°W. | The direction from $O$ to $A$ is closer to south than north. Identify the angle between south and $OA$. Write the bearing as the number of degrees from south towards west. |
| **ii** | $180° + 52° = 232°$<br>True bearing of $A$ from $O$ is 232°T. | The angle between north and south is 180°. Add 180° to 52° to calculate the total number of degrees from north to $OA$.   |
| **b i** | $90° – 37° = 53°$<br>Compass bearing of $A$ from $O$ is S53°E. | The direction from $O$ to $A$ is closer to south than north. Calculate the angle between south and $OA$. Write the bearing as the number of degrees from south towards east. |
| **ii** | $90° + 37° = 127°$<br>True bearing of $A$ from $O$ is 127°T. | The angle between north and east is 90°. Add 90° to 37° to calculate the total number of degrees from north to $OA$.   |
| **c i** | Compass bearing of $A$ from $O$ is N71°W. | The direction from $O$ to $A$ is closer to north than south. Identify the angle between north and $OA$. Write the bearing as the number of degrees from north towards west. |
| **ii** | $360° – 71° = 289°$<br>True bearing of $A$ from $O$ is 289°T. | A full turn is 360°. Subtract 71° from 360° to calculate the total number of degrees from north to $OA$.   |

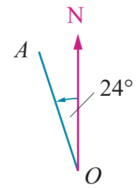
| Apply |
|---|
| For the compass bearing, calculate the angle for $OA$ from north or south, whichever is closer, towards east or west.<br>For the true bearing, calculate the amount of turn in a clockwise direction from north to $OA$. |

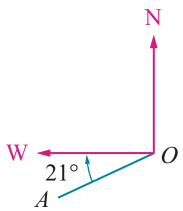
**4** Write the bearing of $A$ from $O$ shown in the following diagrams as:

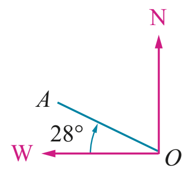
**i** a compass bearing **ii** a true bearing.

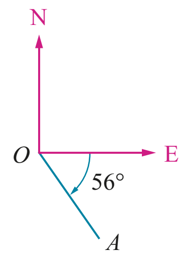
**a**


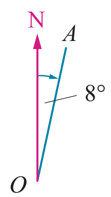

**b**


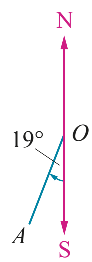

**c**


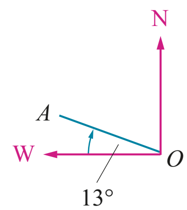

**d**



**e**



**f**


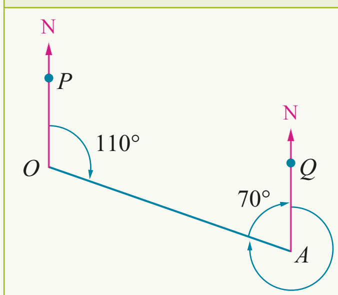

**g**



**h**



**i**



## EXAMPLE 2C–2 Finding the true bearing in the opposite direction

If the bearing of $A$ from $O$ is 110°T, what is the bearing of $O$ from $A$?

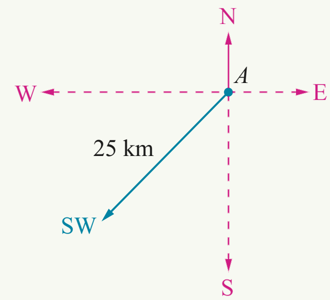
| Solve | Think | Apply |
|---|---|---|
|   <br>$\angle OAQ + \angle POA = 180°$<br>$\angle OAQ + 110° = 180°$<br>$\angle OAQ = 70°$<br>Angle at $A$, clockwise from north to $AO$<br>$= 360° - 70°$<br>$= 290°$<br>Bearing of $O$ from $A$ is 290°T. | Draw a diagram showing a true bearing of 110° from $O$ to $A$, labelling a point $P$ on the north reference line. Construct another north reference line at $A$ and label a point $Q$.<br>Since $OP$ is parallel to $AQ$, angles $OAQ$ and $POA$ are co-interior angles and add to 180°. Hence $\angle OAQ = 70°$.<br>Calculate the angle at $A$, clockwise from north to $AO$, to find the true bearing of $O$ from $A$. | For parallel lines, co-interior angles add up to 180°.<br>Angles at a point add up to 360°. |

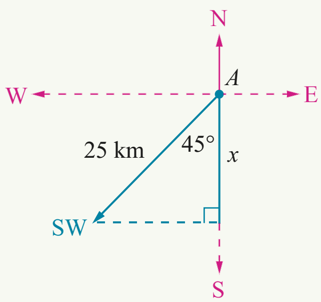
**5** Given these bearings of $A$ from $O$, find the bearing of $O$ from $A$.

**a** 123°T **b** 047°T **c** 231°T **d** 195°T

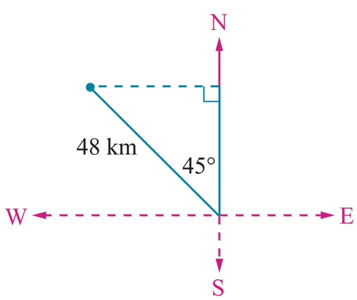
## EXAMPLE 2C–3 Using a compass bearing and trigonometry to calculate a distance

A ship sails for 25 km in a SW direction from port $A$. Find the distance the ship is south of its starting point.

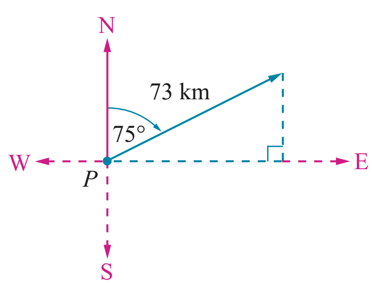



| Solve | Think | Apply |
|---|---|---|
|   $\cos 45° = \frac{x}{25}$<br>$x = 25 \times \cos 45°$<br>$= 17.677...$<br>$\approx 17.7\text{ km}$<br>The ship is about 17.7 km south of its starting position. | Form a right-angled triangle on the diagram. Label the side length representing the distance south with $x$. Label the angle of 45°.<br>In the right-angled triangle, the length of the hypotenuse is 25 km and the unknown side length is on the adjacent side. Use the cosine ratio. | Form a right-angled triangle on the diagram so trigonometry can be used to find the unknown side length. |

**6** A ship sails a distance of 48 km in a NW direction. Find how far north of its starting point it has travelled.

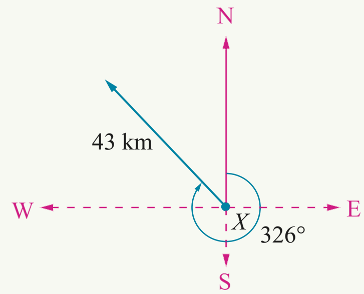


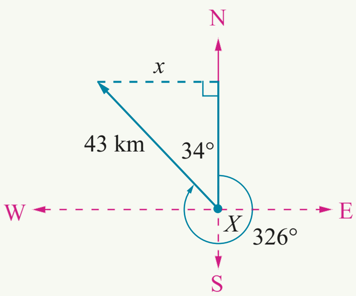

**7** A ship sails 73 km from port $P$ on a bearing of N75°E. Find how far east the ship has sailed.



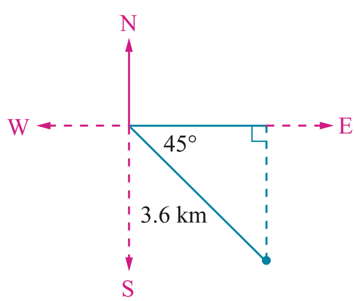
## EXAMPLE 2C–4 Using a true bearing and trigonometry to calculate a distance

A ship sails 43 km from port $X$ on a true bearing of 326°. Find how far the ship is west of port $X$.

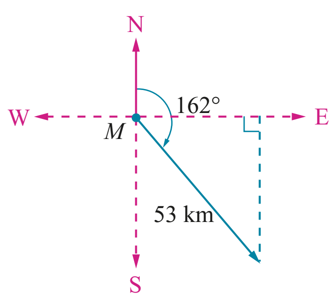



| Solve | Think | Apply |
|---|---|---|
| <br>$\sin 34° = \frac{x}{43}$<br>$x = 43 \times \sin 34°$<br>$= 24.045...$<br>$\approx 24.0\,\text{km}$<br>The ship is about 24.0 km west of port. | Form a right-angled triangle on the diagram. Label the side length representing the distance west with $x$. The angle inside the right-angled triangle is found by subtraction.<br>Angle $= 360° - 326° = 34°$<br>The length of the hypotenuse is 43 km and the unknown side length is on the opposite side. Use the sine ratio. | Form a right-angled triangle on the diagram so trigonometry can be used to find the unknown side length. |

**8** A cross-country skier skis 3.6 km on a true bearing of 135°. Find how far east the skier has travelled from the starting point.

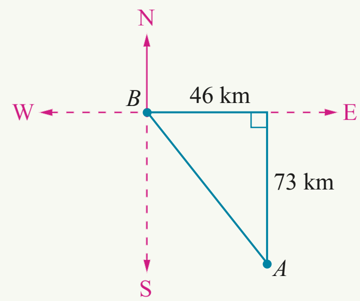


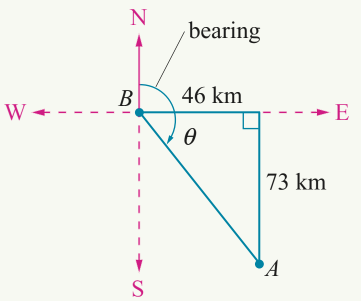

**9** A ship sails 53 km from port $M$ on a bearing of 162°T. Find how far south the ship has sailed.



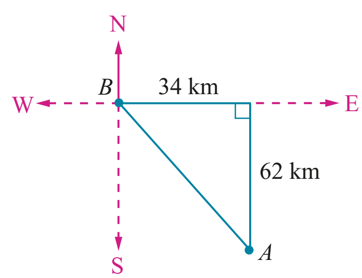
## EXAMPLE 2C–5 Using trigonometry to find a bearing

Town $A$ is 46 km east and 73 km south of town $B$. Find the true bearing of $A$ from $B$, to the nearest minute.

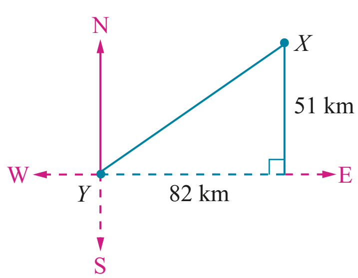



| Solve | Think | Apply |
|---|---|---|
|  <br>$\tan\theta = \frac{73}{46}$<br>$\theta \approx 57°47'$<br>Bearing is $90° + 57°47' = 147°47'$T. | Indicate the angle to be found in the right-angled triangle, $\theta$, to help find the bearing. The opposite side to $\theta$ has a length of 73 km and the adjacent side has a length of 46 km. Use the tangent ratio. | Form a right-angled triangle and label the required angle to be found in the right-angled triangle. Use trigonometry to find the unknown angle and hence work out the bearing. |

**10** Town $A$ is 34 km east and 62 km south of town $B$. Find the true bearing of $A$ from $B$.

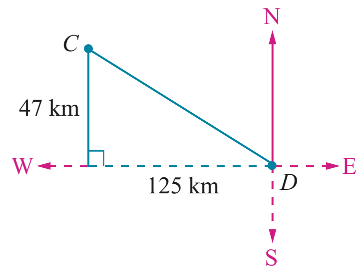



**11** Town $X$ is 82 km east and 51 km north of town $Y$. Find the true bearing of $X$ from $Y$, to the nearest minute.

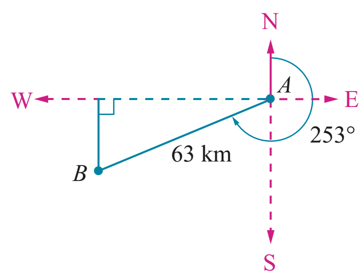


UNDERSTANDING, FLUENCY AND COMMUNICATING

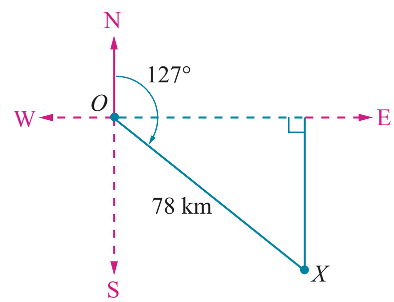
**12** Town $C$ is 125 km west and 47 km north of town $D$. Find the compass bearing of $C$ from $D$.



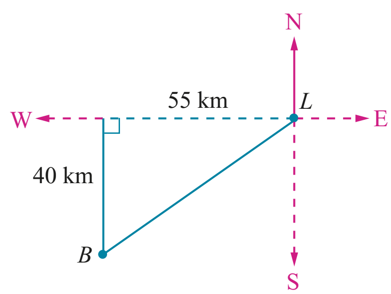

**13** A ship sails 63 km from port $A$ on a bearing of 253°T, to a buoy $B$. Find how far the ship is west and south of $A$.



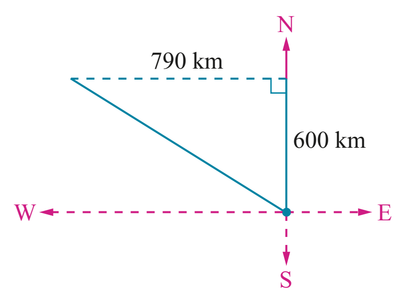

**14** A ship sails 78 km from port $O$ on a bearing of 127°T. Find how far the ship is east and south of $O$.




**15** A ship sails from port $L$ and travels 55 km west then 40 km south to port $B$. Find the compass bearing of $B$ from $L$.




**16** A plane flies 600 km north and 790 km west. Find the compass bearing and distance of the plane from its starting point.

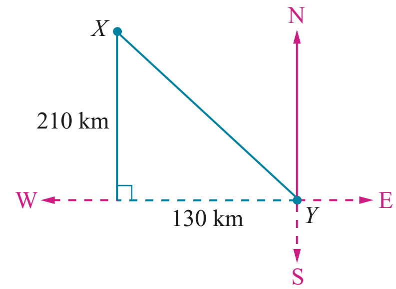



MEASUREMENT

**17** Town $X$ is 130 km west and 210 km north of town $Y$. Find the true bearing of:

**a** $X$ from $Y$

**b** $Y$ from $X$.

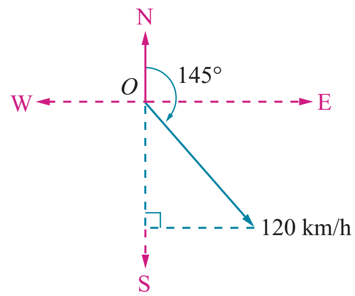



**18** A rally driver heads in a direction of 145° T at a speed of 120 km/h for 3 hours.

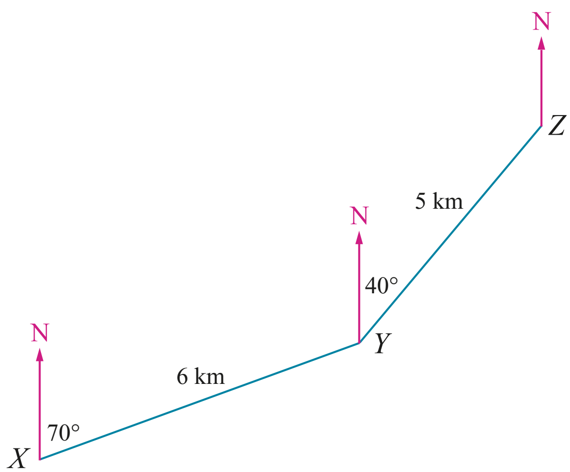
**a** How far does the rally driver travel in 3 hours?

**b** How far east of the starting position is the rally driver now?

**c** The rally driver then turns and drives north until he is due east of the starting point. How long will this take at 120 km/h?




**19** A hiker starts at $X$ and walks 6 km on a true bearing of 070° to $Y$. She then walks on a true bearing of 040° for 5 km from $Y$ to $Z$.




**a** How far is:

  **i** $Y$ east of $X$?

  **ii** $Z$ east of $Y$?

  **iii** $Z$ east of $X$?

**b** How far is:

  **i** $Y$ north of $X$?

  **ii** $Z$ north of $Y$?

  **iii** $Z$ north of $X$?

**c** Using the results from parts **a** and **b**, find:

  **i** the distance from $X$ to $Z$

  **ii** the true bearing of $Z$ from $X$, to the nearest minute.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**20** A yacht sets out in a race with three legs as shown. The first leg is 9.6 km from $A$ on a bearing of 310°T to $B$. The second leg from $B$ to $C$ is 13.2 km on a bearing of 217°T.

**a** How far is:

**i** $B$ north of $A$? **ii** $C$ south of $B$? **iii** $C$ south of $A$?

**b** How far is:

**i** $B$ west of $A$?

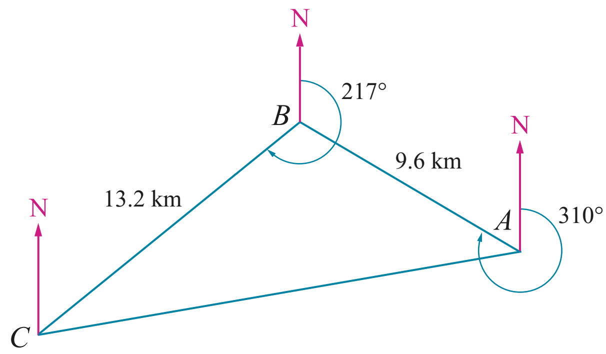
**ii** $C$ west of $B$?

**iii** $C$ west of $A$?

**c** Using the results from parts **a** and **b**, find:

**i** the length of the third leg from $C$ to $A$

**ii** the true bearing from $C$ to $A$, to the nearest minute.




RESEARCH

**21** **a** Define *true north* and *magnetic north*. Explain the difference.

**b** Research the history of magnetic north and find the current location of magnetic north.

**c** Why is true north used on maps? How is magnetic north shown?

**22** Find out more about modern navigational methods including the use of GPS.

**23** Research and investigate the sport of orienteering. Write a short report on your findings.

**24** Some cultures use different navigational methods. For example, since Aboriginal and Torres Strait Islander peoples do not have a written language, they committed their navigational information to memory in the form of songlines that could be passed on to others. Songlines are oral maps of the landscape to enable trade routes to be followed over large distances. In many cases, oral maps of the landscape mirrored patterns of stars in the sky, with these star maps used to help memorise waypoints such as waterholes or turning places along the route.

**a** Find out more about songlines and how they are used as a navigational tool in the Aboriginal and Torres Strait Islander cultures.

**b** Investigate how some modern Australian highways follow the path of Aboriginal songlines.

**c** The concept of cardinal directions (north, south, east and west) is common among Aboriginal language groups in Australia. Find out more about this and how it can relate to sleeping position and burial traditions.

**d** Investigate other navigational methods used by different cultures.

# 2D Area of a triangle

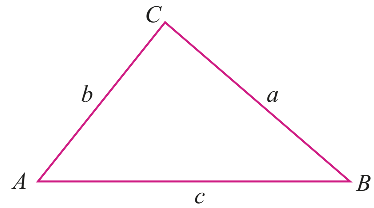
These resources are available on your obook assess:

- **assess quiz 2D:** Test your skills with an auto-correcting multiple-choice quiz

OI
GI

**vertex**
point where two or more lines meet

Trigonometry can also be applied in non-right-angled triangles. The convention is to label the side lengths of the triangle with lowercase letters of the angles opposite these sides. For example, in the diagram on the right, the side length $a$ is opposite the angle $A$, which is found at the **vertex** of the triangle labelled as $A$.

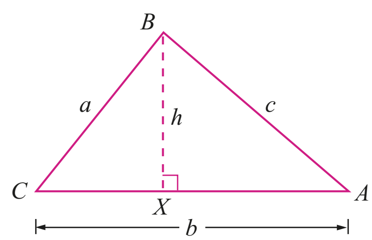



From previous years, you know that the formula for the area of a triangle with base $b$ and perpendicular height $h$ is Area $= \frac{1}{2}bh$. We will now look at finding the area of a triangle using trigonometry.

Consider the area of $\triangle ABC$ shown on the right with base $CA$ (or $b$) and perpendicular height $BX$ (or $h$).

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times CA \times BX\\ &= \frac{1}{2}bh \qquad (1)\end{aligned}$$

$$\begin{aligned}\text{In } \triangle BXC,\ \sin C &= \frac{BX}{BC}\\ &= \frac{h}{a}\\ h &= a \sin C \qquad (2)\end{aligned}$$

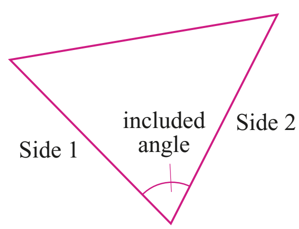



Combining (1) and (2) gives:

$$\begin{aligned}\text{Area} &= \frac{1}{2}b(a \sin C)\\ &= \frac{1}{2}ab \sin C\end{aligned}$$

!

Given the lengths of two sides and the included angle of a triangle, the area is given by: Area $= \frac{1}{2} \times$ length of Side 1 $\times$ length of Side 2 $\times$ sin(included angle)



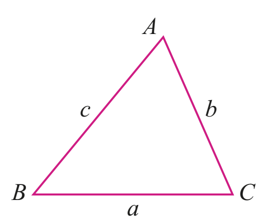

For triangle $ABC$, this can be written as:

$$\text{Area} = \frac{1}{2}ab \sin C$$

$$\text{or Area} = \frac{1}{2}ac \sin B$$

$$\text{or Area} = \frac{1}{2}bc \sin A$$

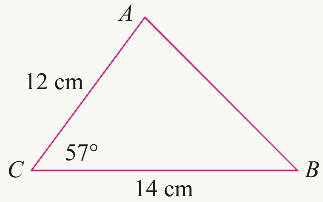
where $a$, $b$ and $c$ are the sides opposite angles $A$, $B$ and $C$ respectively.



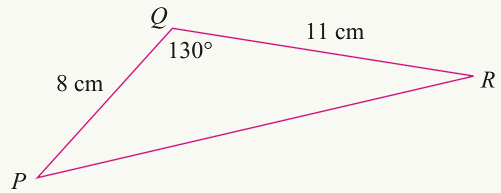
## EXAMPLE 2D–1 Finding the area of a triangle

Find the area of each triangle, correct to one decimal place.

**a**




**b**



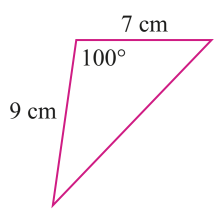

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Area $= \frac{1}{2}ab\sin C$<br>$= \frac{1}{2} \times 14 \times 12 \times \sin 57°$<br>$\approx 70.4\text{ cm}^2$ | In $\triangle ABC$, the known angle is $C$ so the appropriate formula to use is Area $= \frac{1}{2}ab\sin C$ where $a = 14$ cm, $b = 12$ cm and $C = 57°$. | The area of a triangle is half the product of any two sides and the sine of the angle between them. |
| **b** | Area $= \frac{1}{2}pr\sin Q$<br>$= \frac{1}{2} \times 11 \times 8 \times \sin 130°$<br>$\approx 33.7\text{ cm}^2$ | In $\triangle PQR$, the known angle is $Q$ so the appropriate formula to use is Area $= \frac{1}{2}pr\sin Q$ where $p = 11$ cm, $r = 8$ cm and $Q = 130°$. | |

# EXERCISE 2D Area of a triangle

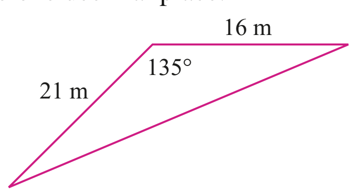
UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Find the area of each triangle, correct to one decimal place.

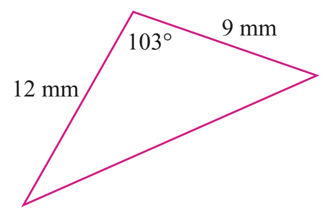
**a**



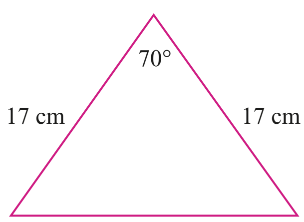

**b**



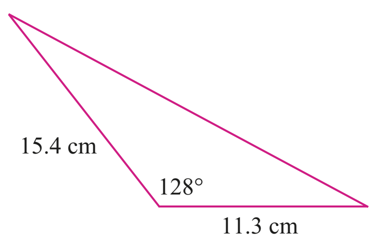

**c**



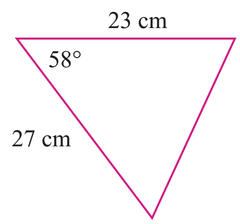

**d**



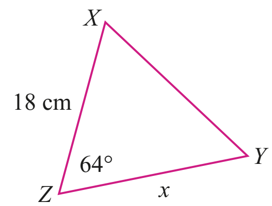

**e**




**f**




**2** Triangle $XYZ$ has an area of 230 cm². Find the value of $x$, correct to one decimal place.




**3** The area of triangle $PQR$ is 14.2 cm². Find $\angle PQR$ to the nearest minute, given $PQ$ is 7 cm and $QR$ is 5 cm.

**4** A give-way sign is close in shape to an equilateral triangle with side lengths of 40 cm. Use the area formula involving sine to estimate the area of the sign to the nearest square centimetre.

**5** For right-angled triangle $ABC$ in which angle $C$ is the right angle, show that the formula Area $= \frac{1}{2}ab\sin C$ becomes Area $= \frac{1}{2}ab$ and hence that it is equivalent to Area $= \frac{1}{2} \times$ base $\times$ height.

# 2E Sine rule

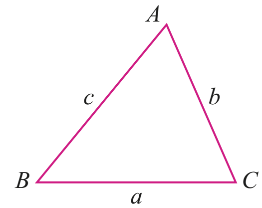
These resources are available on your obook assess:

- **Video tutorial 2E:** Watch and listen to an explanation of Example 2E–1
- **assess quiz 2E:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**sine rule**
rule linking side lengths $a$, $b$ and $c$, and angles $A$, $B$ and $C$ of a triangle $ABC$;
e.g. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
These ratios are used in pairs

The **sine rule** is one method used to find side lengths and angles in non-right-angled triangles.




In $\triangle ABC$ shown on the right, the area may be found in three ways:

Area $= \frac{1}{2}bc \sin A$ Area $= \frac{1}{2}ac \sin B$ Area $= \frac{1}{2}ab \sin C$

Since the three areas are the same for this triangle:

$$\frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

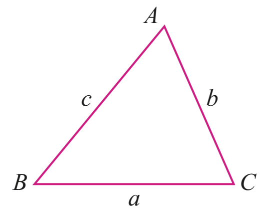
Dividing each of the three expressions by $\frac{1}{2}abc$ gives us:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

!

In any triangle $ABC$ with side lengths and angles as shown, the sine rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

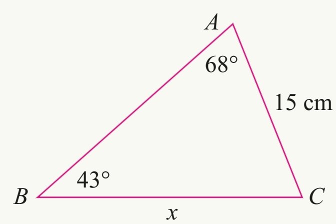



Any two of the three parts (or ratios) of the sine rule are used to find:

- a side length, given two angles and one side length
- an angle, given two side lengths and a non-included angle.

## EXAMPLE 2E–1 Using the sine rule to find an unknown side length

Find the value of $x$ (length of $BC$) in this triangle, correct to one decimal place.



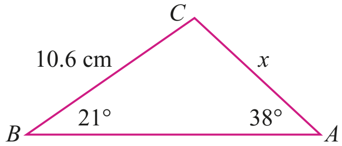

| Solve | Think | Apply |
|---|---|---|
| $\frac{a}{\sin A} = \frac{b}{\sin B}$<br>$\frac{x}{\sin 68°} = \frac{15}{\sin 43°}$<br>$x = \frac{15}{\sin 43°} \times \sin 68°$<br>$= 20.3926...$<br>$\approx 20.4$ cm<br>The length of $BC$ is about 20.4 cm. | Use the sine rule, since we are finding a side length, given two angles and one side length. Write the ratio of the unknown side length ($x$) to the sine of the angle opposite it (sin 68°) and the ratio of the known side length (15 cm) to the sine of the angle opposite it (sin 43°). Solve the resulting equation, multiplying both sides by sin 68° to find $x$. | Use the sine rule involving three known values and one unknown value. |

# EXERCISE 2E Sine rule

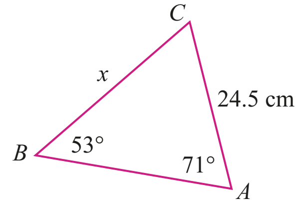


**1** Find the value of $x$ in each of these triangles, correct to one decimal place.

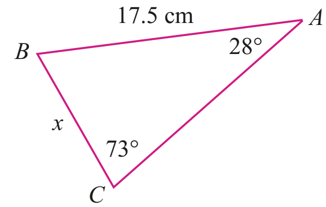
**a**



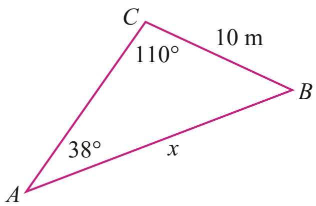

**b**



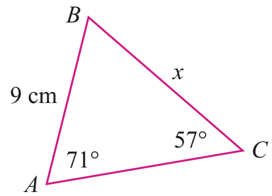

**c**



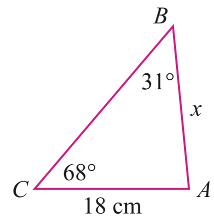

**d**




**e**

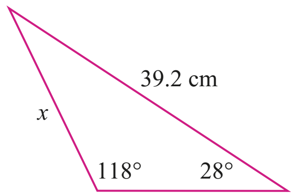



**f**

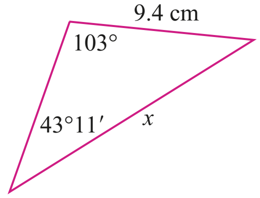



**2** Find the value of $x$ in each of these triangles, correct to one decimal place.

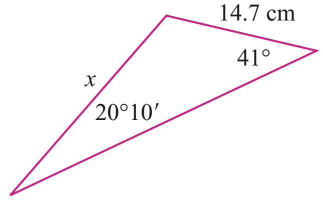
**a**



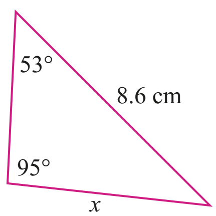

**b**



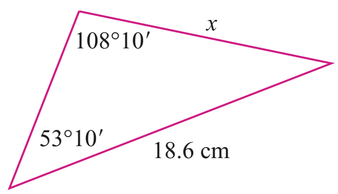

**c**



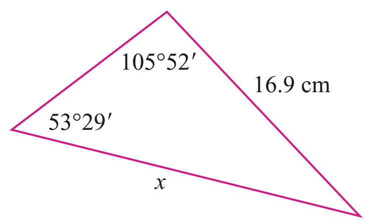

**d**




**e**

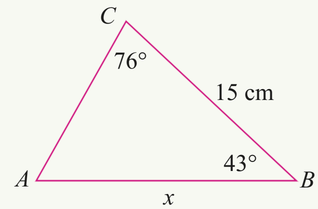



**f**




## EXAMPLE 2E–2 Using the angle sum of a triangle and the sine rule to find an unknown side length

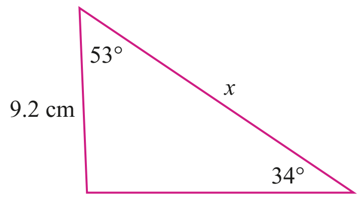
Find the size of the third angle and hence find the value of $x$ (length of $AB$) in this triangle, correct to one decimal place.



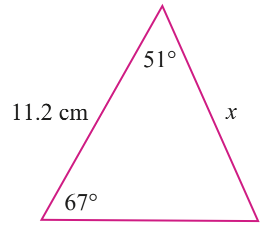

| Solve | Think | Apply |
|---|---|---|
| $A = 61°$<br>$\frac{x}{\sin 76°} = \frac{15}{\sin 61°}$<br>$x = \frac{15}{\sin 61°} \times \sin 76°$<br>$= 16.640\,87...$<br>$\approx 16.6$ cm<br>The length of $AB$ is about 16.6 cm. | In this triangle, the angle opposite the side $CB$ is not given.<br>Use the angle sum of a triangle to find the third angle:<br>$A = 180° - 76° - 43° = 61°$ | Find the angle opposite the known side length, using the fact that the angle sum of a triangle is 180°. Then use the sine rule. |

**3** Find the size of the third angle and hence find the value of $x$ in each of these triangles. Give the answer correct to one decimal place.

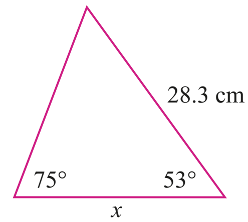
**a**



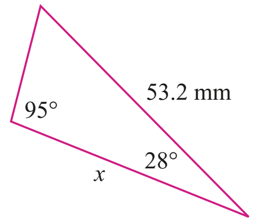

**b**



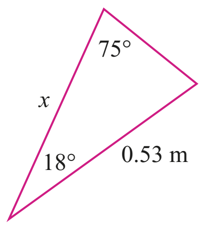

**c**



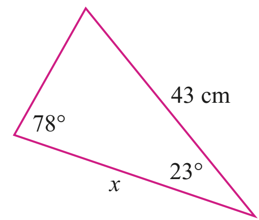

**d**




**e**

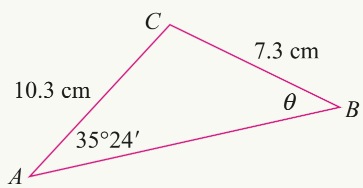



**f**




## EXAMPLE 2E–3 Using the sine rule to find an unknown angle

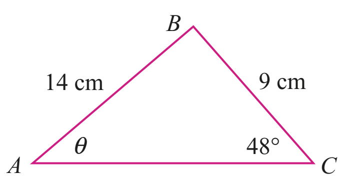
Find the value of $\theta$ in this triangle, correct to the nearest minute.



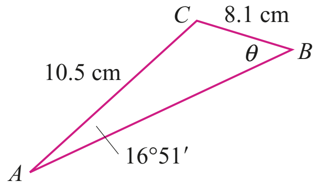

| Solve | Think | Apply |
|---|---|---|
| $\dfrac{10.3}{\sin\theta} = \dfrac{7.3}{\sin 35°24'}$<br>$\dfrac{\sin\theta}{10.3} = \dfrac{\sin 35°24'}{7.3}$<br>$\sin\theta = 10.3 \times \dfrac{\sin 35°24'}{7.3}$<br>$\sin\theta = 0.817...$<br>$\theta = 54.819...°$<br>$\approx 54°49'$ | Substitute into $\dfrac{b}{\sin B} = \dfrac{a}{\sin A}$.<br>To make $\sin\theta$ the subject of the equation, invert both sides then multiply both sides by 10.3.<br>Alternatively, substitute into the other form of the sine rule:<br>$\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$ | Substitute the given measurements into the sine rule. Make $\sin\theta$ the subject of the equation by inverting both sides of the equation, and then solve to find $\theta$. |

**4** Find the value of $\theta$ in each of these triangles, correct to the nearest minute.

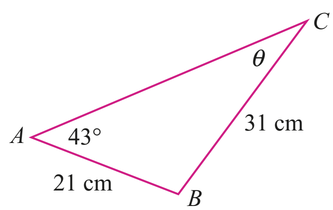
**a**



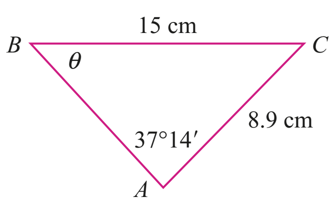

**b**



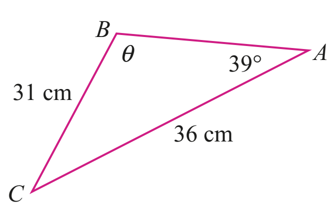

**c**



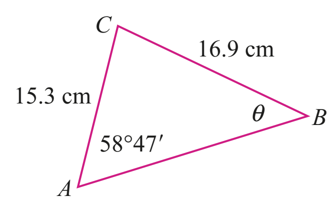

**d**




**e**



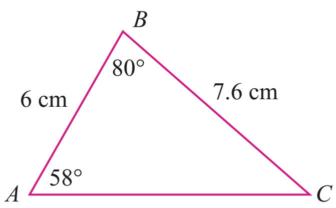

**f**



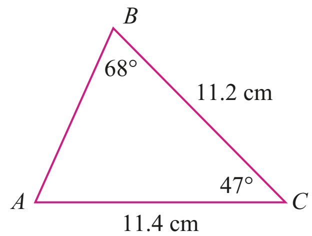
UNDERSTANDING, FLUENCY AND COMMUNICATING

**5** Find the perimeter of each of these triangles, correct to one decimal place.

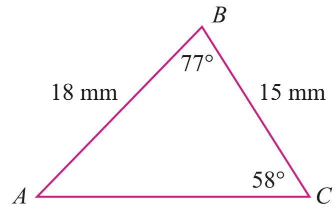
**a**




**b**

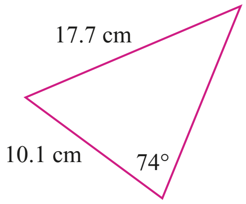



**c**

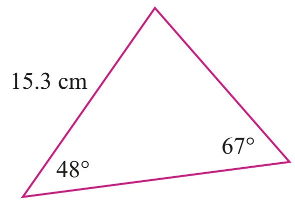



**6** Find the area of each of these triangles, correct to one decimal place. (*Hint:* first use the sine rule and then the angle sum of a triangle.)

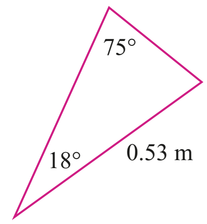
**a**




**b**

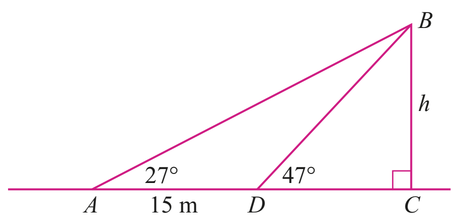



**c**



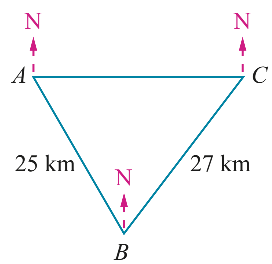

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**7** Tom measures the angle of elevation to the top of a building as 27°. Then he walks 15 m closer to the building and measures the angle of elevation to be 47°. The diagram, shown below, represents this information.




**a** Find the size of angle $ADB$.
**b** Find the size of angle $ABD$.
**c** Calculate the length of $BD$, to one decimal place.
**d** Find the height of the building, $h$.
**e** How far was Tom from the building initially?

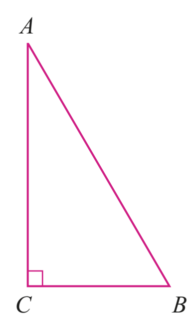
**8** Two bushwalkers start from a point $A$ and hike 25 km to a point $B$ at a true bearing of 150°. They then continue for 27 km to point $C$, which lies due east of their starting point.




**a** Find the size of angle $BAC$.
**b** Find the size of angle $ACB$ to the nearest degree.
**c** Find the size of angle $ABC$.
**d** Find the distance the bushwalkers would need to travel to return directly to their starting point, correct to one decimal place.
**e** Calculate the area enclosed by the bushwalkers' hike.

CHALLENGE

**9** Consider the right-angled triangle $ABC$ shown, and the sine rule in the form $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.




**a** Find the value of $\sin C$ for the triangle.
**b** Show that $\frac{\sin A}{a} = \frac{\sin C}{c}$ becomes $\sin A = \frac{a}{c}$ for this right-angled triangle.
**c** Show that $\frac{\sin B}{b} = \frac{\sin C}{c}$ becomes $\sin B = \frac{b}{c}$ for this right-angled triangle.
**d** Use your results from parts **b** and **c** to explain why the sine rule becomes an application of the sine ratio in a right-angled triangle.

# 2F Cosine rule

These resources are available on your obook assess:
- **Video tutorial 2F:** Watch and listen to an explanation of Example 2F–1
- **assess quiz 2F:** Test your skills with an auto-correcting multiple-choice quiz

OI
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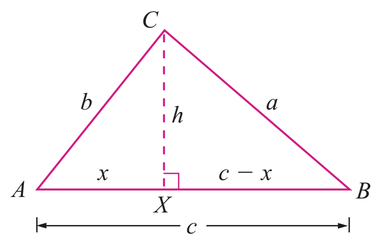
**cosine rule**
rule linking three side lengths $a$, $b$ and $c$, and one angle of a triangle $ABC$; e.g. $a^2 = b^2 + c^2 - 2bc \cos A$

The **cosine rule** is another method used to find side lengths and angles in non-right-angled triangles.




Consider $\triangle ABC$ shown on the right.

Using Pythagoras' theorem in $\triangle ACX$:

$b^2 = h^2 + x^2$

$h^2 = b^2 - x^2$ (1)

Using Pythagoras' theorem in $\triangle BCX$:

$a^2 = h^2 + (c - x)^2$

$h^2 = a^2 - (c - x)^2$ (2)

As (1) and (2) both equal $h^2$, they equal each other:

$$a^2 - (c - x)^2 = b^2 - x^2$$
$$a^2 - c^2 + 2cx - x^2 = b^2 - x^2$$
$$a^2 = b^2 + c^2 - 2cx$$

But, in $\triangle ACX$: $\cos A = \dfrac{x}{b}$

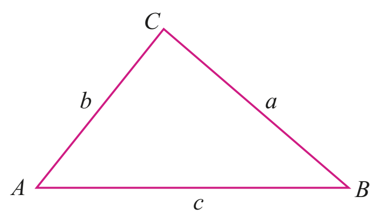
$$x = b \cos A$$

So $a^2 = b^2 + c^2 - 2cx$ becomes $a^2 = b^2 + c^2 - 2bc \cos A$

!

In any triangle $ABC$ with sides and angles as shown:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

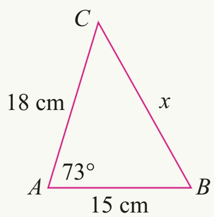



The cosine rule is used to find:
- the third side, given two side lengths and the included angle
- an angle, given three side lengths.

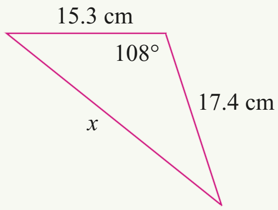
## EXAMPLE 2F–1 Using the cosine rule to find an unknown side length

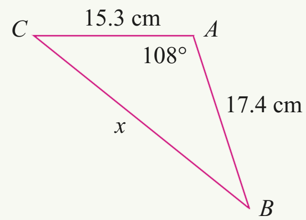
Find the value of $x$ in each of these triangles, correct to one decimal place.

**a**




**b**



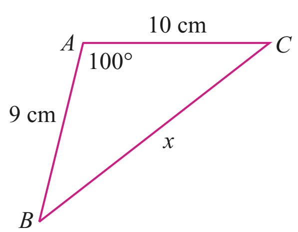
| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $x^2 = 18^2 + 15^2 - 2 \times 18 \times 15 \cos 73°$<br>$= 391.119...$<br>$x = \sqrt{391.119...}$<br>$\approx 19.8\,\text{cm}$ | Since the unknown side relates to the side length $a$ and we have angle $A$, substitute the values into the cosine rule:<br>$a^2 = b^2 + c^2 - 2bc \cos A$ | Choose the appropriate version of the cosine rule and substitute the known measurements, to calculate the unknown side length. |
| **b** | $x^2 = 15.3^2 + 17.4^2 - 2 \times 15.3 \times 17.4 \times \cos 108°$<br>$= 701.383...$<br>$x = \sqrt{701.383...}$<br>$= 26.5\,\text{cm}$ | Label the vertex at the included angle as $A$ and the other vertices as $B$ and $C$ and use:<br>$a^2 = b^2 + c^2 - 2bc \cos A$<br><br> | |

## EXERCISE 2F Cosine rule

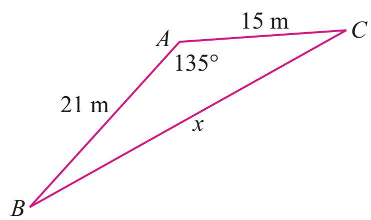
UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Find the value of $x$ in each of these triangles, correct to one decimal place.

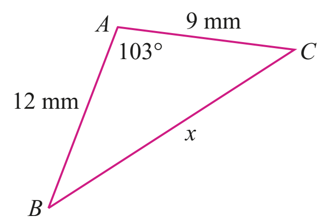
**a**



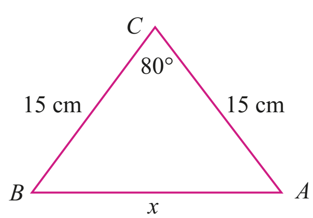

**b**



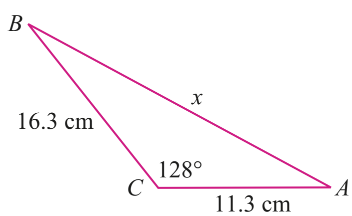

**c**



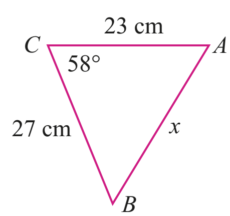

**d**



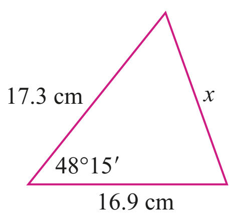

**e**



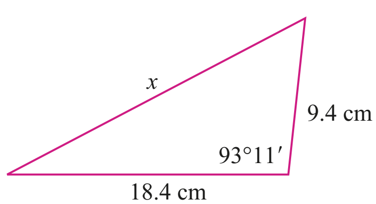

**f**



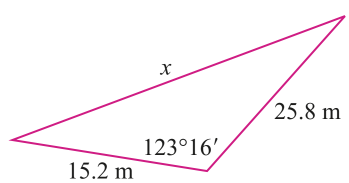

**g**




**h**




**i**

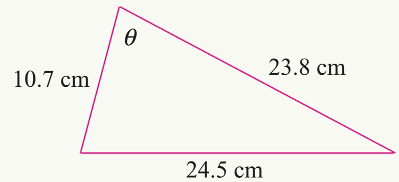


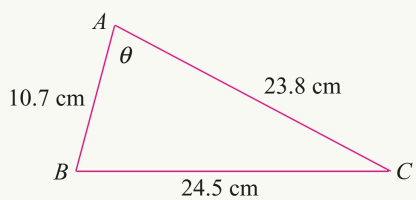

MEASUREMENT

> **!**
>
> Rearranging $a^2 = b^2 + c^2 - 2bc\cos A$
>
> gives $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
>
> which is a more convenient form for finding angles.
>
> Likewise $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

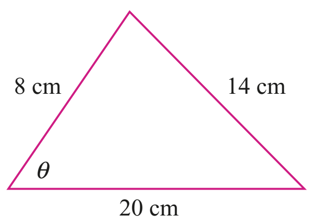
## EXAMPLE 2F–2 Using the cosine rule to find an unknown angle

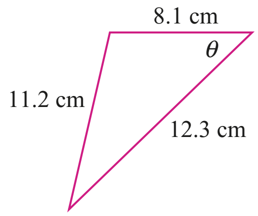
Find the value of $\theta$ in this triangle, correct to the nearest degree.

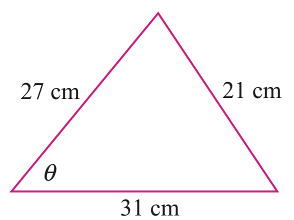


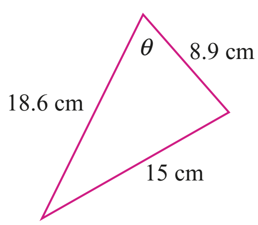

| Solve | Think | Apply |
|---|---|---|
| $\cos\theta = \dfrac{23.8^2 + 10.7^2 - 24.5^2}{2 \times 23.8 \times 10.7}$<br>$= 0.158...$<br>$\theta = 80.885...^\circ$<br>$\approx 81^\circ$ | Label the vertex at the unknown angle as $A$ and the other vertices as $B$ and $C$.<br>Use $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.<br> | Choose the appropriate version of the cosine rule and substitute the known measurements, to calculate the required angle. |

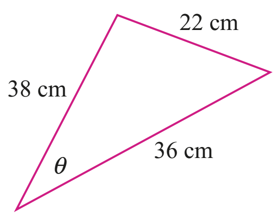
**2** Find the value of $\theta$ in each of these triangles, correct to the nearest degree.

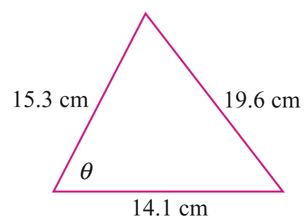
**a**



**b**


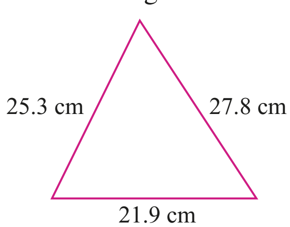

**c**


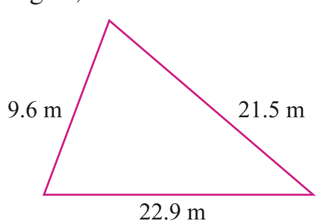

**d**


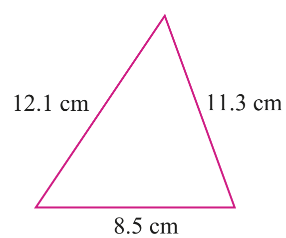

**e**



**f**



**3** Find all three angles in each of these triangles, correct to the nearest minute.

**a**


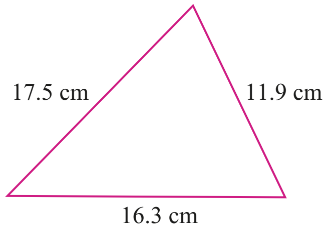

**b**



**c**


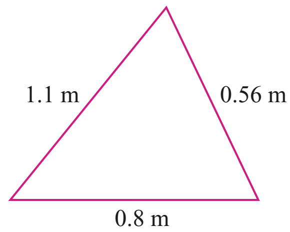
UNDERSTANDING, FLUENCY AND COMMUNICATING

**4** Find the smallest angle in each of these triangles, correct to the nearest minute. (Hint: which side is opposite the smallest angle?)

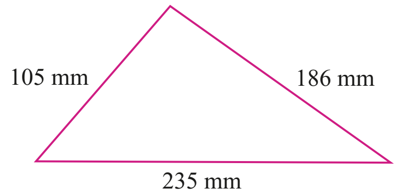
**a**




**b**

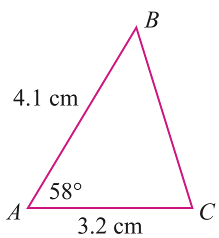



**c**

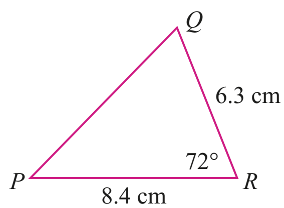



**5** Find the perimeter of each triangle, correct to one decimal place.

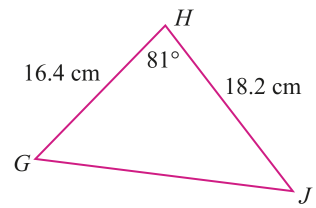
**a**




**b**



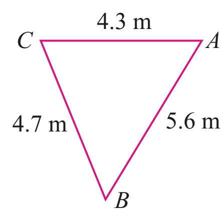

**c**




**6** Find the area of each triangle shown in question **5**.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**7** A triangular garden bed has side lengths 5.6 m, 4.3 m and 4.7 m as shown in the diagram on the right.

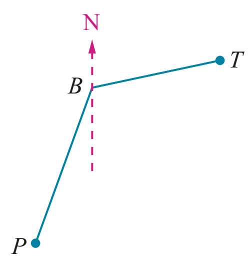



**a** Identify the vertex with the largest angle.

**b** Use the cosine rule to find the size of the largest angle, correct to the nearest minute.

**c** Show how you can use this angle to work out that the area of the garden bed is about 9.8 m$^2$.

**8** From a buoy at sea, Tyson's yacht sails 2.5 km at a true bearing of 080° to point $T$ while Paula's yacht sails 3.5 km at a compass bearing of S20° W to point $P$.




**a** Copy the diagram shown on the right and label all known side lengths and angles.

**b** On your diagram, join points $P$ and $T$ with a straight line to represent the distance between the two yachts.

**c** Find the size of angle $PBT$ within the triangle you have formed.

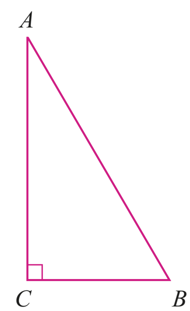
**d** Use the cosine rule to calculate the distance $PT$ between the two yachts.

**e** Use the cosine rule to calculate the size of angle $BPT$, correct to the nearest degree.

**f** Use your answer from part **e** to find the bearing of point $T$ from point $P$.

CHALLENGE

**9** Consider the right-angled triangle $ABC$ shown on the right, and the cosine rule in the form $c^2 = a^2 + b^2 - 2ab \cos C$.




**a** Find the value of cos $C$ from this right-angled triangle.

**b** Substitute your value from part **a** into $c^2 = a^2 + b^2 - 2ab \cos C$ and simplify.

**c** Use your result from part **b** to explain why the cosine rule becomes Pythagoras' theorem in a right-angled triangle.

MEASUREMENT

# 2G More problems involving trigonometry

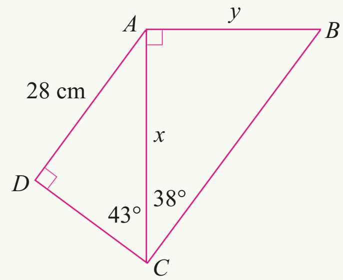
These resources are available on your obook assess:

- **assess quiz 2G:** Test your skills with an auto-correcting multiple-choice quiz

This section uses right-angled triangle trigonometry, bearings and the sine and cosine rules to solve problems.

## EXAMPLE 2G–1 Finding unknown side lengths using two right-angled triangles

In the diagram shown, find the value of the side lengths $x$ and $y$, correct to one decimal place.

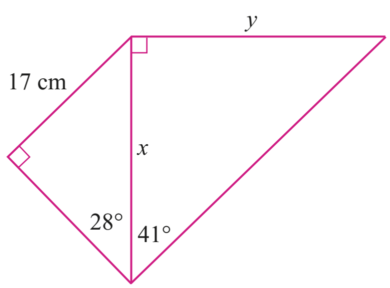



| Solve | Think | Apply |
|---|---|---|
| In $\triangle ACD$, $\sin 43° = \frac{28}{x}$<br>$x = \frac{28}{\sin 43°}$<br>$= 41.055...$<br>$\approx 41\,\text{cm}$ | The trigonometric ratio that connects the sides of length 28 cm and $x$ is the sine ratio. | Use trigonometry to find the unknown side length in one right-angled triangle and then use this result with trigonometry to find the unknown side length in the second right-angled triangle. |
| In $\triangle ABC$, $\tan 38° = \frac{y}{41.056}$<br>$y = 41.056 \times \tan 38°$<br>$= 32.076...$<br>$\approx 32\,\text{cm}$ | Use the result of part **a**. The trigonometric ratio that connects the sides of length $y$ and $x$ is the tangent ratio. | |

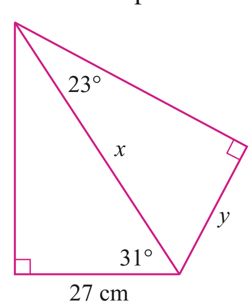
## EXERCISE 2G More problems involving trigonometry

**1** In the following diagrams, find the values of the lengths $x$ and $y$, correct to one decimal place.

**a**

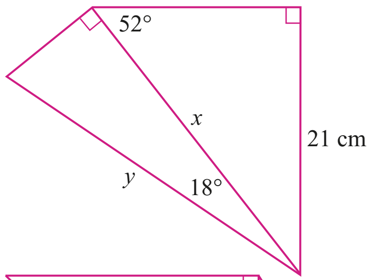



**b**

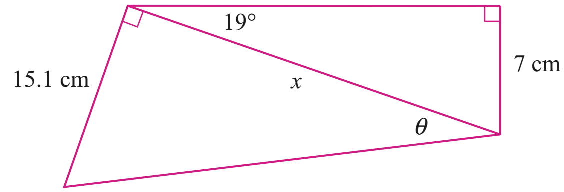


**2** In the following diagrams, find the values of the pronumerals representing a length or an angle. Find all lengths correct to one decimal place and all angles to the nearest minute.

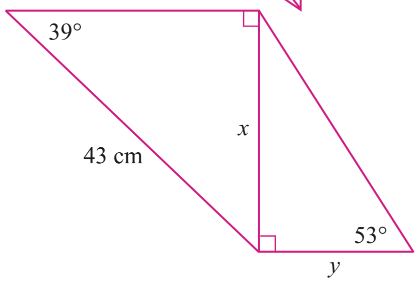
**a**



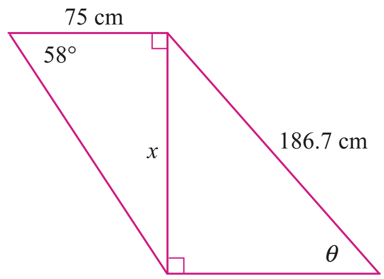

**b**




**c**




**d**



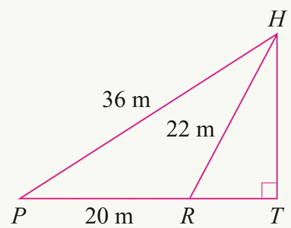

## EXAMPLE 2G–2 Calculating unknown lengths and angles in triangles

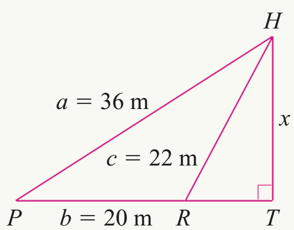
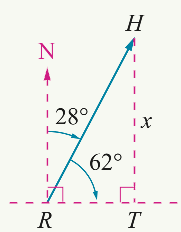
In the diagram, $PR = 20$ m, $RH = 22$ m, $PH = 36$ m, $\angle RTH = 90°$.

**a** Use the cosine rule to find $\angle PRH$ to the nearest degree, and hence find $\angle HRT$.

**b** Find the length of $HT$, correct to one decimal place.

**c** If $TH$ is north, write the true bearing of $H$ from $R$.



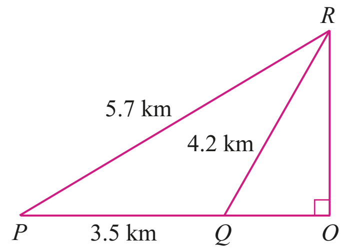

| | Solve | Think | Apply |
|---|---|---|---|
| **a** |  <br>$\cos\angle PRH = \dfrac{20^2 + 22^2 - 36^2}{2 \times 20 \times 22}$<br>$\angle PRH \approx 118°$<br>So $\angle HRT = 180° - 118° = 62°$ | On the diagram, label the side lengths of $\Delta PRH$ as $a$, $b$ and $c$ for the cosine rule to find $\angle PRH$ (angle $A$ in the cosine rule $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$).<br>$\angle PRH$ and $\angle HRT$ make a straight angle. | Calculate the unknown angle using the cosine rule and then use the sine ratio to find the unknown side length. |
| **b** | $\sin 62° = \dfrac{x}{22}$<br>$x = 22 \times \sin 62°$<br>$= 19.424...$<br>So $HT \approx 19.4$ m | On the diagram in part **a**, label the length of $HT$ as $x$.<br>Relative to $\angle HRT$, $x$ is the length of the opposite side and 22 m is the length of the hypotenuse. Use the sine ratio. | |
| **c** | Bearing of $H$ from $R = 90° - 62°$<br>$= 028°$T | Draw a line through $R$ pointing north.<br>$\angle NRH + \angle HRT = 90°$<br>  | |

**3** In the diagram, $PQ = 3.5$ km, $QR = 4.2$ km, $PR = 5.7$ km and $\angle QOR = 90°$.

**a** Use the cosine rule to find $\angle PQR$ to the nearest degree, and hence find $\angle RQO$.

**b** Find $OR$, correct to one decimal place.

**c** If $OR$ is north, find the true bearing of $R$ from $Q$.

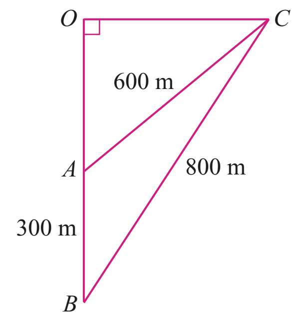



**4** In the diagram, $AB = 300$ m, $AC = 600$ m, $BC = 800$ m and $\angle COA = 90°$.

**a** Find $\angle BAC$ and $\angle OAC$, correct to the nearest degree.

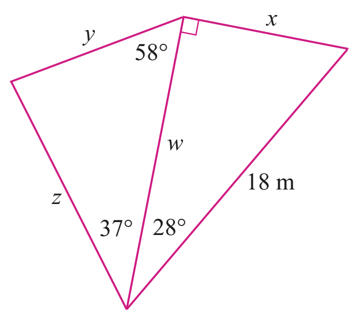
**b** Find $OA$ and $OC$, correct to the nearest 10 metres.

**c** If $BO$ is north, find the true bearing of $C$ from $A$, and $C$ from $B$.

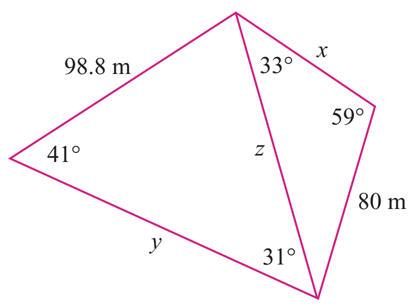



**5** Calculate all the unknown lengths on each diagram, correct to one decimal place.

**a**




**b**



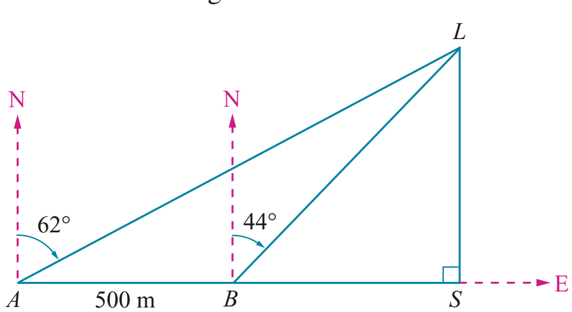



**6** The bearing of a lighthouse, $L$, from a boat is 062°T. The boat travels 500 m east and the bearing of the lighthouse now is 044°T, as shown on the diagram.

**a** Find $\angle LAB$, $\angle LBS$ and $\angle ABL$.

**b** Use the sine rule to find the current distance from the boat to the lighthouse.

**c** Find the distance the boat needs to travel to be due south of the lighthouse.



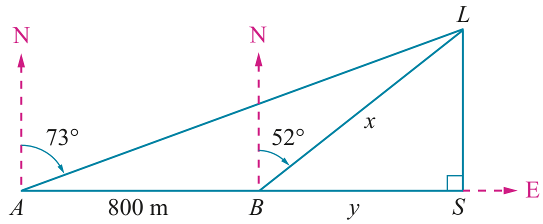

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**7** The bearing of a lighthouse from a boat is 073° T. The boat travels 800 m due east. The bearing of the lighthouse is now 052° T, as shown in the diagram.

**a** Find $\angle ABL$.

**b** Use the sine rule to find the distance $x$ from the boat to the lighthouse.

**c** Find the distance the boat needs to travel to be due south of the lighthouse.

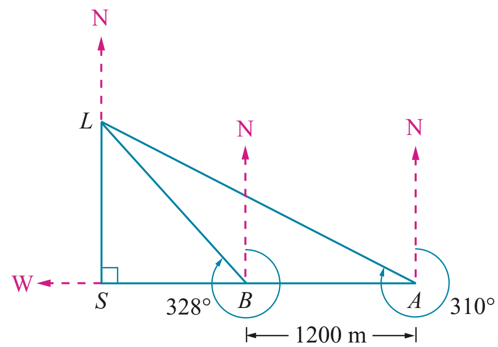



**8** The bearing of a lighthouse from a boat is 310° T. The boat travels 1200 m due west and the bearing of the lighthouse is now 328° T. The diagram shows this information.

**a** Find $\angle ABL$.

**b** Use the sine rule to find the distance from the boat to the lighthouse.

**c** Find the distance the boat needs to travel to be due south of the lighthouse.



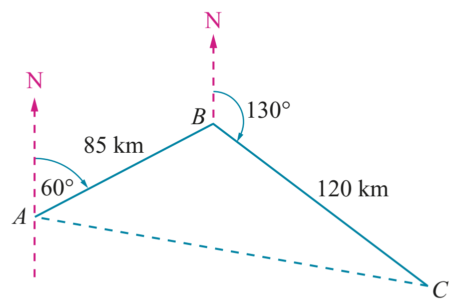

**9** A ship sails 85 km from $A$ to $B$ on a bearing of 060° T. It then turns and sails 120 km to $C$ on a bearing of 130° T.

**a** Find the size of $\angle ABC$.

**b** How far is the ship from its starting point? (*Hint:* use the cosine rule.)

**c** Calculate $\angle BAC$, to the nearest minute. (*Hint:* use the sine rule.)

**d** What is the bearing of the ship from its starting point, to the nearest minute?



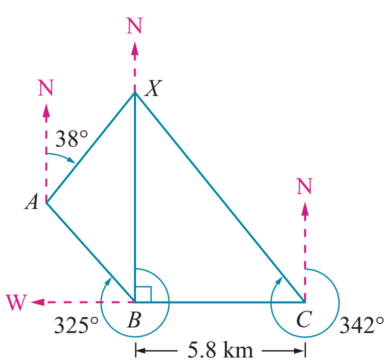

CHALLENGE

**10** A group of roads intersect as shown in the diagram. Roads $BX$ and $BC$ are perpendicular. The bearing of road $BA$ is 325° T. The bearing of road $CX$ is 342° T. The distance $BC$ is 5.8 km. The bearing of road $AX$ is 38° T.

**a** Find $\angle BXC$. Hence find the distance $BX$.

**b** **i** Find $\angle AXB$, $\angle XBA$ and $\angle BAX$.

**ii** Hence find the distance from $B$ to $X$ via $A$.




MEASUREMENT

# 2H Compass radial surveys

These resources are available on your obook assess:
- **Video tutorial 2H:** Watch and listen to an explanation of Example 2H-1
- **assess quiz 2H:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

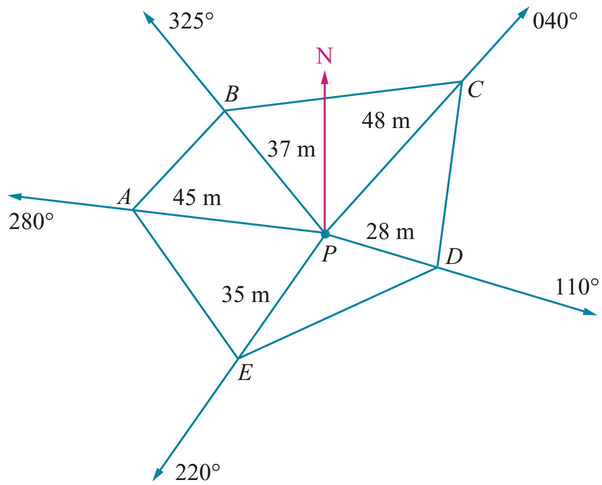
In the Year 11 Mathematics Standard course, we looked at finding the perimeter and area of irregular-shaped fields or blocks of land. When the field had straight edges, it could be divided into triangles, rectangles and/or trapeziums. In some cases, Pythagoras' theorem was used to find an unknown length. When the field had a boundary that was not straight (for example a river), its area could be approximated using the Trapezoidal rule.

**compass radial survey**
survey of a field in which a magnetic compass is used to find the bearing of each vertex along the boundary from a point within the field, and the distance from this point to each vertex is measured

**radial arms**
straight lines radiating or spreading out from a point

In this section you will study a method of surveying known as the **compass radial survey** in which the true bearing of each vertex from a point within the field is found using a magnetic compass.

The sketch below is the result of a compass radial survey for a field *ABCDE*. The chosen point within the field is labelled *P*. Lines called **radial arms** are drawn from *P* to each of the vertices *A*, *B*, *C*, *D* and *E*. Using the north direction line as a reference, the true bearing of each vertex from *P* is measured using a compass. For example, the bearing of *C* from *P* is 040°T and the bearing of *D* from *P* is 110°T. The distance from *P* to each vertex along the radial arm is also measured and included on the sketch. For example, the distance from *P* to *C* is 48 m and the distance from *P* to *D* is 28 m.




As the radial arms divide the field into triangles, we can use our knowledge of trigonometry to help us calculate the perimeter and the area of the field.

!

To calculate the perimeter of a field from a compass radial survey:

- find the length of each straight section of boundary ($c$) using the cosine rule $c^2 = a^2 + b^2 - 2ab\cos C$, where $a$ and $b$ are the lengths along the radial arms forming two sides of the triangle and $C$ is the angle between them at the point $P$
- add the individual lengths.

To calculate the area of a field from a compass radial survey:

- find the area of each triangle using the formula Area $= \frac{1}{2}ab\sin C$ where $a$ and $b$ are the lengths along the radial arms forming two sides of the triangle and $C$ is the angle between them at the point $P$
- add the individual areas.

*Note:* the angle between the radial arms for each triangle, $C$, is calculated from the difference between the bearings.

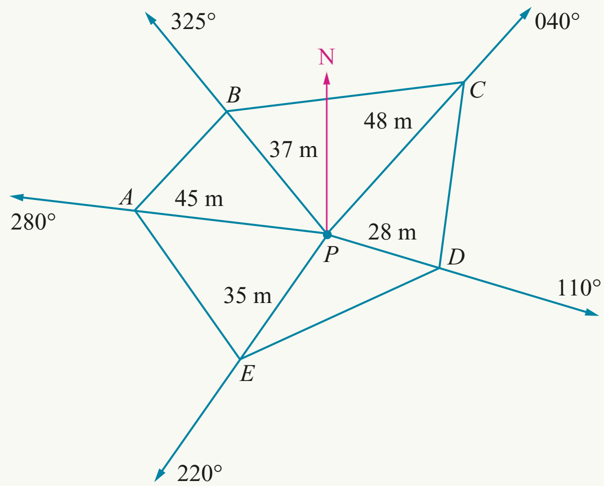
## EXAMPLE 2H–1 Calculating measurements from a compass radial survey

The following diagram is the result of a compass radial survey. Calculate:

**a** the size of angle $BPC$

**b** the length $BC$, correct to one decimal place

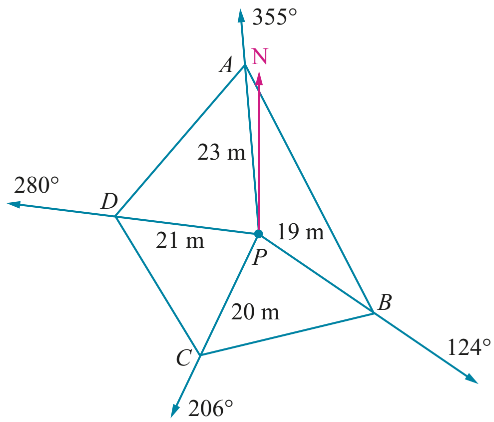
**c** the area of triangle $BPC$, correct to one decimal place.




| | Solve/Think | Apply |
|---|---|---|
| **a** | $\angle BPC = 35° + 40°$ <br> $= 75°$ | Find the angle between the true bearings 325°T and 040°T. First work out the size of the angle between the line $PB$ and the line from $P$ in the north direction ($360° - 325° = 35°$) and add the angle between the line from $P$ in the north direction to the line $PC$ ($40° - 0° = 40°$). |
| **b** | $(BC)^2$ <br> $= 37^2 + 48^2 - 2 \times 37 \times 48 \times \cos 75°$ <br> $\approx 2753.67$ <br> $BC \approx 52.5$ m | Use the cosine rule to find the length of the line $BC$: $c^2 = a^2 + b^2 - 2ab\cos C$ where $a$ and $b$ are the lengths along the radial arms forming two sides of the triangle and $C$ is the angle between them at the point $P$. For $\Delta BPC$, $a = 37$, $b = 48$ and $C = 75°$. |
| **c** | Area of $\Delta BPC$ <br> $= \frac{1}{2} \times 37 \times 48 \times \sin 75°$ <br> $\approx 857.7$ m$^2$ | Use the formula for area of a triangle: Area $= \frac{1}{2}ab\sin C$ where $a$ and $b$ are the lengths along the radial arms forming two sides of the triangle and $C$ is the angle between them at the point $P$. For $\Delta BPC$, $a = 37$, $b = 48$ and $C = 75°$. |

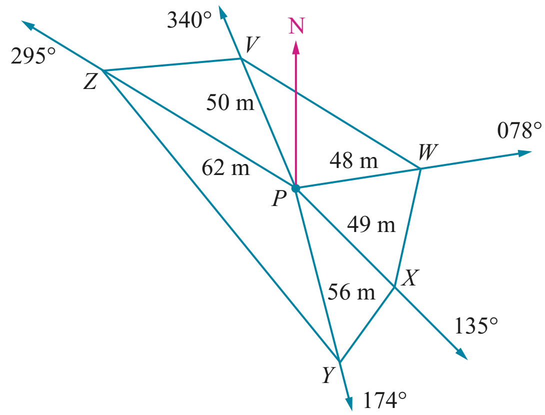
# EXERCISE 2H Compass radial surveys

Unless stated otherwise, find all distances correct to one decimal place.




**1** The diagram on the right shows the result of a compass radial survey. Calculate:
  **a** the size of angle *APB*
  **b** the length *AB*
  **c** the area of triangle *APB*.

**2** Use the diagram in question **1** to calculate:
  **a** the size of angle *BPC*
  **b** the length *BC*
  **c** the area of triangle *BPC*.

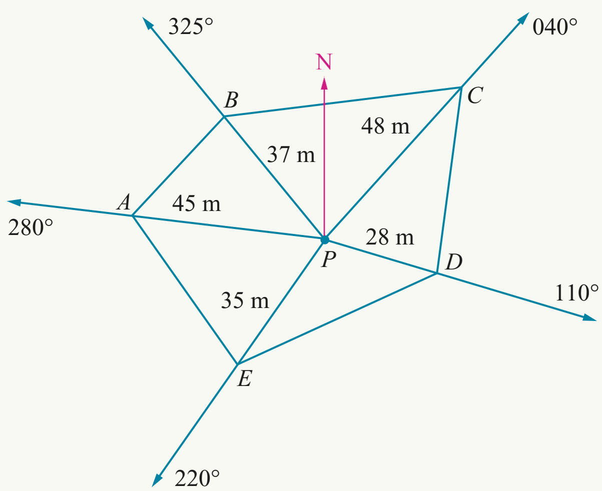



**3** The diagram on the right shows the result of a compass radial survey. Calculate:
  **a** the size of angle *VPW*
  **b** the length *VW*
  **c** the area of triangle *VPW*.

**4** Use the diagram in question **3** to calculate:
  **a** the size of angle *WPX*
  **b** the length *WX*
  **c** the area of triangle *WPX*.

## EXAMPLE 2H–2 Calculating the perimeter of a field from a compass radial survey

Calculate the perimeter of the field *ABCDE* shown in Example 2H–1, to the nearest metre.



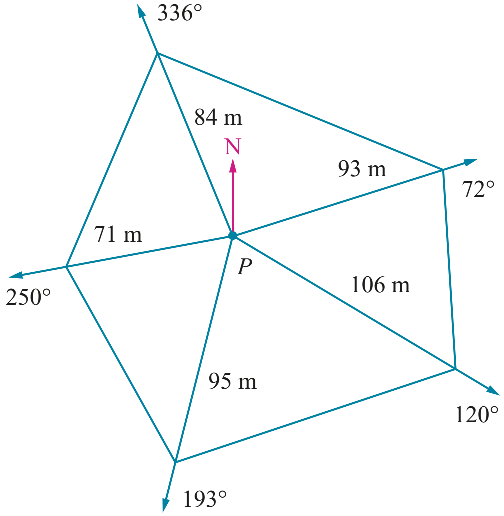
| Solve/Think | Apply |
|---|---|
| $\angle APB = 325° - 280° = 45°$<br>$(AB)^2 = 45^2 + 37^2 - 2 \times 45 \times 37 \times \cos 45°$<br>$\approx 1039.33$<br>$AB \approx 32.2$ m<br>$\angle BPC = 35° + 40° = 75°$<br>$(BC)^2 = 37^2 + 48^2 - 2 \times 37 \times 48 \times \cos 75°$<br>$\approx 2753.67$<br>$BC \approx 52.5$ m<br>$\angle CPD = 110° - 40° = 70°$<br>$(CD)^2 = 48^2 + 28^2 - 2 \times 48 \times 28 \times \cos 70°$<br>$\approx 2168.65$<br>$CD \approx 46.6$ m<br>$\angle DPE = 220° - 110° = 110°$<br>$(DE)^2 = 35^2 + 28^2 - 2 \times 35 \times 28 \times \cos 110°$<br>$\approx 2679.36$<br>$DE \approx 51.8$ m<br>$\angle EPA = 280° - 220° = 60°$<br>$(EA)^2 = 35^2 + 45^2 - 2 \times 35 \times 45 \times \cos 60°$<br>$= 1675$<br>$EA \approx 40.9$ m<br>Perimeter $\approx 32.2 + 52.5 + 46.6 + 51.8 + 40.9$<br>$= 224$ m | Find the angle between the true bearings and use the cosine rule to find the length of each straight side of the field. The cosine rule is $c^2 = a^2 + b^2 - 2ab \cos C$, where $a$ and $b$ are the lengths along the radial arms forming two sides of a triangle and $C$ is the angle between them at the point $P$.<br>Add all the separate lengths of the sides to calculate the perimeter of the field. |

**5** Calculate the perimeter of the field *ABCD*, to the nearest metre, using the compass radial survey shown in question **1**.

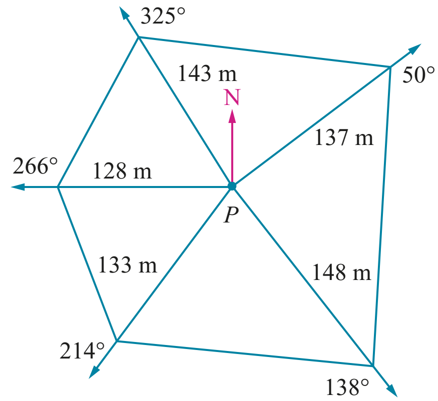
**6** Calculate the perimeter of the field *VWXYZ*, to the nearest metre, using the compass radial survey shown in question **3**.

**7** Calculate the perimeter of each field below, to the nearest metre, using the compass radial survey shown.

**a**

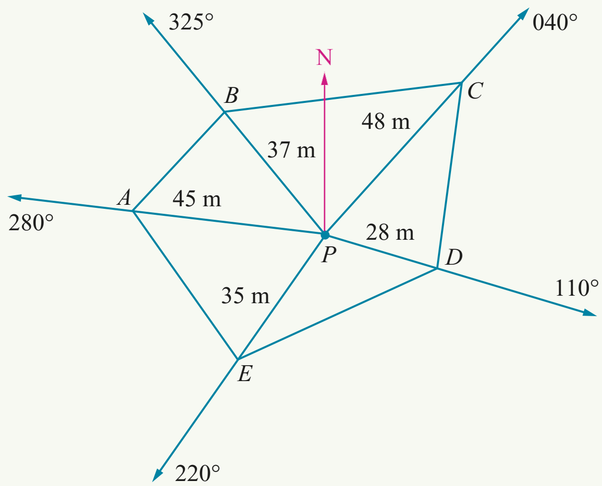



**b**



## EXAMPLE 2H–3 Calculating the area of a field from a compass radial survey

Calculate the area of the field *ABCDE* shown in Example 2H–1, to the nearest square metre.



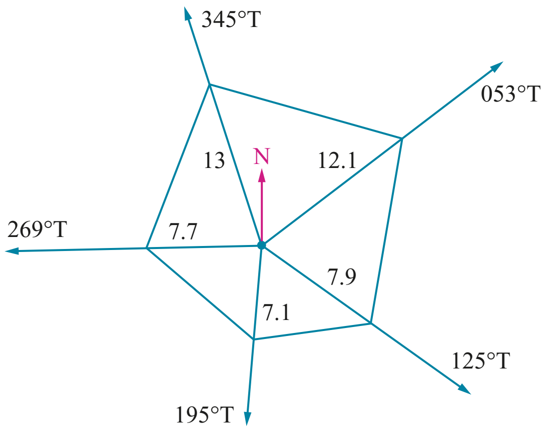

| Solve/Think | Apply |
|---|---|
| $\angle APB = 325° - 280° = 45°$<br>Area of $\Delta APB = \frac{1}{2} \times 45 \times 37 \times \sin 45°$<br>$\approx 588.7 \text{ m}^2$<br>$\angle BPC = 35° + 40° = 75°$<br>Area of $\Delta BPC = \frac{1}{2} \times 37 \times 48 \times \sin 75°$<br>$\approx 857.7 \text{ m}^2$<br>$\angle CPD = 110° - 40° = 70°$<br>Area of $\Delta CPD = \frac{1}{2} \times 48 \times 28 \times \sin 70°$<br>$\approx 631.5 \text{ m}^2$<br>$\angle DPE = 220° - 110° = 110°$<br>Area of $\Delta DPE = \frac{1}{2} \times 28 \times 35 \times \sin 110°$<br>$\approx 460.4 \text{ m}^2$<br>$\angle EPA = 280° - 220° = 60°$<br>Area of $\Delta EPA = \frac{1}{2} \times 35 \times 45 \times \sin 60°$<br>$\approx 682.0 \text{ m}^2$<br>Area of field $\approx 588.7 + 857.7 + 631.5 + 460.4 + 682.0$<br>$= 3220.3$<br>$\approx 3220 \text{ m}^2$ | Find the angle between the true bearings and use the formula for area of a triangle: Area $= \frac{1}{2}ab \sin C$, where $a$ and $b$ are the lengths along the radial arms forming two sides of a triangle and $C$ is the angle between them at the point $P$. Add the separate areas to calculate the total area of the field. |

**8** Calculate the area of the field *ABCD*, to the nearest square metre, using the compass radial survey shown in question **1**.

**9** Calculate the area of the field *VWXYZ*, to the nearest square metre, using the compass radial survey shown in question **3**.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**10** **a** The sketch on the right is the result of a compass radial survey of a field. All measurements are shown in metres.



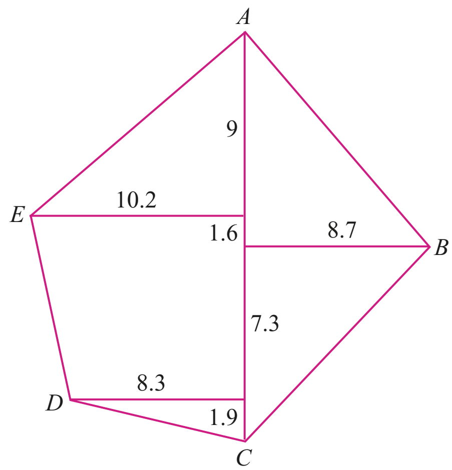

**i** Calculate the perimeter of the field using the cosine rule. Give your answer to the nearest metre.

**ii** Calculate the area of the field. Give your answer to the nearest square metre.

**b** The same field was surveyed using another method known as the traverse survey. The sketch is shown below right. All the measurements are in metres.

**i** Calculate the perimeter of the field using Pythagoras' theorem. Give your answer to the nearest metre.

**ii** Calculate the area of the field. Give your answer to the nearest square metre.




**c** Compare your answers for parts **a** and **b**. Give an explanation of the differences in your answers.

**d** Give a reasonable estimate for the perimeter and area of the field, explaining your reasons.

**e** Produce scale drawings using the information supplied for each survey method and compare the drawings. To create the scale drawings, use a protractor and ruler and decide on a suitable scale (for example 1 : 200 or 1 cm represents 2 m). Discuss any differences in the diagrams and give reasons for these differences.

PRACTICAL TASKS

**11** Work in groups of four or five.

**a** Carry out a compass radial survey of your classroom and sketch the results using the following three steps.

*Step 1:* Choose a suitable point inside the classroom.

*Step 2:* Find the true bearing of each of the vertices of the classroom from this point using a magnetic compass.

*Step 3:* Measure the distance from this point to each vertex.

**b** Use your sketch of the compass radial survey to calculate the perimeter of your classroom.

**c** Use your sketch to calculate the area of your classroom.

**d** Produce a scale drawing of the classroom using measurements from your compass radial survey.

**e** Use your scale drawing to find the perimeter of your classroom and compare to the result obtained in part **b**.

**f** Measure the dimensions of the classroom and use these actual measurements to calculate the perimeter and area.

**g** Compare your perimeter and area results from part **f** with those obtained in parts **b** and **c**.

**12** Repeat parts **a–e** of question **11** but this time carry out a compass radial survey of part of your school grounds.

**13** Carry out a compass radial survey of a field or outdoor space of your own choosing. Give reasonable answers for the perimeter and area.

# CHAPTER 2 REVIEW NON-RIGHT-ANGLED TRIGONOMETRY

You should be able to:

✔ find side lengths and angles in right-angled triangles using sin, cos and tan

✔ solve problems using trigonometric ratios

✔ solve problems involving angles of elevation and depression

✔ understand both true and compass bearings, and solve problems

✔ calculate the area of non-right-angled triangles

✔ use the sine rule to find lengths and angles

✔ use the cosine rule to find lengths and angles

✔ solve problems involving more than one triangle

✔ solve problems involving right-angled and non-right-angled triangle trigonometry

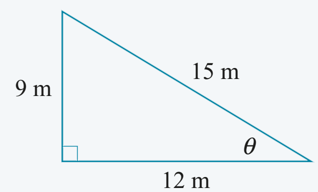
✔ find the perimeter and area of a field from a sketch of a compass radial survey.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

2A **1** In the triangle on the right, what is $\sin\theta$?

**A** $\frac{12}{9}$ **B** $\frac{9}{12}$ **C** $\frac{9}{15}$ **D** $\frac{12}{15}$

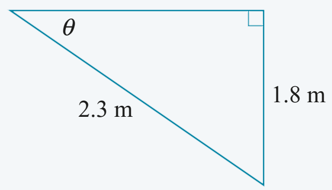


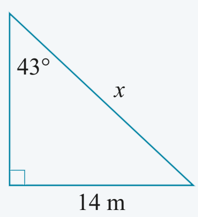

2A **2** What is cos 53°18′, correct to four decimal places?

**A** 0.5976 **B** 0.8018 **C** 0.6018 **D** 1.3416

2A **3** What is the value of $\theta$ in the triangle below, correct to the nearest minute?

**A** 38°29′ **B** 38°30′ **C** 38°3′ **D** 51°30′



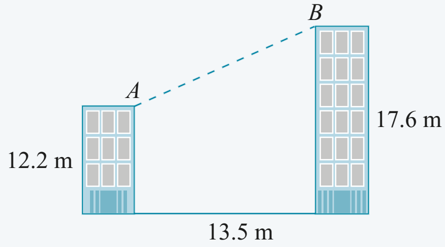




2A **4** What is the value of $x$ in the triangle on the right, to one decimal place?

**A** 20.5 **B** 19.1 **C** 19.2 **D** 15.0

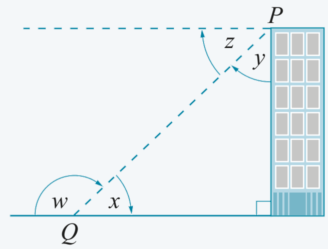
2B **5** What is the angle of elevation from $A$ to $B$, to the nearest degree?

**A** 22° **B** 24° **C** 53° **D** 55°

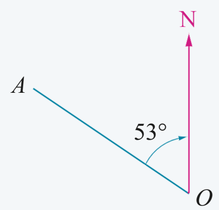


2B ▶ **6** Which angle is the angle of depression from $P$ to $Q$?

**A** $w$ **B** $x$ **C** $y$ **D** $z$




Use this diagram to answer questions **7–9**.




2C ▶ **7** What is the compass bearing of $A$ from $O$ in the diagram?

**A** N53°E **B** N53° W **C** W53°N **D** N37° W

2C ▶ **8** What is the true bearing of $A$ from $O$ in the diagram?

**A** 053° T **B** 127° T **C** 307° T **D** 53°N

2C ▶ **9** A ship sails 57 km from $O$ to $A$ on the bearing shown in the diagram. How far north of its starting point is the ship?

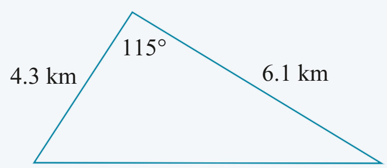
**A** 34.3 km **B** 45.5 km **C** 57.0 km **D** 75.6 km

2C ▶ **10** Town $X$ is 58 km east and 113 km south of town $Y$. The bearing of $X$ from $Y$ is closest to:

**A** S20°E **B** N53°E **C** 063° T **D** 153° T

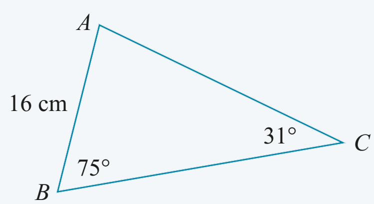
2D ▶ **11** What is the area of this triangle, correct to the nearest km²?

**A** 26.2 km²
**B** 13.1 km²
**C** 11.9 km²
**D** 5.5 km²



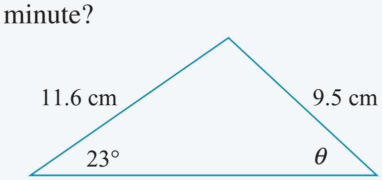

2E ▶ **12** What is the length of $AC$, correct to the nearest cm?

**A** 8.5 cm
**B** 30 cm
**C** 31 cm
**D** 31.06 cm



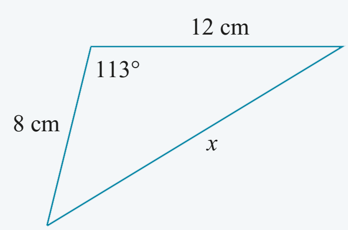

2E ▶ **13** What is the value of $\theta$ in this triangle, correct to the nearest minute?

**A** 40°19′
**B** 18°39′
**C** 18°40′
**D** 28°30′



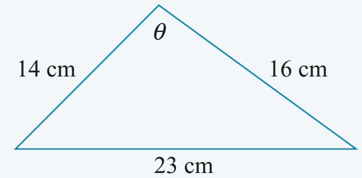

2F ▶ **14** What is the value of $x$, correct to the nearest cm?

**A** 282 cm
**B** 141 cm
**C** 16.8 cm
**D** 11.3 cm

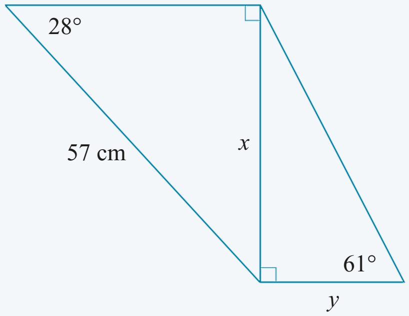


2F **15** What is the value of $\theta$, correct to the nearest minute?

**A** 36°51′
**B** 43°16′
**C** 81°6′
**D** 99°54′




Use the diagram on the right to answer questions **16** and **17**.



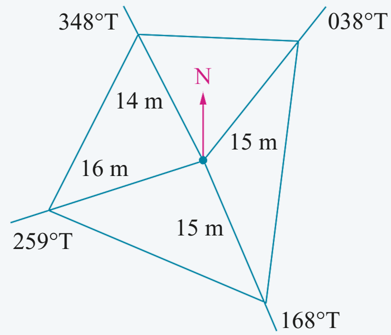

2G **16** What is the value of $x$ closest to?

**A** 27 cm
**B** 65 cm
**C** 107 cm
**D** 121 cm

2G **17** What is the value of $y$ closest to?

**A** 15 cm **B** 74 cm **C** 122 cm **D** 138 cm

Use this diagram of a compass radial survey to answer questions **18** and **19**.




2H **18** Which value is the closest to the perimeter of this field?

**A** 60 m **B** 83 m **C** 110 m **D** 124 m

2H **19** Which value is the closest to the area of this field?

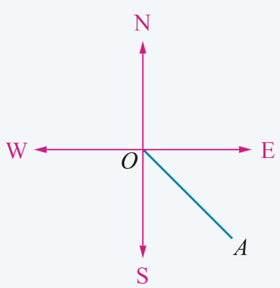
**A** 200 m$^2$
**B** 400 m$^2$
**C** 600 m$^2$
**D** 800 m$^2$

## REVIEW SET 1

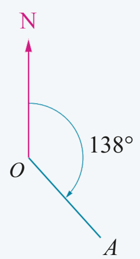
**1** Write the bearing of $A$ from $O$ shown in the following diagrams as:

**i** a compass bearing
**ii** a true bearing.

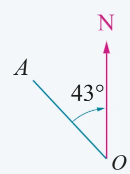
**a**



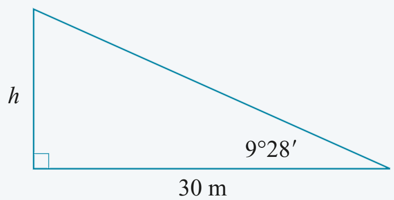

**b**




**c**




**2** The angle of elevation of the top of a bridge from a point 30 m away is 9°28′. Calculate the height of the bridge to the nearest metre.

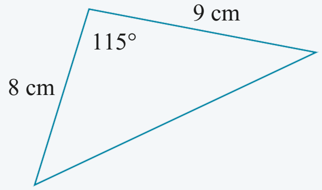



**3** **a** A ship sails 78 km from port $O$ on a bearing of 148°T. How far east has the ship sailed, to the nearest kilometre?

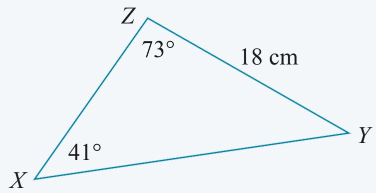
**b** Town $A$ is 43 km east and 74 km south of town $B$. Find the compass bearing and the true bearing of $A$ from $B$, to the nearest degree.

**4** In the following problems, give values for length and area correct to one decimal place and angles to the nearest minute.

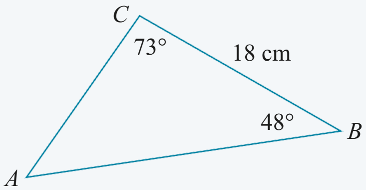
**a** Find the area of this triangle.



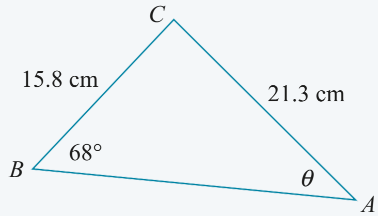

**b** Find the length of $XY$.



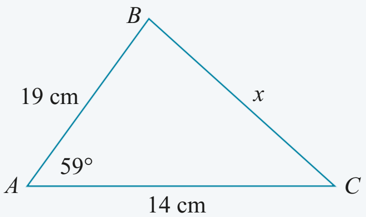

**c** Find the length of $AB$.



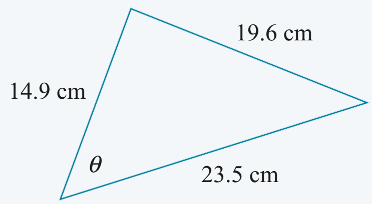

**d** Find the value of $\theta$.




**e** Find the value of $x$.

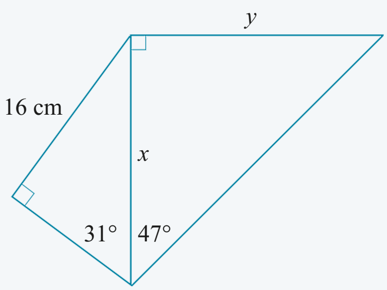



**f** Find the value of $\theta$.

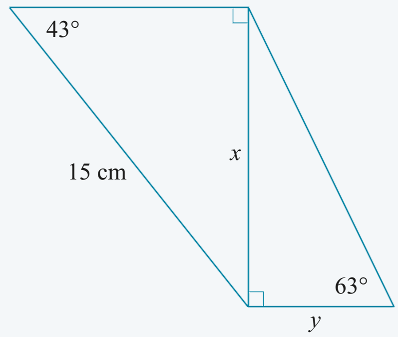



**5** Find the value of $x$ and $y$ in each diagram, correct to one decimal place.

**a**




**b**

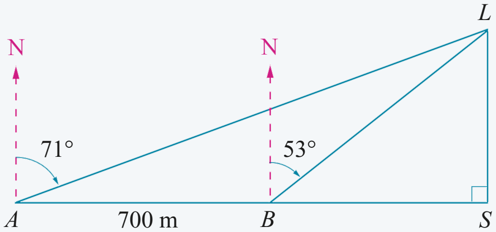



**6** A boat is travelling due east. The bearing of a lighthouse from the boat is 071°T. The boat travels 700 m and the bearing of the lighthouse is now 053°T.

**a** Calculate $\angle LAB$, $\angle LBS$ and $\angle ABL$.

**b** Use the sine rule to find the current distance from the boat to the lighthouse, to the nearest metre.

**c** Find the distance the boat needs to travel on the same course to be due south of the lighthouse, to the nearest metre.

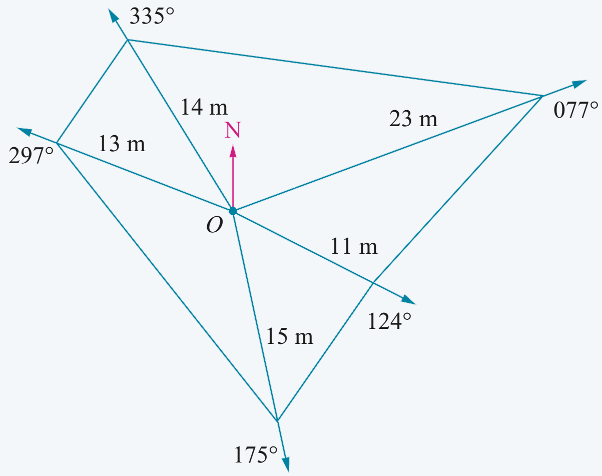



MEASUREMENT

**7** The information from a compass radial survey of a field is shown in the diagram on the right.

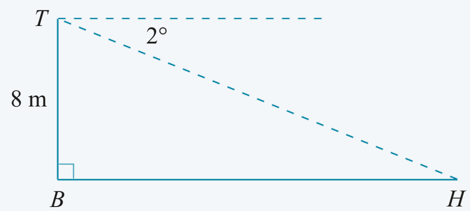
**a** Calculate the perimeter to the nearest metre using the cosine rule.

**b** Calculate the area of the field, to the nearest square metre.




# REVIEW SET 2

**1** An observation tower is 8 m high. From the top of the towers the angle of depression to a small hut is 2°. How far is the hut from the base of the tower, to the nearest metre?

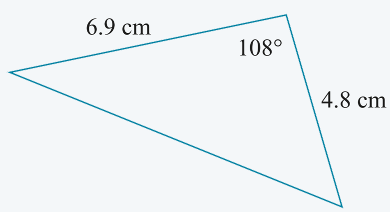



**2** **a** A ship sails 78 km from port $O$ on a bearing of 236°T. How far west has the ship sailed, to the nearest kilometre?

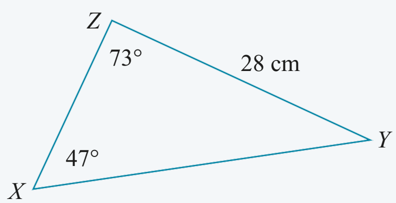
**b** Town $P$ is 48 km west and 94 km south of town $Q$. Find the compass bearing and the true bearing of $P$ from $Q$, to the nearest degree.

**3** In these problems, give values for length and area correct to one decimal place and angles to the nearest minute.

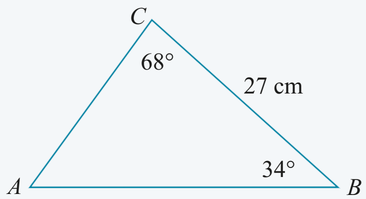
**a** Find the area of this triangle.



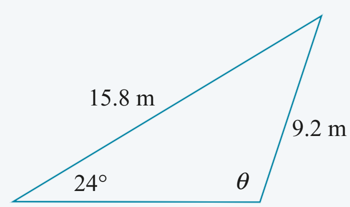

**b** Find the length of $XY$.



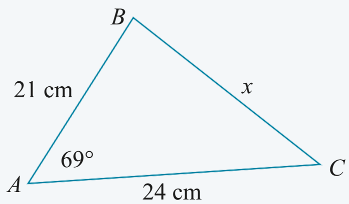

**c** Find the length of $AB$.



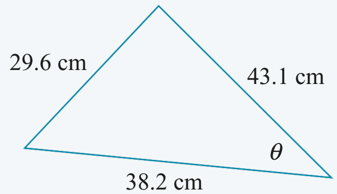

**d** Find the value of $\theta$.




**e** Find the value of $x$.

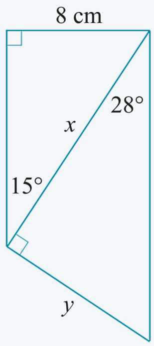



**f** Find the value of $\theta$.

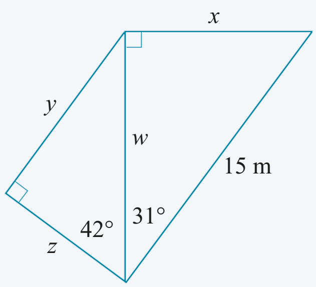


**4** In the following problems, give values for length correct to one decimal place.

**a** Find the value of $x$ and $y$.



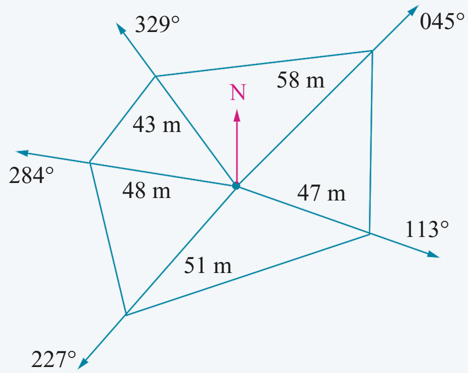

**b** Calculate the value of all pronumerals.




**5** The information from a compass radial survey of a field is shown in the diagram on the right.

**a** Calculate the perimeter of the field, to the nearest metre, using the cosine rule.

**b** Calculate the area of the field, to the nearest square metre.



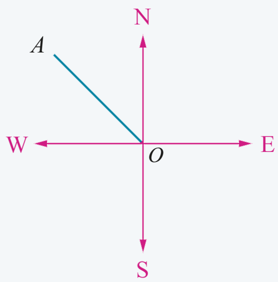

# REVIEW SET 3

**1** Write the bearing of $A$ from $O$ shown in the following diagrams as:

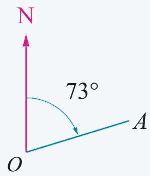
**i** a compass bearing

**ii** a true bearing.

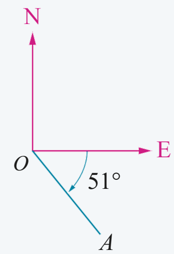
**a**



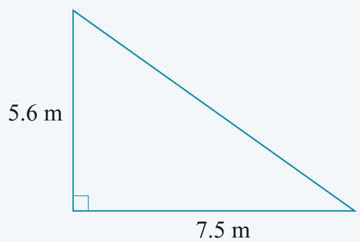

**b**




**c**




**2** A flagpole 5.6 m high casts a shadow 7.5 m long. What is the angle of elevation of the Sun, to the nearest minute?



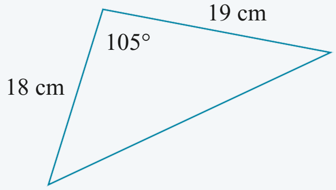

MEASUREMENT

**3** **a** A ship sails 236 km from port $O$ on a bearing of 196° T. How far south has the ship sailed, to the nearest kilometre?

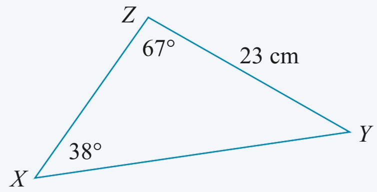
**b** Town $T$ is 93 km west and 73 km north of town $S$. Find the compass bearing and the true bearing of $T$ from $S$, to the nearest degree.

**4** In these problems, give values for length and area correct to one decimal place and angles to the nearest minute.

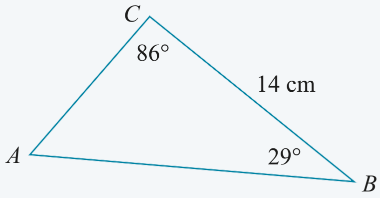
**a** Find the area of this triangle.



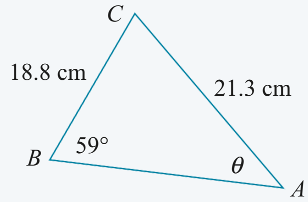

**b** Find the length of $XY$.



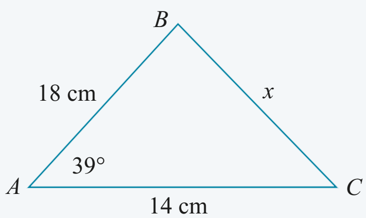

**c** Find the length of $AB$.



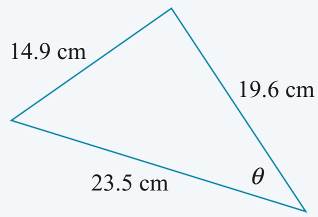

**d** Find the value of $\theta$.




**e** Find the value of $x$.




**f** Find the value of $\theta$.

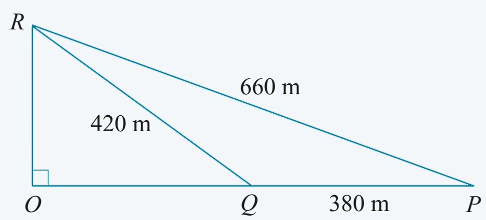



**5** In the diagram on the right, $PQ = 380$ m, $QR = 420$ m, $PR = 660$ m and $\angle QOR = 90°$.

**a** Use the cosine rule to find $\angle PQR$, correct to the nearest minute, and hence $\angle RQO$.

**b** Find the length of $OR$, correct to one decimal place.

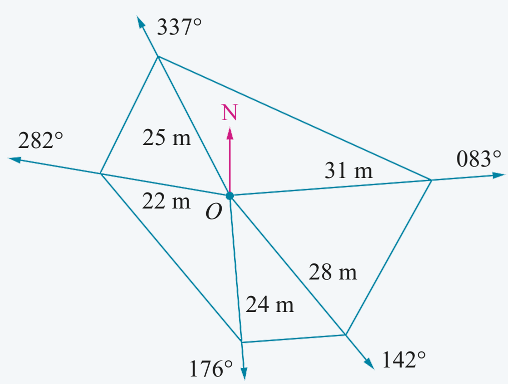
**c** If $OR$ is north, find the true bearing of $R$ from $Q$, correct to the nearest minute.




**6** The information from a compass radial survey of a field is shown in the diagram on the right.

**a** Calculate the perimeter of the field, to the nearest metre, using the cosine rule.

**b** Calculate the area of the field, to the nearest square metre.

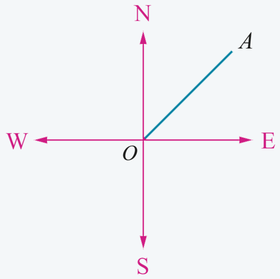


# REVIEW SET 4

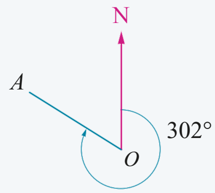
**1** Write the bearing of $A$ from $O$ shown in the following diagrams as:

**i** a compass bearing **ii** a true bearing.

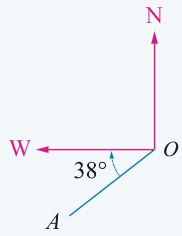
**a**



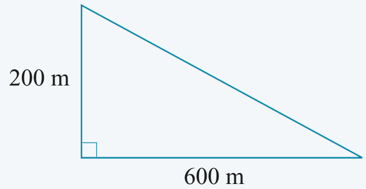

**b**




**c**




**2** A plane flying at an altitude of 200 m is 600 m, horizontally, from an airport runway. Calculate the angle of depression from the plane to the runway, to the nearest minute.

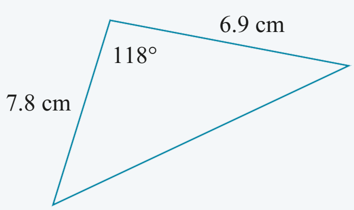



**3** **a** A ship sails 112 km from port $O$ on a bearing of 153°T. How far east has the ship sailed, to the nearest kilometre?

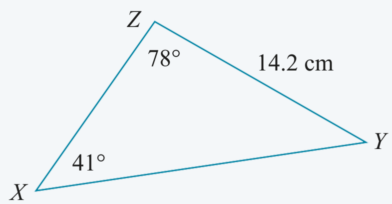
**b** Town $R$ is 125 km east and 87 km north of town $S$. Find the compass bearing and the true bearing of $R$ from $S$, to the nearest degree.

**4** In the following problems, give values for length and area correct to one decimal place and angles to the nearest minute.

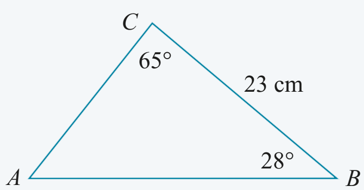
**a** Find the area of this triangle.



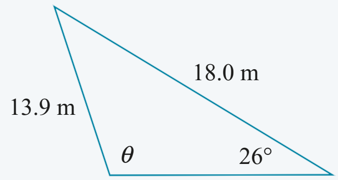

**b** Find the length of $XY$.



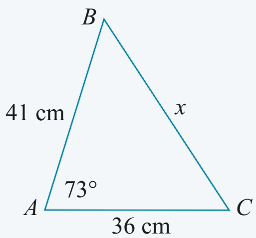

**c** Find the length of $AB$.



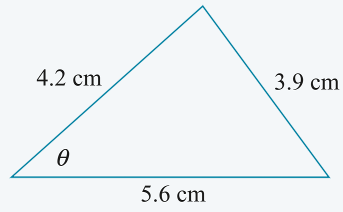

**d** Find the value of $\theta$.




**e** Find the value of $x$.

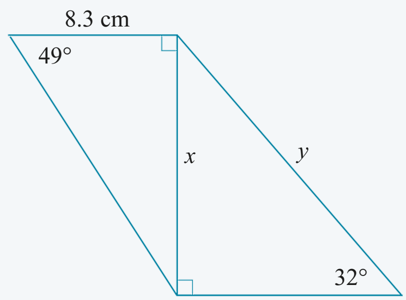



**f** Find the value of $\theta$.




MEASUREMENT

**5** Find the value of $x$ and $y$, correct to one decimal place.



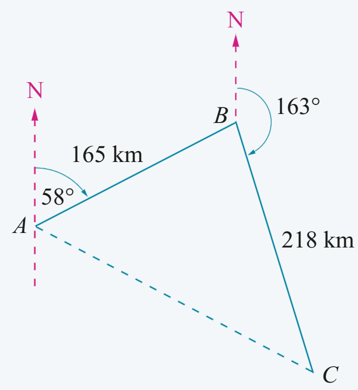

**6** A ship sails 165 km from $A$ to $B$ on a bearing of 058°T. It then turns and sails 218 km to $C$, on a bearing of 163°T.

**a** Find $\angle ABC$.

**b** How far is the ship from its starting point, to the nearest kilometre?

**c** Calculate $\angle BAC$, to the nearest minute.

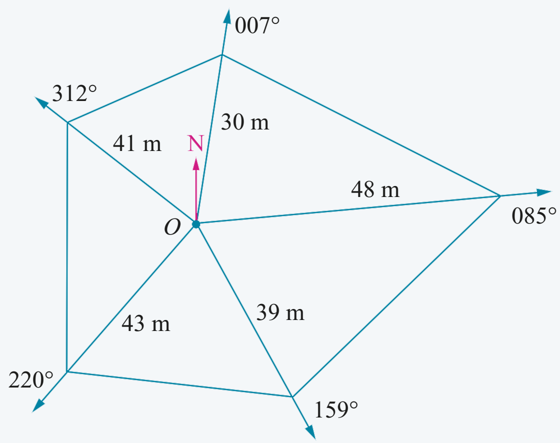
**d** What is the true bearing of the ship from its starting point, to the nearest minute?




**7** The information from a compass radial survey of a field is shown in the diagram on the right.

**a** Calculate the perimeter of the field, to the nearest metre.

**b** Calculate the area of the field, to the nearest square metre.

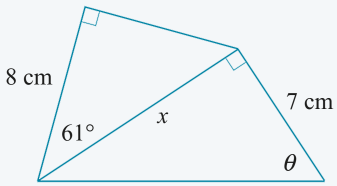



## REVIEW PRACTICE EXAMINATION QUESTION

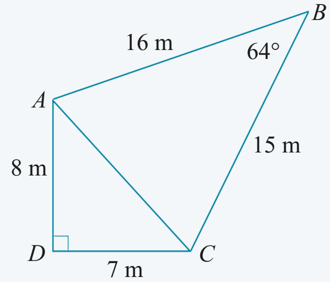
**1** **a** For the diagram below, calculate:

**i** the value of $x$, to one decimal place (1 mark)

**ii** the value of $\theta$, to the nearest minute. (2 marks)



**b** Find the area of the figure $ABCD$, to one decimal point. (2 marks)



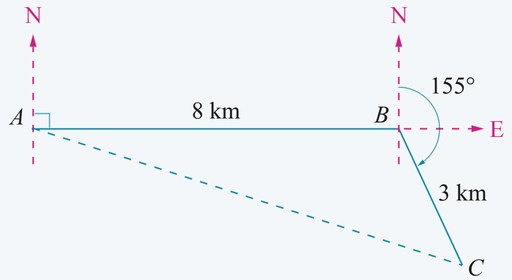

**c** A bushwalker starts at point $A$ and walks 8 km in the direction due east to point $B$. At $B$ he turns and walks on a bearing of 155°T for 3 km to $C$.

**i** Find $\angle ABC$. (2 marks)

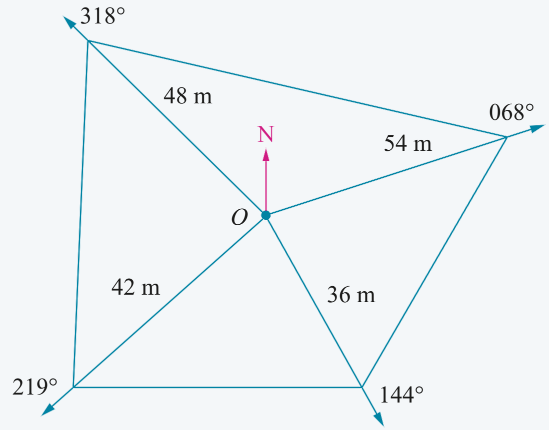
**ii** Calculate the distance $AC$, correct to one decimal place. (2 marks)

**iii** What is the bearing of $C$ from $A$, to the nearest minute? (1 mark)

**iv** What is the bearing of $A$ from $C$, to the nearest minute? (1 mark)




**d** The information from a compass radial survey of a field is shown in the diagram below.




**i** Find the perimeter of the field, to the nearest metre. (2 marks)

**ii** Find the area of the field, to the nearest square metre. (2 marks)

**TOTAL: 15 marks**

MEASUREMENT

# 3

# Rates and ratios

The main mathematical ideas investigated are:

- using rates to solve practical problems (including speed and best buys)
- performing calculations relating to heart rate
- performing calculations relating to fuel consumption rate
- working with power as a rate
- interpreting energy ratings of household appliances
- comparing running costs of appliances
- expressing ratios in simplest form
- dividing quantities in given ratios
- using scale drawings, building plans and maps
- solving practical problems using scale drawings.

# ARE YOU READY?



3A **1** Which measurement is equivalent to 4.5 m?

**A** 45 cm **B** 450 cm
**C** 0.045 km **D** 45 mm

3A **2** Which measurement is equivalent to 260 L?

**A** 0.26 kL **B** 26 kL
**C** 26 000 mL **D** 0.026 ML

3A **3** Which measurement is equivalent to 6 ha?

**A** 600 m$^2$ **B** 6000 m$^2$
**C** 60 000 m$^2$ **D** 6 km$^2$

3A **4** What is the highest common factor of 45 and 72?

**A** 3 **B** 9
**C** 18 **D** 360

3A **5** Which fraction is equivalent to $\frac{1}{3}$?

**A** $\frac{5}{16}$ **B** $\frac{9}{18}$
**C** $\frac{2}{15}$ **D** $\frac{4}{12}$

3A **6** What is 24 000 written in scientific notation?

**A** $24 \times 10^3$ **B** $2.4 \times 10^3$
**C** $2.4 \times 10^4$ **D** $2.4 \times 10^{-4}$

3A **7** What is the result of $40 \times 10^2 \times 10^5$ written in scientific notation?

**A** $40 \times 10^7$ **B** $40 \times 10^{10}$
**C** $4 \times 10^7$ **D** $4 \times 10^8$

3E **8** Which equation is equivalent to $\frac{x}{2} = \frac{5}{7}$?

**A** $x = \frac{5}{7} \div 2$ **B** $x = \frac{5}{7} \times 2$
**C** $x = \frac{5}{7} + 2$ **D** $x = \frac{5}{7} - 2$

3E **9** Which equation is equivalent to $\frac{3}{y} = \frac{11}{2}$?

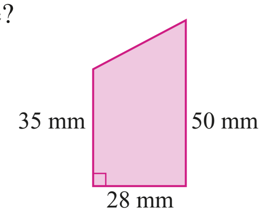
**A** $y = \frac{11}{2} \times 3$ **B** $y = \frac{11}{3} \times 2$
**C** $y = \frac{2}{11} \times 3$ **D** $y = \frac{2}{3} \times 11$

3G **10** What is the area of a rectangle with length 8 m and breadth 3 m?

**A** 11 m$^2$ **B** 22 m$^2$
**C** 24 m$^2$ **D** 48 m$^2$

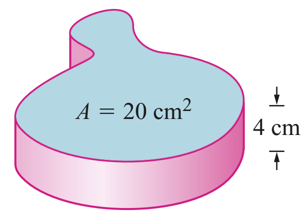
3G **11** What is the area of this shape?

**A** $\frac{1}{2} \times 28 \times (35 + 50)$ mm$^2$
**B** $(28 \times 50)$ mm$^2$
**C** $\frac{1}{2} \times 50 \times (28 + 35)$ mm$^2$
**D** $28 \times (35 + 50)$ mm$^2$



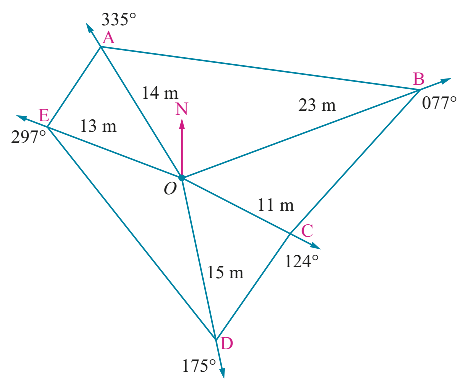

3G **12** What is the volume of this solid?

**A** 5 cm$^3$
**B** 16 cm$^3$
**C** 24 cm$^3$
**D** 80 cm$^3$




3H **13** What is the area of triangle $AOE$ in this radial survey?

**A** $\frac{1}{2} \times 13 \times 14$
**B** $\frac{1}{2} \times 13 \times 14 \times \sin 38°$
**C** $\frac{1}{2} \times 13 \times 14 \times \cos 38°$
**D** $\frac{1}{2} \times 13 \times 14 \times \tan 38°$




If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

| | |
|---|---|
| **Q1–3** | **Support sheet 3A.1 Converting units of measurement** |
| **Q4** | **Support sheet 3A.2 Finding the highest common factor** |
| **Q5** | **Support sheet 3A.3 Simplifying fractions** |
| **Q6–7** | **Support sheet 3A.4 Scientific notation** |
| **Q8–9** | **Support sheet 3E.1 Solving equations involving fractional terms** |
| **Q10–11** | **Support sheet 3G.1 Calculating area** |
| **Q12** | **Support sheet 3G.2 Calculating volume** |
| **Q13** | **Support sheet 3H.1 Calculating area of triangles using radial surveys** |

# 3A Rates

These resources are available on your obook assess:
- **Interactive 3A:** Explore key ideas for rates
- **Video tutorial 3A:** Watch and listen to an explanation of Examples 3A–3, 3A–4
- **Spreadsheet 3A:** Modify a recipe for various numbers of people
- **Worksheet 3A:** Practise your skills with extra problems for writing rates
- **Investigation 3A.1:** Explore rates involved with antipasto platters
- **Investigation 3A.2:** Explore rates when producing different chemical solutions
- **assess quiz 3A:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**rate**
a comparison of one quantity with another; e.g. speed is a rate, as distance is compared to time

A **rate** is a comparison between quantities of different kinds.

The comparison is made by dividing one quantity by the other, in the required order. The rate is then expressed in the form 'the first quantity per unit of the second quantity'.

!

We may compare distance travelled with petrol used, distance travelled in a given time, amount of fertiliser needed with area of land, amount of pay with time worked, etc.

## EXAMPLE 3A–1 Writing a rate to compare two quantities

Wendy types 600 words in 8 minutes.
How many words per minute does she type?

| Solve | Think | Apply |
|---|---|---|
| $\text{Rate} = \dfrac{600\,\text{words}}{8\,\text{min}}$<br>$= \dfrac{(600 \div 8)\ \text{words}}{(8 \div 8)\ \text{min}}$<br>$= \dfrac{75\,\text{words}}{1\ \text{min}}$<br>$= 75\ \text{words/min}$ | This rate is comparing the number of words with time, in that order. So, divide the number of words by the time. Simplify until the rate is expressed as the number of words in 1 minute.<br>Rate $=$ 600 words in 8 min<br>$=$ (600 $\div$ 8) words in 1 min<br>$=$ 75 words in 1 min<br>$=$ 75 words/min<br>This is the number of words per unit of time. | Compare the quantities by dividing one quantity by the other. Simplify until the second quantity is 1 unit. |

# EXERCISE 3A Rates

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Complete the following to write the rate for each scenario.

**a** Jenny typed 300 words in 5 minutes. Find her typing rate per minute.

$$\begin{aligned}\text{Rate} &= \frac{\text{number of words}}{\text{time}}\\ &= \frac{\square\ \text{words}}{\square\ \text{min}}\\ &= \frac{\square\ \text{words}}{1\ \text{min}}\\ &= \_\_\_\_\ \text{words/min}\end{aligned}$$

**b** A 2.5 kg box of soap powder costs $10.90. Find the cost per kg.

$$\begin{aligned}\text{Rate} &= \frac{\text{cost}}{\text{weight}}\\ &= \frac{\$\square}{\square\ \text{kg}}\\ &= \frac{\$\square}{1\ \text{kg}}\\ &= \_\_\_\_\ \$/\text{kg or } \$\_\_\_\_/\text{kg}\end{aligned}$$

**c** Vicki was paid $82.50 for 6 hours of work. Calculate her rate of pay.

$$\begin{aligned}\text{Rate} &= \frac{\text{amount earned}}{\text{hours worked}}\\ &= \frac{\$\square}{\square\ \text{h}}\\ &= \frac{\$\square}{1\ \text{h}}\\ &= \_\_\_\_\ \$/\text{h or } \$\_\_\_\_/\text{h}\end{aligned}$$

**d** Nathan had to pay $55.20 for 120 telephone calls. Calculate the cost per call.

$$\begin{aligned}\text{Rate} &= \frac{\text{total cost}}{\text{number of calls}}\\ &= \frac{\$\square}{\square\ \text{calls}}\\ &= \frac{\$\square}{1\ \text{call}}\\ &= \_\_\_\_\ \$/\text{call or } \$\_\_\_\_/\text{call}\end{aligned}$$

**2** **a** The temperature rose 14°C in $3\frac{1}{2}$ h. At what rate, in degrees per hour, did the temperature rise?

**b** Jeremy spread 24 kg of fertiliser over an area of 60 m$^2$. Calculate the rate of application in kg per m$^2$.

**c** Peggy drove 195 km in $2\frac{1}{2}$ h at a constant speed. Calculate her speed in km/h.

**d** Calculate the flow rate per minute when 119 L of water flows through a pipe in 35 min.

> **!** In some cases, it is more convenient or useful to express a rate using a different set of units. For example, a speed in kilometres per hour (km/h) can be converted to a speed in metres per second (m/s).
>
> We need to convert the given units of each quantity in the rate to the required units. Note that we are still comparing the same quantities; for example, speed is a rate comparing distance and time.

## EXAMPLE 3A–2 Converting units for a rate comparing mass and area

Convert 8 t/ha to:

**a** kg/ha **b** kg/m$^2$ **c** g/m$^2$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $8\ \text{t/ha} = \frac{8\ \text{t}}{1\ \text{ha}} = \frac{8000\ \text{kg}}{1\ \text{ha}} = 8000\ \text{kg/ha}$ | Convert the mass from tonnes to kilograms using 1 t = 1000 kg. There is no change to the area units. | Convert the given units to the required units and divide in the order given. |

MEASUREMENT

| | Solve | Think | Apply |
|---|---|---|---|
| **b** | $8 \text{ t/ha} = \frac{8 \text{ t}}{1 \text{ ha}}$<br>$= \frac{8000 \text{ kg}}{10\,000 \text{ m}^2}$<br>$= (8000 \div 10\,000) \text{ kg/m}^2$<br>$= 0.8 \text{ kg/m}^2$ | Convert the mass from tonnes to kilograms (1 t = 1000 kg) and the area from hectares to square metres (1 ha = 10 000 m$^2$). Divide the mass by the area. | |
| **c** | $8 \text{ t/ha} = \frac{8 \text{ t}}{1 \text{ ha}}$<br>$= \frac{(8 \times 1000 \times 1000) \text{ g}}{10\,000 \text{ m}^2}$<br>$= (8\,000\,000 \div 10\,000) \text{ g/m}^2$<br>$= 800 \text{ g/m}^2$ | Convert the mass from tonnes to grams (1 t = (1000 × 1000) g). Convert the area from hectares to square metres (1 ha = 10 000 m$^2$). Divide the mass by the area. | |

## EXAMPLE 3A–3 Converting each quantity in a rate to smaller units

Convert 72 L/h to mL/s.

| Solve | Think | Apply |
|---|---|---|
| $72 \text{ L/h} = \frac{72 \text{ L}}{1 \text{ h}}$<br>$= \frac{(72 \times 1000) \text{ mL}}{(60 \times 60) \text{ s}}$<br>$= (72\,000 \div 3600) \text{ mL/s}$<br>$= 20 \text{ mL/s}$ | Convert the capacity from litres to millilitres (1 L = 1000 mL). Convert the time from hours to seconds (1 h = (60 × 60) s). Divide the capacity by the time. | Convert the given units to the required units and divide in the order given. |

**3** Complete each of the following to convert:

**a** 5 t/ha to kg/ha

$$\frac{5 \text{ t}}{1 \text{ ha}} = \frac{\square \text{ kg}}{1 \text{ ha}}$$
$$= \_\_\_\_ \text{kg/ha}$$

**b** 16 t/ha to kg/m$^2$

$$\frac{16 \text{ t}}{1 \text{ ha}} = \frac{\square \text{ kg}}{\square \text{ m}^2}$$
$$= \_\_\_\_ \text{kg/m}^2$$

**c** \$3.75/h to cents/min

$$\frac{\$3.75}{1 \text{ h}} = \frac{\square \text{ cents}}{\square \text{ min}}$$
$$= \_\_\_\_ \text{c/min}$$

**d** \$1.80/m to cents/mm

$$\frac{\$1.80}{1 \text{ m}} = \frac{\square \text{ cents}}{\square \text{ mm}}$$
$$= \_\_\_\_ \text{c/mm}$$

**e** 15%/year to %/month

$$\frac{15\%}{1 \text{ year}} = \frac{15\%}{\square \text{ months}}$$
$$= \_\_\_\_ \%/\text{month}$$

**f** 2.4 kg/L to g/mL

$$\frac{2.4 \text{ kg}}{1 \text{ L}} = \frac{\square \text{ g}}{\square \text{ mL}}$$
$$= \_\_\_\_ \text{ g/mL}$$

**4** Convert the following.

**a** 45 L/h to mL/s
**b** \$12/kg to cents/g
**c** 18 km/h to m/s
**d** 27 L/h to mL/s
**e** 7.2 kg/day to g/min
**f** 14 km/h to cm/s

## EXAMPLE 3A–4 Converting each quantity in a rate to larger units

Convert 5 m/s to km/h.

| Solve | Think | Apply |
|---|---|---|
| $5 \text{ m/s} = \frac{5 \text{ m}}{1 \text{ s}}$ <br> $= \frac{(5 \times 60 \times 60) \text{ m}}{1 \text{ h}}$ <br> $= \frac{18\,000 \text{ m}}{1 \text{ h}}$ <br> $= \frac{(18\,000 \div 1000) \text{ km}}{1 \text{ h}}$ <br> $= \frac{18 \text{ km}}{1 \text{ h}}$ <br> $= 18 \text{ km/h}$ | First find the number of metres travelled in 1 hour by multiplying the distance by the number of seconds in an hour. <br> 5 m in 1 s = (5 × 60) m in 1 min <br> = (5 × 60 × 60) m in 1 h <br> = 18 000 m in 1 h <br> Then convert the distance from metres to kilometres. <br> 18 000 m in 1 h <br> = (18 000 ÷ 1000) km in 1 h | Convert the speed from m/s to m/h by multiplying the distance by the number of seconds in an hour. Then convert the rate from m/h to km/h by dividing the distance by the number of metres in a kilometre. |

**5** Complete each of the following to convert:

**a** 6 m/s to km/h

The number of seconds in 1 hour

= ____ × ____

= ____

$\frac{6 \text{ m}}{1 \text{ s}} = \frac{(6 \times \square) \text{ m}}{1 \text{ h}}$

$= \frac{\square \text{ m}}{1 \text{ h}}$

The number of metres in a kilometre = ____

$\frac{\square \text{ m}}{1 \text{ h}} = \frac{(\square \div \square) \text{ km}}{1 \text{ h}}$

$= \frac{\square \text{ km}}{1 \text{ h}}$

So 6 m/s = ____ km/h

**b** 3 cents/minute to \$/day

The number of minutes in 1 day

= 24 × ____

= ____

$\frac{3 \text{ cents}}{1 \text{ minute}} = \frac{(3 \times \square) \text{ cents}}{1 \text{ day}}$

$= \frac{\square \text{ cents}}{1 \text{ day}}$

The number of cents in \$1 = _____

So 3 cents/minute

= ____ \$/day or \$____/day

**6**
- **a** Convert 20 m/s to: **i** m/h **ii** km/h
- **b** Convert 5 g/mL to: **i** g/L **ii** kg/L
- **c** Convert 0.8c/g to: **i** c/kg **ii** \$/kg
- **d** Convert 0.75 kg/m$^2$ to: **i** kg/ha **ii** t/ha
- **e** Convert 0.4 mL/s to: **i** mL/h **ii** L/h
- **f** Convert 0.8c/m to: **i** c/km **ii** \$/km

!

Rates can be used to make comparisons. For example, when comparing two or more purchase options to find the one that is the best value or the **best buy**, we calculate the unit price for each option.

**best buy** found by comparing a number of purchase options to find the one that is the best value for money

## EXAMPLE 3A–5 Using unit prices to make comparisons

A box of 10 pens costs \$6.80. These pens can also be bought in a pack of 15 for \$9.60. Which option provides better value or is the best buy?

| Solve | Think | Apply |
|---|---|---|
| *Option 1:*<br>Rate = \$6.80 for 10 pens<br>= (\$6.80 ÷ 10) for 1 pen<br>= \$0.68/pen<br>*Option 2:*<br>Rate = \$9.60 for 15 pens<br>= (\$9.60 ÷ 15) for 1 pen<br>= \$0.64/pen<br>The best buy is the pack of 15 pens. | For each option, the rate is comparing the cost with the number of pens. Divide the cost by the number of pens to find the unit cost in each case. The option with the lower cost per pen is the best buy. | Compare the unit price for each option; that is, find the rate for cost per one pen (cost/pen). |

**7** At the supermarket, a carton of 12 eggs costs \$5.80. Eggs can also be bought at the Sunday market in a carton of 18 for \$8.95. Which option provides better value or is the best buy?

**8** Which bottle of drink is the better value for money: the 2 L bottle of orange mineral water for \$3.50 or the 1.5 L bottle of cola for \$2.75?

**9** Tyler bought a 400 g pack of tomatoes for \$3.60. Lisa bought 600 g of loose tomatoes for \$5.60.

**a** Calculate the amount paid by Tyler and Lisa for 100 g of tomatoes.

**b** Who made the best buy? Explain.

**c** Why does a store offer both options?

**10** Which is the best buy: 700 g of mince steak for \$10.80 from the butcher or 1.2 kg of mince steak for \$16.50 from the supermarket?

**11** A car travels at 60 km/h.

**a** How far will it travel in $2\frac{1}{2}$ hours?

**b** How long will it take to travel 225 km?

**12** Fertiliser is to be spread at the rate of 0.2 kg/m$^2$.

**a** How much fertiliser would be needed for an area of 600 m$^2$?

**b** If the fertiliser is sold in 50 kg bags, how many bags are needed for an area of 600 m$^2$?

**c** What area could be fertilised with 1 t of fertiliser?

**13** A patient in hospital is given an antibiotic solution intravenously at the rate of 80 mL/h.

**a** How much antibiotic solution will the patient receive in $6\frac{1}{2}$ h?

**b** How often would 600 mL containers of the antibiotic solution need to be changed?

**c** If 1 mL of this solution contains 15 drops, calculate the rate at which the patient receives the antibiotic in drops/minute.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**14** Anna is paid at the rate of \$23.60/h.

**a** How much would she be paid for working 15 h?

**b** How long would she need to work to earn \$472?

**c** Anna wants to save for a trip. If she works 28 h in week 1, and 31 h in the next week, how many more hours does she need to work to earn \$2000?

**15** The conversion rate for Australian dollars (A\$) to American dollars (US\$) is 0.95 A\$/US\$ (US\$1 = A\$0.95).

**a** How many US dollars would I receive for A\$2750?

**b** How many Australian dollars would I receive for US\$1300?

**c** Convert US\$1800 into A\$.

**16** The average distance of the Earth from the Sun is $1.49 \times 10^8$ km. Assume the Earth travels in a circular orbit around the Sun.

**a** Calculate (to three significant figures) the distance travelled by the Earth in one complete orbit. (*Hint:* use $C = 2\pi r$.)

**b** It takes the Earth 1 year (365.25 days) to travel this distance. Using the answer from part **a**, find the average speed at which the Earth travels through space in:

**i** km/h **ii** km/s.

**17** The recipe shown for stir-fried pork serves 4 people.

**a** How much of each ingredient would be needed to make this recipe for one person?

**b** How much of each ingredient would be needed to make this recipe for:

**i** 6 people? **ii** 9 people?

**c** Penny has 1.5 kg of pork. How many people can she feed using this recipe, assuming she has enough of all the other ingredients?

**Stir-fried pork**

*600 g pork*

*3 tbsp peanut oil*

*8 spring onions*

$2\frac{1}{2}$ *tsp grated ginger*

*2 tbsp lime juice*

$\frac{1}{3}$ *cup chicken stock*

*200 g sliced beans*

SPREADSHEET APPLICATION

**18** Enter the quantities for four people for the recipe in question **17** into the cells of a spreadsheet and use it to calculate the quantities of each ingredient for different numbers of people. Print the resulting table with headings. (You may like to use the spreadsheet supplied (Spreadsheet 3A) on your obook assess to help you.)

| | A | B | D | E | F | G |
|---|---|---|---|---|---|---|
| 1 | | | | | | |
| 2 | | | **Stir-fried pork** | | | |
| 3 | **Ingredients** | **Unit of measurement** | **Serves 4** | **Serves 1** | **Serves 2** | **Serves 10** |
| 4 | Pork | grams | 600 | 150 | 300 | |
| 5 | Peanut oil | tablespoons | 3 | 0.75 | 1.5 | |
| 6 | Spring onions | units | 8 | 2 | 4 | |
| 7 | Grated ginger | teaspoons | 2.5 | 0.625 | 1.25 | |
| 8 | Lime juice | tablespoons | 2 | 0.5 | 1 | |
| 9 | Chicken stock | cups | 0.33 | 0.0825 | 0.165 | |
| 10 | Sliced beans | grams | 200 | 50 | 100 | |

MEASUREMENT

# 3B Heart rate

These resources are available on your obook assess:
- **assess quiz 3B:** Test your skills with an auto-correcting multiple-choice quiz

**heart rate**
the number of contractions of the heart per minute measured as beats per minute (bpm)

**Heart rate** is measured by the number of heart beats per minute (bpm). When awake, but relaxed, an 18-year-old male in a good state of health should have a resting heart rate of between 62 and 65 bpm, and an 18-year-old female in a good state of health should have a resting heart rate of between 66 and 69 bpm.

The maximum heart rate, which is the highest heart rate an individual can achieve without severe problems through exercise stress, depends on age. For both an 18-year-old female and for an 18-year-old male, maximum heart rate is about 200 bpm. However, these values can vary for individuals.

The table below shows a health rating for males in a number of age groups based on their resting heart rate (bpm).

| | Age group for males | | | | | |
|---|---|---|---|---|---|---|
| **Health rating** | **18–25** | **26–35** | **36–45** | **46–55** | **56–65** | **65+** |
| Athlete | 49–55 | 49–54 | 50–56 | 50–57 | 51–56 | 50–55 |
| Excellent | 56–61 | 55–61 | 57–62 | 58–63 | 57–61 | 56–61 |
| Good | 62–65 | 62–65 | 63–66 | 64–67 | 62–67 | 62–65 |
| Above-average | 66–69 | 66–70 | 67–70 | 68–71 | 68–71 | 66–69 |
| Average | 70–73 | 71–74 | 71–75 | 72–76 | 72–75 | 70–73 |
| Below-average | 74–81 | 75–81 | 76–82 | 77–83 | 76–81 | 74–79 |
| Poor | 82+ | 82+ | 83+ | 84+ | 82+ | 80+ |

The table below shows a health rating for females in a number of age groups based on their resting heart rate (bpm).

| | Age group for females | | | | | |
|---|---|---|---|---|---|---|
| **Health rating** | **18–25** | **26–35** | **36–45** | **46–55** | **56–65** | **65+** |
| Athlete | 54–60 | 54–59 | 54–59 | 54–60 | 54–59 | 54–59 |
| Excellent | 61–65 | 60–64 | 60–64 | 61–65 | 60–64 | 60–64 |
| Good | 66–69 | 65–68 | 65–69 | 66–69 | 65–68 | 65–68 |
| Above-average | 70–73 | 69–72 | 70–73 | 70–73 | 69–73 | 69–72 |
| Average | 74–78 | 73–76 | 74–78 | 74–77 | 74–77 | 73–76 |
| Below-average | 79–84 | 77–82 | 79–84 | 78–83 | 78–83 | 77–84 |
| Poor | 85+ | 83+ | 85+ | 84+ | 84+ | 84+ |

# EXERCISE 3B Heart rate

UNDERSTANDING, FLUENCY AND COMMUNICATING

Use the tables on the opposite page to answer questions **1–6**.

**1** Find the resting heart rate range for:
- **a** a 50-year-old male with a below-average health rating
- **b** a 34-year-old male with an excellent health rating
- **c** a 25-year-old male with a poor health rating
- **d** a 70-year-old male with a good health rating.

**2** Find the health rating for each male with the given resting heart rate:
- **a** a 44-year-old male, 58 bpm
- **b** a 51-year-old male, 70 bpm
- **c** a 18-year-old male, 80 bpm
- **d** a 58-year-old male, 75 bpm.

**3**
- **a** Find the minimum resting heart rate for a 35-year-old male in below-average health.
- **b** Find the minimum resting heart rate for a 35-year-old male athlete.
- **c** What is the difference in minimum heart rate for these two people?

**4** Find the resting heart rate range for:
- **a** a 34-year-old female with a below-average health rating
- **b** a 25-year-old female with an excellent health rating
- **c** a 50-year-old female with a poor health rating
- **d** a 71-year-old female with a good health rating.

**5** Find the health rating for each female with the given resting heart rate:
- **a** a 44-year-old female, 70 bpm
- **b** an 18-year-old female, 55 bpm
- **c** a 23-year-old female, 63 bpm
- **d** a 58-year-old female, 66 bpm.

**6**
- **a** Find the minimum resting heart rate for a 42-year-old female who has a below-average health rating.
- **b** Find the minimum resting heart rate for a 42-year-old female athlete.
- **c** What is the difference in minimum heart rate for the females in parts **a** and **b**?

**Maximum heart rate** !

There are many different formulas used to calculate maximum heart rate.

For both males and females, the simplest is $220 - $ age in years.

## EXAMPLE 3B–1 Calculating the maximum heart rate

Use the formula 220 – age in years to calculate the maximum heart rate for:

**a** an 18-year-old male **b** a 35-year-old female.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | 202 bpm | $220 - 18 = 202$ | Subtract age in years from 220 to calculate the maximum heart rate in beats per minute. |
| **b** | 185 bpm | $220 - 35 = 185$ | |

**7** Use the formula 220 – age in years to calculate the maximum heart rate for:
- **a** a 40-year-old male
- **b** a 53-year-old male
- **c** a 22-year-old male
- **d** a 21-year-old female
- **e** a 30-year-old female
- **f** a 64-year-old female.

MEASUREMENT

**Target heart rate for exercise**

One method for calculating the target heart rate for exercise is to use a range of values between 50% and 85% of a person's maximum heart rate.

## EXAMPLE 3B–2 Calculating the target heart rate range for exercise

Calculate the target heart rate range for exercise for:

**a** an 18-year-old male **b** a 35-year-old female.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Maximum heart rate is 202 bpm.<br>Lower bound $= 0.5 \times 202 = 101$<br>Upper bound $= 0.85 \times 202 \approx 172$<br>Target heart rate range for exercise is 101 to 172 bpm. | From Example 3B–1, the maximum heart rate is 202 bpm. Find 50% and 85% of 202.<br>$50\% = \frac{50}{100} = 0.5$<br>$85\% = \frac{85}{100} = 0.85$ | First calculate the maximum heart rate. Then find 50% and 85% of that amount. Exercise should be structured so that the heart rate stays within the target range for maximum benefit. |
| **b** | Maximum heart rate is 185 bpm.<br>Lower bound $= 0.5 \times 185 = 93$<br>Upper bound $= 0.85 \times 185 \approx 157$<br>Target heart rate range for exercise is 93 to 157 bpm. | From Example 3B–1, the maximum heart rate is 185 bpm. Find 50% and 85% of 185. | |

**8** Calculate the target heart rate range for a 30-year-old woman with a maximum heart rate of 190 bpm.

**9** Calculate the maximum heart rate and the target heart rate range for:

**a** a 20-year-old male **b** a 43-year-old male **c** a 55-year-old male
**d** a 19-year-old female **e** a 28-year-old female **f** a 58-year-old female.

**Recovery rate number**

One estimate of fitness is the recovery rate number. It is calculated by measuring the heart rate one minute after completing exercise, subtracting this value from the heart rate at the end of the exercise period and dividing the result by 10.

A recovery rate number from 2 to 2.9 is fair, from 3 to 3.9 is good, and from 4 to 6 is excellent. A number over 6 is outstanding and a number below 2 is a sign of poor health or a lack of fitness.

## EXAMPLE 3B–3 Calculating the recovery rate number

Calculate the recovery rate number for a person whose heart rate at the end of exercising is 150 bpm and 90 bpm after one minute. State the level of fitness this indicates.

| Solve | Think | Apply |
|---|---|---|
| Recovery rate number $= \frac{150 - 90}{10}$<br>$= 6$<br>This indicates an excellent level of fitness. | Subtract 90 from 150 then divide the result by 10. | The difference between heart rate at the end of exercise and one minute after exercise is divided by 10 to calculate the recovery rate number. |

**10** Calculate the recovery rate number for a person whose heart rate at the end of exercising is 125 bpm and 92 bpm one minute later. State the level of fitness this indicates.

**11** Calculate the recovery rate number for each set of heart rate results and state the level of fitness this indicates.

**a** heart rate at the end of exercising is 165 bpm and 145 bpm one minute later
**b** heart rate at the end of exercising is 155 bpm and 112 bpm one minute later
**c** heart rate at the end of exercising is 135 bpm and 100 bpm one minute later
**d** heart rate at the end of exercising is 118 bpm and 65 bpm one minute later

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**12** **a** Complete this table for maximum heart rate.

| **Age (years)** | 10 | 30 | 50 | 70 | 90 |
|---|---|---|---|---|---|
| **Maximum heart rate (bpm)** | | | | | |

**b** Plot the points from the table and join them with a line or smooth curve to produce a graph representing this information.
**c** Does the graph show a linear or non-linear relationship between age and maximum heart rate?
**d** Use the graph to estimate the age of a person with maximum heart rate of 175 bpm.

**13** A person trains for 20 minutes with their heart rate at 130 bpm. Calculate the number of heart beats in that time.

**14** If each of the people in question **9** trained for 20 minutes at the lower level of their target heart range, calculate the total number of times their heart would beat in that time.

**15** The average adult heart pumps about 70 mL of blood with each beat. Calculate the volume of blood pumped by the human heart beating for:

**a** 1 min at 65 bpm
**b** 10 min at 120 bpm
**c** 5 min at 95 bpm
**d** 24 h at 70 bpm.

**16** Have two or more students ride an exercise bike, if available, or jog on the spot for 6 minutes. Measure their heart rate every minute for the 6 minutes of exercise and then for the following 4 minutes.

**a** Graph the results.
**b** Compare each person's initial resting heart rate with their heart rate 4 minutes after exercise is completed. What conclusions can you make?
**c** Calculate the recovery rate numbers for each student. What does this indicate about the level of fitness of each student?
**d** Repeat for other students, then graph and compare the results. From the graph, determine whether the relationships between heart rate and exercise time are linear. Explain your answer.

RESEARCH

**17** The formula (220 – age in years) used to calculate the maximum heart rate is the formula attributed to Haskell and Fox. It is used in general circumstances and was not intended to be used in exercise physiology. Elite athletes and sports people need more specific information and further research gives other methods of calculation. Some of these methods are Tanaka, Monahan and Seals; the Gulati formula for women and the Lund study.

**a** Research these formulae and any other formulae you find.
**b** Calculate your maximum heart rate using at least three different formulae. Compare your answers and explain any differences according to the nature of the formula.
**c** Find the average of all the maximum heart rates and compare this to the Haskell and Fox formula results. Explain which value you would use to determine your maximum heart rate.

# 3C Fuel consumption rate

These resources are available on your obook assess:
- **assess quiz 3C:** Test your skills with an auto-correcting multiple-choice quiz
- **Website:** Green Vehicle Guide

OI GI

**fuel consumption rate**
comparison between the two quantities of fuel consumed and distance travelled; usually given as a rate in L/100 km

**Fuel consumption rate** is a comparison between distance travelled and the amount of fuel used. It can be calculated by dividing the distance travelled by the amount of fuel used to give a rate in km/L. It can also be calculated by dividing the amount of fuel used by the distance travelled to give a rate in L/km or L/100 km.

!

To calculate the fuel consumption rate, we can use one of these formulae.

$$\text{Fuel consumption rate (km/L)} = \frac{\text{distance travelled (km)}}{\text{amount of fuel used (L)}}$$

$$\text{Fuel consumption rate (L/km)} = \frac{\text{amount of fuel used (L)}}{\text{distance travelled (km)}}$$

$$\text{Fuel consumption rate (L/100 km)} = \frac{\text{amount of fuel used (L)}}{\text{distance travelled (km)}} \times 100$$

These formulae can be rearranged to help you find the distance travelled or the amount of fuel used when the other two quantities are known.

The most common way of expressing fuel consumption is in L/100 km. A vehicle with a lower rate uses less fuel per 100 km and is said to have a better fuel consumption rate.

## EXAMPLE 3C–1 Calculating the fuel consumption rate

A car used 49 L of petrol on a trip of 500 km.

**a** Calculate the number of kilometres the car travels per litre of petrol consumed.
**b** Calculate the number of litres of petrol the car consumes per kilometre of travel.
**c** Calculate the number of litres of petrol the car consumes per 100 km of travel.

| | Solve | Think |
|---|---|---|
| **a** | Fuel consumption rate<br>$= \frac{500\,\text{km}}{49\,\text{L}}$<br>$= 10.204\ldots$ km/L<br>$= 10.2$ km/L | This rate is comparing the distance travelled with the amount of petrol consumed, in that order.<br>Rate $=$ 500 km for 49 L<br>$= (500 \div 49)$ km for 1 L<br>$\approx 10.2$ km/L<br>The car travels about 10.2 km for every 1 L of petrol used. |

| | Solve | Think |
|---|---|---|
| **b** | Fuel consumption rate<br>$= \dfrac{49\text{ L}}{500\text{ km}}$<br>$= 0.098$ L/km | This rate is comparing the amount of petrol consumed with the distance travelled, in that order.<br>Rate $=$ 49 L for 500 km<br>$= (49 \div 500)$ L for 1 km<br>$= 0.098$ L/km<br>The car uses 0.098 L of petrol for every kilometre. |
| **c** | Fuel consumption rate<br>$= \dfrac{49\text{ L}}{500\text{ km}}$<br>$= 0.098$ L/km<br>$= (0.098 \times 100)$ L/100 km<br>$= 9.8$ L/100 km<br>Or<br>Fuel consumption rate<br>$= \dfrac{49\text{ L}}{500\text{ km}}$<br>$= \dfrac{(49 \div 5)\text{ L}}{100\text{ km}}$<br>$= 9.8$ L/100 km | As the rate in L/km is often small, fuel consumption is usually quoted as the amount of petrol used per 100 km.<br>Rate $=$ 0.098 L for 1 km<br>$= (0.098 \times 100)$ L for 100 km<br>$= 9.8$ L/100 km<br>Alternatively:<br>Rate $=$ 49 L for 500 km<br>$= (49 \div 5)$ L for 100 km<br>$= 9.8$ L/100 km |

| Apply |
|---|
| Determine the order in which the two quantities are being compared and divide one by the other, in that order. Remember to write the appropriate units. |

## EXERCISE 3C Fuel consumption rate

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** A car used 24 L of petrol on a trip of 280 km.

**a** Calculate the number of kilometres the car travels per litre of petrol consumed.

**b** Calculate the number of litres of petrol the car consumes per kilometre of travel.

**c** Calculate the number of litres of petrol the car consumes per 100 km of travel.

**2** Calculate the fuel consumption rate for each trip in:

**i** km/L   **ii** L/km   **iii** L/100 km

**a** A car travels 450 km on 40 L of petrol.

**b** A car travels 220 km on 24 L of petrol.

**c** A motorcycle travels 192 km on 11 L of petrol.

**3** Calculate the fuel consumption rate in L/100 km for each trip.

**a** A truck used 114 L of petrol on a trip of 600 km.

**b** A car used 13.8 L of petrol on a trip of 120 km.

**c** A motorcycle used 22 L of petrol on a trip of 345 km.

> **!** When the rate of fuel consumption is expressed in L/100 km, we can use the following formula to find the distance travelled in kilometres.
>
> $$\text{Distance travelled (km)} = \frac{\text{amount of fuel (L)}}{\text{fuel consumption (L/100 km)}} \times 100$$

MEASUREMENT

## EXAMPLE 3C–2 Calculating distance given the fuel consumption rate and fuel amount

How far can a Holden Astra travel on 42 L of petrol if its fuel consumption is 5.8 L/100 km?

| Solve | Think | Apply |
|---|---|---|
| Distance $= \frac{42}{5.8} \times 100$<br>$\approx 724$ km | The number of 'lots of 5.8 L' used is $\frac{42}{5.8}$. The car travels 100 km for each 'lot of 5.8 L' used, so multiply the fraction $\frac{42}{5.8}$ by 100. | Distance travelled (km)<br>$= \frac{\text{amount of fuel}}{\text{fuel consumption}} \times 100$<br>Fuel consumption is in L/100 km. |

**4** Complete the following to calculate how far a motorcycle can travel on 10 L of petrol if the fuel consumption is 5.6 L/100 km.

Distance $= \frac{10}{\square} \times 100 \approx$ ____ km

**5** How far can a vehicle travel on:

**a** 30 L of petrol if its fuel consumption is 8.6 L/100 km?

**b** 52 L of petrol if its fuel consumption is 9.3 L/100 km?

**c** 98 L of petrol if its fuel consumption is 12.4 L/100 km?

> **!** When the rate of fuel consumption is expressed in L/100 km, we can use the following formula to find the amount of fuel used in litres.
>
> Amount of fuel used (L) $= \frac{\text{distance travelled (km)}}{100} \times$ fuel consumption (L/100 km)

## EXAMPLE 3C–3 Calculating the amount of fuel needed to travel a given distance

Calculate the amount of petrol used (to the nearest litre) by a Toyota Corolla on a trip of 480 km if its fuel consumption is 6.6 L/100 km.

| Solve | Think | Apply |
|---|---|---|
| Amount of petrol used<br>$= \frac{480}{100} \times 6.6$<br>$= 31.68$<br>$\approx 32$ L | The number of 'lots of 100 km' travelled is $\frac{480}{100}$.<br>Each 'lot of 100 km' uses 6.6 L of petrol, so multiply the fraction $\frac{480}{100}$ by 6.6. | Amount of fuel used (L)<br>$= \frac{\text{distance travelled}}{100} \times$ fuel consumption<br>Fuel consumption is in L/100 km and distance travelled is in kilometres. |

**6** Complete the following to calculate the amount of fuel used (to the nearest litre) by a vehicle on a trip of 965 km, if the fuel consumption is 10.2 L/100 km.

Amount of fuel used $= \frac{\square}{100} \times \square =$ ____ $\approx$ ____ L

**7** Calculate the amount of fuel used (to the nearest litre) by a vehicle on a trip of:

**a** 425 km, if the fuel consumption is 8.8 L/100 km

**b** 670 km, if the fuel consumption is 11.2 L/100 km

**c** 315 km, if the fuel consumption is 7.6 L/100 km.

## EXAMPLE 3C–4 Solving problems related to fuel consumption rate

A car uses 22 L of petrol on a trip of 250 km.

**a** Express the fuel consumption in L/100 km.

**b** Assuming the same rate of fuel consumption:

  **i** how much fuel would the car use for a trip of 325 km?

  **ii** how far could the car travel on a full tank of 50 L?

**c** Assuming the cost of petrol was 142.9c/L, what is the difference in cost of the fuel used for each of the two trips described in part **b**?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Fuel consumption rate<br>$= \frac{22\text{ L}}{250\text{ km}}$<br>$= \frac{(22 \div 250)\text{ L}}{(250 \div 250)\text{ km}}$<br>$= \frac{0.088\text{ L}}{1\text{ km}}$<br>$= \frac{(0.088 \times 100)\text{ L}}{(1 \times 100)\text{ km}}$<br>$= \frac{8.8\text{ L}}{100\text{ km}}$<br>$= 8.8$ L/100 km | Write a fraction to compare the number of litres of fuel to the number of kilometres travelled. Multiply or divide the top and the bottom of the fraction by the same number until you obtain 100 km in the denominator. | One way to find fuel consumption rate in L/100 km is to first find the rate in L/km and then multiply the top and the bottom of the fraction by 100. |
| **b i** | Amount of petrol used<br>$= \frac{325}{100} \times 8.8$<br>$= 3.25 \times 8.8$<br>$= 28.6$ L | The number of 'lots of 100 km' travelled is $\frac{325}{100}$; that is, 3.25 lots of 100 km.<br>Each 'lot of 100 km' uses 8.8 L of petrol, so multiply $\frac{325}{100}$ by 8.8. | Since fuel consumption rate is the number of litres used per 100 km, find the number of 'lots of 100 km' travelled and multiply by the fuel consumption rate. |
| **ii** | Distance<br>$= \frac{50}{8.8} \times 100$<br>$= 568.181\ldots$<br>$\approx 568.2$ km | The car travels 100 km for each 8.8 L of petrol used. The number of 'lots of 8.8 L' used is $\frac{50}{8.8}$, so multiply $\frac{50}{8.8}$ by 100. | Since fuel consumption rate is the number of litres used per 100 km, find the number of 'lots of 8.8 L' used and multiply by 100. |
| **c** | Cost of fuel for Trip 1<br>$= 142.9 \times 28.6$<br>$= 4086.94$ cents<br>$= \$40.87$<br>Cost of fuel for Trip 2<br>$= 142.9 \times 50$<br>$= 7145$ cents<br>$= \$71.45$<br>Difference in cost<br>$= \$30.58$<br>Trip 2 costs \$30.58 more in fuel costs. | Cost of petrol is 142.9 cents or \$1.429 for each litre of fuel used, so multiply 142.9 cents or \$1.429 by the number of litres used.<br>For Trip 1, use the number of litres calculated in part **b i**; that is, 28.6 L.<br>For Trip 2, use the number of litres given in part **b ii**; that is, 50 L.<br>Find the difference in fuel costs. | For each trip, multiply the cost per litre by the number of litres of fuel used. Think about whether the cost of the fuel in your calculation is in cents or dollars. |

**8** A car uses 30 L of petrol on a trip of 400 km.

- **a** Express the fuel consumption in L/100 km.
- **b** Assuming the same rate of fuel consumption:
  - **i** how much fuel would the car use for a trip of 500 km?
  - **ii** how far could the car travel on a full tank of 45 L?
- **c** Assuming the cost of petrol was 142.9c/L, what is the difference in cost of the fuel used for each of the two trips described in part **b**?

**9** A courier averages 2800 km of city driving each month in a Ford Fiesta that has a fuel consumption of 6.8 L/100 km (city cycle). Calculate the cost of petrol used in a month in which the average price of unleaded petrol (ULP) is 139.9c/L.

**10** In 2016 the average fuel consumption of Australian passenger vehicles was 10.6 L/100 km and the average yearly distance travelled was 13 716 km. If the average price of fuel was 135.9c/L, what was the average yearly fuel cost?

**11** The table on the right shows the average fuel consumption rates for vehicles with engines containing a different number of cylinders.

| Number of cylinders | Fuel consumption rate |
|---|---|
| 1–3 | 7.7 L/100 km |
| 4 | 10.0 L/100 km |
| 6 | 11.7 L/100 km |
| 8 or more | 12.9 L/100 km |

- **a** If the average price of petrol is 139.9c/L and the annual yearly distance travelled is 10 000 km, calculate the average yearly fuel cost for each type of vehicle.
- **b** What is the annual fuel cost saving of driving a 4-cylinder car compared to a 6-cylinder car?
- **c** An articulated truck has a fuel consumption of 56.3 L/100 km. Compare the annual fuel cost of the truck and an 8-cylinder car if the cost of diesel for the truck is 137.9c/L. Assume each vehicle is driven 10 000 km in a year.

**12** A Mazda MX-5 uses 7.5 L/100 km of ULP and the diesel model of the same car uses 6.0 L/100 km of diesel fuel.

- **a** Calculate the cost of driving the petrol model 800 km if ULP is 142.9c/L.
- **b** Calculate the cost of driving the diesel model 800 km if diesel fuel is 152.2c/L.
- **c** How much cheaper is the diesel option over this distance?

**13** Eric's car uses fuel at the rate of 6.2 L/100 km for country driving and 7.8 L/100 km for city driving. On a trip, he drives 145 km in the country and 25 km in the city.

- **a** Calculate the amount of fuel he used on this trip.
- **b** What is the average fuel consumption rate for this trip?

**14** Liesel is comparing the costs of two different ways of travelling to her workplace. Her car uses one litre of fuel for every 10 km travelled. The cost of fuel is $1.49/L and the distance from her home to her workplace is 38 km. The cost of travelling by bus is $42.50 for 10 single trips. Which way of travelling is cheaper and by how much? Support your answer with calculations.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**15** Tim owns a Holden Commodore that runs on ULP and has a fuel consumption of 8.9 L/100 km. When converted to run on liquid petroleum gas (LPG), the car's fuel consumption is 12.3 L/100 km. Tim averages 18 000 km per year.

**a** Calculate the annual cost for each type of fuel (assuming that the car only runs on one type of fuel for a year) if the average price of ULP is 139.9c/L and LPG is 71.9c/L.

**b** How much does Tim save in fuel costs for the year by converting his car to LPG?

**c** How much does he save per month?

**d** The cost of converting the car to LPG is $2600. How many months would Tim take to break even?

**e** What distance would Tim travel before reaching the break-even point? (The break-even point is the point at which the cost of running the vehicle on each type of fuel is the same.)

**16** Jasmine travels an average of 14 000 km per year and wants to buy a new car. The car she likes is available with a petrol motor or diesel motor. The petrol model has a fuel consumption of 10.8 L/100 km and the diesel model has a fuel consumption of 7.8 L/100 km.

**a** **i** If Jasmine bought the car with the petrol engine, what would be her annual fuel cost if ULP is 139.9c/L?

**ii** If Jasmine bought the car with a diesel engine, what would be her annual fuel cost if diesel fuel is 137.2c/L?

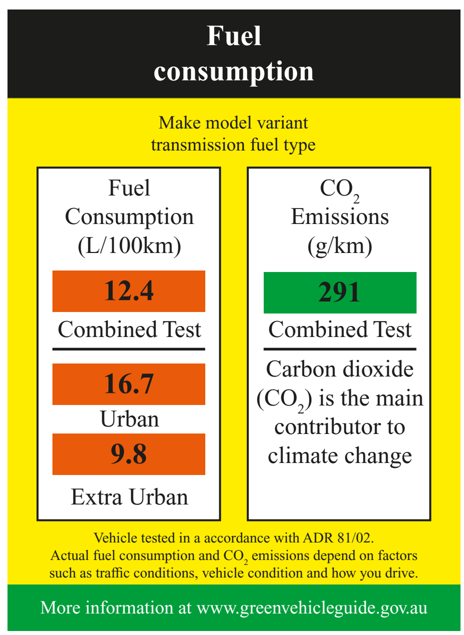
**iii** How much per year would she save on fuel by buying the car with the diesel engine?

**b** What is the average monthly saving with the diesel engine?

**c** The diesel car costs $1200 more to buy than the petrol car. How many months would it take for Jasmine to break even, if she buys the diesel car?

**d** What distance would Jasmine travel before reaching her break-even point?

**17** All new light vehicles sold in Australia are required to display a Fuel Consumption label on the front windscreen. The label indicates the vehicle's fuel consumption in litres of fuel per 100 kilometres (L/100 km) and its emissions of carbon dioxide in grams per kilometre (g/km). The label is designed to help Australian motorists make informed choices about the environmental impact of their new car and the cost of running their vehicle.




**a** Go to the Green Vehicle Guide website to find out why there are three fuel consumption rates given. A link is provided on your obook assess. (*Hint:* locate 'Fuel consumption label' in the menu.)

**b** Use the label shown to calculate the amount of fuel used to travel 200 km in the Combined test.

**c** Calculate the difference in the cost of the fuel used to travel 200 km in Urban conditions compared to Extra Urban conditions when the cost of fuel is $1.42/L.

RESEARCH

**18** Choose at least five different models of new vehicles and compare their fuel consumption rates and carbon dioxide emission rates. One website that you may find useful is Green Vehicle Guide. A link is provided on your obook assess. Write a report that also includes a comparison of fuel consumption rates for different engine sizes and types of fuel used.

**power**
rate at which energy is used, measured in watts (W) or kilowatts (kW)

**power rating**
rating of an electrical appliance to indicate the rate at which the appliance uses electricity, measured in watts (W) or kilowatts (kW)

**kilowatt-hour (kWh)**
unit of measurement for the amount of energy used by an electrical appliance; 1 kWh is the amount of energy used by a 1 kW appliance in 1 hour

# 3D Power and energy consumption rates

These resources are available on your obook assess:
- **assess quiz 3D:** Test your skills with an auto-correcting multiple-choice quiz
- **Website:** Energy Rating

Energy is measured in joules (J) or kilojoules (kJ). **Power** is the rate at which energy is used:

$$\text{Power} = \frac{\text{energy}}{\text{time}}.$$

The International System of Units (SI) unit for power is the watt (W), which is defined as:

$$1 \text{ watt} = 1 \text{ joule/second}$$

$$\text{So } 1 \text{ kilowatt (kW)} = 1000 \text{ W} = 1000 \text{ J/s}.$$

Electricity is a form of energy used in homes and businesses. All electrical appliances have a **power rating**. This is the rate at which the appliance uses electricity.

One watt-hour is the amount of energy used by a 1 watt load for 1 hour. A more satisfactory unit to measure the amount of energy a household uses is the **kilowatt-hour** (kWh or kW-h).

1 kWh is the energy used by a 1 kW appliance operating for 1 hour.

!

The energy consumption of an appliance can be calculated using:

$$\text{Energy (kWh)} = \text{power (kW)} \times \text{time (h)}$$

## EXAMPLE 3D–1 Calculating the cost of running an appliance

Calculate the cost of running a 200 W television set for 7 hours if the domestic electricity rate is 33.25 cents/kWh.

| Solve | Think | Apply |
|---|---|---|
| $\text{Energy} = 0.2 \times 7$<br>$= 1.4 \text{ kWh}$<br>$\text{Cost} = 1.4 \times 0.3325$<br>$= \$0.4655$<br>The cost of running the television for 7 hours is about 47 cents. | Convert power to kilowatts and cost to dollars.<br>200 W = 0.2 kW<br>33.25 cents = \$0.3325 | Substitute the known quantities in the required units:<br>Energy (kWh) = Power (kW) × time (h)<br>and<br>Cost (\$) = energy used (kWh) × rate (\$/kWh) |

## EXERCISE 3D Power and energy consumption rates

**1** Calculate the cost of running each of the following appliances.

**a** A 3000 watt clothes dryer is used for 4 hours. The domestic electricity rate is 43.67 cents/kWh.

**b** An 1800 watt hair dryer is used for 6 minutes every day for 30 days. The rate is 27.6 cents/kWh.

**c** A 450 watt desktop computer is used for 6 hours per day for 5 days. The rate is 16.9 cents/kWh.

**d** A 10 watt LED light bulb is left on for 12 hours per day for a year. The rate is 38.18 cents/kWh.

**2** **a** Calculate the cost of running a 400 watt submersible water pump for 8 hours per day for a year during the peak period if the peak rate is 47.77 cents/kWh.

**b** How much could be saved by programming the pump to come on in the off-peak period of the day when the cost is 19.4 cents/kWh?

**3** If it costs 15 cents for 1 kilowatt (1000 W) of power for one hour, how much does it cost to run a 2400 W heater from 5 pm to 11 pm?

**4** Fred's electricity bill was $294.72 for 1560 kilowatt-hours of power (kWh). What was the cost of electricity per kilowatt-hour?

**5** What is the difference in the annual running costs for each of these two models of dishwasher, given that the energy rate is 34.65 c/kWh?

**a** Gold DL10: energy consumption of 230 kWh/year
Gold DL20: energy consumption of 275 kWh/year

**b** Speedy G123: energy consumption of 231 kWh/year
Speedy G456: energy consumption of 324 kWh/year

**6** What is the difference in the annual running costs for the following models of television sets, given that the energy rate is 42.84 c/kWh?

**a** Sonic LED 138.8 cm screen: energy consumption of 261 kWh/year
Sonic LED 98 cm screen: energy consumption of 141 kWh/year

**b** Yanso LCD 66 cm screen: energy consumption of 241 kWh/year
Yanso LCD 100 cm screen: energy consumption of 561 kWh/year

**7** Many appliances consume electricity even when they are not operating. This is generally referred to as stand-by power consumption. The table shows the average hourly stand-by power consumption for some common household appliances.

**i** Calculate the annual energy consumption for each appliance in kilowatt-hours if they are left on stand-by for 24 hours a day for a year.

**ii** Hence determine the annual cost of stand-by energy used for each, if the cost of electricity is 28.7 cents/kWh.

| | Appliance | Hourly stand-by power consumption (W) |
|---|---|---|
| **a** | Television (off by remote) | 5 |
| **b** | Laser printer (off) | 4 |
| **c** | Microwave oven (on, door closed) | 3 |
| **d** | DVD player (on, not playing) | 14 |
| **e** | Computer speakers (off) | 2 |
| **f** | Computer speakers (on, no sound) | 4 |
| **g** | Games console (active, not playing) | 45 |
| **h** | Desktop computer (off) | 3 |
| **i** | Desktop computer (sleep mode) | 21 |
| **j** | Desktop computer (on, idle) | 74 |

The following table can be used to convert units of power using scientific notation.
For example, 1 mW = $1 \times 10^{-3}$ W and 1 MW = $1 \times 10^{6}$ W.

| Units of power | | |
|---|---|---|
| **Multiple** | **Name** | **Symbol** |
| $10^{-3}$ | milliwatt | mW |
| $10^{0}$ | watt | W |
| $10^{3}$ | kilowatt | kW |
| $10^{6}$ | megawatt | MW |
| $10^{9}$ | gigawatt | GW |

## EXAMPLE 3D–2 Converting units of power

Convert 250 kW to:

**a** megawatts **b** milliwatts.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | 250 kW<br>$= 250 \times 10^3$ W<br>$= 2.5 \times 10^5$ W<br>$= (2.5 \times 10^5 \div 10^6)$ MW<br>$= (2.5 \times 10^{-1})$ MW<br>$= 0.25$ MW | First convert to the smaller unit of watt by multiplying by the conversion factor of $10^3$ (1 kW = $1 \times 10^3$ W).<br>Then convert to the larger unit of megawatts by dividing by the conversion factor of $10^6$ (1 MW = $1 \times 10^6$ W). | To convert to a smaller unit, multiply by the conversion factor.<br>To convert to a larger unit, divide by the conversion factor. Use the multiples in the table to work out the conversion factor. |
| **b** | 250 kW<br>$= 250 \times 10^3$ W<br>$= 2.5 \times 10^5$ W<br>$= (2.5 \times 10^5 \times 10^3)$ mW<br>$= 2.5 \times 10^8$ mW | First convert to the smaller unit of watts by multiplying by the conversion factor of $10^3$ (1 kW = $1 \times 10^3$ W).<br>Then convert to the smaller unit of milliwatts by multiplying by the conversion factor of $10^3$ (1 mW = $1 \times 10^{-3}$ W and so 1 W = $\frac{1}{10^{-3}}$ mW = $1 \times 10^3$ mW). | |

Use the units of power table to answer questions **8–11**.

**8** The Capital Wind farm at Tarago, NSW, has an electricity generation capacity of $1.41 \times 10^8$ W. Convert this to:

**a** gigawatts
**b** megawatts
**c** kilowatts
**d** milliwatts.

UNDERSTANDING, FLUENCY AND COMMUNICATING

**9** The Murray Power Station in the Snowy Mountains has a generation capacity of 1500 MW of hydroelectricity. Convert this to:

**a** watts
**b** kilowatts
**c** gigawatts.

**10** NSW has an electricity generation capacity of approximately 18 000 MW of power. Convert this to:

**a** kilowatts
**b** gigawatts.

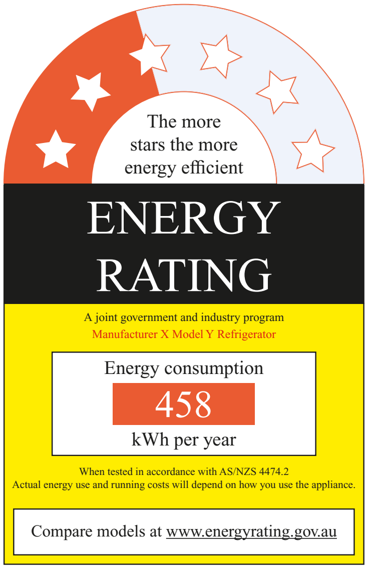
**11** A new mobile phone charger has stand-by power consumption of 96 mW.

**a** Express this in kilowatts, using scientific notation.
**b** What is the energy consumption in kilowatt-hours if the charger is left on stand-by for a week?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**12** The Energy Rating Label allows you to compare the energy efficiency and running costs of electrical appliances. Go to the government website Energy Rating. A link is available on your obook assess. Find the answers to the following questions.

**a** What are the two main features of the energy rating label?
**b** How is the star rating determined?
**c** What is the approximate difference in energy consumption between 2-star and 3-star rated appliances?
**d** What other factors should be considered when buying an appliance?




**13** The cost of buying a new heater is $650. It uses energy according to the label shown in question **12**. If energy is charged at the rate of $0.32/kWh, how much will it cost to purchase and then run this heater for three years?

**14** Sam has a household fridge that uses energy at a rate of 738 kWh per year. She is thinking of buying a bar fridge to store extra bottles of wine. The cost of the bar fridge is $420 and it has an energy consumption rate of 313 kWh per year.

**a** If energy is charged at the rate of $0.32/kWh, what is the energy cost of running the bar fridge for one year?
**b** What would be the percentage increase in energy costs to run both fridges compared to running only the household fridge?
**c** What would be the total cost of buying the bar fridge and running it for five years?

MEASUREMENT

**15** The table shows the approximate electricity consumption for the top 20 users in the world in 2016. The population of each of these countries is also shown.

| Country | Electricity consumption (kWh $\times$ $10^9$) | Population ($\times$ $10^6$) | Electricity consumption per capita (kWh/person, to nearest 100) |
|---|---|---|---|
| Australia | 224 | 23 | |
| Brazil | 518 | 193 | |
| Canada | 528 | 35 | |
| China | 5920 | 1374 | |
| France | 431 | 67 | |
| Germany | 533 | 82 | |
| India | 1409 | 1267 | |
| Iran | 218 | 83 | |
| Italy | 291 | 62 | |
| Japan | 934 | 127 | |
| Korea, South | 495 | 51 | |
| Mexico | 238 | 123 | |
| Russia | 1065 | 142 | |
| Saudi Arabia | 272 | 28 | |
| South Africa | 212 | 54 | |
| Spain | 234 | 49 | |
| Taiwan | 250 | 24 | |
| Turkey | 207 | 80 | |
| United Kingdom | 309 | 64 | |
| USA | 3911 | 324 | |

**a** In which of these countries was the electricity consumption the:
- **i** greatest?
- **ii** least?

**b** What is Australia's rank compared with the rest of the world?

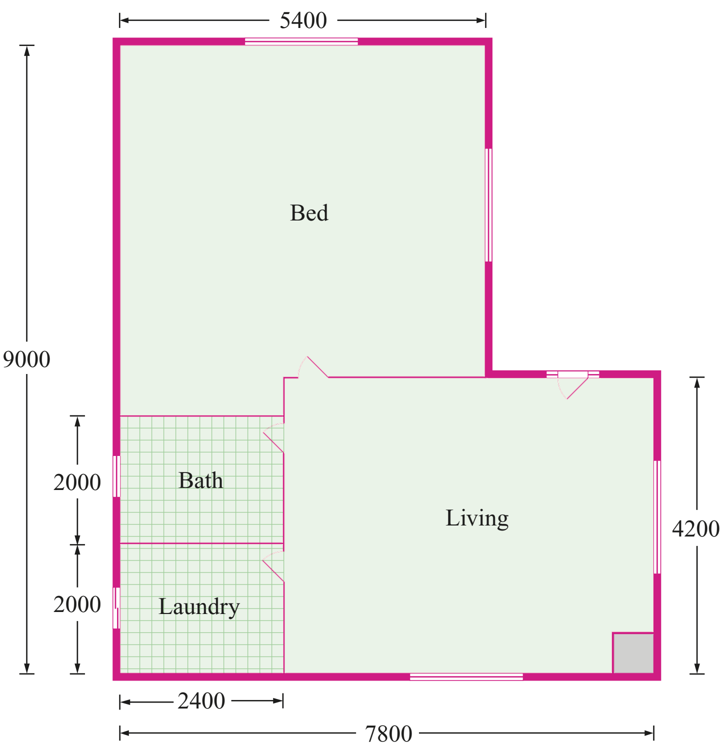
**c** Complete the following statements:
- **i** China's consumption is ___ kWh more than Australia's consumption.
- **ii** China's consumption is ___ times more than Australia's consumption.
- **iii** The total consumption of China and the USA is ___ (< or >) the total consumption of the next 18 biggest users.

**d**
- **i** Complete the column for energy consumption per capita in the table above.
- **ii** Which of these countries has the highest electricity use per person?
- **iii** Which of these countries has the lowest electricity use per person?
- **iv** What is Australia's rank in consumption per person?

**e**
- **i** Does the country with the highest consumption per person have the highest population?
- **ii** Does the country with the lowest population have the lowest consumption per person?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**16** The diagram shows the plan of a small holiday house (all measurements are in millimetres). The owners wish to install air conditioning throughout the house, except for the bathroom and laundry.




**a** Calculate the floor area of the house minus the bathroom and laundry.

**b** The power output of the air conditioning unit required is based on the volume of air to be conditioned.

**i** Calculate the volume of air to be air conditioned in this house if the ceilings are 2400 mm high. (*Hint:* use $V = A \times h$.)

**ii** Use the table below to find the power output of the unit required.

| **Volume (m$^3$)** | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 | 140 | 150 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **kW required** | 3.0 | 3.6 | 4.2 | 4.8 | 5.4 | 6.0 | 6.6 | 7.2 | 7.8 | 8.4 | 9.0 |

**c** Find the approximate power of the air conditioning unit required to air condition the living room of the house.

**d** Find the approximate power of the air conditioning unit required for the living room of the house if the ceiling height is 2700 mm.

**e** Find the approximate power of the air conditioning unit required for the living room and bedroom of the house if the ceiling height is 2700 mm.

RESEARCH

**17** Investigate the price of electricity in the various states and territories of Australia.

**a** Compare the cheapest and most expensive plans.

**b** What proportion of people switch plans each year?

**c** How much has the price increased over the last 1, 2 and 5 years?

MEASUREMENT

# 3E Ratios

These resources are available on your obook assess:

- **Interactive 3E.1:** Explore key ideas for understanding ratios
- **Interactive 3E.2:** Explore simplifying ratios and using equivalent ratios
- **Interactive 3E.3:** Explore key ideas for dividing quantities into ratios
- **Worksheet 3E:** Practise your skills with extra problems on ratios
- **Investigation 3E:** Investigate the ratios of bicycle design
- **assess quiz 3E:** Test your skills with an auto-correcting multiple-choice quiz

**ratio**
comparison between quantities of the same kind, used as a measure of the relative size of the quantities

!

A **ratio** is a comparison between quantities of the same kind.

A ratio can be written using colon notation ( : ) or as a fraction.

A ratio does not have units and usually contains whole numbers.

The order of the quantities in a ratio is important.

Ratios are simplified by multiplying or dividing each term of the ratio by the same number. Unless otherwise specified, the terms should be left as whole numbers with no common factors and no units.

## EXAMPLE 3E–1 Writing comparisons as ratios

Tom's height is 169 cm and Laura's height is 165 cm.

**a** **i** Write the ratio of Tom's height to Laura's height.

**ii** Write the ratio of Laura's height to Tom's height.

**b** Write each ratio from part **a** as a fraction.

| | Solve | Think | Apply |
|---|---|---|---|
| **a i** | Tom's height to Laura's height<br>= 169 cm to 165 cm<br>= 169 : 165 | The quantities compared are 169 cm and 165 cm. These are whole numbers and in the same units so write these quantities in the order given, separated by a colon, leaving out the units. | Check that the quantities are whole numbers and in the same units. Write the quantities in the order given, separated by a colon or as a fraction, leaving out the units. The order is important as $169:165 \neq 165:169$. |
| **ii** | Laura's height to Tom's height<br>= 165 cm to 169 cm<br>= 165 : 169 | | |
| **b i** | $169:165 = \frac{169}{165}$ | Write the first number in the ratio as the numerator of the fraction and the second number as the denominator. | |
| **ii** | $165:169 = \frac{165}{169}$ | | |

# EXERCISE 3E Ratios



**1** The table shows the number of each type of vehicle that passes the front of a school in an hour.
Write the ratio of the number of:

**a** cars to trucks
**b** trucks to cars
**c** motorcycles to cars
**d** trucks to buses
**e** cars to trucks to motorcycles
**f** trucks to motorcycles to buses.

| Type of vehicle | Number |
|---|---|
| Car | 54 |
| Truck | 13 |
| Motorcycle | 7 |
| Bus | 4 |
| Other | 2 |

**2** Write each ratio from parts **a–d** of question **1** as a fraction.

## EXAMPLE 3E–2 Writing comparisons as ratios in simplest form

Write each of these comparisons as a ratio in simplest form.

**a** 24 to 18
**b** 16 to 12 to 20
**c** 1.5 to 2.7
**d** $1\frac{3}{4}$ to $\frac{2}{3}$
**e** 85 cm to 1.2 m

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | 24 to 18<br>$= 24:18$<br>$= 4:3$ | The two quantities are whole numbers, so they can be written as a ratio. The highest common factor (HCF) of 24 and 18 is 6, so divide each term by 6 to simplify the ratio.<br>The fraction key on a calculator can be used when there are two parts to the ratio.<br>Enter: 24 [a b/c] 18 [•] [SHIFT] [a b/c].<br>Display is 4 ⌋ 3. | Ratios must contain whole numbers and can be simplified by multiplying or dividing each term by the same number. This produces an equivalent ratio. |
| **b** | 16 to 12 to 20<br>$= 16:12:20$<br>$= 4:3:5$ | The three quantities are whole numbers, so they can be written as a ratio. HCF of 16, 12 and 20 is 4 so divide each term by 4. | |
| **c** | 1.5 to 2.7<br>$=$ 15 to 27<br>$= 15:27$<br>$= 5:9$ | Ratios must contain whole numbers so first multiply each term by 10.<br>HCF of 15 and 27 is 3, so divide each part by 3. | |
| **d** | $1\frac{3}{4}$ to $\frac{2}{3}$<br>$= \frac{7}{4}$ to $\frac{2}{3}$<br>$= \frac{7}{4} \times 12$ to $\frac{2}{3} \times 12$<br>$=$ 21 to 8<br>$= 21:8$ | Write the mixed number as an improper fraction. Since ratios must contain whole numbers, multiply each term by the lowest common denominator of 12. The ratio 21 : 8 cannot be simplified any further. | |
| **e** | 85 cm to 1.2 m<br>$=$ 85 cm to 120 cm<br>$= 85:120$<br>$= 17:24$ | Write the terms so they have the same unit. Once the two quantities are expressed in the same unit and are whole numbers, they can be written as a ratio. HCF of 85 and 120 is 5, so divide each term by 5. | |

**3** Write each of these comparisons as a ratio in simplest form.

| | | |
|---|---|---|
| **a** 25 to 35 | **b** 27 to 18 | **c** 84 to 48 |
| **d** 100 to 28 to 4 | **e** 12 to 24 to 18 | **f** 36 to 48 to 72 |
| **g** 1.6 to 1.9 | **h** 1.6 to 1.8 | **i** 0.56 to 0.32 |
| **j** 0.93 to 0.6 | **k** 0.256 to 0.8 | **l** $\frac{1}{2}$ to $\frac{1}{3}$ |
| **m** $1\frac{1}{2}$ to $2\frac{1}{2}$ | **n** $\frac{5}{8}$ to $\frac{2}{3}$ | **o** 25 cm to 1.1 m |
| **p** 2.2 kg to 850 g | **q** $1.50 to 80 cents | **r** $1\frac{1}{2}$ h to 40 min |
| **s** 6 min to 2.1 h | **t** 600 mL to 1.5 L to 2.1 L | **u** 40 g/L to 3 mg/mL to 2 mg/mL |

**!** Ratios generally contain whole numbers only. However, it can be useful to write them in the form $n:1$ or $1:n$ where $n$ is not necessarily a whole number. This provides a direct comparison of how much bigger or smaller one quantity is than the other.

## EXAMPLE 3E–3 Expressing ratios in the form $n:1$ or $1:n$

**a** Express the ratio 25 : 10 in the form $n:1$.
**b** Express the ratio 24 : 16 in the form $1:n$.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $25:10 = \frac{25}{10}:\frac{10}{10}$ <br> $= 2.5:1$ | To obtain 1 as the second number of the ratio and maintain an equivalent ratio, divide each number by 10. <br> This means that the first number is 2.5 times the second. | To express the ratio in the form $n:1$, divide each term by the second number. |
| **b** | $24:16 = \frac{24}{24}:\frac{16}{24}$ <br> $= 1:\frac{2}{3}$ or $1:0.\dot{6}$ | To obtain 1 as the first number of the ratio and maintain an equivalent ratio, divide each number by 24. <br> This means that the second number is $\frac{2}{3}$ (or $0.\dot{6}$) times the first. | To express the ratio in the form $1:n$, divide each term by the first number. |

**4** Express each of these ratios in the form $n:1$.

**a** 35 : 10 **b** 72 : 40 **c** 24 : 60 **d** 72 : 80

**5** Write each comparison as a ratio in the form $n:1$. Explain the meaning of the answer.

**a** 3.5 m to 70 cm **b** 1.2 kg to 800 g
**c** $2\frac{1}{2}$ cups to $\frac{1}{2}$ cup **d** 0.04 ha to 500 m$^2$

**6** Express each of these ratios in the form $1:n$.

**a** 50 : 87 **b** 40 : 90 **c** 60 : 48 **d** 125 : 8

**7** Write each comparison as a ratio in the form $1:n$. Explain the meaning of the answer.

**a** 1 cm to 1 m **b** 1 mm to 1 m **c** 20 mL to 1 L **d** 50 mg to 15 g

**EXAMPLE 3E–4** Working with equivalent ratios (unknown quantity in numerator)

The ratio of the number of boys to girls in a school is 7 : 6. If there are 354 girls in the school, how many boys are there?

| Solve | Think | Apply |
|---|---|---|
| number of boys : number of girls $= 7:6$<br>$\frac{\text{number of boys}}{\text{number of girls}} = \frac{7}{6}$<br>$\frac{\text{number of boys}}{354} = \frac{7}{6}$<br>number of boys $= \frac{7}{6} \times 354$<br>$= 413$<br>There are 413 boys. | Write the two equivalent ratios as fractions.<br>number of boys : number of girls $= \frac{\text{number of boys}}{\text{number of girls}}$<br>$7:6 = \frac{7}{6}$<br>As the two equivalent ratios are equal, write as an equation. Solve the equation by multiplying both sides by 354. | Using fraction notation, write two equivalent ratios. (Remember the order of terms is important!)<br>Treat this as an equation and solve. |

**8** The ratio of the number of boys to girls in a school is 9 : 8 and there are 312 girls at the school. Complete the following to find the number of boys.

$$\frac{\text{number of boys}}{\square} = \frac{9}{8}$$

number of boys

$$= \frac{9}{8} \times \_\_\_\_$$

$$= \_\_\_\_$$

**9** For an electrical store the ratio of profit to sales is 2 : 7. Given that the annual sales for the year were \$145 600, what was the annual profit?

**10** The ratio of Ben's net salary to the tax he pays is 10 : 3. Find his net salary if he paid \$14 580 in tax for one year.

**EXAMPLE 3E–5** Working with equivalent ratios (unknown quantity in denominator)

The ratio of boys to girls in a school is 8 : 7. If there are 264 boys in the school, how many girls are there?

| Solve | Think | Apply |
|---|---|---|
| $\frac{264}{\text{number of girls}} = \frac{8}{7}$<br>$\frac{\text{number of girls}}{264} = \frac{7}{8}$<br>number of girls $= \frac{7}{8} \times 264$<br>$= 231$<br>There are 231 girls. | Write two equivalent ratios.<br>number of boys : number of girls $= 8:7$<br>Change the order so the number of girls is the first term in the ratio (and hence the numerator of the fraction).<br>number of girls : number of boys $= 7:8$<br>As the two equivalent ratios are equal, write as an equation using fractions. Solve the equation by multiplying both sides by 264. | Using fraction notation, write two equivalent ratios. Change the order so that the unknown is the first term of the ratio and appears in the numerator of the fraction. Treat this as an equation and solve. |

**11** The ratio of males to females at a basketball match is 10 : 9. Complete the following to find the number of females, given that there are 470 males at the match.

$$\frac{\square}{\text{number of females}} = \frac{10}{9} \text{ so } \frac{\text{number of females}}{\square} = \frac{9}{10}$$

$$\text{number of females} = \frac{9}{10} \times \_\_\_\_ = \_\_\_\_$$

**12** The ratio of a daughter's height to that of her mother is 4 : 5. What is the mother's height if her daughter is 132 cm tall?

**13** The instructions on a fertilising product recommend a ratio of concentrated fertiliser to water of 2 : 7.

**a** How many litres of water are required in a mixture with 10 L of concentrated fertiliser?

**b** How much fertiliser mixture has been made?

**14** A two-stroke fuel mixture is made by mixing oil and petrol in the ratio of 1 : 25.

**a** How many litres of petrol should be added to 750 mL of oil?

**b** How many millilitres of oil should be added to 18 L of petrol?

**15** A builder wants to make concrete in the ratio 1 : 2 : 3; that is, one part cement, two parts sand and three parts screenings.

**a** If the builder buys four bags of sand, how many bags of cement and screenings are needed?

**b** If the builder only uses half a bag of cement, how many bags of sand and screenings are needed?

## EXAMPLE 3E–6 Dividing a quantity in a given ratio

An inheritance of \$24 000 is to be divided between Sam and Jamie in the ratio 2 : 3. How much will each receive?

| Solve | Think | Apply |
| --- | --- | --- |
| The money needs to be divided into 2 + 3 = 5 parts.<br>Sam receives<br>$\frac{2}{5} \times \$24\,000 = \$9600$<br>Jamie receives<br>$\frac{3}{5} \times \$24\,000 = \$14\,400$ | For every \$2 that Sam receives, Jamie receives \$3. Thus Sam receives \$2 out of every \$5, or $\frac{2}{5}$ of the inheritance, and Jamie receives \$3 out of every \$5, or $\frac{3}{5}$ of the inheritance. | Add the terms of the ratio. This is the number of parts into which the quantity is to be divided.<br>The appropriate fraction of the quantity can then be found. |

**16** If \$30 000 is to be divided between Hannah and Rachel in the ratio 5 : 3, how much will each girl receive?

**17** If \$1500 is divided in the ratio 3 : 1, how much is the larger share?

**18** Cordial and water are mixed in the ratio 1 : 8 to make a fruit drink. How much cordial and water would there be in a 180 mL glass of fruit drink?

**19** A metal solder is made by combining lead and tin in the ratio 2 : 3. How much tin is needed to make 10 kg of the metal solder?

**20** The masses of Alison, Vincent and Matthew are in the ratio 4 : 3 : 5. If their combined mass is 168 kg, find the mass of each person.

**21** Explain the difference between a rate and a ratio.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**22** Statistics were gathered on the number of people who develop lung cancer according to their smoking habits. The results are shown in the table below.

| Category | Number who develop lung cancer | Ratio (1 : $n$) |
|---|---|---|
| Males who have never smoked | 10 in 760 | 1 : 76 |
| Males who have smoked | 25 in 300 | |
| Males who currently smoke | 20 in 90 | |
| Females who have never smoked | 3 in 471 | |
| Females who have smoked | 50 in 1150 | |
| Females who currently smoke | 15 in 132 | |

**a** Complete the table to express the risk, in the form 1 : $n$, of developing lung cancer.
**b** Which group of people is most likely to develop lung cancer?
**c** Which group of people is least likely to develop lung cancer?

**23** A farmer planted lemon trees and orange trees in an orchard in the ratio 2 : 3.
**a** In one orchard there are 60 orange trees. How many lemon trees were planted?
**b** In another orchard there are 18 lemon trees. How many orange trees were planted?
**c** How many trees did the farmer plant in total?
**d** What percentage of the trees planted are lemon trees? Explain how this relates to the ratio 2 : 3.

**24** An average residential house construction will have labour costs of about 30% of the total project costs with materials making up the rest.
**a** Write the comparison of labour costs to material costs as a ratio.
**b** If the labour costs for a project are \$87 000, what are the material costs?
**c** Calculate the labour costs for a house construction when the total project costs are \$420 000.

**25** An investment fund has investments in property, shares and government bonds in the ratio 5 : 3 : 2.
**a** If the fund has a total of \$1.8 million invested, find the amount in each of these three investments.
**b** During the next year, the fund manager transfers \$160 000 from investments in government bonds to shares. Find the new ratio of investments in this fund.

CHALLENGE

**26** Kristin and Daniel are to share \$100 between them in either the ratio 5 : 4 or 6 : 5. Which ratio should Kristin choose so as to obtain the larger payment? Explain your answer.

**27** To create two distinct shades of pink paint, white to red paint is mixed in the ratios of 3 : 1 and 7 : 1. A 4 L can of each shade is purchased.
**a** Calculate the amount (in mL) of white and red paint required for each shade.
**b** The two cans are accidently mixed together. What is the new ratio of white to red paint?

MEASUREMENT

# 3F Scale drawings

These resources are available on your obook assess:

- **Interactive 3F:** Explore the key idea of scale
- **Video tutorial 3F:** Watch and listen to an explanation of Examples 3F–1 and 3F–2
- **Worksheet 3F:** Practise finding scale
- **Investigation 3F.1:** Using scale to calculate distances from a map
- **Investigation 3F.2:** Using ratios and scale to investigate travel by motorbike
- **assess quiz 3F:** Test your skills with an auto-correcting multiple-choice quiz

OI
GI

**scale drawing**
drawing of a diagram that is similar to the original but is smaller (or larger) in which the angles are the same and the lengths are in the same ratio as the original

**scale**
the ratio of a length on a drawing to its real length expressed in the form $1:n$ or $\frac{1}{n}$

The aim of making a **scale drawing** is to draw a figure that is similar to the original; that is, the shape of the original is maintained by making the angles the same and the lengths in the same ratio as the original.

!

The scale factor of the length on the drawing to the real length is called the **scale** of the drawing.

$$\text{Scale} = \frac{\text{length on drawing}}{\text{real length}}$$

Scale may be expressed as a ratio, such as $1:1000$, or as a fraction such as $\frac{1}{1000}$.

In general form, this can be written as $1:n$ or $\frac{1}{n}$.

The scale can also be given in words, such as 1 cm to 10 m.

## EXAMPLE 3F–1 Finding the length on a scale drawing given the real length

Determine the length on a scale drawing that would represent a distance of 85 m on land if the scale was:

**a** $1:10\,000$ **b** $1:5000$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\text{Scale} = 1:10\,000 = \frac{1}{10\,000}$<br>So $\frac{\text{length on drawing}}{\text{real length}} = \frac{1}{10\,000}$<br>$\frac{\text{length on drawing}}{85\text{ m}} = \frac{1}{10\,000}$<br>$\text{Length on drawing} = \frac{1}{10\,000} \times 85\text{ m}$<br>$= 0.0085\text{ m}$<br>$= 8.5\text{ mm}$ | Express the ratio scale as a fraction and form an equation using<br>$\text{Scale} = \frac{\text{length on drawing}}{\text{real length}}$<br>or $\frac{\text{length on drawing}}{\text{real length}} = \text{scale}$.<br>Solve the equation by multiplying both sides by 85. | If the scale is $1:n$, then length on drawing or scaled length is $\frac{1}{n} \times$ real length. |
| **b** | $\text{Scale} = 1:5000 = \frac{1}{5000}$<br>So $\frac{\text{length on drawing}}{85\text{ m}} = \frac{1}{5000}$<br>$\text{Length on drawing} = \frac{1}{5000} \times 85\text{ m}$<br>$= 0.017\text{ m}$<br>$= 17\text{ mm}$ | | |

## EXERCISE 3F Scale drawings

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Using the scales below, determine the length on a scale drawing that would be represented by the following distances on land.

**i** 1 : 10 000 **ii** 1 : 5000 **iii** 1 : 1000

**a** 90 m **b** 65 m **c** 160 m
**d** 180 m **e** 240 m **f** 300 m

### EXAMPLE 3F–2 Finding the real length given the length on a scale drawing

Determine the land distance that would be represented by a map distance of 28 mm on a map with a scale of:

**a** 1 : 500 **b** 1 : 1000

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Scale $= 1:500 = \frac{1}{500}$<br>So $\frac{\text{length on drawing}}{\text{real length}} = \frac{1}{500}$<br>$\frac{28\text{ mm}}{\text{real length}} = \frac{1}{500}$<br>$\frac{\text{real length}}{28\text{ mm}} = \frac{500}{1}$<br>Real length<br>$= 500 \times 28$ mm<br>$= 14\,000$ mm<br>$= 14$ m | Express the ratio scale as a fraction and form an equation using $\frac{\text{length on drawing}}{\text{real length}} = \text{scale}$.<br>Take the reciprocal of both sides so the unknown length is in the numerator of the first fraction.<br>Solve the equation by multiplying both sides by 28. | If the scale factor is $1:n$, then real length $= n \times$ length on drawing. |
| **b** | Scale $= 1:1000 = \frac{1}{1000}$<br>$\frac{28\text{ mm}}{\text{real length}} = \frac{1}{1000}$<br>$\frac{\text{real length}}{28\text{ mm}} = \frac{1000}{1}$<br>Real length<br>$= 1000 \times 28$ mm<br>$= 28\,000$ mm<br>$= 28$ m | | |

**2** Using the scales below, determine the distance on land that would be represented by the following map distances.

**i** 1 : 500 **ii** 1 : 1000 **iii** 1 : 20 000

**a** 17 mm **b** 35 mm **c** 4.2 cm
**d** 6 cm **e** 2.6 cm **f** 0.9 cm

## EXAMPLE 3F–3 Writing a scale as a ratio and a fraction

Express each of the scales below as:

**i** a ratio **ii** a fraction.

**a** 1 cm to 5 m **b** 1 mm to 25 m

| | Solve | Think | Apply |
|---|---|---|---|
| **a i** | 1 cm to 5 m<br>= 1 cm to 500 cm<br>= 1 : 500 | Express both measurements in centimetres. As the two quantities are in the same unit, write as a ratio. Remember to leave out the units in the ratio. | Express both measurements in the same unit and simplify to either a ratio or a fraction. |
| **ii** | $1:500 = \frac{1}{500}$ | Express the ratio scale as a fraction with the first number in the numerator and the second number in the denominator. | |
| **b i** | 1 mm to 25 m<br>= 1 mm to 25 000 mm<br>= 1 : 25 000 | Express both measurements in millimetres and write the quantities as a ratio. | |
| **ii** | $1:25\,000 = \frac{1}{25\,000}$ | Express the ratio scale as a fraction. | |

**3** Express each of the scales below as:

**i** a ratio **ii** a fraction.

**a** 1 cm to 10 m **b** 1 cm to 4 m **c** 1 cm to 50 m

**d** 1 mm to 15 cm **e** 1 mm to 5 m **f** 1 mm to 20 m

## EXAMPLE 3F–4 Finding the scale given the length on the drawing and the real length

Find the scale on a drawing given that the length on the drawing is 32 cm and the real length is 1600 m.

| Solve | Think | Apply |
|---|---|---|
| Length on drawing = 32 cm<br>Real length = 1600 m<br>= 160 000 cm<br>$\text{Scale} = \frac{32 \text{ cm}}{160\,000 \text{ cm}}$<br>$= \frac{1}{5000}$ or 1 : 5000 | Write each length in the same unit. Change 1600 m to cm by multiplying by 100. Substitute into $\text{Scale} = \frac{\text{length on drawing}}{\text{real length}}$ and use the fraction key on your calculator to simplify the fraction. | Express both measurements in the same unit and use $\text{Scale} = \frac{\text{length on drawing}}{\text{real length}}$.<br>The scale can be expressed as a fraction or as a ratio. |

**4** Find the scale on a drawing, given the following lengths.

**a** length on the drawing is 52 cm and real length is 520 m

**b** length on the drawing is 52 cm and real length is 5200 m

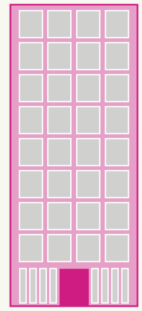
**c** length on the drawing is 16 mm and real length is 64 m

**d** length on the drawing is 15 mm and real length is 7.5 m

**e** length on the drawing is 11.2 cm and real length is 56 km

## EXAMPLE 3F–5 Finding a real length from a scale drawing

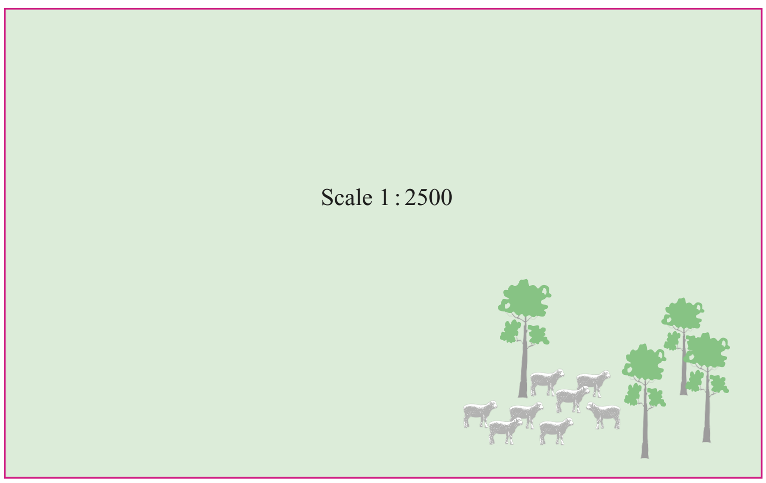
A scale drawing of a building is shown. Use your ruler and the scale given to find the real height of the building.



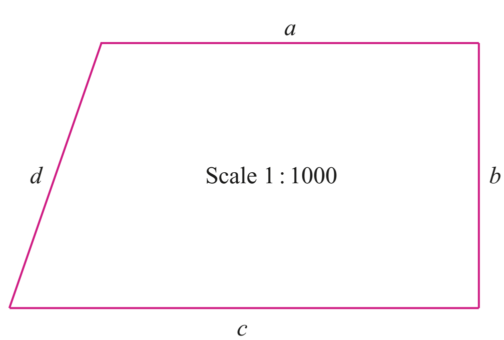
Scale is 1 : 400

| Solve | Think | Apply |
|---|---|---|
| Scale $= 1:400 = \frac{1}{400}$<br>Height on drawing $= 40$ mm<br>$\frac{\text{height on drawing}}{\text{real height}} = \text{scale}$<br>$\frac{40\text{ mm}}{\text{real height}} = \frac{1}{400}$<br>$\frac{\text{real height}}{40\text{ mm}} = \frac{400}{1}$<br>Real height $= 400 \times 40$ mm<br>$= 16\,000$ mm<br>$= 16$ m | Write the scale as a fraction and use a ruler to measure the height on the drawing. Form an equation using $\frac{\text{length on drawing}}{\text{real length}} = \text{scale}$.<br>Take the reciprocal of both sides so the unknown height is in the numerator of the first fraction.<br>Solve the equation by multiplying both sides by 40. | Measure the length on the scale drawing. If the scale is $1:n$, then real length $= n \times$ scaled length. |

**5** This diagram is a scale drawing of a rectangular field. Using your ruler and the scale given, find the real length and breadth of the field.

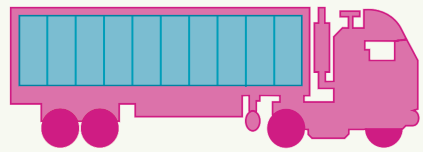



**6** This is a scale drawing of a parking area. Use the scale given to find the real dimensions ($a$, $b$, $c$, $d$) of the parking area.



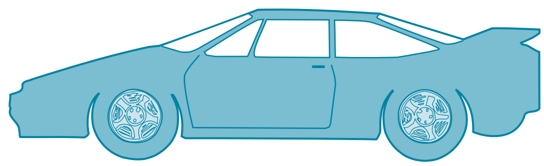
## EXAMPLE 3F–6 Finding the scale from a scale drawing

This is a scale drawing of a truck. The actual length of the truck is 5.4 m. What is the scale used in this drawing?



| Solve | Think | Apply |
|---|---|---|
| Length on drawing $= 54$ mm<br>Real length $= 5.4$ m<br>$= 5400$ mm<br>Scale $= \dfrac{\text{length on drawing}}{\text{real length}}$<br>$= \dfrac{54 \text{ mm}}{5400 \text{ mm}}$<br>$= \dfrac{1}{100}$ or $1:100$ | Use a ruler to measure the length of the truck in the scale drawing. Convert the real length to millimetres. Use the fraction key on a calculator to simplify the fraction. | Measure the length of the truck on the scale drawing. Express both measurements in the same unit and use Scale $= \dfrac{\text{length on drawing}}{\text{real length}}$. |

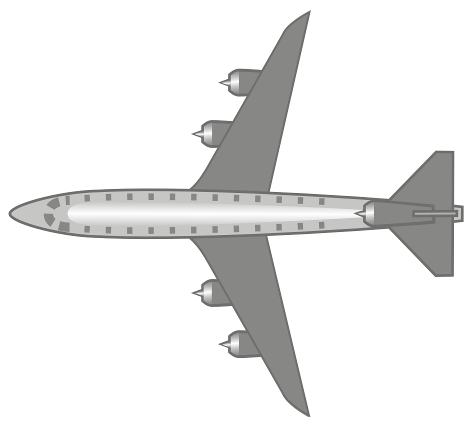
**7** The diagram on the right is a scale drawing of a car.



**a** If the real length of the car is 5.6 m, what is the scale used in the drawing?
**b** Calculate the real height of the car above the ground.
**c** What is the actual diameter of the wheel?

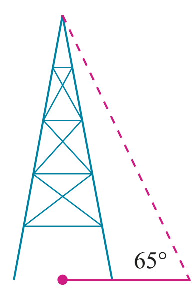
**8** The actual length of the plane shown in the scale drawing is 60 m.

**a** Find the scale used in the drawing.
**b** Find the actual wingspan of the plane.

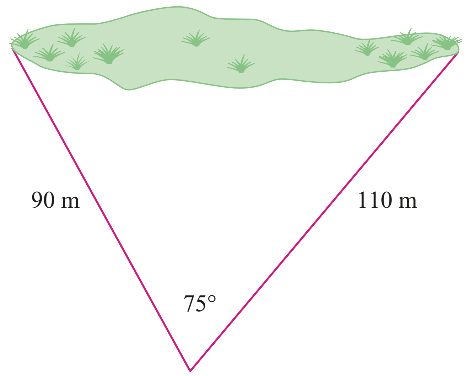


**9** The angle of elevation to the top of a tower from a point on the ground 20 m from the base of the tower is 65°, as shown in the sketch.

**a** Make a scale drawing using a scale of 1 cm to 4 m.
**b** Find the actual height of the tower using the scale drawing.
**c** Confirm your answer to part **b** by calculating the actual height using trigonometry.



**10** A surveyor marked this sketch on the right with measurements to find the distance across a swamp.

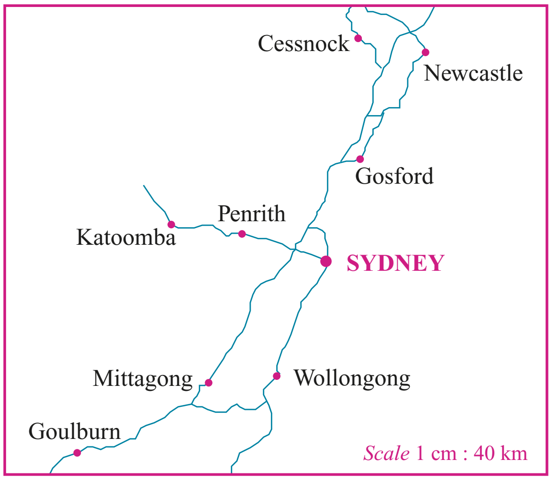



**a** Make a scale drawing of the figure using a scale of 1 : 1000.

**b** From the scale drawing, calculate the actual distance across the swamp.

**c** Confirm your answer to part **b** by calculating the actual distance across the swamp using the cosine rule.

**11** The map on the right shows some routes out of Sydney.




**a** What is the scale on the map?

**b** What is the actual distance (in a straight line) from Sydney to Katoomba?

**c** What is the actual distance from Goulburn to Gosford?

**d** Which town is about 90 km from Sydney?

**e** Which towns are about 40 km apart?

**f** Tina travels by car from Goulburn to Mittagong. The journey takes her 1 hour and 15 minutes. Using the straight-line distance between Goulburn and Mittagong as an estimate of the distance Tina travels, calculate her average speed for the journey.

**g** Estimate how long it would take Tina to drive from Mittagong to Gosford if her average speed is 72 km/h. Use the straight-line distance as an estimate of the distance Tina travels.

**12** The image below shows a football field.

**a** Use the scale shown on the bottom right of the image to work out the length and width of the rectangular football field.

**b** What is the area of the football field to the nearest square metre?

Scale 1 : 1750

**13** Draw a map of your classroom or a netball court (or other playing field) at your school to scale.
Remember to include the scale on your drawing.

CHALLENGE

**14** Download from the internet a map or aerial photograph that contains a feature with known dimensions, for example an Olympic swimming pool, an athletics field, or a playing field for a particular sport. Use this to estimate lengths or distances and to find the scale of the map or photograph.

# 3G Building plans

These resources are available on your obook assess:
- **Investigation 3G:** Using scale to explore the plans for a new house
- **assess quiz 3G:** Test your skills with an auto-correcting multiple-choice quiz
- **Website:** Building and Sustainability Index (BASIX)

OI GI

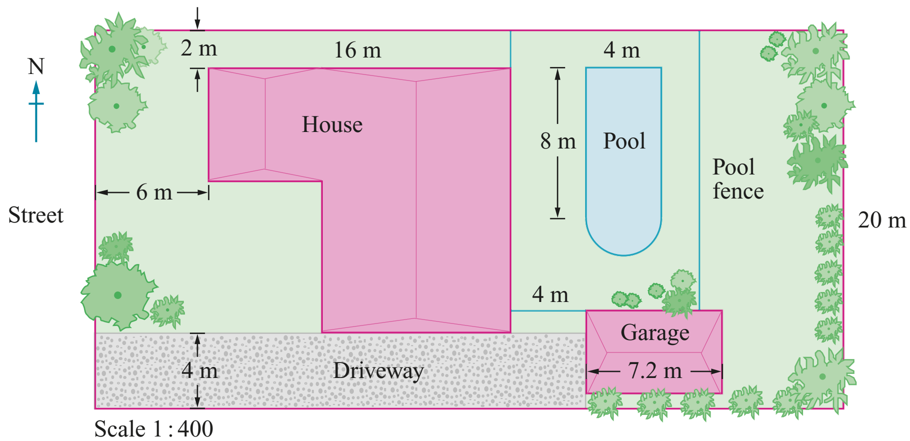
Building plans are scale drawings of houses or other buildings. They include site plans, floor plans and side elevations so that all construction details can be determined.

A site plan is a drawing of the block of land showing the position of the residence and any other buildings, or main features, on the block.

## EXERCISE 3G Building plans

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**1** Answer the following questions using the information given on this site plan and the actual measurements on the scale drawing.




**a** What are the dimensions of the block?
**b** Find the length of the driveway to the garage.
**c** Determine the distance between the house and the fence on:
  **i** the northern boundary
  **ii** the southern boundary.
**d** How far is the house from the street?
**e** How far is the garage wall from the southern boundary?
**f** Calculate the area of the garage, to the nearest square metre.
**g** Find the total length of the pool fences.
**h** Calculate the area of land taken up by the pool, to the nearest square metre.
**i** If the pool has an average depth of 1.2 m, calculate the volume of the pool to the nearest cubic metre. (*Hint:* remember that $V = A \times h$.)
**j** Find the amount of water in the pool, to the nearest 1000 litres ($1 \text{ m}^3 = 1 \text{ kL}$).
**k** How many blocks of land of this size would fit into an area of 1 ha?

**2** The floor plan below of a four-bedroom house is drawn to a scale of 1 : 200.

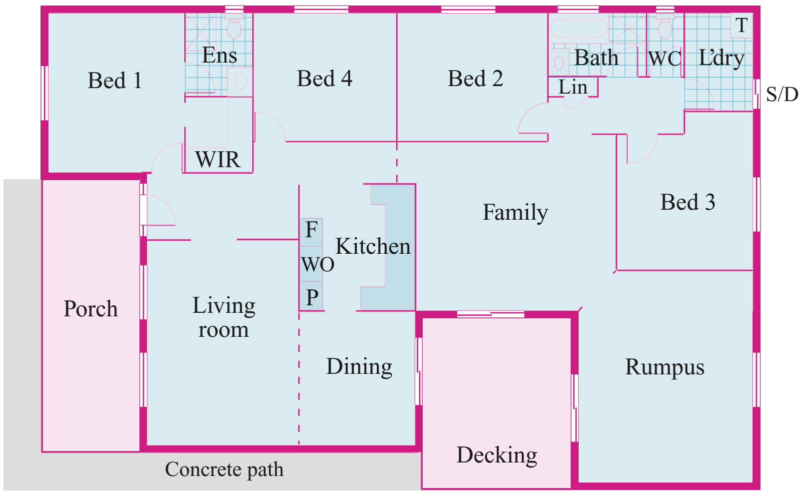
**a** How many of each does the house have?

  **i** windows (shown as )

  **ii** hinged doors (shown as )

  **iii** sliding doors (shown as )

**b** Find the following symbols on the plan and give the meaning of each:
P, WO, F, WC, Ens, WIR, Lin, T




Scale 1 : 200

**c** By measurement and calculation, find the dimensions of the:

  **i** rumpus room (assume rectangular)

  **ii** bedroom 2

  **iii** living room

  **iv** porch

**d** **i** What is the area of land occupied by the whole house, including the porch and decking? (Remember to include the thickness of the walls.) Give your answer to the nearest $m^2$.

  **ii** A 'square' (100 square feet) is an old imperial unit used for measuring the area of a house. If 1 square $\approx$ 9.29 $m^2$, calculate the area of the house, including porch and decking, to the nearest square.

**e** The porch is a reinforced concrete slab 100 mm thick. How much concrete, in cubic metres, was needed for the porch? (It is easier if you first convert all measurements to metres.)

**f** The walls of bedroom 4 are to be repainted.

  **i** Calculate the area of the four walls, including the window and door, given the ceiling height is 2.7 m.

  **ii** Calculate the area to be painted given that the window height is 1.2 m and the door height is 2.1 m.

  **iii** Find the amount of paint needed to apply two coats if 1 L of paint covers 16 $m^2$.

**g** **i** Consider the thickness of the internal walls and the external walls. Discuss why they are different.

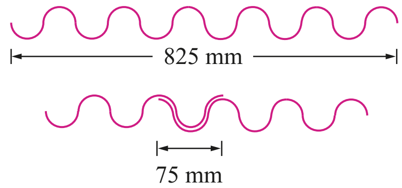
  **ii** Research and list the different ways in which the walls of a house could be constructed. How is this shown on a building plan?

**h** **i** Find the dimensions of the laundry.

  **ii** If the laundry floor is to be tiled with 250 $\times$ 250 mm square tiles, calculate the number of tiles needed.

  **iii** The tiles come in boxes of 16. How many boxes must be bought?

**i** The decking is to be covered with metal roofing that comes in 825 mm widths with the profile shown. How many of these sheets would be needed to cover the decking if they are to be overlapped as shown? The sheets are placed so that the water runs to the side of the house.

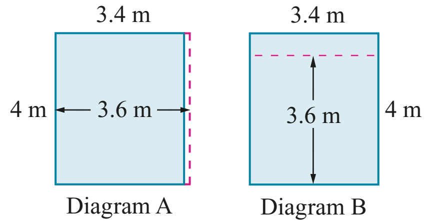



**j** Carpet comes in rolls that are 3.6 m wide.

**i** Calculate the number of metres of carpet needed to cover the floor in bedroom 2 if the carpet is laid as shown in diagram A.

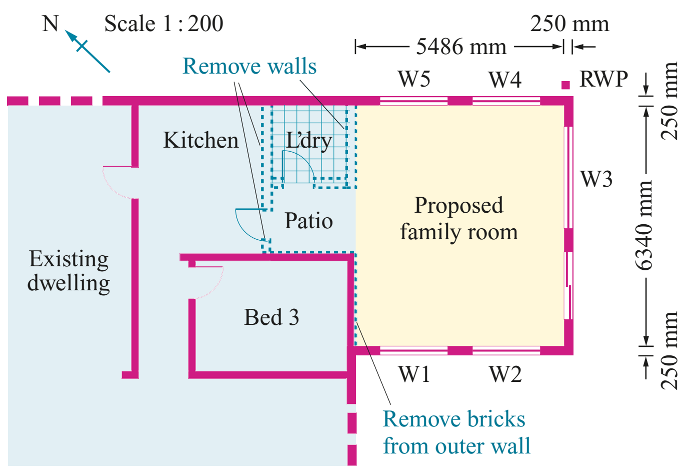
**ii** Calculate how much carpet is needed if it is laid as in diagram B if there is to be only one join and it makes no difference which direction the carpet runs.

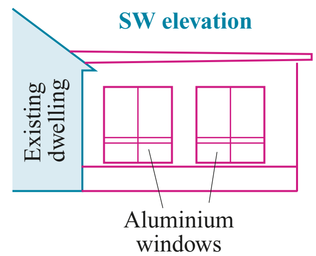
**iii** Which method of laying would be cheaper?

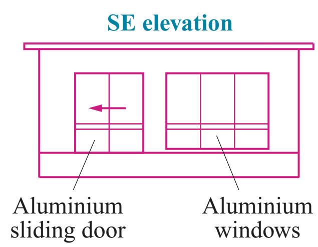


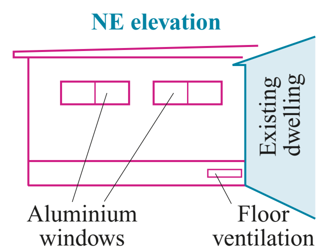

**k** Repeat the method in part **j** to determine the number of metres of carpet needed for bedrooms 1, 3 and 4.

**3** The floor plan and elevations for a proposed extension to a house are shown.




**a** What is the scale on the plan and elevations?

**b** What are the internal dimensions of the proposed family room?

**c** Find the thickness of the walls of the extension.

**d** What length of existing wall indicated is to be removed?

**e** Determine the dimensions of the new kitchen.

**f** What length of guttering is needed across the rear of the family room?

**g** What length of rainwater pipe is needed at the rear of the extension?

**h** Find the height of the ceiling above the floor.

**i** Use the plan and elevations to determine the dimensions of windows W1, W3 and W4.

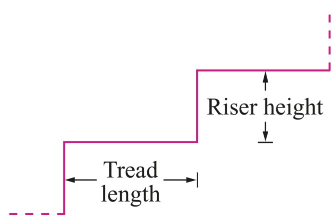
**j** What are the dimensions of the sliding door?

**k** Calculate the area of the new room in square metres.

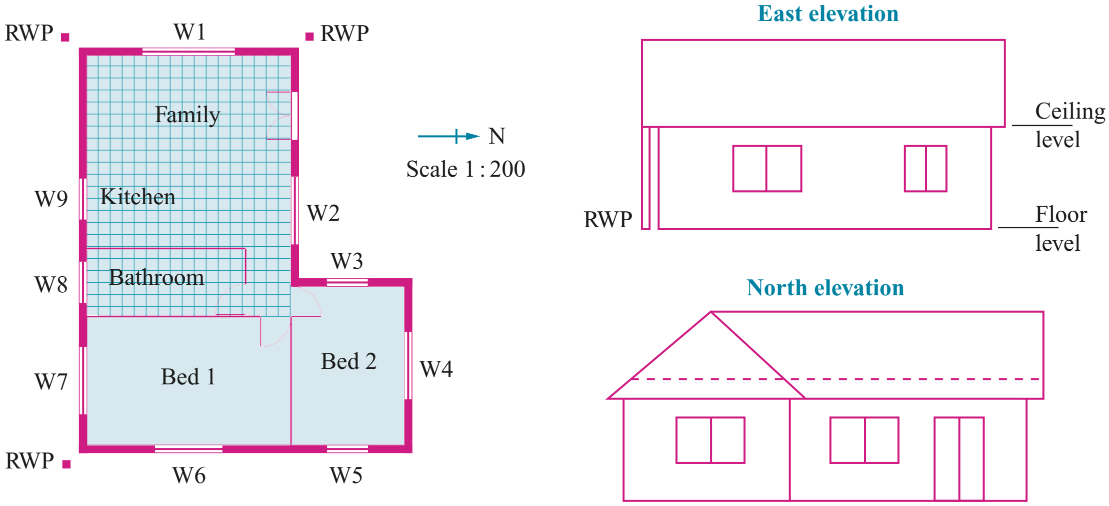
**l** Calculate the area of the new kitchen in square metres.

**m** How much would it cost to cover the floor of the new kitchen and family room with cork tiles if they cost $106/m$^2$ laid (to the nearest dollar)?

**n** What is the minimum number of steps needed from the ground to the rear sliding door if the maximum riser height allowed is 190 mm?



4 The diagram shows the floor plan and elevations of a small holiday house.




**a** Find the following symbols on the plan and give the meaning of each: W, RWP, Bed.

**b** How many of the following are there?

**i** windows **ii** hinged doors

**c** How many windows can be seen in these drawings facing:

**i** north? **ii** east? **iii** south? **iv** west?

**d** What is the scale used?

**e** Find the dimensions of:

**i** the bathroom **ii** bedroom 2 **iii** bedroom 1.

**f** What are the dimensions of windows W2, W5, W6 and the front door?

**g** Find the height of the ceiling above the floor.

**h** What is the maximum height of the roof above floor level?

**i** How far do the eaves extend beyond the walls of the house in the east elevation?

**j** Draw the south elevation.

**k** What area of land does the house occupy?

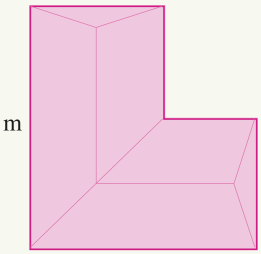
**l** If the house is built on a reinforced concrete slab of depth 250 mm, calculate the amount of concrete needed for the slab, to the nearest cubic metre.

**m** A sewer is required to have a fall of 1 in 40. If a sewer trench is dug along the eastern side of the house, how much deeper should one end be than the other?

> **!**
>
> To find volume of rain collected (in m³) use
>
> $V = C \times A \times R$
>
> where $C$ = the water runoff coefficient, as a decimal
>
> $A$ = roof area (m²)
>
> $R$ = amount of rainfall (m)
>
> *Note:* if 10% of the rainfall is considered wastage, then $C = 90\% = 0.9$.

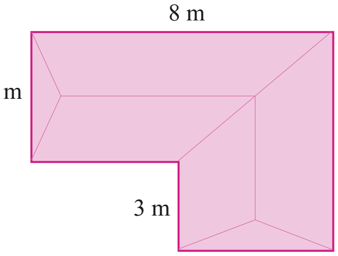
## EXAMPLE 3G–1 Calculating volume of water collected after rain

**a** Calculate the plan-view area of the roof of the house shown.
**b** How much water could be collected from this roof if 6 mm of rain fell on it, allowing 10% for wastage?



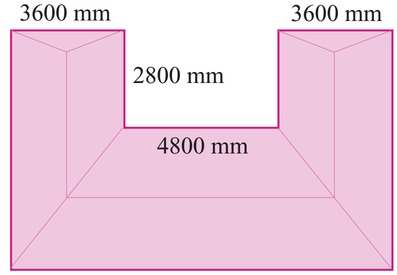

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Area of roof $= 5 \times 9 + 4 \times 3$<br>$= 57\text{ m}^2$ | Divide the plan-view area into two rectangles. | Find the plan-view area of the roof by dividing the shape of the roof into rectangles. |
| **b** | $6\text{ mm} = 0.006\text{ m}$<br>Volume of water collected<br>$= 90\%$ of $(57 \times 0.006)$<br>$= 0.9 \times 57 \times 0.006$<br>$= 0.3078\text{ m}^3$<br>$= 307.8\text{ L}$<br>$\approx 308\text{ L}$ | If all the rain is collected, then $V = (57 \times 0.006)\text{ m}^3$. However, it is usual to allow 10% for wastage, so calculate 90% of this volume. (90% or 0.9 is known as the water runoff coefficient.) | Volume of rain collected<br>$V(\text{m}^3) = C \times A \times R$, where<br>$C$ = the water runoff coefficient, as a decimal<br>$A$ = roof area ($\text{m}^2$)<br>$R$ = amount of rainfall (m) |

*Note:* the plan-view area of the roof in Example 3G–1 includes the width of the eaves and so, in general, is not the same as the floor area. The plan-view area is sometimes referred to as the house's 'footprint' or 'drip line'.




**5** **a** Calculate the plan-view area of the roof of the house shown.
**b** How much water could be collected from this roof if 4 mm of rain fell on it, allowing 10% for wastage?

**6** **a** Calculate the plan-view area of the roof of the house shown on the right.
**b** This house is situated in Newcastle. The average rainfall for Newcastle in January is 91.4 mm. How much water could be expected to be collected from this roof in January next year, allowing 10% for wastage?




**7** **a** Calculate the plan-view area of the roof of the house shown in question **1**.
**b** This house is situated in Wollongong. The average rainfall for Wollongong in spring is 206.7 mm. How much water could be expected to be collected from this roof in spring next year, allowing 15% for wastage?

**8** **a** Calculate the plan-view area of the roof of the house shown in question **4**. Remember to include the eaves in your calculation of the area.
**b** This house is situated in Canberra. The average annual rainfall for Canberra is 633.1 mm. How much water could be expected to be collected from this roof next year, allowing 15% for wastage?

9 A house has a roof area of 160 m$^2$. The average annual rainfall in the locality of this house is 743 mm.

**a** How much water could be collected in a year, allowing 10% for wastage?

**b** For a house of this size with four people, the ideal water consumption is 565 L/day. What is the ideal annual consumption for this household?

**c** What percentage of the ideal annual consumption could be replaced by tank water?

**d** If water costs \$2.05/kL, how much could be saved each year by using the tank water?

10 Use rainfall data available on the internet to calculate how much water could be collected from the roof of your home in a year, allowing 10% for wastage. If you are unable to work out the area of your roof, estimate a possible area and use this in your calculation.

INVESTIGATION

11 To build homes that are energy-efficient, passive design features such as location, orientation, layout, window-sizing, insulation and shading should be taken into consideration. Using materials that provide thermal mass to make the best use of absorbing heat energy at different times of the year is also beneficial. Investigate local council requirements for energy-efficient housing.

12 The aim of the Building and Sustainability Index (BASIX) Certificate is to ensure the efficient use of water and energy in homes across New South Wales by including reduction targets in these areas for all development plans.

Imagine that you are about to build the house with the plans shown on the next two pages. Complete the following in preparation for undertaking a BASIX certificate for this development proposal.

**a** Calculate the:

**i** site area **ii** roof area **iii** floor area

**iv** ratio of floor area for which air conditioning applies to floor area for which air conditioning does not apply. (You will need to decide which rooms you will have air conditioned.)

**b** Prepare a landscape design for the property and calculate the garden and lawn areas.

**c** Assume that you will be installing a water tank and a swimming pool.

**i** Decide on the size of the tank and where it will be situated.

**ii** Calculate the volume of the swimming pool you select. (Keep the design simple.)

**d** What is the thickness of the internal and external walls of the house?

**e** What is the width of the eaves on each side of the house?

**f** Investigate and calculate the amount of roof insulation required for this house.

**g** Determine the orientation of the windows. Determine the breeze path and indicate it on the plan.

Now go to the NSW Government BASIX website. A link is provided on your obook assess. Use the plans given on the next pages and your answers to the questions above to complete each section of the BASIX Certificate. You will need to modify your plans along the way until you have passed.

*Step 1:* On the home page go to the login box and click on ‘Register’ and then ‘next’. Select ‘Education’ as your user type and click on ‘next’.

*Step 2:* Make up a username and password.

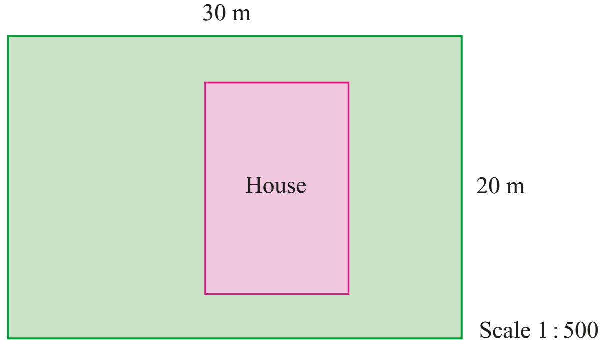
*Step 3:* Enter your name, school (institution) and email address. Click on ‘next’. An email is then sent to you with your verification code.

*Step 4:* Return to the home page. In the login box enter your username and password. Tick ‘… terms and conditions’ and enter the verification code sent to you by email. Submit.

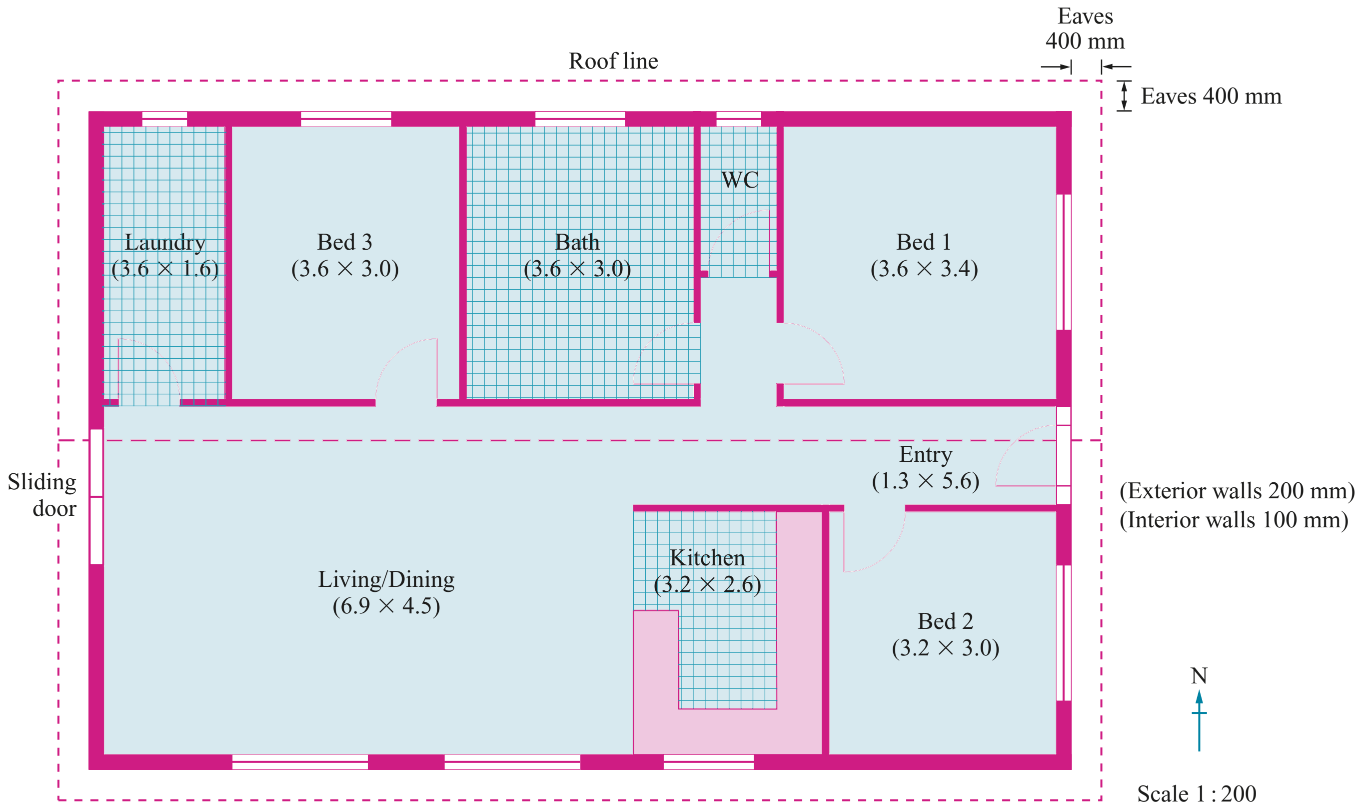
*Step 5:* Select ‘New Dwellings’ and enter.

*Step 6:* Click on ‘Start a new project’.

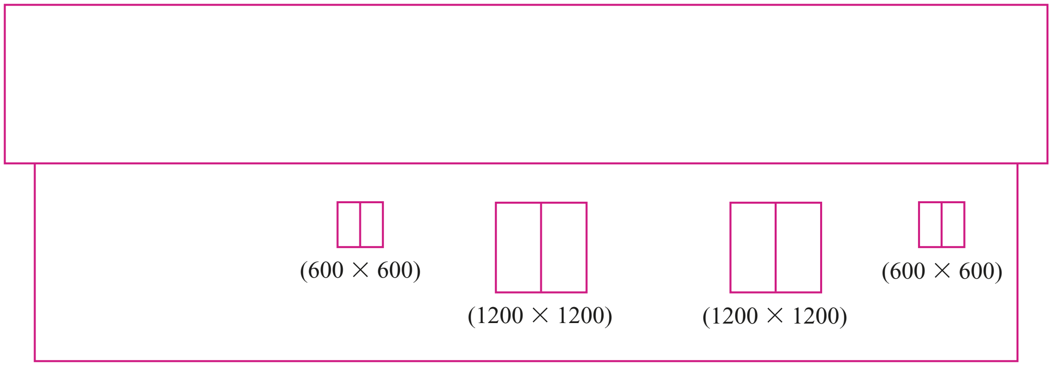
## Sample house plans



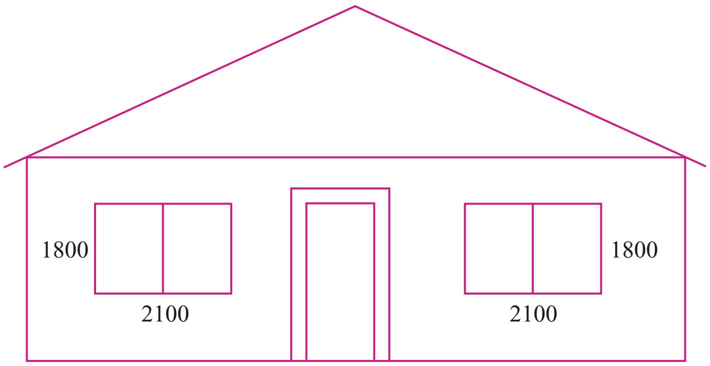

Site plan



**North elevation**

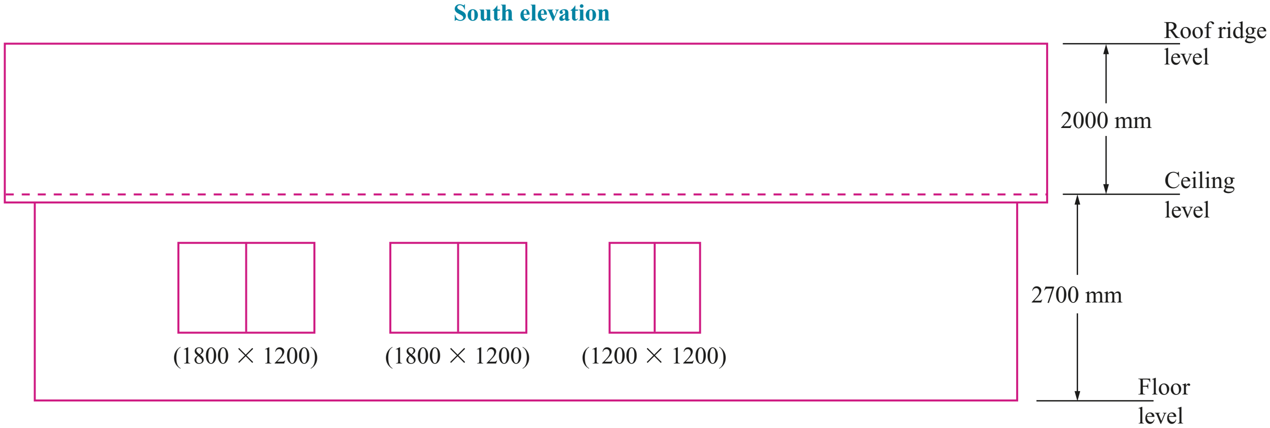



**East elevation**

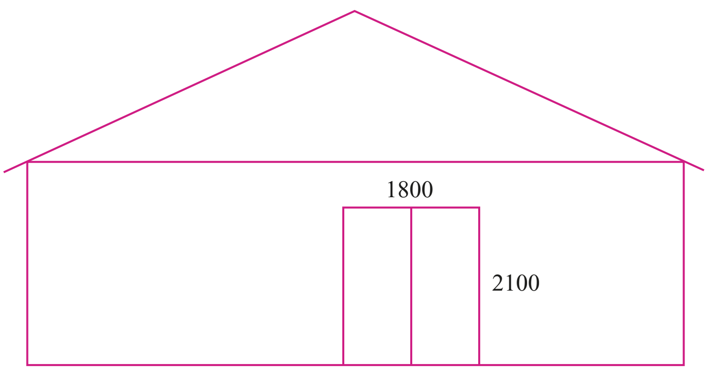



Scale 1 : 100

**South elevation**




**West elevation**




Scale 1 : 100

# 3H Measurements of land using a scale

These resources are available on your obook assess:
- **Video tutorial 3H:** Watch and listen to an explanation of Example 3H–1
- **assess quiz 3H:** Test your skills with an auto-correcting multiple-choice quiz

OI
GI

In the previous topic, the scale shown on building plans allowed us to work out actual lengths and then solve problems by calculating values for perimeter, area and volume. In this topic, we will use a number of methods to calculate perimeter, area and volume for sections of land.

To help us find perimeter and area of sections of land with straight edges, we first use a ruler to measure lengths on the scale drawing and then calculate the matching actual length using the scale factor.

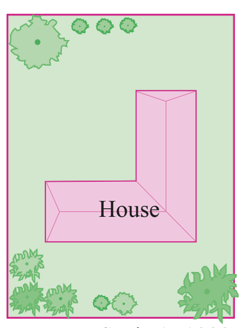
In cases where a section of land has curved edges, we can estimate the area using one of the following methods: the grid-square method, the polygon method or the Trapezoidal rule.

## EXERCISE 3H Measurements of land using a scale

PROBLEM SOLVING, REASONING AND JUSTIFICATION

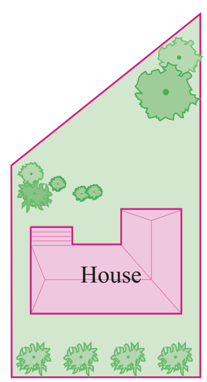
**1** Use measurement and the scale given for the following site plans to find the:
  **i** perimeter of the land  **ii** area of the land.

**a**



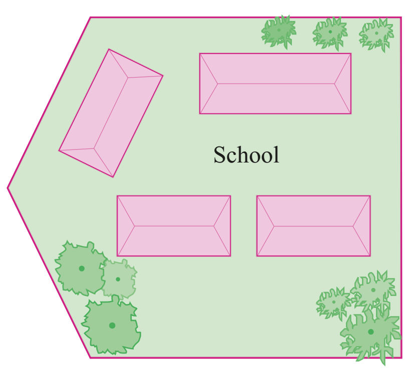

Scale 1 : 1000

**b**



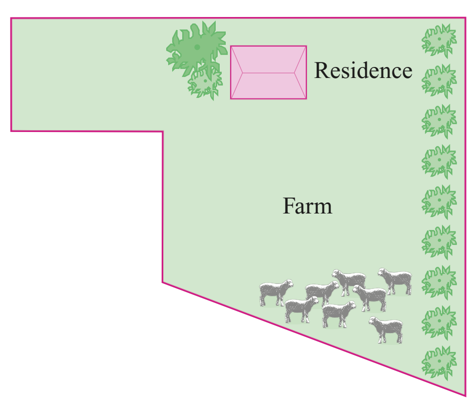

Scale 1 : 800

**c**




Scale 1 : 4000

**d**




Scale 1 : 24 000

**Estimating area using grid squares**

*Step 1:* Place or draw a 1 cm by 1 cm grid over the diagram of the land.

*Step 2:* Count the number of whole grid squares within the boundary.

*Step 3:* Count the grid squares around the edge that are partially inside the boundary. Multiply this number by 0.5 to find the approximate number of whole squares partially inside the boundary.

*Step 4:* Add the numbers found in steps 2 and 3 to find the total number of whole grid squares.

*Step 5:* Use the scale to calculate the area of each grid square.

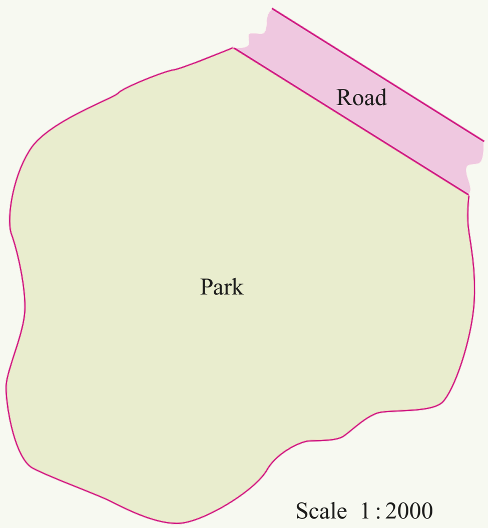
*Step 6:* Use Area = total number of grid squares × area of each grid square.

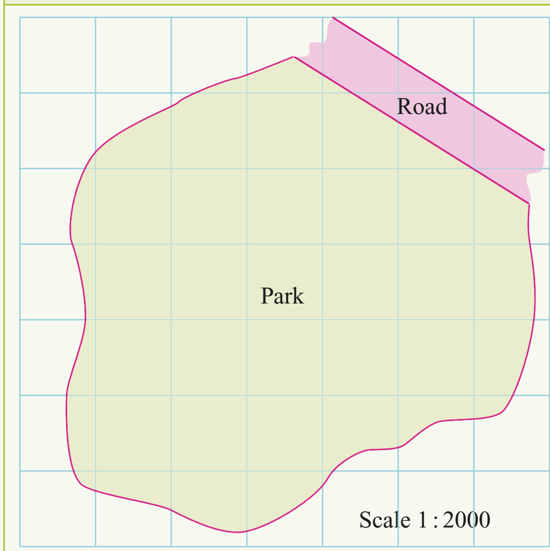
*Note:* in Step 3, various methods can be used to estimate the number of whole grid squares covered by the squares partially inside the boundary. For example, if more than half of any grid square lies within the area, count it as a whole square, and if less than half of a grid square is within the area, ignore it.

Other grid sizes may be used, such as 5 mm by 5 mm or 2 mm by 2 mm. The smaller the grid squares, the more accurate the estimate. To check your answer, lay the grid at a different angle over the area and count squares again.

## EXAMPLE 3H–1 Estimating area of a section of land using grid squares

Use the grid-square method to estimate the area of the park shown.

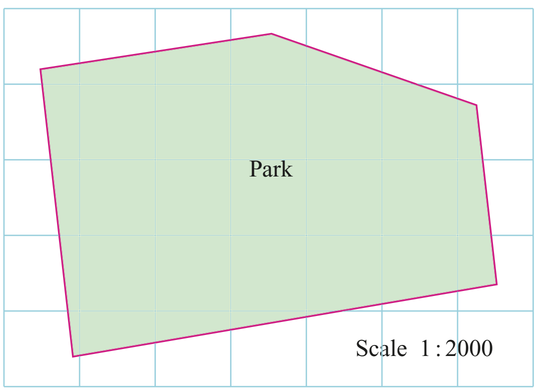



| Solve | Think/Apply |
|---|---|
|   | Place or draw a 1 cm by 1 cm grid over the diagram of the park. |

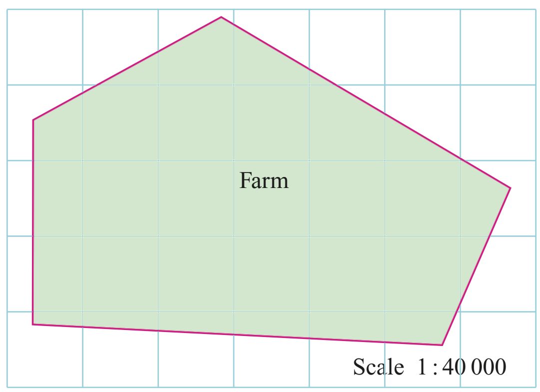
| Solve | Think/Apply |
|---|---|
| Number of whole grid squares within the park's boundary = 19<br>Number of squares around the edge that are partially inside the boundary = 23<br>Number of *whole* grid squares partially within the boundary<br>$\approx 23 \times 0.5$<br>$= 11.5$<br>Total number of whole grid squares $\approx 19 + 11.5 = 30.5$<br>The scale is 1 : 2000, so the actual size of each grid square is 2000 cm by 2000 cm or 20 m by 20 m.<br>Area of each grid square $= 20\text{ m} \times 20\text{ m} = 400\text{ m}^2$<br>Area of park $\approx 30.5 \times 400\text{ m}^2 = 12\,200\text{ m}^2$<br>Approximate area of park is $12\,200\text{ m}^2$. | Count the number of whole grid squares within the boundary.<br>Count the grid squares around the edge that are partially inside the boundary and multiply by 0.5.<br>Add to find the total number of whole grid squares.<br>Use the scale to calculate the area of each grid square in the park.<br>Calculate the approximate area using<br>Area = total number of grid squares × area of each grid square. |

**2** Use the grid-square method to estimate the area of each section of land or water.

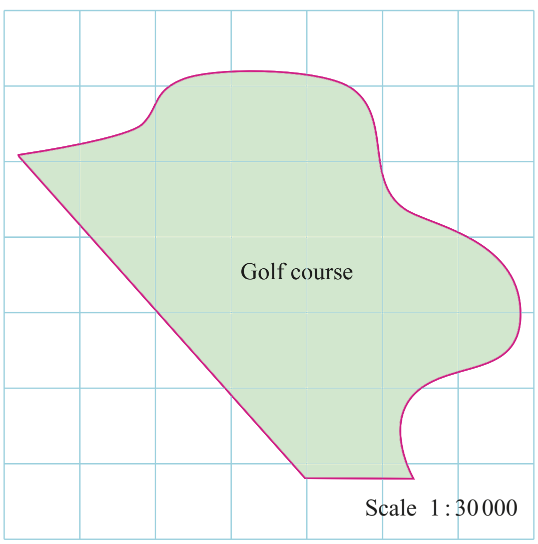
**a**



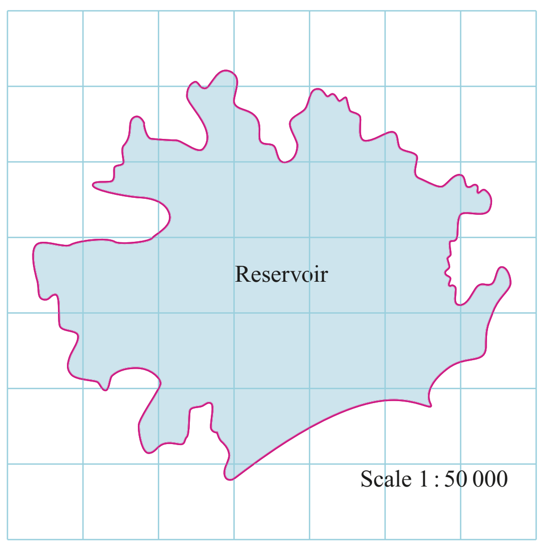

**b**




**c**




**d**



**Estimating area using the polygon method**

The polygon method involves breaking up a polygon into simple plane shapes in order to find its area. One way this can be done is to use the measurements obtained from a radial survey, which divides the polygon into triangles.

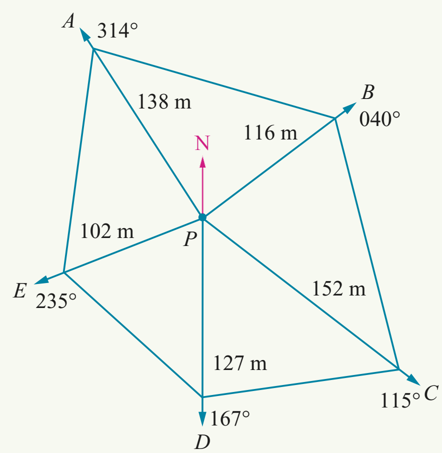
To find the area of each triangle we use the formula Area $= \frac{1}{2} ab \sin C$ where $a$ and $b$ are the lengths along the radial arms forming two sides of the triangle and $C$ is the angle between them at the central point.

For a section of land that has a curved boundary, a polygon can be constructed by joining selected points on the boundary.

*Note:* in Chapter 2 we worked with compass radial surveys, which showed the true bearing to each vertex on the boundary from a centre point of the land along a radial arm. With scaled diagrams we can measure the angles between the radial arms.

## EXAMPLE 3H–2 Using measurements from a radial survey to estimate the area of land with straight edges

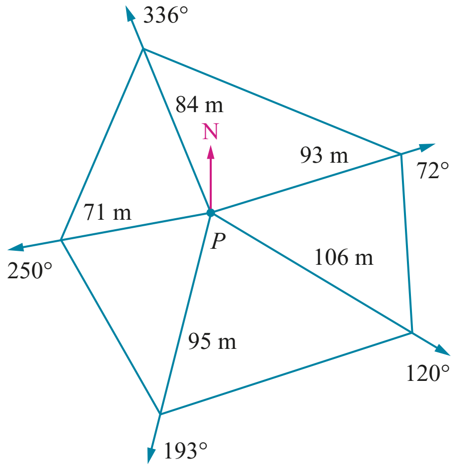
Estimate the area of the land in the diagram, using the measurements obtained from a radial survey.



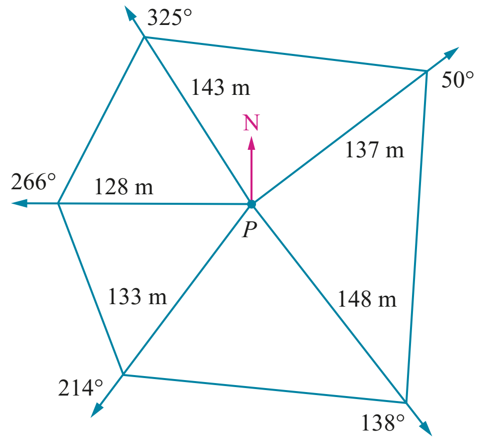

| Solve | Think/Apply |
|---|---|
| $\angle APB = 360° - 314° + 40° = 86°$<br>Area $\triangle PAB = \frac{1}{2} \times 138 \times 116 \times \sin 86° \approx 7984.5 \text{ m}^2$<br>$\angle BPC = 115° - 40° = 75°$<br>Area $\triangle PBC = \frac{1}{2} \times 116 \times 152 \times \sin 75° \approx 8515.6 \text{ m}^2$<br>$\angle CPD = 167° - 115° = 52°$<br>Area $\triangle PCD = \frac{1}{2} \times 152 \times 127 \times \sin 52° \approx 7605.9 \text{ m}^2$<br>$\angle DPE = 235° - 167° = 68°$<br>Area $\triangle PDE = \frac{1}{2} \times 127 \times 102 \times \sin 68° \approx 6005.4 \text{ m}^2$<br>$\angle EPA = 314° - 235° = 79°$<br>Area $\triangle PEA = \frac{1}{2} \times 102 \times 138 \times \sin 79° \approx 6908.7 \text{ m}^2$<br>Total area $\approx 37\,020 \text{ m}^2$ | Find the included angle by subtraction of the true bearings. Find the area of each triangle using Area $= \frac{1}{2} ab \sin C$ and add the individual areas to find the total area. |

**3** Estimate the area of the land in each diagram, using the measurements obtained from a radial survey.

**a**




**b**




**Constructing a radial survey of a scaled diagram to estimate area**

*Step 1:* Choose a point, *P*, inside the shape and label the vertices of the polygon.

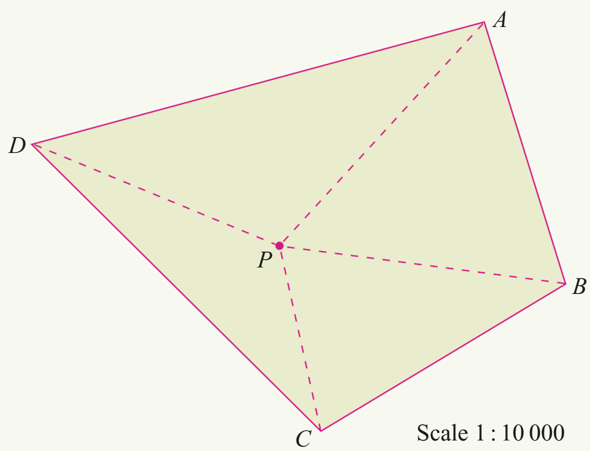
*Step 2:* Draw and measure the length of the radial arms on the diagram. Use the given scale to convert these lengths on the diagram to actual lengths.

*Step 3:* Use a protractor to measure the angles between the radial arms.

*Step 4:* Calculate the area of each triangle and add them to find the total area.

## EXAMPLE 3H–3 Estimating the area from a scaled diagram using a radial survey

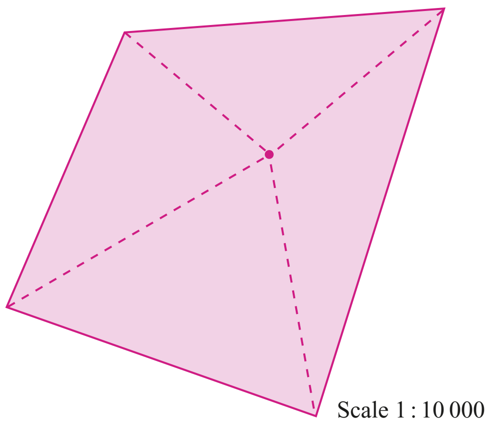
Estimate the area of the land shown in the scaled diagram, using a radial survey. A point, (*P*), has been chosen inside the shape and radial arms drawn to the vertices.



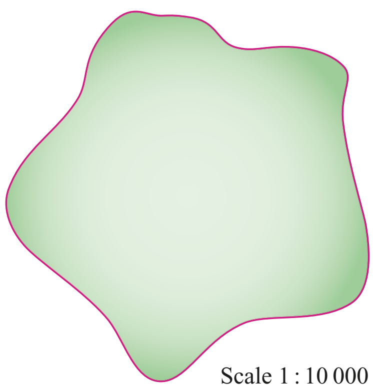

| Solve | Think/Apply |
|---|---|
| Lengths on the diagram:<br>$PA = 40$ mm, $PB = 38$ mm, $PC = 25$ mm, $PD = 35$ mm<br>Actual lengths (using scale of 1 : 10 000):<br>$PA = 40 \times 10\,000$ mm $= 400\,000$ mm or 400 m<br>$PB = 38 \times 10\,000$ mm $= 380\,000$ mm or 380 m<br>$PC = 25 \times 10\,000$ mm $= 250\,000$ mm or 250 m<br>$PD = 35 \times 10\,000$ mm $= 350\,000$ mm or 350 m<br>Angles between the radial arms:<br>$\angle APB = 55°$, $\angle BPC = 70°$, $\angle CPD = 125°$, $\angle DPA = 110°$ | Measure the radial arms on the diagram. Use the scale to convert these to the actual lengths. Use a protractor to measure the angles between the radial arms. |

| Solve | Think/Apply |
|---|---|
| Area of each triangle:<br>Area $\triangle APB = \frac{1}{2} \times 400 \times 380 \times \sin 55° \approx 62\,255.6\ \text{m}^2$<br>Area $\triangle BPC = \frac{1}{2} \times 380 \times 250 \times \sin 70° \approx 44\,635.4\ \text{m}^2$<br>Area $\triangle CPD = \frac{1}{2} \times 250 \times 350 \times \sin 125° \approx 35\,837.9\ \text{m}^2$<br>Area $\triangle DPA = \frac{1}{2} \times 350 \times 400 \times \sin 110° \approx 65\,778.5\ \text{m}^2$<br>Total area $\approx 208\,507\ \text{m}^2$ | Calculate the area of each triangle using Area $= \frac{1}{2} ab \sin C$ and add the individual areas to find the total area. |

**4** Estimate the area of the land shown in the scaled diagram using a radial survey.



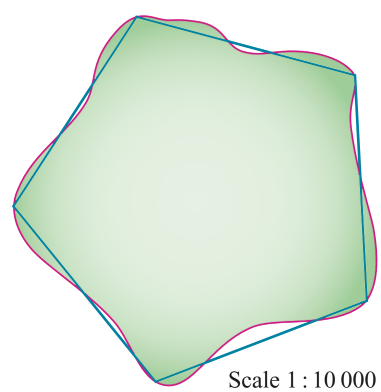

**5** Complete the following steps to estimate the area of this irregular shape using a radial survey.




**a** Copy the diagram. Mark and label five points on the boundary. Join these points to form a polygon.

**b** Choose a point $P$ inside the polygon and use a radial survey to estimate its area.

**c** Choose eight to ten suitable points on the boundary of the shape and determine the area of the polygon formed by joining these points using a radial survey. (*Note:* the more vertices you choose, the better the approximation for the area of the irregular shape.)



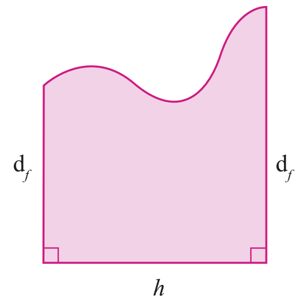

**6** Find an estimate for the area of each section of land or water in question **2** using a radial survey. How do these estimates compare to the values obtained using the grid-square method?

**Estimating area using the Trapezoidal rule**

As seen in Year 11, if a section of land has a boundary that is not straight, then the area can be approximated by replacing the curved boundary with a straight line to form a trapezium. The area of the trapezium is then an estimate of the area of the land. This is known as the Trapezoidal rule and is usually written as

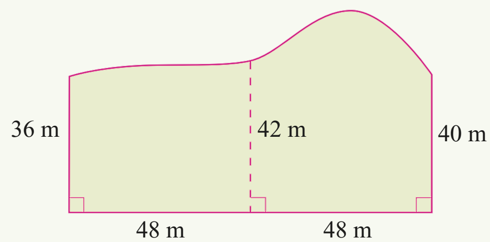
$$A \approx \frac{h}{2}(d_f + d_l)$$

where $d_f$ and $d_l$ are the lengths of the parallel sides of the trapezium and $h$ is the perpendicular distance between them. ($d_f$ is the first length and $d_l$ is the last length.)




## EXAMPLE 3H–4 Estimating area using two applications of the Trapezoidal rule

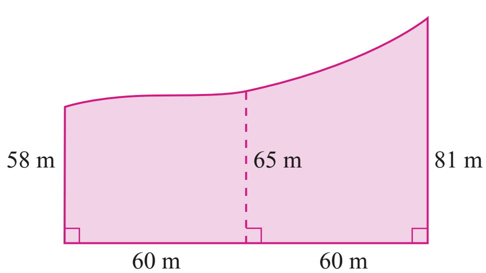
Use two applications of the Trapezoidal rule to estimate the area of the land shown.



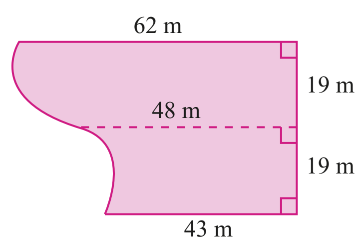

| Solve | Think | Apply |
| --- | --- | --- |
| Area $\approx \frac{48}{2}(36 + 42) + \frac{48}{2}(42 + 40)$<br>$= 24 \times 78 + 24 \times 82$<br>$= 3840\text{ m}^2$<br>Approximate area of the land is $3840\text{ m}^2$. | The land is broken up into two sections or strips. The left strip can be approximated by a trapezium with $h = 48$ m, $d_f = 36$ m and $d_l = 42$ m. The right strip can be approximated by a trapezium with $h = 48$ m, $d_f = 42$ m and $d_l = 40$ m. | Since the shape has been divided into two strips, use two applications of the Trapezoidal rule. Add the approximate areas found for the two strips to estimate the total area of the land. |

**7** Use two applications of the Trapezoidal rule to estimate the area of the land shown.

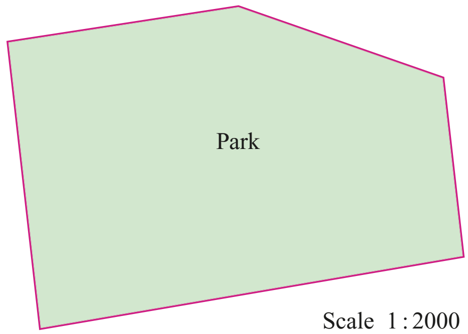
**a**




**b**



**8** Find an estimate for the area of the park shown using two applications of the Trapezoidal rule. Compare your answer to the estimated areas found in questions **2a** and **6a**.




Scale 1 : 2000

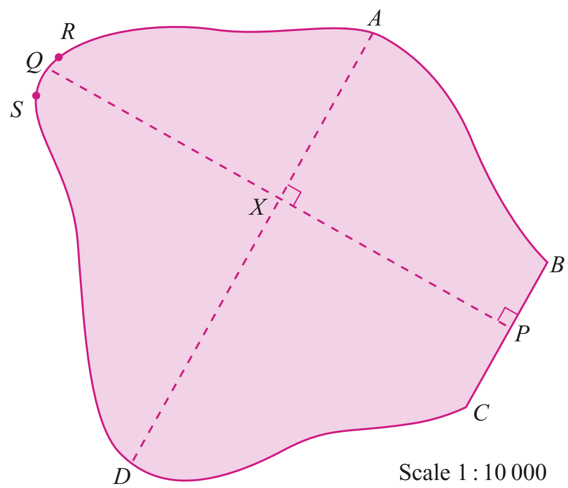
**9** Complete the following to estimate the area in the scaled diagram using the Trapezoidal rule. Approximate the four sections of land as trapeziums with the height of each section measured as $QX$ or $XP$.

By measurement: $PX = XQ = \_\_\_$ mm,

$PB = \_\_\_$ mm, $QR = \_\_\_$ mm,

$XA = \_\_\_$ mm, $XD = \_\_\_$ mm,

$PC = \_\_\_$ mm, $QS = \_\_\_$ mm




Scale 1 : 10 000

The actual lengths are then:

$PX = XQ = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$PB = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$QR = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$XA = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$XD = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$PC = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

$QS = \_\_\_ \times 10\,000 = \_\_\_$ mm or ___ m

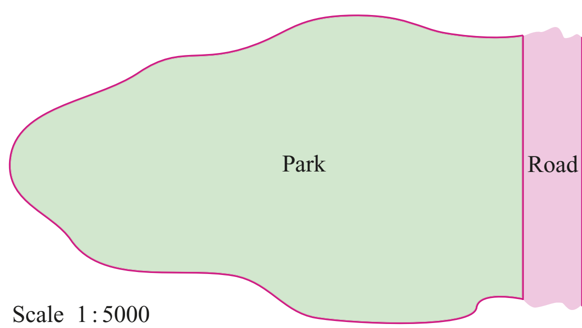
Using the Trapezoidal rule for the four strips:

Area $PBAX \approx \frac{\square}{2}(\_\_\_ + \_\_\_) = \_\_\_$   Area $PCDX \approx \frac{\square}{2}(\_\_\_ + \_\_\_) = \_\_\_$

Area $AXQR \approx \frac{\square}{2}(\_\_\_ + \_\_\_) = \_\_\_$   Area $DXQS \approx \frac{\square}{2}(\_\_\_ + \_\_\_) = \_\_\_$

Total area $\approx \_\_\_$

**10** Repeat the method used in question **9** to estimate the area of the park shown in the scaled diagram.
*Hint:* break the park into four sections.




Scale 1 : 5000

**11** The park shown in question **10** is to have fertiliser spread over it at a rate of 0.2 kg/m$^2$.

**a** How much fertiliser would be needed?

**b** If the fertiliser is sold in 70 kg bags, how many bags are needed to cover the park?

**Estimating volume**

The volume of a lake, dam or reservoir can be estimated if its surface area and average depth are known.

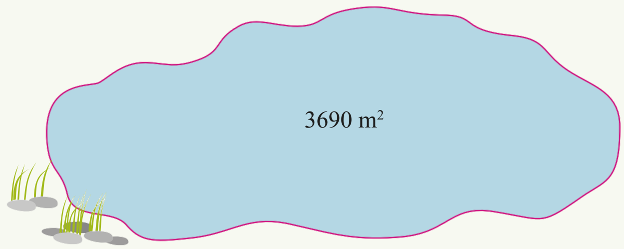
Volume ≈ surface area × average depth or $V \approx A \times h$

*Note:* **1** for $V$ in cubic metres, $A$ must be in square metres and $h$ must be in metres.

**2** 1 m$^3$ = 1000 L and 1 ML = 1 000 000 L

## EXAMPLE 3H–5 Estimating the volume of a reservoir

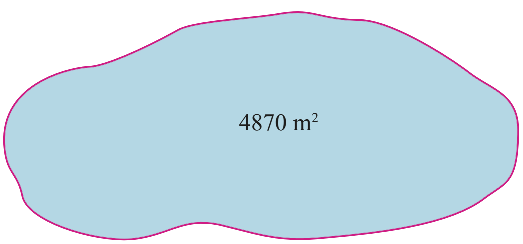
Estimate the volume of the reservoir, given that its average depth is 4.2 m.




| Solve | Think/Apply |
|---|---|
| $V \approx 3690\ \text{m}^2 \times 4.2\ \text{m}$<br>$= 15\,498\ \text{m}^3$<br>$= 15\,498\,000\ \text{L}$<br>$= 15.498\ \text{ML}$<br>Approximate volume is 15.5 ML. | Substitute the given values for surface area and average depth into $V \approx A \times h$. |

**12** Estimate the volume of each of these lakes.

**a**




**b**


## EXAMPLE 3H–6 Estimating the volume increase after rainfall

The surface area of a lake is 460 000 m$^2$. If an average of 6 mm of rain falls on the lake, what will be the increase in the volume of water in the lake? Ignore any runoff from the surrounding area.

| Solve | Think | Apply |
|---|---|---|
| $V = 460\,000 \times 0.006$<br>$= 2760\ \text{m}^3$<br>$= 2\,760\,000$ L or 2.76 ML | 6 mm = 0.006 m<br>1 m$^3$ holds 1000 L.<br>1 ML = 1 000 000 L | Use $V = A \times h$. Check that consistent units are used. |

**13** The surface area of a lake is 785 000 m$^2$ and an average of 4 mm of rain falls on the lake. Find the increase in the volume of water in the lake. Ignore any runoff from the surrounding area.

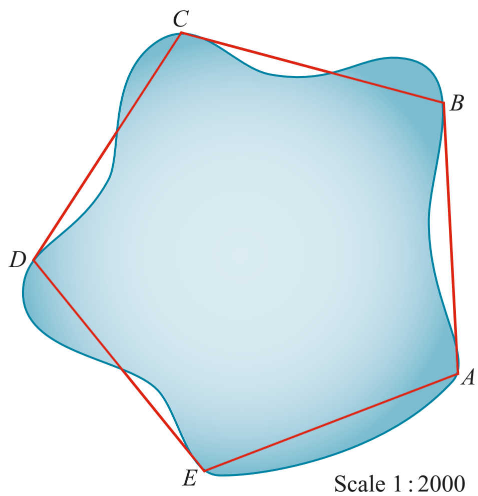
**14** Estimate the volume of water that is added to each of the lakes shown in question **12**, if 4 mm of rain falls over each lake. Ignore any runoff from the surrounding area.

**15** Lake Burragorang is formed by Warragamba dam, which supplies most of Sydney's population with water.

**a** The area of Lake Burragorang is 75 km$^2$. If an average of 3 mm of rain falls over the lake, what will be the increase in the volume of water in the lake?

**b** The total catchment area for Lake Burragorang is 9051 km$^2$. If an average of 1 mm of rain falls over the catchment and 85% of this runs into the lake, what will be the increase in the volume of water in the dam?

**16 a** Estimate the area of the lake shown below using a radial survey.

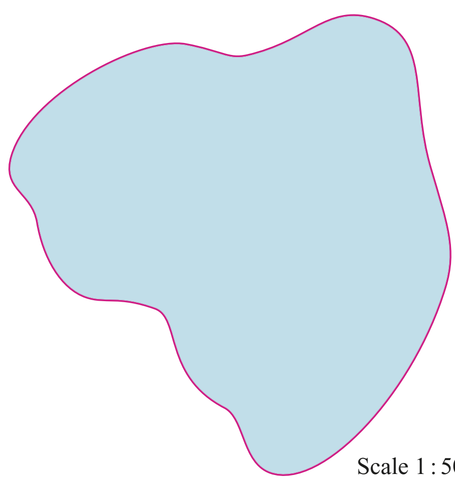



Scale 1 : 2000

**b** Estimate the volume of the lake if the average depth is 2.4 m.

**17 a** Estimate the volume of this reservoir if its average depth is 4.6 m.

**b** If 6 mm of rain falls over the reservoir, estimate the increase in volume of the reservoir. Ignore any runoff from the surrounding area.



Scale 1 : 5000

**18** Download a satellite map or aerial photograph from the internet and identify a geographical feature that has an irregular boundary, for example, a lake or field. Use the scale of the map to calculate the actual dimensions of the feature and hence estimate its area.

# CHAPTER 3 REVIEW RATES AND RATIOS

You should be able to:

- ✔ calculate rates
- ✔ convert between units for rates
- ✔ solve problems involving rates
- ✔ solve problems related to the fuel consumption rate of a motor vehicle
- ✔ compare common appliances in terms of their energy consumption
- ✔ calculate the cost of operating household appliances
- ✔ convert between different units of power
- ✔ find and simplify the ratio of two quantities
- ✔ divide a quantity in a given ratio
- ✔ interpret scale drawings, building plans and maps
- ✔ interpret common symbols and abbreviations on house plans
- ✔ use the scale to calculate actual dimensions on a plan or scale drawing
- ✔ interpret plans and elevations to obtain the internal dimensions of rooms
- ✔ identify the issues and make calculations for the issues addressed in the BASIX
- ✔ calculate perimeter, area and volume based on information on a map, plan or scale drawing.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

3A **1** A garden hose can fill a 5 L bucket in 10 s. What is the rate of flow in litres per hour?

**A** 180 **B** 1800 **C** 30 **D** 200

3A **2** What speed is equivalent to 70 km/h?

**A** 0.02 km/h **B** 1.2 m/s **C** 19.4 m/s **D** 1167 m/s

3A **3** If the current exchange rate is 78 US cents for each A$, how many Australian dollars would you need to exchange to get US$1000?

**A** 12.82 **B** 780 **C** 1000 **D** 1282.05

3B **4** A person's maximum heart rate can be calculated using the formula 220 – age in years. What is the maximum heart rate for a 19-year-old male?

**A** 239 **B** 220 **C** 201 **D** 19

3B **5** During exercise, a person's target heart rate is between 50% and 85% of their maximum heart rate. What is the target heart rate range for a 40-year-old woman with a maximum heart rate of 180?

**A** 90–153 **B** 140–220 **C** 70–119 **D** 0–180

3B **6** The average adult heart pumps 70 mL of blood with each beat. What is the volume of blood pumped in 6 minutes at 115 bpm?

**A** 9.85 mL **B** 630 mL **C** 1341 mL **D** 48 300 mL

3B **7** A person's recovery rate number is calculated by subtracting the heart rate one minute after completing exercise from the heart rate at the end of exercising and dividing the result by 10. What is the recovery number for a person whose heart rate at the end of exercising is 153 bpm and 108 bpm 1 minute later?

**A** 45 **B** 4.5 **C** 261 **D** 2.61

3C **8** What is the fuel consumption rate of a car that travels 500 km on 70 L of petrol?

**A** 0.14 L/100 km **B** 7.14 L/100 km **C** 14 L/100 km **D** 70 L/100 km

3C **9** How far can a motor vehicle travel on 40 L of petrol if its fuel consumption is 8.4 L/100 km?

**A** 4.76 km **B** 21 km **C** 336 km **D** 476 km

3D **10** Given that the cost of electricity is 52.68 cents/kWh, what is the cost of running a 2200 W electric heater for 5 hours a day for 70 days?

**A** $81.13 **B** $579.49 **C** $162.26 **D** $405.64

3D **11** A wind farm generates 2400 megawatts of power. What is this equivalent to?

**A** 2.4 gigawatts **B** 2.4 kilowatts **C** 240 gigawatts **D** 240 kilowatts

3E **12** The ratio of boys to girls in a school is 5 : 7. If there are 364 girls, how many boys are there?

**A** 55 **B** 77 **C** 260 **D** 275

3E **13** When $56 000 is divided in the ratio 5 : 3, how much is the larger amount?

**A** $33 600 **B** $21 000 **C** $35 000 **D** $28 000

3F **14** Using a scale of 1 : 10 000, what length on a scale drawing would represent an actual length of 120 m?

**A** 12 mm **B** 12 cm **C** 1.2 mm **D** 0.12 mm

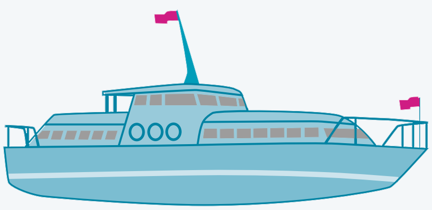
3F **15** What is the scale 1 cm to 5 m equivalent to?

**A** 1 : 5 **B** 1 : 50 **C** 1 : 500 **D** 1 : 5000

3F **16** The actual length of the boat shown in the scale drawing is 11.2 m. What scale has been used?

**A** 1 : 112 **B** 1 : 200

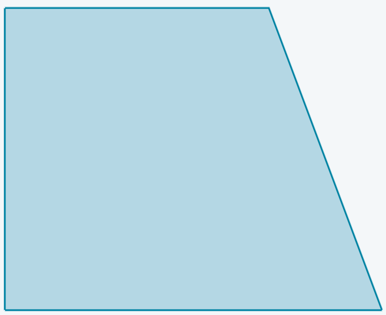
**C** 1 : 500 **D** 1 : 2000



3G **17** Using measurement and the scale given, what is the approximate perimeter of this site plan?

**A** 134.4 m **B** 168 m

**C** 1344 m **D** 1360 m

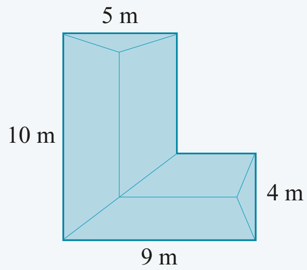


Scale 1 : 800

MEASUREMENT

3G **18** How much water could be collected from this roof when 7 mm of rain falls on it, allowing 10% for wastage?

**A** 0.4158 L
**B** 415.8 L
**C** 0.462 L
**D** 462 L

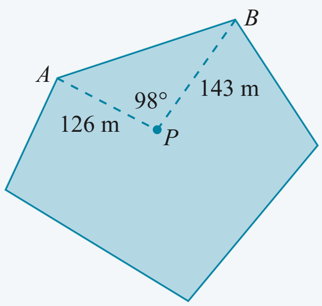



3H **19** A diagram has a scale of 1 : 12 000 and contains a 5 mm by 5 mm grid. What is the real area of each grid square?

**A** 25 $m^2$ **B** 36 $m^2$ **C** 60 $m^2$ **D** 3600 $m^2$

3H **20** What is the area of triangle *PAB* in the diagram?

**A** $\frac{1}{2} \times 126 \times 143$
**B** $\frac{1}{2} \times 126 \times 143 \times \cos 98°$
**C** $\frac{1}{2} \times 126 \times 143 \times \sin 98°$
**D** $\frac{1}{2} \times 126 \times 143 \times \tan 98°$




# REVIEW SET 1

**1** Convert these rates.
**a** 12 t/ha to kg/$m^2$ **b** 9 m/s to km/h

**2** The cost of a pack of 8 pens is \$6.32 and the cost of a box of 5 pens is \$4.10. Which option is the best buy?

**3** A person's maximum heart rate can be calculated using the formula 220 – age in years. Calculate the maximum heart rate for a 25-year-old female.

**4** The average adult heart pumps about 70 mL of blood with each beat. Calculate the volume of blood pumped by a human heart beating for 12 minutes at 118 bpm.

**5** How far can a vehicle travel on 38 L of fuel if its fuel consumption rate is 9.6 L/100 km?

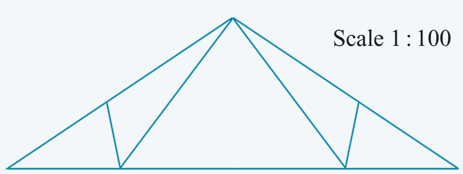
**6** Calculate the cost of running a 350 watt television set for 8 hours when the cost of electricity is 32.77 cents/kWh.

**7** Convert 3.2 MW into kilowatts.

**8** The ratio of boys to girls in a school is 8 : 9. If there are 256 boys, how many girls are there?

**9** Divide \$48 000 into two parts in the ratio 3 : 5.

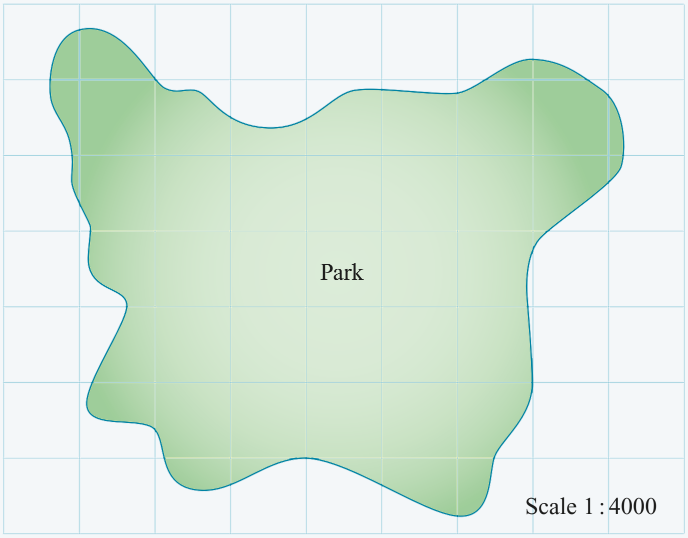
**10** By measurement and calculation, find the total length of timber needed to construct the roof truss shown.



**11** Use the grid-square method, with 1 cm by 1 cm squares, to estimate the area of the park.

**12** Estimate the area of the park shown in Question **11** using a radial survey.

**13** A small farm dam has a surface area of approximately 290 $m^2$. If the average depth of the dam is 2.2 m, calculate the volume of water in the dam.




# REVIEW SET 2

**1** Sue buys 2.8 m of dress material for \$99.68. What is the cost per metre of the material?

**2** Water flows into a tank at the rate of 8 L/min.

**a** How much water will flow into the tank in 3 h and 25 min?

**b** If the tank has a capacity of 4.24 kL, how long will it take to fill the tank?

**3** During exercise, a person's target heart rate is between 50% and 85% of their maximum heart rate. Their maximum heart rate can be calculated using the formula 220 – age in years. Calculate the target heart range for a 32-year-old female.

**4** The average adult heart pumps about 70 mL of blood each beat. Calculate the volume of blood pumped by the human heart beating for 3 minutes at 127 bpm.

**5** A salesperson averages 5400 km of city driving each month in a Fiat 500 that uses 6.3 L/100 km. Calculate the monthly cost of petrol used by the salesperson when the average price of ULP is 139.9c/L.

**6** Calculate the cost of running a 175 watt ceiling fan for 4 hours a day for 60 days when the cost of electricity is 24.6 cents/kWh.

**7** Convert 6.1 GW into:

**a** kilowatts **b** megawatts.

**8** Write each of these comparisons as ratios in simplest form.

**a** 36 to 45 **b** $1\frac{1}{4}$ to 2 **c** 1.5 to 2.4

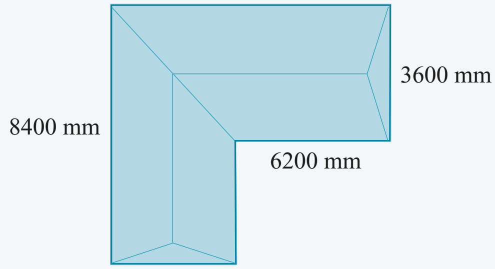
**9** On one day at the school canteen, the ratio of bread rolls to sandwiches sold was 3 : 2 and the ratio of sandwiches to pies sold was 1 : 4. If the canteen sold 72 pies on this day, how many bread rolls were sold?

**10** Using a scale of 1 : 10 000, what is the real length represented by 3.6 cm on a scale drawing?

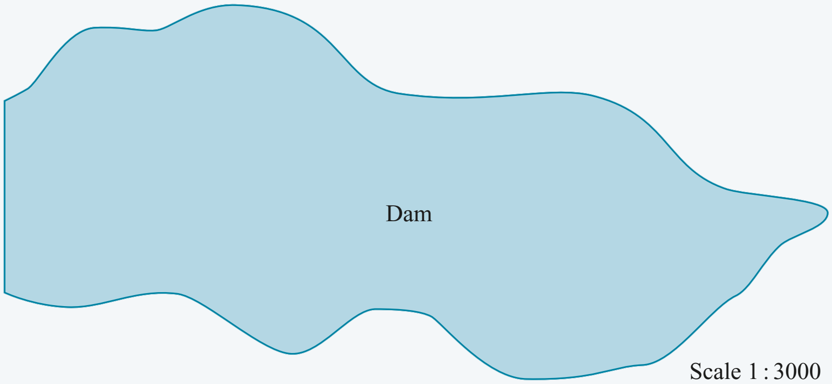
MEASUREMENT

**11** **a** Calculate the plan-view area of the roof of this house.

**b** How much water could be collected from this roof if 3 mm of rain were to fall on it, allowing 15% for wastage?




**12** **a** Use the polygon method (radial survey) to estimate the surface area of the dam shown.




**b** Estimate the volume of water in the dam if its average depth is 1.2 m.

**13** Use the Trapezoidal rule to estimate the surface area of the dam shown in question **12**. (*Hint:* break up the diagram into four sections.) Compare this estimate to the one you found in part **a** of question **12**.

# REVIEW SET 3

**1** Convert each of the following rates, giving the answer to two decimal places where necessary.

**a** 60 km/h to m/s

**b** 50 g/m$^2$ to kg/ha

**2** A person's maximum heart rate can be calculated using the formula 220 – age in years. Calculate the maximum heart rate for a 41-year-old male.

**3** The average adult heart pumps about 70 mL of blood each beat. Calculate the volume of blood pumped by a human heart beating for 5 minutes at 151 bpm.

**4** Calculate the amount of fuel used by a car on a trip of 425 km, if the fuel consumption is 10.6 L/100 km.

**5** **a** A computer monitor has a stand-by power usage of 4 watts. Calculate the annual energy consumption in kilowatt-hours if it is left on stand-by for 20 hours a day for a year.

**b** Hence, determine the cost of stand-by energy used when the cost of electricity is 26.9 c/kWh.

**6** A wind farm has an electricity generation capacity of $1.28 \times 10^8$ watts. Convert this to the following units.

**a** gigawatts

**b** megawatts

**c** kilowatts

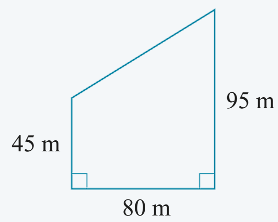
**d** milliwatts

**7** An amount of \$15 000 is divided between two people in the ratio 5 : 3. What is the value of the smaller share?

**8** The weight of meat decreases when it is cooked. If the ratio of the weight of raw meat to cooked meat is 5 : 4, calculate the weight of a 400 g piece of meat after it has been cooked.

**9** **a** Make a scale drawing of the field shown, using a scale of 1 : 1000.

**b** What is the perimeter of the field?




**10** Write the scale 2 cm to 5 m in the form 1 : $n$.

**11** For the site plan shown below, answer the following questions. The plan is drawn to scale.

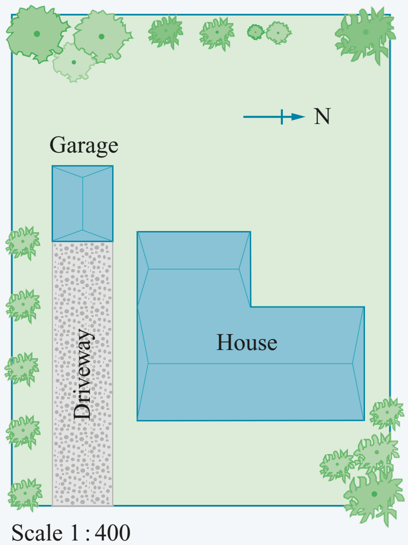
**a** Find the dimensions of the garage.

**b** Find the area of the driveway.

**c** Find the actual dimensions of the house.

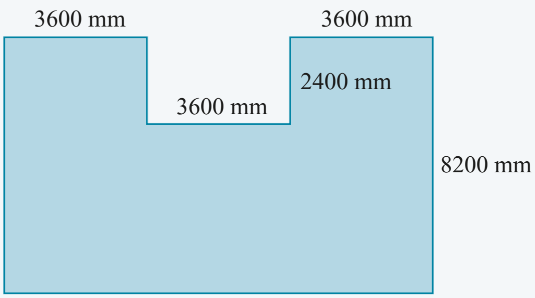
**d** Find the area of the block of land.

**e** If I walk from the front of the driveway to the garage, in which direction do I walk?

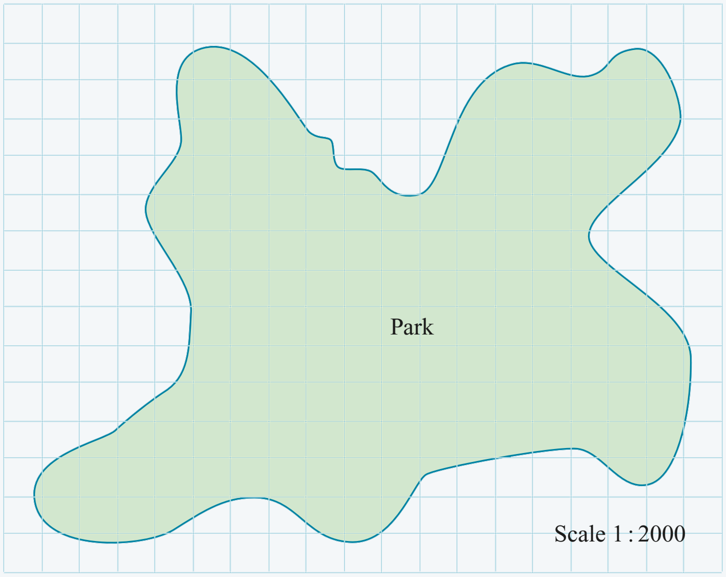



**12** **a** Calculate the plan-view area of the roof of this house.

**b** How much water could be collected from this roof if 6 mm of rain were to fall on it, allowing 15% for wastage?




**13** Use the grid-square method, with 5 mm by 5 mm squares, to estimate the area of land shown.




**14** Estimate the area of the land shown in question **13** using the polygon method (radial survey).

# REVIEW SET 4

**1** The average rate of flow of a bathroom shower with a normal shower rose is 18 L/minute.

**a** Nora has two 10 minute showers each day. Calculate the annual cost of Nora's showers, given the water usage charge is \$2.17/kL. (Use 1 year = 365 days.)

**b** How much could Nora save each year by using a water-efficient shower rose that only uses 8.5 L/minute?

**2** During exercise, a person's target heart rate is between 50% and 85% of their maximum heart rate. Their maximum heart rate can be calculated using the formula 220 – age in years. Calculate the target heart range for a 23-year-old male.

**3** The average adult heart pumps about 70 mL of blood each beat. Calculate the volume of blood pumped by the human heart beating for 7 minutes at 162 bpm.

**4** A car uses 32 L of petrol to travel 250 km.

**a** Calculate the petrol consumption in L/100 km.

**b** At this rate of consumption, how much petrol (to the nearest litre) would be used to travel 650 km?

**c** How far could the car travel on 56 L?

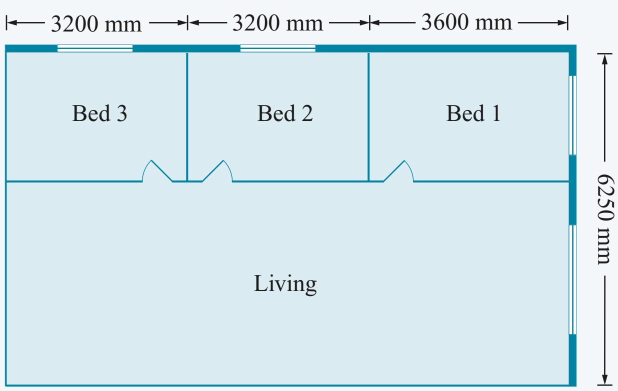
**5** Calculate the annual cost of running the average Sydney home, which uses 21 kWh of energy each day, given that the cost of electricity is 27.77 cents/kWh.

**6** **a** The rooms in the house shown on this plan are to be air conditioned. Calculate the total area of these rooms.

**b** Calculate the volume of air to be conditioned given that the ceilings are 2400 mm high.

**c** Use the table to find the power output of the air conditioning unit required.

| Volume ($m^3$) | kW required |
|---|---|
| 140 | 8.4 |
| 150 | 9.0 |
| 160 | 9.6 |
| 170 | 10.2 |




**7** The profits on an investment are divided between Tony and Jen in the ratio 9 : 11. If the profits are \$12 000, calculate how much each person receives.

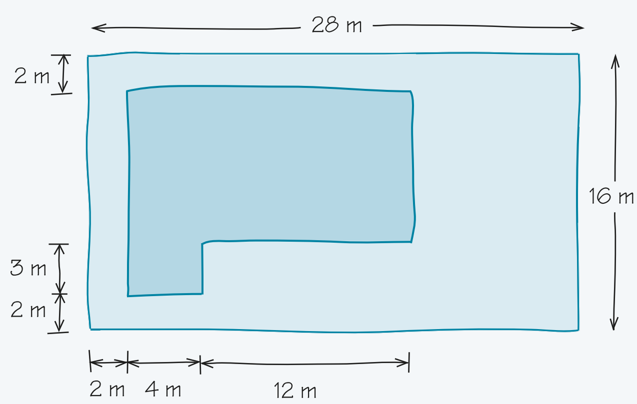
**8** A rough sketch of the site plan of a house is shown.

**a** Draw a scale diagram, using a scale of 1 cm to 2 m.

**b** What is the area of the block of land?

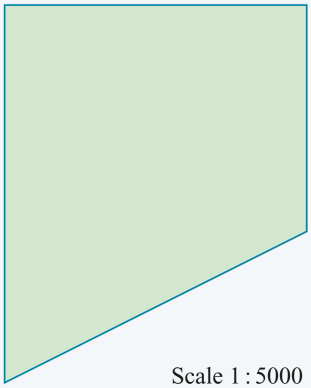
**c** What area of land is taken up by the house?

**d** What percentage of the block of land is occupied by the house?



**9** **a** Find the actual perimeter and area of the section of farmland shown in the scale diagram.

**b** If 2 mm of rain falls on this land and 85% of it flows into a dam, by how much will the capacity of the dam be increased?



Scale 1 : 5000

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** A large park of area 1.2 ha is to be fertilised at the rate of 14 g/m$^2$.

**i** Calculate, in kilograms, the amount of fertiliser needed. (2 marks)

**ii** If the fertiliser is only available in 25 kg bags, how many bags are needed? (1 mark)

**b** Bob is a 37-year-old male.

**i** A person's maximum heart rate can be calculated using the formula 220 – age in years. Calculate Bob's maximum heart rate. (1 mark)

**ii** During exercise, a person's target heart rate is between 50% and 85% of their maximum heart rate. Calculate Bob's target heart range. (2 marks)

**iii** Bob trains for 18 minutes with his heart rate at 127 bpm. Calculate the total number of heart beats in that time. (1 mark)

**c** How far can a car travel on 54 L of fuel if the fuel consumption is 7.8 L/100 km? (1 mark)

**d** **i** Calculate the amount of energy needed to run a 2200 watt heater for 4 hours a day for 90 days. (1 mark)

**ii** How much could be saved in a year by putting the heater on for 3 hours a day instead, given the price of electricity is 33.6 cents/kWh? (1 mark)

**e** The profits of a company are divided between the three directors in the ratio 2 : 2 : 3. If the company makes a profit of $630 000, calculate the largest share. (1 mark)

**f** The diagram shows an aerial plan of a house roof.

Scale 1 : 200

**i** Use the scale shown to calculate the plan-view area of the roof. (2 marks)

**ii** During a storm, 4 mm of rain falls onto the roof and is collected in a water tank. How many litres of water are added to the tank during the storm? (2 marks)

**TOTAL: 15 marks**

MEASUREMENT

# Chapters 1–3 CUMULATIVE REVIEW

**1** Aaron is going on a holiday. He borrows \$5200 over 4 years at a flat interest rate of 10.4% p.a.

**a** Find the simple interest paid.

**b** Find the total to be repaid.

**2** Find the interest payable on a loan of \$7600 at a simple interest rate of 11.9% for these periods of time.

**a** $3\frac{1}{2}$ years

**b** 21 months

**3** Ginger borrows \$3800 to go on a cruise. She repays \$4730 over a period of 2 years. Calculate the simple interest rate.

**4** Leo borrows \$15 000 to buy a car. The simple interest rate is 11.3% and he takes the loan over 4 years.

**a** Find the interest on the loan.

**b** Find the total to be repaid.

**c** What is Leo's monthly payment?

**5** **a** Use the compound interest formula to calculate the future value of a fixed-term investment of \$7580 over 3 years at 4.54% p.a. interest compounding yearly.

**b** Find the total interest earned.

**6** **a** Use the compound interest formula to calculate the future value of a fixed-term investment of \$2300 over 5 years at 3.4% p.a. interest compounding quarterly.

**b** Calculate the amount that must be invested at 3.2% p.a. interest compounding annually to have \$10 000 at the end of 9 years.

**c** Calculate the amount that must be invested at 5.5% p.a. interest compounding quarterly to have \$1040 at the end of 6 years.

**d** Calculate the amount to which \$2000 will grow if invested for 3 years at 6.6% p.a. interest compounding monthly.

**e** Calculate the amount that must be invested now at 3.8% p.a. interest compounding monthly in order to have \$10 000 in 5 years' time.

**7** In three successive years the annual rate of inflation was 2.8%, 2.5% and 2.1% respectively. How much would you expect to pay at the end of these 3 years for a book that cost \$23.99 at the beginning of this period?

**8** A coin collection is purchased for \$2800. It appreciates in value by an average of 3.5% p.a. Find the appreciated value of the coin collection after:

**a** 10 years **b** 20 years.

**9** Calculate the total cost to purchase 1500 shares with a market price of \$3.35. Brokerage is 2.5%.

**10** A dividend of 4.65 cents per share is paid on shares with a market value of \$1.14. Find the dividend yield.

**11** Calculate the amount of dividend on 5000 shares with a face value of 50 cents if the dividend yield is 4.5%.

**12** A car is purchased for \$29 800. It depreciates in value at a rate of 28% per year.

**a** Calculate the salvage value of the car after 3 years.

**b** By what amount has the car depreciated in value after 3 years.

**13** A car depreciates in value from \$42 990 to \$25 820 in 2 years. Use the declining-balance formula to calculate the annual percentage rate of depreciation.

**14** Use the table in Example 1G–1 on page 36 to find the monthly repayment on a reducing-balance loan of \$160 000 over:

**a** 10 years **b** 15 years **c** 25 years **d** 30 years.

**15** Use the table in Example 1G–3 on page 39 to find the monthly repayment on a loan of \$85 000 over:

**a** 15 years at 7.25% **b** 10 years at 8.5%.

**16** The graph in Example 1G–4 on page 40 shows the amount outstanding against repayment period for a home loan of \$240 000 at 8.4% p.a. with a repayment of \$2160 per month. Use the graph to find:

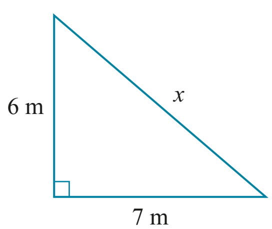
**a** how much is owing after 100 months **b** when the amount owing is \$90 000

**17** The minimum payment on a credit card is the greater of \$10 or 3% of the closing balance. Calculate the minimum payment on a closing balance of \$380.

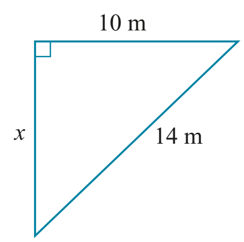
**18** Calculate the interest to be paid on a cash advance of \$2400 if the total amount is repaid after 19 days. The interest rate is 0.043 69% per day.

**19** Find the value of $x$ in the following triangles, correct to one decimal place.

**a**




**b**

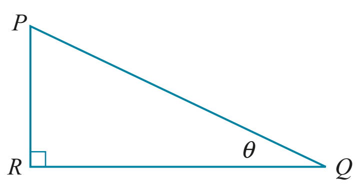



**20** **a** For the angle marked $\theta$ in this triangle, name the:

**i** opposite side

**ii** adjacent side

**iii** hypotenuse.




**b** Write these value as a ratio using the side lengths $PQ$, $PR$ and $RQ$.

**i** $\sin \theta$

**ii** $\cos \theta$

**iii** $\tan \theta$

**21** **a** Find these values, correct to four decimal places.

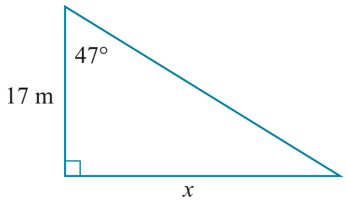
**i** $\sin 73.2°$ **ii** $\cos 71° \ 15'$ **iii** $\tan 18°46'$

**b** Find $\theta$ to the nearest minute, if:

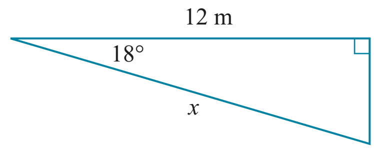
**i** $\sin \theta = 0.271$ **ii** $\cos \theta = 0.389$ **iii** $\tan \theta = 1.41$.

**22** Find the length of the unknown side in these triangles, correct to one decimal place.

**a**

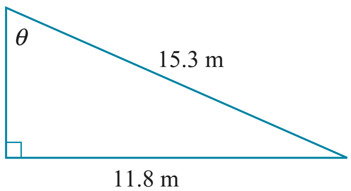



**b**

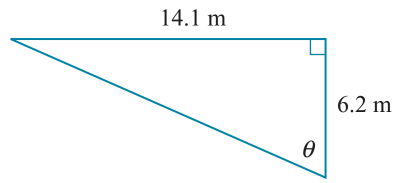



**23** Find the value of $\theta$ to the nearest minute.

**a**



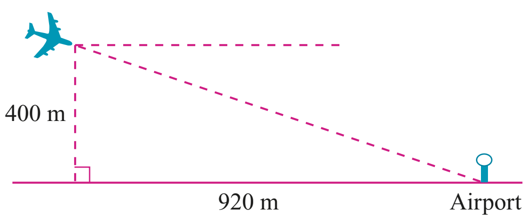

**b**




**24** The pilot of a plane flying at an altitude of 400 m observes an airport runway that is 920 m horizontally from the plane.

**a** What is the angle of depression from the plane to the runway, to the nearest minute?

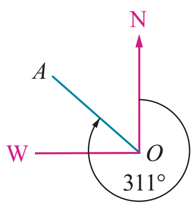
**b** What is the actual distance, to the nearest metre, that the plane has to fly to reach the runway?



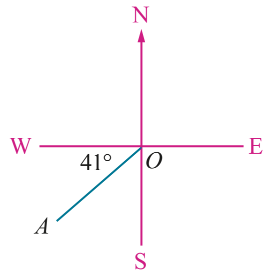

**25** Write the bearing of $A$ in the following diagrams as a:

**i** compass bearing **ii** true bearing.

**a**



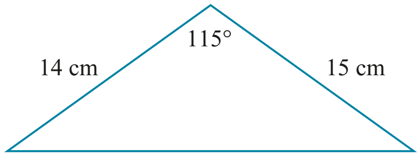

**b**



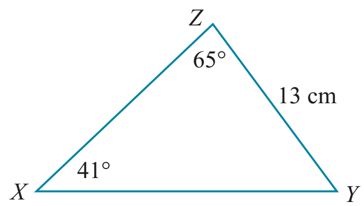

**26** **a** A ship sails 143 km from port $O$ on a bearing of 194°T. How far south has the ship sailed?

**b** Town $P$ is 85 km west and 61 km north of town $Q$. Find the bearing of $P$ from $Q$ to the nearest minute.

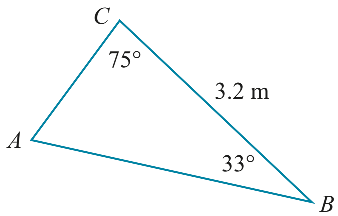
**27** **a** Find the area of this triangle to the nearest cm$^2$.



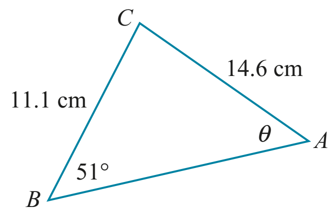

**b** Find $XY$ correct to one decimal place.



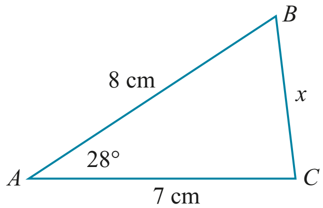

**c** Find $AB$ correct to one decimal place.



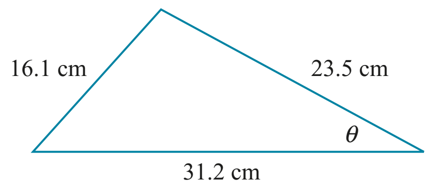

**d** Find $\theta$ correct to the nearest minute.



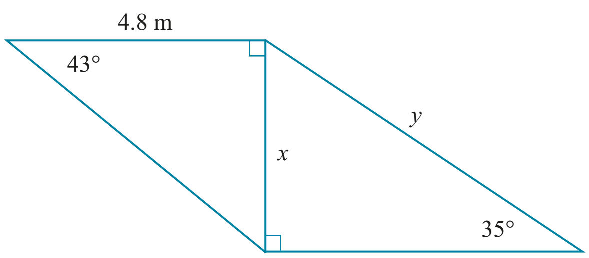

**e** Find $x$ correct to one decimal place.




**f** Find $\theta$ correct to the nearest minute.




**28** Find the values of $x$ and $y$ in this diagram, correct to one decimal place.



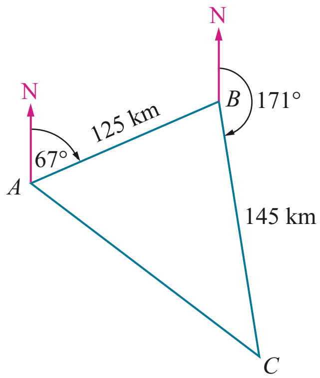
**29** A ship sails 125 km from $A$ to $B$ on a bearing of 67°T. It then turns and sails 145 km to $C$ on a bearing of 171°T.

**a** Find $\angle ABC$.

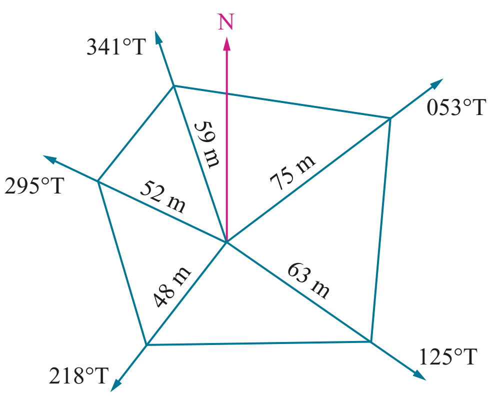
**b** How far is the ship from its starting point to the nearest kilometre?

**c** Calculate $\angle BAC$ to the nearest degree.

**d** What is the bearing of the ship from its starting point?




**30** This diagram shows the result of a compass radial survey of a field. Calculate the perimeter and area of the field.




**31** Convert the following rates.

**a** 24 t/ha to kg/m$^2$

**b** 15 m/s to km/h

**32** The cost of a pack of 5 candles is \$2.95 and the cost of a box of 24 candles is \$15.20. Which option is the better buy?

**33** Calculate the maximum heart rate for a 21-year-old male.

**34** The average adult heart pumps about 70 mL of blood with each beat. Calculate the volume of blood pumped by a human heart beating for 15 minutes at 88 bpm.

**35** How far can a vehicle travel on 42 L of fuel if its fuel consumption rate is 10.1 L/100 km?

**36** Calculate the cost of running a 300 watt television set for 7 hours when the cost of electricity is 35.44 cents/kWh.

**37** Convert 5.7 MW to:

**a** kilowatts **b** milliwatts.

**38** The ratio of boys to girls in a school is 7 : 8. If there are 483 boys, how many girls are there?

**39** Divide $65 000 into two parts in the ratio 3 : 5.

**40** **a** Using the scale 1 : 100 000, what real length would be represented by a length of 2.3 cm on a scale drawing?

**b** Write the scale 4 cm to 5 m in the form 1 : *n*.

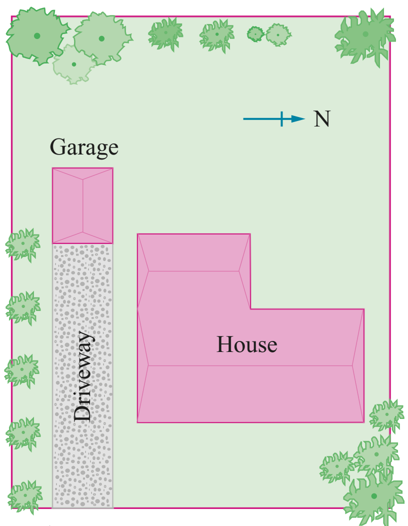
**41** For the site plan shown, answer the following questions. The plan is drawn to scale.

**a** Calculate the length of fencing around the boundary of the block of land. (*Hint:* the entrance to the driveway is not fenced.)

**b** Calculate the plan-view area of the roof of the house.

**c** Calculate the plan-view area of the roof of the garage.

**d** How much water could be collected from both roofs when 5 mm of rain falls on it, allowing 10% for wastage?

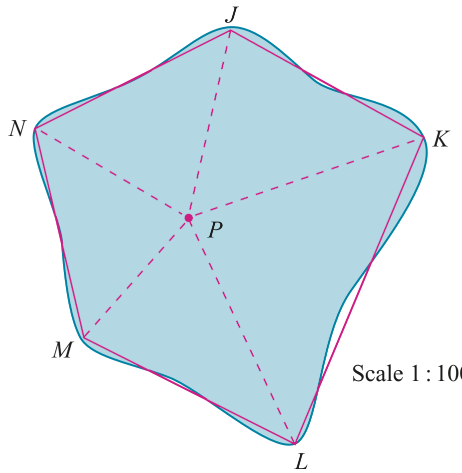



Scale 1 : 400

**42** The boundary of a dam was approximated by the polygon *JKLMN*, as shown in the diagram.

**a** Use a ruler and a protractor to help you find an estimate of the surface area of the dam from the scale drawing.

**b** If the average depth of the dam is 1.8 m, estimate the volume of water in the dam.




Scale 1 : 1000

# 4

# Simultaneous linear equations

The main mathematical ideas in this chapter are:

- using and interpreting graphs of the form $y = mx + c$
- modelling linear relationships
- working with linear models and their graphical representation
- identifying the solution to simultaneous linear equations from tables and graphs
- solving simultaneous linear equations graphically
- solving practical problems using simultaneous linear equations
- conducting a break-even analysis.

ALGEBRA
MS-A4 Types of Relationships A4.1

# ARE YOU READY?



4A **1** What is the value of $y$ in the equation $y = 2x + 5$ when $x = -1$?

**A** $-7$ **B** $-3$
**C** $3$ **D** $7$

4A **2** Which table shows $x$- and $y$-values for $y = 3x - 4$?

**A**

| $x$ | 2 | 3 | 4 |
|---|---|---|---|
| $y$ | 2 | 6 | 9 |

**B**

| $x$ | 1 | 3 | 5 |
|---|---|---|---|
| $y$ | −1 | 5 | 11 |

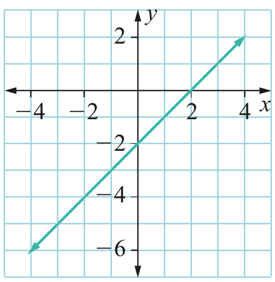
**C**

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 0 | 2 | 8 |

**D**

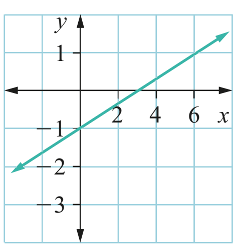
| $x$ | 3 | 4 | 5 |
|---|---|---|---|
| $y$ | 5 | 6 | 7 |

4A **3** Which equation produces the following graph?



**A** $y = x + 2$ **B** $y = 2x - 4$
**C** $y = 2 - x$ **D** $y = x - 2$

Use the following graph for questions **4**, **5** and **6**.



4A **4** What is the gradient of this graph?

**A** $-1$ **B** $\frac{1}{3}$
**C** $\frac{2}{3}$ **D** $3$

4A **5** What is the $y$-intercept of this graph?

**A** 2.5 **B** 3
**C** $-1$ **D** $-3$

4A **6** What is the equation of this graph?

**A** $y = \frac{1}{3}x - 1$ **B** $y = x + \frac{1}{3}$
**C** $y = \frac{1}{3}x + 1$ **D** $y = 3x - 1$

Use the following graph for questions **7**, **8** and **9**.


4B **7** What is the taxi hire charge for travelling 18 km?

**A** \$30 **B** \$40
**C** \$50 **D** \$60

4B **8** How far can you travel by taxi for \$70?

**A** 20 km **B** 22 km
**C** 24 km **D** 26 km

4B **9** Which equation best matches the graph of distance ($d$) and cost ($C$)?

**A** $C = 2.5d + 5$ **B** $C = 5d + 2.5$
**C** $d = 2.5C + 5$ **D** $C = 2.5d - 5$

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1–2** **Support sheet 4A.1 Substituting for pronumerals**
**Q3** **Support sheet 4A.2 Plotting points from a table of values**
**Q4–6** **Support sheet 4A.3 Identifying features of a linear graph**
**Q7–9** **Support sheet 4B.1 Reading values from graphs**

# 4A Graphs of the form $y = mx + c$

These resources are available on your obook assess:

- **Interactive 4A.1:** Explore key ideas for plotting linear graphs
- **Interactive 4A.2:** Explore gradient and intercepts of a linear graph
- **Interactive 4A.3:** Explore the drawing of graphs using gradient and $y$-intercept
- **Spreadsheet 4A.1:** Drawing linear graphs using a table of values
- **Spreadsheet 4A.2:** Drawing linear graphs using the gradient and $y$-intercept
- **assess quiz 4A:** Test your skills with an auto-correcting multiple-choice quiz

**linear relationship** relationship between two variables where the coordinate points describing this relationship lie in a straight line when plotted on a number plane (or Cartesian plane)

In the Year 11 Mathematics Standard course, you used straight-line graphs to model practical situations and make predictions based on the model. Graphical representations are excellent models of practical situations because they provide a visual way to view trends.

This section will revise the generation of tables of values and the graphing of a **linear relationship**. You can also explore the use of technology to produce straight-line graphs.

## EXAMPLE 4A–1 Completing a table of values to graph a linear relationship

**a** Complete this table of values for the equation $y = 5 - 2x$.

**b** Plot each set of ordered pairs from the table for $y = 5 - 2x$ on a number plane and draw a straight line through the points.

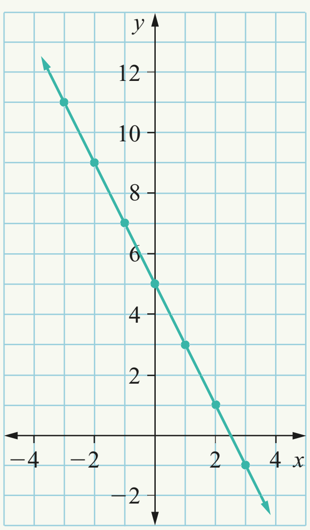
| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | (table below) | Substitute each $x$-value into the equation $y = 5 - 2x$ to complete the table. | Substitute each $x$-value into the linear relationship to find the matching $y$-value. Plot the points on a number plane (also called a Cartesian plane) and join them with a straight line. The line extends beyond the plotted points. The graph of a linear relationship is always a straight line. |
| **b** | (graph below) | Plot the points on a number plane and draw a straight line through them, extending as far as possible in both directions. Include an arrowhead at each end of the line to indicate that the line continues. | |

**a**

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | 11 | 9 | 7 | 5 | 3 | 1 | $-1$ |

**b**



# EXERCISE 4A Graphs of the form $y = mx + c$

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Complete the following tables for the equations provided. Plot each set of ordered pairs on separate number planes and draw a straight line through the points.

**a** $y = 2x + 1$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**b** $y = 2x + 3$

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**c** $y = -3x - 4$

| $x$ | $-4$ | $-2$ | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**d** $y = 4 - 5x$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**e** $y = 2x - 7$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**f** $y = 2 - 2x$

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**2** By using a table of values such as the one shown below, draw graphs of the following linear relationships on separate number planes.

| $x$ | $-3$ | 0 | 3 |
|---|---|---|---|
| $y$ | | | |

**a** $y = x - 3$ **b** $y = x + 2$ **c** $y = 2x - 1$ **d** $y = 1 - 3x$

**e** $y = \frac{1}{2}x$ **f** $y = \frac{2x + 3}{2}$ **g** $y = \frac{2x}{3} - 6$ **h** $y = 4 - 2x$

**3** By using a table of values, draw the graph of $y = \frac{x}{10} + 5$. Explain your choice of $x$-values to use in the table and the scale to use on the axes of the number plane.

> **!** If the equation of a straight line is written in the form $y = mx + c$, the **gradient** of the line is $m$ and the **$y$-intercept** is $c$.
>
> *Note:* $x$ is the **independent variable**, as any value may be used, and $y$ is the **dependent variable**, as it depends on the value of $x$.

**gradient** steepness of a line (defined as $m$) that can be measured using any two points on the line or interval using $m = \frac{\text{rise}}{\text{run}}$

**$y$-intercept** point where a line crosses the $y$-axis of a number plane (or Cartesian plane)

**independent variable** variable whose outcomes are not due to those of another variable; usually represented on the horizontal axis of a graph; e.g. for the linear relationship $y = 2x + 1$, the independent variable is $x$

**dependent variable** variable whose value depends on that of another; usually represented on the vertical axis of a graph; e.g. for the relationship $y = 2x + 1$, the dependent variable is $y$ (as it depends on the value of $x$)

ALGEBRA

## EXAMPLE 4A–2 Writing the gradient and $y$-intercept of a line from its equation

Write the gradient and $y$-intercept of the straight lines with the following equations.

**a** $y = 2x + 7$ **b** $y = -3x + 2$ **c** $y = \frac{2x}{3} - 5$

**d** $y = -x + 3.6$ **e** $y - \frac{x}{5} = \frac{4}{5}$ **f** $4x + 2y = 1$

| | Solve | Think/Apply |
|---|---|---|
| **a** | Gradient $= 2$, $y$-intercept $= 7$ | When the equation is in the form $y = mx + c$, the gradient is $m$, which is the coefficient of $x$ (the number in front of $x$). The $y$-intercept is $c$ (the constant term). If the equation is not in the form $y = mx + c$, it can be rearranged into this form. For part **e**, add $\frac{x}{5}$ to both sides of the equation. For part **f**, subtract $4x$ from both sides of the equation and then divide both sides by 2. |
| **b** | Gradient $= -3$, $y$-intercept $= 2$ | |
| **c** | Gradient $= \frac{2}{3}$, $y$-intercept $= -5$ | |
| **d** | Gradient $= -1$, $y$-intercept $= 3.6$ | |
| **e** | $y - \frac{x}{5} = \frac{4}{5}$ becomes $y = \frac{x}{5} + \frac{4}{5}$<br>Gradient $= \frac{1}{5}$, $y$-intercept $= \frac{4}{5}$ | |
| **f** | $4x + 2y = 1$ becomes $y = -2x + \frac{1}{2}$<br>Gradient $= -2$, $y$-intercept $= \frac{1}{2}$ | |

**4** Write the gradient and $y$-intercept of the straight lines with the following equations.

**a** $y = 4x + 6$ **b** $y = -2x + 9$ **c** $y = \frac{3}{4}x - 8$

**d** $y = x - 2$ **e** $y = -x + 4$ **f** $y = -1.2x - 2.8$

**g** $y = 5.4x + 6.5$ **h** $y = -\frac{3x}{8} + \frac{9}{2}$ **i** $y = \frac{x}{2} + 7$

**j** $y = \frac{x}{6} + \frac{5}{3}$ **k** $y = -\frac{x}{3} - 2$ **l** $y - \frac{x}{2} = 2$

**m** $4x + y = 8$ **n** $3x + 2y = -4$ **o** $2x - y = 5$

## EXAMPLE 4A–3 Using the gradient and $y$-intercept to write the equation of a straight line

Write the equation of the straight line with:

**a** gradient 4 and $y$-intercept $-3$ **b** gradient $-6$ and $y$-intercept 1

**c** gradient $\frac{4}{5}$ and $y$-intercept $-2$ **d** gradient 1 and $y$-intercept 0.9

**e** gradient $\frac{1}{3}$ and $y$-intercept $2\frac{1}{4}$.

| | Solve | Think | Apply |
|---|---|---|---|
| | The equation of a straight line is $y = mx + c$, where $m$ is the gradient and $c$ is the $y$-intercept. | Substitute the values for $m$ and $c$ into $y = mx + c$. | Write the equation in the form $y = mx + c$, where $m$ is the gradient and $c$ is the $y$-intercept. |
| **a** | $y = 4x - 3$ | $m = 4$, $c = -3$ | |
| **b** | $y = -6x + 1$ | $m = -6$, $c = 1$ | |
| **c** | $y = \frac{4}{5}x - 2$ | $m = \frac{4}{5}$, $c = -2$ | |
| **d** | $y = x + 0.9$ | $m = 1$, $c = 0.9$ | |
| **e** | $y = \frac{1}{3}x + 2\frac{1}{4}$ | $m = \frac{1}{3}$, $c = 2\frac{1}{4}$ | |

**5** Write the equation of the straight line with:

**a** gradient 7 and $y$-intercept $-1$

**b** gradient $-2$ and $y$-intercept 5

**c** gradient $\frac{3}{7}$ and $y$-intercept $-2$

**d** gradient 1 and $y$-intercept 4

**e** gradient $-1$ and $y$-intercept 2.7

**f** gradient 0.2 and $y$-intercept 3

**g** gradient $-1.4$ and $y$-intercept $-0.3$

**h** gradient $\frac{5}{6}$ and $y$-intercept $\frac{7}{12}$

**i** gradient $-0.56$ and $y$-intercept $-1.24$

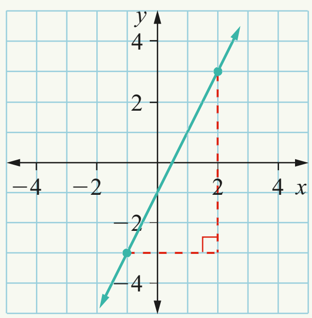
**j** gradient $1\frac{2}{5}$ and $y$-intercept 24.

## EXAMPLE 4A–4 Finding the gradient, $y$-intercept and equation from a linear graph

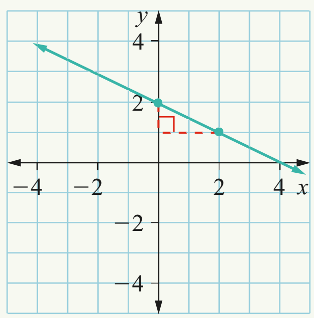
For the straight lines shown below, find the:

**i** gradient **ii** $y$-intercept **iii** equation of the line.

**a**



**b**

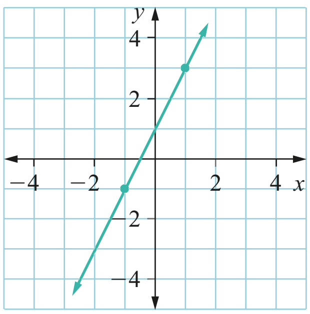


| | Solve | Think | Apply |
|---|---|---|---|
| **a i** | Gradient $= \frac{\text{rise}}{\text{run}}$ <br> $= \frac{6}{3}$ <br> $= 2$ | Choose any two points on the line, say $(-1, -3)$ and $(2, 3)$, and draw a right-angled triangle as shown, to find the gradient of the line. <br> The vertical rise is 6 and the horizontal run is 3. | The working is easier if you can find points whose coordinates are whole numbers. Often using the points where the graph cuts an axis makes calculations easier. |
| **ii** | $y$-intercept $= -1$ | From the graph, the line cuts the $y$-axis at $-1$. | |
| **iii** | The equation of the line is $y = 2x - 1$. | Use $y = mx + c$ and replace $m$ with 2 and $c$ with $-1$. | |
| **b i** | Gradient $= \frac{\text{rise}}{\text{run}}$ <br> $= -\frac{1}{2}$ | Choose any two convenient points on the line, say $(0, 2)$ and $(2, 1)$, and draw a right-angled triangle as shown. The vertical rise is $-1$ and the horizontal run is 2. | The line is sloping ‘downhill’ so the gradient is negative. |
| **ii** | $y$-intercept $= 2$ | From the graph, the line cuts the $y$-axis at 2. | |
| **iii** | The equation of the line is $y = -\frac{1}{2}x + 2$. | Use $y = mx + c$ and replace $m$ with $-\frac{1}{2}$ and $c$ with 2. | |

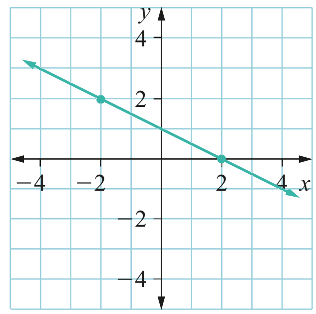
**6** For the straight lines shown, find the:

**i** gradient **ii** $y$-intercept **iii** equation of the line.

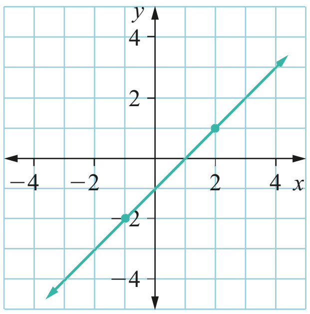
**a**



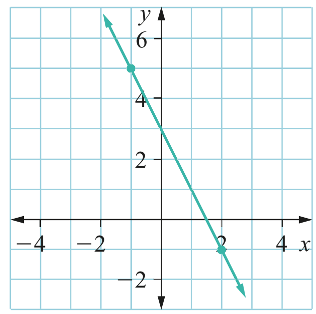
**b**



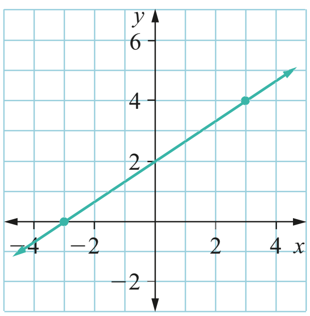
**c**



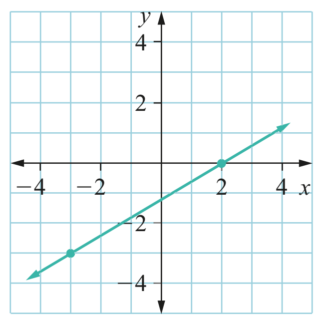
**d**



**e**



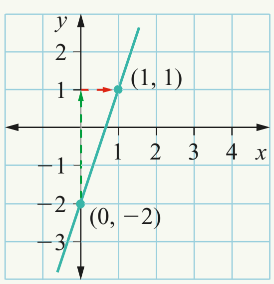
**f**



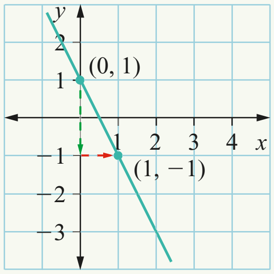
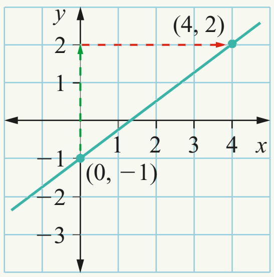
## EXAMPLE 4A–5 Drawing a linear graph using the gradient and $y$-intercept

Draw the following straight lines without using a table of values.

**a** $y = 3x - 2$ **b** $y = -2x + 1$ **c** $y = \frac{3x}{4} - 1$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** |  | Gradient $= 3$, $y$-intercept $= -2$<br>Start at the $y$-intercept $(0, -2)$.<br>As the gradient is $\frac{3}{1}$, move up 3 units then right 1 unit from (0, –2).<br>This gives a second point (1, 1) on the line.<br>Rule a line through these two points. | Find the gradient and $y$-intercept of the line.<br>Use $(0, c)$ as the first point on the line.<br>Use Gradient $= \frac{\text{rise}}{\text{run}}$ to find a second point on the line.<br>Plot the two points and rule a line through them. |

UNDERSTANDING, FLUENCY AND COMMUNICATING

| | Solve | Think | Apply |
|---|---|---|---|
| **b** |  | Gradient $= -2$, $y$-intercept $= 1$<br>Start at the $y$-intercept (0, 1).<br>As the gradient is $-\frac{2}{1}$, move down 2 units then right 1 unit from (0, 1).<br>This gives a second point $(1, -1)$ on the line.<br>Rule a line through these two points. | |
| **c** |  | Gradient $= \frac{3}{4}$, $y$-intercept $= -1$<br>Start at the $y$-intercept $(0, -1)$.<br>As the gradient is $\frac{3}{4}$, move up 3 units then right 4 units from $(0, -1)$.<br>This gives a second point (4, 2) on the line.<br>Rule a line through these two points. | |

**7** Draw each of the lines in question **4 a–f** without using a table of values.

TECHNOLOGY

**8** **a** Use graphing technology to draw the graphs of these equations on the same set of axes.

**i** $y = 2x$ **ii** $y = 2x + 3$ **iii** $y = 2x - 2$ **iv** $y = 2x + 1$

**b** What do you notice about the graphs?

**c** What is the effect of the constant term?

**9** **a** Use graphing technology to draw the graphs of these equations on the same set of axes.

**i** $y = x + 1$ **ii** $y = 2x + 1$ **iii** $y = 3x + 1$ **iv** $y = \frac{1}{2}x + 1$

**b** What do you notice about the graphs?

**c** What is the effect of changing the coefficient of $x$?

**10** Use graphing technology to draw the graphs with the given equations in questions **1**, **2**, **3** and **4** and find specific values. Check the accuracy of your answers from the hand-drawn graphs with those obtained using technology.

**11** Create a spreadsheet to produce the graph of a linear relationship from a table of values. Enter the values for $x$ in the first column and values for $y$ in the second column of the spreadsheet. Highlight the cells containing the $x$- and $y$-values, click **Insert** and **Charts**, and choose **Line graph**. Check some of your answers to questions **1** and **2** using your spreadsheet. You might like to use the prepared spreadsheet (Spreadsheet 4A.1) provided on your obook assess.

**12** Create a spreadsheet to produce the graph of a linear relationship from its equation using the gradient and $y$-intercept. Check some of your answers to questions **1**, **2**, **3** and **4** using your spreadsheet. You might like to use the prepared spreadsheet (Spreadsheet 4A.2) provided on your obook assess.

**production cost**
cost (or expense) of producing an item for sale, which can involve both a fixed cost and a variable cost that depends on the number of items produced

**revenue**
amount of money (or income) received from the sale of items produced by a company, which can depend on the number of items sold

# 4B Linear models

These resources are available on your obook assess:
- **Video tutorial 4B:** Watch and listen to an explanation of Example 4B–1
- **Investigation 4B.1:** Model the height of water flowing into a water tank
- **Investigation 4B.2:** Model the temperature of an icy-pole placed in a freezer
- **assess quiz 4B:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

Linear relationships can be used to model practical situations. The linear equation used as a model will be of the form $y = mx + c$, where the gradient ($m$) and $y$-intercept ($c$) of the corresponding linear graph have practical meanings in each case. Usually the constant $c$ is a fixed amount, and the gradient $m$ is the rate of change. For example, a taxi fare can be calculated by adding a fixed cost to the charge per kilometre travelled.

In this section, we will also look at practical contexts involving **production cost** and **revenue**.

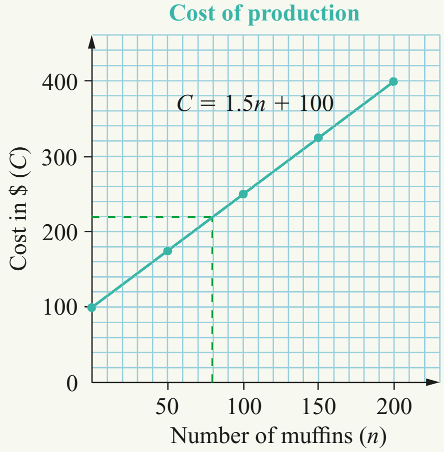
## EXAMPLE 4B–1 Drawing the graph of a cost model from its equation

The Magnificent Muffin Shop has a fixed cost of \$100 per day and a variable production cost of \$1.50 per muffin. The daily cost can be modelled using the equation $C = 1.5n + 100$.

**a** Explain each term in the equation.

**b** Draw the graph of $C = 1.5n + 100$ for $0 \le n \le 200$.

**c** Use the graph to find the number of muffins produced for \$220.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | 1.5 is the cost per muffin, 100 is the fixed cost and $C$ is the total cost. All costs are in dollars. $n$ is the number of muffins made. | The gradient is the coefficient of $n$, which is 1.5. The constant term $c$ is the fixed cost of 100. | The fixed cost is the intercept on the vertical axis. The variable cost relates to the gradient. |
| **b** | Cost of production   | Use graphing technology or complete a table of values for $n$ and $C$ using $C = 1.5n + 100$, and plot the points to produce the graph. | Draw the graph using technology or by completing a table of values and plotting the points. The gradient and $y$-intercept method of drawing a linear graph could also be used. |
| **c** | 80 muffins are produced for \$220. | Draw a line across from the vertical axis at 220 until it meets the line. Draw down to the horizontal axis and read the value of $n$. | Find values by drawing lines from one axis to the linear graph and then to the other axis. |

| $n$ | 0 | 50 | 100 | 150 | 200 |
|---|---|---|---|---|---|
| $C$ | 100 | 175 | 250 | 325 | 400 |

## EXERCISE 4B Linear models

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** The Supertight Tie Company has a fixed cost of \$1200 per day and a variable production cost of \$3 per tie. The daily cost can be modelled using the equation $C = 3n + 1200$.

**a** Explain each term in the equation $C = 3n + 1200$.

**b** Complete this table of values using the equation $C = 3n + 1200$.

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $C$ | | | | | | |

**c** Draw the graph of $C = 3n + 1200$.

**d** Use your graph from part **c** to find the number of ties produced when the daily cost is \$1400.

**2** The distance, $d$ kilometres, travelled by a train in a time of $t$ hours is $d = 80t$.

**a** Complete this table of values using the equation $d = 80t$.

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $d$ | | | | | |

**b** Draw the graph of $d = 80t$.

**c** Where does this graph cut the vertical axis? Why?

**d** What is the gradient of this graph? What does this value represent in this case?

**e** Use the graph to find how far away the train is after $3\frac{1}{2}$ hours.

**f** Use the graph to find when the train is 200 km away.

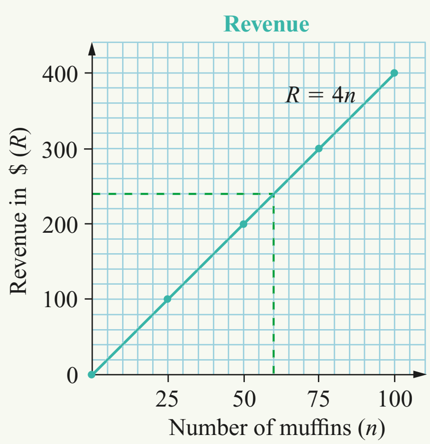
### EXAMPLE 4B–2 Drawing the graph of a revenue model from its equation

The Marvellous Muffin Shop sells its muffins for \$4 each. The sales revenue can be modelled using the equation $R = 4n$.

**a** Explain each term in the equation $R = 4n$.

**b** Graph the straight line with equation $R = 4n$ for $0 \le n \le 100$.

**c** Use the graph to find the number of muffins that are sold if the revenue is \$240.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $R$ is the revenue, $n$ is the number of muffins sold.<br>The gradient is 4, this is the price per muffin in dollars. | There is no fixed amount so the graph starts at the origin. The gradient is the price of each muffin. | If there are no sales then the revenue is zero, so the origin is a point on the graph. |
| **b** |  | Use graphing technology or complete a table of values for $n$ and $R$ using $R = 4n$ and plot the points to produce the graph. | Draw the graph using technology or by completing a table of values and plotting the points. |



| $n$ | 0 | 25 | 50 | 75 | 100 |
|---|---|---|---|---|---|
| $R$ | 0 | 100 | 200 | 300 | 400 |

ALGEBRA

| | Solve | Think | Apply |
|---|---|---|---|
| **c** | 60 muffins must be sold for the revenue to be $240. | Draw a line across from the vertical axis at 240 to the graph and then down to the $n$-axis to read the value of $n$. | Find values by drawing a line from one axis to the graph and then to the other axis. |

UNDERSTANDING, FLUENCY AND COMMUNICATING

**3** The Supertight Tie Company sells ties for $15 each. The sales revenue may be modelled using the equation $R = 15n$.

**a** Explain each term in the equation $R = 15n$.

**b** Complete this table of values for the equation $R = 15n$.

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $R$ | | | | | | |

**c** Draw the straight-line graph of the equation $R = 15n$.

**d** Use the graph to find the number of ties sold when the revenue is $1800.

**e** What is the gradient of the straight line? What does this represent?

**4** Tina travels 200 km to her friend's home at an average speed of 80 km/h. The distance Tina is from her friend's home after travelling for $t$ hours can be modelled using the linear equation $D = 200 - 80t$.

**a** Explain each term in the equation $D = 200 - 80t$.

**b** Draw the straight-line graph of the equation $D = 200 - 80t$ for $t$ values up to 4 hours.

**c** Use the graph to find how far Tina is from her friend's home after 1.5 hours.

**d** Use the graph to find how long it takes for Tina to reach her friend's home.

**e** What is the value of $D$ when $t$ is 4 hours? Explain why this linear model can only be used for particular values of $t$ in this context.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**5** The Munchy Sandwich Shop has a monthly fixed expense of $2000 and the cost of making each sandwich is $1.20.

**a** Write an equation modelling this information. Use $C$ for monthly cost in dollars and $n$ for the number of sandwiches made.

**b** Complete this table of values using your equation from part **a**.

| $n$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|
| $R$ | | | | | | |

**c** Draw the straight-line graph for this linear model.

**d** What is the value of the gradient? What does this value represent?

**e** Use the graph to find the number of sandwiches made in a month if the cost is $3500.

**6** The Tight Squeeze Fruit Juice Company has a weekly fixed cost of $325. The cost to produce a cup of fruit juice is $1.50.

**a** Write a linear equation modelling this information. Use $C$ for the weekly cost in dollars and $n$ for the number of cups of fruit juice.

**b** Complete this table of values using your equation from part **a**.

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $C$ | | | | | | |

**c** Draw the straight-line graph for this linear model.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**d** Where does the graph cut the vertical axis? What does this value represent?

**e** What is the gradient? What does this value represent?

**f** Use the graph to find the number of cups of juice made in a week if the cost is $1000.

**7** The Munchy Sandwich Shop sells sandwiches for $3.50 each.

**a** Write an equation modelling this information. Use $R$ for revenue in dollars and $n$ for the number of sandwiches sold.

**b** Complete this table of values using your equation from part **a**.

| $n$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|
| $C$ | | | | | | |

**c** Draw a straight-line graph for this linear model.

**d** Use the graph to find the number of sandwiches sold when the revenue is $2800.

**e** What is the gradient of this straight line? What does this represent?

**8** The Tight Squeeze Fruit Juice Company sells juice for $4.00 per cup.

**a** Write an equation modelling this information. Use $R$ for revenue in dollars and $n$ for the number of cups of juice sold.

**b** Complete this table of values using your equation from part **a**.

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $R$ | | | | | | |

**c** Draw the straight-line graph for this linear model.

**d** Use the graph to find the number of cups of juice sold when the revenue is $3000.

**e** What is the gradient of this straight line? What does this represent?

**9** The cost of hiring a taxi is $6 flagfall and $3.60 per kilometre travelled.

**a** Write a linear equation modelling this information to relate cost and distance travelled.

**b** Use graphing technology or a table of values to draw the graph of this linear model for distances up to 20 km.

**c** Use the graph to find the cost of travelling 10 km.

**d** Use the graph to estimate how far you could travel for $30.

**10** On a building site, columns with a fixed cross-sectional area of 0.78 $m^2$ are to be constructed by pouring concrete into moulds of different heights.

**a** Write a linear equation modelling this information to relate volume of concrete (in cubic metres) and height of a column (in metres).

**b** Use graphing technology or a table of values to draw the graph of this linear model for height values up to 7 m.

**c** Use the graph to estimate the volume of concrete needed to construct a column that is 4 m high.

**d** Use the graph to estimate the height of a column that can be constructed using 2.4 $m^3$ of concrete.

CHALLENGE

**11** The conversion from temperature in degrees Celsius ($C$) to temperature in degrees Fahrenheit ($F$) is given by the equation $F = \frac{9}{5}C + 32$.

**a** Using graphing technology or by completing a table, draw the graph of this relationship for values of $C$ between 0 and 100.

**b** A 'rule of thumb' conversion is to double the temperature in degrees Celsius and add 30. Graph this model for values of $C$ between 0 and 100.

**c** Compare the two graphs. Describe the accuracy of the rule of thumb. Which would you use? When?

# 4C Identifying solutions to simultaneous linear equations

These resources are available on your obook assess:

- **assess quiz 4C:** Test your skills with an auto-correcting multiple-choice quiz

Consider the following tables of values for the linear equations given.

$y = 2x + 7$

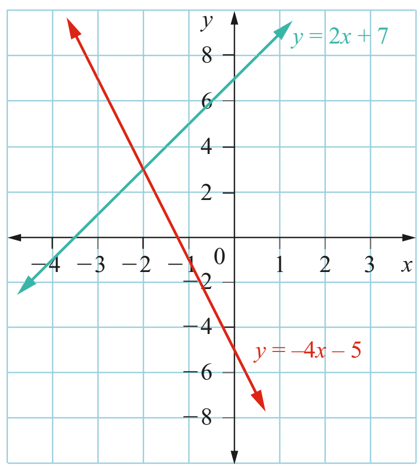
| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 |

$y = -4x - 5$

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 7 | 3 | $-1$ | $-5$ | $-9$ | $-13$ | $-17$ |

The equation $y = 2x + 7$ has a solution of $x = -3$ and $y = 1$ because, when we substitute these values into the equation, the statement $1 = 2 \times -3 + 7$ is true. In fact, this equation has an infinite number of solutions as $x = -2$ and $y = 3$, $x = -1$ and $y = 5$, $x = 0$ and $y = 7$, etc. are also solutions. Similarly the equation $y = -4x - 5$ has an infinite number of solutions as $x = -3$ and $y = 7$, $x = -2$ and $y = 3$, $x = -1$ and $y = -1$, etc. are solutions.

We can see that $x = -2$ and $y = 3$ is a solution of *both* equations. We say that the linear equations $y = 2x + 7$ and $y = -4x - 5$ are solved simultaneously by the values $x = -2$ and $y = 3$. Hence, to solve a pair of linear equations simultaneously, or solve a pair of **simultaneous linear equations**, means to find the values of the variables that make both equations true.




**simultaneous linear equations** two or more linear equations that use the same variables

**point of intersection** point where two or more lines intersect or cross over one another on a graph

The graphs of the two linear equations can be drawn on the same number plane as shown on the right.

What are the coordinates of the **point of intersection** of the two lines? How does this relate to the solution of the simultaneous linear equations $y = 2x + 7$ and $y = -4x - 5$?

## EXERCISE 4C Identifying solutions to simultaneous linear equations

**1** Consider the tables given below to find the $x$-value and the $y$-value that is the solution to each pair of simultaneous linear equations.

**a** $y = 2x$

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $-6$ | $-4$ | $-2$ | 0 | 2 | 4 | 6 |

$x + y = 6$

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 8 | 7 | 6 | 5 | 4 | 3 |

**b** $y = 2x - 1$

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | –1 | 1 | 3 | 5 | 7 |

$x + y = 8$

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | 7 | 6 | 5 | 4 |

**c** $y = 3x - 5$

| $x$ | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|
| $y$ | 10 | 13 | 16 | 19 | 22 |

$x + y = 23$

| $x$ | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|
| $y$ | 18 | 17 | 16 | 15 | 14 |

## EXAMPLE 4C–1 Checking the solution to simultaneous linear equations

Check if each given $x$-value and $y$-value is the solution to the pair of simultaneous linear equations.

**a** $x = 2, y = 4$
$y = 3x - 2$ and $y = -2x + 8$

**b** $x = 5, y = -3$
$y = -2x + 7$ and $y = 2x - 11$

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $y = 3x - 2$<br>When $x = 2$ and $y = 4$,<br>LHS $= 4$ and RHS $= 3 \times 2 - 2 = 4$<br>So LHS = RHS<br>$y = -2x + 8$<br>When $x = 2$ and $y = 4$,<br>LHS $= 4$ and RHS $= -2 \times 2 + 8 = 4$<br>So LHS = RHS<br>As $x = 2$, $y = 4$ satisfies both equations, it is the solution. | Substitute $x = 2$ and $y = 4$ into the left-hand side (LHS) and right-hand side (RHS) of each equation.<br>Check if the LHS and RHS are equal for each equation. They are equal (the values make both equations true statements). Hence $x = 2$, $y = 4$ is the solution to the simultaneous linear equations. | Substitute the given $x$- and $y$-values into each equation and check if they make both equations true statements. Decide if the given $x$-value and $y$-value is the solution to the simultaneous linear equations. |
| **b** | $y = -2x + 7$<br>When $x = 5$ and $y = -3$,<br>LHS $= -3$ and RHS $= -2 \times 5 + 7 = -3$<br>So LHS = RHS<br>$y = 2x - 11$<br>When $x = 5$ and $y = -3$,<br>LHS $= -3$ and RHS $= 2 \times 5 - 11 = -1$<br>So LHS $\neq$ RHS<br>As $x = 5$, $y = -3$ does not satisfy both equations, this is not the solution. | Substitute $x = 5$ and $y = -3$ into the left-hand side (LHS) and right-hand side (RHS) of each equation.<br>Check if the LHS and RHS are equal for each equation. They are equal for the first equation but not the second. Hence $x = 5$, $y = -3$ is not the solution to the simultaneous linear equations. | |

**2** Check if each given $x$-value and $y$-value is the solution to the pair of simultaneous linear equations.

**a** $x = 1, y = 5$
$y = x + 4$ and $y = -2x + 7$

**b** $x = -2, y = 1$
$y = x + 3$ and $y = -3x - 5$

**c** $x = 5, y = -2$
$y = x - 9$ and $y = -2x + 7$

**d** $x = 3, y = -7$
$y = -2x - 1$ and $y = -x - 4$

**e** $x = 4, y = 3$
$y = 2x - 5$ and $y = -x + 1$

**f** $x = -6, y = 0$
$y = x - 6$ and $y = -x - 6$

!
To find the solution of a pair of simultaneous linear equations from a graph, find the point of intersection of the two lines. At the point of intersection, each line will simultaneously have the same $x$-coordinate and $y$-coordinate. The coordinates of the point of intersection provide the $x$-value and the $y$-value of the solution.

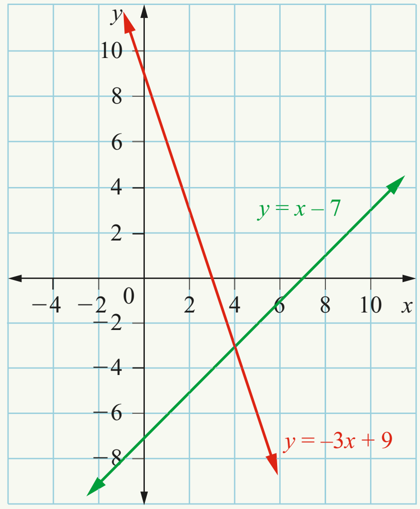
## EXAMPLE 4C–2 Identifying the solution to simultaneous linear equations from graphs

Consider the following pair of simultaneous linear equations and their graphs drawn on the same number plane.

$y = x - 7$ and $y = -3x + 9$

**a** Write the coordinates of the point of intersection of the two lines.

**b** Use the coordinates of this point to write the solution to the simultaneous linear equations.



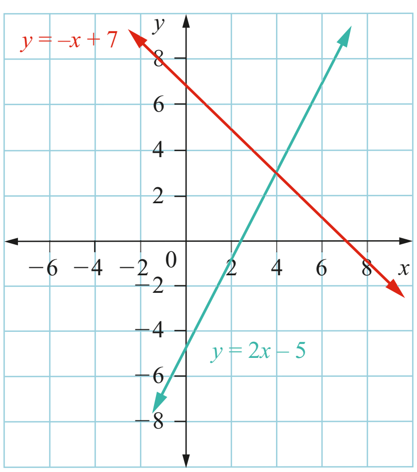

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Point of intersection at (4, –3). | Identify the point where the two lines cross and write the $x$-value and the $y$-value of the point as coordinates. | Identify the point of intersection of the two lines and list the coordinates. |
| **b** | The solution to the simultaneous equations $y = x - 7$ and $y = -3x + 9$ is $x = 4$, $y = -3$. | Use the coordinates of the point to write the solution. | Write the solution as the $x$-value and the $y$-value that makes both equations true. |

**3** Consider each pair of simultaneous linear equations and their graphs.

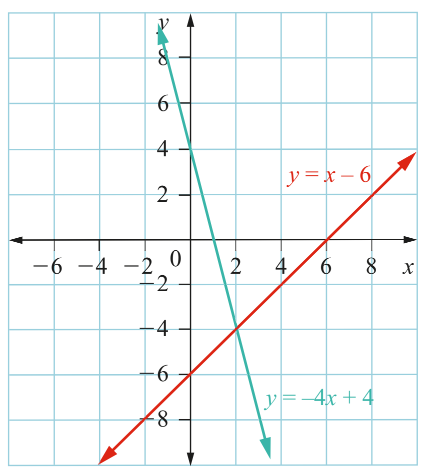
**i** Write the coordinates of the point of intersection of the two lines.

**ii** Use the coordinates of this point to write the solution to the simultaneous linear equations.

**a** $y = 2x - 5$ and $y = -x + 7$

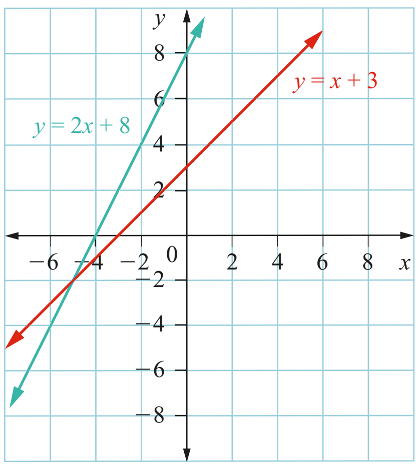



**b** $y = -4x + 4$ and $y = x - 6$

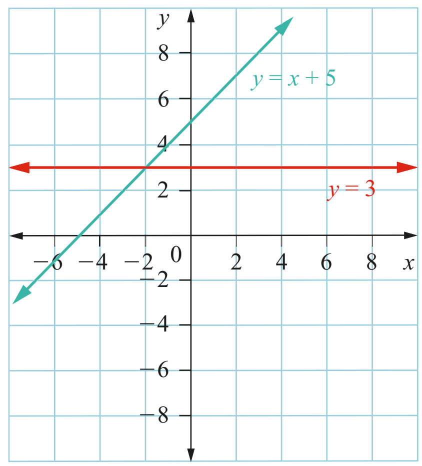


UNDERSTANDING, FLUENCY AND COMMUNICATING

**c** $y = 2x + 8$ and $y = x + 3$




**d** $y = x + 5$ and $y = 3$




**4** Use substitution to check your solutions to the simultaneous linear equations in question **3**.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

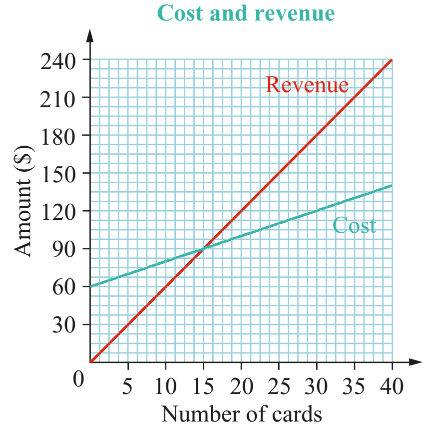
**5** Suzy makes hand-painted cards. She invests \$60 in a set of paintbrushes and enough paint for 40 cards. The cost of each plain white card is \$2 and she sells the painted cards for \$6 each. The production cost and the revenue generated by the sale of the cards can be modelled with two linear equations and shown on a graph.

**a** How many cards does Suzy need to sell to recover all her costs?

**b** How much profit does Suzy make if she sells 30 cards?

**c** What is the maximum profit Suzy can make?

**Cost and revenue**




**6** Daniel is deciding which mobile phone plan to use. He is comparing the Plan 30 and Plan 50 rates. Plan 30 is a minimum of \$30 per month with \$5 of free calls. Calls cost 24 cents/30 s. Plan 50 is a minimum of \$50 per month with \$40 of free calls. Calls cost 19 cents/30 s. This graph shows these plans.

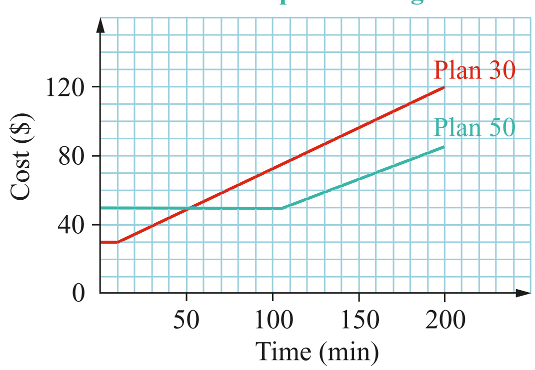
**a** For 35 minutes of calls per month, which plan should he use? Why?

**b** For 120 minutes of calls per month, which plan should he use? Why?

**c** After how many minutes of calls per month are the costs of each plan equal? What is this cost?

**d** Explain why these two lines are not simultaneous linear equations.

**Mobile phone charges**



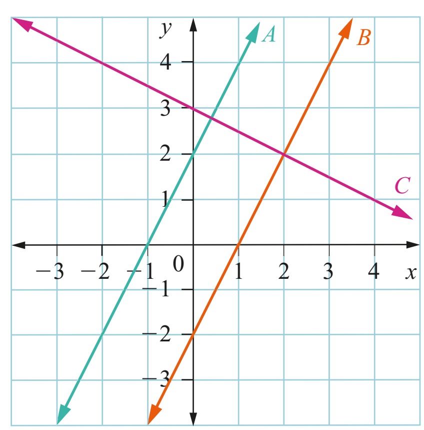

CHALLENGE

**7** Consider the straight lines $A$, $B$ and $C$ shown on this graph.

**a** Write the coordinates of the point of intersection of lines $B$ and $C$ and hence give the solution to the simultaneous equations for these lines.

**b** Use the graph to solve the equations for lines $A$ and $C$ simultaneously.

**c** Can the equations for lines $A$ and $B$ be solved simultaneously? Explain why or why not.

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# 4D Solving simultaneous linear equations graphically

These resources are available on your obook assess:

- **Video tutorial 4D:** Watch and listen to an explanation of Example 4D–1
- **Worksheet 4D:** Practise your skills with extra problems
- **Investigation 4D:** Solve simultaneous linear equations for train travel
- **assess quiz 4D:** Test your skills with an auto-correcting multiple-choice quiz

In Topic 4C, the solution to a pair of simultaneous linear equations was identified from given tables or graphs. In this section you will use your skills in drawing linear graphs to solve simultaneous linear equations graphically. You can use technology or complete tables of values to help you produce the graphs.

If using tables of values, it is easier to have three sets of $x$- and $y$-values. Although only two points need to be plotted to produce a straight-line graph, using three (or four) points provides a check of the substitution calculations. If the plotted points do not lie in a straight line, we can easily see that an error has been made.

## EXAMPLE 4D–1 Drawing graphs from tables of values to solve simultaneous linear equations

**a** Complete the given tables below for the linear equations $y = 2x + 1$ and $y = 4 - x$.

$y = 2x + 1$

| $x$ | –2 | 0 | 2 |
|---|---|---|---|
| $y$ | | | |

$y = 4 - x$

| $x$ | –2 | 0 | 2 |
|---|---|---|---|
| $y$ | | | |

**b** Plot the values from the tables to draw the graphs of $y = 2x + 1$ and $y = 4 - x$ on the same number plane.

**c** Find the point of intersection of the two lines.

**d** Write the solution to the simultaneous linear equations $y = 2x + 1$ and $y = 4 - x$.

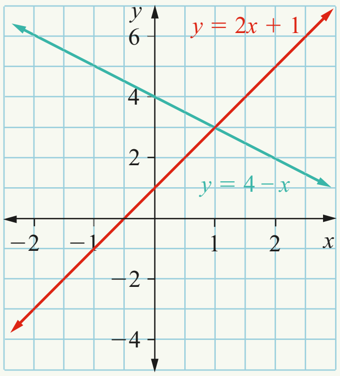
| | Solve | Think/Apply |
|---|---|---|
| **a** | $y = 2x + 1$ | Complete the table of values for each equation by substituting each $x$-value into the equation to find the corresponding $y$-value. |
| | | The coordinates of the point of intersection of the two straight lines give the solution of the simultaneous equations. |

$y = 2x + 1$

| $x$ | –2 | 0 | 2 |
|---|---|---|---|
| $y$ | –3 | 1 | 5 |

$y = 4 - x$

| $x$ | –2 | 0 | 2 |
|---|---|---|---|
| $y$ | 6 | 4 | 2 |

| | Solve | Think/Apply |
|---|---|---|
| **b** |  | Use the tables of values to plot points for each equation on the same number plane and draw straight lines through them. |
| **c** | The point of intersection is (1, 3). | Identify the point where the two lines cross and write the coordinates of this point. |
| **d** | The solution is $x = 1$, $y = 3$. | As the point of intersection lies on both lines, its coordinates satisfy both equations; that is, $x = 1$ and $y = 3$ is the solution of the simultaneous equations. |

## EXERCISE 4D Solving simultaneous linear equations graphically

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Consider the following pairs of linear equations.

**i** Complete the given tables for the linear equations.

**ii** Plot the values from the tables to draw the graphs of the equations on the same number plane.

**iii** Find the point of intersection of the two lines.

**iv** Write the solution to the simultaneous linear equations.

**a** $y = x + 2$

| $x$ | $-2$ | 0 | 2 |
|---|---|---|---|
| $y$ | | | |

$y = 2x - 1$

| $x$ | $-2$ | 0 | 2 |
|---|---|---|---|
| $y$ | | | |

Draw a number plane showing $x$-values from $-4$ to 8 and $y$-values from $-6$ to 8.

**b** $y = 2x + 2$

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | | | |

$y = 3x + 1$

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | | | |

Draw a number plane showing $x$-values from $-4$ to 8 and $y$-values from $-2$ to 14.

**c** $y = 2x - 2$

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $y$ | | | |

$y = 4 - x$

| $x$ | $-1$ | 0 | 1 |
|---|---|---|---|
| $y$ | | | |

Draw a number plane showing $x$-values from $-2$ to 5 and $y$-values from $-4$ to 5.

**2** Solve each of the following pairs of simultaneous linear equations graphically using the approach used in question **1**.

**a** $y = 6 - x$ and $y = 10 - 2x$ **b** $y = x - 3$ and $y = 1 - x$ **c** $y = x - 1$ and $y = 2x$

**d** $y = 4 - x$ and $y = 5 - 2x$ **e** $y = 6 - x$ and $y = 10 - 2x$ **f** $y = x + 1$ and $y = -\frac{2}{3}x + 2$

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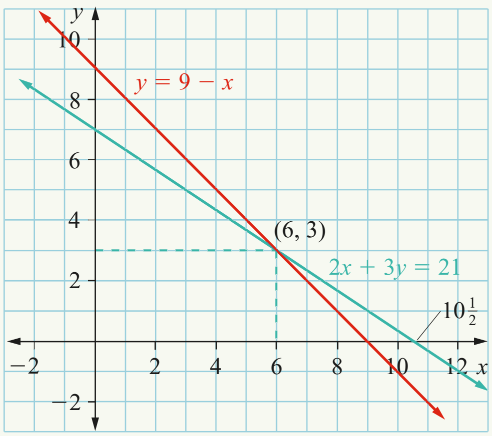
## EXAMPLE 4D–2 Solving simultaneous linear equations graphically

Use a graphical method to solve the simultaneous linear equations $2x + 3y = 21$ and $y = 9 - x$.

**Solve**




The point of intersection is (6, 3), so the solution to the simultaneous linear equations $2x + 3y = 21$ and $y = 9 - x$ is $x = 6$, $y = 3$.

**Think**

Draw the lines for the equations using graphing technology or by completing a table of values and plotting the points.

The equation $2x + 3y = 21$ can be rearranged to become $y = -\frac{2}{3}x + 7$. Choose $x$-values that make the substitution calculations easier.

For $y = -\frac{2}{3}x + 7$:

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 3 | 6 | 9 |
| $y$ | 7 | 5 | 3 | 1 |

For $y = 9 - x$:

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 2 | 4 | 8 |
| $y$ | 9 | 7 | 5 | 1 |

Locate the coordinates of the point where the lines cross and write the solution.

**Apply**

Produce the linear graphs using technology or by completing a table of values and plotting the points. The gradient and $y$-intercept method of drawing a linear graph could also be used.

Find the point of intersection of the two lines and use the coordinates to write the solution to the simultaneous equations.

**3** Use a graphical method to solve these pairs of simultaneous linear equations.

- **a** $y = x - 3$ and $y = 1 - x$
- **b** $x - y = 1$ and $y = 2x$
- **c** $4x + 3y = 10$ and $x - 2y = -3$
- **d** $3x + y = 5$ and $2x - 3y = 18$
- **e** $y = 3x - 8$ and $3x + y = -2$
- **f** $2x + y = 4$ and $2x - y = 8$
- **g** $3x + y = -5$ and $2x + 3y = -8$
- **h** $x - 3y = -9$ and $2x - 3y = -12$
- **i** $x + y = 1$ and $3x + 2y = 1$
- **j** $3x - y = 2$ and $x - y = -4$

**4** Consider the linear equations $y = 2x + 3$ and $y = 8 - 3x$.

- **a** Use graphing technology to draw the graphs of the two linear equations on the same set of axes.
- **b** Find the point of intersection of these lines.
- **c** Hence write the solution to the simultaneous linear equations $y = 2x + 3$ and $y = 8 - 3x$.

**5** Use graphing technology to solve the following pairs of simultaneous linear equations. Compare your answers with those obtained in question **2**.

- **a** $y = 6 - x$ and $y = 10 - 2x$
- **b** $y = x - 3$ and $y = 1 - x$
- **c** $y = x - 1$ and $y = 2x$
- **d** $y = 4 - x$ and $y = 5 - 2x$
- **e** $y = 6 - x$ and $y = 10 - 2x$
- **f** $y = x + 1$ and $y = -\frac{2}{3}x + 2$

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**6** The cost ($\$C$) of hiring a car from company A is \$50 per day; that is, $C = 50d$, where $d$ is the number of days for which the car is hired. Company B charges a flat fee of \$60 plus \$40 per day; that is, $C = 60 + 40d$.

**a** Use graphing technology or complete tables to graph the equations on the same set of axes.

**b** Solve the equations simultaneously using your graph. Explain the meaning of the solution.

**c** After how many days is it cheaper to hire from company B?

**7** Joanna is a salesperson who earns her income from commissions. Her supervisor offers her a choice of two methods of weekly payment.

**i** A straight commission of 10% of her sales: $I = 0.1S$, where $I$ is income and $S$ is sales in dollars

**ii** A retainer of \$300 plus 6% of sales: $I = 300 + 0.06S$

**a** Use graphing technology or complete tables to graph each linear equation on the same set of axes.

**b** Find the solution of the simultaneous linear equations. Explain the meaning of the solution.

**c** What sales would Joanna have to achieve to earn more income using method **ii**?

**8** A new car with a petrol engine uses 10.0 L/100 km. If the cost of petrol is \$1.50/L then the cost ($\$C$) of fuel to drive the car is given by $C = 0.15d$, where $d$ is the distance travelled. The same car with a diesel engine costs \$2000 more to purchase but only uses 6.0 L/100 km. The cost of diesel fuel is \$1.60/L. The cost of driving the diesel model of the car is given by $C = 2000 + 0.096d$.

**a** Find the solution of the simultaneous linear equations using a graphical method.

**b** How far do you have to travel before the cost of the two models is the same?

**9** The perimeter of a printed rectangular photo is 50 cm.

**a** Use $l$ for length and $w$ for width to write a linear equation for the perimeter of the photo.

**b** Use $l$ and $w$ to write a linear equation linking the length and width of the photo, given that the length is 5 cm longer than the width.

**c** Solve the two equations simultaneously using a graphical method, and hence state the dimensions of the photo.

**10** At the cinema, a large drink costs \$2 more than an ice cream. Liam buys five drinks and seven ice creams for \$70.

**a** Write two linear equations to model this information.

**b** Solve the equations simultaneously using a graphical method to find the cost of each item.

**11** Tom rides his bicycle from Town A to Town B at an average speed of 20 km/h. Julia walks from Town B to Town A along the same road at an average speed of 5 km/h. The distance along the road between the two towns is 15 km and they each start off at the same time.

**a** Write two linear equations to model the two journeys. In each case, relate the distance in kilometres *from Town A* to the time travelled in hours.

**b** Solve the equations simultaneously using a graphical method to find the time at which they meet along the road between the two towns.

**c** How far has each travelled before they meet?

**12** Solve the following pairs of simultaneous linear equations graphically and explain your results.

**a** $y = 2x + 1$ and $y = 2x - 3$ **b** $x - 2y = 4$ and $2x - 4y = 8$

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# 4E Break-even analysis

**break-even point**
point at which the revenue (or income) and the cost of production (or expenses) are equal; it can be identified as the point of intersection of the combined graphs of cost and revenue

These resources are available on your obook assess:
- **Video tutorial 4E:** Watch and listen to an explanation of Example 4E–2
- **assess quiz 4E:** Test your skills with an auto-correcting multiple-choice quiz

In Topic 4B, straight-line graphs were drawn to model both production cost and revenue. If we combine graphs for cost and revenue for the same business, we can find the **break-even point**. This is the point at which the graphs intersect and revenue equals cost. Sales exceeding the break-even number result in a profit for the business, and sales below the break-even number result in a loss.

This section uses simultaneous linear equations to look at break-even points, and profits and losses.

## EXAMPLE 4E–1 Determining the break-even point from a given graph

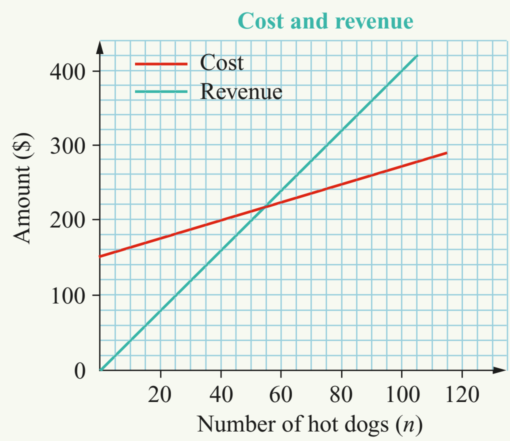
Holly's Hot Dogs makes and sells hot dogs. There is a fixed cost per day of \$150 and each hot dog costs \$1.20 to produce. The hot dogs are sold for \$4 each.

The graphs of the production cost and the revenue made from selling hot dogs are drawn on the same set of axes.

**a** How many hot dogs must be sold to break even?

**b** What is the revenue at the break-even point?

**c** Find the profit made when 90 hot dogs are sold.




| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Approximately 54 hot dogs must be sold to break even. | To break even, the revenue has to equal the production cost. Read the value for $n$ at the point of intersection of the two lines.<br>*Note:* in this case we need to estimate the value. | The break-even point occurs where the lines for revenue and cost intersect. |
| **b** | At the break-even point, revenue is about \$215. | Estimate the value for the revenue from the vertical axis (amount in dollars) when $n \approx 54$. | Read the value for the revenue at the point of intersection of the two lines. |
| **c** | When $n = 90$,<br>revenue = \$360<br>cost = \$260<br>Profit = revenue − cost<br>= 360 − 260<br>= \$100 | Read the values from the vertical axis for the revenue line and the cost line when $n = 90$. Subtract the cost from the revenue to find the profit. | Find the values for the revenue and cost by reading across to the vertical axis from each line for the given number of hot dogs. |

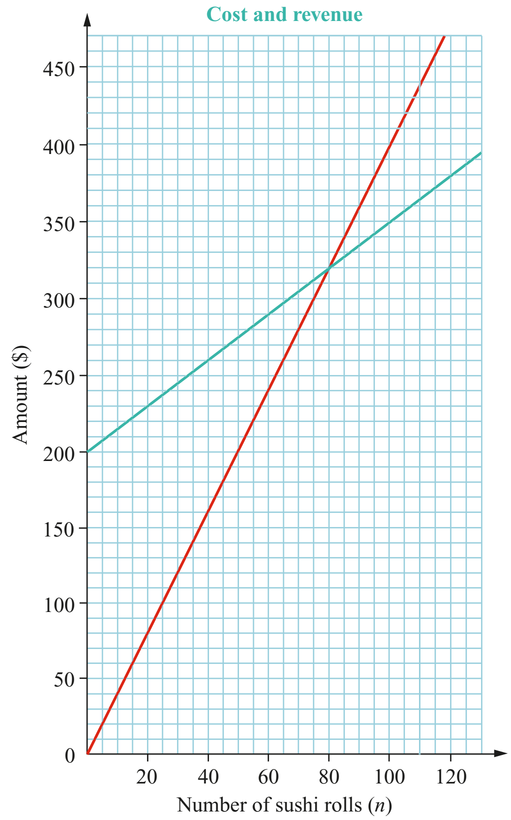
# EXERCISE 4E Break-even analysis

**1** Suki's Sushi Bar makes and sells sushi rolls. There is a fixed cost per day of $200 and each sushi roll costs $1.50 to produce. The sushi rolls are sold for $4 each. The graphs of production cost and revenue are drawn on the same set of axes.

**a** How many sushi rolls must be sold to break even?

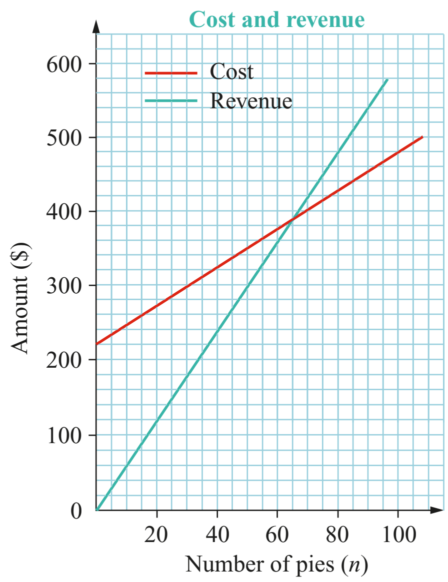
**b** What is the revenue at the break-even point?

**c** Subtract the cost from the revenue to find the profit made when 100 sushi rolls are sold.




**2** Peter's Pie Palace makes and sells gourmet pies. There is a fixed cost per day of $220 and each pie costs $2.60 to produce. The pies are sold for $6 each.

The graphs of the production cost and the revenue made from selling the pies are drawn on the same set of axes.




**a** How many pies must be sold to break even?

**b** What is the revenue at the break-even point?

**c** Find the profit made when 90 pies are sold?

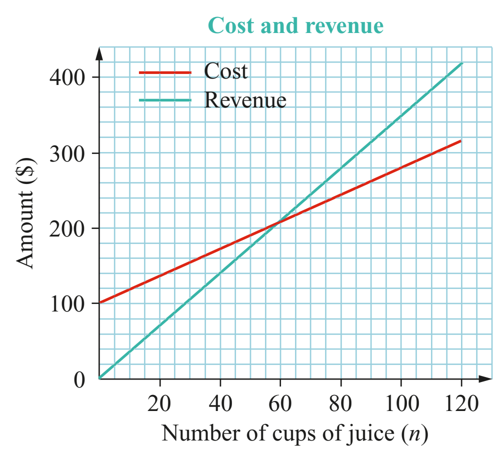
**3** Jenny's Juice Joint makes and sells fresh juices. There is a fixed cost per day of $100 and each cup of juice costs $1.80 to produce. The juices are sold for $3.50 each. The graphs of production cost and revenue are drawn on the same set of axes.

**a** How many cups of juice must be sold to break even?

**b** What is the revenue at the break-even point?

**c** Find the profit made by Jenny's Juice Joint when 110 cups of juice are sold.

**d** Find the difference between the revenue and cost when 30 cups of juice are sold. Does Jenny make a profit? Explain your answer.



UNDERSTANDING, FLUENCY AND COMMUNICATING

**4** A plastics company has a maximum daily production of 700 items.

**a** There is an initial cost of \$3000 per day plus \$8 per item produced. This can be represented by the linear equation $P = 8n + 3000$, where $P$ is the production cost in dollars and $n$ is the number of items. Complete this table of values for the production cost.

| **Number of items ($n$)** | 0 | 100 | 300 | 500 | 700 |
|---|---|---|---|---|---|
| **Production cost ($P$, \$)** | | | | | |

**b** The selling price of each item is \$15. This can be represented by the linear equation $R = 15n$, where $R$ is the revenue in dollars on the sale of $n$ items. Complete this table of values for the revenue.

| **Number of items ($n$)** | 0 | 100 | 300 | 500 | 700 |
|---|---|---|---|---|---|
| **Revenue ($R$, \$)** | | | | | |

**c** Choose a suitable scale and graph these lines on the same set of axes:

**i** production cost versus number of items

**ii** revenue versus number of items.

**d** Find the point of intersection of the two graphs. This is the break-even point.

**e** How many items must be sold each day to break even?

**f** What is the break-even revenue?

**5** The Tin Lid factory produces sports caps. There is a fixed monthly cost of \$1800 and it costs \$3 to produce a cap. The caps are sold for \$8 each. The maximum monthly production of caps is 800.

**a** Use the equation $C = 3n + 1800$, where $C$ is the production cost in dollars to produce $n$ sports caps, to graph the cost of production.

**b** Use the equation $R = 8n$, where $R$ is the revenue in dollars on the sale of $n$ sports caps, to graph the revenue on the same set of axes as the cost equation in part **a**.

**c** Find the point of intersection of the two lines. This is the break-even point.

**d** How many caps must be sold in a month to break even?

**e** What is the break-even revenue?

**6** The Supertight Tie Company has a fixed cost of \$1200 per day and a variable cost of \$3 per tie. This can be represented by the equation $C = 3n + 1200$, where $C$ is the cost of producing $n$ ties. The revenue of \$15 per tie sold can be represented by the equation $R = 15n$, where $R$ is the revenue on the sale of $n$ ties.

**a** Graph these two equations on the same set of axes for $0 \le n \le 250$.

**b** Find the break-even point.

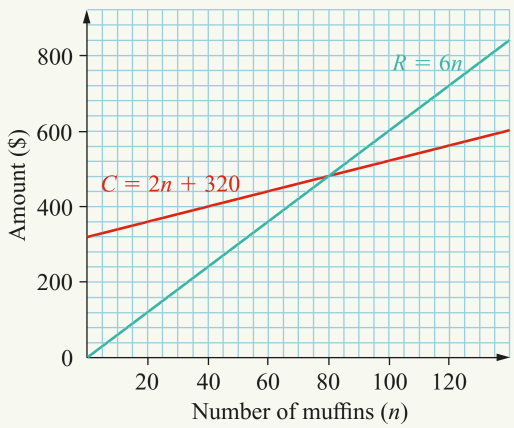
**c** How many ties must be sold to break even?

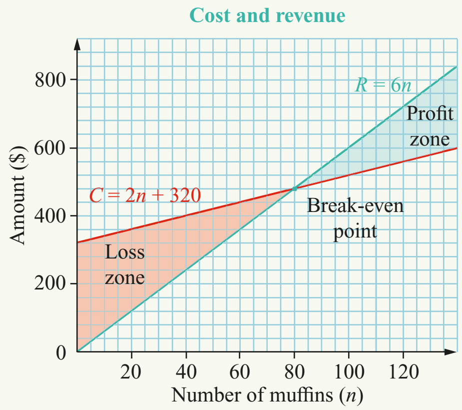
**d** What is the break-even revenue?

## EXAMPLE 4E–2 Modelling revenue and cost for break-even analysis

The Majestic Muffin Shop has a fixed cost of \$320 per day and a variable production cost of \$2 per muffin. The selling price of each muffin is \$6. The maximum daily production is 120 muffins.

**a** Write an equation for the cost, \$$C$, of producing $n$ muffins.
**b** Graph the linear equation for the production cost.
**c** Write an equation for the revenue, \$$R$, for the number of muffins sold, $n$. Draw the straight line for the revenue equation on the same set of axes as the cost equation in part **b**.
**d** Find the point of intersection.
**e** How many muffins must be sold to break even?
**f** What is the break-even revenue?
**g** On the graph drawn for parts **b** and **c**, shade the loss zone and the profit zone.

<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>The cost equation is $C = 2n + 320$.</td><td>The fixed cost is \$320.<br>The variable cost is $2n$ since the cost per muffin (and the gradient) is 2.</td><td>The vertical intercept is the fixed cost. The gradient relates to the variable cost.</td></tr>
<tr><td>b</td><td>Cost and revenue<br><br></td><td>Draw the graph using graphing technology, or the gradient (2) and $y$-intercept (320) method, or by completing a table of values and plotting the points.<br><table><tr><td>$n$</td><td>0</td><td>40</td><td>80</td><td>120</td></tr><tr><td>$C$</td><td>320</td><td>400</td><td>480</td><td>560</td></tr></table></td><td>The maximum number of muffins is 120 so this is the largest value for $n$.</td></tr>
<tr><td>c</td><td>The revenue equation is $R = 6n$.<br>The graph is drawn on the same set of axes as $C = 2n + 320$ in part b.</td><td>Draw the graph using graphing technology, or the gradient (6) and $y$-intercept (0) method, or by completing a table of values and plotting the points on the same set of axes.<br><table><tr><td>$n$</td><td>0</td><td>40</td><td>80</td><td>120</td></tr><tr><td>$R$</td><td>0</td><td>240</td><td>480</td><td>720</td></tr></table></td><td>There is no fixed cost, so the equation has no constant value.<br>Both $C$ and $R$ are values in dollars and so can both be represented on the vertical axis, labelled as ‘Amount (\$)’.</td></tr>
<tr><td>d</td><td>The point of intersection has the coordinates (80, 480).</td><td>Find the values at the point where the lines intersect.</td><td>The break-even point is when revenue equals cost. This is the point of intersection of the two lines.</td></tr>
</table>

| | Solve | Think | Apply |
|---|---|---|---|
| **e** | 80 muffins must be sold to break even. | Read the value for $n$ for the point of intersection of the two lines. | |
| **f** | The break-even revenue is \$480. | Read the amount ($C$ or $R$) in dollars for the point of intersection of the two lines. | |
| **g** |   | For the loss zone, shade the zone between the two lines to the left of the break-even point to indicate where the revenue is less than the cost for making $n$ items. For the profit zone, shade the zone between the two lines to the right of the break-even point to indicate where the revenue is more than the cost for making $n$ items. | The loss zone is the zone between the two lines where the revenue line is below the cost line. The profit zone is the zone between the two lines where the revenue line is above the cost line. |

**7** The Munchy Sandwich Shop (from Exercise 4B) had a fixed expense of \$2000 per month and a cost of \$1.20 per sandwich. The sandwiches were sold for \$3.50 each.

- **a** Write the linear equations that can be used to model the production cost and revenue for $n$ sandwiches.
- **b** Graph straight lines modelling the cost and revenue for $0 \leq n \leq 1000$.
- **c** Find the break-even point.
- **d** How many sandwiches must be sold to break even?
- **e** What is the break-even revenue?
- **f** The maximum number of sandwiches that can be made in a month is 800. Explain the significance of this.
- **g** On the graph drawn for part **b**, shade the loss zone. Explain the significance of this.

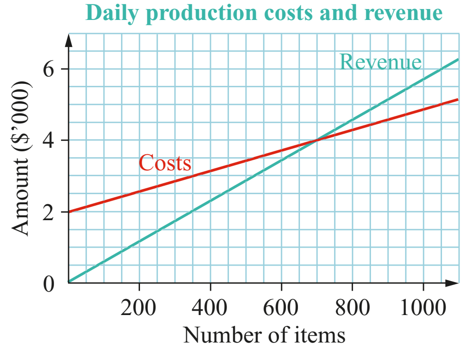
**8** The Full Flavour Juice Company has a weekly fixed cost of \$325 and a cost of \$1.50 per cup of juice. The cups of juice are sold for \$4.

- **a** Graph straight lines modelling the production cost and revenue for up to 200 cups of juice.
- **b** Find the break-even point.
- **c** How many cups of juice must be sold to break even?
- **d** What is the break-even revenue?
- **e** The maximum number of cups of juice per week is 100. Is this a profitable business? Explain.
- **f** On the graph drawn for part **a**, shade the loss zone.

**9** Consider the graphs you have drawn to model the cost and revenue for each company in questions **4**, **5** and **6**.

- **a** Shade the loss zone for the graph drawn in question **4**.
- **b** Shade the profit zone for the graph drawn in question **5**.
- **c** Shade the loss zone for the graph drawn in question **6**.

**10** The daily production costs and revenue from the sale of items is shown on the graph.




**a** Use the graph to determine the number of items that must be sold to break even.

**b** How much profit or loss is made when:

**i** 100 items are sold?

**ii** 1000 items are sold?

**c** How many items need to be sold for the company to make a:

**i** \$1000 profit?

**ii** \$1000 loss?

**11** The cost of producing cotton bags is initially \$2000 plus \$2.50 per item. Each bag is sold for \$5.

**a** Model the production cost and the revenue with linear equations.

**b** Use graphing technology or plot values from tables to draw the linear graphs for the production cost and revenue on the same set of axes, for up to 1200 bags.

**c** How many bags need to be sold to break even?

**d** How much profit or loss is made when:

**i** 100 bags are sold?

**ii** 1000 bags are sold?

**e** How many bags need to be sold for the company to make:

**i** \$1000 profit?

**ii** \$1000 loss?

**12** The production cost of printing books is initially \$8000 plus \$3000 per 1000 books printed. The books are sold for \$6.50 each.

**a** Model the production cost and the revenue with linear equations.

**b** Use graphing technology or plot values from tables to draw the linear graphs for the production cost and revenue on the same set of axes.

**c** How many books need to be sold to break even?

**d** How much profit or loss is made when:

**i** 1000 books are sold?

**ii** 2500 books are sold?

**iii** 3200 books are sold?

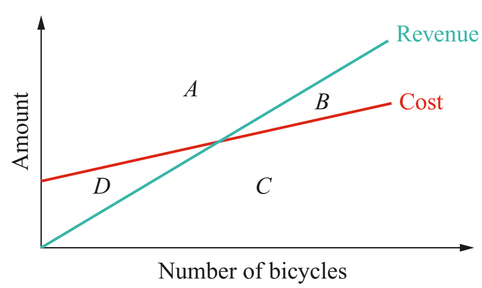
**e** How many books need to be sold for the company to make:

**i** a loss of \$3000?

**ii** a profit of \$2500?

CHALLENGE

**13** A company manufactures bicycles. The company's revenue equation and cost equation are drawn on the same graph. Which region of the graph ($A$, $B$, $C$ or $D$) is the profit zone?




ALGEBRA

# CHAPTER 4 REVIEW SIMULTANEOUS LINEAR EQUATIONS

You should be able to:

✔ complete tables, plot points and draw graphs of the form $y = mx + c$

✔ model situations using linear equations

✔ generate tables and graph linear models

✔ use technology to graph linear models

✔ identify the solution from tables or graphs when solving linear equations simultaneously

✔ find the point of intersection between two straight-line graphs using technology

✔ solve simultaneous linear equations graphically and interpret the solution

✔ solve practical problems using simultaneous linear equations

✔ determine and interpret the break-even point of a simple business problem where cost and revenue are represented by linear equations.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

4A **1** What are the gradient and $y$-intercept respectively of the line $y = -2x + 3$?

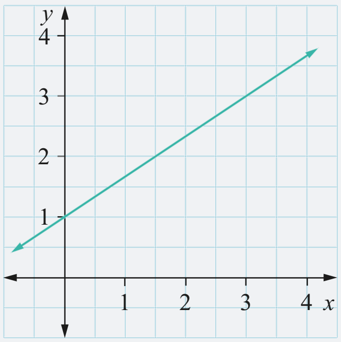
**A** 3 and −2 **B** −2 and 3 **C** −2 and −3 **D** −3 and −2

4A **2** What is the equation of the straight line with gradient of $\frac{1}{2}$ and $y$-intercept −1?

**A** $y = x - \frac{1}{2}$ **B** $y = -x - \frac{1}{2}$

**C** $y = \frac{1}{2}x - 1$ **D** $y = -\frac{1}{2}x + 1$

Use the graph on the right for questions **3** and **4**.




4A **3** What is the gradient of the line shown?

**A** $\frac{2}{3}$ **B** $\frac{3}{2}$

**C** 1 **D** 3

4A **4** What is the equation of the line?

**A** $y = \frac{2}{3}x + 1$ **B** $y = \frac{3}{2}x + 1$

**C** $y = x + \frac{2}{3}$ **D** $y = x + \frac{3}{2}$

4A **5** The Cactus Water Company purifies water. The weekly fixed cost is \$450 and the cost per bottle is \$2. Which equation models this information where $C$ is the total cost and $n$ is the number of bottles?

**A** $C = 2n + 450$ **C** $C = 450n - 2$

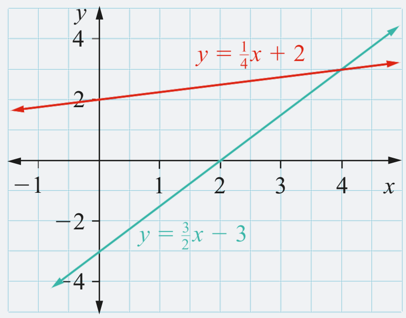
**B** $C = 450n + 2$ **D** $C + 450 = n$

4B **6** The equation $C = 3n + 150$ models the costs for a sandwich shop. What could the 150 represent?

**A** number of sandwiches sold **B** the cost per sandwich

**C** fixed daily cost **D** number of sandwiches made

4C 7 The straight lines for equations $y = \frac{3}{2}x - 3$ and $y = \frac{1}{4}x + 2$ are shown on the graph. What is the solution when the two equations are solved simultaneously?




**A** $x = -8, y = 0$
**B** $x = 0, y = -3$
**C** $x = 3, y = 4$
**D** $x = 4, y = 3$

4C 8 Which pair of simultaneous equations has the solution $x = -5, y = 3$?

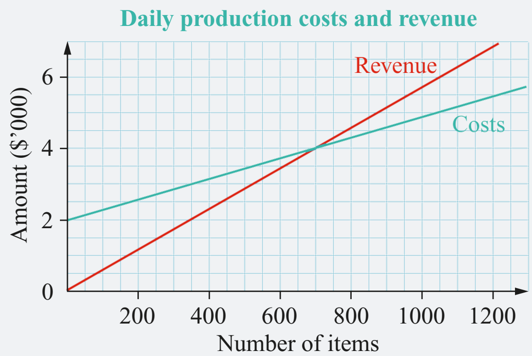
**A** $y = x + 8$ and $4x - 3y = 29$ **B** $y = x + 8$ and $3x - 2y = -21$
**C** $x + y = 2$ and $5x + y = -22$ **D** $y - x = 8$ and $4x - 3y = 26$

4D 9 What is the solution to the simultaneous equations $y = 2x - 1$ and $y = 14 - 3x$?

**A** $x = 5, y = 3$ **B** $x = 3, y = 5$
**C** $x = 2, y = -1$ **D** $x = 14, y = -3$

4E 10 From the graph, determine the number of items that need to be sold for the business to break even.

**A** 4
**B** 4000
**C** 700
**D** 650




# REVIEW SET 1

**1** **a** Find the gradient and $y$-intercept of the line $y = \frac{3}{8}x - 4$.
**b** Write the equation of the line with gradient of 0.08 and $y$-intercept 3.2.

**2** **a** Complete this table of values for $y = 4x + 3$.
**b** Use the table of values to graph $y = 4x + 3$ on a number plane.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**c** Use technology to produce a graph of $y = 4x + 3$ and compare your answer with that for part **b**.

**3** The distance, $d$ kilometres, travelled by a train over time $t$ hours is $d = 70t$.
**a** Complete this table of values for $d = 70t$.

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $d$ | | | | | |

**b** Draw the graph of $d = 70t$.
**c** How far away is the train after $3\frac{1}{2}$ hours?
**d** When is the train 200 km away?

**4** Graphically solve the simultaneous equations $y = \frac{1}{2}x + 1$ and $y = \frac{3}{4}x$.

**5** Solve the equations $y = -3x - 4$ and $x + 2y = 2$ simultaneously using a graphical method.

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**6** The Happy Feet Shoe Company has a weekly fixed cost of \$3725 and a cost of \$7.50 per pair of shoes. The shoes are sold for \$45 a pair.

**a** Graph straight lines modelling the cost and revenue for up to 300 pairs of shows on the same set of axes.

**b** Find the break-even point.

**c** How many pairs of shoes must be sold to break even?

**d** What is the break-even revenue?

**e** Shade the loss zone on the graph you produced in part **a**.

## REVIEW SET 2

**1** By plotting the points (1, 3) and (5, 8), or otherwise, find the gradient of the line through them.

**2** **a** Complete this table of values for $y = 3x + 2$.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Use the table of values to graph $y = 3x + 2$ on a number plane.

**c** Use technology to produce a graph of $y = 3x + 2$ and compare your answer with that for part **b**.

**3** The Smart Tie Company has a fixed cost of \$1400 per day and a variable cost of \$4 per tie. The daily cost can be modelled using the straight-line equation $C = 4n + 1400$.

**a** Explain each term in the equation $C = 4n + 1400$.

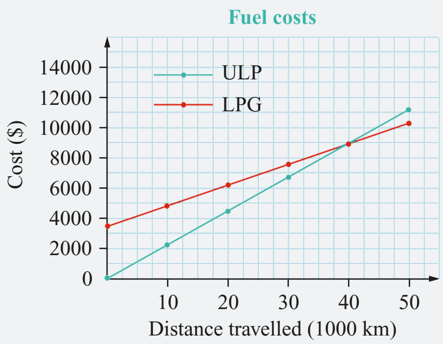
**b** Complete this table of values using the equation $C = 4n + 1400$.

| $n$ | 0 | 50 | 150 | 200 | 250 |
|---|---|---|---|---|---|
| $C$ | | | | | |

**c** Draw the straight-line graph of $C = 4n + 1400$ for $0 \le n \le 250$.

**d** From the graph, determine the number of ties produced when the cost is \$1500.

**4** The fuel cost of driving a large 4WD when running on unleaded petrol (ULP) is shown in the graph on the right. When the vehicle is converted to run on liquid petroleum gas (LPG or autogas), the cost for travelling the same distance was calculated and drawn on the same set of axes. From the graph, estimate the distance travelled before the costs are the same.




**5** Solve the following pairs of simultaneous linear equations using a graphical method.

**a** $y = 2x - 3$ and $y = 6 - x$

**b** $2x + 3y = 0$ and $3x - y = -11$

**6** The Supersheer Stocking Company has a fixed cost of \$1500 per day and a variable cost of \$2 per pair of stockings. This can be modelled by the equation $C = 2n + 1500$, where $C$ is the cost in dollars of producing $n$ pairs of stockings. The revenue of \$12 per pair of stockings sold can be represented by the equation $R = 12n$, where $R$ is the revenue in dollars on the sale of $n$ pairs of stockings.

**a** Graph these two equations on the same set of axes for $0 \le n \le 300$.

**b** Find the break-even point.

**c** How many pairs of stockings must be sold to break even?

**d** What is the break-even revenue?

## REVIEW SET 3

**1** **a** Complete this table of values for $y = 5x - 2$.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Use the table of values to graph $y = 5x - 2$ on a number plane.

**c** Use technology to produce a graph of $y = 5x - 2$ and compare your answer with that for part **b**.

**2** The Silky Scarf Company sells scarves for \$18 each. Sales revenue may be modelled using the equation $R = 18n$.

**a** Explain each term in the equation $R = 18n$.

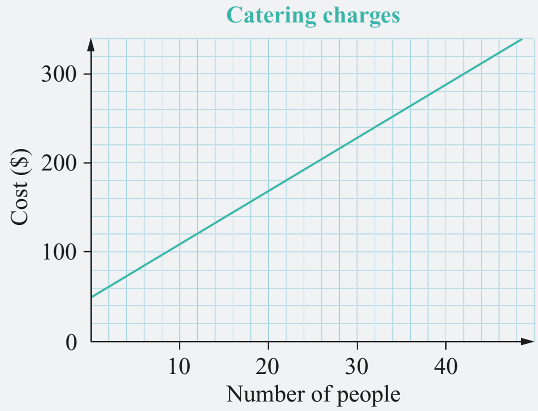
**b** Complete this table of values for $R = 18n$.

| $n$ | 0 | 50 | 100 | 200 | 250 |
|---|---|---|---|---|---|
| $R$ | | | | | |

**c** Draw the straight-line graph of $R = 18n$.

**d** From the graph, find the number of scarves sold when the revenue is \$2070.

**3** This graph models the cost for a company to cater for different numbers of people.




**a** How much would it cost to cater for 35 people?

**b** How many people could be catered for \$300?

**c** Find the gradient. What is its meaning?

**d** Find the intercept on the vertical axis. What is its meaning?

**e** The model for another catering company is represented by the equation $C = 8n$, where $C$ is the cost in dollars to cater for $n$ people. Copy the graph on the right and draw the line for $C = 8n$ on the same set of axes.

**f** For how many people is the cost of catering the same for each company? What is this cost?

**4** Solve the equations $x - 3y = 11$ and $5x - 2y = 16$ simultaneously using a graphical method.

**5** The cost (\$$C$) of hiring a car from company A is given by $C = 60d$, where $d$ is the number of days the car is hired. For company B, the cost of hire is $C = 100 + 40d$.

**a** Draw the graphs of cost versus number of days for each company on the same set of axes, for up to 8 days.

**b** Find the solution of the simultaneous equations $C = 60d$ and $C = 100 + 40d$.

**c** After how many days is the cost the same for both companies?

**6** The Stay Dry Company produces umbrellas. There is a fixed monthly cost of \$2800 and it costs \$7 to produce an umbrella. The umbrellas are sold for \$22 each. The maximum monthly production of umbrellas is 300.

**a** Write two linear equations that can be used to model the production cost and revenue.

**b** Graph straight lines modelling the cost and revenue for $0 \leq n \leq 300$.

**c** Find the point of intersection of the two lines. This is the break-even point.

**d** How many umbrellas must be sold in a month to break even?

**e** What is the break-even revenue?

# REVIEW SET 4

**1** **a** Complete this table of values for $y = 7 - 2x$.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Use the table of values to graph $y = 7 - 2x$ on a number plane.

**c** Use technology to produce a graph of $y = 7 - 2x$ and compare your answer with that for part **b**.

**2** The Tropical Delight Fruit Juice Company has a weekly fixed cost of $400. The cost per cup of juice is $1.75.

**a** Write an equation modelling this information. Use $C$ for cost and $n$ for the number of cups of juice.

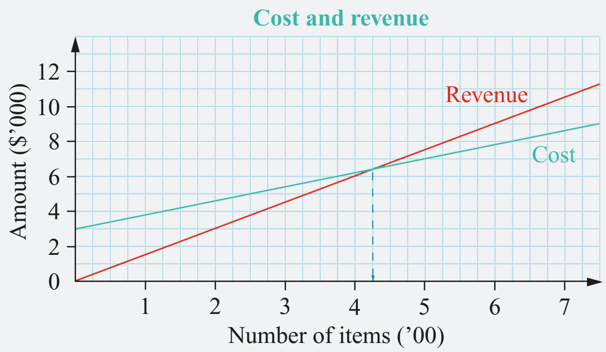
**b** Complete this table of values using the cost equation from part **a**.

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $C$ | | | | | | |

**c** Draw the straight-line graph of your cost equation using the values from the completed table in part **b**.

**d** Use the graph to find the number of cups of juice made when the cost is $1000.

**3** A plastics company has an initial cost of $3000 per day plus $8 per item produced. The selling price of the items is $15 each, and the maximum daily production is 1000 items. The revenue and cost models are represented in the graph on the right.




**a** How many items need to be sold for the company to break even?

**b** What is the break-even revenue?

**4** **a** Draw the graphs of $y = 8 - x$ and $2x + 3y = 23$ on the same number plane.

**b** Find the point of intersection of the two lines.

**c** Hence write the solution to the simultaneous equations $y = 8 - x$ and $2x + 3y = 23$.

**5** The cost ($\$C$) of electricity to run a home is $200 per month. This may be written as $C = 200m$, where $m$ is the number of months. For the same house, the cost of installing a solar system is $4000, but the monthly cost of electricity decreases to $150. The total cost of electricity using a solar system is then $C = 4000 + 150m$.

**a** Use graphing technology or complete the following tables to graph each equation on the same number plane.

$C = 200m$

| $m$ | 0 | 50 | 100 |
|---|---|---|---|
| $C$ | | | |

$C = 4000 + 150m$

| $m$ | 0 | 50 | 100 |
|---|---|---|---|
| $C$ | | | |

**b** Find the simultaneous solution of the linear equations.

**c** After installing the solar system, how long would it take to start saving money?

# REVIEW PRACTICE EXAMINATION QUESTION

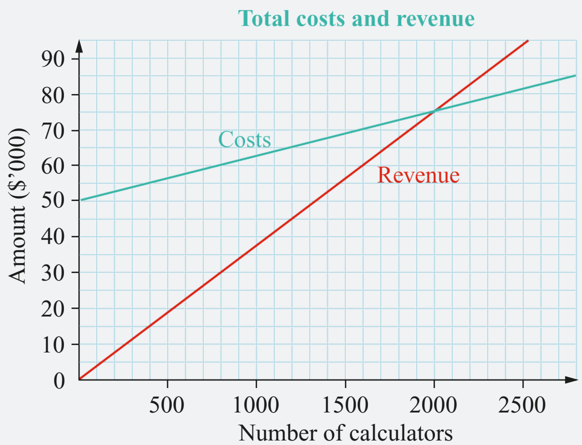
**1** **a** **i** Complete this table and graph $y = 3 - 2x$. (2 marks)

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**ii** What is the gradient of the line? (1 mark)

**iii** What is the $y$-intercept of the line? (1 mark)

**b** The graph shows the total costs and revenue for a manufacturer of calculators.




**i** Find the total cost to the manufacturer of producing 1000 calculators. (1 mark)

**ii** Will the manufacturer make a profit when 1000 calculators are produced? Give reasons for your answers. (1 mark)

**iii** How many calculators must be produced for the manufacturer to break even? (1 mark)

**iv** How many calculators must be produced to make a profit of $10 000? (1 mark)

**v** What is the initial set-up cost to the manufacturer? (1 mark)

**vi** What is the cost to produce each calculator, after the initial set-up costs? (1 mark)

**c** The Creative Cake Company makes cupcakes. The cost of production is $200 per day and the variable production cost is $2.50 per cupcake. Cupcakes are sold for $7.50 each. Maximum daily production is 100 cupcakes.

**i** Write an equation for the cost $\$C$ of producing $n$ cupcakes. (1 mark)

**ii** Write an equation for the revenue $\$R$ of selling $n$ cupcakes. (1 mark)

**iii** Graph both equations for $0 \le n \le 100$ on the same set of axes. (1 mark)

**iv** How many cupcakes must be sold to break even? (1 mark)

**v** If all 100 cupcakes are sold, how much profit is made? (1 mark)

TOTAL:
15 marks

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# 5

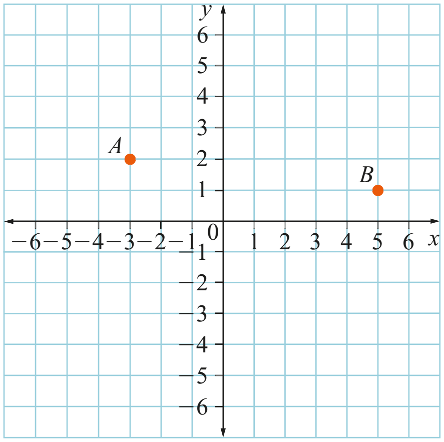
# Bivariate data analysis

The main mathematical ideas investigated are:

- illustrating bivariate data on a scatterplot
- identifying independent and dependent variables within bivariate datasets
- identifying potential outliers in a bivariate dataset
- describing the association between bivariate datasets displayed in a scatterplot in terms of form, direction and strength
- interpreting the value of Pearson's correlation coefficient
- drawing a line of best fit by eye
- adding the regression line to a scatterplot using technology
- interpreting the gradient and vertical intercept of the regression line
- using interpolation and extrapolation to make predictions
- identifying questions that can be answered using statistics
- understanding the statistical investigation process
- demonstrating an awareness around privacy, ethics and bias in statistical investigations.

# ARE YOU READY?

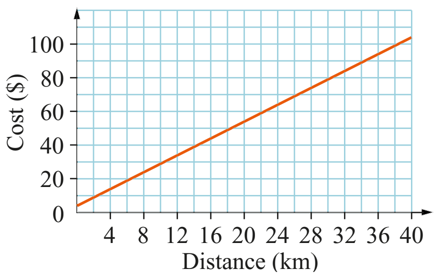
5A **1** What are the coordinates of points $A$ and $B$ respectively in this graph?




**A** (2, 3) and (5, 1) **B** (−2, 3) and (1, 5)
**C** (2, −3) and (5, 1) **D** (−3, 2) and (5, 1)

5A **2** The following graph shows the relationship between distance and cost for hiring a taxi. Which table also details this relationship?

**Taxi hire charges**




**A**

| Distance (km) | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Cost ($) | 4 | 29 | 54 | 79 | 104 |

**B**

| Distance (km) | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Cost ($) | 104 | 79 | 54 | 29 | 4 |

**C**

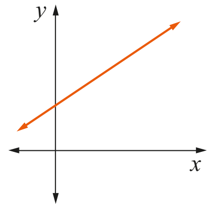
| Distance (km) | 4 | 29 | 54 | 79 | 104 |
|---|---|---|---|---|---|
| Cost ($) | 0 | 10 | 20 | 30 | 40 |

**D**

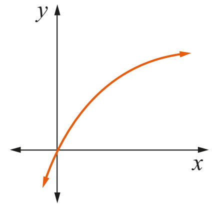
| Distance (km) | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|
| Cost ($) | 8 | 58 | 108 | 158 | 208 |

5A **3** Which of the following graphs shows a linear relationship between the two variables?

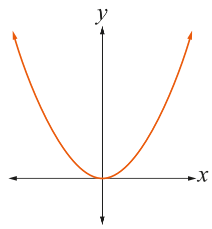
**A**



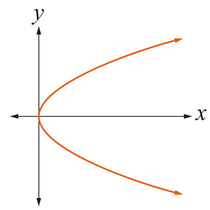
**B**



**C**

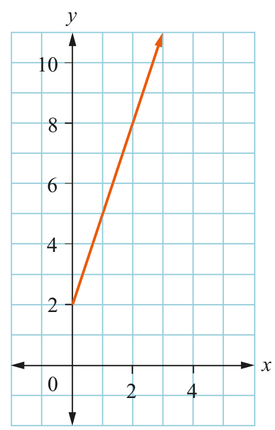


**D**



5C **4** What is the gradient of this line?

**A** 3
**B** −3
**C** $\frac{1}{3}$
**D** $-\frac{1}{3}$




5C **5** What is the gradient of the line $y = 4x - 3$?

**A** 4
**B** −4
**C** −3
**D** 3

5C **6** What is the $y$-intercept of the line $y = -2x + \frac{1}{2}$?

**A** −2 **B** 2 **C** $\frac{1}{2}$ **D** $-\frac{1}{2}$

5D **7** Given the equation $y = 3x - 5$, when $x = 3$, what is the value of $y$?

**A** 28 **B** 4 **C** −4 **D** 3

5D **8** Given the equation $y = -2x + 7$, when $y = 11$, what is the value of $x$?

**A** −15 **B** −9 **C** 2 **D** −2

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1** **Support sheet 5A.1 The Cartesian plane (four quadrants)**
**Q2–3** **Support sheet 5A.2 Plotting relationships**
**Q4–6** **Support sheet 5C.1 Finding the gradient of a linear graph**
**Q4–6** **Support sheet 5C.2 Linear graphs**
**Q7–8** **Support sheet 5D.1 Substituting into and solving simple equations**

# 5A Bivariate scatterplots

These resources are available on your obook assess:
- **Worksheet 5A:** Practise your skills with independent and dependent variables
- **assess quiz 5A:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

The aim of many statistical investigations is to determine whether there is a relationship between the two variables being investigated. For instance, medical researchers might be interested in the relationship, if any, between the amount of a drug administered and the number of patients cured. A business enterprise might be interested in the relationship, if any, between the amount of money spent on advertising and the change in sales.

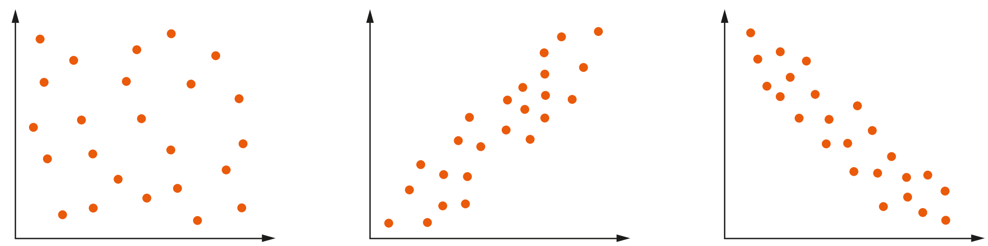
In this chapter we will investigate a way of illustrating data so that a relationship, if it does exist, can be seen. A simple method of illustrating numerical data that relates two variables is to plot the data as ordered pairs on a number plane. The resulting diagram is known as a **scatterplot**.

**scatterplot**
a diagram using Cartesian coordinates to display data values that relate two variables

**correlation**
a measure of the extent to which two variables are related

**bivariate datasets**
a dataset involving two different variables for each data point

In general, if the points are scattered at random over the grid, as in Graph A below, the variables are not mathematically related. If the points are scattered along a straight line, as in Graphs B and C, there may be a mathematical relationship between the two variables. The strength of this relationship, or how closely linked the variables are, is called **correlation**. Correlation will be investigated in more detail in topic 5B.



Graph A Graph B Graph B

When dealing with **bivariate datasets**, we are often working with an independent variable and a dependent variable. We can investigate how the dependent variable changes in response to the independent variable; the expected value of the dependent variable *depends* on the value of the independent variable.

When plotting a bivariate dataset with an independent and dependent variable, we put the independent variable on the horizontal axis and the dependent variable on the vertical axis.

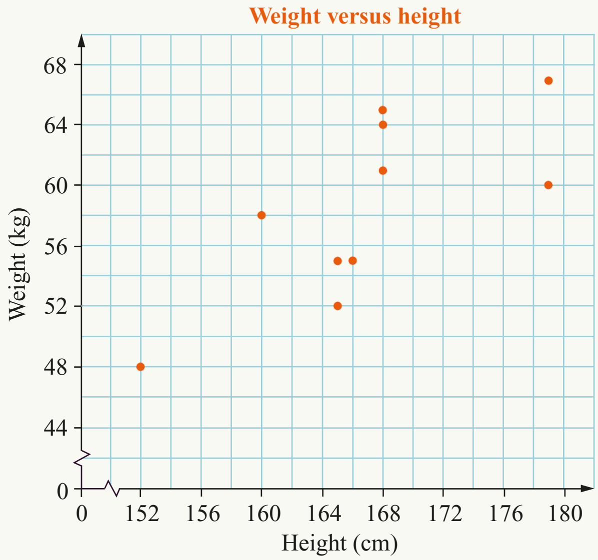
If both variables in a bivariate dataset are independent, then either variable can be put on the horizontal and vertical axes.

## EXAMPLE 5A–1 Illustrating data on a scatterplot

The heights and weights of 10 students were measured and the results shown in the table below.

| Student | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Height (cm) | 179 | 165 | 160 | 179 | 152 | 168 | 168 | 165 | 168 | 166 |
| Weight (kg) | 60 | 55 | 58 | 67 | 48 | 64 | 61 | 52 | 65 | 55 |

Illustrate this data on a scatterplot and determine whether there is a possible relationship between the two variables.

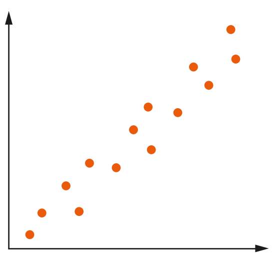
| Solve | Think/Apply |
| --- | --- |
|  <br>From the distribution of points plotted, there appears to be a trend that as height increases so does weight. This might indicate a relationship between these two variables, but because of the scatter of the points, there does not appear to be a strong link. There would not appear to be a mathematical relationship that would allow the weight of a student to be accurately predicted from his or her height. | The data points are plotted as ordered pairs with height on the horizontal axis and weight on the vertical axis.<br>For student 1, go across to 179 on the horizontal axis then up to 60 on the vertical axis and plot the point. Repeat for the other students. |

## EXERCISE 5A Bivariate scatterplots

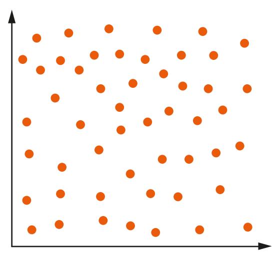
UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** State whether or not there appears to be a linear relationship between the variables plotted on these scatterplots.

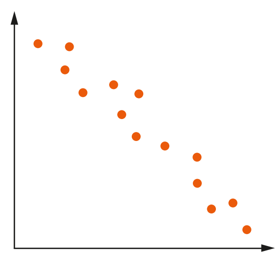
**a**



**b**



**c**



**2** **i** Draw scatterplots for the data in the tables.

**ii** Comment on any possible linear relationship between the variables.

**a**

| $x$ | 40 | 50 | 60 | 70 | 80 | 90 |
| --- | --- | --- | --- | --- | --- | --- |
| $y$ | 42 | 47 | 51 | 59 | 62 | 72 |

**b**

| $x$ | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| --- | --- | --- | --- | --- | --- | --- | --- |
| $y$ | 12 | 17 | 19 | 24 | 29 | 31 | 37 |

**c**

| $x$ | 30 | 40 | 45 | 50 | 60 | 70 | 80 | 90 |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $y$ | 45 | 35 | 85 | 50 | 50 | 40 | 90 | 60 |

STATISTICAL ANALYSIS

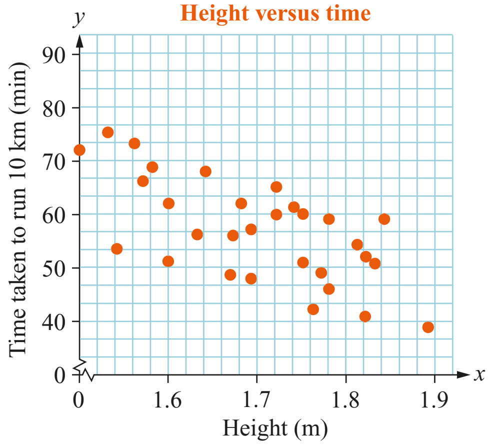
## EXAMPLE 5A–2 Identifying the independent and dependent variables

Cathy wants to investigate if there is a relationship between a person's height and their shoe size. Identify the independent variable and the dependent variable in this situation.

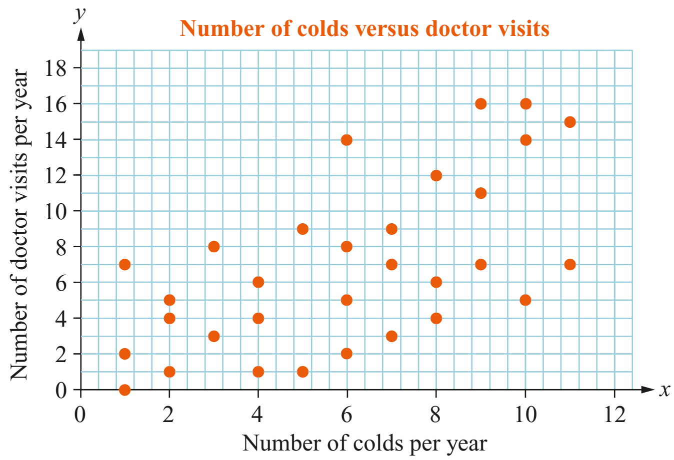
| Solve | Think | Apply |
| --- | --- | --- |
| Independent variable = Height<br>Dependent variable = Shoe size | A person's shoe size is likely to vary depending on the height of the person, so shoe size is the dependent variable and height is the independent variable. | Identify which of the two variables is likely to change depending on the value of the other variable. This is the dependent variable. |

**3** Identify the independent variable and the dependent variable in these bivariate scatterplots.

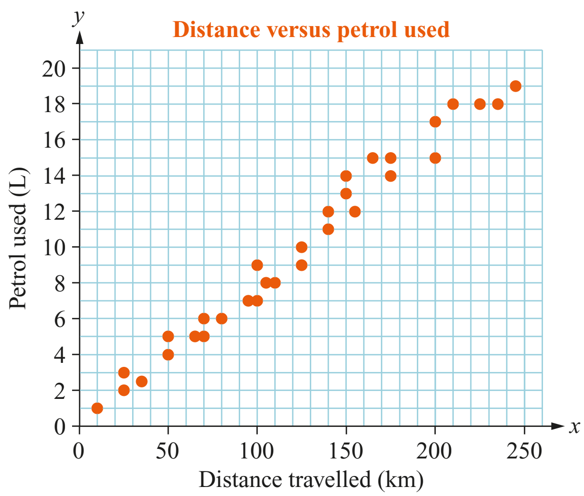
**a**



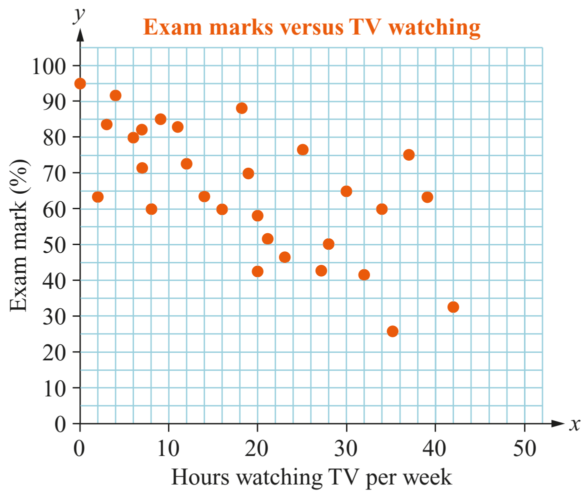

**b**




**c**




**d**




**4** Mika wants to investigate how the amount of sugar in a person's diet affects the size of their waistline. Identify the independent and dependent variables in this situation.

**5** Jelena is looking into the relationship between an NBL player's height and the average number of points they score per game. Identify the independent and dependent variables in this situation.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

An **outlier** is a data point that appears to be an anomaly in a dataset.

When you identify an outlier it makes sense to interrogate it. Some outliers are genuine; for example, one student may score significantly higher in a mathematics test than all of the other students in their class. Other outliers may be the result of a recording or input error. Unless there is a strong reason to believe they are the result of an error, outliers should not be removed from a dataset.

!

**outlier**
a data point that appears to be inconsistent with the other data points in a dataset

**6** The results of a group of students in History and Geography tests are compared.

| Student | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| History test ($H$) | 84 | 53 | 63 | 74 | 68 | 79 | 70 | 61 |
| Geography test ($G$) | 80 | 62 | 75 | 88 | 70 | 54 | 82 | 70 |

**a** Illustrate the given data on a scatterplot.
**b** Which student's results appear to be an outlier?
**c** Is there any reason to believe that this outlier may be the result of an error? Explain your answer.

**7** In your own words explain the difference between an independent variable and a dependent variable.

**8** Javi is using the Australian Bureau of Statistics website to write a report on the life expectancy of females in NSW over time. He collects the following data. The time is measured in years since 1971.

| Year | 1971 | 1976 | 1981 | 1986 | 1991 | 1996 | 2001 | 2006 | 2011 |
|---|---|---|---|---|---|---|---|---|---|
| Time (years) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| Life expectancy | 74.4 | 76.2 | 78.3 | 78.8 | 80.2 | 81.2 | 82.6 | 83.7 | 84.2 |

**a** Identify the independent and dependent variables in this situation.
**b** Draw a scatterplot for this data.
**c** Does there seem to be a linear relationship between the two variables?

SPREADSHEET

A spreadsheet can be used to easily construct a scatterplot by following these steps.

*Step 1*: Put the data values into the spreadsheet, with the independent variable in the first column and the dependent variable in the second column.

*Step 2*: Highlight all of the data values in the spreadsheet, including the two column headings.

*Step 3*: From the Insert menu select X-Y Scatter Chart type.

**9** The data in the table gives the life expectancy, $L$, of Australian males for the years 1881 to 2001. The time, $T$, is measured in years since 1881.

| Year | 1881 | 1901 | 1921 | 1947 | 1961 | 1981 | 2001 |
|---|---|---|---|---|---|---|---|
| Time (years) | 0 | 20 | 40 | 60 | 80 | 100 | 120 |
| Life expectancy (years) | 47.2 | 55.2 | 59.2 | 66.1 | 67.9 | 71.4 | 77.4 |

**a** Draw a scatterplot comparing $T$ and $L$.
**b** Does there seem to be a linear relationship between the two variables? If so, explain the relationship.

# 5B Correlation

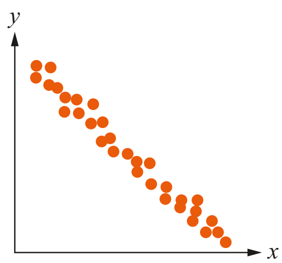
These resources are available on your obook assess:
- **Video tutorial 5B:** Watch and listen to an explanation of Example 5B–1
- **Worksheet 5B:** Practise your skills with scatterplots
- **assess quiz 5B:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

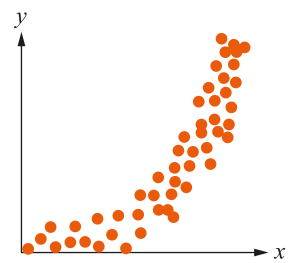
We can describe a bivariate dataset in terms of form, which means determining whether the relationship between the two variables is linear or non-linear.

If the relationship between the two variables is linear, then, in general, as the value of one variable increases, the value of the other variable will increase or decrease at a constant rate.

**Linear relationship**




**Non-linear relationship**



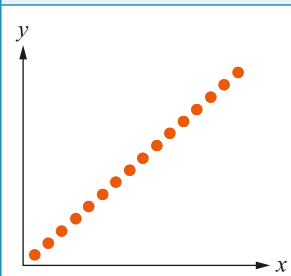

If a relationship between two variables is linear, then we can further describe it in terms of direction (positive and negative) and strength (strong, moderate or weak).

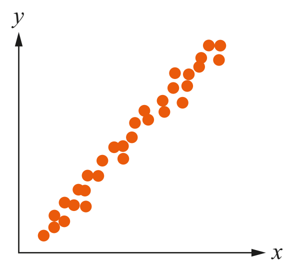
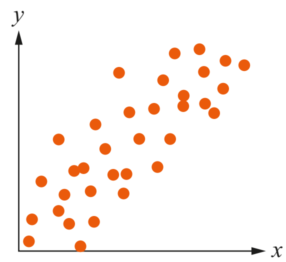
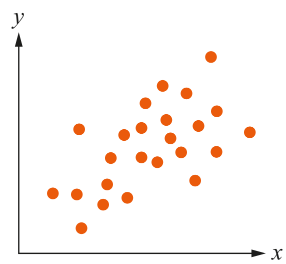
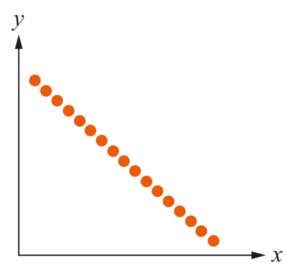
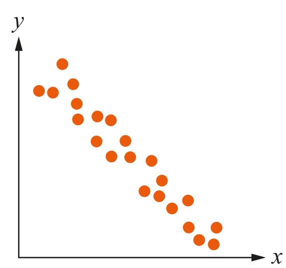
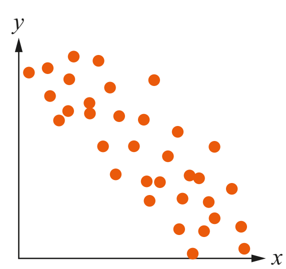
**Pearson's correlation coefficient ($r$)** a statistical measure of the strength of the linear relationship between two numerical variables

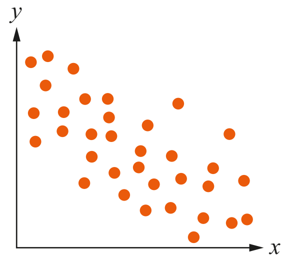
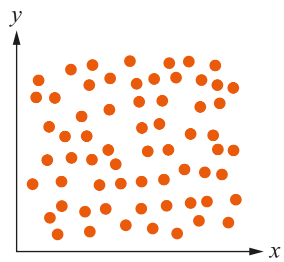
A positive relationship means that as one variable increases, the other variable increases, whereas a negative relationship means that as one variable increases, the other variable decreases.

The strength of the correlation can be measured by calculating **Pearson's correlation coefficient ($r$)**, which assigns a numerical value between −1 and 1 to the correlation. The methods of determining the numerical value of $r$ by hand are beyond the scope this course, but we can use technology to calculate $r$.

Positive values of $r$ represent a positive correlation between the two variables; negative values of $r$ represent a negative correlation between the two variables.

| Type of correlation | Correlation coefficient ($r$) values | Example scatterplot | Description |
|---|---|---|---|
| Perfect positive correlation | $r = 1$ |  | A perfect positive correlation, in which all of the points lie in a straight line, has a correlation coefficient $r = 1$. |

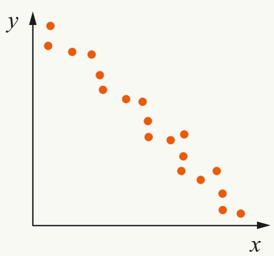
| Type of correlation | Correlation coefficient ($r$) values | Example scatterplot | Description |
|---|---|---|---|
| Strong positive correlation | $0.75 \leq r < 1$ |  | A strong positive correlation has a value of $r$ greater than or equal to 0.75 but less than 1. |
| Moderate positive correlation | $0.5 \leq r < 0.75$ |  | A moderate positive correlation has a value of $r$ greater than or equal to 0.5 but less than 0.75. |
| Weak positive correlation | $0.25 \leq r < 0.5$ |  | A weak positive correlation has a value of $r$ greater than or equal to 0.25 but less than 0.5. |
| Perfect negative correlation | $r = -1$ |  | A perfect negative correlation, in which all of the points lie in a straight line, has a correlation coefficient $r = -1$. |
| Strong negative correlation | $-1 < r \leq -0.75$ |  | A strong negative correlation has a value of $r$ less than or equal to $-0.75$ but greater than $-1$. |
| Moderate negative correlation | $-0.75 < r \leq -0.5$ |  | A moderate negative correlation has a value of $r$ less than or equal to $-0.5$ but greater than $-0.75$. |

| Type of correlation | Correlation coefficient ($r$) values | Example scatterplot | Description |
|---|---|---|---|
| Weak negative correlation | $-0.5 < r \leq -0.25$ |  | A weak negative correlation has a value of $r$ less than or equal to $-0.25$ but greater than $-0.5$. |
| No correlation | $-0.25 < r < 0.25$ |  | No upwards or downwards trend with a value of $r$ between $-0.25$ and $0.25$. |

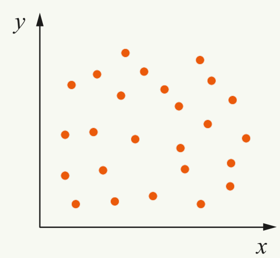
## EXAMPLE 5B–1 Determining the strength of correlation via a scatterplot

State whether the pairs of variables graphed below have perfect, strong, moderate, weak or no correlation.

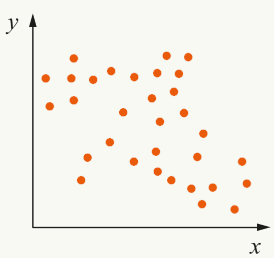
**a**



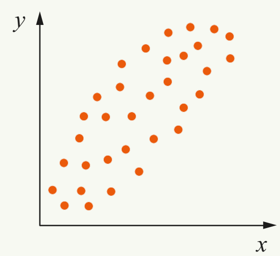

**b**




**c**




**d**

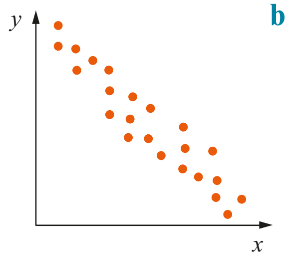


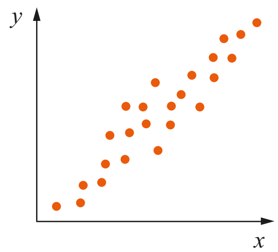

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Strong correlation | All the points are marginally off a straight line, so this is a strong correlation. | For a perfect correlation the points will lie in a straight line.<br>For a strong correlation the points will be marginally off appearing in a straight line.<br>For a moderate correlation there will be an obvious trend in the dataset.<br>For a weak correlation there will be a slight trend in the dataset.<br>For no correlation the points will appear at random. |
| **b** | No correlation | The points appear at random, so there is no correlation. | |
| **c** | Weak correlation | There is a general negative trend, although the points are widely spread, so this is a weak correlation. | |
| **d** | Moderate correlation | There is a definite positive trend with the points being slightly spread, so this is a moderate correlation. | |

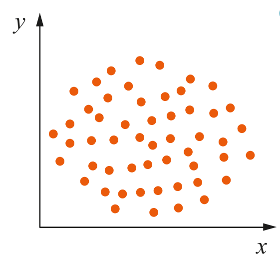
UNDERSTANDING, FLUENCY AND COMMUNICATING

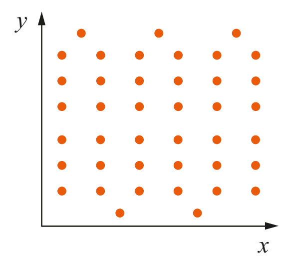
## EXERCISE 5B Correlation

**1** For each of these scatterplots, state whether the correlation coefficient is positive, negative or approximately zero. Give reasons for your answers.

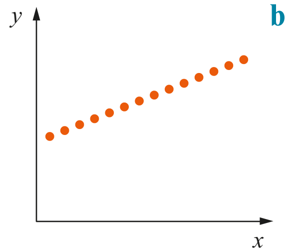
**a**


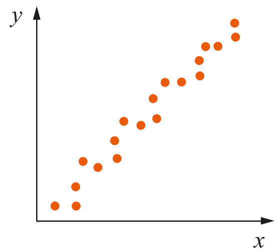
**b**


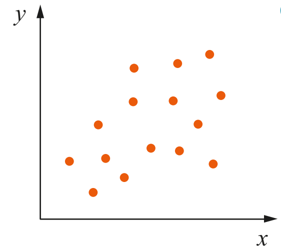
**c**


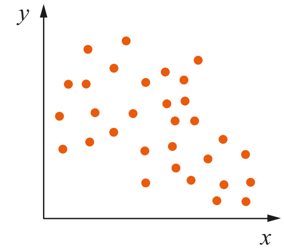
**d**


**2** State whether the pairs of variables graphed below have perfect, strong, moderate, weak or no correlation.

**a**


**b**


**c**


**d**


**3** Draw a scatterplot for two variables that have the following correlations.

**a** strong positive
**b** weak negative
**c** perfect negative
**d** no correlation
**e** perfect positive
**f** moderate negative

### EXAMPLE 5B–2 Interpreting Pearson's correlation coefficient

The members of a sporting club are using a new method of training to increase thigh circumference. Each fortnight, the thigh circumference of each member is measured. At the conclusion of the training period, the correlation coefficient comparing thigh circumference with time is 0.73.

**a** What does this mean?
**b** How strong is the correlation?

| | Solve/Think | Solve/Think | Apply |
|---|---|---|---|
| **a** | The new training increases thigh circumference. | The training method is working to increase thigh circumference; 0.73 indicates a moderate positive correlation. | A positive correlation means that as one variable increases the other increases. A negative correlation means that as one variable increases the other decreases. |
| **b** | The correlation is moderate and positive. | | |

STATISTICAL ANALYSIS

**4** A gymnasium is using a new training program to increase bicep circumference. Each fortnight the bicep circumference of each person is measured. At the end of the program, the bicep measurements are compared with time. The correlation coefficient is 0.64. What does this mean? Explain your answer.

**5** Ten people are on a 6-week diet to lose weight. Their weight is recorded every week. The correlation coefficient of the data is $-0.77$. Does this show that the diet works? Explain the meaning of $r = -0.77$.

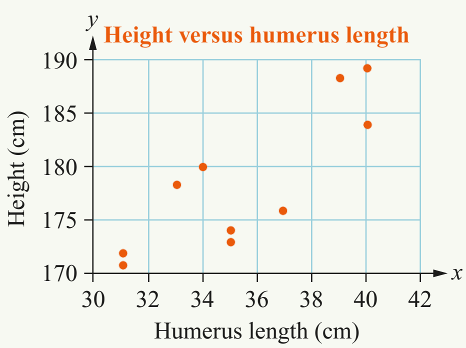
## EXAMPLE 5B–3 Using a spreadsheet to calculate the correlation coefficient

For a group of girls, the humerus length (elbow to shoulder) was measured and compared with height. The results are listed in this table.

| Humerus length (cm) | 37 | 35 | 40 | 31 | 35 | 33 | 31 | 40 | 34 | 39 |
|---|---|---|---|---|---|---|---|---|---|---|
| Height (cm) | 176 | 174 | 184 | 172 | 173 | 178 | 171 | 189 | 180 | 188 |

**a** Enter this data into a spreadsheet and calculate $r$.

**b** Draw a scatterplot of the data.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $r = 0.836$ | Put the data into two columns. Use =CORREL(A2:A11, B2:B11). | Put the heading into first row. Use the built-in formula. |
| **b** |   | From the Insert menu select Scatterplot then the first scatterplot type. | |

**6** The data in the table gives the average height ($H$), in centimetres, of 17-year-old boys for the years 1850 to 1990. The time ($T$) is measured in years since 1850.

| Year | 1850 | 1870 | 1890 | 1910 | 1930 | 1950 | 1970 | 1990 |
|---|---|---|---|---|---|---|---|---|
| Time (years) | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 |
| Height (cm) | 153 | 155 | 159 | 161 | 164 | 168 | 171 | 175 |

**a** Enter this data into a spreadsheet and use the spreadsheet to calculate $r$.

**b** Draw a scatterplot of the data.

**7** The data in the table gives the average height ($H$), in centimetres, of 17-year-old girls for the years 1850 to 1990. The time ($T$) is measured in years since 1850.

| **Year** | **1850** | **1870** | **1890** | **1910** | **1930** | **1950** | **1970** | **1990** |
|---|---|---|---|---|---|---|---|---|
| **Time (years)** | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 |
| **Height (cm)** | 147 | 149 | 155 | 158 | 160 | 161 | 164 | 170 |

**a** Enter this data into a spreadsheet and use the spreadsheet to calculate $r$.

**b** Draw a scatterplot of the data.

## EXAMPLE 5B–4 Discussing the expected correlation between variables

Discuss the expected strength of the relationship (correlation) between these variables.

**a** speed and distance travelled

**b** age and weight of a baby, up to 12 months of age

**c** height and weight of 18-year-old girls

**d** height of 18-year-old girls and mark in Mathematics in the HSC exam

| | Solve/Think | Apply |
|---|---|---|
| **a** | As speed increases, there is a proportional increase in the distance travelled. This is an example of perfect positive correlation. | Sometimes both quantities increase or decrease but are unrelated; that is, there is no correlation. |
| **b** | As a baby's age increases so does its weight. However, this will happen at different rates for different babies; hence, this is an example of strong positive correlation. | |
| **c** | In general, taller girls weigh more than shorter girls; that is, larger heights are associated with larger weights and smaller heights are associated with smaller weights, but there are many exceptions. This is an example of weak positive correlation. | |
| **d** | There is no reason to suspect that there is any relationship between these two variables; that is, height will have no bearing on performance in the HSC or vice versa. This is an example of no correlation. | |

**8** Discuss the expected strength and direction of the relationship between the following variables.

**a** the distance travelled and the cost for a taxi journey

**b** the volume of water remaining in a tank and the time the tap is on

**c** the number of police cars and the number of accidents on a highway

**d** the height and shoe size of male adults

**e** the age of cars and their price

**f** the number of sunny days and the sales of umbrellas for a month

**g** the speed of a car and the stopping distance

**h** family income and the number of family pets

**i** lengths of left arm and right arm of people

**j** eyesight and age

**k** hours spent studying and examination marks

**l** smoking and lung cancer

**cause and effect** the result of the change in one variable is directly responsible for the change in another variable

**spurious correlation** two variables are mathematically related, but the relationship is not due to cause and effect

**coincidence** two or more events occurring simultaneously through natural random variation

!

A high degree of correlation between two variables does not necessarily imply that one causes the other.

A **cause and effect** relationship exists only when the change in one variable causes a direct change in another variable.

A **spurious correlation** is one in which two variables are correlated, but the correlation occurs either due to a third related variable, or due to **coincidence**.

## EXAMPLE 5B–5 Identifying why a relationship between two variables exists

Comment on the following findings. Discuss whether a cause and effect relationship exists, or whether it is a case of spurious correlation.

**a** The heights and reading speeds of children were measured and a strong positive correlation was found.

**b** The number of televisions sold in Newcastle and the number of stray dogs in Wollongong were recorded over several years and a strong positive correlation was found between these variables.

| | Solve/Think | Apply |
|---|---|---|
| **a** | Increases in height were associated with increases in reading speed. However, height does not affect reading speed and reading speed does not affect height. The strong correlation may be attributed to the fact that both variables are closely linked to a third variable, age. That is, as age increases, so do height and reading speed. This is an example of spurious correlation. | These are examples of spurious correlation. The strong correlation occurs because of the existence of a third related variable, or because both variables happen by chance to be increasing or decreasing at the same time. When variables are related such that one variable does cause an effect on the other (i.e. if one is changed, the other will change), we say that a causal relationship exists. |
| **b** | Obviously an increase in the number of televisions sold in Newcastle does not cause an increase in the number of stray dogs in Wollongong, or vice versa. Both variables must have simply happened to be increasing over this period (i.e. coincidentally). This is an example of spurious correlation. | |

**9** The following pairs of variables were measured and a moderate or strong positive correlation between them was found. Discuss whether a cause and effect relationship exists, or whether it is a case of spurious correlation.

**a** the length of a person's left arm and right foot
**b** company expenditure on advertising and sales
**c** daily temperature and ice-cream consumption
**d** the damage caused by a fire and the number of firemen who attend the fire
**e** the number of people unemployed and the price of eggs
**f** the height of parents and the height of adult offspring
**g** the number of hotels and the number of churches in rural towns

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**10** Ms Marker is looking at the amount of time students spent studying and the mark they achieve in the end-of-year Mathematics exam. She records the information in the following table.

| **Student** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Time spent studying (hours)** | 7 | 9 | 14 | 12 | 8 | 3 | 11 | 15 | 5 | 9 |
| **Final mark** | 64 | 67 | 80 | 92 | 73 | 88 | 77 | 89 | 59 | 75 |

**a** Enter this data into a spreadsheet and use the spreadsheet to calculate $r$.

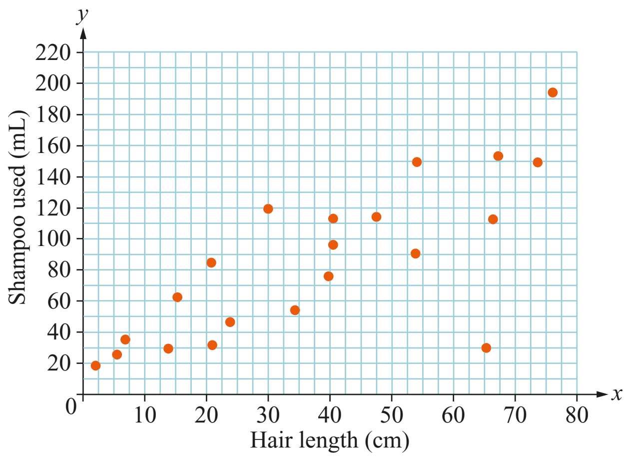
**b** Draw a scatterplot of the data.

**c** Do any of the data points appear to be outliers?

**d** Remove the potential outlier(s) from the dataset and recalculate the value of $r$.

**e** Ms Marker suggests that the outlier should be removed from the dataset. Explain the situations in which it should be removed.

**11** The following scatterplot shows the relationship between the length of hair and the amount of shampoo used each week for 20 adults.




**a** Describe the relationship between the variables in terms of strength and direction.

**b** There is an outlier in the data set. Is there a reason to believe that this outlier might be the result of an error? Explain your answer.

**c** How would removing the outlier from the data set affect the value of Pearson's correlation coefficient?

RESEARCH

**12** Measure the height and foot length of 15–20 students from your school. Plot the data you collect as a scatterplot and calculate the value of $r$. Write a paragraph interpreting the correlation and any possible relationship between height and foot length.

**13** Use the ABS website to write a paragraph on the average weekly earnings of Australian full-time workers over the past 5 years. Calculate the correlation coefficient between time and earnings, and interpret this in your report.

**14** Find a news article that reports on a suggested correlation or connection between two variables. Estimate the value of the correlation based on the article, and discuss whether a cause and effect relationship exists, or whether it is a case of spurious correlation.

# 5C Lines of best fit

These resources are available on your obook assess:

- **Spreadsheet 5C**: Draw a scatterplot with a regression line
- **assess quiz 5C:** Test your skills with an auto-correcting multiple-choice quiz

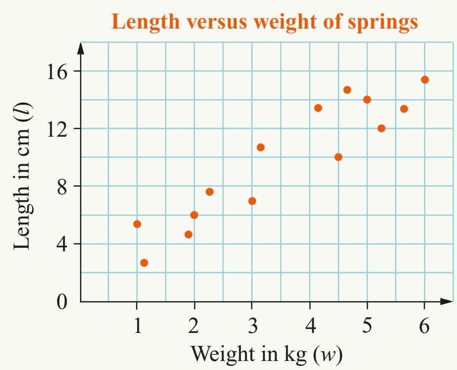
**line of best fit** a straight line that provides a representation of all of the data points in a scatterplot; also known as the trendline

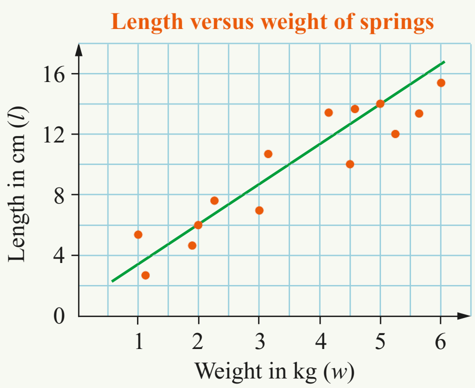
If a pair of variables appears to be related, as indicated by a linear pattern of dots on a scatterplot, then we can draw a straight line that fits the points. This line is known as the **line of best fit** or the trendline.


The simplest way to draw a line of best fit is by eye. The objective is to fit a line to the scatterplot so that the line is as close to as many data points as possible. An approximately equal number of data points should lie above and below the line of best fit, with the distance of the points from the line being minimised. The line of best fit does not need to pass through any data points.

## EXAMPLE 5C–1 Drawing a line of best fit by eye

**a** Draw a line of best fit on the scatterplot.



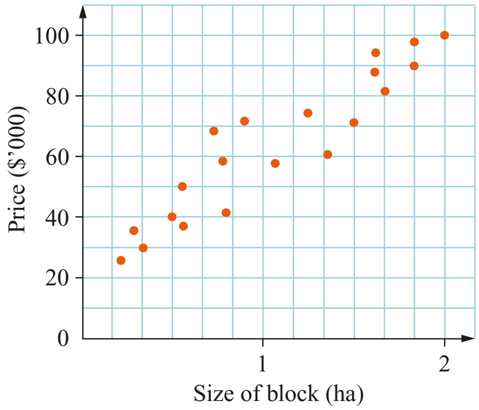

| | Solve | Think | Apply |
|---|---|---|---|
| **a** |   | The line must have about the same number of dots above and below it. | The line need not pass through any of the points but must balance the points above and below it. |

# EXERCISE 5C Lines of best fit

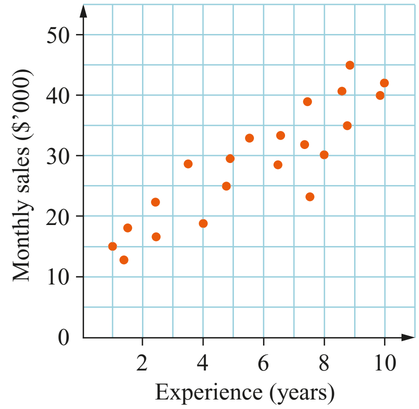


**1** Draw a line of best fit on each of the scatterplots below.

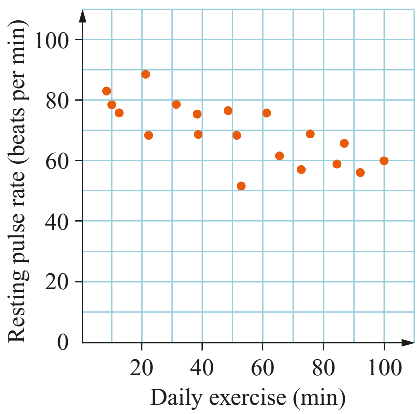
**a**



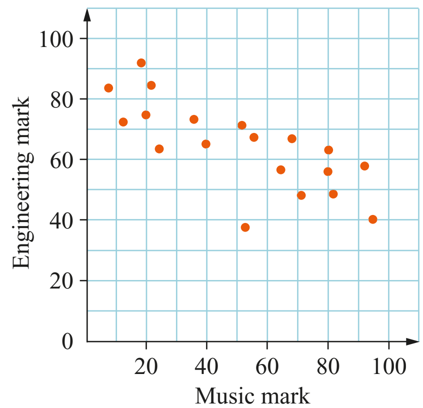

**b**




**c**

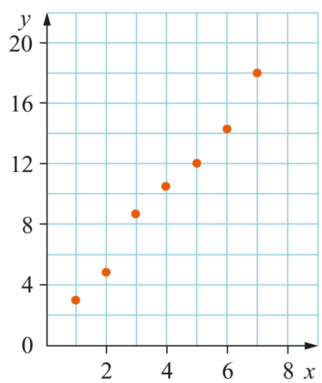



**d**

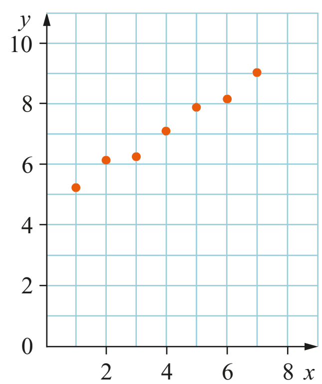



**2** Draw the line of best fit for these bivariate datasets and describe the relationship in terms of direction (positive/negative) and strength (perfect/strong/moderate/weak/no correlation).

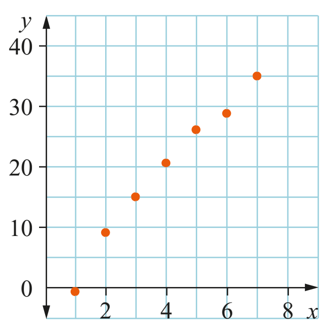
**a**



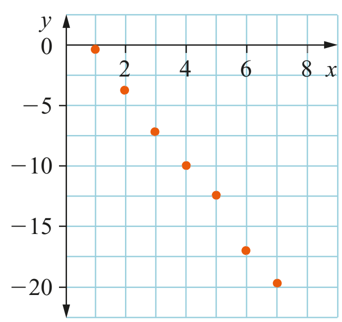

**b**



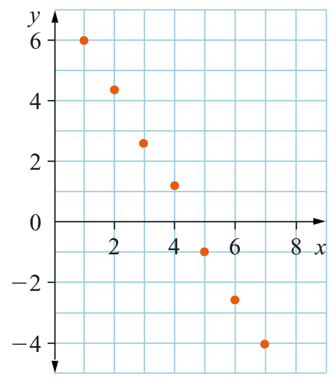

**c**



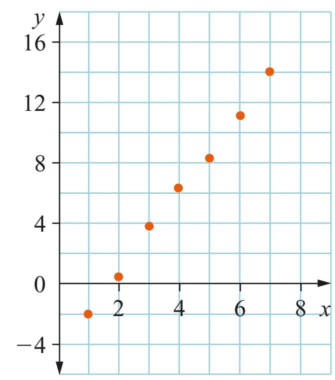

**d**



**e**




**f**




**3** Draw a scatterplot and line of best fit for the following data.

**a**

| $x$ | 100 | 120 | 125 | 140 | 170 | 180 | 190 | 210 | 220 | 240 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 90 | 85 | 100 | 90 | 100 | 115 | 105 | 125 | 110 | 120 |

**b**

| $x$ | 10 | 14 | 20 | 22 | 28 | 35 | 38 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 9 | 15 | 16 | 13 | 24 | 20 | 29 | 22 | 27 |

**c**

| $x$ | 8 | 14 | 17 | 17 | 22 | 27 | 30 | 33 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 11 | 14 | 20 | 16 | 22 | 29 | 28 | 35 |

**d**

| $x$ | 5 | 10 | 14 | 15 | 24 | 27 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 10 | 11 | 12 | 14 | 15 | 17 | 20 | 20 |

**e**

| $x$ | 86 | 95 | 100 | 90 | 96 | 105 | 94 | 98 | 110 | 100 | 93 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 120 | 74 | 20 | 104 | 46 | 50 | 80 | 96 | 10 | 25 | 100 |

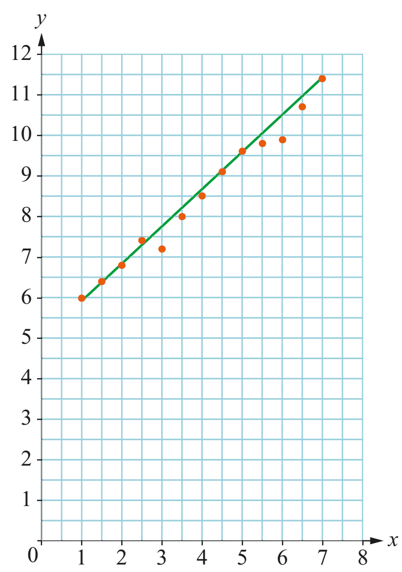
The equation of the **least-squares regression line** will be in the form $y = mx + c$, where $m$ is the value of the gradient of the line and $c$ is the value of the vertical intercept (where the line crosses the vertical axis). $y$ is the variable graphed on the vertical axis and $x$ is the variable graphed on the horizontal axis.

!

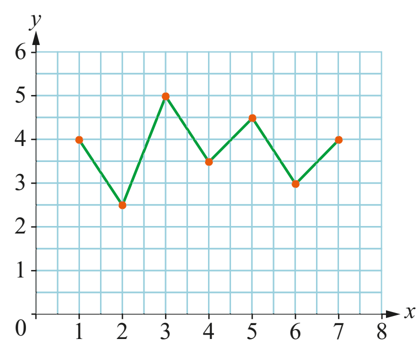
**least-squares regression line** a mathematically determined straight line that best fits a bivariate dataset

**4** Explain why the lines depicted on the following scatterplots do not represent lines of best fit.

**a**




**b**

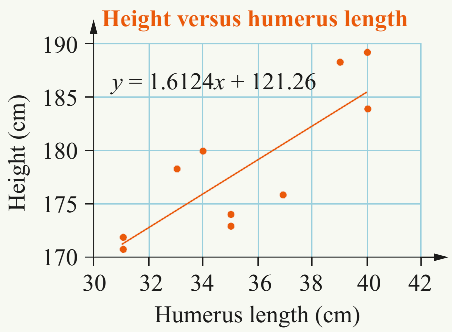


## EXAMPLE 5C–2 Interpreting the gradient and vertical intercept of the regression line

For a group of girls, the humerus length (elbow to shoulder) was measured and compared with height. The results are listed in this table.

| Humerus length (cm) | 37 | 35 | 40 | 31 | 35 | 33 | 31 | 40 | 34 | 39 |
|---|---|---|---|---|---|---|---|---|---|---|
| Height (cm) | 176 | 174 | 184 | 172 | 173 | 178 | 171 | 189 | 180 | 188 |

**a** Enter this data into a spreadsheet.
**b** Draw a scatterplot.
**c** Add the trendline (least-squares regression line) and show the equation.
**d** Interpret the gradient of the fitted line in the context of the data.
**e** Explain why interpreting the vertical intercept does not make sense in the context of the data.

<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>Enter the data into a spreadsheet.</td><td>Put the data into two columns.</td><td>Put a heading into the first row. Use the built-in formula.</td></tr>
<tr><td>b<br>c</td><td>
</td><td colspan="2">From the Insert menu select Scatterplot then first scatterplot type. From the Chart tools select Linear Trendline. Right click on the line and select Format Trendline from the drop-down menu. Check the Display equation on chart box and close.</td></tr>
<tr><td>d</td><td>For each increase in humerus length of 1 cm, the height will increase by an average of 1.6124 cm.</td><td>The gradient/rate is 1.6124. This means that for each increase of 1 unit of the x-variable (humerus length), the y-variable (height) increases by 1.6124 units.</td><td>The gradient/rate is the value of the coefficient in front of x. This tells us how much the y-variable changes for each increase in 1 unit of the x-variable.</td></tr>
<tr><td>e</td><td>The value of the humerus is unlikely to be 0 (except in very rare circumstances), so determining the height in these instances does not make sense in the context of the data.</td><td colspan="2">The vertical intercept tells us the value of the dependent variable when the value of the independent variable in 0.</td></tr>
</table>

**5** The data in the table gives the population, $P$, of Australia for the years 1931 to 2011. The time, $T$, is measured in years since 1931.

| Year | 1931 | 1941 | 1951 | 1961 | 1971 | 1981 | 1991 | 2001 | 2011 |
|---|---|---|---|---|---|---|---|---|---|
| Time (years) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
| Population (millions) | 6.53 | 7.11 | 8.42 | 10.55 | 13.07 | 14.92 | 17.28 | 19.27 | 22.34 |

**a** Enter this data into a spreadsheet.
**b** Draw a scatterplot.
**c** Add the least-squares regression line and show the equation.

**6** The data in the table gives the percentage, $P$, of the Australian population born in the United Kingdom and Ireland for the years 1971 to 2011. The time, $T$, is measured in years since 1971.

| Year | 1971 | 1976 | 1981 | 1986 | 1991 | 1996 | 2001 | 2006 | 2011 |
|---|---|---|---|---|---|---|---|---|---|
| Time (years) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| Population born in the UK and Ireland (%) | 8.53 | 8.22 | 7.72 | 7.18 | 6.91 | 6.33 | 5.79 | 5.48 | 5.43 |

**a** Illustrate the data ($T$ vs $P$) on a scatterplot using a spreadsheet.
**b** Determine the equation of the least-squares regression line and add the line to your scatterplot.
**c** Interpret the gradient and vertical intercept in the context of the data.

**7** The height of a plant $n$ days after being planted is given in the following table.

| Days ($n$) | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|
| Height ($H$ cm) | 4 | 6 | 9 | 11 | 13 | 16 | 18 | 20 |

**a** Illustrate the data ($H$ vs $n$) on a scatterplot using a spreadsheet.
**b** Determine the equation of the least-squares regression line and add the line to your scatterplot.
**c** Interpret the gradient and vertical intercept in the context of the data.

**8** Talula was investigating the relationship between the number of weeks ($n$) people have been on a new diet and their overall weight loss in pounds ($w$). She determines the line of best fit for the relationship as $w = 3.5 + 1.1n$.

**a** Interpret the gradient of the line of best fit in the context of the data.

**b** Does it make sense to interpret the vertical intercept in the context of the data?

**9** Lines of best fit are usually only given if there appears to be a linear relationship between two variables. Why are lines of best fit less useful for non-linear relationships, e.g. exponential relationships?

CALCULATOR APPLICATION

**10** If there is a strong correlation between two variables, will the data points be closer or further away from the line of best fit than if there is weak correlation between two variables? Explain your answer.

For most scientific calculators, the regression line is usually expressed in the form $y = A + Bx$.
*Note*: compared to $y = mx + c$, $A = c$ and $B = m$.
Follow the steps below to find the equation of the regression line for this table of data.

| **Income ('000s)** | 10 | 12 | 16 | 22 | 26 | 29 | 33 | 37 | 42 | 49 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Expenditure ('000s)** | 2.5 | 2.8 | 3 | 3.3 | 3.6 | 3.8 | 3.9 | 4.1 | 4.5 | 4.9 |

First put the calculator in REG mode (regression mode).

On a CASIO fx-82TL press MODE 3 1 for linear regression.

To enter the data from the table, press:
10 , 2.5 DT , 12 , 2.8 DT , 16 , 3 DT , 22 , 3.3 DT , …, 49 , 4.9 DT

To obtain the linear coefficients $A$ and $B$, press SHIFT A = 2.15 (2 decimal places)

SHIFT B = 0.05 (2 decimal places)

The equation of the regression line for this data is $y = 2.15 + 0.05x$, where $x$ represents income and $y$ represents expenditure.

**11** Using your calculator, find the equation of the least-squares regression line for the data below.

**a**

| $x$ | 100 | 120 | 125 | 140 | 170 | 180 | 190 | 210 | 220 | 240 |
|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 90 | 85 | 100 | 90 | 100 | 115 | 105 | 125 | 110 | 120 |

**b**

| $x$ | 10 | 14 | 20 | 22 | 28 | 35 | 38 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 9 | 15 | 16 | 13 | 24 | 20 | 29 | 22 | 27 |

**c**

| $x$ | 8 | 14 | 17 | 17 | 22 | 27 | 30 | 33 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 11 | 14 | 20 | 16 | 22 | 29 | 28 | 35 |

**d**

| $x$ | 5 | 10 | 14 | 15 | 24 | 27 | 32 | 33 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 10 | 11 | 12 | 14 | 15 | 17 | 20 | 20 |

**e**

| $x$ | 86 | 95 | 100 | 90 | 96 | 105 | 94 | 98 | 110 | 100 | 93 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 120 | 74 | 20 | 104 | 46 | 50 | 80 | 96 | 10 | 25 | 100 |

# 5D Interpolation and extrapolation

These resources are available on your obook assess:
- **Video tutorial 5D:** Watch and listen to an explanation of Example 5D–2
- **Worksheet 5D:** Practise your skills with making predictions
- **assess quiz 5D:** Test your skills with an auto-correcting multiple-choice quiz

When the line of best fit has been fitted to a dataset, we can use it to make predictions given a value of one of the two variables.

The magnitude of the correlation coefficient determines the accuracy of the predictions made from the equation of the line of best fit for the data; that is, the closer $r$ is to $+1$ or $-1$, the closer the relationship between the variables and the more accurate the predictions made. The closer $r$ is to 0, the weaker the relationship between the variables and the less accurate the predictions made.

The line of best fit will be more accurate for the population data when we are using a larger sample, therefore predictions made using a line of best fit are also more accurate when there are more data points in the dataset.

Lines of best fit can be extended beyond the range of points in a data set. **Interpolation** is when we make a prediction within the existing range of data points, and **extrapolation** is when we make a prediction outside the existing range of data points.

**interpolation** predicting where a new data point will lie within the existing range of data points

**extrapolation** predicting where a new data point will lie outside the existing range of data points

Interpolation is more accurate than extrapolation, as we cannot be sure that the behaviour of data points we observed will continue outside the range of values in the dataset.

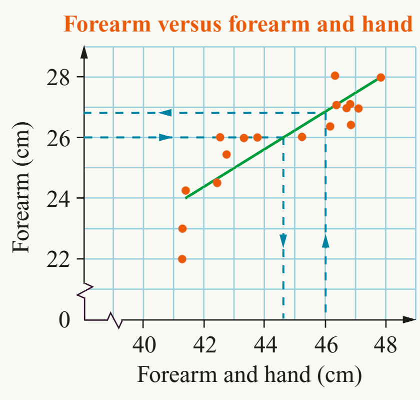
Extrapolation should not be used if the required data value lies far outside the range of the other data values.

## EXAMPLE 5D–1 Making predictions using a line of best fit

This scatterplot shows the forearm and hand against forearm only measurements for a group of students. Use the line of best fit to predict:

**a** the forearm measurement for a student with a forearm and hand measurement of 46 cm

**b** the forearm and hand measurement for a student with a forearm measurement of 26 cm.



| | Solve | Think | Apply |
|---|---|---|---|
| **a** | The forearm measurement is about 26.8 cm. | Draw a vertical line from 46 on the forearm and hand axis to meet the line. From this point of intersection, draw a horizontal line to meet the forearm axis. Read off the approximate forearm measurement. | The line of best fit will approximate the values. It is not a good indicator for values outside the range of the plotted points. The greater the number of points and the closeness of the points to the line, the better the line is as a predictor. |
| **b** | The forearm and hand measurement is about 44.6 cm. | Start at 26 on the forearm axis and reverse the process in part **a**. | |

## EXERCISE 5D Interpolation and extrapolation

UNDERSTANDING, FLUENCY AND COMMUNICATING

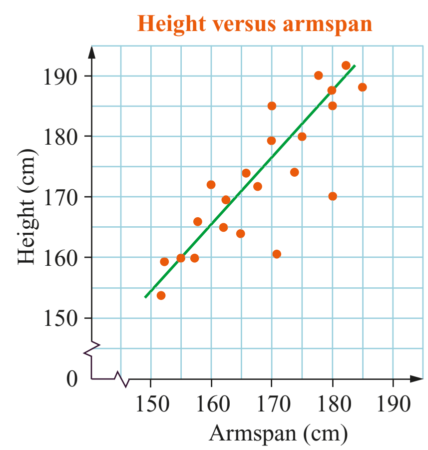
**1** This scatterplot shows the heights and armspans of a group of students. A line of best fit has been drawn for these points. Use the line of best fit to predict:

**a** the heights of students with these armspans

**i** 160 cm **ii** 175 cm **iii** 180 cm

**b** the armspans of students with these heights

**i** 160 cm **ii** 175 cm **iii** 185 cm



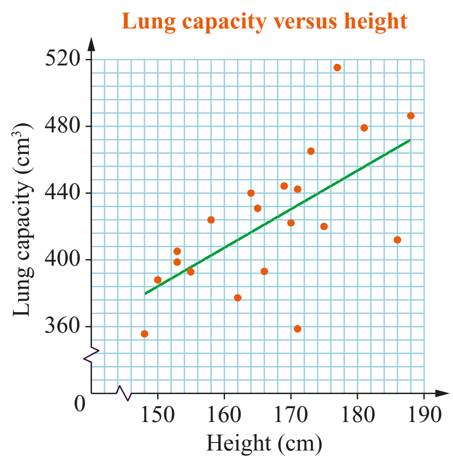

**2** This scatterplot shows the height and lung capacity of a group of students. A line of best fit has been drawn for these points. Use the line of best fit to predict:

**a** the lung capacity of a student with a height of

**i** 160 cm **ii** 178 cm **iii** 172 cm

**b** the height of a student with a lung capacity of

**i** 400 $cm^3$ **ii** 430 $cm^3$ **iii** 450 $cm^3$



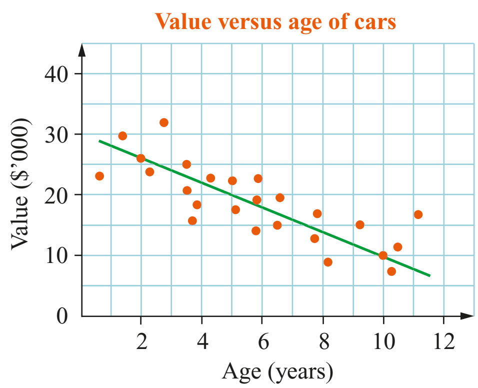

**3** This scatterplot shows the age and value for a sample of cars of a particular model. A line of best fit has been drawn for these points. Use the line to predict:

**a** the value of a car of this model of age

**i** 2 years **ii** 5 years **iii** 10 years

**b** the age of a car of this model with a value of

**i** \$28 000 **ii** \$18 000 **iii** \$12 000




STATISTICAL ANALYSIS

## EXAMPLE 5D–2 Making predictions using the regression line equation

The equation of the line of best fit in Example 5D–1 is $F = 0.6 \times A - 0.7$, where $F$ represents the forearm length and $A$ represents the forearm and hand length. Use the equation to predict:

**a** the forearm length of a student with a forearm and hand length of 45 cm

**b** the forearm and hand length of a student with a forearm length of 25 cm.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $F = 0.6 \times 45 - 0.7$<br>$= 26.3$ cm | Substitute $A = 45$ into the equation and calculate $F$. | Care must be taken if using the equation without viewing the data points. |
| **b** | $25 = 0.6 \times A - 0.7$<br>$25.7 = 0.6A$<br>$A = \frac{25.7}{0.6}$<br>$A = 42.8$ cm | Substitute $F = 25$ then solve the equation for $A$. | |

**4** The equation of the line of best fit connecting height ($H$) and armspan ($A$), plotted on a scatterplot, is $H = 1.2A - 36$. Use the equation to predict:

**a** the height of a student with an armspan length of

**i** 160 cm **ii** 170 cm **iii** 178 cm

**b** the armspan length of a student with a height of

**i** 160 cm **ii** 175 cm **iii** 183 cm.

**5** The equation of the line of best fit connecting hip measurement ($H$), in cm, and waist measurement ($W$), in cm, is $W = 0.7H - 2.1$. Use the equation to predict:

**a** the waist of a person with a hip measurement of

**i** 85 cm **ii** 96 cm **iii** 100 cm

**b** the hip measurement of a person with a waist of

**i** 60 cm **ii** 65 cm **iii** 71 cm.

**6** The equation of the line of best fit connecting grape yield ($G$), in tonnes, of a vineyard and the number of frosts ($n$) during the growing season is $G = -0.14 \times n + 5.6$, when plotted on a scatterplot. Use the equation to predict:

**a** the yield when there are

**i** 5 frosts **ii** 12 frosts **iii** 20 frosts

**b** the number of frosts given that the yield was

**i** 4.2 t **ii** 3.5 t **iii** 2.1 t.

**7** After the age of 30 years a person's height begins to decrease at the rate of approximately 0.06 cm per year. What is the estimated height for the following people?

**a** a 40-year-old who was 175 cm at 30 years

**b** a 70-year-old who was 182 cm at 30 years

**c** a 50-year-old who was 196 cm at 30 years

**d** a 35-year-old who was 154 cm at 30 years

**e** a 67-year-old who was 160 cm at 30 years

**f** an 83-year-old who was 187 cm at 30 years

## EXAMPLE 5D–3 Identifying interpolation and extrapolation

Ahn is doing an experiment measuring the time it takes a range of volunteers to complete a mental arithmetic task. He records the following results.

| Age ($n$) | 18 | 23 | 21 | 16 | 29 | 25 | 33 | 21 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| Time (s) | 9.43 | 11.71 | 8.59 | 13.20 | 11.25 | 14.14 | 17.23 | 9.18 | 13.77 |

He calculates that the line of best fit for the data is $s = 0.318n + 4.636$.

**a** Ahn wants to predict the completion time for a 45-year-old volunteer. Is this an example of interpolation or extrapolation?

**b** Ahn wants to predict the completion time for a 17-year-old volunteer. Is this an example of interpolation or extrapolation?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | This is an example of extrapolation. | The range of values for the ages is from 16 to 29. Predicting values inside this range will be interpolation. Predicting values outside this range will be extrapolation. | If the data value we want to predict lies within the range of the other data values, we are interpolating. If it lies outside the range of the other data values, we are extrapolating. |
| **b** | This is an example of interpolation. | | |

**8** Simone is a swimming coach. The time $t$ it takes a number of her students to swim 25 m freestyle is recorded in the following table.

| Age ($n$) | 18 | 15 | 22 | 17 | 11 | 26 | 14 | 19 | 15 |
|---|---|---|---|---|---|---|---|---|---|
| Time (s) | 24 | 28 | 25 | 21 | 33 | 30 | 29 | 21 | 26 |

**a** If Simone wants to predict the swimming time for an 8-year-old, is this an example of interpolation or extrapolation?

**b** If Simone wants to predict the swimming time for a 30-year-old, is this an example of interpolation or extrapolation?

UNDERSTANDING, FLUENCY AND COMMUNICATING

**9** The results of a group of students on Mathematics and Science tests are compared.

| **Student** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|---|---|---|
| **Maths test ($M$)** | 64 | 67 | 69 | 70 | 73 | 74 | 77 | 82 | 84 | 85 |
| **Science test ($S$)** | 68 | 73 | 68 | 75 | 78 | 73 | 77 | 84 | 86 | 89 |

**a** Illustrate the data on a scatterplot, with $M$ on the horizontal axis and $S$ on the vertical axis.
**b** Fit a least-squares regression line to the data and show the equation of the line.
**c** Use the equation to predict the (average) score in Science of a student who scores 80 in Mathematics.
**d** Use the line of best fit equation to predict the (average) score in Mathematics of a student who scores 60 in Science.
**e** Is either part **c** or part **d** an example of extrapolation?

**10** The population of a town over a period of 10 years is shown in the table. The time is measured in years from the start of 1990; that is, $T = 1$ is the start of 1991, $T = 2$ is the start of 1992, etc.

| **Time (years)** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Population** | 3400 | 4100 | 4500 | 4900 | 5600 | 6100 | 6500 | 6900 | 7400 | 8000 |

**a** Use a spreadsheet to illustrate the data on a scatterplot.
**b** Add the trendline and show the equation of this line.
**c** Use this equation to predict the population:
**i** after 4.5 years **ii** after 7.5 years
**iii** after 12 years **iv** at the start of 2007.
**d** Which of the answers in part **c** is/are the least reliable? Give reasons for your answer.
**e** Use this equation to estimate when the population:
**i** was 5000 **ii** will reach 10 000.

**11** Camille is investigating if there is a link between the number of years someone spends studying ($n$) and their salary at age 35 in \$'000s ($S$). The data she collects is from people who have completed between 12 and 20 years of study, and the line of best fit for her data is $S = 3.5n + 9.8$.
**a** Use this equation to predict the salary at age 35 of someone who has completed:
**i** 10 years of study **ii** 14 years of study **iii** 18 years of study
**b** Which of your answers to part **a** are an example of extrapolation?
**c** The value of Pearson's correlation coefficient for Camille's data is 0.77. Explain what this means in the context of the data.
**d** Considering your answer to part **c**, how accurate do you expect your predictions to be?

**12** April collects data on the shoe sizes of 100 female students from different year levels in her school. Explain why the line of best fit for this data would not be useful for predicting the shoe size of a 30-year-old woman.

**13** May is investigating whether there is a link between the weight and systolic blood pressure of a number of patients at the hospital she works at. She collects the following information.

| **Weight (kg)** | 75 | 83 | 62 | 66 | 98 | 57 | 88 | 111 | 79 | 60 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Systolic blood pressure (mmHg)** | 99 | 115 | 108 | 86 | 127 | 90 | 122 | 143 | 103 | 105 |

**a** Plot this data on a scatterplot.
**b** Give the regression line for this data.
**c** Use the equation of the regression line to predict the systolic blood pressure of patients with the following weights:
**i** 115 kg **ii** 140 kg
**d** Which of the predictions from part **c** is more reliable? Explain your answer.

**14** June is looking to see whether there is a relationship between the number of goals and behinds in AFL football games. The table below is a sample of the data she collects.

| **Goals ($g$)** | 14 | 5 | 14 | 15 | 12 | 18 | 17 | 7 | 11 | 25 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Behinds ($b$)** | 8 | 15 | 13 | 7 | 8 | 9 | 16 | 9 | 12 | 13 |

**a** Plot this data on a scatterplot, with $g$ on the horizontal axis and $b$ on the vertical axis.
**b** Calculate the value of Pearson's correlation coefficient for the data.
**c** Calculate the equation of the regression line for the data.
**d** Considering your answers to parts **b** and **c**, will the regression line be useful in making accurate predictions for this dataset? Explain your answer.

**15** Explain why it will be more accurate to make predictions from a line of best fit that is calculated from many data points than from one that is calculated from just a few data points.

CHALLENGE

**16** Bailey is doing a project about the price of inner-city apartments in Sydney. He collects the following data about apartment sizes (in m$^2$) and their price (in \$'000s).

| **Size (m$^2$)** | 77 | 102 | 52 | 77 | 134 | 68 | 99 | 130 | 61 | 75 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Price (\$'000s)** | 1180 | 1800 | 830 | 1120 | 3050 | 1270 | 1070 | 2600 | 660 | 2350 |

**a** Plot this data on a scatterplot.
**b** Calculate the equation of the regression line for the data.
**c** Use the equation of the regression line to estimate the price of a 90 m$^2$ inner-city apartment in Sydney. Give your answer to the nearest \$10 000.
**d** Bailey re-checks his data and discovers he made a mistake in inputting the size of the final apartment; it is actually 125 m$^2$ instead of 75 m$^2$. Give the equation of the regression line for the corrected data.
**e** Recalculate the estimated price of the 90 m$^2$ apartment from part **c**.
**f** Which factors do you think affect the price of apartments in inner-city Sydney?

# 5E Statistical investigations

These resources are available on your obook assess:
- **Video tutorial 5E:** Watch and listen to an explanation of Example 5E–1
- **Worksheet 5E:** Practise your skills with extra problems for understanding bias
- **assess quiz 5E:** Test your skills with an auto-correcting multiple-choice quiz

Statistics can be used to answer a wide range of real-life questions such as the following:

- Does Sydney receive more sunshine than Melbourne?
- Which drug is most effective in the treatment of the common cold?
- How does time on social media affect the concentration of teenagers?
- Does the location of your school have an impact on your HSC results?

The statistical investigation process can be used to answer these statistical questions. It involves four distinct steps:

1. Identifing your question and planning your investigation
2. Collecting the data
3. Representing and analysing the data
4. Interpreting and communicating the results

The results from a statistical investigation should only ever be used to interpret information about the population that is used in the study. For example, if you are investigating the hip to waist ratio in women, the conclusions you draw will probably not be valid for men.

In this topic we will look at some of the questions that can be answered by using the statistical investigation process, as well as some of the issues that need to be considered when collecting and using data.

## EXAMPLE 5E–1 Identifying questions that can be answered using statistics

Which of these questions can be answered using statistics?

**a** What is the distance between Sydney and Newcastle?
**b** Do students who receive a high mark in Mathematics Standard earn more in their career than students who receive a low mark?
**c** Do teenagers eat more than adults?
**d** What is the average finishing time in the Hobart to Sydney Yacht Race?
**e** Which team scored the least number of tries in the NRL last season?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | No | The answer to this question can be looked up online. | For a question to be answered using statistics it needs to either involve statistical calculations (such as calculating the mean), or a statistical investigation.<br>Questions that can be answered without statistical calculations, e.g. by looking up the information, do not need to be answered using statistics. |
| **b** | Yes | This question would be best answered by conducting a statistical investigation. | |
| **c** | Yes | This question would be best answered by conducting a statistical investigation. | |
| **d** | Yes | The answer to this question would involve calculating the mean or median of finishing times. | |
| **e** | No | The answer to this question can be looked up online. | |

# EXERCISE 5E Statistical investigations

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Which of these questions can be answered using statistics?

- **a** How many more wickets did Shane Warne take in Test cricket than Glenn McGrath?
- **b** Are houses more expensive in Sydney or London?
- **c** Do tall people earn more than short people?
- **d** How hot was the warmest day in Broken Hill last year?
- **e** How tall is the tallest building in Sydney?
- **f** What is the average height of a skyscraper in Sydney?
- **g** How much does it cost to fly from the Central Coast to Perth?
- **h** Who is the most famous movie star in the world?

Samples should be selected so that the sample reflects the population as a whole.

Therefore, if we want to investigate the whole population of Australia, a sample should include people from across Australia, covering religious and cultural differences.

A sample from a population must be of a sufficient size to represent the whole population.

> **!** A good rule of thumb is for the size of a sample to be at least the square root of the population size.
>
> For example, if the size of the population is 10 000, the minimum size of a sample should be $\sqrt{10\,000} = 100$.

**2** Determine the 'rule of thumb' minimum sample size for drawing a sample from the following populations, rounding your answers to the nearest whole number where necessary.

- **a** a school with 900 students
- **b** Australia, with a population of approximately 25 000 000
- **c** a city with a population of 250 000
- **d** a political party with 15 000 members

When handling data we need to be aware of privacy issues. Sensitive data must be stored securely as data breaches in business are becoming increasingly commonplace. Consider who will have access to the data, as well as the participants whose data you are collecting.

**Bias** occurs when one outcome is favoured above other outcomes. Bias can be either intentional or unintentional, and can occur in the selection of the sample, or in the collection of data. Leading questions, which lead the respondent towards a particular response, can introduce bias into an investigation.

> **bias**
> a prejudice in favour of one particular group or outcome

STATISTICAL ANALYSIS

**ethics**
moral principles that guide the way that we should act in a given situation

**Ethics** should be considered when conducting a statistical investigation. Be aware of any cultural sensitivities and act sensitively accordingly. Statistical investigations should be conducted objectively and honestly. You may not always find the outcome that you are either looking for, or expecting to find.

## EXAMPLE 5E–2 Identifying potential bias in sampling methods

Identify why the following sampling methods may result in a biased statistical investigation:

**a** investigating the religious beliefs of Australians by surveying people at shopping malls on Sunday mornings

**b** conducting a survey of the views of Australians about the current Prime Minister by interviewing people from only one electorate.

| | Solve/Think | Think | Apply |
|---|---|---|---|
| **a** | Many religious people attend church on Sunday mornings, so these people will likely be under-represented at shopping malls at this time. | | Identify the population that you are investigating. Does the proposed method take a random sample of that population, or are some people more likely to be included or excluded than others? |
| **b** | Political opinions vary across the country according to a range of factors (urban/rural, income, etc.), so one electorate will not provide a representative sample of the population. | | |

**3** Identify why the following sampling methods may result in a biased statistical investigation:

**a** conducting an investigation on the shopping habits of 17-year-olds by interviewing students from schools in inner-Sydney

**b** researching favourite types of music by interviewing people outside an Adele concert

**c** finding opinions about the new cycle lanes by interviewing members of the local cycling club

**d** looking at the state of the economy by interviewing investment bankers

**e** researching the favourite films of teenagers by conducting interviews at Comic-Con.

**4** Melissa wants to investigate the wages of university graduates. If Melissa only interviewed graduates with business degrees, why would her data not be suitable for estimating the wage of a 24-year-old science graduate?

## EXAMPLE 5E–3 Identifying potential bias in survey questions

Explain why the following survey questions may result in a biased statistical investigation:

**a** Do you agree that lowering the company tax rate will be good for society?

**b** Polls show that 90% of customers prefer the new Strawberry Dream frozen yoghurt range. What do you think?

| | Solve/Think | Apply |
|---|---|---|
| **a** | By asking 'do you agree', the likelihood that the respondent will agree with the survey question is increased. | Leading questions lead the respondent to answer in a particular way. Unbiased questions should not lead the respondent in any way. |
| **b** | By explaining that the majority of people prefer the new frozen yoghurt range, this question is leading people to answer that they also prefer it. | |

**5** Explain why the following survey questions may result in a biased statistical investigation.
- **a** Taking into account the budget deficit, should the government cut military expenditure?
- **b** You don't support the proposal for the new railway line, do you?
- **c** The Avengers was voted the best film of the 21st Century. What's your favourite film of the 21st Century?

**6** The following survey questions may lead to biased responses. Rewrite the questions to try to eliminate any potential bias. The reason for potential bias has been provided in brackets.
- **a** Is hip-hop your favourite genre of music? [There are many genres of music, but only one is mentioned in the question.]
- **b** What is your opinion of the unnecessary rule changes in the NRL? [Not everyone will believe that the rule changes are unnecessary.]

**7** Ryan was researching the health issues that staff at a large company had over the past 10 years. He kept his research in an unlocked spreadsheet, which detailed the names of the staff, as well as their personal information, such as their salary. Explain why there may be a privacy issue with Ryan's handling of the data.

**8** Use the internet to investigate ways that bias can be reduced or eliminated in both sampling methods and in the wording of survey questions.

RESEARCH

**9** **a** Measure the height and handspan of the students in your class (separate results for males and females) and record the information in a table (or directly into a spreadsheet).
- **b** Plot the data as a scatterplot.
- **c** Describe the correlation between these two variables in terms of direction (positive/negative) and strength (perfect/strong/moderate/weak). Is one variable a good predictor of the other? Calculate the value of $r$.
- **d** Use technology to add the least-squares regression line of best fit to the data.
- **e** Measure the heights of some students from another class and predict the handspans of these students, using the equation of the regression line.
- **f** Find the handspans of these students by actual measurement and compare them with your predictions.
- **g** Discuss the results in relation to your answer to part **c**.

**10** Repeat the procedures in question **9** for these measurements:
- **a** height and shoe size
- **b** head circumference and height
- **c** length of femur (thigh bone) and height
- **d** length from hip to the ground and height.

# CHAPTER 5 REVIEW BIVARIATE DATA

You should be able to:

✔ create a scatterplot from a given dataset

✔ identify independent and dependent variables where appropriate

✔ identify potential outliers, and interrogate them

✔ describe the strength and direction of bivariate datasets

✔ calculate Pearson's correlation coefficient using technology

✔ draw a line of best fit for a bivariate dataset by eye

✔ calculate the equation of the least-squares regression line using technology

✔ interpret the gradient and vertical intercept of the line of best fit in the context of the data

✔ make predictions about a bivariate dataset by using either interpolation or extrapolation

✔ understand when predictions are accurate and when they are inaccurate

✔ identify which questions can be answered using statistics

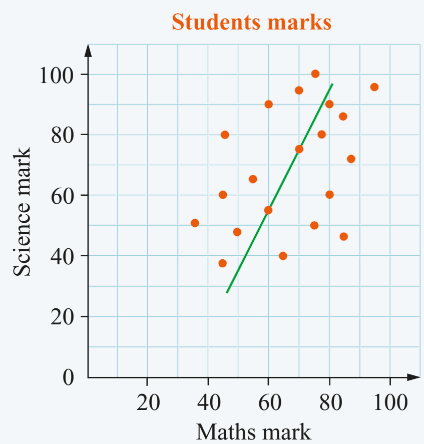
✔ identify how and where bias can appear in both sample selection and data collection

✔ identify any issues of privacy and sensitivities around diverse groups and cultures.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

This graph is a scatterplot, with a line of best fit, of the Mathematics and Science marks for a group of students. Use the graph to answer questions **1–3**.




**1** Using the trendline, what is the predicted Science mark for a student who scored 70 in Mathematics?

**A** 60 **B** 65 **C** 70 **D** 75

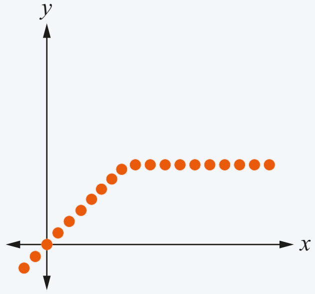
**2** Using the trendline, what is the predicted Mathematics mark for a student who scored 55 in Science?

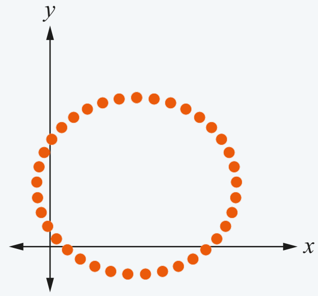
**A** 50 **B** 55 **C** 60 **D** 65

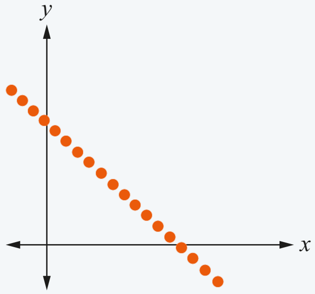
**3** At which score are the predicted values for Mathematics and Science the same?

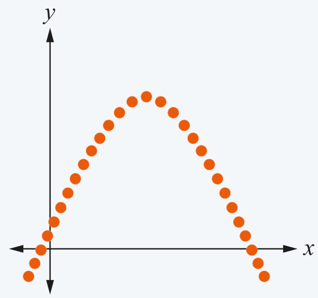
**A** 40 **B** 50 **C** 65 **D** 75

**4** Which of the following scatterplots shows a linear relationship between the variables?

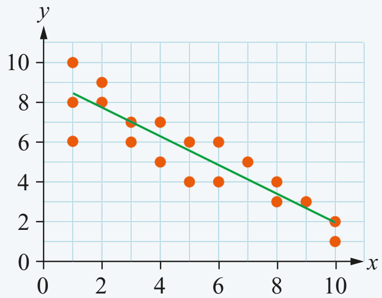
**A**


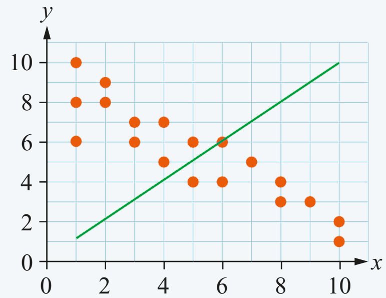

**B**


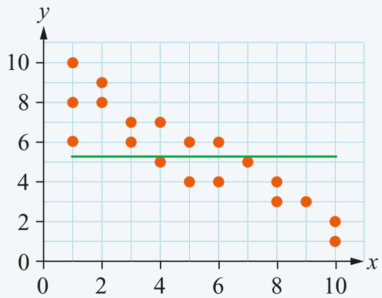

**C**



**D**



**5** Which of the following scatterplots correctly depicts a line of best fit?

**A**



**B**



**C**



**D**



**6** Which of the following values of $r$ represents a strong positive correlation?

**A** –0.99 **B** 0.15 **C** 0.45 **D** 0.80

**7** Which of the following values of $r$ represents a moderate negative correlation?

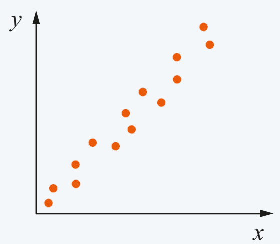
**A** –0.77 **B** –0.52 **C** –0.14 **D** 0.38

**8** Which of the following values of $r$ represents no correlation?

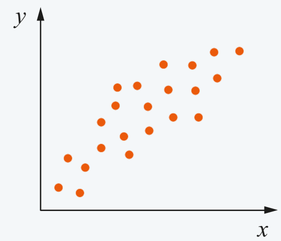
**A** –0.15 **B** 0.25 **C** 0.98 **D** –0.49

**9** Which of the following scatterplots shows a strong negative correlation?

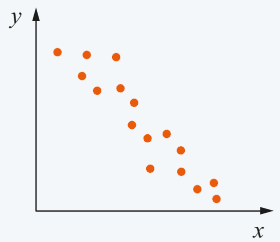
**A**



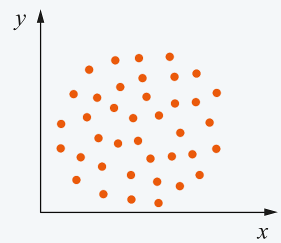

**B**




**C**




**D**




**10** There is a strong degree of correlation between the lengths of the left and right feet of individuals. Which of the following best describes the association between the two variables?

**A** cause and effect
**B** spurious correlation
**C** interpolation
**D** extrapolation

Brooke is investigating the relationship between the number of handballs and kicks players make in AFL football. The line of best fit for her data is $h = 0.85k + 2.5$, where $h$ represents the number of handballs and $k$ represents the number of kicks. Use this equation to answer questions **11** and **12**.

**11** If a player makes 15 kicks in a game, how many handballs would they expect to make (to the nearest whole number)?

**A** 10 **B** 11 **C** 13 **D** 15

**12** If a player makes 325 handballs in a season, how many kicks would they expect to make (to the nearest whole number)?

**A** 274 **B** 278 **C** 379 **D** 385

**13** Using the rule of thumb that the minimum sample size is the square root of the population, what is the minimum sample size for a population of 90 000?

**A** 30 **B** 300 **C** 810 **D** 90 000

**14** The table shows the distance travelled, in 1000 km, versus the servicing costs, in $1000, for a motor vehicle.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Distance ('000 km)** | 50 | 100 | 180 | 200 | 230 | 270 | 330 | 350 | 400 |
| **Cost ($'000)** | 3.2 | 4.1 | 4.4 | 6 | 7.3 | 8.5 | 9.1 | 9.8 | 13.5 |

Which of the following is an example of extrapolation?

**A** estimating the servicing costs for a motor vehicle that has travelled 385 000 km
**B** estimating the distance travelled for a motor vehicle with servicing costs of $5200
**C** estimating the servicing costs for a motor vehicle that has travelled 20 000 km
**D** estimating the distance travelled for a motor vehicle with servicing costs of $10 500

**15** Which of the following values of $r$ is going to give the most accurate predictions from the regression line?

**A** 0.70 **B** $-0.01$ **C** 0.33 **D** $-0.95$

**16** Which of the following values of $r$ is going to give the least accurate predictions from the regression line?

**A** $-0.63$ **B** $-0.30$ **C** 0.41 **D** 0.89

**17** Which of the following questions can be answered without using statistics?

**A** What is the average time it takes year 12 students to run 100 m?

**B** Does it rain more in Sydney or Brisbane?

**C** Which team has won the most NRL premierships?

**D** Do lawyers earn more than doctors?

## REVIEW SET 1

**1** **a** Draw a scatterplot for the following table of data.

| $x$ | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 2.2 | 1.9 | 1.8 | 1.8 | 1.4 | 1.3 | 0.9 | 0.8 |

**b** Is the correlation between $x$ and $y$:

**i** perfect, strong, or weak?

**ii** positive or negative?

**2** The correlation coefficient between the time since a client joins Dr Fitt's weight-loss program and the client's percentage body fat is $-0.42$. Explain what this means in the context of the data.

**3** Discuss the expected strength of the relationship between the following variables.

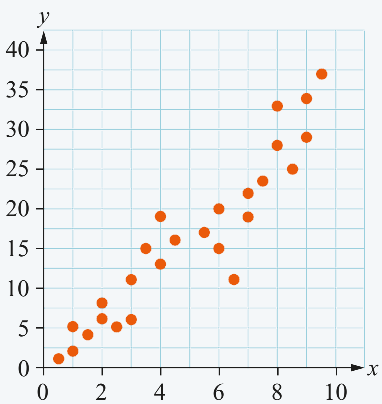
**a** the size of a gold nugget and its value in dollars

**b** the number of chickens on a farm and the number of eggs laid

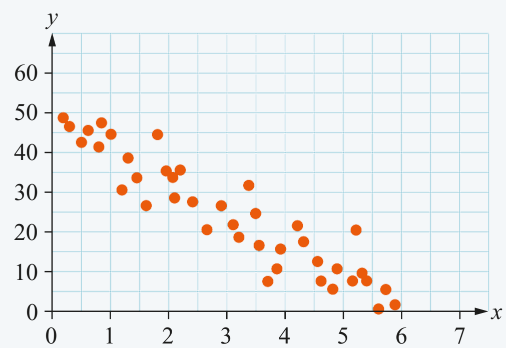
**c** the age of a tractor and its value in dollars

**4** Draw a line of best fit on each of these scatterplots.

**a**




**b**

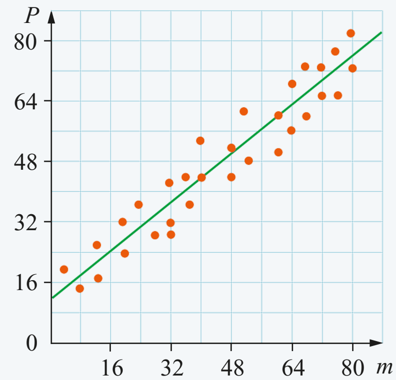



**5** The relationship between time spent training at a gym and the maximum weight that can be bench-pressed is given by the equation $w = 35 + 12.5t$, where $w$ is the weight in kg and $t$ is the time in years.
Use the equation to predict the maximum bench-press weight of someone who has been training for:

**a** 3 years **b** 7 years **c** 15 years

STATISTICAL ANALYSIS

6 Explain why the equation in question **5** will probably not continue to hold for larger values of $t$.




7 Use the graph to estimate:

**a** $P$ when $m = 80$ **b** $m$ when $P = 50$.

8 Identify why the following sampling methods may result in a biased statistical investigation:

**a** interviewing people on the evening of a State of Origin match about what their favourite sport is

**b** asking cinemagoers what their favourite film genre is straight after they have seen the latest *Star Wars* movie.

# REVIEW SET 2

The relationship between the time it takes to run a marathon in minutes ($m$) and the amount of training in hours ($h$) is given by the equation $m = 350 - 0.65h$. Use this equation to answer questions **1** and **2**.

1 Calculate the expected marathon finishing time for athletes who trained for:

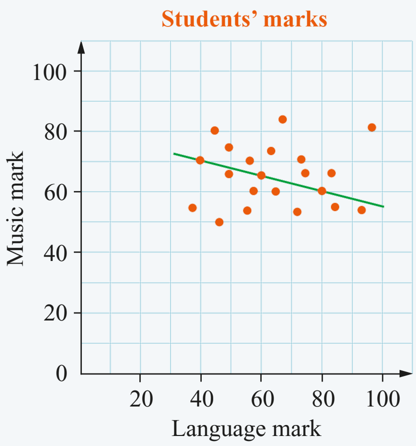
**a** 120 hours **b** 150 hours **c** 200 hours

2 If an athlete finished the marathon in 3 hours and 15 minutes, for how long would you predict they had been training? Give your answer to the nearest hour.

3 The graph shows a scatterplot with a line of best fit for the Language and Music test marks of a group of students. Use the line of best fit to predict:

**a** the Music mark of a student who scores 60 in the Language test

**b** the Language mark of a student who scores 60 in the Music test.




4 Sketch a scatterplot that shows:

**a** a strong positive correlation

**b** a moderate negative correlation.

5 Julie is a marine biologist tracking the population of sea turtles on a Pacific island. Over 24 years she records the following data.

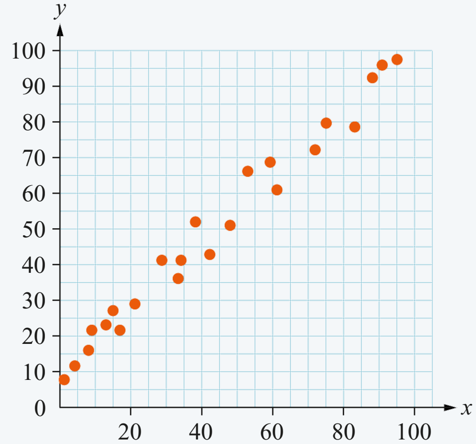
| Year | 1994 | 1997 | 2000 | 2003 | 2006 | 2009 | 2012 | 2015 | 2018 |
|---|---|---|---|---|---|---|---|---|---|
| Population | 3520 | 3314 | 3198 | 2805 | 2487 | 2108 | 1884 | 1539 | 1235 |

**a** Julie wants to know the year in which the population first fell to 2000. Is this an example of interpolation or extrapolation? Explain your answer.

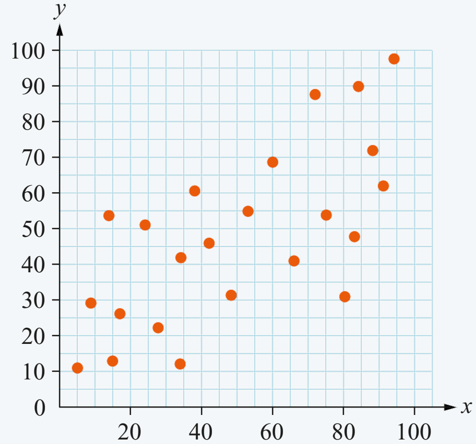
**b** Julie wants to predict the population of sea turtles in 2020. Would this be an example of interpolation or extrapolation? Explain your answer.

**6** Draw a line of best fit on each of these scatterplots.

**a**



**b**



**7** Identify why the following survey questions may result in a biased statistical investigation.

**a** I'm not a fan of the new building design. What do you think?

**b** Do you agree that the new burger joint does the best burgers in town?

## REVIEW SET 3

**1** Draw a scatterplot for the data in this table.

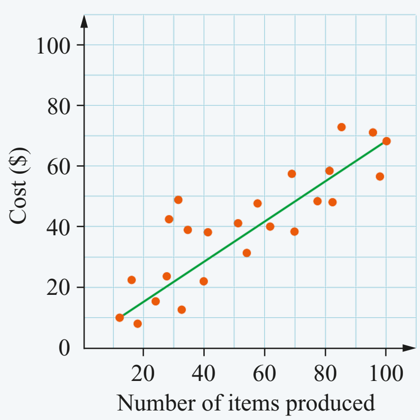
| $x$ | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0 | 8 | 21 | 28 | 43 | 51 | 60 | 69 | 82 | 94 | 102 |

**2** Use this line of best fit to predict:

**a** the cost when the number of items produced is 80

**b** the number of items produced when the cost is $35.

Production costs




**3** The correlation coefficient between marks in an English exam and marks in a French exam was 0.66. Explain what this means in the context of the data.

**4** The following pairs of variables were measured and a strong correlation found. State whether it is a cause and effect relationship or a case of spurious correlation.

**a** the number of umbrellas sold and the number of swimming costumes sold

**b** the number of storks nesting in chimneys and the birth rate

**5** The equation of the regression line connecting the age of a tropical plant in days ($d$) and height of the plant in cm ($h$) is $h = 1.45d - 3.5$.

**a** Use the equation to predict the height of the plant after:

**i** 8 days **ii** 12 days **iii** 20 days

**b** Does the vertical intercept make sense in the context of the data?

6 Sketch a scatterplot that shows:
   a moderate negative correlation
   b perfect positive correlation.

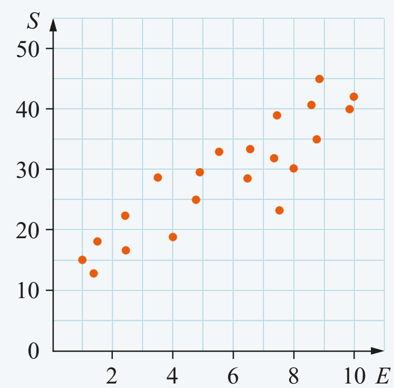
7 Discuss the expected strength of the relationship between the following variables.
   a age of a computer and its processing power
   b height and size of nose
   c experience in a job and salary

## REVIEW SET 4

1 A transport company keeps a record of the annual maintenance costs of its fleet of semitrailers. The distance travelled, (in thousands of kilometres), and cost (in thousands of dollars), is shown in the table below.

| **Distance travelled ('000 km)** | 50 | 100 | 180 | 200 | 230 | 270 | 330 | 350 | 400 |
|---|---|---|---|---|---|---|---|---|---|
| **Cost ($'000)** | 2.3 | 2.7 | 3.3 | 3.5 | 3.7 | 4.1 | 4.5 | 4.7 | 5.1 |

   a Plot the data on a scatterplot.
   b Comment on the correlation between the variables (strong, moderate, weak, positive, negative).
   c Determine the equation of the regression line that fits this data.
   d Predict the annual maintenance cost for a semitrailer that has travelled 300 000 km. Give your answer to the nearest $100.
   e Using your answer for part **b**, comment on the accuracy of this prediction.




2 The scatterplot compares $S$ and $E$.
   a Draw a line of best fit.
   b Find $S$ when $E = 7$.
   c Find $E$ when $S = 40$.

3 The equation of a regression line for a bivariate dataset is $y = 3.93x - 5.7$.
   a What is the value of the gradient?
   b What is the value of the vertical intercept?

4 Karl is tracking the population of kangaroos in a rural area. He records the following data.

| **Year** | 2009 | 2010 | 2011 | 2012 | 2013 | 2014 | 2015 | 2016 | 2017 |
|---|---|---|---|---|---|---|---|---|---|
| **Population** | 45 300 | 46 100 | 46 600 | 47 200 | 47 100 | 48 000 | 48 400 | 48 800 | 49 600 |

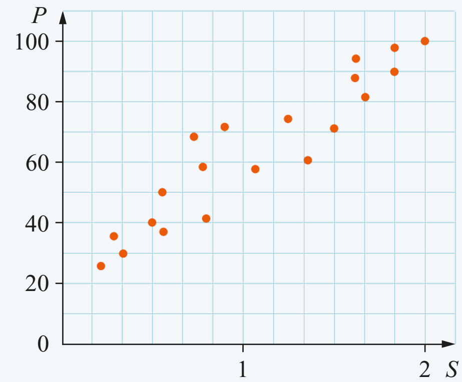
   a If Karl wanted to estimate the population of kangaroos in the area in 2004, would this be an example of interpolation or extrapolation?
   b Explain your answer to part **a**.

**5** Identify why the following sampling methods may result in a biased statistical investigation:

**a** researching bonuses in Australian businesses by interviewing a number of people who work in the finance sector

**b** researching attitudes towards parental leave by speaking to 50 new parents.

**6** Give an approximate value for the correlation coefficient for this scatterplot.




**7** The correlation coefficient between the number of Sudoku puzzles completed and the time taken to complete each puzzle is −0.80.

**a** Explain what this means in the context of the data.

**b** Considering your answer to part **b**, how accurate would you expect predictions made from the regression line for this data set to be?

## REVIEW PRACTICE EXAMINATION QUESTION

**1 a i** Draw a scatterplot for the data in this table. (2 marks)

| $T$ | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 |
|---|---|---|---|---|---|---|---|---|
| $H$ | 38 | 35 | 43 | 54 | 55 | 68 | 72 | 73 |

**ii** Draw a line of best fit by eye and estimate the value of $T$ when $H = 50$. (2 marks)

**iii** The value of the correlation coefficient is 0.97. What is the meaning of a correlation coefficient of 0.97? (1 mark)

**iv** Would predicting the value of $H$ when $T = 160$ be an example of interpolation or extrapolation? (1 mark)

**b** A scientist was investigating the relationship between the amount of medicine administrated in a trial in mL ($m$) and the number of bacteria in millions ($b$).

**i** Identify the independent and dependent variables in this situation. (1 mark)

The line of best fit for the dataset is given by the equation $b = -23m + 6325$.

**ii** What is the value of $b$ when $m = 50$? (1 mark)

**iii** What is the value of $m$ when $b = 4370$? (1 mark)

**iv** What is the value of the gradient for this equation, and what does this mean in the context of the data? (2 marks)

**v** What is the value of the vertical intercept for this equation, and what does this mean in the context of the data? (2 marks)

**vi** Does the vertical intercept make sense in the context of the data? Explain your answer. (2 marks)

**TOTAL: 15 marks**

# 6

# Network concepts

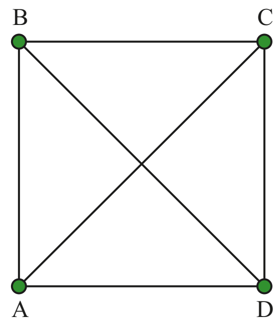
The main mathematical ideas in this chapter are:

- understanding the constituent parts of a network
- drawing network diagrams to represent information
- identifying walks, paths and cycles in a network
- finding trees and minimum spanning trees in a network
- using Kruskal's and Prim's algorithm to find a minimum spanning tree
- finding the shortest path between two vertices
- solving real-life problems using networks.

# ARE YOU READY?

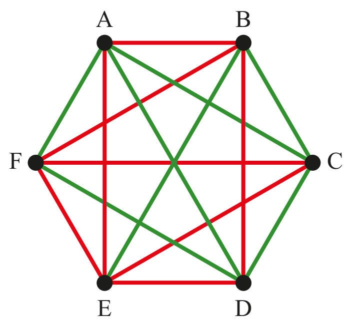


6A **1** How many line segments (straight lines between two points) are shown in the network?




**A** 4 **B** 6
**C** 7 **D** 8

Use the network below to answer questions **2** and **3**.




6A **2** Which of the following is a red triangle?

**A** ADF **B** FCE
**C** AED **D** ACD

6A **3** Which of the following is a green triangle?

**A** ADB **B** FCE
**C** AED **D** ACD

The following table contains the distance (in kilometres) between capital cities in mainland Australia. Use this table to answer questions **4** to **8**.

| | | | | | | |
|---|---|---|---|---|---|---|
| **Adelaide** | | | | | | |
| 2075 | **Brisbane** | | | | | |
| 1209 | 1267 | **Canberra** | | | | |
| 3041 | 3435 | 4034 | **Darwin** | | | |
| 732 | 1813 | 651 | 3773 | **Melbourne** | | |
| 2721 | 4434 | 3930 | 4037 | 3453 | **Perth** | |
| 1419 | 978 | 289 | 3971 | 876 | 3975 | **Sydney** |

6A **4** The distance between Brisbane and Melbourne is:

**A** 651 km **B** 1813 km
**C** 3435 km **D** 4434 km

6A **5** What is the total distance of a trip from Melbourne to Sydney, then Sydney to Canberra?

**A** 876 km **B** 940 km
**C** 1165 km **D** 1527 km

6A **6** What is the total distance of a trip from Melbourne to Canberra, then Canberra to Sydney?

**A** 876 km **B** 940 km
**C** 1165 km **D** 1527 km

6A **7** What is the smallest distance in the table?

**A** 156 km **B** 289 km
**C** 651 km **D** 4434 km

6A **8** A travelling salesman sets out from Melbourne to Brisbane. Along the way he must visit Sydney and Canberra. Which itinerary covers the shortest distance?

**A** Melbourne, Sydney, Canberra, Brisbane
**B** Melbourne, Canberra, Sydney, Brisbane
**C** They cover the same distance
**D** Impossible to tell

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1–3** **Support sheet 6A.1 Lines and line segments**
**Q4–8** **Support sheet 6A.2 Interpreting tables**

OI

# 6A Introduction to networks

These resources are available on your obook assess:
- **Video tutorial 6A:** Watch and listen to an explanation of Example 6A–3
- **Worksheet 6A:** Practise your skills with extra problems for networks
- **assess quiz 6A:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**network**
a collection of objects that are related to each other in some way

**vertex**
a visual representation of some object, usually drawn as a point; sometimes called a node

**edge**
a line between two vertices, which indicates a relationship between the vertices; sometimes called an arc

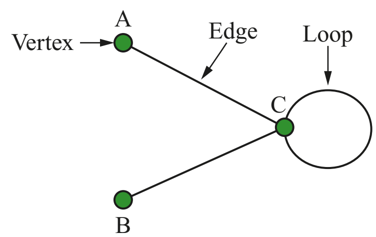
**loop**
an edge that starts and ends at the same vertex

**degree**
(of a vertex) the number of edges connected to a vertex

**Networks** deal with the idea of optimisation, which involves finding the *best possible* solution to a problem. Such as finding the *minimum* spanning tree, or the shortest path between two places.

Before we begin solving practical problems, we need to introduce some terminology.
A **vertex** is a point, and an **edge** is a line drawn between two vertices. An edge drawn from a vertex back to itself is called a **loop**.
The **degree** of a vertex is the number of edges connected to it (a loop adds 2 to the degree of a vertex).

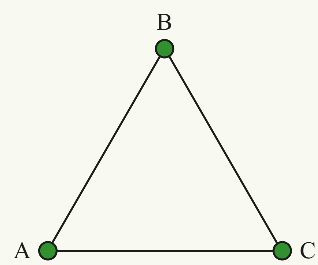
Together vertices and edges form a network.




## EXAMPLE 6A–1 Counting the number of vertices, edges and the degree of each vertex

For the network on the right, count:

**a** the number of vertices
**b** the number of edges
**c** the degree of each vertex.



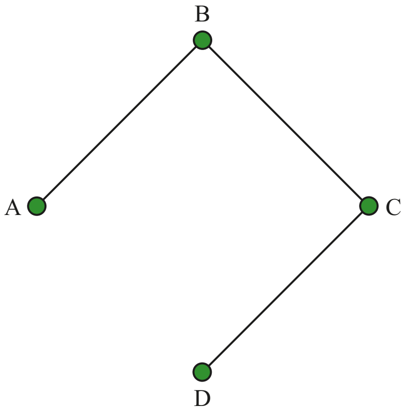

| | Solve/Think | Apply |
|---|---|---|
| **a** | There are three vertices: A, B and C. | Vertices are the points in the network. They are denoted by a single letter. Edges are the lines joining two vertices together. They are denoted by the letters at either end of the edge, or by their own name. The degree of a vertex is the number of edges connected to it. |
| **b** | There are three edges: AB, BC and CA. | |
| **c** | Vertex A is connected to two edges: AB and CA. It has a degree of 2. Vertex B is connected to two edges: AB and BC. It has a degree of 2. Vertex C is connected to two edges: BC and CA. It has a degree of 2. | |

## EXERCISE 6A Introduction to networks

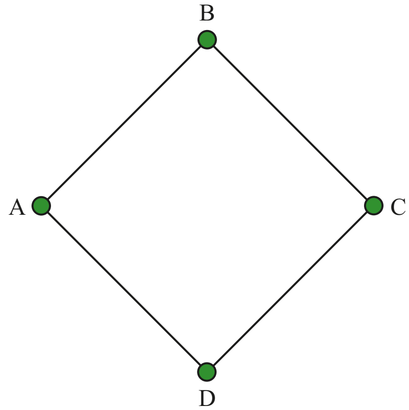
UNDERSTANDING, FLUENCEY AND COMMUNICATING

**1** Count the number of vertices in each network.

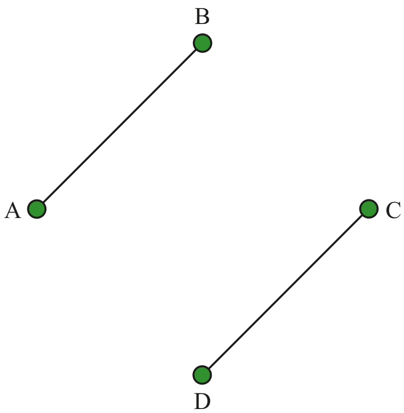
**a**



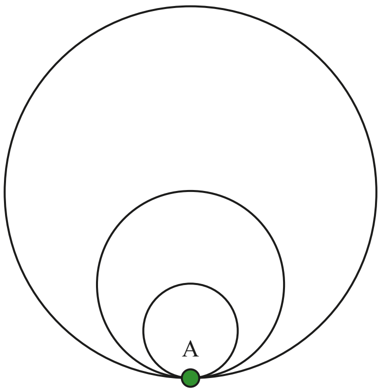

**b**



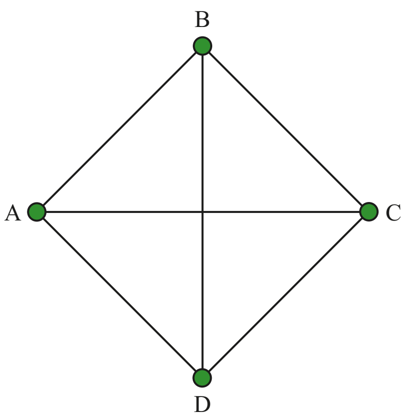

**c**



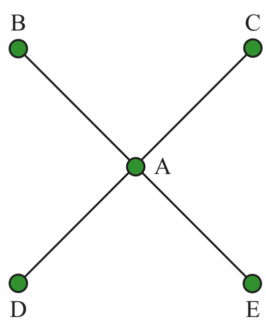

**d**



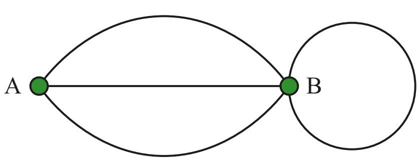

**e**



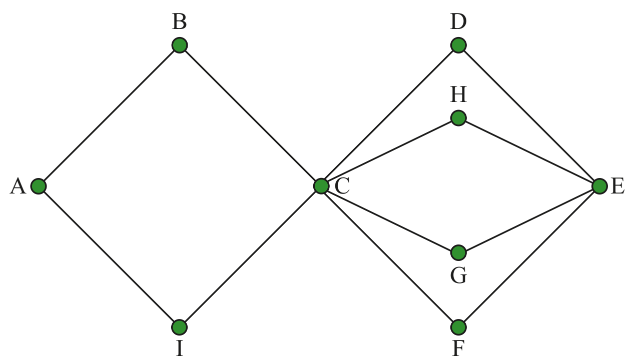

**f**




**g**




**h**

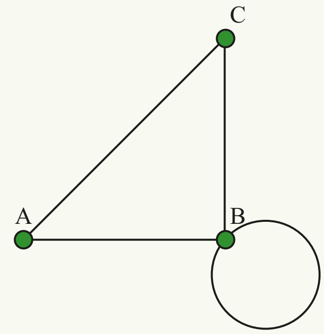



**2** Count the number of edges in each network in question **1**.

NETWORKS

## EXAMPLE 6A–2 Determining the sum total of the vertex degrees of a network

**a** Make a table of the vertex degrees for the network on the right.
**b** How many edges are in this network?
**c** What is the sum total of the vertex degrees?




| | Solve/Think | Apply |
|---|---|---|
| **a** | <table><tr><td>**Vertex**</td><td>A</td><td>B</td><td>C</td></tr><tr><td>**Degree**</td><td>2</td><td>4</td><td>2</td></tr></table> | The loop contributes 2 to the vertex degree of B. |
| **b** | The number of edges is 4. | |
| **c** | The sum of vertex degrees is $2 + 4 + 2 = 8$. | |

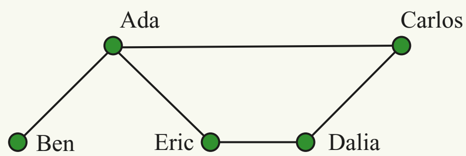
Note that the sum of the vertex degrees (8) is twice the number of edges (4). This is the case for all networks.

**3** **i** Make a table of the vertex degrees for each network in question **1**.
**ii** What is the sum total of the vertex degrees for each network? Verify that the sum total of the vertex degrees is always twice the number of edges.

## EXAMPLE 6A–3 Drawing a network from a table

Here is a list of people and their network of friends. Represent this information as a network.

| **Person** | Ada | Ben | Carlos | Dalia | Eric |
|---|---|---|---|---|---|
| **Friends** | Ben, Carlos, Eric | Ada | Ada, Dalia | Carlos, Eric | Ada, Dalia |

| Solve | Think/Apply |
|---|---|
|  | This information can be represented visually as a network by using a vertex to represent each person, and an edge to indicate that they are friends. |

Note that this is just one of many ways to draw the information in the table. Another way would be to draw the vertex for Ben inside the loop.

**4** Complete the table of the vertex degrees from Example 6A–3.

| **Vertex** | Ada | Ben | Carlos | Dalia | Eric |
|---|---|---|---|---|---|
| **Degree** | 3 | | 2 | | |

**5** The following table represents airports and the destinations serviced by that airport. Draw a network that represents this information.

| **Airport** | Albury (ABX) | Bathurst (BHS) | Coffs Harbour (CFS) | Dubbo City Regional (DBO) | Evans Head (EVH) |
|---|---|---|---|---|---|
| **Destinations** | BHS, DBO | ABX, CFS, DBO | BHS, EVH | ABX, BHS | CFS |

**6** **a** Draw a network that represents the following people and their friends.

| **Person** | Uma | Vanessa | Wasim | Xavier | Yvonne | Zhang |
|---|---|---|---|---|---|---|
| **Friends** | Vanessa, Zhang | Wasim, Uma | Vanessa, Zhang, Yvonne, Xavier | Wasim, Yvonne | Wasim, Xavier | Uma, Wasim |

**b** Complete the table of vertex degrees.

| **Vertex** | Uma | Vanessa | Wasim | Xavier | Yvonne | Zhang |
|---|---|---|---|---|---|---|
| **Degree** | | | | | | |

**c** Uma is telling Vanessa a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is about.

**d** Yvonne is telling Wasim a story about what happened to a friend of a friend of hers. Use your answer to part **a** to find out who the story is *not* about.

**directed network** a network in which the edges have a direction; the orientation of the edge is represented by an arrow

**weighted edge** edges with a weight that represents a numerical quantity associated with the relationship between the vertices

!

A **directed network** is a network in which the edges are given a direction indicated by an arrow. A directed edge indicates a *one-sided* relationship between the vertices, such as the direction of a one-way street or the direction of the flow of water.

A **weighted edge** is an edge with a certain numerical value, such as the length of the street or the size of the flow of water in litres per second. In practice, the meaning of the edge direction and weight is determined by the context of the network.

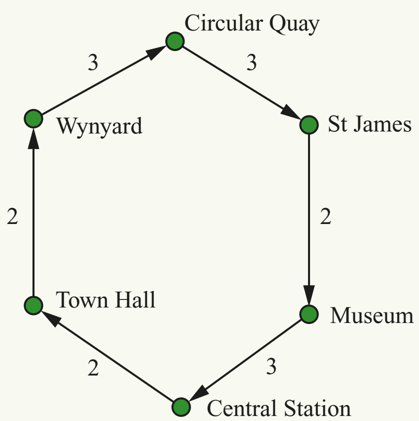
## EXAMPLE 6A–4 Determining the travel time in a network with weighted edges

The Sydney CBD stations are connected by an underground rail network called the City Circle. Trains start at Central Station and travel in the City Circle clockwise (via Town Hall) or anticlockwise (via Museum). The following table shows the stations in the City Circle via the Town Hall line, and the time it takes to travel between them.

| **Station** | Central | Town Hall | Wynyard | Circular Quay | St James | Museum |
|---|---|---|---|---|---|---|
| **Next stop** | Town Hall | Wynyard | Circular Quay | St James | Museum | Central |
| **Time (minutes)** | 2 | 2 | 3 | 3 | 2 | 3 |

**a** Represent the City Circle via Town Hall line as a directed network with weighted edges.

**b** How long does it take to travel from Central to St James using the City Circle via the Town Hall line?

| | Solve/Think | Apply |
|---|---|---|
| **a** |   | Each station is a vertex. The connections between the stations are represented by directed edges and detailed with the time it takes to travel between them. |

| | Solve/Think | Apply |
|---|---|---|
| **b** | It takes 2 + 2 + 3 + 3 = 10 minutes to travel from Central Station to St James. | Travel along this network is only permitted in the direction of the arrows, so although it would be shorter to travel from Central to St James via Museum, the direction of the edges in this network do not permit this route. |

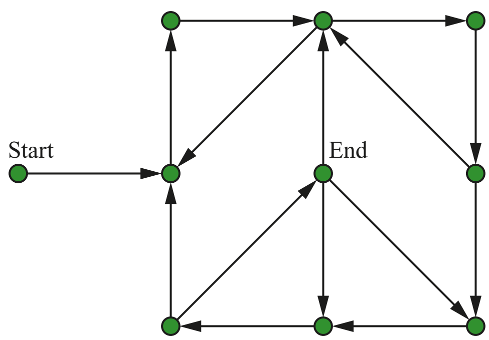
**7** Using the network from Example 6A–4, find the time it takes to travel from Town Hall to Museum using the City Circle via the Town Hall line.

Vertices in a directed network have an indegree and an outdegree, instead of a degree. The **indegree** is the number of edges pointing towards the vertex, and the **outdegree** is the number of edges pointing away from the vertex.

**indegree** the number of edges pointing towards a vertex in a directed network

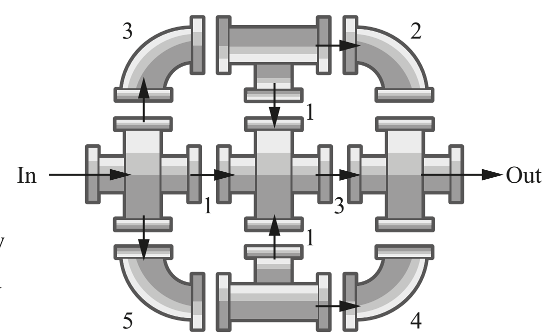
**outdegree** the number of edges pointing away from a vertex in a directed network

**8** Find your way to the centre of the labyrinth. You can only travel along edges in the direction of the arrows.

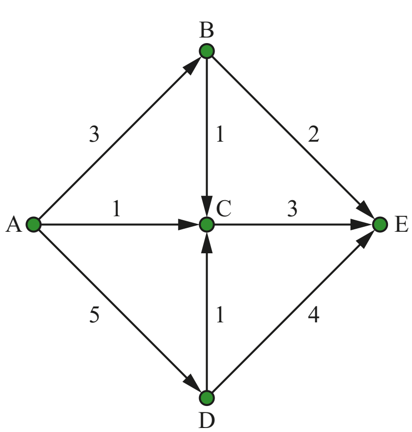



**a** How many vertices did you touch on your way (including the start and end)?
**b** How many edges did you travel along?
**c** What is the indegree of the End vertex?
**d** What is the outdegree of the End vertex?
**e** Is there more than one way from the Start to the End?

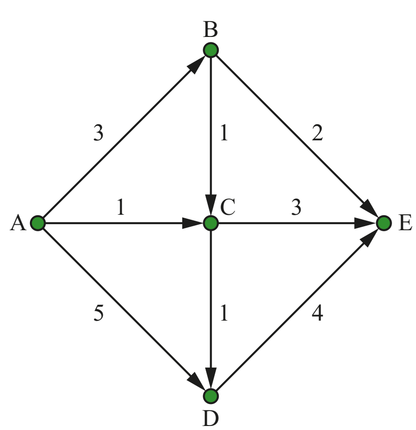
**9** A network of pipes flows from junction to junction as shown. The vertices A, B, C, D and E represent the intersection points of the pipes.
Which of the diagrams below shows the direction of the flow and the amount of the flow, in litres per second, as a directed and weighted network?



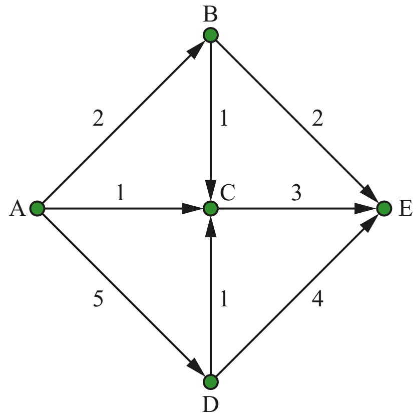

**A**



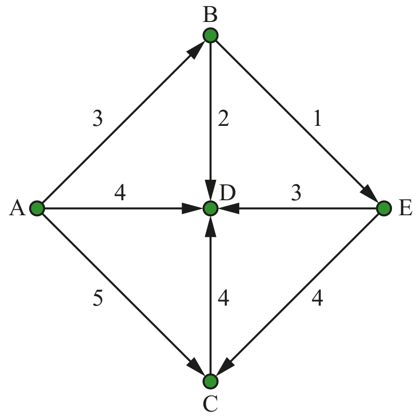

**B**



**C**



**D**



**10** For the network in question **9**, find the total flow, in litres per second that:

**a** flows out of vertex A

**b** flows into vertex E

**c** flows into vertex B

**d** flows out of vertex C

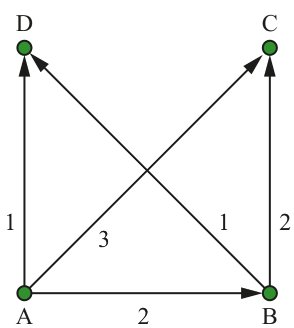
**e** flows out of vertex D.

**11** Here is a table of vertices A, B, C and D, and the edge weights between these vertices.

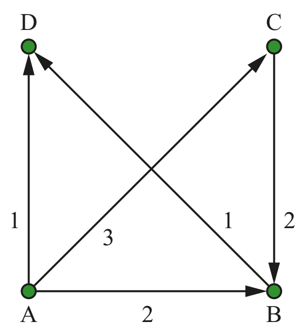
| | To vertex | | | |
|---|---|---|---|---|
| **From vertex** | **A** | **B** | **C** | **D** |
| **A** | – | 2 | 3 | 1 |
| **B** | – | – | – | 3 |
| **C** | – | 2 | – | |
| **D** | – | – | – | – |

Which of the following directed weighted networks represents this information?

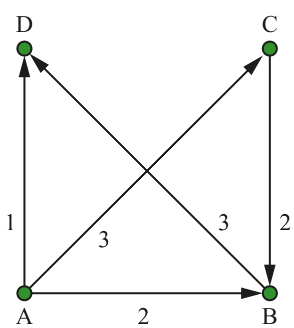
**A**



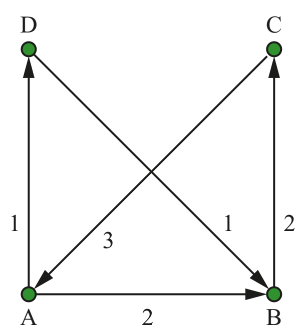
**B**



**C**



**D**



12 The table below shows the edge weights from question **9**. Fill in the blank edge weights.

| From vertex | To vertex | | | | |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** |
| **A** | – | 3 | | | – |
| **B** | – | – | 1 | – | |
| **C** | – | – | – | – | 3 |
| **D** | – | – | 1 | – | |
| **E** | – | – | – | – | – |

13 Here is a table of vertices A, B, C and D, with the edge weights between these vertices. Draw a directed weighted network to represent this information.

| From vertex | To vertex | | | |
|---|---|---|---|---|
| | **A** | **B** | **C** | **D** |
| **A** | – | – | 2 | 5 |
| **B** | 1 | – | 7 | – |
| **C** | – | – | – | 4 |
| **D** | – | 3 | – | – |

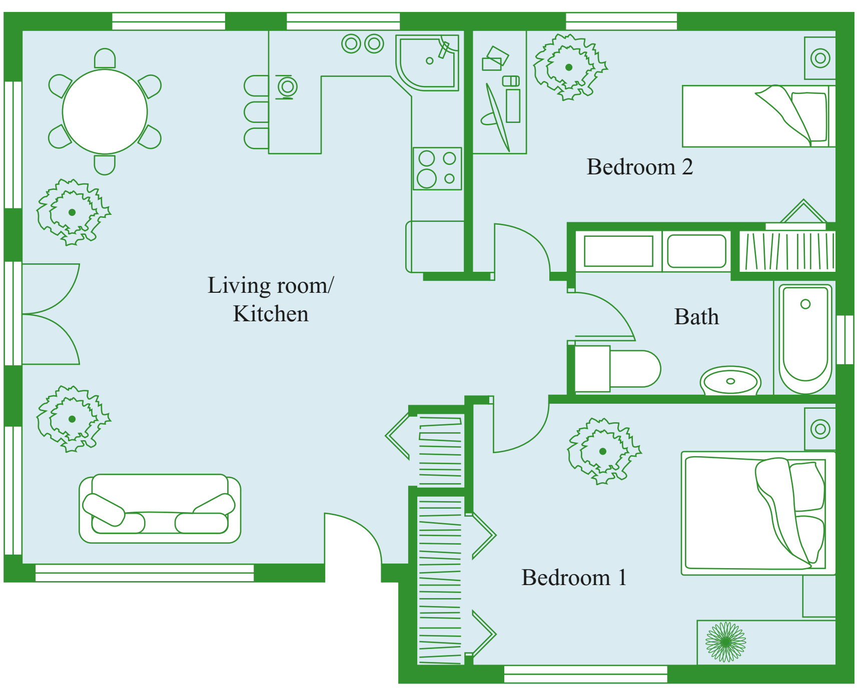
14 Imagine that the networks in question **1** represent people at a party. The people are vertices, and an edge indicates that the two people have shaken hands (a loop means that the person shakes their other hand, and so counts for two handshakes). How many people shook an odd number of hands in each network? Verify that there is always an even number of such people.

15 A house has a printer, computer, TV and gaming console. All these devices must be connected to a single router via ethernet cables.

**a** Draw a network diagram to represent this information. Use vertex names P, C, T, G and R for the printer, computer, TV, gaming console and router respectively, and use edges to represent an ethernet connection between the devices.

**b** How many ethernet connections are required at R?

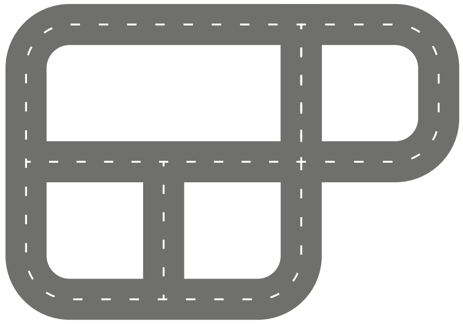
16 Draw the following house plan as a network. Use vertices to represent the rooms and edges to represent the doorways that connect the rooms.



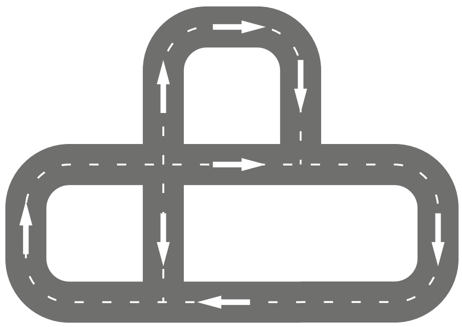

UNDERSTANDING, FLUENCEY AND COMMUNICATING

PROBLEM SOLVING, REASONING AND JUSTIFICATION

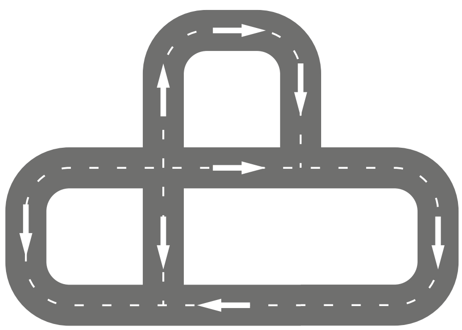
**17** Redraw the roads below as a network using vertices to represent intersections, and edges to represent the span of road connecting two intersections.



**18** Draw a directed network that represents the following street map. Represent the street intersections with vertices, and use directed edges to represent the span of road between intersections.

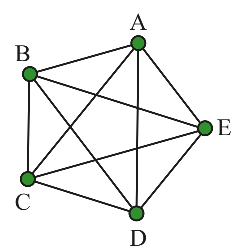


**19** Here is the same road layout, except that the street orientation has changed. Represent this as a directed network, then use the notions of indegree and outdegree to describe the problem at each intersection.



**20** Use a red and blue pen to colour over all the edges in the network on the right. Try to do this without drawing a red triangle or a blue triangle.

*Hint*: first colour as many edges in red as you can without drawing a triangle, then do the same for blue.

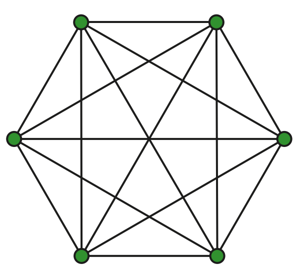





**21** **a** Colour all the edges of the network on the right in blue or red.

**b** There will always be a red triangle or a blue triangle. See if you can find it.

**c** Suppose that the vertices represent people. The blue edges connect people who have met before, and the red edges connect people who have not met before. You are hosting a dinner party and want to ensure that three people will all have met before, or three people will have never met before. What is the smallest number of people you should invite?
Hint: compare your answers to questions **20** and **21b**.



# 6B Paths and cycles

These resources are available on your obook assess:
- **Worksheet 6B:** Practise your skills with walks, paths and cycles
- **assess quiz 6B:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**walk**
a sequence of vertices and the edges between them

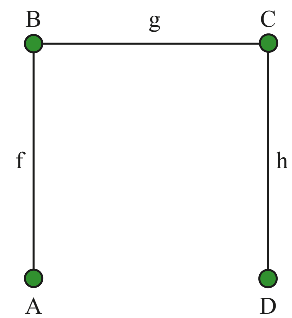
**connected network**
a network in which there is a walk between every pair of vertices

**disconnected network**
a network that is not connected

In this section we will explore walks through a network. A **walk** is a journey through a network that starts at a vertex and travels along the edges of the network, before ending at a vertex.

Trace your finger over the network on the right from vertex A to vertex D. The vertices and edges that you touched, in the order that you touched them, are:

A, f, B, g, C, h, D




This sequence of vertices and edges is a walk. It can be read as follows:

Starting from A, travel along edge f to vertex B, then along edge g to vertex C, then along edge h to vertex D. Vertex A is called the start vertex and vertex D is called the end vertex.

It is sometimes convenient to abbreviate the walk to just the vertices:

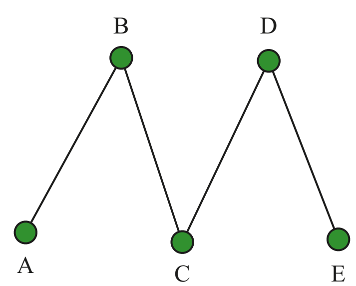
**ABCD**

The edges f, g and h are implicit in the walk that visits vertices A, B, C, D. This is convenient when the edges have no names, and we will use this style of abbreviation often in this chapter. Another style of abbreviation is to use only the edges:

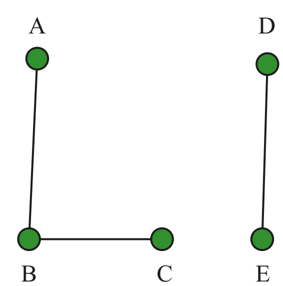
**fgh**

The vertices A, B, C and D are implicit in the walk with edges fgh. This is convenient when the edges do have names.

**Connected networks** are networks in which there is a walk between any pair of vertices. If there is a pair of vertices with no walk between them the network is said to be **disconnected**.




Connected network

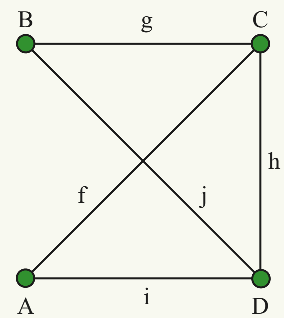



Disconnected network

## EXAMPLE 6B-1 Finding a walk in a network

Which of the following is a walk from A to B?

**A** AB
**B** A, f, C, g, B
**C** A, f, C, h, D
**D** B, j, D, i, A

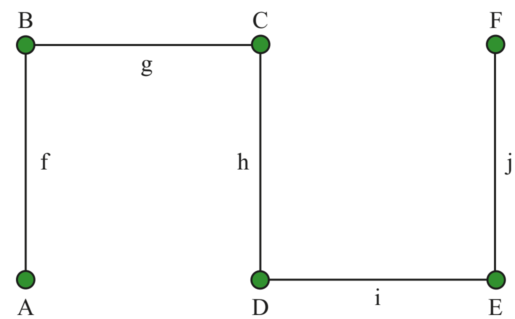



| Solve/Think | Apply |
|---|---|
| Option B (A, f, C, g, B) is a walk from A to B. | Option A is not a walk since there is no edge between A and B.<br>Option C is a walk from A to D.<br>Option D is a walk from B to A. |

# EXERCISE 6B Paths and cycles

UNDERSTANDING, FLUENCEY AND COMMUNICATING

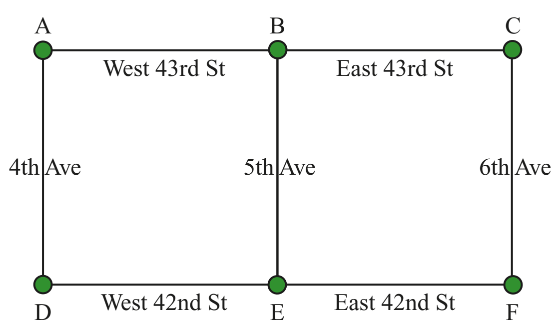
**1** Trace out a walk from A to F in the network below. Write down each vertex and edge that you visit in order.




**2** In the network from question **1**, trace out the walk with edges ihg.

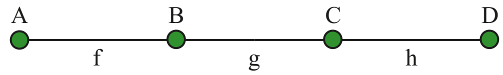
**a** What is the start vertex?
**b** What is the end vertex?

**3** The streets of Manhattan are divided into East and West by 5th Avenue. The following network is a map of Manhattan between 4th and 6th avenues and 43rd and 42nd streets.




You are given the following directions: Start at corner A. Follow West 43rd St to corner B. Follow 5th Avenue to Corner E. Follow East 42nd St to Corner F. Then follow 6th Avenue to corner C. Are these valid directions? Where do they lead?

**4** **a** Consider the walk fgh in the network below. What is the sequence of vertices visited by this walk? What are the start and end vertices?




**b** Consider the walk DCBA. What is the sequence of edges visited by this walk? What are the start and end vertices?

**c** Consider the walk ABCDCBA. What is the sequence of edges visited by this walk? What are the start and end vertices?

**path**
a walk that does not visit any vertex twice (or more)

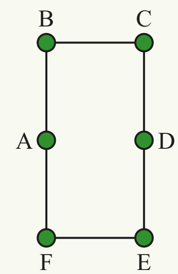
**cycle**
a walk that starts and ends at the same vertex, but does not visit any other vertices twice (or more)

A **path** is a walk that doesn't visit any vertex more than once. A **cycle** is a walk with the same start and end vertex, which doesn't visit any other vertex more than once. **!**

## EXAMPLE 6B–2 Identifying walks, paths and cycles

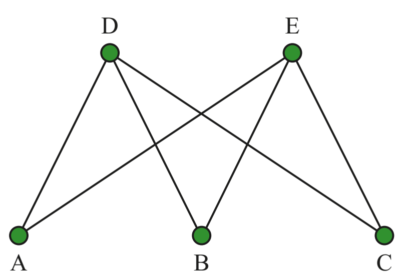
Identify the following as a walk, path, cycle, or other, giving reasons for your choice.

**a** ABCDED
**b** DEFA
**c** ABCDEFA
**d** AD




| | Solve/Think |
|---|---|
| **a** | This is a walk because it follows the edges of the network. It cannot be a path because it visits vertex D twice. |
| **b** | This is a path because it follows the edges of the network but does not visit any vertex more than once. |
| **c** | This is a cycle because it follows the edges of the network, and the start vertex and end vertex are the same. |
| **d** | There is no edge between A and D, hence this is something other than a walk, path or cycle. |
| | **Apply** |
| | In this example, the walks are referred to as a sequence of vertices. This is common practice when the edges do not have labels. |

**5** Which of the following is not a path from A to C?

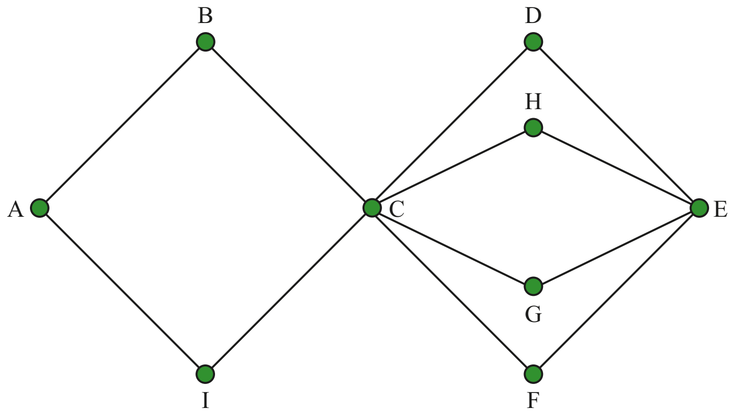



**A** ABC **B** ADC **C** AEC **D** AEBDC

UNDERSTANDING, FLUENCEY AND COMMUNICATING

**6** Which of the paths from question **5** visits every vertex?

Use the network below to answer questions **7** and **8**.




**7** **a** Find a path from A to E.

**b** Find a cycle that starts from vertex A.

**c** Find a path from F to B that includes vertex D.

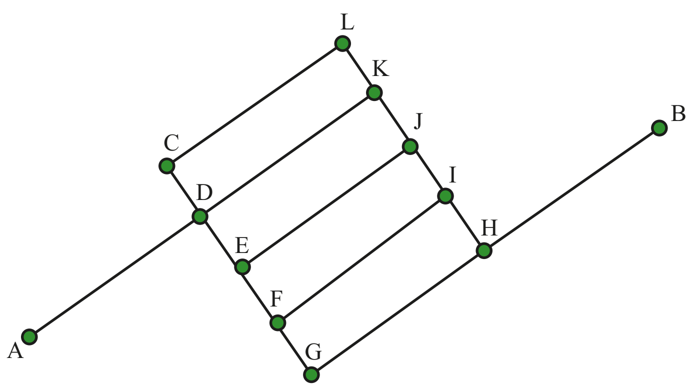
**d** Find a cycle that starts from vertex H and includes vertex F.

**8** Find a walk that visits every edge exactly once. Is your walk a cycle? Is your walk a path?

**9** The following map shows a network of roads between Broken Hill High School (A) and the local park (B).

**a** If it takes one minute to walk down the short side of a block, and three minutes to walk down the long side, how long does it take to walk from A to B?

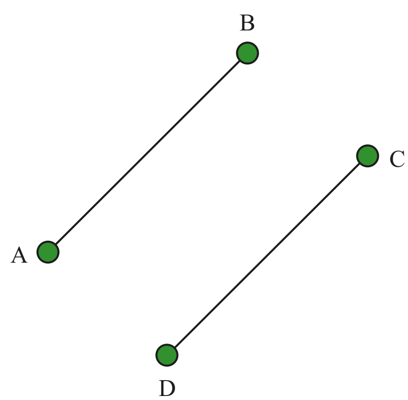
**b** How many different routes from A to B take this amount of time?




**10** The following network is not connected because there is no path between which vertices?

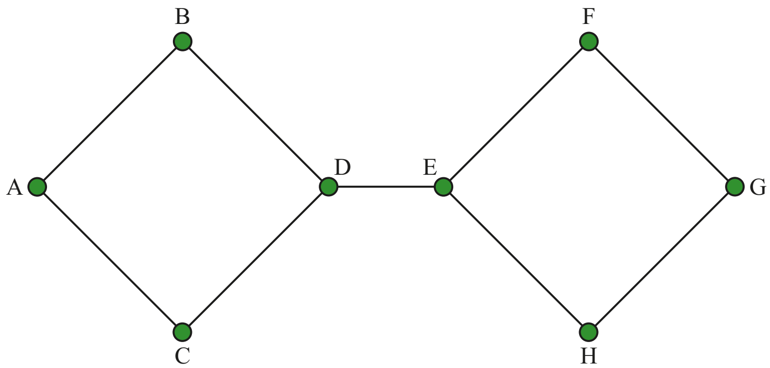
**A** A and B  **B** D and C

**C** A and C  **D** C and C




NETWORKS

Use this network to answer questions **11** and **12**.

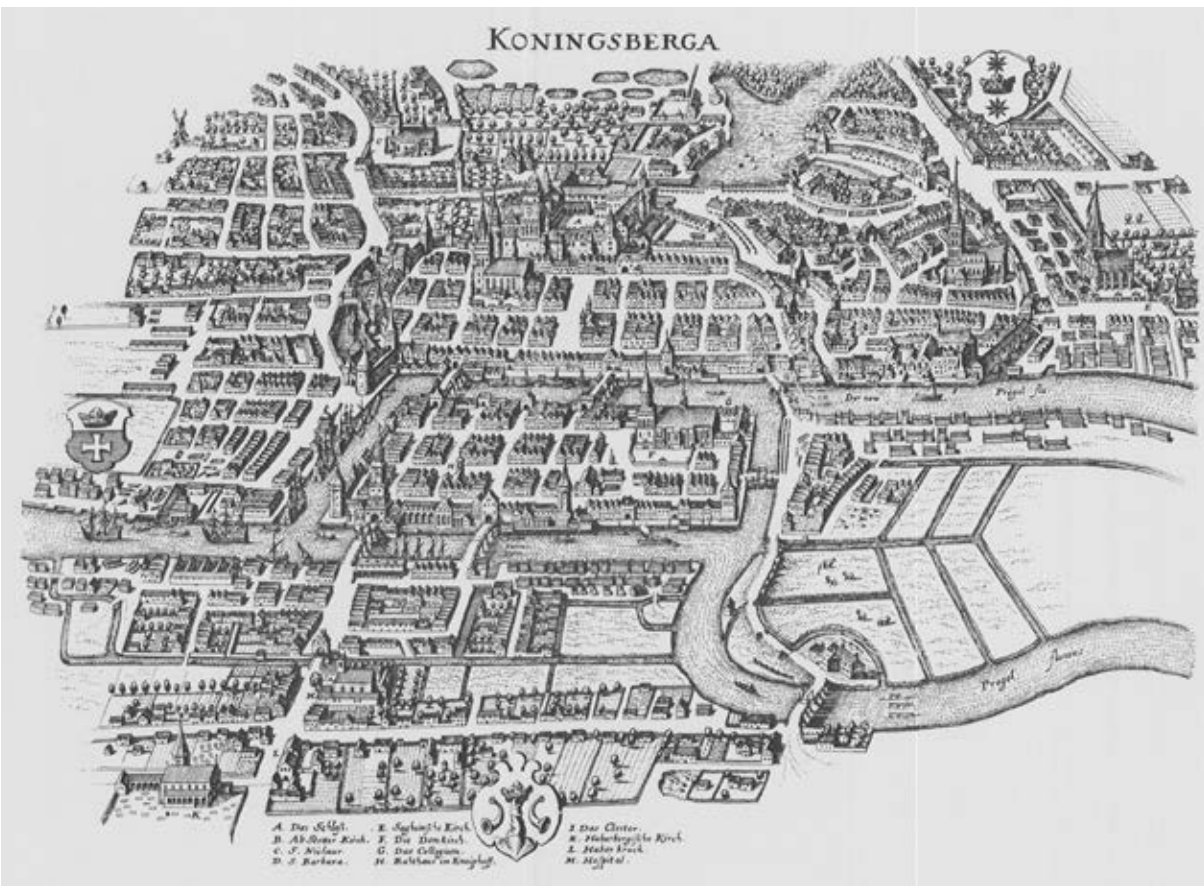



**11** Which single edge should be removed to disconnect the network?

**12** Find a walk in the network that visits every edge exactly once.

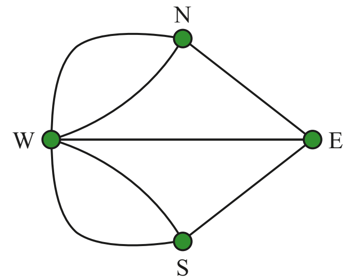
Historically, the first network paths problem is known as the **Seven Bridges of Königsberg**.

The old city of Königsberg lies on a river connected by seven bridges, as shown below. The problem is to plan a walk through Königsberg that crosses every bridge exactly once.




We can represent the city on each side of the river, and the two islands, by vertices. The bridges that connect the city can be represented by edges. This allows us to present a simplified version of the map that contains

only the relevant information. The question can then be rephrased as:
Is there a walk that visits every edge exactly once?




The answer to the seven bridges of Königsberg problem is that there is no such walk. In 1736 Leonhard Euler solved the Königsberg problem, showing that it has only to do with the number of vertices with odd degree. If there are more than two vertices with odd degree, then no walk can visit every edge exactly once.

Let's look at the vertex degrees of this network.

| Vertex | N | E | S | W |
|---|---|---|---|---|
| Degree | 3 | 3 | 3 | 5 |

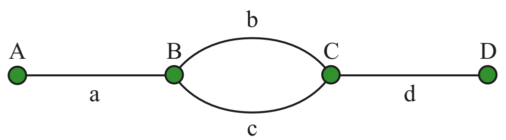
The network for this problem contains four vertices of odd degree, which is two too many! So it is not possible to find a walk that traverses each edge exactly once.

**To find a walk in a network**

If there are exactly two vertices of odd degree, then these vertices must be the start and end.

If there are no vertices of odd degree, then the start and end vertex are the same.

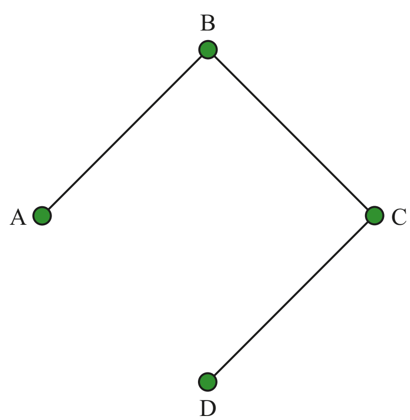
**13** There are two walks from A to D in the network below.



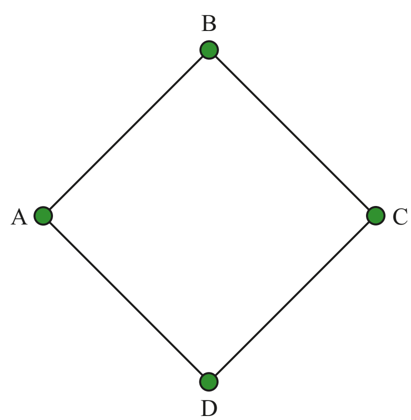

- **a** Write down a walk from A to D as a sequence of vertices and edges.
- **b** Write your answer to part **a** as a sequence of edges only.
- **c** Write your answer to part **a** as a sequence of vertices only.
- **d** Now find a different walk from A to D. Write down your walk as a sequence of vertices and edges.
- **e** Write down your answer to part **d** as a sequence of edges only.
- **f** Write down your answer to part **d** as a sequence of vertices only.
- **g** Compare your answers to parts **c** and **f**. In this example, is it a good idea to abbreviate the walks to a sequence of vertices?

**14** For each of the networks below, first draw a table of vertex degrees and then decide if the network has a walk that traverses every edge exactly once. If such a walk exists then list the vertices of the walk.

**a**

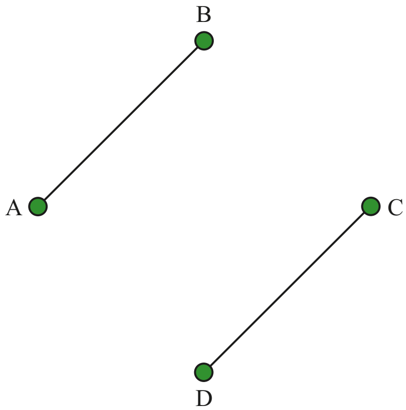



**b**

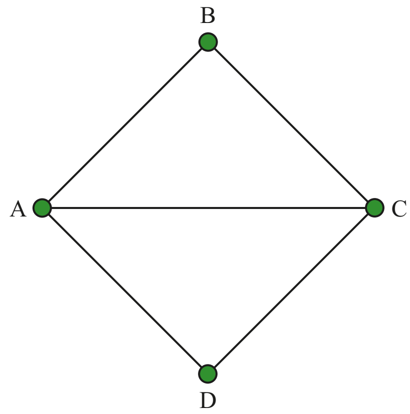


UNDERSTANDING, FLUENCEY AND COMMUNICATING

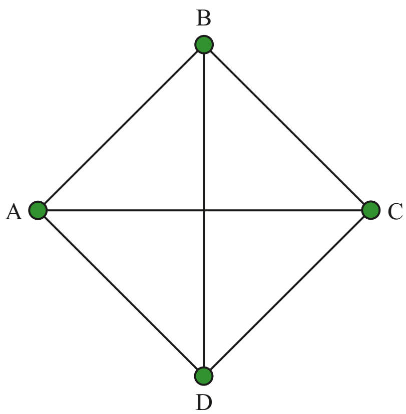
**c**



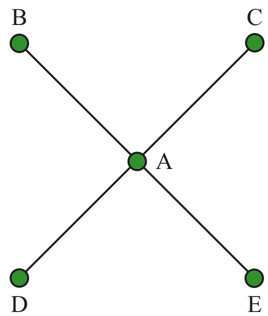

**d**




**e**




**f**



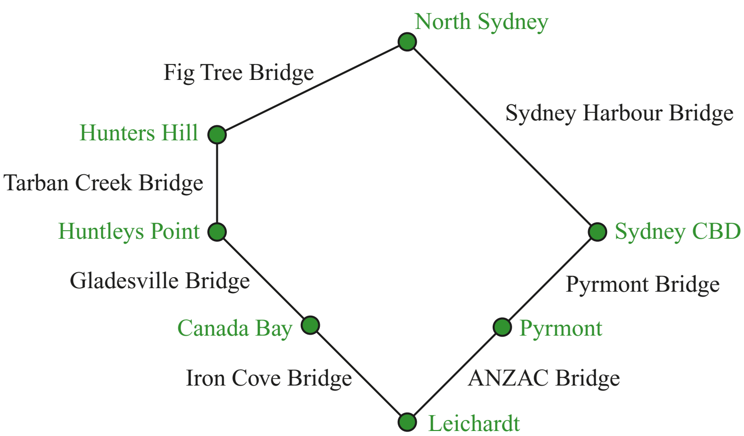

**15** For each of the networks in question **14** parts **a** to **d**, how many different paths can you find from A to D? Remember that a path cannot visit the same vertex twice.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**16** Design a garbage truck route that visits every street in the map below exactly once, or explain why this task is impossible.

**17** Design a garbage truck route that visits every street in the map below exactly once, or explain why this task is impossible.

**18** In 2017 the 'Seven Bridges Walk' toured some of the scenic bridges of Sydney.




Count the vertex degrees, then state why it is possible to cross the seven bridges of Sydney without visiting the same bridge more than once.

# 6C Trees

These resources are available on your obook assess:
- **Worksheet 6C:** Practise your skills with extra problems for trees and forests
- **assess quiz 6C:** Test your skills with an auto-correcting multiple-choice quiz

**tree**
a connected network with no cycles

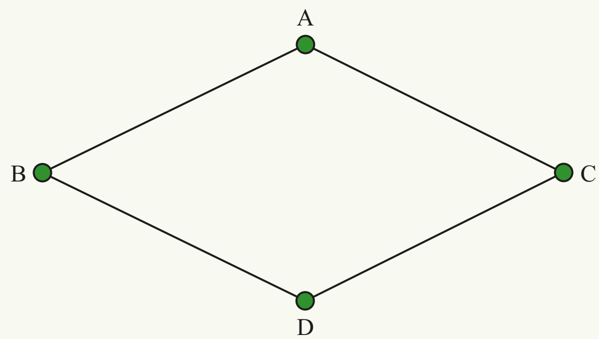
A **tree** is a special kind of network in which there is exactly one path between any two vertices. As such, a tree must be a connected network with no cycles.

## EXAMPLE 6C–1 Identifying trees

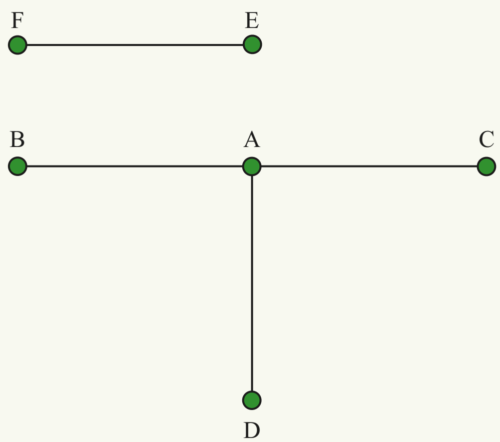
Which of the following networks are trees?

**a**


**b**




**c**

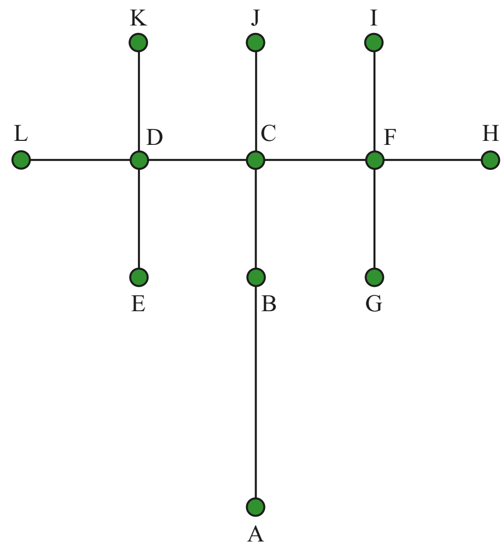



| | Solve/Think | Apply |
|---|---|---|
| **a** | This is a tree because there is exactly one path from A to B: AB, and exactly one path from B to A: BA. | A more visual way to identify a tree is to use the alternative criteria: a tree is a connected network with no cycles. So, **a** is a tree because it has no cycles and is connected, **b** is not a tree because it has a cycle ABCDA, and **c** is not a tree because it is not connected. |
| **b** | This is not a tree because there are two paths from A to D: ABD and ACD. | |
| **c** | This is not a tree because there is no path from A to E. | |

# EXERCISE 6C Trees

UNDERSTANDING, FLUENCEY AND COMMUNICATING

**1** The network given here is a tree, because there is exactly one path between any two vertices.




Find the unique path from:

**a** A to L

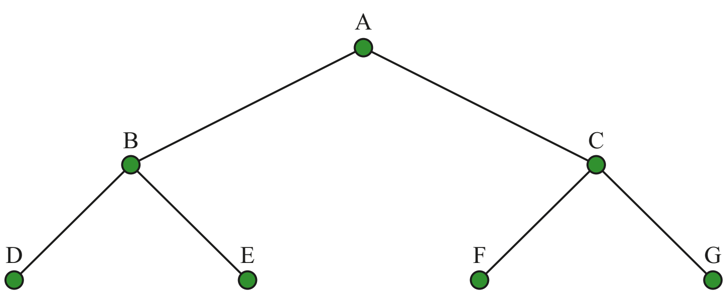
**b** L to E

**c** B to H

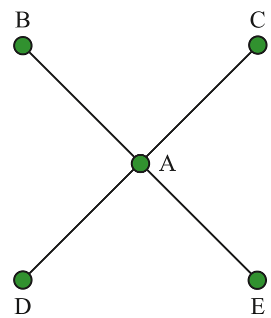
**d** G to K.

**2** Which of the following networks are also trees?

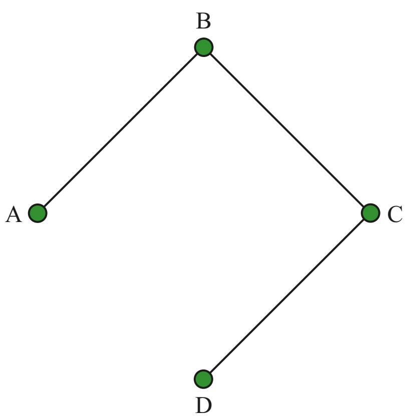
**A**



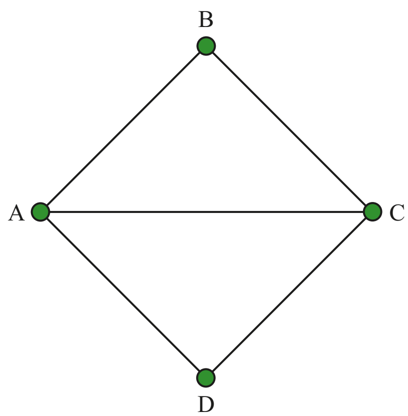

**B**




**C**




**D**




**3** Count the number of edges and vertices in each of the trees from question **2**.

**4** Draw the tree on right as a network. Use vertices to represent the end of a branch (in network terminology, such a vertex is called a **leaf**), as well as the joins between branches. Draw edges to represent the branches. Is your network a tree?

**leaf**
in a tree, any vertex of degree 1

NETWORKS

For a connected network, a **spanning tree** is a tree that contains all the vertices of the original network, and some of the edges. It is helpful to think of a spanning tree as a tree that is carved out of a larger network. Every connected network has at least one spanning tree. A spanning tree can be constructed by following these three steps:

**Step 1:** Find a cycle.
**Step 2:** Remove any edge from that cycle.
**Step 3:** Repeat steps 1 and 2 until there are no more cycles.

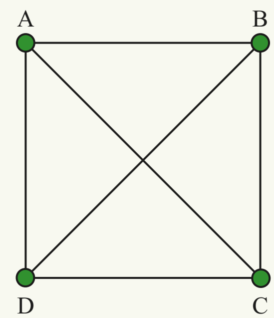
The remaining network will be a spanning tree. Note that a spanning tree always has one less edge than the number of vertices.

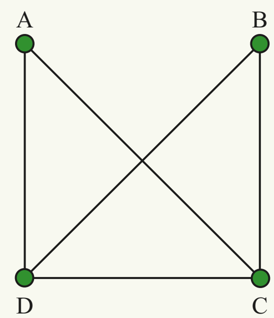
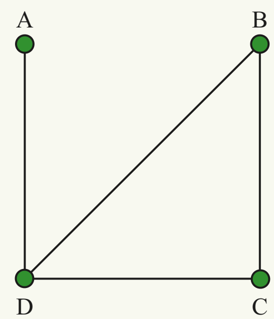
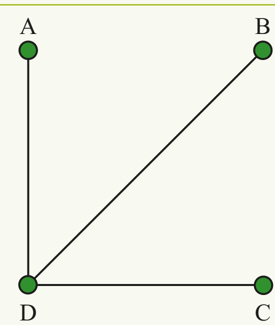
!

**spanning tree**
a tree made from all of the vertices and some of the edges of the original network

## EXAMPLE 6C–2 Constructing a spanning tree

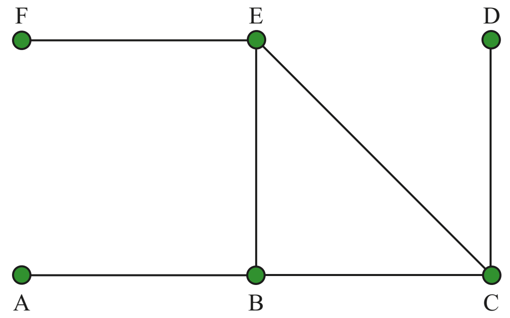
Construct a spanning tree for the following network.




| Step | Solve/Think | Apply |
|---|---|---|
| 1.1 | Identify that ABCA is a cycle. |   |
| 1.2 | Remove any edge from the cycle. In this example we have chosen to remove AB. | |
| 2.1 | Identify that ACDA is a cycle. |   |
| 2.2 | Remove any edge from the cycle. In this example we have chosen to remove AC. | |
| 3.1 | Identify that BCDB is a cycle. |   |
| 3.2 | Remove any edge from the cycle. In this example we have chosen to remove BC. | |
| Stop | There are no more cycles. | |

UNDERSTANDING, FLUENCEY AND COMMUNICATING

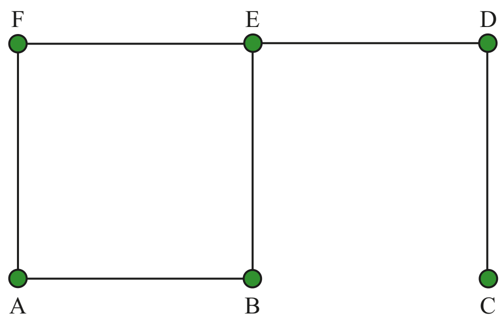
**5** Which edge should be removed to transform this network into a tree?



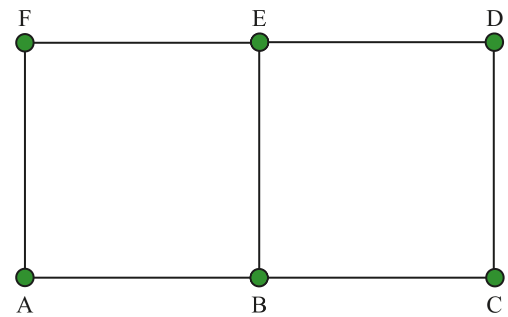

**A** AB **B** BC **C** CD **D** EF

**6** Find a spanning tree for the following networks.

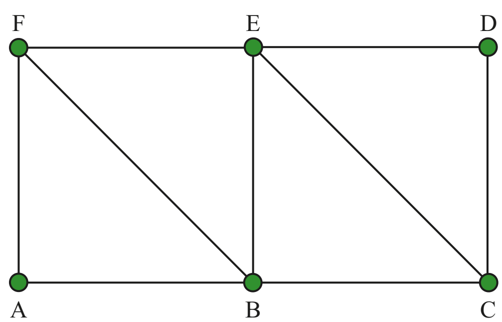
**a**



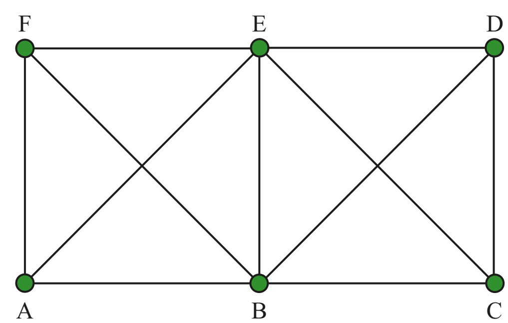

**b**



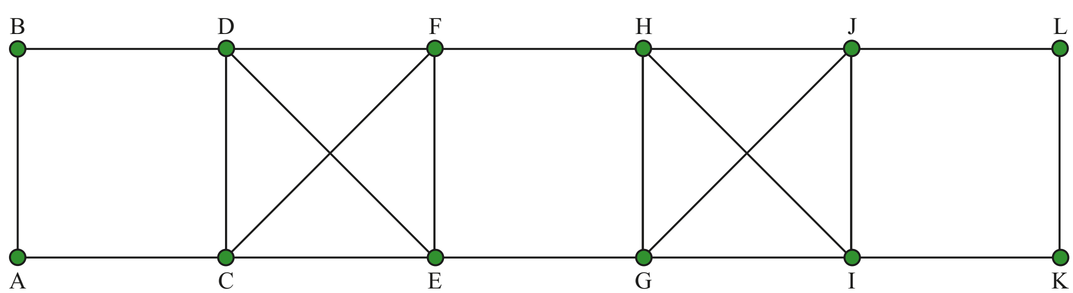

**c**




**d**



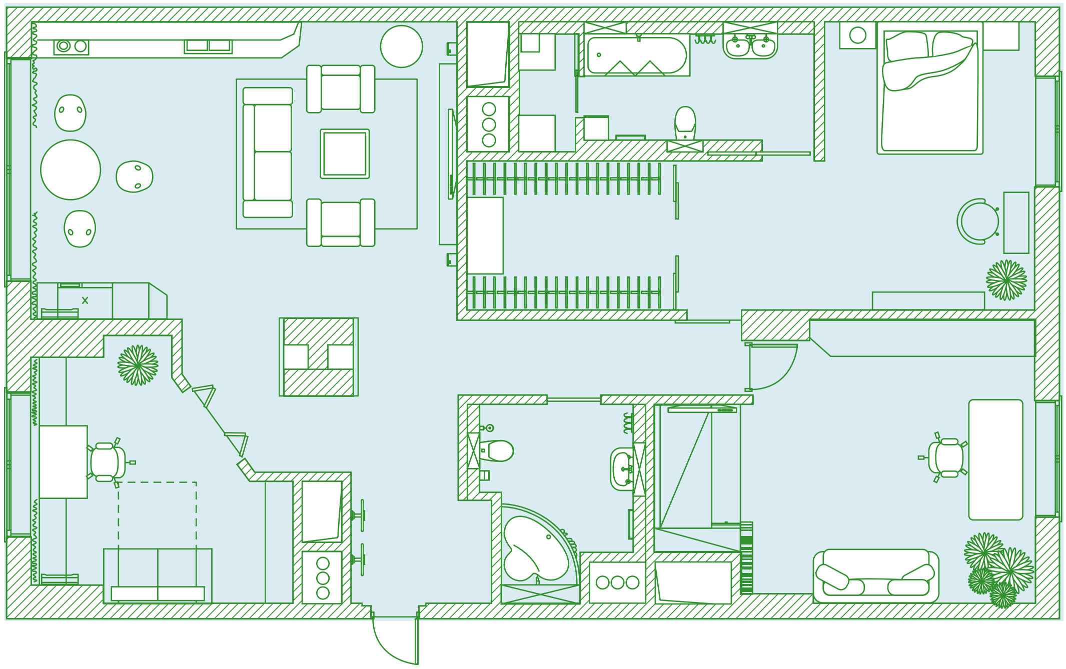

**e**




**7** Find a different spanning tree to the one you found for question **6a** to **6e** (inclusive).

**8** **a** Draw a network that represents the following house plan. Use vertices to represent the rooms and edges to join rooms connected by a door.

**b** Is the network a tree? Does that make it a tree house?



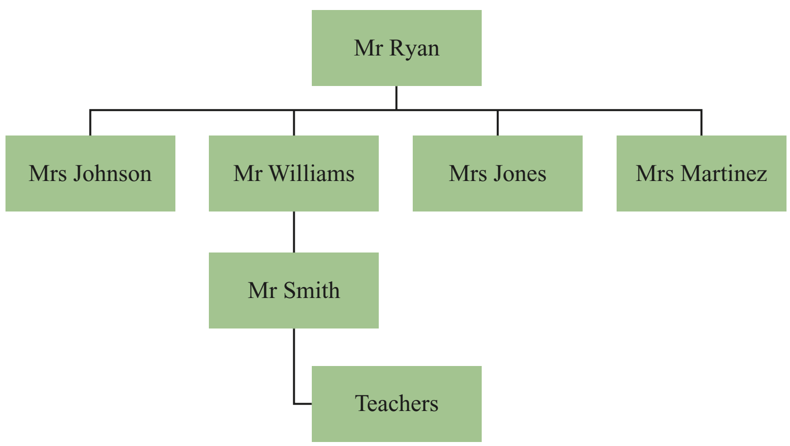
**9** The following organisational chart shows the hierarchy within Beach Road Secondary School.

**a** Draw the organisational chart as a network.

**b** Is the network a tree?

**c** A teacher reports a serious incident to Mr Smith. How many steps will it take for this report to be handed up to Mr Ryan?

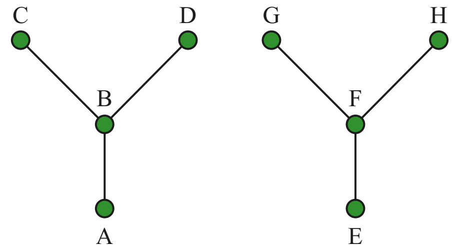
**Beach Road Secondary School**
**Organisation Chart**



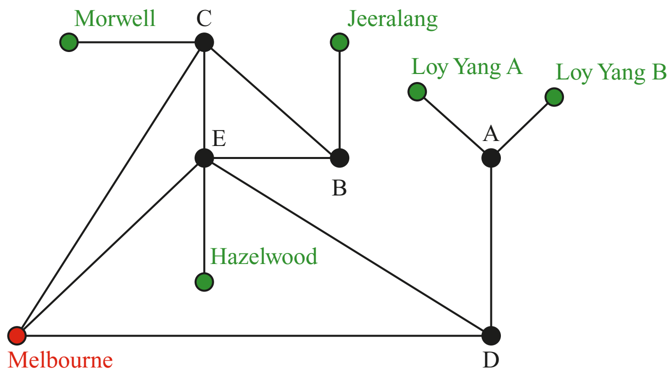

> A forest is a network with no cycles. In other words, a forest is a network made up of separate trees. **!**

**10** Which network in Example 6C–1 is a forest, but not a tree?

**11** Identify the two trees in the forest on the right. Show that the network is not connected by stating two vertices with no path between them, and then turn the forest into a tree by adding an edge between those vertices.

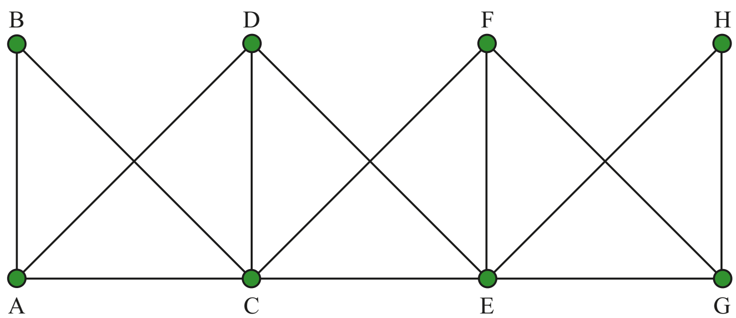



**12** The following network shows several power stations in the Latrobe Valley, Victoria. The power stations, represented by green vertices, are connected to terminals represented by black vertices. Power is transmitted to Melbourne across several transmission lines, represented by edges. Remove as many transmission lines as you can without disconnecting any power station from Melbourne.




**13** Suppose a spanning tree contains 10 vertices. How many edges are in the spanning tree?

**14** For the network shown, find a spanning tree that is also a path.




**15** For the network in question **14**, find a spanning tree that is not a path.

**16** Can a single tree be a forest? Explain your thinking.

**17** Consider a network with five vertices and no edges. Is this a forest?

**18** Consider a network with five vertices and exactly one edge.

**a** If the edge is a loop, is this a forest?

**b** If the edge is not a loop, is this a forest?

CLASS ACTIVITY

**19** Working in pairs, take turns in removing a single edge from the network in question **14**, without disconnecting the network. The winner is the first person to correctly identify when the network becomes a tree.

**20** Draw a network with at most 10 vertices, and challenge one of your classmates to a game of Network Jenga.

# 6D Minimum spanning trees

These resources are available on your obook assess:

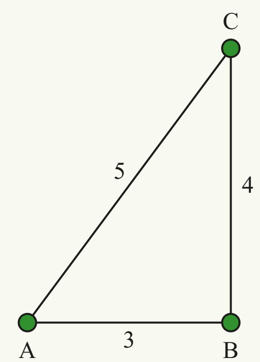
- **Video tutorial 6D:** Watch and listen to an explanation of Example 6D–2
- **Worksheet 6D:** Practise your skills with extra problems for minimum spanning trees
- **assess quiz 6D:** Test your skills with an auto-correcting multiple-choice quiz

**minimum spanning tree** the spanning tree with the smallest total edge weight

For a network without weighted edges there is no notion of a 'good' or 'bad' spanning tree. But if the network is weighted, then the spanning trees also have weight, and some trees might be preferable to others. The spanning tree with the smallest total edge weight is called the **minimum spanning tree**.

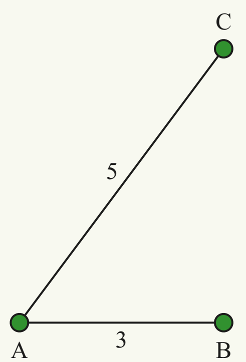
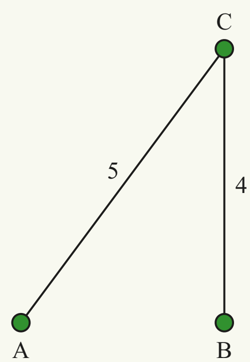
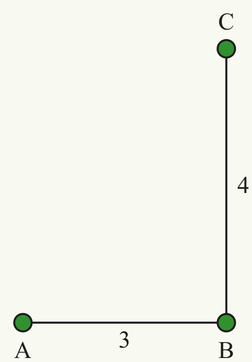
## EXAMPLE 6D–1 Determining the minimum spanning tree by inspection

The network on the right shows the distances (in kilometres) between the three solar power plants A, B and C. The power plants must be connected by transmission lines. Design the shortest possible transmission network that connects power plants A, B and C.




**Solve/Think**

The power plants can be connected by a spanning tree. There are three possible spanning trees.

| | Option 1 | Option 2 | Option 3 |
|---|---|---|---|
| **Spanning tree** |   |   |   |
| **Total distance** | $5 + 3 = 8$ | $5 + 4 = 9$ | $3 + 4 = 7$ |

The spanning tree in option 3 covers the least total distance, and hence it is the minimum spanning tree.

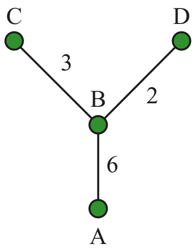
**Apply**

For a small network, we can list all possible spanning trees to determine the minimum spanning tree.

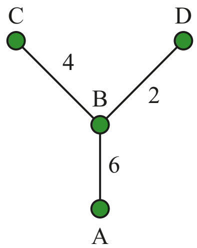
# EXERCISE 6D Minimum spanning trees

**1** What is the total weight of the following trees?

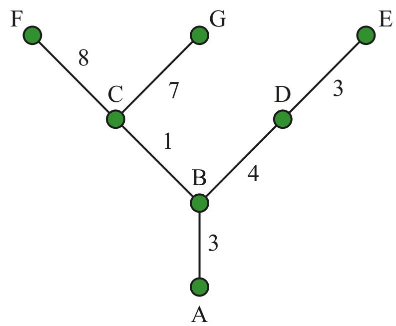
**a**



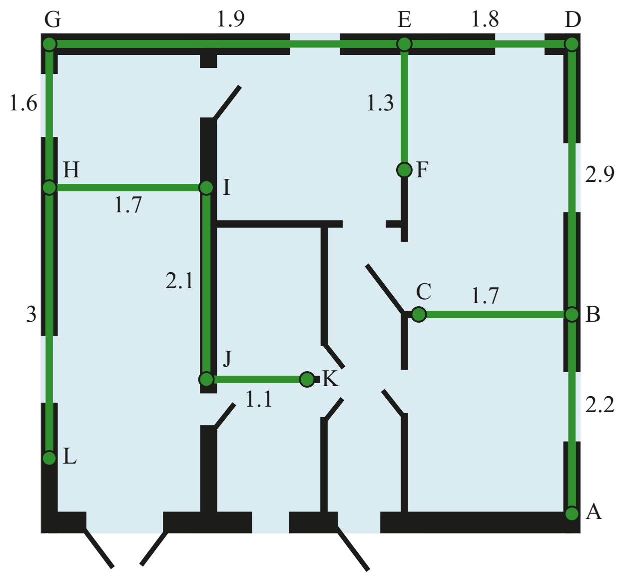

**b**




**c**

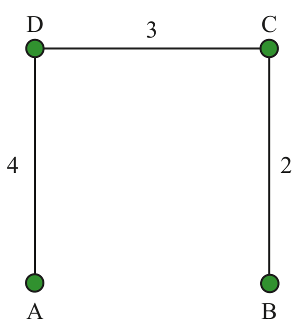



**2** The image on the right shows a house plan and the electrical cables that connect the electricity meter at A to the power outlets throughout the house. The length of wire is shown against each edge. What is the total length of wire required?

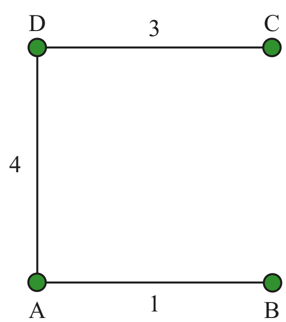



**3** The four vertices A, B, C and D can be connected in four possible configurations. What is the total weight of each configuration below? Which configuration has the smallest total edge weight?

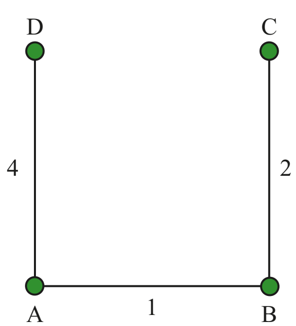
**a**



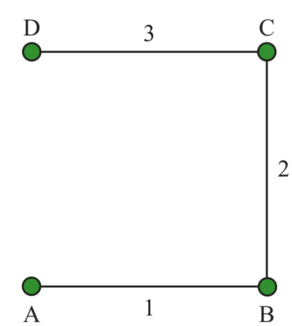

**b**




**c**




**d**



The strategy of finding all possible spanning trees and choosing the one with the least total edge weight becomes impractical for larger networks. Fortunately, there are two ways to find a minimum spanning tree that do not require listing every possible tree: Kruskal's algorithm and Prim's algorithm

**Kruskal's algorithm** was developed by Joseph Kruskal in 1956. Given a weighted network, a minimum spanning tree can be constructed as follows:

1 List all the edges of the network from smallest weight to largest.
2 Choose all of the vertices in the network, but none of the edges.
3 Proceed through your list of edges, starting with the edge of smallest weight. Add edges one by one, provided that your selection does not make a cycle.

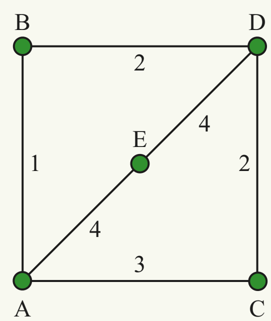
If your original network has $n$ vertices, then you can stop after selecting $n$–1 edges. The outcome is guaranteed to be a spanning tree of minimum total edge weight.

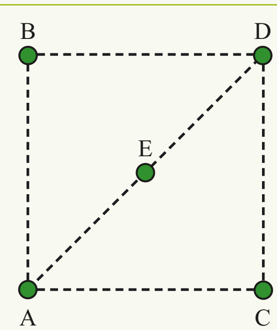
Kruskal's algorithm starts with a forest of vertices, and adds edges starting with the edge of smallest weight. At every step, the network is a forest. The final step is when the forest grows into a single minimum spanning tree.

**Kruskal's algorithm**
a method for finding the minimum spanning tree of a connected, weighted network by selecting the required edges one at a time

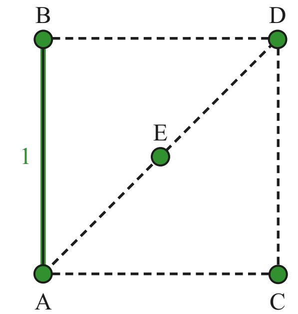
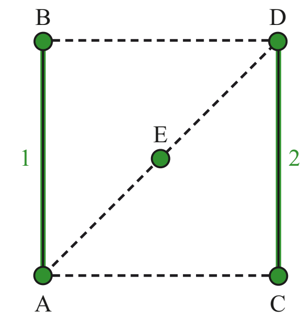
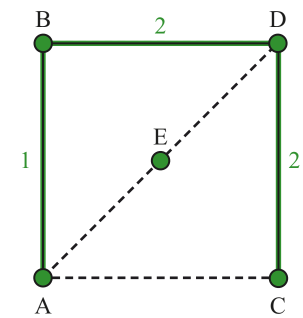
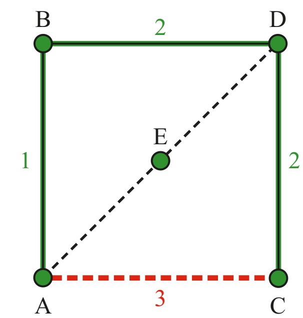
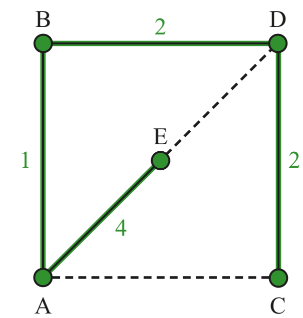
## EXAMPLE 6D–2 Using Kruskal's algorithm

Use Kruskal's algorithm to find the minimum spanning tree for the following weighted network.



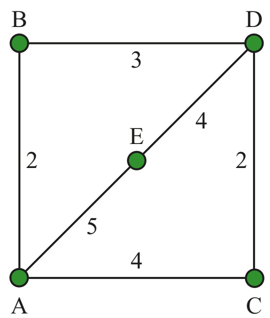

| Step | Solve/Think |
|---|---|
| 1 | The list of edge weights is as follows.<br><br>Edge: AB, DC, BD, CA, AE, ED<br>Weight: 1, 2, 2, 3, 4, 4<br><br>Proceed through this list, adding edges that do not form a cycle. The network has 5 vertices, so we stop once 5 − 1 = 4 edges have been selected. |
| 2 | Choose all the vertices.  |

| Edge | AB | DC | BD | CA | AE | ED |
|---|---|---|---|---|---|---|
| Weight | 1 | 2 | 2 | 3 | 4 | 4 |

| Step | Solve/Think | |
|---|---|---|
| 3.1 | Include the first edge AB. |  |
| 3.2 | Include the second edge DC. |  |
| 3.3 | Include the third edge BD. |  |
| 3.4 | Skip the fourth edge CA, since ABDCA would be a cycle. |  |
| 3.5 | Include the fifth edge AE. |  |
| Stop | The selected vertices and edges form a minimum spanning tree. | |

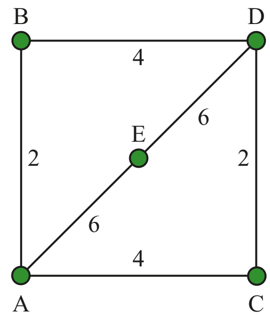
**4** The edges from each network below are listed according to their edge weight. Fill in the remaining edge weights.

**a**



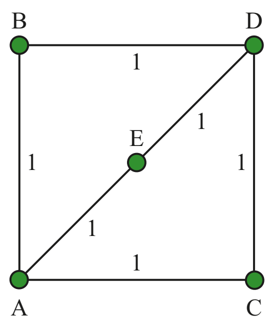

| Edge | AB | DC | BD | DE | AC | AE |
|---|---|---|---|---|---|---|
| Weight | 2 | | 3 | | | |

**b**



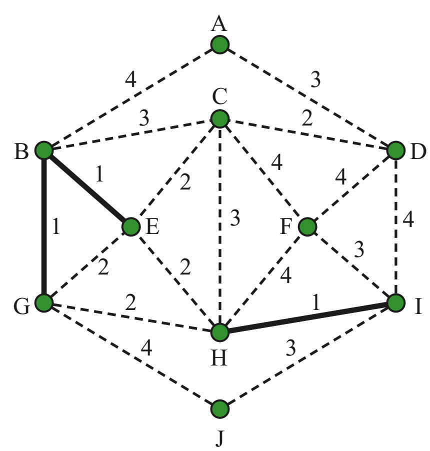

| Edge | AB | DC | BD | AC | ED | AE |
|---|---|---|---|---|---|---|
| Weight | | 2 | | | | 6 |

**c**



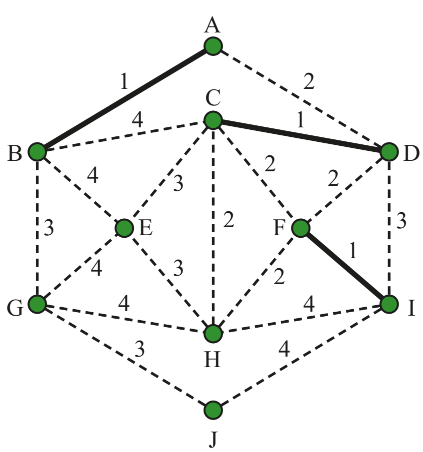

| Edge | | | | | | |
|---|---|---|---|---|---|---|
| Weight | | | | | | |

**5** At every step of Kruskal's algorithm, the selected edges and vertices form a forest of trees. The first three steps of Kruskal's algorithm have been applied to the network on the right, resulting in all edges of weight 1 being added to the forest. Now proceed through all edges of weight 2, and add as many of them as possible to the forest (i.e. without creating a cycle). Then process all edges of weight 3, and then weight 4, until your forest grows into a single tree. The tree you created is a minimum spanning tree. Since this network has 10 vertices, your spanning tree should have 9 edges.

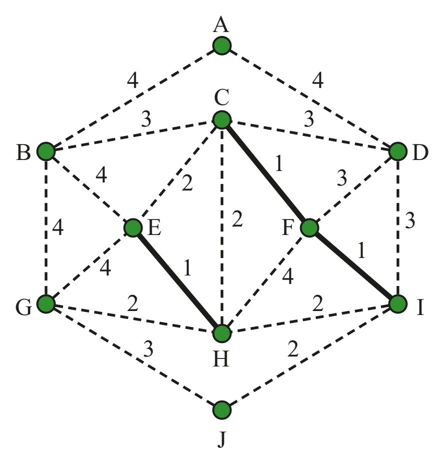



**6** Repeat question **5** for the following networks.

**a**

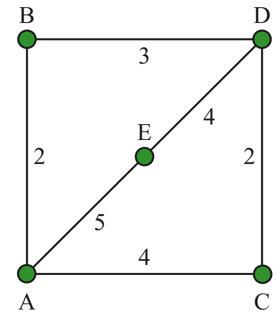



**b**

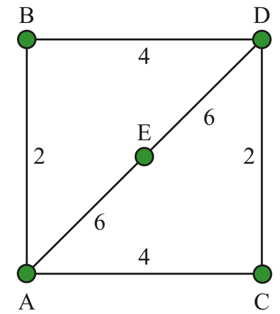


**7** Use Kruskal's algorithm to find a minimum spanning tree for the following networks.

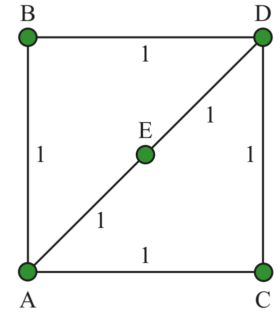
**a**



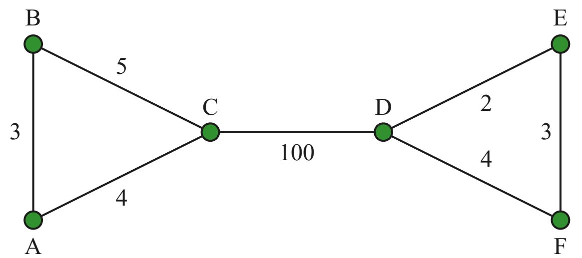

**b**



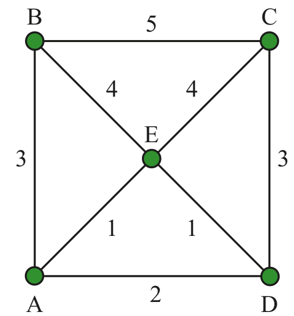

**c**




**d**




**e**




The second procedure for finding a minimum spanning tree is **Prim's algorithm**.

Starting with any vertex, Prim's algorithm selects the next 'best' edge and adds it to the tree without regard for the long-term consequences of that choice. Remarkably, this process still produces a minimum spanning tree.

The steps are as follows:

**1** Choose any vertex as the start vertex. This is the start of the tree.

**2** Of all the edges that join a vertex outside the tree to a vertex in the tree, choose the edge with *minimum weight* and add the vertex and the edge to the tree.

**3** Repeat step 2 until the tree includes all vertices, which is when it becomes a spanning tree.

When there are many edges with the same minimum weight, any one of them can be chosen.

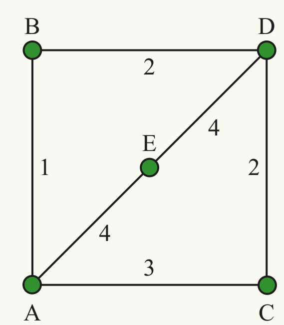
The selected edges and vertices form a tree at every step of Prim's algorithm. The algorithm continues until the tree includes every vertex. In contrast, Kruskal's algorithm contains every vertex right from the start. The two algorithms create minimum spanning trees, but in different ways.

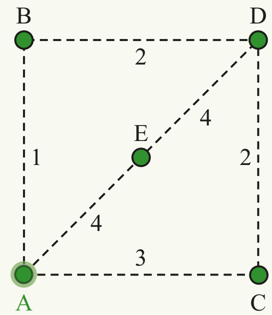
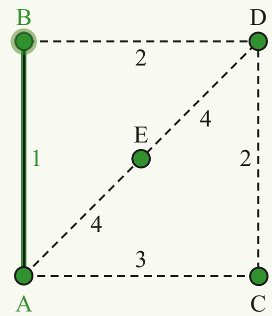
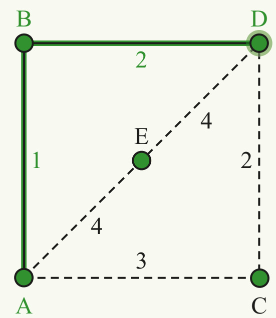
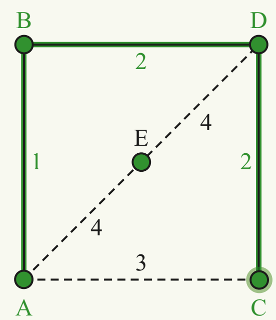
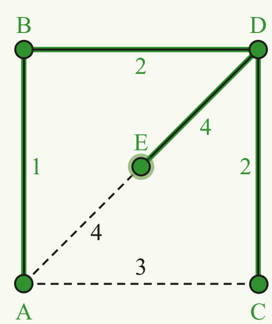
!

**Prim's algorithm** a method for finding the minimum spanning tree of a connected weighted network by iteratively adding the edge of smallest weight

## EXAMPLE 6D–3 Using Prim's algorithm

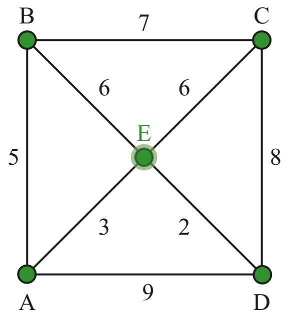
Use Prim's algorithm to find the minimum spanning tree for the weighted network on the right.



<table>
<tr><th>Step</th><th colspan="2">Solve/Think</th></tr>
<tr><td>1</td><td>Select any vertex as the start vertex. In this example vertex A is chosen.</td><td>
</td></tr>
<tr><td>2.1</td><td>Edges AB, AC and AE will connect a new vertex to the tree.
<table><tr><th>Edge</th><td>AB</td><td>AC</td><td>AE</td></tr><tr><th>Weight</th><td>1</td><td>3</td><td>4</td></tr></table>
Connect B to the tree via edge AB because it has the smallest edge weight.</td><td>
</td></tr>
<tr><td>2.2</td><td>Edges BD, AC and AE will connect a new vertex to the tree.
<table><tr><th>Edge</th><td>BD</td><td>AC</td><td>AE</td></tr><tr><th>Weight</th><td>2</td><td>3</td><td>4</td></tr></table>
Connect D to the tree via edge BD because it has the smallest edge weight.</td><td>
</td></tr>
<tr><td>2.3</td><td>Edges AE, AC, DE and DC will connect a new vertex to the tree.
<table><tr><th>Edge</th><td>AE</td><td>AC</td><td>DE</td><td>DC</td></tr><tr><th>Weight</th><td>4</td><td>3</td><td>4</td><td>2</td></tr></table>
Connect C to the tree via edge DC because it has the smallest edge weight.</td><td>
</td></tr>
<tr><td>2.4</td><td>Edges AE and DE (but not AC) will connect a new vertex to the tree.
<table><tr><th>Edge</th><td>AE</td><td>DE</td></tr><tr><th>Weight</th><td>4</td><td>4</td></tr></table>
As both edges have the minimum weight, we are free to choose either. In this example we choose DE.<br>The tree now includes every vertex.</td><td>
</td></tr>
<tr><td>Stop</td><td colspan="2">The vertices and selected edges form a minimum spanning tree.</td></tr>
</table>

**8** The first step of Prim's algorithm has been applied to the network below. In this case vertex E was chosen.

**a** Fill in the table of edges that connect a new vertex to the tree.




| Edge | EB | EC | ED | EA |
|---|---|---|---|---|
| Weight | 6 | | | |

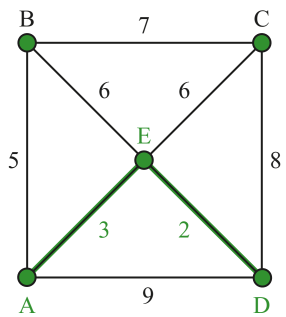
**b** Which of these edges has the smallest weight?

**9** After a few more steps of Prim's algorithm, the selected tree has grown.

Only vertices B and C remain to be connected.

**a** Complete the table of edges that connect these vertices to the tree.

**b** Which of these edges has smallest weight?

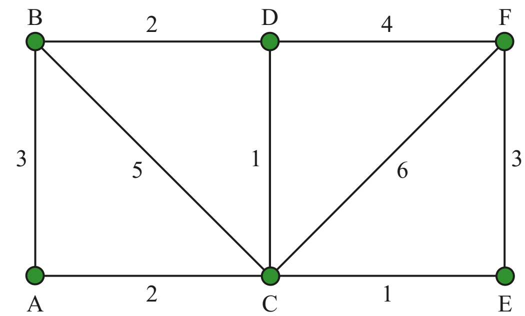



| Edge | EB | EC | AB | CD |
|---|---|---|---|---|
| Weight | | | | 8 |

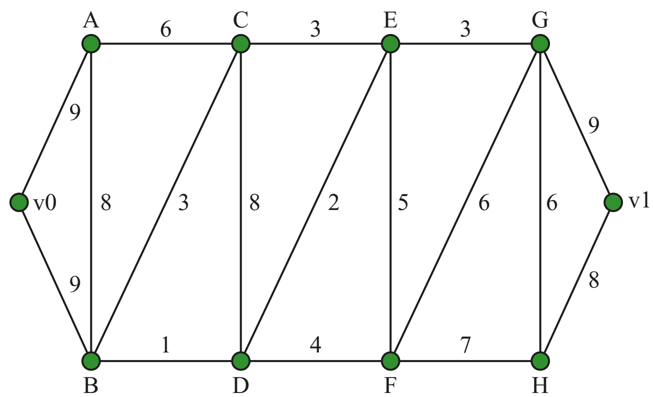
**10** What is the minimum spanning tree for the network in question **8**?

**11** Repeat question **7** with Prim's algorithm instead of Kruskal's.

**12** Find a minimum spanning tree for this network using any method.




**13** Find a minimum spanning tree for this network.

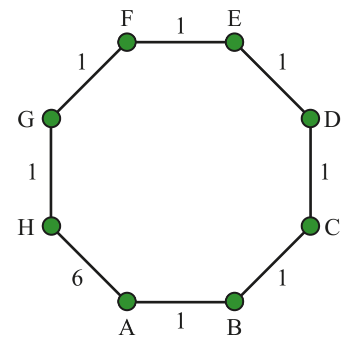


**14** A network of security cameras must be connected by wires. The cost of connecting the cameras is shown, in dollars, as the edge weight in the network on the right.

**a** What is the cheapest way to connect all eight cameras?

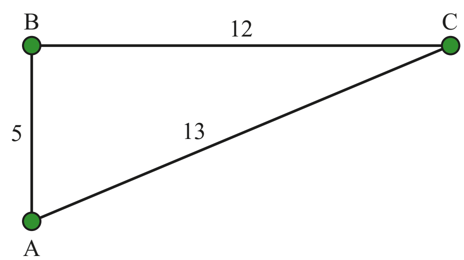
**b** What is the cheapest way to connect only cameras A and H?

**c** What is the cheapest way to connect only cameras A and H if the cost of wire between A and H is changed to 8?

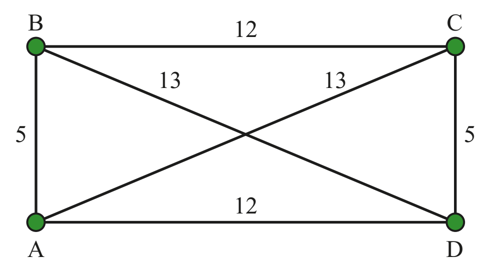



**15** Power plants and the distances between them are shown in the networks below. The power plants must be connected by transmission lines. Design the shortest possible transmission network in each scenario.

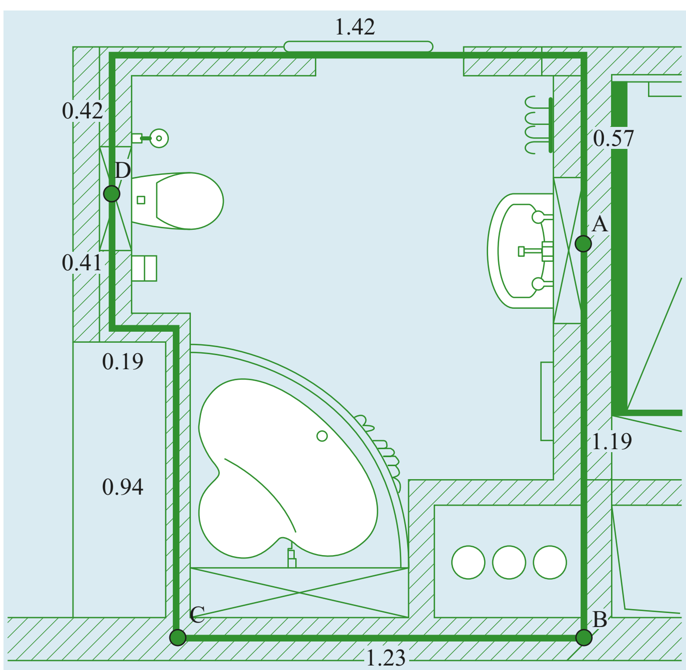
**a**




**b**

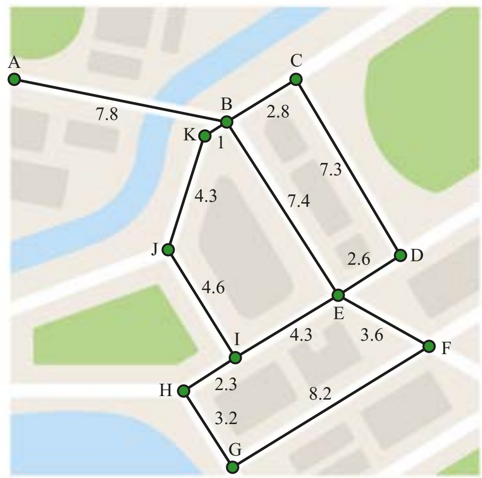



**16** This question will help you decide where to place the water pipes in the following floor plan for a bathroom, so that the sink (at vertex A), bath (at vertex C) and toilet (at vertex D) are connected to the water main (at vertex B). The distances along possible pipelines are shown. First represent this information as a network (i.e. add the distances to find the total distances between A and D, and D and C), and then use Kruskal's algorithm to find a spanning tree that minimises the total distance covered by the pipes.



PROBLEM SOLVING, REASONING AND JUSTIFICATION

**17** The network on the right shows the location of fibre optic nodes that form part of a national high-speed internet network. The possible location of the fibre optic cable is given as edges, and the length of the cable (in kilometres) is specified as the edge weight. Design the most cost-effective possible way of connecting each node (i.e. vertex) with fibre optic cable. How many kilometres of cable does your network require?




**18** The table below shows the distances between towns A, B, C and D in kilometres.

| | A | B | C | D |
|---|---|---|---|---|
| **A** | | 3.3 | 5.8 | 2.5 |
| **B** | 3.3 | | 4.4 | 1.5 |
| **C** | 5.8 | 4.4 | | 3.6 |
| **D** | 2.5 | 1.5 | 3.6 | |

**a** Represent the information in the table as a network.

**b** What would be the most cost-effective way to connect the towns to each other by rail?

**c** If the cost per km of rail is $5 million, how much would the most cost-effective railway cost?

**d** A local politician suggests rail lines along AB, BC and CD. How much would this railway cost?

**e** What is the difference in price between the most cost-effective railway and the local politician's suggested railway?

CHALLENGE

**19** The following table shows the distances (in kilometres) between mainland capital cities in Australia.

| | | | | | | |
|---|---|---|---|---|---|---|
| **Adelaide** | | | | | | |
| 2075 | **Brisbane** | | | | | |
| 1209 | 1267 | **Canberra** | | | | |
| 3041 | 3435 | 4034 | **Darwin** | | | |
| 732 | 1813 | 651 | 3773 | **Melbourne** | | |
| 2721 | 4434 | 3930 | 4037 | 3453 | **Perth** | |
| 1419 | 978 | 289 | 3971 | 876 | 3975 | **Sydney** |

**a** What is the cheapest way to construct a high-speed rail network that connects the mainland capital cities to each other?

**b** How many kilometres of rail is required?

**c** Given the information in tabular form, would you choose to use Kruskal's or Prim's algorithm?

NETWORKS

# 6E The shortest path

**shortest path** (in a network diagram) a path between two vertices in a network for which the sum of the weights of its edges is minimised

These resources are available on your obook assess:

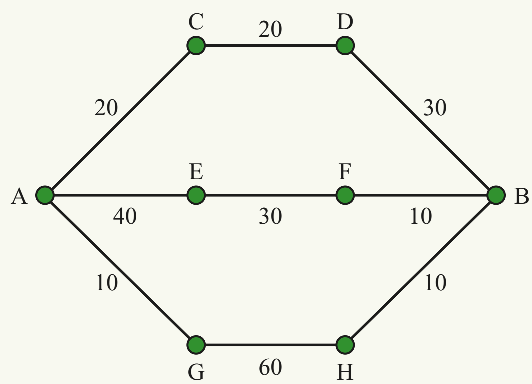
- **Video tutorial 6E:** Watch and listen to an explanation of Example 6E–2
- **Worksheet 6E:** Practise your skills with extra problems for the shortest path
- **assess quiz 6E:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

In a weighted network, the **shortest path** between two vertices is the path with the smallest total edge weight.

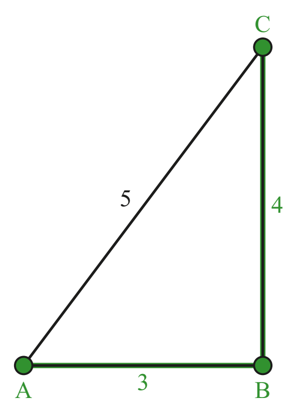
## EXAMPLE 6E–1 Finding the shortest path by inspection

The network below shows cities connected by highways. The length of each highway is indicated by an edge weight. What is the shortest path between cities A and B?




| Solve/Think | Apply |
|---|---|
| There are three paths from A to B.<br>1 Path ACDB has a length of 20 + 20 + 30 = 70.<br>2 Path AEFB has a length of 40 + 30 + 10 = 80.<br>3 Path AGHB has a length of 10 + 60 + 10 = 80.<br>The path with the shortest length is ACDB. | One way to find the shortest path is to calculate the length of every path, and then select the path with the shortest length. This method only works in practice when there is a small number of possible paths to consider. |

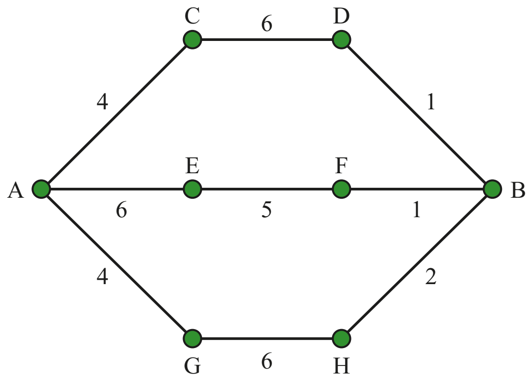
The minimum spanning tree does not always provide the shortest path between two points. For example, in the network on the right the minimum spanning tree contains the path ABC (highlighted in green), whereas the shortest path between vertices A and C is AC.



## EXERCISE 6E The shortest path

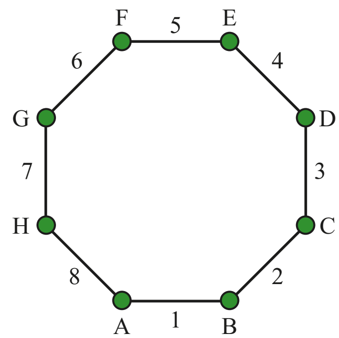


**1** Consider the following network.




**a** What is the shortest path from A to B?
**b** What is the shortest path from F to G?

**2** Consider the network on the right.




**a** Which path from A to F is shorter: ABCDEF or AHGF?
**b** Which path from A to G is shorter: ABCDEFG or AHG?
**c** What is the minimum spanning tree for this network?

For larger networks it is difficult to consider all possible paths between two vertices. Fortunately, **Dijkstra's algorithm** gives us a method to construct a special spanning tree that contains all the shortest paths from a particular start vertex to every other vertex.

Starting with any vertex, let's call it A, Dijkstra's algorithm grows a tree of shortest paths from A. The steps are as follows:

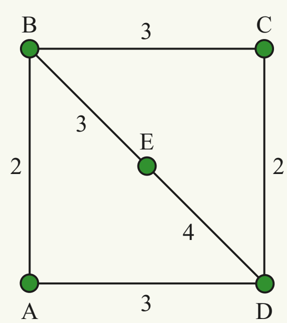
1. Make a tree that only contains the vertex A.
2. Make a list of all paths starting from A that extend the tree by exactly one edge. Choose the path that has the *minimum total weight* and add the vertex and edge to the tree.
3. Repeat step 2 until the tree includes all vertices.

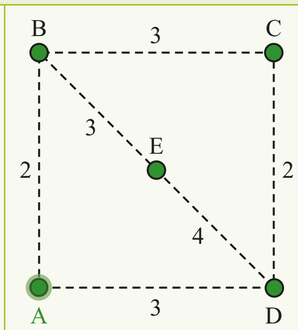
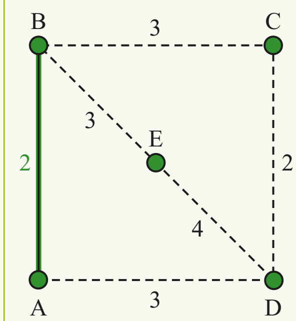
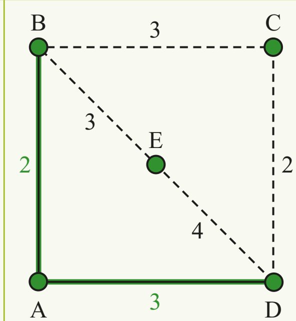
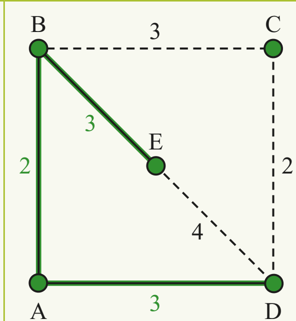
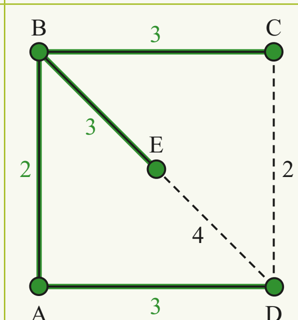
This is very similar to Prim's algorithm, except in step 2 you select a new edge based on the total distance from A rather than the individual edge weight.

**Dijkstra's algorithm**
a method for finding the shortest path between any two vertices of a connected weighted network

### EXAMPLE 6E–2 Using Dijkstra's algorithm

Use Dijkstra's algorithm to find the shortest path from A to E in the following network.



<table>
<tr><th>Step</th><th colspan="2">Solve/Think</th></tr>
<tr><td>1</td><td>Make a tree with only vertex A.</td><td>
</td></tr>
<tr><td>2.1</td><td>The tree contains only the vertex A. The list of all paths from A with exactly one vertex outside this tree is:
<table><tr><th>Path</th><td>AB</td><td>AD</td></tr><tr><th>Total weight</th><td>2</td><td>3</td></tr></table>
Path AB has minimum total weight, so add it to the tree.</td><td>
</td></tr>
<tr><td>2.2</td><td>The tree contains vertices A and B. The list of all paths starting at A with exactly one vertex outside this tree is:
<table><tr><th>Path</th><td>AD</td><td>ABE</td><td>ABC</td></tr><tr><th>Total weight</th><td>3</td><td>5</td><td>5</td></tr></table>
Path AD has minimum total weight, so add it to the tree.</td><td>
</td></tr>
<tr><td>2.3</td><td>The tree contains vertices A, B and D. The list of all paths from A with exactly one vertex outside this tree is:
<table><tr><th>Path</th><td>ABE</td><td>ABC</td><td>ADE</td><td>ADC</td></tr><tr><th>Total weight</th><td>5</td><td>5</td><td>7</td><td>5</td></tr></table>
Paths ABE, ABC and ADC all have the same minimum total weight, so we are free to choose any one and add it to the tree. In this example we have chosen to add ABE.</td><td>
</td></tr>
<tr><td>2.4</td><td>The tree contains vertices A, B, D and E. Only vertex C remains outside the tree. The paths from A with exactly one vertex outside the tree are:
<table><tr><th>Path</th><td>ABC</td><td>ADC</td></tr><tr><th>Total weight</th><td>5</td><td>5</td></tr></table>
All paths have the same weight, so we are again free to choose any one and add it to the tree. In this example we have chosen to add ABC.</td><td>
</td></tr>
<tr><td>2.5</td><td colspan="2">The resulting tree is a spanning tree that contains the shortest path from A to any other vertex. In particular, the shortest path from A to E in this tree is ABE, which has total weight $2 + 3 = 5$.</td></tr>
</table>

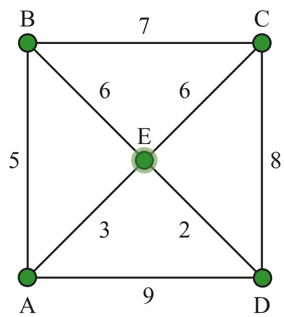
**Apply**

Follow the steps in Dijkstra's algorithm to create a spanning tree that contains all of the shortest paths in the network.

Note: we could have answered the question in step 2.3 when the path from A to E, ABE, was added to the tree of shortest paths.

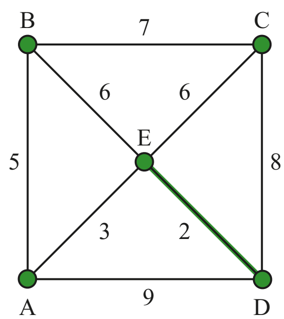
**3** Use Dijkstra's algorithm to find the shortest path from E to B in the network below.

**a** The first step of Dijkstra's algorithm is to create a tree with just the start vertex E. Below is a list of paths with exactly one vertex outside this tree. Calculate the total weight of each of these paths.



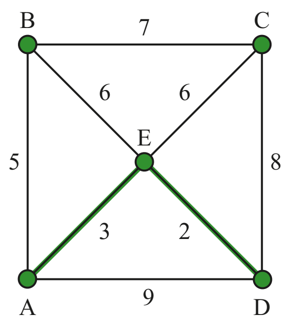

| Path | EA | EB | EC | ED |
|---|---|---|---|---|
| Total weight | 3 | | | |

**b** In the second step, the tree contains the vertices E and D, and the edge ED. Below is a list of all paths from E with exactly one vertex outside this tree. Calculate the total weight of each of these paths.




| Path | EA | EB | EC | EDA | EDC |
|---|---|---|---|---|---|
| Total weight | | | | 11 | |

**c** In the third step, the tree contains vertices A, E and D, and the edges EA and ED. Below is a list of paths from E with exactly one vertex outside this tree. Calculate the total weight of each of these paths.



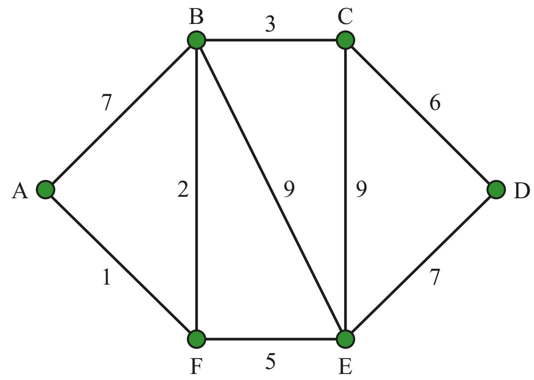

| Path | EB | EC | EAB | EDC |
|---|---|---|---|---|
| Total weight | | | 8 | |

**i** Which of the paths in part **c** has the smallest total weight?

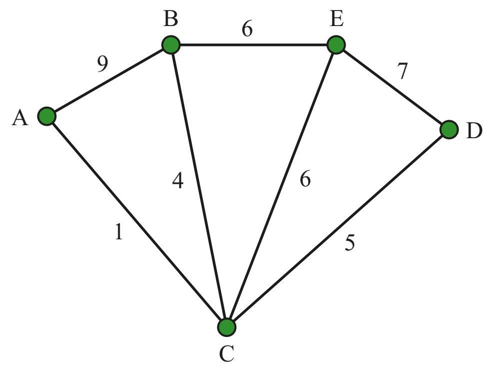
**ii** What is the minimum distance from E to B?

**4** What is the shortest path from B to E in the networks for Exercise 6D question **7**?

**5** What is the shortest path from A to D in the following network?




**6** Find the shortest distance from A to E in the following network.

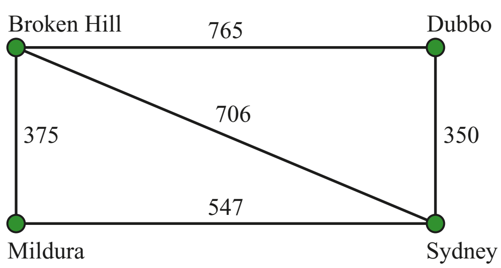




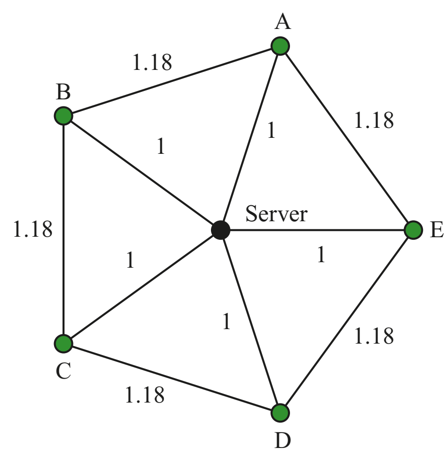

**7** The cost of flying from Sydney to certain regional centres is shown below.

**a** What is the cheapest route to Broken Hill?

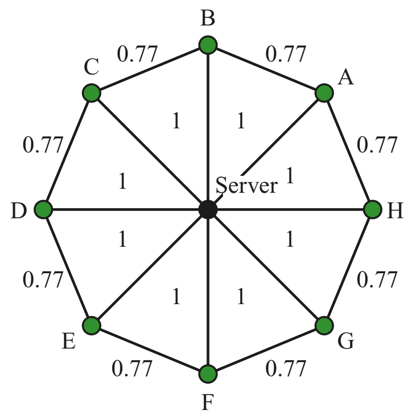
**b** Assuming the cheapest route is booked out, what is the second cheapest route?



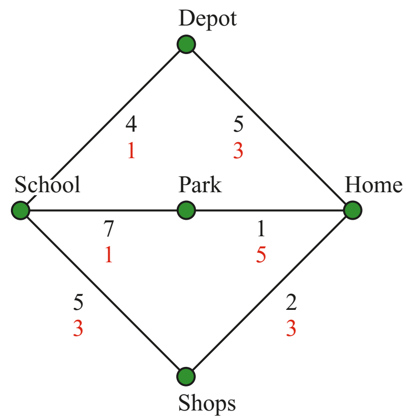

**8** The network on the right shows the distances between the five computers A, B, C, D and E and a central server. Connect each computer to the server, so that the path from the server to the computer is as short as possible. Is your selected network also a minimum spanning tree?



**9** Repeat question **8** for a network with eight computers. Is your selected network also a minimum spanning tree?




**10** The following map shows the possible routes from school to home. The distance (in kilometres) is given by the black edge weight, and the level of danger is given by the red edge weight.

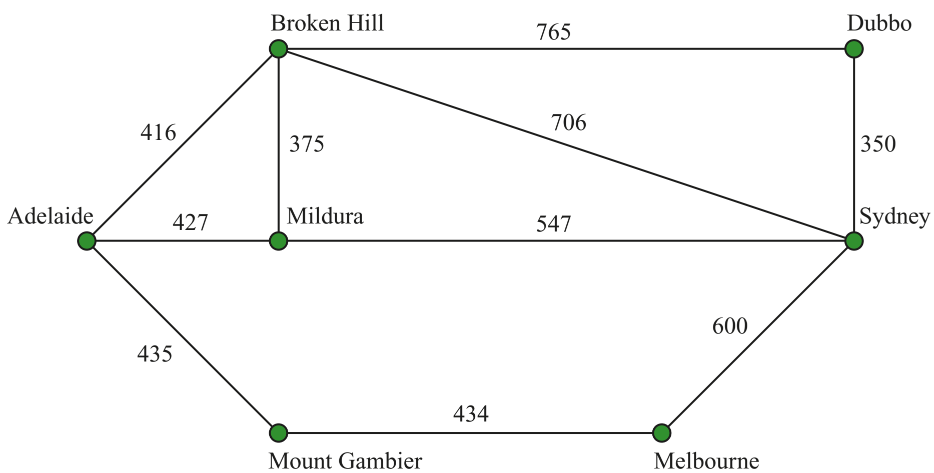



**a** What is the shortest route from school to home?

**b** What is the least dangerous route from school to home?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**11** Here is a larger map of regional flights, along with the cost of each route. What is the cheapest route in the network from Dubbo to Mount Gambier?




CHALLENGE

NETWORKS

# CHAPTER 6 REVIEW NETWORK CONCEPTS

You should now be able to:

✔ convert a table of data into a network

✔ tabulate vertex degrees

✔ understand the difference between a walk, a path and a cycle

✔ determine if a network has a walk that visits every edge exactly once

✔ calculate the weight of a path in a network

✔ find a spanning tree

✔ find a minimum spanning tree using Kruskal's and Prim's algorithms

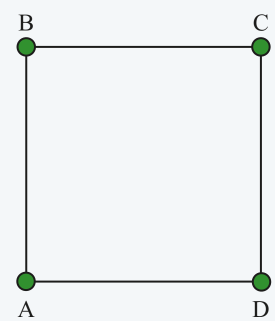
✔ find the shortest path between two vertices using Dijkstra's algorithm

✔ understand that a minimum spanning tree does not necessarily give the shortest path between two vertices.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

Use the diagram on the right to answer questions **1–3**.




6B **1** Which of the following walks is a path?

**A** ABCB **B** ABD **C** DAB **D** ABCDA

6B **2** Which of the following walks is a cycle?

**A** ABAD **B** ABCD **C** ABDA **D** ABCDA

6B **3** Which of the following walks is not a path and not a cycle?

**A** AB **B** ABA **C** BCDAB **D** ABAD

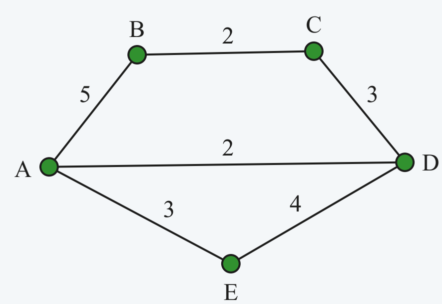
6C **4** A spanning tree contains five edges. How many vertices are in the network?

**A** 3 **B** 4 **C** 5 **D** 6

Use the diagram on the right to answer questions **5–9**.

The diagram shows the possible paths (in km) for laying gas pipes between various locations.

Gas is to be supplied from one location. Any one of the locations can be the source of the supply.




6D **5** What is the minimum total length of the pipes required to provide gas to all the locations?

**A** 10 km **B** 19 km **C** 12 km **D** 11 km

6B **6** What is the total weight of the cycle ADCBA?

**A** 7 **B** 9 **C** 10 **D** 12

6E 7 The shortest path from B to E is:

**A** BAE **B** BE **C** BCDE **D** BCDAE

6E 8 The shortest path from A to C is:

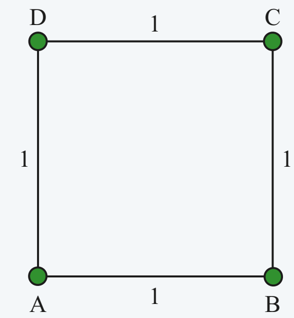
**A** AEDC **B** ABC **C** ADC **D** AC

6A 9 How many vertices have odd degree?

**A** none **B** one **C** two **D** more than two

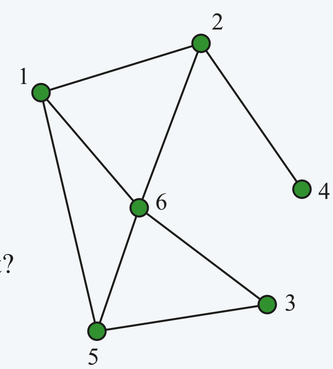
6D 10 How many different minimum spanning trees can you find for the following network?

**A** one **B** two
**C** three **D** four




# REVIEW SET 1

Use the diagram on the right to answer questions **1–3**.




**1** Complete the table of vertex degrees for the network.

| **Vertex** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Degree** | | | | | | |

**2** Is there a walk in the network that visits every edge exactly once? Why or why not?

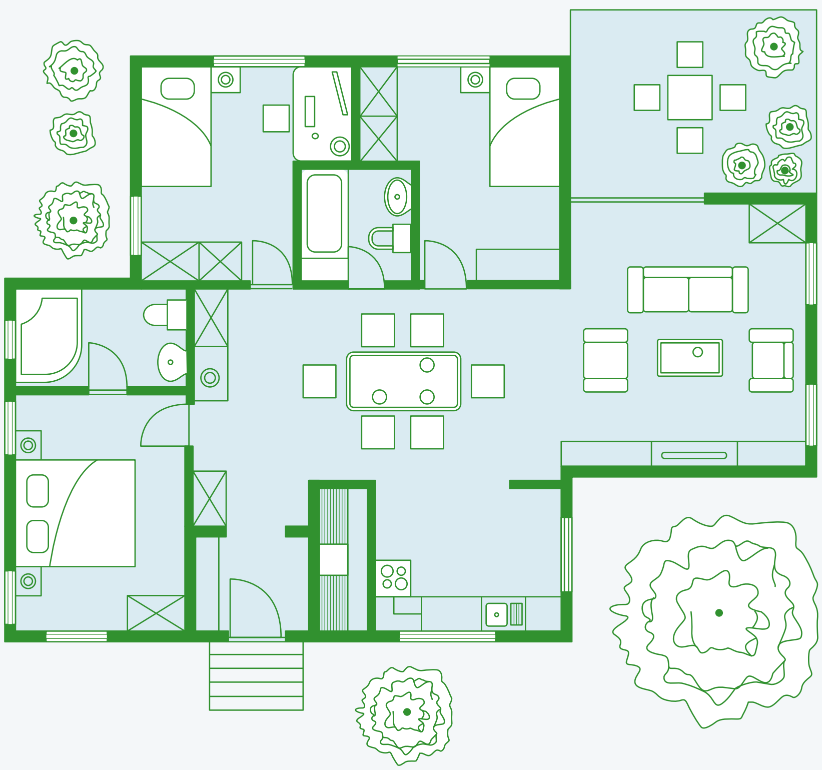
**3** Find a walk in the network that visits every vertex exactly once.

Use the diagram of the floor plan to answer questions **4–6**.

**4** Draw a network that represents the floor plan. Use vertices to represent rooms, and represent rooms connected by an internal door with an edge.

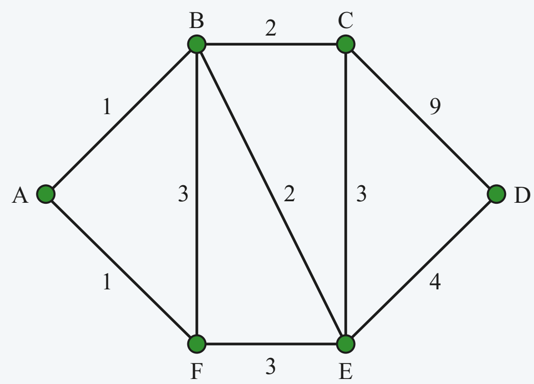
**5** Is your network a tree?

**6** Construct a table of vertex degrees for the network. Does there exist a walk through the house which passes through every internal door exactly once? Why or why not?



NETWORKS

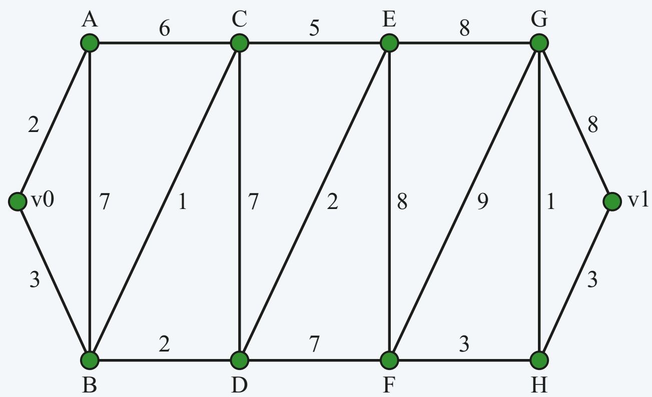
Use the diagram on the right to answer questions **7** and **8**.




**7** Use Prim's algorithm to find the minimum spanning tree for the network.

**8** What is the total weight of the path AFBCD for the network?

**9** Find the shortest path from v0 to v1.

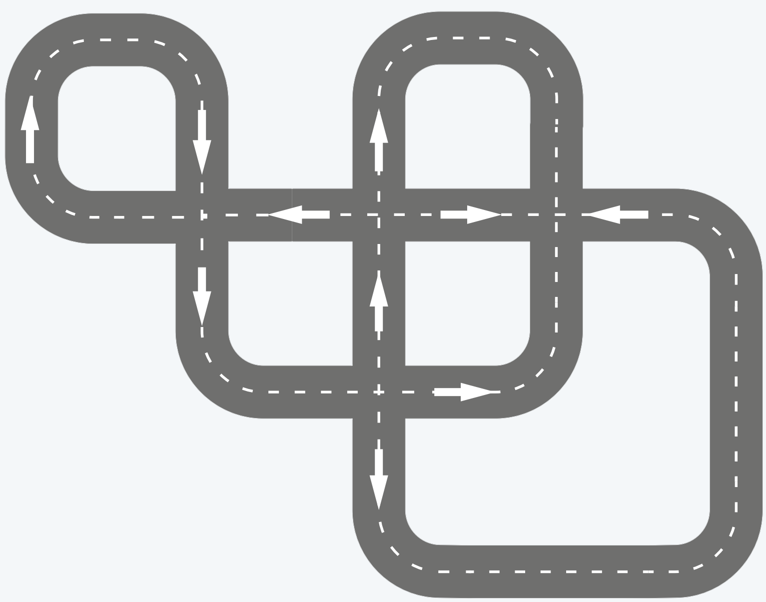



# REVIEW SET 2

**1** Represent the following table as a weighted network.

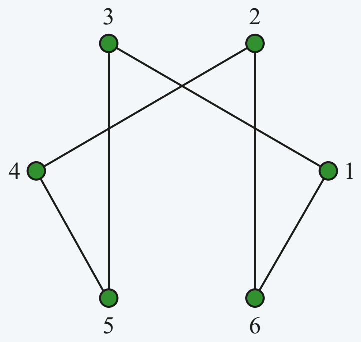
| | A | B | C | D |
|---|---|---|---|---|
| **A** | – | 2 | 8 | 9 |
| **B** | 2 | – | 7 | 4 |
| **C** | 8 | 7 | – | 1 |
| **D** | 9 | 4 | 1 | – |

**2** Draw this system of one way streets as a directed network.



**3** Using the concepts of indegree and outdegree, show that there is something wrong with the road design in question **2**.

**4** Find a cycle with six edges in the following network.



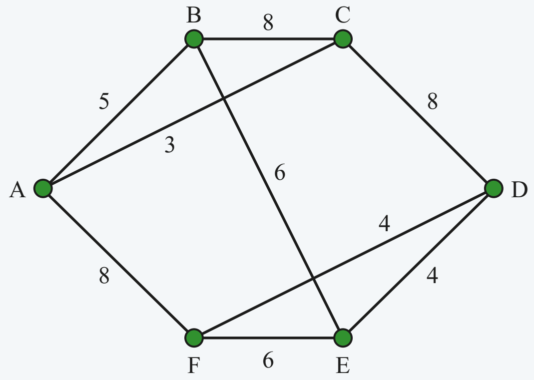

**5** Transform the cycle in question **4** into a spanning tree by removing one or more edges.

**6** Transform the spanning tree in question **5** into a forest of two trees by removing one or more edges.

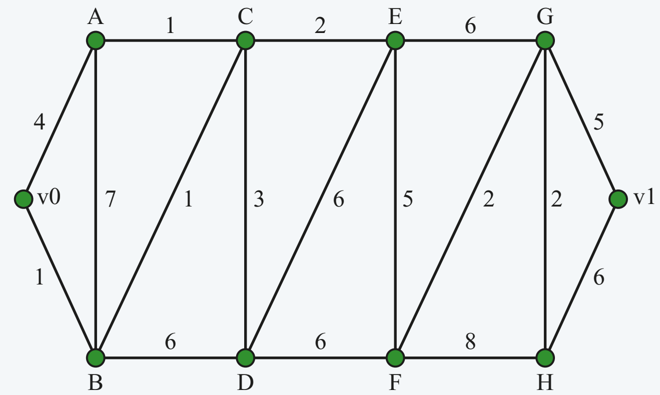
**7** Is there a walk in the network from question **4** that visits every edge exactly once?

**8** Is there a walk in the network from your answer to question **5** that visits every edge exactly once?

**9** Find a minimum spanning tree for the following network.

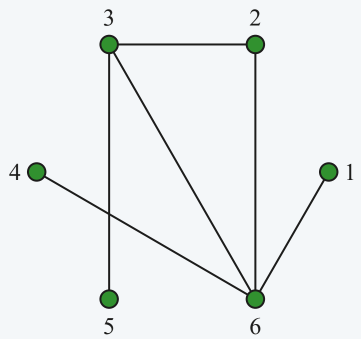



**10** Find the shortest path from v0 to v1 for the following network.



# REVIEW SET 3

Use the diagram below to answer questions **1–2**.

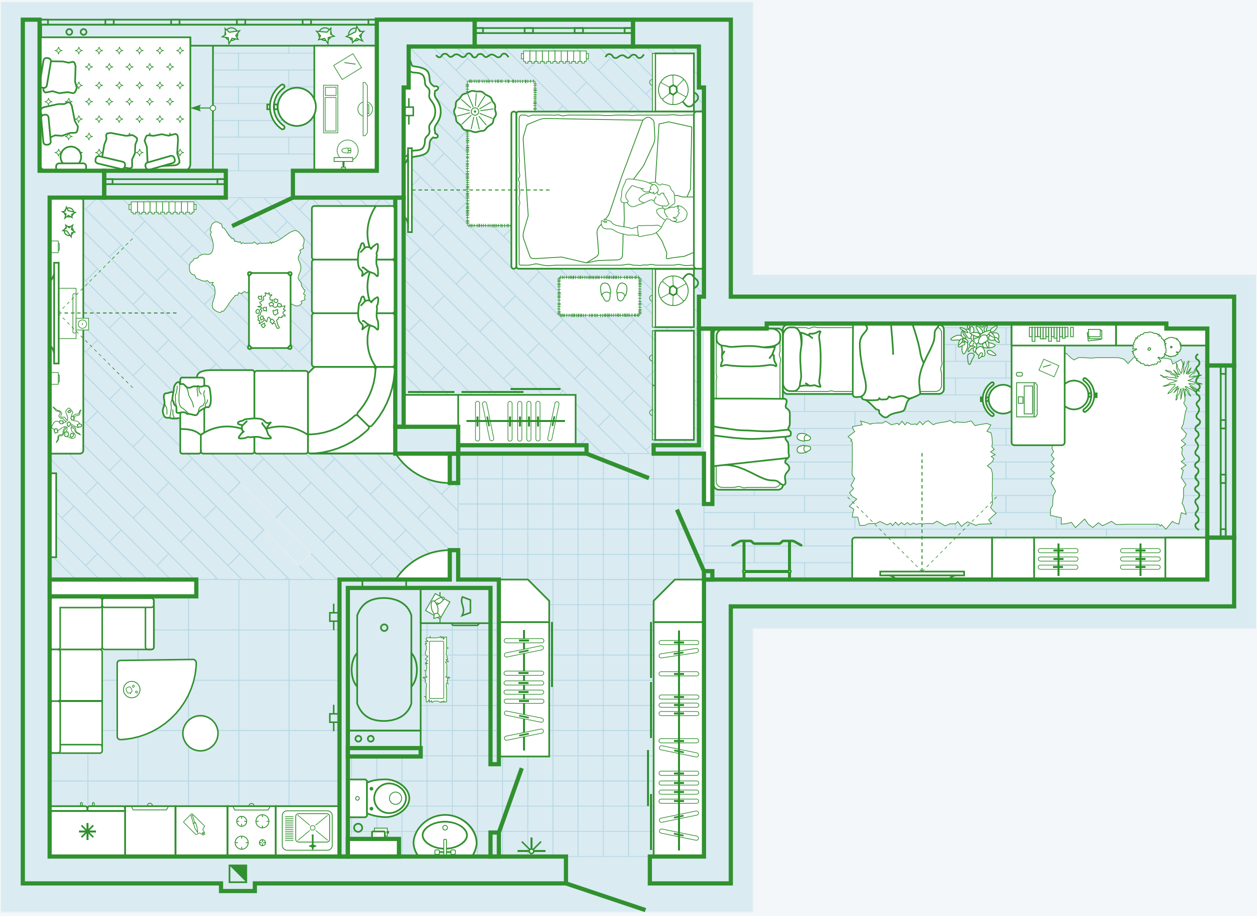



**1** Complete the table of vertex degrees for the network.

| **Vertex** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Degree** | | | | | | |

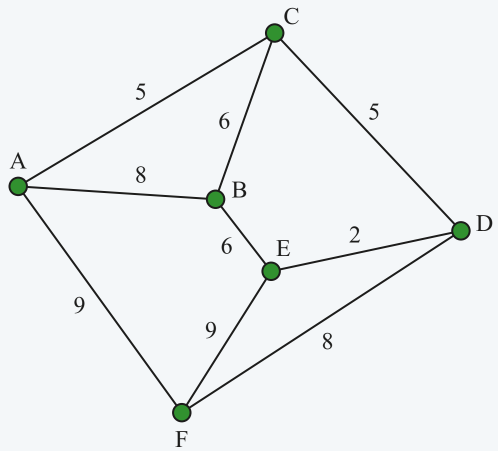
**2** Is there a path in the network that visits every edge exactly once?

**3** Draw a network that represents the following floor plan. Use vertices to represent rooms and represent rooms connected by a door with an edge.



**4** Is the network from question **3** a tree?

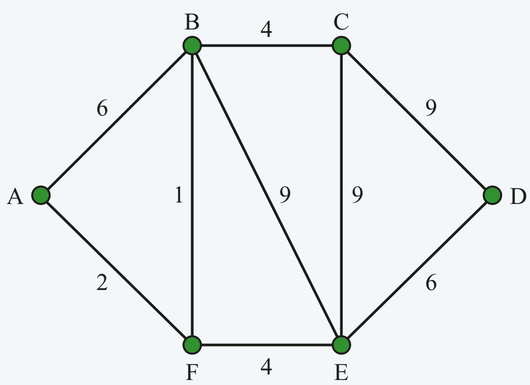
Use the diagram below to answer questions **5** and **6**.




**5** Find a minimum spanning tree for the network.

**6** What is the weight of the walk ABEDC?

Use the diagram below to answer questions **7**, **8** and **9**.



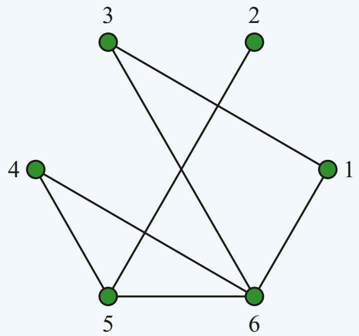

**7** Find the shortest path from A to D in the network.

**8** Find a walk that visits every edge of the network.

**9** Use Kruskal's algorithm to find the minimum spanning tree for the network.

## REVIEW SET 4

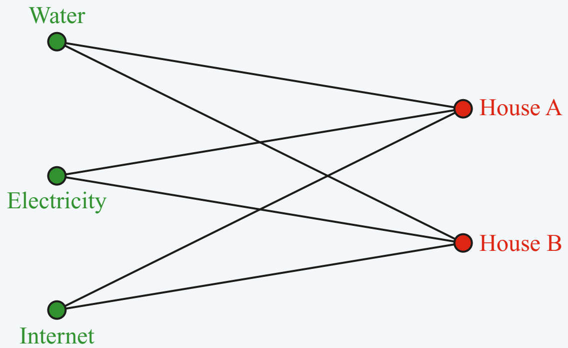
**1** Complete the table of vertex degrees for the following network.




| **Vertex** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Degree** | | | | | | |

NETWORKS

**2** Three essential services (water, electricity and internet) are to be connected to house A and house B.

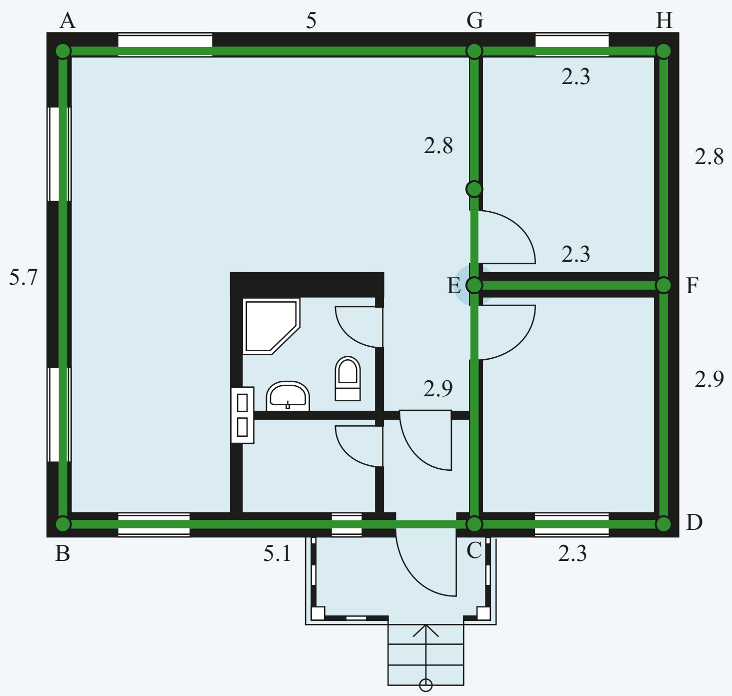



Re-draw this network so that the utility lines do not cross.

**3** Find a walk through the network in question **2** that visits every edge exactly once.

Use the diagram below to answer questions **4–6**.

The following house plan shows the possible locations of power cables.

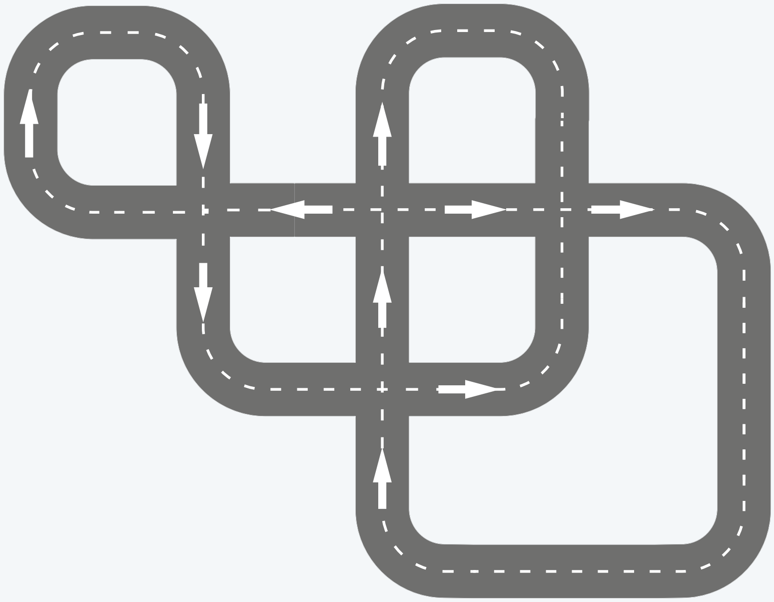



**4** What is the most cost-effective way to connect the points A, B, C, D, E and F to the power meter at G?

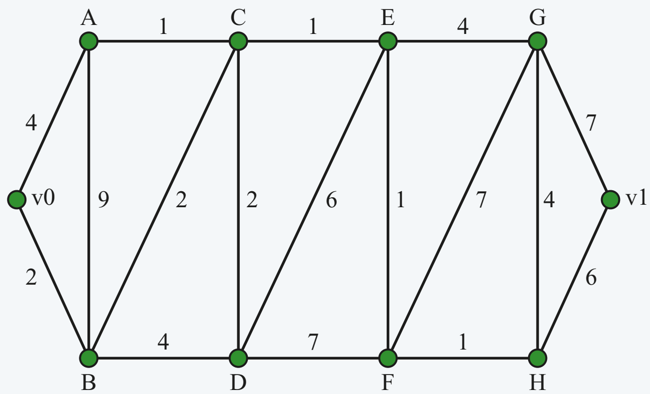
**5** What is the weight of the path ABCDFEG for the network?

**6** Use Prim's algorithm to find the minimum spanning tree for the network.

**7** Draw this system of one way streets as a directed network.

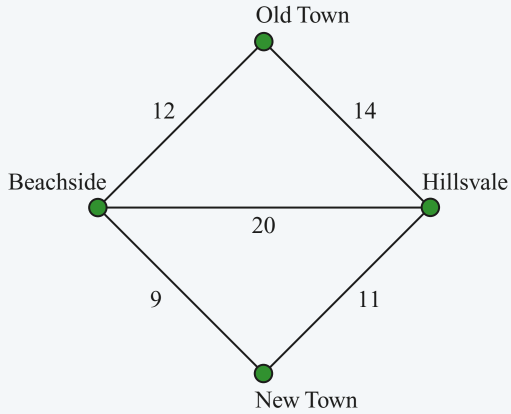


**8** Find the shortest path from v0 to v1.




# REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** This diagram shows the possible paths (in km) for laying gas pipes between various towns.



Gas is to be supplied from one location. Any one of the locations can be the source of the supply.

**i** What is the minimum total length of the pipes required to provide gas to all the locations?

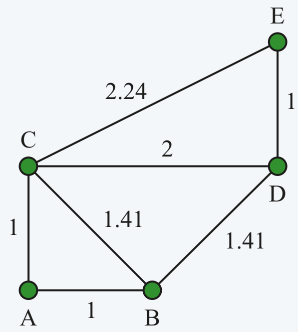
**A** 32 km **B** 33 km **C** 29 km **D** 40 km (1 mark)

**ii** How many different paths are there from New Town to Old Town? (1 mark)

**iii** What is the shortest path from Beachside to Hillsvale? (1 mark)

**iv** Find a walk that visits every edge exactly once. (1 mark)

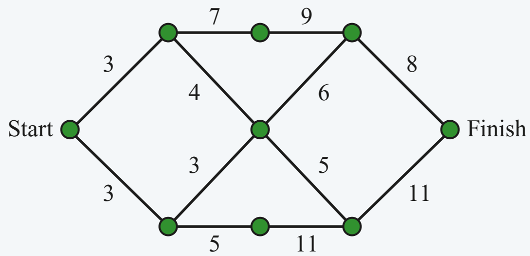
**b** This diagram shows the distance by road (in km) between towns.




**i** What is the shortest path from town A to town E? (1 mark)

**ii** What roads should be resurfaced, to minimise the total length of resurfaced road while still connecting each town with newly resurfaced road? (1 mark)

**c** The following network shows a maze. Distances are in metres.




**i** What is the shortest path from the start of the maze to the finish?

**A** 20 m **B** 30 m **C** 40 m **D** 50 m (1 mark)

**ii** Why does this network not contain a walk that visits every edge exactly once? (1 mark)

**iii** Find a minimum spanning tree for this network. (1 mark)

**d** Consider the road network below.

**i** Is the graph connected? (1 mark)

**ii** Which word best describes the road network?

**A** tree **B** path **C** cycle **D** directed (1 mark)


**e** Consider the following town map as a network of intersections (vertices) connected by roads (edges).

**i** How many vertices are in the network? (1 mark)

**ii** What is the highest vertex degree? (1 mark)

**iii** Can you design a bus route that visits every street exactly once? (2 marks)

TOTAL:
15 marks

NETWORKS

# 7

# Annuities

The main mathematical ideas in this chapter are:

- understanding the nature of an annuity
- performing calculations involving the future value of an annuity
- performing calculations involving the present value of an annuity
- using a table of future value interest factors to perform annuity calculations
- using a table of present value interest factors to perform annuity calculations
- calculating loan repayments
- calculating fees and charges associated with loans
- using technology to model annuities
- using technology to compare annuities
- interpreting graphs that compare repayment options for home loans.

FINANCIAL MATHEMATICS
MS-F5 Annuities

# ARE YOU READY?

7A **1** What is the result of $1.075 \times \$2300$?

**A** \$4025 **B** \$2472.50
**C** \$2 472 500 **D** \$172 500

7A **2** What is the result of $1.05^3$?

**A** 0.35 **B** 1.157 625
**C** 1.053 **D** 1.1025

7A **3** Given that $a = 500$ and $b = 0.03$, what is the value of $a(1 + b)^2$?

**A** 530.45 **B** 253 009
**C** 251 031.0609 **D** 501.032

7A **4** If \$500 is invested in a compound interest account paying 6% p.a. compounding annually, what is the balance at the end of 2 years?

**A** \$561.80 **B** \$18 000
**C** \$6000 **D** \$1060

Use the following table to answer questions **5** and **6**.

| | | Column | | | | |
|---|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** | **5** |
| **Row** | **1** | 15 | 33 | 29 | 70 | 38 |
| | **2** | 43 | 81 | 16 | 47 | 69 |
| | **3** | 22 | 75 | 36 | 41 | 92 |
| | **4** | 18 | 64 | 51 | 68 | 35 |
| | **5** | 49 | 12 | 77 | 56 | 79 |
| | **6** | 85 | 63 | 30 | 48 | 52 |

7B **5** What is the value in row 4 and column 2 of the table?

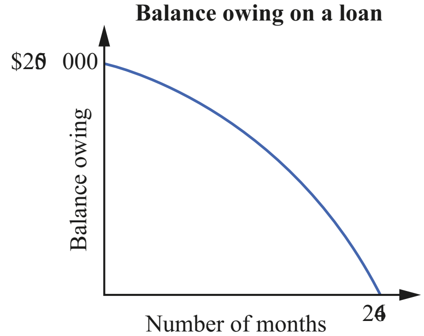
**A** 68 **B** 47 **C** 64 **D** 81

7B **6** In which row and column does the number 12 lie?

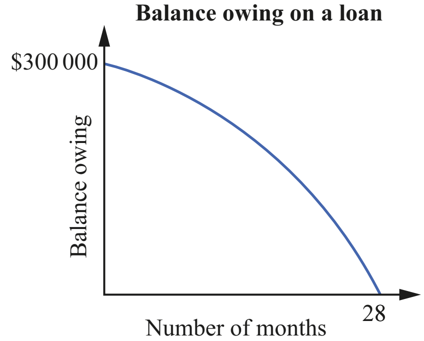
**A** Row 2, column 3
**B** Row 2, column 5
**C** Row 3, column 2
**D** Row 5, column 2

7E **7** In which of the following graphs is the loan repaid quickest?

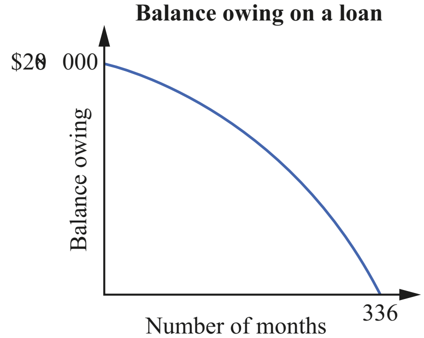
**A**



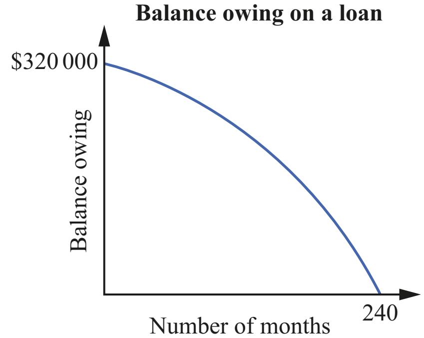

**B**




**C**




**D**




If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1** **Support sheet 7A.1 Multiplying decimals by a whole number**
**Q2** **Support sheet 7A.2 Finding cubes of numbers**
**Q3** **Support sheet 7A.3 Evaluating algebraic expressions involving powers**
**Q4** **Support sheet 7A.4 Compound interest**
**Q5–6** **Support sheet 7B.1 Reading information from a table**
**Q7** **Support sheet 7E.1 Reading graphs**

OB
OA

# 7A Future and present value of an annuity

These resources are available on your obook assess:

- **Video tutorial 7A.1:** Watch and listen to an explanation of Example 7A–1
- **Video tutorial 7A.2:** Watch and listen to an explanation of Example 7A–2
- **Spreadsheet 7A:** Calculate future and present values of an annuity
- **Worksheet 7A:** Practise your skills with extra problems for annuities using graphics calculators
- **assess quiz 7A:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**annuity**
an investment account receiving regular, equal contributions, or a single sum investment from which regular, equal withdrawals are made

In this section we will look at the effect of investments involving periodical, equal contributions to an account, with interest compounding at the end of each period. This is an example of an **annuity**. Another type of annuity is a single sum investment from which periodical, equal withdrawals are made. We will explore this type of annuity later in this chapter. Examples of annuities you are probably already familiar with include superannuation and home loans.

In an annuity, the time at which payments are made corresponds to the time at which the interest is calculated. Annuities can be used for many purposes, including providing a sum of money after a period of time for a major purchase, or providing a lump-sum payment for retirement. An annuity may also refer to income from a capital investment paid out in a series of regular payments.

The *future value* of an annuity is the total value of all contributions and compound interest at the end of the period, when the final payment is made (no interest is paid on the final payment).

The *present value* of an annuity is the single amount of money that, if invested today at the same rate of interest as the annuity, would produce the same result over the same period of time.

For example, \$3000 invested at the end of each year for 6 years at 8% p.a. amounts to \$22 007.79 in 6 years' time. If we invested \$13 868.64 now for 6 years at 8% p.a. compounding annually, we would also have \$22 007.79 in 6 years' time. This means that \$13 868.64 is the present value of an annuity of \$3000 invested per year for 6 years at 8% p.a.

**EXAMPLE 7A–1** Finding the future value of an annuity

A person saving for retirement decides to invest $1000 at the end of each year for 5 years into an account that pays interest of 9% p.a. Find the amount in the account after 5 years.

| Solve | Think | Apply |
|---|---|---|
| **Method 1**<br>$A_1 = \$1000$<br>$A_2 = A_1 \times 1.09 + 1000$<br>$= 1000 \times 1.09 + 1000$<br>$= \$2090$<br>$A_3 = A_2 \times 1.09 + 1000$<br>$= 2090 \times 1.09 + 1000$<br>$= \$3278.10$<br>$A_4 = A_3 \times 1.09 + 1000$<br>$= 3278.10 \times 1.09 + 1000$<br>$= \$4573.13$<br>$A_5 = A_4 \times 1.09 + 1000$<br>$= 4573.13 \times 1.09 + 1000$<br>$= \$5984.71$ | Let $A_n$ be the amount after $n$ years.<br>$1000 is invested at the end of the first year, so no interest is earned in that year.<br>$A_2$ = amount at the end of year 1 + interest on that amount + next $1000<br>$A_3$ = amount at the end of year 2 + interest on that amount + next $1000<br>$A_4$ = amount at the end of year 3 + interest on that amount + next $1000<br>$A_5$ = amount at the end of year 4 + interest on that amount + next $1000<br>To increase by 9%, multiply by 1.09. | Calculate the amount in the account at the end of each year by adding the interest for that year and then adding the next payment. |

**Method 2**

| Payment | Amount | Years invested | Amount and interest | |
|---|---|---|---|---|
| 1 | $1000 | 4 | $1000 \times (1 + \frac{9}{100})^4$ | = $1411.58 |
| 2 | $1000 | 3 | $1000 \times (1 + \frac{9}{100})^3$ | = $1295.03 |
| 3 | $1000 | 2 | $1000 \times (1 + \frac{9}{100})^2$ | = $1188.10 |
| 4 | $1000 | 1 | $1000 \times (1 + \frac{9}{100})^1$ | = $1090.00 |
| 5 | $1000 | 0 | 1000 | = $1000.00 |
| | | | **Total** | $5984.71 |

The first payment earns interest for 4 years, calculated using the compound interest formula. The second payment earns interest for 3 years and so on, with the final payment earning no interest. These amounts are added to obtain the future value of the annuity.

*Apply:* Treat each payment separately and calculate the number of years for which it is invested. Use the compound interest formula from Topic 1B to calculate the amount to which each payment will accrue, and then add each of these amounts to determine the future value of the investment.

## EXERCISE 7A Future and present value of an annuity

**1** Ling invests $1500 per year for 5 years in an annuity. He makes each payment at the end of the year and the interest rate is 8% p.a. Complete the following to find the final value of his annuity. Let $A_n$ be the amount in the account after $n$ years.

$A_1 = \$1500$

$A_2 = A_1 \times \_\_\_ + 1500$

$= 1500 \times \_\_\_ + 1500 = \$\_\_\_$

$A_3 = A_2 \times \_\_\_ + 1500$

$= \_\_\_ \times \_\_\_ + 1500 = \$\_\_\_$

$A_4 = A_3 \times \_\_\_ + 1500$

$= \_\_\_ \times \_\_\_ + 1500 = \$\_\_\_$

$A_5 = \_\_\_ \times \_\_\_ + \_\_\_$

$= \_\_\_ \times \_\_\_ + \_\_\_ = \$\_\_\_$

**2** Use the method of question **1** to find the future value of an annuity in which \$5000 is deposited at the end of each year for 4 years at an interest rate of 10% p.a.

**3** Allie invests \$2000 per year in an annuity and makes the payment at the end of each year. The interest rate is 6% p.a. and she invests for 5 years. Complete the following table to find the final value of her annuity.

| Payment | Amount | Years invested | Amount + interest | Amount |
|---|---|---|---|---|
| 1 | \$2000 | 4 | $\$2000(1+0.06)^4$ | \$2524.95 |
| 2 | \$2000 | 3 | \$___ $(1+0.06)^3$ | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| | | **Total** | | |

**4** Use the method of question **3** to find the future value of an annuity in which \$4000 is deposited at the end of each year for 6 years and the interest rate is 7% p.a.

**!**

The future and present values of an annuity can be calculated using the following formulae:

$$FV = a\left[\frac{(1+r)^n - 1}{r}\right] \text{ and } PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$$

where $a$ = the contribution per period paid at the end of the period,

$r$ = the interest rate per compounding period (expressed as a decimal),

$n$ = the number of periods.

## EXAMPLE 7A–2 Finding the present value of an annuity

**a** Calculate the future value of an annuity if \$2000 is invested at the end of each year for 10 years at 5% p.a.

**b** What single amount of money could be invested now for 10 years at 5% p.a. compounding yearly, to achieve the same result?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $FV = 2000\left[\frac{(1+0.05)^{10}-1}{0.05}\right]$ <br> $= 2000 \times 12.577\,89...$ <br> $= \$25\,155.79$ | $a = 2000$ <br> $r = 0.05$ <br> $n = 10$ | Use $FV = a\left[\frac{(1+r)^n - 1}{r}\right]$. |
| **b** | $PV = 2000\left[\frac{(1+0.05)^{10}-1}{0.05(1+0.05)^{10}}\right]$ <br> $= 2000\left[\frac{0.628\,89...}{0.081\,44...}\right]$ <br> $= 2000 \times 7.721\,73...$ <br> $= \$15\,443.47$ | $a = 2000$ <br> $r = 0.05$ <br> $n = 10$ | Use $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$. |
| | *Note*: we could have also used the compound interest formula, $FV = PV(1+r)^n$, to calculate the answer to part **b**. | | |

**5** **a** Calculate the future value of an annuity if \$6000 per year is invested for 8 years at 4.5% p.a.

**b** What single amount of money could be invested now for 8 years at 4.5% p.a. compounding yearly, to achieve the same result? (This is the present value of the annuity.)

**6** **a** Calculate the future value of an annuity if \$400 per month is invested for 5 years at 6% p.a. compounding monthly.

Hint: first calculate the interest rate per compounding period and the number of compounding periods.

**b** What single amount of money could be invested now for 5 years at 6% p.a. compounding monthly, to achieve the same result? (What is the present value of the annuity?)

**7** **a** Calculate the future value of an annuity if \$4000 per quarter is invested for 5 years at 8% p.a. compounding quarterly.

**b** What is the present value of the annuity?

**8** An annuity involves depositing \$10 000 at the end of each year for 4 years. The interest rate is 5% p.a. Find the present value of the annuity.

**9** An annuity involves depositing \$2000 at the end of each quarter for 5 years at an interest rate of 8% p.a. Find the present value of the annuity.

## EXAMPLE 7A–3 Finding the regular contributions required to meet a savings goal

Claire sets a goal to save \$10 000 over the next three years. How much will she need to put into a savings account at the end of each month if the account has an interest rate of 6.6% p.a. compounding monthly?

| Solve | Think | Apply |
|---|---|---|
| $10\,000 = a\left[\dfrac{(1+0.0055)^{36}-1}{0.0055}\right]$<br>$10\,000 = a\left[\dfrac{(1.0055)^{36}-1}{0.0055}\right]$<br>$\dfrac{10\,000}{\left[\dfrac{(1.0055)^{36}-1}{0.0055}\right]} = a$<br>$a = 251.9454$<br>The required monthly contribution is \$251.95. | $FV = 10\,000$<br>$a =$ monthly contribution<br>$r = 0.066 \div 12$<br>$= 0.0055$<br>$n = 3 \times 12$<br>$= 36$ | Use $FV = a\left[\dfrac{(1+r)^n-1}{r}\right]$, where $a$ is the contribution amount, $r$ is the interest rate per compounding period (expressed as a decimal), and $n$ is the number of periods. |

**10** Paddy would like to save \$12 000 over the next two years. How much will he need to put into a savings account at the end of each month if the account has an interest rate of 6% p.a. compounding monthly?

**11** Svetlana needs to save \$7000 over the next 18 months to pay for a permanent residency visa. How much will she need to put into a savings account at the end of each month if the account pays 7.2% p.a. compounding monthly?

**12** Ella wants to have \$12 000 in 4 years' time. How much would she need to deposit annually into an annuity that pays 8% p.a. interest?

**13** \$2400 is invested into an annuity at the end of each year for 10 years at 4.9% p.a.

**a** Calculate the future value of the annuity.

**b** How much interest was earned over the 10 years?

**14** \$900 is invested into an annuity at the end of each month for 4 years at 9% p.a. interest compounding monthly.

**a** Calculate the future value of the annuity.

**b** How much interest was earned over the 4 years?

**15** Jayne wants to start saving for a home deposit. She prepares a budget and decides that she can put \$1000 each quarter into a savings account paying 10% p.a. compounding quarterly.

**a** How much will be in Jayne's savings account after 4 years?

**b** If Jayne was able to save an extra \$200 each quarter, how much would she have in her savings account after 4 years?

**16** Marie's company paid \$550 into her superannuation account at the end of each month. If she started out with a zero balance, and her superannuation account paid 12% p.a. compounded monthly for 4 years, what would be the balance of her account after 4 years?

**17** Your parents are offered a pension of \$5000 per quarter for the next 12 years or a lump sum now. If interest rates are currently 6% p.a. compounded quarterly, what lump-sum payment would be equivalent to the pension?

**18** Jess wants to save \$15 000 to buy a car in $3\frac{1}{2}$ years' time. She puts \$2000 every 6 months into an annuity that pays 8% p.a. compounding 6-monthly. Will she have enough to buy the car? By how much is she under or over her target?

**19** At the start of 2018 Trevor invested \$300 per month into an annuity paying 5.5% p.a. compounding monthly. A year later his brother, Sam, invested \$380 per month into an annuity paying 6% p.a. compounding monthly. Who will have the most money in their annuity at the start of 2022, and by how much?

**20** Max invests \$1200 at the end of each year into an annuity that pays interest of 8% p.a. Darrel invests \$2400 at the end of each year into an annuity that pays 4% p.a. Who will have more money in their fund after 25 years?

**21** **a** Eli needs a minimum of \$8000 to go on a holiday in 4 years' time. How much would she have to deposit at the end of each month in an annuity that pays 6% p.a. compounding monthly, to have this amount?

**b** If Eli had another \$2000 to spend, she could extend her holiday by a week. How much extra would she need to deposit each month to have enough to stay another week?

**22** Khalia wants to have \$500 000 in her superannuation fund when she retires in 25 years' time. The fund pays 10% p.a. interest, compounded quarterly. Her employer makes quarterly deposits of \$1000 into her fund. How much extra must Khalia contribute herself, each quarter, to reach her goal?

## EXAMPLE 7A–1 Comparing compound interest investments and annuities

Mina would like to have \$30 000 in her savings account in 5 years' time as a house deposit.

**a** How much would she need to invest now as a lump sum if she invested her money into a savings account paying 6% p.a. compounding monthly to have \$30 000 in 5 years?

**b** If Mina had no savings at present, but was able to put \$400 each month into the savings account, would she reach her target in 5 years?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $PV = \dfrac{30\,000}{(1 + 0.005)^{30}}$<br>$= \dfrac{30\,000}{1.005^{30}}$<br>$= \$22\,241.17$<br>Mina would need to invest \$22 241.17 now to have \$30 000 in 5 years. | This is a compound interest problem, so use the rearranged compound interest formula from chapter 1.<br>$FV = 30\,000$<br>$r = 0.06 \div 12 = 0.005$<br>$n = 5 \times 12 = 60$ | Use the formula $PV = \dfrac{FV}{(1 + r)^n}$ to calculate the present value of the investment. |
| **b** | $FV = 400\left[\dfrac{(1 + 0.005)^{60} - 1}{0.005}\right]$<br>$= 400 \times 69.770\,03...$<br>$= \$27\,908.01$<br>Mina would not reach her target. | $a = 400$<br>$r = 0.005$<br>$n = 60$ | Use $FV = a\left[\dfrac{(1 + r)^n - 1}{r}\right]$. |

**23** Mustafi wants to have \$50 000 in his savings account in 5 years' time.

**a** How much would he need to invest now as a lump sum if he invested his money into a savings account paying 6.6% p.a. compounding quarterly?

**b** If Mustafi has no savings at present, but was able to put \$2000 each quarter into the savings account, would he reach his target in 5 years?

CHALLENGE

**24** Kristen wins a competition, and is given the option of either receiving a lump sum of \$60 000 now, or \$1000 at the end of each month for 6 years. If she is able to invest the money into an account paying 5.7% p.a. compounding monthly, which option would give her the greater return after the 6 years, and by how much?

**25** Venus wants to save a \$50 000 house deposit over the next 3 years. She already has \$16 000 in her savings account, which pays 6% p.a. compounding monthly.

**a** How much will Venus have in her account after 3 years if she deposits \$700 into her account at the end of each month?

**b** By how much will Venus be short of her target at the end of the 3 years?

Hint: treat Venus' savings account as a compound interest investment for her savings + an annuity for the deposits each month.

# 7B Future value using a table

These resources are available on your obook assess:

- **assess quiz 7B:** Test your skills with an auto-correcting multiple-choice quiz

Calculations to determine the future value of an annuity can be simplified by using prepared tables of values. The following tables show the future value of $1 invested at the end of each time period at the rate indicated. These tables are easily produced using a spreadsheet and the future value annuity formula.

**Future value of $1 invested**

| Period | Interest rate per period | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2% | 2.5% | 3% | 3.5% | 4% | 4.5% | 5% | 5.5% | 6% | 8% |
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 2.0200 | 2.0250 | 2.0300 | 2.0350 | 2.0400 | 2.0450 | 2.0500 | 2.0550 | 2.0600 | 2.0800 |
| 3 | 3.0604 | 3.0756 | 3.0909 | 3.1062 | 3.1216 | 3.1370 | 3.1525 | 3.1680 | 3.1836 | 3.2464 |
| 4 | 4.1216 | 4.1525 | 4.1836 | 4.2149 | 4.2465 | 4.2782 | 4.3101 | 4.3423 | 4.3746 | 4.5061 |
| 5 | 5.2040 | 5.2563 | 5.3091 | 5.3625 | 5.4163 | 5.4707 | 5.5256 | 5.5811 | 5.6371 | 5.8666 |
| 6 | 6.3081 | 6.3877 | 6.4684 | 6.5502 | 6.6330 | 6.7169 | 6.8019 | 6.8881 | 6.9753 | 7.3359 |
| 7 | 7.4343 | 7.5474 | 7.6625 | 7.7794 | 7.8983 | 8.0192 | 8.1420 | 8.2669 | 8.3938 | 8.9228 |
| 8 | 8.5830 | 8.7361 | 8.8923 | 9.0517 | 9.2142 | 9.3800 | 9.5491 | 9.7216 | 9.8975 | 10.6366 |
| 9 | 9.7546 | 9.9545 | 10.1591 | 10.3685 | 10.5828 | 10.8021 | 11.0266 | 11.2563 | 11.4913 | 12.4876 |
| 10 | 10.9497 | 11.2034 | 11.4639 | 11.7314 | 12.0061 | 12.2882 | 12.5779 | 12.8754 | 13.1808 | 14.4866 |
| 11 | 12.1687 | 12.4835 | 12.8078 | 13.1420 | 13.4864 | 13.8412 | 14.2068 | 14.5835 | 14.9716 | 16.6455 |
| 12 | 13.4121 | 13.7956 | 14.1920 | 14.6020 | 15.0258 | 15.4640 | 15.9171 | 16.3856 | 16.8699 | 18.9771 |
| 15 | 17.2934 | 17.9319 | 18.5989 | 19.2957 | 20.0236 | 20.7841 | 21.5786 | 22.4087 | 23.2760 | 27.1521 |
| 18 | 21.4123 | 22.3863 | 23.4144 | 24.4997 | 25.6454 | 26.8551 | 28.1324 | 29.4812 | 30.9057 | 37.4502 |
| 20 | 24.2974 | 25.5447 | 26.8704 | 28.2797 | 29.7781 | 31.3714 | 33.0660 | 34.8683 | 36.7856 | 45.7620 |
| 21 | 25.7833 | 27.1833 | 28.6765 | 30.2695 | 31.9692 | 33.7831 | 35.7193 | 37.7861 | 39.9927 | 50.4229 |
| 24 | 30.4219 | 32.3490 | 34.4265 | 36.6665 | 39.0826 | 41.6892 | 44.5020 | 47.5380 | 50.8156 | 66.7648 |
| 25 | 32.0303 | 34.1578 | 36.4593 | 38.9499 | 41.6459 | 44.5652 | 47.7271 | 51.1526 | 54.8645 | 73.1059 |
| 30 | 40.5681 | 43.9027 | 47.5754 | 51.6227 | 56.0849 | 61.0071 | 66.4388 | 72.4355 | 79.0582 | 113.2832 |
| 36 | 51.9944 | 57.3014 | 63.2759 | 70.0076 | 77.5983 | 86.1640 | 95.8363 | 106.7652 | 119.1209 | 187.1021 |
| 42 | 64.8622 | 72.8398 | 82.0232 | 92.6074 | 104.8196 | 118.9248 | 135.2318 | 154.1005 | 175.9505 | 304.2435 |
| 48 | 79.3535 | 90.8596 | 104.4084 | 120.3883 | 139.2632 | 161.5879 | 188.0254 | 219.3684 | 256.5645 | 490.1322 |
| 54 | 95.6731 | 111.7570 | 131.1375 | 154.5381 | 182.8454 | 217.1464 | 258.7739 | 309.3625 | 370.9170 | 785.1141 |
| 60 | 114.0515 | 135.9916 | 163.0534 | 196.5169 | 237.9907 | 289.4980 | 353.5837 | 433.4504 | 533.1282 | 1253.2133 |
| 100 | 312.2323 | 432.5487 | 607.2877 | 862.6117 | 1237.6237 | 1790.8560 | 2610.0252 | 3826.7025 | 5638.3681 | 27484.5157 |

| Period | Interest rate per period | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.2% | 0.25% | 0.3% | 0.4% | 0.5% | 0.8% | 1% |
| 12 | 12.1329 | 12.1664 | 12.2000 | 12.2676 | 12.3356 | 12.5423 | 12.6825 |
| 24 | 24.5602 | 24.7028 | 24.8465 | 25.1371 | 25.4320 | 26.3432 | 26.9735 |
| 36 | 37.2890 | 37.6206 | 37.9559 | 38.6381 | 39.3361 | 41.5287 | 43.0769 |
| 48 | 50.3268 | 50.9312 | 51.5451 | 52.8016 | 54.0978 | 58.2380 | 61.2226 |
| 60 | 63.6809 | 64.6467 | 65.6316 | 67.6602 | 69.7700 | 76.6239 | 81.6697 |
| 72 | 77.3590 | 78.7794 | 80.2337 | 83.2478 | 86.4089 | 96.8545 | 104.7099 |
| 84 | 91.3691 | 93.3419 | 95.3703 | 99.6004 | 104.0739 | 119.1151 | 130.6723 |
| 96 | 105.7191 | 108.3474 | 111.0608 | 116.7553 | 122.8285 | 143.6093 | 159.9273 |
| 108 | 120.4173 | 123.8093 | 127.3256 | 134.7521 | 142.7399 | 170.5612 | 192.8926 |
| 120 | 135.4722 | 139.7414 | 144.1857 | 153.6320 | 163.8793 | 200.2175 | 230.0387 |
| 180 | 216.4071 | 226.9727 | 238.2067 | 262.8712 | 290.8187 | 399.5728 | 499.5802 |
| 240 | 307.6500 | 328.3020 | 350.7400 | 401.6750 | 462.0409 | 721.1312 | 989.2554 |
| 300 | 410.5137 | 446.0078 | 485.4305 | 578.0448 | 692.9940 | 1239.8020 | 1878.8466 |
| 360 | 526.4783 | 582.7369 | 646.6408 | 802.1475 | 1004.5150 | 2076.4132 | 3494.9641 |

!

The formula $FV = a \times X$ can be used to determine the future value of an annuity,
where $a$ = the contribution made each time period,
$X$ = the interest factor from the table for the given rate and given number of time periods.

## EXAMPLE 7B–1 Using a future value annuity table

**a** Use the tables of future values to find the future value of an annuity of \$800 invested at the end of each year for 15 years at an interest rate of 5.5% p.a.

**b** Calculate the amount of interest earned on the investment.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $FV = 800 \times 22.4087$<br>$= \$17\,926.96$ | $a = 800$<br>The interest rate is 5.5% p.a., and the number of years is 15, giving an interest factor of 22.4087. | Use the formula $FV = a \times X$. |
| **b** | Interest earned<br>$= 17\,926.96 - 15 \times 800$<br>$= \$5926.96$ | The final value of the investment is \$17 926.96. The total of the contributions made is 15 × \$800. The difference is the amount of interest earned. | The amount of interest earned is the difference between the final value of the investment and the total contributions made. |

*Note:* calculating the future value of the annuity in Example 7B–1 using the formula from topic 7A gives \$17 926.93, which is almost identical to the value calculated using the future value table. The slight difference is due to rounding, as the values in the future value table are given to four decimal places.

## EXERCISE 7B Future value using a table

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** **a** Use the tables of future values of $1 to find the future value of the annuities listed below. The contribution is made at the end of each year.

**b** Calculate the amount of interest earned on each investment.

| | Amount ($) | Number of years | Interest rate p.a. |
|---|---|---|---|
| **a** | 2500 | 5 | 6.0% |
| **b** | 650 | 8 | 8.0% |
| **c** | 1190 | 3 | 3.5% |
| **d** | 3000 | 11 | 4.0% |
| **e** | 10 000 | 20 | 3.0% |
| **f** | 3200 | 9 | 2.5% |
| **g** | 4800 | 25 | 6.0% |
| **h** | 20 000 | 8 | 5.5% |

**2** When Robbie was born his grandparents began an annuity for him. They deposited $1000 on his first birthday and at each birthday up to and including his 21st birthday. If the interest rate was 6% p.a., what was the value of the annuity when Robbie turned 21?

**3** **a** Madeline's employer invested $5000 at the end of each year into a superannuation fund that pays 4.5% p.a. interest. What would be the value of this annuity in 30 years' time?

**b** After 10 years Madeline is advised to make her own contributions of $2000 at the end of each year into the same fund. How much extra would she have in her retirement fund if she decides to take this advice?

### EXAMPLE 7B–2 Finding the future value with a non-annual compounding period

Find the future value of an annuity if $200 is invested at the end of each month for 2 years at an interest rate of 3.6% p.a.

| Solve | Think | Apply |
|---|---|---|
| $FV = 200 \times 24.8465$<br>$= \$4969.30$ | 3.6% p.a. = 0.3% per month<br>2 years = 24 months<br>From the table, for 0.3% and 24 time periods, $X = 24.8465$. | Use $FV = a \times X$.<br>As the contributions are made monthly, convert the interest to rate per month and the time periods to months. |

**4** Use the tables of future value of $1 to find the future value of the following annuities. The contribution is made at the end of each time period.

| | Amount ($) | Invested | Number of years | Interest rate p.a. |
|---|---|---|---|---|
| **a** | 300 | monthly | 3 | 4.8% |
| **b** | 100 | monthly | 4 | 6.0% |
| **c** | 1000 | quarterly | 5 | 8.0% |
| **d** | 3000 | quarterly | 6 | 4.0% |
| **e** | 15 000 | 6-monthly | 5 | 5.0% |
| **f** | 4600 | 6-monthly | 18 | 2.0% |

UNDERSTANDING, FLUENCY AND COMMUNICATING

**5** Ginny's employer deposits \$176 per month into her superannuation fund, which pays interest of 12% p.a. compounding monthly. She also makes personal contributions of \$150 per month into the same fund. How much will she have in her superannuation fund after 20 years?

**6** Grant wants to travel overseas in 3 years' time and estimates that he will need \$10 000. He begins paying \$650 per quarter into an annuity at 10% p.a. compounding quarterly. Will he have enough to go on his trip? By how much is he under or over his target?

## EXAMPLE 7B–3 Determining the annual contribution required

George wants to make regular annual payments into an annuity that pays interest at 8% p.a. How much will each annual contribution need to be if George wants to have \$10 000 after 7 years?

| Solve | Think | Apply |
|---|---|---|
| $10\,000 = a \times 8.9228$<br>$a = \dfrac{10\,000}{8.9228}$<br>$a = 1120.724...$<br>The annual contribution would need to be \$1120.72. | The future value of the annuity is to be \$10 000. From the table, the interest factor for 8% and 7 time periods is 8.9228. | From the table, determine the interest factor for the given interest rate and number of time periods.<br>Substitute this and the required future value into the formula $FV = a \times X$.<br>Solve the resulting equation for $a$. |

**7** Eliza wants to have \$5000 in 6 years' time. How much would she need to deposit annually into an annuity that pays 5% p.a. interest?

**8** Adrian wants to have \$8000 in 5 years' time. How much would he need to deposit annually into an annuity that pays 6% p.a. interest?

## EXAMPLE 7B–4 Determining the non-annual contribution required

Ingrid wants to have \$5000 in 3 years' time. She invests in an annuity that pays 10% p.a. compounding quarterly. How much does she need to deposit at the end of each quarter to achieve this result?

| Solve | Think | Apply |
|---|---|---|
| $5000 = a \times 13.7956$<br>$a = \dfrac{5000}{13.7956}$<br>$a = 362.434...$<br>The quarterly contribution would need to be \$362.43. | The future value of the annuity is to be \$5000.<br>$10\%$ p.a. $= \dfrac{10}{4}\%$ per quarter<br>$= 2.5\%$ per quarter<br>3 years $= 12$ quarters<br>From the table, the interest factor for 2.5% and 12 time periods is 13.7956. | Convert the interest rate and time period to quarters. From the table, determine the interest factor for the given interest rate and number of time periods.<br>Substitute this and the required future value into the formula $FV = a \times X$.<br>Solve the resulting equation for $a$. |

**9** Robert and Jo want to present their new baby with \$25 000 on her 21st birthday. They decide to put a fixed amount every 6 months after her birth into an annuity that pays 9% p.a. compounding 6-monthly. How much do they need to deposit each 6 months?

CHALLENGE

**10** Use the formula $FV = a\left[\dfrac{(1 + r)^n - 1}{r}\right]$ to verify that the future value of \$1 invested for each time and interest rate given below is the amount shown in the future value tables.

**a** 8 years at 6% p.a.
**b** 7 years at 8% p.a.
**c** 48 months at 0.25% per month
**d** 6 years at 3.5% p.a.

# 7C Present value using a table

These resources are available on your obook assess:
- **Spreadsheet 7C:** Present value tables
- **assess quiz 7C:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

The present value calculations in Topic 7A can also be simplified by using prepared tables of values. The tables on the following pages show the present value of an annuity of $1 invested at the end of each time period at the rate indicated.

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The formula $PV = a \times X$ can be used to determine the present value of an annuity, where:

$a$ = the contribution made each time period

$X$ = the interest factor from the table for the given rate and given number of time periods.

**Present value of an annuity of $1**

| Period | Interest rate per period | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2% | 2.5% | 3% | 3.5% | 4% | 4.5% | 5% | 5.5% | 6% | 8% |
| 2 | 1.9416 | 1.9274 | 1.9135 | 1.8997 | 1.8861 | 1.8727 | 1.8594 | 1.8463 | 1.8334 | 1.7833 |
| 3 | 2.8839 | 2.8560 | 2.8286 | 2.8016 | 2.7751 | 2.7490 | 2.7232 | 2.6979 | 2.6730 | 2.5771 |
| 4 | 3.8077 | 3.7620 | 3.7171 | 3.6731 | 3.6299 | 3.5875 | 3.5460 | 3.5052 | 3.4651 | 3.3121 |
| 5 | 4.7135 | 4.6458 | 4.5797 | 4.5151 | 4.4518 | 4.3900 | 4.3295 | 4.2703 | 4.2124 | 3.9927 |
| 6 | 5.6014 | 5.5081 | 5.4172 | 5.3286 | 5.2421 | 5.1579 | 5.0757 | 4.9955 | 4.9173 | 4.6229 |
| 7 | 6.4720 | 6.3494 | 6.2303 | 6.1145 | 6.0021 | 5.8927 | 5.7864 | 5.6830 | 5.5824 | 5.2064 |
| 8 | 7.3255 | 7.1701 | 7.0197 | 6.8740 | 6.7327 | 6.5959 | 6.4632 | 6.3346 | 6.2098 | 5.7466 |
| 9 | 8.1622 | 7.9709 | 7.7861 | 7.6077 | 7.4353 | 7.2688 | 7.1078 | 6.9522 | 6.8017 | 6.2469 |
| 10 | 8.9826 | 8.7521 | 8.5302 | 8.3166 | 8.1109 | 7.9127 | 7.7217 | 7.5376 | 7.3601 | 6.7101 |
| 20 | 16.3514 | 15.5892 | 14.8775 | 14.2124 | 13.5903 | 13.0079 | 12.4622 | 11.9504 | 11.4699 | 9.8181 |
| 24 | 18.9139 | 17.8850 | 16.9355 | 16.0584 | 15.2470 | 14.4955 | 13.7986 | 13.1517 | 12.5504 | 10.5288 |
| 36 | 25.4888 | 23.5563 | 21.8323 | 20.2905 | 18.9083 | 17.6660 | 16.5469 | 15.5361 | 14.6210 | 11.7172 |
| 48 | 30.6731 | 27.7732 | 25.2667 | 23.0912 | 21.1951 | 19.5356 | 18.0772 | 16.7902 | 15.6500 | 12.1891 |
| 60 | 34.7609 | 30.9087 | 27.6756 | 24.9447 | 22.6235 | 20.6380 | 18.9293 | 17.4499 | 16.1614 | 12.3766 |
| 120 | 45.3554 | 37.9337 | 32.3730 | 28.1111 | 24.7741 | 22.1093 | 19.9427 | 18.1524 | 16.6514 | 12.4988 |
| 180 | 48.5844 | 39.5304 | 33.1703 | 28.5130 | 24.9785 | 22.2142 | 19.9969 | 18.1806 | 16.6662 | 12.5000 |
| 240 | 49.5686 | 39.8933 | 33.3057 | 28.5640 | 24.9980 | 22.2216 | 19.9998 | 18.1818 | 16.6667 | 12.5000 |
| 248 | 49.6318 | 39.9124 | 33.3115 | 28.5658 | 24.9985 | 22.2218 | 19.9999 | 18.1818 | 16.6667 | 12.5000 |
| 300 | 49.8685 | 39.9757 | 33.3286 | 28.5705 | 24.9998 | 22.2222 | 20.0000 | 18.1818 | 16.6667 | 12.5000 |
| 360 | 49.9599 | 39.9945 | 33.3325 | 28.5713 | 25.0000 | 22.2222 | 20.0000 | 18.1818 | 16.6667 | 12.5000 |

| Period | Interest rate per period | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.4% | 0.5% | 0.6% | 0.7% | 0.8% | 1% | 1.5% |
| 1 | 0.9960 | 0.9950 | 0.9940 | 0.9930 | 0.9921 | 0.9901 | 0.9852 |
| 2 | 1.9881 | 1.9851 | 1.9821 | 1.9792 | 1.9763 | 1.9704 | 1.9559 |
| 3 | 2.9762 | 2.9702 | 2.9644 | 2.9585 | 2.9526 | 2.9410 | 2.9122 |
| 4 | 3.9603 | 3.9505 | 3.9407 | 3.9310 | 3.9213 | 3.9020 | 3.8544 |
| 5 | 4.9406 | 4.9259 | 4.9112 | 4.8967 | 4.8822 | 4.8534 | 4.7826 |
| 6 | 5.9169 | 5.8964 | 5.8760 | 5.8557 | 5.8355 | 5.7955 | 5.6972 |
| 7 | 6.8893 | 6.8621 | 6.8350 | 6.8080 | 6.7813 | 6.7282 | 6.5982 |
| 8 | 7.8579 | 7.8230 | 7.7882 | 7.7538 | 7.7195 | 7.6517 | 7.4859 |
| 9 | 8.8226 | 8.7791 | 8.7358 | 8.6929 | 8.6503 | 8.5660 | 8.3605 |
| 10 | 9.7835 | 9.7304 | 9.6778 | 9.6255 | 9.5737 | 9.4713 | 9.2222 |
| 20 | 19.1841 | 18.9874 | 18.7936 | 18.6025 | 18.4142 | 18.0456 | 17.1686 |
| 24 | 22.8405 | 22.5629 | 22.2899 | 22.0216 | 21.7578 | 21.2434 | 20.0304 |
| 36 | 33.4659 | 32.8710 | 32.2907 | 31.7247 | 31.1723 | 30.1075 | 27.6607 |
| 48 | 43.5942 | 42.5803 | 41.5988 | 40.6486 | 39.7284 | 37.9740 | 34.0426 |
| 60 | 53.2489 | 51.7256 | 50.2621 | 48.8559 | 47.5042 | 44.9550 | 39.3803 |
| 120 | 95.1560 | 90.0735 | 85.3666 | 81.0035 | 76.9552 | 69.7005 | 55.4985 |
| 180 | 128.1370 | 118.5035 | 109.8845 | 102.1569 | 95.2139 | 83.3217 | 62.0956 |
| 240 | 154.0933 | 139.5808 | 127.0084 | 116.0760 | 106.5336 | 90.8194 | 64.7957 |
| 248 | 157.1078 | 141.9441 | 128.8616 | 117.5296 | 107.6740 | 91.5219 | 65.0058 |
| 300 | 174.5210 | 155.2069 | 138.9683 | 125.2349 | 113.5515 | 94.9466 | 65.9009 |
| 360 | 190.5977 | 166.7916 | 147.3214 | 131.2616 | 117.9023 | 97.2183 | 66.3532 |

## EXAMPLE 7C–1 Using a present value interest factor table

An annuity involves depositing \$3000 at the end of each year for 6 years. The interest rate is 8% p.a. Find the present value of the annuity.

| Solve | Think | Apply |
|---|---|---|
| $PV = 3000 \times 4.6229$<br>$= \$13\,868.70$ | We need to find the single amount of money that could be invested now for 6 years at 8% p.a., to produce the same amount as an annuity of \$3000 per year for 6 years at 8% p.a. In this case $a = 3000$.<br>From the table of present values, the interest factor for 8% p.a. for 6 years is 4.6229. | From the present value table opposite, find the interest rate factor for the given rate and number of time periods.<br>Use the formula $PV = a \times X$ to find the present value of the annuity. |

A single amount of \$13 868.70 invested now for 6 years at 8% p.a. compound interest would accumulate to the same amount as a \$3000 payment invested at the end of each year for 6 years at 8% p.a. compound interest. This can be checked as shown on the next page.

- If $13 868.70 is invested for 6 years at 8% p.a. compound interest, using the compound interest formula we get:

  $FV = PV(1 + r)^n$

  $= 13\,868.70 \times (1 + 0.08)^6$

  $= \$22\,007.88$

- Using the future value tables, the future value of an annuity of $3000 paid at the end of each year for 6 years at 8% p.a. is:

  $FV = a \times X$

  $= 3000 \times 7.3359$

  $= \$22\,007.70$

## EXERCISE 7C Present value using a table

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** An annuity involves depositing $2000 at the end of each year for 7 years. The interest rate is 8% p.a.

**a** Find the single amount of money that could be invested now at the same rate for the same period of time to produce the same final amount.

**b** Using the compound interest formula, find the final amount of the investment.

**c** Using the future value table, check that the annuity gives the same final amount.

**2** An annuity involves depositing $5000 at the end of each year for 8 years. The interest rate is 5.5% p.a.

**a** Find the single amount of money that could be invested now at the same rate for the same period of time to produce the same final amount.

**b** Using the compound interest formula, find the final amount of the investment.

**c** Using the future value table, check that the annuity gives the same final amount.

**3** An annuity involves depositing $500 at the end of each month for 4 years. The interest rate is 6% p.a.

**a** Find the present value of the annuity.

**b** Using the compound interest formula, find the final amount of the investment.

**c** Using the future value annuity formula, check that the annuity gives the same final amount.

### EXAMPLE 7C–2 Determining the required investment to receive a fixed annuity payment

Malcolm needs $400 per month for 3 years while he studies at university. What amount of money do his parents need to invest now at 6% p.a. compounding monthly, to provide the money Malcolm needs?

| Solve | Think | Apply |
|---|---|---|
| $PV = 400 \times 32.8710$<br>$= \$13\,148.40$<br>Malcolm's parents would need to invest $13 148.40 now to provide Malcolm with $400 per month for the next 3 years. | We need to find the single amount of money that invested now at 6% p.a. compounding monthly for 3 years will allow $400 to be withdrawn every month for 3 years, with the balance at the end of the 3 years being $0.<br>The loan can be thought of as the present value of an annuity of $400 per month over 36 months at 0.5% per month.<br>From the present value table $X = 32.8710$. | Use $PV = a \times X$, where $a$ = the regular amount to be withdrawn and $X$ = the interest factor, from the present values table, for the given rate of interest and number of time periods. |

UNDERSTANDING, FLUENCY AND COMMUNICATING

**4** Stuart estimates that he will need \$500 per month for the 4 years of his university course. If the current interest rate is 7.2% p.a. compounding monthly, how much will Stuart's parents need to deposit now to meet his needs?

**5** Angelica is planning an overseas trip for 3 years and estimates that she will need \$600 per month for expenses. How much should she deposit in an account paying 4.8% p.a. compounding monthly to have enough money for her trip?

**6** Mick opens a motorbike repair shop and signs a lease for \$17 500 per year for 5 years. The money is deposited into the landlord's account, which pays 6% p.a. interest. The landlord asks Mick to pay the rent as a lump sum in advance for the 5 years. How much should he pay?

**7** Geoff wants to provide a regular quarterly income of \$20 000 for the next 9 years. If the current interest rate is 6% p.a., how much does he need to invest now to provide the income he wants?

**8** On retirement Joanne receives a lump sum. She wants to invest part of the lump sum in an annuity that will provide \$5000 per month for the next 15 years. If the best interest rate available is 7.2% p.a. compounding monthly, how much of her lump sum must she invest now to provide this monthly income?

**9** Glenn retires and receives a lump-sum payment. He wants to convert part of the lump sum into an annuity that will provide a monthly income of \$7000 for the first 10 years of his retirement. If the best interest rate available is 4.8% p.a. compounding monthly, how much of his lump sum must he invest now to provide this monthly income?

### EXAMPLE 7C–3 Calculating the monthly income from an annuity

In order to provide a regular income when he retires, Paul invests \$300 000 for 15 years in an annuity that pays 6% p.a. compounding monthly. He plans to make regular equal monthly withdrawals for 15 years until there is nothing left in his account. What will be his monthly income from the annuity?

| Solve | Think | Apply |
| --- | --- | --- |
| $PV = a \times X$<br>$300\,000 = a \times 118.5035$<br>$\frac{300\,000}{118.5035} = a$<br>$a = 2531.570$<br>Paul can withdraw \$2531.57 each month. | $PV = 300\,000$<br>$X$ is the present value interest factor for 180 months at 0.5% per month = 118.5035. | The amount invested is the $PV$ of an annuity invested for 15 years at 6% p.a. compounding monthly. Use $PV = a \times X$ to find $a$. |

**10** In order to have a regular income when he retires, Sam invests \$550 000 for 15 years in an annuity that pays 7.2% p.a. compounding monthly. He plans to make regular equal monthly withdrawals for 15 years until there is nothing left in his account. What will be his monthly income from the annuity?

**11** Julie invests \$600 000 for 20 years in an annuity that pays 4.8% p.a. compounding monthly. She plans to make regular equal monthly withdrawals for 20 years until there is nothing left in her account. What will be her monthly income from the annuity?

CHALLENGE

**12** Use the present value formula $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$ to verify that the present value of \$1, invested for the period and interest rate given below, is the amount shown in the present value table.

**a** 5 years at 6% p.a.
**b** 6 years at 8% p.a.
**c** 48 months at 0.4% per month
**d** 120 months at 0.6% per month

# 7D Repaying loans

These resources are available on your obook assess:
- **Video tutorial 7D:** Watch and listen to an explanation of Example 7D–2
- **assess quiz 7D:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

Many financial institutions have different loans with different rates. Some offer no-frills loans with lower interest rates, some have monthly fees, and others have low start loans with lower replayment amounts at the start of the loan.

We can use the table of present value interest factors and the present value formula, $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$, to help us answer questions about repaying loans.

### EXAMPLE 7D–1 Calculating the maximum home loan amount

Jenny and Peter are looking for a loan to buy a house. They have prepared a budget and have calculated that they can afford to pay \$1800 per month. If the current interest rate is 6% p.a. compounding monthly, use the table of present values to determine how much they can afford to borrow over 20 years. Give your answer to the nearest dollar.

| Solve | Think | Apply |
|---|---|---|
| $PV = 1800 \times 139.5808$<br>$= \$251\,245$ (to the nearest dollar) | $a = 1800$, $X = 139.5808$ | The loan can be thought of as the present value of an annuity of \$1800 per month over 240 months at 0.5% per month. |

## EXERCISE 7D Repaying loans

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Lisa and Jack are looking for a loan to buy a house. They have worked out that they can afford to pay \$1000 per month. If the current interest rate is 4.8% p.a. compounding monthly, use the table of present values to determine how much, to the nearest dollar, they can afford to borrow over 20 years.

**2** Zara and Will have calculated that they can afford to pay \$1200 per month on a home loan.

**a** If the current interest rate is 7.2% p.a., compounding monthly, use the table of present values to determine how much, to the nearest dollar, they can afford to borrow over 15 years.

**b** How much more could they borrow if they took the loan over 20 years? Give your answer to the nearest dollar.

### EXAMPLE 7D–2 Calculating loan repayments using the table of present value interest factors

To buy a car, John obtains a bank loan for \$15 000 over 5 years at 7.2% p.a. compounding monthly.

**a** Use the table of present value interest factors to calculate the monthly repayment.

**b** Calculate the total amount to be repaid over the term of the loan.

**c** What is the amount of interest to be paid?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $PV = a \times X$<br>$15\,000 = a \times 50.2621$<br>$\frac{15\,000}{50.2621} = a$<br>$a = 298.4356$<br>The monthly repayment is \$298.44. | $PV = 15\,000$<br>$X$ is the present value interest factor for \$1 over 60 months at 0.6% per month $= 50.2621$. | The amount borrowed can be thought of as the present value of an annuity.<br>Use $PV = a \times X$ to find $a$, where $a$ is the repayment, $X$ is the present value interest factor for the given time period at the given interest rate (per time period). |
| **b** | Total amount repaid $= 298.44 \times 60$<br>$= \$17\,906.40$ | | |
| **c** | Interest paid $= 17\,906.40 - 15\,000$<br>$= \$2906.40$ | | |

**3** Gillian borrows \$12 000 over 4 years at 4.8% p.a. compounding monthly to go on a holiday.

**a** Use the table of present value interest factors to calculate the monthly repayment.

**b** Calculate the total amount to be repaid over the term of the loan.

**c** What is the amount of interest to be paid?

**4** Peter borrows \$20 000 over 5 years at 6% p.a. compounding quarterly to renovate his bathroom.

**a** Use the table of present value interest factors to calculate the quarterly repayment.

**b** Calculate the total amount to be repaid over the term of the loan.

**c** What is the amount of interest to be paid?

**5** Gurpreet obtains a business loan from a bank for \$65 000 at 9% p.a. She agrees to repay the loan in 6-monthly instalments over 10 years.

**a** Use the table of present value interest factors to calculate the 6-monthly instalment.

**b** Calculate the total amount to be repaid over the term of the loan.

**c** What is the amount of interest to be paid?

## EXAMPLE 7D–3 Calculating the monthly repayment using the present value formula

To buy a home, the Ryan's obtain a bank loan for \$435 000 over 25 years at 7.2% p.a. compounding monthly.

**a** Use the present value formula to calculate the monthly repayment.

**b** Calculate the total amount to be repaid over the term of the loan.

**c** What is the amount of interest to be paid?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $435\,000 = a\left[\frac{(1+0.006)^{300}-1}{0.006(1+0.006)^{300}}\right]$<br>$435\,000 = a\left[\frac{(1.006)^{300}-1}{0.006(1.006)^{300}}\right]$<br>$\frac{435\,000}{\left[\frac{(1.006)^{300}-1}{0.006(1.006)^{300}}\right]} = a$<br>$a = 3130.2108$<br>The monthly repayment is \$3130.21. | $PV = 435\,000$<br>$a$ is the monthly repayment.<br>$r = 0.072 \div 12$<br>$= 0.006$<br>$n = 25 \times 12$<br>$= 300$ | The amount borrowed can be thought of as the present value of an annuity.<br>Use $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$, where $a$ is the repayment amount, $r$ is the interest rate per compounding period (expressed as a decimal), and $n$ is the number of periods. |
| **b** | Total amount repaid $= 3130.21 \times 300$<br>$= \$939\,063$ | | |
| **c** | Interest paid $= 939\,063 - 435\,000$<br>$= \$504\,063$ | | |

UNDERSTANDING, FLUENCY AND COMMUNICATING

**6** A home loan is taken out at an interest rate of 7.2% p.a. compounding monthly. The loan of $320 000 is to be borrowed over 25 years.

- **a** Use the present value formula to calculate the monthly repayment.
- **b** Calculate the total amount to be repaid.
- **c** What is the amount of interest to be paid over the term of the loan?

**7** A home loan is taken out at an interest rate of 4.8% p.a. compounding monthly. The $250 000 loan is to be borrowed over 30 years.

- **a** Use the present value formula to find the monthly repayment.
- **b** Calculate the total amount to be repaid.
- **c** Calculate the interest to be paid over the term of the loan.

**8** A loan for $360 000 is taken over 25 years.

- **a** Use the table of present value interest factors to calculate the monthly payment if the interest rate is 8.4% p.a.
- **b** Find the total to be repaid at this rate.
- **c** Use the table of present value interest factors to calculate the monthly payment if the interest rate is 7.2%.
- **d** Find the total repaid at this rate.
- **e** Calculate the amount saved on a $360 000 loan over 25 years if the interest rate charged is 7.2% p.a. instead of 8.4% p.a.

**9** Use the table of present value interest factors to calculate the amount saved on a $280 000 loan over 30 years if the interest rate is 6% p.a. compounded monthly instead of 7.2% p.a. compounded monthly.

**10** Use the present value formula to calculate the amount saved on a $150 000 loan at 9.6% p.a. compounded monthly, if the term is 20 years instead of 25 years.

**11** Use the present value formula to calculate the amount saved on a $120 000 loan at 12% p.a. compounded monthly, if the term is 15 years instead of 25 years.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**12** Fung borrows $290 000 to buy a home unit. The interest rate is 4.8% p.a. compounding monthly and he agrees to repay the loan over 20 years.

- **a** Use the present value formula to calculate the monthly repayment on the amount borrowed.
- **b** The bank charges a management fee of $2.80 and loan protection insurance of $9.50 each month. What is the total monthly repayment he must make?
- **c** There is a mortgage discharge fee of $350 after the loan is paid. What is the total amount paid for the loan?

**13** Anuj and Laura borrow $340 000 over 25 years at 7.2% p.a. compounding monthly to buy a house.

- **a** Use the table of present value interest factors to calculate the monthly repayment on the amount borrowed.
- **b** The bank also charges a monthly loan insurance fee of 38 cents for each $10 000 originally borrowed. Calculate the monthly loan insurance fee.
- **c** What is the total monthly repayment they must make?
- **d** The bank charges a one-off loan establishment fee of 0.6% of the amount of the loan. Calculate this fee.
- **e** At the completion of the loan there is a mortgage discharge fee of $600. What is the total paid for the loan?

**14** The Federal Bank of Australia offers a ‘low start’ loan. Allan and Natalie take out a loan of $220 000 at 7.2% p.a. compounding monthly over 30 years. The monthly repayments on this loan are then $1493. However, the low start repayment for the first year is $1100 per month.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**a** Complete the table to find the amount owing after 12 months.

| Month | Balance at start of month | Interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | \$220 000 | \$1320 | \$1100 | \$220 220 |
| 2 | \$220 220 | \$1321.32 | \$1100 | |
| 3 | | | | |
| 4 | \$220 663.97 | \$1323.98 | \$1100 | \$220 887.95 |
| 5 | \$220 887.95 | \$1325.33 | \$1100 | \$221 113.28 |
| 6 | \$221 113.28 | \$1326.68 | \$1100 | \$221 339.96 |
| 7 | \$221 339.96 | \$1328.04 | \$1100 | \$221 568.00 |
| 8 | \$221 568.00 | \$1329.41 | \$1100 | \$221 797.41 |
| 9 | \$221 797.41 | \$1330.78 | \$1100 | \$222 028.19 |
| 10 | \$222 028.19 | \$1332.17 | \$1100 | \$222 260.36 |
| 11 | | | | |
| 12 | | | | |

**b** What is the disadvantage of the low start loan? What is an advantage of the low start loan?

**c** After the first 12 months the monthly loan repayment reverts to \$1493. It will then take a further 31 years and 5 months to pay off the loan completely. What is the total amount paid for the low start loan?

**d** How much extra is repaid using the low start loan?

**15** The St Harry bank offers an option on its home loan of paying interest only for a period of time up to 5 years, after which normal repayments to complete the loan begin. The interest rate is 8.4% p.a. compounding monthly.

**a** Use the compound interest formula to calculate the monthly interest on a \$310 000 loan. This is the monthly repayment for the interest-only period.

**b** Calculate the normal loan repayment on \$310 000 for a loan that is taken over 25 years.

**c** How much lower is the interest-only repayment than the normal repayment?

**d** Interest-only repayments are made for 3 years, then normal repayments are made for the next 25 years. How much is repaid in total?

**e** How much extra is repaid by taking up the interest-only repayment option?

**f** Give an advantage of using the 'interest-only' option. Give a disadvantage of using the 'interest-only' option.

**g** Give an advantage of the 'interest-only' option over the 'low start' option of question **14**.

Home owners with a mortgage often make additional **lump-sum payments** to reduce the balance of their mortgage and the time taken to repay it.

!

**lump sum payment**
a one-off payment, usually of a significant amount of money

CHALLENGE

**16** Brad and Nailah take out a loan of \$190 000 over 15 years at 8.4% p.a. compounding monthly. Their monthly repayment on this loan is \$1860.

**a** Calculate the total amount they will repay for this loan.

**b** After 36 months of repaying the loan, the amount still owing is \$168 386. Brad and Nailah inherit \$50 000, which they decide to pay into their loan. How much is still owing after making this payment?

**c** If they continue to pay off the loan at \$1860 each month, it will take them another 7 years to repay the loan completely. How much do they save by making the lump-sum payment?

# 7E Using technology to model annuities

These resources are available on your obook assess:

- **Spreadsheet 7E:** Modelling annuities as recurrence relations
- **assess quiz 7E:** Test your skills with an auto-correcting multiple-choice quiz

**recurrence relation** an equation that defines the relationship between terms of a sequence

A **recurrence relation** is an equation that defines a sequence. In a recurrence relation, we define the relationship between terms of the sequence. As shown in this topic, we can model annuities as a recurrence relation.

For example, if \$1000 is invested at the end of each year for 5 years into an account that pays interest of 7% p.a., the balance at the end of each year will be as shown in the table on the right.

| End of year ($n$) | Balance (\$) |
|---|---|
| 1 | 1000 |
| 2 | 2070 |
| 3 | 3214.90 |
| 4 | 4439.94 |
| 5 | 5750.74 |

If we denoted the balance at the end of year $n$ as $A_n$, then:

$A_1 = 1000, A_2 = 2070, A_3 = 3214.90, A_4 = 4439.94, A_5 = 5750.74$

We would then be able to create the following recurrence relation between consecutive terms of the sequence:

$A_{n+1} = 1.07A_n + 1000$

This means that to calculate the balance at the end of one year, we can multiply the balance at the end of the previous year by 1.07 and add \$1000.

Example 7E–1 will show you how to set up a recurrence relation to model an annuity using a spreadsheet.

## EXAMPLE 7E–1 Using a spreadsheet to model an annuity as a recurrence relation

\$5000 is invested at the end of each year for 10 years into an account that pays interest of 8.5% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model this annuity.

**b** What will be the balance in the account at the end of 10 years?

**Solve**

**a**

| | A | B |
|---|---|---|
| 1 | **End of year** | **Balance** |
| 2 | 1 | \$5,000.00 |
| 3 | 2 | = B2*1.085+5000 |
| 4 | 3 | |
| 5 | 4 | |
| 6 | 5 | |

**Think/Apply**

Enter the headings into cells A1 and B1 as shown.

Enter 1 into cell A2 to represent the first year, 2 into cell A3 to represent the second year, and so on.

As the values in column B will be currency values, we can set this in the system. Select this column by clicking on the B, right-click and select 'Format Cells…'. Choose 'Currency' under the Category options.

We know that the balance at the end of the first year will be \$5000, so enter '5000' into cell B2.

To calculate the balance at the end of the second year we need to multiply the balance at the end of the first year by 1.085 (which represents the 8.5% interest) and add 5000. To represent this in the spreadsheet we can use the formula '=B2*1.085+5000'. Enter this formula into cell B3 as shown.

**b**

| | A | B |
|---|---|---|
| 1 | **End of year** | **Balance** |
| 2 | 1 | $5,000.00 |
| 3 | 2 | $10,425.00 |
| 4 | 3 | $16,311.13 |
| 5 | 4 | $22,697.57 |
| 6 | 5 | $29,626.86 |
| 7 | 6 | $37,145.15 |
| 8 | 7 | $45,302.49 |
| 9 | 8 | $54,153.20 |
| 10 | 9 | $63,756.22 |
| 11 | 10 | $74,175.50 |

The balance at the end of 10 years will be $74 175.50.

To calculate the balance for subsequent years we need to follow the same procedure as we did for the second year: multiply the previous year's balance by 1.085 and add 5000.
When the formula has been entered into cell B3 from part **a**, and you've pressed 'Enter', the value for year 2 should appear. Click on this cell, and drag and copy the content to the following cells in this column, up to at least the 10th year (row 11).

To help you, you might like to use the prepared spreadsheet file (Spreadsheet 7E) provided on your obook assess.

## EXERCISE 7E Using technology to model annuities

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** $7000 is invested at the end of each year for 20 years into an account that pays interest of 6.7% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model this annuity.

**b** What will be the balance in the account at the end of 20 years?

**2** Each year, since his daughter's first birthday, Lucas has put $1500 into a savings account that can be accessed when his daughter turns 21. The savings account pays interest of 5% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model the amount in the savings account after $n$ years.

**b** How much will be in the savings account on his daughter's 21st birthday? Assume that Lucas contributes a final $1500 to the account on this day.

**3** On her son's fifth birthday, Emma sets up an annuity to help pay for her son's university education. She starts by putting $2000 into a savings account, and puts an additional $2000 into the account on each further birthday up until her son's eighteenth. The savings account pays interest of 5.8% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model this annuity.

**b** How much will be in the savings account after the final payment?

**4** Andreas puts $1400 into an annuity at the end of each quarter for 5 years. The annuity pays 6.8% p.a. compounding quarterly.

**a** Andreas does not have enough in his annuity after 5 years to purchase his dream car, which retails for $42 000. By how much is he short?

**b** If Andreas continues to put $1400 into his annuity at the end of each quarter, after how many additional quarters will he be able to purchase his dream car?

## EXAMPLE 7E–2 Using a spreadsheet to model a home loan

Kieran obtains a \$300 000 mortgage to buy a one-bedroom apartment. The mortgage has a fixed rate of 6% p.a. compounded monthly, and he makes a mortgage payment of \$1800 each month.

**a** Use a spreadsheet to set up a recurrence relation to model this home loan.

**b** Assuming that Kieran doesn't make any additional payments on his mortgage, how long will it take him to pay off the home loan?

**Solve** | **Think/Apply**

**a**

| | A | B |
|---|---|---|
| 1 | **Month** | **Mortgage balance** |
| 2 | 0 | \$300,000.00 |
| 3 | 1 | = B2*1.005–1800 |
| 4 | 2 | |
| 5 | 3 | |
| 6 | 4 | |
| 7 | 5 | |

Enter the headings into cells A1 and B1 as shown.
Enter 0 into cell A2 to represent the time when the mortgage was taken, 1 into cell A3 to represent the first month, and so on. By highlighting cells A2 and A3 together and dragging the content down, you can fill in the rest of this column.
Set the cells in column B to be currency options, as in Example 7E–1.
We know that the balance at the start of the mortgage is \$300 000, so enter '300 000' into cell B2.
The monthly interest rate can be calculate by dividing 6% by 12: $6 \div 12 = 0.5$.
To calculate the balance at the end of the first month we need to multiply the balance at the end of the first year by 1.005 (which represents the 0.5% interest) and subtract 1800. To represent this in the spreadsheet we can use the formula '=B2*1.005-1800'.
Enter this formula into cell B3 as shown.

**b**

| | A | B |
|---|---|---|
| 1 | **Month** | **Mortgage balance** |
| 2 | 0 | \$300,000.00 |
| 3 | 1 | \$299,700.00 |
| 4 | 2 | \$299,398.50 |
| 5 | 3 | \$299,095.49 |
| 6 | 4 | \$298,790.97 |
| 7 | 5 | \$298,484.92 |

| | | |
|---|---|---|
| 357 | 355 | \$7,545.41 |
| 358 | 356 | \$5,783.13 |
| 359 | 357 | \$4,012.05 |
| 360 | 358 | \$2,232.11 |
| 361 | 359 | \$443.27 |
| 362 | 360 | –\$1,354.51 |

The mortgage will be paid off after 360 months (30 years).

To calculate the balance for subsequent years we need to follow the same procedure as we did for the first month: multiply the previous month's balance by 1.005 and subtract 1800.
When the formula has been entered into cell B3 from part **a**, and you've pressed 'Enter', the value for month 1 should appear. Click on this cell, and drag and copy the content to the following cells in this column, until the value in this column is negative. This is the month when the mortgage will be paid off.

**5** George and Georgina buy their first home with a \$650 000 mortgage. The mortgage has a fixed rate of 4.8% p.a. compounded monthly, and they make a mortgage payment of \$3600 each month.

**a** Use a spreadsheet to set up a recurrence relation to model this home loan.

**b** Assuming that George and Georgina don't make any additional payments on their mortgage, how long will it take them to pay off the home loan?

UNDERSTANDING, FLUENCY AND COMMUNICATING

**6** Sari buys a house with a \$750 000 mortgage. The mortgage has a fixed rate of 5.4% p.a. compounded monthly, and she makes a mortgage payment of \$4200 each month.

- **a** What will be the balance of her mortgage after 10 years?
- **b** What will be the balance of her mortgage after 20 years?
- **c** What will be the balance of her mortgage after 30 years?

## EXAMPLE 7E–3 Using a spreadsheet to compare annuity investments

Jane is looking to set up an annuity to pay for her children's university education. After speaking to a financial adviser she decides that she can put between \$3500 and \$4000 into an annuity at the end of each year for 10 years. The savings account pays interest of 7.75% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model the following two scenarios:
Scenario 1: Jane invests \$3500 each year.
Scenario 2: Jane invests \$4000 each year.

**b** How much more will there be in the annuity after 10 years if Jane decides to go with scenario 2 as opposed to scenario 1?

| | Solve | Think/Apply |
|---|---|---|
| **a** | (spreadsheet below) | Follow the instructions in Example 7E–1 to set up your spreadsheet.<br>The formula for cell B3 will be '=B2*1.0775+3500', and the formula for cell C3 will '=C2*1.0775+4000'. Copy these formulae into the other cells in the columns to complete you model. |
| **b** | 57 262.83 – 50 104.97 = \$7157.86<br>There will be \$7157.86 more in the annuity if Jane decides to go with scenario 2 as opposed to scenario 1. | Calculate the difference between scenarios 1 and 2. |

| | A | B | C |
|---|---|---|---|
| 1 | **Year** | **Scenario 1** | **Scenario 2** |
| 2 | 1 | \$3,500.00 | \$4,000.00 |
| 3 | 2 | \$7,271.25 | \$8,310.00 |
| 4 | 3 | \$11,334.77 | \$12,954.03 |
| 5 | 4 | \$15,713.22 | \$17,957.96 |
| 6 | 5 | \$20,430.99 | \$23,349.70 |
| 7 | 6 | \$25,514.39 | \$29,159.31 |
| 8 | 7 | \$30,991.76 | \$35,419.15 |
| 9 | 8 | \$36,893.62 | \$42,164.14 |
| 10 | 9 | \$43,252.88 | \$49,431.86 |
| 11 | 10 | \$50,104.97 | \$57,262.83 |

**7** Peter is looking to set up an annuity to save for a future home loan. After speaking to his financial adviser he decides that he can put between \$5000 and \$6000 into an annuity at the end of each year for 12 years. The account pays interest of 6.8% p.a.

**a** Use a spreadsheet to set up a recurrence relation to model the following two scenarios:
Scenario 1: Peter invests \$5000 each year.
Scenario 2: Peter invests \$6000 each year.

**b** How much more will Peter have in his account after 12 years if he decides to go with scenario 2 instead of scenario 1?

**8** \$3000 was invested into a savings account at the end of each year for 8 years. Use technology to determine how much more will be in the account if it receives an interest rate of 6.9% p.a. as opposed to an interest rate of 6.5% p.a.

**9** Use technology to determine how much longer, in months, it will take an annuity to reach $100 000 in a savings account paying 8.1% p.a. if $3750 is invested at the end of each month, instead of $3250 being invested at the end of each month.

**10** Use technology to determine how much longer, in months, it will take to pay back a mortgage of $420 000 with monthly repayments of $3000 if the interest rate is 6% p.a., instead of to 5.4% p.a.?

## EXAMPLE 7E–4 Using a spreadsheet to compare home loans graphically

Harriet is a mortgage broker and is setting up a fixed rate mortgage for one of her clients. Her client wants to borrow $520 000 at a fixed mortgage rate of 6% p.a. compounded monthly, and is willing to pay $3400 per month. Harriet suggests that her client pays back $3600 to reduce the length of the mortgage.

**a** Use a spreadsheet to set up a recurrence relation to model the following two scenarios:
Scenario 1: The client pays back $3400 per month.
Scenario 2: The client pays back $3600 per month.

**b** How much quicker will it be to repay the mortgage under scenario 2?

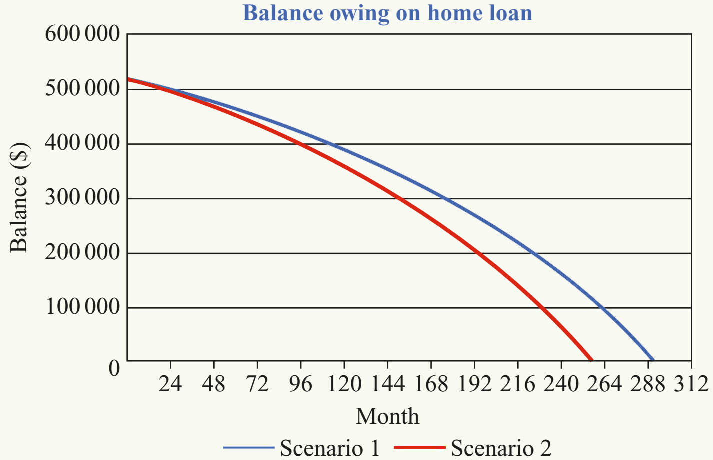
**c** Use a spreadsheet to graphically represent the two scenarios.

| | Solve | Think/Apply |
|---|---|---|
| **a** | (spreadsheet below) | Follow the instructions in Example 7E–2 to set up your spreadsheet.<br>The formula for cell B3 will be '=B2*1.005-3400', and the formula for cell C3 will '=C2*1.005-3600'.<br>Copy these formulae into the other cells in the columns to complete your model. |
| **b** | 291 – 257 = 34 months<br>It will be 34 months quicker to repay the mortgage under scenario 2. | Find the value where the values in each column first reach 0 to determine when the mortgage is repaid. For scenario 1 this will be after 291 months and for scenario 2 this will be after 257 months. |
| **c** | (graph below) | Change the final value in columns B and C to 0, as the value of the mortgage will not go below 0 (the final repayment will not be the full amount).<br>Highlight all of the cells in your spreadsheet which contain data, including the headings.<br>Select 'Insert', then 'Line' and choose the first option. Use the available options to change the scales, add axis labels and a title to your graph. |

**a**

| | A | B | C |
|---|---|---|---|
| 1 | **Month** | **Scenario 1** | **Scenario 2** |
| 2 | 0 | $520,000.00 | $520,000.00 |
| 3 | 1 | $519,200.00 | $519,000.00 |
| 4 | 2 | $518,396.00 | $517,995.00 |
| 5 | 3 | $517,587.98 | $516,984.98 |
| 6 | 4 | $516,775.92 | $515,969.90 |
| 7 | 5 | $515,959.80 | $514,949.75 |

**c**



**11** Francesca and Paulo have finally found their dream home, and set up a fixed-rate mortgage for $800 000 with interest of 4.8% p.a. compounded monthly. Francesca wants to pay back $4400 per month, whereas Paulo suggests they pay back $4500 per month.

**a** Use a spreadsheet to set up a recurrence relation to model the following two scenarios:

Scenario 1: Francesca and Paulo pay back $4400 per month.

Scenario 2: Francesca and Paulo pay back $4500 per month.

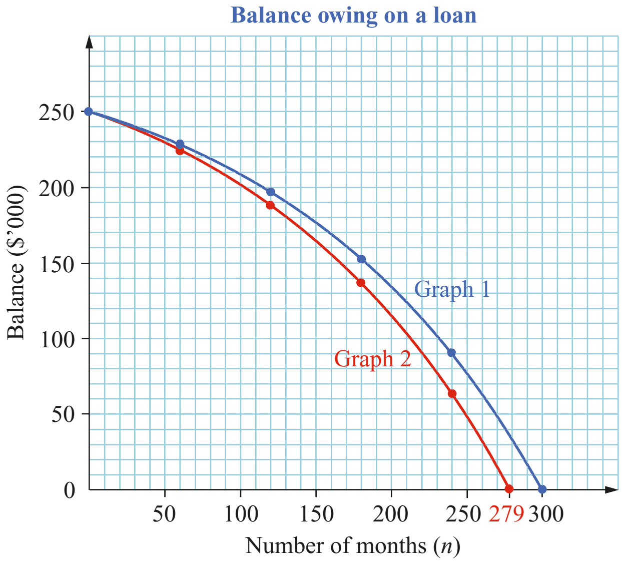
**b** How much quicker will it be to repay the mortgage under scenario 2?

**c** Use a spreadsheet to graphically represent the two scenarios.


**12** Graph 1 shows the balance owing on a loan of $250 000 at 7.2% p.a. compounding monthly over 25 years. The monthly repayment is $1799. Graph 2 shows the balance owing on this loan if an extra $50 per month is paid off.

**a** By how many months is the loan reduced by paying an extra $50 per month?

**b** How much is saved on the total cost of the loan?



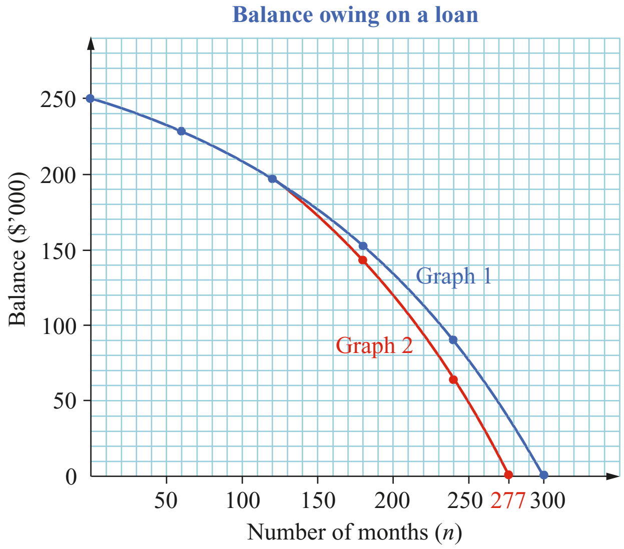

**13** Graph 1 shows the balance owing on a loan of $250 000 at 7.2% p.a. compounding monthly over 25 years. The monthly repayment is $1799. Graph 2 shows the balance if the borrower starts paying an extra $100 each month during the term of the loan.

**a** When does the borrower start paying the extra $100 each month?

**b** By how many months is the loan reduced by paying the extra $100 per month?

**c** For how many months does the borrower pay the extra $100?

**d** How much is saved on the total cost of the loan?



UNDERSTANDING, FLUENCY AND COMMUNICATING

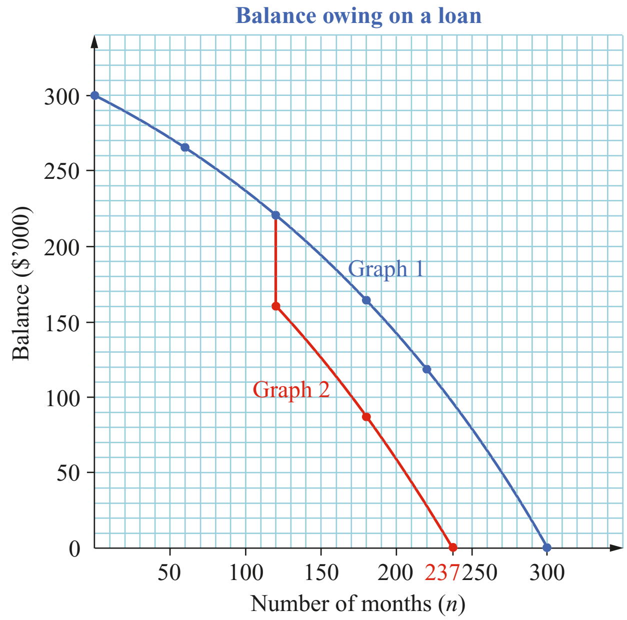
**14** Graph 1 shows the balance owing on a loan of \$300 000 at 4.8% p.a. compounding monthly over 25 years. The monthly repayment is \$1719. Graph 2 shows the balance owing when a lump-sum payment is made during the term of the loan and the repayments are kept the same.

**a** When was the lump-sum payment made?

**b** How much was this payment?

**c** By how many months was the loan reduced by paying the lump sum?

**d** How much was saved on the total cost of the loan by making the lump-sum payment?




PROBLEM SOLVING, REASONING AND JUSTIFICATION

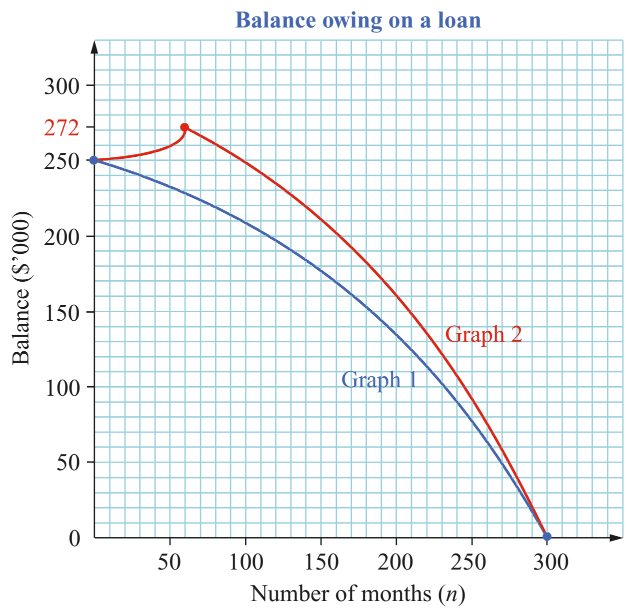
**15** Graph 1 shows the balance owing on a normal loan of \$250 000 at 7.2% p.a. compounding monthly over 25 years. The monthly repayment is \$1799. Graph 2 shows the balance owing on the same loan if only \$1200 per month is paid for the first 5 years, and the loan is then paid off over the next 20 years. Consider graph 2.

**a** What happens to the balance owing in the first 5 years? Explain why.

**b** What is the balance after 5 years?

**c** Use the table of present value interest factors to calculate the monthly repayment now required to pay off the loan. Give your answer to the nearest dollar.

**d** How much extra does it cost to take this 'low start' loan compared with the normal loan?




**16** Given the same starting balance and monthly repayments, will a loan with a high interest rate be paid back faster or slower than a loan with a low interest rate? Explain your reasoning.

**17** Explain, in your own words, how varying the payment amount and the interest rate will affect the future value of an annuity.

**18** Graph 1 in each of the following diagrams shows the balance owing on a normal reducing-balance home loan with regular equal repayments over a period of time. Match each graph with its description.

**a** A lump-sum payment is made during the period of the loan.

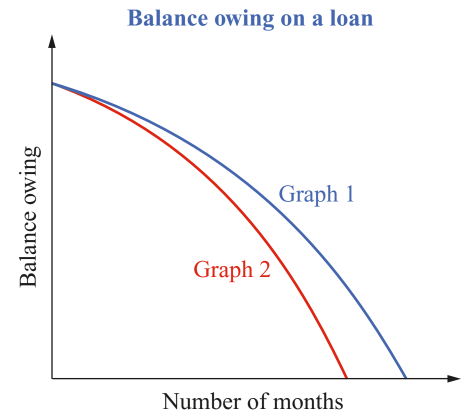
**b** During the period of the loan the borrower starts making higher repayments than those needed to pay off the loan in the agreed period of time.

**c** There is an interest-only period then regular payments are made.

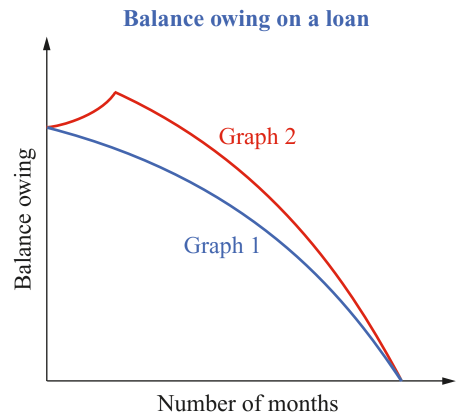
**d** For the first 3 years the monthly repayments are less than the interest on the loan and then regular repayments are made to pay off the loan.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

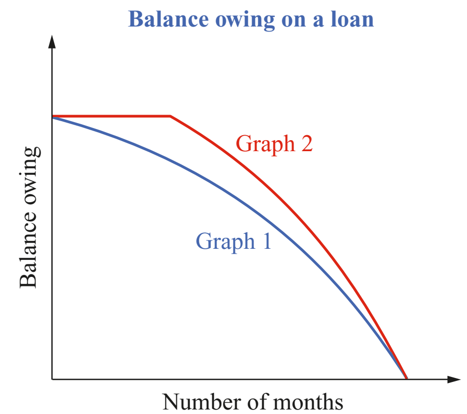
A



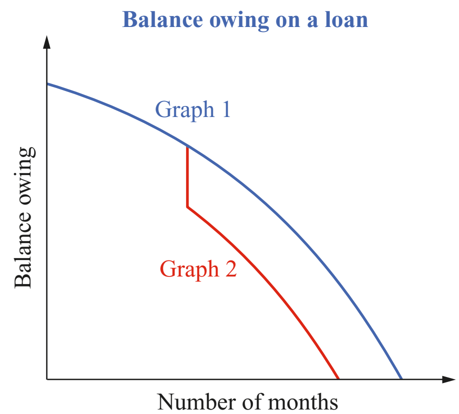

B




C




D




CHALLENGE

**19** Olivia is able to put \$3000 of her salary each year into a savings account that pays interest of 6.6% p.a.

How much more will be in Olivia's account at the end of 10 years if she invests \$250 at the end of each month with the interest compounded monthly, as opposed to \$3000 at the end of each year with the interest compounded annually?

Hint: use technology to create two tables representing the two scenarios.

**20** Davide takes out a \$40 000 loan to purchase a new sports car. The loan has an interest rate of 5.2% p.a.

Davide can pay back \$7800 each year. Use technology to determine how much money he will save by paying the loan back weekly (with the interest compounded weekly) rather than monthly (with the interest compounded monthly). *Note*: assume there are 52 weeks in a year.

Hint: to calculate the total repayment amount multiply the number of repayments by the repayment amount, then subtract the amount that has been overpaid with the final repayment.

FINANCIAL MATHEMATICS

# CHAPTER 7 REVIEW ANNUITIES

You should now be able to:

✔ understand that an annuity is a form of investment or loan

✔ calculate the future value of an annuity

✔ calculate the present value of an annuity

✔ solve problems involving annuities

✔ use the present value formula to calculate loan repayments, and the total amount repaid

✔ calculate fees and charges on loans

✔ use technology to model and compare annuities

✔ interpret graphs that compare repayment options for home loans.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

Use the future value annuity formula $FV = a\left[\frac{(1+r)^n - 1}{r}\right]$ to answer questions **1–3**.

7A **1** The future value of an annuity with \$4600 invested at the end of each year for 6 years at 6% p.a. interest is:

**A** \$1656 **B** \$6256 **C** \$32 086.47 **D** \$397 480.74

7A **2** An annuity has \$170 invested at the end of each month for 3 years. The interest rate is 12% p.a. compounded monthly. The future value of the annuity is:

**A** \$568.90 **B** \$568.91 **C** \$6883.78 **D** \$7323.07

7A **3** \$5000 is put into an annuity at the end of each year for 8 years. How much more interest will be earned if the annuity pays 5.5% p.a. compounding yearly compared to 4.5% p.a. compounding yearly?

**A** \$170.78 **B** \$469.09 **C** \$1238.71 **D** \$1707.80

Use the future value tables on pages 308 and 309 in Topic 7B to answer questions **4** and **5**.

7B **4** Eudora wants to have \$8000 in 6 years' time. The amount she would need to deposit annually into an annuity that pays 5.5% p.a. interest is:

**A** \$1146.90 **B** \$1290.17 **C** \$1161.42 **D** \$1176.14

7B **5** The amount that Eric would need to deposit at the end of each quarter in an annuity that pays 8% p.a. compounding quarterly to have \$7000 after 5 years is:

**A** \$1698.37 **B** \$1553.45 **C** \$288.10 **D** \$152.97

Use the present value annuity formula $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$ to answer question **6**.

7A **6** An annuity has \$400 invested at the end of each month for 3 years. The interest rate is 9.6% p.a. The single amount of money that could be invested at the same rate for the same period of time to produce the same final amount is:

**A** \$1001.78 **B** \$1001.79 **C** \$12 468.93 **D** \$12 468.94

Use the present value tables on pages 312 and 313 in Topic 7C to answer questions **7–10**.

7C **7** Selim estimates that while at university he will need \$320 per month for the 5 years of his course. If current interest rates are 6% p.a. compounding monthly, the amount Selim's parents need to deposit now to meet his needs is:

**A** \$16 552.19 **B** \$6057.37 **C** \$1576.28 **D** \$1347.96

7C **8** Maureen wants to buy a home unit. She has worked out that she can afford to pay \$1600 per month. If the interest rate is 7.2% p.a. compounding monthly, how much can she afford to borrow over 20 years?

**A** \$206 179 **B** \$203 213 **C** \$185 722 **D** \$175 815

7C **9** Shen invests \$550 000 for 10 years in an annuity that pays 4.8% p.a. compounding monthly. He plans to make regular equal monthly withdrawals until there is nothing left in the account. What will be his monthly income from the annuity?

**A** \$3580.00 **B** \$4292.28 **C** \$56 217.67 **D** \$5779.98

7D **10** A home loan of \$290 000 is taken over 25 years at 6% p.a. compounding monthly. The monthly repayment is:

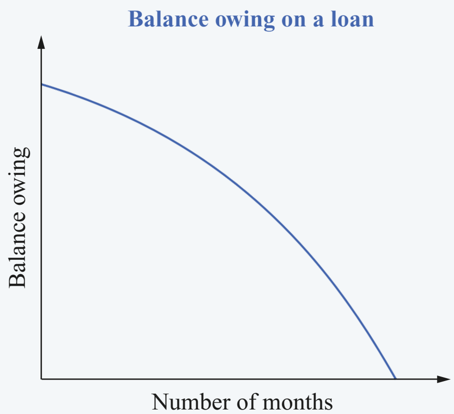
**A** \$418.47 **B** \$1738.70 **C** \$1868.47 **D** \$2043.06

7E **11** How much quicker will a \$250 000 home loan at 4.8% p.a. compounding monthly take to repay if the repayments are \$2600 per month instead of to \$2300 per month?

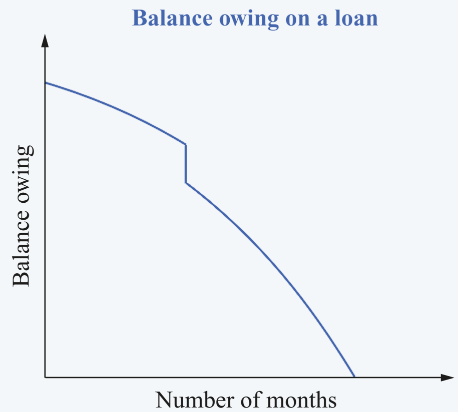
**A** 21 months **B** 143 months **C** 14 months **D** 3 months

7E **12** Which of the following graphs of amount owing on a home loan versus time shows that a lump sum has been paid off the loan during the period of the loan?

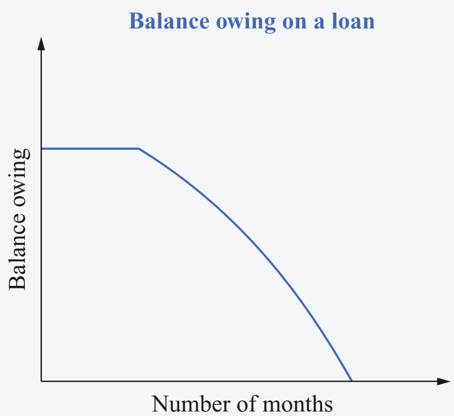
**A**



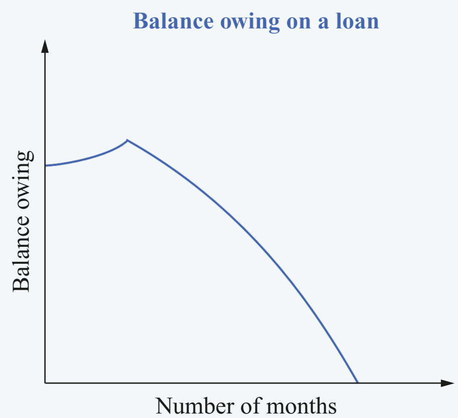

**B**




**C**




**D**




FINANCIAL MATHEMATICS

# REVIEW SET 1

1 Find the future value of an annuity when $2500 is invested at the end of each year for 7 years at 6% p.a. interest.

2 Find the future value of an annuity when $400 is invested at the end of each month for 4 years. The interest rate is 4.8% p.a. compounded monthly.

3 Mirka wants to have $6000 in 4 years' time. How much would she need to deposit annually into an annuity that pays 8% p.a. interest?

4 How much would Karen need to deposit at the end of each month in an annuity that pays 6% p.a. compounding monthly, to have $11 000 after 6 years?

5 An annuity involves depositing $8000 at the end of each year for 6 years. The interest rate is 4.5% p.a.
   **a** Find the single amount of money that could be invested at the same rate for the same period of time to produce the same final amount.
   **b** Using the compound interest formula, find the final amount of the investment.
   **c** Using the value future tables on pages 308 and 309 in Topic 7B, check that the annuity produces the same final amount.

6 Marlin opens a video repair shop and signs a lease for $4500 per year for 5 years. The money is deposited into the landlord's account, which pays 5.5% p.a. interest. The landlord asks Marlin to pay the rent as a lump sum in advance for the 5 years. How much should he pay?

7 A home loan is taken out at an interest rate of 7.2% p.a. compounding monthly. The $410 000 loan is to be repaid over 20 years.
   **a** Find the monthly repayment using the present value formula.
   **b** What is the total amount to be repaid?
   **c** What is the interest to be paid over the term of the loan?

8 $3000 is put into an annuity at the end of each year for 10 years. How much more will the balance be if the interest rate is 7.5% p.a. instead of 6.5% p.a.?

9 Colin and Cass take out a loan of $240 000 over 20 years at 7.2% p.a. compounding monthly. Their monthly repayment on this loan is $1890.
   **a** Calculate the total amount they will pay for this loan.
   **b** After 36 months of repaying the loan, the amount still owing is $221 977. Colin and Cass then pay a lump sum of $30 000 off their loan. How much is still owing after making this lump-sum payment?
   **c** If they continue to pay off the loan at $1890 each month it will take them another 13 years and 1 month to repay the loan completely. How much do they save by making the lump-sum payment?

# REVIEW SET 2

1 Use the tables on pages 308 and 309 in Topic 7B to find the future value of an annuity when $3600 is invested at the end of each year for 7 years at 6% p.a. interest.

2 Find the future value of an annuity when $150 is invested at the end of each month for 2 years. The interest rate is 6% p.a. compounded monthly.

3 Emily wants to have $10 000 in 5 years' time. How much would she need to deposit annually into an annuity that pays 4.5% p.a. interest?

4 How much would Ewan need to deposit at the end of each quarter in an annuity that pays 8% p.a. compounding quarterly, to have $6000 after 5 years?

5 An annuity has $300 invested at the end of each month for 4 years. The interest rate is 9.6% p.a.

a Find the lump sum that could be invested at the same rate for the same amount of time to produce the same final amount.

b Using the compound interest formula, find the final amount of the investment.

c Using the future value tables on pages 308 and 309 in Topic 7B, check that the annuity produces the same final amount.

6 Stan estimates that while at university he will need $650 per month for the 4 years of his course. If current interest rates are 8.4% p.a. compounding monthly, how much will Stan's parents need to deposit now to meet his needs?

7 Use the present value formula to calculate the amount saved on a $270 000 loan over 30 years if the interest rate is 6% p.a. instead of 7.2%, assuming monthly repayments.

8 The Eastern bank offers a 'no start' loan. This home loan allows customers to make no payments for the first year; however, interest still accrues over the first year on the amount borrowed. The interest rate is 7.2% p.a. compounded monthly. Jan and George borrow $325 000 using this no start loan.

a Calculate the amount owing at the end of the first year. Give your answer to the nearest dollar.

b They agree to pay back the amount owing at the end of the first year over 25 years. Calculate the monthly repayment.

c What is the total cost of this no start loan?

d If Jan and George had taken a normal loan of $325 000 at 7.2% p.a. compounded monthly over 25 years, what would the monthly repayment have been?

e How much extra does the no start loan cost?

## REVIEW SET 3

1 Use the tables on pages 308 and 309 in Topic 7B to find the future value of an annuity when $1600 is invested at the end of each year for 5 years at 3.5% p.a. interest.

2 Find the future value of an annuity that has $200 invested at the end of each month for 3 years. The interest rate is 6% p.a. compounded monthly.

3 Ardley wants to have $7000 in 5 years' time. How much would he need to deposit annually into an annuity that pays 5.5% p.a. interest?

4 How much would Esteban need to deposit at the end of each quarter in an annuity that pays 8% p.a. compounding quarterly, to have $7000 after 3 years?

5 An annuity has $3000 invested at the end of each year for 8 years. The interest rate is 5.5% p.a.

a Find the single amount of money that could be invested at the same rate for the same amount of time to produce the same final amount.

b Using the compound interest formula, find the final amount of the investment.

c Using the future value tables on pages 308 and 309 in Topic 7B, check that the annuity produces the same final amount.

6 Avril is planning an overseas trip for 2 years and estimates that she will need $800 per month for expenses. How much should she deposit in an account paying 6% p.a. compounding monthly, to have the money for her trip?

7 A home loan is taken out at an interest rate of 8.4% p.a. compounding monthly. The amount of \$340 000 is to be borrowed over 25 years.

- **a** Find the monthly repayment using the present value formula.
- **b** Calculate the total amount to be repaid.
- **c** What is the interest to be paid over the term of the loan?

8 Kate is looking to start an annuity and finds an account that pays 6.9% p.a. compounding annually. How much more will be in her account at the end of 5 years if she puts \$4500 rather than \$4000 into the account at the end of each year?

9 Lucia and Andrew take out a loan of \$380 000 over 25 years at 4.8% p.a. compounding monthly. Their monthly repayment on this loan is \$2177.

- **a** Calculate the total amount they will repay for this loan.
- **b** After 48 months of repaying the loan, the amount still owing is \$345 309. Lucia and Andrew then pay a lump sum of \$90 000 off their loan. How much is still owing after this lump-sum payment?
- **c** If they continue to pay off the loan at \$2177 each month it will take them another 13 years and 3 months to repay the loan completely. How much do they save by making the lump-sum payment?

## REVIEW SET 4

1 Use the tables on pages 308 and 309 in Topic 7B to find the future value of an annuity when \$2455 is invested at the end of each year for 9 years at 4.5% p.a. interest.

2 Find the future value of an annuity when \$350 is invested at the end of each month for 5 years. The interest rate is 9.6% p.a. compounded monthly.

3 Eloise wants to have \$4000 in 7 years' time. How much would she need to deposit annually into an annuity that pays 5.5% p.a. interest?

4 How much does Keisha need to deposit at the end of each month in an annuity that pays 6% p.a. compounding monthly, to have \$9000 after 5 years?

5 An annuity has \$4300 invested at the end of each year for 5 years. The interest rate is 8% p.a.

- **a** Find the lump sum that could be invested at the same rate for the same amount of time to produce the same final amount.
- **b** Using the compound interest formula, find the final amount of the investment.
- **c** Using the future value tables on pages 308 and 309 in Topic 7B, check that the annuity produces the same final amount.

6 Calculate the amount saved on a \$330 000 loan at 7.2% p.a. if the term is 20 years instead of 25 years, assuming monthly repayments.

7 Eastpac bank offers an option on its home loan to pay interest only for a period of time up to 5 years, after which normal repayments to complete the loan begin. The interest rate is 4.8% p.a. compounding monthly.

- **a** Calculate the monthly interest on a \$400 000 loan. This is the monthly repayment for the interest-only period.
- **b** Calculate the normal loan repayment on \$400 000 if the loan is taken over 25 years.
- **c** How much lower is the interest-only repayment than the normal repayment?
- **d** The loan is paid as interest only for 3 years, then normal repayments are made for the next 25 years. How much is repaid in total?
- **e** How much extra is repaid by taking up the interest-only repayment option?

**8** Maynard opens a computer repair shop and signs a lease for \$9400 per year for 7 years. The money is deposited into the landlord's account, which pays 4.5% p.a. interest. The landlord asks Maynard to pay the rent as a lump sum in advance for the 7 years. How much should he pay?

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** Greg pays \$1500 at the end of each quarter for 12 years into an annuity that pays 8% p.a. compounded quarterly. Calculate the future value of the annuity (to the nearest dollar). (1 mark)

**b** Chris wants to have \$15 000 in 5 years' time to be able to buy a car. She invests in an annuity that pays 12% p.a. compounding monthly. How much must she invest each month to reach her target? (1 mark)

**c** What lump sum must be invested now to have \$25 000 in 10 years' time, if the interest rate is 7.5% p.a. compounding monthly? Give your answer to the nearest dollar. (1 mark)

**d** When he retires, Rob receives a lump-sum payment. He wants to convert part of the lump sum into an annuity that will provide a monthly income of \$8000 for the first 10 years of his retirement. If the best interest rate available is 6% p.a. compounding monthly, how much of his lump sum, to the nearest dollar, must he invest now to provide this monthly income? (1 mark)

**e** Zac and Wendy have calculated that they can afford to pay \$1400 per month on a home loan.

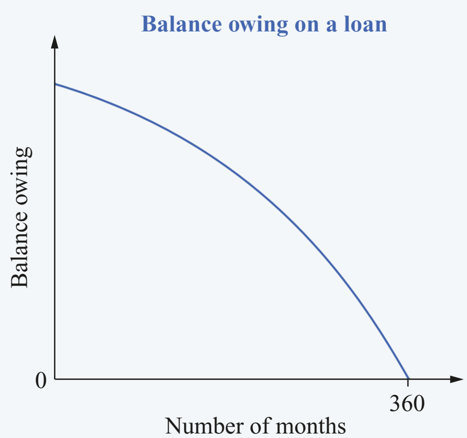
- **i** If the current interest rate is 7.2% p.a., how much can they afford to borrow over 25 years? (1 mark)
- **ii** How much more could they borrow if they took the loan over 30 years? (1 mark)
- **iii** How much more could they borrow if they could get a loan with an interest rate of 6% over 25 years? (1 mark)

**f** Kylie and Rick are buying a home unit. The purchase price is \$560 000 and they have saved a deposit of \$150 000. A bank offers them a loan for the balance of the purchase price over 30 years at 7.2% p.a. compounding monthly. The bank fees are:

| | |
|---|---|
| Loan establishment fee | \$600* |
| Loan insurance | 1.4c per month per \$1000 borrowed |
| Management fee | \$3.80 per month |
| Mortgage discharge fee | \$720 |

**Note*: this is an additional fee to the loan amount.

- **i** Calculate the monthly repayment on the amount borrowed. (1 mark)
- **ii** Calculate the amount of interest paid on the loan. (1 mark)
- **iii** Calculate the loan insurance fee per month. (1 mark)
- **iv** Calculate the total monthly repayment on the loan. (1 mark)
- **v** Calculate the total amount paid to the bank. (2 marks)
- **vi** Calculate the total cost of the loan (including fees). (1 mark)
- **vii** The graph shows the balance owing over the term of the loan. Copy the graph and sketch another graph on the same axes to show the effect on the balance of making a lump-sum payment off the loan after 10 years. (1 mark)

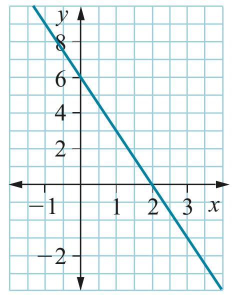



**TOTAL: 15 marks**

FINANCIAL MATHEMATICS

# Chapters 4–7 CUMULATIVE REVIEW

**1** Find the gradient and $y$-intercept of this straight-line graph.




**2** **a** Complete this table of values for $y = 3x - 2$.

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Graph $y = 3x - 2$ on a number plane.

**3** Draw the straight-line graph $y = -3 - 4x$.

**4** The graph models taxi charges.

**a** Find the cost of travelling 75 km.

**b** How far can you travel for \$50?

**c** Find the gradient. What is the meaning of the gradient in this context?

**d** Find the intercept on the vertical axis. What is its meaning?

**Taxi hire charges**



**5** The Super Sock Company has fixed costs of \$800 per day and a variable cost of \$6 per pair of socks. The daily cost may be modelled using the straight-line equation $C = 6n + 800$.

**a** Explain each term in the formula $C = 6n + 800$.

**b** Complete this table of values.

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $C$ | | | | | | |

**c** Draw the straight-line graph of $C = 6n + 800$.

**d** Use the graph to find the number of pairs of socks produced when the cost is \$1580.

**6** A mobile telephone plan charges calls at 75 cents per minute.

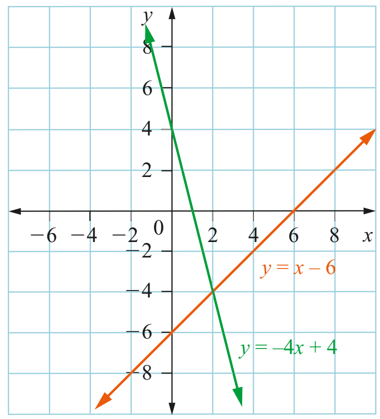
**a** Write a linear equation to model this information. Relate the cost in dollars to the time in minutes.

**b** Draw the graph of time versus cost. Which is the dependent variable?

**c** Use the graph to find the cost of a call of 2.5 min.

**d** What is the maximum length of a call that costs \$2.50?

**7** The graphs of the pair of simultaneous linear equations $y = x - 6$ and $y = -4x + 4$ are drawn on the same number plane.




**a** Write the coordinates of the point of intersection of the two lines.

**b** Use the coordinates of this point to write the solution to the simultaneous linear equations.

**8** Solve each pair of simultaneous equations using a graphical method.

**a** $y = 7x - 5$ and $y = 5 - 3x$

**b** $4x - 3y = 27$ and $y = 1 - 2x$

**9** The Gossimer Stocking Company has fixed costs of \$900 per day and a variable cost of \$3 per pair of stockings, $C = 3n + 900$, and revenue of \$10 per pair sold, $R = 10n$.

**a** Graph these two equations on the same number plane.

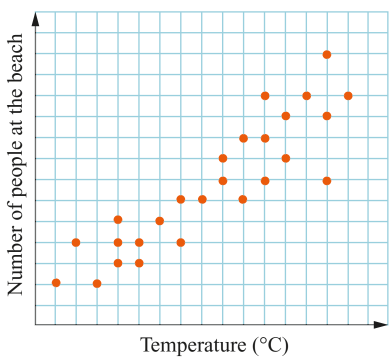
**b** Find the break-even point.

**c** How many pairs of stockings must be sold to break even?

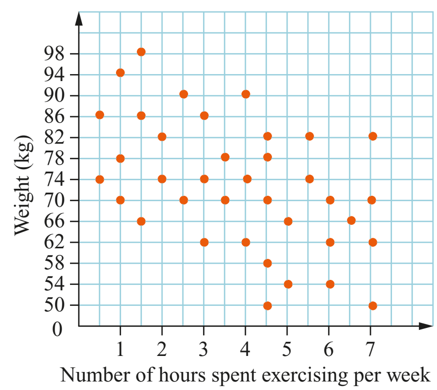
**d** What is the break-even income?

**10** Consider the following bivariate scatterplots.

**i**




**ii**

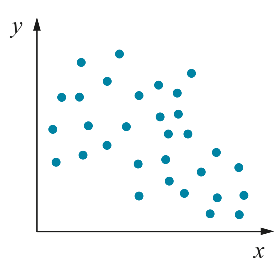



**a** Identify the independent variable and the dependent variable in each scatterplot.

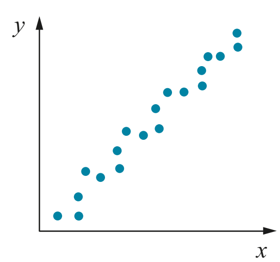
**b** State whether or not there appears to be a linear relationship between the variables in each scatterplot.

**11** State whether the pairs of variables graphed below have perfect, strong, moderate, weak or no correlation.

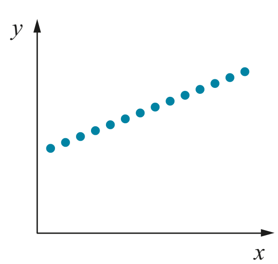
**a**



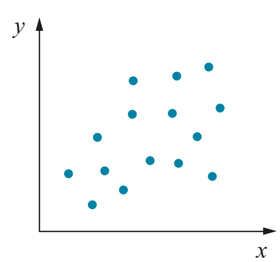

**b**




**c**




**d**



**12** For each of the scatterplots in question **11**, state whether the correlation coefficient is positive, negative or approximately zero.

**13** Explain what a correlation coefficient of 0.85 means.

**14** **a** Enter the data shown in the table into a spreadsheet and calculate the correlation coefficient, $r$.

| $x$ | 15 | 25 | 36 | 11 | 32 | 25 | 18 | 33 | 16 | 27 | 41 | 8 | 20 | 31 | 28 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 25 | 16 | 2 | 42 | 12 | 11 | 51 | 5 | 33 | 19 | 3 | 35 | 21 | 10 | 15 |

**b** Draw a scatterplot of the data.

**c** Describe the trend shown in the scatterplot.

**15** The following pairs of variables were measured and a moderate or strong positive correlation between them was found. Discuss whether a cause and effect relationship exists, or whether it is a case of spurious correlation.

**a** the height of a person and the length of their left foot

**b** the number of people working at a supermarket and the price of milk

**c** daily temperature and cold drink consumption

**16** The graph shows a scatterplot with a line of best fit for the price and number of jeans sold. Use the line of best fit to predict:

**a** the number of jeans sold when the price of jeans is $80

**b** the price of jeans when the number of jeans sold is 150.


**17** The Maths test score obtained by a student after studying for a number of hours is shown in this table.

| **Hours of study** | 2 | 1 | 4 | 5 | 2 | 3 | 3 | 2 | 4 | 1 |
|---|---|---|---|---|---|---|---|---|---|---|
| **Maths test score** | 67 | 55 | 87 | 81 | 70 | 75 | 59 | 60 | 93 | 62 |

**a** Use a spreadsheet to illustrate the data on a scatterplot.

**b** Calculate the correlation coefficient, $r$.

**c** Draw the trendline and find the equation of this line.

**d** Interpret the gradient of the trendline in the context of the data.

**e** Use this equation to predict a student's Maths test score:

**i** after 3 hours of study

**ii** after 4.5 hours of study

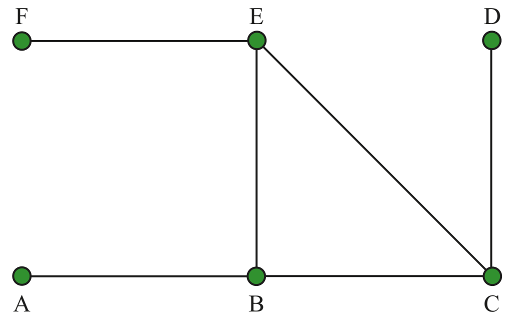
**iii** after 30 minutes of study

**iv** after 7 hours of study.

**f** Which of the answers in part **e** are the least reliable? Give reasons using your understanding of interpolation and extrapolation.

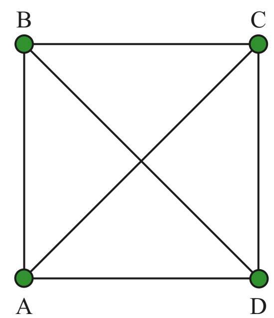
**18** Lisa wants to investigate the most popular pet and she asks people she meets in the street the following survey question: Do you like cats or dogs? Explain why this survey question may result in a biased statistical investigation.

**19** The network below shows vertices connected by edges. Fill out the table, which lists the vertices and the connections between them.




| Vertex | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Connected to | B | A, E, C | | | | |

**20** Inspector Poirot has four suspects in his criminal investigation: A, B, C and D. Any one suspect may have committed the crime, or the suspects could be working together. Poirot indicates the possibility that two suspects are working together by drawing an edge between the corresponding vertices in the network below.




Evidence emerges that suspects A and B are not working together, and that suspect D is completely innocent. What does Poirot's network of suspects look like with this new evidence?

**21** The five boroughs of New York – Manhattan, the Bronx, Queens, Brooklyn and Staten Island – are connected by many bridges, as listed in the table. Queens and Brooklyn are connected by land, and so for this question are considered the same.

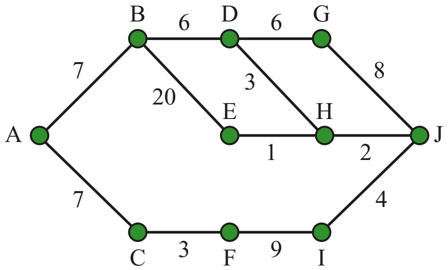
| | Manhattan | The Bronx | Queens/Brooklyn | Staten Island |
|---|---|---|---|---|
| Manhattan | – | 11 | 8 | 0 |
| The Bronx | | – | 3 | 0 |
| Queens/Brooklyn | | | – | 1 |
| Staten Island | | | | – |

Is it possible to cross every bridge in New York exactly once?

**22** Find a spanning tree for the network in question **20**.

CUMULATIVE REVIEW

Use the network below to answer questions **23–26**.




**23** Use Prim's algorithm to find a minimal spanning tree for this network.

**24** Use Kruskal's algorithm to find a minimal spanning tree for this network.

**25** Use Dijkstra's algorithm to find the shortest path between A and J in this network.

**26** What is the shortest path between B and E in this network?

**27** Find the future value of an annuity when \$1500 is invested at the end of each year for 4 years at 6% p.a. interest.

**28** Find the future value of an annuity when \$700 is invested at the end of each month for 5 years. The interest rate is 9% p.a. compounded monthly.

**29** Marie wants to have \$8000 in 5 years' time. How much would she need to deposit annually into an annuity that pays 4.9% p.a. interest?

**30** How much would Tyna need to deposit at the end of each month in an annuity that pays 3.4% p.a. compounding monthly, to have \$9000 after 7 years?

**31** An annuity has \$5000 invested at the end of each year for 9 years. The interest rate is 4.7% p.a.

**a** Find the lump of sum money that could be invested at the same rate for the same period of time to produce the same final amount.

**b** Using the compound interest formula, find the final amount of the investment.

**c** Calculate the future value of the annuity using the future value formula.

**32** Zac opens a menswear store and signs a lease for \$7500 per year for 4 years. The money is deposited into the landlord's account, which pays 4.2% p.a. interest. The landlord asks Zac to pay the rent as a lump sum in advance for the 4 years. How much should he pay?

**33** A home loan is taken out at an interest rate of 9.6% p.a. compounding monthly. The loan of \$130 000 is to be repaid over 20 years.

**a** Find the monthly repayment using the present value formula.

**b** Calculate the total amount to be repaid.

**c** What is the interest to be paid over the term of the loan?

**34** Use the future value tables on pages 308 and 309 in Topic 7B to find the value of these annuities.

**a** \$4000 is invested at the end of each year at 3% p.a. for 4 years.

**b** \$650 is invested at the end of each month for 12 months at 1% per month.

**c** \$900 is invested in an annuity paying 6% per month for 3 months.

**35** Use the future value tables on pages 308 and 309 in Topic 7B to find the annual interest rate for an annuity with annual payments of \$6500 p.a. for 42 years, amounting to \$681 327.40.

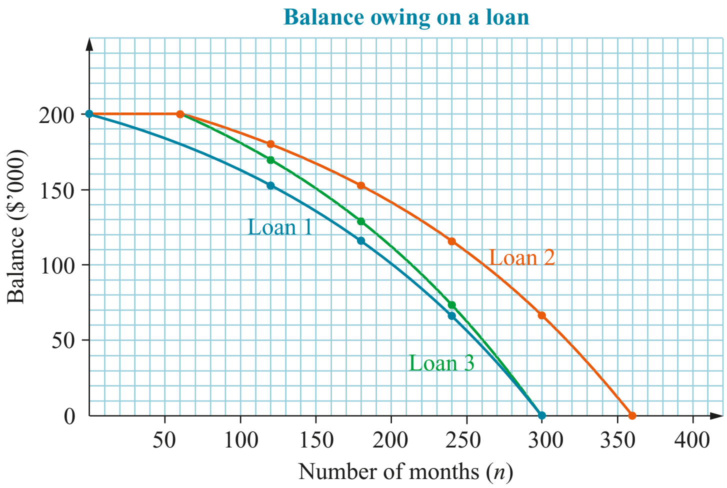
**36** Shae and Brent take out a loan of \$120 000 over 20 years at 7.25% p.a. interest compounding monthly. Their repayment is \$949 per month. After 5 years of repaying the loan, Brent wins \$70 000. They pay this amount off their loan.

**a** How long will it now take to repay the loan?

**b** Calculate the total amount that would have been paid without the lump sum.

**c** How much is saved by making the lump-sum payment?

**37** The graphs show the balance owing on three home loans.




**a** How much has been borrowed in each case?

**b** How long is needed to pay off each loan?

**c** What is the balance owing on loans 2 and 3 after 5 years?

**d** What types of loans are these?

**e** How much is still owing on loan 2 when loans 1 and 3 are paid off?

**f** Compare the monthly repayments on loans 2 and 3. Which loan would have the higher monthly repayment?

**g** Compare the total cost of each loan.

**i** Which loan would cost the least? **ii** Which loan would cost the most?

# 8

# Non-linear relationships

The main mathematical ideas investigated are:

- working with non-linear relationships and their graphical representation
- interpreting the meaning of the intercepts of graphs in a variety of contexts
- finding and interpreting the maximum and minimum values of quadratic relationships
- solving problems involving exponential relationships, including modelling exponential growth and decay
- recognising that reciprocal relationships represent inverse variation and graph as hyperbolas
- developing equations from descriptions of how one quantity relates to another
- using algebraic relationships to model physical phenomena
- recognising the limits of models.

ALGEBRA
MS-A4 Types of Relationships A4.2

# ARE YOU READY?



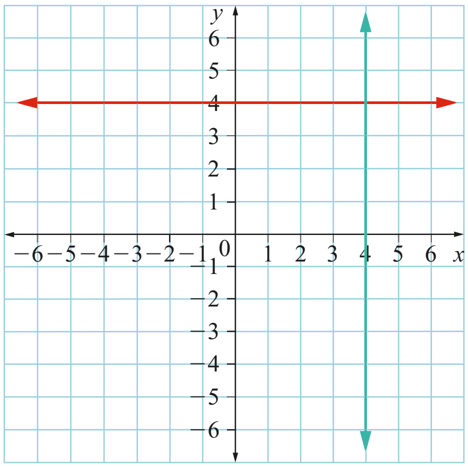
8A **1** When $x = -2$, what is the value of $3x^2$?

**A** $-12$ **B** $-6$ **C** $12$ **D** $36$

8A **2** If $y = -x^2 + 4x - 7$, what is the value of $y$ when $x = 3$?

**A** $-4$ **B** $-1$ **C** $10$ **D** $14$

Use the diagram below to answer questions **3** and **4**.




8A **3** What is the equation of the vertical line shown?

**A** $y = 0$ **B** $x = 0$

**C** $y = 4$ **D** $x = 4$

8A **4** What is the equation of the horizontal line shown?

**A** $y = 0$ **B** $x = 0$

**C** $y = 4$ **D** $x = 4$

8B **5** If $y = ax^2$, what is the value of $a$ when $x = 6$ and $y = 18$?

**A** $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** $2$ **D** $3$

8B **6** If $y = \frac{a}{x}$, what is the value of $a$ when $x = 5$ and $y = 20$?

**A** $\frac{1}{4}$ **B** $4$ **C** $25$ **D** $100$

8C **7** When $x = 4$, what is the value of $3^x$?

**A** $7$ **B** $12$ **C** $64$ **D** $81$

8C **8** If $y = \frac{1}{2}(4)^x$, what is the value of $y$ when $x = 3$?

**A** $6$ **B** $8$ **C** $32$ **D** $64$

8C **9** If $y = 1000(1.05)^x$, what is the approximate value of $y$ when $x = 12$?

**A** $1000$ **B** $1796$

**C** $12\,600$ **D** $129\,746$

8E **10** When $x = 30$, what is the value of $\frac{480}{x}$?

**A** $\frac{1}{16}$ **B** $16$ **C** $160$ **D** $450$

8E **11** Which table of values matches the relationship $y = \frac{72}{x}$?

**A**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 72 | 36 | 24 | 18 |

**B**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 72 | 32 | 22 | 12 |

**C**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 36 | 24 | 18 | 12 |

**D**

| $x$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$ | 72 | 144 | 216 | 288 |

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1–2** **Support sheet 8A.1 Substitution in quadratic relationships**
**Q3–4** **Support sheet 8A.2 Equations for horizontal and vertical lines**
**Q5–6** **Support sheet 8B.1 Solving equations after substitution**
**Q7–9** **Support sheet 8C.1 Substitution in exponential relationships**
**Q10–11** **Support sheet 8E.1 Substitution in reciprocal relationships**

# 8A The parabola

**quadratic relationship**
relationship between two variables where the highest power of the independent variable is 2; e.g. the general equation $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$, describes the quadratic relationship between the independent variable $x$ and the dependent variable $y$

**parabola**
graph of a quadratic relationship

**turning point**
a point on the parabola at which the curve changes direction

**axis of symmetry**
an imaginary line that divides a symmetrical graph so that one side is a reflection of the other. A parabola has a vertical axis of symmetry through its turning point; e.g. if the coordinates of the turning point are (3, 2) then the equation of the axis of symmetry is $x = 3$

These resources are available on your obook assess:
- **Interactive 8A:** Explore key ideas for plotting quadratic relationships
- **assess quiz 8A:** Test your skills with an auto-correcting multiple-choice quiz

From previous work, we have observed that the graph of a linear relationship is a straight line. There are many non-linear relationships that can also be graphed. The graph of a **quadratic relationship** is called a **parabola**.

This section will revise the generation of tables of values to graph quadratic relationships. You will also explore the use of technology to produce parabolas and observe the features of these curves.

!

The parabola is the curved shape resulting from a quadratic relationship.

The general equation of the parabola is $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$.

The simplest parabola has the equation $y = x^2$.

Every parabola is symmetrical about an axis of symmetry. The **turning point** of a parabola will always lie on the **axis of symmetry**.

## EXERCISE 8A The parabola

**1** **a** Substitute into the equation $y = x^2$ to complete this table of values.

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

**b** Plot the points and join with a smooth curve to draw the parabola for $y = x^2$.
**c** Use graphing technology to produce the parabola for $y = x^2$.
**d** What are the coordinates of the vertex or turning point of the parabola?
**e** What is the equation of the axis of symmetry?

**2** **a** Use graphing technology to draw the graph of $y = x^2$ for values of $x$ from −4 to 4.
**b** On the same set of axes used for part **a**, draw the parabola for each of the following equations. Use values of $x$ from −4 to 4.
**i** $y = 2x^2$ **ii** $y = 3x^2$ **iii** $y = \frac{1}{2}x^2$ **iv** $y = \frac{1}{4}x^2$
**c** Compare each graph from part **b** with the parabola for $y = x^2$. Which features are different? Which features are the same? Comment on the turning point, the axis of symmetry and the general shape of each parabola.
**d** What is the effect of changing the coefficient of the $x^2$ term in $y = x^2$?

**3** **a** Use graphing technology to draw the graph of $y = x^2$ for values of $x$ from −4 to 4.
**b** On the same set of axes used for part **a**, draw the parabola for each of the following equations. Use values of $x$ from −4 to 4.
**i** $y = x^2 + 1$ **ii** $y = x^2 + 3$ **iii** $y = x^2 - 2$ **iv** $y = x^2 - 4$

UNDERSTANDING, FLUENCY AND COMMUNICATING

**c** Compare each graph from part **b** with the parabola for $y = x^2$. Which features are different? Which features are the same? Comment on the turning point, the axis of symmetry and the general shape of each parabola.

**d** What is the effect of adding a constant term to $x^2$ in $y = x^2$?

**4** **a** Use graphing technology to draw the graph of $y = x^2$ for values of $x$ from −4 to 4.

**b** On the same set of axes used for part **a**, draw the parabola for each of the following equations. Use values of $x$ from −4 to 4.

**i** $y = (x + 1)^2$ **ii** $y = (x + 2)^2$ **iii** $y = (x - 2)^2$ **iv** $y = (x - 3)^2$

**c** Compare each graph from part **b** with the parabola for $y = x^2$. Comment on the turning point, the axis of symmetry and the general shape of each parabola. (*Note:* if it is difficult to see the shape for any of the graphs in part **b**, change the interval of $x$-values you use.)

**d** What is the effect of adding a constant term to $x$ before squaring in $y = x^2$?

**5** **a** Use graphing technology to draw the graph of $y = x^2$ and $y = -x^2$ on the same set of axes for values of $x$ from −4 to 4.

**b** Compare the two parabolas. Comment on the turning point, the axis of symmetry and the general shape of each parabola.

**6** **a** Consider the parabola for each of the following equations and describe how it would look compared to the parabola for $y = x^2$.

**i** $y = -2x^2$ **ii** $y = -3x^2$

**iii** $y = -\frac{1}{2}x^2$ **iv** $y = -\frac{1}{4}x^2$

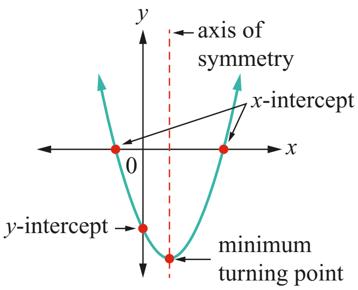
**b** Check your answer to part **a** using graphing technology.

**!**

When the coefficient of the $x^2$ term in $y = ax^2 + bx + c$ is positive ($a > 0$), the parabola is upright (∪-shaped) with a turning point at its **minimum value**. (The turning point is described as a minimum turning point.) The minimum value is the $y$-value at the lowest point of the parabola.

The **$y$-intercept** is the point where the parabola crosses the $y$-axis.

An **$x$-intercept** is a point where the parabola crosses the $x$-axis.




When the coefficient of the $x^2$ term in $y = ax^2 + bx + c$ is negative ($a < 0$), the parabola is inverted (∩-shaped) with a turning point at its **maximum value**. (The turning point is described as a maximum turning point.) The maximum value is the $y$-value at the highest point of the parabola.

The simplest parabola with equation $y = x^2$ is upright with a turning point at (0, 0), and a minimum value of 0.

**minimum value** the smallest value of $y$ in the relationship between $x$ and $y$ for a given interval of $x$-values

**$y$-intercept** point at which a line crosses the $y$-axis of a number plane; e.g. if a line crosses the $y$-axis at 2 then the $y$-intercept is 2 and the coordinates of the $y$-intercept are (0, 2)

**$x$-intercept** point at which a line crosses the $x$-axis of a number plane; e.g. if a line crosses the $x$-axis at −3 then the $x$-intercept is −3 and the coordinates of the $x$-intercept are (−3, 0)

**maximum value** the largest value of $y$ in the relationship between $x$ and $y$ for a given interval of $x$-values

ALGEBRA

## EXAMPLE 8A–1 Identifying features of a parabola

The graph shows the parabola $y = x^2 - 4x - 5$.

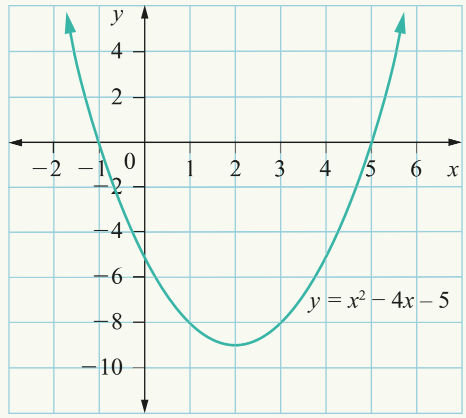
**a** Find the coordinates of the turning point.

**b** Write the equation of the axis of symmetry.

**c** State the coordinates of the $y$-intercept.

**d** State the coordinates of the $x$-intercepts.

**e** What is the minimum value of the quadratic relationship?




| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Turning point is at (2, −9). | Identify the turning point and write its coordinates. | Find the point on the parabola where the curve changes direction. |
| **b** | Equation of axis of symmetry is $x = 2$. | Imagine a vertical line through the turning point. Write the equation in the form $x = \ldots$ | A parabola has a vertical axis of symmetry through its turning point. |
| **c** | $y$-intercept is −5.<br>The coordinates are (0, −5). | Identify the point where the parabola crosses the $y$-axis. | Write the coordinates of the point where the parabola crosses the $y$-axis. |
| **d** | $x$-intercepts are −1 and 5.<br>The coordinates are (−1, 0) and (5, 0). | Identify the points where the parabola crosses the $x$-axis. In this case, there are two $x$-intercepts. | Write the coordinates of the points where the parabola crosses the $x$-axis. |
| **e** | Minimum value is −9. | The minimum value is the smallest value of $y$ on the parabola. | The minimum value occurs at the lowest point on the curve. |

**7** For each of the following graphs, find:

**i** the coordinates of the turning point

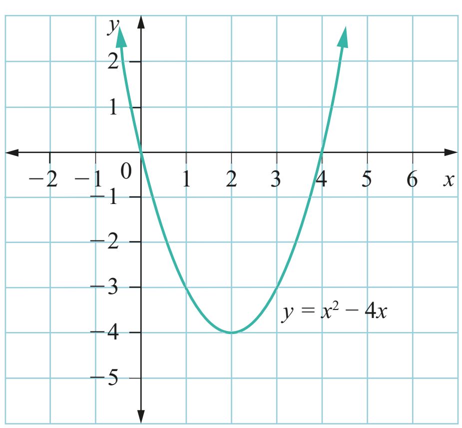
**ii** the equation of the axis of symmetry

**iii** the coordinates of the $y$-intercept

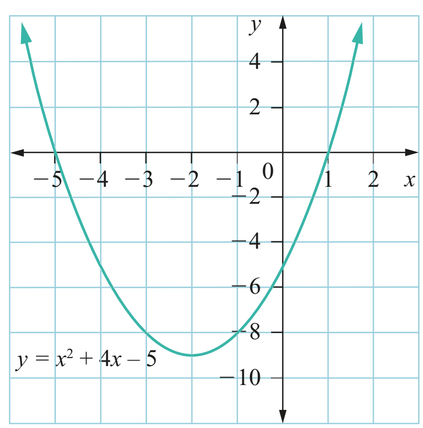
**iv** the coordinates of the $x$-intercepts

**v** the minimum value of the quadratic relationship.

**a**




**b**



8 For each of the following graphs, find:

i the coordinates of the turning point

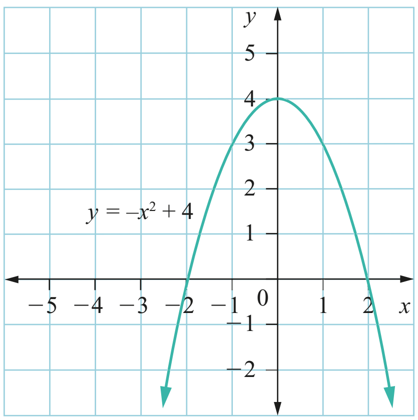
ii the equation of the axis of symmetry

iii the coordinates of the $y$-intercept

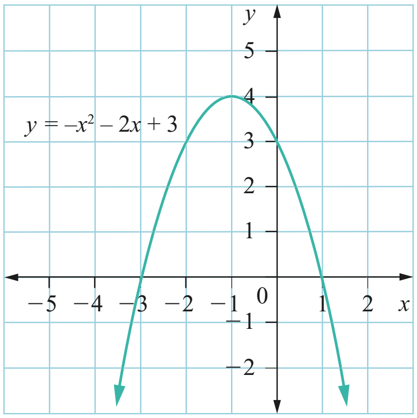
iv the coordinates of the $x$-intercepts

v the maximum value of the quadratic relationship.

**a**




**b**




9 For the graph shown on the right, find:

**a** the coordinates of the turning point

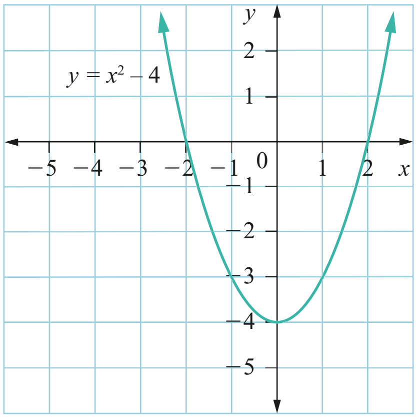
**b** the equation of the axis of symmetry

**c** the coordinates of the $y$-intercept

**d** the coordinates of the $x$-intercepts

**e** the value of $y$ when $x = 1$

**f** the minimum value of the quadratic relationship.




10 For the graph shown on the right, find:

**a** the coordinates of the turning point

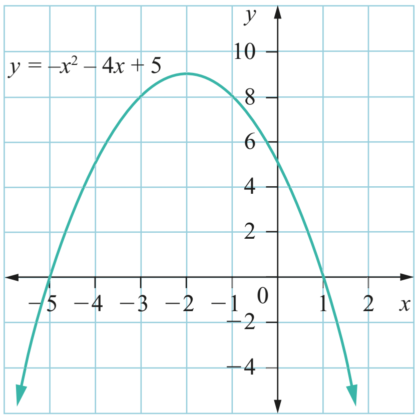
**b** the equation of the axis of symmetry

**c** the coordinates of the $y$-intercept

**d** the coordinates of the $x$-intercepts

**e** the value of $y$ when $x = -1$

**f** the maximum value of the quadratic relationship.




11 For the graph in question **9**:

**a** estimate the value of $y$ when $x = 1.5$

**b** check your result to part **a** by substituting into the quadratic equation.

12 For the graph in question **10**:

**a** estimate the value of $y$ when $x = -3.5$

**b** check your result to part **a** by substituting into the quadratic equation.

## EXAMPLE 8A–2 Drawing a parabola from its equation

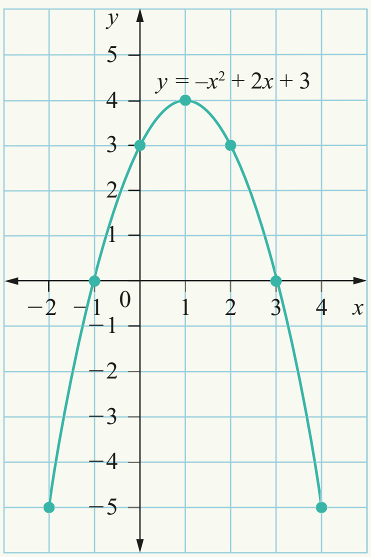
**a** Complete this table of values for $y = -x^2 + 2x + 3$.

**b** Draw the parabola with equation $y = -x^2 + 2x + 3$ for values of $x$ from −2 to 4.

**c** What is the maximum value of the quadratic relationship?

**d** What is the value of $x$ that gives this maximum value?

| $x$ | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | The table of values for $y = -x^2 + 2x + 3$ is:<br><br>$x$: −2, −1, 0, 1, 2, 3, 4<br>$y$: −5, 0, 3, 4, 3, 0, −5 | Substitute the $x$-values.<br>For $x = -2$: $y = -(-2)^2 + 2(-2) + 3 = -5$<br>For $x = -1$: $y = -(-1)^2 + 2(-1) + 3 = 0$<br>For $x = 0$: $y = -(0)^2 + 2(0) + 3 = 3$<br>For $x = 1$: $y = -(1)^2 + 2(1) + 3 = 4$<br>For $x = 2$: $y = -(2)^2 + 2(2) + 3 = 3$<br>For $x = 3$: $y = -(3)^2 + 2(3) + 3 = 0$<br>For $x = 4$: $y = -(4)^2 + 2(4) + 3 = -5$ | Substitute each value of $x$ into the equation to find the corresponding value of $y$. |
| **b** |  | Plot the points from the table on a number plane and join with a smooth curve. Along the axes of the number plane, use $x$-values from −2 to 4 and $y$-values from at least −5 to 4. | Draw the graph of the parabola using the table of values. Graphing technology can also be used. |
| **c** | Maximum value is 4. | The maximum value is the largest value of $y$ on the parabola. It is the $y$-coordinate of the turning point. | The maximum value occurs at the highest point on the curve. |
| **d** | Maximum value occurs when $x$ is 1. | Find the $x$-value that corresponds to the $y$-value of 4 on the parabola. It is the $x$-coordinate of the turning point. | |



**13 a** Complete this table of values for $y = -x^2 + 6x - 5$.

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

**b** Draw the parabola with equation $y = -x^2 + 6x - 5$ for values of $x$ from −1 to 7.

**c** What is the maximum value of the quadratic relationship?

**d** What is the value of $x$ that gives this maximum value?

**14** **a** Complete this table of values for $y = 2x^2 - 4x - 6$.

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Draw the parabola with equation $y = 2x^2 - 4x - 6$ for values of $x$ from $-2$ to 4.

**c** What is the minimum value of the quadratic relationship?

**d** What is the value of $x$ that gives this minimum value?

**15** Use graphing technology to draw the parabolas for the quadratic relationships in questions **13** and **14**. Check your answers for parts **c** and **d** of each question.

**16** Use graphing technology or a table of values to draw the graphs of the following quadratic relationships for the given $x$-values.

**a** $y = x^2 - 4$ ($x$-values from $-3$ to 3)

**b** $y = x^2 - 2x - 15$ ($x$-values from $-4$ to 6)

**c** $y = -3x^2 - 6x$ ($x$-values from $-3$ to 1)

**d** $y = 2x^2 - 4x - 6$ ($x$-values from $-3$ to 5)

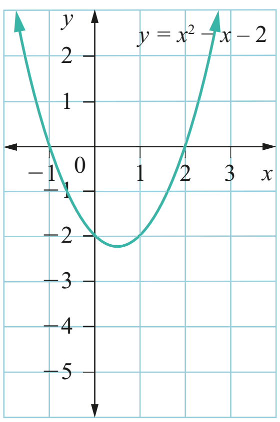
**e** $y = -\frac{1}{2}x^2 + 4x - 6$ ($x$-values from 0 to 8)

**f** $y = 2x^2 - 3x - 5$ ($x$-values from 0 to 5)

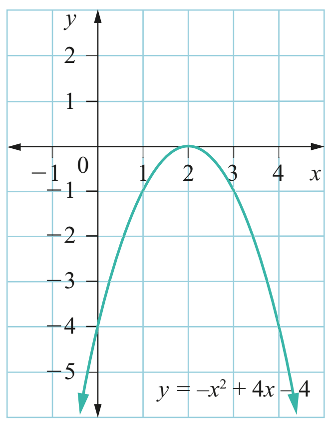
CHALLENGE

**17** Consider the graphs of these three quadratic relationships.

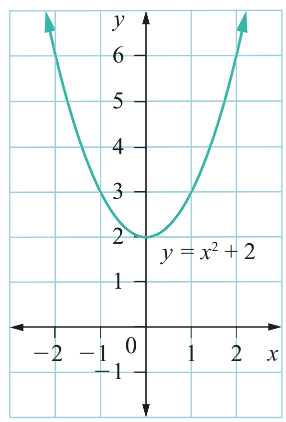
**i**




**ii**




**iii**




**a** How many $x$-intercepts does each parabola have?

**b** Can a parabola have more than two $x$-intercepts? Explain.

**c** How many $y$-intercepts does each parabola have?

**d** Can you draw a parabola with a different number of $y$-intercepts? Explain.

**e** How many turning points does each parabola have?

**f** Can you draw a parabola with a different number of turning points? Explain.

**g** Write the coordinates of the turning point for each parabola and state whether it is a maximum or minimum turning point.

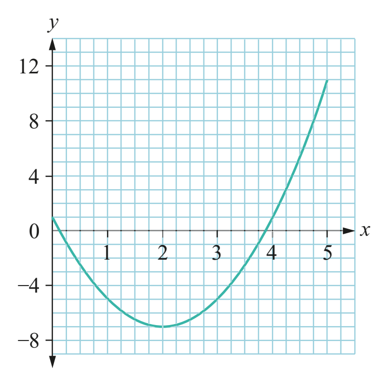
**18** Which one of the equations listed below matches the graph shown on the right?

**A** $y = 2x^2 - 4x - 7$

**B** $y = 2x^2 - 4x + 1$

**C** $y = 2x^2 - 8x + 1$

**D** $y = 2x^2 - 8x + 4$



ALGEBRA

# 8B Quadratic models

These resources are available on your obook assess:

- **Video tutorial 8B:** Watch and listen to an explanation of Example 8B–1
- **Spreadsheet 8B:** Model the distance an object falls over time
- **Investigation 8B:** Analyse the path of the ball during a soccer match
- **assess quiz 8B:** Test your skills with an auto-correcting multiple-choice quiz

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GI

Quadratic relationships can be used to model practical situations. For example, parabolic curves are used in headlight and torch reflectors, solar furnaces, satellite dishes and other applications.

In this section, we will look at simple scenarios in which the relationship between two variables can be modelled by a quadratic equation. For example, we can write a quadratic equation to represent the relationship between the height of a falling object and the time it takes the object to fall.

The quadratic equation used as a model will be of the form $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$. As we are dealing with practical situations, there may be restrictions on the values used for the two variables. For example, for a quadratic model involving time $t$, we know that $t \geq 0$.

!

Graphs of quadratic models are useful for finding the maximum (or minimum) value of a relationship. The maximum (or minimum) turning point of the parabola provides information on the maximum (or minimum) value and for what value of the independent variable this occurs.

## EXAMPLE 8B–1 Interpreting the graph of a quadratic model

A girl throws a ball in the air. The height of the ball can be modelled using the quadratic equation $h = -5t^2 + 25t + 1$, where $h$ is the height of the ball in metres and $t$ is the time in seconds. The graph of this model is shown at right.

**a** Find the maximum height of the ball.

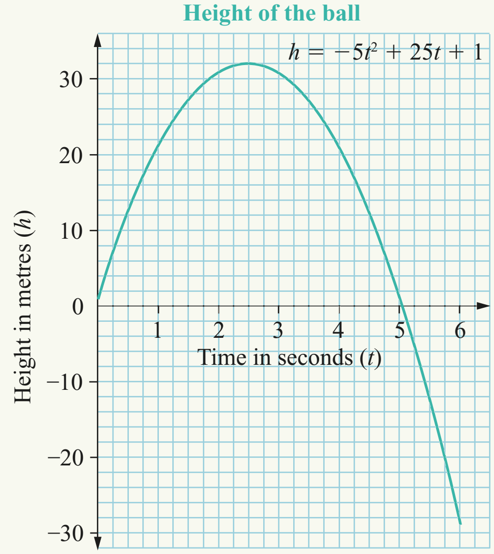
**b** When is the ball at its maximum height?

**c** At which two times is the ball 31 m above the ground?

**d** At what time is the height zero?

**e** What is the $h$-value when $t = 6$? What limitations are there on this model?

Height of the ball



<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>The maximum height of the ball is 32 m.</td><td>Locate the highest point on the graph; this is the maximum value.</td><td rowspan="5">Use the graph to answer the questions. It is important to note that the model may only be appropriate for particular values. The answers must make sense in the context of the question. Negative heights, for example, do not make sense in the context of this question. Also time cannot have negative values.</td></tr>
<tr><td>b</td><td>The ball is at its maximum height after 2.5 s.</td><td>Find the value of $t$ from the graph when $h$ is 32 m.</td></tr>
<tr><td>c</td><td>The ball is 31 m above the ground after 2 s and 3 s.</td><td>Draw a line across from 31 m on the height axis cutting the parabola twice. Draw lines down from the curve to the time axis and read the two values.</td></tr>
<tr><td>d</td><td>The height is zero at about 5 s.</td><td>The height is zero where the parabola cuts the horizontal axis.</td></tr>
<tr><td>e</td><td>When $t = 6$, $h = -29$ m.<br>The model has negative values for the height, but in reality the lowest point is at $h = 0$. In this practical context, the model only works for $t$-values from 0 to just over 5.</td><td>Find the value of $h$ when $t = 6$. From the graph, $h = -29$ m means 29 m below the ground level. This is not possible in this context and shows that models have limitations.</td></tr>
</table>

## EXERCISE 8B Quadratic models

**1** A ball is thrown into the air. The height of the ball, $h$, in metres, can be modelled using the quadratic equation $h = -5t^2 + 20t + 1$, where $t$ is time in seconds. The graph of this model is shown below.

**a** Find the maximum height of the ball.

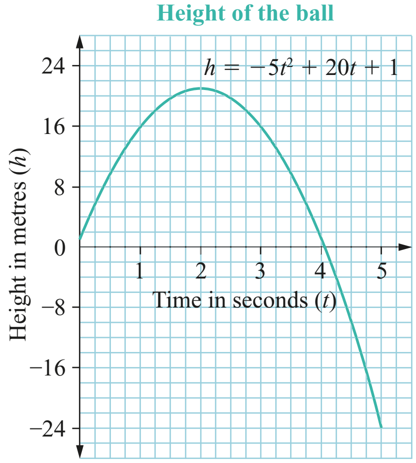
**b** When is the ball at its maximum height?

**c** At which two times is the ball 16 m above the ground.

**d** At what time is the height zero?

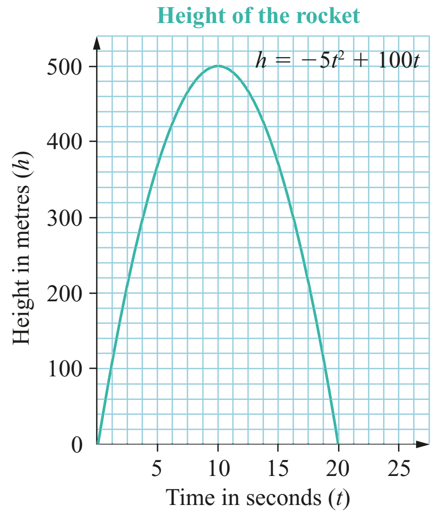
**e** What is the value of $h$ when $t = 5$? What limitations are there on this model?

**Height of the ball**



**2** A model rocket is launched into the air. The height of the rocket can be modelled using the quadratic equation $h = -5t^2 + 100t$, where $h$ is the height in metres and $t$ is the time in seconds. The graph of this model is shown.

**a** Find the maximum height of the rocket. When does this occur?
**b** At which two times is the ball 375 m above the ground?
**c** At what time is the height zero?
**d** What limitations are there on this model?

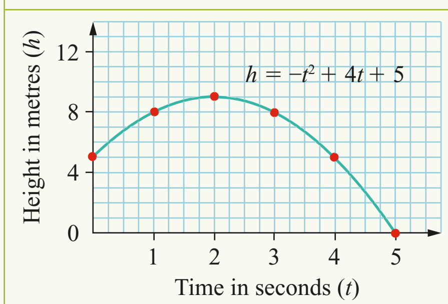



## EXAMPLE 8B–2 Drawing a quadratic model

An object is thrown into the air. The height of the object can be modelled using the equation $h = -t^2 + 4t + 5$, where $h$ is the height in metres and $t$ is the time in seconds.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $h$ | | | | | | |

**a** Complete the table of values for $h = -t^2 + 4t + 5$.
**b** Use graphing technology or the table of values to draw the parabola with equation $h = -t^2 + 4t + 5$ for values of $t$ from 0 to 5.
**c** From the graph, find the maximum height of the object.
**d** When does the maximum height occur?
**e** What was the initial height of the object?
**f** When does the object hit the ground?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $h = -t^2 + 4t + 5$<br><br>$t$: 0, 1, 2, 3, 4, 5<br>$h$: 5, 8, 9, 8, 5, 0 | For $t = 0$:<br>$y = -1(0) + 4(0) + 5 = 5$<br>For $t = 1$:<br>$h = -1(1)^2 + 4(1) + 5 = 8$<br>For $t = 2$:<br>$h = -1(2)^2 + 4(2) + 5 = 9$<br>For $t = 3$:<br>$h = -1(3)^2 + 4(3) + 5 = 8$<br>For $t = 4$:<br>$h = -1(4)^2 + 4(4) + 5 = 5$<br>For $t = 5$:<br>$h = -1(5)^2 + 4(5) + 5 = 0$ | Substitute each value of $t$ into the equation to find the corresponding value of $h$. |
| **b** |   | Plot these points on a number plane and join with a smooth curve. The full parabola is not drawn in this case. The curve is only shown for $t$-values from 0 to 5. | Draw the parabola for the given interval of $t$-values. |

| | | | |
|---|---|---|---|
| **c** | The maximum height is 9 m. | Find the maximum value of the parabola within the given interval of $t$-values. | The maximum value is at the highest point. This is the $h$-coordinate of the turning point. |
| **d** | The maximum height occurs after 2 s. | Find the corresponding $t$-value for the maximum height; that is, the maximum value occurs when $t = 2$. | Identify the $t$-coordinate of the turning point. |
| **e** | The initial height of the object was 5 m. | Find the height when $t = 0$. | The initial height occurs when time is zero. |
| **f** | The object hits the ground after 5 s. | Find the time when $h = 0$. | The object hits the ground when the height is zero. |

**3** An object is thrown into the air. The height of the object can be modelled using the equation $h = -t^2 + 2t + 3$, where $h$ is the height in metres and $t$ is the time in seconds.

**a** Complete this table of values for $h = -t^2 + 2t + 3$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $h$ | | | | | | |

**b** Use graphing technology or the table of values to draw the parabola with equation $h = -t^2 + 2t + 3$ for values of $t$ from 0 to 5.

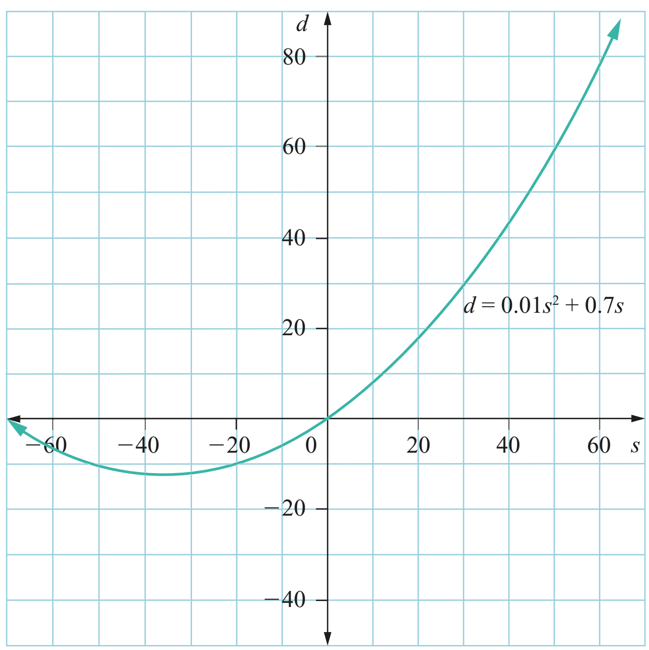
**c** From the graph, find the maximum height of the object.

**d** When does the maximum height occur?

**e** What was the initial height of the object?

**f** When does the object hit the ground?

**4** Nadia is investigating stopping distances for a car travelling at different speeds. To model this she uses the equation $d = 0.01s^2 + 0.7s$, where $d$ is the stopping distance in metres and $s$ is the speed of the car in km/h. The graph of this equation is drawn on the right.




**a** Nadia knows that only part of this curve applies to her model for stopping distances. Draw the part of this curve that applies for stopping distances.

**b** What is the difference between the stopping distances in a school zone when travelling at a speed of 40 km/h and when travelling at a speed of 60 km/h?

**5** From question **4**, we can see that the equation $d = 0.01s^2 + 0.7s$ can be used to model the distance taken for a car to stop (in metres) without skidding when travelling at a given speed (in km/h). This quadratic model applies to a car travelling on a dry road.

**a** Complete this table of values using $d = 0.01s^2 + 0.7s$.

| $s$ | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| $d$ | | | | | | |

**b** Plot the values from the table and extrapolate (or extend) the graph to include values of $s$ up to 100 km/h.

**c** Use graphing technology to draw the graph of this quadratic model for speeds up to 100 km/h.

**d** Many residential areas have a 50 km/h speed limit. How much shorter is the stopping distance for a car travelling at 50 km/h than for a car travelling at 60 km/h? Does the 10 km/h reduction make a significant difference? Explain your answer.

**e** In the wet, the model becomes $d = 0.015s^2 + 0.7s$. Repeat parts **a**, **b**, **c** and **d** using this equation.

**f** Compare the 100 km/h stopping distances for dry and wet roads.

**6** An amateur golfer hits a ball up into the air. The path of the ball follows the relationship $h = -d^2 + 20d$, where $h$ is the height of the ball for a horizontal distance $d$ from where the ball was hit. Both $h$ and $d$ are in metres.

**a** Use graphing technology or a table of values to draw the graph of this relationship from when the ball was hit to when it landed. (*Hint:* use the fact that a parabola is symmetrical.)

**b** What was the maximum height of the golf ball?

**c** How far from the golfer did the ball land?

**7** The height above the ground for a bungee jumper is measured from the start of the first downwards movement to just before the start of the second downwards movement. These measurements are modelled by the equation $h = 5t^2 - 40t + 90$ where $h$ is the height in metres after time $t$ seconds.

**a** Use graphing technology or a table of values to draw the graph of this relationship.

**b** Use the graph to find the height of the person above the ground at the start of the jump.

**c** Use the graph to find the lowest height above the ground that the person falls to in the first downwards movement of the jump.

**8** A ball is kicked off the ground. Its path can be modelled using the equation $h = -0.05d^2 + 1.6d$, where $h$ is the height in metres and $d$ is the horizontal distance in metres.

**a** Draw the graph of this relationship and use it to answer the questions that follow.

**b** What was the maximum height of the ball?

**c** What horizontal distance had the ball travelled when it reached its maximum height?

**d** What horizontal distance did the ball travel before hitting the ground?

**e** If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

**9** The amount of money in Zac's bank account over a 3-week interval can be modelled using the equation $a = t^2 - 20t + 84$, where $a$ is the account balance in dollars after $t$ days.

**a** Draw the graph of this relationship and use it to answer the questions that follow.

**b** How much money was in Zac's account at the start of the 3 weeks?

**c** How much money was in his account after 2 days?

**d** When was Zac's account first overdrawn?

**e** What was the highest amount that he owed the bank during the 3 weeks?

**f** When was his account balance back to zero?

**g** What was the highest amount in Zac's account over the 3 weeks?

## EXAMPLE 8B–3 Writing the equation for a quadratic model

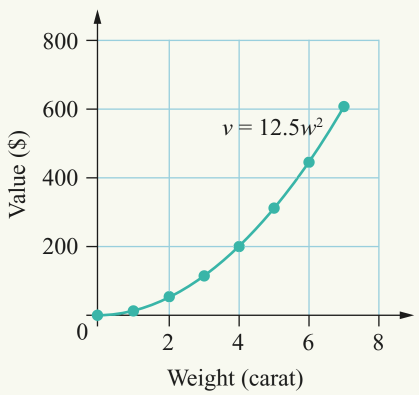
The value of a gemstone is directly related to the square of its weight and can be modelled using a quadratic equation of the form $v = aw^2$, where $v$ is the value in dollars, $w$ is the weight in carats and $a$ is a constant.

**a** If a 4-carat stone is valued at \$200, write the equation to model the relationship between value and weight of a gemstone.

**b** Use graphing technology or a table of values to draw the graph of this quadratic model for values of $w$ up to 7 carats.

**c** Use the graph to find the value of a 5.5-carat gemstone.

**d** Use the graph to find the weight of a gemstone that is worth \$500.

| | Solve | Think/Apply |
|---|---|---|
| **a** | $v = aw^2$<br>When $w = 4$ and $v = 200$,<br>$200 = a \times 4^2$<br>$a = \dfrac{200}{16}$<br>$= 12.5$<br>The equation is $v = 12.5w^2$. | Use the known information to find the value of the constant in the general equation. Substitute the known values for $w$ and $v$ into the equation and solve for $a$. |
| **b** |  | Use graphing technology or plot the points using a table of values to produce the parabola. |
| **c** | From the graph:<br>when $w = 5.5$, $v \approx 380$.<br>The value of a 5.5-carat gemstone is about \$380. | Read the value for $v$ from the graph.<br>Move up from 5.5 on the $w$-axis and then across to the $v$-axis. (Using the equation, $v = \$378.13$.) |
| **d** | From the graph:<br>when $v = 500$, $w \approx 6.3$.<br>The weight of a gemstone valued at \$500 is about 6.3 carats. | Read the value for $w$ from the graph.<br>Move across from 500 on the $v$-axis and then down to the $w$-axis. (Using the equation, $w \approx 6.325$.) |

| $w$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $v$ | 0 | 12.5 | 50 | 112.5 | 200 | 312.5 | 450 | 612.5 |

**10** When a stone falls freely, the distance it falls is directly related to the square of the time taken and can be modelled by a quadratic equation of the form $d = at^2$, where $d$ is the distance in metres, $t$ is the time in seconds and $a$ is a constant.

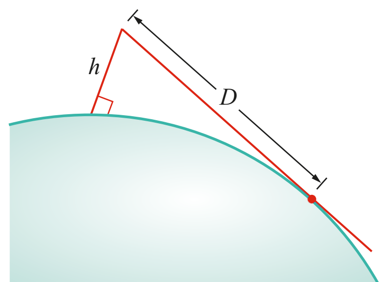
- **a** If it takes 4 s for a stone to fall 78.4 m, write the equation to model the relationship between distance and time.
- **b** Use graphing technology or a table of values to draw the graph of this quadratic model for the first 12 s.
- **c** Use the graph to find the distance a stone falls in 7 s.
- **d** Use the graph to find how long it would take for a stone to fall 500 m.

**11** The volume of blood flowing in a blood vessel, $V$, in millilitres, is directly related to the square of the internal diameter of the blood vessel, $d$, in centimetres. A volume of 100 mL flows through a 0.34 cm blood vessel.

- **a** If the equation to model this relationship is of the form $V = ad^2$, where $a$ is a constant, use the known information to write the equation connecting volume of blood and internal diameter of blood vessel.
- **b** Draw the graph of this quadratic model for values of $d$ up to 0.40 cm.
- **c** Use the graph to find the volume flowing through a blood vessel with internal diameter 0.26 cm.
- **d** Use the graph to find the internal diameter of a blood vessel that has a volume of 80 mL of blood flowing through it.
- **e** Confirm your answers for parts **c** and **d** by substituting into the quadratic equation.

**12** A quadratic model can be used to relate the distance of the visible horizon, $D$, in kilometres, at sea to the height of an observer's eye above sea level, $h$, in metres. It is known that the horizon is 9 km away when the observer's eye is 5.4 m above sea level.

- **a** If the equation to model this situation is of the form $h = \frac{1}{p}D^2$, where $p$ is a constant, use the known information to find the value of $p$.
- **b** Write the equation and then draw the graph of this quadratic model for values of $D$ up to 15 km.
- **c** Use the graph to find the height of a person's eye above sea level for the distance to the horizon to be 7 km.
- **d** Use the graph to find how far a person can see at a height of 10 m.
- **e** Confirm your answers for parts **c** and **d** by substituting into the quadratic equation.



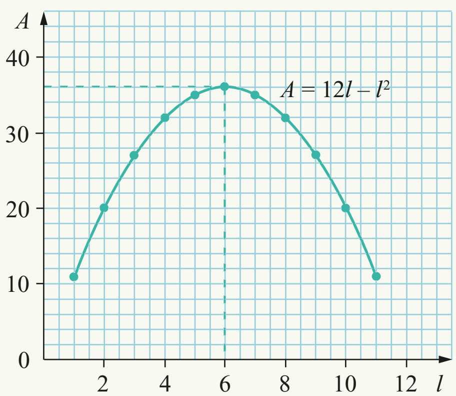

## EXAMPLE 8B–4 Finding the maximum area of a rectangle with given perimeter

A rectangle has a perimeter of 24 cm.

- **a** Write an expression for the perimeter involving length $l$ (in cm) and breadth $b$ (in cm) and hence write an equation for $b$ in terms of $l$.
- **b** Substitute the expression for $b$ in the formula for area $A$ (in cm$^2$) to obtain an equation in terms of $A$ and $l$ only. (*Hint:* use $A = l \times b$.)
- **c** Complete this table comparing length $l$ (in cm) and area $A$ (in cm$^2$) using the equation you found in part **b**.

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | | | | | | | | | | | |

- **d** Draw the graph of $l$ versus $A$ for this rectangle.
- **e** What is the maximum area? When does it occur?
- **f** Can $l$ be 13 cm or 0 cm?
- **g** Find the value of $A$ when $l = 3.5$ cm.

<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>Perimeter $= 2l + 2b$<br>$2l + 2b = 24$<br>$l + b = 12$<br>$b = 12 - l$</td><td>Write an expression for perimeter. Use $2l + 2b$ or $2(l + b)$. Equate this to 24 and rearrange to make $b$ the subject of the equation.</td><td>Substitute 24 in the perimeter formula and rearrange to make $b$ the subject of the equation.</td></tr>
<tr><td>b</td><td>$A = l \times b$<br>As $b = 12 - l$,<br>$A = l \times (12 - l)$<br>$= 12l - l^2$</td><td>Write the formula for area and replace $b$ with $12 - l$. Simplify the quadratic equation.</td><td>Form an equation for area in terms of $A$ and $l$ only by replacing $b$ in the formula $A = l \times b$ with an equivalent expression involving $l$.</td></tr>
<tr><td>c</td><td><table><tr><td>$l$</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td></tr><tr><td>$A$</td><td>11</td><td>20</td><td>27</td><td>32</td><td>35</td><td>36</td><td>35</td><td>32</td><td>27</td><td>20</td><td>11</td></tr></table></td><td>Substitute each value for $l$ in the equation $A = 12l - l^2$ to calculate the matching value for $A$.</td><td>Complete the table of values by substituting each value of $l$ in the equation to find the corresponding value of $A$.</td></tr>
<tr><td>d</td><td>
</td><td>Plot each of the points for $A$ and $l$ showing $l$ along the horizontal axis and $A$ along the vertical axis of the number plane.</td><td>Plot $l$ versus $A$ on the number plane. Graphing technology can also be used.</td></tr>
<tr><td>e</td><td>Maximum area $= 36$ cm$^2$ when $l = 6$ cm.<br>When $l = 6$ cm, $b = 12 - 6 = 6$ cm.<br>So the rectangle with maximum area is a square.</td><td>The maximum value is at the highest point on the parabola.</td><td>The maximum value for the area can be seen at the maximum turning point of the parabola.</td></tr>
<tr><td>f</td><td>If $l = 13$ cm then $b = 12 - 13 = -1$ cm.<br>It is not possible to have a side length of $-1$ cm, so $l \neq 13$ cm.<br>If $l = 0$ cm, then the side length would be zero and there would be no rectangle at all.</td><td>Side length $l$ must be greater than zero.</td><td>The values for $l$ must lie between 0 and 12 cm in this context.</td></tr>
<tr><td>g</td><td>From the graph, when $l = 3.5$ cm, $A \approx 30$ cm$^2$.<br>Using the equation, when $l = 3.5$ cm,<br>$A = 12 \times 3.5 - 3.5^2$<br>$= 29.75$ cm$^2$</td><td>Using the graph, go up from 3.5 on the $l$-axis to the curve and across to the $A$-axis to read an estimate of 30. The equation for area can be used to obtain a more accurate answer.</td><td>The values for $l$ and $A$ can be read from the graph or calculated by substituting into the equation $A = 12l - l^2$.</td></tr>
</table>

**13** A rectangular garden bed has a perimeter of 20 m.

**a** Write an expression for the perimeter involving length $l$ (in m) and breadth $b$ (in m) and hence write an equation for $b$ in terms of $l$.

**b** Substitute the expression for $b$ in the formula for area $A$ (in $m^2$) to obtain an equation in terms of $A$ and $l$ only. (*Hint:* use $A = l \times b$.)

**c** Complete this table comparing length $l$ (in m) and area $A$ (in $m^2$) using the equation you found in part **b**.

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | | | | | | | | | |

**d** Draw the graph of $l$ versus $A$ for this rectangle.

**e** Find the maximum area. When does it occur?

**f** Find the value of $A$ when $l = 3.5$ m.

**14** Repeat question **13** using rectangles with the following perimeters.

**a** 18 cm  **b** 28 cm  **c** 22 m  **d** 26 m  **e** 30 mm

SPREADSHEET APPLICATION

**15** **a** On Earth, the equation $d = 4.9t^2$ can be used to model the distance, $d$, in metres that an object falls in time, $t$, in seconds, if air resistance is ignored.

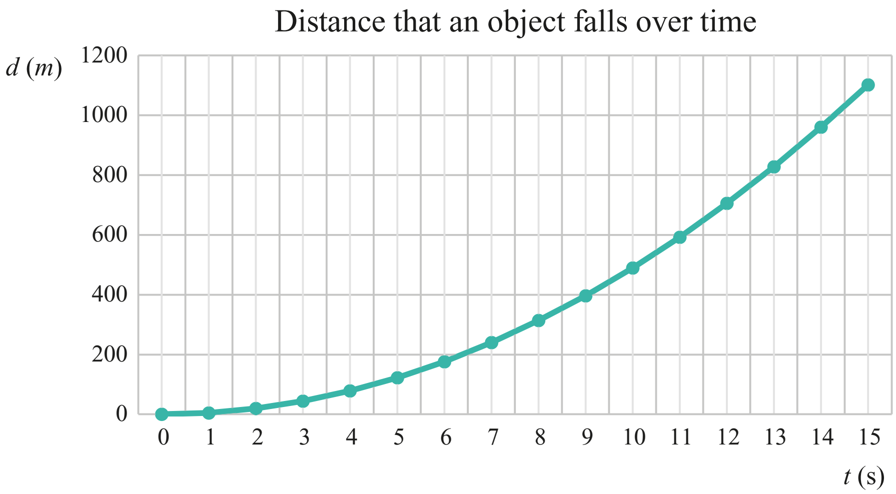
**i** Create a table of values for the equation $d = 4.9t^2$ using a spreadsheet with $t$ in the first column and $d$ in the second column. Use the **Fill Down** function to generate values for $d$ from your equation for $t$-values up to 15 seconds.

**ii** Highlight the cells containing the $t$ and $d$ values, click **Insert** and **Charts**, and choose **Line graph**. Adjust the style of the chart to make it easier to read values. (If needed, Spreadsheet 8B is provided on your obook assess to help you with this task.)

**iii** Use the table or graph to find the distance an object has fallen after 12 s.

**iv** Use the table or graph to find the time it takes for an object to fall 300 m.

**v** The coefficient of $t^2$ in the equation $d = 4.9t^2$ is half the value of the acceleration due to gravity. What is the acceleration due to gravity on Earth (in $m/s^2$)?



**b** On the Moon, the equation $d = 0.8t^2$ can be used to model the distance, $d$, in metres, that an object falls in time, $t$, in seconds.

**i** Repeat the steps in part **a** to create a table of values for the equation $d = 0.8t^2$ and produce the graph using a spreadsheet.

**ii** Use the table or graph to find the distance an object has fallen on the Moon after 12 s. Compare this result to the distance obtained on Earth in part **a**.

**iii** Use the table or graph to find the time it takes for an object to fall 300 m on the Moon. Compare this result to the time found on Earth in part **a**.

**iv** If the coefficient of $t^2$ in the equation $d = 0.8t^2$ is half the value of the acceleration due to gravity, what is the acceleration due to gravity on the Moon (in m/s$^2$)?

**c** An experiment is conducted on another planet to find a model that represents the relationship between the distance that an object falls in metres and the time it takes in seconds. It is assumed that the equation will be of the form $d = at^2$ where $a$ is a constant. The table of values obtained is given below.

| $t$ (s) | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $d$ (m) | 0 | 3 | 11 | 23 | 42 | 66 |

**i** Use the experimental data in the table of values to calculate an average value for $a$. Use this value of $a$ to write the equation that models the relationship between distance and time.

**ii** Enter the values from the table into a spreadsheet. Extend the table using the equation to estimate the distance an object has fallen on this planet after 12 s. Compare your result to the distance obtained on Earth and the Moon in parts **a** and **b**.

**iii** Use the table to find the time it takes for an object to fall 300 m on this planet. Compare your result to the time found on Earth and the Moon in parts **a** and **b**.

**iv** Give an estimate for the value of the acceleration due to gravity on this planet.

**v** Given that the greater the mass of the planet the larger the acceleration due to gravity, is the planet larger or smaller than Earth? Explain your answer.

**d** Investigate the quadratic models that can be used for other planets in our solar system.

**i** Use the information in the table to write the equation for each planet that models the relationship between the distance that an object falls and the time it takes.

| Planet | Acceleration due to gravity (m/s$^2$) |
|---|---|
| Venus | 8.87 |
| Mars | 3.7 |
| Jupiter | 24.5 |

**ii** Compare the time it takes for an object to fall 300 m on each planet.

**iii** Discuss how you can rank the planets in order of mass using the values given for the acceleration due to gravity.

# 8C The exponential graph

These resources are available on your obook assess:

- **assess quiz 8C:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**exponential relationship** relationship between two variables in which the independent variable is the exponent or power or index of a term in index form; e.g. the equation $y = 2^x$ describes the relationship between the independent variable $x$ and the dependent variable $y$

**exponential graph** graph of an exponential relationship

**asymptote** a straight line or boundary that a curve approaches (gets closer to) but does not cross

Another example of a non-linear relationship is an **exponential relationship** in which the independent variable is the exponent or power or index of a term in index form. The graph of an exponential relationship is called an **exponential graph**. The method for drawing the exponential graph is the same as that used to draw a parabola. Graphing technology can be used, or a table of values completed and points plotted to produce the curve.

**exponential growth** relationship between two variables in which, as one variable increases, the other increases at an increasing rate of change; the graph curves upwards from left to right

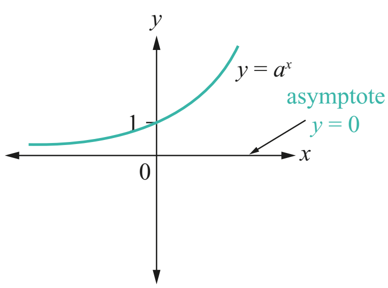
**exponential decay** relationship between two variables in which, as one variable increases, the other decreases at a decreasing rate of change; the graph curves downwards from left to right

!

The equation for an exponential graph is of the general form:

$y = ka^x + c$ where $k$, $a$ and $c$ are constants, and $a > 0$ and $a \neq 1$

The simplest exponential graph has the equation $y = a^x$, where $a > 1$. The curve has a $y$-intercept of 1 and a horizontal **asymptote** along the $x$-axis. The equation of the asymptote is $y = 0$. The asymptote is sometimes shown as a dashed line.




Depending on the given equation, an exponential relationship will either show **exponential growth** or **exponential decay**.

## EXERCISE 8C The exponential graph

**1** **a** Complete this table of values for the equation $y = 2^x$.

| $x$ | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

**b** Plot these points on a number plane and join with a smooth curve.

**c** Use graphing technology to produce the curve for values of $x$ from $-4$ to 4.

**d** What happens to the value of $y$ as the value of $x$ increases? This is known as exponential growth.

**e** Are there any negative values for $y$? Explain.

**f** What is the $y$-intercept of this exponential graph?

**g** Are there any $x$-intercepts?

**h** This curve has a horizontal asymptote. What is its equation?

UNDERSTANDING, FLUENCY AND COMMUNICATING

**2** **a** Use graphing technology to draw the graphs of $y = 2^x$ and $y = 3^x$ on the same number plane for values of $x$ from −3 to 3.

**b** Compare the two curves. Which features are different? Which features are the same? Comment on the $y$-intercept, the position of the asymptote and the general shape of each curve.

**c** Predict how the graph of $y = 5^x$ would compare to the two curves you have drawn. Check your answer using graphing technology.

**3** **a** Complete a table of values for the equation $y = \left(\frac{1}{2}\right)^x$ using $x$-values from −4 to 4.
(*Note:* $y = \left(\frac{1}{2}\right)^x$ can also be written as $y = (0.5)^x$.)

**b** Plot these points on a number plane and join with a smooth curve.

**c** Use graphing technology to produce the curve for values of $x$ from −4 to 4.

**d** What happens to the value of $y$ as the value of $x$ increases? This is known as exponential decay.

**e** Are there any negative values for $y$? Explain.

**f** What are the $x$- and $y$-intercepts of this exponential graph?

**g** What is the equation of the asymptote?

**4** **a** Use graphing technology to draw the graphs of $y = 3x$ and $y = \left(\frac{1}{3}\right)^x$ on the same number plane.

**b** Compare the two curves. Which features are different? Which features are the same? Comment on the $y$-intercept, the position of the asymptote and the general shape of each curve.

**c** Predict how the graph of $y = \left(\frac{1}{5}\right)^x$ would compare to the graph of $y = 5^x$. Check your answer using graphing technology.

**5** **a** Use graphing technology to draw the graph of $y = 2^x$ for values of $x$ from −4 to 4.

**b** On the same set of axes used for part **a**, draw the exponential graph for each of the following equations.

**i** $y = 2(2^x)$ **ii** $y = 3(2^x)$ **iii** $y = 0.5(2^x)$ **iv** $y = 0.2(2^x)$

**c** List the $y$-intercept for each curve in part **b**.

**d** Write the equation of the asymptote for each curve in part **b**.

**e** Compare each graph from part **b** with the exponential graph for $y = 2^x$. What is the effect of multiplying $2^x$ by a constant term in $y = 2^x$?

**6** Using your answers to questions **4** and **5**, draw a sketch of what the graph of each of these exponential relationships would look like. Clearly show the $y$-intercept and the position of the asymptote.

**a** $y = 2\left(\frac{1}{2}\right)^x$ **b** $y = 3(0.5)^x$ **c** $y = 0.5\left(\frac{1}{2}\right)^x$ **d** $y = 0.2(0.5)^x$

**7** Check your graphs from question **6** using graphing technology.

**8** Use graphing technology or a table of values to draw the graph of each of the following exponential relationships. Adjust the interval of $x$-values you use so that you can see the shape of the graph clearly. Indicate the $y$-intercept for each graph.

**a** $y = 1.4(3^x)$ **b** $y = 3.5(0.6)^x$ **c** $y = 200(1.02)^x$ **d** $y = 700(0.95)^x$

**9** Which of your graphs from question **8** show exponential growth and which show exponential decay?

**10** Use graphing technology to further explore the effect of changing the values of $k$, $a$ and $c$ on the exponential graph of $y = ka^x + c$, where $k$, $a$ and $c$ are constants and $a > 0$. Comment on how the $y$-intercept, the position of the asymptote and the general shape of the curve are affected.

# 8D Exponential models

These resources are available on your obook assess:

- **Video tutorial 8D:** Watch and listen to an explanation of Example 8D–1
- **Investigation 8D:** Explore the exponential growth of one grain of rice
- **assess quiz 8D:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

Exponential relationships can be used to model practical situations. Population growth, compound interest, declining-balance depreciation and radioactive decay are just a few examples of exponential growth or decay.

In this section, we will focus on exponential models that can be represented by the general equation $y = ka^x$, where $a > 0$ (and $a \neq 1$). As with quadratic models for practical situations, there may be restrictions on the values used for the two variables. For example, for an exponential model involving time $t$, we know that $t \geq 0$.

## EXAMPLE 8D–1 Interpreting the graph of an exponential model

The population of an island can be modelled using the equation $P = 5000(1.09)^t$, where $P$ is the population and $t$ is the time in years. The graph of the equation $P = 5000(1.09)^t$ is drawn on the right.

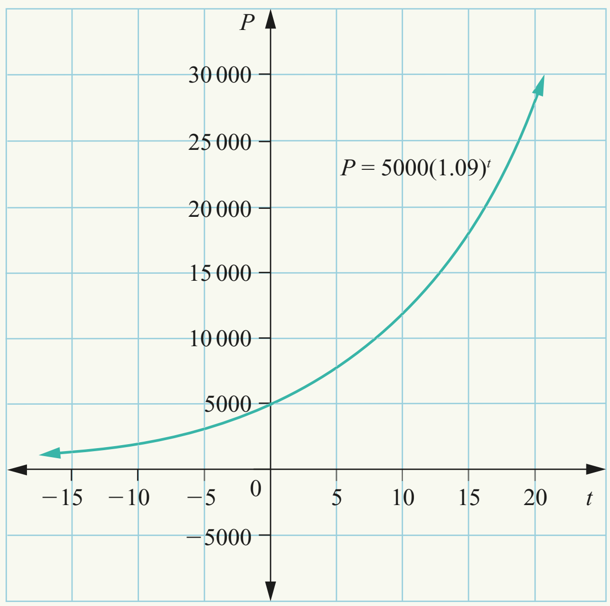
**a** Which part of the exponential curve applies to the population model in this case?

**b** Does the graph show exponential growth or decay?

**c** What was the initial population of the island?

**d** Use the graph to estimate the population after 12 years.

**e** How long did it take for the population to double?



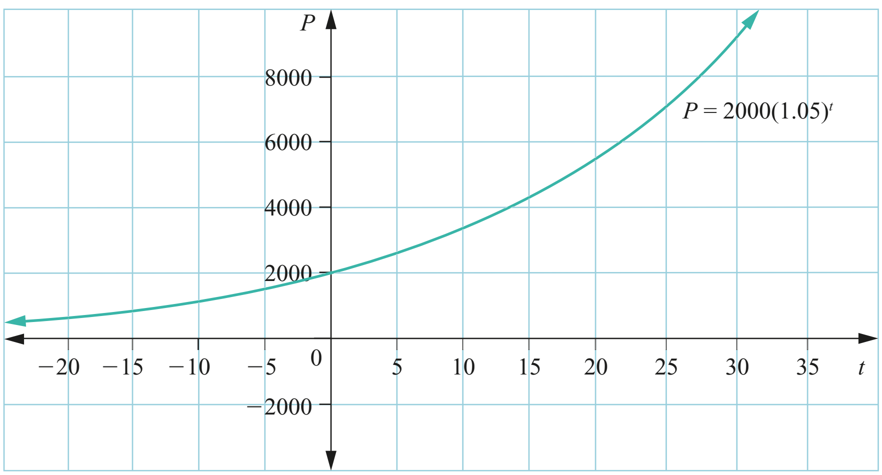

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | The curve matches the population model for $t \geq 0$. | Time cannot be negative, so the values for $t$ must be greater than or equal to zero. | Identify the values of the variables that apply to the practical context. |

| | Solve | Think | Apply |
|---|---|---|---|
| **b** | Exponential growth | As $t$ increases, $P$ increases at an increasing rate. | The curve for exponential growth moves upwards at an increasing rate from left to right. |
| **c** | When $t = 0$, $P = 5000$. The initial population was 5000. | Find the population when $t = 0$. Read the value for $P$ where the curve cuts the vertical axis (i.e. find the $P$-intercept). | The initial population is when time is zero. |
| **d** | When $t = 12$, $P \approx 14\,000$. After 12 years, the population is about 14 000. | Find the population when $t = 12$. Move up from 12 on the horizontal axis to the curve and then move left to the vertical axis to read the value for $P$. | Use the graph to read the value for the population when time is 12 seconds. |
| **e** | For double the initial population, $P = 10\,000$. When $P = 10\,000$, $t \approx 8$. The population has doubled after about 8 years. | Find the time when $P = 10\,000$. Move across from 10 000 on the vertical axis to the curve and then move down to the horizontal axis to read the value for $t$. | Use the graph to read the value for the time when the population is 10 000. |

## EXERCISE 8D Exponential models

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** The population of a mining town can be modelled using the equation $P = 2000(1.05)^t$, where $P$ is the population and $t$ is the time in years. The graph of the equation $P = 2000(1.05)^t$ is drawn below.




**a** Which part of the exponential curve applies to the population model in this case?
**b** Does the graph show exponential growth or decay? Explain.
**c** What was the initial population of the town?
**d** From the graph, estimate the population after 26 years.
**e** How long did it take for the population to double?

ALGEBRA

**2** The equation $m = 1.5(1.2)^x$ can be used to model the mass of a baby orang-utan up to 6 months of age, where $m$ is the mass in kilograms and $x$ is the age in months.

**a** Complete this table of values using the equation $m = 1.5(1.2)^x$ for values of $x$ up to 6.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $m$ | | | | | | | |

**b** Use graphing technology or the table of values to draw the exponential graph with equation $m = 1.5(1.2)^x$ for values of $x$ from 0 to 6, putting the values of $m$ on the vertical axis and $x$ on the horizontal axis.

**c** Estimate the mass of the orang-utan at age $3\left(\frac{1}{2}\right)$ months.

**d** At about what age would the orang-utan weigh 3.4 kg?

**e** Each month, the mass increases by a factor of 1.2. What is the monthly growth rate of the baby orang-utan's mass? Write your answer as a percentage.

**f** Why is this model only useful up to age 6 months? Explain your answer.

**3** The population of an island is modelled using the equation $P = 30\,000(1.02)^t$, where $P$ is the population and $t$ is the time in years.

**a** Complete this table of values using the equation $P = 30\,000(1.02)^t$.

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $P$ | | | | | | |

**b** Use graphing technology or the table of values to draw the graph of the population model for values of $t$ from 0 to 25.

**c** What does the intercept of the curve with the vertical axis represent?

**d** Use the graph to estimate the population after 18 years.

**e** Estimate the time taken for the population to reach 39 000.

**f** Extrapolate (or extend) your graph to obtain an estimate of the time taken for the population to double.

**g** What is the percentage increase in population each year? Explain how you can work this out from looking at the equation $P = 30\,000(1.02)^t$.

**4** Radioactive decay is a physical phenomenon that can be modelled using an exponential relationship. A piece of radioactive material decays according to the equation $m = 800(0.95)^t$, where $m$ is the mass of radioactive substance in grams and $t$ is the number of years.

**a** Complete this table of values using $m = 800(0.95)^t$.

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $m$ | | | | | | |

**b** Use graphing technology or the table of values to draw the graph of $m = 800(0.95)^t$ for values of $t$ from 0 to 25.

**c** Explain how we can see that this model represents exponential decay.

**d** What was the initial mass of radioactive material?

**e** Use the graph to estimate the mass of radioactive substance after 23 years.

**f** Estimate the time taken for the mass of radioactive substance to reach 600 g.

**g** Extrapolate your graph to obtain an estimate of the time taken for the mass of radioactive substance to reach 100 g.

**h** What is the percentage rate of decay each year? Explain how you can work this out from looking at the equation $m = 800(0.95)^t$.

**5** Depreciation, appreciation and inflation may all be modelled using graphs. Depreciation is an example of exponential decay, and appreciation and inflation are examples of exponential growth.

**a** Draw the graph of $y = 1000(1.08)^x$ using values of $x$ from 0 and 10.

**b** Draw the graph of $y = 1000(0.92)^x$ using values of $x$ from 0 and 10.

**c** Which of these graphs would be used to model depreciation? Why?

**d** Which of these graphs would be used to model inflation? Why?

**e** What would be the difference in the value of two items after 8 years, if both were purchased for $1000 and one appreciated at 8% p.a. and the other depreciated at 8% p.a.?



**6** A new motorcycle is purchased for $20 000 and depreciates in value at a rate of 25% per year. The salvage value of the motorcycle can be modelled using the equation $S = 20\,000(0.75)^n$, where $S$ is the salvage value in dollars after $n$ years.

**a** Use graphing technology or a table of values to draw the graph of this exponential model for values of $n$ from 0 to 6.

**b** Does this graph represent exponential growth or decay? Explain your answer.

**c** What does the intercept of the curve with the vertical axis represent?

**d** Use the graph to find the salvage value of the motorcycle after 2 years.

**e** From the graph, estimate the number of years it takes for the salvage value of the motorcycle to be less than $6000.

**7** The formula for the declining-balance method of depreciation is $S = V_0(1 - r)^n$, where $S$ is the salvage value of the asset in dollars, $V_0$ is the purchase price of the asset in dollars, $r$ is the depreciation rate per time period as a decimal, and $n$ is the total number of time periods. The value of Nick's work tools depreciates at the rate of 20% per year and can be represented by an exponential model.

**a** If Nick bought the tools for $2500, show that the equation of this exponential model is $S = 2500(0.8)^n$.

**b** Use technology or a table of values to draw the graph of $S = 2500(0.8)^n$ for values of $n$ from 0 to 10.

**c** Use the graph to find the salvage value of Nick's tools after 3 years.

**d** From the graph, estimate the number of years it takes for the salvage value to be less than $500.

**8** A new car is purchased for \$39 000. It depreciates in value at a rate of 28% per year.

**a** Write an equation to model the value of the car over time.

**b** Does this equation represent exponential growth or decay?

**c** Use technology to draw the graph of this exponential model.

**d** From the graph, find the salvage value of the car after 5 years.

**e** Estimate the number of years it takes for the car to be worth half of its original value.

**9** Simone invests \$3000 at 10% p.a. interest compounded annually. After investing a principal, $P$, (in dollars) for $n$ years at an annual compound interest rate, $r$ (as a decimal), the amount (in dollars) of the investment, $A$, can be represented by the formula $A = P(1 + r)^n$.

**a** Show that the model for this context can be represented by the equation $A = 3000(1.1)^n$.

**b** Use technology or a table of values to draw the graph of $A = 3000(1.1)^n$ for values of $n$ from 0 to 10.

**c** What does the intercept of the curve with the vertical axis represent?

**d** From the graph, find the amount of the investment after 4 years.

**e** From the graph, estimate the number of years it takes for the investment amount to be more than \$6000.

**f** Simone invests \$4000 instead of \$3000. Write an equation for this new investment model and hence compare the investment amount after 4 years.

**10** The compound interest formula is $FV = PV(1 + r)^n$, where $FV$ is the future value of the investment in dollars, $PV$ is the present value of the investment in dollars, $r$ is the percentage interest rate per compounding period as a decimal, and $n$ is the total number of compounding periods.

**a** Use the compound interest formula to write an equation to represent the value of an investment of \$10 000 at 9% p.a. compounded annually.

**b** Draw a graph of this equation and use it to estimate the time it takes for the investment to grow to \$15 000.

**c** Consider the scenario in which the interest is compounded every six months.

**i** Write an equation to represent the value of the investment.

**ii** Use the graph of this equation to estimate the time it takes for the investment to grow to \$15 000.

**d** Repeat part **c** for the scenario in which the interest is compounded monthly.

**e** Repeat part **c** for the scenario in which the interest is compounded daily.

**f** Use your answers to discuss the effect on the value of the investment as the length of the compounding time period shortens.

**11** A model for the population of Earth is given by $P = 3.03(1.02)^t$, where $P$ is the population in billions and $t$ is the time in years since 1960.

**a** Use graphing technology or a table of values to draw the graph of this population model for values of $t$ from 0 to 50.

**b** Use the graph to estimate the population after 15 years. This was the population in the year 1975. Check this against the actual population in 1975.

**c** Use the model to estimate the population in the year 2001. Check this against the actual population.

**d** Use the graph to estimate the year in which the population would reach:

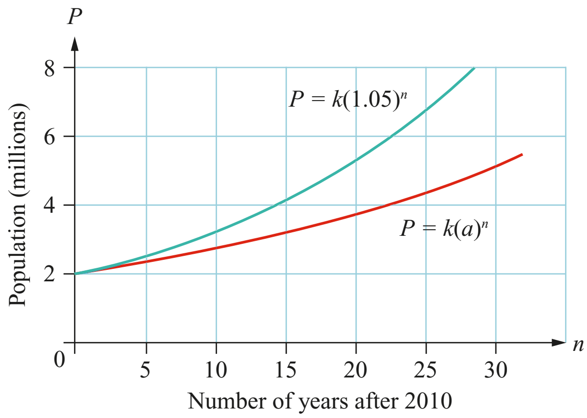
**i** 4 billion **ii** 5 billion **iii** 7 billion **iv** 10 billion

You may need to extrapolate (or extend) your graph.

**e** What factors could influence the actual population?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**12** In 2010, the predicted population of a city was modelled using the equation $P = k(1.05)^n$. That year, a new policy was introduced to slow the population growth. The new predicted population was modelled using the equation $P = k(a)^n$. In both equations, $P$ is the predicted population and $n$ is the number of years after 2010. The graph shows the two predicted populations.




**a** Use the graph to find the predicted population of the city in 2025 if the population policy had not been introduced.

**b** In each of the two equations, the value of $k$ is 2 000 000. What does $k$ represent?

**c** A guess-and-check method is to be used to find the value of $a$ in $P = k(a)^n$.

**i** Explain, with or without calculations, why 1.06 is not a suitable first estimate for $a$.

**ii** With $n = 10$ and $P = 2\,690\,000$, use the guess-and check method and the equation $P = k(a)^n$ to estimate the value of $a$ to two decimal places. Show at least two estimate values for $a$, including calculations and conclusions.

**d** The city was aiming to have a population under 7 000 000 in 2050. Does the model indicate that the city will achieve this aim? Justify your answer with suitable calculations.

CHALLENGE

**13** After the water boils in an electric kettle, the temperature of the water can be modelled using the equation $T = 80(0.8)^t + 20$, where $T$ is the temperature in degrees Celsius after $t$ minutes.

**a** Does the equation $T = 80(0.8)^t + 20$ represent exponential growth or decay?

**b** What was the initial temperature of the water in the kettle?

**c** Use graphing technology to draw the graph of this model.

**d** Use the graph to estimate the number of minutes it takes for the temperature of the water to be half the initial temperature.

**e** What is the lowest temperature that this kettle of water can reach?

**14** Exponential growth and decay models are used in many fields. The constant $e$ is used extensively. Research the history of $e$ and its use in practical circumstances. Determine the type of number that $e$ is and who discovered it.

**15** Rabbits were introduced to Australia in 1859 when a farmer imported 24 wild English rabbits and set them free on his land. An equation to model the growth of this rabbit population is $P = 24e^{0.19t}$, where $P$ is the number of rabbits after $t$ months. Explore this model and determine how long it took for these 24 rabbits to become 22 million rabbits. Do you think the current rabbit population continues to grow at this exponential rate? Explain your answer.

# 8E The rectangular hyperbola

**reciprocal relationship** relationship between two variables that involves the reciprocal of the independent variable; e.g. the equation $y = \frac{1}{x}$ describes the relationship between the independent variable $x$ and the dependent variable $y$

**hyperbola** graph of a reciprocal relationship

**rectangular hyperbola** hyperbola with a vertical asymptote and a horizontal asymptote

These resources are available on your obook assess:

- **assess quiz 8E:** Test your skills with an auto-correcting multiple-choice quiz

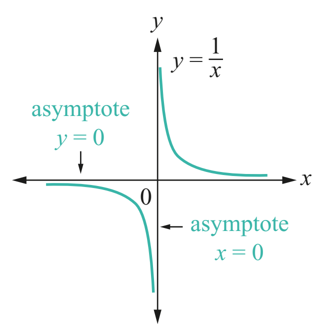
A further example of a non-linear relationship is a **reciprocal relationship** that involves the reciprocal of the independent variable. The graph of a reciprocal relationship is called a **hyperbola**. The method for drawing the hyperbola is the same as that used to draw a parabola or an exponential graph. Graphing technology can be used, or a table of values completed and points plotted to produce the curve.

In this section, we will focus on a particular type of hyperbola called a **rectangular hyperbola**.

The equation for a hyperbola is of the general form:

$$y = \frac{k}{x} \text{ where } k \text{ is a constant and } x \neq 0.$$

The simplest hyperbola has the equation $y = \frac{1}{x}$ and is made up of two curves. Its graph has two asymptotes and no $x$-intercepts or $y$-intercepts. As one asymptote is horizontal and the other vertical, the graph is a rectangular hyperbola.




The equation of the horizontal asymptote is $y = 0$ and the equation of the vertical asymptote is $x = 0$. The asymptotes form boundaries for the two curves.

## EXERCISE 8E The rectangular hyperbola

**1** **a** Complete this table of values for the equation $y = \frac{1}{x}$.

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | | | | | | |

**b** What happens at $x = 0$?

**c** Is it possible for $y$ to equal 0?

**d** Plot the points from the table of values on a number plane. The graph has two separate arms that do not cut the $x$-axis or the $y$-axis. Join the points for each arm with a smooth curve.

**e** Use graphing technology to produce the graph of $y = \frac{1}{x}$ for values of $x$ from $-4$ to 4.

**f** What happens to the value of $y$ as the value of $x$ increases? (Consider $y$ for $x = 10, 100, 1000, \ldots$)

**g** What happens to the value of $y$ as the value of $x$ gets closer and closer to 0 approaching from the right? (Consider $y$ for $x = \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$)

**h** What happens to the value of $y$ as the value of $x$ decreases? (Consider $y$ for $x = -10, -100, -1000, \ldots$)

**i** What happens to the value of $y$ as the value of $x$ gets closer and closer to 0 approaching from the left? (Consider $y$ for $x = -\frac{1}{10}, -\frac{1}{100}, -\frac{1}{1000}, \ldots$)

**j** Are there any $x$-intercepts or $y$-intercepts?

**k** An asymptote is a line that the curve gets closer and closer to but never crosses. Write the equation of the horizontal asymptote and the equation of the vertical asymptote. Use dashed lines to mark the asymptotes on your graph in part **d**.

**2** **a** Complete this table of values for the equation $y = \frac{6}{x}$.

| $x$ | −24 | −18 | −12 | −6 | −4 | −3 | −2 | −1 | 1 | 2 | 3 | 4 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | | | | | | | | |

**b** Plot these points on a number plane. Join the points for each arm with a smooth curve.

**c** Repeat parts **a** and **b** using the equation $y = \frac{12}{x}$.

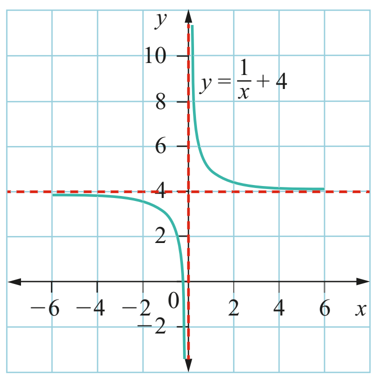
**d** Use graphing technology to produce the graphs of $y = \frac{6}{x}$ and $y = \frac{12}{x}$ on the same number plane. (If necessary, adjust the scale on the vertical axis so that the hyperbolas can be seen clearly.)

**e** Compare the two graphs. Which features are different? Which features are the same? Comment on the position of the asymptotes and the general shape of the arms of each hyperbola.

**f** Predict how the graph of $y = \frac{24}{x}$ would compare to the two hyperbolas you have drawn. Check your answer using graphing technology.

**3** **a** Use graphing technology to draw the graph of $y = \frac{1}{x}$ for values of $x$ from −6 to 6.

**b** Use graphing technology to produce the rectangular hyperbola for each of the following equations. Use values of $x$ from −6 to 6. Make a sketch of each hyperbola in your book. *Note:* when drawing a sketch of a hyperbola in your book, indicate each asymptote with a dashed line. For example, the graph drawn on the right for $y = \frac{1}{x} + 4$ shows a horizontal dashed line at $y = 4$ and a vertical dashed line at $x = 0$.




**i** $y = \frac{1}{x} + 3$

**ii** $y = \frac{1}{x} + 1$

**iii** $y = \frac{1}{x} - 2$

**iv** $y = \frac{1}{x} - 5$

**c** From the graph, list the $x$-intercept for each hyperbola in part **b**.

**d** Write the equations of the asymptotes for each hyperbola in part **b**.

**e** Compare each hyperbola from part **b** with the hyperbola for $y = \frac{1}{x}$. What is the effect of adding a constant term to $\frac{1}{x}$ in $y = \frac{1}{x}$?

**4** **a** Use graphing technology to draw the graphs of $y = \frac{1}{x}$ and $y = -\frac{1}{x}$ on the same number plane. (If necessary, adjust the scale on the axes so that the hyperbolas can be seen clearly.)

**b** Compare the two hyperbolas. Which features are different? Which features are the same? Comment on the positions of the asymptotes and the general shape of each curve.

**c** Predict how the graph of $y = -\frac{24}{x}$ would compare to the graph of $y = \frac{24}{x}$.

**5** Using your answers to questions **2–4**, draw a sketch of what the graph of each of these reciprocal relationships would look like. Use a dashed straight line to show the position of each asymptote.

**a** $y = \frac{20}{x}$ **b** $y = \frac{8}{x} + 16$ **c** $y = -\frac{5}{x}$ **d** $y = \frac{100}{x} - 50$

**6** Check your graphs from question **5** using graphing technology. Adjust the scale on the axes so that you can see the shape of the graph clearly. List any $x$-intercepts you see.

**7** Use graphing technology to further explore the effect of changing the values of $k$ and $c$ on the graph of $y = \frac{k}{x} + c$, where $k$ and $c$ are constants and $x \neq 0$. Comment on how the position of the asymptotes and the general shape of the hyperbola are affected.

# 8F Reciprocal models

These resources are available on your obook assess:

- **Video tutorial 8F:** Watch and listen to an explanation of Example 8F–1
- **Worksheet 8F:** Model further problems involving inverse proportion
- **assess quiz 8F:** Test your skills with an auto-correcting multiple-choice quiz

**inverse variation** relationship between two variables in which, as one variable increases, the other decreases in the same proportion, and vice versa; we say $y$ varies inversely as $x$, or $y$ varies as $\frac{1}{x}$

Reciprocal relationships can be used to model practical situations and represent **inverse variation**. If two quantities vary *inversely*, then an increase in one quantity results in a decrease in the other quantity in the same proportion, and vice versa. For example, if two painters can paint a room in 6 hours, then four painters could complete the same job in 3 hours working at the same rate; that is, as the number of painters increases, the number of hours to paint the same area decreases.

Suppose $x$ is the number of painters and $y$ is the number of hours they take to complete a job, then the following table shows the possibilities.

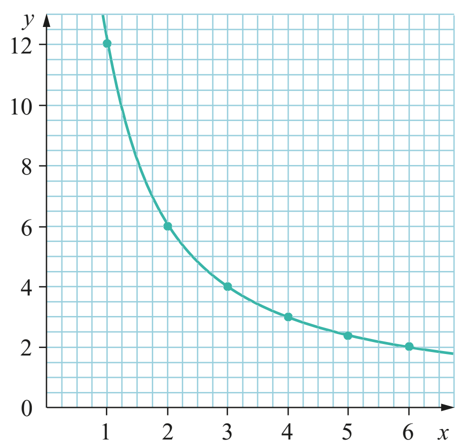
×3

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 12 | 6 | 4 | 3 | 2.4 | 2 |

÷3

Notice that when $x$ is multiplied by 3, $y$ is divided by 3.

The equation for the reciprocal relationship connecting $x$ and $y$ is $y = \frac{12}{x}$. The graph of $y = \frac{12}{x}$ is a rectangular hyperbola. However, only one arm of the hyperbola would be drawn to show the trend of the plotted points for this practical situation. Why do you think this is the case?




In this section, we will focus on reciprocal models that can be represented by the general equation $y = \frac{k}{x}$, where $k$ is a constant and $x > 0$. As with quadratic and exponential models for practical situations, there may be restrictions on the values used for the two variables.

## EXERCISE 8F Reciprocal models

**1** Complete the tables of values using the equations and the given values of $x$.

**a** $y = \frac{1}{x}$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**b** $y = \frac{2}{x}$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**c** $y = \frac{6}{x}$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**d** $y = \frac{8}{x}$

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | | | | | |

**2** For the equations in question **1**, plot the points in each table and join them with a smooth curve to produce the graph. Are you drawing the full rectangular hyperbola in each case? Explain your answer.

## EXAMPLE 8F–1 Interpreting a reciprocal model

The equation $s = \frac{240}{t}$ can be used to model the average speed necessary to travel 240 km in a particular time, where $t$ is the time in hours and $s$ is the speed in kilometres per hour.

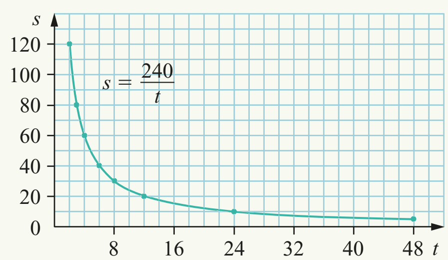
**a** Complete this table of values using the equation $s = \frac{240}{t}$.

| $t$ | 2 | 3 | 4 | 6 | 8 | 12 | 24 | 48 |
|---|---|---|---|---|---|---|---|---|
| $s$ | | | | | | | | |

**b** Use graphing technology or the table of values to draw the graph of $s = \frac{240}{t}$ for values of $t$ from 2 to 48, with $t$ on the horizontal axis and $s$ on the vertical axis.

**c** Use the graph to find the average speed required to complete the 240 km trip in 5 hours.

**d** About how long would it take to complete the trip at 95 km/h?

<table>
<tr><th></th><th>Solve</th><th>Think/Apply</th></tr>
<tr><td>a</td><td>
<table>
<tr><td>t</td><td>2</td><td>3</td><td>4</td><td>6</td><td>8</td><td>12</td><td>24</td><td>48</td></tr>
<tr><td>s</td><td>120</td><td>80</td><td>60</td><td>40</td><td>30</td><td>20</td><td>10</td><td>5</td></tr>
</table>
</td><td>Substitute each value of $t$ into the equation to find the corresponding value of $s$.</td></tr>
<tr><td>b</td><td>
</td><td>Draw the graph of $s = \frac{240}{t}$. This is the arm of the rectangular hyperbola when $t > 0$.</td></tr>
<tr><td>c</td><td>From the graph, when $t = 5$, $s \approx 50$.<br>The average speed required to complete the trip in 5 hours is about 50 km/h.</td><td>From 5 on the $t$-axis, move up to the curve and then across to read the value on the $s$-axis. (Using the equation, $s = 48$.)</td></tr>
<tr><td>d</td><td>From the graph, when $s = 95$, $t \approx 2.5$.<br>At 95 km/h, the trip would take about 2.5 hours.</td><td>From 95 on the $s$-axis, move across to the curve and then down to read the value on the $t$-axis.</td></tr>
</table>

**3** The equation $s = \frac{360}{t}$ can be used to model the average speed necessary to travel 360 km in a particular time, where $t$ is the time in hours and $s$ is the speed in kilometres per hour.

**a** Complete this table of values using the equation $s = \frac{360}{t}$.

| $t$ | 3 | 4 | 6 | 9 | 12 | 18 | 36 | 72 |
|---|---|---|---|---|---|---|---|---|
| $s$ | | | | | | | | |

**b** Use graphing technology or the table of values to draw the graph of $s = \frac{360}{t}$ for values of $t$ from 3 to 72, with $t$ on the horizontal axis and $s$ on the vertical axis.

**c** What average speed is required to complete the 360 km trip in 7 hours?

**d** How long would it take to complete the trip at 75 km/h?

**4** Paul rents a guesthouse from the landlord for \$720 per week. In order to save money he has other people stay and divides the expense equally. The equation that models this situation is $C = \frac{720}{n}$, where $C$ is the cost per person and $n$ is the number of people staying per week.

**a** Complete this table of values for $C = \frac{720}{n}$.

| $n$ | 1 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $C$ | | | | | |

**b** Use graphing technology or the table of values to draw the graph of $C = \frac{720}{n}$ for $n$ values from 1 to 8.

**c** Use the graph to find the cost per person per week when 3 people are staying.

**d** Use the graph to find the number of people staying in a week when the cost per person is \$144.

**e** Three people stay all week and one person stays for half a week. Using the graph, how much does it cost for a 'full-week' person to stay? How much does it cost for a 'half-week' person to stay?

**f** Two people stay all week and five people stay for half a week. Estimate the cost per person per week.

**g** Explain how the cost could be \$288 per week for each of the people who stay all week.

**5** A household has \$306 in bills. Create a table and draw a graph to show how much each person pays if there are 2, 3, 4 or 5 people contributing equally to paying the bills.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

## EXAMPLE 8F–2 Writing an equation for a reciprocal model

A preschool centre pays \$210 to hire a bus for an excursion. Let $C$ be the cost per person and $N$ the number of people going on the excursion.

**a** Find the cost per person, when 70 people go on the bus.

**b** Write the equation connecting $C$ and $N$.

**c** How much would each person pay when 50 people go on the bus?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $\frac{\$210}{70} = \$3$ each person | Divide the cost, \$210, by 70 people. | As the number of people increases the cost per person decreases. This is a reciprocal relationship showing inverse variation. |
| **b** | $C = \frac{210}{N}$ | The equation is of the form $C = \frac{k}{N}$ where $k$ is 210. | |
| **c** | When $N = 50$, $C = \frac{210}{50} = 4.2$<br>Each person pays \$4.20. | Substitute $N = 50$ into the equation to find $C$. | |

**6** A school pays \$1200 to hire a campsite. Let $C$ be the cost per person and $N$ the number of people going on the school camp.

**a** Find the cost per person for the campsite, when 25 people go on the camp.

**b** Write the equation connecting $C$ and $N$.

**c** How much would each person pay when:

**i** 40 people go on the camp? **ii** 75 people go on the camp?

**d** Give one limitation of the equation in relation to this context.

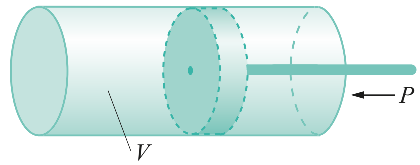
**e** Is it possible for the cost per person to be \$42? Support your answer with appropriate calculations.

**7** The weekly rental of a house is shared by a number of people. The rent per person ($R$) varies inversely with the number of people ($N$) living in the house. This means $R = \frac{k}{N}$, where $k$ is a constant.

**a** If each of the four occupants is paying \$72 per week, write the equation describing the relationship between $R$ and $N$. (*Hint:* use the given information to find the value of $k$.)

**b** How much would each person pay if there were: **i** 5 occupants? **ii** 3 occupants?

**8** Consider the gas trapped inside an airtight cylinder. For a given mass of gas, kept at a constant temperature, the volume is inversely proportional to the pressure. This means that the volume can be modelled using an equation of the form $V = \frac{k}{P}$, where $V$ is the volume, $P$ is the pressure and $k$ is a constant. When $V = 10\text{ cm}^3$, the pressure is 40 units.




- **a** Write the equation connecting $V$ and $P$.
- **b** Use graphing technology to draw the graph of the reciprocal model.
- **c** Use the graph to find the:
  - **i** pressure when the volume is $20\text{ cm}^3$
  - **ii** volume when the pressure is 100 units.
- **d** Check your answers to part **c** by substituting into the equation.

**9** The time ($t$) taken to drive from Sydney to Port Macquarie varies inversely with the average speed ($s$) of the car. The trip takes $4\frac{1}{2}$ hours travelling at an average speed of 90 km/h.

- **a** Write the equation connecting $t$ and $s$.
- **b** Use graphing technology to draw the graph for this reciprocal model.
- **c** Use the graph to find the average speed required to complete the trip in 6 hours.
- **d** Use the graph to find the time it takes to complete the trip at 100 km/h.

**10** The area of a rectangular block of land of length $l$ (in m) and width $w$ (in m) is $1200\text{ m}^2$.

- **a** Write an equation to describe the reciprocal relationship between $l$ and $w$.
- **b** Graph the relationship of length versus width. What happens to the width as the length increases?

**11** The number of people who can attend a concert varies inversely with the amount of space allowed per person. If each person is allowed $3500\text{ cm}^2$, the venue can hold 3500 people.

- **a** Write an equation to model the relationship between the number of people who can attend a concert and the amount of space allowed per person.
- **b** How many people could the venue hold if only $2000\text{ cm}^2$ were allowed per person?
- **c** Give at least one limitation of the equation in relation to this context.

**12** At a bank, the average waiting time of customers in a queue is related to the number of bank tellers on duty and can be modelled using the equation $w = \frac{12}{n}$, where $w$ is the average waiting time in minutes when there are $n$ bank tellers on duty.

- **a** Sketch the shape of the graph for this model.
- **b** On average, how long does a customer wait when the number of bank tellers on duty is:
  - **i** 2? **ii** 3? **iii** 4? **iv** 6? **v** 12? **vi** 24?
- **c** What is the maximum waiting time for a customer? When does this occur?
- **d** According to this model, will there be occasions when the waiting time is zero? Explain.

**13** The time taken to complete a certain job varies inversely with the number of workers doing the task.

- **a** Given that 20 workers could do the job in 6 days, write an equation to model this situation.
- **b** How long would it take 15 workers to complete the job, working at the same rate?
- **c** How long would it take 24 workers to complete the job, working at the same rate?

CHALLENGE

**14** Peter noticed that his 15 zebra finches took 6 days to empty their seed container. He added 3 more zebra finches to the cage. How long would it now take the zebra finches to empty the seed container?

**15** Squeaky Clean Windows can clean all windows of a building in 5 days when it uses a team of 8 people. This year the client wants the job completed in 4 days. How many cleaners are required?

# CHAPTER 8 REVIEW NON-LINEAR RELATIONSHIPS

Having completed this chapter you should be able to:

✔ generate tables and plot the graphs of non-linear relationships

✔ use technology to graph non-linear relationships

✔ recognise, draw and interpret a parabola

✔ interpret the turning point and intercepts of a parabola in a practical context

✔ model quadratic relationships for practical problems

✔ recognise, draw and interpret an exponential graph

✔ interpret the intercept of an exponential graph in a practical context

✔ model exponential relationships for practical problems

✔ understand the difference between exponential growth and decay

✔ recognise, draw and interpret a rectangular hyperbola

✔ model reciprocal relationships for practical problems

✔ recognise that reciprocal relationships of the form $y = \frac{k}{x}$, where $k$ is a constant and $x > 0$, represent inverse variation

✔ recognise the limitations of an algebraic model.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

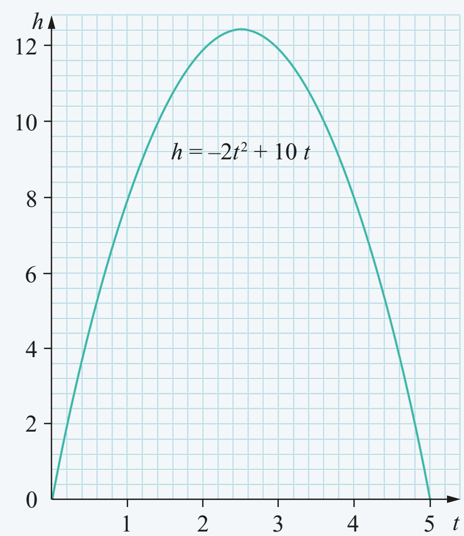
# REVIEW MULTIPLE-CHOICE QUESTIONS

8A **1** What are the coordinates of the turning point for the parabola with equation $y = x^2 + 4$?

**A** (4, 0) **C** (−1, 5)
**B** (0, 4) **D** (3, 13)

8B **2** The height, $h$ metres, of a toy hot air balloon after $t$ seconds is modelled by the equation $h = -2t^2 + 10t$. What is its maximum height?

**A** 2.5 m **C** 12 m
**B** 5 m **D** 12.5 m




8B **3** The value of a gemstone is related to the square of its weight. A stone weighing 3 carats is valued at $2000. What is the approximate value of a stone weighing 5 carats?

**A** $1000 **C** $3000
**B** $2000 **D** $6000

8C **4** Which equation could represent exponential decay?

**A** $y = (0.98)^x$ **B** $y = (1.02)^x$ **C** $y = \frac{6}{x}$ **D** $y = (0.5)x^2$

8D 5 The population of a mining town can be modelled using the equation $P = 1000(1.02)^t$, where $P$ is the population after $t$ years. According to the model, what was the initial population of the town?

**A** 1.02 **B** 500 **C** 1000 **D** 1020

8E 6 Using the equation $y = \frac{420}{n}$, what is the value of $y$ when $n = 8$?

**A** 50 **B** 52.5 **C** 55 **D** 3360

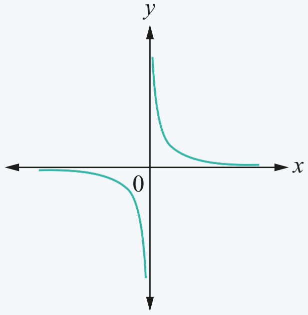
8F 7 The time taken to complete a job is inversely proportional to the number of workers. Given that 16 workers could complete a task in 6 days, how long would it take 25 workers to complete the task if they worked at the same rate?

**A** 2.67 days **B** 3.84 days **C** 7 days **D** 9.375 days

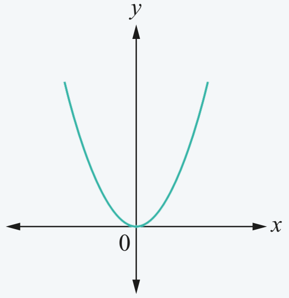
# REVIEW SET 1

**1** Match each of the following graphs with its equation.

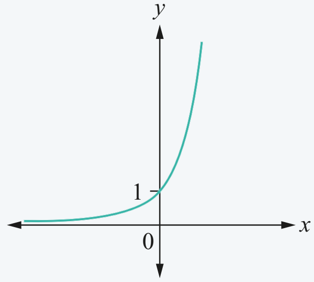
**A**




**B**




**C**




**a** $y = x^2$ **b** $y = 2^x$ **c** $y = \frac{1}{x}$

**2** **a** Complete this table of values for $y = 3x^2 - 4x + 3$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 3 | 2 | 7 | 18 | | |

**b** Use graphing technology or the table of values to draw the parabola with equation $y = 3x^2 - 4x + 3$ for values of $x$ from 0 to 5.

**c** What is the minimum value of this relationship?

**3** The population of an island is modelled using the equation $P = 40\,000(1.03)^t$, where $P$ is the population and $t$ is the time in years.

**a** What was the initial population of the island?

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $P$ | | | | | | |

**b** Complete this table of values using the equation $P = 40\,000(1.03)^t$.

**c** Draw the population graph with $t$ on the horizontal axis and $P$ on the vertical axis.

**d** Use the graph to estimate the population after 18 years.

**e** Use the graph to estimate the time taken for the population to reach 59 000.

**f** Extrapolate (or extend) your graph to obtain an estimate of the time taken for the population to reach 100 000.

**g** What is the percentage increase in population each year? Explain how you worked this out from the equation.

**4** Jenny rents a guesthouse from the landlord for \$420 per week. In order to save money she has other people to stay and divides the expense equally. The equation that models this situation is $C = \frac{420}{n}$, where $C$ is the cost per person per week and $n$ is the number of people staying.

**a** Complete this table of values for $C = \frac{420}{n}$.

| $n$ | 1 | 2 | 4 | 6 | 7 |
|---|---|---|---|---|---|
| $C$ | | | | | |

**b** Use graphing technology or the table of values to graph the equation $C = \frac{420}{n}$.

**c** Use the graph to find the cost per person per week when 5 people are staying.

**d** Use the graph to find the number of people staying per week if the cost per person is \$140.

**e** Check your answers to parts **c** and **d** by substituting into the equation.

**f** Give one limitation of the equation used to model this situation.

## REVIEW SET 2

**1** A ball is thrown into the air. The height of the ball, $h$ metres, can be modelled using the quadratic equation $h = -5t^2 + 30t + 3$, where $t$ is the time in seconds.

**a** Complete this table of values for the equation $h = -5t^2 + 30t + 3$.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $h$ | | | | | | | |

**b** Draw a graph modelling the height of the ball. Put the $t-1$ values on the horizontal axis and the $h-$ values on the vertical axis.

**c** From the graph find the maximum height of the ball. When does this occur?

**2** The volume of blood flowing in a blood vessel is directly related to the square of the internal diameter of the blood vessel and can be modelled by an equation in the form $V = ad^2$, where $V$ is the volume, $d$ is the internal diameter and $a$ is a constant. A volume of 100 mL flows through a blood vessel that has an internal diameter of 0.31 cm.

**a** Find the equation connecting volume of blood with internal diameter.

**b** Find the volume when the internal diameter is 0.36 cm.

**c** What is the internal diameter of a blood vessel that has a flow of 70 mL?

**3 a** Complete this table of values for the equation $y = \frac{1}{8}(2)^x$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | |

**b** Plot these points on a number plane and join with a smooth curve.

**c** What type of curve has been drawn?

**d** What happens to the value of $y$ as the value of $x$ increases?

**4 a** Complete this table of values for the equation $y = \frac{24}{x}$.

**b** Plot these points on a number plane and join them with a smooth curve.

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | |

**c** What type of curve has been drawn?

# REVIEW SET 3

**1** **a** Complete this table of values for $y = -x^2 + 6x - 2$.

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | | | | | | |

**b** Draw the parabola with equation $y = -x^2 + 6x - 2$ for values of $x$ from 0 to 5.

**c** What is the maximum value of this relationship?

**2** The surface area, $S$, of a cube is directly proportional to the square of its side length, $l$, and can be modelled by an equation in the form $S = kl^2$, where $k$ is a constant. A cube with surface area 121.5 cm$^2$ has side length 4.5 cm.

**a** Find the equation connecting $S$ and $l$.

**b** Draw the graph of the quadratic model.

**c** Find the surface area when the side length is 3.8 cm.

**d** What is the side length of a cube with a surface area of 200 cm$^2$?

**3** Radioactive decay can be modelled using the equation $m = 500(0.92)^t$, where $m$ is the mass of radioactive substance in grams and $t$ is the number of years.

**a** Complete this table of values using the equation $m = 500(0.92)^t$.

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $m$ | | | | | | |

**b** Draw the graph of this model with $t$ on the horizontal axis and $m$ on the vertical axis.

**c** What was the initial mass of radioactive material?

**d** Use the graph to estimate the mass of radioactive substance after 23 years.

**e** Estimate the time taken for the mass of radioactive substance to reach 400 g.

**f** Extrapolate your graph to obtain an estimate of the time taken for the mass of radioactive substance to reach 50 g.

**g** What is the percentage rate of decay each year? Explain how you worked this out from the equation.

**4** The time taken to complete a certain job is inversely proportional to the number of workers doing the task.

**a** If 25 workers can complete a job in 4 days, write an equation to model the relationship between the time taken to complete a job and the number of workers doing the task.

**b** How long would it take 11 workers to complete the task if they worked at the same rate?

ALGEBRA

# REVIEW SET 4

**1** A skyrocket is launched into the air. The height of the rocket, $h$, in metres, can be modelled using the equation $h = -5t^2 + 80t$, where $t$ is the time in seconds.

**a** Complete this table of values using the equation $h = -5t^2 + 80t$.

| $t$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $h$ | | | | | | | |

**b** Draw a graph modelling the height of the rocket. Put the $t$-values on the horizontal axis and the $h$-values on the vertical axis.

**c** From the graph find the maximum height of the rocket. When does this occur?

**2** A new car is purchased for \$28 000 and depreciates in value at a rate of 24% per year. The salvage value of the car can be modelled using the equation $S = 28\,000(0.76)^n$, where $S$ is the salvage value in dollars after $n$ years.

**a** Use graphing technology or a table of values to draw the graph of this exponential model for values of $n$ from 0 to 6.

**b** Does this graph represent exponential growth or decay? Explain your answer.

**c** What does the intercept of the curve with the vertical axis represent?

**d** Use the graph to find the salvage value of the car after 3 years.

**e** Use the graph to estimate the number of years it takes for the salvage value of the car to be less than \$10 000.

**3** The equation $s = \frac{300}{t}$ can be used to calculate the average speed required to travel 300 km in a particular time, where $s$ is the speed in kilometres per hour and $t$ is the time in hours.

**a** Complete this table of values using the equation $s = \frac{300}{t}$.

| $t$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $s$ | 100 | 80 | 60 | 40 | 30 | 20 | 10 | 5 |

**b** Draw the graph of $s = \frac{300}{t}$, with $t$ on the horizontal axis and $s$ on the vertical axis.

**c** What average speed is required to complete the 300 km trip in 7 hours?

**d** How long would the trip take at 95 km/h?

**4** The time taken to write a computer program is inversely proportional to the number of people working on the task. A team of 6 people can write a computer program in 30 days. The schedule requires that the program be written in 18 days. How many people need to be on the team?

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** A ball is thrown into the air. The height of the ball can be modelled using the equation $h = -3t^2 + 18t + 1$, where $h$ is the height in metres after $t$ seconds.

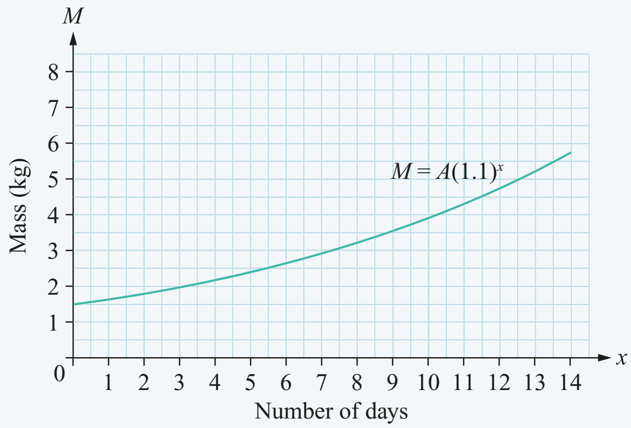
**i** Complete this table of values using the equation $h = -3t^2 + 18t + 1$. (2 marks)

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $h$ | | | | | |

**ii** Draw a graph modelling the height of the ball. (1 mark)

**iii** From the graph, find the maximum height of the ball and when this occurs. (2 marks)

**b** The mass $M$ kg of a puppy at age $x$ days is given by $M = A(1.1)^x$, where $A$ is a constant. The graph of this equation is shown.




**i** Does this model show exponential growth or decay? Explain your answer. (1 mark)

**ii** What is the value of $A$? (1 mark)

**iii** Estimate the mass of the puppy after 6 days. (1 mark)

**iv** Estimate the number of days before the puppy weighs 4.5 kg. (1 mark)

**v** What is the daily growth rate of the puppy's mass? Write your answer as a percentage. (1 mark)

**c** The time taken to paint a house varies inversely with the number of workers doing the task.

**i** Given that 10 workers could do the job in 4 days, write an equation to model this situation. (1 mark)

**ii** Draw a graph to represent this model. (2 marks)

**iii** How long would it take 8 workers to complete the job, working at the same rate? (1 mark)

**iv** How long would it take 20 workers to complete the job, working at the same rate? (1 mark)

**TOTAL: 15 marks**

ALGEBRA

# 9

# The normal distribution

The main mathematical ideas investigated are:

- drawing frequency distribution curves
- identifying normal distributions
- determining standardised scores ($z$-scores)
- using $z$-scores to compare data sets
- using the properties of normally distributed data
- determining the probability of an event using $z$-score(s) and the standard normal distribution table.

# ARE YOU READY?

Use the set of scores below to answer questions **1–3**.

40, 42, 56, 42, 30, 62, 38, 50, 76, 50, 44, 42, 73

9A **1** What is the mean of this set of scores, to the nearest whole number?

**A** 42 **B** 44 **C** 46 **D** 50

9A **2** What is the median of this set of scores?

**A** 42 **B** 44 **C** 46 **D** 50

9A **3** What is the mode of this set of scores?

**A** 42 **B** 44 **C** 46 **D** 50

Use the frequency distribution table below to answer questions **4–6**.

| Score | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|
| Frequency | 7 | 3 | 25 | 14 | 6 | 12 | 16 |

9A **4** What is the mean of this set of scores?

**A** 6 **B** 14 **C** 15 **D** 15.3

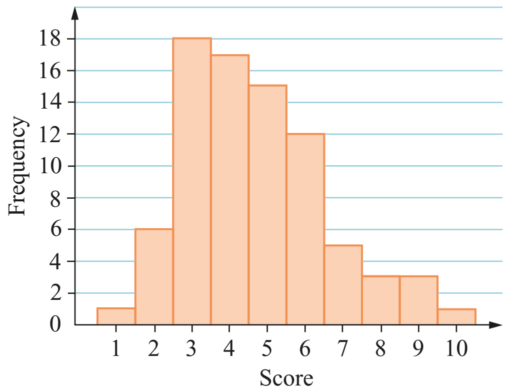
9A **5** What is the median of this set of scores?

**A** 6 **B** 14 **C** 15 **D** 15.3

9A **6** What is the mode of this set of scores?

**A** 6 **B** 14 **C** 15 **D** 15.3

Use the frequency histogram below to answer questions **7–12**.




9A **7** How many scores were recorded?

**A** 10 **B** 57 **C** 80 **D** 81

9A **8** How many scores of 5 were recorded?

**A** 5 **B** 12 **C** 15 **D** 17

9A **9** How many scores of 5 or less were recorded?

**A** 17 **B** 24 **C** 42 **D** 57

9A **10** What is the mean of the dataset shown in the frequency histogram?

**A** 3 **B** 4 **C** 4.7 **D** 9

9A **11** What is the median of the dataset shown in the frequency histogram?

**A** 3 **B** 4 **C** 4.7 **D** 9

9A **12** What is the mode of the dataset shown in the frequency histogram?

**A** 3 **B** 4 **C** 4.7 **D** 9

9A **13** Which statement is *not* true for a frequency histogram?

**A** The columns of a histogram are of equal width.

**B** The first column starts one column space from the vertical axis.

**C** The frequency polygon passes through the middle of each column and joins the horizontal axis.

**D** The tallest column represents the score with the highest frequency.

9A **14** Using a calculator or other technology, what is the sample standard deviation of the dataset for question **1**?

**A** 13.7 **B** 18 **C** 46 **D** 49.6

If you had difficulty with any of these questions or would like further practice, complete one or more of the matching Support sheets available on your obook assess.

**Q1–3** **Support sheet 9A.1 Mean, median and mode for data in a list**

**Q4–6** **Support sheet 9A.2 Mean, median and mode for data in a frequency table**

**Q7–13** **Support sheet 9A.3 Frequency histograms**

**Q14** **Support sheet 9A.4 Standard deviation of a dataset**

oi

# 9A Frequency distribution graphs

**symmetrical** description of a distribution graph of a data set for which an imaginary vertical line divides the curve so that one side is a reflection of the other

**skewed** description of a distribution graph of a data set for which the shape is not symmetrical

**negatively skewed** description of a skewed graph that has a longer tail to the left

**positively skewed** description of a skewed graph that has a longer tail to the right

These resources are available on your obook assess:
- **Interactive 9A.1:** Explore mean, median, mode and range
- **Interactive 9A.2:** Explore different shapes of frequency distribution graphs
- **assess quiz 9A:** Test your skills with an auto-correcting multiple-choice quiz

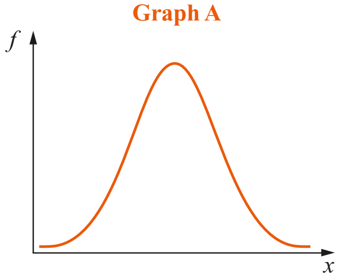
OI GI

Frequency distributions may be displayed using histograms and polygons. If instead of a frequency polygon we draw a smooth curve, we get a frequency graph whose general shape and properties can be used to make judgements about the data.

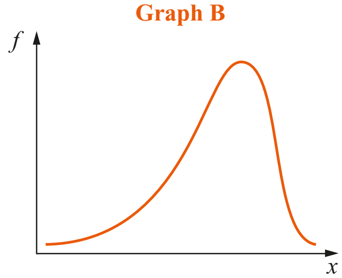
In Year 11, you described the shape of a frequency distribution graph in terms of smoothness, symmetry and number of modes. Smooth frequency graphs with one mode can also be described as **symmetrical** or **skewed**. The mode is the score with the highest frequency, so it is the score with the highest value on the graph.

Of the three graphs below, graph A is symmetrical, graph B is **negatively skewed** and graph C is **positively skewed**.

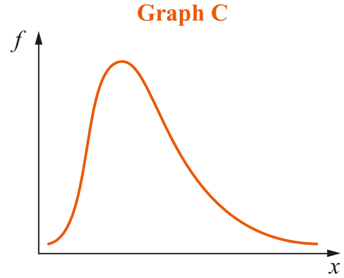
Graph A




Graph B




Graph C

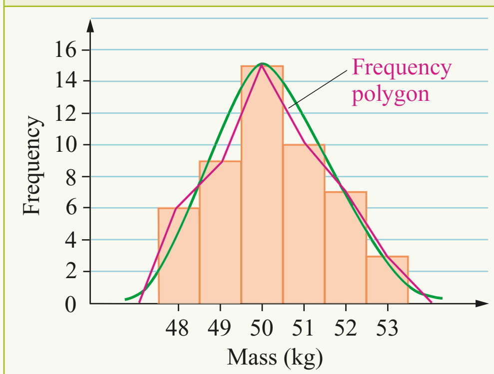



In this section, we will focus on drawing and describing the graphs of various frequency distributions.

## EXAMPLE 9A–1 Drawing the graph of a frequency distribution

On the same diagram, draw a frequency histogram, a frequency polygon and a smooth curve for the data given.

| Mass (kg) | 48 | 49 | 50 | 51 | 52 | 53 |
|---|---|---|---|---|---|---|
| Frequency | 6 | 9 | 15 | 10 | 7 | 3 |

| Solve | Think | Apply |
|---|---|---|
|  | Plot the points from the table. Leave a half column space from the frequency axis. Columns are drawn with equal widths. The polygon (pink line) passes through the middle of the top of each column and joins to the horizontal axis. Draw a smooth curve (green line) that follows the general shape of the polygon. | The frequency polygon can be drawn by plotting the points and joining them, but usually both the columns for the histogram and the line for the polygon are drawn. The frequency curve usually gives an indication of shape. |

# EXERCISE 9A Frequency distribution graphs

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** On the same diagram, draw a frequency histogram, a polygon and a smooth curve for each set of data.

**a**

| **Height (cm)** | 155 | 156 | 157 | 158 | 159 |
|---|---|---|---|---|---|
| **Frequency** | 5 | 25 | 26 | 24 | 3 |

**b**

| **Mark** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| **Frequency** | 1 | 2 | 4 | 5 | 8 | 10 | 7 |

**c**

| **Mark** | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| **Frequency** | 5 | 8 | 9 | 4 | 3 | 2 | 1 |

**d**

| **Shoe size** | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| **Frequency** | 4 | 5 | 9 | 6 | 7 | 9 | 5 |

**e**

| **Number of minutes late** | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| **Frequency** | 55 | 16 | 10 | 7 | 4 | 2 |

**f**

| **Age** | 0–9 | 10–19 | 20–29 | 30–39 | 40–49 | 50–59 | 60–69 |
|---|---|---|---|---|---|---|---|
| **Number of pedestrians killed** | 85 | 30 | 11 | 22 | 25 | 41 | 62 |

**2** Describe the curve for each frequency distribution in question **1** in terms of symmetry and number of modes.

**3** For each set of data in question **1**, find:

**i** the mean **ii** the median **iii** the mode/s.

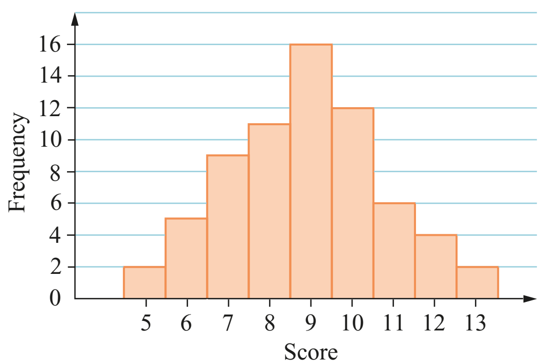
**4** Which of the frequency distributions in question **1** has one mode and looks symmetrical? What do you notice about the mean, median and mode of this distribution?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

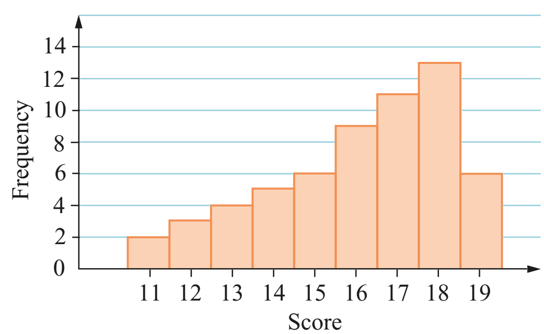
**5** Describe the curve for the frequency distribution in Example 9A–1 in terms of symmetry and number of modes. Find the mean, median and mode of the frequency distribution. What do you notice?

**6** Consider the graphs of the frequency distributions shown.

**i**




**ii**




**a** Write the mode for each frequency distribution.

**b** Predict whether the mean for each distribution will be less than, greater than or the same as the mode.

**c** Predict whether the median for each distribution will it be less than, greater than or the same as the mode.

**d** Use technology to check the predictions you made in parts **b** and **c**.

CHALLENGE

**7** Use the frequency distribution in question **1a** to answer the following.

**a** Find the standard deviation, correct to one decimal place.

**b** The lowest value for the height is 155 cm. How far away is this score from the mean?

**c** How many 'lots' of standard deviation is 155 cm from the mean?

**d** The highest value for the height is 159 cm. How far away is this score from the mean?

**e** How many 'lots' of standard deviation is 159 cm from the mean?

STATISTICAL ANALYSIS

# 9B The normal distribution

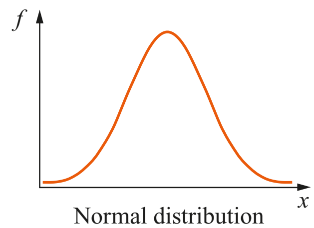
These resources are available on your obook assess:
- **assess quiz 9B:** Test your skills with an auto-correcting multiple-choice quiz

**normal curve**
the shape of a frequency curve that is symmetrical with one mode; a bell-shaped curve

**normal distribution**
a frequency distribution of scores that produces a bell-shaped curve (normal curve) and where the mean, median and mode are equal

When the frequency polygon for a distribution of scores is drawn as a smooth curve, one curve in particular occurs many times, especially for large distributions of naturally occurring characteristics such as height or weight. The shape of this particular curve is symmetrical with one mode and is known as the **normal curve**.

A distribution of scores that produces a curve with this shape is known as a **normal distribution**.

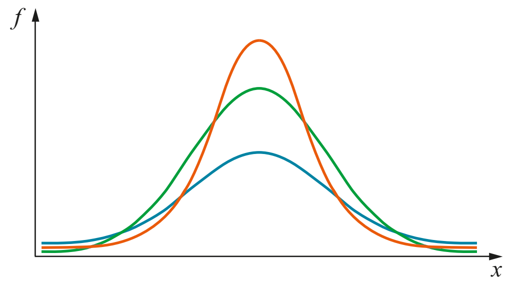


Normal distribution

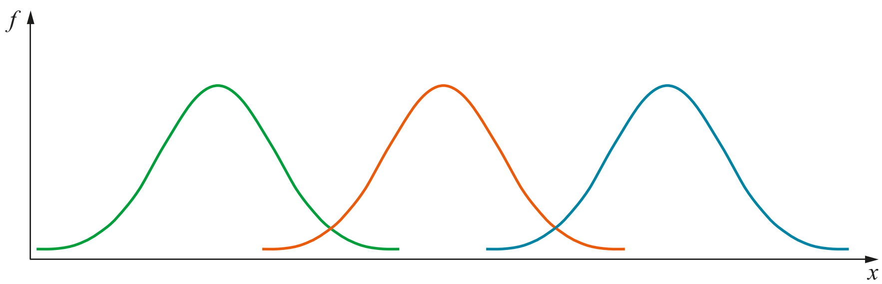
For a normal distribution:

- the frequency curve is bell-shaped and symmetrical about the mean
- the mean, median and mode are equal.

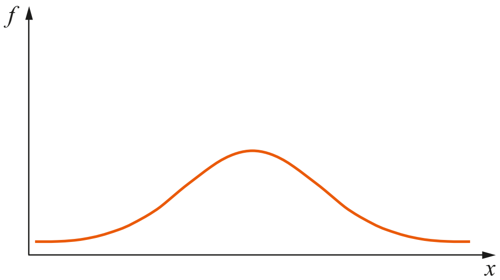
The size and shape of the bell are determined by the mean and the standard deviation of the distribution. These three normal curves have the same mean but different standard deviations.



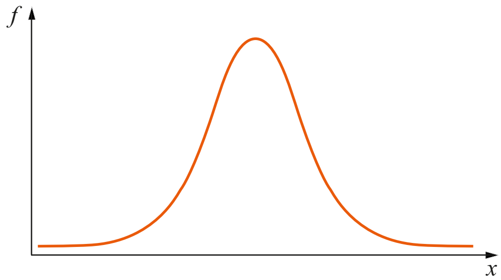

The three normal curves below have the same standard deviation but different means.




If a normal distribution of scores has a large standard deviation, the scores are widely spread from the mean and the graph will be short and widely spread (below left). If the standard deviation of the scores is small, most of the scores are close to the mean and the graph will be tall and narrow (below right).



Normal distribution with large $s$

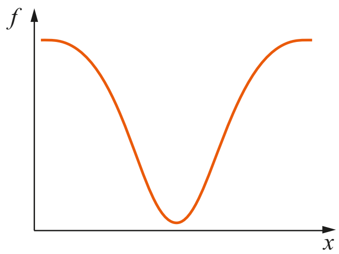


Normal distribution with small $s$

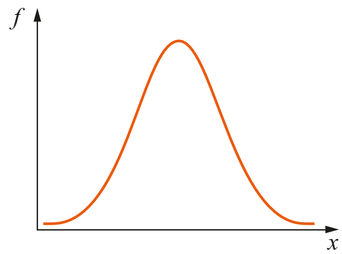
# EXERCISE 9B The normal distribution

**1** Which of the following is a normal curve?

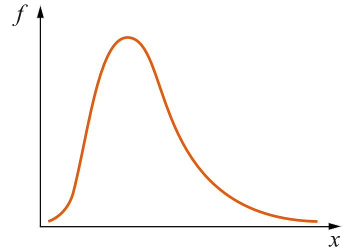
**A**




**B**




**C**




**2** Draw two normal curves with:

**a** the same mean but different standard deviations

**b** different means but the same standard deviation.

## EXAMPLE 9B-1 Determining whether a given frequency distribution is a normal distribution

Given the frequency distributions below, determine whether or not each is normally distributed by:

**i** first drawing a frequency distribution graph

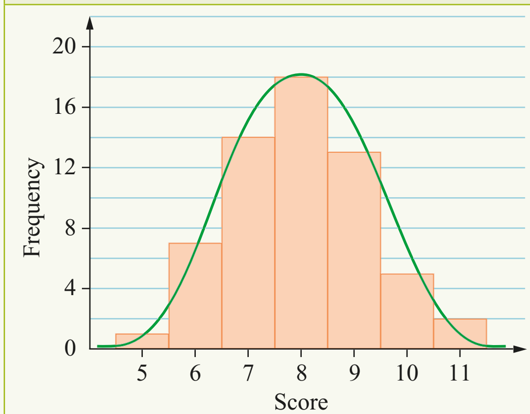
**ii** then, if necessary, finding the mean, mode and median.

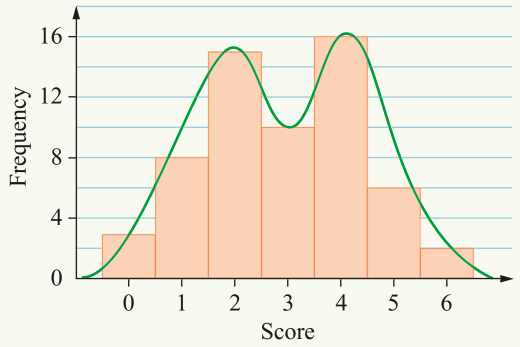
**a**

| Score | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|
| Frequency | 1 | 7 | 14 | 18 | 13 | 5 | 2 |

**b**

| Score | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Frequency | 3 | 8 | 15 | 10 | 16 | 6 | 2 |

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | <br>The curve appears to be symmetrical and bell-shaped, but not perfectly so.<br>Mean $\approx$ 7.97, Mode = 8, Median = 8<br>As the mean, mode and median are all almost exactly the same, the distribution is approximately normally distributed. | Draw a frequency histogram and sketch the frequency curve for this distribution.<br>Calculate the mean, mode and median for the distribution. | A normal distribution is symmetrical with mean, mode and median equal.<br>In this case all three measures equal 8, so it is a normal distribution. |

| | Solve | Think | Apply |
|---|---|---|---|
| **b** |   The graph is not bell-shaped so the distribution is not normal. | Draw a frequency histogram and sketch the frequency curve for this distribution. There is no need to find the mean, mode and median. | This distribution is bimodal and not a normal distribution. |

**3** Given the frequency distributions below, determine whether or not each is normally distributed by:

**i** first drawing a frequency distribution graph

**ii** then, if necessary, finding the mean, mode and median.

**a**

| Score | 4 | 5 | 7 | 6 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|
| Frequency | 7 | 9 | 21 | 14 | 20 | 2 | 1 |

**b**

| Score | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|
| Frequency | 2 | 9 | 14 | 25 | 16 | 7 | 2 |

**c**

| Score | 30 | 31 | 32 | 33 | 34 | 35 |
|---|---|---|---|---|---|---|
| Frequency | 10 | 28 | 6 | 4 | 2 | 1 |

**d**

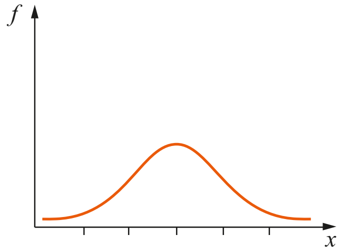
| Score | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 |
|---|---|---|---|---|---|---|---|---|
| Frequency | 3 | 12 | 18 | 31 | 30 | 19 | 13 | 1 |

**4** The normal curves below are drawn with the same scale on each horizontal axis and the same scale on each vertical axis. Which distribution has the:

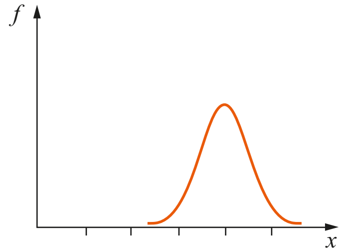
**a** largest standard deviation? **b** smallest standard deviation?

**c** largest mean? **d** smallest mean?

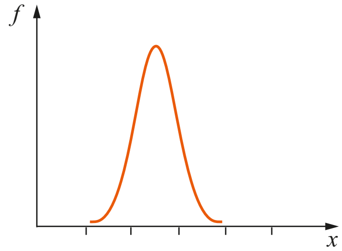
**A**




**B**

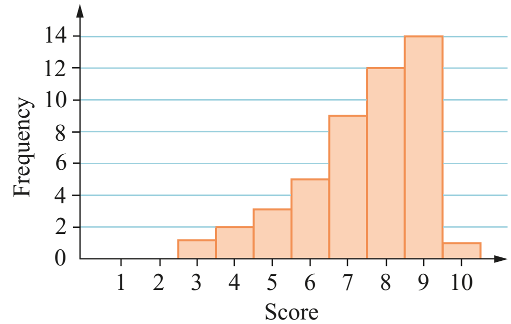



**C**



PROBLEM SOLVING, REASONING AND JUSTIFICATION

**5** The frequency histogram shows the results of 47 students in a test marked out of 10.



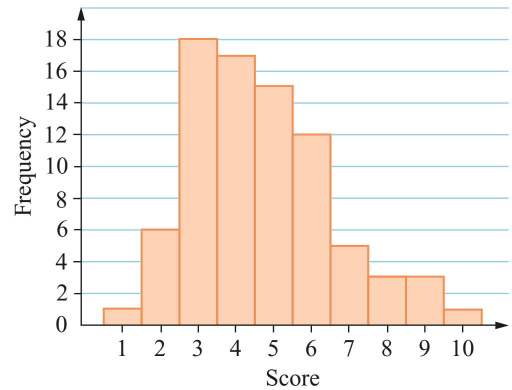

**a** Draw a smooth curve on the histogram.

**b** Describe the shape of the curve.

**c** What comments can be made about the test based on the shape of the curve?

**d** Can this distribution be classified as a normal distribution? Explain.

**6** The histogram shows the results of 81 students in a test marked out of 10.



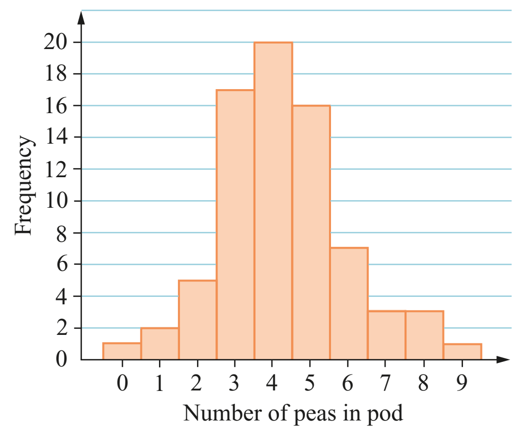

**a** Draw a smooth curve on the histogram.

**b** Describe the shape of the curve.

**c** What comments can be made about the test based on the shape of the curve?

**d** Can this distribution be classified as a normal distribution? Explain.

**7** The histogram shows the number of peas in 75 pods.




**a** Draw a smooth curve on the histogram.

**b** Describe the shape of the curve.

**c** What comments can be made about the number of peas in a pod based on the shape of the curve?

**d** Can this distribution be classified as a normal distribution? Explain your answer.

**8** The number of tomatoes on each plant of a large sample of a particular species of tomato is shown in the table. Determine whether or not the distribution is normal. Justify your answer.

| **Number of tomatoes on a plant** | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|
| **Number of plants** | 20 | 32 | 76 | 142 | 160 | 146 | 70 | 38 | 16 |

INVESTIGATION

**9** Investigate whether the heights of the students in your class are normally distributed by completing the following.

**a** Obtain the measurements for the height of each student in your class and display the results in a frequency distribution table.

**b** Draw a frequency distribution histogram and curve for this dataset.

**c** Is this distribution normally distributed? Justify your answer using appropriate calculations.

**10** Repeat question **9** but this time use a dataset of your own choosing.

# 9C The standardised score

These resources are available on your obook assess:
- **Video tutorial 9C:** Watch and listen to an explanation of Example 9C–2
- **assess quiz 9C:** Test your skills with an auto-correcting multiple-choice quiz

An important consideration when analysing sets of data is the relative significance of each score. For example, suppose Kim scores 40 marks in a Mathematics test. If the mark is out of a total of 50, then it is likely that it is a good mark; however, if it is out of 100 then it is probably a poor mark.

These conclusions are by no means certain. If the test was out of 50 and most students in the class scored more than 40 out of 50 then it is a relatively poor mark. Similarly, if the test was out of 100 and most students scored less than 40 then it is a relatively good mark: it is the relative value of a score in a data set that indicates its significance.

There are a number of ways in which the relative significance of a score can be indicated. The rank position of the score, or the decile or quartile in which the score lies could be given. A convenient method, which is also useful for comparing scores from different data sets, is to indicate the distance of the score from the mean of the data in terms of the standard deviation of the data. For example, suppose the mean score in the Mathematics test above was 56 and the standard deviation was 8.

The difference between Kim's score and the mean is 16, which is two times the standard deviation; that is, the distance of Kim's score from the mean is twice that of the 'typical' distance of all the scores from the mean.

Further, as Kim's mark is below the mean, the relative significance of her mark could be given as a score $-2$.

**standardised score ($z$-score)** the number of standard deviations a score lies above or below the mean

This is known as the **standardised score** or **$z$-score** for Kim's mark on this test.

Remember, the standard deviation is the 'typical' distance of all the scores from the mean.

$$\text{Standardised score} = \frac{\text{raw score} - \text{mean}}{\text{standard deviation}}$$

or $$z = \frac{x - \bar{x}}{s}$$

where $x$ = the raw score, $\bar{x}$ = the mean and

$s$ = the standard deviation ($s = \sigma_n$ for population standard deviation).

Standardised scores from different tests can be compared to find the better relative performance, or added to find the better total performance.

## EXAMPLE 9C–1 Writing the $z$-score given the number of standard deviations the raw score is from the mean

Write the $z$-score corresponding to a raw score that is:

**a** 0.8 standard deviations above the mean
**b** equal to the mean
**c** 1.3 standard deviations below the mean.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | $z = 0.8$ | The $z$-score is positive if the raw score is above the mean. | The $z$-score is the number of standard deviations the raw score is from the mean. The sign of the $z$-score is very important, as it describes whether the score is above (+) or below (−) the mean. |
| **b** | $z = 0$ | The $z$-score is zero if the raw score is equal to the mean. | |
| **c** | $z = -1.3$ | The $z$-score is negative if the raw score is below the mean. | |

# EXERCISE 9C The standardised score

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Write the standardised score corresponding to a score that is:

**a** 1 standard deviation above the mean
**b** 1 standard deviation below the mean
**c** 2 standard deviations above the mean
**d** 2 standard deviations below the mean
**e** 3 standard deviations above the mean
**f** 3 standard deviations below the mean
**g** equal to the mean
**h** 1.5 standard deviations above the mean
**i** 0.8 standard deviations below the mean
**j** 2.6 standard deviations below the mean.

## EXAMPLE 9C–2 Calculating $z$-scores

In an English examination the mean was 60 and the standard deviation was 10.

**a** Calculate the $z$-score (standardised score) for marks of:
**i** 50 **ii** 58 **iii** 60 **iv** 64 **v** 83
**b** Explain the meaning of these scores.

| | Solve | Think | Apply |
|---|---|---|---|
| **a i** | $z = \frac{50 - 60}{10} = -1$ | Use $z = \frac{x - \bar{x}}{s}$ where $\bar{x} = 60$ and $s = 10$. Subtract the mean of 60 from the score. Divide the answer by the standard deviation of 10. | If the score is less than the mean, the $z$-score will be negative. If the score is greater than the mean, the $z$-score will be positive. If the score is the same as the mean, the $z$-score will be zero. |
| **ii** | $z = \frac{58 - 60}{10} = -0.2$ | | |
| **iii** | $z = \frac{60 - 60}{10} = 0$ | | |
| **iv** | $z = \frac{64 - 60}{10} = 0.4$ | | |
| **v** | $z = \frac{64 - 60}{10} = 2.3$ | | |

STATISTICAL ANALYSIS

| | | | |
|---|---|---|---|
| **b** **i** | $z$-score $= -1$. This mark is 1 standard deviation below the mean. | A $z$-score indicates the number of standard deviations the score is from the mean. If the $z$-score is negative, then the score is below the mean. If the $z$-score is positive, the score is above the mean. | Positive $z$-scores are above the mean, negative $z$-scores are below the mean. A $z$-score of 0 is equal to the mean. The larger the magnitude of the $z$-score, the greater the distance from the mean. |
| **ii** | $z$-score $= -0.2$. This mark is 0.2 standard deviations below the mean. | | |
| **iii** | $z$-score $= 0$. This mark is equal to the mean. | | |
| **iv** | $z$-score $= 0.4$. This mark is 0.4 standard deviations above the mean. | | |
| **v** | $z$-score $= 2.3$. This mark is 2.3 standard deviations above the mean. | | |

**2** In a Science test the mean was 63 and the standard deviation 10.

**a** Calculate the standardised scores for these marks.

**i** 43 **ii** 53 **iii** 60 **iv** 63

**v** 69 **vi** 73 **vii** 78 **viii** 85

**b** Explain the meaning of these scores.

**3** In a History examination the mean was 56 and the standard deviation 8. Complete the table, which shows the marks scored by some students.

| Student | Mark | Standardised score |
|---|---|---|
| Julie | 65 | |
| Imran | 48 | |
| Sevvy | 72 | |
| Brad | 52 | |
| Katrina | 71 | |
| Sam | 40 | |
| Megumi | 56 | |
| Sasha | 38 | |

## EXAMPLE 9C–3 Comparing performance in test results

Harry scored 55 in an English test for which the mean was 50 and the standard deviation was 6. He scored 64 in a Mathematics test for which the mean was 59 and the standard deviation 9.

**a** Calculate his standardised score for each subject.

**b** In which subject did he perform better, given that the classes are of equal ability?

**c** What mark would Harry have had to score in Mathematics for his performance to be equivalent to that for English?

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | English:<br>$z = \dfrac{55 - 50}{6}$<br>$= 0.833...$<br>$\approx 0.8$<br>Mathematics:<br>$z = \dfrac{64 - 59}{9}$<br>$= 0.555...$<br>$\approx 0.6$ | Scores for both subjects are above the mean, so the $z$-scores are both positive. | The $z$-score allows comparisons between results with different means and standard deviations.<br>The number of students in each class does not have to be the same. |
| **b** | Harry performed better in English. | Harry's score for English was 0.8 standard deviations above the mean.<br>His score for Mathematics was 0.6 standard deviations above the mean.<br>He performed slightly better in English than in Mathematics. | |
| **c** | Mathematics score, $x$<br>$= 59 + 0.8 \times 9$<br>$= 66.2$<br>or<br>$\dfrac{x - 59}{9} = 0.8$<br>$x - 59 = 7.2$<br>$x = 66.2$<br>He would have had to score 66. | Harry's score would have to be 0.8 standard deviations above the mean for Mathematics.<br>That is, mean + 0.8 × standard deviation,<br>or substitute into $z = \dfrac{x - \bar{x}}{s}$, where $z = 0.8$, $\bar{x} = 59$ and $s = 9$, and solve to find $x$. | |

**4** Janelle scored 62 in a History test for which the mean was 52 and the standard deviation was 8. She scored 67 in a Geography test for which the mean was 55 and the standard deviation 12.

**a** Calculate her standardised score in each subject.

**b** In which subject did she perform better, given that the classes are of equal ability?

**c** What mark would Janelle have to have scored in Geography for her performance to be equivalent to that for History?

**5** In an examination the mean was 65 and the standard deviation 10. Write the raw examination mark corresponding to these standardised scores.

**a** 1    **b** −1    **c** 2    **d** −2    **e** 3    **f** −3

**g** 0    **h** 1.5    **i** −1.5    **j** 0.8    **k** −2.3    **l** 2.7

**6** Tom scored 78% on a Science test and 72% on a Mathematics test.

**a** In which test does Tom appear to have performed better?

**b** The mean score for the Science test was 66% and the mean score for the Mathematics test was 62%. Using this information and Tom's test scores, in which test does he appear to have performed better?

**c** The standard deviation of the scores for the Science test was 6% and the standard deviation of the scores for the Mathematics test was 4%. Using all the information you now have, in which test has Tom performed better? Explain your answer.

**7** **a** Complete the following table, which shows Hassan's marks in the yearly examinations.

| Subject | Mark | Mean | Standard deviation | $z$-score |
|---|---|---|---|---|
| English | 51 | 55 | 5 | |
| Mathematics | 64 | 60 | 12 | |
| Biology | 62 | 58 | 8 | |
| Art | 76 | 56 | 9 | |
| Economics | 58 | 58 | 14 | |
| History | 51 | 60 | 11 | |

**b** In which subject did he perform best?
**c** In which subject was his relative performance the worst?
**d** List all his subjects in order of relative performance.

**8** **a** The table shows Melissa's marks in four Mathematics tests. In which term was her relative performance best?

| Test | Mark | Mean | Standard deviation |
|---|---|---|---|
| Term 1 | 74 | 60 | 12 |
| Term 2 | 71 | 58 | 10 |
| Term 3 | 78 | 61 | 15 |
| Term 4 | 72 | 60 | 8 |

**b** Did her performance improve from:
**i** term 1 to term 2? **ii** term 2 to term 3? **iii** term 3 to term 4?

**9** **a** The marks of five students in the half-yearly Physics examination for which the mean was 60 and standard deviation 16 are shown in the first table. Complete the table.
**b** The marks of the same five students in the yearly Physics examination for which the mean was 58 and the standard deviation 12 are shown in the second table. Complete the table.
**c** Which student(s) improved their relative performance from the first to the second examination?
**d** Which student improved the most?
**e** Which students' performances decreased in the second examination?
**f** Which student's performance decreased the most?
**g** Which student's performance was the most consistent over the two examinations?

| Half-yearly Physics exam | | |
|---|---|---|
| Student | Mark | $z$-score |
| Phil | 73 | |
| Evie | 41 | |
| Tina | 84 | |
| Han Jin | 54 | |
| Kyla | 62 | |

| Yearly Physics exam | | |
|---|---|---|
| Student | Mark | $z$-score |
| Phil | 69 | |
| Evie | 40 | |
| Tina | 80 | |
| Han Jin | 59 | |
| Kyla | 52 | |

**10** The marks of the five students from question **9** in the half-yearly Physics examination for which the mean was 60 and standard deviation 16 are: 73, 41, 84, 54, 62.
**a** How many of the marks are within 1 standard deviation of the mean? (That is, how many students had a $z$-score between $-1$ and 1?)
**b** What percentage of the marks lie within 1 standard deviation of the mean?

PROBLEM SOLVING, REASONING AND JUSTIFICATION

## EXAMPLE 9C–4 Finding the better total performance

In the yearly examination Kylie scored 60 in Mathematics and 71 in Science. Elena scored 68 in Mathematics and 63 in Science. If the means and standard deviations for Mathematics and Science were 50 and 8 and 58 and 10 respectively, which student had the better total performance in these subjects?

**Solve**

| Student | Mathematics | | Science | | Total | |
|---|---|---|---|---|---|---|
| | Mark | $z$-score | Mark | $z$-score | Mark | $z$-score |
| Kylie | 60 | 1.25 | 71 | 1.3 | 131 | 2.55 |
| Elena | 68 | 2.25 | 63 | 0.5 | 131 | 2.75 |

Elena had the better total performance in these subjects.

**Apply**

$z$-scores can be added to obtain an overall result. The higher the total $z$-score, the better the result.

**Think**

By completing the table, we see that both students had the same total number of raw marks. However, by adding the standardised scores for each subject, we can conclude that Elena's total performance was better than Kylie's.

**11** In an examination, Brian scored 65 in English and 71 in Mathematics. Trevor scored 72 in English and 64 in Mathematics. If the means and standard deviations for English and Mathematics were 60 and 6, and 65 and 8 respectively, which student had the better total performance in these subjects?

**12** The marks scored by five students in English, Mathematics, Science and Art are shown in the table below. Place the students in order of merit on the basis of their performance in all subjects.

| Student | English | Mathematics | Science | Art |
|---|---|---|---|---|
| Melanie | 74 | 71 | 84 | 52 |
| Josh | 67 | 70 | 75 | 76 |
| Ming | 53 | 85 | 83 | 64 |
| Vera | 76 | 67 | 64 | 78 |
| Jake | 71 | 70 | 71 | 69 |
| **Mean** | 64 | 66 | 59 | 64 |
| **Standard deviation** | 10 | 15 | 12 | 13 |

CHALLENGE

**13** The following are marks scored in a Geography test:
57, 58, 56, 47, 62, 74, 60, 55, 33, 85, 63, 71, 58, 40, 55

**a** Calculate the mean and standard deviation of the marks in this test.

**b** Calculate the $z$-score for each mark.

**c** How many of the marks are within 1 standard deviation of the mean? (That is, how many had a $z$-score between $-1$ and 1?)

**d** What percentage of the marks lie within 1 standard deviation of the mean?

**e** What percentage of the marks are within 2 standard deviations of the mean?

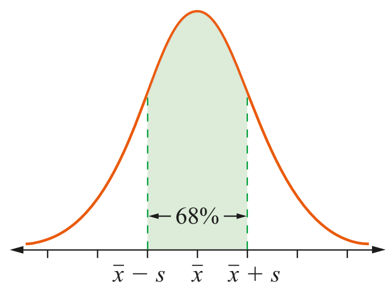
**f** What percentage of the marks are more than 2 standard deviations from the mean?

# 9D Properties of the normal distribution

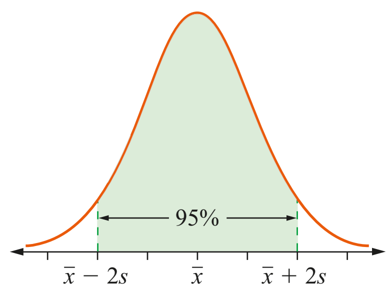
These resources are available on your obook assess:
- **Video tutorial 9D:** Watch and listen to an explanation of Example 9D–1
- **assess quiz 9D:** Test your skills with an auto-correcting multiple-choice quiz

The normal distribution has been studied in some detail by mathematicians and it can be shown that for a normal distribution, approximately:

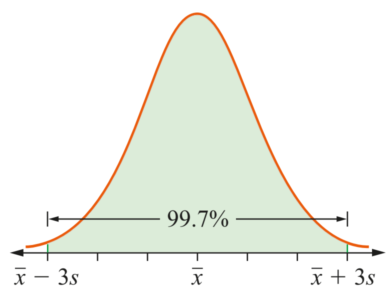
- 68% of the scores will lie within one standard deviation of the mean: $-1 < z\text{-score} < 1$




- 95% of the scores will lie within two standard deviations of the mean: $-2 < z\text{-score} < 2$




- 99.7% of the scores will lie within three standard deviations of the mean: $-3 < z\text{-score} < 3$



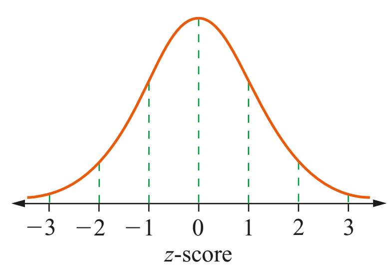

**empirical rule** describes the proportion of scores that lie within one standard deviation of the mean (68%), within two standard deviations of the mean (95%) and within three standard deviations of the mean (99.7%)

**standard normal distribution** a normal distribution with a mean of 0 and a standard deviation of 1; it uses standardised scores or *z*-scores

The percentages 68–95–99.7 are known as the **empirical rule**. This rule relates to the proportion of scores that lie within one, two and three standard deviations of the mean.

A normal distribution is described as a **standard normal distribution** when the mean is 0 and the standard deviation is 1. The standard normal distribution uses standardised scores or *z*-scores.

The standard normal distribution curve shows *z*-scores along its horizontal axis.



# EXERCISE 9D Properties of the normal distribution

UNDERSTANDING, FLUENCY AND COMMUNICATING

Use the graph of a normal distribution on the opposite page to answer questions **1** and **2**.

**1** What is the equivalent $z$-score for:

**a** $\bar{x} - s$ **b** $\bar{x} + s$ **c** $\bar{x} - 2s$ **d** $\bar{x}$

**e** $\bar{x} + 2s$ **f** $\bar{x} - 3s$ **g** $\bar{x} + 3s$

**2** Copy the three normal distribution graphs on the opposite page and include the appropriate $z$-scores along the horizontal scale.

**3** Use the graphs of standard normal distributions on the right to answer the following questions.

**a** What percentage of scores have a $z$-score:

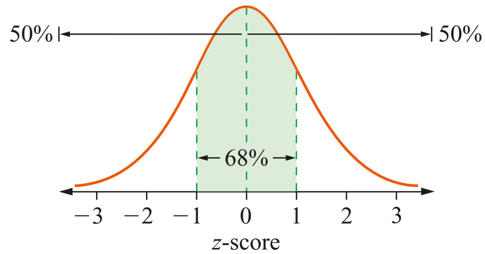
**i** between −1 and 1? **ii** between 0 and 1?

**iii** between 0 and −1? **iv** greater than 0?

**v** less than 0? **vi** greater than 1?

**vii** less than −1? **viii** greater than −1?

**ix** less than 1?



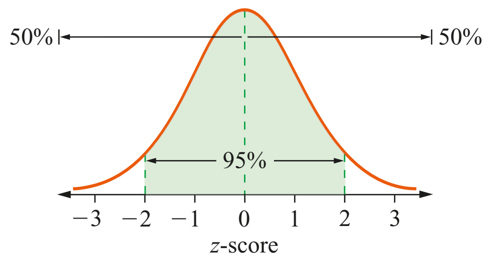

**b** What percentage of scores have a $z$-score:

**i** between −2 and 2? **ii** between 0 and 2?

**iii** between 0 and −2? **iv** greater than 2?

**v** less than −2? **vi** greater than −2?

**vii** less than 2?



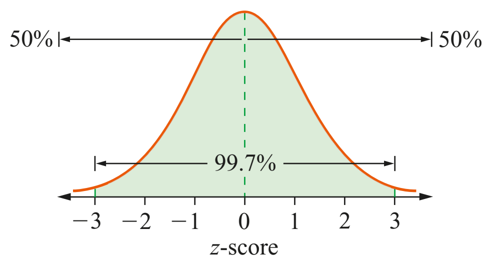

**c** What percentage of scores have a $z$-score:

**i** between −3 and 3? **ii** between 0 and 3?

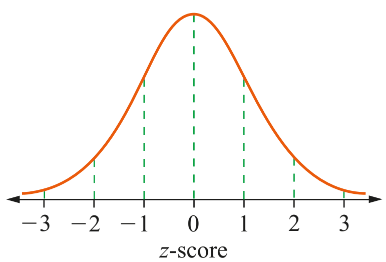
**iii** between 0 and −3? **iv** greater than 3?

**v** less than −3? **vi** greater than −3?

**vii** less than 3?




**4** A data set is normally distributed. Draw a standard normal distribution graph similar to the one shown on the right and shade the region under the graph to illustrate each of the following proportions.




**a** 50% of the scores lie below the mean

**b** 50% of the scores lie above the mean

**c** 34% of the scores lie between the mean and one standard deviation above the mean

**d** 47.5% of the scores lie between the mean and two standard deviations below the mean

**e** 13.5% of the scores lie between one and two standard deviations above the mean

**f** 84% of the scores lie below one standard deviation above the mean

**5** What percentage of scores have a $z$-score between:

**a** 1 and 2? **b** 1 and 3? **c** 2 and 3?

**d** −1 and 2? **e** −1 and 3? **f** −2 and 1?

**g** −2 and 3? **h** −3 and 1? **i** −3 and 2?

STATISTICAL ANALYSIS

## EXAMPLE 9D–1 Finding a test mark that is a whole number of standard deviations away from the mean

The marks on a test are normally distributed with a mean of 65 and a standard deviation of 8.

**a** What mark is one standard deviation:
  **i** above the mean? **ii** below the mean?

**b** What proportion of students scored a mark between 57 and 73?

**c** What mark is two standard deviations:
  **i** above the mean? **ii** below the mean?

**d** What proportion of students scored a mark between 49 and 81?

**e** What mark is three standard deviations:
  **i** above the mean? **ii** below the mean?

**f** What proportion of students scored a mark between 41 and 89?

| | Solve | Think | Apply |
|---|---|---|---|
| **a i** | $65 + 8 = 73$ | Add the standard deviation (8) to the mean (65). | One standard deviation above the mean is a $z$-score of $+1$. One standard deviation below the mean is a $z$-score of $-1$. Similarly for $z$-scores of $\pm 2$, $\pm 3$. |
| **ii** | $65 - 8 = 57$ | Subtract the standard deviation (8) from the mean (65). | |
| **b** | Approximately 68% of the students scored a mark between 57 and 73. | It is known that 68% of scores are between $z$-scores of $\pm 1$. | |
| **c i** | $65 + 2 \times 8 = 81$ | Add and subtract 2 times the standard deviation from the mean. | |
| **ii** | $65 - 2 \times 8 = 49$ | | |
| **d** | Approximately 95% of the students scored a mark between 49 and 81. | It is known that 95% of scores are between $z$-scores of $\pm 2$. | |
| **e i** | $65 + 3 \times 8 = 89$ | Add and subtract 3 times the standard deviation from the mean. | |
| **ii** | $65 - 3 \times 8 = 41$ | | |
| **f** | Approximately 99.7% of the students scored a mark between 41 and 89. | It is known that 99.7% of scores are between $z$-scores of $\pm 3$. | |

**6** The marks on a test are normally distributed with a mean of 62 and a standard deviation of 5.

**a** What mark is one standard deviation:
  **i** above the mean? **ii** below the mean?

**b** Approximately what proportion of students scored a mark within 1 standard deviation of the mean?

**c** What mark is two standard deviations:
  **i** above the mean? **ii** below the mean?

**d** Approximately what proportion of students scored a mark within 2 standard deviations of the mean?

**e** What mark is three standard deviations:
  **i** above the mean? **ii** below the mean?

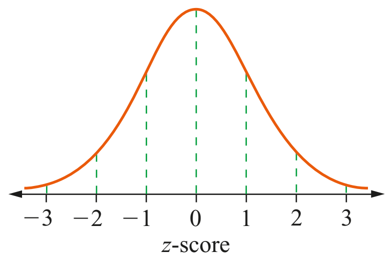
**f** Approximately what proportion of students scored a mark within 3 standard deviations of the mean?

**7** **a** Draw a normal distribution curve similar to the one shown.

**b** Use your answers from question **6** to write the corresponding mark underneath each $z$-score along the horizontal axis of your diagram.

**c** Where is the value of the mean written? Show this on your diagram if you haven't already done so.

**d** What is the median mark and the modal mark on this test?




**8** The life of a certain brand of torch batteries is normally distributed with a mean of 56 hours and standard deviation of 2.6 hours.

**a** Complete the following table.

| **Life (hours)** | 48.2 | 50.8 | 53.4 | 56 | 58.6 | 61.2 | 63.8 |
|---|---|---|---|---|---|---|---|
| **$z$-score** | | | | 0 | | | |

**b** Draw a normal distribution curve similar to the one shown in question **7a** and show both the number of hours and the corresponding $z$-scores along the horizontal axis.

**c** Approximately what proportion of batteries have a life between:

**i** 48.2 hours and 63.8 hours?

**ii** 50.8 hours and 61.2 hours?

**iii** 53.4 hours and 58.6 hours?

**d** What percentage of batteries have a life that is:

**i** more than 56 hours?

**ii** less than 53.4 hours?

**iii** between 56 hours and 61.2 hours?

**9** The heights of 1000 Year 12 males are approximately normally distributed with a mean of 168 cm and standard deviation of 9 cm.

**a** Draw a normal distribution curve similar to the one shown in question **7a**.

**b** Use the mean and standard deviation to write the corresponding height underneath each $z$-score along the horizontal axis of your diagram.

**c** What percentage of students have heights between:

**i** 159 cm and 177 cm?

**ii** 150 cm and 186 cm?

**iii** 141 cm and 195 cm?

**d** What percentage of students have heights:

**i** greater than 177 cm?

**ii** less than 150 cm?

**iii** greater than 159 cm?

**10** Packets of rice are filled and labelled as having a mass of 1 kg. The mass of these packets is normally distributed with a mean of 1.02 kg and a standard deviation of 0.01 kg.

**a** Complete the following table.

| **Mass (kg)** | 1.00 | 1.01 | 1.02 | 1.03 | 1.04 |
|---|---|---|---|---|---|
| ***z*-score** | | | 0 | | |

**b** What percentage of packets will have a mass less than 1.02 kg?

**c** What percentage of packets will have a mass between 1.01 kg and 1.03 kg?

**d** What percentage of packets will have a mass between 1.02 and 1.03 kg?

**e** What percentage of packets will have a mass between 1.00 kg and 1.04 kg?

**f** What percentage of packets will have a mass greater than 1.03 kg?

**g** What percentage of packets will have a mass less than the labelled mass?

**11** Tim scored 53 in a test for which the marks were normally distributed with a mean of 65 and a standard deviation of 12. What percentage of students scored higher marks than Tim?

**12** The ages of students in an athletic training squad are normally distributed. If 95% of the students have ages between 15.4 and 18.6 years, find the mean and standard deviation.

**13** The average life of a particular brand of car battery is 4 years and 4 months with a standard deviation of 8 months. The manufacturer guarantees to replace any battery that does not last 3 years. If the life of these batteries is normally distributed, approximately what percentage could the manufacturer expect to replace?

**14** The heights of a large group of female university students are found to be normally distributed with a mean of 160 cm and a standard deviation of 6 cm. Between what limits would you expect the heights of:

**a** the middle 68% of these students to lie?

**b** the middle 95% of these students to lie?

**c** almost all, 99.7%, of these students to lie?

**15** A machine is set to produce metal cylinders with a mean diameter of 5.00 cm and a standard deviation of 0.02 cm.

**a** If the diameters are normally distributed, within what interval will 99.7% of the diameters lie?

**b** A cylinder is chosen at random and its diameter measured to be 5.07 cm. Discuss what conclusions could be drawn.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**16** A machine fills packets of lollies. The mean number of lollies in each packet is 40 with a standard deviation of 1.5 lollies. The number of lollies is normally distributed. Rory claims that he bought a packet that had 48 lollies in it. Discuss Rory's claim.

**17** When a machine that produces nails is set correctly, the lengths of the nails are normally distributed with a mean of 4.000 cm and a standard deviation of 0.030 cm. To confirm the setting of the machine, three nails are randomly selected. In one sample the lengths are 3.950 cm, 3.985 cm and 4.132 cm. The setting of the machine needs to be checked when the lengths of two or more nails in a sample lie more than one standard deviation from the mean. Does the setting on the machine need to be checked? Justify your answer with suitable calculations.

**18** The number of peas in each pod of a large sample of a particular species of pea are normally distributed and shown in the table below.

| Number of peas/pod | Number of pods |
|---|---|
| 2 | 10 |
| 3 | 16 |
| 4 | 38 |
| 5 | 71 |
| 6 | 80 |
| 7 | 73 |
| 8 | 35 |
| 9 | 19 |
| 10 | 8 |

**a** Determine the mean of the distribution.

**b** Find the standard deviation.

**c** What percentage of the pods in this sample lie within:

**i** one standard deviation of the mean?

**ii** two standard deviations of the mean?

**iii** three standard deviations of the mean?

**d** What is the probability that the number of peas in a pod chosen at random lies within one standard deviation of the mean? (*Hint:* use your answer to part **c i** to help you.)

CHALLENGE

**19** A normal distribution has a standard deviation of 3. If 16% of the scores are at least 29, find the mean.

**20** A normal distribution has a mean of 95. If 2.5% of the scores are below 73, find the standard deviation.

**21** A dataset is normally distributed and has a mean of 40 and a standard deviation of 9. One of the scores is 81. Is this score an outlier? Explain.

# 9E Probability and normal distributions

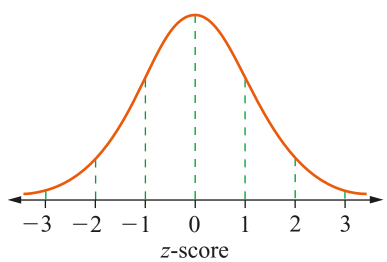
These resources are available on your obook assess:

- **Video tutorial 9E:** Watch and listen to an explanation of Example 9E–2
- **assess quiz 9E:** Test your skills with an auto-correcting multiple-choice quiz

OI
GI

In the last section we worked with the empirical rule that:

- 68% of data will have $z$-scores between –1 and 1
- 95% of data will have $z$-scores between –2 and 2
- 99.7% of data will have $z$-scores between –3 and 3.




We can use these percentages to state the probability a randomly chosen score will lie within one, two or three standard deviations of the mean. For example, the probability that a randomly chosen score lies:

- within one standard deviation of the mean is 0.68: $P(-1 < z < 1) = 0.68$
- within two standard deviations of the mean is 0.95: $P(-2 < z < 2) = 0.95$
- within three standard deviations of the mean is 0.997: $P(-3 < z < 3) = 0.997$.

As the region under the curve represents 100% of the scores or a probability of 1, $P(z < 0) = 0.5$ and $P(z > 0) = 0.5$.

It is important to note that the area under the curve which represents the probability at a particular $z$-score, say $c$, is considered to be zero (zero width for the area), so $P(z = c) = 0$. This means that $P(z = 0) = 0$ so $P(z < 0) = P(z \leq 0) = 0.5$ and $P(z > 0) = P(z \geq 0) = 0.5$. We will use $<$ and $>$ rather than $\leq$ and $\geq$ for the probability calculations in this chapter.

Also, as $P(z < c) = P(z \leq c)$ and $P(z > c) = P(z \geq c)$, we can write $P(z < c) + P(z > c) = 1$ or $P(z > c) = 1 - P(z < c)$ where $c$ is a particular $z$-score.

For example, as $P(z < 2) + P(z > 2) = 1$, we can find $P(z > 2)$ by using the relationship $P(z > 2) = 1 - P(z < 2)$.

*Remember:* we can make judgements more easily if raw scores ($x$) are converted to $z$-scores. The formula to use is $z = \frac{x - \bar{x}}{s}$ where $\bar{x}$ is the mean and $s$ is the standard deviation.

In summary:

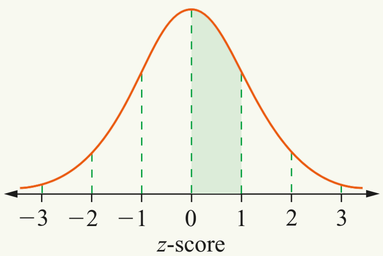
!

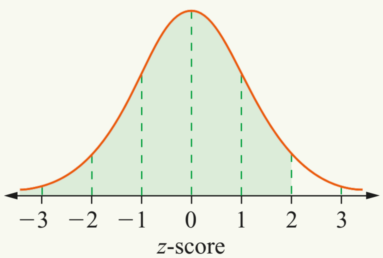
- $P(z < 0) = 0.5$ and $P(z > 0) = 0.5$
- $P(-1 < z < 1) = 0.68$
- $P(-2 < z < 2) = 0.95$
- $P(-3 < z < 3) = 0.997$
- $P(z < c) + P(z > c) = 1$ or $P(z > c) = 1 - P(z < c)$ where $c$ is a particular $z$-score

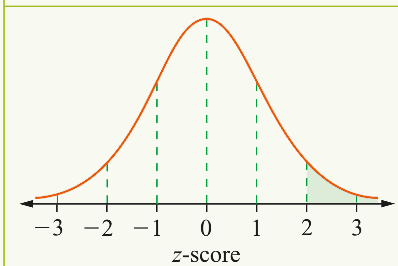
## EXAMPLE 9E-1 Finding the probability of a randomly chosen score using whole number $z$-scores

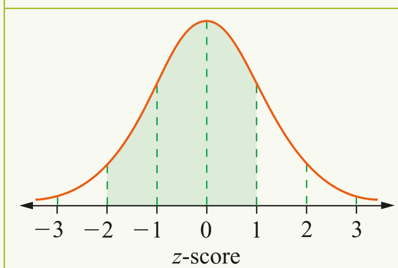
For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

**a** between 0 and 1 **b** less than 2 **c** greater than 2 **d** between –2 and 1.

<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>

$P(0 < z < 1) = 0.34$</td><td>Draw a quick sketch to represent the probability you want to find.<br>By symmetry, $P(0 < z < 1)$ will be half of $P(–1 < z < 1)$; i.e.<br>$\frac{1}{2} \times 0.68 = 0.34$.</td><td rowspan="4">Draw a quick sketch of a standard normal distribution curve and shade the appropriate region under the curve. This also helps you to check whether the answer you obtain seems reasonable.<br>Use symmetry and the probabilities associated with the empirical rule:<br>$P(–1 < z < 1) = 0.68$<br>$P(–2 < z < 2) = 0.95$<br>$P(–3 < z < 3) = 0.997$<br>Also, use the fact that 50% of the scores lie below the mean and 50% of the scores lie above the mean; i.e.<br>$P(z < 0) = 0.5$ and<br>$P(z > 0) = 0.5$.<br>The fact that 100% of the scores are represented by the region under the curve means that<br>$P(z > c) = 1 – P(z < c)$,<br>where $c$ is a particular $z$-score.</td></tr>
<tr><td>b</td><td>

$P(z < 2)$<br>$= P(z < 0) + P(0 < z < 2)$<br>$= 0.5 + 0.475$<br>$= 0.975$</td><td>Draw a quick sketch to represent the probability you want to find.<br>$P(z < 0) = 0.5$ as 50% of the scores lie below the mean ($z = 0$).<br>By symmetry, $P(0 < z < 2)$ will be half of $P(–2 < z < 2)$; i.e.<br>$\frac{1}{2} \times 0.95 = 0.475$.</td></tr>
<tr><td>c</td><td>

$P(z > 2)$<br>$= 1 – P(z < 2)$<br>$= 1 – 0.975$<br>$= 0.025$</td><td>Draw a quick sketch to represent the probability you want to find.<br>Rearranging<br>$P(z < 2) + P(z > 2) = 1$<br>gives<br>$P(z > 2) = 1 – P(z < 2)$.<br>Use the value of $P(z < 2)$ from part b and subtract from 1.</td></tr>
<tr><td>d</td><td>

$P(–2 < z < 1)$<br>$= P(–2 < z < 0) + P(0 < z < 1)$<br>$= 0.475 + 0.34$<br>$= 0.815$</td><td>Draw a quick sketch to represent the probability you want to find.<br>By symmetry, $P(–2 < z < 0)$ will be half of $P(–2 < z < 2)$; i.e.<br>$\frac{1}{2} \times 0.95 = 0.475$.<br>Similarly, $P(0 < z < 1)$ will be half of $P(–1 < z < 1)$; i.e.<br>$\frac{1}{2} \times 0.68 = 0.34$.<br>Add the two probabilities.</td></tr>
</table>

# EXERCISE 9E Probability and normal distributions

**1** For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

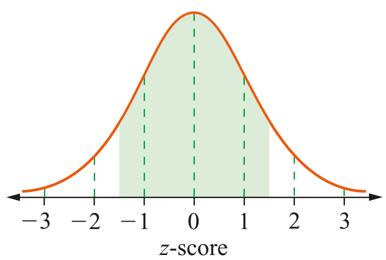
**a** between 0 and 2
**b** less than 1
**c** between –1 and 2
**d** between –3 and 0
**e** less than –2
**f** more than 1
**g** between 2 and 3
**h** more than –2
**i** between –3 and –1.

**2** For a dataset that is normally distributed with a mean of 50 and a standard deviation of 6, find the probability that a randomly chosen score is:

**a** between 44 and 56
**b** between 50 and 62
**c** less than 56
**d** between 38 and 56
**e** less than 38
**f** more than 44
**g** between 56 and 62
**h** between 32 and 44
**i** more than 68.

(*Hint:* to make it easier, first find the equivalent $z$-score for each raw score.)

**3** For a dataset that is normally distributed, what is the probability that a randomly chosen score has a $z$-score between –1.5 and 1.5? Use the diagram to help you estimate a value for the probability. Explain how you obtained your answer.

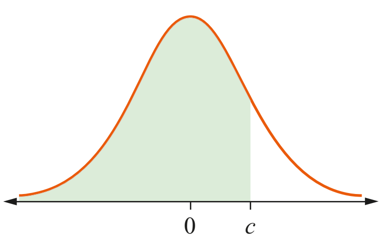



!

To find the probability of a $z$-score between, less than or more than a whole number of standard deviations from the mean, we can use symmetry of the normal curve and the empirical rule.

However, for $z$-scores that are not a whole number of standard deviations from the mean, we need to use a table of values or technology to help us find the appropriate probability.

The standard normal distribution table (or cumulative normal distribution table) shown on page 401 represents the probability $P(z < c)$ for positive values of $c$. This means that we can read the probability for any $c$ value from 0.00 to 3.99 from the table.




In the table, the far left column titled $z$ represents the ones digit and the first digit after the decimal point (the tenths digit) of $c$ values and the top row represents the second digit after the decimal point (the hundredths digit) of $c$. The body of the table shows the matching probability.

For example, if $c$ is 0.82, go to the row for 0.8 and read across to the column labelled .02 in the top row. The intersection of the row for 0.8 and the column for .02 shows .79389. So $P(z < 0.82) \approx 0.79$.

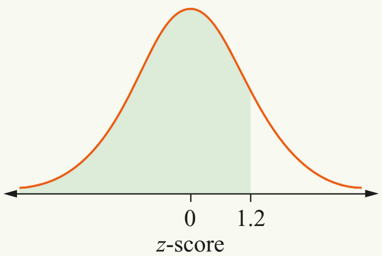
**Standard normal distribution: Table values represent the area under the normal curve to the left of the $z$-score**

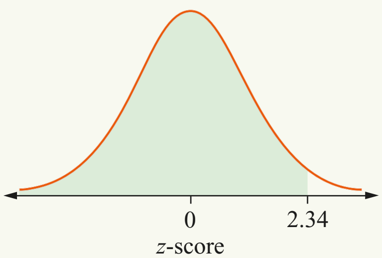
| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| 0.1 | .53983 | .54380 | .54776 | .55172 | .55567 | .55966 | .56360 | .56749 | .57142 | .57535 |
| 0.2 | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| 0.3 | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| 0.4 | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| 0.5 | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| 0.6 | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| 0.7 | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| 0.8 | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| 0.9 | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| 1.0 | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| 1.1 | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| 1.2 | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| 1.3 | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91308 | .91466 | .91621 | .91774 |
| 1.4 | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| 1.5 | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| 1.6 | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| 1.7 | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| 1.8 | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| 1.9 | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| 2.0 | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| 2.1 | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| 2.2 | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| 2.3 | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| 2.4 | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| 2.5 | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| 2.6 | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| 2.7 | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| 2.8 | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| 2.9 | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| 3.0 | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| 3.1 | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| 3.2 | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| 3.3 | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| 3.4 | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| 3.5 | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| 3.6 | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| 3.7 | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| 3.8 | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| 3.9 | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |

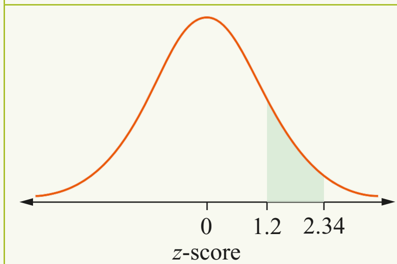
## EXAMPLE 9E–2 Finding the probability of a randomly chosen score using $z$-scores and a standard normal distribution table (positive $c$)

For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

**a** less than 1.2 **b** more than 2.34 **c** between 1.2 and 2.34.

<table>
<tr><th></th><th>Solve</th><th>Think</th><th>Apply</th></tr>
<tr><td>a</td><td>

$P(z < 1.2)$<br>$= 0.884\ 93$<br>$\approx 0.88$</td><td>Draw a quick sketch to represent the probability you want to find. Use the table of values to find the probability when $c = 1.20$ by reading across the row for 1.2 to the column labelled .00 in the top row. The intersection of the row for 1.2 and the column for .00 shows the value .884 93.</td><td rowspan="3">First draw a quick sketch of a standard normal distribution curve and shade the appropriate region under the curve. This also helps you to check whether the answer you obtain seems reasonable.<br>Use the standard normal distribution table to find the appropriate probability. Remember that this table only applies to finding $P(z < c)$ where $c$ is positive.<br>Also, use the fact that 100% of the scores are represented by the region below the normal curve. That is,<br>$P(z < c) + P(z > c) = 1$,<br>so<br>$P(z > c) = 1 - P(z < c)$.</td></tr>
<tr><td>b</td><td>

$P(z > 2.34)$<br>$= 1 - P(z < 2.34)$<br>$= 1 - 0.990\ 36$<br>$\approx 0.01$</td><td>Draw a quick sketch to represent the probability you want to find. $P(z > 2.34)$ cannot be found directly from the table. However, $P(z > 2.34) = 1 - P(z < 2.34)$, so we can find $P(z < 2.34)$ and subtract this from 1. The intersection of the row for 2.3 and the column for .04 shows the value .990 36.</td></tr>
<tr><td>c</td><td>

$P(1.2 < z < 2.34)$<br>$= P(z < 2.34) - P(z < 1.2)$<br>$= 0.990\ 36 - 0.884\ 93$<br>$= 0.105\ 43$<br>$\approx 0.11$</td><td>Draw a quick sketch to represent the probability you want to find. To obtain the required shaded region we consider the shaded region for $P(z < 2.34)$ and remove the shaded region for $P(z < 1.2)$. This leaves the shaded region between 1.2 and 2.34.</td></tr>
</table>

**4** For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

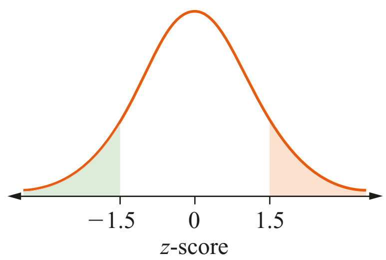
**a** less than 0.67 **b** less than 1.83 **c** less than 2.15

**d** more than 1.23 **e** more than 0.99 **f** more than 2.81

**g** between 0.67 and 1.83 **h** between 0.86 and 1.77 **i** between 1.05 and 1.92.

**5** A dataset is normally distributed with a mean of 75 and a standard deviation of 12. Using standardised scores and the standard normal distribution table, find the probability that a randomly chosen score is:

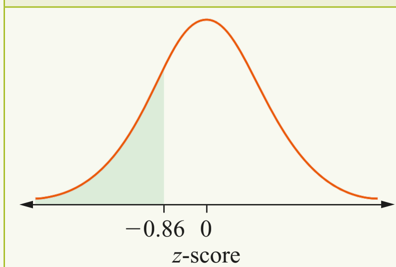
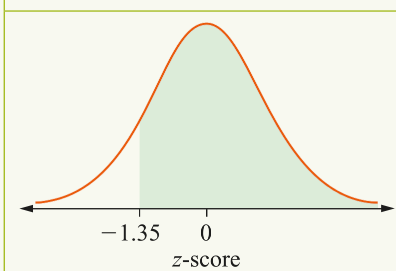
**a** less than 84 **b** less than 79 **c** less than 100
**d** more than 76 **e** more than 90 **f** more than 87
**g** between 79 and 84 **h** between 80 and 100 **i** between 87 and 106.

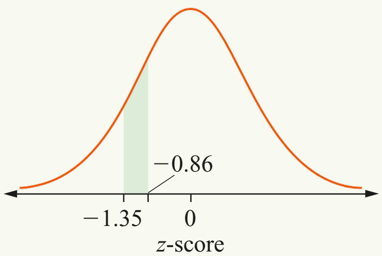
> **!**
>
> The standard normal distribution table is used to find $P(z < c)$ when $c$ is positive.
>
> For negative values of $c$, we use the symmetry of the standard normal distribution curve to find an equivalent probability expression so that the table of values can be used.
>
> For example, the green shaded region representing $P(z < -1.5)$ has the same area as the orange shaded region representing $P(z > 1.5)$, so $P(z < -1.5) = P(z > 1.5)$.
>
> From earlier work, we know that $P(z > 1.5) = 1 - P(z < 1.5)$ so we can continue with this calculation using the standard normal distribution table.
>
> 
> 

## EXAMPLE 9E–3 Finding the probability of a randomly chosen score using z-scores and a standard normal distribution table (negative c)

For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

**a** less than –0.86 **b** more than –1.35 **c** between –1.35 and –0.86.

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | <br>$P(z < -0.86)$<br>$= P(z > 0.86)$<br>$= 1 - P(z < 0.86)$<br>$= 1 - 0.805\ 11$<br>$\approx 0.19$ | Draw a quick sketch to represent the probability you want to find. Use symmetry to write an equivalent expression for $P(z < -0.86)$ so that the expression contains a positive $z$-score. In this case, write another equivalent expression so that the expression contains $z <$ positive value. Use the standard normal distribution table to find $P(z < 0.86)$ and complete the calculation. | First draw a quick sketch of a standard normal distribution curve and shade the appropriate region under the curve. This also helps you to check whether the answer you obtain seems reasonable. Remember that the standard normal distribution table only applies to finding $P(z < c)$ where $c$ is positive. Use symmetry of the normal distribution curve to write equivalent probability expressions. |
| **b** | <br>$P(z > -1.35)$<br>$= P(z < 1.35)$<br>$= 0.911\ 49$<br>$\approx 0.91$ | Draw a quick sketch to represent the probability you want to find. Use symmetry to write an equivalent expression for $P(z > -1.35)$ so that the expression contains a positive $z$-score. As we obtain an expression that contains $z <$ positive score, we can use the standard normal distribution table to find $P(z < 1.35)$. | |

| | Solve | Think | Apply |
|---|---|---|---|
| **c** | <br>$P(-1.35 < z < -0.86)$<br>$= P(0.86 < z < 1.35)$<br>$= P(z < 1.35) - P(z < 0.86)$<br>$= 0.911\,49 - 0.805\,11$<br>$\approx 0.11$ | Draw a quick sketch to represent the probability you want to find. Use symmetry to write an equivalent expression for $P(-1.35 < z < -0.86)$ so that the expression contains positive $z$-scores. To obtain the shaded region for $P(0.86 < z < 1.35)$, we consider the shaded region for $P(z < 1.35)$ and remove the shaded region for $P(z < 0.86)$. This leaves the shaded region between 0.86 and 1.35. | |

**6** For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

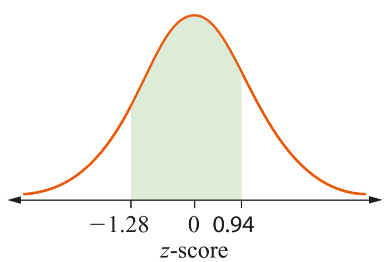
**a** less than –0.78 **b** less than –1.42 **c** less than –2.21
**d** more than –0.33 **e** more than –1.19 **f** more than –2.07
**g** between –1.42 and –0.78 **h** between –0.95 and –0.52 **i** between –1.96 and –1.24.

**7** A dataset is normally distributed with a mean of 98 and a standard deviation of 9. Using standardised scores and the standard normal distribution table, find the probability that a randomly chosen score is:

**a** less than 90 **b** less than 82 **c** less than 78
**d** more than 95 **e** more than 76 **f** more than 84
**g** between 79 and 96 **h** between 74 and 89 **i** between 78 and 92.

**8** Complete the following working to calculate $P(-1.28 < z < 0.94)$.

$P(-1.28 < z < 0.94)$
$= P(z < 0.94) - P(z < \_\_\_\_)$
$= P(z < 0.94) - P(z > \_\_\_\_)$
$= P(z < 0.94) - [1 - P(z < \_\_\_\_)]$
$= \_\_\_\_ - (1 - \_\_\_\_)$
$\approx \_\_\_\_$




**9** For a dataset that is normally distributed, find the probability that a randomly chosen score has a $z$-score:

**a** between –0.67 and 1.83
**b** between –1.41 and 2.15
**c** between –0.82 and 1.36.

**10** A dataset is normally distributed with a mean of 100 and a standard deviation of 15. Using standardised scores and the standard normal distribution table, find the probability that a randomly chosen score is:

**a** between 95 and 110
**b** between 87 and 102
**c** between 72 and 120.

PROBLEM SOLVING, REASONING AND JUSTIFICATION

**11** Adriana scored 70 in a test for which the marks were normally distributed with a mean of 64 and a standard deviation of 12. What is the probability that a randomly chosen student scored a lower mark than Adriana?

**12** A machine produces cylindrical pipes. The mean of the diameters of the pipes is 7.00 cm and the standard deviation is 0.03 cm. Assuming a normal distribution, what is the probability that a randomly chosen pipe has a diameter less than 6.95 cm?

**13** The lengths jumped by Year 8 long-jump competitors follow a normal distribution with a mean of 432 cm and a standard deviation of 14 cm.

**a** Find the probability that a competitor jumps:

- **i** more than 445 cm
- **ii** at least 410 cm
- **iii** between 420 cm and 458 cm.

**b** A competitor jumps 486 cm. Is this unusual? Explain your answer.

**14** The heights of 1000 Year 12 males are found to be approximately normally distributed with a mean of 168 cm and standard deviation of 9 cm.

**a** What is the probability that a randomly chosen Year 12 male has a height greater than 175 cm?
**b** What is the probability that a randomly chosen Year 12 male has a height less than 165 cm?
**c** What is the probability that a randomly chosen Year 12 male has a height between 170 cm and 175 cm?
**d** How many Year 12 males in the sample have a height between 170 cm and 175 cm?
**e** Brad is a Year 12 male whose height puts him just inside the top 10%. Find Brad's height.

**15** The weights of 10 000 newborn babies in NSW are normally distributed. These weights have a mean of 3.1 kg and a standard deviation of 0.35 kg.

**a** What is the probability that a randomly chosen baby has a weight between 2.65 kg and 3.75 kg?
**b** How many of the 10 000 newborn babies have a weight between 2.65 kg and 3.75 kg?

**16** Packets of sugar are filled and labelled as having a weight of 1 kg. The weights of these packets are normally distributed with a mean of 1.01 kg and a standard deviation of 0.01 kg. A packet of sugar is rejected if its weight is lower than 985 g.

**a** What is the probability that a randomly chosen packet of sugar is rejected?
**b** From a batch of 1000 packets of sugar, how many packets would you expect to be rejected?

CHALLENGE

**17** Speedometers of cars may not be accurate. Suppose that when the speedometer of a randomly chosen car registers 60 km/h, the actual speed follows a normal distribution with mean 61 km/h and standard deviation of 2 km/h.

**a** What is the probability that the randomly chosen car is travelling at more than 65 km/h when the speedometer registers 60 km/h?
**b** What proportion of cars exceed 60 km/h when their speedometers register 60 km/h?

# CHAPTER 9 REVIEW THE NORMAL DISTRIBUTION

Having completed this chapter you should be able to:

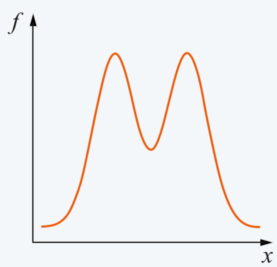
- ✔ draw frequency distribution graphs
- ✔ make judgements about the nature of data based on the shape of the display
- ✔ identify normal distributions
- ✔ determine $z$-scores
- ✔ use $z$-scores to compare different data sets
- ✔ identify and use the properties of normally distributed data
- ✔ understand the difference between a normal distribution and a standard normal distribution
- ✔ find the probability that a score lies within one, two or three standard deviations of the mean
- ✔ use $z$-scores and the standard normal distribution table to find the probability of an event.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

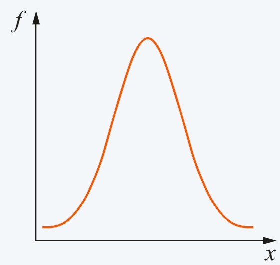
# REVIEW MULTIPLE-CHOICE QUESTIONS

9B **1** Which of the graphs below could represent a normal distribution?

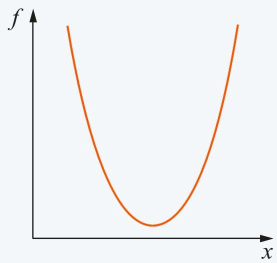
**A**



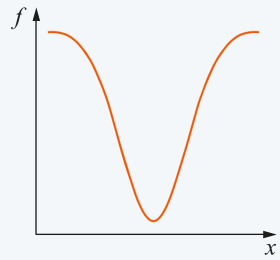

**B**




**C**




**D**



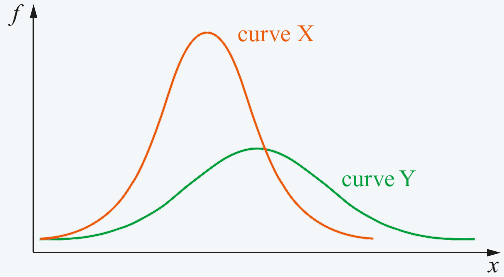

9B **2** Compare the curve X and curve Y. Which statement is correct?

**A** X has a larger mean and larger standard deviation than Y.

**B** X has a larger mean and smaller standard deviation than Y.

**C** X has a smaller mean and larger standard deviation than Y.

**D** X has a smaller mean and smaller standard deviation than Y.



9B 3 Which of the following distributions could be normal?

A

| $x$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|
| $f$ | 2 | 5 | 10 | 18 | 12 | 6 | 1 |

B

| $x$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $f$ | 25 | 18 | 12 | 9 | 11 | 19 | 24 |

C

| $x$ | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|---|---|---|---|---|---|---|---|
| $f$ | 19 | 28 | 16 | 8 | 4 | 2 | 1 |

D

| $x$ | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|---|---|---|---|---|---|---|---|
| $f$ | 5 | 7 | 10 | 13 | 15 | 26 | 38 |

9C 4 What is the $z$-score for a mark of 56 in a test with a mean of 60 and standard deviation of 8?

**A** 4 **B** $-4$ **C** 0.5 **D** $-0.5$

9C 5 In an examination the mean was 65 and the standard deviation 15. What is the raw examination mark corresponding to a standardised score of 1.4?

**A** 86 **B** 79 **C** 80 **D** 66.4

9D 6 The marks on a test are normally distributed with a mean of 55 and a standard deviation of 9. What is the approximate proportion of scores that lie between 37 and 73?

**A** 68% **B** 86% **C** 95% **D** 99.7%

9D 7 The number of matches in a box is approximately normally distributed with a mean of 50 and a standard deviation of 1. How many matches will be in approximately 95% of boxes?

**A** between 49 and 51 matches **B** between 48 and 52 matches
**C** between 47 and 53 matches **D** between 46 and 54 matches

9D 8 For a normal distribution, what is the percentage of scores between $z = -2$ and $z = -1$?

**A** 27 **B** 34 **C** 13.5 **D** 2.35

9E 9 The results of a class test are found to be normally distributed with a mean of 68 cm and a standard deviation of 10 cm. What is the probability that a randomly chosen student obtained a result between 58 and 78?

**A** 0.34 **B** 0.68 **C** 0.95 **D** 0.997

9E 10 The heights of Year 12 girls are normally distributed with a mean of 165 cm and a standard deviation of 5.5 cm. What is the probability, correct to two decimal places, that a randomly chosen Year 12 girl is taller than 170 cm?

**A** 0.09 **B** 0.18 **C** 0.82 **D** 0.91

## REVIEW SET 1

1 Draw a frequency histogram, polygon and curve for the height data given in the table.

| **Height (cm)** | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|
| **Frequency** | 4 | 9 | 15 | 21 | 16 | 10 | 3 |

2 Draw three normal curves with different means but the same standard deviation.

3 For a normal distribution, which of the following is true?

**A** the median $<$ the mean $<$ the mode **B** the median $=$ the mean $=$ the mode
**C** the median $>$ the mean $>$ the mode **D** none of the above

**4** **a** Determine whether the frequency distribution given below is normal by first drawing a frequency graph.

**b** Then, if necessary, find the mean, mode and median to confirm your answer.

| $x$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $f$ | 2 | 6 | 12 | 19 | 25 | 18 | 11 | 6 | 3 |

**5** In an Art test the mean was 68 and the standard deviation was 8. Calculate the standardised score for a mark of:

**a** 60 **b** 76 **c** 48 **d** 80

**6** Skye scored 48 in a Geography test for which the mean was 58 and the standard deviation 4. She scored 49 in a History test, for which the mean was 61 and the standard deviation 8.

**a** In which subject did she perform better, given that the classes are of equal ability?

**b** What mark would Skye have to have scored in Geography for her performance to be equivalent to that in History?

**7** The following table shows Ben's marks in the term tests. In which test did he perform the best?

| Test | Mark | Mean | Standard deviation |
|---|---|---|---|
| Term 1 | 65 | 55 | 8 |
| Term 2 | 69 | 60 | 6 |
| Term 3 | 66 | 58 | 10 |
| Term 4 | 65 | 62 | 4 |

**8** The marks on a test are normally distributed with a mean of 63 and a standard deviation of 5.

**a** What mark is one standard deviation:

**i** above the mean? **ii** below the mean?

**b** What percentage of scores lie between 58 and 68?

**c** What percentage of scores lie between 53 and 73?

**d** What percentage of scores lie between 48 and 78?

**9** For a normal distribution, what percentage of scores have a $z$-score:

**a** between $z = 0$ and $z = 1$? **b** between $z = -1$ and $z = 0$?

**c** greater than $z = 0$? **d** greater than $z = 1$?

**10** The speed limit outside a school is 40 km/h. Students measured the speed of passing vehicles over a period of time. They found the set of data to be normally distributed with a mean speed of 36 km/h and a standard deviation of 2 km/h.

**a** What percentage of the vehicles passed the school at a speed greater than 40 km/h?

**b** What is the probability that a randomly chosen vehicle passed the school at a speed between 34 km/h and 38 km/h?

**c** What is the probability that a randomly chosen vehicle passed the school at a speed between 33 km/h and 40 km/h?

# REVIEW SET 2

**1** Draw two normal curves with the same mean but different standard deviations.

**2** In an examination the mean mark was 70 and the standard deviation was 10. Write the examination mark corresponding to a $z$-score of:

**a** 2 **b** $-1$ **c** 1.3 **d** $-2.6$

**3** Matthew scored 62 in a Science test for which the mean was 50 and the standard deviation 12. He scored 58 in an English test for which the mean was 51 and the standard deviation 8.

**a** In which subject did he perform better, given that the classes are of equal ability?

**b** What mark would Matthew have to have scored in English for his performance to be equivalent to that in Science?

**4** Rebecca scored 72 in Mathematics and 69 in Science. Serena scored 66 in Mathematics and 75 in Science. The mean in Mathematics was 60 and the standard deviation was 8. For Science the mean was 65 and the standard deviation was 10. By completing the table below, determine who had the better overall performance.

| | Mathematics | | Science | | Total | |
|---|---|---|---|---|---|---|
| | Mark | $z$-score | Mark | $z$-score | Mark | $z$-score |
| **Rebecca** | 72 | | 69 | | | |
| **Serena** | 66 | | 75 | | | |

**5** Toothpicks are packaged in boxes with a mean of 98 and standard deviation of 4 toothpicks. What proportion of boxes will contain between:

**a** 94 and 102 toothpicks?

**b** 90 and 106 toothpicks?

**c** 86 and 110 toothpicks?

**6** A machine produces bolts with a mean length of 25.0 mm and standard deviation 0.6 mm. Any bolts shorter than 23.2 m or longer than 26.8 mm are rejected. What percentage of bolts made by this machine will be rejected?

**7** The weights of a large sample of 18-year-old males is normally distributed with a mean of 74 kg and standard deviation of 6.5 kg. Between what limits would you expect the weights of:

**a** the middle 68% of these males to lie?

**b** the middle 95% of these males to lie?

**c** almost all, 99.7%, of these males to lie?

**8** For a normal distribution, state the percentage of scores that have a $z$-score:

**a** between $z = -2$ and $z = 2$ **b** greater than $z = 2$ **c** less than $z = 2$

**d** less than $z = -2$ **e** greater than $z = -2$.

**9** A dataset is normally distributed with a mean of 60 and a standard deviation of 10. Using standardised scores and the standard normal distribution table, find the probability that a randomly chosen score is:

**a** less than 67 **b** more than 42 **c** between 65 and 75.

**10** Kyle scored 78 in a test for which the marks were normally distributed with a mean of 65 and a standard deviation of 11.

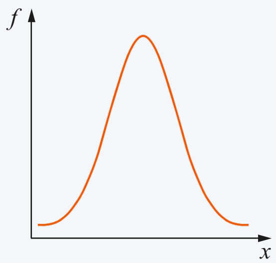
**a** What is the probability that a randomly chosen student scored a higher mark than Kyle?

**b** What is the probability that a randomly chosen student scored at least 65?

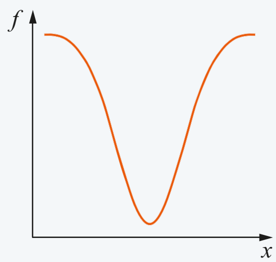
## REVIEW SET 3

**1** Which of the following is a normal curve?

**A**



**B**



**C**

**2** **a** Determine whether the frequency distribution given below is normal by first drawing a frequency graph.
**b** Then, if necessary, find the mean, mode and median to confirm your answer.

| $x$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|
| $f$ | 21 | 16 | 8 | 5 | 9 | 15 | 19 |

**3** In a Commerce test the mean was 63 and the standard deviation was 12. Calculate the standardised scores for test marks of:
**a** 81 **b** 39 **c** 66 **d** 58

**4** The results for an examination had a mean score of 55 and a standard deviation of 10. Write the raw examination mark corresponding to a standardised score of:
**a** 0.6 **b** 1.3 **c** −0.9 **d** −2.1

**5** The following are the scores in a Language test:
60 61 59 50 65 77 63 58 36 88 66 74 61 43 58
**a** Calculate the mean and standard deviation of these marks.
**b** How many scores lie within one standard deviation of the mean?
**c** How many scores are more than two standard deviations above the mean?

**6** Francis scored 52 in the half-yearly English test for which the mean was 68 and the standard deviation 11. In the yearly examination for which the mean was 60 and the standard deviation 6, she scored 56. Determine whether or not Francis improved her performance in English over the year. Justify your answer.

**7** The life of a brand of electric light globe is normally distributed with a mean of 1400 hours and a standard deviation of 120 hours. Within what limits would you expect:
**a** the middle 68% of globes to last?
**b** the middle 95% of globes to last?
**c** nearly all, 99.7%, of globes to last?

**8** **a** Packets of sugar are labelled as having a mass of 500 g. The mass of these packets is normally distributed with a mean of 510 g and standard deviation of 10 g. Complete the following table.

| $z$**-score** | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| **Mass (g)** | | | | 510 | | | |

**b** What percentage of packets have a mass that lies between:
**i** 500 g and 520 g? **ii** 490 g and 530 g? **iii** 480 g and 540 g?
**c** A packet is chosen at random and its mass is 450 g. What conclusion could be drawn?

**9** A dataset is normally distributed with a mean of 46 and a standard deviation of 7. Using standardised scores and the standard normal distribution table, find the probability that a randomly chosen score is:
**a** more than 50 **b** less than 38 **c** between 45 and 55.

**10** A machine produces metal screws. The mean of the lengths of the screws is 4.4 cm and the standard deviation is 0.1 cm. Assuming a normal distribution, what is the probability that a randomly chosen metal screw has a length less than 4.25 cm?

# REVIEW SET 4

**1** In a test the mean was 64 and the standard deviation was 6. Calculate the standardised score for a mark of:

**a** 60 **b** 76 **c** 48 **d** 80

**2** Melanie scored 67 in a Science test for which the mean was 56 and the standard deviation 11. She scored 60 in an English test for which the mean was 54 and the standard deviation 8.

**a** In which subject did she perform better, given that the classes are of equal ability?

**b** What mark would Melanie have to have scored in English for her performance to be equivalent to that in Science?

**3** The mean score for an examination was 50 and the standard deviation was 15. Write the raw examination mark corresponding to a standardised score of:

**a** 0.6 **b** 1.4 **c** −0.8 **d** −2.2

**4** The marks scored by three students in tests in English, Mathematics, Science and Art are shown in the table below. List the students in order of merit on the basis of their total performance in all subjects.

| Student | English | Mathematics | Science | Art |
|---|---|---|---|---|
| **Michael** | 74 | 71 | 84 | 52 |
| **Janelle** | 67 | 70 | 75 | 76 |
| **Melissa** | 53 | 85 | 83 | 64 |
| **Mean** | 65 | 62 | 60 | 65 |
| **Standard deviation** | 12 | 15 | 14 | 12 |

**5** Jelly babies are produced in monster packs with a mean of 50 and a standard deviation of 2. What proportion of packets will contain between:

**a** 48 and 52 jelly babies?

**b** 46 and 54 jelly babies?

**c** 44 and 56 jelly babies?

**6** The life of a brand of torch battery is normally distributed with a mean of 42 hours and a standard deviation of 3.2 hours. Within what limits would you expect:

**a** the middle 68% of batteries to last?

**b** the middle 95% of batteries to last?

**c** nearly all, 99.7%, of batteries to last?

**7** For a normal distribution, state the percentage of scores that have a $z$-score between:

**a** $z = 0$ and $z = 1$ **b** $z = 0$ and $z = 2$ **c** $z = 1$ and $z = 2$ **d** $z = 2$ and $z = 3$.

**8** The average life of a particular brand of car battery is 4 years and 6 months with a standard deviation of 9 months. The manufacturer guarantees to replace any battery that does not last 3 years. If the life of these batteries is normally distributed, approximately what percentage could the manufacturer expect to replace?

**9** Packets of rice are filled and labelled as having a mass of 1 kg. The mass of these packets is normally distributed with a mean of 1.05 kg and a standard deviation of 0.02 kg.

**a** Complete the following table.

| **Mass (kg)** | 0.99 | 1.01 | 1.03 | 1.05 | 1.07 | 1.09 | 1.11 |
|---|---|---|---|---|---|---|---|
| ***z*-score** | | | | 0 | 1 | | |

**b** What percentage of packets will have a mass of less than 1.05 kg?

**c** What percentage of packets will have a mass between 1.03 kg and 1.07 kg?

**d** What percentage of packets will have a mass between 1.05 and 1.07 kg?

**e** What percentage of packets will have a mass between 1.01 kg and 1.09 kg?

**f** What percentage of packets will have a mass greater than 1.07 kg?

**g** What is the probability of a randomly chosen packet having a mass less than the labelled mass?

**10** The heights of 2000 Year 10 females are found to be approximately normally distributed with a mean of 162 cm and standard deviation of 8 cm.

**a** What is the probability that a randomly chosen Year 10 female has a height greater than 165 cm?

**b** What is the probability that a randomly chosen Year 10 female has a height less than 160 cm?

**c** What is the probability that a randomly chosen Year 10 female has a height between 160 cm and 165 cm?

**d** How many Year 10 females in the sample have a height between 156 cm and 176 cm?

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** The table shows the frequency distribution of data.

| **Score** | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|
| **Frequency** | 2 | 6 | 12 | 13 | 14 | 11 | 7 | 3 |

**i** Draw a frequency histogram and polygon. (2 marks)

**ii** Calculate the mean, mode and median. (2 marks)

**iii** Is the distribution normal? Explain your answer. (1 mark)

**b** Ally scored 54 in the half-yearly exam for which the mean was 70 and the standard deviation was 11. She scored 58 in the yearly exam for which the mean was 62 and standard deviation was 6. Did Ally improve? Justify your answer. (2 marks)

**c** A normal distribution has a mean of 20 and standard deviation of 3.

**i** What percentage of scores are less than 20? (1 mark)

**ii** What percentage of scores lie between 17 and 23? (1 mark)

**iii** What percentage of scores are greater than 23? (1 mark)

**d** The results of a test are normally distributed with a mean of 38 and standard deviation of 4.

**i** Use standardised scores and the standard normal distribution table to find the probability that a randomly chosen person obtains a result of at least 45. (1 mark)

**ii** If 500 people sit the test, how many would be expected to obtain a result between 35 and 45. (2 marks)

**e** Pancakes are made by a machine and their weights are normally distributed with a mean of 39 g and a standard deviation of 1.5 g. A pancake weighing 44 g is produced. Is the machine functioning properly? Explain your answer. (2 marks)

**TOTAL: 15 marks**

# 10

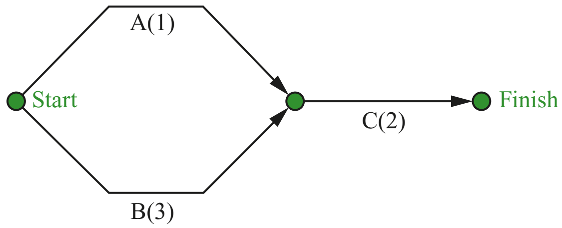
# Critical path analysis

The main mathematical ideas in this chapter are:

- understanding activity tables
- making activity charts from activity tables
- determining the earliest start and finish times of an activity using a forward scan
- determining the latest finish and start times of an activity using a backward scan
- identifying the critical path
- determining the maximum flow through a network
- finding the minimum cut of a network.

# ARE YOU READY?

Use the activity chart below to answer questions **1–5**.




10A **1** Which of the following is a path from the start vertex to the finish vertex (only edges are specified)?

**A** AB **B** BA **C** CA **D** AC

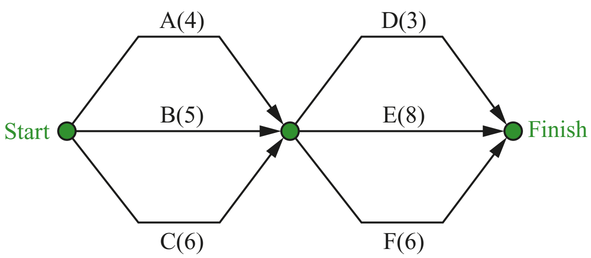
10A **2** What is the weight of the path BC?

**A** 2 **B** 3 **C** 4 **D** 5

10A **3** How many paths are there from start to finish?

**A** 2 **B** 3 **C** 4 **D** 5

Use the activity chart below to answer questions **4–6**.




10A **4** Which of the following is a path from the start vertex to the finish vertex (only edges are specified)?

**A** ACBE **B** BDEF
**C** EB **D** CD

10A **5** What is the weight of the path AD?

**A** 4 **B** 10 **C** 7 **D** 13

10A **6** How many different paths are there from start to finish?

**A** 3 **B** 6 **C** 9 **D** 12

10A **7** Which two words describe the networks in questions **1** and **4**?

**A** directed **B** small
**C** weighted **D** cyclic

10E **8** What is the weight of the shortest path from start to finish in the network from question 4?

**A** 4 **B** 7
**C** 10 **D** 13

10E **9** What is the weight of the longest path from start to finish in the network from question 4?

**A** 4 **B** 7
**C** 10 **D** 14

ARE YOU READY?

If you had difficulty with any of these questions or would like further practice, complete the matching Support sheet available on your obook assess.

**Q1–7** **Support sheet 10A Weighted and directed networks**

# 10A Activity tables and charts

These resources are available on your obook assess:

- **Video tutorial 10A:** Watch and listen to an explanation of Example 10A–2
- **Worksheet 10A:** Practise your skills with activity tables and charts
- **assess quiz 10A:** Test your skills with an auto-correcting multiple-choice quiz

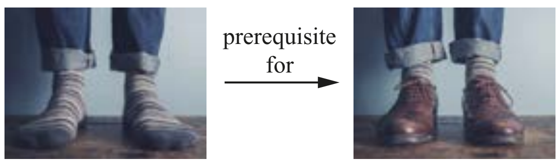
This topic is about representing information as a network. By the end of this chapter you will be able to use networks to make important real-life decisions about complex tasks.

Consider, for example, that you must be wearing socks and jeans before you put on your shoes. Then we say that the activities of putting on socks and putting on jeans are **prerequisite** activities for putting on your shoes.

**prerequisite** anything that must be done before an activity can start

**activity table** a table of activities specifying prerequisite activities and the expected timeframe for each activity

**activity chart** a weighted and directed network representing the dependencies in a set of activities




An **activity table** is a list of activities that includes a list of prerequisite activities and how long each activity will take.

We can represent an activity table as an **activity chart**, which is a directed and weighted network. Each edge represents the actual doing of a single activity. For example, suppose that you want to put on jeans, socks and shoes. There are three activities involved, so the activity chart will have three edges. The jeans and the socks are prerequisites of the shoes, and we represent this dependency through the direction of the edges.

Vertices represent completed activities. An activity chart will always have a start vertex (which represents zero completed activities), and a finish vertex (which represents all completed activities). In general, a vertex represents the completion of every single activity on any path between it and the start vertex.


## EXAMPLE 10A–1 Interpreting information from an activity table

The table lists the activities required for getting ready in the morning.

Answer the following questions according to the table.

**a** What activities are a prerequisite for activity C?

**b** Activity A is necessary for which activities?

**c** Is activity B necessary for any activity?

**d** Make a list of all activities that have no prerequisites.

| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Put on socks | 2 | None |
| B | Brush teeth | 3 | None |
| C | Put on shoes | 1 | A |

| | Solve | Think | Apply |
|---|---|---|---|
| **a** | Activity A | This is the only prerequisite for activity C. | Use the prerequisites column to identify which activities need to be completed before other activities can begin. |
| **b** | Activity C | Activity C has activity A as a prerequisite, so A is necessary for C. | |
| **c** | No | No activity includes B as a prerequisite. | |
| **d** | Activities A and B | These activities have no prerequisites. | |

## EXERCISE 10A Activity tables and charts

UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Consider the following table, which lists things that can be crafted in a game.

| Name | Activity | Cost | Prerequisite |
|---|---|---|---|
| A | Make a craft bench | 4 wood | None |
| B | Make a shield | 6 wood, 1 iron | A |

Which of the following statements are correct according to the table?

**a** Making a craft bench has a prerequisite of making a shield.
**b** Making a craft bench is necessary for making a shield.
**c** Making a shield has a prerequisite of making a craft bench.
**d** Making a shield is necessary for making a craft bench.

**2** Here is a list of the usual activities involved in buying a house.

| Name | Activity | Cost | Prerequisite |
|---|---|---|---|
| A | Find house | 2 | None |
| B | Obtain loan approval | 4 | A |
| C | Building inspection | 2 | A |
| D | Buy house | 1 | C, B |

Answer the following according to the table above.

**a** Make a list of all the activities that are directly dependent upon A.
**b** If you have obtained loan approval, have you necessarily found a house?
**c** Must you find a house before obtaining a building inspection for that house?
**d** Is A necessary for B?
**e** Can you obtain loan approval without a building inspection?
**f** Can you obtain a building inspection without loan approval?
**g** Can you buy a house without first obtaining a building inspection?

!

A **dummy activity** is an activity that takes no time, has no name, and is drawn as a dashed edge in an activity chart. Dummy activities are used to create extra paths in the activity chart. Activity charts should be simplified to use as few dummy activities as possible.

A dummy activity is removed by **contraction**: shrink the edge and the two vertices it touches into a single vertex. This must be done whenever the dummy edge is the only edge that leaves a vertex or the only edge that enters a vertex.

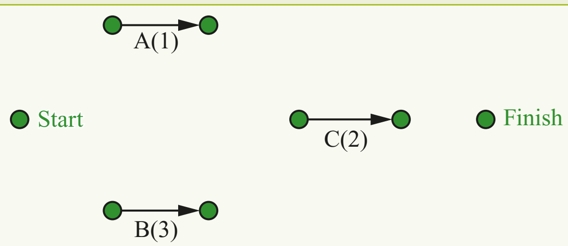
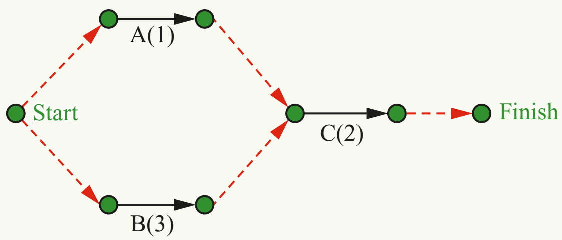
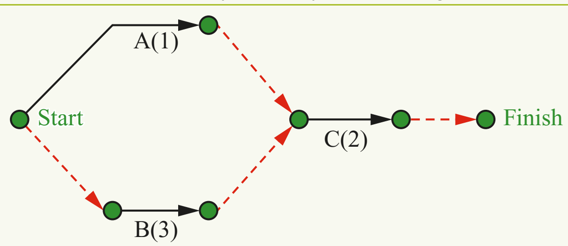
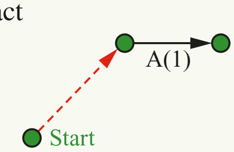
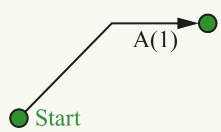
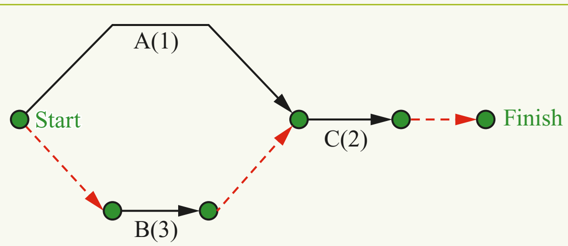
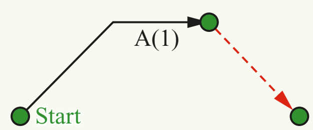
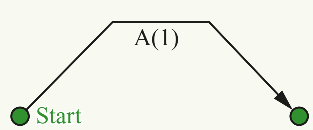
In most (but not all) circumstances, edge contraction means you just ‘fill in’ the dummy edge, so it is a good idea to use a pencil.

**dummy activity**
an edge in an activity chart with no name or weight, usually indicated as dashed directed edge

## EXAMPLE 10A–2 Constructing an activity chart from an activity table

Here is the table of activities involved in putting on socks, jeans and shoes.
Construct an activity chart for this table.

| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Put on socks | 1 | None |
| B | Put on jeans | 3 | None |
| C | Put on shoes | 2 | A, B |

| Step | Solve | Think/Apply |
|---|---|---|
| **1** |  | Draw the three activities as isolated edges, along with a start and finish vertex. To make things as neat as possible, try to put activities on the right of their prerequisites. In this example, C is on the right of its prerequisites A and B. |
| **2** |  | Connect each activity to its prerequisites using a dummy activity. Activities that have 'None' as the prerequisite are connected to the Start vertex, and any activity that is not a prerequisite for other activities is connected to the Finish vertex. |
| **3.1** | Technically, this is an activity chart. But it has too many dummy activities and should be simplified. If a dummy activity is the only edge that leaves or enters a vertex, then we can contract the dummy activity into a single vertex. | |
| **3.2** |  | Contract<br><br>into<br> |
| **3.3** |  | Contract<br><br>into<br> |

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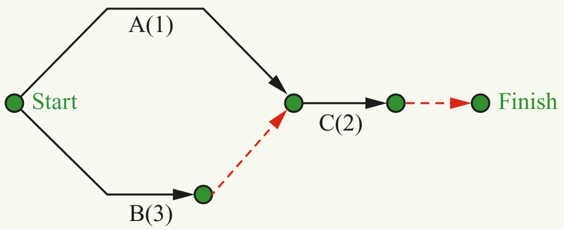
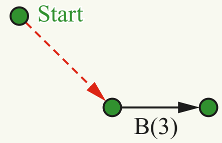
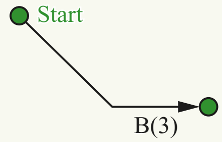
<tr><td>3.4</td><td>
</td><td>Contract<br>
<br>into<br>
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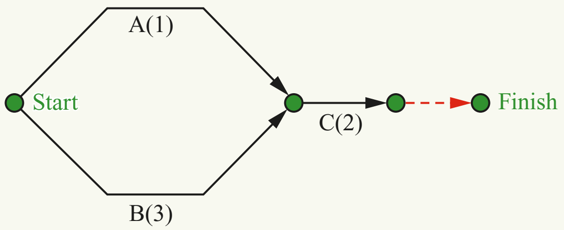
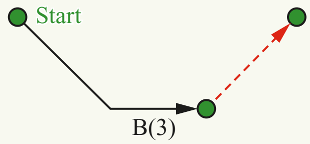
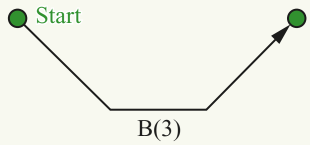
<tr><td>3.5</td><td>
</td><td>Contract<br>
<br>into<br>
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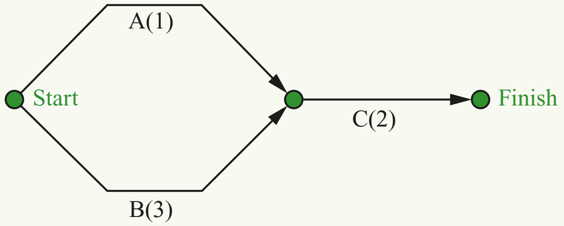
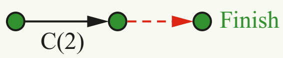
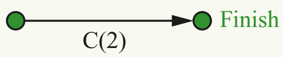
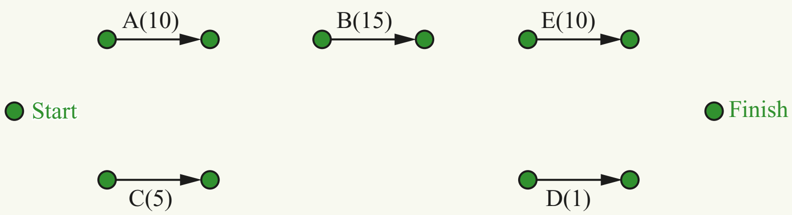
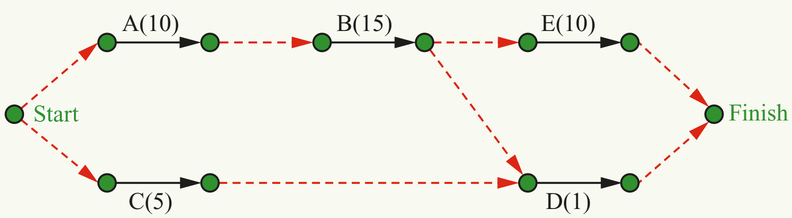
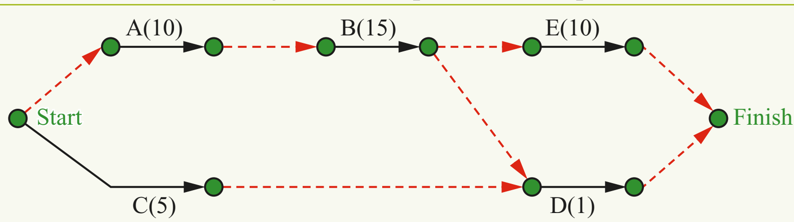
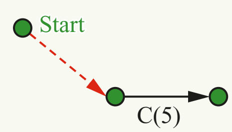
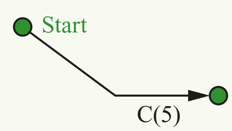
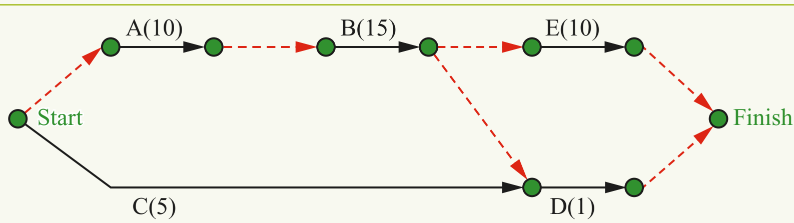
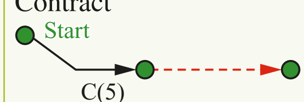
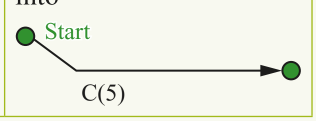
<tr><td>3.6</td><td>
</td><td>Contract<br>
<br>into<br>
</td></tr>
<tr><td></td><td colspan="2">This is the activity chart that represents the activities of putting on your socks, jeans and shoes, and the order in which those activities must be performed.</td></tr>
</table>

It is not always possible to remove all the dummy activities from an activity chart. !

## EXAMPLE 10A–3 Removing dummy activities

Here is the morning routine of a certain textbook author, who must drop off his two children, Quentin and Imogen, on his way to work. Construct an activity for this table.

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Drop off Quentin at school | 10 | None |
| B | Drop off Imogen at day care | 15 | A |
| C | Drink coffee | 5 | None |
| D | Relax | 1 | B, C |
| E | Go to work | 10 | B |

<table>
<tr><th>Step</th><th>Solve</th><th>Think/Apply</th></tr>
<tr><td>1</td><td colspan="2">Draw each activity as an isolated edge, along with start and finish vertices. To make things as neat as possible, try to put activities on the right of their prerequisites.<br>In this example:<br>• A and C are to the right of the start vertex because they have no prerequisites.<br>• B is on the right of A because A is a prerequisite of B.<br>• D is on the right of B and C because they are the prerequisites of D.<br>• E is on the right of B.<br><br></td></tr>
<tr><td>2</td><td colspan="2">Connect each activity to its prerequisites using a dummy activity. Activities that have ‘None’ as the prerequisite are connected to the Start vertex, and any activity that is not a prerequisite for other activities is connected to the Finish vertex.<br><br></td></tr>
<tr><td>3.1</td><td colspan="2">Technically, this is an activity chart, but it has too many dummy activities. We can contract many (but not all) of these edges as in the previous example.</td></tr>
<tr><td>3.2</td><td><br></td><td>Contract<br><br><br>into<br><br></td></tr>
<tr><td>3.3</td><td><br></td><td>Contract<br><br><br>into<br><br></td></tr>
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| Step | Solve | Think/Apply |
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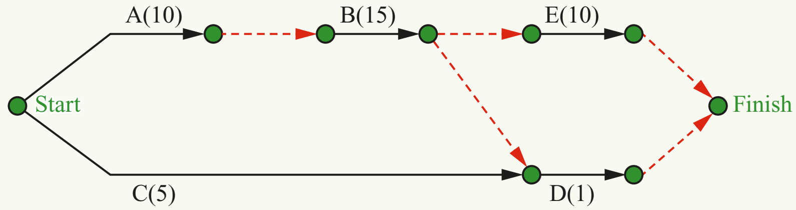
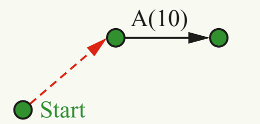
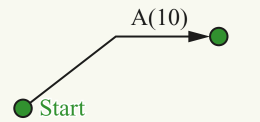
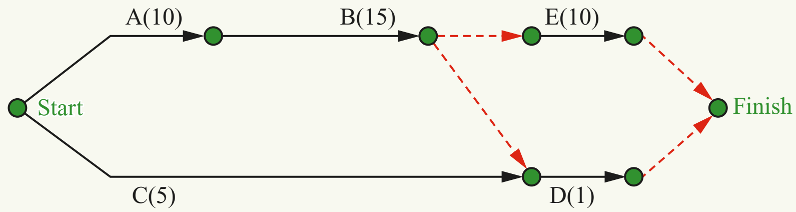
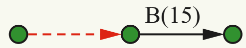
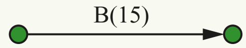
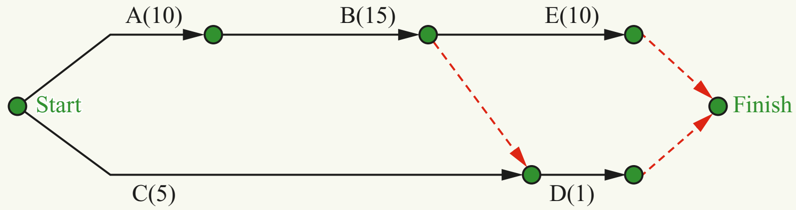
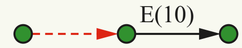
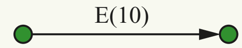
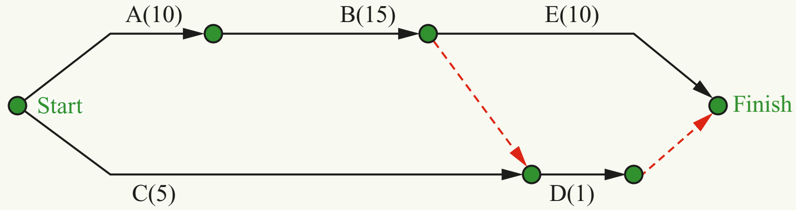
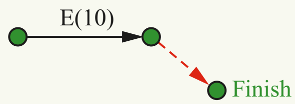
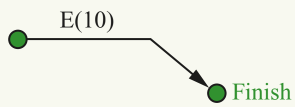
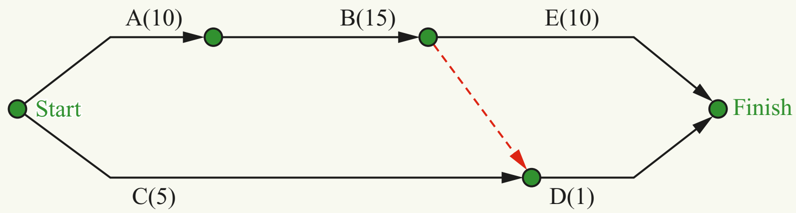
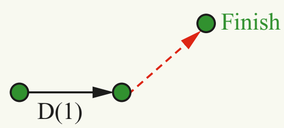
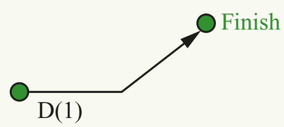
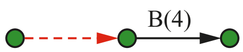
| 3.4 |  | Contract  into  |
| 3.5 |  | Contract  into  |
| 3.6 |  | Contract  into  |
| 3.7 |  | Contract  into  |
| 3.8 |  | Contract  into  |
| | Notice that the last remaining dummy activity cannot be removed because the criteria for contraction is not satisfied. | |

**3** Contract the dummy activities in the following networks where possible. When it is not possible to contract the dummy activity, explain why.

**a**
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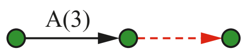


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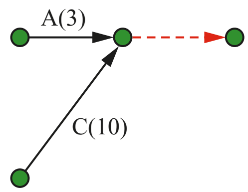


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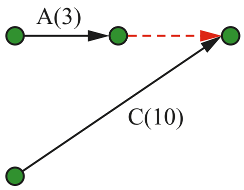


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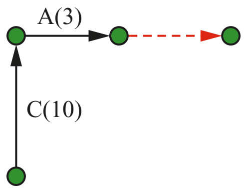


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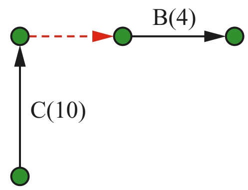


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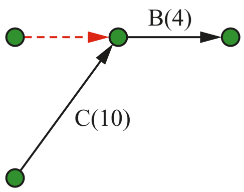


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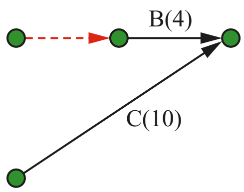
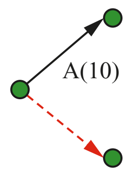
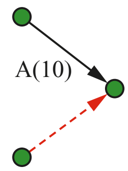


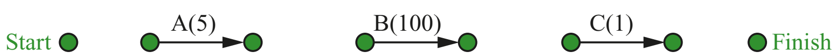

**i**



**j**



**4** Consider the following activity table.

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Wake up | 5 | None |
| B | Work | 100 | A |
| C | Get paid | 1 | B |

The first step of constructing the activity chart is to draw all the activities as edges, with activities drawn on the right of their prerequisites. This is done below.



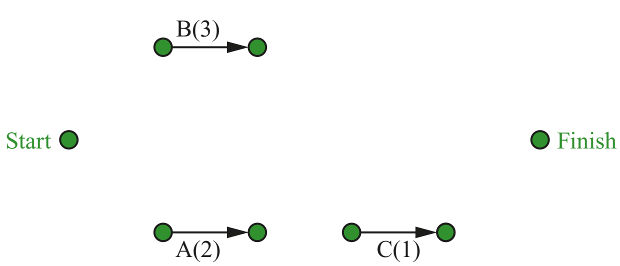

**a** Add dummy activities to connect activities to their prerequisites.

**b** Simplify the network by contracting all dummy activities.

**5** Consider the following activity table from Example 10A–1.

| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Put on socks | 2 | None |
| B | Brush teeth | 3 | None |
| C | Put on shoes | 1 | A |

The activities of this table are drawn below, along with the start and finish vertices.




**a** Add dummy activities to connect activities to their prerequisites.

**b** Simplify the network by contracting all dummy activities.

**6** Draw an activity chart for the following activity table involving two tasks. Your final activity chart should contain no dummy activities.

**a**

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Buy fruit | 10 | None |
| B | Eat fruit | 5 | A |

**b**

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Buy fruit | 10 | None |
| B | Buy veg | 5 | None |

**7** Draw an activity chart for the following activity tables involving three tasks. Your final activity chart should contain no dummy activities.

**a**

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Travel to greengrocer | 5 | None |
| B | Buy fruit | 10 | A |
| C | Eat fruit | 5 | B |

**b**

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Travel to greengrocer | 5 | None |
| B | Buy fruit | 10 | A |
| C | Buy veg | 10 | A |

**c**

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Travel to greengrocer | 5 | None |
| B | Travel to the beach | 10 | None |
| C | Travel to the park | 10 | None |

**8** Draw an activity chart for the following activity table involving five tasks. Your activity chart should contain one dummy activity.

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Travel to greengrocer | 5 | None |
| B | Buy fruit | 10 | A |
| C | Find salad bowl | 10 | None |
| D | Make fruit salad | 15 | B, C |
| E | Travel back home | 5 | B |

**9** Draw an activity chart for the following activity table involving five tasks. It will necessarily contain two dummy activities.

| Name | Activity | Time (minutes) | Prerequisites |
|---|---|---|---|
| A | Buy fruit | 5 | None |
| B | Find bowls | 10 | None |
| C | Buy soup ingredients | 10 | None |
| D | Make fruit salad | 15 | A, B |
| E | Make soup | 30 | C, B |

**10** The following activity table contains the activities required in a project, but not the time it takes to complete each activity. Draw an activity chart for this table (without the edge weights).

| Name | Prerequisites |
|---|---|
| A | None |
| B | None |
| C | None |
| D | A |
| E | B |
| F | D, E |
| G | C |

**11** Here is a list of courses in the Bachelor of Early Childhood Education and Care degree (slightly adapted).

| Course code | Name | Length | Prerequisite |
|---|---|---|---|
| ECCWC101A | Nutrition, Health, Safety and Wellness | 40 | None |
| ECSFC101A | Professional, Cultural and Academic Practice | 40 | None |
| ECTPP101A | Holistic Child Development | 40 | None |
| ECTPP102A | Play and Documentation | 40 | None |
| ECPPL101A | Professional Experience 1 (Birth–2 years) | 110 | None |
| ECCDD101A | Health and Physical Wellbeing | 40 | ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A |
| ECCDD102A | Creativity in the Early Years | 40 | ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A, ECCDD101A |
| ECCDD103A | Children's Self-identity and Family Contexts | 40 | ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A, ECCDD101A |
| ECSFC102A | Sustainability in Early Childhood | 40 | ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A, ECCDD101A |
| ECPPL102A | Professional Experience 2 (2–3 years) | 110 | ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A, ECCDD101A |

**a** What are the prerequisite activities for ECCDD102A – Creativity in the Early Years?

**b** Can you begin ECPPL101A – Professional Experience 1 (Birth–2 years) without first completing ECCDD102A – Creativity in the Early Years?

**c** Can you begin ECPPL102A – Professional Experience 2 (2–3 years) without first completing ECCDD102A – Creativity in the Early Years?

**d** What classes have ECTPP101A – Holistic Child Development as a prerequisite?

**12** Here is a list of courses offered in a civil engineering degree.

| Course code | Name | Length | Prerequisites |
|---|---|---|---|
| MATH1131 | Mathematics 1A | 72 hours | None |
| MATH1231 | Mathematics 1B | 72 hours | MATH1131 |
| PYHS1121 | Physics 1A | 72 hours | None |
| CVEN1300 | Engineering Mechanics | 72 hours | MATH1131, PHYS1121 |
| CVEN2002 | Engineering Computation | 72 hours | MATH1231 |

**a** Which courses have no prerequisites?

**b** Which courses have MATH1131 as a prerequisite?

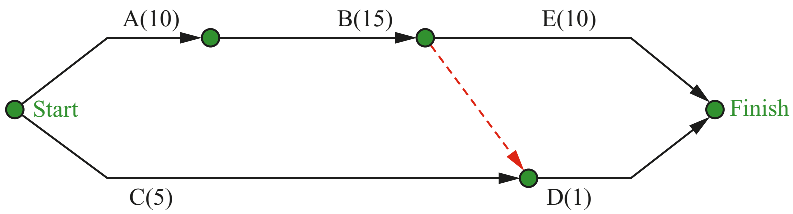
**c** MATH1131 is necessary for which courses?

**d** If you are currently studying CVEN1300, have you necessarily completed PHYS1121?

**e** If you are currently studying CVEN2002, have you necessarily completed PHYS1121?

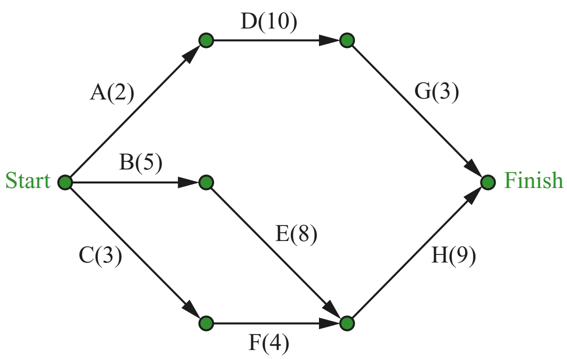
**13** Consider the activity chart from Example 10A–3 (reproduced below).

- **a** There are three paths from the Start to Finish vertex. Two such paths are ABE and ABD (using the dummy activity between B and D). What is the other path?
- **b** What is the total weight of each of the paths from part **a**?
- **c** Which path has the largest weight?
- **d** What activities must be completed before activity D can begin?
- **e** Can activity B begin before activity C?



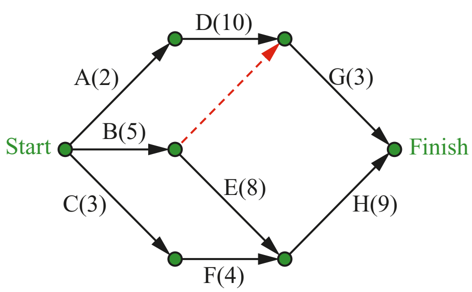

**14** Consider the activity chart below.

- **a** What are the prerequisites of activity E?
- **b** What are the prerequisites of activity H?
- **c** Which activities have no prerequisites?
- **d** What are the prerequisites of activity G?
- **e** Write an activity table for this activity chart.

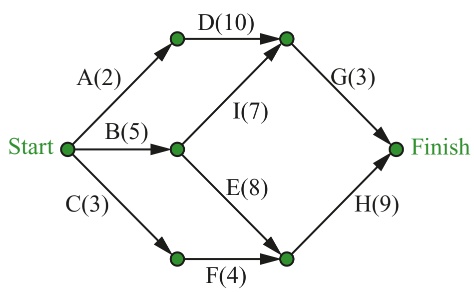



**15** How does your answer to question **14** part **d** change when the activity chart is modified as follows?

**a**

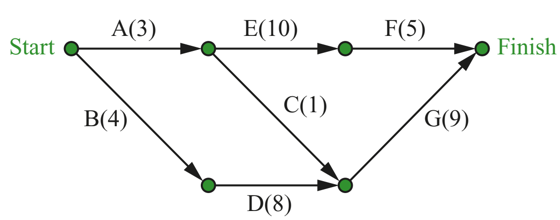



**b**




**16** Consider the activity chart on the right.

- **a** What are the prerequisites of activity F?
- **b** What are the prerequisites of activity D?
- **c** What are the prerequisites of activity G?
- **d** Draw an activity table for this activity chart.



PROBLEM SOLVING, REASONING AND JUSTIFICATION

**17** Cooking risotto is a delicate task that involves performing the following activities exactly.

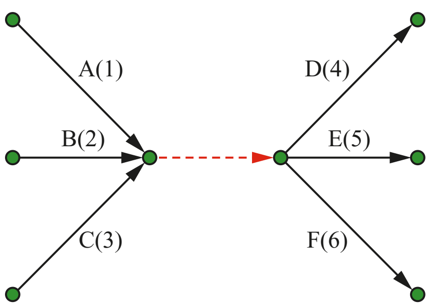
| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Put 1.7 litres of chicken stock in pot on medium heat. | 10 | None |
| B | Finely chop 1 onion. | 2 | None |
| C | Sauté onion in deep frypan with 60 g of butter. | 5 | B |
| D | Add 2 cups of Arborio rice. | 1 | C |
| E | Stir rice constantly, using a ladle to add stock to keep rice covered. | 20 | A, D |
| F | Add juice and zest of one lemon, 60 g of butter and parsley. | 5 | E |

Represent these tasks as an activity chart.

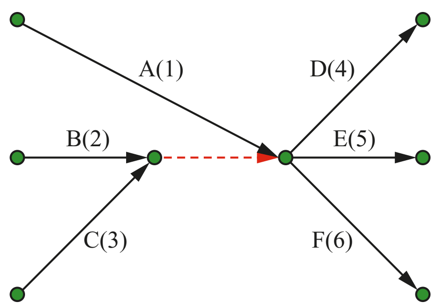
CHALLENGE

**18** In most situations dummy edges can be contracted by simply 'filling in' the dashed line. More complicated situations are rarely encountered, but they may require you to re-draw the entire network. Contract the dummy activities in the following networks where possible. When it is not possible to contract the dummy activity, explain why.

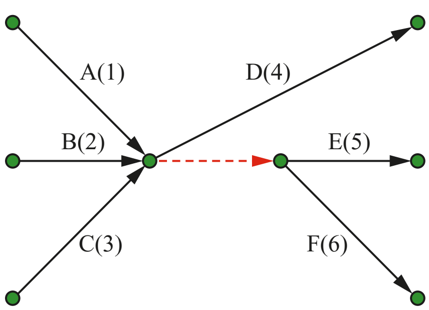
**a**



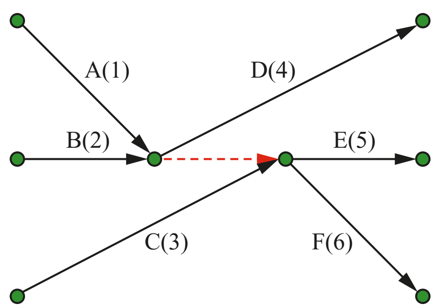

**b**




**c**




**d**




**19** Find a recipe and write the steps as an activity table. Give it to your classmates to represent as an activity chart.

NETWORKS

# 10B The earliest starting and finishing times

These resources are available on your obook assess:

- **Video tutorial 10B:** Watch and listen to an explanation of Example 10B–2
- **Worksheet 10B:** Practise your skills with earliest starting and finishing times
- **assess quiz 10B:** Test your skills with an auto-correcting multiple-choice quiz

OI GI

**earliest starting time (EST)** the minimum amount of time it takes to complete all prerequisite activities

**critical time** the shortest amount of time required to complete all activities

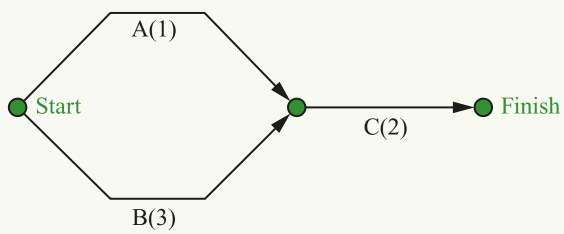
The **earliest starting time (EST)** of an activity is the minimum amount of time it takes to complete *all* prerequisite activities. Alternatively, this is the moment when all prerequisites activities have finished, or it is the length of the *longest* path from the Start vertex (of the chart) to the start vertex for the activity.

Edges that share the same start vertex also share the same EST.

The start and finish vertices also have an EST. The EST of the Start vertex is always zero, and the EST of the finish vertex is the time it takes to complete all activities in the project. This is called the **critical time** for the project.

## EXAMPLE 10B–1 Finding the Earliest starting time

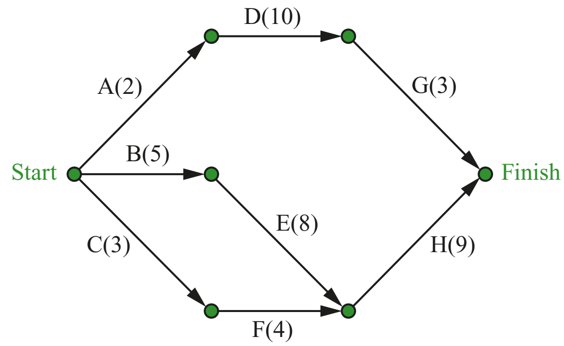
Calculate the EST for each activity in the following activity chart.




| Solve | Think/Apply |
|---|---|
| Activity A and B have an EST of 0.<br>Activity C has an EST of 3. | Any vertex connected to the Start vertex will have an EST of 0. Recall that these activities have no prerequisites and can begin immediately.<br>The earliest that activity A can be finished is at time 1, and the earliest that activity B can be finished is at time 3. Activity C cannot begin until both A and B have finished. So the EST for activity C is the maximum of these weights: 3.<br>Even if activities A and B are done concurrently, this is the soonest that activity C can begin. |

# EXERCISE 10B The earliest starting and finishing times

**1** Consider the activity chart from Exercise 10A question **14** (reproduced below).




**a** Which activities can start immediately?

**b** What is the earliest time that activity A can be finished and activity D can begin?

**c** What is the earliest time that activity B can be finished and activity E can begin?

**d** What is the earliest time that activity C can be finished and activity F can begin?

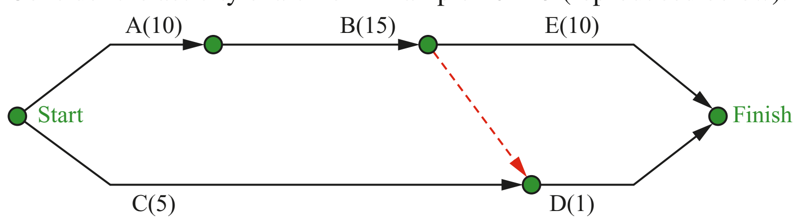
**e** What is the earliest time that activity D can be finished and activity G can begin?

**f** Activity H can start once *both* activities E and F are finished. What is the earliest time that both activities E and F can be finished?

**g** Use your answers to parts **a–f** to fill out the table of activities and their ESTs.

| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| EST | | | | | | | | |

**2** Consider the activity chart from Example 10A–3 (reproduced below).




**a** Which activities have an EST of zero?

**b** What is the earliest time that activity A can be finished and activity B can begin?

**c** What is the earliest time that activity B can be finished and activity E can begin?

**d** Activity D can start once both activities C and B are complete. What is the earliest time that both activities C and B can be finished?

**e** Use your answers to parts **a–d** to complete the table of activities and their ESTs.

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| EST | | | | | |

**f** Which activities share the same EST?

The **earliest finishing time (EFT)** for an activity is the soonest an activity can be completed. This is the time it takes to complete the activity as well as *all* prerequisite activities:

$$\text{EFT} = \text{weight} + \text{EST}$$

Unlike the EST, the EFT is not shared by all activities with the same start vertex; instead it depends upon the weight of the individual activity.

!

**earliest finishing time (EFT)** the minimum amount of time it takes to complete the activity plus all prerequisite activities, and is calculated by adding the edge weight to the EST

**3** **a** Activity A has an EST of 6 and a weight of 12. What is its EFT?
**b** Activity B has an EST of 2 and a weight of 3. What is its EFT?
**c** Activity C has an EST of 11 and a weight of 9. What is its EFT?
**d** A dummy activity has an EST of 100. What is its EFT?
**e** Activity E has an EST of 3 and an EFT of 7. What is its weight?
**f** Activity F has a weight of 9 and an EFT of 11. What is its EST?

**4** Add a new row to the tables in questions **1g** and **2e**. Fill in the EFT of each activity.

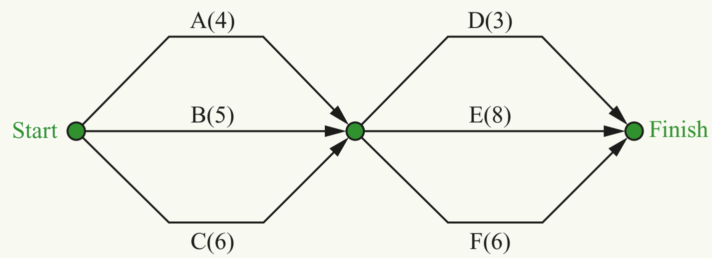
A fast, systematic method for finding the EST for all activities is called the **forward scan**. Instead of looking at all possible paths, this algorithm begins at the Start vertex and moves along the edges of the activity chart. It calculates the EST at each vertex as the maximum EFT of all activities that end at that vertex.

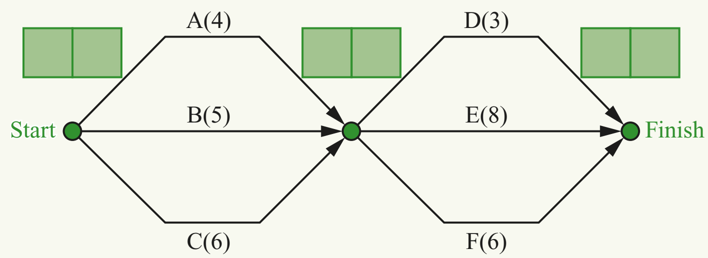
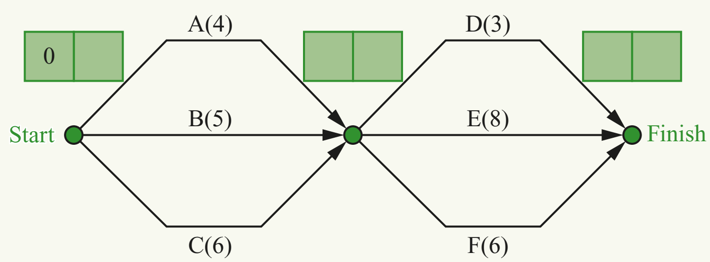
!

**forward scan** the procedure to determine the EST for each activity in an activity chart

## EXAMPLE 10B–2 Performing a forward scan

Perform a forward scan to determine the EST for each activity in the following activity chart.

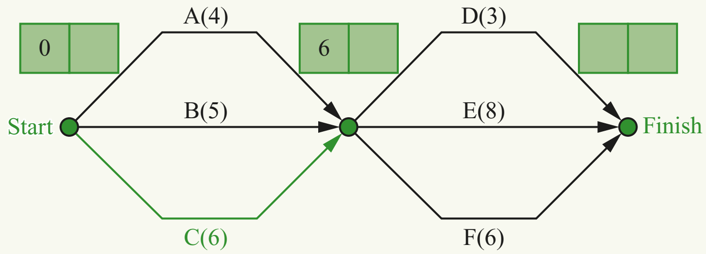



| Step | Solve/Think/Apply |
|---|---|
| 1 | Decorate each vertex of the activity chart with two boxes. The leftmost box will be populated with the EST, which is shared by all activities that start from that vertex. The rightmost box will contain the latest finishing time of the activity, but we will leave it empty until the next topic.   |
| 2.1 | Give the Start vertex an EST of zero. This means that activities A, B and C can begin straight away.   |

**2.2** To find the EST of the middle vertex, take the maximum EFT over all edges which end at this vertex.

| Activity | A | B | C |
|---|---|---|---|
| EST | 0 | 0 | 0 |
| Weight | 4 | 5 | 6 |
| EFT | 4 | 5 | 6 |

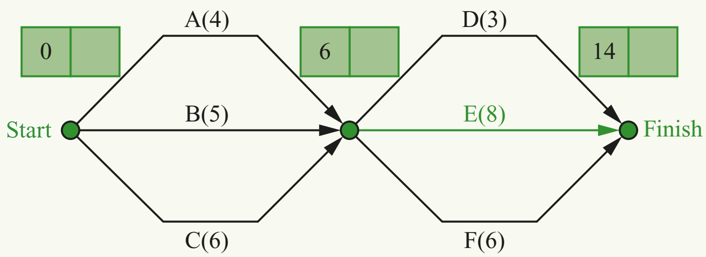
The EST of the middle vertex is the largest EFT: 6. Decorate the vertex with this number.
This number represents the earliest time that *all* activities A, B and C can be completed, which is also the soonest that D, E and F can begin. So it is actually the EST of activities D, E and F.




**2.3** Now we use the EST of the middle vertex to find the EST of the Finish vertex. Again, look at each prerequisite activity to find the maximum EFT.

| Activity | D | E | F |
|---|---|---|---|
| EST | 6 | 6 | 6 |
| Weight | 3 | 8 | 6 |
| EFT | 9 | 14 | 12 |

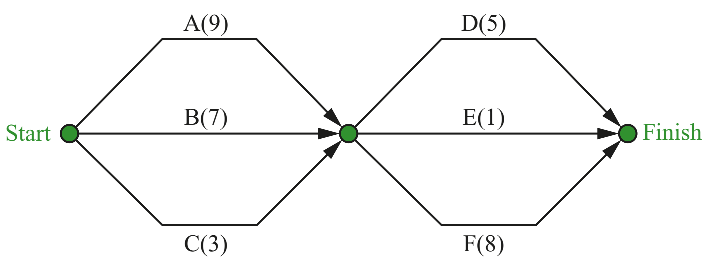
The EST of the finish vertex is the largest EFT: 14.



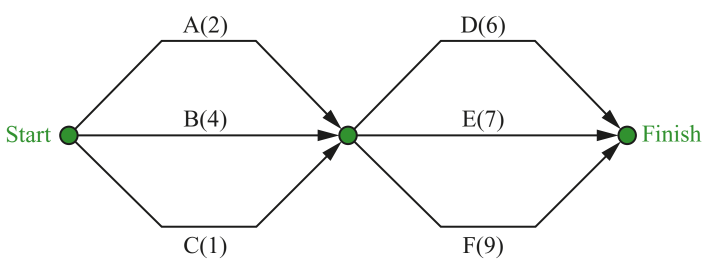

Note: the EST of the Finish vertex is the critical time for the project.

**5** Use a forward scan to find the EST of all activities in the following networks.

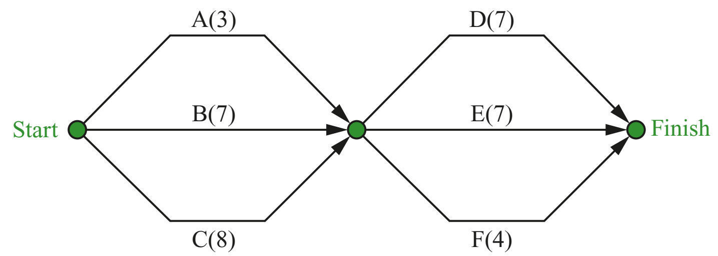
**a**



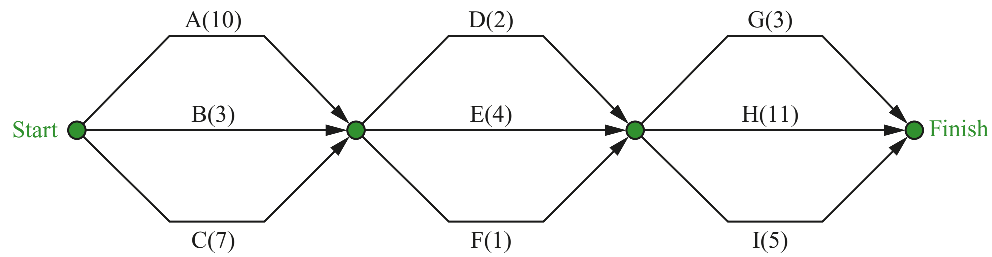

**b**



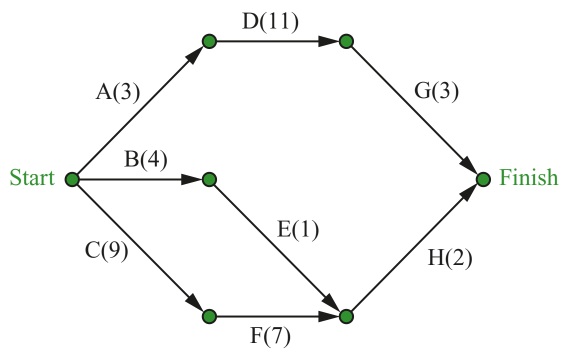
**c**




**d**

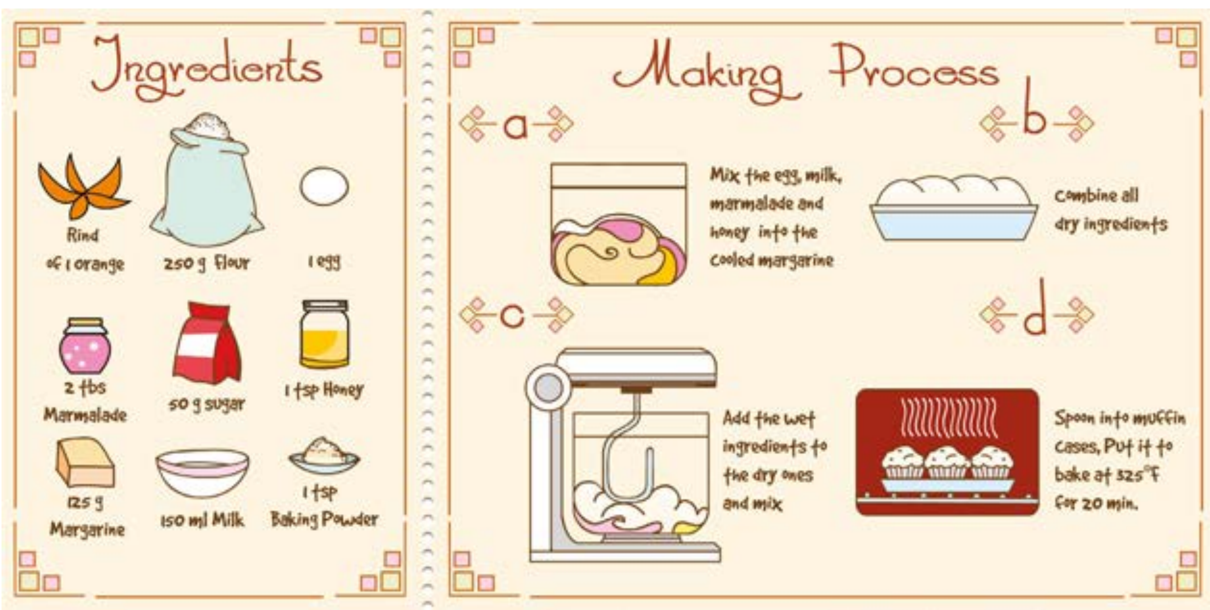



**6** Complete the table of earliest start times for all the activities in the network below.




| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| EST | | | | | | | | |

**7** Consider the following recipe for perfect muffins.




**a** Estimate the time it will take you to perform activities a, b, c and d.

**b** What is the EST of activity d?

**c** How soon can the muffins be ready?

**8** In question **1** there are three paths from the Start vertex to the Finish vertex.

**a** What is the weight of each path?

**b** What is the largest value in your answer to part **a**?

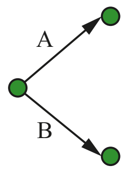
**c** What is critical time for this project?

**9** Repeat question **8** for the activity chart for question **2**.

**10** Suppose two activities A 'Buy lunch' and B 'Buy coffee' in an activity chart share the same start vertex. This vertex represents your arrival at the food hall, which is the prerequisite of A and B.

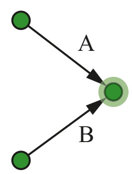
**a** Will the activities have the same EST?

**b** Under what circumstances will the activities have the same EFT?




**11** Activities A 'book airfares' and B 'book accommodation' are the prerequisites of your holiday. (These prerequisites are represented as the highlighted vertex.) The EST and weight of each activity is given in the table.

| **Activity** | **A** | **B** |
|---|---|---|
| **Weight** | 4 | 6 |
| **EST** | 5 | 4 |

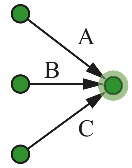



**a** Calculate the EFT of each activity.

**b** What is the maximum EFT of activities A and B?

**12** Activities A ‘book airfares’, B ‘book accommodation’ and C ‘change currency’ are the prerequisites of your holiday. (These prerequisites are represented as the highlighted vertex.) The EST and weight of each activity is given in the table.

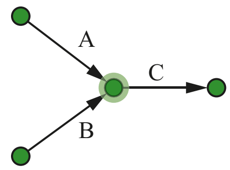
| Activity | A | B | C |
|---|---|---|---|
| Weight | 3 | 5 | 4 |
| EST | 9 | 5 | 8 |




**a** Calculate the EFT of each activity.
**b** What is the maximum EFT of all activities that end at the green vertex?

**13** Activities A ‘book airfares’ and B ‘book accommodation’ are the prerequisites required to board the plane. (These prerequisites are represented as the highlighted vertex.) Activity C ‘holiday’ starts from the highlighted vertex. The EST of activities A and B are known.

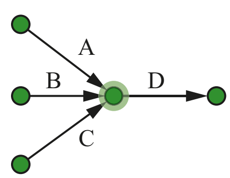
| Activity | A | B | C |
|---|---|---|---|
| Weight | 2 | 1 | 4 |
| EST | 3 | 5 | |




**a** Calculate the EFT of activities A and B.
**b** What is the maximum EFT of activities A and B?
**c** Using your answer to part **b**, what is the earliest time that all the prerequisites of C will be met?

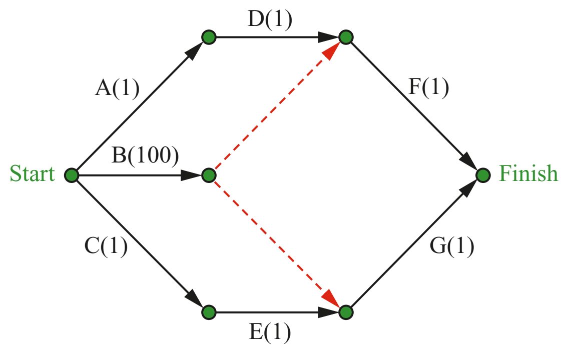
**14** Activities A ‘book airfares’ and B ‘book accommodation’ and C ‘change currency’ are the prerequisites required for your holiday. (These prerequisites are represented as the highlighted vertex.) Activity D ‘holiday’ starts from the green vertex. The EST of activities A, B and C are known.

| Activity | A | B | C | D |
|---|---|---|---|---|
| Weight | 4 | 7 | 6 | 5 |
| EST | 8 | 3 | 6 | |




**a** Calculate the EFT of activities A, B and C.
**b** What is the maximum EFT of activities A, B and C?
**c** Using your answer to part **b**, what is the earliest time that all the prerequisites of D will be met?

**15** The activity chart shows six short activities that each take one hour to complete, and one long activity (B) which takes 100 hours.



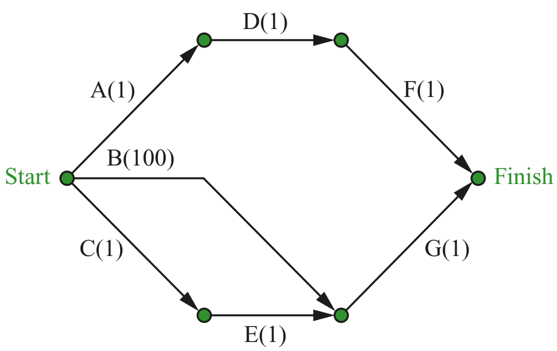

**a** Which activities depend upon activity B?

**b** All the activities are outsourced to separate contractors. Assume that each contractor finishes their activity in the shortest possible time. How soon can activity F start?

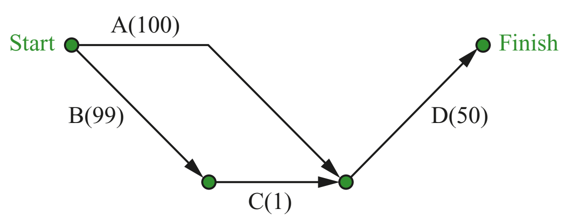
**c** The contractor assigned to activity B offers to complete the activity in half the allocated time, for an additional fee. How soon could the whole project be finished if you take up this offer?

**d** Task A is more complicated than initially expected and will actually take 20 hours. Will this affect the critical time of the project?

**16** Repeat question **15** for the following activity network.




**17** The following activity chart shows four activities, and the time required to complete them (in hours). All the activities are outsourced to separate contractors. Assume that each contractor finishes their activity in the shortest possible time.




**a** How soon can activity D begin?

**b** The contractor for activity A offers to complete the activity in half the allocated time, for an additional fee. How soon could the whole project be finished if you take up this offer?

**c** The contractor you hired for activity D is incompetent and takes an extra 10 hours to complete the task. How does this affect the critical time of the project?

**18** Consider the risotto recipe from Exercise 10A question **17**. Assuming that you can outsource all tasks in your kitchen, how soon can the risotto be ready?

# 10C The latest finishing and starting times

These resources are available on your obook assess:
- **Video tutorial 10C:** Watch and listen to an explanation of Example 10C–2
- **Worksheet 10C:** Practise your skills latest finishing and starting times
- **assess quiz 10C:** Test your skills with an auto-correcting multiple-choice quiz

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**latest finishing time (LFT)**
the latest time the activity can be finished without delaying the finish time of the project

The **latest finishing time (LFT)** is the latest time an activity can be finished without causing delay in the critical time of the project. Alternatively, it is the last possible moment that the activity can finish with the project still remaining on schedule.

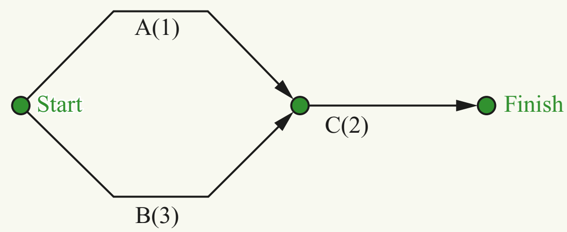
Activities that share the same end vertex are all part of the same collection of prerequisites, and so will have the same LFT.

The Start and Finish vertices of the activity chart also have an LFT. The LFT of the Finish vertex is the critical time of the project, and the LFT of the Start vertex is always zero.

The final activities in a project all have an LFT equal to the critical time of the project.

## EXAMPLE 10C–1 Determining the latest finishing time

Consider the activity chart below. Weights are in minutes, and the critical time is 5 minutes.

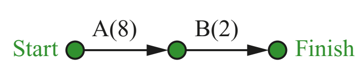



**a** What is the LFT of activity C?
**b** What is the LFT of activities A and B?

| | Solve | Think/Apply |
|---|---|---|
| **a** | The LFT of C is 5. | Activity C is the final activity in the project, so this must be completed by the critical time (5). |
| **b** | The LFT of A and B is 3. | To finish by time 5, activity C must have started by time 3.<br>But activity C can only start once *both* A and B are finished, so both A and B must be finished by time 3. |

## EXERCISE 10C The latest finishing and starting times

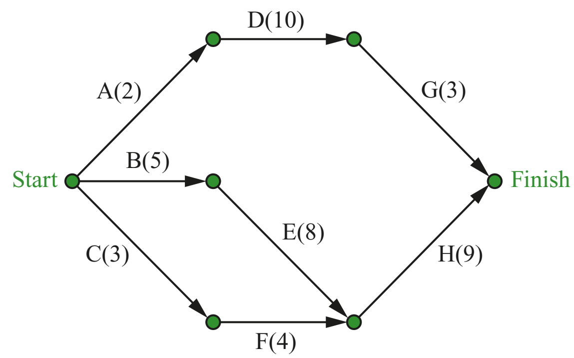
**1** Consider the activity chart on the right, with activity times given in minutes.




**a** What is the critical time for this project?
**b** What is the LFT of activity B?
**c** What is the LFT of activity A?

UNDERSTANDING, FLUENCY AND COMMUNICATING

**2** Consider the activity chart from 10A question **14** (reproduced here). Weights are in minutes, and the critical time of this network is 22 minutes.




**a** What is the latest time that activities G and H can finish?

**b** What is the latest time that activity D can finish and still give enough time to do activity G?

**c** What is the latest time that activity A can finish and still give enough time to do activity D?

**d** What is the latest time that activities E and F can finish and still give enough time to do activity H?

**e** What is the latest time that activity B can finish and still give enough time to do activity E?

**f** What is the latest time that activity C can finish and still give enough time to do activity F?

**g** Populate the table of activities and their LFTs.

| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| LFT | | | | | | | | |

The **latest starting time (LST)** for an activity is the latest an activity can be started. This is calculated as the latest time it can be finished, minus the time taken to complete the activity:

$$\text{LST} = \text{LFT} - \text{weight}$$

Unlike the LFT, the LST is not shared by all activities with the same end vertex. Instead it depends upon the weight of the individual activity.

**latest starting time (LST)** the latest time the activity can be started without delaying the finish time of the project

**3** **a** Activity A has an LFT of 9 and a weight of 2. What is its LST?

**b** Activity B has an LFT of 2 and a weight of 2. What is its LST?

**c** Activity C has an LFT of 11 and a weight of 9. What is its LST?

**d** A dummy activity has an LFT of 100. What is its LST?

**e** Activity E has an LST of 5 and an LFT of 9. What is its weight?

**f** Activity F has a weight of 9 and an LST of 11. What is its LFT?

**4** Answer True or False to the following statements.

**a** The earliest starting time is always less than or equal to the latest starting time.

**b** The earliest finishing time is always less than or equal to the latest finishing time.

**c** The latest finishing time is always less than or equal to the latest starting time.

**5** The LFT and weight for each activity in Example 10C–1 is given in the table below. Calculate the LST for each activity.

| Activity | A | B | C |
|---|---|---|---|
| LFT | 3 | 3 | 5 |
| Weight | 1 | 3 | 2 |
| LST | | | |

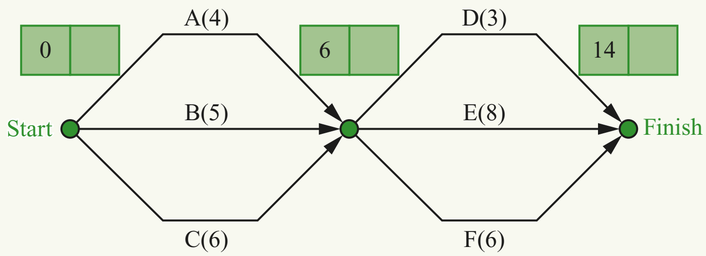
**6** Use your answer to question **2g** and the weights given in the activity chart to find the LST for each activity.

A **backward scan** can be performed to calculate the LFT. The backward scan algorithm begins at the Finish vertex and works backwards along the edges of the activity chart. It calculates the LFT at each vertex as the smallest LST of all activities that depend upon (i.e. start from) that vertex.

**backward scan**
the algorithm used to determine the LFT of each activity

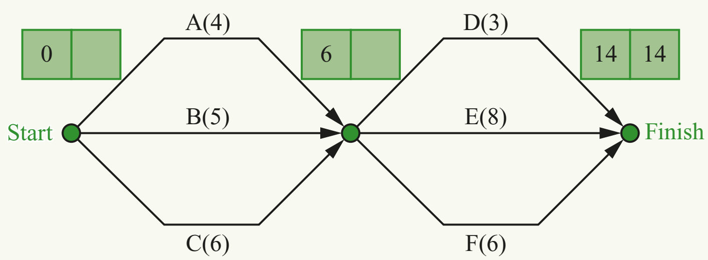
## EXAMPLE 10C–2 Performing a backward scan

A forward scan has been completed on the activity chart below. Perform a backward scan and then calculate the LFT for each activity in the activity chart.




**Step 1**

**Solve**



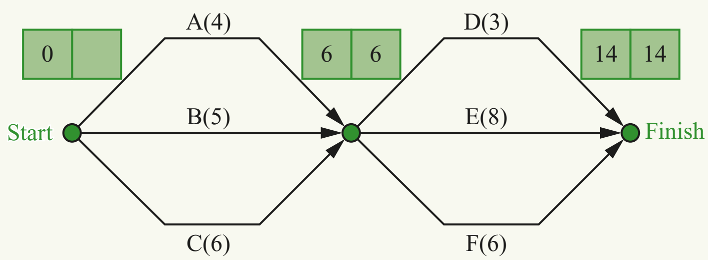

**Think/Apply**

Begin at the Finish vertex and copy the EST over to the box on the right.

This number is now the LFT for every activity that points towards the vertex: D, E and F.

**Step 2.1**

**Solve**




**Think/Apply**

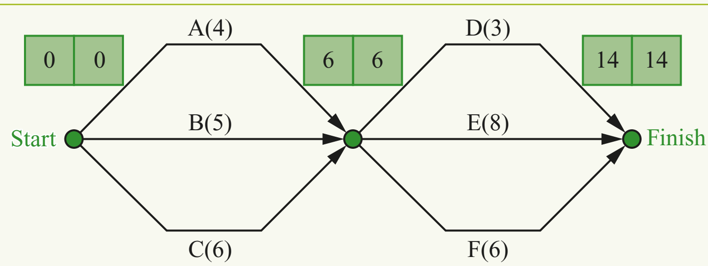
Proceed towards the left and consider the LST of every edge that starts from the middle vertex.

| Activity | D | E | F |
|---|---|---|---|
| LFT | 14 | 14 | 14 |
| Weight | 3 | 8 | 6 |
| LST | 11 | 6 | 8 |

The LFT at this vertex is *smallest* LST: 6. Decorate the vertex with this number.

**Step 2.3**

**Solve**




**Think/Apply**

Now consider the Start vertex. Calculate the LST of each edge that begins at this vertex.

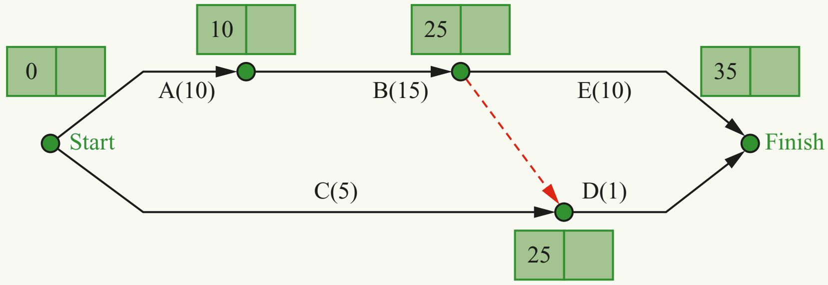
| Activity | A | B | C |
|---|---|---|---|
| LFT | 6 | 6 | 6 |
| Weight | 4 | 5 | 6 |
| LST | 2 | 1 | 0 |

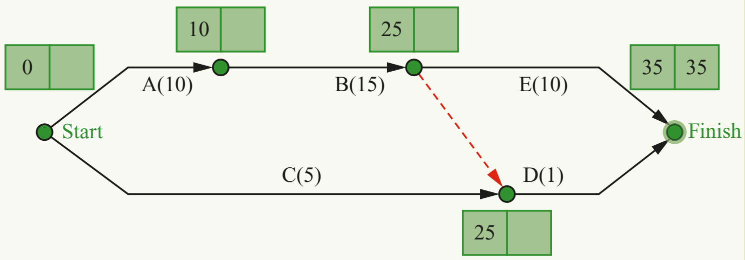
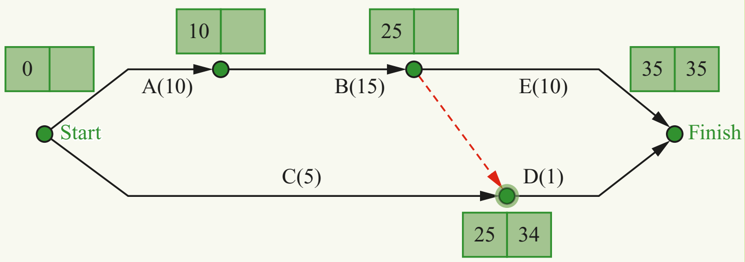
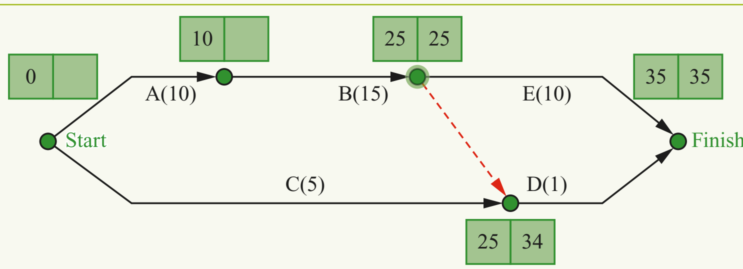
The LFT of the Start vertex is the smallest LST: 0.

*Note*: the LFT of the Start vertex should always be zero. Checking this is an easy way to check your working.

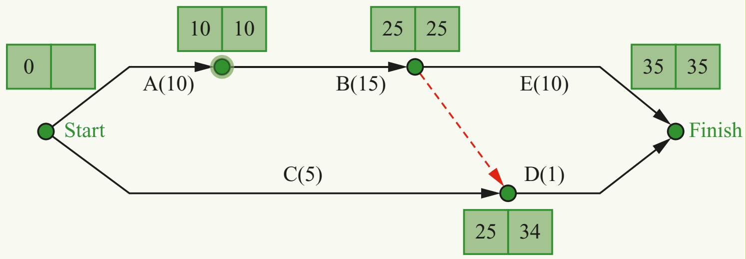
## EXAMPLE 10C–3 Performing a backward scan with a dummy activity

A forward scan has been completed on the activity chart below. Perform a backward scan and then calculate the LFT for each activity in the activity chart.




<table>
<tr><th>Step</th><th>Solve</th><th>Think/Apply</th></tr>
<tr><td>1</td><td></td><td>Begin at the Finish vertex and copy the EST over to the box on the right.</td></tr>
<tr><td>2.1</td><td></td><td>Consider the vertex at the start of activity D. There is only one activity that begins at this vertex, and the LFT and LST are:
<table>
<tr><th>Activity</th><th>D</th></tr>
<tr><td>LFT</td><td>35</td></tr>
<tr><td>Weight</td><td>1</td></tr>
<tr><td>LST</td><td>34</td></tr>
</table>
The LFT at this vertex is the smallest LST in this list: 34. Decorate the vertex with this number.</td></tr>
<tr><td>2.2</td><td></td><td>Consider the vertex at the start of activity E. There are two activities that begin at this vertex, and the LFT and LST are:
<table>
<tr><th>Activity</th><th>E</th><th>Dummy</th></tr>
<tr><td>LFT</td><td>35</td><td>34</td></tr>
<tr><td>Weight</td><td>10</td><td>0</td></tr>
<tr><td>LST</td><td>25</td><td>34</td></tr>
</table>
The LFT is the smallest LST in this list: 25. Decorate the vertex with this number.</td></tr>
</table>

**2.3**

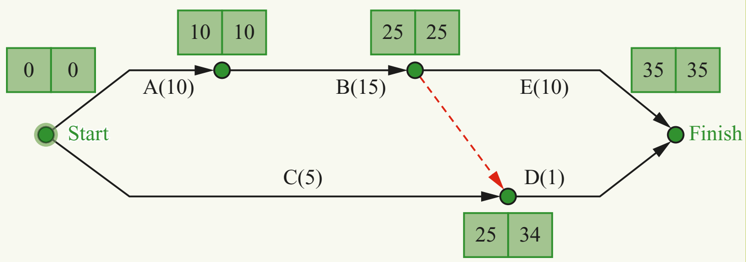



Consider the vertex at the start of activity B. There is only one activity that begins at this vertex, and the LFT and LST are:

| **Activity** | **B** |
|---|---|
| **LFT** | 25 |
| **Weight** | 15 |
| **LST** | 10 |

The LFT is the smallest LST in this list: 10. Decorate the vertex with this number.

**2.4**



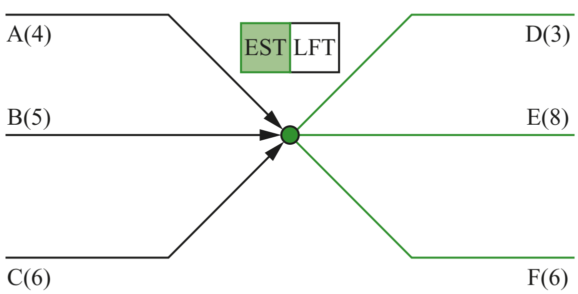

Finally, consider the Start vertex. There are two activities that begin at this vertex, and their LFT and LST are:

| **Activity** | **A** | **C** |
|---|---|---|
| **LFT** | 10 | 34 |
| **Weight** | 10 | 5 |
| **LST** | 0 | 29 |

The LFT is the smallest LST in this list: 0. Decorate the vertex with this number.

**!**

The following picture may help you to remember the difference between the EST and LFT.

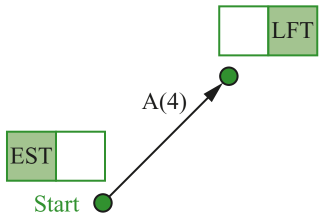


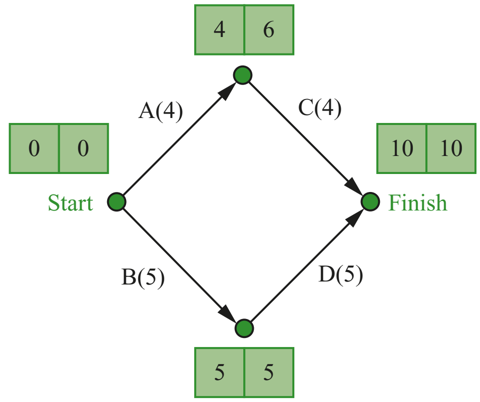

- The LFT is common to all activities with the same end vertex. It is determined by a backward scan.
- The EST is common to all activities with the same start vertex. It is determined by a forward scan.
- Add the edge weight to a starting time to find the corresponding finishing time:

  EST + weight = EFT

  LST + weight = LFT

**7** Here is an activity chart, together with the EST and LFT of each activity.
The EST of the activity is the left number at the start vertex.
The LFT of the activity is the right number at the end vertex.



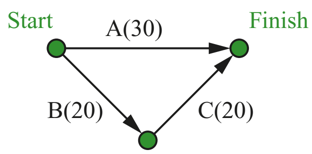




**a** Use the information in the chart to complete the following table.

| Activity | A | B | C | D |
|---|---|---|---|---|
| EST | | | | |
| EFT | | | | |
| LFT | | | | |
| LST | | | | |

**b** Which activities have an EFT that is later than its LST?

**8** The critical time of the following network is 40 minutes. Use a backwards scan to find the LFT of each activity.



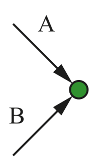

**9** Perform a backward scan on the activity chart from Exercise 10B question **16**.

**10** Here is a table of activities involved in calculating the LFT of an activity table.

| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Draw activity chart | 3 | None |
| B | Forward scan | 4 | A |
| C | Backward scan | 4 | B |
| D | Draw a neat activity chart | 9 | A |

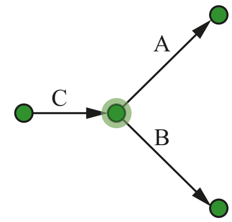
**a** Draw an activity chart for the activity table.
**b** Perform a forward scan to determine the EST of each activity.
**c** Perform a backward scan to determine the LFT of each activity.
**d** Redraw the activity chart with all this information (optional).

**11** Suppose activities A ‘Buy plane tickets’ and B ‘Pack bags’ share the same end vertex in an activity chart.
**a** Do the activities have the same LFT?
**b** Under what circumstances will the activities have the same LST?



**12** Activities A 'buy food' and B 'buy coffee' start at the same vertex (highlighted). This vertex represents your arrival at the food hall, which is the prerequisite of A and B. The LFT of activities A and B is known. The LFT of activity C 'drive to food hall' is not yet known.

| **Activity** | **A** | **B** |
|---|---|---|
| **Weight** | 10 | 5 |
| **LFT** | 24 | 20 |



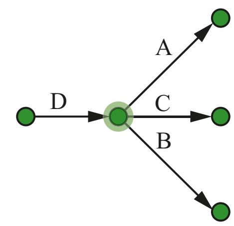

**a** What are the LSTs of activities A and B?

**b** What is the smallest LST of all activities that start at the highlighted vertex?

**c** Given the latest possible time for you to start buying food and coffee was given in part **b**, what is the latest time you can arrive at the food hall?

**13** Activities A 'buy food', B 'buy coffee' and C 'find a table' all share the same start vertex (highlighted). This vertex represents your arrival at the food hall, which is the prerequisite of A, B and C. The weight and LFT of these activities is given below. The LFT of activity D 'drive to food hall' is not yet known.

| **Activity** | **A** | **B** | **C** |
|---|---|---|---|
| **Weight** | 5 | 7 | 3 |
| **LFT** | 10 | 13 | 14 |




**a** What is the LST of each activity?

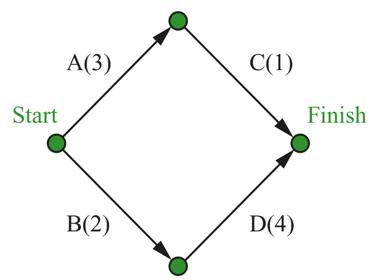
**b** What is the smallest LST of all activities that start at the highlighted vertex?

**c** The latest possible time for you to start buying food or coffee and find a table was given in part **b**. what is the latest time you can arrive at the food hall?

**14** Consider the activity chart below.

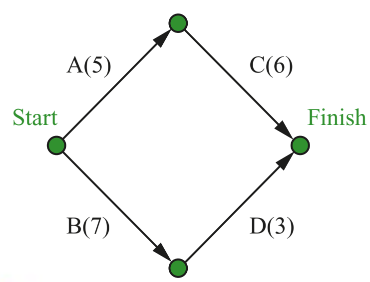
**a** List the EST, EFT, LFT and LST of each activity in the table.

| **Activity** | **A** | **B** | **C** | **D** |
|---|---|---|---|---|
| **EST** | | | | |
| **EFT** | | | | |
| **LFT** | | | | |
| **LST** | | | | |




**b** Which activities have the same earliest and latest start times?

**15** Consider the task of outsourcing each activity in the activity chart on the right. Weights are in days.




**a** Calculate the EST and LST of each activity in the chart.

**b** What is the critical time of the project?

**c** Unfortunately, one of your contractors is not very reliable and will take an extra day to complete any given task. Which tasks could be given to this contractor, without changing the critical time of the project?

**16** Consider the following after-school routine.

| Name | Activity | Time (hours) | Prerequisites |
|---|---|---|---|
| **A** | Relax | 1 | None |
| **B** | Homework | 2 | None |
| **C** | Walk dog | 0.5 | None |
| **D** | Dinner | 0.5 | B |
| **E** | Clean teeth and shower | 0.5 | D |

**a** If you outsource all of these activities to external contractors, how soon can your afternoon routine be complete?

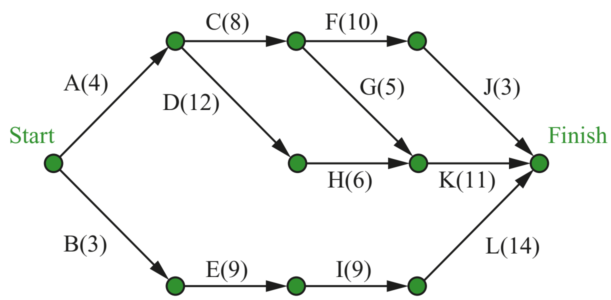
**b** Unfortunately, one of your contractors is not very reliable and may take an extra hour to complete any task. What tasks could you give to this contractor, without changing the critical time of this project?

**c** You hire an intern, who is cheap but takes an extra 2.5 hours to do any activity. Can you use your intern on this project without changing the critical time?

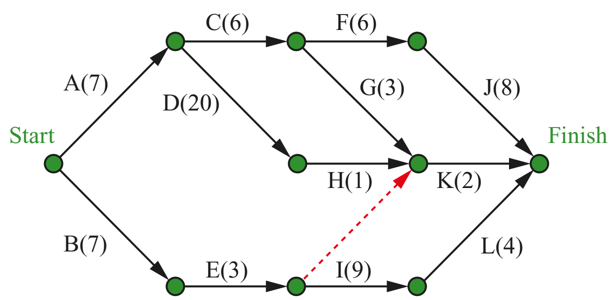


**17** Perform a forward scan and then a backward scan on each of the activity charts below.

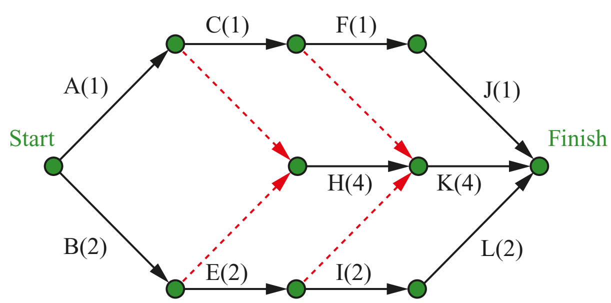
**a**




**b**




**c**



**critical path**
the longest path between the Start and Finish vertices; usually found by looking for activities with zero float time

**critical step**
any activity on the critical path is called a critical step; delay to a critical step will cause delay to the whole project

# 10D The critical path

These resources are available on your obook assess:

- **Worksheet 10D:** Practise your skills with critical paths
- **assess quiz 10D:** Test your skills with an auto-correcting multiple-choice quiz

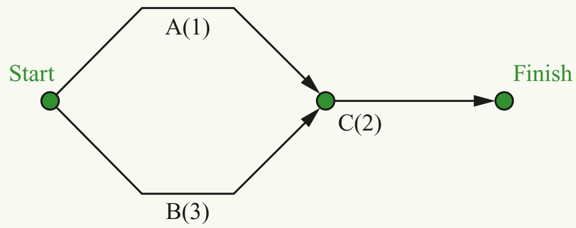
OI GI

This leads us to the major definition of this chapter: the **critical path**. The critical path is the longest path from the Start vertex to the Finish vertex. It represents the sequence of activities that are time-critical, in that any delay to these activities will cause a delay to the entire project. Any activity on the critical path is called a **critical step**.

There will always be at least one critical path. There can also be more than one.

## EXAMPLE 10D–1 Finding the critical path

Find the critical path in the following activity chart.



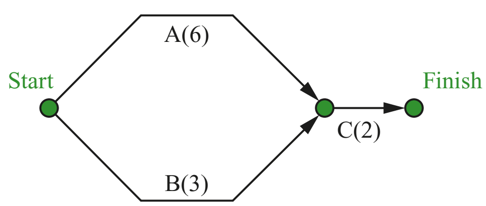

| Solve | Think/Apply |
|---|---|
| The critical path is BC. | AC and BC are the only two paths from Start to Finish. BC is the critical path because it is the longest. |

## EXERCISE 10D The critical path

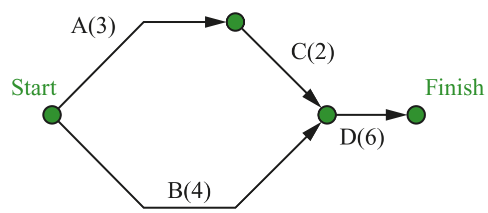
UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Find the critical path of the following activity charts.

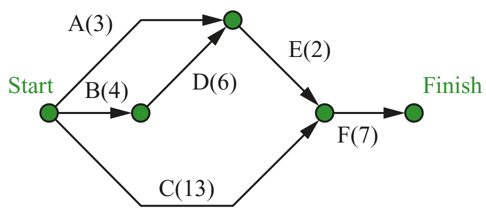
**a**



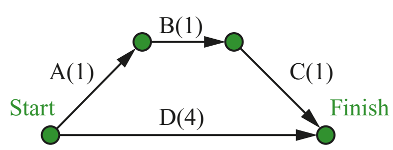

**b**




**c**




**2** Which path is the longest from Start to Finish?



UNDERSTANDING, FLUENCY AND COMMUNICATING

The **float time** measures the variability in the start time of an activity. It is calculated as the difference between the latest possible and earliest possible start times:

$$\text{LST} - \text{EST} = \text{float time}$$

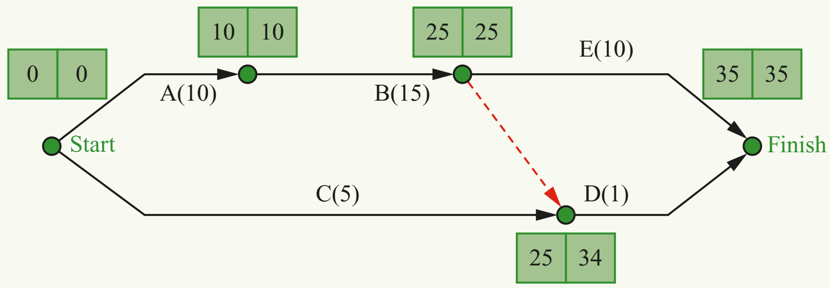
The critical path can be easily identified by the float time of the activities, because every activity in the critical path has a float time of zero.

**float time** the difference between the LST and the EST of an activity

## EXAMPLE 10D–2 Calculating float times

Complete the following task for the activity chart below.

**a** List the EST of each activity.
**b** List the LFT and LST of each activity.
**c** Find the float times of each activity.
**d** Find the critical path.




**Solve/Think/Apply**

**a** The EST for each activity is given as the left number at the start vertex of each activity.

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| EST | 0 | 10 | 0 | 25 | 25 |

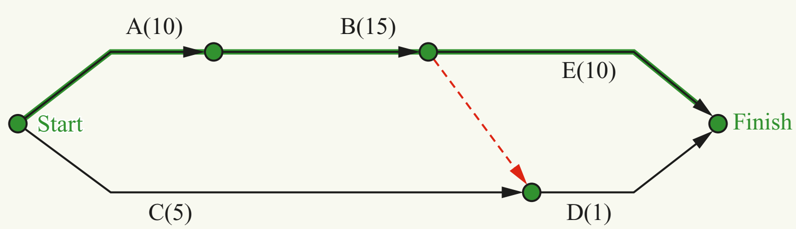
**b** The LFT for each activity is given as the right number at the end vertex of each activity. The LST can be calculated by subtracting the activity weight from the LFT.

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| LFT | 10 | 25 | 34 | 35 | 35 |
| LST | 10 − 10 = 0 | 25 − 15 = 10 | 34 − 5 = 29 | 35 − 1 = 34 | 35 − 10 = 25 |

**c** The float time is the difference between the late and early start times: LST − EST.

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| Float time | 0 − 0 = 0 | 10 − 10 = 0 | 29 − 0 = 29 | 34 − 25 = 9 | 25 − 25 = 0 |

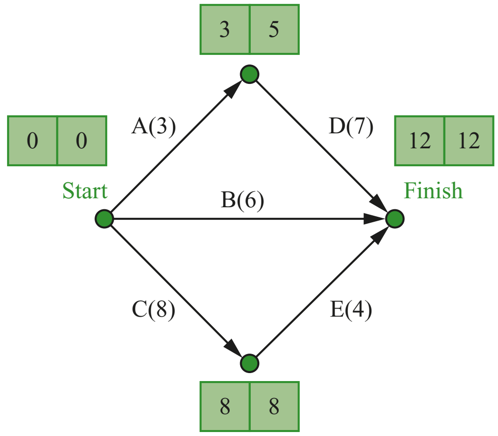
Edges with zero float time are coloured green in the network below.




**d** The critical path is ABE, as this is the only path from Start to Finish where the edges have zero float time.

NETWORKS

**3** Consider the activity chart with EST and LFT given on the right.




**a** List the EST, LFT and LST in the table.

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| EST | | | | | |
| LFT | | | | | |
| LST | | | | | |

**b** Find the float time for each activity.

**c** Find the critical path.

**4** Consider the following activity table.

| Name | Activity | Time (hours) | Prerequisites |
|---|---|---|---|
| A | Activity tables and charts | 3 | None |
| B | EST and EFT | 2 | A |
| C | LFT and LST | 2 | B |
| D | The critical path | 2 | C |
| E | Max flow/Min cut | 2 | None |

**a** Draw an activity chart for the activity table.

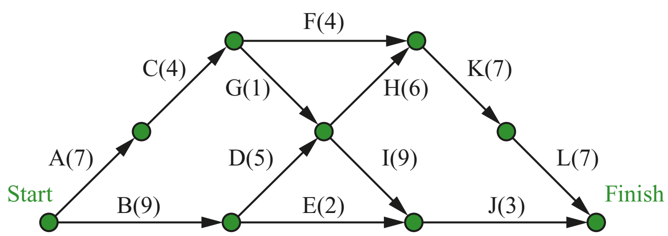
**b** Use a forward scan to determine the EST of each activity, and the critical time.

**c** Use a backward scan to determine the LFT of each activity.

**d** Which are the critical steps?

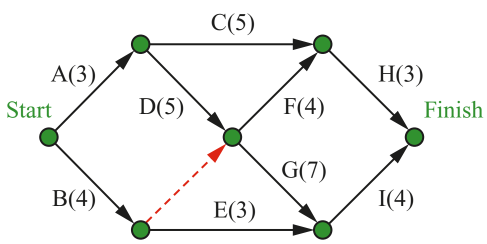
**e** What is the float time of activity E?

**5** Perform a forward scan then a backward scan to find the EST and LFT for the following network. Then find the critical path.

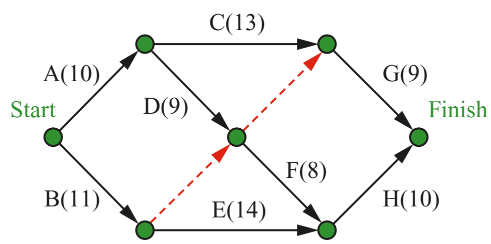



**6** Find the critical path of the following activity charts.

**a**

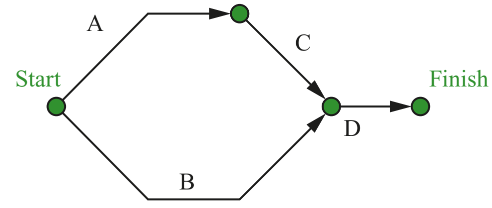



**b**

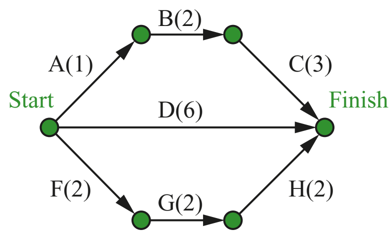



**7** Use your previous answers to Exercise 10C question **17** to find the critical path for each network.

**8** Without knowing the weights on the activity chart, it is impossible to determine the whole critical path. Name one activity in this chart that will be on the critical path regardless of the weights.




**9** How many critical paths are in the following network?

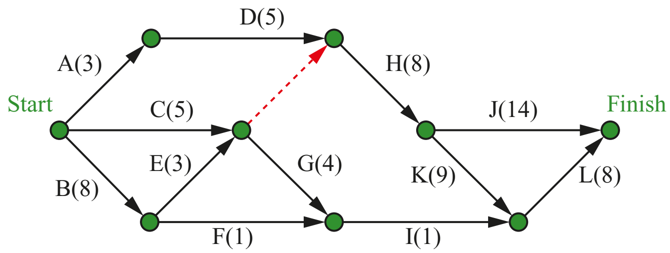



**10** The following activities are required to make a simple pasta dish.

| Name | Activity | Time | Prerequisites |
|---|---|---|---|
| A | Boil water in a pot (with a little salt and olive oil). | 8 | None |
| B | Cook noodles. | 7 | A |
| C | Finely chop onions and garlic. Cut cherry tomatoes in half. | 5 | None |
| D | Sauté onions and garlic in olive oil. Then add capers, anchovies, Kalamata olives and cherry tomatoes. | 8 | C |
| E | Add passata (and chilli, if desired). | 4 | D |
| F | Set table. | 4 | None |

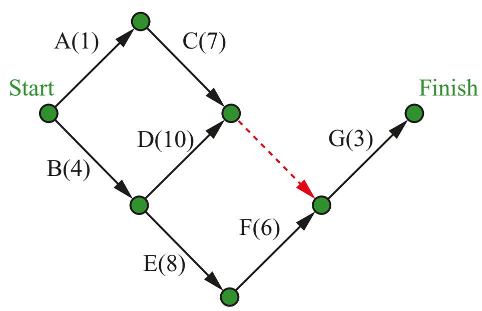
**a** Draw an activity chart for this simple but delicious pasta.
**b** What needs to be done first to ensure that your pasta is ready as soon as possible?
**c** To ensure that you eat your pasta as soon as the noodles are cooked (which is the best time to eat them), should you start cooking the noodles at the EST or the LST?

**11** Consider the activity chart below.




**a** Find the critical path for the project.
**b** What is the float time of activity J?

**12** Consider the activity chart on the right.
**a** Calculate the float times of each activity.
**b** What is the critical path?
**c** Which activity has the largest float time?



# 10E The maximum-flow minimum-cut theorem

These resources are available on your obook assess:

- **Worksheet 10E:** Practise your skills with maximum flows and minimum cuts
- **assess quiz 10E:** Test your skills with an auto-correcting multiple-choice quiz

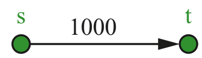
**flow diagram**
a weighted and directed network used to represent the transport of objects from a source to a sink

**maximum flow**
a flow capacity that is as large as possible

**flow capacity**
the total flow from source to sink

A **flow diagram** is a weighted and directed network with a source vertex (labelled s) and a sink vertex (labelled t). Unlike an activity chart, a flow diagram represents a transportation network of objects moving from the source to the sink. The direction of the edges indicates the direction of movement, and the weight of an edge represents the capacity, or a maximum rate of movement, along the edge.

The simplest example has only a source and a sink, with one edge between them.



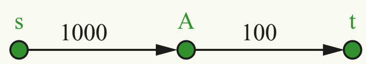

This network could represent two computers with a gigabit ethernet connection between them, where data can flow from computer s to computer t at a rate of up to 1000 megabits per second.

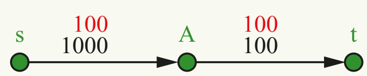
The network gives the capacities of each connection. But how much can actually flow from the source to the sink? This is called the **maximum flow** or the **flow capacity**.

## EXAMPLE 10E–1 Determining the maximum flow

Two computers are connected by ethernet cables of different speeds. The source computer (s) can transfer data to router A at a rate of 1000 Mbps (megabits per second). The router can transfer data to the sink computer (t) at a rate of 100 Mbps.

What is the maximum flow from source to sink?




| Solve | Think/Apply |
|---|---|
| 100 Mbps | Data can flow along edge sA at a rate of 1000 Mbps, but it will get slowed down at vertex A because no more than 100 Mbps can flow along the edge At. So the maximum flow from s to t is 100 Mbps.<br>We can represent this visually by writing the actual flow along each edge (in red), on top of the capacity of the edge.  |

**saturated**
an edge where the flow is equal to the capacity

**inflow**
the total flow of all edges pointing towards the vertex

**outflow**
the total flow of all edges pointing away from the vertex

In this topic, an edge has two numbers associated with it: a capacity (given by the question) and a flow (which is to be determined). For each edge, the flow must be a number between zero and the capacity of the edge.

$$0 \leq \text{flow} \leq \text{capacity}$$

In Example 10E–1, we would say that edge At is **saturated**, meaning that the flow along this edge is equal to the capacity of the edge.

Calculating the maximum flow in a network is a game: move as much as you can from s to t using as many edges as you like. The only rule is that there can be no storage of flow at the vertices. In other words, the **inflow** of a vertex must be equal to the **outflow**. This rule does not apply to the source (which has only outflow and no inflow) or the sink (which has only inflow and no outflow).

# EXERCISE 10E The maximum-flow minimum-cut theorem

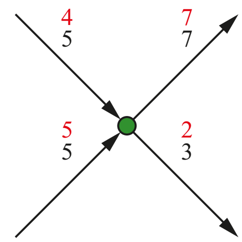
UNDERSTANDING, FLUENCY AND COMMUNICATING

**1** Consider the vertex for the flow diagram on the right, where the red edge weight is the flow, and the black edge weight is the capacity for each edge.

**a** Calculate the inflow to this vertex.

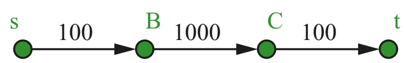
**b** Calculate the outflow from this vertex.

**c** Which edges are saturated (bottom left, bottom right, top left or top right)?

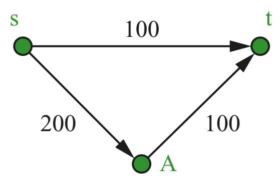



**2** Calculate the maximum flow in the following networks.

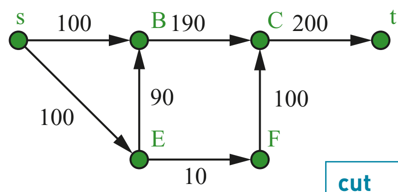
**a**




**b**




**c**




A **cut** in a flow diagram is a selection of edges that, once removed, disconnects the source from the sink. One of the best ways to visualise a cut is as a fence that completely encloses the source vertex with the sink vertex on the outside. Not every edge that crosses the fence is considered part of the cut, only those edges that start on the source side and end on the sink side are included. The **cut capacity** is the sum of the capacities of only those included edges.

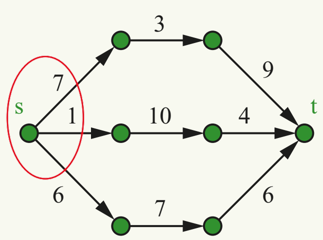
**cut**
a selection of edges that disconnects the flow from the source to the sink

**cut capacity**
the sum of all capacities in the cut

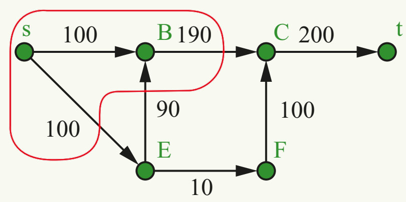
## EXAMPLE 10E–2 Finding the capacity of a cut

Find the capacity of the following cuts.

**a**




**b**

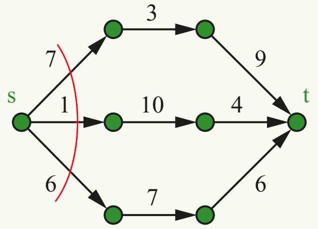



| | Solve | Think/Apply |
|---|---|---|
| **a** | The capacity of this cut is 14. | All three edges start on the source side and finish on the sink side, and so are part of the cut. The capacity of this cut is the sum of the capacities: $7 + 1 + 6 = 14$. |
| **b** | The capacity of this cut is 290. | Only edges BC and sE are part of this cut. The edge EB begins on the sink side and ends on the source side, and so is not part of the cut because it points the wrong way. The capacity of this cut is the sum of the capacities of BC and sE only: $190 + 100 = 290$. |

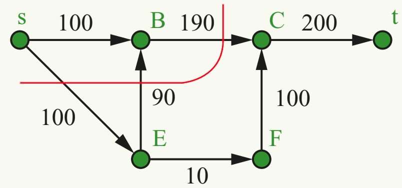
NETWORKS

| Solve | Think/Apply |
|---|---|

A shorter and more common way to draw a cut is as a curve that has the source on the left and the sink on the right. So, the examples above would usually be drawn as shown.




and




Even so, it is useful to think of the cut as enclosing the source vertex because then there is an inside and an outside, which helps you to decide which edges are part of the cut.

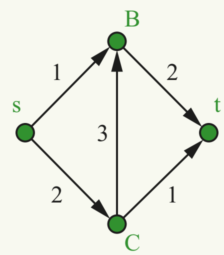
A **minimum cut** is a cut that has the smallest possible capacity. Finding the minimum cut is seemingly quite a different game to finding the maximum flow, because the minimum cut only concerns capacity and not flow.

**!**

**minimum cut**
a cut with the smallest possible capacity

## EXAMPLE 10E–3 Finding the minimum cut capacity

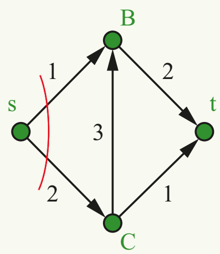
Find every possible cut in the network on the right. What is the minimal cut capacity?




### Solve/Think/Apply

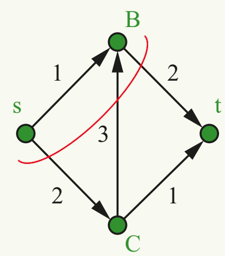
There are four possible cuts.

**a**



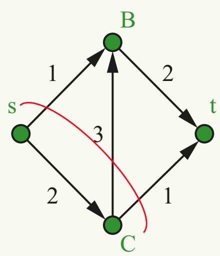

This has a cut capacity of 1 + 2 = 3.

**b**



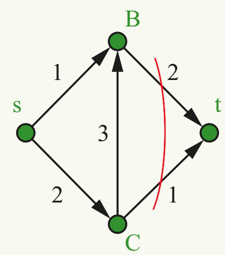

This has a cut capacity of 2 + 2 = 4. (Note that edge BC is not considered part of this cut.)

**c**




This has a cut capacity of 1 + 3 + 1 = 5.
(Note that edge BC is considered part of this cut.)

**d**

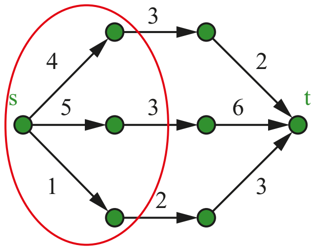



This has a cut capacity of 2 + 1 = 3.

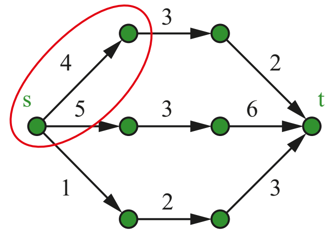
The minimal cut capacity is 3, which is the capacity of both cuts **a** and **d**.

**3** What is the capacity of the following cuts?

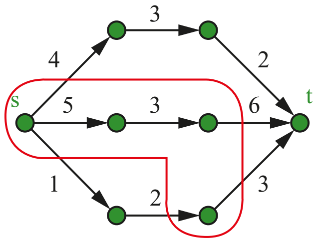
**a**



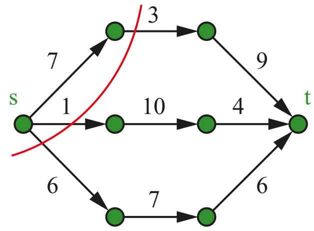

**b**




**c**

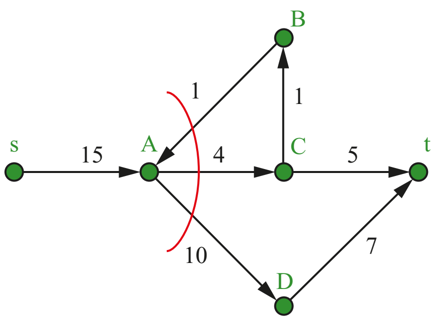



**d**



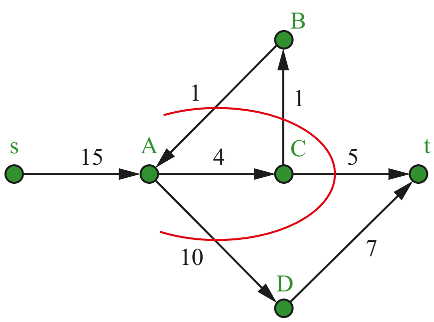

**4** Which edges are included by the cut?

**a**



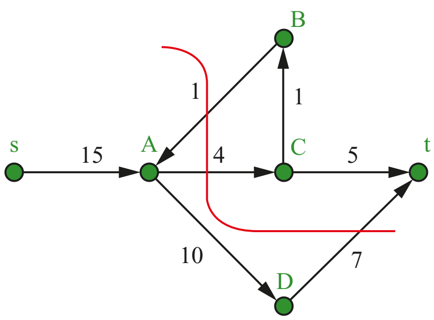

| **Edge** | sA | BA | AC | AD | CB | Ct | Dt |
|---|---|---|---|---|---|---|---|
| **Included in cut?** | No | | | | | | |

**b**




| **Edge** | sA | BA | AC | AD | CB | Ct | Dt |
|---|---|---|---|---|---|---|---|
| **Included in cut?** | | | | | | | |

**c**




| **Edge** | sA | BA | AC | AD | CB | Ct | Dt |
|---|---|---|---|---|---|---|---|
| **Included in cut?** | | | | | | | |

**d** Which of the cuts in parts **a**, **b** or **c** has the smallest capacity?

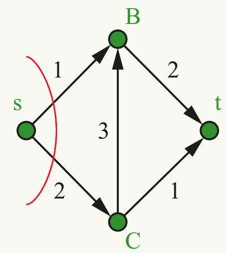
The maximum-flow minimum-cut theorem states that there is always some flow and cut with *equal* capacity. When this happens, the flow is the largest possible and the cut is the smallest possible.

This theorem allows you to 'guess and check' a maximum flow or a minimum cut, without checking every possible flow or cut.

- A cut is minimal when there exists a flow with the same capacity.
- A flow is maximal when there exists a cut with the same capacity.

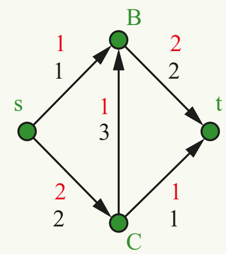
## EXAMPLE 10E–4 Finding the minimum cut

Reconsider the cut from Example 10E–3 part **a**. Show that this is the minimum cut without considering every cut possible.




### Solve/Think/Apply

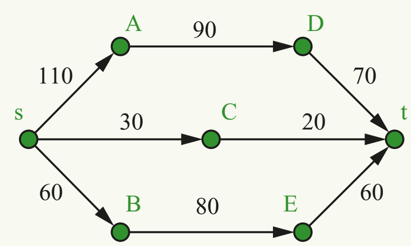
The total flow in the network is 3. So by the maximum-flow minimum-cut theorem, the cut with capacity 3 is minimal.




## EXAMPLE 10E–5 Finding the maximum flow

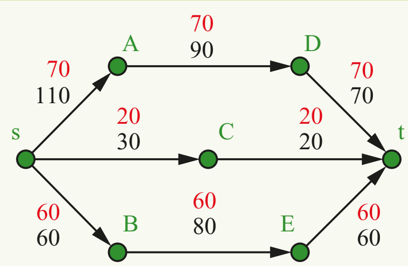
**a** Find a flow of 150 units from the source to the sink.

**b** Prove that this flow is the maximum possible.




### Solve/Think/Apply

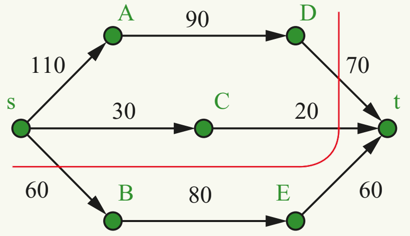
**a**



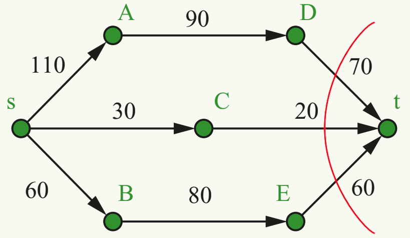

The total inflow to the sink vertex is $70 + 20 + 60 = 150$.

Note that the edges sB, Et, Ct and Dt are saturated.

**b** To prove that the flow is maximal by the maximum-flow minimum-cut theorem, we need only find a cut with capacity 150. There are two possibilities:




or

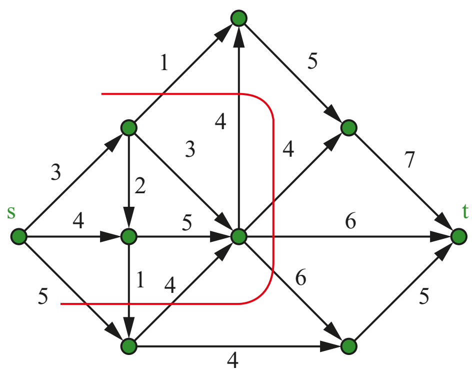



Both cuts have a capacity of $60 + 20 + 70 = 150$.

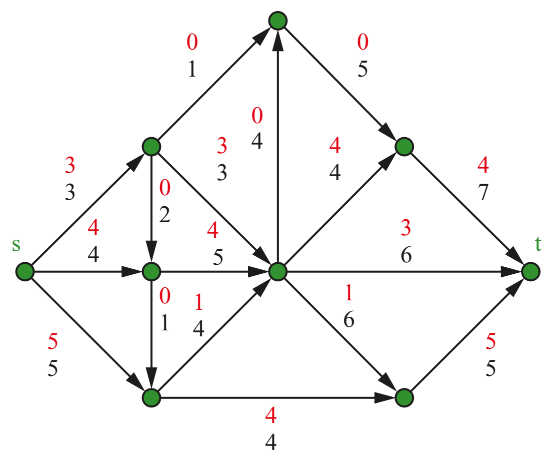
Therefore, by the maximum-flow minimum-cut theorem, 150 is the maximum flow.

Hint: when you already have a flow and are considering where to make a cut, consider only the edges that are saturated. !

**5** Find the capacity of the cut in the flow diagram.




**6** The flow along each edge is given in red above the capacity of the edge in black. What is the flow capacity specified below?



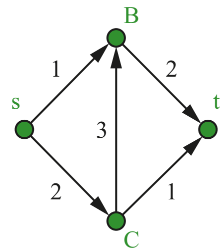
**7** Consider the flow diagram on the right.

**a** What is the maximum flow?

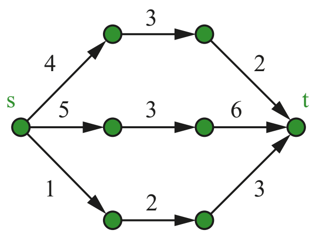
**b** From your answer to part **a**, which edges are saturated?

**c** What is the inflow of vertex B?

**d** What is the outflow of vertex B?



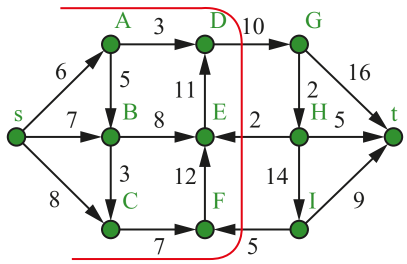

**8** What is the maximum flow in the following diagram?




**9** An electrician needs to isolate a broken power terminal (t) from the power source (s). The terminal is connected to the source by multiple connections. The direction of the flow of electricity, and the capacity of each connection, is given on the right.

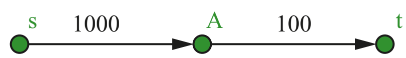
**a** The electrician removes only edge DG. Explain why the source s is no longer connected to the terminal t.

**b** Hence explain why the cut only contains one edge.

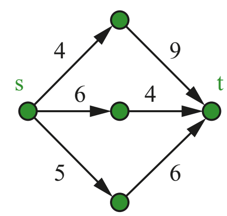



**10** Find the minimum cut in the following networks.

**a**



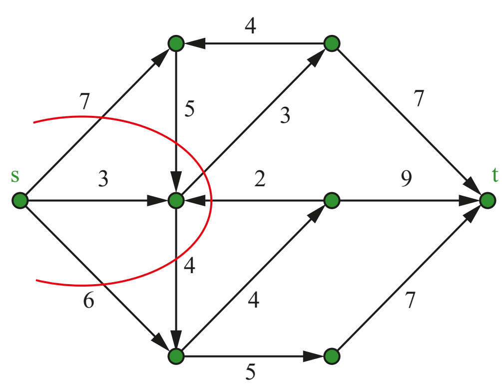

**b**




**11** Find the minimum cut in each network from question **2**.

**12** A plumber needs to fix a sink (t) that is connected to the water source (s) by an interconnected series of one-directional pipes. The capacity of each pipe is shown below.

**a** What is the capacity of the cut below?



**b** Explain why edges that start on the source side and point towards the sink are the only edges included in the cut.

**c** Find a cut with capacity 12.

**d** The pipes are really hard to cut and the plumber wants to know if this is the smallest cut possible (in terms of capacity). Prove to yourself, and the plumber, that this is a minimum cut.

**13** Show that the flow capacity in question **12** is a maximum flow, by creating a cut with the same capacity.

**14** A network of roads flows from the fairground (the source s) into the train station (the sink t). The capacity of each road is shown as an edge weight in cars per minute, and an intersection is shown as a vertex.


**a** At the end of the fair, the cars drive to the train station. What is the flow capacity of the traffic?

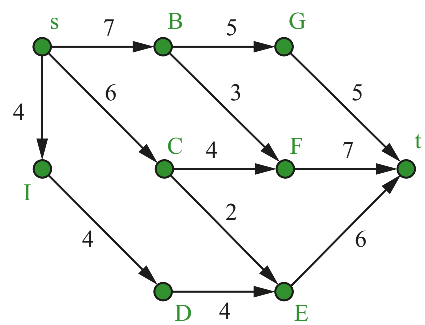
**b** A national gas pipe is being constructed. The pipe must cut across the town somewhere between the source and the sink. Find a line to cut across town with minimal capacity.

**c** Which roads are fully saturated?

**d** The council plans to widen one road to increase the flow capacity from source to sink. Which road should be widened?

**e** What is the largest capacity that the widened road should have?

**15** A shopping centre carpark (the sink) is connected to a main road (the source) by a series of one-way ramps. The direction and capacity of the ramps is shown on the right.




**a** What is the maximum flow from the main road to the carpark?

**b** Find a cut with the same capacity as your flow from part **a**. Does this make your cut minimal?

**c** What edges are saturated?

**d** Increase the capacity of one edge, so that every edge is saturated.

**16** The table below lists the capacities of creeks from a water source (the source s) to a lagoon (the sink t) measured in litres per second.

| **From** | s | s | s | A | B | C |
|---|---|---|---|---|---|---|
| **To** | A | B | C | t | t | t |
| **Capacity** | 1 L/s | 3 L/s | 5 L/s | 2 L/s | 4 L/s | 6 L/s |

Draw this information as a network diagram and answer the following questions.

**a** What is the maximum flow from the water source to the lagoon?

**b** What edges are included in the minimum cut for this network?

# CHAPTER 10 REVIEW CRITICAL PATH ANALYSIS

You should now be able to:

✔ construct an activity chart from an activity table

✔ determine the earliest start and finish times of an activity using a forward scan

✔ determine the latest finish and start times of an activity using a backward scan

✔ use float times to identify the critical path

✔ calculate the capacity of a flow and a cut through a network

✔ demonstrate that a flow is maximal, or that a cut is minimal, using the maximum-flow minimum-cut theorem.

Create a summary overview of this chapter. Include your own descriptions of key terms and strategies.

# REVIEW MULTIPLE-CHOICE QUESTIONS

10C **1** What number is always the smallest?

**A** EST **B** EFT **C** LST **D** LFT

10C **2** What number is always the largest?

**A** EST **B** EFT **C** LST **D** LFT

10D **3** The float time is the difference between which of the following?

**A** the start times **B** the finish times

**C** the late start and early finish **D** the late finish and early start times.

10D **4** Complete the sentence. The critical path is the ______ path from start to finish.

**A** shortest **B** longest **C** fastest **D** most important

10D **5** How many critical paths are in a network?

**A** zero **B** one **C** at least one **D** more than one

10E **6** If you have a flow capacity that is equal to a cut capacity, then by the maximum-flow minimum-cut theorem (select two):

**A** the flow is minimal **B** the flow is maximal **C** the cut is minimal **D** the cut is maximal.

10E **7** In a network flow diagram, any flow is always ___ any cut.

**A** smaller than **B** smaller than or equal to

**C** greater than or equal to **D** greater than

10D **8** Which formula does **not** give the float time of an activity?

**A** LST − EST **B** LFT − weight − EST

**C** LST − EST − weight **D** LFT − EFT

# REVIEW SET 1

**1** Construct an activity chart from the table on the right.

| Name | Cost (hours) | Prerequisites |
|---|---|---|
| A | 12 | None |
| B | 18 | None |
| C | 7 | A, B |
| D | 13 | C |

Use the activity chart from question **1** to answer questions **2–7**.

**2** Determine the EST and EFT of each activity.

**3** What is the critical time of the project?

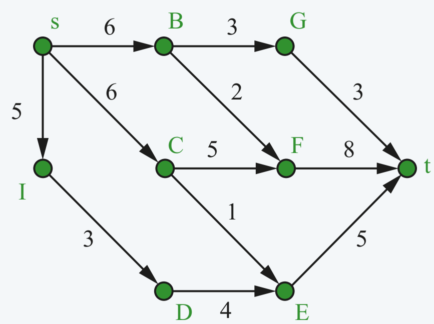
**4** Determine the LFT and LST of each activity.

**5** Which activities have a float time of zero?

**6** What is the critical path?

**7** A contractor offers to complete activity A in 6 hours instead of 12, for an additional fee. How does this affect the critical time of the project?

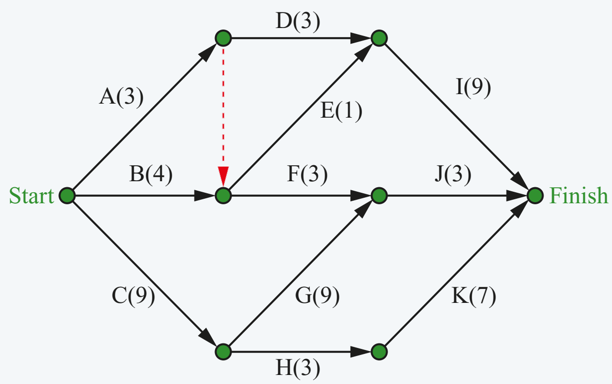
**8** Find the minimum cut in the network on the right.




**9** Find the maximum flow in the network from question **8**.

# REVIEW SET 2

**1** Construct an activity table from the activity chart on the right. Weights are in hours.




Use the activity chart from question **1** to answer questions **2–7**.

**2** Determine the EST and EFT of each activity.

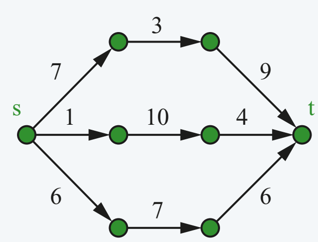
**3** What is the critical time of the project?

**4** Determine the LFT and LST of each activity.

**5** Which activities have a float time of zero?

**6** What is the critical path?

**7** If the time taken for activity D changes to 9 hours, does this change the critical time? How does this change the critical path?




**8** Find the maximum flow in the network on the right.

**9** Find the minimum cut for the network from question **8**.

NETWORKS

# REVIEW SET 3

**1** Draw an activity chart for the (incomplete) activity table on the right.

The critical path is B, C, F, G. The critical time for the project is 22 days.

| Activity | Duration (days) | Prerequisites |
|---|---|---|
| A | ? | None |
| B | 6 | None |
| C | 5 | B |
| D | 3 | A |
| E | 3 | D |
| F | 8 | A, C |
| G | ? | E, F |
| H | ? | A, C |

Use the activity chart from question **1** to answer questions **2–6**.

**2** Find the weight of activity G.

**3** The float time of activity H is 1 day. Find the weight of activity H.

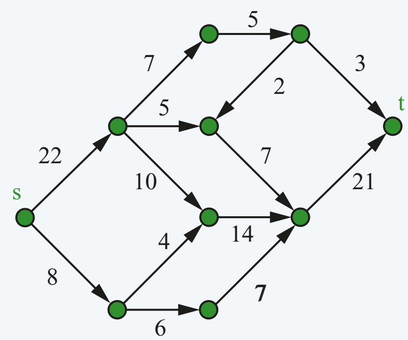
**4** What is the longest possible duration of activity A?

**5** The duration of activity F is reduced to 6 days. What happens to the critical time of the project?

**6** Will a further reduction of the duration of activity F to 5 days reduce the critical time even more?

**7** Find the maximum flow through the network on the right.

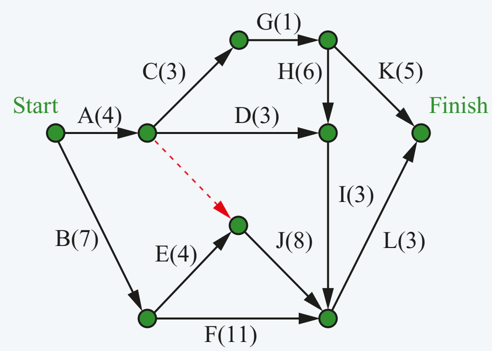
**8** Find a minimum cut for the network in question **7**.




# REVIEW SET 4

**1** Construct an activity table from the activity chart on the right. Weights are in days.

Use the activity chart from question **1** to answer questions **2–7**.




**2** Determine the EST and EFT of each activity.

**3** What is the critical time of the project?

**4** Determine the LFT and LST of each activity.

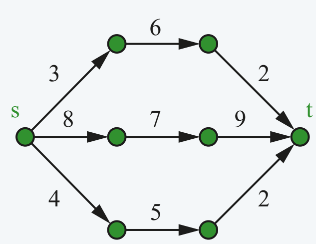
**5** Which activities have a float time of zero?

**6** What is the critical path?

**7** The time taken for activity D changes to 9 days. Does this change the critical time?

**8** Find the maximum flow in the network on the right.

**9** Find a minimum cut for the network in question **8**.



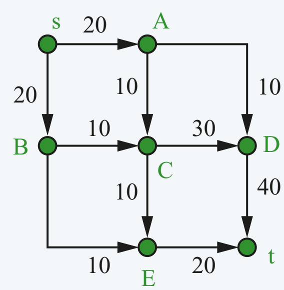
**1** **a** Consider the following activity chart.

| Activity | Duration (days) | Prerequisites |
|---|---|---|
| A | 12 | None |
| B | ? | None |
| C | ? | A |
| D | 13 | A, B |
| E | 10 | D |
| F | ? | C, E |
| G | ? | D |

The critical path is A, D, E, F. The critical time for the project is 46 days.

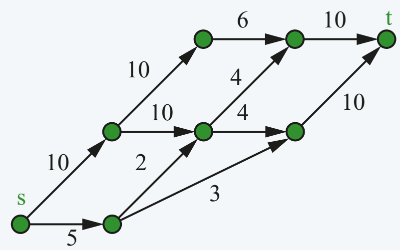
**i** Draw an activity chart for this activity table, with weights for activities A, D and E only. (1 mark)
**ii** Find the weight of activity F. (1 mark)
**iii** The float time of activity G is 17 days. Find the weight of activity G. (1 mark)
**iv** The float time of activity C is 14 days. Find the weight of activity C. (1 mark)
**v** What is the longest possible duration of activity B? (1 mark)

**b** A shopping centre carpark (the sink) is connected to a main road (the source) by a series of one-way ramps. The direction and capacity of the ramps is shown on the right.




**i** What is the flow capacity of this network? (1 mark)
**ii** Which edges are not saturated by your flow from part **a**? (1 mark)
**iii** Find a minimum cut for this network. (1 mark)
**iv** The shopping centre wants to optimise carpark access. Which of the following options will increase the flow capacity, to the point where every road is saturated?

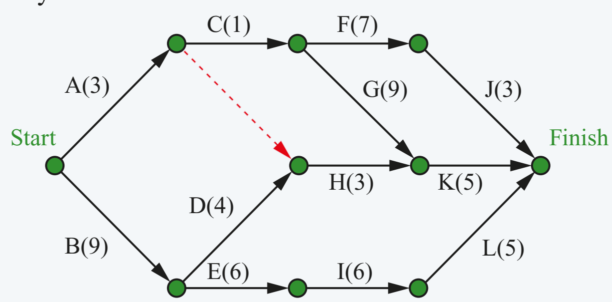
**A** Increase sA to capacity 40 and AC to capacity 40.
**B** Increase sB to capacity 30 and BC to capacity 20.
**C** Increase sB to capacity 40 and BC to capacity 40.
**D** Increase sA to capacity 40 and AC to capacity 30. (1 mark)




**c** Which one or more of the following is too big to be a flow in the network above right?

**A** 10 **B** 15 **C** 20 **D** 21 (1 mark)

**d** Consider the activity chart on the right. Weights are in days.




**i** What is the EFT of activity A? (1 mark)
**ii** What is the EST of activity H? (1 mark)
**iii** What is the critical path of the project? (1 mark)
**iv** What is the LST of activity K? (1 mark)
**v** What is the LFT of activity E? (1 mark)

**TOTAL: 15 marks**

# Chapters 8–10 CUMULATIVE REVIEW

**1** **a** Complete this table of values for $y = x^2 - 2x - 3$.

| $x$ | –2 | –1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | |

**b** Use the table of values to draw the parabola with equation $y = x^2 - 2x - 3$ for values of $x$ from –2 to 4.

**c** What are the coordinates of the $y$-intercept?

**d** What are the coordinates of the $x$-intercepts?

**e** What are the coordinates of the turning point?

**f** What is the equation of the axis of symmetry?

**g** What is the minimum value of the quadratic relationship?

**h** What is the value of $x$ that gives this minimum value?

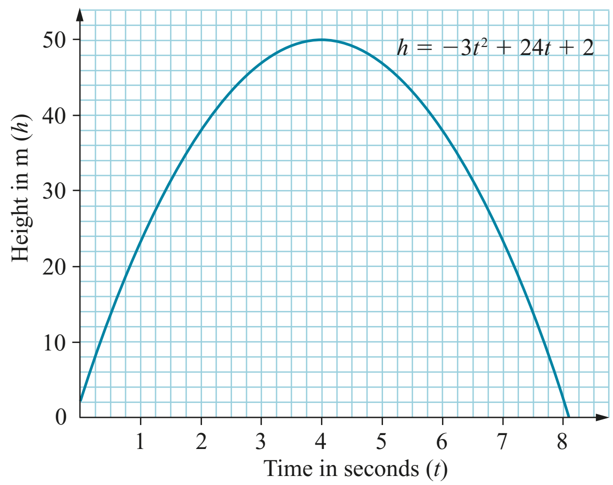
**2** A ball is thrown into the air. The height of the ball, $h$ metres, can be modelled by the quadratic equation $h = -3t^2 + 24t + 2$, where $t$ is the time in seconds.

**a** What is the maximum height of the ball?

**b** When does the ball reach its maximum height?

**c** When is the ball at a height of 38 m?

**d** What is the value of $h$ when $t = 9$? What limitation is there on this model?




**3** When a stone falls freely, the distance it falls is directly related to the square of the time taken and can be modelled by a quadratic equation of the form $d = at^2$, where $d$ is the distance in metres, $t$ is the time in seconds and $a$ is a constant.

**a** It takes 3 s for a stone to fall 44.1 m. Write the equation to model the relationship between distance and time.

**b** Use graphing technology or a table of values to draw the graph of this quadratic model for the first 10 s.

**c** Use the graph to find the distance a stone falls down a well in 6 s.

**d** Use the graph to find how long it would take for a stone to fall 300 m down a well.

**4** **a** Use graphing technology or a table of values to draw the graph of each of the following exponential relationships. Indicate the $y$-intercept for each graph.

**i** $y = 1.8(2^x)$ **ii** $y = 2.5(0.7)^x$ **iii** $y = 1000(1.03)^x$

**b** Which of your graphs from part **a** show exponential growth and which show exponential decay?

**5** The population of an island is modelled using the equation $P = 10\,000(1.01)^t$, where $P$ is the population and $t$ is the time in years.

**a** Complete this table of values using the equation $P = 10\,000(1.05)^t$.

| $t$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $P$ | | | | | |

**b** Use graphing technology or the table of values to draw the graph of the population model for values of $t$ from 0 to 20.

**c** What does the intercept of the curve with the vertical axis represent?

**d** Use the graph to estimate the population after 12 years.

**e** Estimate the time taken for the population to reach 22 000.

**f** Extrapolate (or extend) your graph to obtain an estimate of the time taken for the population to triple.

**g** What is the percentage increase in population each year? Explain how you can work this out by looking at the equation $P = 10\,000(1.05)^t$.

**6** The formula for the declining-balance method of depreciation is $S = V_0(1 - r)^n$, where $S$ is the salvage value of the asset in dollars, $V_0$ is the purchase price of the asset in dollars, $r$ is the interest rate per time period as a decimal, and $n$ is the total number of time periods.
The value of Melita's work tools depreciates at the rate of 25% per year and can be represented by an exponential model.

**a** If Melita bought the tools for \$1200, show that the equation of this exponential model is $S = 1200(0.75)^n$.

**b** Use technology or a table of values to draw the graph of $S = 1200(0.75)^n$ for values of $n$ from 0 to 10.

**c** Use the graph to find the salvage value of Melita's tools after 3 years.

**d** Use the graph to estimate the number of years it takes for the salvage value to be less than \$300.

**e** Does this model represent exponential growth or exponential decay? Explain.

**7** **a** Complete this table of values for the curve $y = \frac{48}{x}$.

| $x$ | –24 | –16 | –12 | –6 | –4 | –3 | –2 | –1 | 1 | 2 | 3 | 4 | 6 | 12 | 16 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | | | | | | | | | | | | | | | | |

**b** Plot these points on a number plane and join with a smooth curve.

**c** What type of curve has been drawn?

**8** The equation $s = \frac{480}{t}$ can be used to model the average speed necessary to travel 480 km in a particular time, where $t$ is the time in hours and $s$ is the speed in kilometres per hour.

**a** Complete this table of values using the equation $s = \frac{480}{t}$.

| $t$ | 4 | 6 | 8 | 12 | 16 | 24 |
|---|---|---|---|---|---|---|
| $s$ | | | | | | |

**b** Draw the graph of $s = \frac{480}{t}$ for values of $t$ from 4 to 24, with $t$ on the horizontal axis and $s$ on the vertical axis.

**c** Use the graph to find the average speed required to complete the 480 km trip in 7 hours.

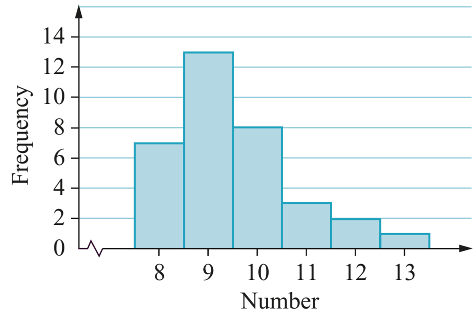
**d** Use the graph to estimate the time it would take to complete the trip at an average speed of 45 km/h.

**9** The time taken to complete a task varies inversely with the number of workers doing the task.

**a** If 27 workers can complete the task in 12 days, write an equation to model this situation.

**b** How long would it take 17 workers to complete the task, working at the same rate?

**10** **a** Draw a smooth curve representing the data in this histogram.




**b** Describe the shape of this curve.

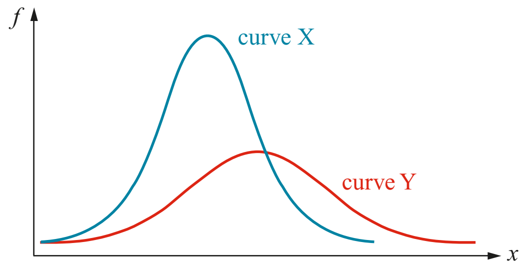
**c** Find the mean, median and mode of this frequency distribution.

**d** Can this distribution be classified as a normal distribution? Explain.

**11** Determine which curve has:

**a** the larger mean

**b** the larger standard deviation.




**12** The following table shows Katrina's marks in four tests.

| Test | Mark | Mean | Standard deviation |
|---|---|---|---|
| Term 1 | 46 | 56 | 8 |
| Term 2 | 52 | 60 | 5 |
| Term 3 | 53 | 59 | 10 |
| Term 4 | 58 | 61 | 4 |

**a** Calculate the standardised score for each test.

**b** In which test did Katrina perform best?

**13** The results for a test have a mean of 54 and standard deviation of 10. Write the raw mark corresponding to a $z$-score of:

**a** $-1$ **b** 0.8 **c** 0

**14** The heights of a large sample of Year 12 students are normally distributed with a mean of 168 cm and standard deviation of 11 cm. Between what limits would you expect the heights of:

**a** the middle 68% of Year 12 students to lie?

**b** the middle 95% of Year 12 students to lie?

**c** almost all, 99.7%, of Year 12 students to lie?

**15** For the normal distribution, state the percentage of data with the following $z$-score.

**a** between 0 and 1 **b** greater than 1

**c** between 0 and $-2$ **d** less than $-2$

**e** between 0 and 3 **f** less than 3

**16** The heights of a large sample of seedlings are normally distributed with a mean of 180 mm and a standard deviation of 8 mm.

**a** Calculate the height that is three standard deviations above the mean.

**b** Mati claims to have grown a seedling with height 210 mm. Is this likely? Explain your answer.

**17** Luke scored 58 in a state-wide test for which the marks were normally distributed with a mean of 65 and a standard deviation of 11.

**a** What is the probability, to two decimal places, that a randomly chosen student scored:

**i** between 54 and 76 on the test?

**ii** at least 70 on the test?

**iii** a lower mark than Luke?

**b** What percentage of students scored at least 80?

**c** If 10 000 students completed this state-wide test, how many do you expect to score at least 75?

**18** Inspector Poirot conducts a methodical criminal investigation. The steps he follows in his investigation are given in the table below.

| Name | Activity | Weight (hours) | Prerequisites |
|---|---|---|---|
| A | Investigate crime scene | 3 | None |
| B | Interrogate witnesses | 5 | None |
| C | Sit down and think | 4 | A, B |
| D | Solve the mystery | 1 | C |

Draw an activity chart for this activity table.

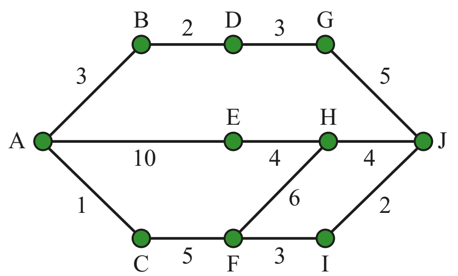
**19** Perform a forward scan to determine the EST of each activity in question **18**.

**20** Perform a backward scan to determine the LFT of each activity in question **18**.

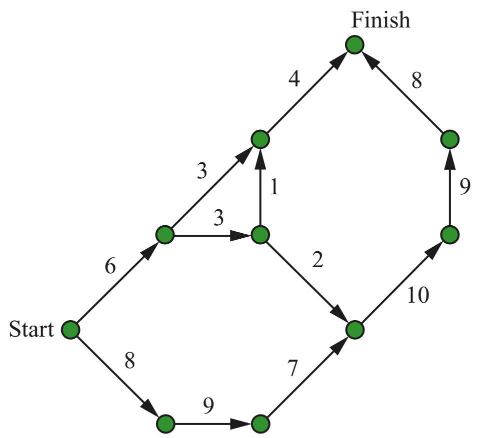
**21** What is the longest path from the start vertex to the end vertex in the network from question **18**?

**22** What is the shortest path from the start vertex to the end vertex in the network from question **18**?

**23** Find a minimal spanning tree for the following network.




**24** What is the maximum flow for the network below?




**25** Find a cut with the same capacity as your flow for question **24**.

# ANSWERS

## CHAPTER 1 INVESTMENTS, DEPRECIATION AND LOANS

### ARE YOU READY?

| | | | |
|---|---|---|---|
| 1 D | 2 B | 3 C | 4 C |
| 5 B | 6 B | 7 B | 8 D |
| 9 A | 10 D | 11 A | 12 A |
| 13 D | 14 A | | |

### EXERCISE 1A COMPARING SIMPLE AND COMPOUND INTEREST INVESTMENTS

Note: answers may vary slightly due to rounding.

**1** **a** \$1044 **b** \$457.60 **c** \$3.71

**2** **a** \$970.67 **b** \$27.41 **c** \$1345.25 **d** \$65.80

**3** **a** \$6255 **b** \$6526.25 **c** \$7141.82 **d** \$6339.18

**4** **a** Interest on the loan $= 14\,300 \times \frac{12.5}{100} \times 3$
$= \$5362.50$

**b** Total to be repaid $= 14\,300 + 5362.50$
$= \$19\,662.50$

**c** Monthly repayment $= \frac{19\,662.50}{36}$
$= \$546.18$

**5** **a** \$1995.20 **b** \$7795.20 **c** \$162.40

**6** **a**

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | \$2800 | \$196 | \$2996 |
| 2 | \$2996 | \$209.72 | \$3205.72 |
| 3 | \$3205.72 | \$224.40 | \$3430.12 |

**b** Interest $= 3403.12 - 2800$
$= \$603.12$

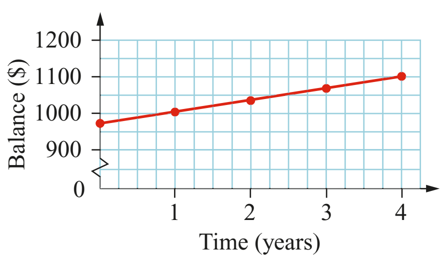
**7** **a** \$8362.03 **b** \$1862.03

**8** **a**

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | \$980 | \$29.40 | \$1009.40 |
| 2 | \$1009.40 | \$30.28 | \$1039.68 |
| 3 | \$1039.68 | \$31.19 | \$1070.87 |
| 4 | \$1070.87 | \$32.13 | \$1103.00 |

**b** \$123.00

**c**




**9** **a**

| Year | Balance at start of year | Interest | Balance at end of year |
|---|---|---|---|
| 1 | \$10 350 | \$796.95 | \$11 146.95 |
| 2 | \$11 146.95 | \$858.32 | \$12 005.27 |
| 3 | \$12 005.27 | \$924.41 | \$12 929.68 |
| 4 | \$12 929.68 | \$995.59 | \$13 925.27 |
| 5 | \$13 925.27 | \$1072.25 | \$14 997.52 |

**b** Yes, she will have \$14 997.52 when her investment matures, which is enough to buy the car.

**c** \$502.52

**10** **a**

| Time | Simple interest at 5% p.a. | Compound interest at 5% p.a. |
|---|---|---|
| Start of first year | \$5000 | \$5000 |
| Start of second year | \$5250 | \$5250 |
| Start of third year | \$5500 | \$5512.50 |
| Start of fourth year | \$5750 | \$5788.13 |
| Start of fifth year | \$6000 | \$6077.54 |
| Start of sixth year | \$6250 | \$6381.42 |

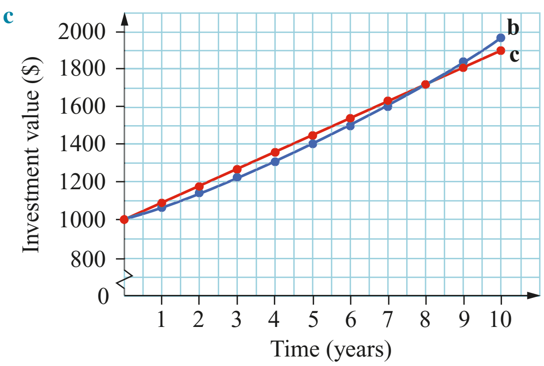
**b** \$131.42

**11** **a** Bank A by \$3.22 **b** Bank B by \$31.60

**c** The balance for bank A increases by the same amount each year, whereas the balance for bank B increases by a greater amount each and every month.

**12** **a** Year 1: \$1070, Year 2: \$1144.90, Year 3: \$1225.04, Year 4: \$1310.80, Year 5: \$1402.55, Year 6: \$1500.73, Year 7: \$1605.78, Year 8: \$1718.19, Year 9: \$1838.46, Year 10: \$1967.15

**b, c**

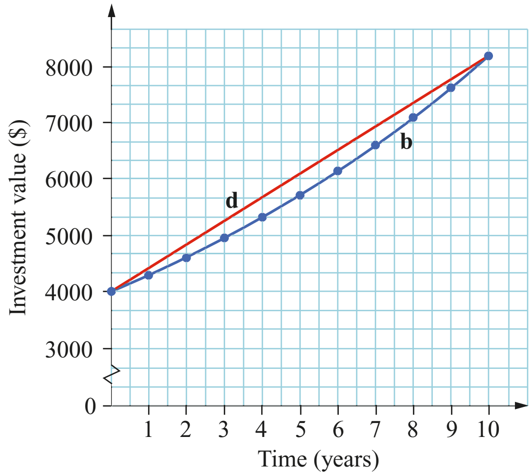



**d** 8 years

**13** **a**

| Time (end of year) | Investment (\$) | Time (end of year) | Investment (\$) |
|---|---|---|---|
| 1 | \$4296.00 | 6 | \$6138.83 |
| 2 | \$4613.90 | 7 | \$6593.10 |
| 3 | \$4955.33 | 8 | \$7080.99 |
| 4 | \$5322.03 | 9 | \$7604.99 |
| 5 | \$5715.86 | 10 | \$8167.76 |

**b, d**

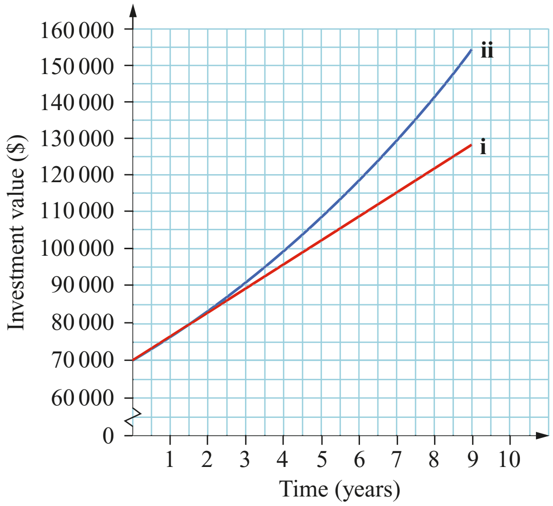



**c** Just under 7 years **e** 10.4% p.a.

**14 a i** $57 960 **ii** $84 561.66

**b** The simple interest investment produces the same amount of interest each year, whereas the compound interest investment produces a higher amount of interest each year (when compared to the previous year).

**c**




**d** The graph for simple interest is a straight line (and hence showing a linear relationship), as the investment is increasing by a constant amount each year.

**e** The graph for compound interest shows an exponential relationship, as the investment is increasing at a faster rate each year.

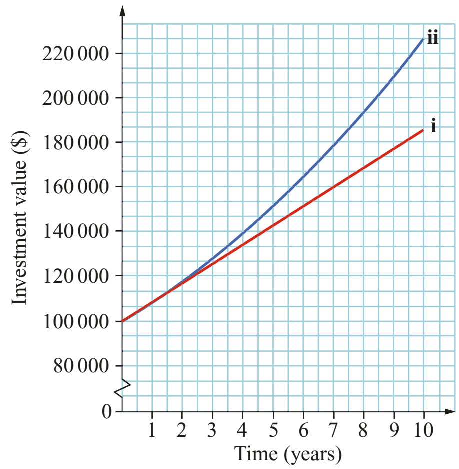
**f** The equation $A = Prn + P$ will give a straight line graph, with $P$ being the value of the vertical intercept (equivalent to $c$) and $Pr$ being the value of the value of the gradient (equivalent to $m$).

**g** The equation $A = P(1 + r)^n$ will give an exponential graph, with $P$ being equivalent to $k$ and $(1 + r)$ being equivalent to $a$ in the general equation for exponential equations, $y = ka^x$.

**15 a i** $185 000 **ii** $226 098.34

**b** The simple interest investment increases by the same amount each year, whereas the compound interest investment increases by a higher amount each year (when compared to the previous year).

**c**




**d** The simple interest graph shows a linear relationship, as the value of the investment increases at a constant rate.

**e** The compound interest graph shows an exponential relationship, as the value of the investment increases at an ever-increasing rate.

**f** Investors would prefer to invest their money using compound interest, as the value of the investment will rise at an ever-increasing rate throughout the lifetime of the investment.

## EXERCISE 1B THE COMPOUND INTEREST FORMULA

Note: answers may vary slightly due to rounding.

**1 a** $PV = \$6500 \quad r = 4.2 \div 100 = 0.042 \quad n = 7$

$$\begin{aligned} FV &= PV(1 + r)^n \\ &= 6500(1 + 0.042)^7 \\ &= 6500(1.042)^7 \\ &= \$8669.37 \end{aligned}$$

**b** Interest = 8669.37 − 6500 = $2169.37

**2 a** $6173.21 **b** $2173.21

**3 a** $7459.49 **b** $1006.49

**4** $PV = 1200 \quad n = 5 \times 4 = 20 \quad r = 0.084 \div 4 = 0.021$

$$\begin{aligned} FV &= PV(1 + r)^n \\ &= 1200(1 + 0.021)^{20} \\ &= 1200(1.021)^{20} \\ &= \$1818.43 \end{aligned}$$

**5** $1073.67

**6 a** $836.93 **b** $670.43 **c** $13 245.75
**d** $894.66 **e** $12 573.94 **f** $2566.36

**7** Option B gives the best return. It returns $1.59 more than option A and $1.12 more than option C.

**8** $FV = 6000 \quad r = 0.07 \quad n = 5$

$$\begin{aligned} PV &= \frac{FV}{(1 + r)^n} \\ &= \frac{6000}{(1 + 0.07)^5} \\ &= \frac{6000}{1.402...} \\ &= \$4277.92 \end{aligned}$$

**9** $1137.09

**10** $5288.97

**11** Quarterly interest rate $= 0.085 \div 4 = 0.02125$
Number of quarters $= 7 \times 4 \quad n = 28$

$$PV = \frac{FV}{(1+r)^n} = \frac{2300}{(1+0.02125)^{28}} = \frac{2300}{1.801...} = \$1276.53$$

**12** **a** \$1120.05 **b** \$6683.89 **c** \$2292.45 **d** \$663.24 **e** \$985.90 **f** \$5332.12
**13** \$2155.66 **14** \$16427.35 **15** Option A
**16** Option A
**17** **a** \$6945.75 **b** \$7045.45 **c** \$7146.10
**18** **a** \$29591.12 **b** \$32520.65 **c** \$35740.19
**19** \$93.08
**20** **a** \$29 530.99 **b** 9.53%
**21** \$5302.20
**22** **a** Option A: \$15666.41, Option B: \$16275, Option C: \$14977.81
**b** \$1297.19
**23** **a** Option A: \$66155.07, Option B: \$79796.11, Option C: \$78711.94
**b** The stock market is more volatile than the other two options, with the possibility of losing money as well as gaining money.
**24** **a** Option C, \$61242.29
**b** Option A, \$82000.02
**c** Option A: \$5907.28, Option B: \$3280, Option C: \$5575.33
**25** 6.6%
**26** The interest earned will only be doubled if there is only one compounding period. If there is more than one compounding period, then the doubled interest rate will earn more than double the interest of the lower interest rate.
**27** **a** **i** \$5624.32 **ii** \$5955.08 **iii** \$6298.56 **iv** \$6655
**b** A higher interest rate will produce a higher future value of the investment.
**28** **a** **i** \$5512.50 **ii** \$5522.43 **iii** \$5524.71 **iv** \$5525.82
**b** A shorter compounding period will produce a higher future value of the investment.
**29** **a** **i** \$10 616.78 **ii** \$11271.60 **iii** \$12704.89 **iv** \$16141.43
**b** No, the interest earned increases by more than double, as with a compound interest investment you also earn interest on the interest you have previously earned.

## EXERCISE 1C USING A COMPOUNDED VALUE TABLE

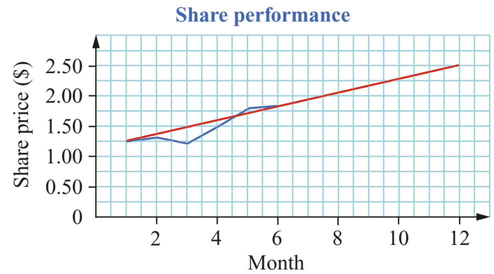
**1** **a** 1.158, \$9264 **b** \$9745 **c** \$21264 **d** 1.062, \$1189.44 **e** \$11581.50 **f** \$1836.54 **g** \$1661.60 **h** \$81635.22
**2** **a** \$7060 **b** \$5579 **c** \$11280 **d** \$1599 **e** \$7432.40 **f** \$3745.90 **g** \$916.70 **h** \$12975.31
**3** **a** 1.217, \$4930.16 **b** \$2327.13 **c** \$2794.12 **d** \$1500, \$810.37 **e** \$2382.91 **f** \$1877.13 **g** \$1041.67 **h** \$3510.64
**4** \$4342.80 **5** \$1723.50
**6** \$1953.82 **7** \$840

## EXERCISE 1D INFLATION AND APPRECIATED VALUE

**1** Price after 4 years $= 12.95 \times \left(1 + \frac{3.8}{100}\right)^4 = \$15.03$
**2** **a** \$2.99 **b** \$3.14 **c** \$3.30 **d** \$3.46 **e** \$3.64
**3** **a** \$548.45 **b** \$912.87 **c** \$584.40 **d** \$155.22 **e** \$421.66
**4** \$3.77 **5** \$33.22 **6** \$36.39
**7** \$63 251 **8** \$70 501
**9** **a** \$3882.40 **b** \$6029.30 **c** \$22 581.60
**10** **a** \$4057.45 **b** \$4703.70 **c** \$6321.40
**11** \$59211 **12** \$1794.78
**13** Under the expected value by \$438.66
**14** 2.53% **15** 3.77%
**16** **a** 8.51% **b** 4.56%
**17** **b** No; $1.03 \times 1.04 \neq 1.035^2$

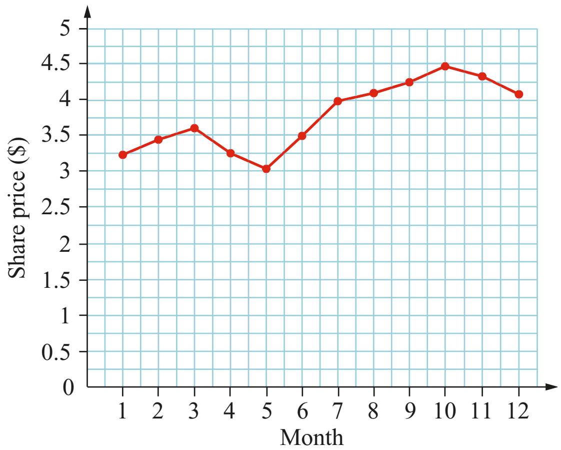
## EXERCISE 1E SHARES

**1** Cost of shares $= 600 \times \$7.97 = \$4782$
Brokerage $= 2.5\%$ of \$4782 $= 2.5 \div 100 \times 4782 = \$119.55$
Total cost $= 4782 + 119.55 = \$4901.55$
**2** **a** \$2859.75 **b** \$4627.88 **c** \$1314.56 **d** \$2982.75
**3** Selling income $= 800 \times 4.85 = \$3880$
Amount obtained $= 3880 - 29 = \$3851$
**4** **a** \$1447 **b** \$1672.50 **c** \$1728 **d** \$3417
**5** **a** \$7385.86 **b** \$704.16 **c** \$3711.51 **d** \$3227.40 **e** \$2581.92
**6** 73, 8.66% **7** 7.58%
**8** 5.11% **9** 3.23%
**10** 11%, $\frac{11}{100}$, 5.5, 1200, 6600, \$66
**11** **a** \$50 **b** \$42 **c** \$161
**12** **a** \$32 **b** \$10.50 **c** \$10 **d** \$11
**13** **a** \$4625 **b** \$625 **c** 13.51%
**14** **a** \$7326 **b** \$1089 **c** 14.86%
**15** **a** \$9968.13 **b** \$13138.13 **c** \$3170 **d** 31.8%
**16** **a** 5% **b** 18 cents **c** 3.23% **d** 12 cents **e** 80 cents
**17** **a** Abigail **b** Abigail **c** Stephen
**18** **a** \$4165 **b** \$4256.63 **c** \$287 **d** 6.74% **e** A loss of \$77.03
**19** **a**




**b** No. Given the trend they may only reach \$2.50.

**20** **a**




**b** We do not have enough information to know for certain. Share prices are calculated daily, and it is possible that the share price dropped below $3.00 at some point during the year.

## EXERCISE 1F DECLINING-BALANCE METHOD OF DEPRECIATION

**1**

| Year | Value ($) | Depreciation ($) | Depreciated value ($) |
|---|---|---|---|
| 1 | 24 900 | $0.22 \times 24\,900 = 5478$ | 19 422 |
| 2 | 19 422 | $0.22 \times 19\,422 = 4272.84$ | 15 149.16 |
| 3 | 15 149.16 | $0.22 \times 15\,149.16 = 3332.82$ | 11 816.34 |

**2**

| Year | Value ($) | Depreciation ($) | Depreciated value ($) |
|---|---|---|---|
| 1 | 34 800 | $0.18 \times 34\,800 = 6264$ | 28 536 |
| 2 | 28 536 | $0.18 \times 28\,536 = 5136.48$ | 23 399.52 |
| 3 | 23 399.52 | 4211.91 | 19 187.61 |
| 4 | 19 187.61 | 3453.77 | 15 733.84 |

**3** **a** $S = 35\,000\,(1 - 0.24)^3 = \$15\,364$
**b** Change in value $= 35\,000 - 15\,364 = \$19\,636$
**4** **a** $3867.90 **b** $16 122.10
**5** **a** $9996.35 **b** $46 003.65
**6**

$20\,462 = 29\,000(1 - r)^2 \quad \frac{20\,462}{29\,000} = (1 - r)^2$

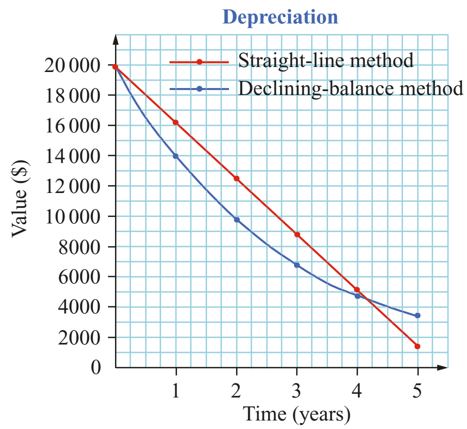
$\sqrt{\frac{20\,462}{29\,000}} = 1 - r = 0.839\ldots \quad r = 1 - 0.839\ldots \approx 0.16$

$\therefore$ Rate of depreciation $= 16\%$

**7** 27% **8** 15%
**9** 33% **10** 23%
**11** **a** $1610 **b** $690 **c** $483
**12** $19 600
**13** **a** $16 452.48 **b** 24%
**14** **a** $15 000
**b** **i** $12 200 **ii** $11 000
**c** **i** $2\frac{1}{2}$ years **ii** 2 years
**d** **i** 3.7 years **ii** $4600
**e** After $1\frac{1}{2}$ to 2 years ≈ $1500
**f** $2800
**15** **a** $6500
**b** **i** $4100 **ii** $3400
**c** **i** 2.1 years **ii** $1\frac{1}{2}$ years
**d** 4 years, $1700
**e** **i** $500 **ii** $1250
**f** $1200
**16** **a** $37 000
**b** **i** $30 000 **ii** $23 000
**c** **i** Just over 5 years **ii** Just less than 3 years
**d** 9 years, $4500 **e** $3600 **f** 21%
**17** **a**

| Year | Straight-line value ($) | Declining-balance value ($) |
|---|---|---|
| 0 | 19 900 | 19 900 |
| 1 | 16 200 | 13 930 |
| 2 | 12 500 | 9751 |
| 3 | 8800 | 6826 |
| 4 | 5100 | 4778 |
| 5 | 1400 | 3345 |

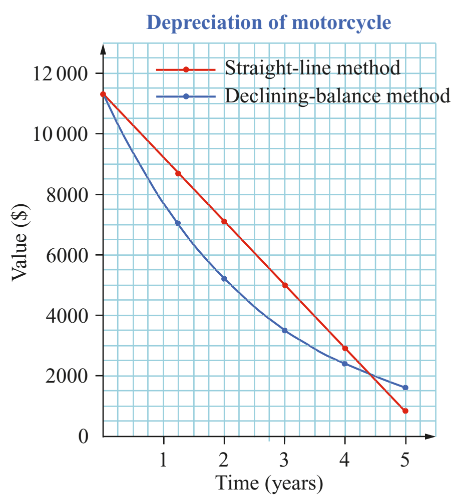
**b**




**c** $12 500, $9751
**d** $7000, $5500
**e** ≈ 4.2 years ≈ 4 years 2 months
**18** **a**

| Year | Straight-line value ($) | Declining-balance value ($) |
|---|---|---|
| 0 | 11 350 | 11 350 |
| 1 | 9250 | 7718 |
| 2 | 7150 | 5248 |
| 3 | 5050 | 3569 |
| 4 | 2950 | 2427 |
| 5 | 850 | 1650 |

**b**




**c** After about 4.4 years
**d** Straight-line: $4000, declining balance: $3000
**e** ≈ 2.7 years, ≈ 1.8 years

## EXERCISE 1G REDUCING-BALANCE LOANS

Note: answers may vary slightly due to rounding.

**1** a $3967.54 b $3151.16
c $2576.72 d $2460.52

**2** a $3471.28 b $3124.16 c $347.12

**3** a $240 000 b $280 000
c $320 000 d $320 000

**4** $120 000

**6**

| Month | Principal at the start of the month | Monthly interest | Monthly ® | Balance at end of month |
|---|---|---|---|---|
| 1 | $420 000 | $2520 | $3306.83 | $419 213.17 |
| 2 | $419 213.17 | $2515.28 | $3306.83 | $418 421.62 |
| 3 | $418 421.62 | $2510.53 | $3306.83 | $417 625.32 |
| 4 | $417 625.32 | $2505.75 | $3306.83 | $416 824.24 |
| 5 | $416 824.24 | $2500.94 | $3306.83 | $416 018.36 |
| 6 | $416 018.36 | $2496.11 | $3306.83 | $415 207.64 |

**7**

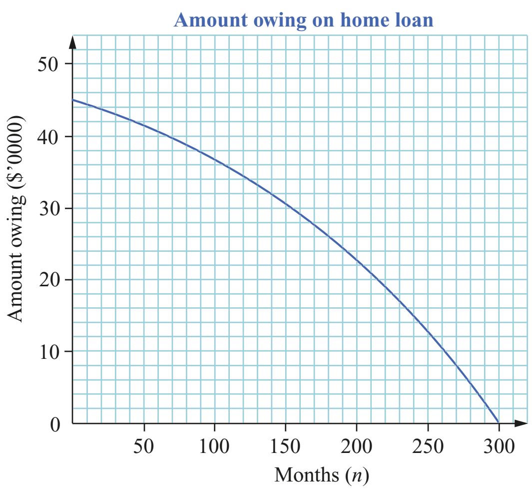
| Month | Principal at the start of the month | Interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 1 | $350 000 | $2216.67 | $2471.26 | $349 745.41 |
| 2 | $349 745.41 | $2215.05 | $2471.26 | $349 489.20 |
| 3 | $349 489.20 | $2213.43 | $2471.26 | $349 231.37 |
| 4 | $349 231.37 | $2211.80 | $2471.26 | $348 971.91 |
| 5 | $348 971.91 | $2210.16 | $2471.26 | $348 710.81 |
| 6 | $348 710.81 | $2208.50 | $2471.26 | $348 448.05 |

**8** a $1879.31 b $2328.55 c $5090.10
d $3250.96 e $5041.11 f $6003.10

**9** a $170 000 b $140 000
c 160 months d 180 months

**10** a $300 000 b $250 000 c $150 000
d $210 000 e 160 months f 260 months
g 200 months h 285 months

**11**




a $450 000 b $370 000 c $200 000
d 150 months e 200 months f 300 months

**12** a $227.35 b $509.88
c $1145.63 d $487.18

**13** a $164.76 b $190.96
c $282.33 d $273.54

**14** a i $395.96 ii $612.63 iii $429.38
iv $604.61 v $323.56
b i $20 872.80 ii $3872.80 iii 5.70%

**15**

| Month | Principal at start of the month | Monthly interest | Account-keeping fee | Monthly repayment | Balance at end of month |
|---|---|---|---|---|---|
| 1 | 15 000 + 390 = $15 390 | $184.68 | $10 | $413 | $15 171.68 |
| 2 | $15 171.68 | $182.06 | $10 | $413 | $14 950.74 |
| 3 | $14 950.74 | $179.41 | $10 | $413 | $14 727.15 |
| 4 | $14 727.15 | $176.73 | $10 | $413 | $14 500.88 |
| 5 | $14 500.88 | $174.01 | $10 | $413 | $14 271.89 |
| 6 | $14 271.89 | $171.26 | $10 | $413 | $14 040.15 |

**16**

| Month | Principal at start of the month | Monthly interest | Account-keeping fee | Monthly repayment | Balance at end of month |
|---|---|---|---|---|---|
| 1 | 18 000 + 260 =$18 260 | $237.38 | $8 | $434 | $18 071.38 |
| 2 | 18 071.38 | $234.93 | $8 | $434 | $17 880.31 |
| 3 | 17 880.31 | $232.44 | $8 | $434 | $17 686.75 |
| 4 | 17 686.75 | $229.93 | $8 | $434 | $17 490.68 |
| 5 | 17 490.68 | $227.38 | $8 | $434 | $17 292.06 |
| 6 | 17 292.06 | $224.80 | $8 | $434 | $17 090.86 |

**17** a 30 months b Same time

**18** a 60 months b $770
c No, he will still pay it back in 3 years

## EXERCISE 1H CREDIT CARDS

Note: answers may vary slightly due to rounding.

**1** a $10 b $11.70 c $37.35 d $10

**2** a 25 July b 24 January c 27 March

**3** a 45 b 26 c 47 d 39

**4** **September statement**
Opening balance: $0
Closing balance: $134
Minimum payment due: $10
Due date: 25 Oct

**October statement**
Opening balance: $134
Closing balance: $1147
Minimum payment due: $45.88
Due date: 24 Nov

**November statement**
24 Nov −1147
Opening balance: $1147
Closing balance: $114
Minimum payment due: $10
Due date: 25 Dec

**5** Daily interest rate $= 22.9\% \div 365 = 0.062\ 74\%$
$= 0.000\ 627\ 4$

Balance $= 500 \times (1 + 0.000\ 627\ 4)^{23}$
$= \$507.27$

Cash advance fee $= \frac{1.5}{100} \times 500 = \$7.50$

Total amount due $= 507.27 + 7.50$
$= \$514.77$

**6** $461.15 **7** $154.22 **8** $271.88

**9** $233.22 **10** $418.67 **11** $822.59

**12** **a**

| November statement | | |
|---|---|---|
| **Date** | **Details** | **Amount ($)** |
| 1 Nov | Opening balance | 517 |
| 6 Nov | Purchase | 29 |
| 7 Nov | Payment | −300 |
| 12 Nov | Cash advance | 200 |
| 13 Nov | Purchase | 134 |
| 19 Nov | Purchase | 97 |
| 20 Nov | Payment | −100 |
| 24 Nov | Purchase | 166 |
| 25 Nov | Cash advance | 150 |

**b** $907.55 **c** $14.55 **d** $36.30

**15** Card B (by $3.25)
**16** Card A by $4.20
**17** **a** $50 **b** $30.03
**18** $6.26

## REVIEW MULTIPLE-CHOICE QUESTIONS

**1** D **2** B **3** C **4** A **5** B
**6** B **7** C **8** A **9** C **10** A
**11** C **12** B **13** D **14** C **15** B

## REVIEW SET 1

**1** **a** $7548.92 **b** $1548.92
**2** $3239.81 **3** $3698.77 **4** $5.75
**5** **a** $28 483.67 **b** $34 516.33
**6** $4243.50 **7** $57.20
**8** **a** $160 000 **b** $240 000 **c** $320 000 **d** $320 000
**9** **a** $574.67 **b** $552.13
**10** **a** $20.80 **b** $10
**11** **a** 26 July **b** 25 November
**12** $122 214.72

## REVIEW SET 2

**1** $4014.78 **2** $1250.20 **3** $7357.07
**4** $27.45 **5** 3.63% **6** $144
**7** **a** $14 298.13 **b** $14 701.87
**8** 24%
**9** **a** $2429.90 **b** $2777.04 **c** $347.14
**10** **a** $125 000 **b** 120 months
**11** $614.97

## REVIEW SET 3

**1** **a** $2811.53 **b** $353.53
**2** $5973.17 **3** $5.59
**4** **a** $9877.55 **b** $5122.46
**5** 19% **6** $5672.50
**7** $121.50 **8** **a** $4959.44 **b** $3075.64
**9** **a** $175 000 **b** 170 months
**10** **a** $13.80 **b** $10
**11** $67 049.05 **12** $22 055.79

## REVIEW SET 4

**1** $3612.92 **2** $190 **3** $1419.79
**4** $8000.28 **5** $3.65 **6** 26%
**7** $17 988.75 **8** 7.07%
**9** **a** $2460.52 **b** $2576.72
**10** **a** $255 000 **b** 220 months
**11** $385.76

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** **i** 0.02 **ii** 20 **iii** $9287.17 **iv** $3037.17
**b** **i** $11 631.51 **ii** 82 cents **iii** 2.16%
**c** **i** $1503.63
**d** $4633.00
**e** $1317.95
**f** $319 610.92

# CHAPTER 2 NON-RIGHT-ANGLED TRIGONOMETRY

## ARE YOU READY?

**1** B **2** A **3** D **4** B
**5** C **6** D **7** B **8** A
**9** C **10** B **11** B **12** C

## EXERCISE 2A REVIEW OF RIGHT-ANGLED TRIGONOMETRY

**1** **a** $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$
**b** $\sin\theta = \frac{24}{26}$, $\cos\theta = \frac{10}{26}$, $\tan\theta = \frac{24}{10}$
**c** $\sin\theta = \frac{24}{25}$, $\cos\theta = \frac{7}{25}$, $\tan\theta = \frac{24}{7}$
**2** **a** 0.7314 **b** 0.6157 **c** 2.2460
**d** 0.9755 **e** 0.2438 **f** 0.2096
**g** 3.7062 **h** 0.6057 **i** 0.6915
**j** 0.4376 **k** 0.6856 **l** 0.4861
**m** 3.0208 **n** 0.8699 **o** 1.4469 **p** 0.9835
**3** **a** 35°22′ **b** 81°48′ **c** 28°48′
**d** 26°21′ **e** 19°41′ **f** 58°21′
**g** 47°45′ **h** 24°2′ **i** 2°51′
**4** **a** 39°10′ **b** 47°1′ **c** 20°33′
**d** 52°8′ **e** 61°3′ **f** 62°58′
**g** 56°21′ **h** 55°22′ **i** 39°15′
**5** **a** 25°23′ **b** 20°33′ **c** 62°58′
**d** 28°18′ **e** 58°28′ **f** 46°47′
**g** 70°15′ **h** 59°56′ **i** 65°41′
**6** **a** 22.7 m **b** 23.9 m **c** 70.4 m
**d** 74.3 cm **e** 16.6 m **f** 21.7 mm
**g** 74.0 cm **h** 82.3 m **i** 34.0 m
**7** **a** 56.7 m **b** 19.0 cm **c** 9.5 m
**d** 77.6 m **e** 15.4 cm **f** 26.8 km
**g** 214.0 cm **h** 10.7 m **i** 147.8 mm
**8** **a** 5.7 m **b** 2.0 m
**9** 11.7 m **10** 69°54′ **11** 49°18′
**12** 38.2 m **13** 25°46′ **14** 27°55′
**15** 14 371 m **16** 5.0 m **17** 275 cm or 2.75 m

## EXERCISE 2B ANGLES OF ELEVATION AND DEPRESSION

**1** 204 m **2** 42.8 m **3** 60°39′
**4** 17°13′ **5** 446.7 m **6** 19°48′
**7** 3°49′
**8** **a** 23°12′ **b** 761.5 m
**9** **a** 16.2 m **b** 17.8 m
**10** 10.4 m

## EXERCISE 2C BEARINGS AND NAVIGATIONAL METHODS

**1** a NE b E c SW
**2** a N27°W b S51°W c N31°E
**3** a 175°T b 215°T c 046°T
**4** a i S44°E ii 136°T
b i S59°W ii 239°T
c i N24°W ii 336°T
d i S69°W ii 249°T
e i N62°W ii 298°T
f i S34°E ii 146°T
g i N8°W ii 008°T
h i S19°W ii 199°T
i i N77°W ii 283°T
**5** a 303°T b 227°T
c 051°T d 015°T
**6** 33.9 km **7** 70.5 km
**8** 2.5 km **9** 50.4 km
**10** 151°T **11** 058°7′ T
**12** N69°W **13** 60.2 km west, 18.4 km south
**14** 62.3 km east, 46.9 km south
**15** S54°W **16** N53°W, 992 km
**17** a 328°T b 148°T
**18** a 360 km b 206.5 km c 2 h 28 min
**19** a i 5.6 km ii 3.2 km iii 8.8 km
b i 2.1 km ii 3.8 km iii 5.9 km
c i 10.6 km ii 056°10′
**20** a i 6.2 km ii 10.5 km iii 4.3 km
b i 7.4 km ii 7.9 km iii 15.3 km
c i 15.9 km ii 074°18′

## EXERCISE 2D AREA OF A TRIANGLE

**1** a 31.0 $cm^2$ b 118.8 $m^2$
c 52.6 $mm^2$ d 135.8 $cm^2$
e 68.6 $cm^2$ f 263.3 $cm^2$
**2** 28.4 cm **3** 54°14′
**4** 693 $cm^2$
**5** Angle $C$ is 90° so $\sin C = 1$.
Area $= \frac{1}{2}ab \sin C$ becomes Area $= \frac{1}{2}ab \times 1$ or Area $= \frac{1}{2}ab$.
This is equivalent to Area $= \frac{1}{2} \times$ base $\times$ height, where $a$ is the base and $b$ is the height of the triangle.

## EXERCISE 2E SINE RULE

**1** a 6.2 cm b 29.0 cm c 8.6 cm
d 15.3 m e 10.1 cm f 32.4 cm
**2** a 20.8 cm b 13.4 cm c 28.0 cm
d 6.9 cm e 15.7 cm f 20.2 cm
**3** a 16.4 cm b 11.7 cm c 23.1 cm
d 44.8 mm e 0.5 m f 43.2 cm
**4** a 28°32′ b 22°4′ c 27°31′
d 21°2′ e 46°57′ f 50°44′
**5** a 22.4 cm b 31.6 cm c 53.7 mm
**6** a 85.4 $cm^2$ b 85.6 $cm^2$ c 448.7 $cm^2$
**7** a 133° b 20° c 19.9 m
d 14.6 m e 28.6 m
**8** a 60° b 53° c 67°
d 28.7 km e 310.7 $km^2$

**9** a $\sin C = \sin 90° = 1$

b $\frac{\sin A}{a} = \frac{\sin C}{c}$

$\frac{\sin A}{a} = \frac{1}{c}$

$\sin A = \frac{1 \times a}{c}$

$\sin A = \frac{a}{c}$

c $\frac{\sin B}{b} = \frac{\sin C}{c}$

$\frac{\sin B}{b} = \frac{1}{c}$

$\sin B = \frac{1 \times b}{c}$

$\sin B = \frac{b}{c}$

d For reference angle $A$ in the right-angled triangle:

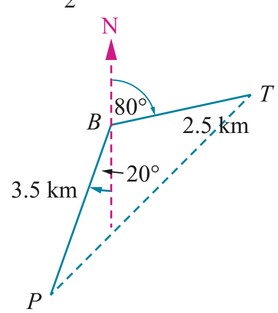
$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}$

For reference angle $B$ in the right-angled triangle:

$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$

## EXERCISE 2F COSINE RULE

**1** a 14.6 cm b 33.3 m c 16.5 mm
d 19.3 cm e 24.9 cm f 24.5 cm
g 14.0 cm h 21.1 cm i 36.4 m
**2** a 33° b 63° c 42°
d 53° e 34° f 84°
**3** a 71°46′, 59°48′, 48°26′
b 24°43′, 69°27′, 85°50′
c 42°26′, 63°45′, 73°49′
**4** a 41°2′ b 29°12′ c 25°40′
**5** a 10.9 cm b 23.5 cm c 22.5 cm
**6** a 5.6 $cm^2$ b 25.2 $cm^2$ c 147.4 $cm^2$
**7** a $C$ b 76°49′
c Area $= \frac{1}{2} \times 4.3 \times 4.7 \times \sin 76°49' \approx 9.8\ m^2$
**8** a, b




c 120° d 5.2 km
e 25° f 045°T or NE
**9** a Angle $C$ is 90° so $\cos C = 0$.
b When $\cos C = 0$:
$c^2 = a^2 + b^2 - 2ab \cos C$
becomes $c^2 = a^2 + b^2 - 0$
or $c^2 = a^2 + b^2$
c In a right-angled triangle, one of the angles is 90°. The cosine rule that involves this 90° angle simplifies to become Pythagoras' theorem, as $\cos 90° = 0$.

## EXERCISE 2G MORE PROBLEMS INVOLVING TRIGONOMETRY

**1** **a** $x = 36.2$ cm, $y = 31.5$ cm
**b** $x = 31.5$ cm, $y = 12.3$ cm
**2** **a** $x = 26.6$ cm, $y = 28.0$ cm
**b** $x = 21.5$ cm, $\theta = 35°5'$
**c** $x = 27.1$ cm, $y = 20.4$ cm
**d** $x = 120.0$ cm, $\theta = 40°0'$
**3** **a** 95°, 85° **b** 4.2 km **c** 005°T
**4** **a** 122°, 58° **b** 320 m, 510 m **c** 058°T, 040°T
**5** **a** $w = 15.9$ m, $x = 8.5$ m, $y = 9.6$ m, $z = 13.5$ m
**b** $x = 146.8$ m, $y = 182.4$ m, $z = 125.9$ m
**6** **a** $\angle LAB = 28°$, $\angle LBS = 46°$, $\angle ABL = 134°$
**b** 760 m **c** 528 m
**7** **a** 142° **b** 653 m **c** 514 m
**8** **a** 122° **b** 2496 m **c** 1323 m
**9** **a** 110° **b** 169 km
**c** 41°51′ **d** 101°51′ T
**10** **a** 18°, 17.9 km
**b** **i** 38°, 35°, 107° **ii** 22.2 km

## EXERCISE 2H COMPASS RADIAL SURVEYS

**1** **a** 129° **b** 37.9 m **c** 169.8 m²
**2** **a** 82° **b** 25.6 m **c** 188.2 m²
**3** **a** 98° **b** 74.0 m **c** 1188.3 m²
**4** **a** 57° **b** 46.3 m **c** 986.3 m²
**5** 115 m
**6** 303 m
**7** **a** 521.5 m **b** 809.3 m
**8** 793 m²
**9** 5622 m²
**10** **a** **i** 57 m **ii** 220 m²
**b** **i** 58 m **ii** 222 m²
**c** Different measurements/errors
**d** 57 m, 220 m²

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | |
|---|---|---|---|---|
| **1** C | **2** A | **3** D | **4** A | **5** A |
| **6** D | **7** B | **8** C | **9** A | **10** D |
| **11** C | **12** B | **13** D | **14** C | **15** D |
| **16** A | **17** A | **18** B | **19** B | |

## REVIEW SET 1

**1** **a** **i** SE or S45°E **ii** 135°T
**b** **i** S42°E **ii** 138°T
**c** **i** N43°W **ii** 317°T
**2** 5 m
**3** **a** 41 km **b** S30°E, 150°T
**4** **a** 32.6 cm² **b** 26.2 cm
**c** 20.1 cm **d** 43°27′
**e** 16.8 cm **f** 56°9′
**5** **a** $x = 31.1$ cm, $y = 33.3$ cm
**b** $x = 10.2$ cm, $y = 5.2$ cm
**6** **a** $\angle LAB = 19°$, $\angle LBS = 37°$, $\angle ABL = 143°$
**b** 737 m **c** 589 m
**7** **a** 92 m **b** 453 m²

## REVIEW SET 2

**1** 229 m
**2** **a** 65 km **b** S27°W, 207°T
**3** **a** 15.7 cm² **b** 36.6 cm
**c** 25.6 cm **d** 135°41′
**e** 25.6 cm **f** 42°10′
**4** **a** $x = 30.9$ cm, $y = 16.4$ cm
**b** $w = 12.9$ m, $x = 7.7$ m, $y = 8.6$ m, $z = 9.6$ m
**5** **a** 287 m **b** 5325 m²

## REVIEW SET 3

**1** **a** **i** NW or N45°W **ii** 315°T
**b** **i** N73°E **ii** 073°T
**c** **i** S39°E **ii** 141°T
**2** 36°45′
**3** **a** 227 km **b** N52°W, 308°T
**4** **a** 165.2 cm² **b** 34.4 cm
**c** 15.4 cm **d** 49°10′
**e** 11.3 cm **f** 39°9′
**5** **a** $\angle PQR = 111°5'$, $\angle RQO = 68°55'$
**b** 391.9 m **c** 338°55′T
**6** **a** 149 m **b** 1412 m²

## REVIEW SET 4

**1** **a** **i** NE or N45°E **ii** 045°T
**b** **i** N58°W **ii** 302°T
**c** **i** S52°W **ii** 232°T
**2** 18°26′
**3** **a** 51 km **b** N55°E, 055°T
**4** **a** 23.8 cm² **b** 21.2 cm **c** 20.9 cm
**d** 145°25′ **e** 46.0 cm **f** 44°5′
**5** $x = 9.5$ cm, $y = 17.9$ cm
**6** **a** 75° **b** 237 km
**c** 62°41′ **d** 120°41′T
**7** **a** 240 m **b** 3722 m²

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** **i** 16.5 cm **ii** 67°1′
**b** 135.9 m²
**c** **i** 115° **ii** 9.7 km
**iii** 106°21′ using $AC = 9.6584\ldots$ km or 106°16′ using $AC$ 9.7 km
**iv** 286°21′ using $\angle BAC = 106°21'$ or 286°16′ using $\angle BAC = 106°16'$
**d** **i** 257 m **ii** 3887 m²

# CHAPTER 3 RATES AND RATIOS

## ARE YOU READY?

| | | | |
|---|---|---|---|
| **1** B | **2** A | **3** C | **4** B |
| **5** D | **6** C | **7** D | **8** B |
| **9** C | **10** C | **11** A | **12** D |
| **13** B | | | |

## EXERCISE 3A RATES

**1** **a** $\frac{300 \text{ words}}{5 \text{ min}} = 60$ words/min

**b** $\frac{\$10.90}{2.5 \text{ kg}} = 4.36$ \$/kg or \$4.36/kg

**c** $\frac{\$82.50}{6 \text{ h}} = 13.75$ \$/h or \$13.75/h

**d** $\frac{\$55.20}{120 \text{ calls}} = 0.46$ \$/call or \$0.46/call

**2** **a** 4 degrees/h **b** 0.4 kg/m$^2$
**c** 78 km/h **d** 3.4 L/min

**3** **a** $\frac{5000 \text{ kg}}{1 \text{ ha}} = 5000$ kg/ha

**b** $\frac{16\,000 \text{ kg}}{10\,000 \text{ m}^2} = 1.6$ kg/m$^2$

**c** $\frac{375 \text{ cents}}{60 \text{ min}} = 6.25$ c/min

**d** $\frac{180 \text{ cents}}{1000 \text{ mm}} = 0.18$ c/mm

**e** $\frac{15\%}{12 \text{ months}} = 1.25\%$/month

**f** $\frac{2400 \text{ g}}{1000 \text{ mL}} = 2.4$ g/mL

**4** **a** 12.5 mL/s **b** 1.2 c/g **c** 5 m/s
**d** 7.5 mL/s **e** 5 g/min **f** 388.9 cm/s

**5** **a** $60 \times 60 = 3600, \frac{6 \text{ m}}{1 \text{ s}} = \frac{(6 \times 3600) \text{ m}}{1 \text{ h}} = \frac{21\,600 \text{ m}}{1 \text{ h}},$
$1000, \frac{21\,600 \text{ m}}{1 \text{ h}} = \frac{(21\,600 \div 1000) \text{ km}}{1 \text{ h}} = \frac{21.6 \text{ km}}{1 \text{ h}},$
6 m/s = 21.6 km/h

**b** $24 \times 60 = 1440, \frac{3 \text{ cents}}{1 \text{ minute}} = \frac{(3 \times 1440) \text{ cents}}{1 \text{ day}}$
$= \frac{4320 \text{ cents}}{1 \text{ day}}$, 100, 43.2 \$/day or \$43.20/day

**6** **a** **i** 72 000 m/h **ii** 72 km/h
**b** **i** 5000 g/L **ii** 5 kg/L
**c** **i** 800 c/kg **ii** \$8/kg
**d** **i** 7500 kg/ha **ii** 7.5 t/ha
**e** **i** 1440 mL/h **ii** 1.44 L/h
**f** **i** 800 c/km **ii** \$8/km

**7** Supermarket: \$0.48/egg, Sunday market: \$0.50/egg; cheaper to buy carton of 12 eggs at the supermarket

**8** Mineral water: \$1.75/L, cola: \$1.83/L; 2-L bottle of orange mineral water is better value

**9** **a** Tyler: \$0.90, Lisa: \$0.93
**b** Tyler paid less per 100 g so he made the best buy.
**c** Some customers may prefer to buy only what they need rather than a full pack.

**10** Butcher: \$1.54/100 g, supermarket: \$1.38/ 100 g; supermarket offers the best buy

**11** **a** 150 km **b** 3.75 h

**12** **a** 120 kg **b** 3 bags **c** 5000 m$^2$

**13** **a** 520 mL **b** Every 7.5 h **c** 20 drops/min

**14** **a** \$354 **b** 20 h **c** 26 h

**15** **a** US\$2894.74 **b** A\$1235 **c** A\$1710

**16** **a** $9.36 \times 10^8$ km
**b** **i** 107 000 km/h **ii** 29.7 km/s

**17** **a** 150 g pork, $\frac{3}{4}$ tbsp peanut oil, 2 spring onions, $\frac{5}{8}$ ($\approx \frac{1}{2}$) tsp ginger, $\frac{1}{2}$ tbsp lime juice, $\frac{1}{12}$ cup chicken stock, 50 g beans

**b** **i** 900 g pork, $4\frac{1}{2}$ tbsp peanut oil, 12 spring onions, $3\frac{3}{4}$ tsp ginger, 3 tbsp lime juice, $\frac{1}{2}$ cup chicken stock, 300 g beans

**ii** 1350 g pork, $6\frac{3}{4}$ tbsp peanut oil, 18 spring onions, $5\frac{5}{8}$ ($\approx 5\frac{1}{2}$) tsp ginger, $4\frac{1}{2}$ tbsp lime juice, $\frac{3}{4}$ cup chicken stock, 450 g beans

**c** 10

## EXERCISE 3B HEART RATE

**1** **a** 77–83 bpm **b** 55–61 bpm
**c** 82 + bpm **d** 62–65 bpm

**2** **a** Excellent **b** Above-average
**c** Below-average **d** Average

**3** **a** 75 bpm **b** 49 bpm **c** 26 bpm

**4** **a** 77–82 bpm **b** 61–65 bpm
**c** 84 + bpm **d** 65–68 bpm

**5** **a** Above-average **b** Athlete
**c** Excellent **d** Good

**6** **a** 79 bpm **b** 54 bpm **c** 25 bpm

**7** **a** 180 bpm **b** 167 bpm **c** 198 bpm
**d** 199 bpm **e** 190 bpm **f** 156 bpm

**8** 95–162 bpm

**9** **a** 200 bpm, 100–170 bpm **b** 177 bpm, 89–150 bpm
**c** 165 bpm, 83–140 bpm **d** 201 bpm, 101–171 bpm
**e** 192 bpm, 96–163 bpm **f** 162 bpm, 81–138 bpm

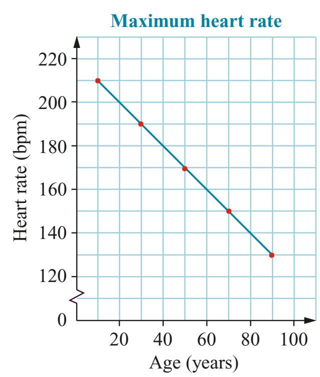
**10** 3.3, good

**11** **a** 2, fair **b** 4.3, excellent **c** 3.5, good
**d** 5.3, excellent

**12** **a**

| Age (years) | 10 | 30 | 50 | 70 | 90 |
|---|---|---|---|---|---|
| Maximum heart rate (bpm) | 210 | 190 | 170 | 150 | 130 |

**b**




**c** Linear relationship
**d** 45

**13** 2600

**14** **a** 2000 **b** 1780 **c** 1660
**d** 2020 **e** 1920 **f** 1620

**15** **a** 4550 mL or 4.5 L **b** 84 000 mL or 84 L
**c** 33 250 mL or 33.25 L **d** 7 056 000 mL or 7056 L

## EXERCISE 3C FUEL CONSUMPTION RATE

**1** **a** 11.7 km/L **b** 0.086 L/km **c** 8.6 L/100 km

**2** **a** **i** 11.25 km/L **ii** 0.089 L/km **iii** 8.9 L/100 km
**b** **i** 9.2 km/L **ii** 0.109 L/km **iii** 10.9 L/100 km
**c** **i** 17.5 km/L **ii** 0.057 L/km **iii** 5.7 L/100 km

**3** **a** 19 L/100 km **b** 11.5 L/100 km **c** 6.4 L/100 km

**4** Distance $= \frac{10}{5.6} \times 100 \approx 179$ km

**5** **a** 349 km **b** 559 km **c** 790 km

6 Fuel used $= \frac{965}{100} \times 10.2 = 98.43 \approx 98$ L

7 a 37 L b 75 L c 24 L

8 a 7.5 L/100 km

b i 37.5 L ii 600 km

c Trip 2 costs $10.72 more

9 $266.37 10 $1975.84

11 a 1–3 cylinders: $1077.23, 4 cylinders: $1399.00, 6 cylinders: $1636.83, 8+ cylinders: $1804.71

b $237.83

c Annual fuel cost for truck is $7763.77 which is more than 4 times as much as the cost for an 8-cylinder car.

12 a $85.74 b $73.06 c $12.68

13 a 10.94 L b About 6.4 L/100 km

14 Cheaper to travel by bus; cost per one-way trip: bus $4.25, car $5.66; saving of about $1.41 per one-way trip

15 a ULP: $2241.20, LPG: $1591.87

b $649.33 c $54.11

d Approx. 48 months e About 72 000 km

16 a i $2115.29 ii $1498.22 iii $617.07

b $51.42

c Approx. 23.3 months

d Approx. 27 183 km

17 a The standardised test comprises two cycles: Urban cycle represents stop-start city driving, Extra Urban cycle is a high-speed test to represent freeway or highway driving. The Combined test result is based on a weighting of the Urban and Extra Urban figures according to the distance travelled in each cycle.

b 24.8 L c $19.60

## EXERCISE 3D POWER AND ENERGY CONSUMPTION RATES

1 a $5.24 b $1.49

c $2.28 d $16.72

2 a $557.95 b $331.36

3 $2.16

4 18.9 cents/kWh

5 a $15.59 b $32.22

6 a $51.41 b $137.09

7 a i 43.8 kWh ii $12.57

b i 35.04 kWh ii $10.06

c i 26.28 kWh ii $7.54

d i 122.64 kWh ii $35.20

e i 17.52 kWh ii $5.03

f i 35.04 kWh ii $10.06

g i 394.2 kWh ii $113.14

h i 26.28 kWh ii $7.54

i i 183.96 kWh ii $52.80

j i 648.24 kWh ii $186.04

8 a 0.141 GW b 141 MW

c $1.41 \times 10^5$ kW d $1.41 \times 10^{11}$ mW

9 a $1.5 \times 10^9$ W b $1.5 \times 10^6$ kW c 1.5 GW

10 a $1.8 \times 10^7$ kW b 18 GW

11 a $9.6 \times 10^{-5}$ kW b 0.016 kWh

12 a Energy consumption in kWh per year for an appliance (compare products of any size), and its star rating (compare products of the same size or capacity)

b Rating comparing how energy efficient a model is compared to other models of the same size; an appliance with more stars is more energy efficient; most appliances are given between 1 and 6 stars (though super-efficient models can have up to 10)

c For refrigerators, there is a saving of about 23% of running costs

d Size of appliance that suits your needs, running costs related to energy consumption

13 $1089.68

14 a $100.16 b About 42% c $920.80

15 a i China ii Turkey

b 17th

c i $5.696 \times 10^{12}$ kWh ii over 26 times

iii > ($9.831 \times 10^{12}$ kWh > $8.368 \times 10^{12}$ kWh)

d i Australia 9700, Brazil 2700, Canada 15 100, China 4300, France 6400, Germany 6500, India 1100, Iran 2600, Italy 4700, Japan 7400, Korea South 9700, Mexico 1900, Russia 7500, Saudi Arabia 9700, South Africa 3900, Spain 4800, Taiwan 10 400, Turkey 2600, United Kingdom 4800, USA 12 100

ii Canada iii India

iv Equal 4th with South Korea and Saudi Arabia

e i No ii No

16 a 49.08 $m^2$

b i Approx. 120 $m^3$ (117.8 $m^3$) ii 7.2 kW

c 3.6 kW d 4.2 kW e 8.4 kW

## EXERCISE 3E RATIOS

1 a 54 : 13 b 13 : 54 c 7 : 54

d 13 : 4 e 54 : 13 : 7 f 13 : 7 : 4

2 a $\frac{54}{13}$ b $\frac{13}{54}$

c $\frac{7}{54}$ d $\frac{13}{4}$

3 a 5 : 7 b 3 : 2 c 7 : 4

d 25 : 7 : 1 e 2 : 4 : 3 f 3 : 4 : 6

g 16 : 19 h 8 : 9 i 7 : 4

j 31 : 20 k 8 : 25 l 3 : 2

m 3 : 5 n 15 : 16 o 5 : 22

p 44 : 17 q 15 : 8 r 9 : 4

s 1 : 21 t 2 : 5 : 7 u 40 : 3 : 2

4 a 3.5 : 1 b 1.8 : 1 c 0.4 : 1 d 0.9 : 1

5 a 5 : 1; 3.5 m is 5 times bigger than 70 cm

b 1.5 : 1; 1.2 kg is $1\frac{1}{2}$ times bigger than 800 g

c 5 : 1; $2\frac{1}{2}$ cups is 5 times bigger than $\frac{1}{2}$ cup

d 0.8 : 1; 0.04 ha is 0.8 times the size of 500 $m^2$

6 a 1 : 1.74 b 1 : 2.25 c 1 : 0.8 d 1 : 0.064

7 a 1 : 100; 1 cm is 100 times smaller than 1 m (or 1 m is 100 times bigger than 1 cm)

b 1 : 1000; 1 mm is 1000 times smaller than 1 m (or 1 m is 1000 times bigger than 1 mm)

c 1 : 50; 20 mL is 50 times smaller than 1 L (or 1 L is 50 times bigger than 20 mL)

d 1 : 300; 50 mg is 300 times smaller than 15 g (or 15 g is 300 times bigger than 50 mg)

8 $\frac{\text{number of boys}}{312} = \frac{9}{8}$; number of boys $= \frac{9}{8} \times 312 = 351$

9 $41 600 10 $48 600

11 $\frac{470}{\text{number of females}} = \frac{10}{9}$ so $\frac{\text{number of females}}{470} = \frac{9}{10}$

number of females $= \frac{9}{10} \times 470 = 423$

12 165 cm

13 a 35 L b 45 L

14 a 18.75 L b 720 mL

15 a 2 bags of cement, 6 bags of screenings

b 1 bag of sand, $1\frac{1}{2}$ bags of screenings

16 Hannah $18 750, Rachel $11 250

**17** $1125 **18** 20 mL, 160 mL **19** 6 kg

**20** Alison 56 kg, Vincent 42 kg, Matthew 70 kg

**21** A ratio compares two (or more) quantities of the same kind, whereas a rate compares two different quantities.

**22** **a** Males who never smoked 1 : 76, Males who have smoked 1 : 12, Males who currently smoke 1 : 4.5, Females who never smoked 1 : 157, Females who have smoked 1 : 23, Females who currently smoke 1 : 8.8

**b** Males who currently smoke

**c** Females who have never smoked

**23** **a** 40 lemon trees **b** 27 orange trees **c** 145 trees

**d** 40%; 2 : 3 is equivalent to 40 : 60 or 40% to 60%

**24** **a** 3 : 7 **b** $203 000 **c** $126 000

**25** **a** Property $900 000, shares $540 000, bonds $360 000

**b** 9 : 7 : 2

**26** 5 : 4, as this ratio gives her a larger fraction of $100; that is $\frac{5}{9}$ compared to $\frac{6}{11}$ ($55.56 compared to $54.55)

**27** **a** Dark pink: white 3000 mL, red 1000 mL; light pink: white 3500 mL, red 500 mL

**b** 13 : 3

## EXERCISE 3F SCALE DRAWINGS

Note: answers may vary slightly due to measurement.

**1** **a i** 9 mm **ii** 18 mm **iii** 90 mm

**b i** 6.5 mm **ii** 13 mm **iii** 65 mm

**c i** 16 mm **ii** 32 mm **iii** 160 mm

**d i** 18 mm **ii** 36 mm **iii** 180 mm

**e i** 24 mm **ii** 48 mm **iii** 240 mm

**f i** 30 mm **ii** 60 mm **iii** 300 mm

**2** **a i** 8.5 m **ii** 17 m **iii** 340 m

**b i** 17.5 m **ii** 35 m **iii** 700 m

**c i** 21 m **ii** 42 m **iii** 840 m

**d i** 30 m **ii** 60 m **iii** 1200 m

**e i** 13 m **ii** 26 m **iii** 520 m

**e i** 4.5 m **ii** 9 m **iii** 180 m

**3** **a i** 1 : 1000 **ii** $\frac{1}{1000}$

**b i** 1 : 400 **ii** $\frac{1}{400}$

**c i** 1 : 5000 **ii** $\frac{1}{5000}$

**d i** 1 : 150 **ii** $\frac{1}{150}$

**e i** 1 : 5000 **ii** $\frac{1}{5000}$

**f i** 1 : 20 000 **ii** $\frac{1}{20\,000}$

**4** **a** 1 : 1000 **b** 1 : 10 000 **c** 1 : 4000

**d** 1 : 500 **e** 1 : 500 000

**5** Length = 250 m, breadth = 155 m

**6** $a = 50$ m, $b = 35$ m, $c = 62$ m, $d = 37$ m

**7** **a** 1 : 80 **b** 1.6 m **c** 0.88 m

**8** **a** 1 : 1000 **b** 54 m

**9** **b** 43 m **c** $20 \times \tan 65° \approx 42.9$ m

**10** **b** 122 m

**c** $\sqrt{90^2 + 110^2 - 2 \times 90 \times 110 \times \cos 75°} \approx 122.8$ m

**11** **a** 1 : 4 000 000 **b** 84 km **c** 216 km

**d** Mittagong **e** Katoomba and Penrith

**f** 64 km/h **g** About 2 hours

**12** **a** Length 82.25 m, width 54.25 m

**b** 4462 m$^2$

## EXERCISE 3G BUILDING PLANS

Note: answers may vary slightly due to measurement.

**1** **a** 39.6 m by 20 m **b** 26 m

**c i** 2 m **ii** 4 m

**d** 6 m **e** 0.8 m **f** 32 m$^2$ **g** 20.8 m

**h** 38 m$^2$ **i** 46 m$^3$ **j** 46 000 L **k** About 12.6

**2** **a i** 12 **ii** 10 **iii** 4

**b** P: pantry, WO: wall oven, F: fridge, WC: water closet (toilet), Ens: ensuite, WIR: walk in robe, Lin: linen cupboard, T: tub

**c i** 5.6 m by 4.6 m **ii** 4.0 m by 3.4 m

**iii** 5.4 m by 4.0 m **iv** 7.2 m by 2.6 m

**d i** 233.2 m$^2$ **ii** 25 squares

**e** 1.9 m$^3$

**f i** 38.9 m$^2$ **ii** 35.5 m$^2$ **iii** 4.4 L

**h i** 2600 mm by 1800 mm

**ii** 75 tiles **iii** 5 boxes

**i** 6 sheets

**j i** 4.0 m **ii** 3.8 m **iii** B

**k** Bedroom 1 **i** 4.2 m **ii** 4.2 m **iii** Either

Bedroom 3 **i** 4.2 m **ii** 4.2 m **iii** Either

Bedroom 4 **i** 3.8 m **ii** 3.6 m **iii** B

**3** **a** 1 : 200 **b** 6340 mm by 5486 mm

**c** 250 mm **d** 6200 mm or 6.2 m

**e** 5600 mm by 3800 mm **f** 6340 mm

**g** 3.4 m **h** 2.6 m

**i** W1 1800 mm by 2000 mm, W3 2700 mm by 2000 mm, W4 1800 mm by 600 mm

**j** 1800 mm by 2100 mm

**k** 34.8 m$^2$ **l** 21.3 m$^2$

**m** $5947 **n** 4 steps

**4** **a** W: window, RWP: rainwater pipe, Bed: bedroom

**b i** 9 **ii** 5

**c i** 2 **ii** 2 **iii** 3 **iv** 2

**d** 1 : 200

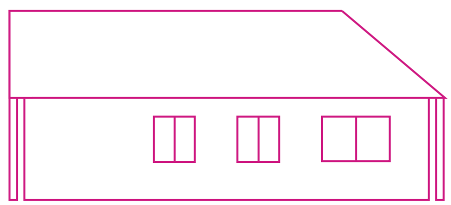
**e i** 4200 mm by 1800 mm **ii** 4200 mm by 3000 mm

**iii** 5400 mm by 3400 mm

**f** W2 1800 mm by 1200 mm, W5 1100 mm by 1200 mm, W6 1800 mm by 1200 mm, Door 1300 mm by 2200 mm

**g** 2700 mm **h** 5000 mm **i** 400 mm

**j**



South elevation

**k** 76.4 m$^2$ **l** 19 m$^3$ **m** 220 mm

**5** **a** 44 m$^2$ **b** about 158 L

**6** **a** 65.76 m$^2$ **b** 5409 L

**7** **a** 176 m$^2$ **b** 30 922 L

**8** **a** 92.8 m$^2$ **b** 49 939 L

**9** **a** 106 992 L **b** 206 225 L

**c** 52% **d** about $219

**12** **a i** 600 m$^2$ **ii** about 524 m$^2$ **iii** about 413 m$^2$

**d** internal walls 100 mm, exterior walls 200 mm

**e** 400 mm

## EXERCISE 3H MEASUREMENTS OF LAND USING A SCALE

Note: answers may vary slightly due to measurement.

**1** **a** **i** 140 m **ii** 1200 $m^2$
**b** **i** 106.4 m **ii** 608 $m^2$
**c** **i** 708 m **ii** 33 480 $m^2$
**d** **i** 4992 m **ii** 1 152 000 $m^2$
**2** **a** About 8600 $m^2$ **b** About 3.3 $km^2$
**c** About 1.8 $km^2$ **d** About 5.1 $km^2$
**3** **a** 18 166 $m^2$ **b** 43 992 $m^2$
**4** 195 160 $m^2$ **5** Answers will vary.
**6** Answers will vary.
**7** **a** 8070 $m^2$ **b** 1909.5 $m^2$
**8** About 8000 $m^2$
**9** $PX = XQ = 35$ mm, $PB = 10$ mm, $QR = 4$ mm, $XA = 25$ mm, $XD = 40$ mm, $PC = 12$ mm, $QS = 4$ mm
The actual lengths are:
$PX = XQ = 35 \times 10000 = 350000$ mm or 350 m
$PB = 10 \times 10000 = 100000$ mm or 100 m,
$QR = 4 \times 10000 = 40000$ mm or 40 m,
$XA = 25 \times 10000 = 250000$ mm or 250 m,
$XD = 40 \times 10000 = 400000$ mm or 400 m,
$PC = 12 \times 10000 = 120000$ mm or 120 m,
$QS = 4 \times 10000 = 40000$ mm or 40 m

Area $PBAX \approx \frac{350}{2}(250 + 100) = 61250$ $m^2$
Area $PCDX \approx \frac{350}{2}(400 + 120) = 91000$ $m^2$
Area $AXQR \approx \frac{350}{2}(40 + 250) = 50750$ $m^2$
Area $DXQS \approx \frac{350}{2}(40 + 400) = 77000$ $m^2$

Total area ≈ 280 000 $m^2$
**10** About 48 375 $m^2$
**11** **a** 9675 kg **b** 139 bags
**12** **a** 18.993 ML **b** 92.69 ML
**13** 3 140 000 L or 3.14 ML
**14** **a** 19 480 L **b** 57 040 L
**15** **a** 225 ML **b** 7.693 GL
**16** **a** Answers will vary; about 8670 $m^2$
**b** $8670 \times 2.4 = 20808$ $m^3$ ≈ 20.8 ML
**17** **a** Surface area ≈ 60 000 $m^2$; volume ≈ 276 000 $m^3$ or 276 ML
**b** 360 000 L or 360 kL

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | |
|---|---|---|---|---|
| **1** B | **2** C | **3** D | **4** C | **5** A |
| **6** D | **7** B | **8** C | **9** D | **10** D |
| **11** A | **12** C | **13** C | **14** A | **15** C |
| **16** B | **17** A | **18** B | **19** D | **20** C |

## REVIEW SET 1

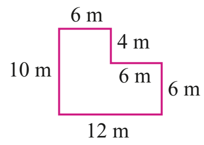
**1** **a** 1.2 kg/$m^2$ **b** 32.4 km/h
**2** Pack of 8 pens **3** 195 bpm
**4** 99 120 mL or 99.12 L **5** About 396 km
**6** $0.92 **7** 3200 kW **8** 288 girls
**9** $18 000, $30 000
**10** $1 \times 6$ m, $2 \times 3.6$ m, $2 \times 2.5$ m, $2 \times 0.9$ m; total length 20 m
**11** About 48 000 $m^2$ **12** Answers will vary, about 48 000 $m^2$
**13** 638 000 L or 638 kL

## REVIEW SET 2

**1** $35.60/m
**2** **a** 1640 L **b** 8 h 50 min
**3** 94–160 bpm **4** 26 670 mL or 26.67 L
**5** $475.94 **6** $10.33
**7** **a** $6.1 \times 10^6$ kW **b** $6.1 \times 10^3$ MW
**8** **a** 4 : 5 **b** 5 : 8 **c** 5 : 8
**9** 27 **10** 360 m
**11** **a** 59.28 $m^2$ **b** 151 L
**12** **a** About 25 000 $m^2$ **b** 30 ML
**13** Answers will vary; about 25 000 $m^2$

## REVIEW SET 3

**1** **a** 16.67 m/s **b** 500 kg/ha
**2** 179 bpm **3** 52 850 mL or 52.85 L
**4** About 45 L
**5** **a** 29.2 kWh **b** $7.85
**6** **a** 0.128 GW **b** 128 MW
**c** $1.28 \times 10^5$ kW **d** $1.28 \times 10^{11}$ mW
**7** $5625 **8** 320 g
**9** **b** 314 m **10** 1 : 250
**11** **a** 4 m by 3.2 m **b** 44.8 $m^2$
**c**




**d** 520 $m^2$ **e** West
**12** **a** 79.92 $m^2$ **b** 408 L
**13** About 14 300 $m^2$
**14** Answers will vary; about 110 000 $m^2$

## REVIEW SET 4

**1** **a** $285.14 **b** $150.49
**2** 99–167 bpm **3** 79 380 mL or 79.38 L
**4** **a** 12.8 L/100 km **b** 83 L **c** 437.5 km
**5** $2128.57
**6** **a** 62.5 $m^2$ **b** 150 $m^3$ **c** 9.0 kW
**7** Tony $5400, Jen $6600
**8** **b** 448 $m^2$ **c** 156 $m^2$ **d** 34.8%
**9** **a** $P = 825$ m, $A = 40000$ $m^2$
**b** 68 000 L or 68 kL

## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** **i** 168 kg **ii** 7 bags
**b** **i** 183 bpm **ii** 92–156 bpm **iii** 2286 beats
**c** 692 km
**d** **i** 792 kWh **ii** $66.53
**e** $270 000
**f** **i** 84 $m^2$ **ii** 336 L

## CHAPTERS 1–3 CUMULATIVE REVIEW

**1** **a** $2163.20 **b** $7363.20
**2** **a** $3165.40 **b** $1582.70
**3** 12.2%
**4** **a** $6780 **b** $21 780 **c** $453.75
**5** **a** $8659.98 **b** $1079.98
**6** **a** $2724.24 **b** $7531.52 **c** $749.36
**d** $2436.60 **e** $8272.07

**7** \$25.81
**8** **a** \$3949.68 **b** \$5571.41
**9** \$5150.63
**10** 4.1%
**11** \$112.50
**12** **a** \$11 122.79 **b** \$18 677 21
**13** 22.5%
**14** **a** \$1983.78 **b** \$1575.58 **c** \$1288.36 **d** \$1230.26
**15** **a** \$775.93 **b** \$1053.87
**16** **a** \$170 000 **b** About 165 months
**17** \$11.40
**18** \$19.92
**19** **a** 9.2 m **b** 9.8 m
**20** **a** **i** $PR$ **ii** $RQ$ **iii** $PQ$
**b** **i** $\sin\theta = \dfrac{PR}{PQ}$ **ii** $\cos\theta = \dfrac{RQ}{PQ}$ **iii** $\tan\theta = \dfrac{PR}{RQ}$
**21** **a** **i** 0.9573 **ii** 0.3214 **iii** 0.3398
**b** **i** 15°43′ **ii** 67°6′ **iii** 54°39′
**22** **a** 18.2 m **b** 12.6 m
**23** **a** 50°28′ **b** 66°16′
**24** **a** 23°30′ **b** 1003 m
**25** **a** **i** N49°W **ii** 311°T
**b** **i** S49°W **ii** 229°T
**26** **a** 139 km **b** N54°20′W or 305°40′T
**27** **a** 95 $cm^2$ **b** 18.0 cm **c** 3.3 m
**d** 36°13′ **e** 3.8 cm **f** 30°16′
**28** $x = 4.5$ cm, $y = 7.8$ cm
**29** **a** $\angle ABC = 76°$ **b** 167 km
**c** $\angle BAC = 57°$ **d** 124°T
**30** Perimeter ≈ 350 m, area ≈ 8180 $m^2$
**31** **a** 2.4 kg/$m^2$ **b** 54 km/h
**32** Pack of 5 candles (59c each)
**33** 199 bpm
**34** 92 400 mL or 92.4 L
**35** 415.8 km
**36** \$0.74
**37** **a** $5.7 \times 10^3$ kW **b** $5.7 \times 10^9$ mW
**38** 552 girls
**39** 24 375, 40 625
**40** **a** 2.3 km **b** 1:125
**41** **a** 88.8 m **b** 96 $m^2$ **c** 12.8 $m^2$ **d** 489.6 L
**42** **a** About 1698 $m^2$ **b** About 3056 $m^2$

# CHAPTER 4 SIMULTANEOUS LINEAR EQUATIONS

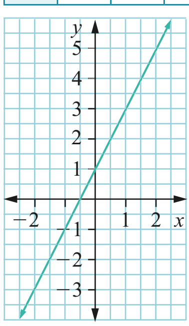
## ARE YOU READY?

**1** C **2** B **3** D **4** B
**5** C **6** A **7** C **8** D
**9** A

## EXERCISE 4 A GRAPHS OF THE FORM $Y = MX + C$

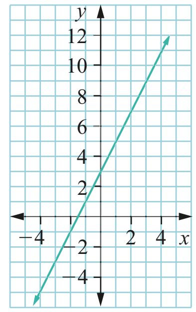
**1** **a** $y = 2x + 1$

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | −3 | −1 | 1 | 3 | 5 |



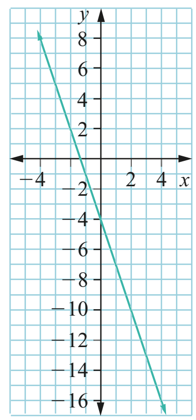
**b** $y = 2x + 3$

| $x$ | −4 | −2 | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | −5 | −1 | 3 | 7 | 11 |



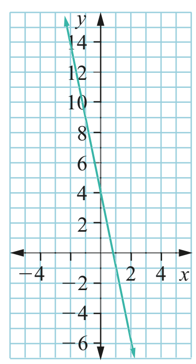
**c** $y = -3x - 4$

| $x$ | −4 | −2 | 0 | 2 | 4 |
|---|---|---|---|---|---|
| $y$ | 8 | 2 | −4 | −10 | −16 |



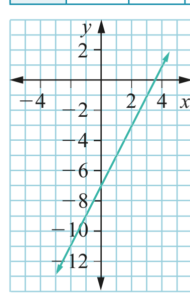
**d** $y = 4 - 5x$

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | 14 | 9 | 4 | −1 | −6 |



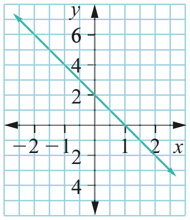
**e** $y = 2x - 7$

| $x$ | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | −11 | −9 | −7 | −5 | −3 |



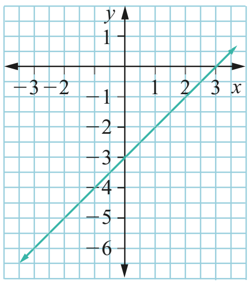
**f** $y = 2 - 2x$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $6$ | $4$ | $2$ | $0$ | $-2$ |



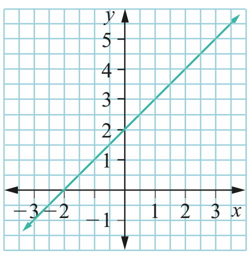

**2** **a** $y = x - 3$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-6$ | $-3$ | $0$ |



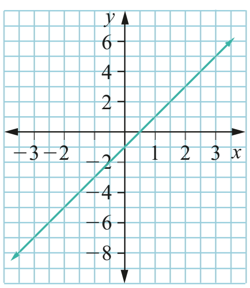

**b** $y = x + 2$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-1$ | $2$ | $5$ |



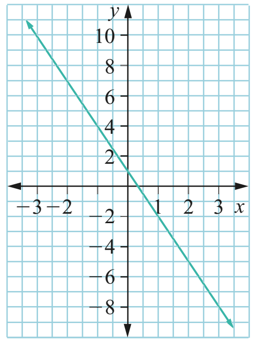

**c** $y = 2x - 1$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-7$ | $-1$ | $5$ |



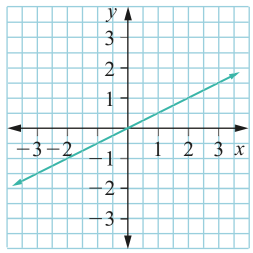

**d** $y = 1 - 3x$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $10$ | $1$ | $-8$ |



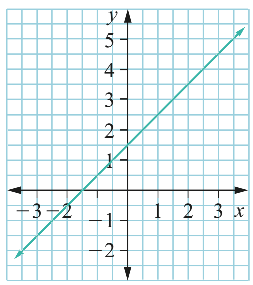

**e** $y = \frac{1}{2}x$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-1\frac{1}{2}$ | $0$ | $1\frac{1}{2}$ |



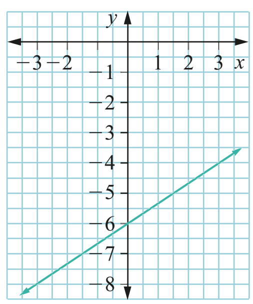

**f** $y = \frac{2x + 3}{2}$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-\frac{3}{2}$ | $\frac{3}{2}$ | $\frac{9}{2}$ |



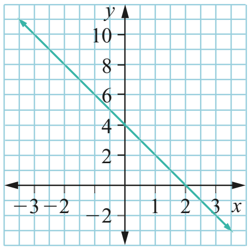

**g** $y = \frac{2x}{3} - 6$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $-8$ | $-6$ | $-4$ |



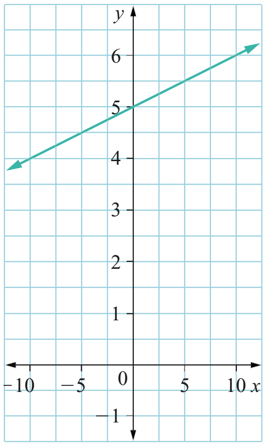
**h** $y = 4 - 2x$

| $x$ | $-3$ | $0$ | $3$ |
|---|---|---|---|
| $y$ | $10$ | $4$ | $-2$ |




**3** One possible answer is:

| $x$ | $-10$ | $0$ | $10$ |
|---|---|---|---|
| $y$ | $4$ | $5$ | $6$ |

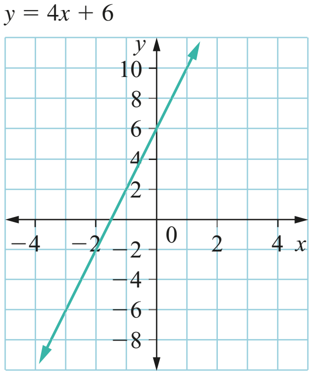



**4** **a** $4, 6$ **b** $-2, 9$ **c** $\frac{3}{4}, -8$ **d** $1, -2$
**e** $-1, 4$ **f** $-1.2, -2.8$ **g** $5.4, 6.5$ **h** $-\frac{3}{8}, 4\frac{1}{2}$
**i** $\frac{1}{2}, 7$ **j** $\frac{1}{6}, 1\frac{2}{3}$ **k** $-\frac{1}{3}, -2$ **l** $\frac{1}{2}, 2$
**m** $-4, 8$ **n** $-1\frac{1}{2}, -2$ **o** $2, -5$

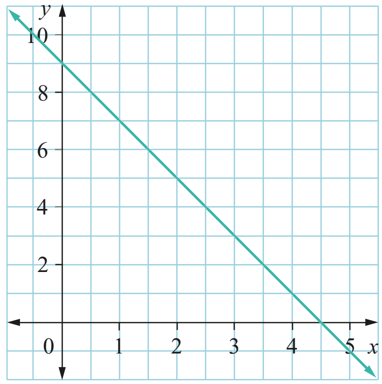
**5** **a** $y = 7x - 1$ **b** $y = -2x + 5$
**c** $y = \frac{3}{7}x - 2$ **d** $y = x + 4$
**e** $y = -x + 2.7$ **f** $y = 0.2x + 3$
**g** $y = -1.4x - 0.3$ **h** $y = \frac{5}{6}x + \frac{7}{12}$
**i** $y = -0.56x - 1.24$ **j** $y = \frac{7}{5}x + 24$

**6** **a i** $2$ **ii** $1$ **iii** $y = 2x + 1$
**b i** $-\frac{1}{2}$ **ii** $1$ **iii** $y = -\frac{1}{2}x + 1$
**c i** $1$ **ii** $-1$ **iii** $y = x - 1$
**d i** $-2$ **ii** $3$ **iii** $y = -2x + 3$
**e i** $\frac{2}{3}$ **ii** $2$ **iii** $y = \frac{2}{3}x + 2$
**f i** $0.6$ **ii** $-1.2$ **iii** $y = 0.6x - 1.2$

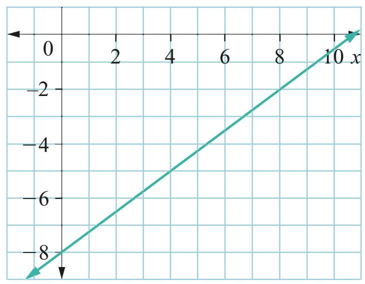
**7** **a** $y = 4x + 6$



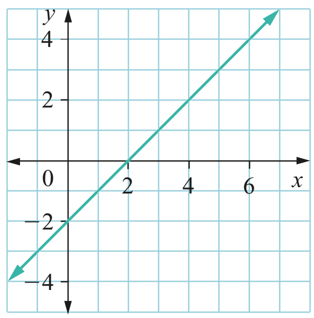

**b** $y = -2x + 9$



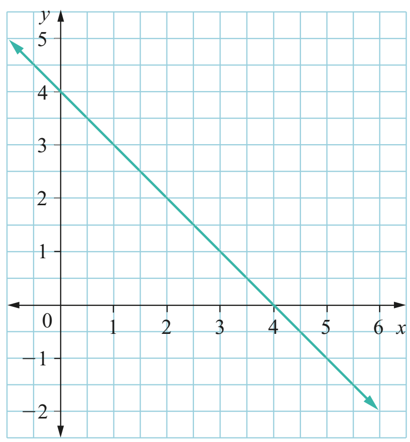

**c** $y = \frac{3}{4}x - 8$



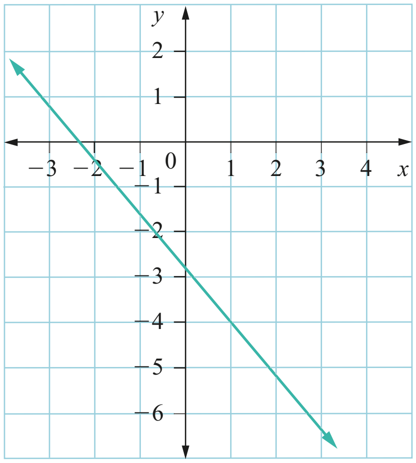

**d** $y = x - 2$



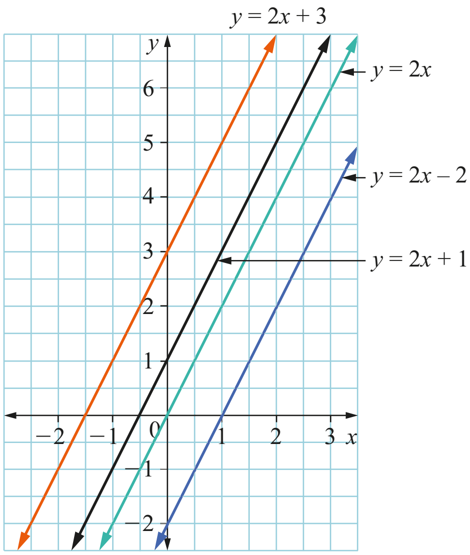

**e** $y = -x + 4$



f $y = -1.2x - 2.8$



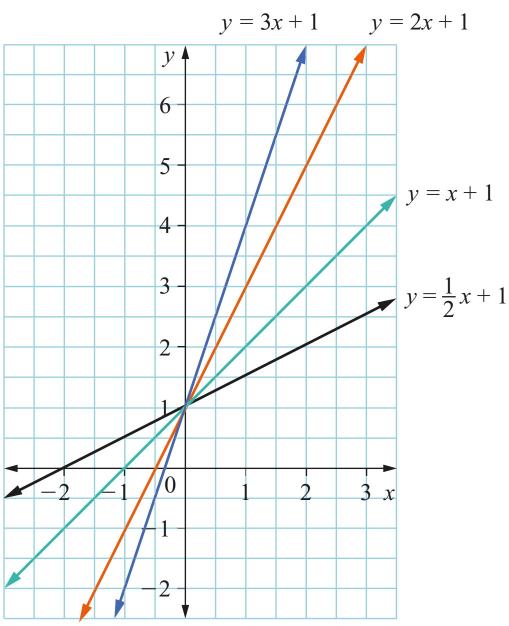

8 a




b The four lines are parallel; they have the same gradient but different $y$-intercepts.

c The constant term added to $2x$ moves the line for $y = 2x$ up or down. For example, adding 3 moves the line for $y = 2x$ up 3 units. The constant term is also the $y$-intercept for each line.

9 a




b The four lines pass through the same $y$-intercept of 1 but have different gradients.

c The larger the size of the coefficient, the steeper the line. For example, the line with equation $y = 3x + 1$ is steeper than the line with equation $y = 2x + 1$.

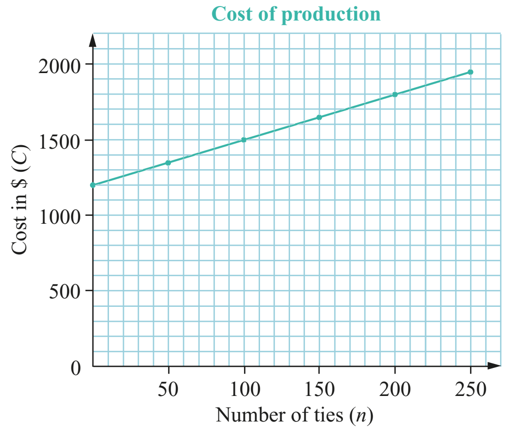
## EXERCISE 4B LINEAR MODELS

1 a $C$ is the total cost of the day's production.
3 is the cost per tie.
$n$ is the number of ties produced.
1200 is the fixed cost.

b

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $C$ | 1200 | 1350 | 1500 | 1650 | 1800 | 1950 |

c

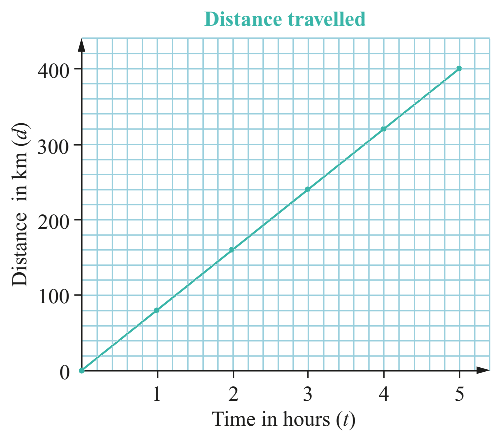



d About 65 ties

2 a

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $d$ | 0 | 80 | 160 | 240 | 320 | 400 |

b




c $y$-intercept is 0. At time 0 h the train has travelled 0 km.

d Gradient is 80; it is the speed of the train (80 km/h).

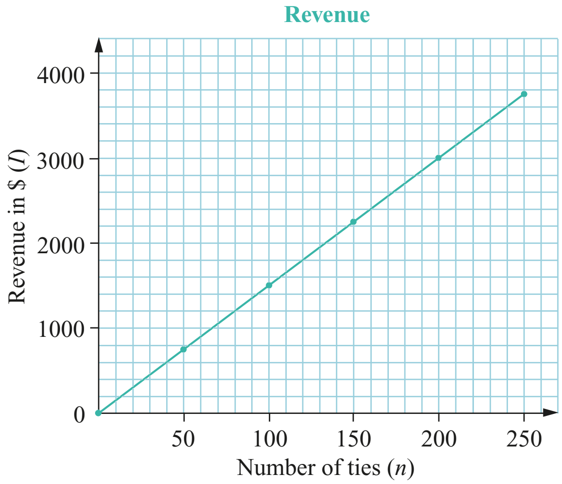
e 280 km f 2.5 h

3 a $R$ is the total revenue from sales.
15 is the revenue per tie.
$n$ is the total number of ties sold.

b

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $R$ | 0 | 750 | 1500 | 2250 | 3000 | 3750 |

**c**

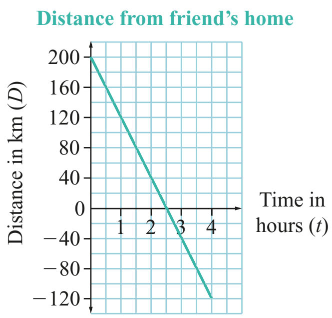



**d** 120 ties

**e** Gradient is 15; it is the sale price of each tie ($15).

**4** **a** $D$ is the distance Tina is from her friend's home. 200 is the total distance Tina travels to reach her friend's home. 80 is the average speed. $t$ is the time spent travelling.

**b**




**c** 80 km **d** 2.5 hours

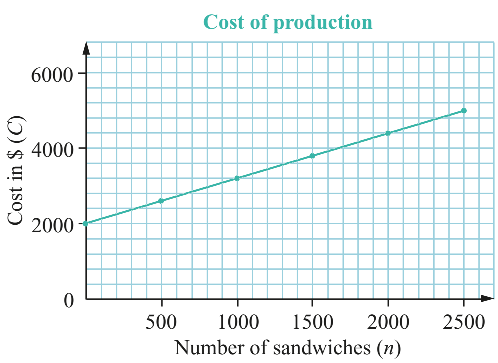
**e** When $t = 4$, $D = -120$. The negative distance indicates that Tina has driven a further 120 km past her friend's home, which does not match this context.

**5** **a** $C = 1.2n + 2000$

**b**

| $n$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|
| $C$ | 2000 | 2600 | 3200 | 3800 | 4400 | 5000 |

**c**




**d** Gradient is 1.2; it is the cost per sandwich ($1.20).

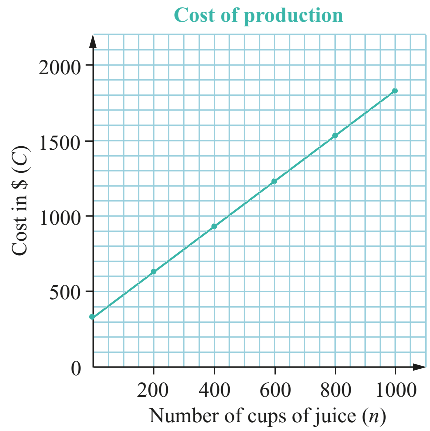
**e** 1250 sandwiches

**6** **a** $C = 1.5n + 325$

**b**

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $C$ | 325 | 625 | 925 | 1225 | 1525 | 1825 |

**c**




**d** $y$-intercept is 325, the weekly cost ($325).

**e** Gradient is 1.5; it is the cost per cup of juice ($1.50).

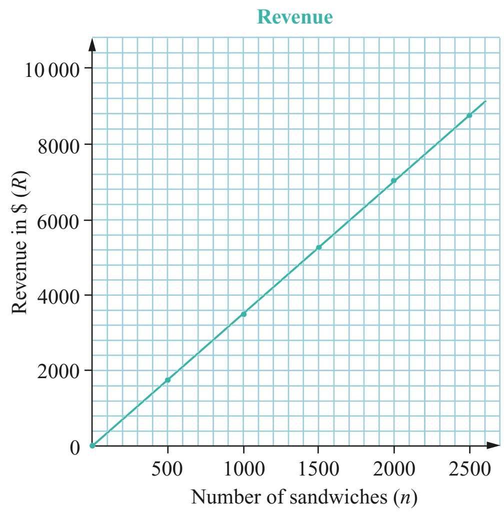
**f** 450 cups

**7** **a** $R = 3.5n$

**b**

| $n$ | 0 | 500 | 1000 | 1500 | 2000 | 2500 |
|---|---|---|---|---|---|---|
| $R$ | 0 | 1750 | 3500 | 5250 | 7000 | 8750 |

**c**




**d** 800 sandwiches

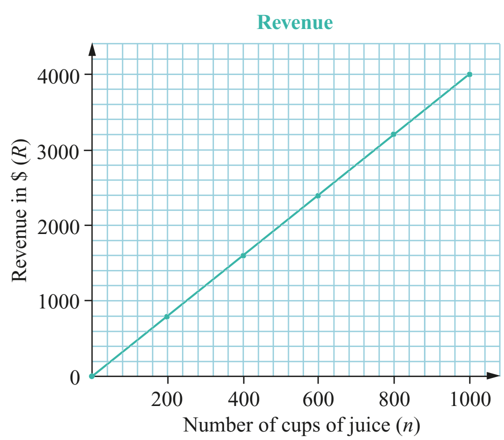
**e** Gradient is 3.5; it is the price of each sandwich ($3.50).

**8** **a** $R = 4n$

**b**

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $R$ | 0 | 800 | 1600 | 2400 | 3200 | 4000 |

**c**

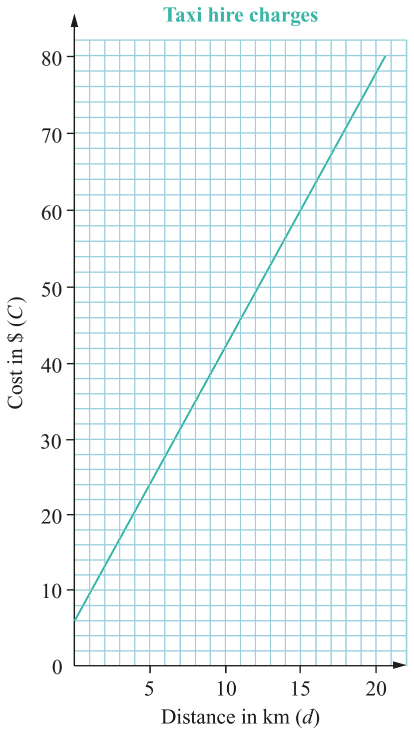



**d** 750 cups

**e** Gradient is 4; it is the price of each cup of juice ($4).

9 a $C = 3.6d + 6$

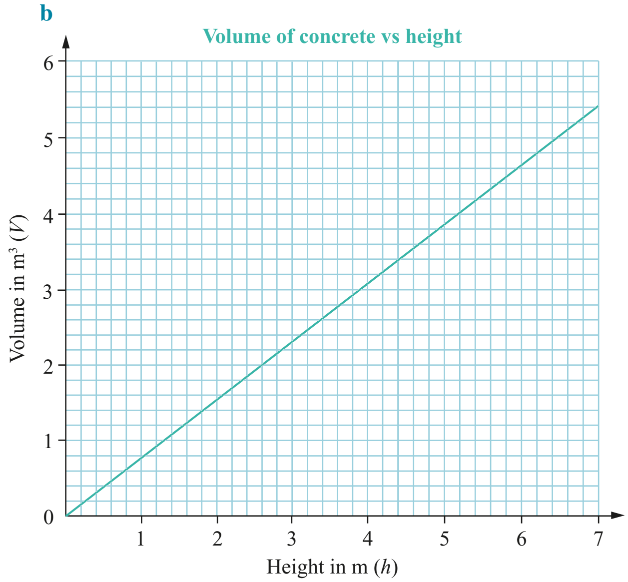
b




c $42 d About 7 km

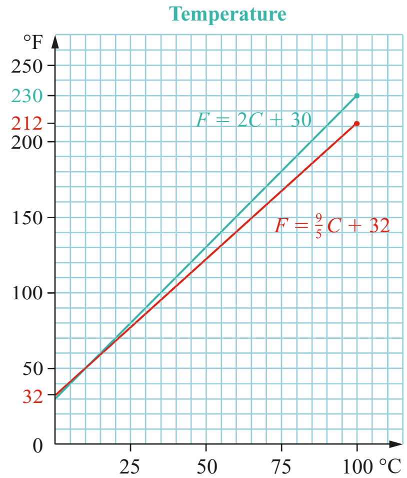
10 a $V = 0.78h$

b




c About 3 $m^3$ d About 3 m

11 a, b




c Rule of thumb is mostly a reasonable approximation for low temperatures.

## EXERCISE 4C IDENTIFYING SOLUTIONS TO SIMULTANEOUS LINEAR EQUATIONS

1 a $x = 2, y = 4$ b $x = 3, y = 5$ c $x = 7, y = 16$

2 a Yes b Yes c No
d Yes e No f No

3 a i (4, 3) ii Solution is $x = 4, y = 3$.
b i (2, −4) ii Solution is $x = 2, y = -4$.
c i (−5, −2) ii Solution is $x = -5, y = -2$.
d i (−2, 3) ii Solution is $x = -2, y = 3$.

5 a 15 cards b $60 c $100

6 a Plan 30: the graph shows it costs about $10 less.
b Plan 50: the graph shows it costs about $25 less.
c About 55 min, cost is $50.
d Each plan is a piecewise relationship that is made up of two lines with different equations.

7 a (2, 2), solution is $x = 2, y = 2$.
b Solution is $x = 0.4, y = 2.8$.
c No, lines $A$ and $B$ are parallel and do not intersect.

## EXERCISE 4D SOLVING SIMULTANEOUS LINEAR EQUATIONS GRAPHICALLY

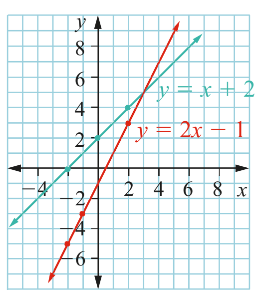
1 a i $y = x + 2$

| $x$ | −2 | 0 | 2 |
|---|---|---|---|
| $y$ | 0 | 2 | 4 |

$y = 2x - 1$

| $x$ | −2 | 0 | 2 |
|---|---|---|---|
| $y$ | −5 | −1 | 3 |

ii




iii (3, 5) iv Solution is $x = 3, y = 5$.

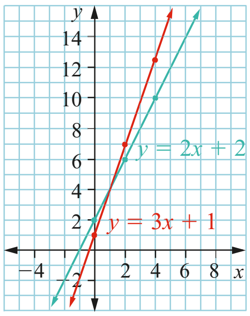
b i $y = 2x + 2$

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 2 | 6 | 10 |

$y = 3x + 1$

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 1 | 7 | 13 |

ii




iii (1, 4) **iv** Solution is $x = 1, y = 4$.

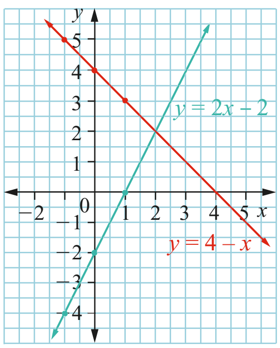
**c i** $y = 2x - 2$

| $x$ | −1 | 0 | 1 |
|---|---|---|---|
| $y$ | −4 | −2 | 0 |

$x + y = 4$

| $x$ | −1 | 0 | 1 |
|---|---|---|---|
| $y$ | 5 | 4 | 3 |

ii

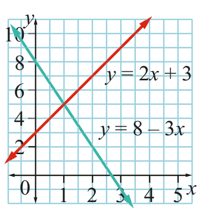



iii (2, 2) **iv** Solution is $x = 2, y = 2$.

**2** **a** Solution is $x = 4, y = 2$.
**b** Solution is $x = 2, y = -1$.
**c** Solution is $x = -1, y = -2$.
**d** Solution is $x = 1, y = 3$.
**e** Solution is $x = 4, y = 2$.
**f** Solution is $x = \frac{3}{5}$ or 0.6, $y = 1\frac{3}{5}$ or 1.6.

**3** **a** Solution is $x = 2, y = -1$.
**b** Solution is $x = -1, y = -2$.
**c** Solution is $x = 1, y = 2$.
**d** Solution is $x = 3, y = -4$.
**e** Solution is $x = 1, y = -5$.
**f** Solution is $x = 3, y = -2$.
**g** Solution is $x = -1, y = -2$.
**h** Solution is $x = -3, y = 2$.
**i** Solution is $x = -1, y = 2$.
**j** Solution is $x = 3, y = 7$.

**4** **a**



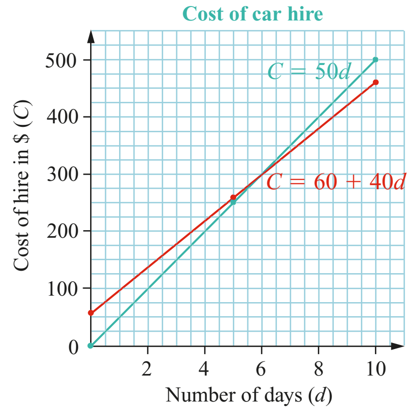

**b** (1, 5)
**c** Solution is $x = 1, y = 5$.

**6** **a** $C = 50d$

| $d$ | 0 | 5 | 10 |
|---|---|---|---|
| $C$ | 0 | 250 | 500 |

$C = 60 + 40d$

| $d$ | 0 | 5 | 10 |
|---|---|---|---|
| $C$ | 60 | 260 | 460 |



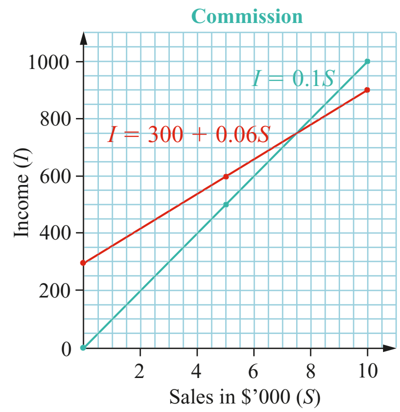

**b** $d = 6$, $C = 300$; the cost of hiring a car for 6 days is the same for both companies, cost is \$300.
**c** 6 days

**7** **a** $I = 0.1S$

| $S$ | 0 | 5000 | 10 000 |
|---|---|---|---|
| $I$ | 0 | 500 | 1000 |

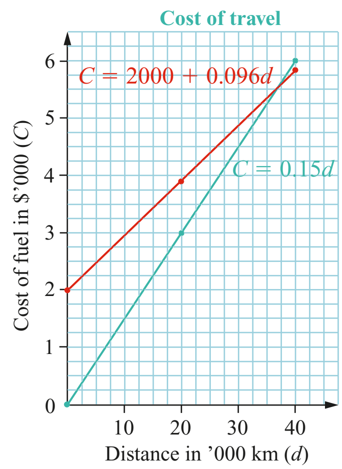
$I = 300 + 0.06S$

| $S$ | 0 | 5000 | 10 000 |
|---|---|---|---|
| $I$ | 300 | 600 | 900 |




**b** $S = 7500$, $I = 750$ when sales are \$7500, the two payment options give the same income of \$750.
**c** Sales greater than \$7500.

**8** **a**




d ≈ 37 000, $C \approx 5500$
**b** About 37 000 km

**9** **a** $2l + 2w = 50$ **b** $l = w + 5$
**c** Solution is $w = 10$, $l = 15$. Dimensions of the photo are 15 cm by 10 cm.

**10 a** If $d$ is the cost of a large drink and $c$ is the cost of an ice cream, $d = c + 2$ and $5d + 7c = 70$. All costs are in dollars.

**b** Solution is $c = 5$, $d = 7$; large drink costs \$7 and an ice cream costs \$5.

**11 a** If $d$ is the distance in km from Town A after $t$ hours, $d = 20t$ (for Tom) and $d = 15 - 5t$ (for Julia).

**b** They meet after travelling 0.6 hours or 36 minutes.

**c** Tom has travelled 12 km and Julia has travelled 3 km.

**12 a** No solution as the two lines are parallel and do not intersect.

**b** There is an infinite number of solutions, as the graphs of the two equations lie on top of one another; i.e. they are the same line.

## EXERCISE 4E BREAK-EVEN ANALYSIS

**1 a** 80 **b** \$320 **c** \$50

**2 a** About 65 **b** About \$390 **c** About \$85

**3 a** About 60 **b** About \$210 **c** About \$85

**d** Revenue − cost ≈ −50. There is a loss of \$50. Jenny makes a loss not a profit.

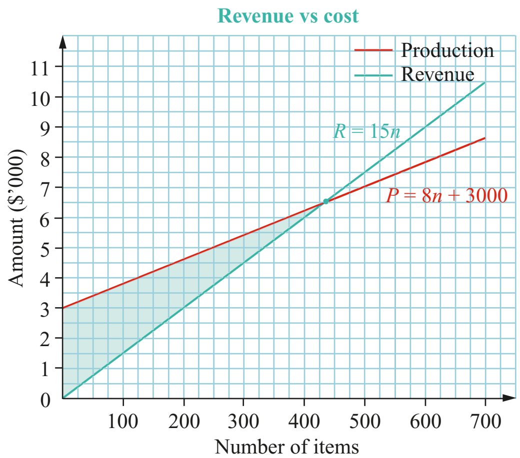
**4 a** $P = 8n + 3000$

| Number of items ($n$) | 0 | 100 | 300 | 500 | 700 |
|---|---|---|---|---|---|
| Production cost ($P$, \$) | 3000 | 3800 | 5400 | 7000 | 8600 |

**b** $R = 15n$

| Number of items ($n$) | 0 | 100 | 300 | 500 | 700 |
|---|---|---|---|---|---|
| Revenue ($R$, \$) | 0 | 1500 | 4500 | 7500 | 10 500 |

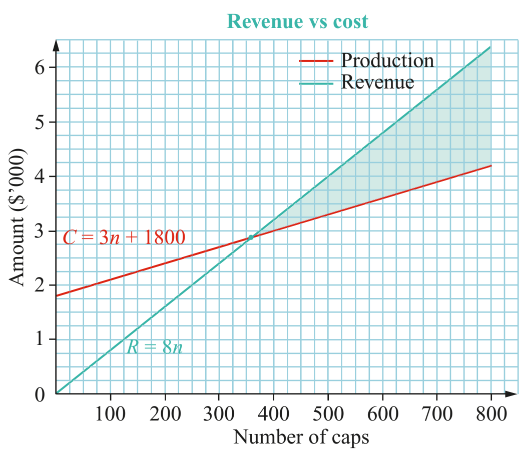
**c**




**d** About (430, 6500); actual coordinates are close to (428.57, 6428.57)

**e** About 430 **f** About \$6500

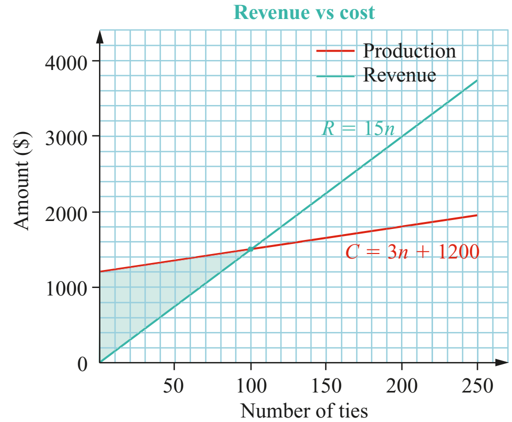
**5 a, b**




**c** About (360, 2900); actual coordinates are (360, 2880)

**d** 360 **e** About \$2900

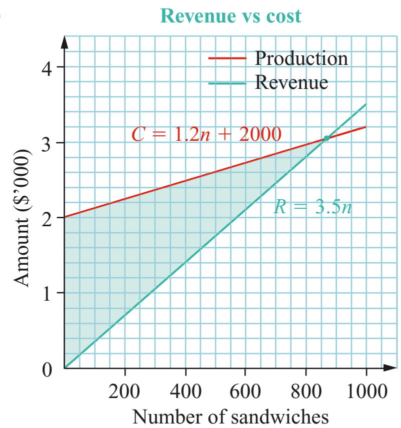
**6 a**




**b** (100, 1500) **c** 100 **d** \$1500

**7 a** $C = 1.2n + 2000$ and $R = 3.5n$

**b**



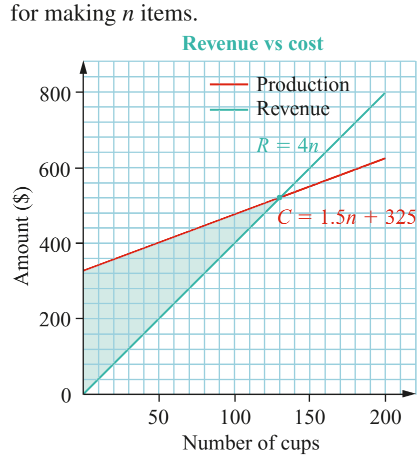

**c** About (870, 3050); actual coordinates are close to (869.57, 3043.48)

**d** About 870 **e** About \$3050

**f** Need to make 870 sandwiches to break even, therefore must always make a loss. Something needs to change to become profitable.

**g** See green shading on graph for part **b**. This indicates where a loss is made; i.e. the revenue is less than the cost for making $n$ items.

**8 a**




**b** (130, 520) **c** 130 **d** \$520

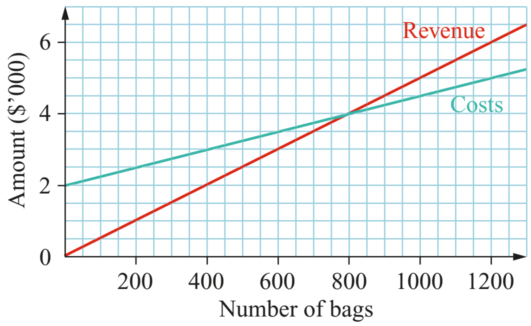
**e** Need to make 130 cups to break even, therefore operating at a loss. Need to increase price or production numbers or decrease costs to make a profit.

**f** See green shading on graph for part **a**.

**9** **a** See green shading on graph for question **4c**.
**b** See green shading on graph for question **5a, b**.
**c** See green shading on graph for question **6a**.

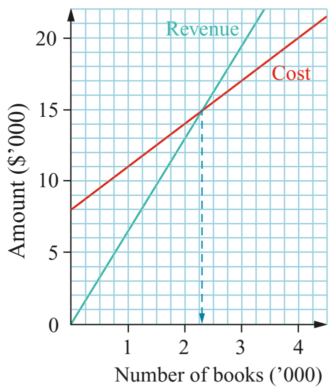
**10** **a** 700
**b** **i** Loss of about \$1700 **ii** Profit of about \$800
**c** **i** 1050 **ii** 350

**11** **a** $C = 2.5n + 2000$, $R = 5n$ where $C$ is the production cost in dollars and $R$ is the revenue for $n$ bags. Both $C$ and $R$ are in dollars.
**b**




**c** 800 bags
**d** **i** Loss of \$1750 **ii** Profit of \$500
**e** **i** 1200 bags **ii** 400 bags

**12** **a** $C = 3n + 8000$, $R = 6.5n$ where $C$ is the production cost and $R$ is the revenue for $n$ books. Both $C$ and $R$ are in dollars.
**b**




**c** About 2300 books
**d** **i** Loss of about \$4500 **ii** Profit of about \$750
**iii** Profit of about \$3200
**e** **i** Just under 1500 books ($\approx$ 1430)
**ii** About 3000 books

**13** Region B

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | |
|---|---|---|---|---|
| **1** B | **2** C | **3** A | **4** A | **5** A |
| **6** C | **7** D | **8** B | **9** B | **10** C |

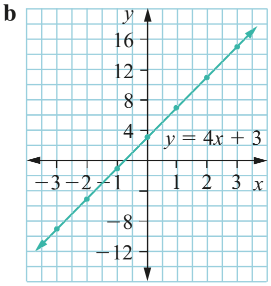
## REVIEW SET 1

**1** **a** $m = \frac{3}{8}$, $y$-intercept $= -4$
**b** $y = 0.08x + 3.2$

**2** **a**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $-9$ | $-5$ | $-1$ | 3 | 7 | 11 | 15 |

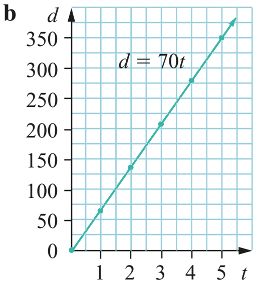
**b**




**3** **a**

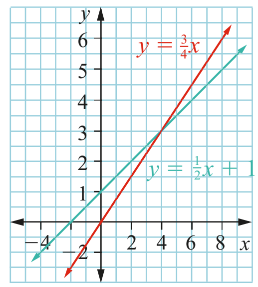
| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $d$ | 70 | 140 | 210 | 280 | 350 |

**b**




**c** Just under 250 km (245 km) **d** About 2 h 50 min

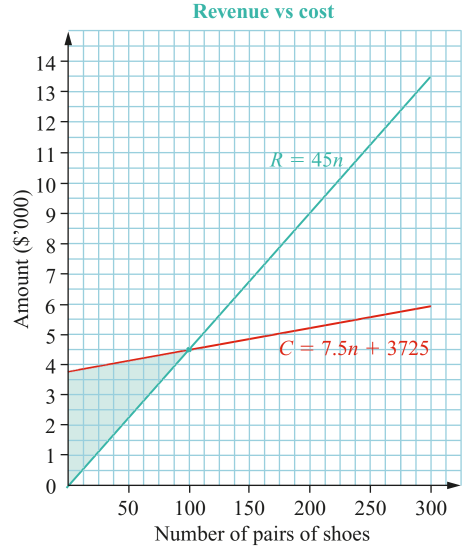
**4**




Solution is $x = 4$, $y = 3$.

**5** Solution is $x = -2$, $y = 2$.

**6** **a**




**b** About (100, \$4500); actual coordinates are close to (99.3, 4470)
**c** About 100 pairs **d** About \$4500
**e** See shading on graph in part **a**.

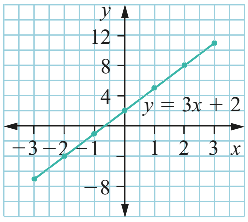
## REVIEW SET 2

**1** Gradient $= \frac{5}{4}$

**2** **a**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $-7$ | $-4$ | $-1$ | 2 | 5 | 8 | 11 |

**b**

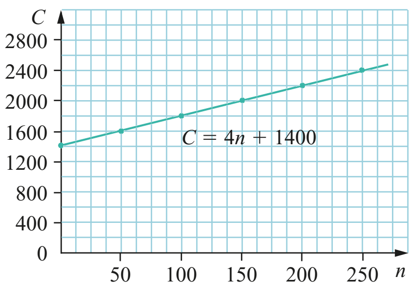



**3** **a** $C$ is the cost per day to make the ties.
4 is the cost per tie.
$n$ is the number of ties made.
1400 is the fixed cost.

**b**

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $C$ | 1400 | 1600 | 1800 | 2000 | 2200 | 2400 |

**c**

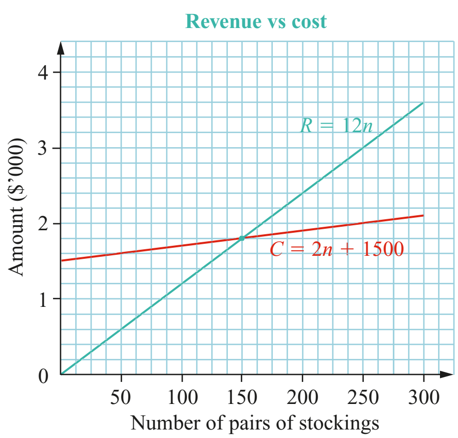



**d** 25 ties

**4** 40 000 km

**5** **a** Solution is $x = 3$, $y = 3$.
**b** Solution is $x = -3$, $y = 2$.

**6** **a**



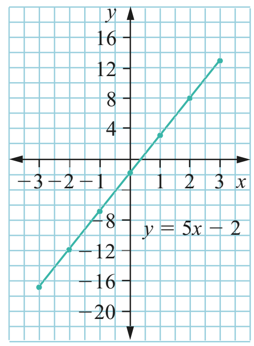

**b** (150, 1800) **c** 150 pairs **d** $1800

## REVIEW SET 3

**1** **a**

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | $-17$ | $-12$ | $-7$ | $-2$ | 3 | 8 | 13 |

**b**

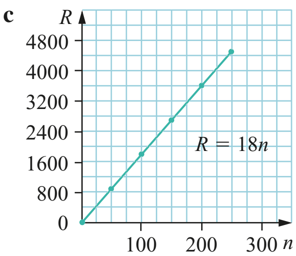



**2** **a** $R$ is the revenue, $n$ is the number of scarves sold and 18 is the price of each scarf.

**b**

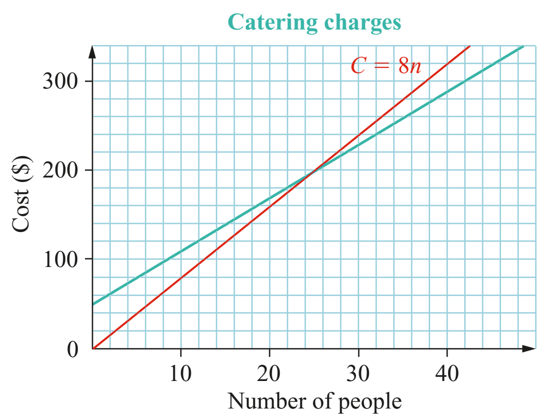
| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $R$ | 0 | 900 | 1800 | 2700 | 3600 | 4500 |

**c**




**d** 115 scarves

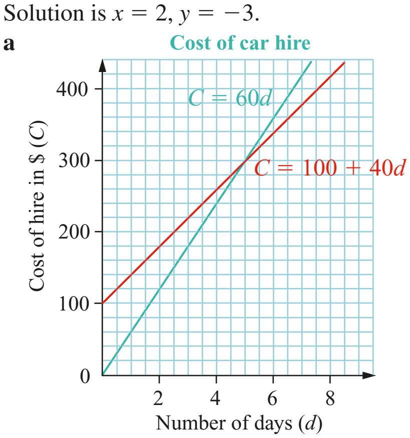
**3** **a** About $260 **b** 42
**c** Gradient = 6, this is the cost per person.
**d** $50, this is the set up cost of $50.
**e**




**f** 25 people at a cost of $200

**4** Solution is $x = 2$, $y = -3$.

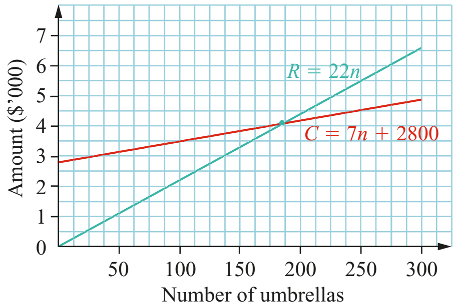
**5** **a**




**b** $d = 5$, $C = 300$ **c** 5 days

**6 a** $C = 7n + 2800$ and $R = 22n$, where $C$ is the production cost and $R$ is the revenue for $n$ umbrellas.

**b** Revenue vs cost




**c** About (185, 4100); actual coordinates close to (186.67, 4106.67)

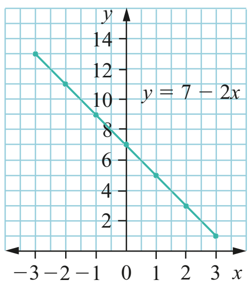
**d** About 185 **e** About $4100

## REVIEW SET 4

**1 a**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 13 | 11 | 9 | 7 | 5 | 3 | 1 |

**b**

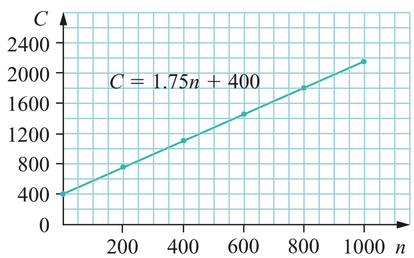



**2 a** $C = 1.75n + 400$

**b**

| $n$ | 0 | 200 | 400 | 600 | 800 | 1000 |
|---|---|---|---|---|---|---|
| $C$ | 400 | 750 | 1100 | 1450 | 1800 | 2150 |

**c**

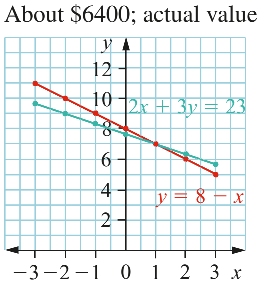



**d** About 340 cups

**3 a** About 425; actual value close to 429

**b** About $6400; actual value close to $6430

**4 a**




**b** (1, 7) **c** Solution is $x = 1$, $y = 7$.

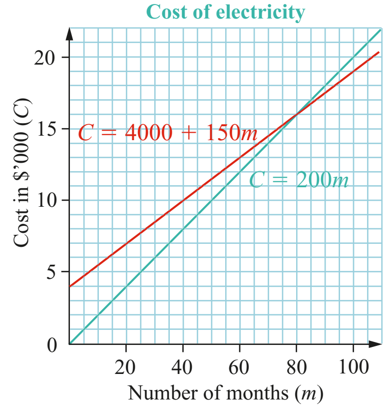
**5 a** $C = 200m$

| $m$ | 0 | 50 | 100 |
|---|---|---|---|
| $C$ | 0 | 10 000 | 20 000 |

$C = 4000 + 150m$

| $m$ | 0 | 50 | 100 |
|---|---|---|---|
| $C$ | 4000 | 11 500 | 19 000 |

Cost of electricity



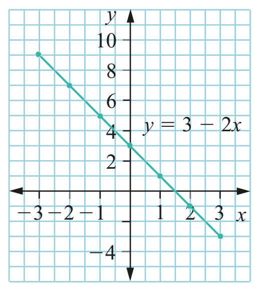

**b** Solution is $m = 80$, $C = 16\,000$.

**c** 80 months

## REVIEW PRACTICE EXAMINATION QUESTION

**1 a i**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | 9 | 7 | 5 | 3 | 1 | −1 | −3 |




**ii** Gradient is −2. **iii** $y$-intercept is 3.

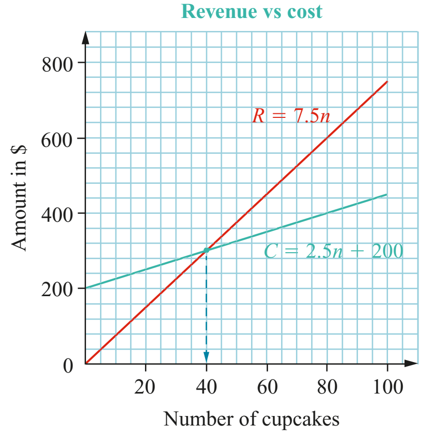
**b i** About $63 000 **ii** No, revenue is less than cost.

**iii** 2000 **iv** 2400

**v** $50 000 **vi** $12.50

**c i** $C = 2.5n + 200$ **ii** $R = 7.5n$

**iii** Revenue vs cost




**iv** 40 **v** $300

# CHAPTER 5 BIVARIATE DATA ANALYSIS

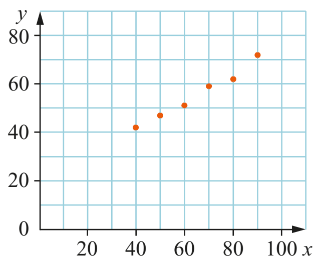
## ARE YOU READY?

| | | | |
|---|---|---|---|
| **1** D | **2** A | **3** A | **4** A |
| **5** A | **6** C | **7** B | **8** D |

## EXERCISE 5A BIVARIATE SCATTERPLOTS

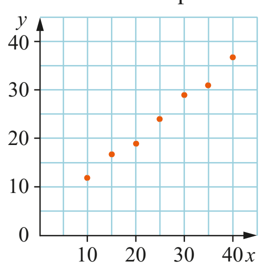
**1** **a** Yes **b** No **c** Yes

**2** **a i**



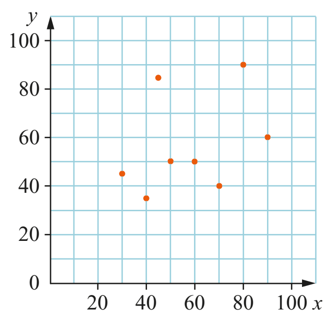

**ii** Linear relationship

**b i**




**ii** Linear relationship

**c i**



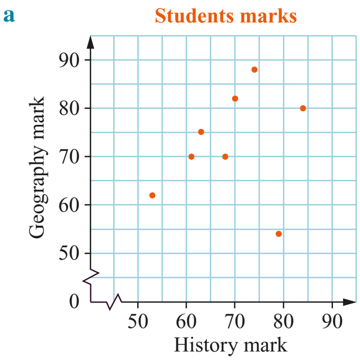

**ii** Non-linear relationship

**3** **a** Independent variable = Height
Dependent variable = Time taken to run 10 km
**b** Independent variable = Number of colds per year,
Dependent variable = Number of doctor visits per year
**c** Independent variable = Distance travelled
Dependent variable = Petrol used
**d** Independent variable = Hours watching TV per week
Dependent variable = Exam mark

**4** Independent variable = Amount of sugar
Dependent variable = Size of waistline

**5** Independent variable = Height,
Dependent variable = Points per game

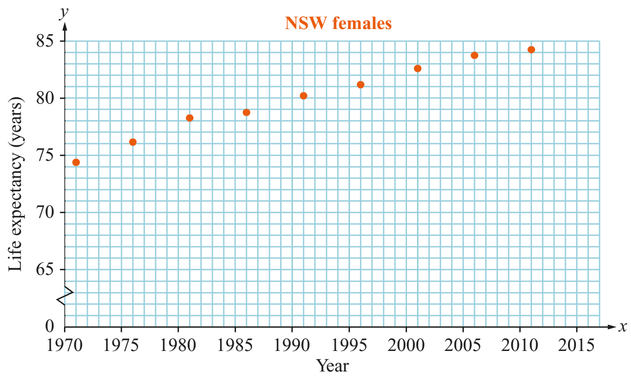
**6** **a**




**b** Student 6
**c** There is no reason to believe this outlier is the result of an error. This student probably just performed badly in the Geography test.

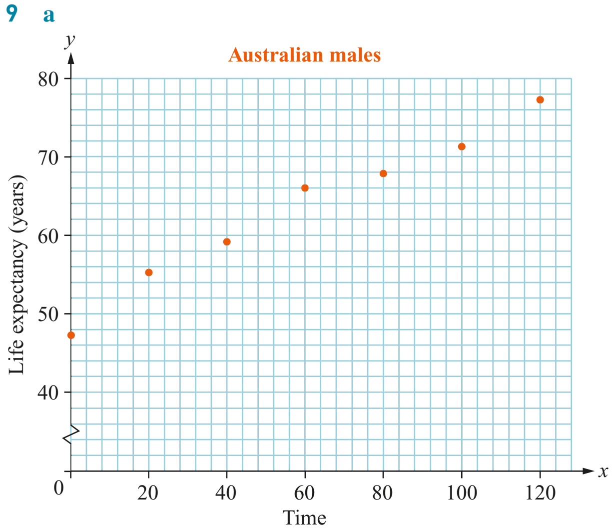
**7** **a** An independent variable isn't affected by the value of the dependent variable. The value of the dependent variable depends on the value of the independent variable.

**8** **a** Independent variable = Time
Dependent variable = Life expectancy
**b**




**c** Yes

**9** **a**




**b** Yes, the average life expectancy of Australian males increased steadily from 1881 to 2001.

## EXERCISE 5B CORRELATION

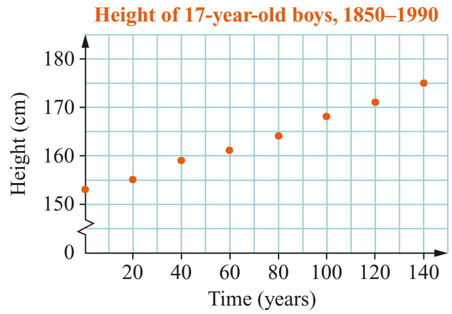
1 a Negative b Positive
c Approximately zero
d Approximately zero

2 a Perfect b Strong
c Moderate d Weak

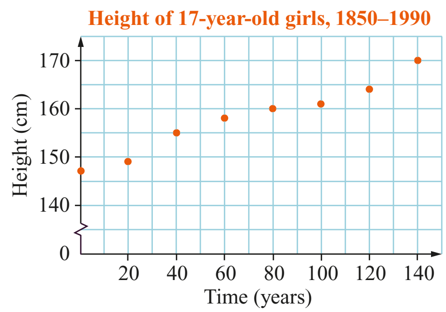
4 As time increases, bicep size increases. The correlation is moderate.

5 Yes, the negative correlation means that as the time spent on a diet increases the weight decreases. $r = -0.77$ indicates a strong negative correlation.

6 a $r = 0.997$
b



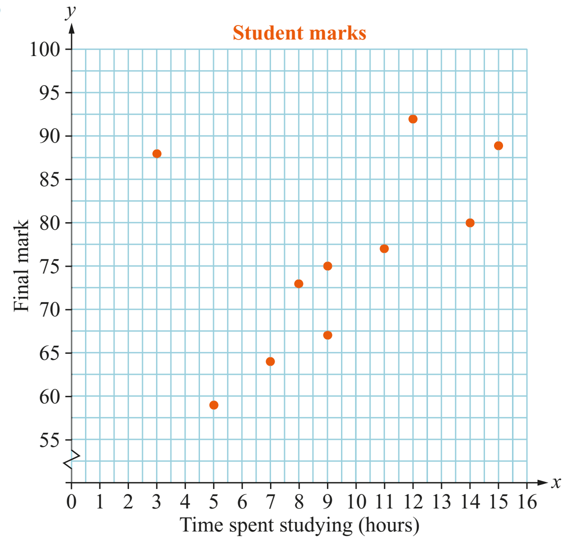

7 a 0.983
b




8 a Perfect positive correlation
b Perfect positive correlation
c Moderate negative correlation
d Moderate positive correlation
e Strong negative correlation
f Moderate negative correlation
g Strong positive correlation
h Zero correlation
i Strong positive correlation
j Weak negative correlation
k Moderate positive correlation
l Strong positive correlation

9 a Spurious correlation b Cause and effect
c Cause and effect d Spurious correlation
e Spurious correlation f Cause and effect
g Spurious correlation

10 a $r = 0.447$
b

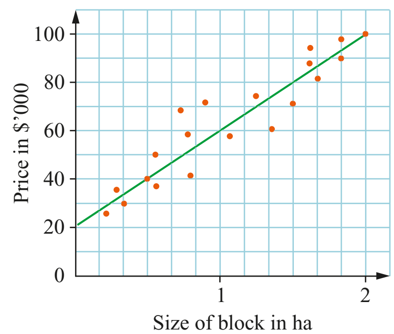



c Yes, student 6 (3, 88) d $r = 0.874$
e The outlier should only be removed if it is the result of an error.

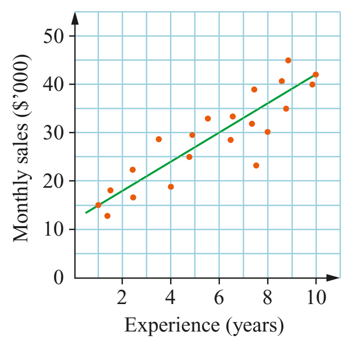
11 a Moderate positive correlation
b There is no reason to believe that this outlier is the result of an error. We would not expect everyone with long hair to use a large amount of shampoo.
c Removing the outlier would increase Pearson's correlation coefficient.

## EXERCISE 5C LINES OF BEST FIT

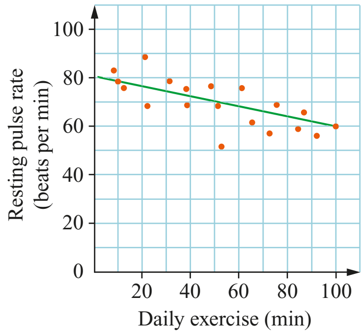
1 a



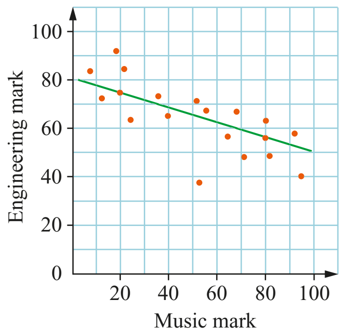

b



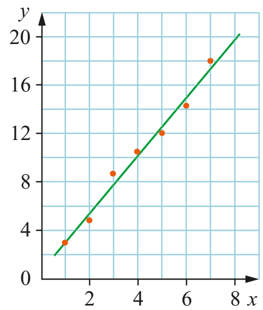
**c**



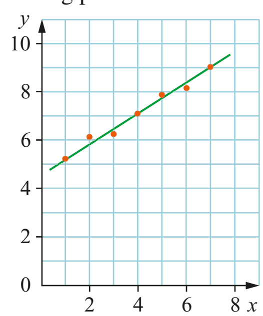

**d**



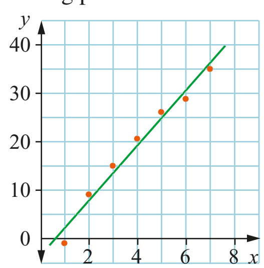

**2** **a** Strong positive correlation



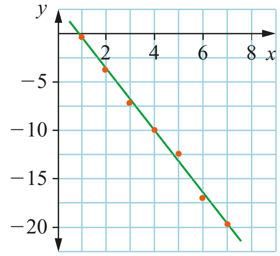

**b** Strong positive correlation



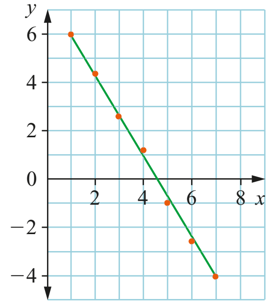

**c** Strong positive correlation



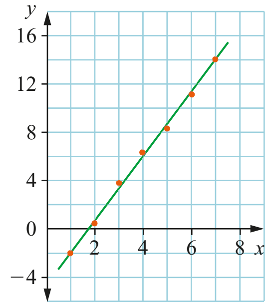

**d** Strong negative correlation



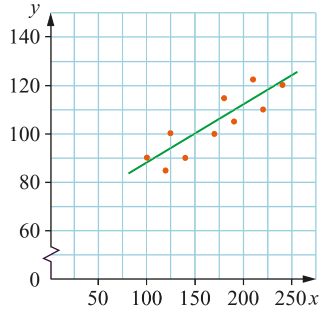

**e** Strong negative correlation



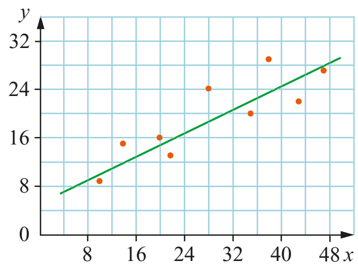

**f** Strong positive correlation



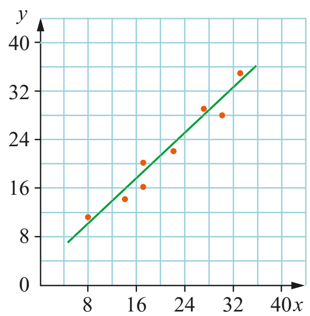

**3** **a**



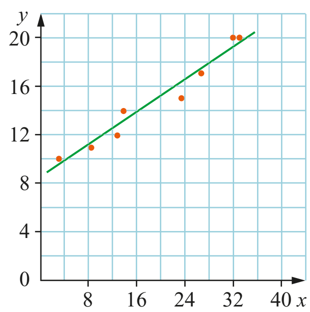

**b**



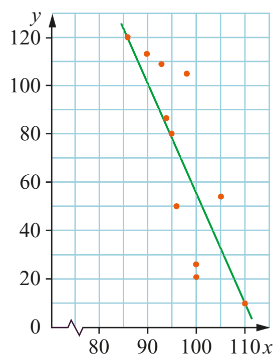

**c**



**d**



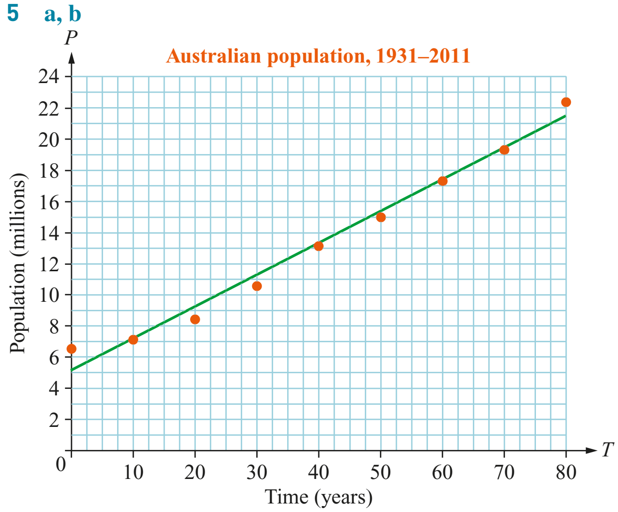

**e**




**4** **a** An approximately equal number of points should lie above and below the line of best fit

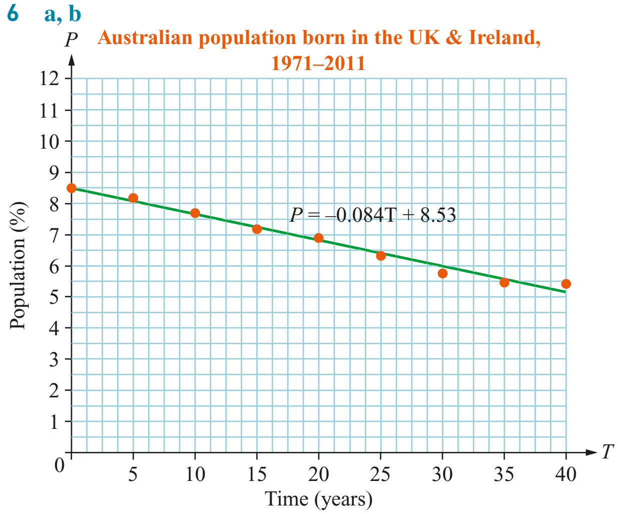
**b** The line of best fit is a straight line

**5** **a, b**



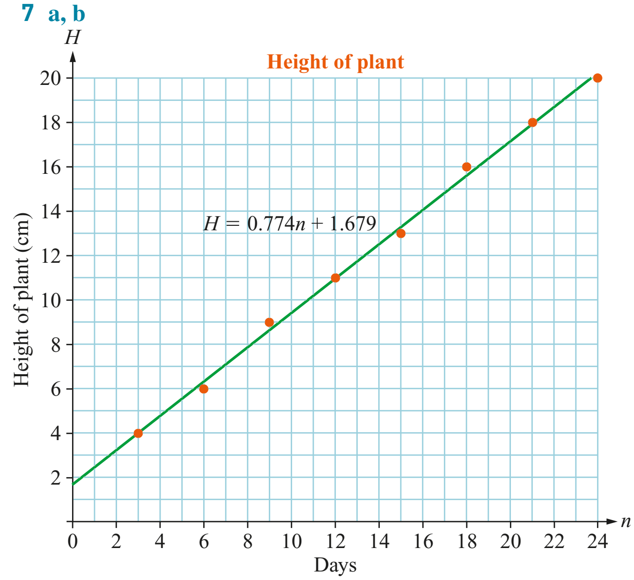

**c** $P = 0.203T + 5.156$

**6** **a, b**




**c** The vertical intercept tells the percentage of the population born in the UK and Ireland at the time when $T = 0$ (i.e. 8.53% in 1971), and the gradient tells us the expected change in percentage each year (i.e. on average for each increase of 1 year, the population born in the UK and Ireland decreases by 0.084%).

**7** **a, b**




**c** The vertical intercept tells us the height on the day the plant is planted (i.e. 1.679 cm), and the gradient tells us the change in height each day (i.e. on average for each day the planet grows by 0.774 cm).

**8** **a** The gradient tells us the average weight loss for each week on the diet (i.e. 1.1 pounds).

**b** No, as the vertical intercept tells us the weight loss at the start of the diet, and we would not expect a person to have lost any weight before starting the diet.

**9** Lines of best fit are straight lines that represent the data values in a scatterplot, so they are less useful when the data does not appear in a straight line (i.e. is not linear).

**10** With a strong correlation, the data values will be closer to the line of best fit. This is because the correlation measures how well the data values fit the line of best fit.

**11** **a** $y = 0.25x + 61.9$ **b** $y = 0.44x + 6.8$

**c** $y = 0.94x + 2.1$ **d** $y = 0.36x + 7.6$

**e** $y = -4.69x + 521$

## EXERCISE 5D INTERPOLATION AND EXTRAPOLATION

| | | | |
|---|---|---|---|
| **1** | **a i** 165 cm | **ii** 182 cm | **iii** 187 cm |
| | **b i** 155 cm | **ii** 168 cm | **iii** 177 cm |
| **2** | **a i** 408 cm$^3$ | **ii** 448 cm$^3$ | **iii** 435 cm$^3$ |
| | **b i** 157 cm | **ii** 170 cm | **iii** 178 cm |
| **3** | **a i** \$26 000 | **ii** \$20 000 | **iii** \$10 000 |
| | **b i** 1 year | **ii** 6 years | **iii** 9 years |
| **4** | **a i** 156 cm | **ii** 168 cm | **iii** 177.6 cm |
| | **b i** 163.3 cm | **ii** 175.8 cm | **iii** 182.5 cm |
| **5** | **a i** 57.4 cm | **ii** 65.1 cm | **iii** 67.9 cm |
| | **b i** 88.7 cm | **ii** 95.9 cm | **iii** 104.4 cm |
| **6** | **a i** 4.9 t | **ii** 3.92 t | **iii** 2.8 t |
| | **b i** 10 frosts | **ii** 15 frosts | **iii** 25 frosts |

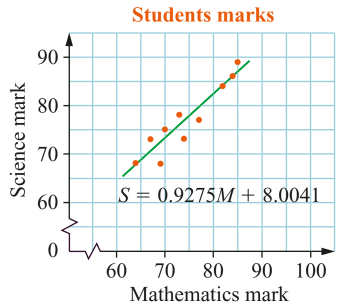
**7** **a** 174.4 cm **b** 179.6 cm

**c** 194.8 cm **d** 153.7 cm

**e** 157.8 cm **f** 183.8 cm

**8** **a** Extrapolation **b** Extrapolation

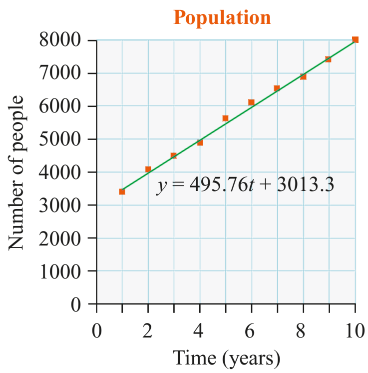
**9** **a, b**




**c** 82 **d** 56

**e** Part **d** is an example of extrapolation.

**10** **a, b**




**c** **i** 5244 **ii** 6732 **iii** 8962 **iv** 11 441

**d** Answers **iii** and **iv** because they are based on values outside the measured range.

**e** **i** During 1994 **ii** During 2004

**11** **a** **i** \$44 800 **ii** \$58 800 **iii** \$72 800

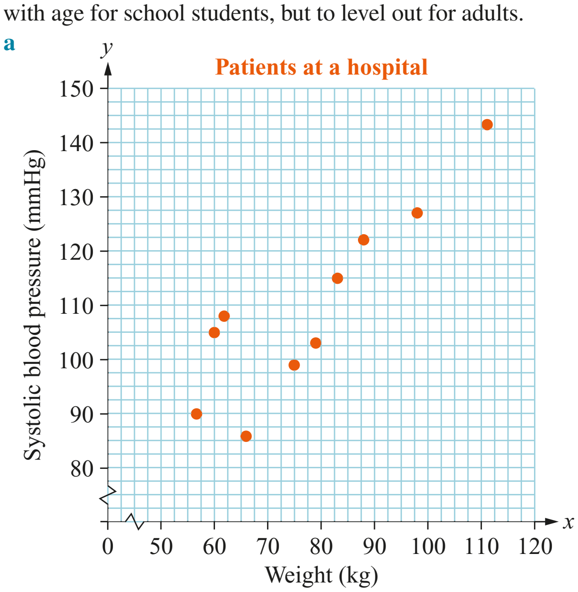
**b** Part **i** is an example of extrapolation.

**c** This means there is a strong positive correlation between the variables, and the data values are close to the line of best fit.

**d** Predictions made using interpolation (parts **ii** and **iii**) should be accurate. The prediction to part i might not be as accurate.

**12** The range of ages in April's dataset will be the ages of school children, so a 30-year-old will be well outside the range of this dataset and the prediction would not be useful. In practical terms, we would expect shoe sizes to increase with age for school students, but to level out for adults.

**13** **a**

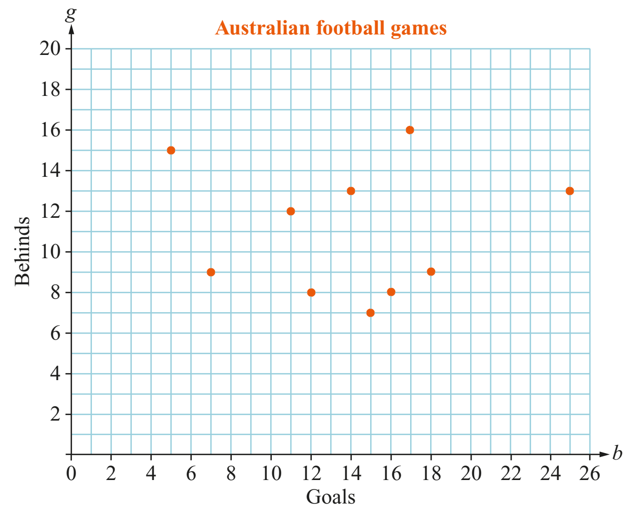



**b** $y = 0.871x + 41.93$

**c** **i** 142 mmHg **ii** 164 mmHg

**d** **i**, as it is closer to the range of values in the dataset.

**14** **a**



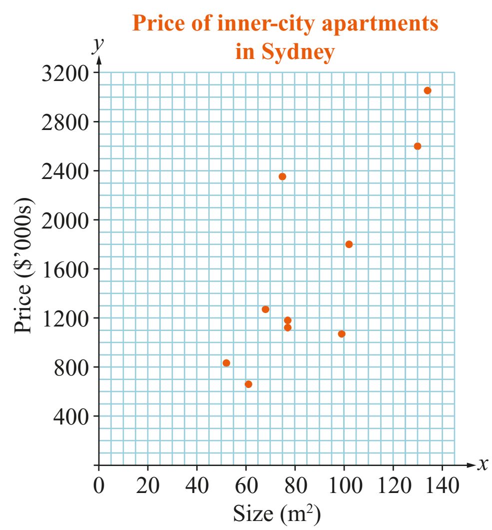

**b** 0.037

**c** $b = 0.021g + 10.714$

**d** No; the value of Pearson's correlation coefficient indicates that there is no correlation between the number of goals and the number of behinds scored, so predictions from the regression line will not be accurate.

**15** A line of best fit calculated from many data points is going to be more representative of the population data; therefore, predictions made from this line of best fit will be more accurate.

**16** **a**




**b** $y = 23.538x - 466.589$

**c** \$1 650 000

**d** $y = 25.320x - 749.132$

**e** \$1 530 000

## EXERCISE 5E STATISTICAL INVESTIGATIONS

**1** **a** No **b** Yes **c** Yes **d** No
**e** No **f** Yes **g** Yes **h** Yes

**2** **a** 30 **b** 5000 **c** 500 **d** 122

**3** **a** Students from schools in inner Sydney are more likely to come from wealthy families.
**b** The favourite music of people outside an Adele concert is more likely to be Adele.
**c** Members of a local cycling club are likely to be in favour of more cycle lanes.
**d** Investment bankers will tend to be optimistic about the state of the economy.
**e** Teenagers at Comic-Con will probably favour particular genres of film.

**4** The jobs that graduates with business degrees will be applying for will be different from those that graduates with science degrees will apply for, so their expected wages are likely to be different.

**5** **a** This question is leading people to say that military expenditure should be cut.
**b** The wording of this question suggests that it would be unusual to support the proposal for the new railway.
**c** By stating the film that won in a recent poll, people are more likely to also name this film when asked this question.

**6** **a** What is your favourite genre of music?
**b** What is your opinion of the rule changes in the NRL?

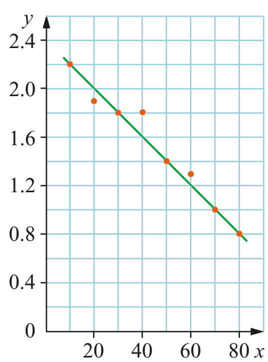
**7** Ryan's spreadsheet could be easily accessed by someone within his company, or by a hacker. The spreadsheet is unlocked, so if someone obtains the file they will have access to the personal information of the staff.

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1** | D | **2** | C | **3** | C | **4** | C | **5** | A |
| **6** | D | **7** | B | **8** | A | **9** | C | **10** | B |
| **11** | D | **12** | C | **13** | B | **14** | C | **15** | D |
| **16** | B | **17** | C | | | | | | |

## REVIEW SET 1

**1** **a**

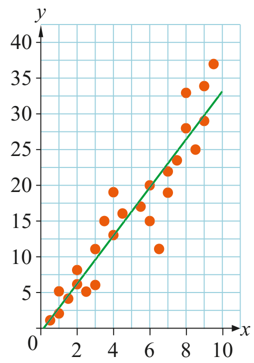



**b** **i** Strong **ii** Negative

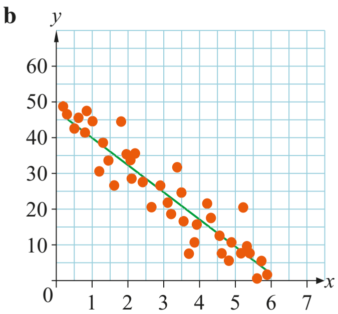
**2** This means that a client's percentage body fat decreases during the time they are in Dr Fitt's weight-loss programme: $r = -0.42$ is a weak negative correlation.

**3** **a** Strong positive correlation
**b** Strong positive correlation
**c** Strong negative correlation

**4** **a**




**b**




**5** **a** 72.5 kg **b** 122.5 kg **c** 222.5 kg

**6** It is not reasonable to expect the weight a person can bench press to continue to increase at a steady rate. It is more likely that their ability would level off. It is also possible that older people will start to lose some of their strength, and hence bench-pressing ability, due to other factors.

**7** **a** 75 **b** 48

**8** **a** On the evening of a State of Origin match, people will be more likely to have rugby league on their mind, and the media will have been full of stories about rugby league for the past week, so they will be more likely to choose it as their favourite sport.
**b** People who have just seen the latest Star Wars movie are more likely to be fans of the sci-fi/action genres, so are going to be biased in choosing their favourite film genre.

## REVIEW SET 2

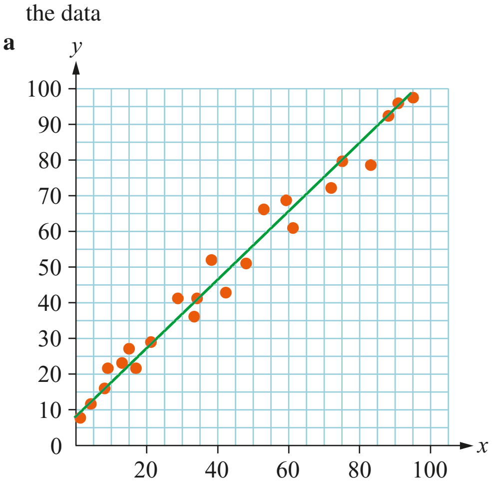
**1** **a** 272 minutes **b** 252.5 minutes **c** 220 minutes

**2** 238 hours

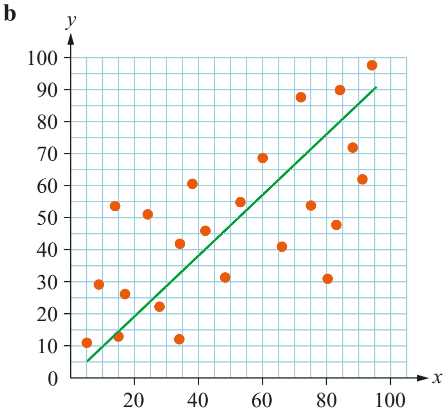
**3** **a** 65 **b** 80

**5** **a** Interpolation, as the population fell below 2000 within the range of the data
**b** Extrapolation, as the year 2020 is outside the range of the data

**6** **a**



**b**

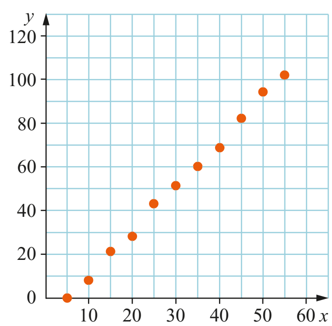


**7** **a** The wording of this question suggests that the new building design is bad, thereby leading people to say that they also don't like it.

**b** Asking 'Do you agree …', is more likely to lead people to say that they do agree than that they don't agree.

## REVIEW SET 3

**1**



**2** **a** \$55 **b** 50

**3** The higher mark a student receives in an English exam, the higher the mark they are likely to receive in a French exam: $r = 0.66$ is a moderate positive correlation.

**4** **a** Spurious correlation

**b** Spurious correlation

**5** **a** **i** 8.1 cm **ii** 13.9 cm **iii** 25.5 cm

**b** No, as it suggests that the initial height of the plant is a negative value ($-3.5$ cm).

**7** **a** Strong negative correlation

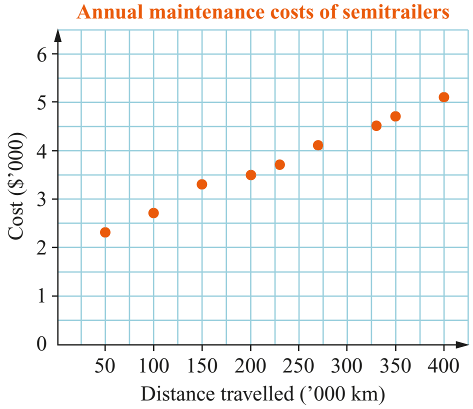
**b** No correlation

**c** Moderate positive correlation

## REVIEW SET 4

**1** **a**

Annual maintenance costs of semitrailers



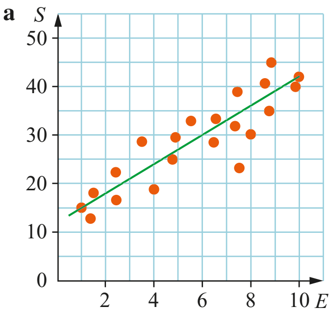

**b** Strong positive correlation

**c** $y = 0.008x + 1.891$

**d** \$4300

**e** This prediction should be accurate as it is uses interpolation and there is a strong correlation between the variables.

**2** **a**



**b** 33 **c** 9.5

**3** **a** 3.93 **b** $-5.7$

**4** **a** Extrapolation

**b** 2004 is outside the range of the years given in the dataset.

**5** **a** People who work in the finance sector often receive higher bonuses, which will lead to biased results when wanting to research bonuses in all Australian businesses.

**b** New parents are likely to have very favourable attitudes towards parental leave, which will lead to biased results in this research.

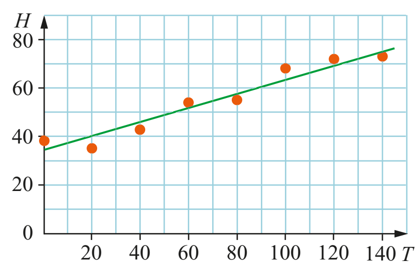
**6** 0.7

**7** **a** This means that there is a strong negative correlation between the variables, and the data values are close to the line of best fit.

**b** Predictions made from the regression line should be accurate for this data set.

## REVIEW PRACTICE EXAMINATION QUESTION

**a** **i**



**ii** 50

**iii** Strong positive correlation, i.e. data points are close to a straight line.

**iv** Extrapolation

**b** **i** Independent variable : amount of medicine administrated, dependent variable : number of bacteria

**ii** 5175 **iii** 85

**iv** $-23$; for each additional mL of medicine administered, the number of bacteria reduces by 23 million.

**v** 6325; the number of bacteria in millions before any medicine has been administered.

**vi** Yes, there will be bacteria before any medicine has been administered, so the intercept makes sense in the context of the data.

## ARE YOU READY?

**1** B **2** B **3** D **4** B
**5** C **6** B **7** B **8** B

## EXERCISE 6A INTRODUCTION TO NETWORKS

**1** **a** 4 **b** 4 **c** 4 **d** 1
**e** 4 **f** 5 **g** 2 **h** 9

**2** **a** 3 **b** 4 **c** 2 **d** 3
**e** 6 **f** 4 **g** 4 **h** 12

**3** **a** **i**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 1 | 2 | 2 | 1 |

**ii** 6

**b** **i**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 2 | 2 | 2 | 2 |

**ii** 8

**c** **i**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 1 | 1 | 1 | 1 |

**ii** 4

**d** **i**

| Vertex | A |
|---|---|
| Degree | 6 |

**e** **i**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 3 | 3 | 3 | 3 |

**ii** 12

**f** **i**

| Vertex | A | B | C | D | E |
|---|---|---|---|---|---|
| Degree | 4 | 1 | 1 | 1 | 1 |

**ii** 8

**g** **i**

| Vertex | A | B |
|---|---|---|
| Degree | 3 | 5 |

**ii** 8

**h** **i**

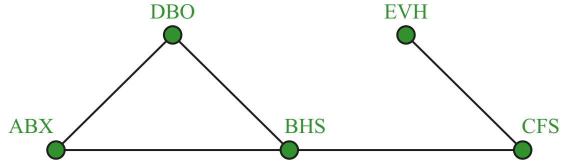
| Vertex | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| Degree | 2 | 2 | 6 | 2 | 4 | 2 | 2 | 2 | 2 |

**ii** 24

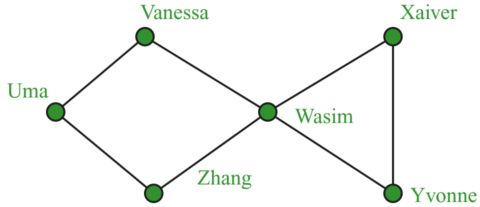
**4**

| Vertex | Ada | Ben | Carlos | Dalia | Eric |
|---|---|---|---|---|---|
| Degree | 3 | 1 | 2 | 2 | 2 |

**5**




**6** **a**




**b**

| Vertex | Uma | Vanessa | Wasim | Xavier | Yvonne | Zhang |
|---|---|---|---|---|---|---|
| Degree | 2 | 2 | 4 | 2 | 2 | 2 |

**c** Wasim or Uma **d** Uma

**7** 10 minutes

**8** **a** 10 (including the start and end)
**b** 9 **c** 1 **d** 3
**e** Yes (look for a cycle)

**9** A

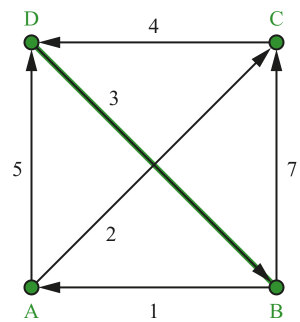
**10** **a** 9 litres per second **b** 8 litres per second
**c** 3 litres per second **d** 4 litres per second
**e** 5 litres per second

**11** C

**12**

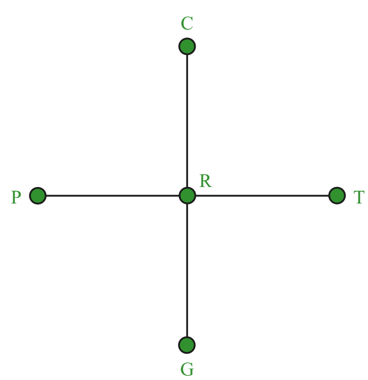
| From vertex | To | | | | |
|---|---|---|---|---|---|
| | **A** | **B** | **C** | **D** | **E** |
| **A** | – | 3 | 1 | 5 | – |
| **B** | – | – | 1 | – | 2 |
| **C** | – | – | – | – | 3 |
| **D** | – | – | 1 | – | 4 |
| **E** | – | – | – | – | – |

**13**



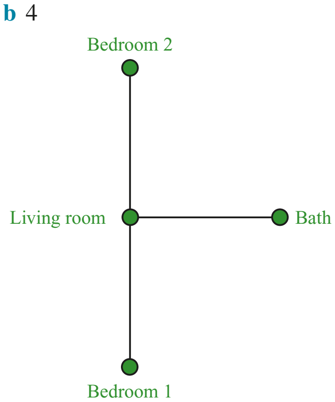

**14** **a** 2 **b** 0 **c** 4 **d** 0
**e** 4 **f** 4 **g** 2 **h** 0

**15** **a**

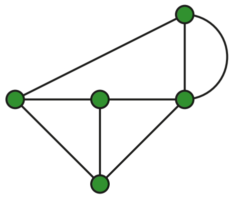



**b** 4

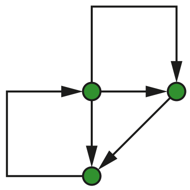
**16**



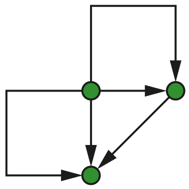
**17**



**18**

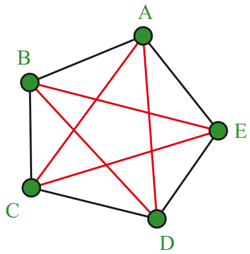


**19**



One vertex has zero indegree (cars cannot enter the intersection) and another has zero outdegree (cars cannot exit the intersection).

**20**




**21** **c** 6

## EXERCISE 6B PATHS AND CYCLES

**1** A, f, B, g, C, h, D, i, E, j, F

**2** **a** E **b** B

**3** Corner C

**4** **a** A, B, C, D. The start vertex is A and the end vertex is D.
**b** h, g, f. The start vertex is D and the end vertex is A.
**c** f, g, h, h, g, f. A is both the start and end vertex.

**5** A **6** D

**7** **a** ABCDE **b** ABCIA
**c** FEDCB **d** HCFEH

**8** ABCDEFCHEGCIA. This is neither a cycle nor a path.

**9** **a** 12 minutes
**b** There are four paths that each take 12 minutes.

**10** C **11** DE

**12** DBACDEFGH

**13** **a** A, a, B, b, C, d, D **b** abd
**c** ABCD **d** A, a, B, c, C, d, D
**e** acd **f** ABCD
**g** No, because the walks are different but answers **c** and **f** are the same.

**14** **a**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 1 | 2 | 2 | 1 |

This network has the walk ABCD.

**b**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 2 | 2 | 2 | 2 |

This network has the walk ABCDA.

**c**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 1 | 1 | 1 | 1 |

This network has no such walk because more than two vertices have an odd degree.

**d**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 3 | 2 | 3 | 2 |

This network has walk ABCADC.

**e**

| Vertex | A | B | C | D |
|---|---|---|---|---|
| Degree | 3 | 3 | 3 | 3 |

This network has no such walk because more than two vertices have an odd degree.

**f**

| Vertex | A | B | C | D | E |
|---|---|---|---|---|---|
| Degree | 4 | 1 | 1 | 1 | 1 |

This network has no such walk because more than two vertices have an odd degree.

**15** **a** 1 **b** 2 **c** 0 **d** 3

**16**


The garbage truck route is fgkjhi.

**17** This task is not possible because there are more than two intersections with an odd degree.

**18** There are zero vertices with odd degree so it is possible to cross every bridge exactly once.

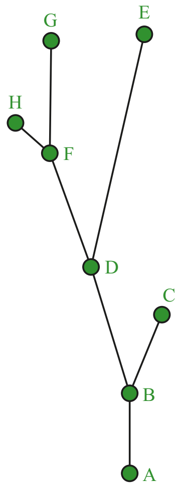
## EXERCISE 6C TREES

**1** **a** ABCDL **b** LDE
**c** BCFH **d** GFCDK

**2** A, B and C

**3** A: 7 vertices and 6 edges
B: 5 vertices and 4 edges
C: 4 vertices and 3 edges

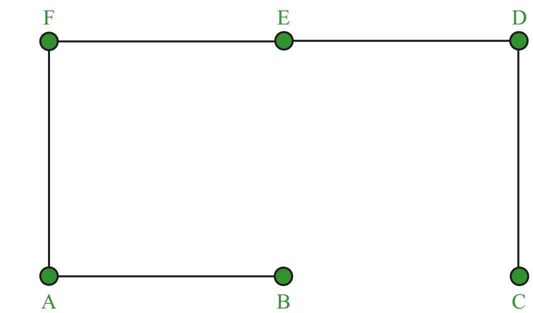
**4**



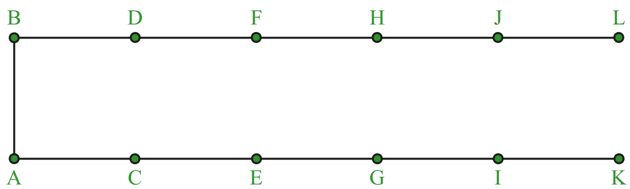

It is a tree.

**5** B

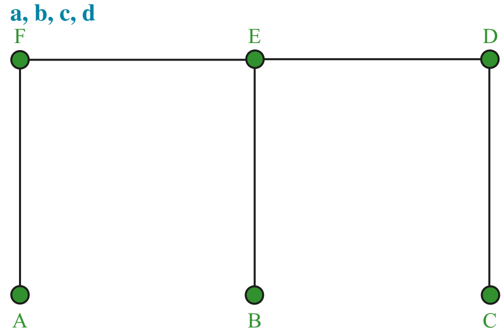
**6** There are many correct answers. For **a**, **b**, **c** and **d** the same spanning tree can be chosen.



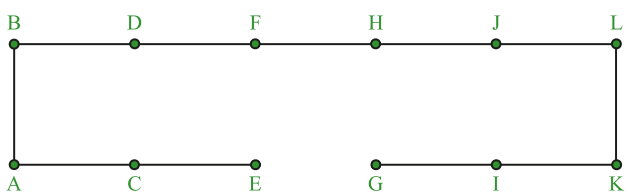

**e**



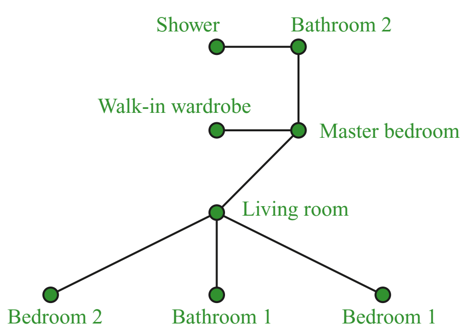

**7** **a, b, c, d**



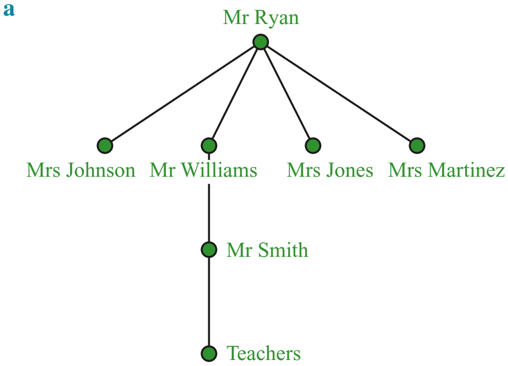

**e**




**8** **a**




**9** **a**



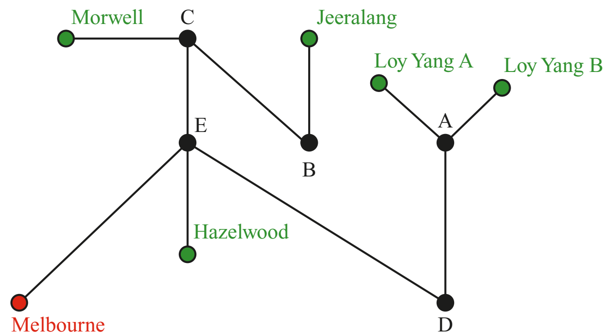

**b** It is a tree.

**c** 2

**10** Part **c**

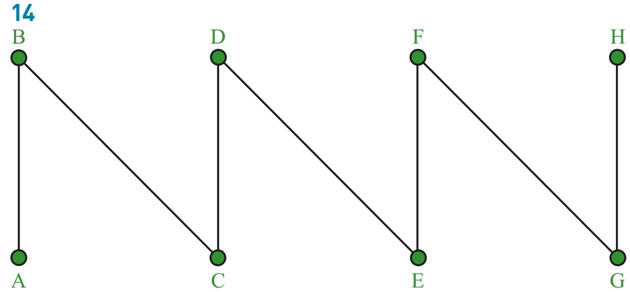
**11** There is a left tree (with vertices A, B, C and D) and a right tree (with vertices E, F, G, H).
Connecting D and G with an edge will turn the forest into a tree.

**12** At most 3 edges can be removed.

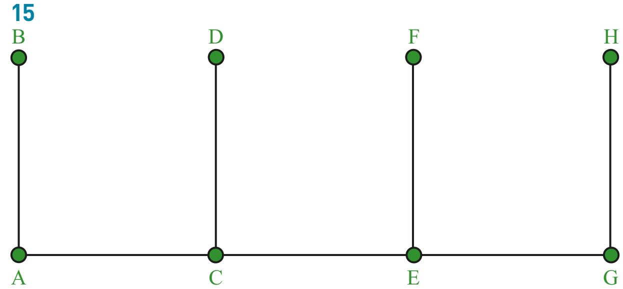



**13** 9

**14**




**15**




**16** Yes. Every tree is a forest, but not every forest is a tree.

**17** Yes

**18** **a** No **b** Yes

## EXERCISE 6D MINIMUM SPANNING TREES

**1** **a** 11 **b** 12 **c** 26

**2** 21.3 m

**3** **a** 9 **b** 8 **c** 7

**d** 6 (smallest total edge weight)

**4** **a**

| Edge | AB | DC | BD | DE | AC | AE |
|---|---|---|---|---|---|---|
| Weight | 2 | 2 | 3 | 4 | 4 | 5 |

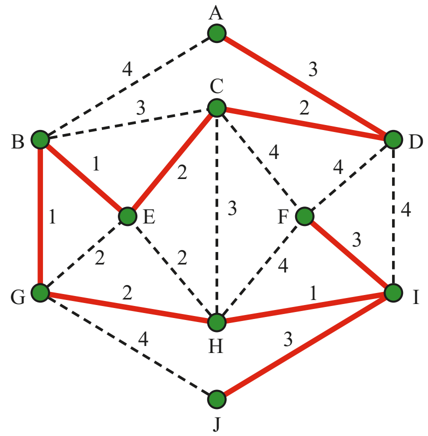
**b**

| Edge | AB | DC | BD | AC | ED | AE |
|---|---|---|---|---|---|---|
| Weight | 2 | 2 | 4 | 4 | 6 | 6 |

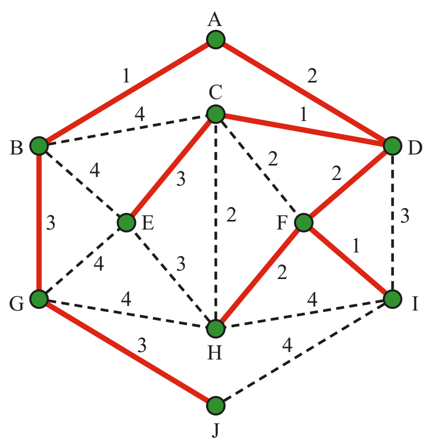
**c**

| Edge | AB | AC | BD | CD | AE | ED |
|---|---|---|---|---|---|---|
| Weight | 1 | 1 | 1 | 1 | 1 | 1 |

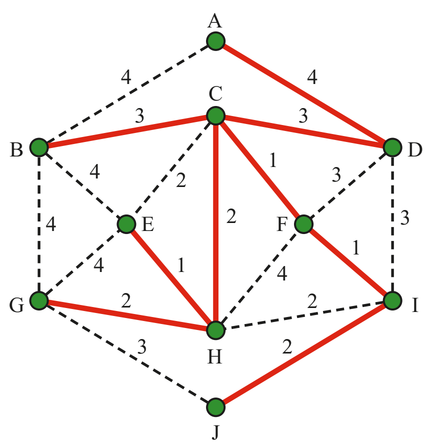
**5**



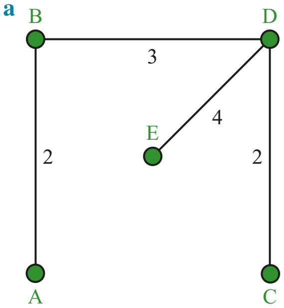

**6** **a**



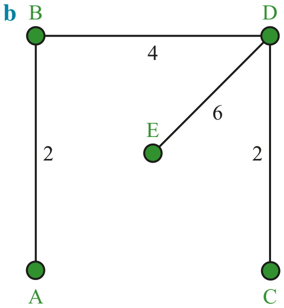

**b**



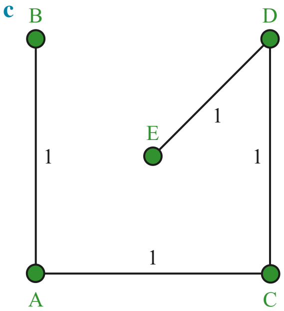

**7** **a**



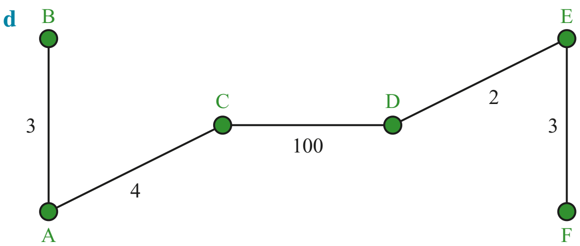

**b**



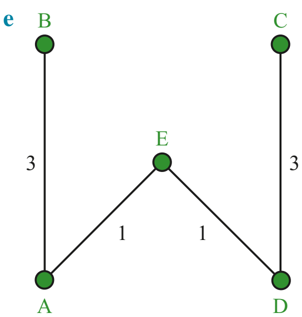

**c**




**d**




**e**




**8** **a**

| Edge | EB | EC | ED | EA |
|---|---|---|---|---|
| Weight | 6 | 6 | 2 | 3 |

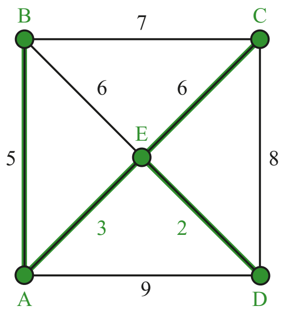
**b** Edge ED has the smallest weight.

**9** **a**

| Edge | EB | EC | AB | CD |
|---|---|---|---|---|
| Weight | 6 | 6 | 5 | 8 |

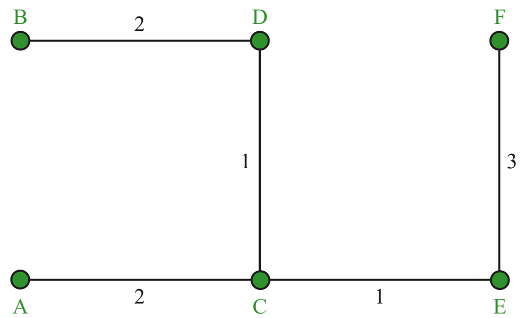
**b** Edge AB has the smallest weight.

**10**

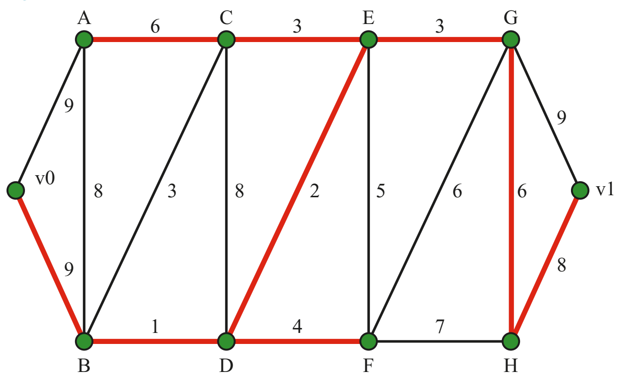



**11** See answer to question **7**.

**12**

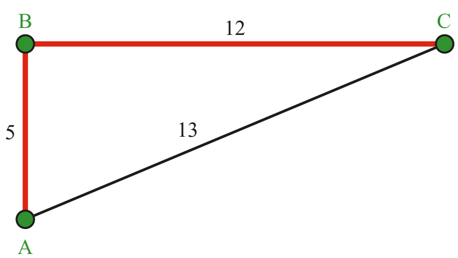



**13**

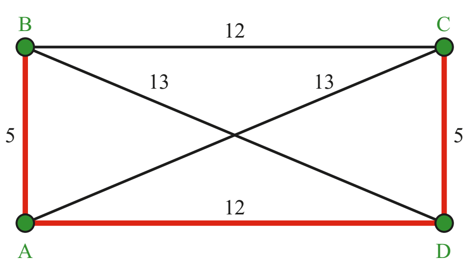



**14** **a** \$7 **b** \$6 **c** \$7

**15** **a**

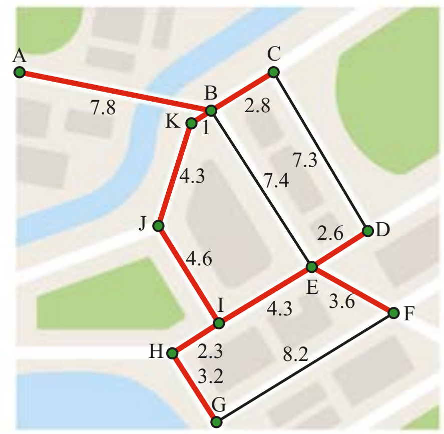



**b**



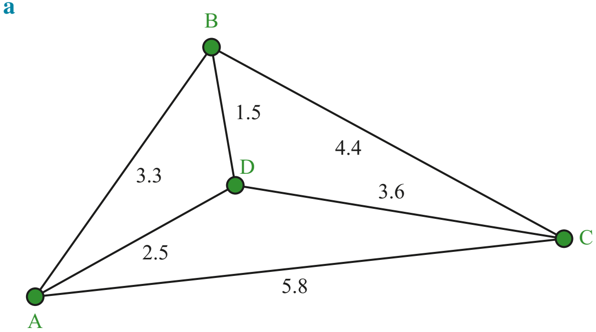

**16** Connect AB, BC and CD.

**17**




36.5 km

**18** **a**




**b** Connect rail lines along AD, BD and DC.
**c** \$38 million
**d** \$56.5 million
**e** \$18.5 million

**19** **a** Perth to Adelaide, Adelaide to Melbourne, Melbourne to Canberra, Canberra to Sydney, Sydney to Brisbane, Adelaide to Darwin
**b** 8412 km

## EXERCISE 6E THE SHORTEST PATH

**1** **a** 11 **b** 9

**2** **a** ABCDEF **b** AHG **c** ABCDEFGH

**3** **a**

| Path | EA | EB | EC | ED |
|---|---|---|---|---|
| Total weight | 3 | 6 | 6 | 2 |

**b**

| Path | EA | EB | EC | EDA | EDC |
|---|---|---|---|---|---|
| Total weight | 3 | 6 | 6 | 11 | 10 |

**c**

| Path | EB | EC | EAB | EDC |
|---|---|---|---|---|
| Total weight | 6 | 6 | 8 | 10 |

**i** EB and EC both have the smallest weight.
**ii** 6

**4** **a** 7 **b** 8 **c** 2
**d** 107 **e** 4

**5** 12 **6** 7

**7** **a** Direct flight (\$706) **b** Via Mildura (\$922)

**8** Yes **9** No

**10** **a** Via the shops **b** Via the depot

**11** Dubbo, Sydney, Melbourne, Mount Gambier

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1** | C | **2** | D | **3** | D | **4** | D | **5** | A |
| **6** | D | **7** | A | **8** | C | **9** | C | **10** | D |

## REVIEW SET 1

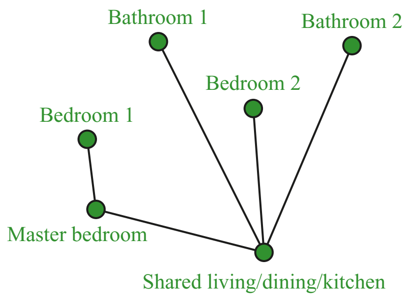
**1**

| **Vertex** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Degree** | 3 | 3 | 2 | 1 | 3 | 4 |

**2** No, there are more than two edges with odd degree.

**3** 4, 2, 1, 6, 5, 3

**4**




**5** Yes

**6**

| **Vertex** | Shared living/ dining/ kitchen | Master bedroom | Bathroom 1 | Bedroom 1 | Bathroom 2 | Bedroom 2 |
|---|---|---|---|---|---|---|
| **Degree** | 4 | 2 | 1 | 1 | 1 | 1 |

No, as there are more than two vertices with odd degree.

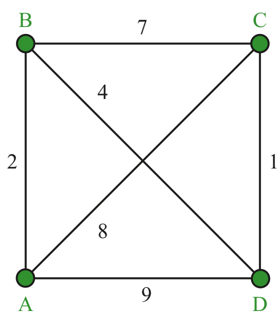
**7** If you start with vertex A then Prim's algorithm chooses edges: AF, AB, BC, BE, ED.

**8** 15

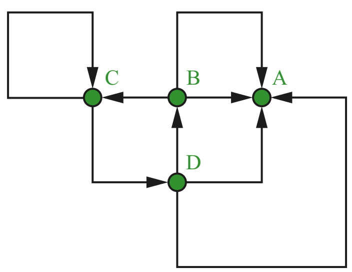
**9** The shortest path is v0, B, D, F, H, v1.

## REVIEW SET 2

**1**




**2**




**3** Vertex A has indegree 4 but outdegree 0, so cars cannot exit this intersection.

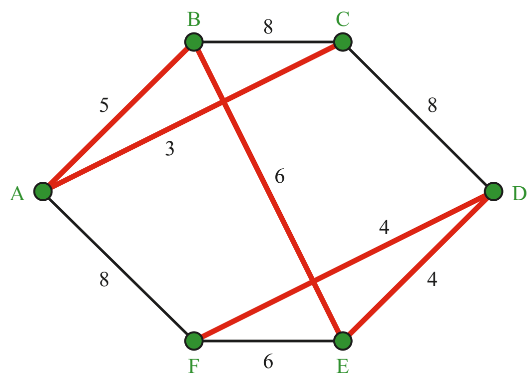
**4** 1, 3, 5, 4, 2, 6, 1

**5** Remove any single edge.

**6** Remove another edge (any one will do).

**7** Yes **8** Yes

**9**




**10** The shortest path is v0, B, C, E, G, v1

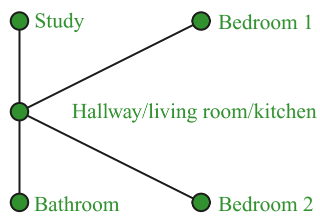
## REVIEW SET 3

**1**

| **Vertex** | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| **Degree** | 1 | 2 | 3 | 1 | 1 | 4 |

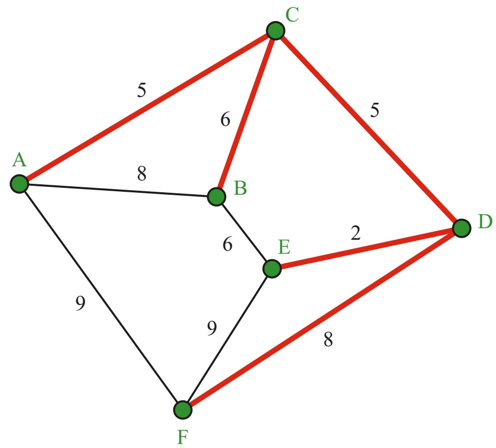
**2** No, there are more than two vertices with odd degree.

**3**




**4** Yes

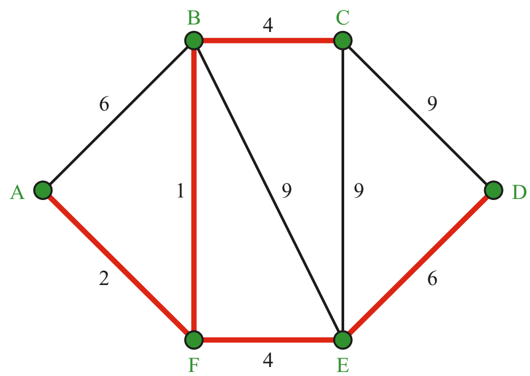
**5**




**6** 21 **7** AFED

**8** FBAFEBCEDC

**9**

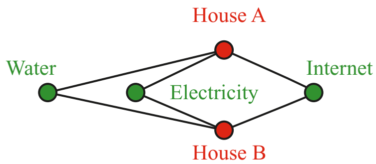


## REVIEW SET 4

1

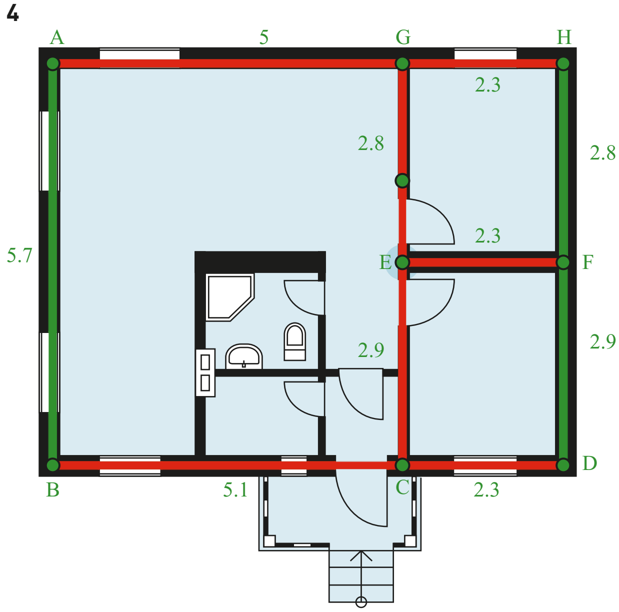
| Vertex | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Degree | 2 | 1 | 2 | 2 | 3 | 4 |

2




3 House A, Electricity, House B, Internet, House A, Water, House B

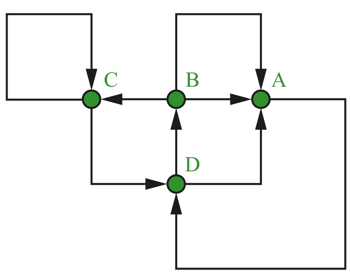
4




5 21.1

6 See answer to question **4**

7




8 v0, B, C, E, F, H, v1

## REVIEW PRACTICE EXAMINATION QUESTIONS

**a i** A **ii** 4

**iii** Either directly, or via New Town (both 20 km)

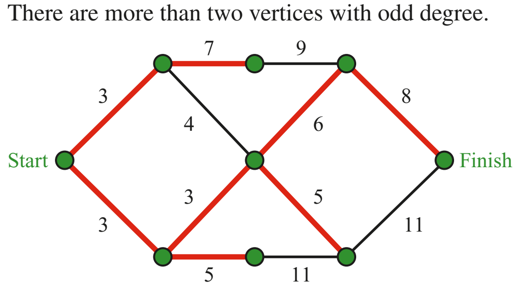
**iv** Beachside, Hillsvale, Old Town, Beachside, New Town, Hillsvale

**b i** ACE **ii** AC, AB, DE and BD

**c i** A

**ii** There are more than two vertices with odd degree.

**iii**




**d i** Yes **ii** A

**e i** 3 **ii** 4

**iii** Yes, since there are only two vertices with odd degree.

# CHAPTER 7 ANNUITIES

## ARE YOU READY?

**1** B **2** B **3** A **4** A **5** C **6** D **7** D

## EXERCISE 7A FUTURE AND PRESENT VALUE OF AN ANNUITY

**1** $A_1 = \$1500$

$A_2 = A_1 \times 1.08 + \$1500$
$= \$1500 \times 1.08 + \$1500$
$= \$3120$

$A_3 = A_2 \times 1.08 + \$1500$
$= \$3120 \times 1.08 + \$1500$
$= \$4869.60$

$A_4 = A_3 \times 1.08 + \$1500$
$= \$4869.60 \times 1.08 + \$1500$
$= \$6759.17$

$A_5 = A_4 \times 1.08 + \$1500$
$= \$6759.17 \times 1.08 + \$1500$
$= \$8799.90$

**2** $23 205

**3**

| Payment | Amount | Years invested | Amount + interest | Amount |
|---|---|---|---|---|
| 1 | $2000 | 4 | $\$2000(1 + 0.06)^4$ | $2524.95 |
| 2 | $2000 | 3 | $\$2000(1 + 0.06)^3$ | $2382.03 |
| 3 | $2000 | 2 | $\$2000(1 + 0.06)^2$ | $2247.20 |
| 4 | $2000 | 1 | $2000(1 + 0.06) | $2120 |
| 5 | $2000 | 0 | | $2000 |
| | | | **Total** | $11 274.18 |

**4** $28 613.16
**5 a** $56 280.08 **b** $39 575.32
**6 a** $27 908.01 **b** $20 690.22
**7 a** $97 189.48 **b** $65 406.73
**8** $35 459.51
**9** $32 702.87
**10** $471.85
**11** $369.43
**12** $2663.05
**13 a** $30 046.42 **b** $6046.42
**14 a** $51 768.64 **b** $8568.64
**15 a** $19 380.22 **b** $23 256.27
**16** $33 672.43
**17** $170 212.77
**18** Yes, she will be $796.59 over her target.
**19** Trevor by $1118.14
**20** Darrel
**21 a** $147.88 **b** $36.97
**22** $155.94
**23 a** $36 043.12 **b** No
**24** Receiving $1000 at the end of each month, by $1207.15
**25 a** $46 682.16 **b** $33 17.84

## EXERCISE 7B FUTURE VALUE USING A TABLE

**1** **a** **i** $14 092.75 **ii** $1592.75
**b** **i** $6913.79 **ii** $1713.79
**c** **i** $3696.38 **ii** $126.38
**d** **i** $40 459.20 **ii** $7459.20
**e** **i** $268 704 **ii** $68 704
**f** **i** $31 854.40 **ii** $3054.40
**g** **i** $263 349.60 **ii** $143 349.60
**h** **i** $194 432 **ii** $34 432
**2** $39 992.70
**3** **a** $305 035.50 **b** $62 742.80
**4** **a** $11 591.43 **b** $5409.78 **c** $24 297.40
**d** $80 920.50 **e** $168 051 **f** $198 153.74
**5** $322 497.26
**6** $8967.14. No, he will be $1032.86 below his target.
**7** $735.09
**8** $1419.17
**9** $210.22
**10** **a** $9.90 **b** $8.92
**c** $50.93 **d** $6.55

## EXERCISE 7C PRESENT VALUE USING A TABLE

**1** **a** $10 412.80 **b** $17 845.71
**2** **a** $31 673 **b** $48 608.13
**3** **a** $21 290.15 **b** $27 048.90
**4** $20 799.40
**5** $20 079.54
**6** $73 717
**7** $553 214
**8** $549 422.50
**9** $666 092
**10** $5005.26
**11** $3893.74
**12** **a** $4.21 **b** $4.62
**c** $43.59 **d** $85.37

## EXERCISE 7D REPAYING LOANS

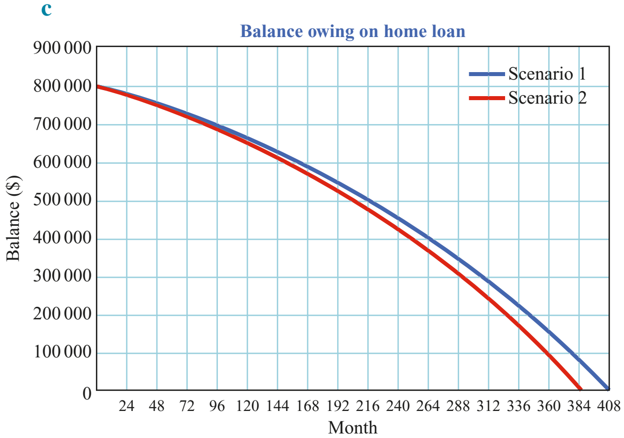
**1** $154 093.30
**2** **a** $131 861 **b** $20 549
**3** **a** $275.27 **b** $13 212.96 **c** $1212.96
**4** **a** $1164.92 **b** $23 298.40 **c** $3298.40
**5** **a** $4996.96 **b** $99 939.20 **c** $34 939.20
**6** **a** $2302.68 **b** $690 804 **c** $370 804
**7** **a** $1311.66 **b** $472 197.60 **c** $222 197.60
**8** **a** $2874.60 **b** $862 380 **c** $2590.52
**d** $777 156 **e** $85 224
**9** $79 873.20
**10** $58 374.60
**11** $119 925
**12** **a** $1881.98 **b** $1894.28 **c** $454 977.20
**13** **a** $2446.60 **b** $12.92 **c** $2459.52
**d** $2040 **e** $740 496
**14** **a**

| Month | Balance at start of month | Interest | Monthly repayment | Balance at end of month |
|---|---|---|---|---|
| 2 | $220 220 | $1321.32 | $1100 | $220 441.32 |
| 3 | $220 441.32 | $1322.65 | $1100 | $220 663.97 |
| 11 | $222 260.36 | $1333.56 | $1100 | $222 493.92 |
| 12 | $222 493.92 | $1334.96 | $1100 | $222 728.88 |

**b** Disadvantage: the principal increases because the repayment does not cover the interest charged for the month.
Advantage: repayments are lower for first 12 months.
**c** $576 061 **d** $38 581
**15** **a** $2170 **b** $2475.35 **c** $305.35
**d** $820 725 **e** $78 120
**f** Advantage: lower repayments for 3 years.
Disadvantage: you pay more for the loan.
**g** The amount owing does not increase.
**16** **a** $334 800 **b** $118 386 **c** $61 600

## EXERCISE 7E USING TECHNOLOGY TO MODEL ANNUITIES

**1** **b** $277 740.82
**2** **b** $53 578.88
**3** **b** $41 445.13
**4** **b** $8981.54 **b** 5
**5** **b** 321 months (26 years, 9 months)
**6** **a** $619 112.94 **b** $394 781.76 **c** $10 293.96
**7** $17 679.28
**8** $438.21
**9** 2 months
**10** 20 months
**11** **b** 14 months
**c**




**12** **a** 21 months **b** $23 829
**13** **a** After 10 years (120 months) **b** 23 months
**c** 157 months **d** $25 677
**14** **a** After 10 years **b** $60 000
**c** 63 months **d** $48 297
**15** **a** The balance owing increases because the repayment is not enough to cover the interest for the month.
**b** $272 000 **c** $2142 **d** $46 280
**16** A loan with a high interest rate will be paid back slower than a loan with a low interest rate, as the high interest rate will increase the balance by more each time interval.
**17** An increase in the payment amount will either increase the value of an annuity investment, or reduce the time it takes to pay back an annuity loan. An increase in the interest rate will either increase the value of an annuity investment, or increase the time it takes to pay back an annuity loan.
**18** **a** D **b** A **c** C **d** B
**19** $1657.22
**20** $77.73

## REVIEW MULTIPLE-CHOICE QUESTIONS

**1** C **2** D **3** D **4** C **5** C **6** D
**7** A **8** B **9** D **10** C **11** A **12** B

## REVIEW SET 1

1 $20 984.95
2 $21 120.66
3 $1331.52
4 $127.30
5 **a** $41 262.98 **b** $53 735.13
6 $19 216.28
7 **a** $3228.13 **b** $774 751.20 **c** $364 751.20
8 $1957.99
9 **a** $453 600 **b** $191 977 **c** $58 830

## REVIEW SET 2

1 $30 217.68
2 $3814.79
3 $1827.92
4 $246.94
5 **a** $11 918.52 **b** $17 471.41
6 $26 421.56
7 $77 018.40
8 **a** $349 187.85 **b** $2512.72
**c** $753 816 **d** $2338.66
**e** $52 218

## REVIEW SET 3

1 $8580
2 $7867.22
3 $1254.24
4 $521.92
5 **a** $19 003.70 **b** $29 164.72
6 $18 050.29
7 **a** $2714.90 **b** $814 470 **c** $474 470
8 $2869.64
9 **a** $653 100 **b** $255 309 **c** $112 461

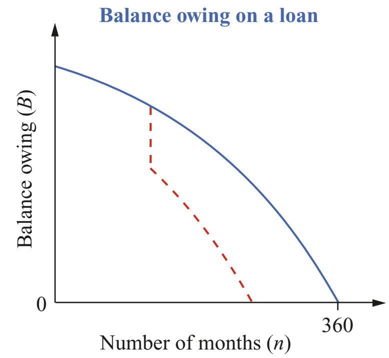
## REVIEW SET 4

1 $26 519.16
2 $26 818.35
3 $483.86
4 $129.00
5 **a** $17 168.65 **b** $25 226.38
6 $88 812
7 **a** $1600 **b** $2291.99 **c** $691.99
**d** $745 197 **e** $57 600
8 $55 391.39

## REVIEW PRACTICE EXAMINATION QUESTION

**a** $119 030
**b** $183.67
**c** $11 837
**d** $720 588
**e i** $194 556 **ii** $11 694 **iii** $22 734
**f i** $2783.03 **ii** $591 890.80
**iii** $5.74 **iv** $2792.57
**v** $1 006 645.20 **vi** $596 645.20

**vii**



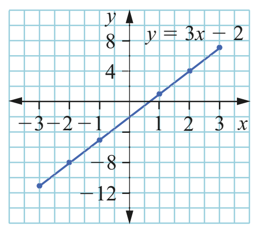

## CHAPTERS 4–7 CUMULATIVE REVIEW

1 Gradient is −3, $y$-intercept is 6.

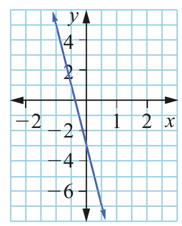
2 **a**

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $y$ | −11 | −8 | −5 | −2 | 1 | 4 | 7 |

**b**




3



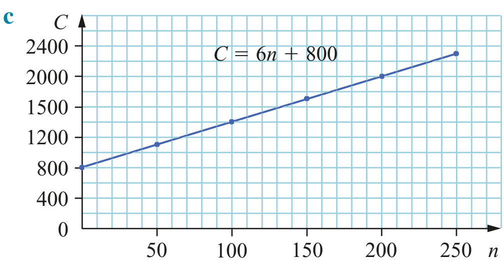

4 **a** $40 **b** 100 km
**c** Gradient is 0.4; cost per km.
**d** Vertical intercept is $10; flag fall.

5 **a** $C$ = total cost.
$n$ is number of pairs of socks produced.
$800 is fixed cost.
$6 is the cost per pair of socks.

**b**

| $n$ | 0 | 50 | 100 | 150 | 200 | 250 |
|---|---|---|---|---|---|---|
| $C$ | 800 | 1100 | 1400 | 1700 | 2000 | 2300 |

**c**

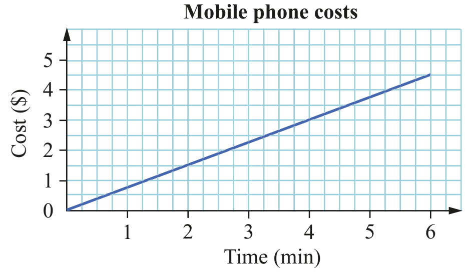



**d** 130 pairs of socks

6 **a** $C = 0.75t$, where $C$ is the total cost in dollars after $t$ minutes

**b** Dependent variable is cost.

**Mobile phone costs**



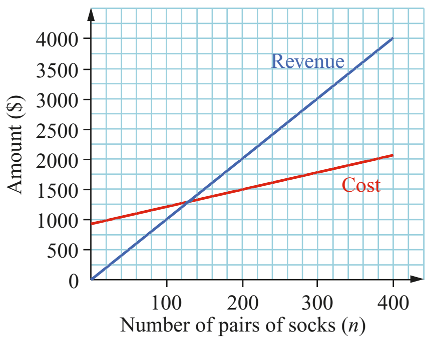

**c** About $1.90 **d** About $3\frac{1}{3}$ min

7 **a** $(2, -4)$ **b** Solution is $x = 2$, $y = -4$.

8 **a** Solution is $x = 1$, $y = 2$.

**b** Solution is $x = 3$, $y = -5$.

9 **a**




**b** About (130, 1300)

**c** About 130 **d** About $1300

10 **a** **i** Independent variable is temperature and dependent variable is number of people at the beach.

**ii** Independent variable is number of hours spent exercising per week and dependent variable is weight.

**b** **i** Yes **ii** yes

11 **a** Moderate correlation **b** Strong correlation

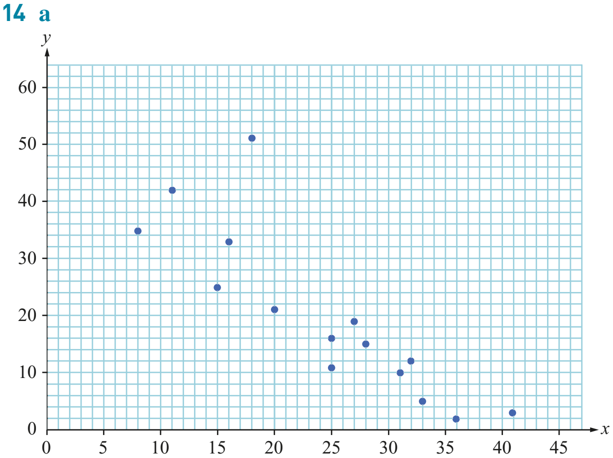
**c** Perfect correlation **d** Weak correlation

12 **a** Negative **b** Positive

**c** Positive **d** Positive

13 Strong positive correlation

14 **a**




**b** $-0.85835$

**c** Strong negative correlation

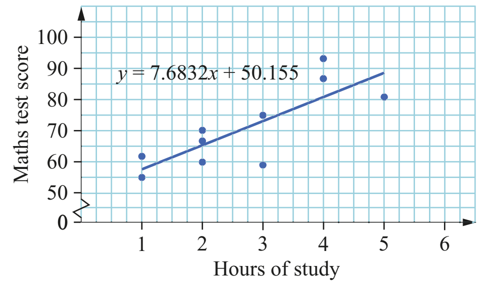
15 **a** Spurious correlation

**b** Spurious correlation

**c** Cause and effect relationship

16 **a** About 130 **b** About 63

17 **a**




**b** 0.802741

**c** $y = 7.6832x + 50.155$ where $x$ is the number of hours of study and $y$ is the maths test score

**d** Gradient is 7.6832. This means that for each increase of 1 hour of study, the Maths test score increases by about 7.7 marks.

**e** **i** 73 **ii** 85 **iii** 54 **iv** 104

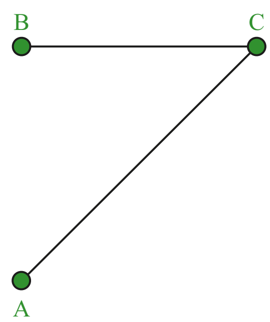
**f** Parts **iii** and **iv** are the least reliable as the predicted results are outside the range of data; extrapolation was needed.

18 People who have a dog as a pet regularly need to walk their dog, so will be more likely to be found walking outside, whereas most cat owners don't take their cats outside for walks. This question may target more dog owners than cat owners.

19

| Vertex | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| Connected to | B | A, E, C | D, E, B | C | F, B, C | E |

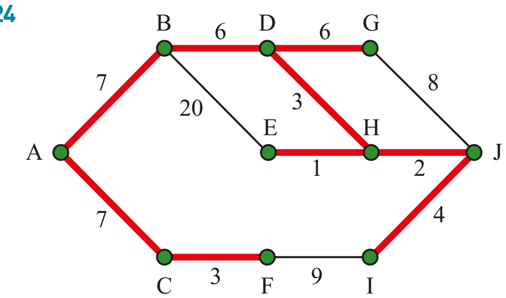
20




21 Yes

22 Many possible answers, for example edges AB, BC and CD.

23, 24




25 ABDHJ

26 BDHE

27 $6561.92

28 $52 796.90

29 $1450.69

30 $95.05

31 **a** $36 018.74 **b** $54 456.37 **c** $54 456.36

32 $27 096.38

33 **a** $1220.27 **b** $292 864.80 **c** $162 864.80

34 **a** $16 734.40 **b** $8243.63 **c** $2865.24

**35** 4%

**36 a** Paid in the 101st month, i.e. 8 years 5 months

**b** \$227 760 **c** \$61 911

**37 a** \$200 000

**b** Loans 1 and 3 take 25 years, loan 2 takes 30 years.

**c** \$200 000 **d** 'Interest only' (for 5 years)

**e** ≈ \$66 000 **f** Loan 3

**g i** Loan 1 **ii** Loan 2

## CHAPTER 8 NON-LINEAR RELATIONSHIPS

### ARE YOU READY?

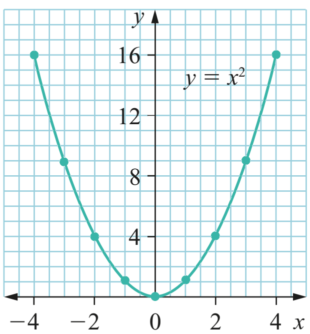
**1** C **2** A **3** D **4** C **5** B **6** D
**7** D **8** C **9** B **10** B **11** A

### EXERCISE 8A THE PARABOLA

**1 a**

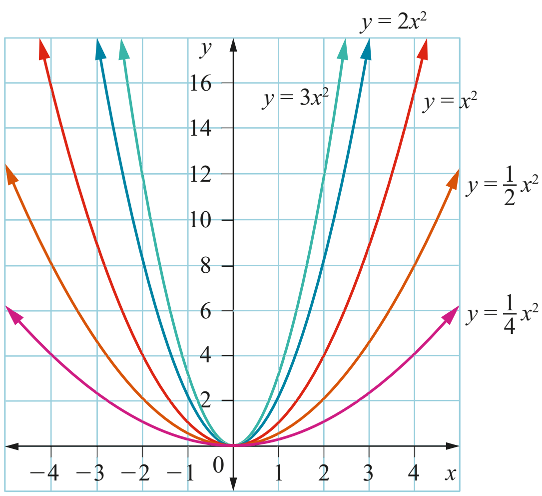
| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 |

**b, c**




**d** (0, 0) **e** $x = 0$

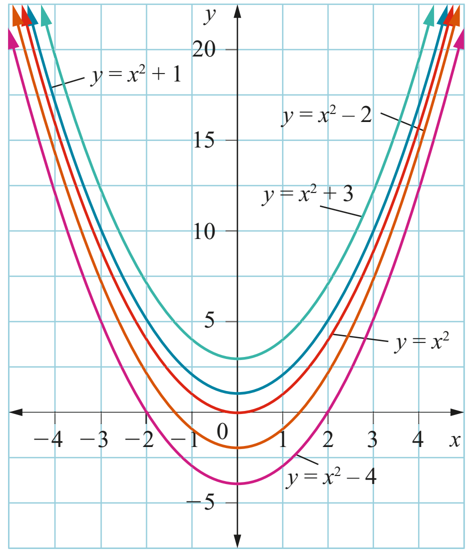
**2 a, b**




**c** Each parabola has the same turning point and axis of symmetry, but the general shape varies between wide and narrow.

**d** When the coefficient of $x^2$ is greater than 1, the parabola is narrower than the parabola for $y = x^2$. The larger the coefficient, the narrower the parabola becomes. When the coefficient of $x^2$ is between 0 and 1, the parabola is wider than the parabola for $y = x^2$. The smaller the coefficient, the wider the parabola becomes.

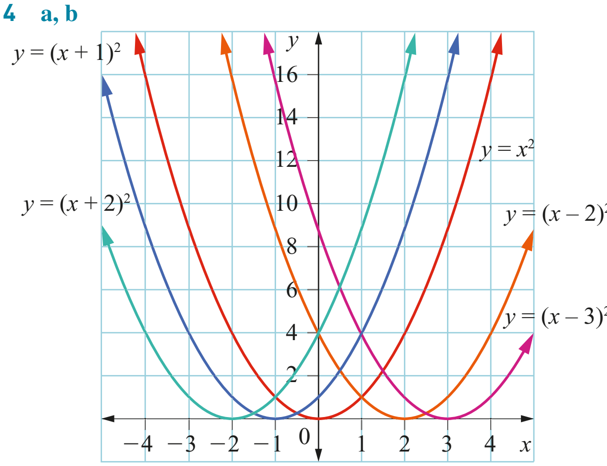
**3 a, b**




**c** Each parabola has the same shape and axis of symmetry, but a different turning point. Turning points: (0, 0) for $y = x^2$, (0, 1) for $y = x^2 + 1$, (0, 3) for $y = x^2 + 3$, (0, −2) for $y = x^2 - 2$ and (0, −4) for $y = x^2 - 4$.

**d** When a constant term is added to $x^2$ in $y = x^2$, the parabola for $y = x^2$ is moved up if the constant is positive and down if the constant is negative. The size of the constant term indicates how many units the parabola for $y = x^2$ is moved. For example, '+ 3' means the parabola for $y = x^2$ is moved 3 units up and '− 4'means the parabola for $y = x^2$ is moved 4 units down.

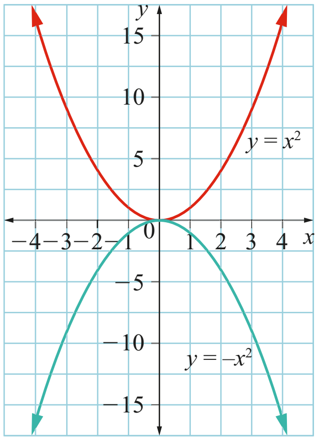
**4 a, b**




**c** Each parabola has the same shape but a different axis of symmetry and turning point. Turning points: (0, 0) for $y = x^2$, (−1, 0) for $y = (x + 1)^2$, (−2, 0) for $y = (x + 2)^2$, (2, 0) for $y = (x - 2)^2$ and (3, 0) for $y = (x - 3)^2$.

**d** When a constant term is added to $x^2$ before squaring in $y = x^2$, the parabola for $y = x^2$ is moved left if the constant is positive and right if the constant is negative. The size of the constant term indicates how many units the parabola for $y = x^2$ is moved. For example, '+ 2' means the parabola for $y = x^2$ is moved 2 units left and '− 3'means the parabola for $y = x^2$ is moved 3 units right.

**5** **a**




**b** Each parabola has the same axis of symmetry, turning point and general shape, but $y = -x^2$ is the reflection of $y = x^2$ in the $x$-axis.

**6** **a** **i** Narrower than the parabola for $y = x^2$ and reflected in the $x$-axis (inverted) but with same turning point and axis of symmetry

**ii** Narrower than the parabola for $y = x^2$ (and narrower than the parabola for $y = 2x^2$) and reflected in the $x$-axis (inverted) but with same turning point and axis of symmetry

**iii** Wider than the parabola for $y = x^2$ and reflected in the $x$-axis (inverted) but with same turning point and axis of symmetry

**iv** Wider than the parabola for $y = x^2$ (and wider than the parabola for $y = \frac{1}{2}x^2$) and reflected in the $x$-axis (inverted) but with same turning point and axis of symmetry

**7** **a** **i** (2, −4) **ii** $x = 2$ **iii** (0, 0)
**iv** (0, 0) and (4, 0) **v** −4

**b** **i** (−2, −9) **ii** $x = -2$ **iii** (0, −5)
**iv** (−5, 0) and (1, 0) **v** −9

**8** **a** **i** (0, 4) **ii** $x = 0$ **iii** (0, 4)
**iv** (−2, 0) and (2, 0) **v** 4

**b** **i** (−1, 4) **ii** $x = -1$ **iii** (0, 3)
**iv** (−3, 0) and (1, 0) **v** 4

**9** **a** (0, −4) **b** $x = 0$ **c** (0, −4)
**d** (−2, 0) and (2, 0) **e** −3 **f** −4

**10** **a** (−2, 9) **b** $x = -2$ **c** (0, 5)
**d** (−5, 0) and (1, 0) **e** 8 **f** 9

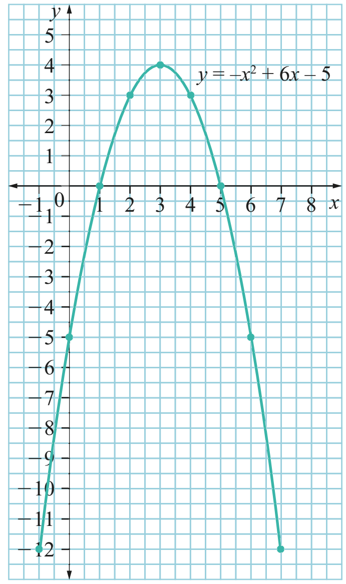
**11** **a, b** −1.75

**12** **a, b** 6.75

**13** **a** $y = -x^2 + 6x - 5$

| $x$ | −1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | −12 | −5 | 0 | 3 | 4 | 3 | 0 | −5 | −12 |

**b**

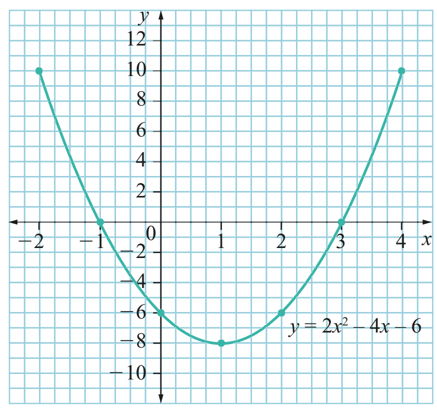



**c** 4 **d** 3

**14** **a** $y = 2x^2 - 4x - 6$

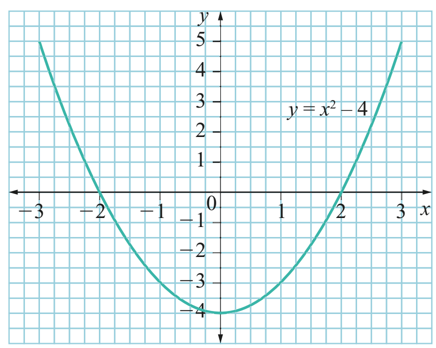
| $x$ | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 10 | 0 | −6 | −8 | −6 | 0 | 10 |

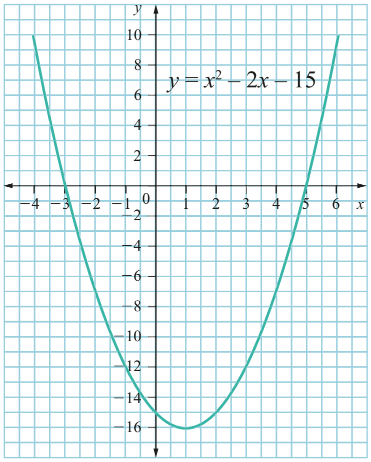
**b**

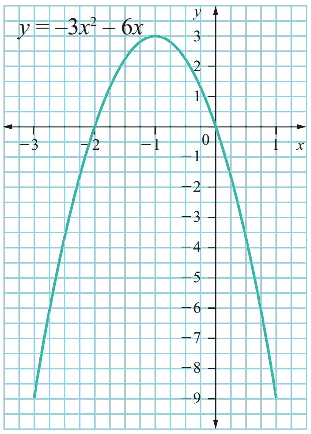


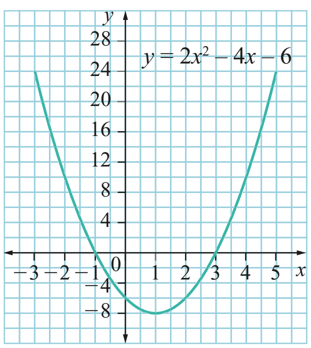

**c** −8 **d** 1

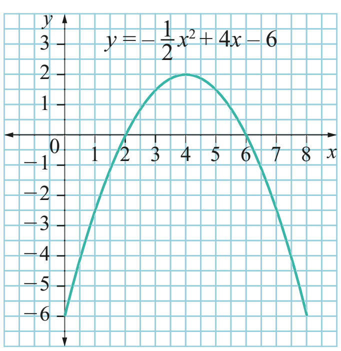
**16** **a**

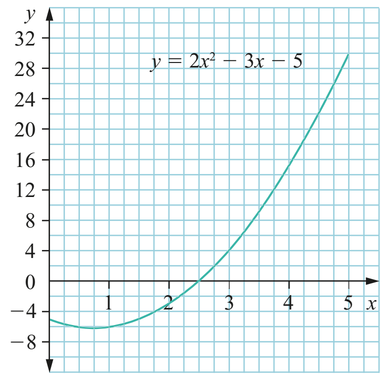


b



c



d



e



f



**17 a i** Two **ii** One **iii** None

**b** No, a horizontal line will only cross a parabola twice at the most.

**c** One

**d** No, a vertical line will only cross a parabola once.

**e** One

**f** No, a parabola is symmetrical and changes direction only once.

**g i** Minimum turning point $(0.5, -2.25)$

**ii** Maximum turning point $(2, 0)$

**iii** Minimum turning point $(0, 2)$

**18** C

## EXERCISE 8B QUADRATIC MODELS

**1 a** 21 m **b** 2 s

**c** 1 s and 3 s **d** 4.1 s

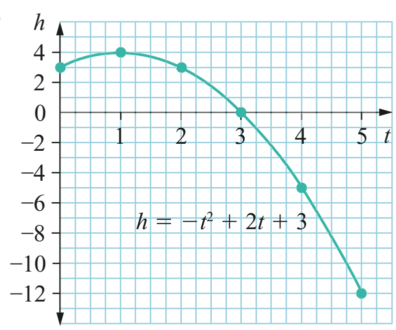
**e** When $t = 5$, $h = -24$ m. The model has negative values for the height, but in reality the lowest point is at $h = 0$. The model only works for $t$-values from 0 to just after 4 s (about 4.1 s).

**2 a** 500 m at 10 s **b** 5 s and 15 s **c** 20 s

**d** The model allows negative values for height, but in reality the lowest point is at $h = 0$. The model only works for $t$-values from 0 to 20 s.

**3 a**

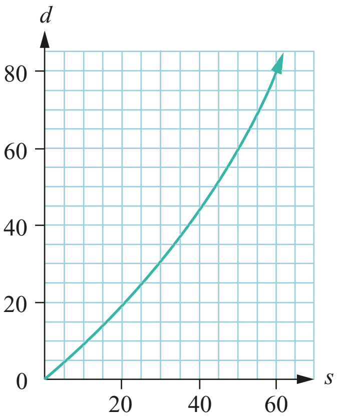
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $h$ | 3 | 4 | 3 | 0 | −5 | −12 |

**b**



**c** 4 m **d** 1 s

**e** 3 m **f** 3 s

**4** **a**

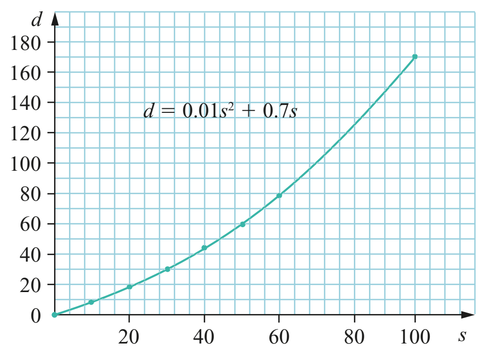



**b** $78 - 44 = 34$ m

**5** **a**

| *s* | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| *d* | 8 | 18 | 30 | 44 | 60 | 78 |

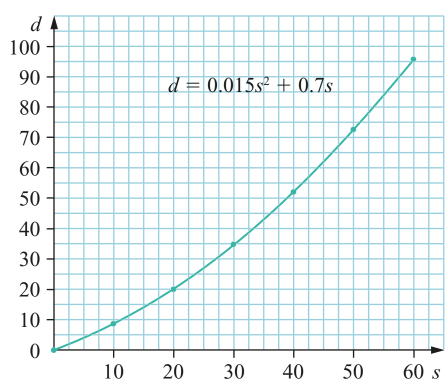
**b, c**




**d** $78 - 60 = 18$ m; the reduction means cars can stop more quickly.

**e**

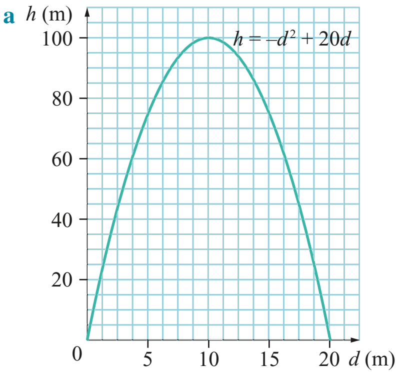
| *s* | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|
| *d* | 8.5 | 20 | 34.5 | 52 | 72.5 | 96 |




Stopping distance is 23.5 m shorter.

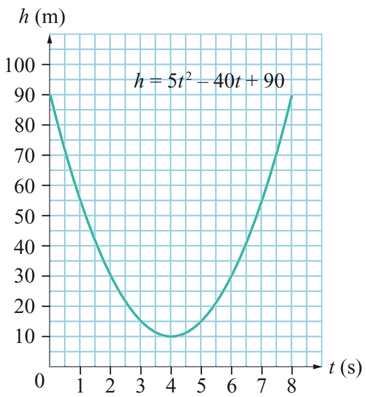
**f** For dry roads, $d = 170$ m, for wet roads $d = 220$ m. Need an extra 50 m to stop on wet roads.

**6** **a**



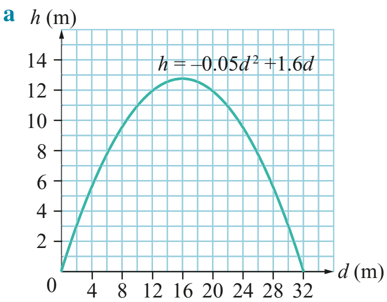

**b** 100 m **c** 20 m

**7** **a**




**b** 90 m **c** 10 m

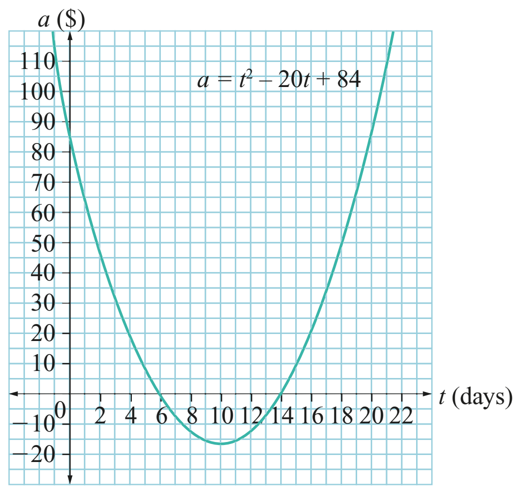
**8** **a**




**b** 12.8 m **c** 16 m

**d** 32 m **e** 30.8 m

**9** **a**



**b** \$84 **c** \$48

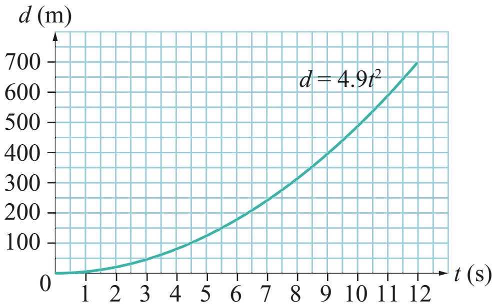
**d** Just after sixth day

**e** \$16 **f** After 14th day

**g** \$105 (at end of the three weeks)

**10 a** $d = 4.9t^2$

**b**

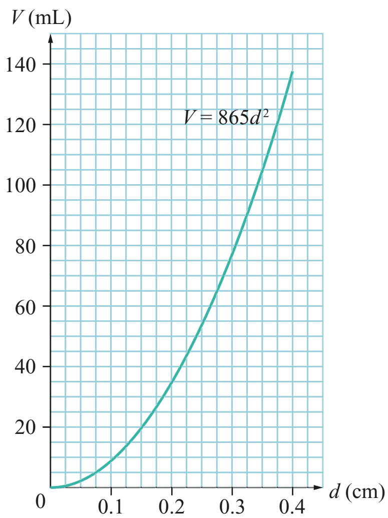



**c** About 240 m (240.1 m)

**d** About 10 s (10.1 s)

**11 a** $V = 865d^2$

**b**



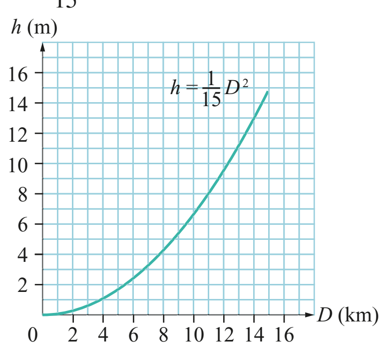

**c** About 60 mL

**d** About 0.3 cm

**e** $865 \times (0.26)^2 = 58.474$ mL; $\sqrt{\frac{80}{865}} \approx 0.3041$ cm$^2$

**12 a** $p = 15$

**b** $h = \frac{1}{15}D^2$




**c** About 3.25 m

**d** About 12.25 km

**e** $\frac{1}{15} \times 7^2 \approx 3.267$ m; $\sqrt{10 \times 15} \approx 12.247$ km

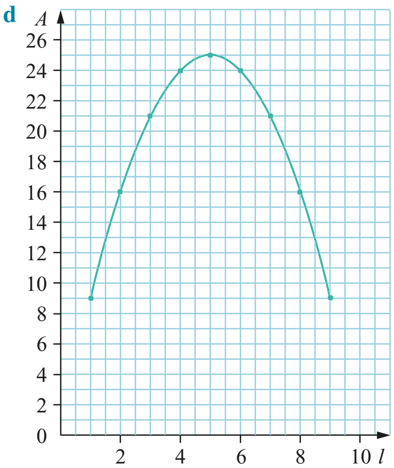
**13 a** $2l + 2b = 20$, $b = 10 - l$

**b** $A = 10l - l^2$

**c**

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | 9 | 16 | 21 | 24 | 25 | 24 | 21 | 16 | 9 |

**d**




**e** Maximum area of 25 m$^2$ when $l = b = 5$ m

**f** About 22.75 m$^2$

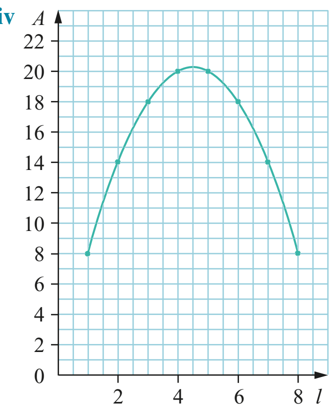
**14 a i** $2l + 2b = 18$, $b = 9 - l$

**ii** $A = 9l - l^2$

**iii**

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $A$ | 8 | 14 | 18 | 20 | 20 | 18 | 14 | 8 |

**iv**




**v** Maximum area of 20.25 m$^2$ when $l = b = 4.5$ m

**vi** About 19.5 m$^2$

**b i** $2l + 2b = 28$, $b = 14 - l$

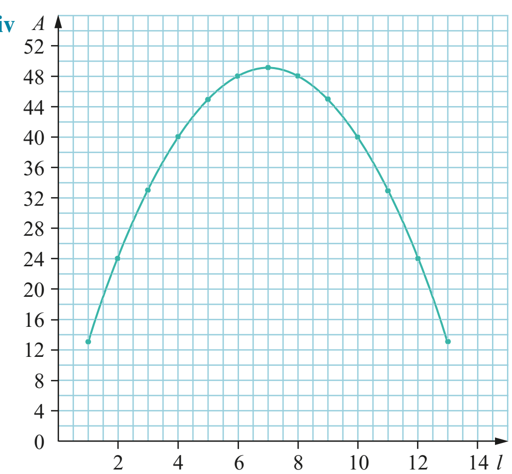
**ii** $A = 14l - l^2$

**iii**

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $A$ | 13 | 24 | 33 | 40 | 45 | 48 | 49 |

| $l$ | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|
| $A$ | 48 | 45 | 40 | 33 | 24 | 13 |

**iv**



**v** Maximum area of 49 cm$^2$ when $l = b = 7$ cm

**vi** About 36.75 cm$^2$

**c i** $2l + 2b = 22$, $b = 11 - l$

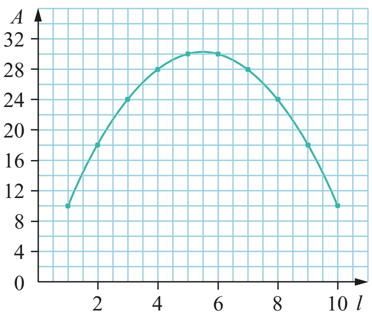
**ii** $A = 11l - l^2$

**iii**

| $l$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $A$ | 10 | 18 | 24 | 28 | 30 |

| $l$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $A$ | 30 | 28 | 24 | 18 | 10 |

**iv**



**v** Maximum area of 30.25 m$^2$ when $l = b = 5.5$ m

**vi** About 26.25 m$^2$

**d i** $2l + 2b = 26$, $b = 13 - l$

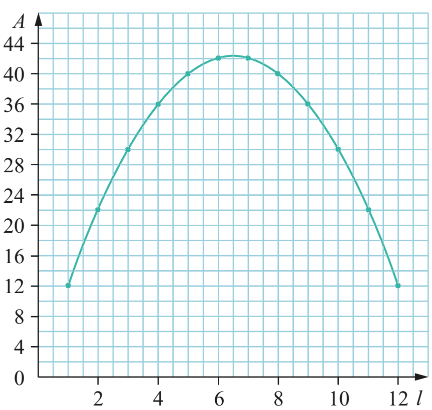
**ii** $A = 13l - l^2$

**iii**

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $A$ | 12 | 22 | 30 | 36 | 40 | 42 |

| $l$ | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|
| $A$ | 42 | 40 | 36 | 30 | 22 | 12 |

**iv**



**v** Maximum area of 42.25 m$^2$ when $l = b = 6.5$ m

**vi** About 33.25 m$^2$

**e i** $2l + 2b = 30$, $b = 15 - l$

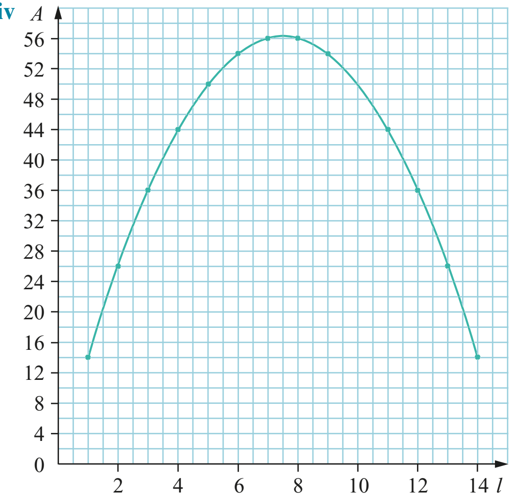
**ii** $A = 15l - l^2$

**iii**

| $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $A$ | 14 | 26 | 36 | 44 | 50 | 54 | 56 |

| $l$ | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|
| $A$ | 56 | 54 | 50 | 44 | 36 | 26 | 14 |

**iv**



**v** Maximum area of 56.25 mm$^2$ when $l = b = 7.5$ mm

**vi** About 40.25 mm$^2$

**15 a iii** About 700 m (705.6 m)

**iv** About 8 s (7.8 s)

**v** 9.8 m/s$^2$

**b ii** About 115 m (115.2 m)

**iii** About 20 s (19.4 s)

**iv** 1.6 m/s$^2$

**c i** $d = 2.72t^2$

**ii** 391.68 m; object falls less distance than on Earth (700 m) and more than on the Moon (115 m).

**iii** 10.50 s; time to fall 300 m is more than the time on Earth (8 s) and less than on the Moon (20 s).

**iv** 5.44 m/s$^2$

**v** Planet is smaller than Earth as 5.44 m/s$^2$ < 9.8 m/s$^2$.

**d i** Venus: $d = 4.435t^2$, Mars: $d = 1.85t^2$, Jupiter: $d = 12.25t^2$

**ii** Venus: about 8.2 s, Mars: about 12.7 s, Jupiter: about 4.9 s; object falls fastest on Jupiter.

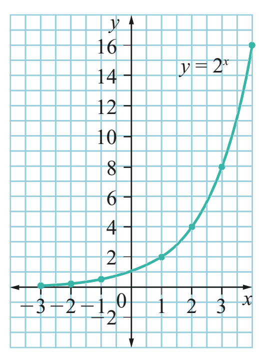
**iii** Ascending order of mass: Mars, Venus, (Earth), Jupiter

## EXERCISE 8C THE EXPONENTIAL GRAPH

**1 a**

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.0625 | 0.125 | 0.25 | 0.5 | 1 | 2 | 4 | 8 | 16 |

**b, c**



**d** $y$ increases at an increasing rate of change.

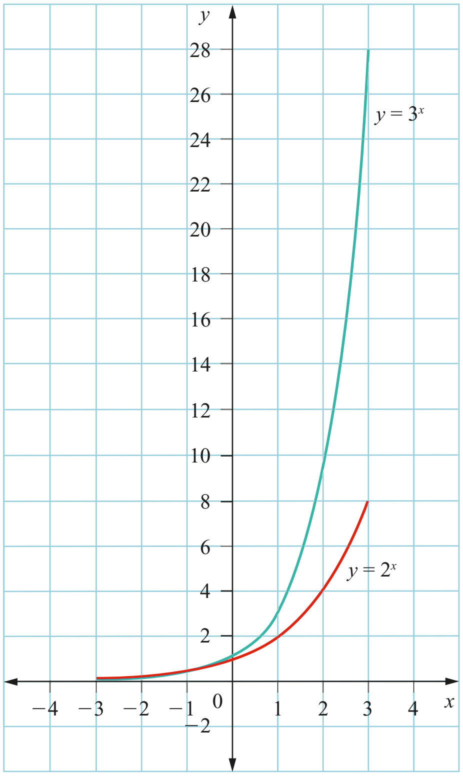
**e** No, all $y$-values are positive as 2 to any power always gives a positive result.

**f** 1

**g** No $x$-intercepts

**h** $y = 0$

**2 a**



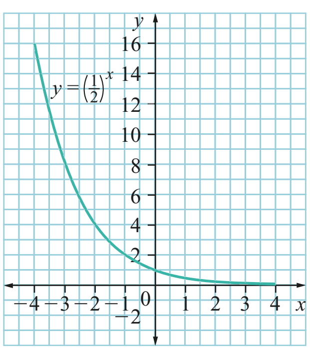

**b** The curve for $y = 3^x$ rises more steeply (as the $x$-values increase, the $y$-values increase at a higher rate) though the general shape is the same. For $x < 0$, the curve of $y = 3^x$ is below the curve for $y = 2^x$ but, for $x > 0$, the curve of $y = 3^x$ is above the curve for $y = 2^x$. The curves have the same $y$-intercept of 1 and the same asymptote (equation is $y = 0$).

**c** The curve for $y = 5^x$ would rise more steeply than the curves for $y = 2^x$ and $y = 3^x$.

**3 a**

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 16 | 8 | 4 | 2 | 1 | 0.5 | 0.25 | 0.125 | 0.0625 |

**b, c**



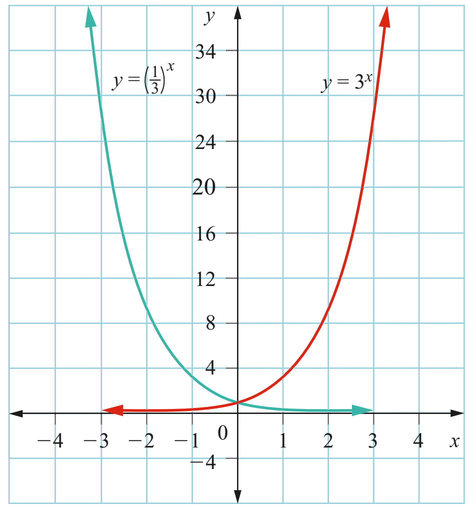

**d** $y$ decreases at a decreasing rate of change.

**e** No, all $y$-values are positive as 2 to any power always gives a positive result.

**f** No $x$-intercepts, $y$-intercept of 1

**g** $y = 0$.

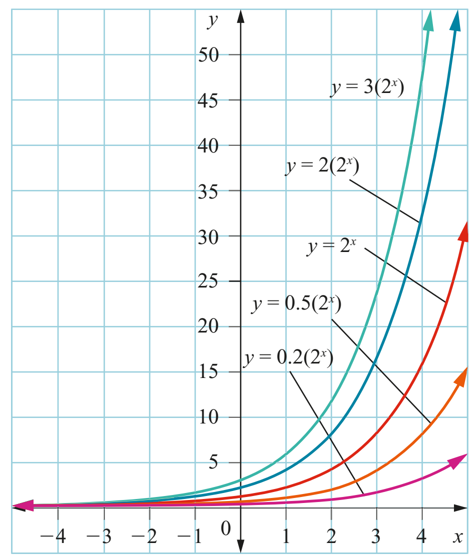
**4 a**




**b** The curve for $y = \left(\frac{1}{3}\right)^x$ has the same shape but is the reflection of $y = 3^x$ in the $y$-axis. The curves have the same $y$-intercept of 1 and the same asymptote (equation is $y = 0$).

**c** The curve for $y = \left(\frac{1}{5}\right)^x$ would have the same shape but will be the reflection of $y = 5^x$ in the $y$-axis. The curves will have the same $y$-intercept of 1 and the same asymptote (equation is $y = 0$).

**5 a, b**

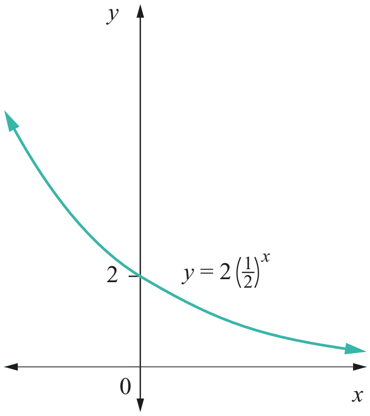



**c i** 2 **ii** 3 **iii** 0.5 **iv** 0.2

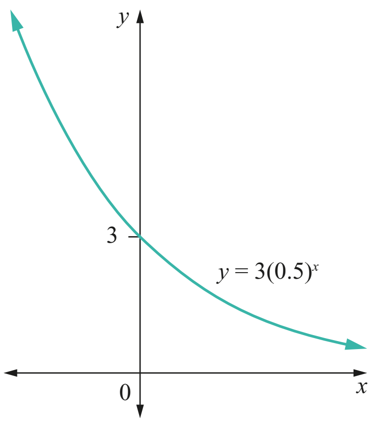
**d** $y = 0$ for each curve

**e** When $2^x$ is multiplied by a constant term that is greater than 1, the curve rises more steeply than the curve for $y = 2^x$. When $2^x$ is multiplied by a constant term that is between 0 and 1, the curve rises less steeply than the curve for $y = 2^x$.

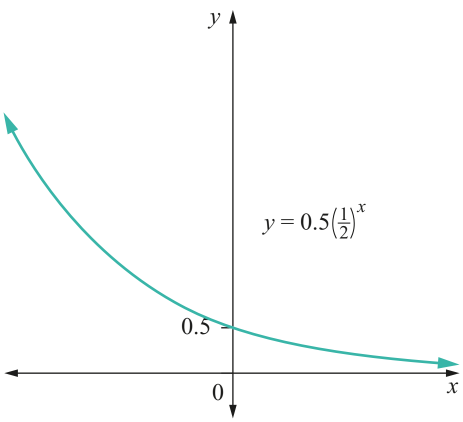
**6** **a**



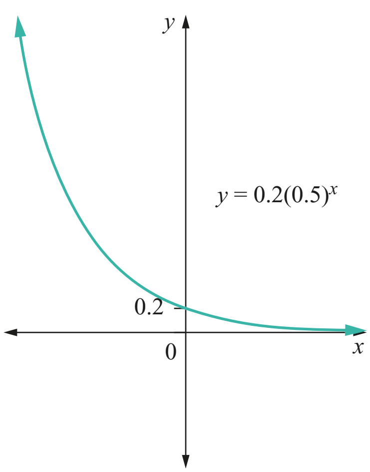

**b**



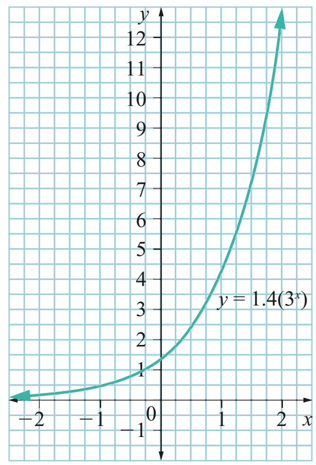

**c**




**d**

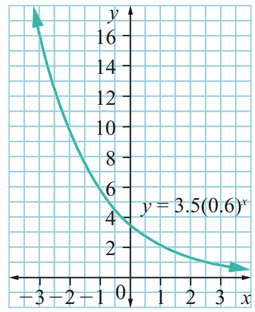



**8** **a**



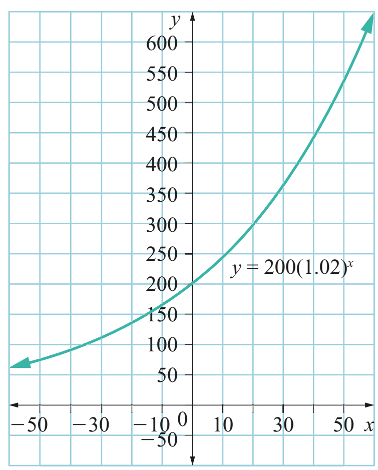

$y$-intercept at 1.4

**b**



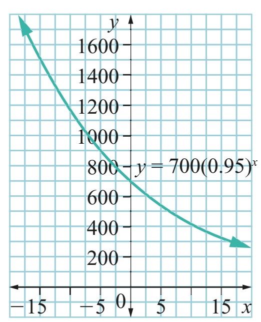

$y$-intercept at 3.5

**c**




$y$-intercept at 200

**d**




$y$-intercept at 700

**9** Exponential growth: parts **a** and **c**; exponential decay: parts **b** and **d**

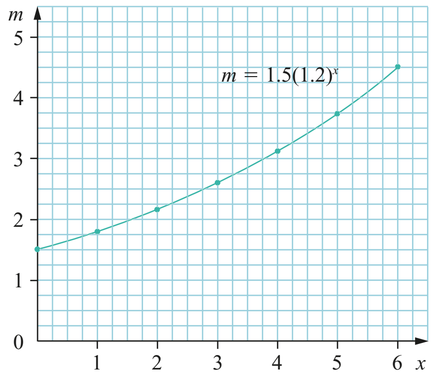
## EXERCISE 8D EXPONENTIAL MODELS

**1** **a** Population model matched for $t \geq 0$.
**b** Exponential growth
**c** 2000
**d** About 7000 (7111)
**e** About 14 years

**2** **a**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $m$ | 1.5 | 1.8 | 2.2 | 2.6 | 3.1 | 3.7 | 4.5 |

**b**

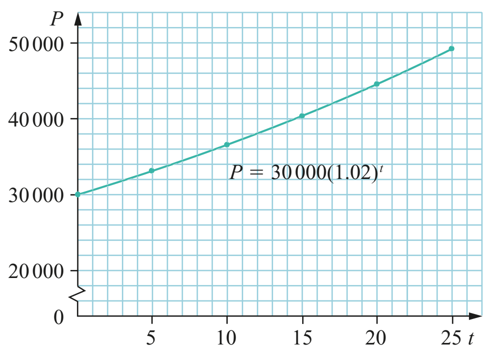



**c** About 2.8 kg (2.839 kg)
**d** About 4.5 months
**e** Mass increases by 20% each month.
**f** Growth rate may change after 6 months.

**3** **a**

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $P$ | 30 000 | 33 122 | 36 570 | 40 376 | 44 578 | 49 218 |

**b**

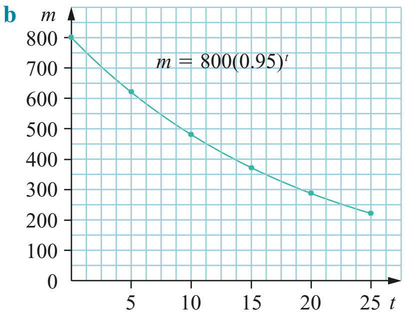



**c** Initial population
**d** About 43 000 (42 847)
**e** About 13.2 years
**f** 35 years
**g** Population increases by 2% each year; this can be seen from factor of 1.02 in the equation, which means that the following year's population is 1.02 times or 102% of the current year's population. This is an increase of 2% per year.

**4** **a**

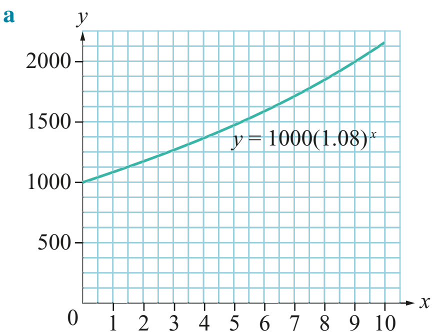
| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $m$ | 800 | 619 | 479 | 371 | 287 | 222 |

**b**

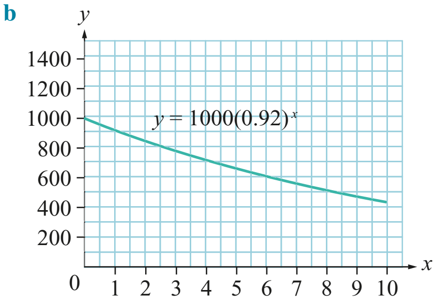



**c** As $t$ increases, $m$ decreases at a decreasing rate of change.
**d** 800 g
**e** About 250 g (246 g)
**f** About 5.6 years
**g** About 41 years
**h** 5% per year; this can be seen from factor of 0.95 in the equation, which means that the following year's mass is 0.95 times or 95% of the current year's mass. This is a decrease of 5% per year.

**5** **a**

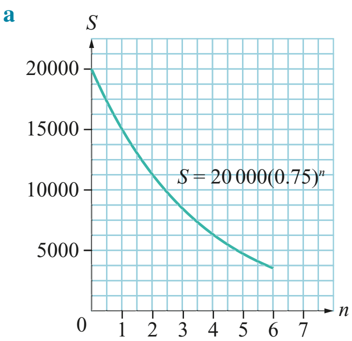



**b**




**c** $y = 1000(0.92)^x$, as graph shows exponential decay
**d** $y = 1000(1.08)^x$, as graph shows exponential growth
**e** \$1337.71

**6** **a**



**b** Exponential decay

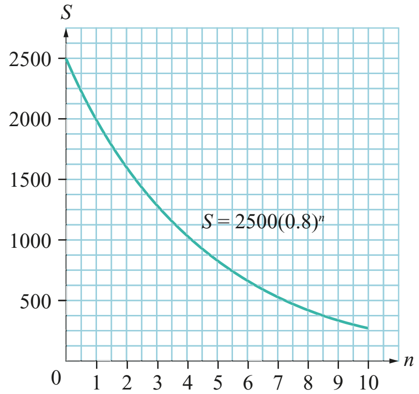
**c** Initial value (purchase price) of new motorcycle

**d** $11 250

**e** About 4.2 years

**7** **a** $V_0 = \$2500$, $r = 20\% = 0.2$ so $S = V_0(1 - r)^n$ becomes $S = 2500(1 - 0.2)^n$ or $S = 2500(0.8)^n$

**b**



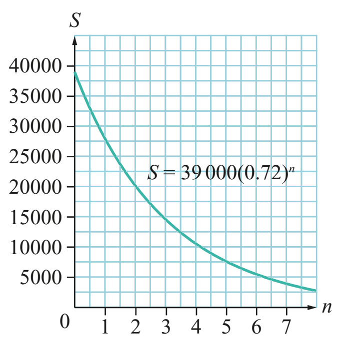

**c** About $1300 ($1280)

**d** About 7.2 years

**8** **a** $S = 39\,000(0.72)^n$, where $S$ is the salvage value in dollars after $n$ years

**b** Exponential decay

**c**

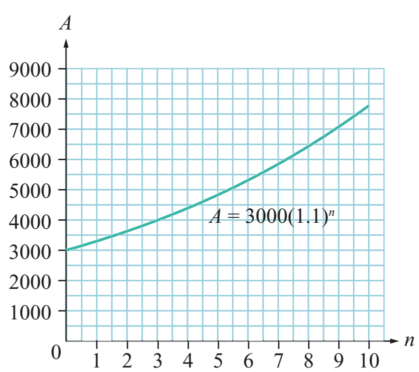



**d** About $7500 ($7546.18)

**e** About 2.1 years

**9** **a** $P = 3000$, $r = 10\% = 0.1$ so $A = P(1 + r)^n$ becomes $A = 3000(1 + 0.1)^n$ or $A = 3000(1.1)^n$

**b**




**c** Initial investment

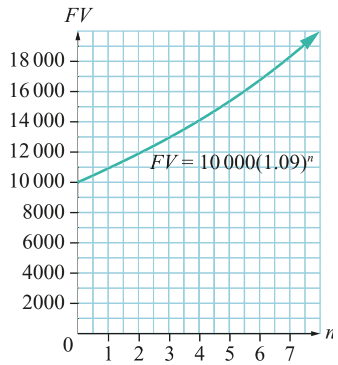
**d** About $4500 ($4392.30)

**e** About 7.3 years

**f** $A = 4000(1.1)^n$. After 4 years, the $3000 investment becomes $4392.30 while the $4000 investment becomes $5856.40. The extra $1000 investment yields an extra $464.10.

**10** **a** $FV = 10\,000(1.09)^n$ where $n$, is the number of yearly compounding periods

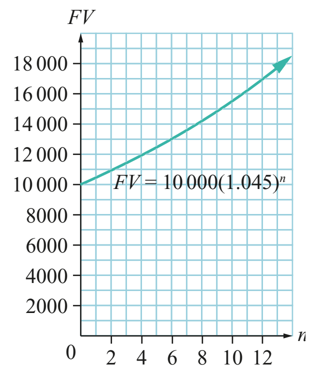
**b**




Investment grows to $15 000 after about 4.7 years.

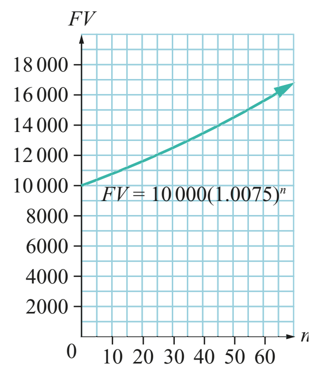
**c** **i** $FV = 10\,000(1.045)^n$, where $n$ is the number of six-monthly compounding periods

**ii**



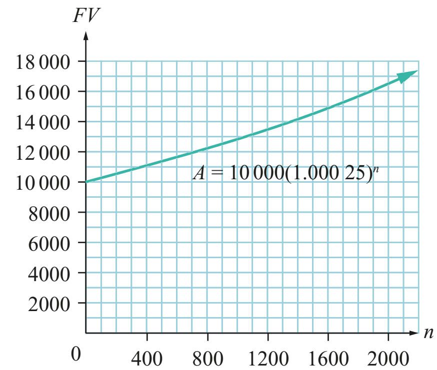

Investment grows to $15 000 after about 9.2 compounding periods or 4.6 years.

**d** **i** $FV = 10\,000(1.0075)^n$, where $n$ is the number of monthly compounding periods




**ii** Investment grows to $15 000 after about 54 compounding periods or 4.5 years.

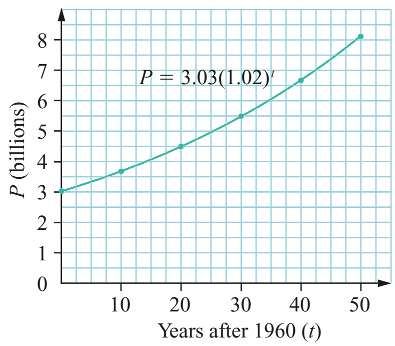
**e i** $FV = 10\,000(1.000\,25)^n$, where $n$ is the number of daily compounding periods




**ii** Investment grows to \$15 000 after about 1622 compounding periods or 4.44 years.

**f** Investment grows more quickly as the length of the compounding period shortens.

**11 a**




**b** About 4 billion (4.078 billion)

**c** About 6.8 billion (6.824 billion)

**d i** $t = 14.03$ years; 1975 **ii** $t = 25.3$ years; 1986

**iii** $t = 42.3$ years; 2003 **iv** $t = 60.3$ years; 2021

**e** Disease, war, health, etc.

**12 a** About 4 200 000

**b** Population in 2010 when the new policy was introduced

**c i** $a \neq 1.06$ as the population would increase at a greater rate than currently predicted without the new policy.

**ii** $2\,690\,000 = 2\,000\,000(a)^{10}$ so $a^{10} = 1.345$. Try $a = 1.02$, $a = 1.03$ and $a = 1.04$: $(1.02)^{10} \approx 1.219$, $(1.03)^{10} \approx 1.344$ and $(1.04)^{10} \approx 1.480$, so value of $a$ must be between 1.03 and 1.04 (but closer to 1.03).

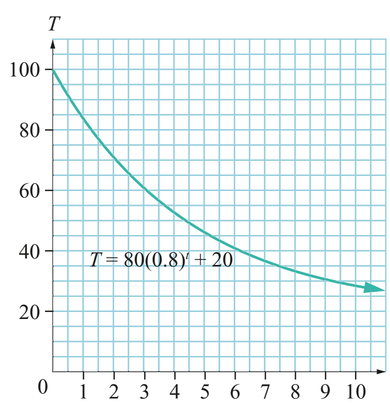
**d** Yes, assuming $a = 1.03$, the population will be under 7 000 000.

When $n = 40$, $P = 2\,000\,000(1.03)^{40} \approx 6\,524\,076$

**13 a** Exponential decay

**b** 100°C

**c**




**d** About 4.4 minutes

**e** Just above 20°C

**15** About 72.3 months or about 6 years

## EXERCISE 8E THE RECTANGULAR HYPERBOLA

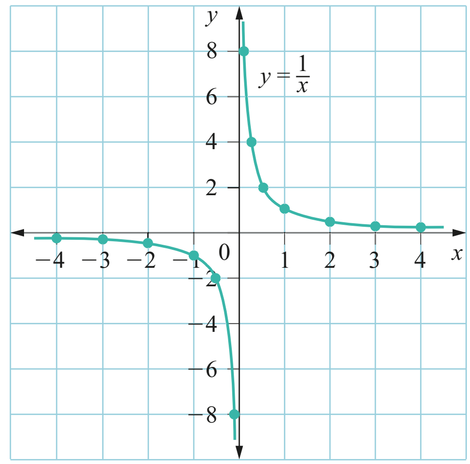
**1 a**

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{1}{8}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | $-\frac{1}{4}$ | $-\frac{1}{3}$ | $-\frac{1}{2}$ | $-1$ | $-2$ | $-4$ | $-8$ | 8 | 4 | 2 | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ |

**b** There is no value for $y$ when $x = 0$; the relationship is undefined at $x = 0$.

**c** No

**d, e**




**f** The value of $y$ is positive and decreases towards 0.

**g** The value of $y$ is positive and increases towards large positive numbers.

**h** The value of $y$ is negative and increases towards 0.

**i** The value of $y$ is negative and decreases towards large negative numbers.

**j** No $x$- or $y$-intercepts

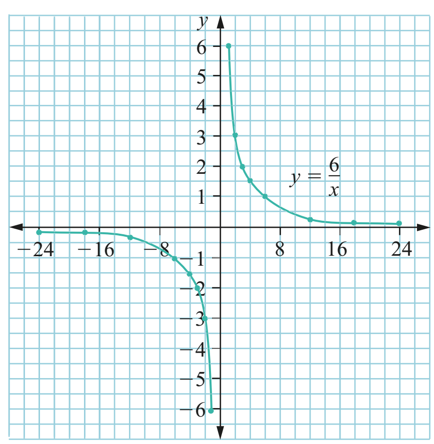
**k** Horizontal asymptote has equation $y = 0$, vertical asymptote has equation $x = 0$.

**2 a**

| $x$ | $-24$ | $-18$ | $-12$ | $-6$ | $-4$ | $-3$ | $-2$ | $-1$ |
|---|---|---|---|---|---|---|---|---|
| $y$ | $-\frac{1}{4}$ | $-\frac{1}{3}$ | $-0.5$ | $-1$ | $-1.5$ | $-2$ | $-3$ | $-6$ |

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 6 | 3 | 2 | 1.5 | 1 | 0.5 | $\frac{1}{3}$ | $\frac{1}{4}$ |

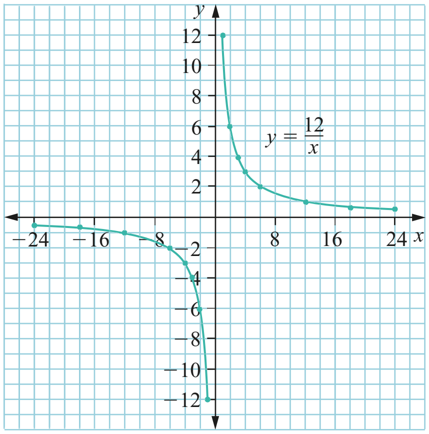
**b**



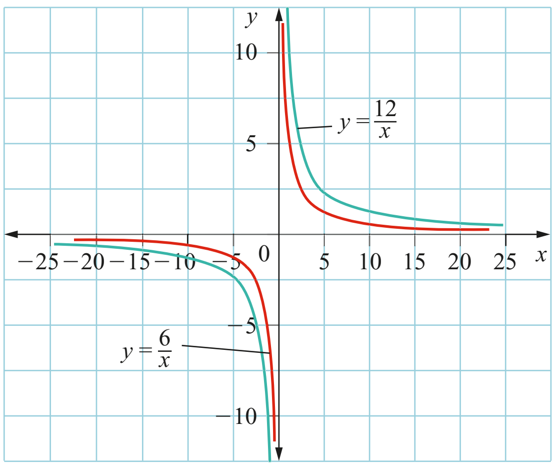
**c**

| $x$ | $-24$ | $-18$ | $-12$ | $-6$ | $-4$ | $-3$ | $-2$ | $-1$ |
|---|---|---|---|---|---|---|---|---|
| $y$ | $-\frac{1}{2}$ | $-\frac{2}{3}$ | $-1$ | $-2$ | $-3$ | $-4$ | $-6$ | $-12$ |

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 12 | 6 | 4 | 3 | 2 | 1 | $\frac{2}{3}$ | $\frac{1}{2}$ |



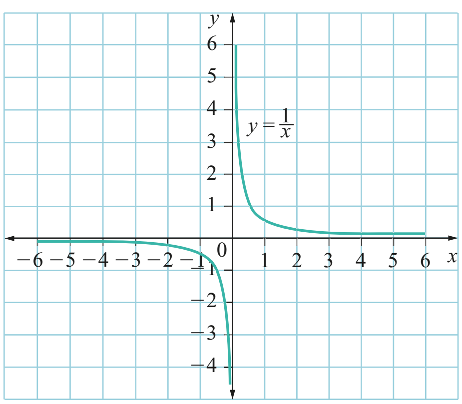

**d**



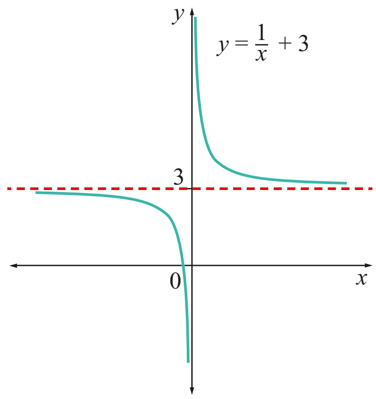

**e** General shape is the same but rate of change of curve is different. They have the same asymptotes.

**f** The graph of $y = \frac{24}{x}$ would have the same general shape and the same asymptotes but the right arm of the curve would appear above the right arm of $y = \frac{12}{x}$ (and $y = \frac{6}{x}$) and the left arm of the curve would appear below the left arm of $y = \frac{12}{x}$ (and $y = \frac{6}{x}$).

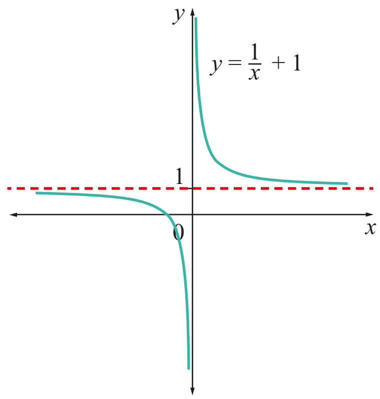
**3 a**



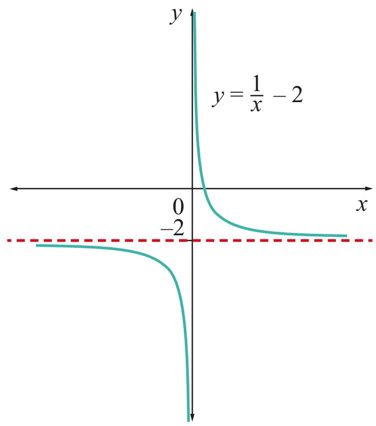

**b i**



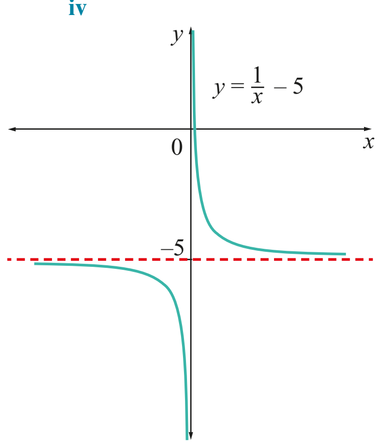

**ii**




**iii**




**iv**



**c** **i** $-\frac{1}{3}$ **ii** $-1$ **iii** 0.5 **iv** 0.2

**d** **i** $x = 0, y = 3$

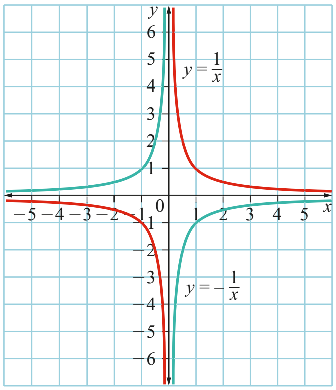
**ii** $x = 0, y = 1$

**iii** $x = 0, y = -2$

**iv** $x = 0, y = -5$

**e** When a constant term is added to $\frac{1}{x}$, the hyperbola for $y = \frac{1}{x}$ is moved up if the constant is positive and down if the constant is negative. The size of the constant term indicates how many units the hyperbola for $y = \frac{1}{x}$ is moved. For example, '+ 3' means the hyperbola for $y = \frac{1}{x}$ is moved 3 units up and '− 5' means the hyperbola for $y = \frac{1}{x}$ is moved 5 units down.

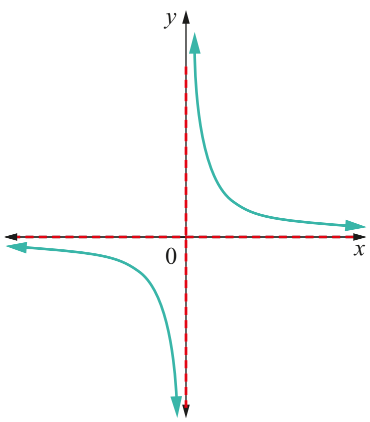
**4** **a**



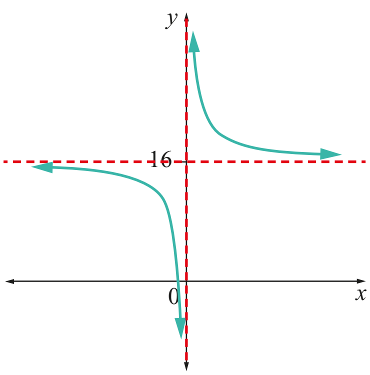

**b** The graph of $y = -\frac{1}{x}$ has the same general shape as the graph of $y = \frac{1}{x}$ but the two arms are reflected in the $x$-axis. The two hyperbolas have the same asymptotes.

**c** The graph of $y = -\frac{24}{x}$ would have the same general shape as the graph of $y = \frac{24}{x}$ but the two arms would be in quadrants 2 and 4 rather than in quadrants 1 and 3 of the number plane. The asymptotes would be the same.

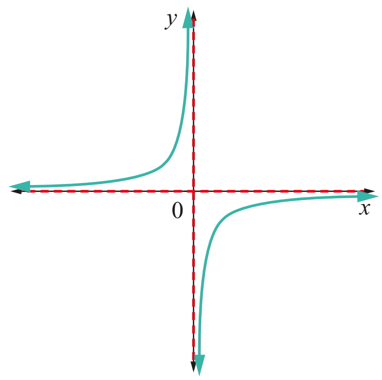
**5** **a**



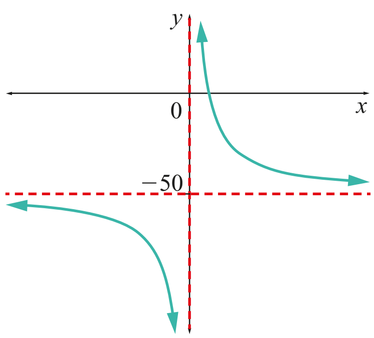

**b**




**c**




**d**




**6** **a** No $x$-intercept **b** $-0.5$

**c** No $x$-intercept **d** 2

## EXERCISE 8F RECIPROCAL MODELS

**1** **a**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 1 | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{5}$ |

**b**

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 2 | 1 | $\frac{2}{3}$ | $\frac{2}{4}$ | $\frac{2}{5}$ |

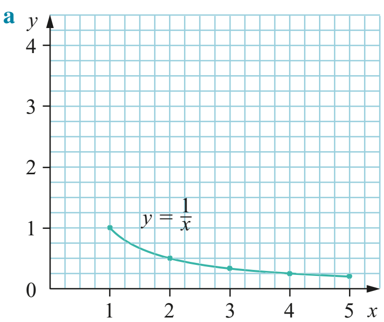
**c**

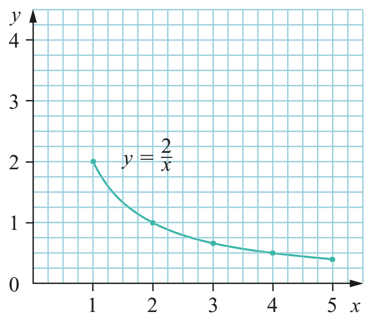
| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 6 | 3 | 2 | $\frac{6}{4}$ | $\frac{6}{5}$ |

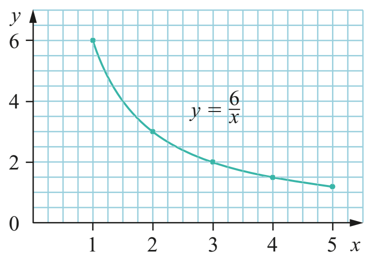
**d**

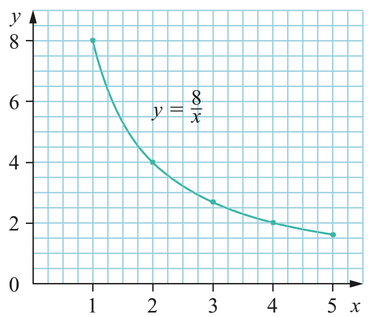
| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 8 | 4 | $\frac{8}{3}$ | $\frac{8}{4}$ | $\frac{8}{5}$ |

**2** **a**



**b** 

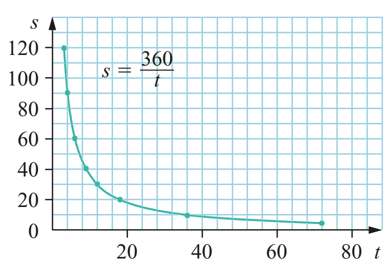

**c** 


**d** 


The full rectangular hyperbola is not drawn in each case as the values of $x$ have been restricted.

**3** **a**

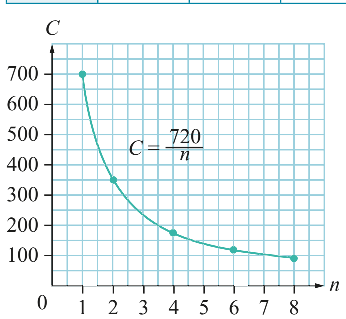
| $t$ | 3 | 4 | 6 | 9 | 12 | 18 | 36 | 72 |
|---|---|---|---|---|---|---|---|---|
| $s$ | 120 | 90 | 60 | 40 | 30 | 20 | 10 | 5 |

**b** 


**c** About 50 km/h (51.4 km/h)
**d** 4.8 hours

**4** **a**

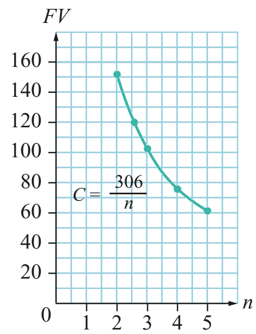
| $n$ | 1 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $C$ | 720 | 360 | 180 | 120 | 90 |

**b** 


**c** \$240 **d** 5 people
**e** Full week: \$205.71, half week: \$102.86
**f** Full week: \$160, half week: \$80
**g** Two people stay for a full week and one person stays for half a week.

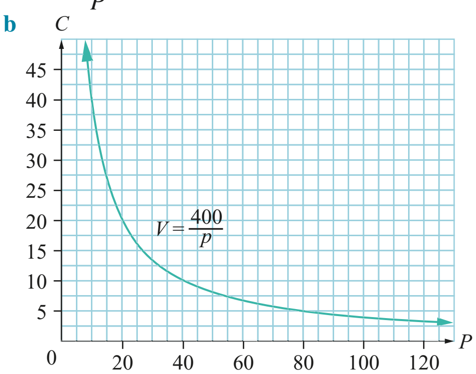
**5**

| $n$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $C$ | 153 | 102 | 76.50 | 61.20 |



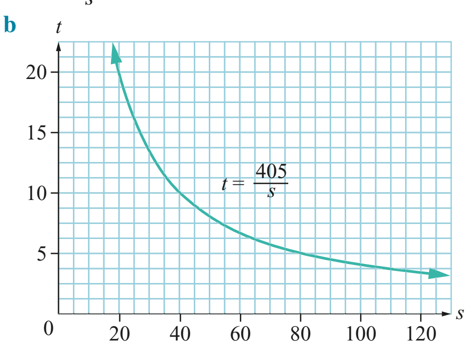

**6** **a** \$48 **b** $C = \frac{1200}{N}$
**c i** \$30 **ii** \$16
**d** Cost ($C$) cannot be zero, or the number of people ($N$) cannot be zero, or the number of people ($N$) must be positive whole number values only.
**e** No, as $N$ must be a whole number. $C = \frac{1200}{N}$ becomes $42 = \frac{1200}{N}$, $42N = 1200$, $N \approx 28.6$.

**7** **a** $R = \frac{288}{N}$
**b i** \$57.60 **ii** \$96

**8** **a** $V = \frac{400}{P}$
**b** 


**i** 20 units
**ii** 4 cm$^3$

**d** $20 = \frac{400}{P}$, $20P = 400$, $P = 20$ units
$V = \frac{400}{100}$, $V = 4$ cm$^3$

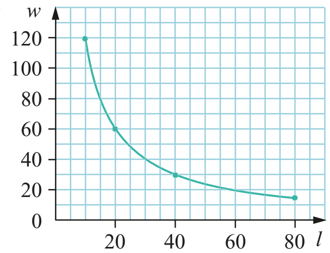
**9** **a** $t = \frac{405}{s}$
**b** 

**c** About 68 km/h (67.5 km/h)

**d** About 4 hours (4.05 hours)

**10 a** $l = \frac{1200}{w}$ or $w = \frac{1200}{l}$ or $lw = 1200$

**b**



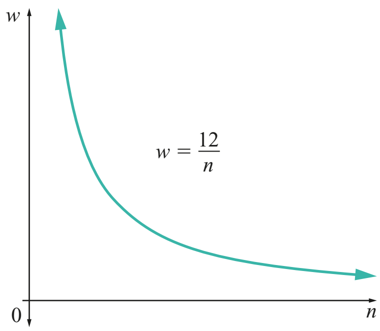

**c** As the length increases, the width decreases.

**11 a** $n = \frac{12\,250\,000}{a}$ where $n$ is the number of people and $a$ is the amount of space in $cm^2$

**b** 6125

**c** Amount of space ($a$) cannot be zero or negative, or the amount of space ($a$) must be above a minimum value that is equivalent to the floor area taken up by a standing person, or the number of people ($n$) cannot be zero, or the number of people ($n$) must be positive whole number values only.

**12 a**




**b i** 6 min **ii** 4 min **iii** 3 min

**iv** 2 min **v** 1 min **vi** 0.5 min

**c** 12 min when there is one bank teller on duty

**d** No, as $\frac{12}{n} \neq 0$; however, $w$ approaches 0 as $n$ increases.

**13 a** $t = \frac{120}{n}$, where $t$ is the time taken in days with $n$ workers doing the task

**b** 8 days **c** 5 days

**14** 5 days

**15** 10

## REVIEW MULTIPLE-CHOICE QUESTIONS

**1** B **2** D **3** D **4** A **5** C **6** B **7** B

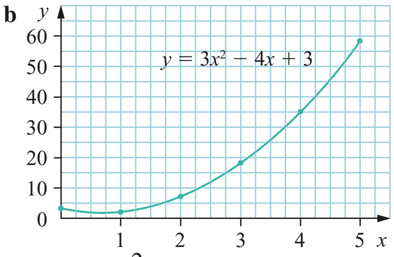
## REVIEW SET 1

**1 a** B **b** C **c** A

**2 a**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | 3 | 2 | 7 | 18 | 35 | 58 |

**b**



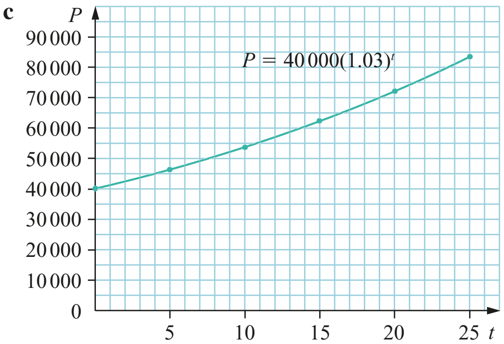

**c** About 2 ($1\frac{2}{3} \approx 1.7$)

**3 a** 40 000

**b**

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $P$ | 40 000 | 46 371 | 53 757 | 62 319 | 72 244 | 83 751 |

**c**




**d** About 68 000 (68 097)

**e** About 13 years

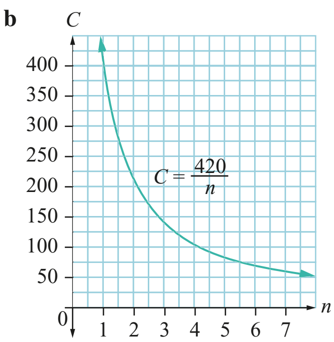
**f** About 31 years

**g** 3% per year; this can be seen from factor of 1.03 in the equation, which means that the following year's population is 1.03 times or 103% of the current year's population. This is an increase of 3% per year.

**4 a**

| $n$ | 1 | 2 | 4 | 6 | 7 |
|---|---|---|---|---|---|
| $C$ | 420 | 210 | 105 | 70 | 60 |

**b**




**c** \$84

**d** 3 people

**e** $C = \frac{420}{5} = \$84$

$140 = \frac{420}{n}$, $140n = 420$, $n = \frac{420}{140} = 3$

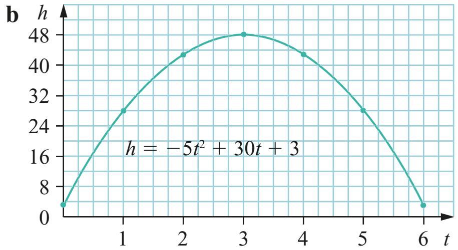
**f** Cost ($C$) cannot be zero or negative or more than \$420, or the number of people ($n$) cannot be zero, or the number of people ($n$) must be positive whole number values only.

## REVIEW SET 2

**1 a**

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $h$ | 3 | 28 | 43 | 48 | 43 | 28 | 3 |

**b**



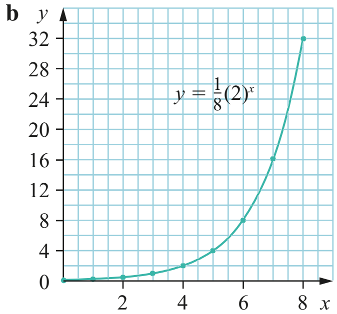

**c** Maximum height is 48 m when $t = 3$ s.

**2 a** $V = 1041d^2$ **b** 135 mL **c** 0.26 cm

**3 a**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 | 32 |

**b**



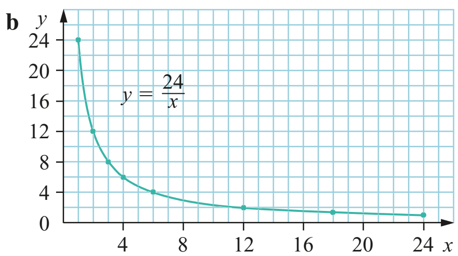

**c** Exponential graph

**d** Increases rapidly

**4 a**

| $x$ | 1 | 2 | 3 | 4 | 6 | 12 | 18 | 24 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 24 | 12 | 8 | 6 | 4 | 2 | $1\frac{1}{3}$ | 1 |

**b**



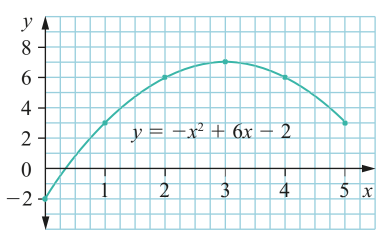

**c** Rectangular hyperbola

## REVIEW SET 3

**1 a**

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $y$ | $-2$ | 3 | 6 | 7 | 6 | 3 |

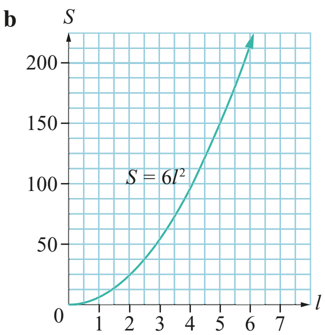
**b**




**c** Maximum value is 7 when $x = 3$.

**2 a** $S = 6l^2$

**b**

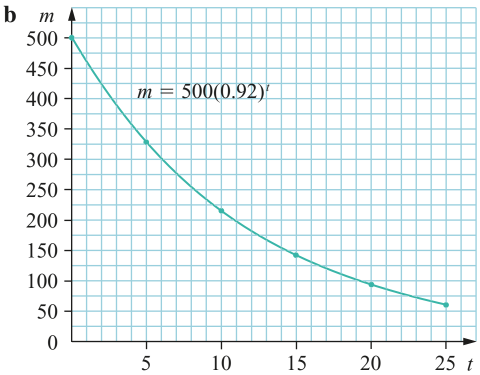



**c** 86.64 cm$^2$ **d** 5.8 cm

**3 a**

| $t$ | 0 | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|---|
| $m$ | 500 | 330 | 217 | 143 | 94 | 62 |

**b**




**c** 500 g

**d** About 75 g (73.467 g)

**e** About 3 years (2.67 years)

**f** About 28 years (27.6 years)

**g** 8% per year; this can be seen from factor of 0.92 in the equation, which means that the following year's mass is 0.92 times or 92% of the current year's mass. This is a decrease of 8% per year.

**4 a** $t = \frac{100}{n}$ where $t$ is the time taken in days with $n$ workers doing the task

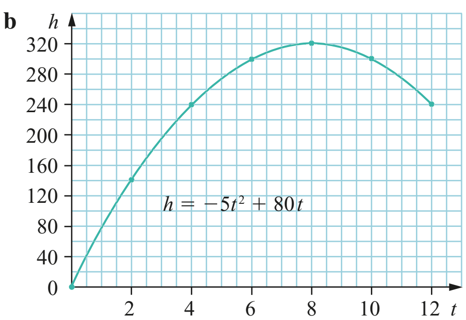
**b** $9\frac{1}{11}$ days

## REVIEW SET 4

**1 a**

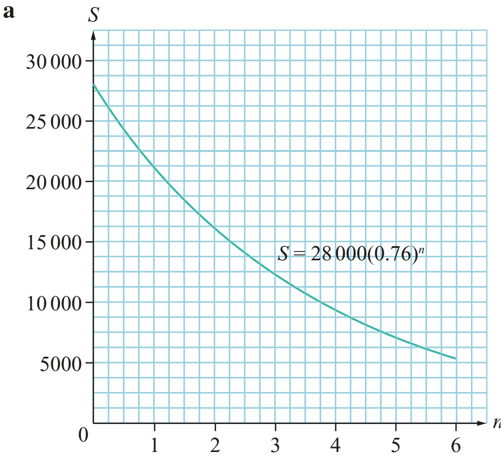
| $t$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|---|
| $h$ | 0 | 140 | 240 | 300 | 320 | 300 | 240 |

**b**




**c** Maximum height is 320 m when $t = 8$ s.

**2 a**




**b** Exponential decay; as $n$ increases, $S$ decreases at a decreasing rate.

**c** Initial value (purchase price) of the new car

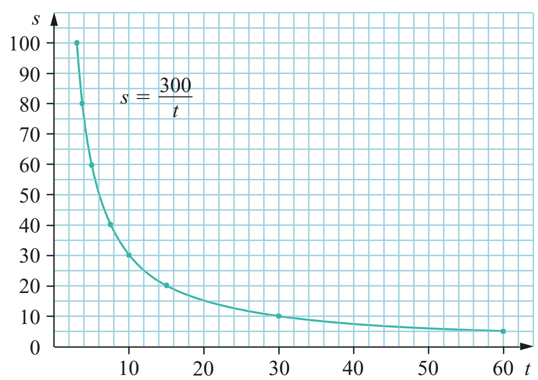
**d** About \$12 300 (\$12 291.33)

**e** After about 3.75 years

**3 a**

| $t$ | 3 | 3.75 | 5 | 7.5 | 10 | 15 | 30 | 60 |
|---|---|---|---|---|---|---|---|---|
| $s$ | 100 | 80 | 60 | 40 | 30 | 20 | 10 | 5 |

**b**




**c** About 43 km/h **d** 3.16 hours

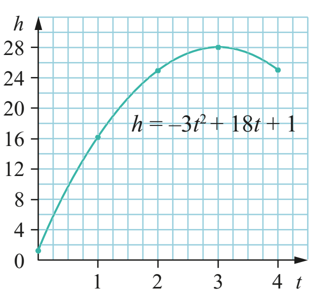
**4** 10 people

## REVIEW PRACTICE EXAMINATION QUESTION

**1 a i**

| $t$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $h$ | 1 | 16 | 25 | 28 | 25 |

**ii**




**iii** Maximum height is 28 m when $t = 3$ s.

**b i** Exponential growth; as $x$ increases, $M$ increases at an increasing rate

**ii** 1.5

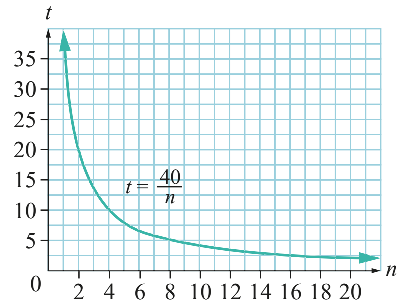
**iii** About 2.7 kg (2.657 kg)

**iv** About 11.5 days (11.527 days)

**v** 10%

**c i** $t = \frac{40}{n}$ where $t$ is the time taken in days with $n$ workers doing the task

**ii**




**iii** 5 days

**iv** 2 days

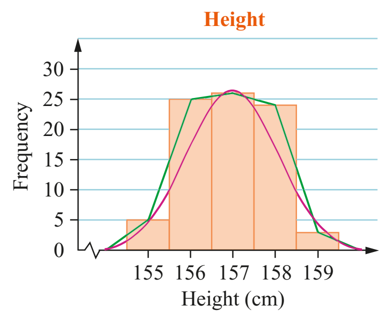
# CHAPTER 9 THE NORMAL DISTRIBUTION

## ARE YOU READY?

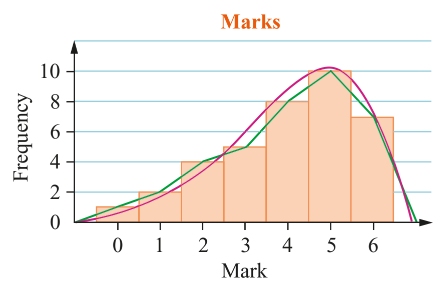
**1** D **2** B **3** A **4** D **5** C **6** B **7** D
**8** C **9** D **10** C **11** B **12** A **13** B **14** A

## EXERCISE 9A FREQUENCY DISTRIBUTION GRAPHS

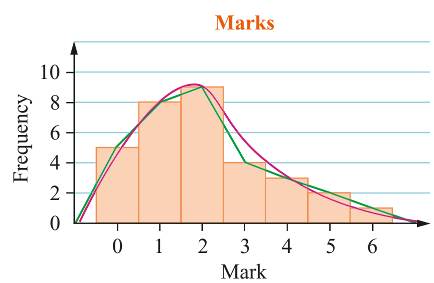
**1 a**




**b**

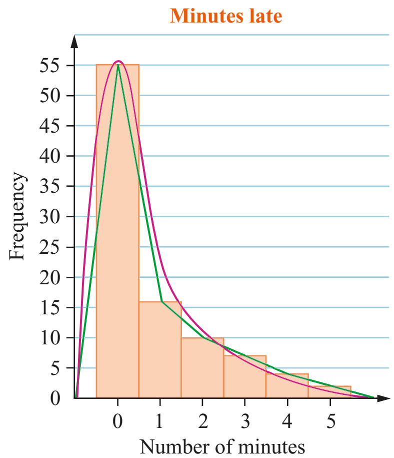



**c**

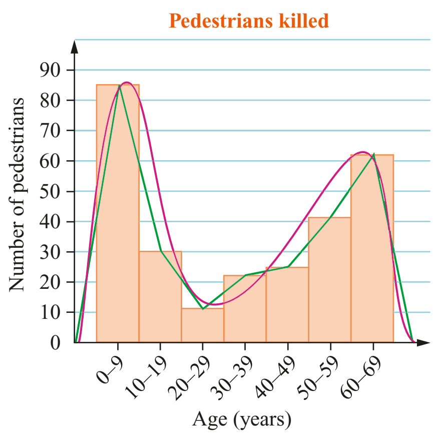


**d**


**e**




**f**




**2** **a** Symmetrical, unimodal (one mode)
**b** Negatively skewed, unimodal
**c** Positively skewed, unimodal
**d** Symmetrical, bimodal (two modes)
**e** Positively skewed, unimodal
**f** Not symmetrical, bimodal

**3** **a i** Mean ≈ 156.94
**ii** Median = 157
**iii** Mode = 157
**b i** Mean ≈ 4.03 **ii** Median = 4 **iii** Mode = 5
**c i** Mean ≈ 2.06, **ii** Median = 2, **iii** Mode = 2
**d i** Mean = 6.2, **ii** Median = 6, **iii** Mode = 5 and 8
**e i** Mean ≈ 0.88, **ii** Median = 0, **iii** Mode = 0
**f i** Mean ≈ 33.30, **ii** Median = 34.5, **iii** Mode = 4.5 (modal class 0−9) and 64.5 (modal class 60−69)

**4** Frequency distribution in part **a**; mean, median and mode are almost exactly the same.

**5** It is symmetrical, unimodal; mean ≈ 50.24, median = 50, mode = 50. The mean, median and mode are almost exactly the same.

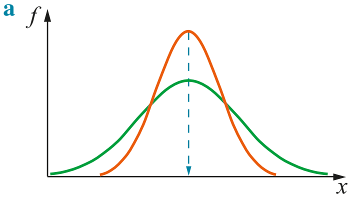
**6** **a i** 9 **ii** 18
**b i** Same as mode **ii** Less than mode
**c i** Same as mode **ii** Less than mode
**d i** Mean ≈ 8.88, median = 9
**ii** Mean ≈ 16.08, median = 17

**7** **a** 1.0
**b** About 1.94 **c** About two
**d** About 2.06 **e** About two

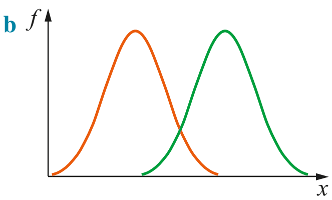
## EXERCISE 9B THE NORMAL DISTRIBUTION

**1** B

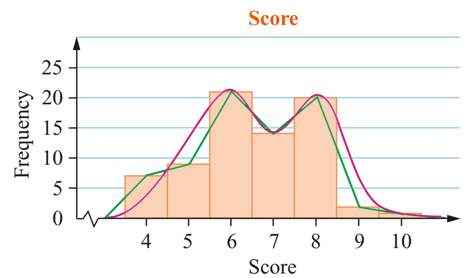
**2** **a**




**b**

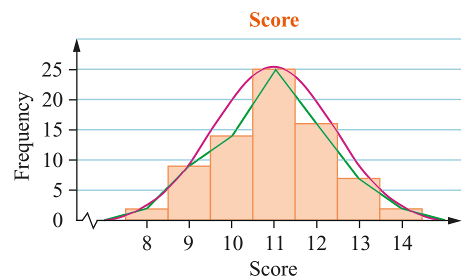



**3** **a i**



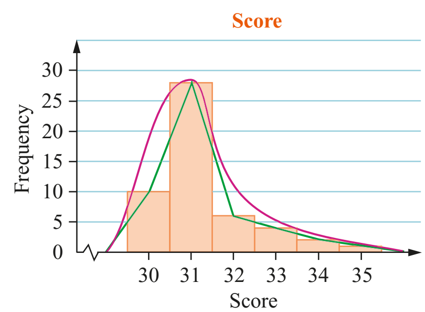

Curve is not bell-shaped, so the distribution is not normal.

**b i**



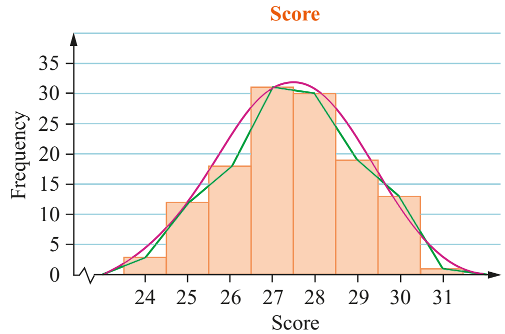

**ii** Mean = 11, mode = 11, median = 11
Distribution is approximately normal.

**c i**




Curve is not bell-shaped, so the distribution is not normal.

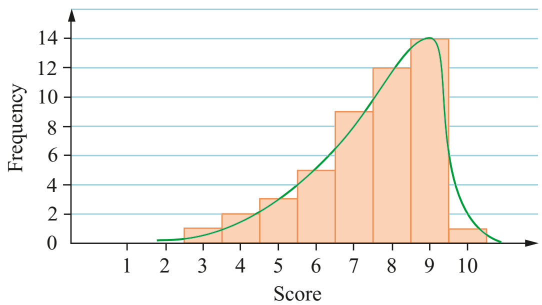
**d i**




**ii** Mean = 27.5, mode = 27, median = 27
Distribution is approximately normal.

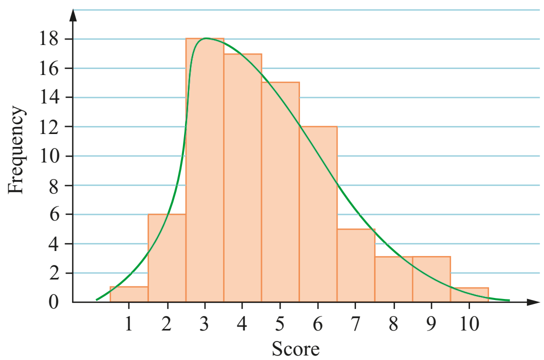
**4** **a** A **b** C **c** B **d** C

**5** **a**



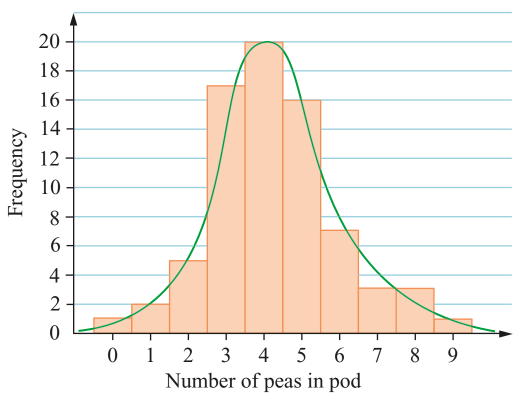

**b** Negatively skewed **c** An easy test
**d** Curve is not bell-shaped, so the distribution is not normal.

**6** **a**



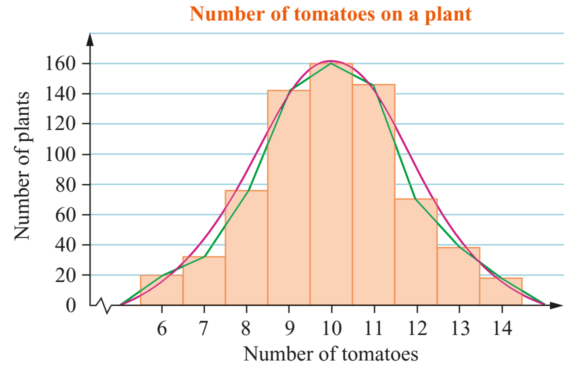

**b** Positively skewed **c** A hard test
**d** Curve is not bell-shaped, so the distribution is not normal.

**7** **a**




**b** Curve is symmetrical.
**c** The curve is centred around 4 peas in a pod.
**d** Curve is bell-shaped and the mean (≈ 4.25), median (4) and mode (4) are almost the same, so the distribution is normal.

**8**




Curve is bell-shaped and the mean (≈ 9.99), median (10) and mode (10) are almost exactly the same, so the distribution is normal.

## EXERCISE 9C THE STANDARDISED SCORE

**1** **a** 1 **b** −1 **c** 2 **d** −2 **e** 3
**f** −3 **g** 0 **h** 1.5 **i** −0.8 **j** −2.6

**2** **a i** −2 **ii** −1 **iii** −0.3 **iv** 0
**v** 0.6 **vi** 1 **vii** 1.5 **viii** 2.2

**3** Julie 1.125, Imram −1, Servy 2, Brad −0.5, Katrina 1.875, Sam −2, Megumi 0, Sasha −2.25

**4** **a** History 1.25, Geography 1.0
**b** History **c** 70

**5** **a** 75 **b** 55 **c** 85 **d** 45 **e** 95
**f** 35 **g** 65 **h** 80 **i** 50 **j** 73
**k** 42 **l** 92

**6** **a** Science test **b** Science test
**c** Mathematics test; $z$-score for Science test is 2, $z$-score for Maths test is 2.5.

**7** **a** English −0.8, Mathematics 0.3, Biology 0.5, Art 2.2, Economics 0, History −0.82
**b** Art **c** History
**d** Art, Biology, Maths, Economics, English, History

**8** **a** Term 4
**b i** Yes **ii** No **iii** Yes

**9** **a** Phil 0.8, Evie −1.2, Tina 1.5, Han Jin −0.4, Kyla 0.1
**b** Phil 0.9, Evie −1.5, Tina 1.8, Han Jin 0.1, Kyla −0.5
**c** Phil, Tina, Han Jin **d** Han Jin
**e** Evie, Kyla **f** Kyla
**g** Phil

**10** **a** 3 **b** 60%

**11** Trevor

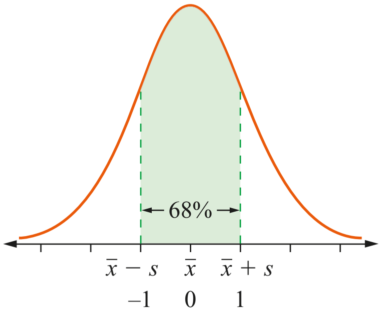
**12** Josh, Vera, Melanie, Jake, Ming

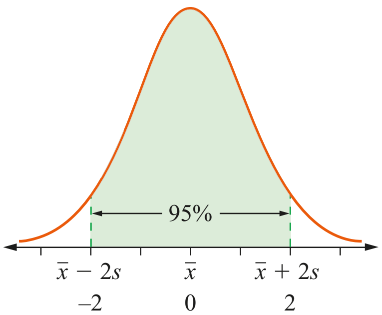
**13** **a** $\bar{x} = 58.3$, $s = 12.4$
**b** −0.1, −0.02, −0.2, −0.9, 0.3, 1.3, 0.1, −0.3, −2.0, 2.2, 0.4, 1.0, −0.02, −1.5, −0.3
**c** 10 **d** 67% **e** 87% **f** 7%

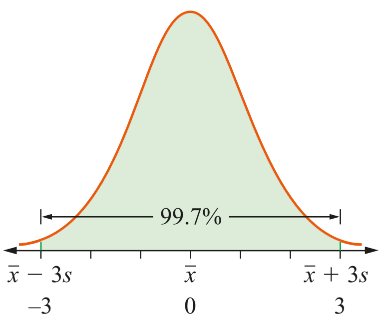
## EXERCISE 9D PROPERTIES OF THE NORMAL DISTRIBUTION

**1** **a** −1 **b** 1 **c** −2 **d** 0
**e** 2 **f** −3 **g** 3

**2**

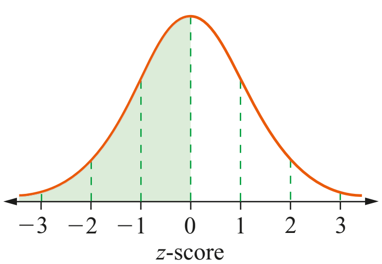


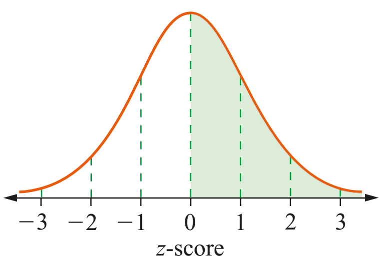




**3** **a** **i** 68% **ii** 34% **iii** 34%
**iv** 50% **v** 50% **vi** 16%
**vii** 16% **viii** 84% **ix** 84%
**b** **i** 95% **ii** 47.5% **iii** 47.5%
**iv** 2.5% **v** 2.5% **vi** 97.5%
**vii** 97.5%
**c** **i** 99.7% **ii** 49.85% **iii** 49.85%
**iv** 0.15% **v** 0.15% **vi** 99.85%
**vii** 99.85%

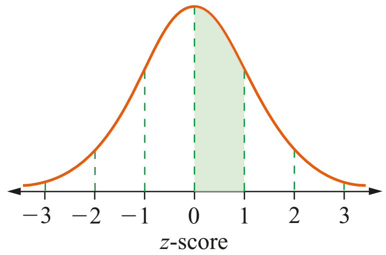
**4** **a**



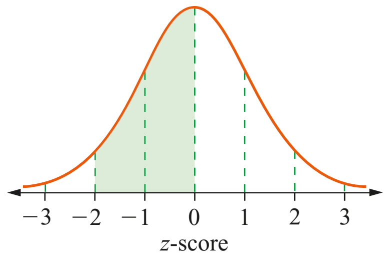

**b**



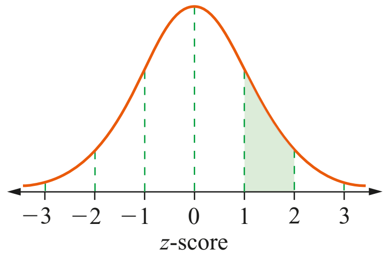

**c**



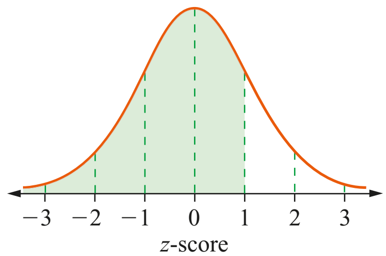

**d**




**e**



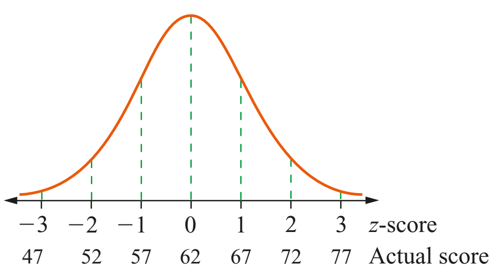

**f**




**5** **a** 13.5% **b** 15.85% **c** 2.35%
**d** 81.5% **e** 83.85% **f** 81.5%
**g** 97.35% **h** 83.85% **i** 97.35%

**6** **a** **i** 67 **ii** 57
**b** 68%
**c** **i** 72 **ii** 52
**d** 95%
**e** **i** 77 **ii** 47
**f** 99.7%

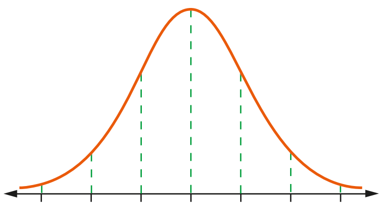
**7** **a–c**




**d** For a normal distribution, the median mark and the modal mark will be the same as the mean mark of 62.

**8** **a** $z = -3, -2, -1, 0, 1, 2, 3$

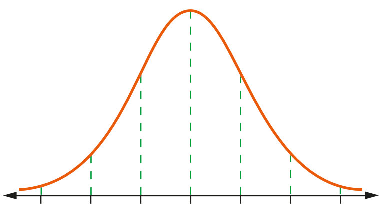
**b**




**c** **i** 99.7% **ii** 95% **iii** 68%

**d** **i** 50% **ii** 16% **iii** 47.5%

**9** **a, b**




**c** **i** 68% **ii** 95% **iii** 99.7%

**d** **i** 16% **ii** 2.5% **iii** 84%

**10** **a** $z = -2, -1, 0, 1, 2$ **b** 50%

**c** 68% **d** 34%

**e** 95% **f** 16%

**g** 2.5%

**11** 84%

**12** Mean 17, standard deviation 0.8

**13** 2.5%

**14** **a** 154 cm and 166 cm

**b** 148 cm and 172 cm

**c** 142 cm and 178 cm

**15** **a** 4.94 cm and 5.06 cm

**b** Statistically this is very unlikely to happen as 5.07 cm is more than three standard deviations above the mean. Check measurement, adjust machine.

**16** Very unlikely to occur as 48 is more than three standard deviations above the mean.

**17** Lengths that are within one standard deviation of the mean lie between 3.970 cm and 4.030 cm. Two of the lengths in the sample lie outside this interval (3.950 cm and 4.132 cm) so the setting on the machine will need to be checked.

**18** **a** ≈5.99

**b** 1.72

**c** **i** 64% **ii** 95% **iii** 100%

**d** 0.64

**19** 26

**20** 11

**21** Yes, as it is considerably more than three standard deviations above the mean.

## EXERCISE 9E PROBABILITY AND NORMAL DISTRIBUTIONS

**1** **a** 0.475 **b** 0.84 **c** 0.815

**d** 0.4985 **e** 0.025 **f** 0.16

**g** 0.0235 **h** 0.975 **i** 0.1585

**2** **a** 0.68 **b** 0.475 **c** 0.84

**d** 0.815 **e** 0.025 **f** 0.84

**g** 0.135 **h** 0.1585 **i** 0.0015

**3** The probability value will be somewhere between 0.68 ($z$-score between −1 and 1) and 0.95 ($z$-score between −2 and 2). Half-way between 0.68 and 0.95 is 0.815. However, because of the shape of the curve, the area under the curve and hence the probability value will be greater than 0.815. The actual value is about 0.87.

**4** **a** 0.748 57 ≈ 0.75 **b** 0.966 38 ≈ 0.97

**c** 0.984 22 ≈ 0.98 **d** 0.109 35 ≈ 0.11

**e** 0.161 09 ≈ 0.16 **f** 0.002 48 ≈ 0.00

**g** 0.217 81 ≈ 0.22 **h** 0.156 53 ≈ 0.16

**i** 0.119 43 ≈ 0.12

**5** **a** 0.773 37 ≈ 0.77 **b** 0.629 30 ≈ 0.63

**c** 0.981 24 ≈ 0.98 **d** 0.468 12 ≈ 0.47

**e** 0.105 65 ≈ 0.11 **f** 0.158 66 ≈ 0.16

**g** 0.144 07 ≈ 0.14 **h** 0.318 48 ≈ 0.32

**i** 0.153 72 ≈ 0.15

**6** **a** 0.217 70 ≈ 0.22 **b** 0.077 80 ≈ 0.08

**c** 0.013 55 ≈ 0.01 **d** 0.629 30 ≈ 0.63

**e** 0.882 98 ≈ 0.88 **f** 0.980 77 ≈ 0.98

**g** 0.139 90 ≈ 0.14 **h** 0.130 47 ≈ 0.13

**i** 0.082 49 ≈ 0.08

**7** **a** 0.186 73 ≈ 0.19 **b** 0.037 54 ≈ 0.04

**c** 0.013 21 ≈ 0.01 **d** 0.629 30 ≈ 0.63

**e** 0.992 66 ≈ 0.99 **f** 0.940 62 ≈ 0.94

**g** 0.395 51 ≈ 0.40 **h** 0.154 87 ≈ 0.15

**i** 0.238 22 ≈ 0.24

**8** $P(-1.28 < z < 0.94)$

$= P(z < 0.94) - P(z < -1.28)$

$= P(z < 0.94) - P(z > 1.28)$

$= P(z < 0.94) - [1 - P(z < 1.28)]$

$= 0.826\,39 - (1 - 0.899\,73) \approx 0.73$

**9** **a** 0.714 95 ≈ 0.71 **b** 0.904 95 ≈ 0.90

**c** 0.706 98 ≈ 0.71

**10** **a** 0.377 87 ≈ 0.38 **b** 0.359 57 ≈ 0.36

**c** 0.877 50 ≈ 0.88

**11** $P(x < 70) = P(z < 0.50) = 0.691\,46 \approx 0.69$

**12** $P(x < 6.95) = P(z < -1.67) = 0.047\,46 \approx 0.05$

**13** **a** **i** $P(x > 445) = P(z > 0.93) = 0.176\,19 \approx 0.18$

**ii** $P(x > 410) = P(z > -1.57) = 0.941\,79 \approx 0.94$

**iii** $P(420 < x < 458) = P(-0.86 < z < 1.86)$ $= 0.773\,67 \approx 0.77$

**b** This jump is unusual as the $z$-score for a length of 486 cm is 3.86. This means that the length is nearly four standard deviations above the mean.

**14** **a** $P(x > 175) = P(z > 0.78) = 0.217\,70 \approx 0.22$

**b** $P(x < 165) = P(z < -0.33) = 0.370\,70 \approx 0.37$

**c** $P(170 < x < 175) = P(0.22 < z < 0.78)$ $= 0.195\,24 \approx 0.20$

**d** $0.195\,24 \times 1000 \approx 195$; 195 Year 12 males

**e** 180 cm

**15** **a** $P(2.65 < x < 3.75) = P(-1.29 < z < 1.86)$ $= 0.870\,03 \approx 0.87$

**b** $0.870\,03 \times 10\,000 \approx 8700$; 8700 babies

**16** **a** $P(x < 0.985) = P(z < -2.5) = 0.006\,21 \approx 0.01$

**b** $0.006\,21 \times 1000 \approx 6$; expect 6 packets to be rejected

**17** **a** $P(x > 65) = P(z > 2.00) = 0.022\,75 \approx 0.02$

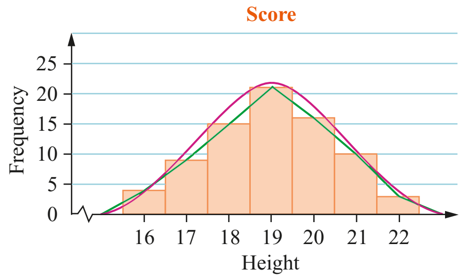
**b** $P(x > 60) = P(z > -0.50) = 0.691\,46 \approx 0.69$; 69% of cars

## REVIEW MULTIPLE-CHOICE QUESTIONS

| | | | | |
|---|---|---|---|---|
| **1** B | **2** D | **3** A | **4** D | **5** A |
| **6** C | **7** B | **8** C | **9** B | **10** B |

## REVIEW SET 1

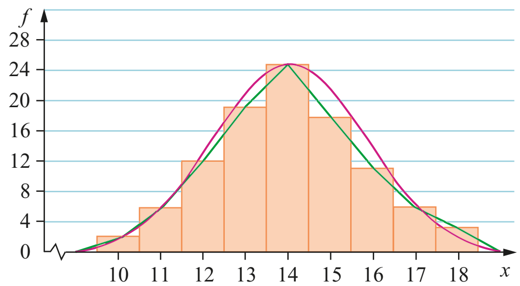
**1**




**2**


**3** B

**4** **a**




**b** Mean = 14, mode = 14, median = 14
Distribution is normal.

**5** **a** −1 **b** 1 **c** −2.5 **d** 1.5

**6** **a** History **b** 52

**7** Term 2

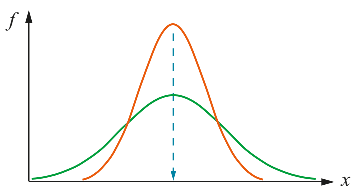
**8** **a** **i** 68 **ii** 58
**b** 68% **c** 95% **d** 99.7%

**9** **a** 34% **b** 34% **c** 50% **d** 16%

**10** **a** 2.5%
**b** $P(34 < x < 38) = P(-1 < z < 1) = 0.68\,269 \approx 0.68$
**c** $P(33 < x < 40) = P(-1.5 < z < 2) = 0.91\,044 \approx 0.91$

## REVIEW SET 2

**1**




**2** **a** 90 **b** 60 **c** 83 **d** 44

**3** **a** Science **b** 59

**4** Mathematics: Rebecca $z = 1.5$, Serena $z = 0.75$
Science: Rebecca $z = 0.4$, Serena $z = 1$
Total: Rebecca $z = 1.9$, Rebecca $z = 1.75$
Serena had the better overall performance.

**5** **a** 68% **b** 95% **c** 99.7%

**6** 0.3%

**7** **a** 67.5 kg and 80.5 kg **b** 61 kg and 87 kg
**c** 54.5 kg and 93.5 kg

**8** **a** 95% **b** 2.5% **c** 97.5%
**d** 2.5% **e** 97.5%

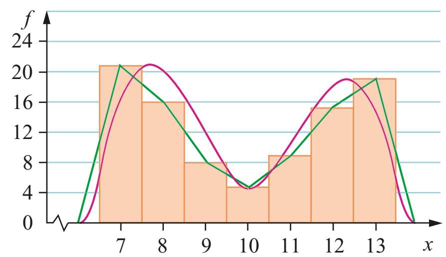
**9** **a** $P(x < 67) = P(z < 0.70) = 0.758\,04 \approx 0.76$
**b** $P(x > 42) = P(z > -1.80) = 0.964\,07 \approx 0.96$
**c** $P(65 < x < 75) = P(0.50 < z < 1.50) = 0.241\,73 \approx 0.24$

**10** **a** $P(x > 78) = P(z > 1.18) = 0.119\,00 \approx 0.12$
**b** $P(x > 65) = P(z > 0) = 0.5$

## REVIEW SET 3

**1** A

**2**




It is not a normal distribution.

**3** **a** 1.5 **b** −2 **c** 0.25 **d** −0.42

**4** **a** 61 **b** 68 **c** 46 **d** 34

**5** **a** Mean = 61.3, standard deviation 12.4
**b** 10 **c** 1

**6** Her performance improved ($z = -1.5$ to $z = -0.7$).

**7** **a** 1280 to 1520 hours **b** 1160 to 1640 hours
**c** 1040 to 1760 hours

**8** **a** Mass (g): 480, 490, 500, 510, 520, 530, 540
**b** **i** 68% **ii** 95% **iii** 99.7%
**c** Error in measurement of mass or machine needs adjusting.

**9** **a** $P(x > 50) = P(z > 0.57) = 0.284\,34 \approx 0.28$
**b** $P(x < 38) = P(z < -1.14) = 0.127\,14 \approx 0.13$
**c** $P(45 < x < 55) = P(-0.14 < z < 1.29)$
$= 0.457\,14 \approx 0.46$

**10** $P(x < 4.25) = P(z < -1.5) = 0.066\,81 \approx 0.07$

## REVIEW SET 4

**1** **a** −0.67 **b** 2 **c** −2.67 **d** 2.67

**2** **a** Science **b** 62

**3** **a** 59 **b** 71 **c** 38 **d** 17

**4** Janelle, Melissa, Michael

**5** **a** 68% **b** 95% **c** 99.7%

**6** **a** 38.8 to 45.2 hours **b** 35.6 to 48.4 hours
**c** 32.4 to 51.6 hours

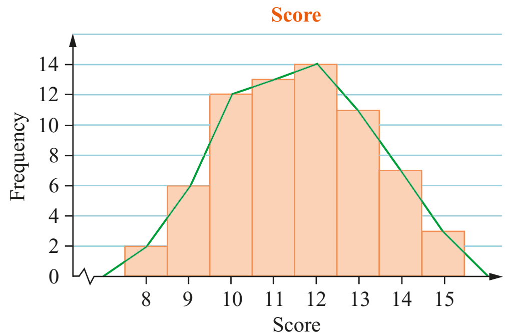
**7** **a** 34% **b** 47.5% **c** 13.5% **d** 2.35%

**8** 2.5%

**9** **a** $z$-score: −3, −2, −1, 0, 1, 2, 3
**b** 50% **c** 68% **d** 34%
**e** 95% **f** 16% **g** 2.5%

**10** **a** $P(x > 165) = P(z > 0.38) = 0.351\,97 \approx 0.35$
**b** $P(x < 160) = P(z < -0.25) = 0.401\,29 \approx 0.40$
**c** $P(160 < x < 165) = P(-0.25 < z < 0.38)$
$= 0.246\,74 \approx 0.25$
**d** $P(156 < x < 176) = P(-0.75 < z < 1.75)$
$= 0.733\,31 \approx 0.73$
$2000 \times 0.733\,31 \approx 1467$; 1467 Year 10 females

**1** **a** **i**




**ii** Mean ≈ 11.6, mode = 12, median = 12

**iii** Yes, mean = mode = median and frequency curve is bell-shaped.

**b** Half-yearly $z = -1.45$, yearly $z = -0.67$
She improved as her $z$-score increased.

**c** **i** 50% **ii** 68% **iii** 16%

**d** **i** $P(x > 45) = P(z > 1.75) = 0.040\,06 \approx 0.04$

**ii** $P(35 < x < 45) = P(-0.75 < z < 1.75) = 0.733\,31 \approx 0.73$
$500 \times 0.733\,31 \approx 367$; expect 367 people to obtain a result between 35 and 45

**e** 44 g pancake $z$-score = 3.33; since the weight is more than three standard deviations from the mean, it is likely that the machine is faulty.

# CHAPTER 10 CRITICAL PATH ANALYSIS

## ARE YOU READY?

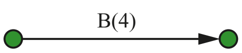
| | | | | |
|---|---|---|---|---|
| **1** D | **2** D | **3** A | **4** D | **5** C |
| **6** C | **7** A, C | **8** B | **9** D | |

## EXERCISE 10A ACTIVITY TABLES AND CHARTS

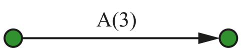
**1** b, c

**2** **a** B, C **b** Yes **c** Yes **d** Yes
**e** Yes **f** Yes
**g** No

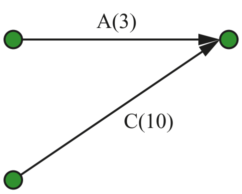
**3** **a**




**b**

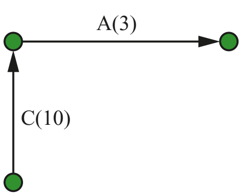



**c**

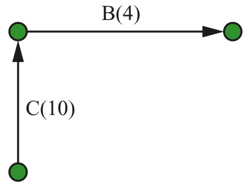



**d** Same as **c**

**e**

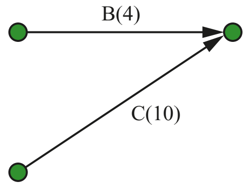



**f**

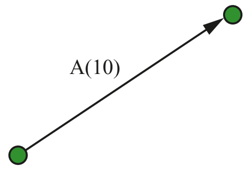



**g** Same as **f**

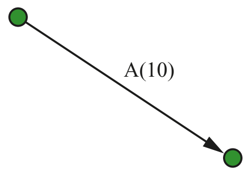
**h**



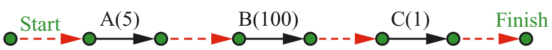

**i**



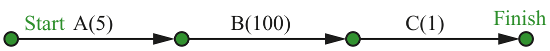

**j**



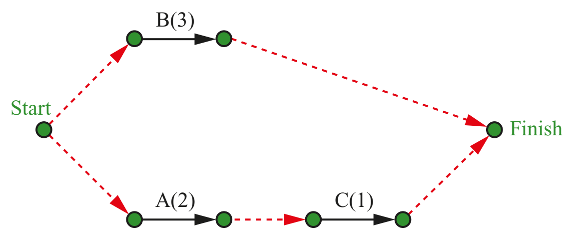

**4** **a**



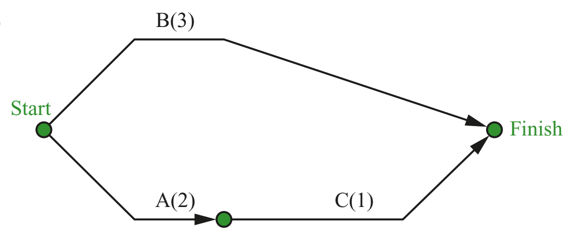

**b**



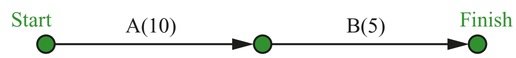

**5** **a**



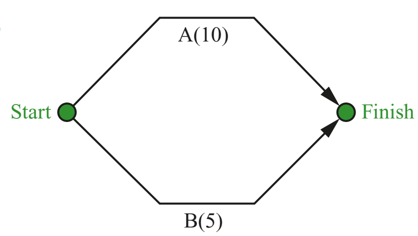

**b**



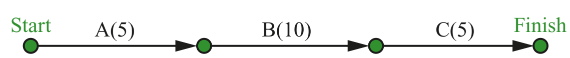

**6** **a**



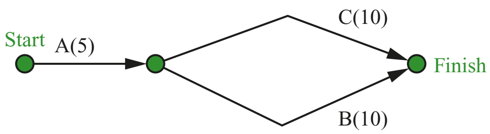

**b**



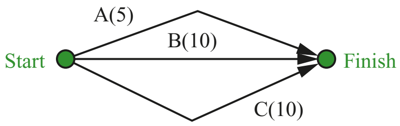

**7** **a**



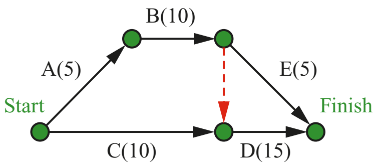
**b**



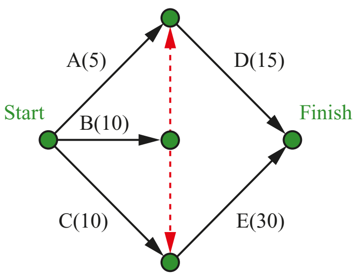

**c**



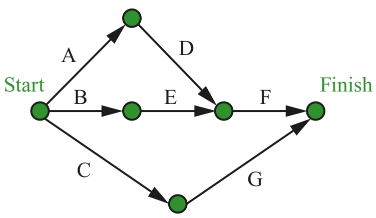

**8**




**9**




**10**




**11** **a** ECCWC101A, ECSFC101A, ECTPP101A, ECTPP102A, ECCDD101A
**b** Yes
**c** Yes
**d** ECCDD101A, ECCDD102A, ECCDD103A, ECSFC102A, ECPPL102A

**12** **a** MATH1131 and PHYS1121
**b** MATH1231 and CVEN1300
**c** Same as **b**
**d** Yes
**e** No

**13** **a** CD
**b**

| Path | ABE | ABD | CD |
|---|---|---|---|
| Total weight | 35 | 26 | 6 |

**c** ABE
**d** B and C
**e** Yes

**14** **a** B
**b** E and F
**c** A, B and C
**d** D
**e**

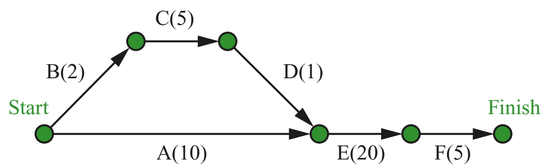
| Activity | Length | Prerequisites |
|---|---|---|
| A | 2 | None |
| B | 5 | None |
| C | 3 | None |
| D | 10 | A |
| E | 8 | B |
| F | 4 | C |
| G | 3 | D |
| H | 9 | E, F |

**15** **a** D, B
**b** D, I

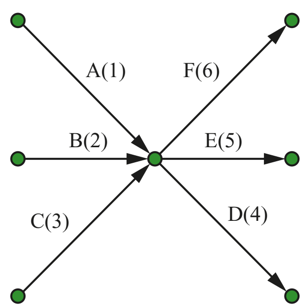
**16** **a** E
**b** B
**c** C and D
**d**

| Activity | Length | Prerequisites |
|---|---|---|
| A | 3 | None |
| B | 4 | None |
| C | 1 | A |
| D | 8 | B |
| E | 10 | A |
| F | 5 | E |
| G | 9 | D, C |

**17**




**18** **a**




**b** See part **a**.
**c** See part **a**.
**d** Not possible. The dummy edge is one of two edges leaving its start vertex, and one of two edges ending at its end vertex.

## EXERCISE 10B THE EARLIEST STARTING AND FINISHING TIMES

**1** **a** A, B, C **b** 2 **c** 5
**d** 3 **e** 12 **f** 13
**g** See question **4**.

**2** **a** A, C **b** 10 **c** 25 **d** 25
**e** See question **4**.
**f** A and C, and D and E

**3** **a** 18 **b** 5 **c** 20
**d** 100 **e** 4 **f** 2

**4** (for question **1g**)

| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| EST | 0 | 0 | 0 | 2 | 5 | 3 | 12 | 13 |
| EFT | 2 | 5 | 3 | 12 | 13 | 7 | 15 | 22 |

(for question **2e**)

| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| EST | 0 | 10 | 0 | 25 | 25 |
| EFT | 10 | 25 | 5 | 26 | 35 |

**5** **a** EST of A, B and C is zero, EST of D, E and F is 9.
**b** EST of A, B and C is zero, EST of D, E and F is 4.
**c** EST of A, B and C is zero, EST of D, E and F is 8.
**d** EST of A, B and C is zero, EST of D, E and F is 10, EST of G, H and I is 14.

**6**

| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| EST | 0 | 0 | 0 | 3 | 4 | 9 | 14 | 16 |

**7** **a**

| Activity | A | B | C | D |
|---|---|---|---|---|
| Weight (your estimate may differ) | 3 | 4 | 5 | 20 |
| EST | 0 | 3 | 7 | 12 |

**b** 12
**c** 32

**8** **a**

| Path | ADG | BEH | CFH |
|---|---|---|---|
| Total weight | 15 | 22 | 16 |

**b** 22
**c** 22

**9** **a**

| Path | ABE | ABD | CD |
|---|---|---|---|
| Total weight | 35 | 26 | 6 |

**b** 35
**c** 35

**10** **a** Yes
**b** When the activities have equal weight

**11** **a** EFT of A is 9, EFT of B is 10.
**b** 10

**12** **a** EFT of A is 12, EFT of B is 10, EFT of C is 12.
**b** 12

**13** **a** EFT of A is 5, EFT of B is 6.
**b** 6 **c** 6

**14** **a** EFT of A is 12, EFT of B is 10, EFT of C is 12
**b** 12 **c** 12

**15** **a** F and G **b** After 100 hours
**c** 51 hours **d** No

**16** **a** G **b** After 2 hours
**c** 51 hours **d** No

**17** **a** After 100 hours
**b** 150 hours
**c** Adds 10 hours

**18** 35 minutes

## EXERCISE 10C THE LATEST FINISHING AND STARTING TIMES

**1** **a** 10 minutes **b** 10 minutes **c** 8 minutes
**2** **a** 22 minutes **b** 19 minutes **c** 9 minutes
**d** 13 minutes **e** 5 minutes **f** 9 minutes
**g** See question **6**.
**3** **a** 7 **b** 0 **c** 2
**d** 100 **e** 4 **f** 20
**4** **a** True **b** True **c** False

**5**

| Activity | A | B | C |
|---|---|---|---|
| LST | 2 | 0 | 3 |

**6**

| Activity | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| LFT | 9 | 5 | 9 | 19 | 13 | 13 | 22 | 22 |
| Weight | 2 | 5 | 3 | 10 | 8 | 4 | 3 | 9 |
| LST | 7 | 0 | 6 | 9 | 5 | 9 | 19 | 13 |

**7** **a**

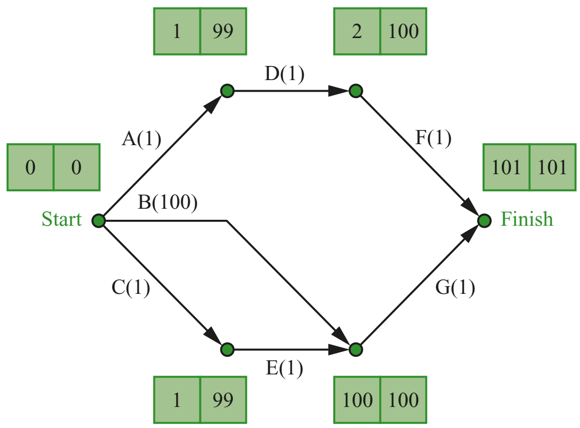
| Activity | A | B | C | D |
|---|---|---|---|---|
| EST | 0 | 0 | 4 | 5 |
| EFT | 4 | 5 | 8 | 10 |
| LFT | 6 | 5 | 10 | 10 |
| LST | 2 | 0 | 6 | 5 |

**b** All activities have an EFT later than their LST.

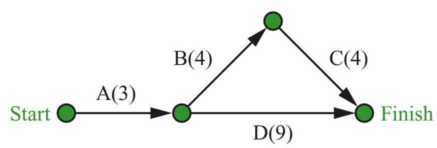
**8**

| Activity | A | B | C |
|---|---|---|---|
| LFT | 40 | 20 | 40 |

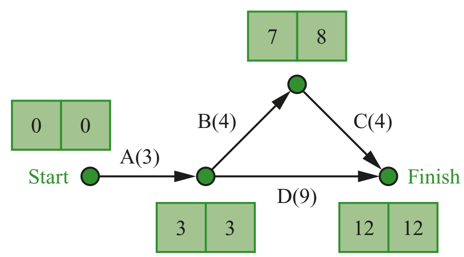
**9**




**10** **a**




**b–d**




**11** **a** Yes
**b** When the activities have the same weight.

**12** **a** The LST of A is 14 and the LST of B is 15.
**b** 14
**c** 14

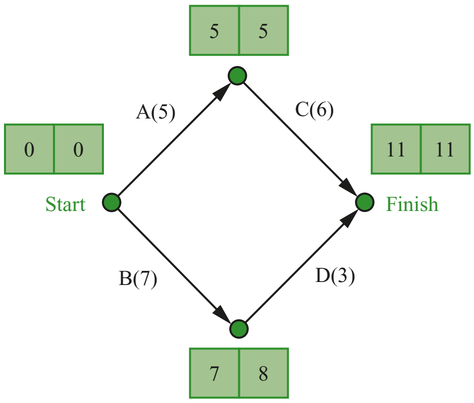
**13** **a** The LST of A is 5, the LST of B is 6, and the LST of C is 11.
**b** 5
**c** 5

**14** **a**

| Activity | A | B | C | D |
|---|---|---|---|---|
| EST | 0 | 0 | 3 | 2 |
| EFT | 3 | 2 | 4 | 6 |
| LFT | 5 | 2 | 6 | 6 |
| LST | 2 | 0 | 5 | 2 |

**b** B and D

**15 a**




**b** 11 days

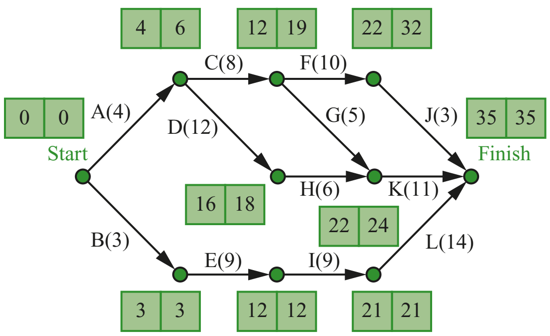
**c** B or D

**16 a** 3 hours

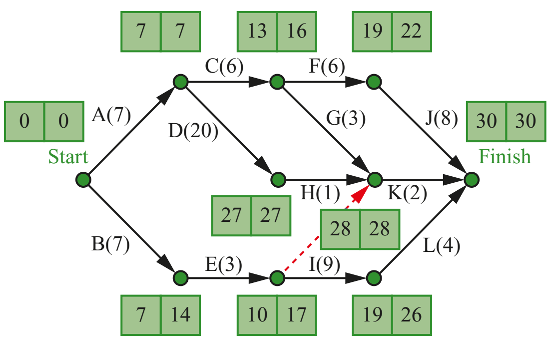
**b** A or C

**c** Yes, activity C

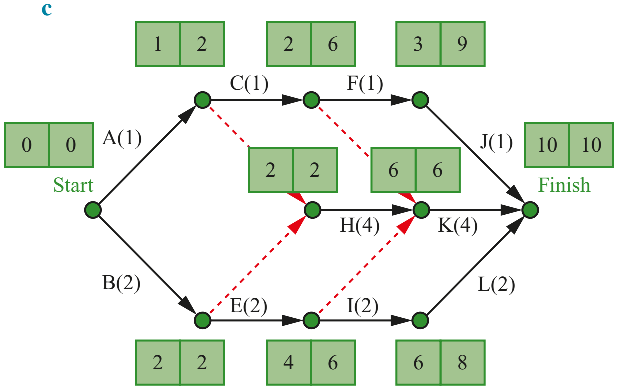
**17 a**




**b**




**c**




## EXERCISE 10D THE CRITICAL PATH

**1 a** AC

**b** ACD

**c** CF

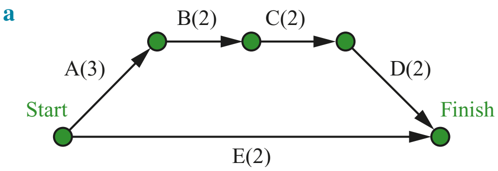
**2** D

**3 a, b**

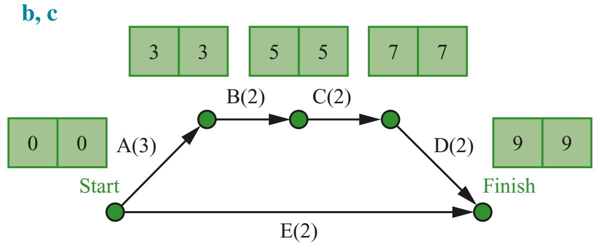
| Activity | A | B | C | D | E |
|---|---|---|---|---|---|
| EST | 0 | 0 | 0 | 3 | 8 |
| LFT | 5 | 12 | 8 | 12 | 12 |
| LST | 2 | 6 | 0 | 5 | 8 |
| Float time | 2 | 6 | 0 | 2 | 0 |

**c** CE

**4 a**



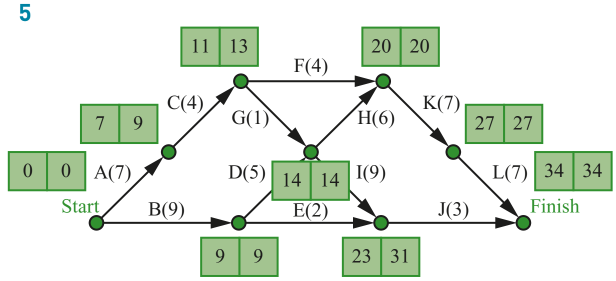

**b, c**




**d** ABCD

**e** 7

**5**




The critical path is BDHKL.

**6 a** ADGI

**b** ADFH

**7 a** BEIL

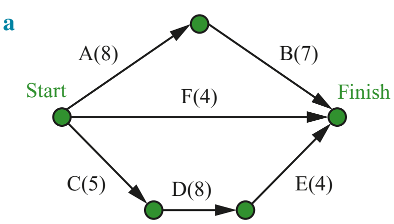
**b** ADHK

**c** BHK

**8** D

**9** 3

**10 a**




**b** C

**c** LST

**11 a** BEHKL

**b** 3

**12 a**

| Activity | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| Weight | 1 | 4 | 7 | 10 | 8 | 6 | 3 |
| EST | 0 | 0 | 1 | 4 | 4 | 12 | 18 |
| LFT | 11 | 4 | 18 | 18 | 12 | 18 | 21 |
| LST | 10 | 0 | 11 | 8 | 4 | 12 | 18 |
| Float time (LST–EST) | 10 | 0 | 10 | 4 | 0 | 0 | 0 |

**b** BEFG
**c** A and C

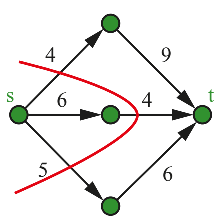
## EXERCISE 10E THE MAXIMUM-FLOW MINIMUM-CUT THEOREM

**1 a** 9
**b** 9
**c** Bottom left and top right
**2 a** 100
**b** 200
**c** 200
**3 a** 8
**b** 9
**c** 14
**d** 10

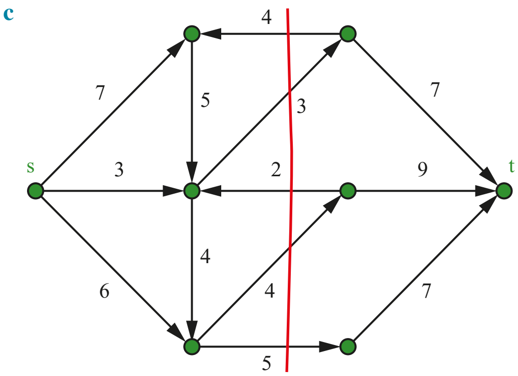
**4**

| Edge | | sA | BA | AC | AD | CB | Ct | Dt |
|---|---|---|---|---|---|---|---|---|
| Included in cut? | a | No | No | Yes | Yes | No | No | No |
| | b | No | No | No | Yes | Yes | Yes | No |
| | c | No | No | Yes | No | No | No | Yes |

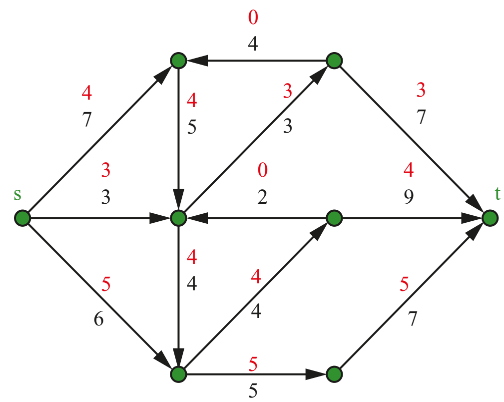
**d** c
**5** 27
**6** 12
**7 a** 3
**b** sB, SC, Bt and Ct
**c** 2
**d** 2
**8** 6
**9 a** Because electricity flows the other way in edges HE and IF.
**b** Because edges HE and IF are not considered part of the cut.
**10 a** At
**b**




**11 a** sB or Ct
**b** St and At
**c** sB and sE, or BC and EF, or Ct
**12 a** 20
**b** Because only these pipes need to be removed to stop the flow from the source to the sink.
**c**




**d** It is possible to create a flow of 12, so by the max-flow min-cut theorem the cut is minimal.



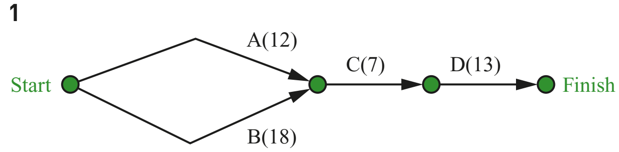

**13** See **12c**.
**14 a** 29
**b** The pipe should cross the three roads directly connected to the sink.
**c** sA, BA, Ct, and the two roads from A to t.
**d** Ct
**e** 20
**15 a** 17
**b** sB, sC and sI; yes
**c** Every edge except BG, Gt (or BG, Ft)
**d** Increasing the capacity of sB by one will saturate all edges.
**16 a** 9 L/s
**b** sA, sB and sC.

## REVIEW MULTIPLE-CHOICE QUESTIONS

**1** A
**2** D
**3** A
**4** B
**5** C
**6** B and C
**7** B
**8** C

## REVIEW SET 1

**1**



**2**

| Activity | A | B | C | D |
|---|---|---|---|---|
| EST | 0 | 0 | 18 | 25 |
| EFT | 12 | 18 | 25 | 38 |

**3** 38 hours

**4**

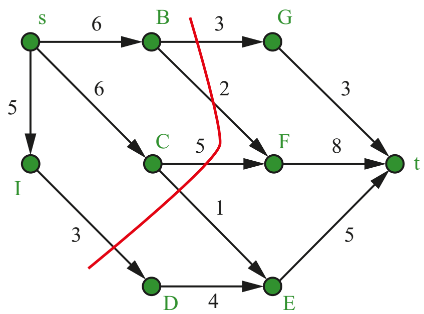
| Activity | A | B | C | D |
|---|---|---|---|---|
| LFT | 18 | 18 | 25 | 38 |
| LST | 6 | 0 | 18 | 25 |

**5** B, C and D

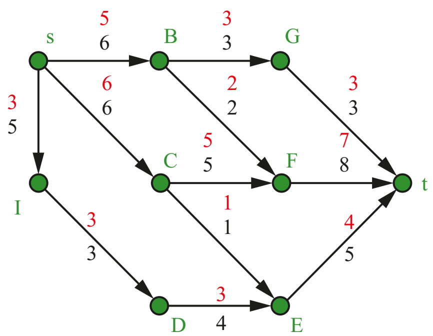
**6** BCD

**7** No change

**8**




**9** The maximum flow is 14.

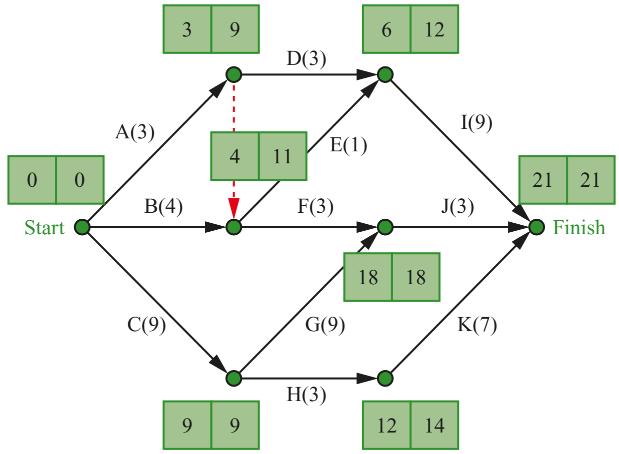



## REVIEW SET 2

**1**

| Activity | Weight (hours) | Prerequisites |
|---|---|---|
| A | 3 | None |
| B | 4 | None |
| C | 9 | None |
| D | 3 | A |
| E | 1 | A, B |
| F | 3 | A, B |
| G | 9 | C |
| H | 3 | C |
| I | 9 | D, E |
| J | 3 | F, G |
| K | 7 | H |

**2**




| Activity | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| EST | 0 | 0 | 0 | 3 | 4 | 4 | 9 | 9 | 6 | 18 | 12 |
| EFT | 3 | 4 | 9 | 6 | 5 | 7 | 18 | 12 | 15 | 21 | 19 |

**3** 21 hours

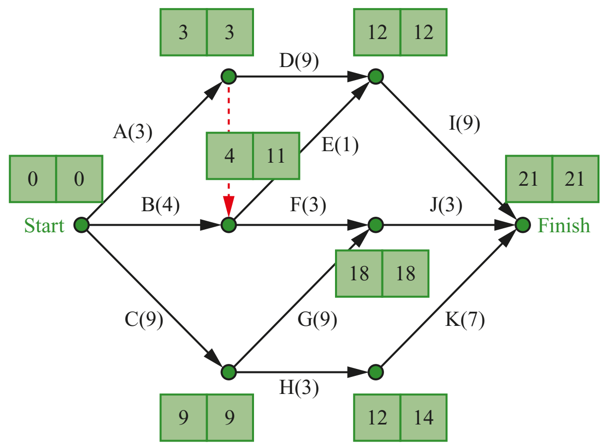
**4**

| Activity | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| LFT | 9 | 11 | 9 | 12 | 12 | 18 | 18 | 14 | 21 | 21 | 21 |
| LST | 6 | 7 | 0 | 9 | 11 | 15 | 9 | 11 | 12 | 18 | 14 |

**5** C, G and J

**6** CGJ

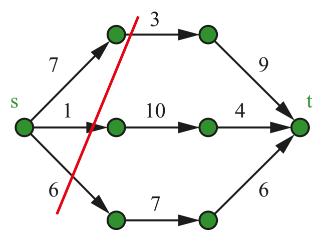
**7**




This does not change the critical time, but now there are two critical paths: CGJ and ADI.

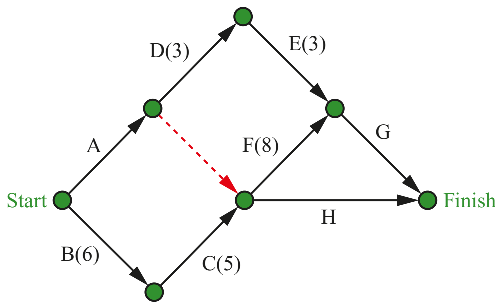
**8** Maximum flow is 10.

**9**

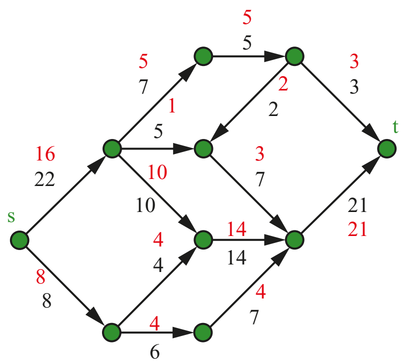


## REVIEW SET 3

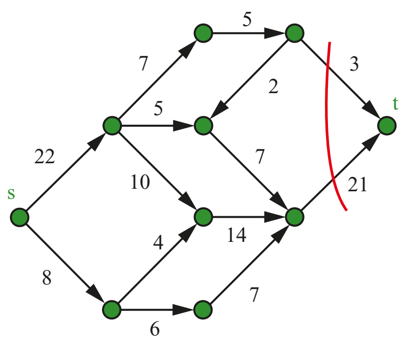
**1**



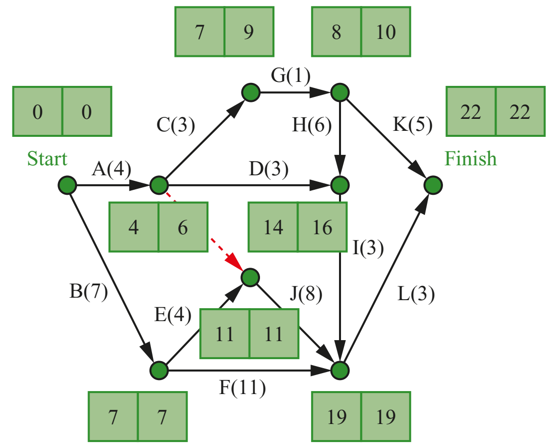

**2** 3
**3** 10
**4** 11 days (10 days if we insist BCFG is the only critical path)
**5** The critical time is reduced by one.
**6** No
**7** 24



**8**




**2**



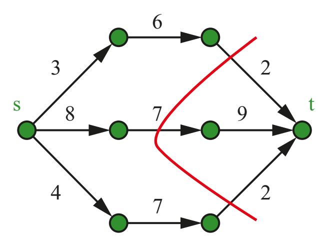

| Activity | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| EST | 0 | 0 | 4 | 4 | 7 | 7 | 7 | 8 | 14 | 11 | 8 | 19 |
| EFT | 4 | 7 | 7 | 7 | 11 | 18 | 8 | 14 | 17 | 19 | 13 | 22 |

**3** 22 days

**4**

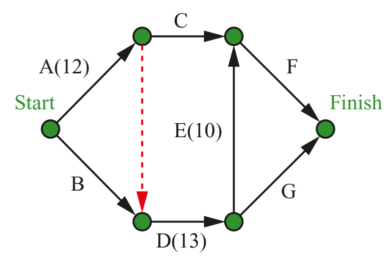
| Activity | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LFT | 6 | 7 | 9 | 16 | 11 | 19 | 10 | 16 | 19 | 19 | 22 | 22 |
| LST | 2 | 0 | 6 | 13 | 7 | 8 | 9 | 10 | 16 | 11 | 16 | 19 |

**5** B, E, J, L
**6** BEJL
**7** No
**8** Maximum flow is 11.
**9**




## REVIEW PRACTICE EXAMINATION QUESTION

**1** **a** **i**




**ii** 11 days
**iii** 4 days
**iv** 9 days
**v** 12 days (or 11 days if we insist that ADEF is the only critical path).

**b** **i** 40
**ii** CD and Dt (and possibly CE and Et)
**iii** sA and sB
**iv** D

**c** C and D

## REVIEW SET 4

**1**

| Activity | Weight (hours) | Prerequisites |
|---|---|---|
| A | 4 | None |
| B | 7 | None |
| C | 3 | A |
| D | 3 | A |
| E | 4 | B |
| F | 11 | B |
| G | 1 | C |
| H | 6 | G |
| I | 3 | D, H |
| J | 8 | A, E |
| K | 5 | G |
| L | 3 | I, J, F |

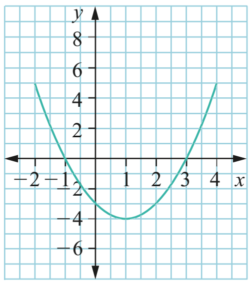
**d i** 3
**ii** 13
**iii** BEIL
**iv** 21
**v** 15

## CHAPTERS 8–10 CUMULATIVE REVIEW

**1 a**

| $x$ | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| $y$ | 5 | 0 | −3 | −4 | −3 | 0 | 5 |

**b**

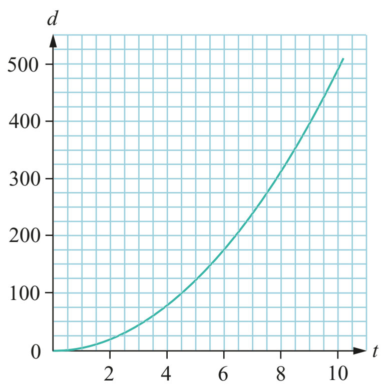



**c** (0, −3)
**d** (−1, 0) and (3, 0)
**e** (1, −4)
**f** $x = 1$
**g** −4
**h** 1

**2 a** 50 m
**b** 4 s
**c** 2 s and 6 s
**d** −25, model only applies for values of $t$ between 0 and about 8.1 s

**3 a** $d = 4.9t^2$

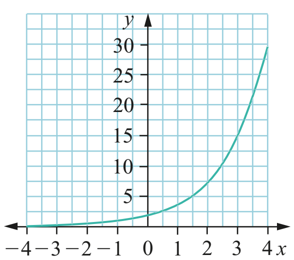
**b**




**c** About 180 m (176.4 m)
**d** About 8 s

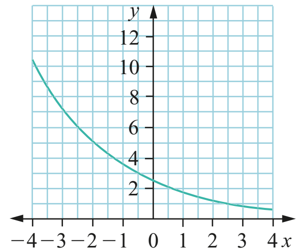
**4 a i** $y$-intercept is 1.8

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 0.1125 | 0.225 | 0.45 | 0.9 | 1.8 | 3.6 | 7.2 | 14.4 | 28.8 |



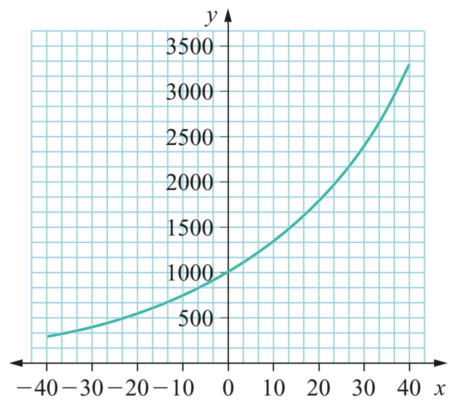

**ii** $y$-intercept is 2.5

| $x$ | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 10.41 | 7.29 | 5.10 | 3.57 | 2.5 | 1.75 | 1.23 | 0.86 | 0.60 |




**iii** $y$-intercept is 1000

| $x$ | −40 | −30 | −20 | −10 | 0 | 10 | 20 | 30 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 307 | 412 | 554 | 744 | 1000 | 1344 | 1806 | 2427 | 3262 |

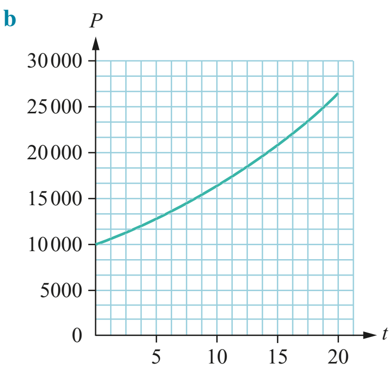



**b i** Exponential growth
**ii** Exponential decay
**iii** Exponential growth

**5 a**

| $t$ | 0 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| $P$ | 10 000 | 12 763 | 16 289 | 20 789 | 26 533 |

**b**



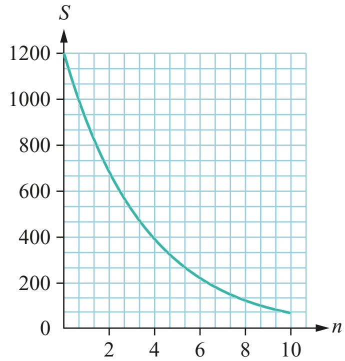

**c** The population when $t = 0$
**d** 18 000
**e** 16 years
**f** $22\frac{1}{2}$ years
**g** 5%; the population increases by a factor of 1.05 each year, which means that the population in the following year is 105% of the current year, an increase of 5%

**6 a** Substituting $V_0 = 1200$, $r = 0.25$ (from 25%) into $S = V_0(1 - r)^n$ gives $S = 1200(1 - 0.25)^n$ or $S = 1200(0.75)^n$.

**b**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $S$ | 1200 | 900 | 675 | 506.25 | 379.69 | 284.77 |

| $n$ | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|
| $S$ | 213.57 | 160.18 | 120.14 | 90.10 | 67.58 |




**c** About \$500 (\$506.25)

**d** About 4.8 years (4.82 years)

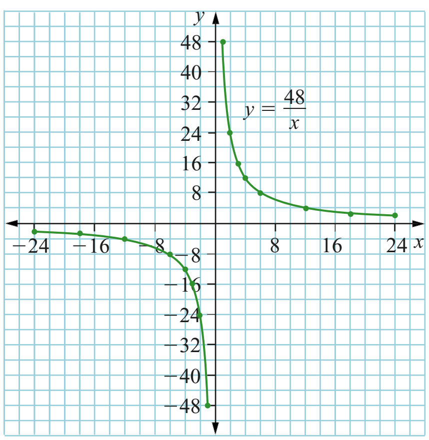
**e** Exponential decay, $S$ decreases as $n$ increases

**7 a**

| $x$ | −24 | −16 | −12 | −6 | −4 | −3 | −2 |
|---|---|---|---|---|---|---|---|
| $y$ | −2 | −3 | −4 | −8 | −12 | −16 | −24 |

| $x$ | −1 | 1 | 2 | 3 | 4 | 6 | 12 | 16 | 24 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | −48 | 48 | 24 | 16 | 12 | 8 | 4 | 3 | 2 |

**b**

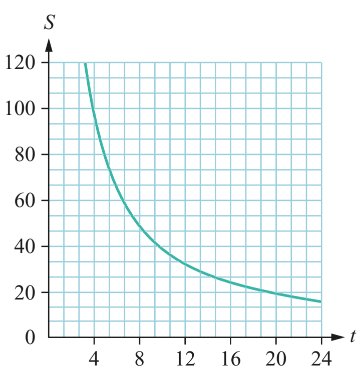



**c** Rectangular hyperbola

**8 a**

| $t$ | 4 | 6 | 8 | 12 | 16 | 24 |
|---|---|---|---|---|---|---|
| $s$ | 120 | 80 | 60 | 40 | 30 | 20 |

**b**



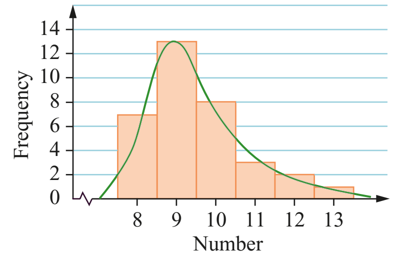

**c** About 70 km/h (68.6 km/h)

**d** About 10 h (10.7 h)

**9 a** $t = \frac{324}{n}$ where $t$ is the time in days to complete the task with $n$ workers

**b** 19 days

**10 a**




**b** Positively skewed, unimodal

**c** Mean 9.5, median 9, mode 9

**d** No, as the frequency curve is not symmetrical (bell-shaped) and the mean, median and mode are not equal.

**11 a** Curve Y **b** Curve Y

**12 a** Term 1: −1.25, Term 2: −1.6, Term 3: −0.6, Term 4: −0.75

**b** Term 3

**13 a** 44 **b** 62 **c** 54

**14 a** 157 to 179

**b** 146 to 190

**c** 135 to 201

**15 a** 34% **b** 16% **c** 47.5%

**d** 2.5% **e** 49.85% **f** 99.85%

**16 a** 204 mm

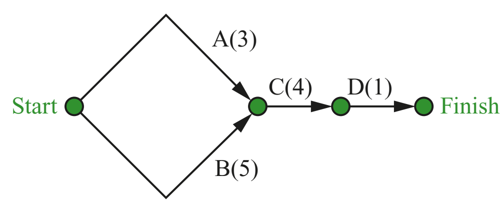
**b** Not likely as 210 mm is considerably more than three standard deviations above the mean.

**17 a i** 0.68 **ii** 0.33 **iii** 0.26

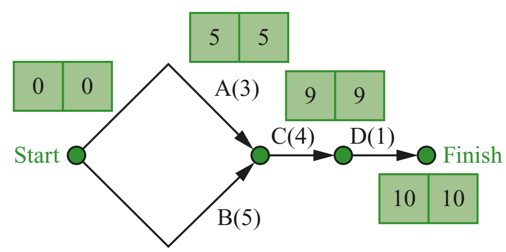
**b** About 8.7%

**c** About 1814 students

**18**



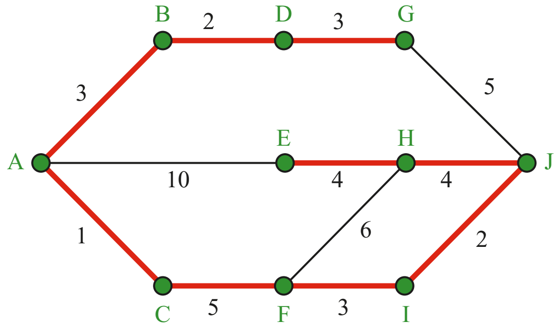

**19, 20**



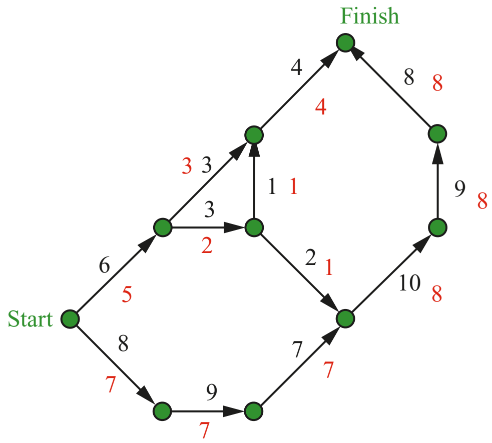
**21** BCD

**22** ACD

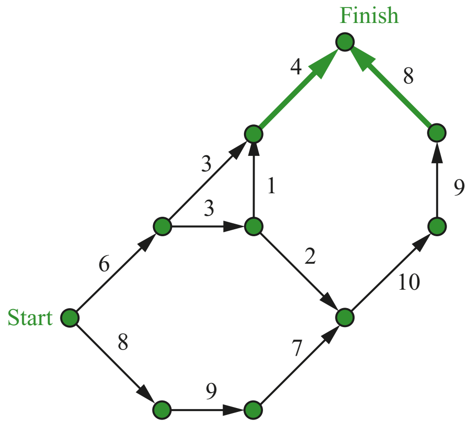
**23**




**24** 12




**25**



# GLOSSARY

**activity chart** a weighted and directed network representing the dependencies in a set of activities

**activity table** a table of activities specifying prerequisite activities and the expected timeframe for each activity

**adjacent side** side that is adjacent or next to the reference angle in a right-angled triangle (but is not the hypotenuse)

**angle of depression** angle between a horizontal line and the line of sight to an object below the horizontal

**angle of elevation** angle between a horizontal line and the line of sight to an object above the horizontal

**annuity** an investment account receiving regular, equal contributions, or a single sum investment from which regular, equal withdrawals are made

**appreciate** a rise in the value or price of an item over a period of time

**asymptote** a straight line or boundary that a curve approaches (gets closer to) but does not cross

**axis of symmetry** an imaginary line that divides a symmetrical graph so that one side is a reflection of the other; for example, a parabola has a vertical axis of symmetry through its turning point. If the coordinates of the turning point are (3, 2) then the equation of the axis of symmetry is $x = 3$

**backward scan** the algorithm used to determine the LFT of each activity

**bearing** direction from one position to another, using the angles around a magnetic compass

**best buy** found by comparing a number of purchase options to find the one that is the best value for money

**bias** a prejudice in favour of one particular group or outcome

**bivariate dataset** a dataset involving two different variables for each data point

**break-even point** point at which the revenue (or income) and the cost of production (or expenses) are equal; it can be identified as the point of intersection of the combined graphs of cost and revenue

**brokerage** a fee charged by stockbrokers to carry out trades on behalf of an investor

**cause and effect** the result of the change in one variable is directly responsible for the change in another variable

**coincidence** two or more events occurring simultaneously through natural random variation

**compass bearing** indicates direction as an angle measured from north or south towards east or west; e.g. N40°W represents a bearing of 40° from north towards west and S75°E represents a bearing of 75° from south towards east

**compass radial survey** survey of a field in which a magnetic compass is used to find the bearing of each vertex along the boundary from a point within the field, and the distance from this point to each vertex is measured

**compound interest** interest that is calculated on the current balance of an investment, including the interest from the previous time period

**connected network** a network in which there is a walk between every pair of vertices

**correlation** a measure of the extent to which two variables are related

**cosine rule** rule linking three side lengths *a, b* and *c,* and one angle of a triangle *ABC*; e.g. $a^2 = b^2 + c^2 - 2bc \cos A$

**credit card** plastic card that allows the holder to purchase goods and services on credit

**critical path** the longest path between the Start and Finish vertices; usually found by looking for activities with zero float time

**critical step** any activity on the critical path is called a critical step; delay to a critical step will cause delay to the whole project

**critical time** the shortest amount of time required to complete all activities

**cut capacity** the sum of all capacities in the cut

**cut** a selection of edges that disconnects the flow from the source to the sink

**cycle** a walk that starts and ends in the same vertex, but does not visit any other vertices twice (or more)

**degree** (of a vertex) the number of edges connected to a vertex

**dependent variable** variable whose value depends on that of another; usually represented on the vertical axis of a graph; e.g. for the relationship $y = 2x + 1$, the dependent variable is $y$ (as it depends on the value of $x$)

**depreciation** a loss in the value of an item over time

**Dijkstra's algorithm** a method for finding the shortest path between any two vertices of a connected weighted network

**directed network** a network in which the edges have a direction; the orientation of the edge is represented by an arrow

**disconnected network** a network that is not connected

**dividend yield** the percentage of a current share's market price that is paid as a dividend

**dividend** a payment made regularly from a company to its shareholders

**dummy activity** an edge in an activity chart with no name or weight, usually indicated as dashed directed edge

**earliest finishing time (EFT)** the minimum amount of time it takes to complete the activity plus all prerequisite activities, and is calculated by adding the edge weight to the EST

**earliest starting time (EST)** the minimum amount of time it takes to complete all prerequisite activities

**edge** a line between two vertices, which indicates a relationship between the vertices; sometimes called an arc

**empirical rule** describes the proportion of scores that lie within one standard deviation of the mean (68%), within two standard deviations of the mean (95%) and within three standard deviations of the mean (99.7%)

**ethics** moral principles that guide the way that we should act in a given situation

**exponential decay** relationship between two variables in which, as one variable increases, the other decreases at a decreasing rate of change; the graph curves downwards from left to right

**exponential graph** graph of an exponential relationship

**exponential growth** relationship between two variables in which, as one variable increases, the other increases at an increasing rate of change; the graph curves upwards from left to right

**exponential relationship** relationship between two variables in which the independent variable is the exponent or power or index of a term in index form; e.g. the equation $y = 2x$ describes the relationship between the independent variable $x$ and the dependent variable $y$

**extrapolation** predicting where a new data point will lie outside the existing range of data points

**face (or par) value** the value of a share when it is first issued

**float time** the difference between the LST and the EST of an activity

**flow capacity** the total flow from source to sink

**flow diagram** a weighted and directed network used to represent the transport of objects from a source to a sink

**forward scan** the procedure to determine the EST for each activity in an activity chart

**fuel consumption rate** comparison between the two quantities fuel consumed and distance travelled; usually given as a rate in L/100 km

**gradient** steepness of a line (defined as $m$) that can be measured using any two points on the line or interval using $m = \frac{\text{rise}}{\text{run}}$

**heart rate** the number of contractions of the heart per minute measured as beats per minute (bpm)

**hyperbola** graph of a reciprocal relationship

**hypotenuse** side that is opposite the right angle in a right-angled triangle

**indegree** the number of edges pointing towards a vertex in a directed network

**independent variable** variable whose outcomes are not due to those of another variable; usually represented on the horizontal axis of a graph; e.g. for the linear relationship $y = 2x + 1$, the independent variable is $x$

**inflation** an increase in the price of goods and services over a period of time

**inflow** the total flow of all edges pointing towards the vertex

**interpolation** predicting where a new data point will lie within the existing range of data points

**inverse variation** relationship between two variables in which, as one variable increases, the other decreases in the same proportion, and vice versa; we say $y$ varies inversely as $x$, or $y$ varies as $\frac{1}{x}$

**kilowatt-hour (kWh)** unit of measurement for the amount of energy used by an electrical appliance; 1 kWh is the amount of energy used by a 1 kW appliance in 1 hour

**Kruskal's algorithm** a method for finding the minimum spanning tree of a connected, weighted network by selecting the required edges one at a time

**latest finishing time (LFT)** the latest time the activity can be finished without delaying the finish time of the project

**latest starting time (LST)** the latest time the activity can be started without delaying the finish time of the project

**leaf** in a tree, any vertex of degree 1

**least-squares regression line** the mathematically determined straight line that best fits a bivariate dataset

**line of best fit** a straight line that provides a representation of all of the data points in a scatterplot; also known as the trendline

**linear relationship** relationship between two variables where the coordinate points describing this relationship lie in a straight line when plotted on a number plane (or Cartesian plane)

**loop** an edge that starts and ends at the same vertex

**lump sum payment** a one-off payment, usually of a significant amount of money

**market price** the current price of a share

**maximum flow** a flow capacity that is as large as possible

**maximum value** the largest value of $y$ in the relationship between $x$ and $y$ for a given interval of $x$-values

**minimum cut** a cut with the smallest possible capacity

**minimum spanning tree** the spanning tree with smallest total edge weight

**minimum value** the smallest value of $y$ in the relationship between $x$ and $y$ for a given interval of $x$-values

**negatively skewed** description of a skewed graph that has a longer tail to the left

**network** a collection of objects that are related to each other in some way

**normal curve** the shape of a frequency curve that is symmetrical with one mode; a bell-shaped curve

**normal distribution** a frequency distribution of scores that produces a bell-shaped curve (normal curve) and where the mean, median and mode are equal

**opposite side** side that is opposite the reference angle in a right-angled triangle

**outdegree** the number of edges pointing away from a vertex in a directed network

**outflow** the total flow of all edges pointing away from the vertex

**outlier** a data point that appears to be inconsistent with the other data points in a dataset

**parabola** graph of a quadratic relationship

**path** a walk that does not visit any vertex twice (or more)

**Pearson's correlation coefficient (*r*)** a statistical measure of the strength of the linear relationship between two numerical variables

**point of intersection** point where two or more lines intersect or cross over one another on a graph

**positively skewed** description of a skewed graph that has a longer tail to the right

**power rating** rating of an electrical appliance to indicate the rate at which the appliance uses electricity, measured in watts (W) or kilowatts (kW)

**power** rate at which energy is used, measured in watts (W) or kilowatts (kW)

**prerequisite** anything that must be done before an activity can start

**Prim's algorithm** a method for finding the minimum spanning tree of a connected weighted network by iteratively adding the edge of smallest weight

**production cost** cost (or expense) of producing an item for sale, which can involve both a fixed cost and a variable cost that depends on the number of items produced

**quadratic relationship** relationship between two variables where the highest power of the independent variable is 2; for example, the general equation $y = ax^2 + bx + c$, where $a$, $b$ and $c$ are constants and $a \neq 0$, describes the quadratic relationship between the independent variable $x$ and the dependent variable $y$

**radial arms** straight lines radiating or spreading out from a point

**rate** comparison of one quantity with another; e.g. speed is a rate as distance is compared to time

**ratio** comparison between quantities of the same kind, used as a measure of the relative size of the quantities

**reciprocal relationship** relationship between two variables that involves the reciprocal of the independent variable; e.g. the equation $y = \frac{1}{x}$ describes the relationship between the independent variable $x$ and the dependent variable $y$

**rectangular hyperbola** hyperbola with a vertical asymptote and a horizontal asymptote

**recurrence relation** an equation that defines the relationship between terms of a sequence

**reducing-balance loan** a loan for which the interest is calculated on the balance owing, which gradually reduces over time

**revenue** amount of money (or income) received from the sale of items produced by a company, which can depend on the number of items sold

**salvage value** the reduced value of an item after depreciation

**saturated** an edge where the flow is equal to the capacity

**scale drawing** a drawing of a diagram that is similar to the original but is smaller (or larger) in which the angles are the same and the lengths are in the same ratio as the original

**scale** the ratio of a length on a drawing to its real length expressed in the form 1 : n or $\frac{1}{n}$

**scatterplot** a diagram using Cartesian coordinates to display data values that relate two variables

**share** a proportion of a company given a monetary value, that can be sold or exchanged

**shortest path** (in a network diagram) a path between two vertices in a network for which the sum of the weights of its edges is minimised

**simple interest** interest that is calculated on the original principal for the lifetime of the investment/loan; also known as flat rate interest

**simultaneous linear equations** two or more linear equations that use the same variables

**sine rule** rule linking side lengths $a$, $b$ and $c$, and angles $A$, $B$ and $C$ of a triangle $ABC$; e.g. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ or $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$. These ratios are used in pairs

**skewed** description of a distribution graph of a data set for which the shape is not symmetrical

**spanning tree** a tree made from some of the edges and all of the vertices of the original network

**spurious correlation** two variables are mathematically related, but the relationship is not due to cause and effect

**standard normal distribution** a normal distribution with a mean of 0 and a standard deviation of 1; it uses standardised scores or $z$-scores

**standardised score ($z$-score)** the number of standard deviations a score lies above or below the mean

**symmetrical** description of a distribution graph of a data set for which an imaginary vertical line divides the curve so that one side is a reflection of the other

**tree** a connected network with no cycles

**trigonometry** study of relationships between angles and side lengths of a triangle

**true bearing** indicates direction as an angle measured from north in a clockwise direction and written with three digits; e.g. east is represented by 090°T and south-west is represented by 225°T

**turning point** a point on the parabola at which the curve changes direction

**vertex** a visual representation of some object, usually drawn as a point; sometimes called a node

**vertex** point where two or more lines meet

**walk** a sequence of vertices and the edges between them

**weighted edge** edges with a weight that represents a numerical quantity associated with the relationship between the vertices

***x*-intercept** point at which a line crosses the $x$-axis of a number plane; e.g. if a line crosses the $x$-axis at –3 then the $x$-intercept is –3 and the coordinates of the $x$-intercept are (–3, 0)

***y*-intercept** point at which a line crosses the $y$-axis of a number plane; e.g. if a line crosses the $y$-axis at 2 then the $y$-intercept is 2 and the coordinates of the $y$-intercept are (0, 2)

# INDEX

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ISBN 978-0-19-031214-5

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