NEW SENIOR MATHEMATICS

EXTENSION 1 FOR YEARS 11 & 12

SECOND EDITION

J. B. Fitzpatrick
Bob Aus
Merv Curran

ALWAYS LEARNING

PEARSON

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Pearson Australia
(a division of Pearson Australia Group Pty Ltd)
707 Collins Street, Melbourne, Victoria 3008
PO Box 23360, Melbourne, Victoria 8012
www.pearson.com.au

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First published 2013 by Pearson Australia
2016 2015 2014 2013
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Publisher: Vebica Evans
Project Managers: Aisling Coughlan, Shivani Singh
Editor: Tim Carruthers
Designer: Jay Urbanic
Copyright & Pictures Editor: Siân Bradfield
Typesetters: Aptara
Printed in ** by **

National Library of Australia Cataloguing-in-Publication entry:
Author: Fitzpatrick, J. B. (John Bernard), author.
Title: New Senior Mathematics Extension 1 for Years 11 & 12 / J. B. Fitzpatrick, Bob Aus, Merv Curran.
Edition: 2nd edition.
ISBN 978 1 4425 6618 7 (paperback)
Target Audience: For senior secondary school students.
Subjects: Mathematics.
Other Authors/Contributors: Aus, Bob, author. Curran, Merv, author.
Dewey Number: 510

Pearson Australia Group Pty Ltd ABN 40 004 245 943

Acknowledgements
GeoGebra tasks provided by Deb Bridge.

We would like to thank the following teachers for their reviews:
Evie Apfelbaum, Moriah College, Queens Park
John Cascun-Valencic, Northholm Grammar School, Arcadia
Trisha Goss, Carroll College, Broulee
Michelle Hill, Kinross Wolaroi School, Orange
Piv Huot, Our Lady of Mercy College, Parramatta
Nicola Kovacs, Gungahlin College, Gungahlin
Rodney Miller, Normanhurst Boys High School, Normanhurst
Peter Muddle, Covenant Christian School, Belrose
Maryanne O'Donohue, Saint Ignatius' College, Riverview
Dijana Popin, Moriah College, Queens Park
John Sowden, The King's School, Parramatta
Trevor Wykes, The King's School, Parramatta
Heather Brown
David Coffey
John McMillan

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INTRODUCTION AND DEDICATION

It is interesting to wonder whether J. B. Fitzpatrick ('Bernie') realised in 1983 just how popular his book *New Senior Mathematics* would be. That first edition of *New Senior Mathematics* was to remain in print for almost 30 years. It has stood the test of time thanks to the quality, rigour and variety of its questions, its accuracy and its high mathematical standards.

As Fitzpatrick wrote in 1983: 'Mathematics, like many other things, is best learnt by doing. A student begins to appreciate the power of mathematics when he or she has achieved a mastery of basic techniques, not after reading lengthy explanations … The emphasis throughout the book is on the understanding of mathematical concepts' (Introduction, *New Senior Mathematics* 1984).

Fitzpatrick was only willing to entrust the writing of a new edition to people who would understand what makes this series so important. We enlisted the skills of respected mathematics author and teacher Bob Aus and experienced teacher Merv Curran, who together have 86 years of senior maths education experience.

In this revision, Bob Aus and Merv Curran have updated and improved aspects of *New Senior Mathematics* to make it better suited to the modern-day classroom. While maintaining the challenging and demanding questions, they have improved the question grading to better scaffold students' skill development. New revision sections at the end of each chapter and five sample exams also provide more opportunities for students to consolidate skills.

We hope all loyal users of *New Senior Mathematics*, or 'Fitzy's book' as it is fondly known, enjoy all the new improvements.

Vebica Evans

Pearson

J. B. Fitzpatrick passed away in 2008. Fitzpatrick was a respected author, teacher and figurehead of mathematics education.

NEW SENIOR MATHEMATICS

SECOND EDITION

New Senior Mathematics Extension 1 for Years 11 & 12 is a new edition of the well-known mathematics series for New South Wales. We've completely updated the series for today's classrooms, continuing the much-loved approach to deliver mathematical rigour with challenging student questions.

Student Book

The first three chapters of the first student book contain revision material that provides the necessary foundation for the development of senior mathematics concepts. In the new edition you'll also find:

- better grading and scaffolding of questions
- new HSC-style multiple-choice questions
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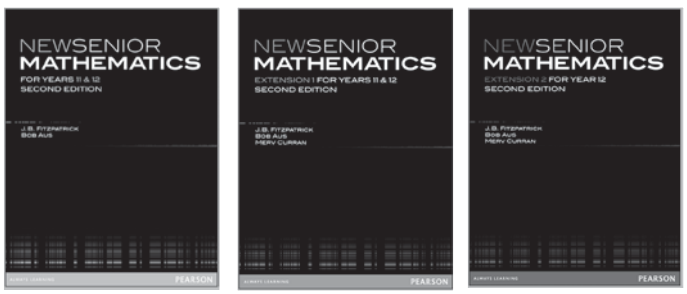

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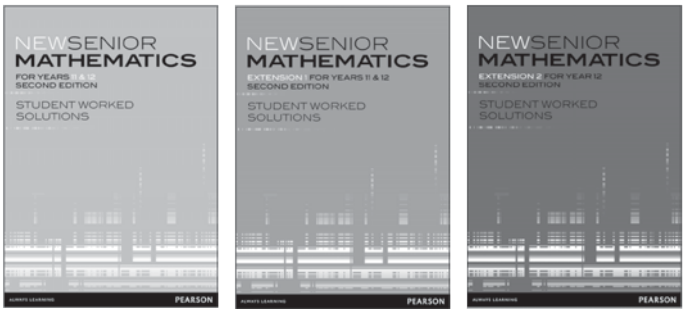

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CONTENTS

Introduction and dedication . . . . . **iii**

PRELIMINARY

SYLLABUS REFERENCE

CHAPTER 1 Circle geometry **1** **2.6–2.10E**

Assumed knowledge 1
1.1 Simple angle properties of a circle 1
1.2 Chord properties 1 9
1.3 Chord properties 2 11
1.4 Angles in circles 14
1.5 Cyclic quadrilaterals 18
1.6 Tests for concyclic points 21
1.7 Angles in alternate segments 21
1.8 Tangents from an external point 24
1.9 Intercepts of chords 27
1.10 Tangents and secants 30
1.11 Touching circles 35
Summary of results 38
Chapter review 1 39

CHAPTER 2 Polynomials **41** **16.1–16.3E**

2.1 Polynomials 41
2.2 Division of polynomials and the remainder theorem 43
2.3 The factor theorem 47
2.4 Polynomial functions 49
2.5 Relationship between roots and coefficients 55
Chapter review 2 60

CHAPTER 3 Permutations and combinations **61** **18.1E**

3.1 Fundamental counting principle 61
3.2 Permutations 63
3.3 Arrangement of *n* objects when some are identical 69
3.4 Combinations 71
Chapter review 3 78

CHAPTER 4 **Further trigonometry** 79 5.6–5.9E
4.1 Applied trigonometry 79
4.2 Special angle formulae 83
4.3 Half-angle formulae 88
4.4 Using radians. 90
4.5 Simple trigonometric equations 92
4.6 Quadratic trigonometric equations 94
4.7 Trigonometric equations involving angle formulae 95
4.8 Auxiliary angle method 96
4.9 General solution of trigonometric equations 100
Chapter review 4 106

CHAPTER 5 **Miscellaneous** 107
5.1 Inequalities 107 1.4E
5.2 Angle between two intersecting lines 116 6.6E
5.3 Division in a ratio 117 6.7E
5.4 Equations reducible to quadratics 119 9.4E
Chapter review 5 122

CHAPTER 6 **Parametric equations and the parabola** 123 9.6E
6.1 Parametric representation 123
6.2 Parametric equations of the parabola 124
6.3 Locus problems and the parabola $x^2 = 4ay$ 131
Chapter review 6 136

HSC

SYLLABUS REFERENCE

CHAPTER 7 **Mathematical induction** 137 7.4E
7.1 Proving sums or products by induction 138
7.2 Proving divisibility by induction 142
7.3 Proving inequalities by induction 144
7.4 Other proofs by induction 145
Chapter review 7 147

CHAPTER 8 Iterative methods 148 16.4E

8.1 Approximate solutions of equations 148
8.2 Halving the interval 149
8.3 Newton's method 151
Chapter review 8 156

CHAPTER 9 Integration techniques 157 11.5E, 13.6E

9.1 Indefinite integrals and substitution 157
9.2 Definite integrals and substitution 160
9.3 Integration of $\sin^2 x$ and $\cos^2 x$ 163
9.4 Integrals of the type $\int f'(x)(f(x))^n \, dx$ 165
9.5 Integrals involving trigonometric substitution 167
Chapter review 9 170

CHAPTER 10 Inverse functions 171 15.1–15.5E

10.1 Inverse functions 171
10.2 Inverse trigonometric functions 180
10.3 Differentiation of inverse trigonometric functions 190
10.4 Integration involving inverse trigonometric functions 194
Chapter review 10 198

CHAPTER 11 Binomial theorem 200 17.1–17.3E

11.1 Expansion of $(1 + x)^n$, Pascal's triangle 200
11.2 More Pascal's triangle expansions 203
11.3 Pascal's triangle relations and the binomial theorem 204
11.4 Harder binomial expansions 208
11.5 Relationships between binomial coefficients 213
Chapter review 11 218

CHAPTER 12 Further probability 219 18.1E, 18.2E

12.1 Counting techniques in probability 219
12.2 Binomial probability 227
Chapter review 12 235

CHAPTER 13 Applications of calculus to the physical world **237** **14.1–14.4E**

13.1 Related rates 237
13.2 Exponential change 243
13.3 Velocity and acceleration as functions of x 249
13.4 Simple harmonic motion (SHM) 254
13.5 Projectile motion 265
Chapter review 13 276

CHAPTER 14 Harder applications **277**

14.1 Graphing rational algebraic functions 277 **10.5E**
14.2 The limit $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ 280 **13.4E**
14.3 Harder applications of Mathematics course topics 281
Chapter review 14 283

Standard integrals **284**
Summary—Preliminary **285**
Summary—HSC **293**
Mathematics Extension 1 course outcomes **304**
Mathematics Extension 1 performance bands **305**
Answers **307**
Glossary **358**

CHAPTER 1
CIRCLE GEOMETRY

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE2 uses multi-step deductive reasoning in a variety of contexts

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

ASSUMED KNOWLEDGE

Tests for congruent triangles

SSS Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other triangle.

SAS Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle.

AAS Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle.

RHS Two right-angled triangles are congruent if the hypotenuse and one side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.

Tests for similar triangles

- Two triangles are similar if the three angles of one triangle are respectively equal to the three angles of the other triangle. (**AAA**)
- Two triangles are similar if the three pairs of corresponding sides are proportional.
- Two triangles are similar if two pairs of corresponding sides are proportional and the angles included by these sides are equal.

Pythagoras' theorem

In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

1.1 SIMPLE ANGLE PROPERTIES OF A CIRCLE

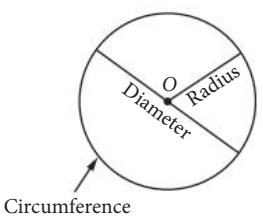
Terms related to circles

A **circle** is a set of all points in a plane that are the same distance from a fixed point in the plane.

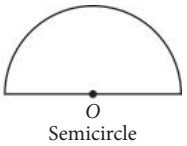
The fixed point O is called the **centre** of the circle.

The fixed distance is called the **radius** of the circle.

The **diameter** of a circle is the line through the centre joining two points on the circle.

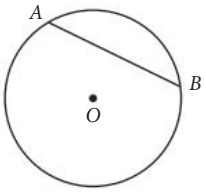


A **semicircle** is a half-circle, bounded by a diameter and the circumference of the circle.



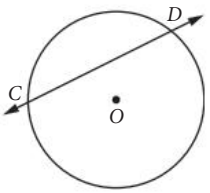
Semicircle

A **chord** is an interval joining two points on the circle.
AB is a chord.
A chord that passes through the centre is a diameter.



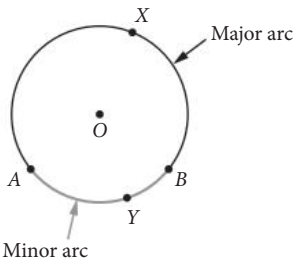
Chord

A **secant** is a straight line that cuts the circle at two points.
CD is a secant.

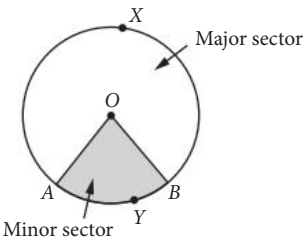


Secant

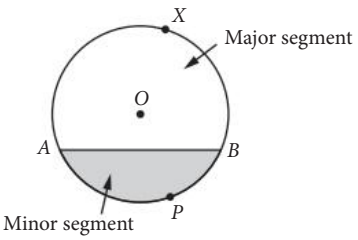
An **arc** is a part of the circumference of a circle.
AYB is a **minor arc** (less than half a circumference).
AXB is a **major arc** (more than half a circumference).



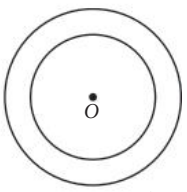
A **sector** is part of the plane bounded by two radii and an arc.
AYBO is a **minor sector** (less than half of the area of the circle).
AXBO is a **major sector** (more than half of the area of the circle).



A **segment** is part of the plane bounded by a chord and an arc.
APB is a **minor segment** (less than half of the area of the circle).
AXB is a **major segment** (more than half of the area of the circle).



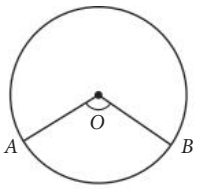
Concentric circles are circles that have the same centre but different radii.



Concentric circles

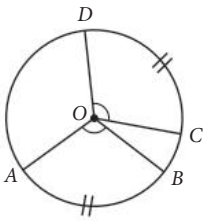
$\angle AOB$ is the angle that the arc AB **subtends** at the centre O of the circle. A and B are the endpoints of the arc.

Note: The verb *subtend* means 'to extend under', 'to be opposite to', or 'to be held by'.

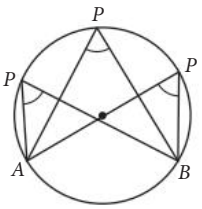


Angle at the centre

In a given circle (or in equal circles), equal arcs subtend equal angles at the centre. Conversely, equal angles at the centre stand on equal arcs.

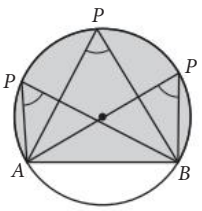


These angles $\angle APB$ are the angles that the arc AB subtends at the circumference of the circle, for various positions of P on the major arc APB.



Angles at the circumference

These angles $\angle APB$ are angles in the same segment for different positions of the point P. The angles at P are equal in each case.
(Notice that this diagram is the same as the previous one.)

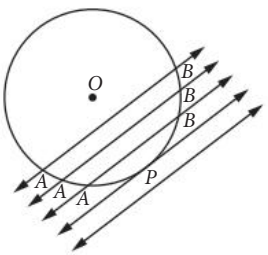


Angles in the same segment

This diagram shows a series of parallel secants AB. As they move further away from the centre of the circle, the two points A and B move closer together until they become the single point P.

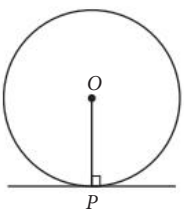
The secant in this position (P) is called a **tangent** to the circle at P. We say that the tangent **touches** the circle at P.

The parallel line beyond the circle is **not** a secant, nor a tangent.



A tangent to a circle is perpendicular to the radius at the point of contact: OP is perpendicular to the tangent at P.

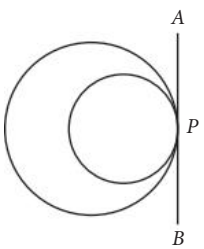
$OP \perp$ the tangent at P

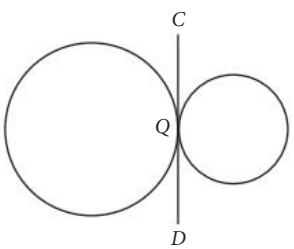


If two circles touch, then they have a common tangent at the point of contact.

In the left diagram, two circles touch internally at P and APB is the common tangent.

In the right diagram, two circles touch externally at Q and CQD is the common tangent.

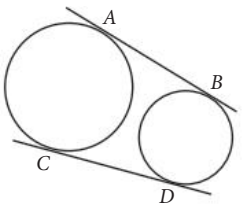


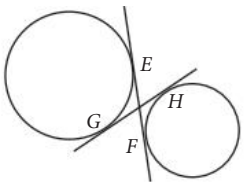


Two circles that do **not** touch will also have common tangents.

The left diagram shows the **direct** common tangents AB and CD.

The right diagram shows the **transverse** common tangents EF and GH.

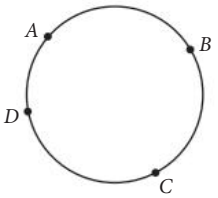




A circle can be drawn to pass through any three non-collinear points.

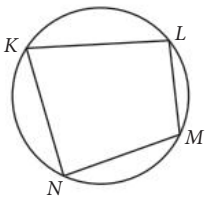
If a circle can be drawn through four points, then the points are called **concyclic points**.

Here A, B C and D are concyclic points.



If a circle can be drawn through the vertices of a quadrilateral, then the quadrilateral is called a **cyclic quadrilateral** and its vertices are concyclic points.

Here $KLMN$ is a cyclic quadrilateral.

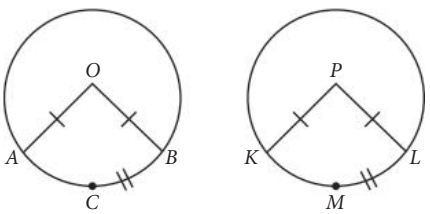


To help prove some theorems about circle geometry, we need to make two assumptions. We will not prove these assumptions here, but you should see that they seem to be true.

Assumption 1

Equal arcs on circles of equal radii subtend equal angles at the centre.

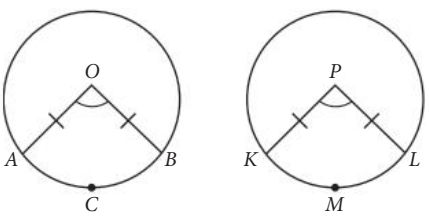
As $AO = KP$, the circles have equal radii, and as arc ACB = arc KML, then by our assumption $\angle AOB = \angle KPL$.



Assumption 2

If two arcs of circles of equal radii subtend equal angles at their centres, then the arcs are equal.

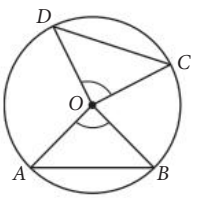
As $AO = KP$, the circles have equal radii, and as $\angle AOB = \angle KPL$, then by our assumption arc ACB = arc KML.



The proofs to the next five theorems do not have to be reproduced.

Theorem 1

Equal angles at the centre of a circle stand on equal chords.



Given: $\angle AOB = \angle DOC$

Aim: To prove that $AB = CD$.

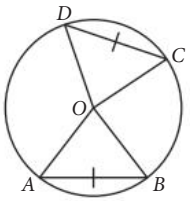
Proof: In ΔAOB and ΔCOD:

	$AO = CO$	(equal radii)
	$BO = DO$	(equal radii)
	$\angle AOB = \angle COD$	(given)
$\therefore$	$\Delta AOB \equiv \Delta COD$	(SAS)
$\therefore$	$AB = CD$	(matching sides of congruent triangles)

The converse of theorem 1 follows.

Theorem 2

Equal chords subtend equal angles at the centre of a circle.



Given: $AB = CD$

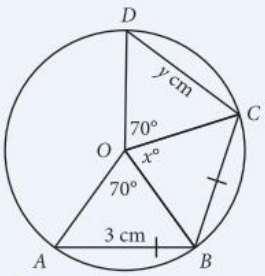
Aim: To prove that $\angle AOB = \angle DOC$.

Proof: In ΔAOB and ΔCOD:

	$AO = CO$	(equal radii)
	$BO = DO$	(equal radii)
	$AB = CD$	(given)
$\therefore$	$\Delta AOB \equiv \Delta COD$	(SSS)
$\therefore$	$\angle AOB = \angle COD$	(matching angles of congruent triangles)

Example 1

Calculate the value of each pronumeral in the diagram.
Give reasons for your answers.



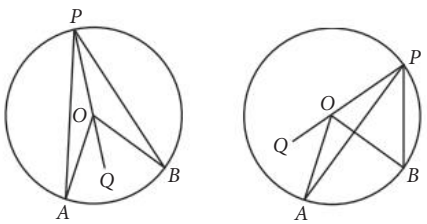
Solution

$AB = BC$, so $\angle AOB = \angle BOC$ (equal chords subtend equal angles at the centre)
$\therefore x = 70$

$\angle AOB = \angle COD$, so $AB = CD$ (equal angles at the centre stand on equal chords)
$\therefore y = 3$

Theorem 3

The angle at the centre of a circle is twice the angle at the circumference that is subtended by the same arc.



Given: AB is an arc of a circle with centre O, and P is any point on the circle that is not on the minor arc AB.

Aim: To prove that $\angle AOB = 2\angle APB$.

Construction: Join PO and produce it to Q.

Proof:	$AO = OP$	(equal radii)
$\therefore$	ΔAOP is isosceles	
$\therefore$	$\angle OAP = \angle OPA$	(base angles of an isosceles triangle are equal)
But	$\angle AOQ = \angle OAP + \angle OPA$	(exterior angle of triangle is sum of interior opposite angles)
$\therefore$	$\angle AOQ = 2\angle APO$	

Similarly, ΔBOP is isosceles so $\angle BOQ = 2\angle BPO$.

In the left diagram on previous page:

$\angle APB = \angle APO + \angle BPO$

and $\angle AOB = \angle AOQ + \angle QOB$

$\therefore \angle AOB = 2\angle APO + 2\angle BPO$

$\therefore \angle AOB = 2\angle APB$

In the right diagram on previous page:

$\angle APB = \angle BPO - \angle APO$

and $\angle AOB = \angle QOB - \angle AOQ$

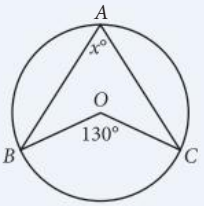
$\therefore \angle AOB = 2\angle BPO - 2\angle APO$

$\therefore \angle AOB = 2\angle APB$

As an exercise, you can prove this theorem when AOP is a diameter.

Example 2

Find the value of x, giving reasons for your answer.



Solution

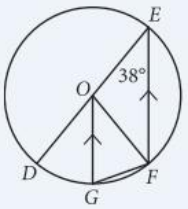
$\angle BOC = 2\angle BAC$ (angle at the centre is twice the angle at the circumference subtended by the same arc)

$\therefore \angle BAC = 65°$

Example 3

DOE is a diameter of the circle with centre O. $OG \parallel EF$. $\angle DEF = 38°$.

Find the size of $\angle OGF$, giving reasons for your answer.



Solution

$\angle DOG = \angle OEF$ (corresponding angles, $OG \parallel EF$)

$\angle DOF = 2\angle DEF$ (angle at centre is twice angle at circumference)

$\therefore \angle DOF = 76°$

But $\angle GOF = \angle DOF - \angle DOG$

$\therefore \angle GOF = 38°$

Now $OG = OF$ (equal radii)

$\therefore \Delta OGF$ is isosceles

$\angle OGF = \angle OFG$ (base angles of isosceles triangle)

$\therefore 2\angle OGF + 38° = 180°$ (angle sum of ΔOGF)

$\therefore 2\angle OGF = 142°$

$\angle OGF = 71°$

TECHNOLOGY EXPLORATION

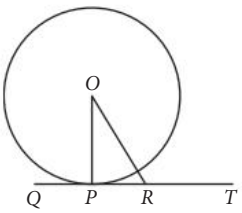
Angles in circles

1. In GeoGebra, draw a circle using the **Circle with Centre through Point** tool.
2. Use the **Interval between Two Points** tool to draw an angle at the centre and an angle at the circumference of the circle, as shown.
3. Use the **Angle** tool to measure the angle at the centre and the angle at the circumference. The measured angles will be displayed as α and β in the **Algebra** view at the left of the screen.

4. Create a new measurement γ by entering $\gamma=2^*\beta$ in the input bar at the bottom of the screen. (Note that you can enter Greek letters using the α symbol at the end of the input bar.)
5. Looking at the displayed values for α, β and γ, what conclusion can you make about the angle at the centre and the angle at the circumference of the circle?

Theorem 4

The tangent to a circle is perpendicular to the radius drawn to the point of contact.



Given: TPQ is a tangent to the circle with centre O, touching it at P.
OP is the radius drawn to the point of contact.

Aim: To prove that $TP \perp OP$.

Construction: Let R be any point on TQ. Join OR.

Proof: Every point on TQ except P is outside the circle.

$\therefore$ $OP < OR$
This is true for every point R on TQ.

$\therefore$ OP is the shortest line from O to TQ

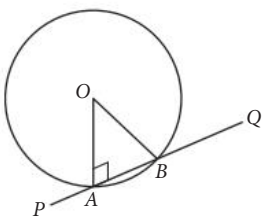
$\therefore$ $OP \perp TQ$

i.e. $TP \perp OP$

The converse of theorem 4 follows.

Theorem 5

A straight line drawn through a point on a circle, perpendicular to the radius at that point, is a tangent to the circle.



Given: OA is a radius of the circle with centre O.
PAQ is a straight line perpendicular to OA.

Aim: To prove that PAQ is a tangent to the circle at A.

Construction: Suppose that PAQ cuts the circle again at B (i.e. is *not* a tangent). Join OB.

Proof: $OA = OB$ (radii), so ΔAOB is isosceles.

$\therefore$ $\angle OBA = \angle OAB$ (base angles of isosceles triangle)

But $\angle OAB = 90°$ ($OA \perp PAQ$)

$\therefore$ $\angle OBA = 90°$

$\therefore$ The sum of the angles of ΔAOB is greater than 180°, which is impossible.

$\therefore$ PAQ cannot cut the circle again

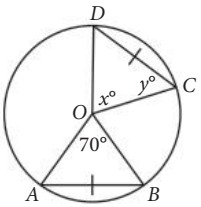
$\therefore$ PAQ is a tangent to the circle at A.

EXERCISE 1.1 SIMPLE ANGLE PROPERTIES OF A CIRCLE

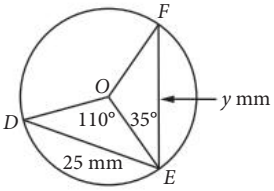
In all diagrams, O is the centre of the circle.

1 Calculate the value of the pronumerals. Give reasons for your answers.

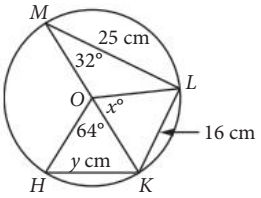
(a)



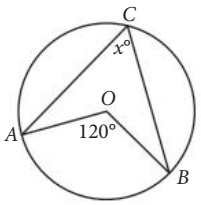
(b)



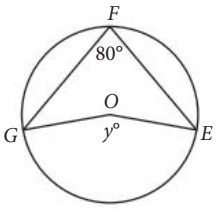
(c)



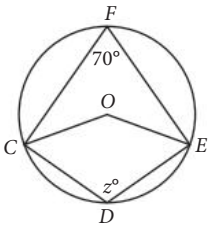
(d)



(e)

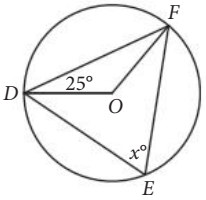


(f)



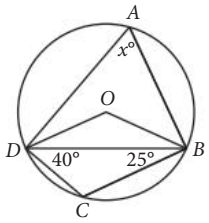
2 The value of x is:

A 25 **B** 50
C 65 **D** 130

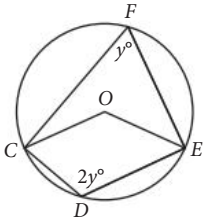


3 Calculate the value of the pronumerals. Give reasons for your answers.

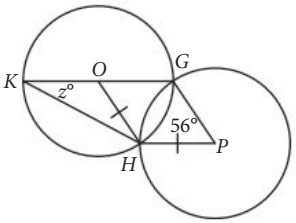
(a)



(b)

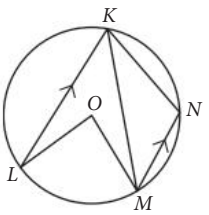


(c)



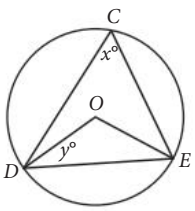
4 $LK \parallel MN$, $\angle KNM = 110°$, $\angle NKM = 45°$. Calculate the size of $\angle LOM$.
Indicate whether each statement about the diagram is correct or incorrect.

(a) $\angle KMN = 25°$ (angle sum of ΔKMN is 180°)
(b) $\angle LKM = 25°$ (alternate angles equal, $LK \parallel MN$)
(c) $\angle LKN = 70°$ (cointerior angles supplementary, $LK \parallel MN$)
(d) $\angle LOM = 50°$ (angle at centre is twice angle at circumference on same arc)

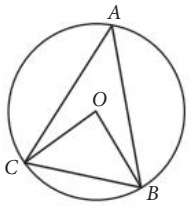


5 D, E and F are three points on the circumference of a circle such that $DE = EF$. If the minor arc DE subtends an angle of 40° at the centre of the circle, what angle does the minor arc DF subtend at the circumference?

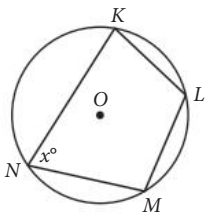
6 Prove that $x + y = 90°$.



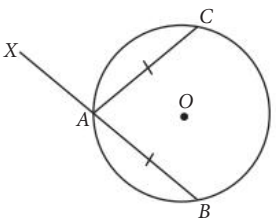
7 If $\angle BOC = 104°$ and $\angle ACB = 71°$, calculate the size of $\angle OAC$. Give reasons for your answer.



8 $KLMN$ is a quadrilateral inscribed in a circle with centre O. By joining KO and OM, prove that $\angle KLM + \angle KNM = 180°$.



9 If $AC = AB$ and O is the centre of the circle, prove that $\angle XAC = \angle AOC$.

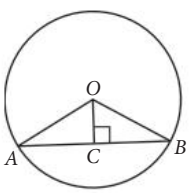


1.2 CHORD PROPERTIES 1

You must be able to prove theorems 6 to 21 below.

Theorem 6

The perpendicular from the centre of a circle to a chord bisects the chord.



Given: AB is a chord of a circle with centre O.
C is a point on AB such that $OC \perp AB$.

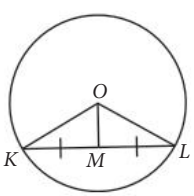
Aim: To prove that $AC = CB$.

Proof: In ΔAOC and ΔBOC:

$\angle OCA = \angle OCB = 90°$	($OC \perp AB$)
$AO = OB$	(equal radii)
OC is a common side	
$\therefore\ \Delta AOC \equiv \Delta BOC$	(RHS)
$\therefore\ AC = CB$	(corresponding sides of congruent triangles)

Theorem 7

The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.



Given: KL is a chord of a circle with centre O. M is the midpoint of KL.

Aim: To prove that $OM \perp KL$.

Construction: Join OK and OL.

Proof: In ΔOKM and ΔOLM:

$OK = OL$	(equal radii)
OM is a common side	
$MK = ML$	(M is midpoint of KL)
$\therefore\ \Delta OKM \equiv \Delta OLM$	(SSS)

$\therefore \quad \angle OMK = \angle OML$ (corresponding angles of congruent triangles)

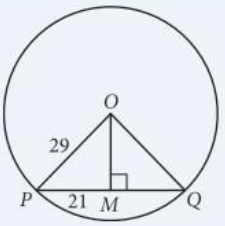
But $\angle OMK + \angle OML = 180°$ (KML is a straight line)

$\therefore \quad \angle OMK = \angle OML = 90°$

$\therefore \quad OM \perp KL$

Example 4

PQ is a chord of a circle with centre O. If $PQ = 42$ mm and $OP = 29$ mm, calculate the distance of PQ from O. Give reasons for your answer.



Solution

Let M be the foot of the perpendicular from O to PQ.

$\therefore \quad PM = MQ$ (line from centre of circle perpendicular to chord bisects chord)

$\therefore \quad PM = 21$ mm

But $OM^2 = 29^2 - 21^2$ (Pythagoras' theorem)

$OM^2 = 841 - 441$

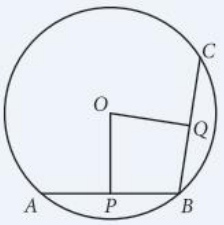
$= 400$

$\therefore \quad OM = 20$ mm

$\therefore \quad PQ$ is 20 mm from O.

Example 5

O is the centre of the circle ACB. P and Q are the midpoints of AB and BC respectively. Prove that $\angle POQ + \angle ABC = 180°$.



Solution

$OP \perp AB$ (P is the midpoint of AB)

$\therefore \quad \angle OPB = 90°$

Similarly $OQ \perp BC$

$\therefore \quad \angle OQB = 90°$

$\therefore \quad \angle OPB + \angle OQB = 180°$

But $\angle OPB + \angle PBQ + \angle OQB + \angle QOP = 360°$ (angle sum of a quadrilateral)

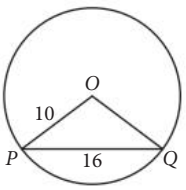
$\therefore \quad \angle PBQ + \angle QOP = 180°$

i.e. $\angle ABC + \angle POQ = 180°$

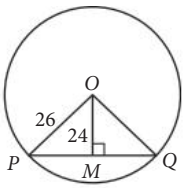
EXERCISE 1.2 CHORD PROPERTIES 1

In all diagrams, O is the centre of the circle.

1 If $PQ = 16$ mm and $OP = 10$ mm, calculate the distance from O to PQ.



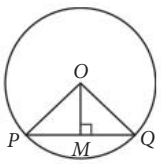
2 If PQ is 24 mm from the centre O of a circle with radius 26 mm, calculate the length of PQ.



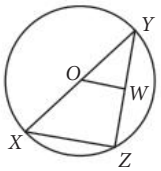
3 If PQ is 16 mm long and is 15 mm from the centre, then the radius is:

A $\sqrt{31}$ mm **B** $\sqrt{161}$ mm

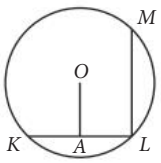
C 17 mm **D** $\sqrt{481}$ mm



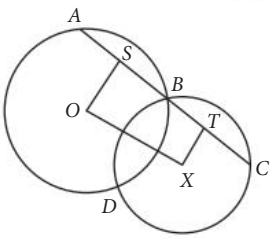
4 XY is a diameter of the circle that passes through X, Y and Z. W is the midpoint of YZ. Prove that $OW \parallel XZ$ and hence that $\angle XZY = 90°$.



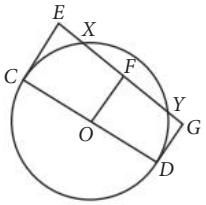
5 O is the centre of the circle KML. $KA = AL$ and $\angle KLM = 90°$. Prove that ML is twice OA.



6 ABC is a straight line. S and T are the midpoints of AB and BC respectively. O is the centre of the circle ABD and X is the centre of the circle BCD. Prove that $\angle SOX$ and $\angle OXT$ are supplementary angles.



7 CD is a diameter of the circle. CE, OF, DG are each perpendicular to the line $EXFYG$. Prove that $EX = YG$.

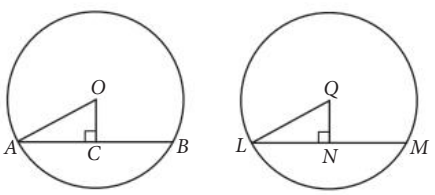


8 PAB is an isosceles triangle inscribed in a circle of radius 26 mm. If $PA = PB$ and $AB = 48$ mm, calculate the length of AP.

1.3 CHORD PROPERTIES 2

Theorem 8

Equal chords in equal circles are equidistant from the centres.



Given: AB and LM are equal chords in equal circles with centres O and Q respectively. $OC \perp AB$, $QN \perp LM$.

Aim: To prove that $OC = QN$.

Construction: Join OA and QL.

Proof: C is the midpoint of AB (perpendicular from the centre of a circle to a chord bisects the chord)

Similarly N is the midpoint of LM

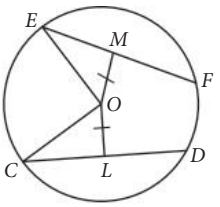
$\therefore$ $AC = CB = LN = NM$ ($AB = LM$, given)

In ΔAOC and ΔLQN:

	$AO = LQ$	(radii of equal circles)
	$AC = LN$	(proved above)
	$\angle OCA = \angle QNL = 90°$	($OC \perp AB$, $QN \perp LM$)
$\therefore$	$\Delta AOC \equiv \Delta LQN$	(RHS)
$\therefore$	$OC = QN$	(corresponding sides of congruent triangles)

Theorem 9

Chords in a circle that are equidistant from the centre are equal.



Given: In a circle with centre O, OL and OM are perpendiculars from O to the chords CD and EF respectively. $OL = OM$.

Aim: To prove that $CD = EF$.

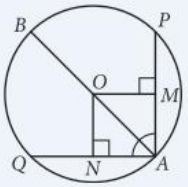
Construction: Join OC and OE.

Proof: In ΔOCL and ΔOEM:

	$\angle OLC = \angle OME = 90°$	($OL \perp CD$, $OM \perp EF$)
	$OC = OE$	(equal radii)
	$OL = OM$	(given)
$\therefore$	$\Delta OCL \equiv \Delta OEM$	(RHS)
$\therefore$	$CL = EM$	(corresponding sides of congruent triangles)
	L is the midpoint of CD	(perpendicular from the centre of a circle to a chord bisects the chord)
$\therefore$	$CD = 2CL$	
Similarly	M is the midpoint of EF	
$\therefore$	$EF = 2EM$	
$\therefore$	$CD = EF$	(as we have proved that $CL = EM$)

Example 6

AP and AQ are two chords of a circle with centre O. AB is a diameter. If $\angle PAB = \angle QAB$, prove that $AP = AQ$.



Solution

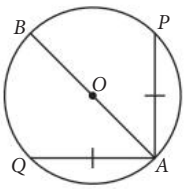
Let M and N be the midpoints of AP and AQ respectively. Join OM and ON.

	$OM \perp PA$	(line from centre of circle to midpoint of chord $\perp$ chord)
Similarly	$ON \perp AQ$	
	In ΔOAM and ΔOAN:	
	$\angle OMA = \angle ONA = 90°$	(proved above)
	$\angle OAM = \angle OAN$	(given)
$\therefore$	$\Delta OAM \equiv \Delta OAN$	(AAS)
$\therefore$	$AM = AN$	(corresponding sides of congruent triangles)
But	$AP = 2AM$	(M is midpoint of PA)
and	$AQ = 2AN$	(N is midpoint of QA)
$\therefore$	$AP = AQ$	(as $AM = AN$)

EXERCISE 1.3 CHORD PROPERTIES 2

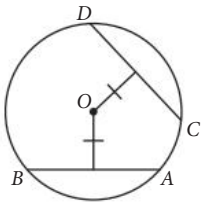
In all diagrams, O is the centre of the circle.

1 AP and AQ are two equal chords of a circle. Prove that the diameter bisects $\angle PAQ$.

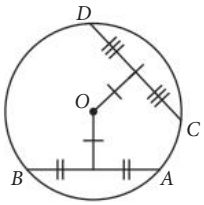


2 In which diagram is $AB = CD$?

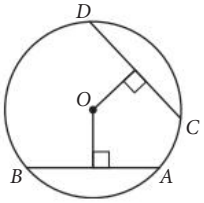
A



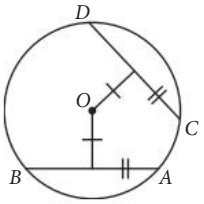
B



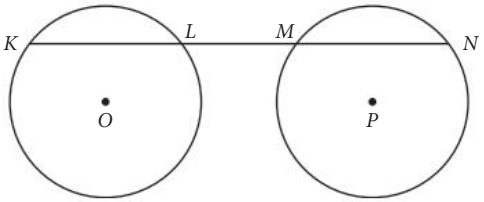
C



D

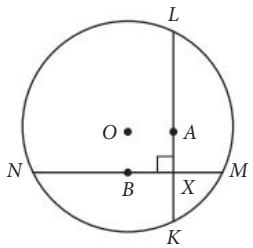


3 The circles with centres O and P have equal radii. If $KLMN$ is a straight line and $KL = MN$, prove that $KN \parallel OP$.

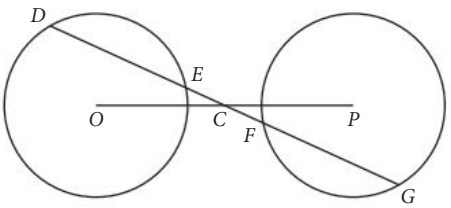


4 CD and XY are two equal chords of a circle that also cut a smaller concentric circle at E, F and V, W respectively. If O is the centre of the circles, prove that $EF = VW$.

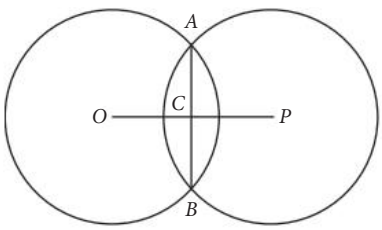
5 KL and MN are equal chords of a circle with centre O. A and B are the midpoints of KL and MN respectively. KL and MN intersect at X. If $KL \perp MN$, prove that $AOBX$ is a square.



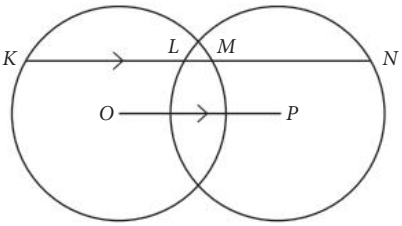
6 $DEFG$ is a straight line intersecting the equal circles centred at O and P. If $DE = FG$, prove that DG bisects OP.



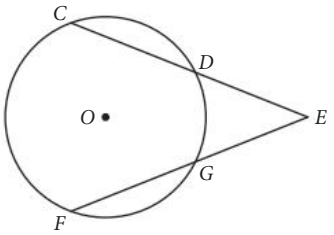
7 Two equal circles, with centres O and P, intersect at A and B. If AB cuts OP at C, prove that $OC = CP$.



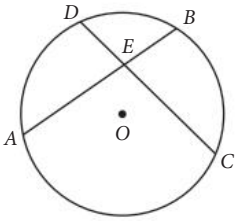
8 O and P are the centres of two equal intersecting circles. If $KLMN \parallel OP$, prove that $KL = MN$.



9 In the circle with centre O, $CD = GF$. Prove that $DE = GE$.

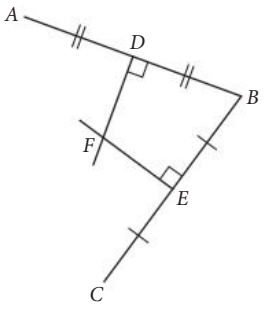


10 In the circle with centre O, CD and AB intersect at E. $DC = AB$. By drawing the perpendiculars from O to each chord, prove that $AE = EC$.



1.4 ANGLES IN CIRCLES

Theorem 10



Any three non-collinear points are on a unique circle whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.
(The 'point of concurrency' is the point where the lines meet).

Given: Three points A, B and C that are not collinear.

Aim: To prove that there is only one circle through A, B and C whose centre is the point of concurrency of the perpendicular bisectors of AB, BC and CA.

Construction: Join AB and BC. Draw the perpendicular bisectors of AB and BC, where D is the midpoint of AB and E the midpoint of BC.

Proof: AB and BC are not in the same straight line, so their perpendicular bisectors are not parallel and hence must intersect (at F in the diagram).

DF is the perpendicular bisector of AB, so all points on DF are equidistant from A and B.

$\therefore$ $AF = BF$

EF is the perpendicular bisector of BC, so all points on EF are equidistant from B and C.

$\therefore$ $BF = CF$

$\therefore$ F is the only point that is equidistant from A, B and C.

$\therefore$ The circle with centre F and radius FA passes through A, B and C, and there is no other circle which passes through A, B and C.

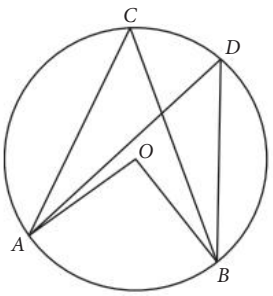
The perpendicular bisector of AC must pass through F.
(perpendicular bisector of a chord passes through the centre of the circle)

Hence there is only one circle through A, B and C whose centre is the point of concurrency of the perpendicular bisectors of AB, BC and CA.

Note: This circle is called the **circumcircle** of a triangle. Its centre is the point of intersection of the perpendicular bisectors of the sides of the triangle ABC.

Theorem 11

Angles in the same segment are equal.



Given: $\angle ACB$ and $\angle ADB$ are angles in the same segment, subtended by the chord AB of the circle with centre O.

Aim: To prove that $\angle ACB = \angle ADB$.

Construction: Join OA and OB.

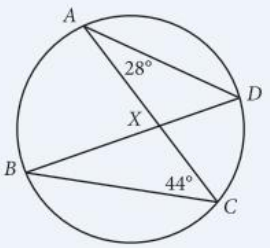
Proof: $\angle AOB = 2\angle ACB$ (angle at centre is twice angle at circumference)

Similarly $\angle AOB = 2\angle ADB$

$\therefore$ $\angle ACB = \angle ADB$ (each is half of $\angle AOB$)

Example 7

If $\angle CAD = 28°$ and $\angle ACB = 44°$, calculate the size of $\angle AXD$.
Give reasons for your answer.



Solution

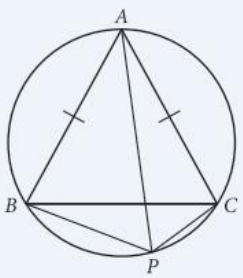
$\angle ADB = \angle ACB = 44°$ (angles in the same segment are equal)

$\therefore \angle AXD + 44° + 28° = 180°$ (angle sum of ΔAXD is 180°)

$\therefore \angle AXD = 108°$

Example 8

A, B and C are three points on the circumference of a circle such that chord AB = chord AC. P is any point on the circle such that A and P are on opposite sides of BC. Prove that $\angle APB = \angle APC$.



Solution

$\angle ABC = \angle ACB$ (base angles of isosceles triangle)

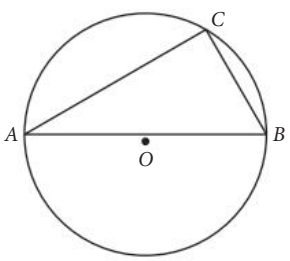
$\angle ABC = \angle APC$ (angles in the same segment)

$\angle ACB = \angle APB$ (angles in the same segment)

$\therefore \angle APB = \angle APC$ (from above)

Theorem 12

The angle in a semicircle is a right angle.



Given: AB is a diameter of the circle with centre O.
$\angle ACB$ is any angle in the semicircle ACB.

Aim: To prove that $\angle ACB = 90°$.

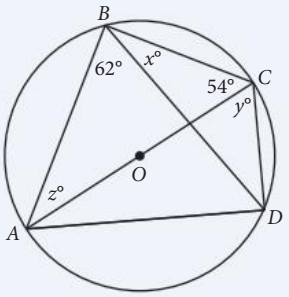
Proof: $\angle AOB = 2\angle ACB$ (angle at centre is twice angle at circumference)

But $\angle AOB = 180°$ (AOB is a straight line)

$\therefore$ $\angle ACB = 90°$

Example 9

AC is a diameter of the circle with centre *O*. $\angle ABD = 62°$, $\angle ACB = 54°$. Find the value of each pronumeral, giving reasons for your answers.



Solution

$\angle ABC = 90°$ (angle in a semicircle)

$\therefore\ x + 62 = 90$

$x = 28$

$\angle ACD = \angle ABD$ (angles in the same segment)

$\therefore\ y = 62$ (angle sum of ΔABC is 180°)

$z = 180 - (90 + 54)$

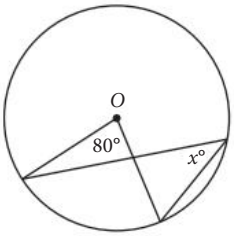
$\therefore\ z = 36$

EXERCISE 1.4 ANGLES IN CIRCLES

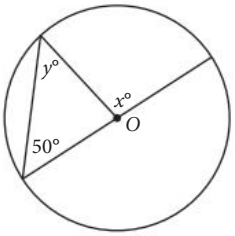
In all diagrams, *O* is the centre of the circle.

1 *O* is the centre of the circle. Find the value of the pronumerals, giving reasons for your answers.

(a)



(b)



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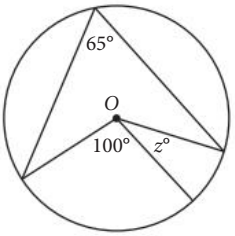

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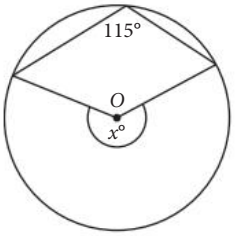

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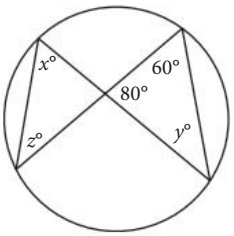

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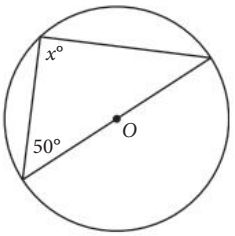

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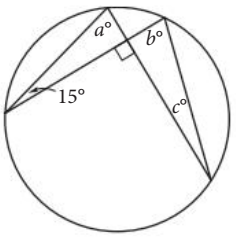

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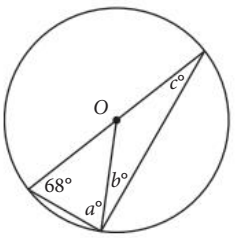
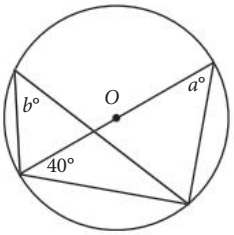


(i)



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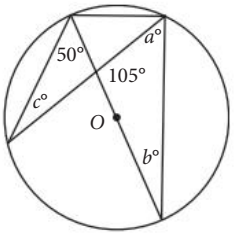

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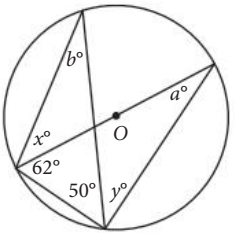

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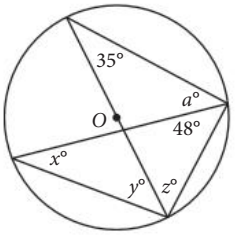

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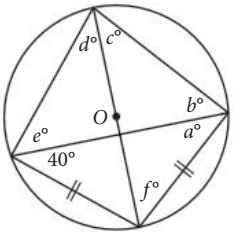

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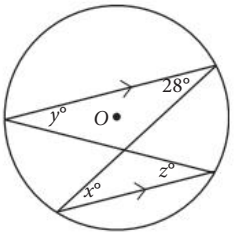

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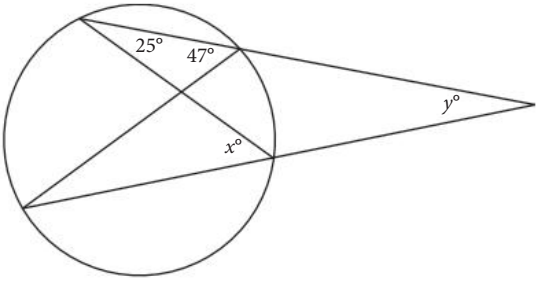
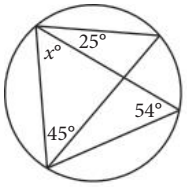


2 The value of x in the diagram is:

A 25 **B** 45
C 54 **D** 56

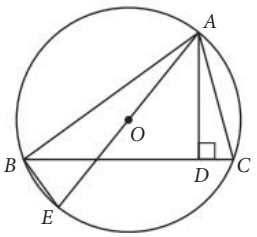


3 Two chords of a circle, AB and CD, intersect at a point E inside the circle. Prove that the triangles AEC and DEB are equiangular.

4 Two circles intersect at A and B. AX is a diameter of one circle and AY is a diameter of the other. Prove that the points X, B and Y are collinear.
(Construction: Join the common chord AB.)

5 ABC is a triangle inscribed in a circle with centre O. AD is perpendicular to BC.
State the reasons **(a)**, **(b)**, **(c)**, **(d)** for the steps below to prove that $\angle BAE = \angle DAC$.

Proof: $\angle ABE = 90°$ **(a)**
In ΔABE and ΔADC:
$\angle ABE = \angle ADC$ **(b)**
$\angle AEB = \angle ACD$ **(c)**
$\angle BAE = \angle DAC$ **(d)**

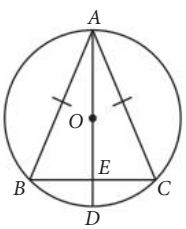


6 AB is a chord of a circle with centre O. The bisector of $\angle OAB$ meets the circle at D. Prove that $OD \parallel AB$.

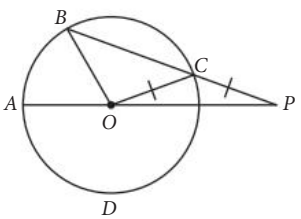
7 A, B, C, D are four points in order on a circle. Chords AC and BD intersect at E. If $AD \parallel BC$, prove that triangles AED and BEC are isosceles.

8 ABC is an isosceles triangle inscribed in a circle. AD is a diameter. State the reasons **(a)**, **(b)**, **(c)**, **(d)**, **(e)** for the steps below to prove that $\angle BED = 90°$.

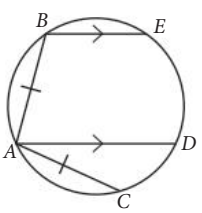
Proof: $\angle ABC = \angle ACB$ **(a)**
$\angle ACB = \angle ADB$ **(b)**
$\therefore\ \angle ADB = \angle ABC$
$\angle ABE + \angle EBD = \angle ABD$ **(c)**
$\therefore\ \angle EDB + \angle EBD = \angle ABD$
But $\angle ABD = 90°$ **(d)**
$\therefore\ \angle EDB + \angle EBD = 90°$
$\therefore\ \angle BED = 90°$ **(e)**



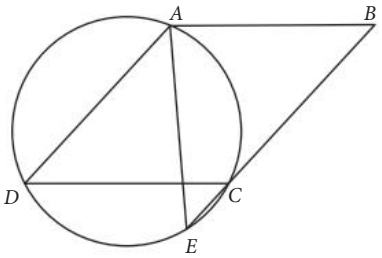
9 A chord BC and the diameter from A meet at a point P such that $OC = CP$. If $\angle CPO = a°$, prove that $\angle AOB = 3a°$.
(This is a method of trisecting an angle by construction.)



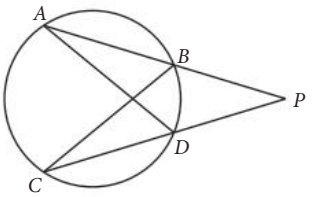
10 AB and AC are equal chords of a circle. AD and BE are parallel chords through A and B respectively. Prove that AE is parallel to CD.



11 $ABCD$ is a parallelogram. Prove that ΔABE is isosceles.

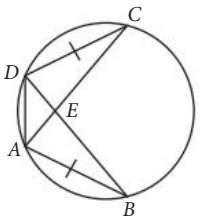


12 Two chords AB and CD are produced to intersect at point P outside the circle. Prove that $\Delta APD \,|||\, \Delta CPB$.



13 AB and DC are equal chords of a circle. Prove that:

(a) ΔCDE and ΔBAE are congruent
(b) ΔEDA is isosceles
(c) ΔADC and ΔDAB are congruent
(d) AC and BD are equal chords.



14 AB is a chord of a circle with centre O. The circle with AO as diameter cuts AB at C. Prove that C is the midpoint of AB.

15 In ΔAPB, the side AB remains fixed and the vertex P moves in a plane. If $\angle APB$ is always the same size, what can you say about the points A, P and B?

16 In ΔAPB, $\angle APB = 90°$. What can you say about the points A, P and B?

17 A, B, C and D are four points in order on a circle with centre O. If $\angle BAC = 28°$ and $\angle OBA = 53°$, calculate the size of $\angle OAC$, $\angle ACB$ and $\angle ADC$.

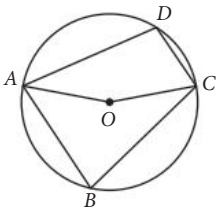
18 Two perpendicular chords of a circle, AB and CD, intersect at E. Let $\angle CBE = x°$, $\angle OBA = y°$, $\angle ODE = z°$.

(a) Prove that $\angle BOC = 180° - 2(x + y)°$.
(b) Prove that $\angle AOD = 180° - 2(x + z)°$.
(c) Hence prove that $\angle AOD + \angle BOC = 180°$.

1.5 CYCLIC QUADRILATERALS

Theorem 13

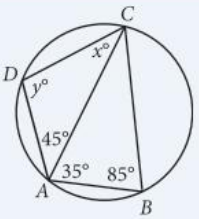
The opposite angles of a cyclic quadrilateral are supplementary.



Given:	$ABCD$ is a cyclic quadrilateral.	
Aim:	To prove that: (i) $\angle ABC + \angle ADC = 180°$ (ii) $\angle BAD + \angle BCD = 180°$	
Construction:	Join OA and OC.	
Proof:	$\angle AOC = 2\angle ABC$	(angle at centre is twice angle at circumference)
Similarly	reflex-$\angle AOC = 2\angle ADC$	
But	$\angle AOC$ + reflex-$\angle AOC = 360°$	(angles at a point)
$\therefore$	$2\angle ABC + 2\angle ADC = 360°$	
	$\angle ABC + \angle ADC = 180°$	(i)
Now	$\angle ABC + \angle BCD + \angle ADC + \angle BAD = 360°$	(angle sum of a quadrilateral)
$\therefore$	$\angle BAD + \angle BCD = 180°$	(ii)

Example 10

If $\angle DAC = 45°$, $\angle BAC = 35°$ and $\angle ABC = 85°$, find the values of x and y.



Solution

$\angle DAB = 45° + 35° = 80°$

$\angle DCB = 100°$ (opposite angles of a cyclic quadrilateral are supplementary)

$\angle ACB = 60°$ (sum of angles of ΔABC is 180°)

$\therefore \angle DCA = \angle DCB - \angle ACB$
$= 100° - 60°$

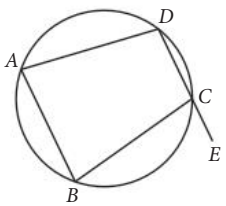
$\therefore x = 40$

$y = 95$ (opposite angles B and D of a cyclic quadrilateral are supplementary)

Note: This problem can be solved in fewer steps. Can you see how?

Theorem 14

The exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.



Given: $ABCD$ is a cyclic quadrilateral. DC is produced to E.

Aim: To prove that $\angle BCE = \angle BAD$.

Proof: Let $\angle BAD = x°$

$\therefore \angle BCD = (180 - x)°$ (opposite angles of a cyclic quadrilateral are supplementary)

and $\angle BAD + \angle BCE = 180°$ (DCE is a straight line)

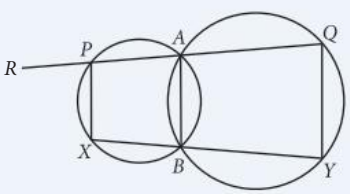
$\therefore (180 - x)° + \angle BCE = 180°$

$\therefore \angle BCE = x°$

$\therefore \angle BCE = \angle BAD$ (both angles are $x°$)

Example 11

Two circles intersect at A and B. PAQ and XBY are straight lines through A and B that cut the circles at P, Q and X, Y respectively. Prove that $PX \parallel QY$.



Solution

Construction: Join AB. Produce AP to R.

Proof: $PABX$ is a cyclic quadrilateral

$\therefore \angle RPX = \angle ABX$ (exterior angle equals interior opposite angle)

$QABY$ is a cyclic quadrilateral

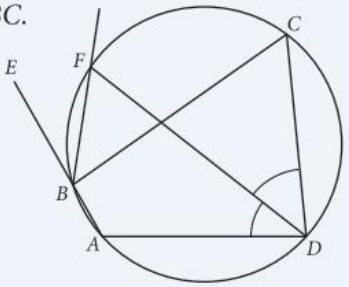
$\therefore \angle ABX = \angle AQY$ (exterior angle equals interior opposite angle)

$\therefore \angle RPX = \angle AQY$

$\therefore PX \parallel QY$ (corresponding angles equal)

Example 12

ABCD is a cyclic quadrilateral and *AB* is produced to *E*. The bisector of $\angle ADC$ cuts the circle around *ABCD* at *F*. Prove that *BF* (produced, if necessary) bisects $\angle EBC$.



Solution

Proof: $\angle FBC = \angle CDF$ (angles in the same segment are equal)

$\angle EBF = \angle ADF$ (exterior angle of cyclic quadrilateral *FBAD* equals interior opposite angle)

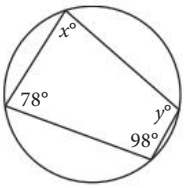
But $\angle ADF = \angle CDF$ (*FD* bisects $\angle ADC$, given)

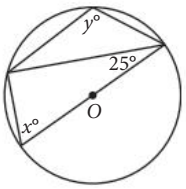
$\therefore$ $\angle FBC = \angle EBF$

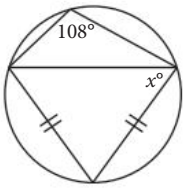
EXERCISE 1.5 CYCLIC QUADRILATERALS

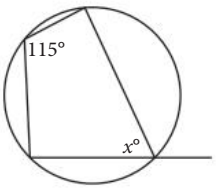
In all diagrams, *O* is the centre of the circle.

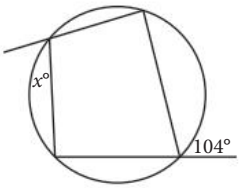
1 Find the value of each pronumeral, giving reasons for your answers.

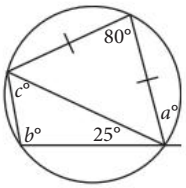
(a) 

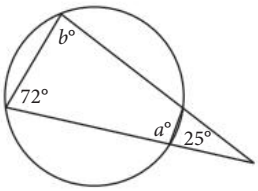
(b) 

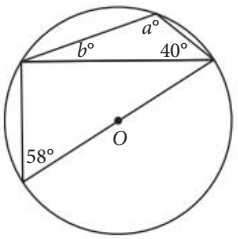
(c) 

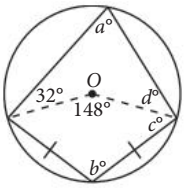
(d) 

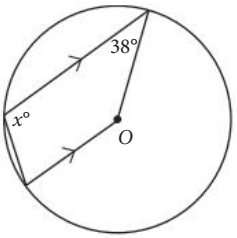
(e) 

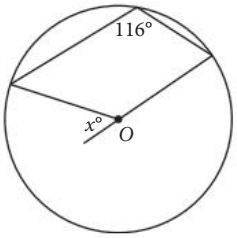
(f) 

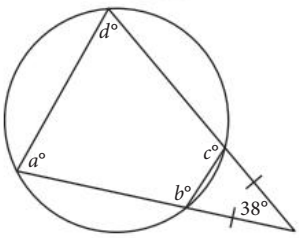
(g) 

(h) 

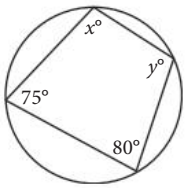
(i) 

(j) 

(k) 

(l) 

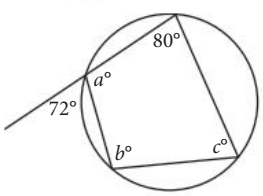
2 Which statement about the diagram is correct?



A $x = 80, y = 75$ **B** $x = 75, y = 80$

C $x = 100, y = 105$ **D** $x = 105, y = 100$

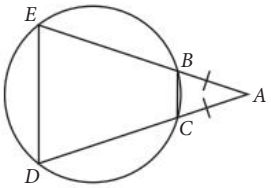
3 Indicate whether each statement is correct or incorrect.



(a) $a = 108$ (b) $b = 100$

(c) $c = 108$ (d) $c = 72$

4 State the reasons **(a)**, **(b)**, **(c)**, **(d)** for the steps of the following proof.



Given: $AB = AC$

Aim: To prove that $BC \parallel ED$.

Proof: $\angle ABC = \angle ACB$ **(a)**

$\angle ABC + \angle EBC = 180°$ **(b)**

$\angle EBC + \angle EDC = 180°$ **(c)**

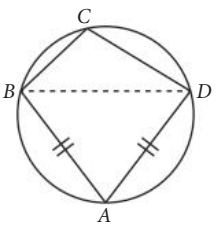
$\therefore \angle ABC = \angle EDC$

$\therefore \angle ACB = \angle ADE$

$\therefore BC \parallel ED$ **(d)**

5 Two circles intersect at A and B. PAQ and XBY are parallel straight lines through A and B that cut the circles at P, Q and X, Y respectively. Join AB and prove that $\angle PXY = \angle PQY$.

6 $ABCD$ is a cyclic quadrilateral and $AB = AD$. If $\angle ABD = x°$, prove that $\angle DCB = 2x°$.



7 $ABCD$ is a cyclic quadrilateral in which $AD = DC = CB$. Prove that:

(a) $\angle ADC = \angle DCB$ **(b)** $DC \parallel AB$

8 C is any point on a circle with diameter AB. P and Q are points on the minor arcs AC and BC. Prove that $\angle APC + \angle CQB = 3$ right angles.

9 $ABCD$ is a cyclic quadrilateral and $AB \parallel DC$. Prove that $\angle DAB = \angle ABC$.

10 A, B, C and D are four points in order on a circle. If AB is a diameter and $\angle ADC = \angle BCD$, prove that:

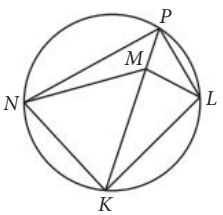
(a) $DC \parallel AB$ **(b)** $\angle DBA = \angle CAB$

11 A, B and C are three points on a circle and ΔABC is acute-angled. AD is parallel to BC. CD is parallel to BA. Prove that D cannot be on the circle. (D is the point of intersection of the two parallels.)

1.6 TESTS FOR CONCYCLIC POINTS

Theorem 15

If the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic. (This is also a test for four points to be concyclic.)



Given: $KLMN$ is a quadrilateral in which $\angle KLM + \angle KNM = 180°$.

Aim: To prove that $KLMN$ is a cyclic quadrilateral.

Construction: Join KM, which is between KL and KN. Draw the circle passing through K, L and N. Suppose this circle does not pass through M. Thus it must cut KM (or KM produced at P). Join LP and NP.

Proof: By construction, $KLPN$ is a cyclic quadrilateral.

$\therefore \angle KLP + \angle KNP = 180°$ (opposite angles of cyclic quadrilateral are supplementary)

But $\angle KLM + \angle KNM = 180°$ (given)

$\therefore \angle KLP + \angle KNP = \angle KLM + \angle KNM$ (each add to 180°)

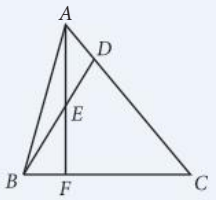
But this is impossible, because the angles KLP and KNP are each respectively greater than the angles KLM and KNM.

$\therefore$ The circle passing through K, L and N must also pass through M.

$\therefore$ $KLMN$ is a cyclic quadrilateral.

Example 13

In ΔABC, $AF \perp BC$ and $BD \perp AC$. Prove that $CDEF$ is a cyclic quadrilateral.



Solution

$\angle AFC = 90°$ $(AF \perp BC)$

$\angle BDC = 90°$ $(BD \perp AC)$

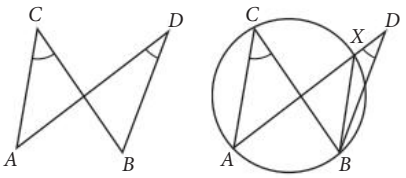
$\therefore \angle AFC + \angle BDC = 180°$

But $\angle AFC$ and $\angle BDC$ are opposite angles of the quadrilateral $CDEF$.

$\therefore$ $CDEF$ is a cyclic quadrilateral (quadrilateral with pair of opposite angles supplementary)

Theorem 16

If an interval subtends equal angles at two points on the same side of the interval, then the two points and the endpoints of the interval are concyclic. (This is also a test for a cyclic quadrilateral.)



Given: C and D are points on the same side of AB such that $\angle ACB = \angle ADB$.

Aim: To prove that A, B, D and C are concyclic.

Construction: Draw the circle passing through A, B and C. Let this circle cut AD (or AD produced at X). Join BX.

Proof: By construction, A, C, X and B are concyclic.

$\therefore$ $\angle ACB = \angle AXB$ (angles in the same segment)

But $\angle ACB = \angle ADB$ (given)

$\therefore$ $\angle AXB = \angle ADB$ (both equal $\angle ACB$)

In ΔBXD: $\angle AXB = \angle XDB + \angle DBX$ (exterior angle of a triangle equals the interior opposite angle)

$\therefore$ $\angle AXB = \angle ADB + \angle DBX$ $(\angle ADB = \angle XDB)$

This result is possible only if $\angle DBX = 0°$, because we have already shown that $\angle AXB = \angle ADB$.

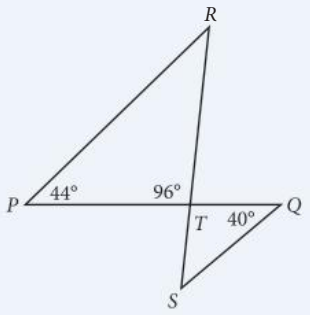
$\therefore$ D and X must be the same point

$\therefore$ D is on the same circle as X

$\therefore$ A, B, D and C are concyclic.

Example 14

PQ and RS are two lines that intersect at T. If $\angle RPT = 44°$, $\angle RTP = 96°$ and $\angle TQS = 40°$, prove that the points P, R, Q and S are concyclic.



Solution

$\angle PRT = 40°$ (angle sum of ΔPRT is 180°)

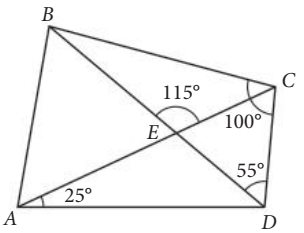
$\therefore \angle PRS = \angle PQS$ (both 40°)

But these are a pair of equal angles on the same side of PS:

$\therefore$ P, R, Q and S are concyclic.

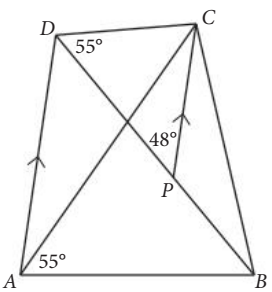
EXERCISE 1.6 TESTS FOR CONCYCLIC POINTS

1 $ABCD$ is a quadrilateral and its diagonals intersect at E. $\angle EAD = 25°$, $\angle EDC = 55°$, $\angle BEC = 115°$, $\angle BCD = 100°$. Calculate the size of $\angle ABD$, giving reasons for your answer.



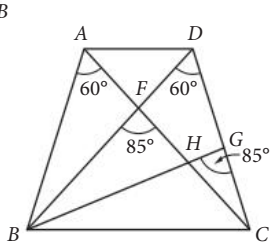
2 $ABCD$ is a quadrilateral. $\angle BAC = \angle BDC = 55°$, $\angle DPC = 48°$ and $AD \parallel PC$. The size of $\angle ACB$ is:

A 48°
B 55°
C 77°
D 103°



3 For the given diagram, indicate whether each statement is correct or incorrect.

(a) A, D, C and B are concyclic
(b) A, D, G and F are concyclic
(c) B, F, G and C are concyclic
(d) A, D, H and B are concyclic

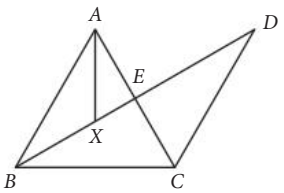


4 Draw a quadrilateral $PQRS$, joining PR and QS. If $\angle PQR = 70°$, $\angle PRQ = 35°$ and $\angle QSR = 75°$, prove that P, Q, R and S are concyclic and find the size of $\angle PSQ$.

5 $ABCD$ is a quadrilateral. $AB = AD$, $DB = DC$. If $\angle DBA = x°$ and $\angle DBC = 2x°$, prove that $ABCD$ is a cyclic quadrilateral.

6 LMN and LMP are two right-angled triangles on the same side of their shared hypotenuse LM. Prove that L, M, N and P are concyclic.

7 ABC is an equilateral triangle. AX and BX bisect $\angle BAC$ and $\angle ABC$ respectively, and $BC = CD$. Find the size of all the angles in the figure and write four concyclic points, giving reasons for your answers.



8 ABC is an isosceles triangle with $AB = AC$. P and Q are points on AB and AC respectively. PQ is parallel to BC. Prove that the quadrilateral $PBCQ$ is cyclic.

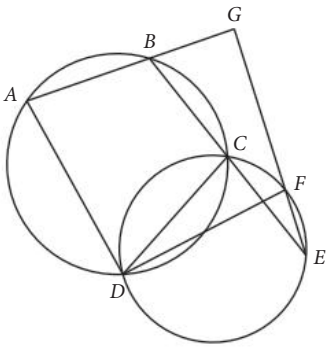
9 Prove that a parallelogram that is cyclic must be a rectangle.

10 ABC is an isosceles triangle with $AB = AC$. A straight line through C perpendicular to AC meets the bisector of $\angle BAC$ at D. Prove that the points A, B, D and C are concyclic.

11 AB and AC are two chords of a circle, where B and C are on opposite sides of the radius OA. P and Q are the midpoints of AB and AC respectively. Prove that A, P, O and Q are concyclic.

12 $ABCD$ is a rectangle. The line through C perpendicular to AC meets AB and AD produced at P and Q respectively. Prove that P, B, D and Q are concyclic.

13 Two circles intersect at C and D. $ABCD$ is a cyclic quadrilateral in one circle. BC produced meets the other circle at E. The points C, F, E and D are concyclic. AB produced meets EF produced at G. Prove that $GFDA$ is a cyclic quadrilateral.

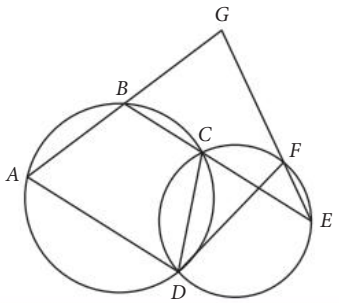


14 ABC and DBC are two isosceles triangles on the same base BC, such that D is within ΔABC. BD is produced to meet AC at E. CD is produced to meet AB at F. Prove that:

(a) triangles BFD and EDC are equiangular

(b) $BFEC$ is a cyclic quadrilateral.

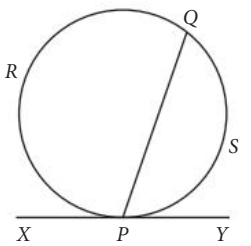
15 Two circles intersect at C and D. $ABCD$ is a cyclic quadrilateral in one circle. BC produced meets the other circle at E. The points C, F, E and D are concyclic. AB produced meets EF produced at G. Prove that $GFDA$ is a cyclic quadrilateral.



1.7 ANGLES IN ALTERNATE SEGMENTS

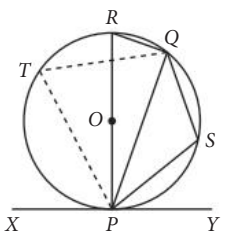
XY is a tangent to the circle at P. The chord PQ divides the circle into two segments, PQR and PQS.

- The segment PQR is said to be **alternate** to $\angle QPY$.
  $\angle PRQ$ is called the angle in the alternate segment corresponding to $\angle QPY$.
- The segment PQS is said to be **alternate** to $\angle QPX$.
  $\angle PSQ$ is called the angle in the alternate segment corresponding to $\angle QPX$.



Theorem 17

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.



Given: XY is a tangent at P to the circle with centre O. PQ is a chord through P.

Aim: To prove that: **(a)** $\angle QPY$ = any angle in the alternate segment PTQ
(b) $\angle QPX$ = any angle in the alternate segment PSQ.

Construction: Draw the diameter PR and join RQ.

(a) Proof: $\angle QPY + \angle QPR = 90°$ (tangent $\perp$ radius drawn to point of contact)

and $\angle PQR = 90°$ (angle in a semicircle)

$\therefore\ \angle PRQ + \angle QPR = 90°$ (angle sum of ΔPQR)

$\therefore\ \angle QPY = \angle PRQ$

But $\angle PRQ = \angle PTQ$ (angles in the same segment)

$\therefore\ \angle QPY = \angle PTQ$, any angle in the alternate segment

(b) Proof: $\angle QPX + \angle QPY = 180°$ (straight angle)

$\therefore\ \angle QPX = 180° - \angle QPY$

and $\angle PTQ + \angle PSQ = 180°$ (opposite angles of a cyclic quadrilateral are supplementary)

$\therefore\ \angle PSQ = 180° - \angle PTQ$

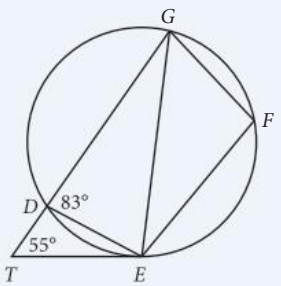
But $\angle PTQ = \angle QPY$ (proved in **(a)**)

$\therefore\ \angle PSQ = 180° - \angle QPY$

$\therefore\ \angle QPX = \angle PSQ$, any angle in the alternate segment

Example 15

TE is a tangent to the circle. $\angle DTE = 55°$, $\angle GDE = 83°$. Find the size of $\angle DGE$, giving reasons for your answer.



Solution

$\angle TED + 55° = 83°$ (exterior angle of triangle is sum of interior opposite angles)

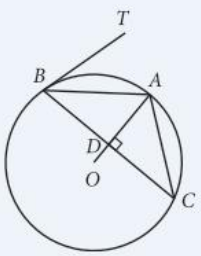
$\therefore\ \angle TED = 28°$

But $\angle TED = \angle DGE$ (angle in the alternate segment)

$\therefore\ \angle DGE = 28°$.

Example 16

TB is a tangent to the circle with centre *O*. *BC* is a chord of the circle. *OA* is the radius perpendicular to *BC* that intersects *BC* at *D*. Prove that *AB* bisects $\angle TBC$.



Solution

Proof: In ΔABD and ΔACD:

$BD = DC$ (radius perpendicular to chord bisects chord)

$\angle BDA = \angle ADC$ ($OA \perp BC$)

DA is a common side

$\therefore\ \Delta ABD \equiv \Delta ACD$ (SAS)

$\therefore\ \angle CBA = \angle BCA$ (corresponding angles of congruent triangles)

But $\angle TBA = \angle BCA$ (angle in the alternate segment)

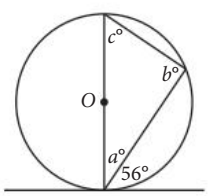
$\therefore\ \angle TBA = \angle CBA$

$\therefore$ *AB* bisects $\angle TBC$.

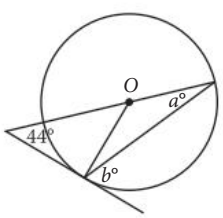
EXERCISE 1.7 ANGLES IN ALTERNATE SEGMENTS

1 Find the values of the pronumerals if *O* is the centre of the circle. Give reasons for your answers.

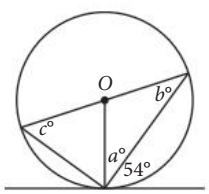
(a)



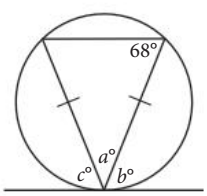
(b)

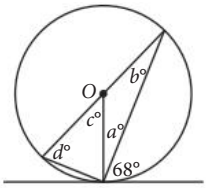


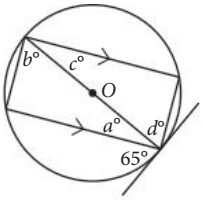
(c)

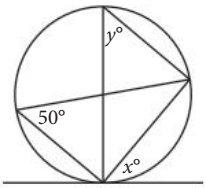


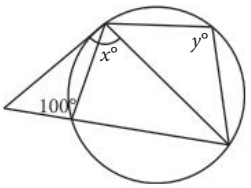
(d)



(e) 

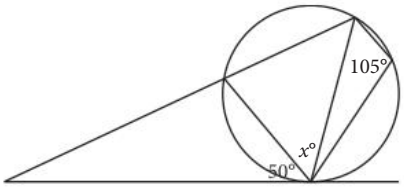
(f) 

(g) 

(h) 

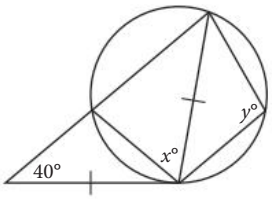
2 The value of x is:

A 50 **B** 55 **C** 80 **D** 105



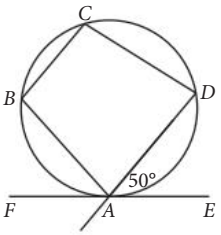
3 For the given diagram, indicate whether each statement is correct or incorrect.

(a) $x = 40$ **(b)** $x = 60$ **(c)** $y = 80$ **(d)** $y = 100$



4 $ABCD$ is a cyclic quadrilateral and FAE is a tangent at A. $\angle DAE = 50°$. If $BD \parallel FE$, calculate the size of:

(a) $\angle BAF$ **(b)** $\angle BCD$

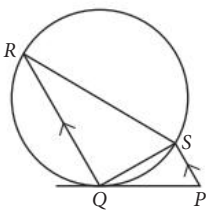


5 From a point P outside a circle, two straight lines are drawn. The first line PAB cuts the circle at A and B. The second line PTX is a tangent to the circle at T. AT and BT are joined. Prove that $\angle BTX = \angle APT + \angle ATP$.

6 Two points A and B are on a circle. C is the other end of the diameter through A. If AE is the perpendicular from A to the tangent at B, prove that AB bisects $\angle CAE$.

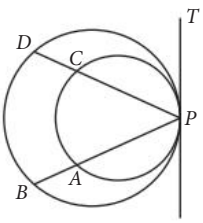
7 Two circles intersect at A and B. The tangent to the first circle at A cuts the second circle at C. The tangent to the second circle at A cuts the first circle at D. Prove that ΔABC and ΔDBA are similar triangles. If D, B and C are collinear, what can you say about AD and AC?

8 PQ is a tangent at Q and $PS \parallel QR$. Prove that $\angle SPQ = \angle QSR$.



9 Two circles intersect at A and B. The tangent to the second circle at A cuts the first circle at C and the tangent to the first circle at B cuts the second circle at D. Prove that $AD \parallel CB$.

10 Two circles touch internally at P. The lines PAB and PCD cut the smaller circle at A and C, and cut the larger circle at B and D. By drawing a common tangent at P, prove that $AC \parallel BD$.

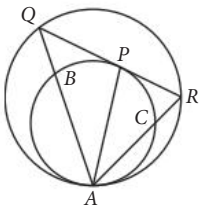


11 Two circles touch internally at A. The tangent at P on the smaller circle cuts the larger circle at Q and R. QA cuts the smaller circle at B. RA cuts the smaller circle at C. $\angle RPC = y°$.

(a) Let the tangent at A meet QR produced at T. Join BP and PC. Let $\angle TAR = x°$. Find the size of $\angle AQR$ and $\angle APC$.

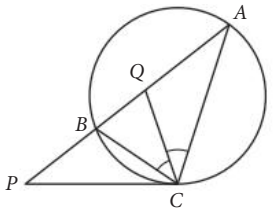
(b) Let $\angle RPC = y°$. Find the size of $\angle PAC$.

(c) Prove that AP bisects $\angle RAQ$.



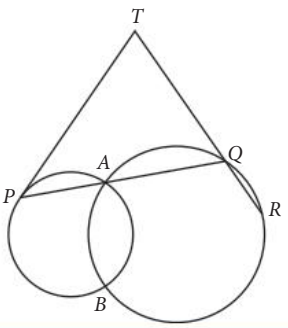
12 Two circles intersect at X and Y. The tangent to the first circle at X cuts the second circle at A. AY produced cuts the first circle at B. Prove that XB is parallel to the tangent to the second circle at A.

13 ABC is a triangle inscribed in a circle. The tangent at C meets AB produced at P. The bisector of $\angle ACB$ meets AB at Q. Prove that $PC = PQ$.



14 AB and CD are parallel chords of a circle. The tangent at B meets CD produced at E. Prove that ΔABC and ΔDBE are similar.

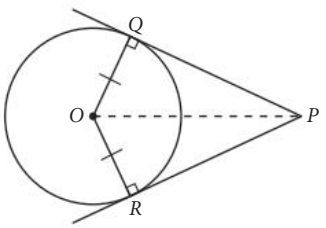
15 Two circles intersect at A and B. A line through A cuts the first circle at P and the second circle at Q. From an external point T, a tangent TP is drawn. TQ produced meets the second circle again at R. Prove that the points P, T, R and B are concyclic.



1.8 TANGENTS FROM AN EXTERNAL POINT

Theorem 18

Tangents to a circle from an external point are equal.



Given: PQ and PR are tangents to the circle with centre O.

Aim: To prove that $PQ = PR$.

Construction: Join OQ, OP and OR.

Proof: In ΔPOQ and ΔPOR:

	$OQ = OR$	(equal radii)
	$OP = OP$	(common side)
	$\angle OQP = \angle ORP$	(tangent $\perp$ radius drawn to point of contact)
$\therefore$	$\Delta POQ \equiv \Delta POR$	(RHS)
$\therefore$	$PQ = PR$	(corresponding sides of congruent triangles)
Also:	$\angle OPQ = \angle OPR$	(matching angles in congruent triangles)

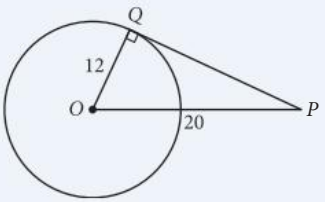
Thus the tangents are equally inclined to the line joining the external point to the centre of the circle.

Also: $\angle POQ = \angle POR$ (matching angles in congruent triangles)

Thus the tangents from an external point subtend equal angles at the centre of the circle.

Example 17

From a point P, 20 cm from the centre of a circle of radius 12 cm, a tangent is drawn. Calculate the length of the tangent to the circle.



Solution

$\angle OQP = 90°$ (tangent $\perp$ radius drawn to point of contact)

$OP^2 = OQ^2 + QP^2$ (Pythagoras' theorem)

$400 = 144 + PQ^2$

$PQ^2 = 256$

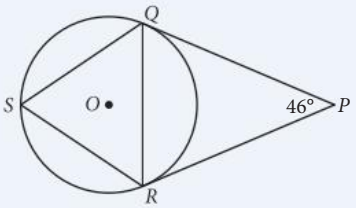
$PQ = \sqrt{256}$

$= 16$

The length of the tangent is 16 cm.

Example 18

PQ and PR are tangents to the circle with centre O. If $\angle QPR = 46°$, calculate the size of $\angle QSR$.



Solution

$PQ = PR$ (tangents to a circle from an external point are equal)

$\therefore$ ΔPQR is isosceles

$\therefore$ $\angle PQR = \angle PRQ$ (base angles of isosceles triangle)

and $\angle PQR + \angle PRQ = 180° - 46°$ (angle sum of ΔPQR is 180°)

$\therefore$ $2\angle PQR = 134°$

$\angle PQR = 67°$

But $\angle QSR = \angle PQR$ (angle in the alternate segment)

$\therefore$ $\angle QSR = 67°$

TECHNOLOGY EXPLORATION

Tangents from an external point

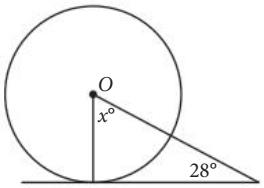
1. In GeoGebra, draw a circle using the **Circle with Centre through Point** tool.
2. Use the **Tangents** tool to draw tangents from an external point. (Click to select an external point, then click on the circle.)
3. Use the **Intersect Two Objects** tool to locate the points of intersection of the tangents and the circle.
4. Use the **Distance or Length** tool to measure the length of each interval between the circle and the external point.
5. Now use the **Move** tool to move the external point. What can you conclude about the lengths of the tangents?

EXERCISE 1.8 TANGENTS FROM AN EXTERNAL POINT

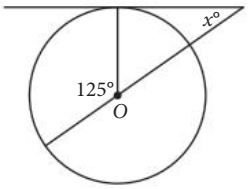
In all diagrams, O is the centre of the circle.

1 Find the value of each pronumeral. Give reasons for your answers.

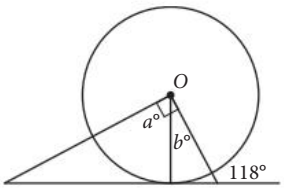
(a)



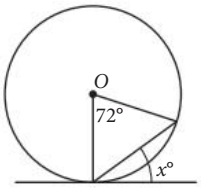
(b)



(c)



(d)



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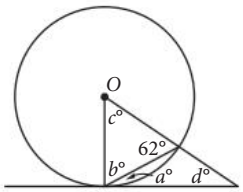

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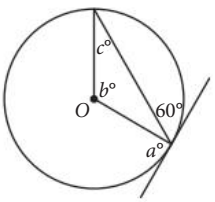
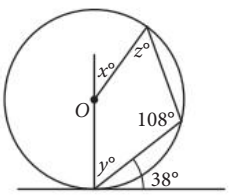
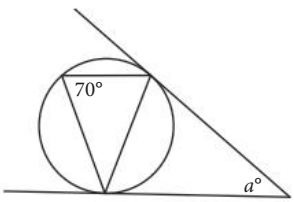


(g)

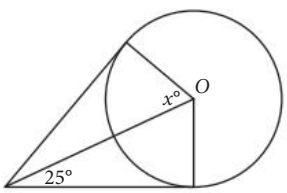


(h)

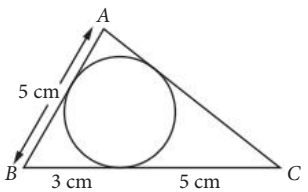


2 The value of x is:

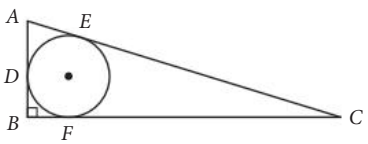
A 25 **B** 50 **C** 65 **D** 90



3 The sides of ΔABC are tangents to the circle. Find the perimeter of ΔABC.



4 In the right-angled triangle ABC, $BC = 24$ cm, $DB = 3$ cm, $AD = x$ cm. Use Pythagoras' theorem to form an equation in x. Solve this equation.

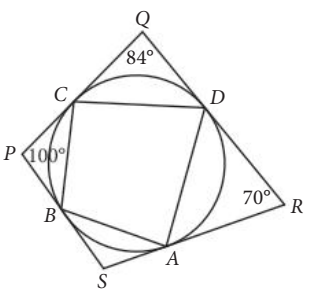


5 Calculate:

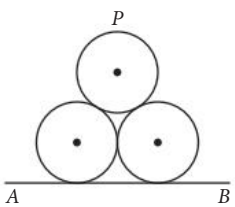
(a) the distance of a chord of length 24 cm from the centre of a circle of radius 13 cm

(b) the length of the tangents drawn from an external point to the extremities of the chord.

6 The sides of the quadrilateral $PQRS$ are tangents to the circle. Calculate the size of each angle of the quadrilateral $ABCD$.

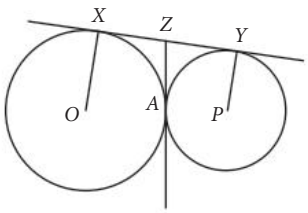


7 The diagram at right shows 3 circles, each of radius 2 cm. Calculate the height of the highest point P on a circle above AB.



8 Two circles, centred at O and P, touch externally at A. The direct common tangent touches the circles at X and Y respectively. The common tangent at A meets the direct common tangent at Z. Prove that:

(a) $ZX = ZY$ **(b)** $\angle OZP = 90°$

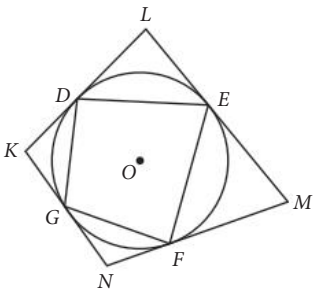


9 Ten equal circular discs fit exactly in an equilateral triangle frame. Nine discs touch the sides of the frame, while the tenth disc (in the centre) touches six of the other discs. If the internal sides of the frame are each of length R, express the radius of each circular disc in terms of R. Show that the outside perimeter of this shape without the frame (i.e. just the outside border of all the discs together) is $\frac{11}{12}\left(3-\sqrt{3}\right)\pi R$.

10 Six equal circular discs are placed so that their centres are on the circumference of a given circle and each disc touches its two neighbours. If the radius of the given circle is r, find:

(a) the radius of a seventh disc that touches all the other discs
(b) the length of the outside perimeter of the given figure.

11 $KLMN$ (at right) is a quadrilateral whose sides are tangents to the circle. Prove that $KN + ML = KL + NM$.



12 Inside a circle K of radius R, three circular discs A, B and C (each of radius r) are placed so that each disc touches the other two discs as well as K. Express R in terms of r.

13 A playground consists of three equal circular areas, each of which touches the other two areas at their edge, as well as the area between the circles. If the circles have radius r, find:

(a) the length of the playground's perimeter **(b)** the playground's total area.
(c) A child stands at the centre of the playground. What is the greatest distance that another child in the playground can be from her?

14 Three circular discs, with radii 10 cm, 20 cm and 30 cm, touch each other to form a single shape. Calculate:

(a) the length of the perimeter of this shape **(b)** the area enclosed between the discs.

15 Three circular discs, each of radius a units, touch each other. In terms of a, calculate:

(a) the area enclosed between the discs
(b) the radius of the largest circle that could fit into the space between the three discs.

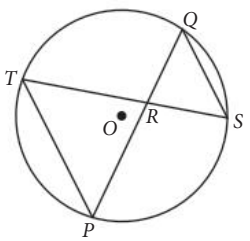
1.9 INTERCEPTS OF CHORDS

Theorem 19

The products of the intercepts of two intersecting chords are equal.

Given: PQ and ST are two chords of a circle with centre O, which intersect at R.

Aim: To prove that $PR \times RQ = SR \times RT$.



Method 1

Construction: Join PT and SQ.

Proof: In ΔPTR and ΔSQR:

$\angle PTR = \angle SQR$ (angles in the same segment)

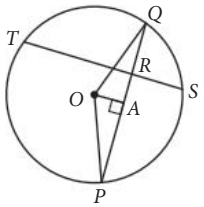
Similarly $\angle TPR = \angle QSR$

$\therefore \Delta PTR \,|||\, \Delta SQR$ (equiangular)

$\therefore \frac{PR}{RS} = \frac{TR}{RQ}$ (corresponding sides of similar triangles are proportional)

$\therefore PR \times RQ = SR \times RT$

Method 2



Construction: Draw the perpendicular OA from the centre O to PQ.
Join OP and OQ.

Proof: $PA = AQ$ (perpendicular from centre bisects the chord)

Now $PR \times RQ = (PA + AR)(AQ - AR)$
$= (PA + AR)(PA - AR)$
$= PA^2 - AR^2$

But $PA^2 = OP^2 - OA^2$ (Pythagoras' theorem)

and $AR^2 = OR^2 - OA^2$ (Pythagoras' theorem)

$\therefore\ PA^2 - AR^2 = OP^2 - OR^2$

$\therefore\ PR \times RQ = OP^2 - OR^2$

Similarly, by drawing the perpendicular from O to ST, it can be proved that:

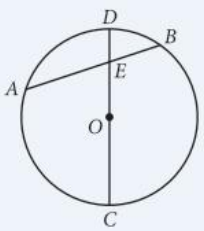
$SR \times RT = OS^2 - OR^2$

But $OP = OS$ (radii)

$\therefore PR \times RQ = SR \times RT.$

Example 19

AB is a chord of length 8 cm in a circle with centre O. CD is a diameter of length 10 cm. AB and CD intersect at E. $BE = 3$ cm. Find the length of OE.



Solution

Let $OE = x$ cm. $AB = 8$ cm, $BE = 3$ cm, so $AE = 8 - 3 = 5$ cm.

$\therefore\ OD = 5$ cm (radius is half diameter, $CD = 10$ cm)

$\therefore\ CE = 5 + x$ and $ED = 5 - x$

But $CE \times ED = AE \times EB$ (products of intercepts of intersecting chords are equal)

$\therefore\ (5 + x)(5 - x) = 5 \times 3$

$25 - x^2 = 15$

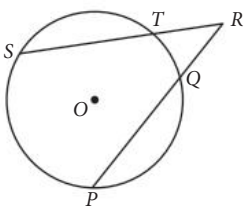
$x^2 = 10$

$x = \sqrt{10}$

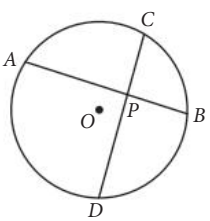
$\therefore\ OE = \sqrt{10}$ cm

Note:

- Theorem 19 also holds when the two chords intersect outside the circle. It can be proved as an application of theorem 20 (see page 32 below). For example: PQ and ST intersect at R, which is outside the circle.



- The converse of theorem 19 gives another test for concyclic points: If two straight lines AB and CD are divided both internally or both externally at the same point P such that $PA \times PB = PC \times PD$, then the four points A, C, B and D are concyclic. (This result is not included in this course.)

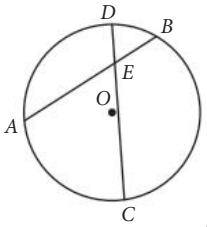


Historical note

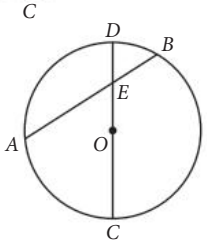
Theorem 19 was originally expressed as follows: 'If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other'. It is an interesting exercise to check that the two different expressions of this theorem actually mean the same thing.

EXERCISE 1.9 INTERCEPTS OF CHORDS

In all diagrams, O is the centre of the circle.



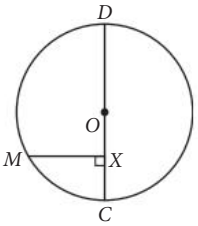
1 **(a)** Calculate the length of CE if $AE = 5$ cm, $EB = 4$ cm and $ED = 2.5$ cm.
(b) Calculate the length of CD if $AE = 6$ cm, $EB = 5$ cm and $CE = 10$ cm.
(c) Calculate the length of AB if $AE = 3$ cm, $EB = CD$ and $CE = 4$ cm.
(d) Calculate the length of CD if $AE = 9$ cm, $EB = 4$ cm and $CE = ED$.



2 CD is a diameter of the circle. AB is a chord of the circle that intersects CD at E. If $CD = 12$ cm, $AB = 9$ cm, $BE = 4$ cm, then the length of OE is:

A 3 cm **B** 4 cm
C 5 cm **D** 6 cm

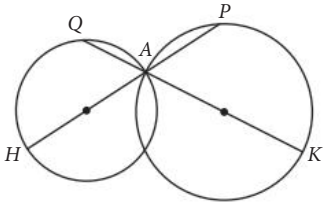
3 In a circle of radius 9 cm, X is a point 5 cm from the centre. AB is any chord through X. Find the value of $AX \times XB$.



4 X is a point on the diameter CD of a circle. XM is drawn perpendicular to CD, meeting the circumference of the circle at M. Show that $CX \times XD = MX^2$.

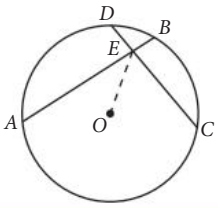
5 AB and AC are equal chords of a circle. AP is another chord of this circle and cuts BC at Q. Prove that:

(a) $\Delta AQC \,|||\, \Delta APC$ **(b)** $AB^2 = AQ \times AP$



6 AH and AK are diameters of circles AQH and APK respectively. If PAH and QAK are straight lines, prove that:

(a) H, Q, P and K are concyclic
(b) $PA \times AH = QA \times AK$

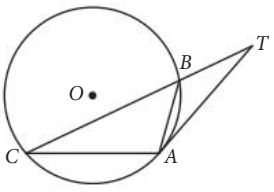


7 **(a)** If $OA = 6$ cm, $AE = 5$ cm, $OE = 4$ cm, find BE.
(b) If $OA = 7$ cm, $OE = 5$ cm, find $AE \times EB$.

1.10 TANGENTS AND SECANTS

Theorem 20

The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.



Given: TA is a tangent to the circle with centre O. TBC is a secant drawn from T to intersect the circle at B and C.

Aim: To prove that $TA^2 = TB \times TC$.

Construction: Join AB and AC.

Proof: In ΔTAB and ΔTCA:

$\angle TAB = \angle TCB$ (angles in the alternate segment)

$\angle ATB = \angle CTA$ (common angle)

$\therefore$ $\Delta TAB ||| \Delta TCA$ (two angles equal)

$\therefore$ $\frac{TA}{TC} = \frac{TB}{TA}$ (corresponding sides of similar triangles are proportional)

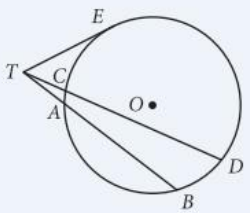
$\therefore$ $TA^2 = TB \times TC$

Example 20

TE is a tangent to the circle with centre O. TAB and TCD are secants from T that intersect the circle at A, B and C, D respectively.

If $TA = 6$ cm, $AB = 2$ cm, $TC = 3$ cm, find:

(a) TB **(b)** TE **(c)** CD **(d)** TD



Solution

(a) $TB = 6 + 2 = 8$ cm

(b) $TE^2 = TA \times TB$ (square on tangent equals product of intercepts on secant)

$TE^2 = 6 \times 8 = 48$

$TE = \sqrt{48} = 4\sqrt{3}$ cm

(c) $TE^2 = TC \times TD$ (square on tangent equals product of intercepts on secant)

$TD = 3 + CD$

$\therefore$ $48 = 3 \times (3 + CD)$

$16 = 3 + CD$

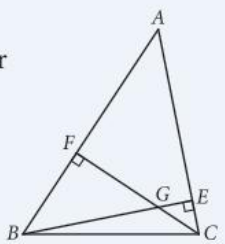
$CD = 13$ cm

(d) $TD = 16$ cm

Example 21

In ΔABC, points E and F are on the sides of the triangle so that BE is perpendicular to AC and CF is perpendicular to AB. BE and CF intersect at G. Prove that:

(a) $EG \times GB = FG \times GC$ **(b)** $AF \times AB = AE \times AC$



Solution

(a) $\angle BFC = \angle BEC = 90°$ ($BE \perp AC$, $CF \perp AB$)

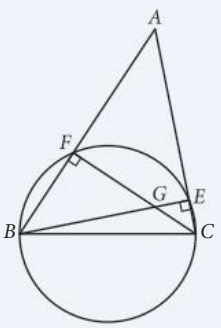
$\angle BFC$ and $\angle BEC$ are equal angles subtended on the same side of BC.

$\therefore$ B, F, E and C are concyclic points.

Draw a circle though B, F, E and C.

Hence BE and CF are chords of a circle that intersect at G.

$\therefore$ $EG \times GB = FG \times GC$ (product of intercepts of intersecting chords)

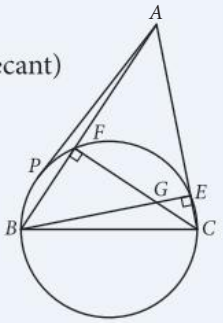


(b) Draw the tangent AP to the circle through B, F, E and C.

$AP^2 = AF \times AB$ (square on tangent equals product of intercepts on secant)

Similarly $AP^2 = AE \times AC$

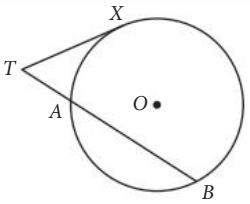
$\therefore$ $AF \times AB = AE \times AC$ (both products equal AP^2)



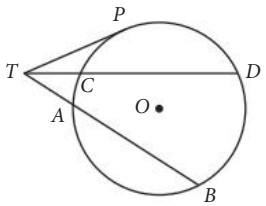
Example **21** shows how to prove that when two chords of a circle intersect outside the circle, the products of their intercepts are equal.

EXERCISE 1.10 TANGENTS AND SECANTS

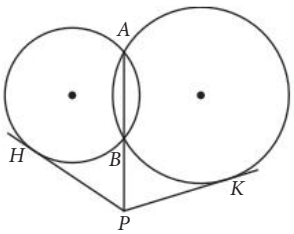
1 **(a)** If $TX = 12$ cm, $AB = 7$ cm, find the length of TA.
(b) If $TA = 4$ cm, $AB = 5$ cm, find the length of TX.
(c) If $TX = 8$ cm, $TA = 4$ cm, find the length of AB.
(d) If $TX = 10$ cm, $TA = AB$, find the length of AB as a surd.



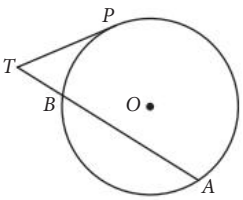
2 For the given diagram, indicate whether each statement is correct or incorrect.
(a) $TP = 9$ cm, $TC = 3$ cm, $CD = 24$ cm
(b) $TA = 3$ cm, $AB = 5$ cm, $TC = 4$ cm, $CD = 6$ cm
(c) $TP = 7$ cm, $TA = 5$ cm, $AB = 5$ cm
(d) $TP = 5$ cm, $TC = 2$ cm, $TD = 12.5$ cm



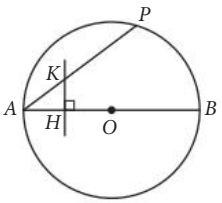
3 Two circles intersect at A and B. P is a point on AB produced. Prove that the tangents to these circles from P are equal.



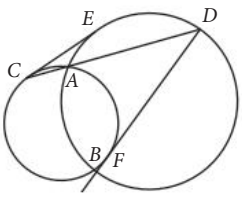
4 The tangent at P to the circle APB meets AB produced at T. If $TA = 2TP$, prove that $AB = 3BT$.



5 AB is a diameter of the circle APB. A line perpendicular to AB cuts AB and AP at H and K respectively. Prove that:
(a) K, P, B and H are concyclic
(b) $AH \times AB = AK \times AP$ (draw a tangent from A to the circle $KPBH$).



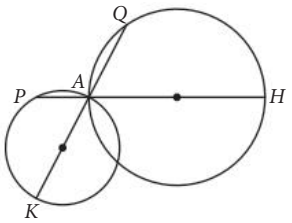
6 CE touches the circle $BAED$ and DF touches the circle CAB. If CAD is a straight line, prove that $CE^2 + DF^2 = CD^2$.



7 AH and AK are diameters of the circles AQH and APK respectively. If P, A, H and Q, A, K are collinear, prove that:

(a) K, P, Q and H are concyclic

(b) $PA \times AH = QA \times AK$

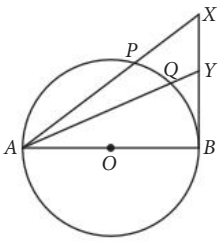


8 AB is a diameter of the circle $APQB$. The tangent at B meets AP and AQ produced at X and Y respectively.

(a) If $YQ = AY - AQ$, prove that $YB^2 = AY^2 - AQ \times AY$.

(b) If $XP = AX - AP$, prove that $XB^2 = AX^2 - AP \times AX$.

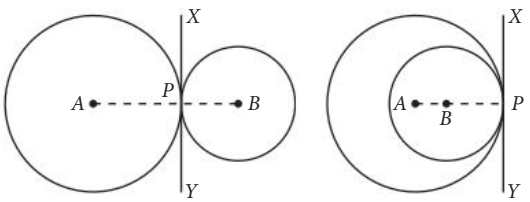
(c) Hence prove that $AP \times AX = AQ \times AY = AB^2$.



1.11 TOUCHING CIRCLES

Theorem 21

When two circles touch, the line through both centres passes through the point of contact.



Given: A and B are the centres of two circles that touch at P
(in the left diagram, touching externally; in the right diagram, touching internally).

Aim: To prove that A, P and B are collinear.

Construction: Join AP and PB.

Proof: XY is a tangent to the circle with centre A

$\therefore$ $\angle XPA = 90°$ (tangent is perpendicular to the radius drawn to point of contact)

XY is a tangent to the circle with centre B

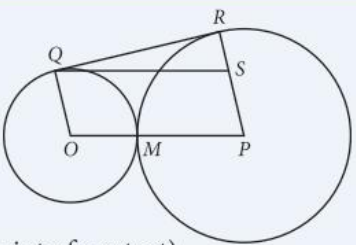
$\therefore$ $\angle XPB = 90°$ (tangent is perpendicular to the radius drawn to point of contact)

$\therefore$ PA and PB are both perpendicular to XY at P.

$\therefore$ A, P and B are collinear.

Example 22

Two circles of radii 5 cm and 8 cm touch externally.
Calculate the length of the common tangent.



Solution

O and P are the centres of the two circles that touch externally at M.

$\therefore$ OMP is a straight line (line through centres passes through point of contact)

QR is the common tangent to the two circles.
Draw QS parallel to OP: $OQ = 5$ cm, $PR = 8$ cm

$\therefore$ $RS = 3$ cm and $OP = QS = 13$ cm

$\angle QRP = 90°$ (tangent $\perp$ radius drawn to point of contact)

$\therefore$ $QR^2 = QS^2 - RS^2$ (Pythagoras' theorem)

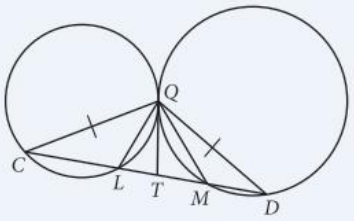
$QR^2 = 169 - 9$

$QR^2 = 160$

$QR = 4\sqrt{10}$ cm

Example 23

Two circles touch externally at Q. QC and QD are two equal chords of the respective circles. CD cuts the circles again at L, M. Prove that TQ, the tangent at Q, bisects $\angle LQM$.



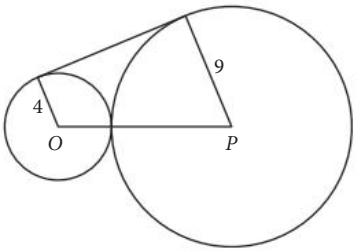
Solution

	$CQ = QD$	(given)
$\therefore$	ΔCQD is isosceles	
$\therefore$	$\angle QCD = \angle QDC$	(base angles of isosceles triangle)
Let	$\angle QCD = \angle QDC = x°$	
$\therefore$	$\angle TQL = \angle QCL = x°$	(angle in the alternate segment)
and	$\angle TQM = \angle QDM = x°$	(angle in the alternate segment)
$\therefore$	$\angle TQL = \angle TQM$	(both equal $x°$)
$\therefore$	TQ bisects $\angle LQM$	

Example **23** could be solved using the property of a common tangent and the line of the centres. When questions involve common tangents, this is often the case.

EXERCISE 1.11 TOUCHING CIRCLES

1 Two circles of radii 4 cm and 9 cm touch externally. Calculate the length of the common tangent.



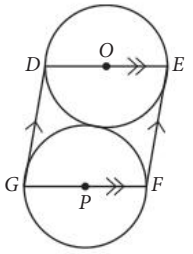
2 If the radii of two intersecting circles are 17 cm and 10 cm, and the length of the common chord is 16 cm, calculate:

(a) the length of the line joining the centres of the circles
(b) the length of the common tangent.

3 If the radii of two intersecting circles are 51 cm and 74 cm, and the length of the common chord is 48 cm, calculate:

(a) the length of the line joining the centres of the circles
(b) the length of the common tangent.

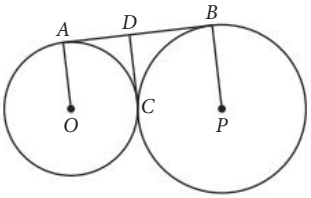
4 $DEFG$ is a parallelogram. The circles with DE and FG as diameters touch each other. O is the centre of the circle on DE and P is the centre of the circle on FG. Prove that $DE = EF$.



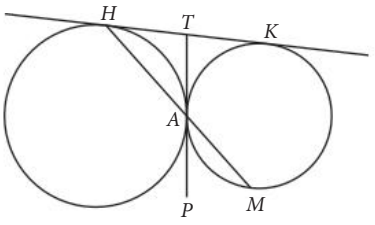
5 Four coins (10c, 20c, \$1 and \$2) are placed on a flat surface so that each coin touches exactly two of the other coins. Prove that the four points of contact are concyclic.

6 Two circles, centred at O and P, touch externally at C. AB is a common tangent to the two circles. The common tangent at C meets AB at D.

(a) Prove that the quadrilaterals $ADCO$ and $DBPC$ are cyclic.
(b) Hence prove that $\angle ODP = 90°$.



7 HK and TAP are common tangents to two circles that touch externally at A. HA is produced to M. Indicate whether each statement is correct or incorrect.

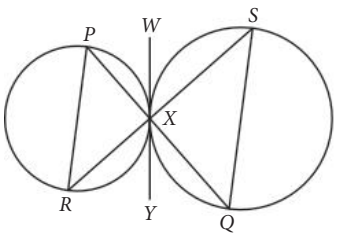


(a) $TH = TK$
(b) $\angle HAK = 90°$
(c) KM is a diameter of the circle KAM
(d) $\angle HKM = 90°$

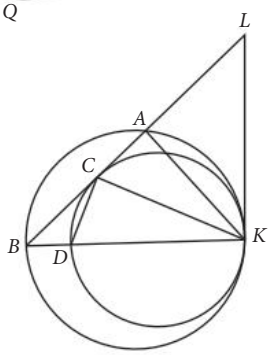
8 Three equal circles are drawn so that each one touches the other two externally.

(a) Prove that the triangle formed by joining the centres is equilateral.
(b) Prove that the triangle formed by joining the points of contact is equilateral.

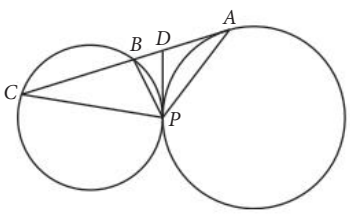
9 X is the point of contact of two circles. WXY is the common tangent. Prove that $PR \parallel SQ$.



10 Two circles touch internally at K. LK and LC are tangents to the small circle. A and B are the points of intersection of LC and the large circle. KB cuts the small circle at D. Prove that $\angle AKC = \angle CKB$.



11 Two circles touch externally at P. The tangent at A on one circle cuts the other circle at B and C. PD is the common tangent to the circles meeting AC at D. Prove that $\angle APB + \angle APC = 180°$.



SUMMARY OF RESULTS

Assumption 1 Equal arcs on circles of equal radii subtend equal angles at the centre.

Assumption 2 If two arcs of circles of equal radii subtend equal angles at their centres, then the arcs are equal.

The proofs to theorems 1 to 5 do not have to be reproduced.

Theorem 1 Equal angles at the centre of a circle stand on equal chords.

Theorem 2 Equal chords subtend equal angles at the centre of a circle.

Theorem 3 The angle at the centre of a circle is twice the angle at the circumference that is subtended by the same arc.

Theorem 4 The tangent to a circle is perpendicular to the radius drawn to the point of contact.

Theorem 5 A straight line drawn through a point on a circle, perpendicular to the radius at that point, is a tangent to the circle.

You must be able to prove theorems 6 to 21.

Theorem 6 The perpendicular from the centre of a circle to a chord bisects the chord.

Theorem 7 The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

Theorem 8 Equal chords in equal circles are equidistant from the centres.

Theorem 9 Chords in a circle that are equidistant from the centre are equal.

Theorem 10 Any three non-collinear points are on a unique circle whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.

Theorem 11 Angles in the same segment are equal.

Theorem 12 The angle in a semicircle is a right angle.

Theorem 13 The opposite angles of a cyclic quadrilateral are supplementary.

Theorem 14 The exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.

Theorem 15 If the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic.

Theorem 16 If an interval subtends equal angles at two points on the same side of the interval, then the two points and the endpoints of the interval are concyclic.

Theorem 17 The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

Theorem 18 Tangents to a circle from an external point are equal.

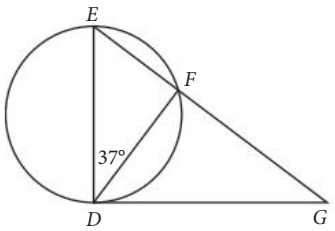
Theorem 19 The products of the intercepts of two intersecting chords are equal.

Theorem 20 The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.

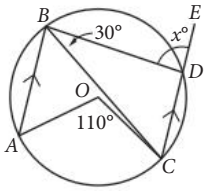
Theorem 21 When two circles touch, the line through both centres passes through the point of contact.

CHAPTER REVIEW 1

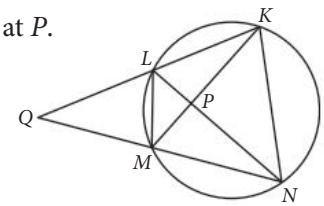
1 DE is a diameter of the circle. DG is a tangent. If $\angle EDF = 37°$, calculate the size of $\angle EGD$. Give reasons for your answer.



2 O is the centre of the circle, $AB \| CE$, $\angle CBD = 30°$, $\angle BDE = x°$. Calculate the value of x, giving reasons for your answer.



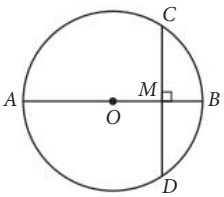
3 $KLMN$ is a quadrilateral inscribed in a circle. KM and LN intersect at P. KL produced and NM produced intersect at Q. Prove that $\angle LPM + \angle LQM = 2\angle KLN$.



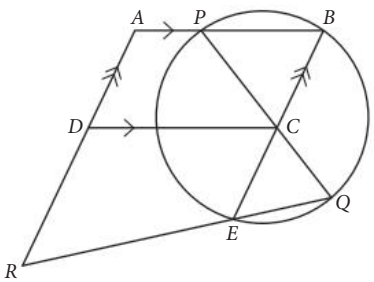
4 The length of the tangent from a point T to a circle with radius 2.5 cm is 6 cm. Calculate the radius of the circle with centre T that touches the given circle externally.

5 AB and AC are two equal chords of a circle with centre O. Prove that the bisector of $\angle BAC$ passes through O.

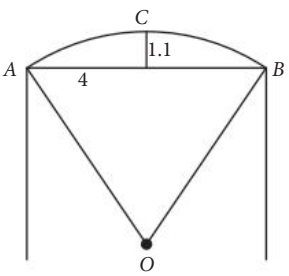
6 AB is a diameter of a circle with centre O. CD is a chord perpendicular to AB that cuts AB at M. Prove that $BD^2 = BM \times BA$.



7 $ABCD$ is a parallelogram. AD produced and QE produced meet at R. Prove that $\angle PCD = \angle DRQ$.

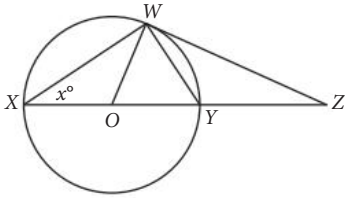


8 A bridge is a circular arc ACB resting on supports at A and B, which are at the same level. The highest point C of the bridge is 1.1 m above AB. If $AB = 8$ m, calculate the radius of the bridge's arc to the nearest cm.

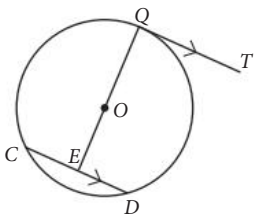


9 Two equal circles intersect at C and D. PDQ is a straight line cutting the circles at P and Q. Prove that $PC = CQ$.

10 ZW is a tangent to the circle with centre O. XY is a diameter. Express each of the angles in the figure in terms of x. Give reasons for your answers.



11 QT is a tangent to the circle with centre O. $QT \parallel CD$. Prove that QE bisects CD.



12 $CDEF$ is a quadrilateral inscribed in a circle. The sides CD and EF subtend right angles at the circle's centre O. Prove that the diagonals CE and DF cut at right angles.

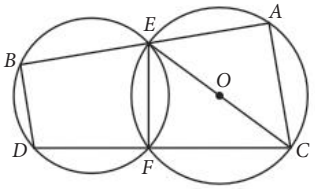
13 The straight line joining the centre O of a circle to an external point T cuts the circumference at Z. If $TZ = 32$ cm and the tangent to the circle from T is 40 cm, calculate the radius of the circle. Give reasons for your answer.

14 At a point F on the circumference of a circle with centre O, a tangent FG is drawn the same length as the circle's radius. GO produced cuts the circumference at H. Calculate the size of $\angle FHG$.

15 EC is a diameter. Prove that:

(a) $DB \perp AB$

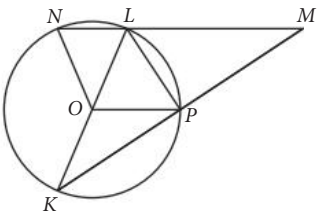
(b) $\angle AFB = \angle CED$



16 KL is a diameter of a circle with centre O. $KL = LM$.

(a) Prove that PL bisects $\angle KLM$.

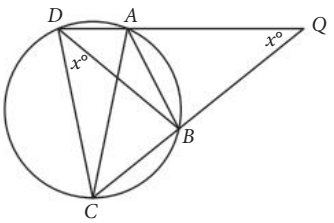
(b) If $\angle LMP = 40°$, calculate the size of $\angle NOK$. Give reasons for your answer.



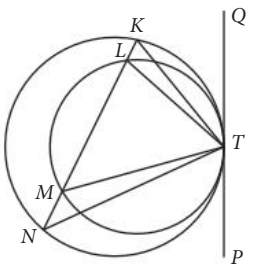
17 D, E and F are three points on the minor arc DEF of a circle with centre O. DX is perpendicular to FE produced, meeting it at X. EO produced meets the circumference at G. Prove that:

(a) $GF \parallel DX$ **(b)** $\angle DEG = \angle EDX + \angle EGF$

18 DA produced and CB produced meet at Q. If $\angle CDB = \angle DQC = x°$, prove that $CD = CA$. (Hint: Let $\angle CBD = y°$.)



19 Two circles touch internally at T. QTP is the common tangent. KN is a chord of the larger circle and intersects the smaller circle at L and M respectively. Prove that $\angle KTL = \angle MTN$.



CHAPTER 2
POLYNOMIALS

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE2 uses multi-step deductive reasoning in a variety of contexts

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

2.1 POLYNOMIALS

A real polynomial $P(x)$ is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where $n, n-1, \ldots$ are all positive integers and $a_n, a_{n-1}, \ldots$ are the coefficients, which for convenience will usually be chosen as real integers.

The term $a_n x^n$ is the **leading term**. If $a_n \neq 0$, the polynomial is said to be of the n-th degree, or 'degree n'.

e.g. $P(x) = 4x^5 - 2x^4 + x^2 + 5x - 6$ is a polynomial of degree 5.

A real polynomial $P(x)$ is defined for all real x and is a continuous and differentiable function of x.

$P(x) = 0$ is a polynomial equation of degree n. Real numbers x that satisfy this polynomial equation are called the real **roots** of the equation or the real **zeros** of the corresponding polynomial.

In your Mathematics course (see *New Senior Mathematics for Years 11 & 12*) you have already investigated the roots of the quadratic equation $ax^2 + bx + c = 0$ and found that there are at most two real roots, but there may be only one or none.

Similarly, any polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ will have at most n real roots. (Finding these real roots will become easier after we study the factor theorem in section 2.3, page 47 below.)

- $2x^3 + 4x^2 - 3x - 5$ is a polynomial of the 3rd degree in x, because the highest power of x is 3.
- $x^4 - 2x + 7$ is a polynomial of the 4th degree in x.
- $x^{\frac{1}{2}} + 2 + x$ and $\frac{1}{x} + 2x^{-3}$ are not polynomials, because they include powers of x that are not positive integers.

When the leading coefficient $a_n = 1$, the polynomial is said to be **monic**. $x^4 - 2x + 7$ is a monic polynomial.
We can also describe a polynomial according to which subset of real numbers contains the coefficients $a_n, a_{n-1}, \ldots, a_0$.

- $2x^2 - 7x + 1$ is a 2nd-degree polynomial over the integers.
- $\frac{1}{4}x^5 + 2x + \frac{3}{8}$ is a 5th-degree polynomial over the rational numbers (or 'the rational number field').
- $\sqrt{3}x - \sqrt{6}$ is a 1st-degree polynomial over the irrational numbers.
- $4x^2 + \pi x - \frac{1}{2}$ is a 2nd-degree polynomial over the real numbers.

Example 1

Express the polynomial $P(x) = x^3 - x^4 + 6x^2$ in standard form. Then write:

(a) its degree
(b) the constant term
(c) the coefficient of x^2
(d) the leading term
(e) the greatest number of real zeros possible.
(f) Hence solve the equation $P(x) = 0$.

Solution

$P(x) = -x^4 + x^3 + 6x^2$. In standard form (fully expanded): $P(x) = -x^4 + x^3 + 6x^2 + 0x + 0$

(a) degree = 4
(b) constant term = 0
(c) coefficient of $x^2 = 6$
(d) leading term $= -x^4$
(e) The polynomial is of degree 4, so there are at most 4 zeros.
(f)
$$-x^4 + x^3 + 6x^2 = 0$$
$$-x^2(x^2 - x - 6) = 0$$
$$x^2(x - 3)(x + 2) = 0$$
$$\therefore x = 0, 3, -2$$

The equation has 3 real roots. $x = 0$ is a double root.

A graph of $y = x^3 - x^4 + 6x^2$ shows that at $x = 0$, the curve touches the x-axis.

Operations with polynomials

You have added, subtracted and multiplied simple algebraic expressions before. With more complicated polynomials the only difference is that more terms are involved, so you must be more careful that you don't miss any terms.

Example 2

If $A(x) = x^2 + 2x + 3$, $B(x) = 2x - 5$, $C(x) = 3x^4 - 3x^2 + 5x + 6$ and $D(x) = x^5 - 3x^2 + 1$, simplify:

(a) $A(x) + C(x)$ **(b)** $B(x) \times D(x)$ **(c)** $D(x) - C(x)$ **(d)** $A(x) \times D(x)$ **(e)** $A(x) + 2C(x) - 3B(x)$

Solution

(a)
$$\begin{aligned} A(x) + C(x) &= x^2 + 2x + 3 + 3x^4 - 3x^2 + 5x + 6 \\ &= 3x^4 - 2x^2 + 7x + 9 \end{aligned}$$

(b)
$$\begin{aligned} B(x) \times D(x) &= (2x - 5)(x^5 - 3x^2 + 1) \\ &= 2x^6 - 6x^3 + 2x - 5x^5 + 15x^2 - 5 \\ &= 2x^6 - 5x^5 - 6x^3 + 15x^2 + 2x - 5 \end{aligned}$$

(c)
$$\begin{aligned} D(x) - C(x) &= x^5 - 3x^2 + 1 - (3x^4 - 3x^2 + 5x + 6) \\ &= x^5 - 3x^2 + 1 - 3x^4 + 3x^2 - 5x - 6 \\ &= x^5 - 3x^4 - 5x - 5 \end{aligned}$$

(d)
$$\begin{aligned} A(x) \times D(x) &= (x^2 + 2x + 3)(x^5 - 3x^2 + 1) \\ &= x^7 - 3x^4 + x^2 + 2x^6 - 6x^3 + 2x + 3x^5 - 9x^2 + 3 \\ &= x^7 + 2x^6 + 3x^5 - 3x^4 - 6x^3 - 8x^2 + 2x + 3 \end{aligned}$$

(e)
$$\begin{aligned} A(x) + 2C(x) - 3B(x) &= x^2 + 2x + 3 + 2(3x^4 - 3x^2 + 5x + 6) - 3(2x - 5) \\ &= x^2 + 2x + 3 + 6x^4 - 6x^2 + 10x + 12 - 6x + 15 \\ &= 6x^4 - 5x^2 + 6x + 30 \end{aligned}$$

EXERCISE 2.1 POLYNOMIALS

1 For the polynomial $P(x) = 3x^4 + 2x^3 + 7$, which statement is correct?

A degree = 3 **B** leading term = 3 **C** leading coefficient = 3 **D** constant term = 3

2 Express the polynomial $P(x) = x^2 - x^3 + 6x$ in standard form. Then write:

(a) its degree **(b)** the constant term
(c) the coefficient of x^2 **(d)** the leading term
(e) the greatest number of real zeros possible. **(f)** Hence solve the equation $P(x) = 0$.

3 Write the following polynomials in standard form and then state:

(i) the degree **(ii)** the constant term **(iii)** the coefficient of x^2
(iv) whether or not it is monic **(v)** the greatest number of real zeros possible.

(a) $x^2 + 5x^3 + 7 - 6x$ **(b)** $27 - x^3$ **(c)** $ax^3 + bx + cx^2 - d$
(d) $x(5x^3 - 3x^2 + 2)$ **(e)** $(3x + 1)(2x^3 - 5)$ **(f)** $(3x^2 - 2)^2$

4 State whether each expression is a polynomial or not. If it is not a polynomial, explain why.

(a) $x^2 - 6x + 3$ **(b)** $x + 4$ **(c)** $\sqrt{3}x - 4$
(d) $x^9 + 1$ **(e)** $4 - \frac{1}{x}$ **(f)** $\frac{6x + 2}{3}$
(g) $x^2 + 3x^{\frac{1}{2}} - 4x^{-1}$ **(h)** $\frac{3x + 2}{3x - 1}$ **(i)** $2^x + 3x - 5$

5 If $A(x) = x^2 - 5x + 1$ and $B(x) = 3x^4 - 2x^2 + 5x + 3$, then $B(x) - A(x) = \ldots$

A $-3x^4 + 3x^2 - 10x - 2$ **B** $3x^4 - 3x^2 + 2$ **C** $3x^4 - 3x^2 - 10x + 2$ **D** $3x^4 - 3x^2 + 10x + 2$

6 If $A(x) = 3x^2 - 2x + 1$, $B(x) = 5x - 2$, $C(x) = 2x^4 - 5x^2 + 3x + 4$ and $D(x) = 2x^5 - 4x^2 - 3$, simplify:

(a) $A(x) + C(x)$ **(b)** $B(x) \times D(x)$ **(c)** $D(x) - C(x)$
(d) $A(x) \times B(x)$ **(e)** $A(x) - 3C(x) + 2B(x)$

7 If $E(x) = x^2 - 3$, $F(x) = 3x + 2$, $G(x) = x^2 + 2x + 1$ and $H(x) = x^2 - 3x + 2$, find the polynomial for:

(a) $E(x) \times F(x)$ **(b)** $F(x) \times G(x)$ **(c)** $3G(x) - 4H(x)$
(d) $(x - 3)G(x)$ **(e)** $[F(x)]^2$ **(f)** $E(x) \times G(x) + F(x) \times H(x)$

2.2 DIVISION OF POLYNOMIALS AND THE REMAINDER THEOREM

If $P(x) = 3x^2 + 7x + 4$, then $P(1)$ is the value of $P(x)$ when $x = 1$, which is found by substituting $x = 1$ into the polynomial. For example:

$P(1) = 3 + 7 + 4 = 14$

$P(-1) = 3 \times (-1)^2 + 7 \times (-1) + 4 = 0$

$P(a) = 3a^2 + 7a + 4$

You already know how to perform long division using integers (e.g. $532 \div 19 = 28$). We can similarly perform long division of a polynomial by a linear expression or by another polynomial. For example:

$\frac{x^2 + 7x + 12}{x + 3}, x \neq -3$

$(x^2 + 7x + 12) \div (x + 3), x \neq -3$

The two ways of writing the division above are equivalent.

Example 3

Find $\frac{x^2+7x+12}{x+3}$, $x \neq -3$.

Solution

We have the condition $x \neq -3$ because we cannot divide by zero.

Divide the leading term of the dividend (x^2) by the leading term of the divisor (x) and write the answer (x) above the x^2.

Multiply the divisor ($x+3$) by the result of $x^2 \div x$ (i.e. x) to obtain x^2+3x.

Subtract (x^2+3x) from (x^2+7x) to get $4x$, then bring down the $+12$ from above.

Divide this new leading term ($4x$) by x and write the answer ($+4$) above $7x$.

Multiply ($x+3$) by 4 to obtain $4x+12$.

Subtract the last two lines to get a remainder of 0, completing the division.

$$\begin{array}{r} x+4 \\ x+3\overline{)\,x^2+7x+12} \\ \underline{x^2+3x} \\ 4x+12 \\ \underline{4x+12} \\ 0 \end{array}$$

Hence we have $\frac{x^2+7x+12}{x+3} = x+4$, with no remainder.

We could also write $x^2+7x+4 = (x+3)(x+4)+0$:

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

Example 4

If $P(x) = 3x^3 - 7x^2 + 2x + 4$, divide $P(x)$ by $x-3$.

Solution

Divide $3x^3$ by x and write the answer ($3x^2$) above the $3x^3$.

Multiply ($x-3$) by $3x^2$ to obtain $3x^3 - 9x^2$ and write it under $3x^3 - 7x^2$.

Subtract ($3x^3 - 9x^2$) from ($3x^3 - 7x^2$) to get $2x^2$, then bring down the $+2x$ from above.

Divide ($2x^2+2x$) by x and write the answer ($+2x$) above $-7x^2$.

Multiply ($x-3$) by $2x$ to obtain $2x^2 - 6x$ and write it under $2x^2+2x$.

Subtract ($2x^2 - 6x$) from ($2x^2+2x$) to get $8x$, then bring down the $+4$ from above.

Divide $8x$ by x and write the answer ($+8$) above $+2x$.

Multiply ($x-3$) by 8 to obtain $8x-24$ and write it under $8x+4$.

Subtract ($8x-24$) from ($8x+4$) to get 28.

$$\begin{array}{r} 3x^2+2x+8 \\ x-3\overline{)\,3x^3-7x^2+2x+4} \\ \underline{3x^3-9x^2} \\ 2x^2+2x \\ \underline{2x^2-6x} \\ 8x+4 \\ \underline{8x-24} \\ 28 \end{array}$$

The number 28 is a constant with a degree less than the divisor, so it is not divisible by ($x-3$). Thus the remainder is 28.

Hence: $\frac{3x^3-7x^2+2x+4}{x-3} = 3x^2+2x+8+\frac{28}{x-3}$

or $3x^3 - 7x^2 + 2x + 4 = (x-3)(3x^2+2x+8) + 28$

i.e. $P(x) = (x-3)Q(x) + R$

where $Q(x)$ is the quotient and R is the remainder (i.e. in this case, 28). *Note:*

The degree of the remainder < the degree of the divisor.
The degree of the quotient < the degree of the dividend.

When the divisor is a linear function (first degree), the remainder will be a constant.

Note also that $P(x) = (x-3)(3x^2+2x+8)+28$, so:

$P(3) = (3-3) \times Q(3) + 28 = 0 + 28 = 28$

This suggests that we may be able to find the remainder of division by $(x-3)$ just by finding the value of $P(3)$, without doing the long division.

Example 5

Divide $4x^3 - 19x + 9$ by $2x - 3$.

Solution

When writing out the dividend, include all missing terms by writing them with zero coefficients.

$$
\begin{array}{r}
2x^2+3x\ -\ 5 \\
2x-3\overline{)\,4x^3+0x^2-19x+9} \\
\underline{4x^3-6x^2} \\
6x^2-19x \\
\underline{6x^2-\ 9x} \\
-10x+9 \\
\underline{-10x+15} \\
-6
\end{array}
$$

The quotient is $2x^2 + 3x - 5$ and the remainder is -6.

$$\frac{4x^3-19x+9}{2x-3} = 2x^2+3x-5-\frac{6}{2x-3}$$

or $(4x^3 - 19x + 9) = (2x-3)(2x^2+3x-5) - 6$

$$P(x) = (2x-3) \times Q(x) - 6$$

If $2x - 3 = 0$ then $x = \frac{3}{2}$.

$$P\left(\frac{3}{2}\right) = 0 \times Q\left(\frac{3}{2}\right) - 6 = -6$$

This again suggests a simpler method for finding the remainder.

Example 6

Divide $x^4 + x^3 - 7x^2 - x + 6$ by $x^2 - 1$.

Solution

$$
\begin{array}{r}
x^2+x\ -6 \\
x^2-1\overline{)\,x^4+x^3-7x^2-x+6} \\
\underline{x^4\quad\ -x^2} \\
x^3-6x^2-x \\
\underline{x^3\quad\ -x} \\
-6x^2\quad +6 \\
\underline{-6x^2\quad +6} \\
0
\end{array}
$$

The remainder is zero, so this means that $(x^2 - 1)$ is a factor of $x^4 + x^3 - 7x^2 - x + 6$. This then means that $(x + 1)$ and $(x - 1)$ are also factors of $x^4 + x^3 - 7x^2 - x + 6$.

The remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$ until the remainder R does not contain x, then $R = P(a)$.

For any polynomial, $P(x) = (x-a)Q(x) + R$ where $Q(x)$ is a polynomial.

Thus $P(a) = (a-a)Q(a) + R$

i.e. $P(a) = 0 \times Q(a) + R$

$\therefore \quad R = P(a)$

Important considerations:

1 If $P(x)$ is divided by $x+a$, as $x+a=x-(-a)$, then $R=P(-a)$.

2 If $P(x)$ is divided by $ax-b$, as $ax-b=a\left(x-\frac{b}{a}\right)$, then $R=P\left(\frac{b}{a}\right)$.

Example 7

Find the remainder when $P(x)=2x^3-6x^2+4x+3$ is divided by the following.

(a) $x-2$ **(b)** $x+3$ **(c)** $2x-1$ **(d)** $3x+2$

Solution

(a) $R=P(2)=2\times 2^3-6\times 2^2+4\times 2+3=16-24+8+3=3$

(b) $R=P(-3)=2\times(-3)^3-6\times(-3)^2+4\times(-3)+3=-54-54-12+3=-117$

(c) $R=P\left(\frac{1}{2}\right)=2\times\left(\frac{1}{2}\right)^3-6\times\left(\frac{1}{2}\right)^2+4\times\frac{1}{2}+3=\frac{1}{4}-\frac{3}{2}+2+3=3\frac{3}{4}$

(d) $R=P\left(-\frac{2}{3}\right)=2\times\left(-\frac{2}{3}\right)^3-6\times\left(-\frac{2}{3}\right)^2+4\times\left(-\frac{2}{3}\right)+3=-\frac{16}{27}-\frac{8}{3}-\frac{8}{3}+3=-\frac{79}{27}$

EXERCISE 2.2 DIVISION OF POLYNOMIALS AND THE REMAINDER THEOREM

1 Perform the following long divisions.

(a) $(3x^2-2x+5)\div(x-2)$
(b) $(x^3-x^2+x-1)\div(x-1)$
(c) $(4x^3-8x^2+8x-1)\div(2x+1)$
(d) $(2x^4-3x^3+6x^2-5x-4)\div(2x-3)$
(e) $(x^3-27)\div(x-3)$
(f) $(x^4+3x^3+5x^2-6x-8)\div(x+2)$
(g) $(-2x^3+7x^2-7x+1)\div(2x+3)$
(h) $(x^4-3x^2+2)\div(x-1)$

2 If $P(x)=2x^3-4x^2+2x-1$, find $P(1)$, $P(-3)$, $P(a)$, $P(-2)$.

3 If $P(x)=x^4-2x^3+x^2-4$, find $P(2)$, $P(-1)$, $P(a)$.

4 If $P(x)=x^3-3x^2+kx-7$ and $P(2)=0$, then $k=\ldots$

A 5.5 **B** 0 **C** −5.5 **D** −6.5

5 $P(x)=x^4-x^3+px^2-4x+q$. Find p and q if $P(0)=3$ and $P(-1)=11$.

6 $P(x)=ax^3-2x^2+bx+c$. Find a, b and c if $P(0)=12$, $P(-1)=3$ and $P(2)=36$.

7 Using the remainder theorem, find the remainder when:

(a) x^3+3x^2+2x-7 is divided by $(x+2)$
(b) $3x^2+7x-2$ is divided by $(x-4)$
(c) $2y^3+y^2-2y+3$ is divided by $(y+1)$
(d) $x^4-3x^3+5x^2-2$ is divided by $(x-4)$
(e) $x^4+2x^2-13x-60$ is divided by $(2x+3)$
(f) x^4-3x^3+x+1 is divided by $(2x-1)$.

8 When x^3+3x^2+7x-b is divided by $(x-2)$ the remainder is 20. Find b.

9 When x^4+5x^3-ax+b and ax^2+bx-1 are each divided by $(x+1)$, the remainders are 7 and −6 respectively. Find a and b.

10 When the polynomial x^3+3x^2-mx+n is divided by $(x+2)$ the remainder is 9; when divided by $(x-3)$ the remainder is 49. Find m and n.

11 If x^3+2x^2+3x+p and x^3+x^2+9 have the same remainder when divided by $x+2$, find p.

12 When $3x^3 - ax^2 - bx + 1$ is divided by $(x - 2)$ the remainder is 5; when divided by $(x - 1)$ there is no remainder. Find a and b.

13 When $2x^3 + 7x^2 + ax + b$ is divided by $(x - 3)$ the remainder is 120; when divided by $(x + 1)$ the remainder is -8. Find the values of a and b.

14 Let $P(x) = x^3 - ax^2 + x$ be a polynomial where a is a real number. When $P(x)$ is divided by $(x - 2)$ the remainder is -2. Find the remainder when $P(x)$ is divided by $(x + 1)$.

2.3 THE FACTOR THEOREM

When the remainder of a division is zero, the divisor is a factor of the expression being divided. For example, 2 is a factor of 14 because $14 = 2 \times 7 + 0$. Similarly, $(x + 3)$ is a factor of $x^2 + 4x + 3$ because $x^2 + 4x + 3 = (x + 3)(x + 1) + 0$.

The **factor theorem** states:

For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

The converse of this result is also true:

If $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

By finding the zeros of a polynomial (i.e. values of a such that $P(a) = 0$), we can **factorise** the polynomial. We can usually find the first factor by trial and error, and then by long division find the quotient, which will have a degree of one less than the original polynomial. We may then be able to find factors of the quotient, which must also be factors of the original polynomial.

For example, if the original polynomial is cubic then the quotient will be a quadratic polynomial, which you can then factorise into real factors (if they exist). For higher degree polynomials, further trial and error may be needed after long division until you can reach a factorisable quotient.

Example 8

$P(x) = x^3 - x^2 - 14x + 24$. Find one zero of the polynomial and then express $P(x)$ as a product of linear factors.

Solution

$P(x) = x^3 - x^2 - 14x + 24$ is monic, so we can look at the factors of 24 as the only possible zeros, namely $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$:

$P(1) = 1 - 1 - 14 + 24 = 10 \neq 0$, $(x - 1)$ is not a factor

$P(-1) = -1 - 1 + 14 + 24 = 36 \neq 0$, $(x + 1)$ is not a factor

$P(2) = 8 - 4 - 28 + 24 = 0$, $(x - 2)$ is a factor

Hence $P(x) = (x - 2)(x^2 + x - 12)$

$= (x - 2)(x + 4)(x - 3)$

The factor theorem could also have been used to find the zeros 3 and -4. You should use the method that you find easiest and quickest in each case.

$$
\begin{array}{r}
x^2 + x - 12 \\
x - 2 \overline{)\, x^3 - x^2 - 14x + 24} \\
\underline{x^3 - 2x^2} \\
x^2 - 14x \\
\underline{x^2 - 2x} \\
-12x + 24 \\
\underline{-12x + 24} \\
0
\end{array}
$$

Example 9

Find the linear factors of $x^4 + x^3 - 7x^2 - x + 6$.

Solution

$P(x) = x^4 + x^3 - 7x^2 - x + 6$

$P(1) = 1 + 1 - 7 - 1 + 6 = 0$ Hence $(x - 1)$ is a factor.

$P(-1) = 1 - 1 - 7 + 1 + 6 = 0$ Hence $(x + 1)$ is a factor.

We now know that $(x - 1)(x + 1) = x^2 - 1$ is a factor, so we can use this as the divisor in the long division.

$$\begin{array}{r} x^2 + x - 6 \\ x^2 - 1 \overline{)\, x^4 + x^3 - 7x^2 - x + 6} \\ \underline{x^4 \quad - x^2} \\ x^3 - 6x^2 - x \\ \underline{x^3 \quad - x} \\ -6x^2 + 6 \\ \underline{-6x^2 + 6} \\ 0 \end{array}$$

Hence $P(x) = x^4 + x^3 - 7x^2 - x + 6 = (x - 1)(x + 1)(x^2 + x - 6)$

$= (x - 1)(x + 1)(x + 3)(x - 2)$

EXERCISE 2.3 THE FACTOR THEOREM

1 If $P(x) = x^4 + x^2 - 2$, indicate whether each statement is correct or incorrect.

(a) $P(1) = 0$ **(b)** $P(2) = 0$ **(c)** $P(-1) = 0$ **(d)** $P(-2) = 0$

2 Use the factor theorem to factorise each polynomial over the rational numbers.

(a) $x^3 + 4x^2 - 7x - 10$ **(b)** $x^3 + 2x^2 - 41x - 42$ **(c)** $2x^3 + x^2 - 22x + 24$

(d) $6x^3 - 5x^2 - 12x - 4$ **(e)** $3x^3 + 10x^2 + 9x + 2$ **(f)** $2x^3 + 7x^2 - 10x - 24$

3 Use the factor theorem to find the linear factors of each polynomial over the rational number field.

(a) $5 - 6x - 29x^2 + 6x^3$ **(b)** $4x^3 + 12x^2 - x - 3$ **(c)** $6x^3 + 35x^2 + 34x - 40$

(d) $6x^3 - 29x^2 - 40x - 12$ **(e)** $x^3 - 5x^2 + 6x$ **(f)** $x^3 + 2x^2 - 4x - 8$

4 Use the factor theorem to find the linear factors of each polynomial over the rational number field.

(a) $6x^3 - 5x^2 - 2x + 1$ **(b)** $3x + 7x^2 - 6x^3$ **(c)** $2x^3 - 7x^2 + 7x - 2$

(d) $2x^3 + 3x^2 - 9x$ **(e)** $6 + 5x - 2x^2 - x^3$ **(f)** $x^3 - x^2 - 10x - 8$

5 The only linear factor of $x^3 - 8$ is:

A $x - 1$ **B** $x + 4$ **C** $x + 2$ **D** $x - 2$

6 Find the values of a and b that make $x^4 + 4x^3 + ax^2 - b$ divisible by $(x - 1)$ and $(x + 2)$.

7 Given that $(x - 3)$ and $(x + 2)$ are factors of $x^3 - 6x^2 + px + q$, find the values of p and q.

8 If $5x^2 - 6x - 56$ and $3x^2 - 14x + a$ have a common factor $(x + b)$, find the values of a and b.

9 Given that $(x - 1)$ and $(x + 2)$ are factors of $6x^4 + ax^3 - 17x^2 + bx - 4$, find a and b and the other two factors.

10 Find the values of a and b that make $2x^3 + ax^2 - 13x + b$ exactly divisible by $x^2 - x - 6$.

11 Find the linear factors over the real number field of $x^3 - 3x^2 - 8x + 4$.

12 Let $P(x) = (x - 1)(x + 3)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $(x + 3)$. When $P(x)$ is divided by $(x - 1)$ the remainder is 8.

(a) Find the values of a and b.

(b) Find the remainder when $P(x)$ is divided by $(x - 1)(x + 3)$.

13 The polynomial $P(x)$ is given by $P(x) = ax^3 + 12x^2 + cx - 60$, where a and c are constants. The three zeros of $P(x)$ are 2, -3 and β. Find the value of β.

14 When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$. What is the remainder when $P(x)$ is divided by $x - 1$?

2.4 POLYNOMIAL FUNCTIONS

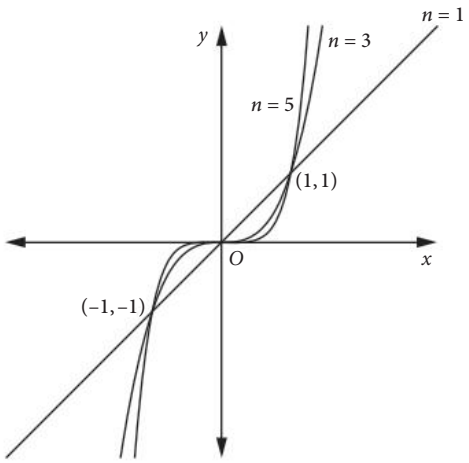
The function f where $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \ \ (a_n \neq 0)$ is called the **general polynomial function**. This function is defined for all real values of x and is continuous and differentiable. Following our earlier notation, $P(x)$ is often written instead of $f(x)$ when defining rules for polynomial functions. For $a \neq 0$, we have:

- $f(x) = ax + b$ (general linear function)
- $f(x) = ax^2 + bx + c$ (general quadratic function)
- $f(x) = ax^3 + bx^2 + cx + d$ (general cubic function)

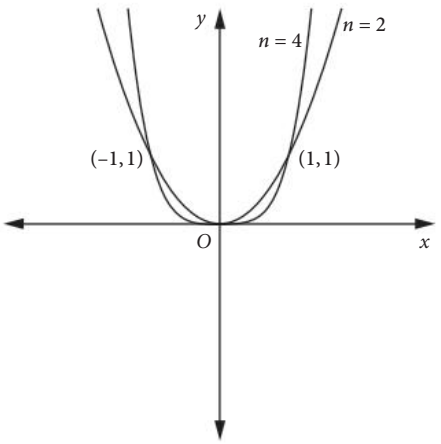
If the coefficient of the highest power of x is unity (1), the polynomial is said to be monic.

Graphs of polynomial functions

The simplest polynomial function of degree n is $f(x) = x^n$. Graphs of these basic polynomials for $n = 1, 2, 3, 4, 5$ are shown below (with $n = 1, 3, 5$ on the left and $n = 2, 4$ on the right).



Odd functions $f(-x) = -f(x)$



Even functions $f(-x) = f(x)$

Important features of $f(x) = x^n$

1. The x-axis is a tangent to each graph at the origin ($n \neq 1$).
2. $f(x) = x^n$ for even values of n defines **even functions** (see above right).
   For even functions, $f(-x) = f(x)$, so their graphs are symmetrical about the y-axis.
   For example, if $f(x) = x^4$, $f(-x) = (-x)^4 = x^4 = f(x)$.
3. $f(x) = x^n$ for odd values of n defines **odd functions** (see above left).
   For odd functions, $f(-x) = -f(x)$. Because $f(-x)$ and $f(x)$ are opposite in sign, the graph of f for $x \leq 0$ can be obtained by rotating the graph of f for $x \geq 0$ through 180° about the origin.
   For example, if $f(x) = x^3$, $f(-x) = (-x)^3 = -x^3 = -f(x)$.
4. Recognising that a function is odd or even means that you only need to draw half of the graph in detail. The other half can then be drawn using the symmetry properties.
5. Odd functions will have an inverse function, although some may require a restriction on the domain.
6. Even functions will not have a single inverse function, but can be split into two parts (by restricting the domain) so that each part has an inverse function. (Inverse functions are covered in Chapter 10.)
7. Note that most functions are neither even nor odd, e.g. $f(x) = x^2 + x$, $f(x) = e^x$.

Graphs and graphing software

It is a good idea to use GeoGebra or other graphing software to check your sketches. Software-generated graphs will also allow you to zoom in and to easily find the coordinates of important points.

Example 10

Sketch the graph of the following.

(a) $f(x) = x^3$ **(b)** $f(x) = (x-2)^3$ **(c)** $f(x) = (x-2)^3 + 1$

Solution

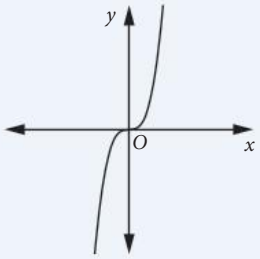
(a) $f(x) = x^3 = 0$ for $x = 0$

For all $x < 0, f(x) < 0$

For all $x > 0, f(x) > 0$

$f(-a) = (-a)^3, f(a) = a^3$

$\therefore\ f(-x) = -f(x)$ so the function is odd.



(b) $f(x) = (x-2)^3$

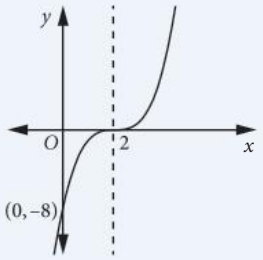
$f(x) = 0$ at $x = 2$

The graph of $f(x) = (x-2)^3$ can be obtained from the graph of $f(x) = x^3$ by a translation of 2 units to the right, parallel to the x-axis.

For all $x < 2, f(x) < 0$

For all $x > 2, f(x) > 0$

The function does not pass through the origin, so it cannot be odd.

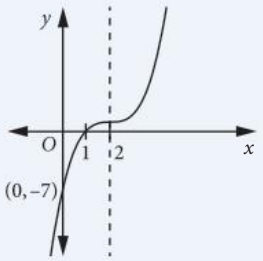


(c) $f(x) = (x-2)^3 + 1$

$f(x) = 0$ at $x = 1$

The graph of $f(x) = (x-2)^3 + 1$ can be obtained from the graph of $f(x) = (x-2)^3$ by a translation of 1 unit upwards, parallel to the y-axis.

In general, the graph of $f(x) = (x+b)^n + c$ will have the same general shape as the graph of $f(x) = x^n$.



Cubic functions

A general cubic function is a polynomial function f of the third degree, defined by $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$. Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$, where α is a real number. Factors like this can be found using the factor theorem (see section 2.3, page 47).

Example 11

Sketch the graph of f where $f(x) = (x+1)(x-2)(x-3)$.

Solution

At the points where the graph of f crosses the x-axis, $f(x) = 0$:

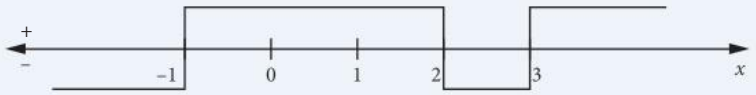
i.e. $(x+1)(x-2)(x-3) = 0$

$x = -1, 2, 3$

The function value will change sign from positive to negative or from negative to positive at these points ($x = -1, x = 2, x = 3$).

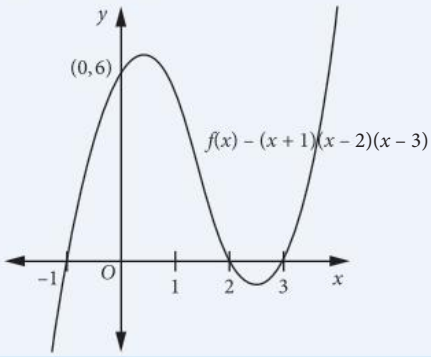
- For all $x < -1$, each of the three factors is negative, so $f(x) < 0$. e.g. $x = -2$: $f(-2) = -20 < 0$
- For $-1 < x < 2$, the factors $(x + 1) > 0$, $(x - 2) < 0$, $(x - 3) < 0$, so $f(x) > 0$. e.g. $x = 0$: $f(0) = 6 > 0$
- For $2 < x < 3$, the factors $(x + 1) > 0$, $(x - 2) > 0$ and $(x - 3) < 0$, so $f(x) < 0$. e.g. $x = 2.5$: $f(2.5) = -0.875 < 0$
- For all $x > 3$, each of the factors is positive, so $f(x) > 0$. e.g. $x = 4$: $f(4) = 20 > 0$

This information can be summarised in a sign diagram:



At the point where the graph of f crosses the y-axis, $x = 0$, so $f(0) = 1 \times (-2) \times (-3) = 6$.
Thus the graph crosses the y-axis at $(0, 6)$.

Note that this cubic equation has a positive coefficient of x^3, so that as x increases, y increases also (except in the domain between the two turning points). For $x > 3$, $f(x) > 0$ for all values of x.



Example 12

Sketch the graph of f, where $f(x) = 2x^3 - x^2 - 13x - 6$, and find the values of x for which $2x^3 - x^2 - 13x - 6 > 0$.

Solution

It is useful to find the linear factors of $2x^3 - x^2 - 13x - 6$. (Remember there will always be at least one.)

$f(1) = 2 - 1 - 13 - 6 = -18 \neq 0$ so $(x - 1)$ is not a factor
$f(-1) = -2 - 1 + 13 - 6 = 4 \neq 0$ so $(x + 1)$ is not a factor
$f(2) = 16 - 4 - 26 - 6 = -20 \neq 0$ so $(x - 2)$ is not a factor
$f(-2) = -16 - 4 + 26 - 6 = 0$ so $(x + 2)$ is a factor

To get the other linear factors (if any), we can divide $2x^3 - x^2 - 13x - 6$ by $(x + 2)$ and then factorise the quotient.

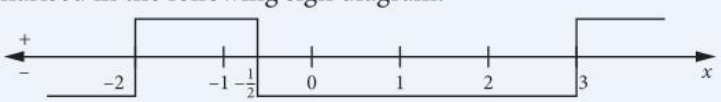
$$
\begin{array}{r}
2x^2 - 5x - 3 \\
x+2 \overline{)\,2x^3 - x^2 - 13x - 6} \\
\underline{2x^3 + 4x^2} \\
-5x^2 - 13x \\
\underline{-5x^2 - 10x} \\
-3x - 6 \\
\underline{-3x - 6} \\
0
\end{array}
$$

The factors of $2x^2 - 5x - 3$ are $(2x + 1)$ and $(x - 3)$:

$\therefore$ $f(x) = (x + 2)(2x + 1)(x - 3)$
$f(x) = 0$ at $x = -2, -\frac{1}{2}, 3$

- For all $x < -2$, $f(x) < 0$ e.g. $x = -3$: $f(-3) = -30 < 0$
- For $-2 < x < -\frac{1}{2}$, $f(x) > 0$ e.g. $x = -1$: $f(-1) = 4 > 0$
- For $-\frac{1}{2} < x < 3$, $f(x) < 0$ e.g. $x = 0$: $f(0) = -6 < 0$
- For all $x > 3$, $f(x) > 0$ e.g. $x = 4$: $f(4) = 54 > 0$

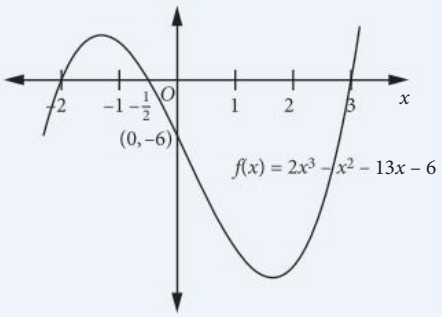
This information is summarised in the following sign diagram.



Hence $f(x) > 0$ for $-2 < x < -\frac{1}{2}$ and for $x > 3$.

At $x = 0$, $f(0) = -6$, so the graph crosses the y-axis at $(0, -6)$.

Note that this cubic equation has a positive coefficient of x^3, so that as x increases, y increases also (except in the domain between the two turning points). For $x > 3$, $f(x) > 0$ for all values of x.



Example 13

Without showing too much detail, sketch graphs of the polynomial functions defined by the following rules.

(a) $y = (x+2)(x-1)^2(x+1)$ **(b)** $y = x^2(x-3)(x+1)^3$

Solution

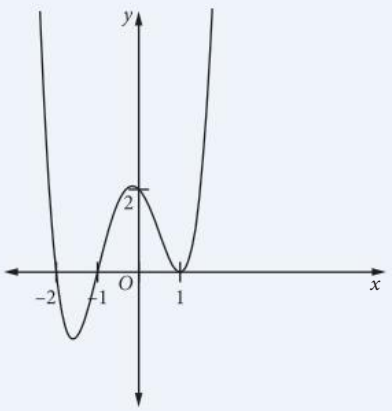
(a) $y = (x+2)(x-1)^2(x+1)$. When $|x|$ is large, y behaves like x^4.

$(x-1)^2 \geq 0$, so the sign of y is determined by $(x+2)(x+1)$.

Hence $y \geq 0$ for $x < -2$ and for $x > -1$ (equality at $x = 1$);
$y < 0$ for $-2 < x < -1$

Near $x = -2$, y behaves like a multiple of $(x+2)$ and cuts the x-axis like a straight line at $x = -2$. Similarly at $x = -1$.

Near $x = 1$, y behaves like a multiple of $(x-1)^2$ so that $x = 1$ is a stationary point (minimum turning point) and the x-axis is a tangent to the curve at $x = 1$.



(b) $y = x^2(x-3)(x+1)^3$. When $|x|$ is large, y behaves like x^5.

$x^2 \geq 0$, $(x+1)^2 \geq 0$, so the sign of y is determined by $(x-3)(x+1)$.

Hence $y \geq 0$ for $x < -1$ and for $x > 3$ (equality at $x = -1$ and $x = 3$);
$y \leq 0$ for $-1 < x < 3$ (equality at $x = 0$).

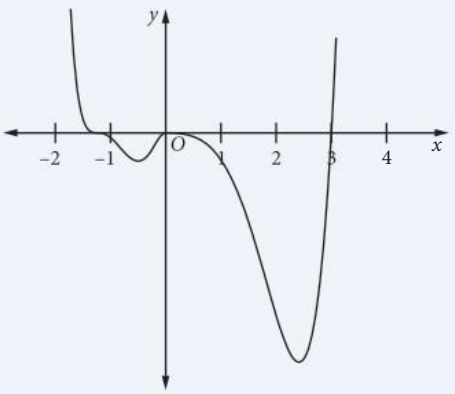
Near $x = 3$, y behaves like a multiple of $(x-3)$ and cuts the x-axis like a straight line at $x = 3$.

Near $x = 0$, y behaves like a multiple of x^2 and the graph touches the x-axis at $x = 0$.

Near $x = -1$, y behaves like a multiple of $(x+1)^3$ and the graph cuts the x-axis at $x = -1$ like a basic cubic curve, i.e. with a horizontal point of inflexion.

This sketch is **not** drawn to scale. The minimum between $x = -1$ and $x = 0$ cannot be shown at a reasonable scale without the other minimum being below the page.

You can use GeoGebra or other graphing software to draw this graph. As you zoom out to find the minimum turning point between $x = 0$ and $x = 3$, the curve between $x = -1$ and $x = 0$ appears to become a straight line.



Summary of polynomial functions

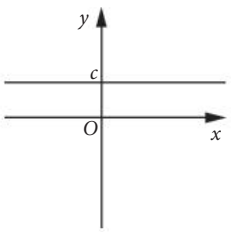
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, then:

1. For very large $|x|$, $P(x) \approx a_n x^n$.
2. A polynomial of odd degree always has at least one real zero (i.e. its graph cuts the x-axis at least once).
3. At least one maximum or minimum value of P occurs between any two distinct real zeros.
4. For a polynomial of odd degree, the ends of the graph go in opposite directions.
5. For a polynomial of even degree, the ends of the graph go in the same direction.
6. When the graph of a polynomial function meets the x-axis, it may cut it (single zero), touch it (double zero) or cut it at a point of inflexion (triple zero).

Polynomial graphs, standard forms

Degree 0 (constant)

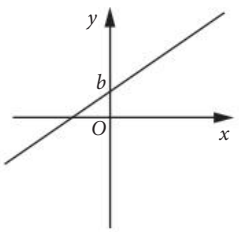
$P(x) = c$



Does not cut
No zeros

Degree 1 (linear)

$P(x) = mx + b$

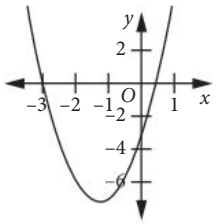


Cuts once
One zero

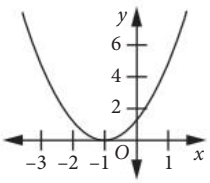
Degree 2 (quadratic)

$P(x) = ax^2 + bx + c$

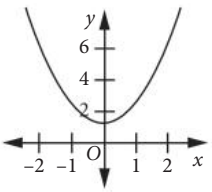
***a* > 0:**



Cuts twice
$\Delta > 0$
Two zeros

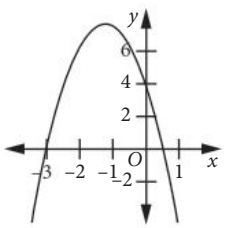


Touches
$\Delta = 0$
Double zero

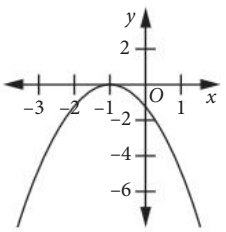


Does not cut
$\Delta < 0$
No real zeros

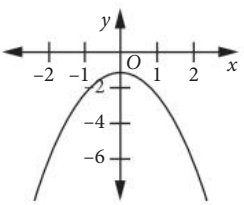
***a* < 0:**



Cuts twice
$\Delta > 0$
Two zeros



Touches
$\Delta = 0$
Double zero



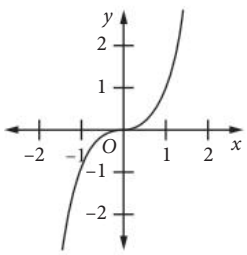
Does not cut
$\Delta < 0$
No real zeros

A quadratic polynomial may have two, one or no real zeros.

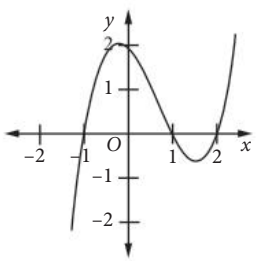
Degree 3 (cubic)

$P(x) = ax^3 + bx^2 + cx + d$

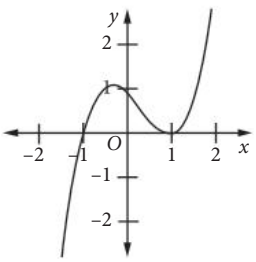
***a* > 0:**



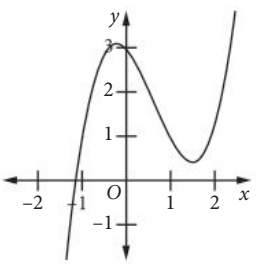
Cuts once
Triple zero



Cuts three times
Three zeros

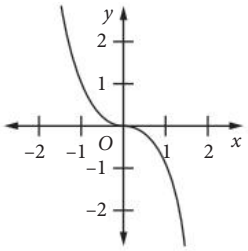


Cuts once and touches
One zero, one double zero

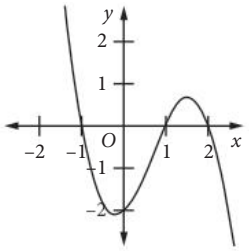


Cuts once
One zero

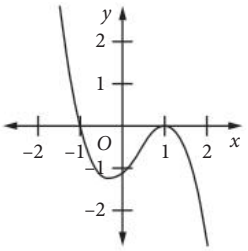
$a < 0$:



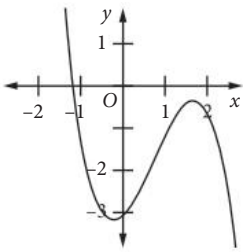
Cuts once
Triple zero



Cuts three times
Three zeros



Cuts once and touches
One zero, one double zero



Cuts once
One zero

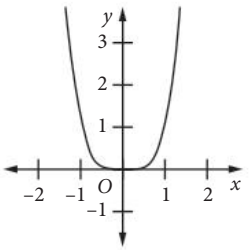
A cubic polynomial may have one, two or three real zeros.

A cubic polynomial always has at least one real zero.

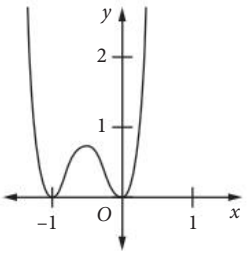
Degree 4 (quartic)

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

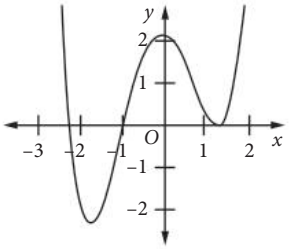
$a > 0$:



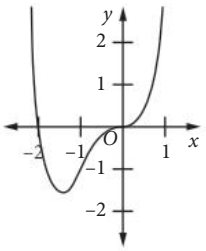
Touches
Quadruple zero



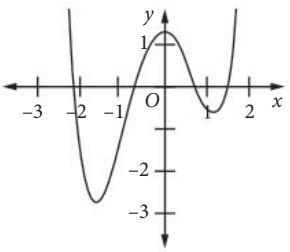
Touches twice
Two double zeros



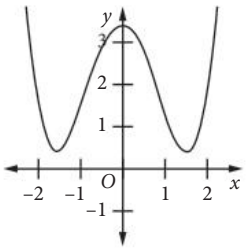
Touches once, cuts twice
Two zeros, one double zero



Cuts twice
One zero, one triple zero



Cuts four times
Four zeros



Does not cut
No real zeros

The addition of an appropriate constant to each equation can create a polynomial with no real zeros.

$a < 0$:

A negative a inverts each of the six graphs above so that they open downwards, with the properties of their zeros the same.

Summary of quartic polynomials

1. A quartic polynomial may have four, three, two, one or no real zeros.
2. If a quartic polynomial has only one real zero then it must be a quadruple zero.
3. If a quartic polynomial has only two distinct real zeros, then they are either a triple zero and a single zero or they are both double zeros.
4. If a quartic polynomial has only three distinct real zeros, then they are a double zero and two single zeros.
5. If a quartic polynomial has four distinct real zeros, then it can be factorised into four real linear factors.
6. If a quartic polynomial has no real zeros, then it cannot be factorised into any real linear factors.

EXERCISE 2.4 POLYNOMIAL FUNCTIONS

1 Sketch the graph of each function and find the values of x for which: **(i)** $f(x) = 0$ **(ii)** $f(x) > 0$

(a) $f(x) = x(x-2)(x+3)$ **(b)** $f(x) = 2(x-\frac{1}{2})(x+1)(2x+3)$ **(c)** $f(x) = x(x-2)^2$

2 Sketch the graph of f where $f(x)$ is the polynomial given. Draw a sign diagram and label your sketch showing the points of intersection with the axes. Find the values of x for which $f(x) \geq 0$.

(a) $f(x) = x^3 - 4x$ **(b)** $f(x) = x^3 + 4x^2 + 4x$ **(c)** $f(x) = 2x^2 - x^3$ **(d)** $f(x) = x^3 - x^2 - 10x - 8$

3 **(a)** Find the linear factors of $6 + 5x - 2x^2 - x^3$.
(b) Find the values of x for which: **(i)** $6 + 5x - 2x^2 - x^3 = 0$ **(ii)** $6 + 5x - 2x^2 - x^3 > 0$
(c) Sketch the graph of f where $f(x) = 6 + 5x - 2x^2 - x^3$.

4 **(a)** Find the linear factors of $x^3 - 5x^2 + 8x - 4$.
(b) Find the values of x for which: **(i)** $x^3 - 5x^2 + 8x - 4 = 0$ **(ii)** $x^3 - 5x^2 + 8x - 4 > 0$
(c) Sketch the graph of f where $f(x) = x^3 - 5x^2 + 8x - 4$.

5 **(a)** Find the linear factors of $-x^3 + 2x^2 + x - 2$.
(b) Draw a sign diagram to find the values of x for which $-x^3 + 2x^2 + x - 2 > 0$.
(c) Sketch the graph of f where $f(x) = -x^3 + 2x^2 + x - 2$.

6 The number of zeros of the polynomial $f(x) = x^3 - x$ is:

A 0 **B** 1 **C** 2 **D** 3

7 Describe how the graph of $f(x) = x^3$ can be transformed into the graph of the following.

(a) $f(x) = (x+3)^3$ **(b)** $f(x) = (x+3)^3 - 2$

8 Show that the graph of f, where $f(x) = x^3 - 8$, cuts the x-axis at one point only.

9 Show that the graph of f, where $f(x) = x^3 - x^2 - 8x + 12$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

10 Show that the graph of f, where $f(x) = x^3 - 4x^2 + 8x - 8$, cuts the x-axis at one point only.

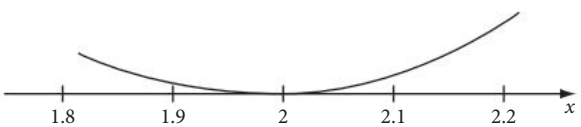
11 Sketch graphs to show the general form of the following functions.

(a) $y = (x+1)(x-2)(x+3)$ **(b)** $y = x(x-1)(x-2)(x-3)$ **(c)** $y = x(x-1)^2$ **(d)** $y = x^2(x-1)$
(e) $y = x(x+3)^2(2-x)$ **(f)** $y = (x-1)(2x-3)(x+4)^2$ **(g)** $y = x(2x-1)(x-1)^3$
(h) $y = x(x+2)(x-1)^4$ **(i)** $y = (3x-5)^3(x-1)^2$ **(j)** $y = x(x-1)^2(2-x)^3$

12 Part of the graph of $y = P(x)$ is shown, where $P(x)$ is a polynomial of degree four:

Which of the following could be the polynomial $P(x)$?

A $P(x) = x^2(x+2)^2$ **B** $P(x) = (x-2)^4$
C $P(x) = x(x-2)^3$ **D** $P(x) = (x-1)^2(x+2)^2$



13 Sketch graphs to show the basic features of the following functions.

(a) $y = x(x-2)^3(x+1)^4$ **(b)** $y = x(x^2-1)(x-1)^2$
(c) $y = (3-x)(1-x)^4(x+2)$ **(d)** $y = (2-x)^3(1+x)^2(x-3)$

2.5 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

Quadratic equations

The general quadratic equation is: $ax^2 + bx + c = 0, a \neq 0$

Dividing both sides by a: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ [1]

Let the roots of this equation be α and β.

Hence: $(x-\alpha)(x-\beta)=0$

$$x^2-(\alpha+\beta)x+\alpha\beta=0 \quad [2]$$

i.e. $x^2-(\text{sum of roots})x+(\text{product of roots})=0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha+\beta=-\frac{b}{a}$ [3]

Product of roots: $\alpha\beta=\frac{c}{a}$ [4]

The equations [3] and [4] show the relationships between the roots α, β and the coefficients a, b, c of a quadratic equation.

Example 14

Write the quadratic equation with roots that are the squares of the roots of $2x^2+3x+5=0$.

Solution

We can answer this without solving the original equation.

Let the roots of $2x^2+3x+5=0$ be α and β.

Now $a=2$, $b=3$, $c=5$: $\alpha+\beta=-\frac{3}{2}$ and $\alpha\beta=\frac{5}{2}$

The new equation is of the form: $x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$

But $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ so $\alpha^2+\beta^2=\left(-\frac{3}{2}\right)^2-2\times\frac{5}{2}=\frac{9}{4}-5=-\frac{11}{4}$

and $\alpha^2\beta^2=\frac{25}{4}$

Hence the required equation is: $x^2+\frac{11}{4}x+\frac{25}{4}=0$

i.e. $4x^2+11x+25=0$

Two useful identities:

1. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
2. $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

Cubic equations

The general cubic equation is: $ax^3+bx^2+cx+d=0$, $a\neq 0$

Dividing both sides by a: $x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$ [1]

Let the roots of this equation be α, β and γ.

Hence: $(x-\alpha)(x-\beta)(x-\gamma)=0$

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0 \quad [2]$$

i.e. $x^3-(\text{sum of roots})x^2+(\text{sum of products of each pair of roots})x-(\text{product of roots})=0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha+\beta+\gamma=-\frac{b}{a}$ [3]

Sum of products of each pair of roots: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$ [4]

Product of roots: $\alpha\beta\gamma = -\frac{d}{a}$ [5]

The equations [3], [4] and [5] show the relationships between the roots α, β, γ and the coefficients a, b, c, d of a cubic equation.

Note:
These relationships between the roots and the coefficients are not enough to find the roots of an equation without some additional information.

Example 15

Show that $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$.

Solution

$$\begin{aligned}\text{RHS} &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \alpha\beta + \alpha\gamma + \alpha\beta + \beta^2 + \beta\gamma + \alpha\gamma + \beta\gamma + \gamma^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= \alpha^2 + \beta^2 + \gamma^2 = \text{LHS}\end{aligned}$$

Example 16

If α, β, γ are the roots of the equation $x^3 + 2x^2 + 3x + 4 = 0$, find the value of the following.

(a) $\alpha + \beta + \gamma$ **(b)** $\alpha\beta + \alpha\gamma + \beta\gamma$ **(c)** $\alpha\beta\gamma$ **(d)** $(\alpha - 1)(\beta - 1)(\gamma - 1)$

(e) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(f)** $\alpha^2 + \beta^2 + \gamma^2$ **(g)** $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}$

Solution

(a) Sum of roots $= \alpha + \beta + \gamma = -\frac{b}{a} = -2$

(b) Sum of products of pairs of roots $= \alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = 3$

(c) Product of roots $= \alpha\beta\gamma = -\frac{d}{a} = -4$

(d) $$\begin{aligned}(\alpha - 1)(\beta - 1)(\gamma - 1) &= \alpha\beta\gamma - (\alpha\beta + \alpha\gamma + \beta\gamma) + (\alpha + \beta + \gamma) - 1\\ &= -4 - 3 - 2 - 1\\ &= -10\end{aligned}$$

(e) $$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}\\ &= -\frac{3}{4}\end{aligned}$$

(f) $$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)\\ &= (-2)^2 - 2 \times 3\\ &= -2\end{aligned}$$

(g) $$\begin{aligned}\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} &= \frac{\gamma + \beta + \alpha}{\alpha\beta\gamma}\\ &= \frac{-2}{-4}\\ &= \frac{1}{2}\end{aligned}$$

Another useful identity:

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

Example 17

Solve the equation $2x^3 - 7x^2 - 12x + 45 = 0$ given that two of its roots are equal.

Solution

Let the roots be α, α, β.

Sum of roots: $2\alpha + \beta = \frac{7}{2}$ [1]

Sum of products of pairs of roots: $\alpha^2 + 2\alpha\beta = -6$ [2]

Product of roots: $\alpha^2\beta = -\frac{45}{2}$ [3]

From [1]: $\beta = \frac{7}{2} - 2\alpha$

Substitute into [2]:
$$\alpha^2 + 2\alpha\left(\frac{7}{2} - 2\right) = -6$$
$$\alpha^2 + 7\alpha - 4\alpha^2 = -6$$
$$3\alpha^2 - 7\alpha - 6 = 0$$
$$(3\alpha + 2)(\alpha - 3) = 0$$
$$\alpha = 3 \text{ or } -\frac{2}{3}$$

Substitute into [1]: $\beta = -\frac{5}{2}$ or $\frac{29}{6}$

Substitute $\alpha = 3, \beta = -\frac{5}{2}$ into [3]: $\text{LHS} = 3^2 \times \left(-\frac{5}{2}\right) = -\frac{45}{2} = \text{RHS}$

Substitute $\alpha = -\frac{2}{3}, \beta = \frac{29}{6}$ into [3]: $\text{LHS} = \left(-\frac{2}{3}\right)^2 \times \frac{29}{6} = -\frac{58}{27} \neq \text{RHS}$

Therefore the roots of the equation are 3, 3, –2.5.

Example 18

If the roots of the equation $x^3 + px^2 + qx + r = 0$ are consecutive terms of an arithmetic series, find an equation linking p, q and r.

Solution

Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$ (i.e. three terms with the common difference β).

Sum of roots: $3\alpha = -p$ [1]

Sum of products of pairs of roots: $\alpha(\alpha - \beta) + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = q$

$3\alpha^2 - \beta^2 = q$ [2]

Product of roots: $\alpha(\alpha^2 - \beta^2) = -r$ [3]

From [1]: $\alpha = -\frac{p}{3}$

Substitute into [2]:
$$\frac{p^2}{3} - \beta^2 = q$$
$$\beta^2 = \frac{p^2}{3} - q$$

Substitute into [3]:
$$-\frac{p}{3}\left(\frac{p^2}{9} - \left(\frac{p^2}{3} - q\right)\right) = -r$$
$$p\left(\frac{p^2}{9} - \frac{p^2}{3} + q\right) = 3r$$
$$p(p^2 - 3p^2 + 9q) = 27r$$
$$p(9q - 2p^2) = 27r$$

Quartic equations (4th degree)

The general quartic equation is: $ax^4 + bx^3 + cx^2 + dx + e = 0,\ a \neq 0$

Dividing both sides by a: $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$ [1]

Let the roots of this equation be α, β, γ and δ.

Hence: $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$

$x^4 - (\alpha+\beta+\gamma+\delta)\,x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta\theta+\beta\gamma+\beta\delta+\gamma\delta)\,x^2 - (\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)\,x + \alpha\beta\gamma\delta = 0$ [2]

i.e. $x^4 - (\text{sum of roots})x^3 + (\text{sum of products of each pair of roots})x^2$
$- (\text{sum of products of each triplet of roots})x + (\text{product of roots}) = 0$

Equating the coefficients of like powers of x in [1] and [2]:

Sum of roots: $\alpha+\beta+\gamma+\delta = -\frac{b}{a}$ [3]

Sum of products of pairs of roots: $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$ [4]

Sum of products of triplets of roots: $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$ [5]

Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$ [6]

EXERCISE 2.5 RELATIONSHIP BETWEEN ROOTS AND COEFFICIENTS

1 If α and β are roots of the equation $x^2 + 8x - 5 = 0$, find the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

2 If α and β are roots of the equation $x^2 + 4x + 1 = 0$, find the value of:

(a) $\alpha+\beta$ **(b)** $\alpha\beta$ **(c)** $\alpha^2+\beta^2$ **(d)** $\alpha^3+\beta^3$

3 If α and β are roots of the equation $x^2 + mx + n = 0$, find the roots of $nx^2 + (2n - m^2)x + n = 0$ in terms of α and β.

4 If α and β are roots of the equation $px^2 + qx + r = 0$, find the following in terms of p, q and r.

(a) $\frac{1}{\alpha}+\frac{1}{\beta}$ **(b)** $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

5 If α and β are roots of the equation $x^2 + 5x + 7 = 0$, the quadratic equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$ is:

A $x^2 + 22x + 75 = 0$ **B** $x^2 + 22x - 75 = 0$ **C** $x^2 - 22x - 75 = 0$ **D** $x^2 - 22x + 75 = 0$

6 Solve the equation $x^3 - 3x^2 - 4x + 12 = 0$ given that the sum of two of its roots is zero.

7 Solve the equation $2x^3 - 21x^2 + 42x - 16 = 0$ given that the roots are consecutive terms of a geometric series. (Hint: Let the roots be $\frac{\alpha}{\beta}$, α, $\alpha\beta$.)

8 Solve the equation $4x^3 + 32x^2 + 79x + 60 = 0$ given that one root is equal to the sum of the other two roots.

9 Solve the equation $3x^3 - 17x^2 - 8x + 12 = 0$ given that the product of two of the roots is 4.

10 If the roots of the equation $x^3 + px^2 + qx + r = 0$ are consecutive terms of a geometric series, prove that $q^3 = p^3r$. Show that this condition is satisfied for the equation $8x^3 - 100x^2 + 250x - 125 = 0$ and solve this equation.

11 Find two values of m such that the roots of the equation $x^3 + 2x^2 + mx - 16 = 0$ are α, β, $\alpha\beta$. Using these values of m, find α and β.

12 If the roots of the equation $24x^4 - 52x^3 + 18x^2 + 13x - 6 = 0$ are α, $-\alpha$, β, $\frac{1}{\beta}$, find the values of α and β.

13 If two of the roots of the equation $x^3 + qx + r = 0$ are equal, show that $4q^3 + 27r^2 = 0$.

14 If the roots of the equation $x^3 + 3x^2 - 2x + 1 = 0$ are α, β, γ, find the value of:

(a) $\alpha^2(\beta+\gamma) + \beta^2(\gamma+\alpha) + \gamma^2(\alpha+\beta)$ **(b)** $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

15 Solve the equation $4x^3 - 12x^2 + 9x - 2 = 0$ given that two of its roots are equal.

16 Solve the equation $6x^4 - 11x^3 - 26x^2 + 22x + 24 = 0$ given that the product of two of the roots is equal to the product of the other two roots.

17 Solve the equation $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$ given that the product of two of the roots is 2.

18 Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$ given that the roots are consecutive terms of an arithmetic series.

19 Solve the equation $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$ given that the sum of two of its roots is zero.

20 If α, β, γ are the roots of $3x^3 + 8x^2 - 1 = 0$, find the value of: $\left(\beta + \frac{1}{\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right)\left(\alpha + \frac{1}{\beta}\right)$

21 Solve the equation $x^4 - 2x^3 - 3x^2 + 4x + 2 = 0$ given that the sum of two of its roots is zero.

22 If the roots of the equation $x^4 - 6x^3 + 3x^2 + 12x + 4 = 0$ are $\alpha, 2\alpha, \beta, 2\beta$, find the values of α and β.

CHAPTER REVIEW 2

1 Perform the following polynomial divisions.

(a) $(x^3 + 2x^2 - 3x + 4) \div (x - 1)$ **(b)** $(4x^4 - 6x^2 + 10x - 40) \div (x + 3)$

2 Use the remainder theorem to find the remainder of the following.

(a) $x^3 - 4x^2 + 3x - 5$ divided by $(x - 2)$ **(b)** $x^4 + x^3 - 5x^2 + 4x - 2$ divided by $(x + 1)$

3 Use the factor theorem to find the linear factors (over the rational number field) of each polynomial.

(a) $x^3 - 2x^2 - 5x + 6$ **(b)** $x^3 + 7x^2 + 14x + 8$ **(c)** $x^3 + 5x^2 - x - 5$ **(d)** $x^4 - x^3 - 16x^2 + 4x + 48$

4 Let $P(x) = (x - 1)(x + 2)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are real numbers. The polynomial $P(x)$ has a factor of $x + 2$. When $P(x)$ is divided by $x - 1$ the remainder is 6.

(a) Find the values of a and b.

(b) Find the remainder when $P(x)$ is divided by $(x - 1)(x + 2)$.

5 Let $P(x) = x^3 + ax^2 - x + 1$ be a polynomial where a is a real number. When $P(x)$ is divided by $x - 2$ the remainder is 15. Find the remainder when $P(x)$ is divided by $x + 3$.

6 The polynomial $P(x) = x^3 + ax - b$ has a remainder of 5 when divided by $(x + 1)$ and a remainder of 2 when divided by $(x - 2)$. Find the values of a and b and hence find the remainder when $P(x)$ is divided by $(x - 3)$.

7 The polynomial $P(x)$ is given by $P(x) = ax^3 + 15x^2 + cx - 72$, where a and c are constants. The three zeros of $P(x)$ are -3, 2 and α. Find the value of α.

8 The cubic polynomial $P(x) = x^3 + bx^2 + cx + d$ (where b, c, d are real numbers) has three real zeros: -1, α and $-\alpha$.

(a) Find the value of b. **(b)** Find the value of $c - d$.

9 The polynomial $P(x) = x^3 - 4x^2 + kx + 12$ has zeros α, β, γ.

(a) Find the value of $\alpha + \beta + \gamma$. **(b)** Find the value of $\alpha\beta\gamma$.

(c) Two of the three zeros are equal in magnitude but opposite in sign. Find the third zero and hence find the value of k.

10 Sketch graphs of each function. For what values of x is each function positive?

(a) $y = (x - 1)(x + 2)(x - 3)$ **(b)** $y = (x - 2)(x + 2)^2$ **(c)** $y = x(x^2 - 1)(x + 2)$ **(d)** $y = x^2(x - 2)^2$

CHAPTER 3
PERMUTATIONS AND COMBINATIONS

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE2 uses multi-step deductive reasoning in a variety of contexts

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

3.1 FUNDAMENTAL COUNTING PRINCIPLE

If an outcome can happen in m different ways, and a second outcome can happen in n different ways, then the total number of ways in which the two outcomes can happen together is $m \times n$.

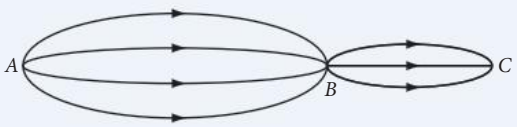
Total number of ways $= m \times n$ where m = number of ways first outcome can happen
n = number of ways second outcome can happen

For example, this simply means that if you have 5 different shirts and 7 different hats, then the total number of different selections of a shirt and hat is $5 \times 7 = 35$.

This is also known as the **multiplication principle**.

Example 1

There are four roads from town A to town B, and three roads from town B to town C.
How many different ways are there to travel by road from A to B to C?



Solution

Consider the first road from A to B. After this road there are three ways to travel from B to C.

Similarly, after taking the second road from A to B there are then three ways to go from B to C.

As there are 4 different ways from A to B, and after each of these there are 3 different ways to go from B to C, there are in total $4 \times 3 = 12$ ways to go from A to B to C (i.e. from A to C via B).

Example 2

The old New South Wales black-and-yellow number plates showed three letters and three digits. How many different number plates could be made?

Solution

There are 26 letters of the alphabet and 10 digits to choose from.

Number of number plates $= 26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 \times 10^3 = 17\,576\,000$

Example 3

In how many different ways can six people A, B, C, D, E, F be arranged:

(a) in a row of 3 **(b)** in a row of 6?

Solution

(a) There is a choice of 6 people for the first place.

There is a choice of 5 people for the second place, as one person already occupies the first place.

There is a choice of 4 people for the third place, as two people have already been used.

Hence there are $6 \times 5 \times 4 = 120$ different ways to fill the three places in the row.

This problem could also be considered using a diagram to represent the places:

6	5	4

Each number shows how many ways there are to fill each place.

Using the fundamental counting principle, this gives $6 \times 5 \times 4 = 120$ different ways.

(b) The same method as in part **(a)**, continued for six places:

6	5	4	3	2	1

Number of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Factorial notation

Factorial notation uses an exclamation mark **!** as a short way to write the product of consecutive positive integers. The answer to Example **3(b)** above can be written as 6!, which means the product of the consecutive integers from 6 to 1.

6! is read as '6 factorial' or 'factorial 6'. In general, the product of n consecutive positive integers from n to 1 is written $n!$ and read as 'n factorial'.

$$n! = n(n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$$

Make sure you know how to use the $n!$ key on your calculator.

Example 4

A group of 6 boys and 5 girls decide to go to the movies.

(a) In how many ways can the boys sit together in a row?

(b) In how many ways can the girls sit together in a row?

(c) In how many ways can the whole group sit together in a row?

(d) Why is the answer to part **(c)** not the product of the answers in parts **(a)** and **(b)**?

Solution

(a) The 6 boys can sit together in 6! ways, i.e. 720 ways.

(b) The 5 girls can sit together in 5! ways, i.e. 120 ways.

(c) The group of 11 people can sit together in 11! ways, i.e. 39 916 800 ways.
(*Note:* This answer is best written simply as '11!' rather than as '39 916 800'.)

(d) In part **(c)** there is no grouping by gender, so we are arranging 11 different people in one big group. But multiplying the answers to parts **(a)** and **(b)** would keep the two groups separate, only finding half the number of ways a group of 6 people can sit in a row with a group of 5 people.

EXERCISE 3.1 FUNDAMENTAL COUNTING PRINCIPLE

1 There are five roads from town A to town B, and two roads from town B to town C. In how many different ways can you travel by road from A to B to C?

2 A man has three pairs of shoes, four suits and six ties. How many different sets of shoes, suits and ties can he wear?

3 In how many ways can seven books be arranged in a row?

4 A Mathematics test contains 20 multiple-choice questions. Each question has four possible answers, A, B, C and D. If a student guesses every answer, in how many different ways can the answers be given?

5 A restaurant menu has three choices of soup, five choices of main course and three choices of dessert. How many different meals of soup, main course and dessert are possible?

6 There are 10 candidates for school captain and vice-captain. The number of different ways they might be selected is:

A 90 **B** 45 **C** 10 **D** 9

7 New South Wales black-and-white number plates consist of 3 letters, 2 digits and 1 letter. How many different number plates can be made?

8 How many arrangements of the letters of the word PENCIL are possible?

9 In how many different ways can A, B, C, D, E be arranged:

(a) in a row of three **(b)** in a row all together?

10 The newer New South Wales black-and-yellow number plates consist of two letters, two digits and two letters.

(a) How many different number plates can be made?

(b) What is the reason for changing the number plates from 3 letters and 3 digits to 2 letters, 2 digits, 2 letters?

11 The Olympic teams from 8 countries are swimming in the 200-metre relay final. How many different finishing results are possible? (Assume no tied results.)

12 The digits 0 to 9 are used to make 10-digit numbers (not beginning with zero). How many different numbers are possible if:

(a) each digit can be used only once

(b) each digit can be used any number of times?

3.2 PERMUTATIONS

A **permutation** is an ordered selection or arrangement of all or part of a set of objects.

It is very important to remember that permutations are used *when the **order** is important!*

Example **3(b)** above is an example of the number of arrangements of a set of six objects. Example **3(a)** is an example of the number of arrangements of three members of a set of six objects. Note that in general:

- the number of arrangements of n different objects is $n!$
- the number of arrangements of 3 objects taken from a set of n different objects is $n(n-1)(n-2)$.

Consider the following statement: $n(n-1)(n-2) = \dfrac{n(n-1)(n-2)(n-3)(n-4)\times \quad \times 3\times 2\times 1}{(n-3)(n-4)\times \quad \times 3\times 2\times 1} = \dfrac{n!}{(n-3)!}$

Example 5

From a standard pack of 52 playing cards, the set of hearts (♥) are taken. In how many ways:

(a) can the 13 cards be arranged in a row

(b) can 6 of the 13 cards be arranged in a row?

Solution

(a) 13 cards can be arranged in 13! ways.

(b) 6 out of 13 cards can be arranged in $13 \times 12 \times 11 \times 10 \times 9 \times 8 = 1\,235\,520$ ways.

This could be written as $\frac{13!}{(13-6)!} = \frac{13!}{7!} = 1\,235\,520$ ways.

Number of permutations ${}^nP_r$

The symbol ${}^nP_r$ means the number of arrangements (i.e. ordered subsets, permutations) of any n different objects taken r at a time (or in other words, the arrangements of r objects taken from a group of n objects). This denotes the number of ways of filling r places with the n different objects that are available ($r \leq n$).

Using the diagram notation from Example **3** above:

${}^nP_r =$	n	$n-1$	$n-2$	…	$n-r+1$

The first place can be filled in n ways, because any one of the n objects can occupy this place. When the first place has been filled in any one of these ways, there remain $(n-1)$ objects, any one of which can occupy the second place. By the fundamental counting principle, each way of filling the first place can be associated with each way of filling the second place, so the number of ways of filling the first two places is $n(n-1)$.

Similarly, after the first two places have been filled in any one of these ways there remain $(n-2)$ objects, any one of which can occupy the third place. Hence the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing this pattern, the number of ways of filling r places is given by:

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\ldots(n-r+1) \\ &= \frac{n(n-1)(n-2)\quad(n-r+1)(n-r)!}{(n-r)!} \qquad [1] \end{aligned}$$

$$\therefore \quad {}^nP_r = \frac{n!}{(n-r)!} \qquad [2]$$

Similarly, note that the symbol ${}^nP_n$ would denote the number of ways of filling n places with n objects available:

From [1]: $${}^nP_n = n(n-1)(n-2)\ldots(n-n+1)$$
$$= n(n-1)(n-2)\times\ldots\times 3\times 2\times 1 = n!$$

But from [2]: $${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

Hence we have: $$n! = \frac{n!}{0!}$$

For consistency, we therefore define 0! to be equal to 1: $0! = 1$

Example 6

(a) In how many ways can a first, second and third prize be awarded in a class of 10 students?

(b) In how many ways can a Mathematics prize, a Physics prize and a Chemistry prize be awarded in a class of 10 students?

Solution

(a) This question is asking us to find how many different arrangements of 3 students can be selected from 10 students.

Number of ways $= {}^{10}P_3 = 10 \times 9 \times 8 = 720$

(b) In this question each student could win any or all of the prizes, so permutations are not used.

Number of ways $= 10 \times 10 \times 10 = 1000$

Arrangements with restrictions

Example 7

In how many ways can six students and two teachers be arranged in a row if:

(a) the two teachers are together

(b) the two teachers are not together

(c) there are at least three students separating the teachers?

Solution

(a) If the teachers are together they can be regarded as one 'unit'. Thus there are seven 'units' (six students plus the unit of two teachers) to be arranged in a row. This can be done in 7! ways.

However, the two teachers (label them A and B) can be arranged among themselves in 2! (i.e. two) ways, AB or BA.

$\therefore$ Number of arrangements $= 2! \times 7! = 10\,080$

(b) Number of arrangements without restriction $= 8!$

Number of arrangements with the teachers together $= 2! \times 7!$

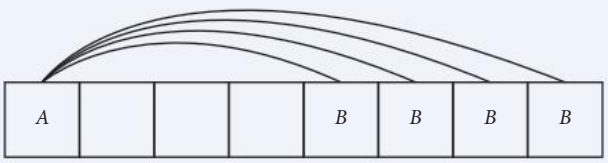
$\therefore$ Number of arrangements with the teachers not together $= 8! - 2! \times 7!$

$= 8 \times 7! - 2 \times 7!$

$= 6 \times 7! = 30\,240$

(c) There is a restriction on the places to be occupied by the teachers, so it is easiest to arrange the teachers first.

If teacher A is in the first position then teacher B can occupy the fifth, sixth, seventh or eighth position, as shown:



Similarly, for all the possible first positions of A there are:

- 4 arrangements of A and B with A in first position
- 3 arrangements of A and B with A in second position
- 2 arrangements of A and B with A in third position

- 1 arrangement of A and B with A in fourth position
- 1 arrangement of A and B with A in fifth position
- 2 arrangements of A and B with A in sixth position
- 3 arrangements of A and B with A in seventh position
- 4 arrangements of A and B with A in eighth position

$\therefore$ There are 20 different ways of arranging A and B (i.e. the sum of the above arrangements).

For each of these 20 arrangements there are six places left, which can be filled by the students in 6! ways.

$\therefore$ Number of arrangements (total) $= 20 \times 6! = 14\,400$

Example 8

How many different arrangements of the letters of the word TUESDAY are possible if:

(a) the three vowels are together

(b) the first and last places are consonants

(c) the consonants and vowels occupy alternate positions?

Solution

The vowels are A, E, U; the consonants are D, S, T, Y.

(a) As the three vowels are together we can regard them as one unit.

Thus there are five units (the four consonants plus the group of vowels) to be arranged in a row. This can be done in 5! ways.

Also, the three vowels can be arranged in 3! ways.

$\therefore$ Total number of arrangements $= 3! \times 5! = 720$

(b) There are restrictions on the first and last places, so fill them first.

- There are 4 consonants, so the first place can be filled in 4 ways.
- After the first place is filled, the last place can be filled in 3 ways.
- The remaining 5 places can be filled without restriction in 5! ways.

4	5	4	3	2	1	3

$\therefore$ Total number of arrangements $= 4 \times 3 \times 5! = 1440$

(c) The consonants (C) and vowels (V) occupy alternate positions, as shown in the table below. There are more consonants than vowels, so the consonants must occupy the end places.

C	V	C	V	C	V	C

The 4 consonants can fill their places in 4! ways. The 3 vowels can fill their places in 3! ways.

$\therefore$ Total number of arrangements $= 4! \times 3! = 144$

Arrangements in a circle

The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, can be shown to be:

$$\frac{n!}{n} = (n-1)!$$

(This is shown in Example **9** below.)

In a circular arrangement there is no start or finish, so to count the arrangements we need to fix one object as our 'starting' object.

Example 9

In how many ways can the four people Alexia, Bronwyn, Carmen and Divya be arranged in a circle?

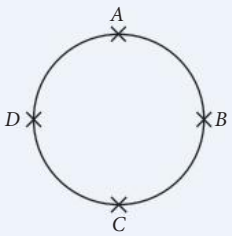
Solution

Method 1

There is no first place to fill, so we can fix any one person as our 'starting' person (or our 'fixed position') and arrange the other three people around her. This can be done in 3! ways.

Method 2

The diagram at right shows an arrangement of the four people (labelled with the first letter of each name, *A*, *B*, *C*, *D*).



If we keep *A*, *B*, *C*, *D* in the same position relative to each other and move them all one place clockwise, we still have the same arrangement—the arrangement has not changed.

If *A*, *B*, *C*, *D* were arranged in a straight line and we moved them all along one position to the right (so that the fourth position moved to the first position), then the arrangement would have changed.

A	*B*	*C*	*D*

D	*A*	*B*	*C*

C	*D*	*A*	*B*

B	*C*	*D*	*A*

You should be able to see that for every 1 arrangement of the group in a circle, there are 4 different arrangements of the same group in a straight line.

∴ Number of circular arrangements = Number of linear arrangements ÷ Number of elements

$$= \frac{4!}{4} = 3! = 6$$

Mutually exclusive operations—the addition principle

In Example **7(c)** above, *A* could occupy the first position and *B* the fifth position *or* *A* the first and *B* the sixth *or* *A* the first and *B* the seventh, and so on. Arrangements like these are said to be **mutually exclusive**, because they cannot both be true at the same time. For example, '*A* is first and *B* is fifth' excludes the possibility that '*A* is first and *B* is sixth'—these two descriptions cannot both be true at the same time.

Example 10

How many numbers greater than 6000 can be formed using the digits 3, 4, 6, 8, 9 if no digit can be used more than once per number?

Solution

Numbers containing either four digits *or* five digits can be formed.

For four-digit numbers, the first place can be filled in three ways by either 6, 8 or 9.

After the first place is filled (in any one of these three ways), there remain 4 digits, any one of which can occupy second place. Following the multiplication principle, the second place can be filled in 4 ways, the third place in 3 ways and the fourth place in 2 ways:

∴ Number of arrangements = $3 \times 4 \times 3 \times 2 = 72$

3	4	3	2

or Number of arrangements = $3 \times {}^4P_3 = 3 \times \frac{4!}{(4-3)!} = 72$

For five-digit numbers the first place can be filled in 5 ways, the second in 4 ways, and so on.

5	4	3	2	1

$\therefore$ Number of arrangements $= 5 \times 4 \times 3 \times 2 \times 1 = 120$

or Number of arrangements $= {}^5P_5 = \dfrac{5!}{(5-5)!} = 120$

We cannot form a four-digit number and a five-digit number at the same time. These two operations are mutually exclusive.

$\therefore$ Total number of arrangements = 4-digit arrangements + 5-digit arrangements $= 72 + 120 = 192$

EXERCISE 3.2 PERMUTATIONS

1 Simplify:

(a) ${}^5P_3$ **(b)** ${}^4P_2$ **(c)** ${}^6P_5$ **(d)** ${}^8P_4$ **(e)** ${}^9P_1$ **(f)** ${}^{10}P_{10}$ **(g)** ${}^nP_2$ **(h)** ${}^nP_4$

2 Simplify:

(a) 5! **(b)** 4! **(c)** 8! **(d)** 7! **(e)** $\dfrac{5!}{2!}$ **(f)** $\dfrac{7!}{6!}$ **(g)** $\dfrac{10!}{8!}$ **(h)** $\dfrac{n!}{(n-2)!}$

3 How many different arrangements can be made using three of the letters of the word SUNDAY?

4 In how many different ways can 5 people be arranged in a row?

A 20 **B** 60 **C** 120 **D** 720

5 In how many ways can a first, second and third prize be awarded in a class of 8 students?

6 How many different arrangements of the letters of the word MINOR are possible if:

(a) the two vowels are next to each other **(b)** the first and last letters are consonants?

7 In how many ways can four different consonants and four different vowels be arranged in a row, if:

(a) the four vowels are together **(b)** the first and last letters are consonants

(c) the vowels and consonants occupy alternate positions?

8 In how many ways can four girls and three boys be arranged in a row so that:

(a) the boys are always together **(b)** the girls and boys occupy alternate places?

9 How many five-digit numbers can be formed using the digits 2, 3, 5, 6, 8, 9 if no digit can be used more than once per number? How many even numbers can be formed?

10 Five different magazines and four different books are arranged in a row with the books standing together. Indicate whether each statement below is a correct or incorrect step in the calculation of the total number of arrangements.

(a) $5! \times 5!$ **(b)** $6! \times 4!$ **(c)** 9! **(d)** 17 280

11 Five swimmers and five runners are to be seated alternately in a row (i.e. alternating so that no two swimmers or two runners sit next to each other). In how many different ways can this be done?

12 How many four-digit even numbers can be formed using the digits 3, 4, 7, 8 if:

(a) no digit is repeated **(b)** digits may be repeated?

13 How many numbers greater than 4000 can be formed using the digits 3, 5, 7, 8, 9 if repetition is not allowed?

14 How many numbers of seven digits can be formed using the digits 1, 2, 3, 4, 5, 6, 7 if:

(a) the numbers all begin with 1 and end with 2 **(b)** the numbers are all odd?

15 In how many ways can five different Mathematics books, four different Physics books and two different Chemistry books be arranged on a shelf if the books in each subject must be together?

16 In how many ways can three doctors, three nurses and three patients be arranged in a row if the three patients must be together?

17 Find the number of possible arrangements of the letters in the word PENCILS if:

(a) 'E' is next to 'I' **(b)** 'E' comes before 'I' **(c)** there are three letters between 'E' and 'I'.

18 How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5 if the same digit may be used more than once per number and each number is:

(a) divisible by 5 **(b)** an even number **(c)** an odd number **(d)** divisible by 4?

19 A car holds three people in the front seat and four in the back seat. In how many ways can seven people be seated in the car if two particular people must sit in the back seat and one particular person is the driver?

20 If ${}^{6}P_{r} = 120$, find the value of r.

21 In how many ways can 5 lawyers and 3 judges be arranged in a circle if the judges are always to stand together?

22 In how many ways can six people be arranged in a circle if two particular people are always:

(a) together **(b)** separated?

23 A father, a mother and six children stand in a ring. In how many ways can they be arranged if the father and the mother do not stand together?

24 Four actors and four comedians are seated alternately at a round table (i.e. alternating so that no two actors or two comedians sit next to each other). In how many ways can this be done?

25 The ratio of the number of arrangements of $(2n + 2)$ different objects taken n at a time to the number of arrangements of $2n$ different objects taken n at a time is 14 : 5. Find the value of n.

26 If ${}^{2n}P_{n} = 8 \times {}^{2n-1}P_{n-1}$, find the value of n.

27 Prove from the formula for ${}^{n}P_{r}$ that: ${}^{n+1}P_{r} = {}^{n}P_{r} + r \times {}^{n}P_{r-1}$

28 Show that: ${}^{n}P_{r} = {}^{n-2}P_{r} + 2r \times {}^{n-2}P_{r-1} + r(r-1) \times {}^{n-2}P_{r-2}$

29 In how many ways can five writers and five artists be arranged in a circle so that the writers are separated? In how many ways can this be done if two particular artists must not sit next to a particular writer?

3.3 ARRANGEMENT OF *n* OBJECTS WHEN SOME ARE IDENTICAL

The number of ways of arranging n objects in a row when p of the objects are identical, and q of the objects are identical (but different to the others), and so on, is not $n!$ but instead:

$$\frac{n!}{p!q!\ldots}$$

(This is shown in Example **11** below.)

Example 11

In how many ways can the six letters of the word MAMMAL be arranged in a line?

Solution

There are six letters, of which three are 'M' and two are 'A'.

$$\therefore \text{Number of arrangements} = \frac{6!}{3! \times 2!}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2}{3 \times 2 \times 2} = 60$$

One of the 60 arrangements is MMMAAL. If the letters 'M' were different, e.g. M_1, M_2, M_3, then this arrangement could be six different arrangements (3!):

$M_1M_2M_3AAL$ $M_1M_3M_2AAL$ $M_2M_1M_3AAL$ $M_2M_3M_1AAL$ $M_3M_1M_2AAL$ $M_3M_2M_1AAL$

Without the subscripts, all these arrangements are the same. This is why 6! is divided by 3! in the calculation of identical object arrangements.

Similarly, the two letters 'A' have 2! arrangements the same, so the result is also divided by 2!.

EXERCISE 3.3 ARRANGEMENT OF *n* OBJECTS WHEN SOME ARE IDENTICAL

1 In how many ways can the letters of MOTOR be arranged in a line?

2 In how many ways can the letters of NEWCASTLE be arranged in a line?

A 4! **B** 8! **C** 9! **D** $\frac{9!}{2!}$

3 The letters of PRINCIPLE are arranged in a line. In how many ways:

(a) can this be done **(b)** can the letters 'P' be next to each other?

4 In how many ways can the letters of PERMUTE be arranged if:

(a) consonants occupy the first and the last places
(b) the vowels and the consonants occupy alternate places?

5 **(a)** In how many ways can the letters of PRECISION be arranged?
(b) In how many of these arrangements do the vowels occupy all the 'even' places (second, fourth, sixth, eighth)?

6 How many arrangements can be made of the letters of DEFINITION if:

(a) the letters 'I' do not occupy the first or last place **(b)** the letters 'I' are together?

7 How many arrangements of the letters of TOMATO are possible if the letters 'O' are never next to each other? Indicate whether each statement is correct or incorrect.

(a) $\frac{6!}{2! \times 2}$ **(b)** $\frac{5!}{2!}$ **(c)** $\frac{6!}{2! \times 2!} - \frac{5!}{2!}$ **(d)** 120

8 How many arrangements of the letters of PARRAMATTA are possible?

9 Seven cubes, identical except that four are red and three are black, are arranged in a row. How many different arrangements are possible?

10 Three blue, three white and three red balls are placed in a row.

(a) How many different arrangements are possible?
(b) In how many of these arrangements are the red balls together?

11 How many seven-digit numbers can be formed that contain all the digits 2, 3, 3, 3, 4, 5, 6?

12 Find the number of different ways in which *n* students can stand in a row when two are boys, the rest are girls, and the boys all stand together.

13 Five drummers and five singers sit together in a row.

(a) In how many ways can this be done?
(b) In how many ways can this be done if a particular drummer must not sit between two particular singers?

14 Prove from the formula for ${}^nP_r$ that: ${}^{n+1}P_r = {}^nP_r + r \times {}^nP_{r-1}$

15 In how many ways can four people be accommodated at a hotel if there are four rooms available? (Assume that each room has enough beds for everyone.)

3.4 COMBINATIONS

A **combination** (or 'selection') is an unordered permutation of all or part of a set of objects.

The order is not important.

Consider a set of four people, *A*, *B*, *C* and *D*. In groups of two people at a time, they can be arranged as follows: *AB*, *AC*, *AD*, *BC*, *BD*, *BA*, *CD*, *CA*, *CB*, *DA*, *DB*, *DC*.

There are ${}^4P_2 = 12$ arrangements.

However, the number of combinations (selections) of two people at a time is only 6. For example, *AB* and *BA* are two arrangements, but only one combination—the order in which they are combined does not matter. *A* and *B* can be arranged together in 2! ways, but combined in only 1 way.

The six different combinations (selections) are as follows: *AB*, *AC*, *AD*, *BC*, *BD*, *CD*.

Thus there are $\frac{{}^4P_2}{2!} = \frac{12}{2} = 6$ combinations.

The number of combinations of 4 people taken 2 at a time is represented by the symbols ${}^4C_2$ or $\binom{4}{2}$ and we can write: ${}^4C_2 = \frac{{}^4P_2}{2!} = \frac{12}{2} = 6$

Example 12

How many groups of three can be selected from the four people *A*, *B*, *C* and *D*?

Solution

The selections are: *ABC*, *ABD*, *ACD*, *BCD*.

Number of combinations $= {}^4C_3 = \frac{{}^4P_3}{3!} = \frac{4\times3\times2}{3\times2} = 4$

Number of combinations ${}^nC_r$ or $\binom{n}{r}$

The symbol ${}^nC_r$ or $\binom{n}{r}$ denotes the number of combinations of n different objects taken r at a time. Each combination consists of a group of r different elements that can be ordered in $r!$ ways.

Note the difference between ordered and unordered sets of objects:

- An ordered sample of r objects taken from n different objects can be chosen in ${}^nP_r$ ways.
- An unordered sample of r objects taken from n different objects can be chosen in ${}^nC_r$ ways.

An unordered sample can be ordered in $r!$ ways.

Hence $r! \times {}^nC_r = {}^nP_r$

$$\therefore \quad {}^nC_r = \binom{n}{r} = \frac{{}^nP_r}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

This result can be written differently using factorial notation:

$${}^nC_r = \binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)\times(n-r)!}{r!\times(n-r)!}$$

$$\therefore \quad {}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Hence, for example: ${}^7C_3 = \frac{7\times(7-1)\times\ldots\times(7-3+1)}{3!} = \frac{7\times6\times5}{3\times2\times1} = 35$ or ${}^7C_3 = \frac{7!}{3!(7-3)!} = \frac{7\times6\times5}{3\times2\times1} = 35$

Example 13

In how many ways can a group of four people be selected from ten people if:

(a) there are no restrictions

(b) the eldest person is included in the group

(c) the eldest person is excluded from the group?

(d) What proportion of all possible groups contain the eldest person?

Solution

(a) Number of ways $= {}^{10}C_4 = \dfrac{10!}{4!(10-4)!} = \dfrac{10\times9\times8\times7}{4\times3\times2\times1} = 210$

(b) Automatically including the eldest, the selection is now for 3 out of 9 people.

Number of ways $= {}^{9}C_3 = \dfrac{9!}{3!(9-3)!} = \dfrac{9\times8\times7}{3\times2\times1} = 84$

(c) Excluding the eldest, the selection is now for 4 out of 9 people.

Number of ways $= {}^{9}C_4 = \dfrac{9!}{4!(9-4)!} = \dfrac{9\times8\times7\times6}{4\times3\times2\times1} = 126$

(d) 84 out of the 210 possible sets contain the eldest person.

Proportion $= \dfrac{84}{210} = \dfrac{2}{5} = 40\%$

You can use the ${}^{n}C_r$ button on your calculator to check your answers to this work.

It is a good idea to write the numerical expression for the ${}^{n}C_r$ result and to practise the numerical simplification. This practice will help you to simplify the algebraic expressions that can occur in this work.

Important result

From the definition for ${}^{n}C_r$:

$${}^{n}C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and} \quad {}^{n}C_{n-r} = \binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$$

Thus: ${}^{n}C_r = {}^{n}C_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

For each set of r objects selected, there is left behind a set containing $(n-r)$ objects. Thus there must be the same number of sets containing $(n-r)$ objects as there are containing r objects.

For example, if a group of 6 is selected from a set of 9 people, then a group containing 3 people is left behind:

$${}^{9}C_6 = \frac{9!}{6!(9-6)!} = \frac{9\times8\times7\times6!}{6!\times3!} = \frac{9\times8\times7}{3!} = {}^{9}C_3$$

This useful result allows you to simplify your calculations by calculating smaller factorials:

$${}^{9}C_6 = {}^{9}C_3 = \frac{9!}{3!(9-3)!} = \frac{9\times8\times7}{3!} = 84$$

Example 14

A class consists of 15 students, of whom 5 are prefects. How many committees of 8 students can be formed if:

(a) there are no restrictions

(b) each committee contains exactly 2 prefects

(c) each committee contains at least 2 prefects?

Solution

(a) Number of committees $= \binom{15}{8} = \binom{15}{7} = \frac{15!}{7!(15-7)!} = \frac{15\times14\times13\times12\times11\times10\times9}{7!} = 6435$

(b) The class contains 5 prefects and 10 non-prefects. We have to select 2 prefects and 6 non-prefects.

Number of ways of selecting 2 prefects $= \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\times4}{2} = 10$

Number of ways of selecting 6 non-prefects $= \binom{10}{6} = \binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10\times9\times8\times7}{4\times3\times2} = 210$

Number of committees $= \binom{5}{2} \times \binom{10}{6} = 10 \times 210 = 2100$

(c) The committees may consist of:

- 2 prefects and 6 non-prefects
- 3 prefects and 5 non-prefects
- 4 prefects and 4 non-prefects
- 5 prefects and 3 non-prefects

These results are mutually exclusive, so the results are added.

Number of committees $= \binom{5}{2} \times \binom{10}{6} + \binom{5}{3} \times \binom{10}{5} + \binom{5}{4} \times \binom{10}{4} + \binom{5}{5} \times \binom{10}{3}$

$$= \frac{5\times4}{2\times1} \times \frac{10\times9\times8\times7}{4\times3\times2\times1} + \frac{5\times4\times3}{3\times2\times1} \times \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1} + \frac{5}{1} \times \frac{10\times9\times8\times7}{4\times3\times2\times1} + 1 \times \frac{10\times9\times8}{3\times2\times1}$$

$$= 2100 + 2520 + 1050 + 120 = 5790$$

Doing the numerical calculations by showing this working-out will help you to develop the skills you will need to simplify expressions involving $\binom{n}{r}$. It is tempting to just write the first line of working and then use your calculator, but you should not!

Note: Part **(c)** could also be calculated by finding how many committees contain only 1 or 0 prefects, then subtracting this from the total number of committees without restrictions.

Number of committees $= \binom{15}{8} - \binom{10}{8} - \binom{10}{7} \times \binom{5}{1} = 5790$

Example 15

In how many ways can four runners and three swimmers be arranged in a row if there are eight runners and five swimmers to select from?

Solution

First select the total group of seven runners and swimmers, then arrange the group in a row:

- 4 runners and 3 swimmers can be selected in ${}^8C_4 \times {}^5C_3$ ways.
- The group of 7 can be arranged in 7! ways.

Number of arrangements $= {}^8C_4 \times {}^5C_3 \times 7!$

$$= \frac{8\times7\times6\times5}{4\times3\times2\times1} \times \frac{5\times4}{2\times1} \times 7! = 3\,528\,000$$

This answer is not ${}^8P_4 \times {}^5P_3$, because the group has to be selected before it can be arranged. When selecting the group, the order is not important, so the number of ways the group of seven can be selected is ${}^8C_4 \times {}^5C_3$. After the group is selected they can then be arranged in a row.

Example 16

Prove that: ${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^mC_2\,{}^nC_{r-2} + {}^mC_1\,{}^nC_{r-1} + {}^nC_r$

Solution

The symbol ${}^{m+n}C_r$ denotes the number of selections, each of r objects, when we select from two sets containing m and n objects respectively.

The r objects can be selected as follows (assuming r is not greater than either m or n):

r objects from m and no objects from n in ${}^mC_r$ ways

or $(r-1)$ objects from m and 1 object from n in ${}^mC_{r-1}\,{}^nC_1$ ways

or $(r-2)$ objects from m and 2 objects from n in ${}^mC_{r-2}\,{}^nC_2$ ways

and so on until:

no objects from m and r objects from n in ${}^nC_r$ ways.

These selections are mutually exclusive, so they are added to obtain the final result:

$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \ldots + {}^mC_2\,{}^nC_{r-2} + {}^mC_1\,{}^nC_{r-1} + {}^nC_r$$

Example 17

Prove that: ${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$

Solution

Method 1

The symbol ${}^{n+1}C_k$ denotes the number of selections, each of k objects, when we select from two sets containing n objects and 1 object respectively.

The k objects can be selected as follows (assuming $k \leq n$):

- k objects from the set of n and none from the set of one in ${}^nC_k\,{}^1C_0 = {}^nC_k$ ways
- $(k-1)$ objects from the set of n and the other one object in ${}^nC_{k-1}\,{}^1C_1 = {}^nC_{k-1}$ ways.

Note that ${}^1C_0 = 1$ and ${}^1C_1 = 1$.

These two selections are mutually exclusive, so they are added to obtain the final result:

$${}^{n+1}C_k = {}^nC_k + {}^nC_{k-1}$$

Method 2

Use the result for ${}^nC_r$ in factorial notation:

$$\begin{aligned}
{}^nC_k + {}^nC_{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= n!\left(\frac{1}{k!(n-k)!} + \frac{1}{(k-1)!(n-k+1)!}\right) \\
&= n!\left(\frac{n-k+1}{k(k-1)!(n-k+1)!} + \frac{k}{k(k-1)!(n-k+1)!}\right) \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= {}^{n+1}C_k
\end{aligned}$$

from $k! = k(k-1)!$

and $(n-k+1)! = (n-k+1)\times(n-k)!$

Note: $(n-k+1) = (n+1-k)$

Combinations of any number of objects

The symbol ${}^nC_r$ can also be described as representing the number of r-subsets in a given set, where $0 \leq r \leq n$.

If no specific value of r is stated, r can assume any value between 0 and n inclusive. Hence the number of subsets that each contain at least one element, i.e. the number of non-empty subsets in an n-set, is ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r$.

From a set of n objects, each object can be dealt with in two ways: it can be included or it can be excluded. The number of ways of dealing with the n objects is thus $2 \times 2 \times 2 \times \ldots$ to n factors, or 2^n. But this includes the case when all n objects are excluded, so the number of combinations of at least one object is $2^n - 1$.

$\therefore {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r = 2^n - 1$

Example 18

How many selections can be made from five different books, taking any number of books at a time?

Solution

The books can be selected one at a time, two at a time, or up to five at a time.

$$\begin{aligned}\text{Number of selections} &= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 \\ &= 5 + 10 + 10 + 5 + 1 \\ &= 31 = 2^5 - 1\end{aligned}$$

Example 19

In how many ways can four people be divided into:

(a) an A pair and then a B pair **(b)** any pairs?

Solution

(a) Number of ways the A pair can be selected $= {}^4C_2 = 6$

Number of ways the B pair can now be selected $= 1$

$\therefore$ Number of ways an A pair and a B pair can be selected $= 6 \times 1 = 6$

After the first pair is selected, there is only one pair left for the B pair.

In this question we are distinguishing between the two pairs selected by calling them A and B.

(b) Number of ways of selecting two people from four $= {}^4C_2 = 6$

Let the people be W, X, Y, Z, so the six possible pairs are WX, WY, WZ, XY, XZ, YZ.

Unlike in part **(a)**, the order of selecting the two pairs now does not matter, so WX and YZ is not considered different to YZ and WX.

Thus the number of ways of dividing four people into two pairs $= \dfrac{{}^4C_2}{2!} = \dfrac{6}{2} = 3$

Example 20

In how many ways can six people be placed into three groups containing:

(a) 3, 2 and 1 people respectively **(b)** 2 people in each group?

Solution

(a) 3 people can be selected from the 6 people in ${}^6C_3$ ways.

From the remaining 3 people, 2 people can be selected in ${}^3C_2$ ways.

This leaves 1 person for the last group.

Number of ways $= {}^6C_3 \times {}^3C_2 \times 1$

$$= \frac{6!}{3!(6-3)!} \times \frac{3!}{2!(3-2)!} \times 1 = \frac{6 \times 5 \times 4}{2} = 60$$

(b) 2 people can be selected from 6 people in ${}^6C_2$ ways.

From the remaining 4 people, 2 people can be selected in ${}^4C_2$ ways.

This leaves 2 people for the last group.

We might distinguish between the groups, e.g. by calling them X, Y, Z.

Number of ways people can be put into 3 distinct groups of $2 = {}^6C_2 \times {}^4C_2 \times 1 = 90$

If the groups are indistinguishable (i.e. do not have names), then we have to divide the total number of arrangements by 3! (the number of arrangements of the 3 groups).

Number of ways people can be put into any 3 groups of $2 = \frac{{}^6C_2 \times {}^4C_2 \times 1}{3!} = \frac{90}{6} = 15$

EXERCISE 3.4 COMBINATIONS

1 Simplify, without using your calculator:

(a) ${}^7C_2$ **(b)** ${}^9C_1$ **(c)** ${}^8C_4$ **(d)** ${}^{12}C_{10}$ **(e)** $\binom{12}{4}$ **(f)** $\binom{15}{1}$

(g) $\binom{9}{9}$ **(h)** $\binom{6}{4}$ **(i)** ${}^nC_2$ **(j)** $\binom{n}{4}$ **(k)** ${}^nC_n$ **(l)** $\binom{n}{k}$

2 Evaluate:

(a) ${}^{10}C_5$ **(b)** ${}^{20}C_9$ **(c)** ${}^{25}C_{10}$ **(d)** ${}^nC_5$

(e) $\binom{15}{10}$ **(f)** $\binom{18}{11}$ **(g)** $\binom{16}{8}$ **(h)** $\binom{48}{45}$

3 In how many ways can three books be selected from eight different books?

4 How many different hands of five cards can be dealt from a standard pack of 52 playing cards?

5 In how many ways can a jury of 12 people be chosen from 20 people?

6 The number of ways of picking 6 numbers from 45 numbers is:

A ${}^{45}C_6$ **B** ${}^{45}P_6$ **C** $6 \times {}^{45}C_1$ **D** $6 \times {}^{45}P_1$

7 In how many ways can a set of 2 cooks and 3 waiters be selected from 5 cooks and 4 waiters?

8 How many different selections can be made by taking three of the digits 4, 5, 6, 7, 8, 9?

9 In how many ways can a committee of 4 teachers and 5 parents be formed from 8 teachers and 7 parents?

10 From 8 lawyers, 7 clerks and 5 judges, how many different groups could be formed that contain 5 lawyers, 4 clerks and 3 judges? Indicate whether each answer is correct or incorrect.

(a) ${}^{20}C_{12}$ **(b)** ${}^{8}P_5 \times {}^{7}P_4 \times {}^{5}P_3$ **(c)** ${}^{8}C_5 \times {}^{7}C_4 \times {}^{5}C_3$ **(d)** 19 600

11 In how many ways can eight different rabbits be divided into a group of five and a group of three?

12 A committee of six is selected from ten people, of whom A and B are two. How many committees can be formed:

(a) containing both A and B **(b)** excluding A if B is included?

13 In how many ways can three cards be selected from a standard pack of 52 playing cards if:

(a) at least one selected card is an Ace

(b) no more than one selected card is an Ace?

14 In how many ways can a team of 3 runners and 4 hurdlers be chosen from 6 runners and 7 hurdlers?

15 A committee of 7 politicians is chosen from 10 Liberal members, 8 Labor members and 5 independents. In how many ways can this be done to include exactly 1 independent, at least 3 Liberal members and at least 1 Labor member?

16 A team of 11 is chosen from 15 cricketers. Five of the 15 cricketers are bowlers only, two are wicketkeepers only and the rest are batters only. How many possible teams can be chosen that contain:

(a) 4 bowlers, 1 wicketkeeper and 6 batters

(b) at least 4 bowlers and at least 1 wicketkeeper?

17 From seven teachers and five students, a committee of seven is formed. How many different committees can be selected if teachers and students are both represented and the teachers are in a majority?

18 From four oranges, three bananas and two apples, how many selections of five pieces of fruit can be made, taking at least one of each kind?

19 In horse racing, a 'trifecta' is when the first three horses are picked in the correct order, while a 'quinella' is when the first two horses are picked in any order. In a ten-horse race, what is the number of possible:

(a) trifecta combinations **(b)** quinella combinations?

20 In how many ways can a jury of 12 people be chosen from 10 women and 7 men so that there are at least 6 women and not more than 4 men?

21 In how many ways can a group of three or more be selected from nine?

22 In how many ways can a committee of three women and four girls be chosen from seven women and six girls so that if the eldest woman is serving on the committee then the youngest girl is not?

23 How many **(a)** selections **(b)** arrangements consisting of three consonants and two vowels can be made from eight different consonants and four different vowels?

24 In how many ways can four Physics books and three Mathematics books be arranged on a shelf if a selection is made from six different Physics books and five different Mathematics books? In how many of these arrangements are all the Physics books next to each other?

25 In how many ways can three glasses and two plates be arranged in a row if a selection is made from five different glasses and four different plates? In how many of these arrangements does a glass occupy the middle position?

26 How many words (i.e. any arrangements of letters) containing three consonants and two vowels can be formed from the letters of the word PROMISE?

27 If, ${}^{n}C_6 = {}^{n}C_4$, find the value of n.

28 In how many ways can nine books be distributed among a teacher, a parent and a child, if the teacher receives four, the parent three and the child two?

29 In how many ways can eight different toys be divided into two unequal groups?

30 In how many ways can eight basketball players be divided into four groups of two?

31 In how many ways can n objects be shared between two people? ('Shared' means that each person gets at least one.)

32 From the definition of ${}^nC_r$, prove each of the following:

(a) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

(b) ${}^nC_k + 2 \times {}^nC_{k-1} + {}^nC_{k-2} = {}^{n+2}C_k$

(c) ${}^nC_r = \dfrac{n-r+1}{r} \times {}^nC_{r-1}$

(d) ${}^{m+n}C_3 = {}^mC_3 + {}^mC_2\,{}^nC_1 + {}^mC_1\,{}^nC_2 + {}^nC_3$

(e) ${}^{n+3}C_r = {}^nC_r + 3 \times {}^nC_{r-1} + 3 \times {}^nC_{r-2} + {}^nC_{r-3}$

33 The ratio of the number of combinations of $(2n + 2)$ different objects taken n at a time to the number of combinations of $(2n - 2)$ different objects taken n at a time is 99 : 7. Find the value of n.

CHAPTER REVIEW 3

1 A coin is tossed, a die is rolled, and a card is selected from a standard pack of 52 playing cards and its suit noted. Which would be the best form to display the possible outcomes of this experiment: table, list or tree diagram?

2 A bag contains four discs, each of a different colour. How many different arrangements can be made of four discs if:

(a) they are selected at random one at a time and placed in a row until the bag is empty

(b) one disc is selected at random, its colour is noted and it is put back in the bag before the next disc is selected?

3 How many different arrangements can be made of the five vowels a, e, i, o, u?

4 How many different four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, 7, 8 if:

(a) none of the digits are repeated

(b) the digits may be repeated?

5 How many four-digit or five-digit numbers greater than 4000 can be formed using 0, 2, 4, 5, 7 if:

(a) none of the digits are repeated

(b) the digits may be repeated

(c) none of the digits are repeated and the numbers formed are even?

6 **(a)** How many arrangements of the letters in the word ISOSCELES are possible?

(b) How many arrangements of the letters in the word ISOSCELES are possible if the consonants and vowels must alternate?

7 In how many ways can a cycling team of four riders be selected from a squad of ten riders?

8 A committee of six is chosen from 14 people. In how many ways can this be done?

9 A mixed volleyball team of eight players is selected from eight males and nine females. In how many ways can this be done if the team must have an equal number of male and female players?

10 A committee of five is to be selected from a class of 30 students.

(a) In how many ways can this be done?

(b) After the committee is selected, one person is elected chairperson and another is elected secretary. In how many ways can these positions be filled?

11 In how many ways can five people be seated in a five-seater car if only two people have a licence to legally occupy the driver's seat?

CHAPTER 4
FURTHER TRIGONOMETRY

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

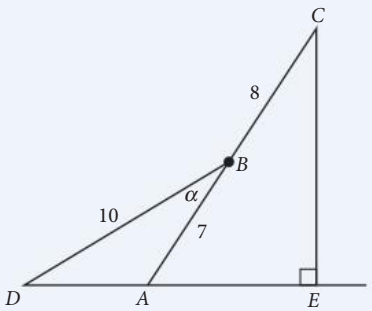
4.1 APPLIED TRIGONOMETRY

This section uses the skills from Chapter 5 of the previous book (*New Senior Mathematics for Years 11 & 12*) to solve harder two-dimensional problems and problems in three dimensions. In three-dimensional problems it is important that you break the 3D diagram into its 2D parts.

Example 1

In this diagram, $AC = 15$ cm, $AB = 7$ cm, $DB = 10$ cm and $\angle DBA = \alpha$.

Find the perpendicular distance CE of C from DA in terms of α and evaluate it for $\alpha = 25°$.



Solution

Use the cosine rule in ΔDBA: $DA^2 = 10^2 + 7^2 - 2 \times 10 \times 7 \cos\alpha$

$DA^2 = 149 - 140\cos\alpha$

$\therefore \quad DA = \sqrt{149 - 140\cos\alpha}$ [1]

Use the sine rule in ΔDBA: $\dfrac{10}{\sin(\angle DAB)} = \dfrac{DA}{\sin\alpha}$

$\therefore \quad \sin(\angle DAB) = \dfrac{10\sin\alpha}{DA}$ [2]

In ΔCAE: $CE = 15\sin(\angle CAE)$ [3]

Now: $\angle DAB = 180° - \angle CAE$

Hence: $\sin(\angle DAB) = \sin(\angle CAE)$

Using [1], [2] and [3]: $CE = \dfrac{150\sin\alpha}{\sqrt{149 - 140\cos\alpha}}$

For $\alpha = 25°$: $CE = \dfrac{150\sin 25°}{\sqrt{149 - 140\cos 25°}}$

$= 13.48$

Perpendicular distance = 13.48 cm

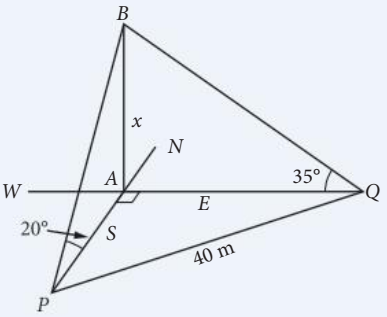
Example 2

From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20°; from a point Q due east of the tower, the angle of elevation is 35°. The distance from P to Q is 40 metres. Find the height of the tower.

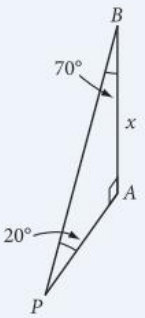
Solution

(Trigonometric values are rounded to 3 decimal places.)

Draw a 3D diagram to represent this information. Let the height of the tower be x m.



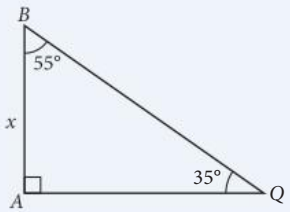
There are three right-angled triangles: ΔPAB and ΔQAB in the vertical plane, and ΔPAQ in the horizontal plane. Draw each right-angled triangle separately as you use it.



In ΔPAB: $\tan 20° = \dfrac{x}{PA}$ *or* $\tan 70° = \dfrac{PA}{x}$

$\therefore \quad PA = \dfrac{x}{\tan 20°}$ $\qquad PA = x \tan 70°$

$\therefore \quad PA = 2.747x$ $\qquad PA = 2.747x$



In ΔQAB: $\tan 35° = \dfrac{x}{QA}$ *or* $\tan 55° = \dfrac{QA}{x}$

$\therefore \quad QA = \dfrac{x}{\tan 35°}$ $\qquad QA = x \tan 55°$

$\therefore \quad QA = 1.428x$ $\qquad QA = 1.428x$

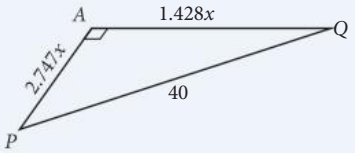
In ΔPAQ, use Pythagoras' theorem: $(2.747x)^2 + (1.428x)^2 = 40^2$

$$7.546x^2 + 2.039x^2 = 1600$$

$$9.585x^2 = 1600$$

$$x = \sqrt{\frac{1600}{9.585}}$$

$$x = 12.920$$



The height of the tower is 12.92 m.

To avoid errors due to rounding, it is better if you can avoid using your calculator until the last step. Without rounding, the calcuation is: $(x \tan 70°)^2 + (x \tan 55°)^2 = 1600$

$$x^2(\tan^2 70° + \tan^2 55°) = 1600$$

$$x^2 = \frac{1600}{\tan^2 70° + \tan^2 55°}$$

$$x = \frac{40}{\sqrt{\tan^2 70° + \tan^2 55°}} \qquad x = 12.918 \approx 12.92 \text{ m}$$

EXERCISE 4.1 APPLIED TRIGONOMETRY

1 P, A, B, C are four points in a plane such that the angles BPA and CPA are obtuse and on opposite sides of PA. $PA = 8$ cm, $BP = 10$ cm, $PC = 12$ cm, $AB = 14$ cm and $AC = 18$ cm. Calculate the length of BC and the area of the triangle ABC.

2 P, A, B, C are four points (in order) on a straight road that runs up a hill at a constant angle of 10° to the horizontal. A flagpole with a height of h m stands at P. From A and B, the top of the flagpole has elevations of 30° and 5° respectively above the horizontal. If AB is 100 m long, what is the height of the flagpole? If BC is also 100 m long, what is the elevation of the top of the flagpole from C?

3 From a point P, a person observes that the angle of elevation of the top of a cliff A is 40°. After walking 100 m towards A along a straight road that inclines upwards at an angle of 15° to the horizontal, the angle of elevation of A is observed to be 50°. Find the vertical height of A above P.

4 A ship sailing in a direction 065° T observes two lighthouses in a line due north. After the ship has travelled 4 km, one of the lighthouses bears 285° T and the other 315° T. The distance between the lighthouses is given by which two of the following expressions?

A $4\left(\frac{\sin 75°}{\sin 45°} - \frac{\sin 40°}{\sin 70°}\right)$ **B** $4\left(\frac{\sin 70°}{\sin 45°} - \frac{\sin 40°}{\sin 75°}\right)$ **C** $\frac{2\sin 65°}{\sin 75° \sin 45°}$ **D** $\frac{2\sin 75°}{\sin 65° \sin 45°}$

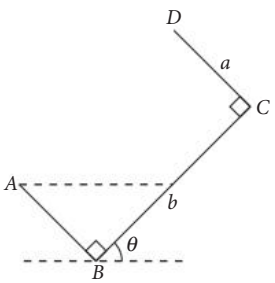
5 A and B are the bases of two vertical towers each 30 m high. B is due north of A. From a point P that is due east of A and in the same horizontal plane, the angles of elevation of the tops of the towers A and B are 45° and 36° 52′ respectively. Calculate the distance:

(a) from P to the bases of the two towers **(b)** between the two towers.

6 From a point P, an observer finds that the angle of elevation of the top of a vertical tower is $\alpha°$. After walking x metres horizontally towards the base of the tower, the observer finds the angle of elevation of the top of the tower is now $\beta°$. If the height of the tower is h metres, show that $x = h(\cot \alpha° - \cot \beta°)$.

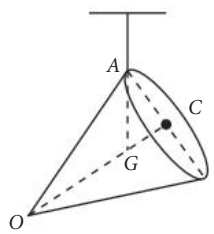
7 A flagpole 5 m high stands on top of a vertical tower. From a point on the ground, the angles of elevation of the top and bottom of the flagpole are 68° and 62° respectively. Show that the height of the tower (h metres) is given by $h = \frac{5\tan 62°}{\tan 68° - \tan 62°}$.

8 An open rectangular tank a units deep and b units wide holds water and is tilted so that the base BC makes an angle θ with the horizontal. When BC is returned to the horizontal, show that the depth of the water is $\frac{a^2 \cot \theta}{2b}$ units.



Cross-section of tank

9 A conical vessel with vertex at O and semi-vertical angle α is suspended from a point A on the rim of the base. G is a point on the axis OC of the cone such that $OG = \frac{2}{3}OC$. If the vessel rests with G vertically below A, show that the acute angle β that AO makes with the vertical is given by $\tan \beta = \frac{2\tan \alpha}{1 + 3\tan^2 \alpha}$.



10 In ΔABC, $AC = 7$ cm. A circle with centre B and radius BC cuts AB internally at D. $AD = 5$ cm, $DC = 4$ cm. The length of BC is:

A 10 cm **B** 5 cm **C** 7 cm **D** impossible to find

11 A and B are two towers, with B 4 km due east of A. A flagpole C has true bearings from A and B that are $\alpha°$ east of north and $\alpha°$ west of north respectively. A second flagpole D has true bearings from A and B that are $(\alpha + \beta)°$ east of north and $(\alpha - \beta)°$ west of north respectively. Draw a sketch to indicate the positions of A, B, C and D. Assuming that A, B, C and D are on level ground and that $\alpha = 25$, $\beta = 10$, find the distance between C and D.

12 A, B, C, D are four points equally spaced on the circumference of a horizontal circle of radius 3 cm. P is a point above the circle such that $PA = PD = 4$ cm and $PB = PC = 6$ cm. Calculate the size of $\angle APC$.

13 The angle of elevation of the top of a building from a point P due east of it is 40°, and from a point Q due south of P is 30°. If the distance from P to Q is 20 metres, find the height of the building.

14 A railway line runs south-west between two railway stations A and B. Two spires 3 km apart are both directly north-west of A. From B, the bearings of the spires are N 7.5° E and N 37.5° E. Show that $AB = 6\sin 52.5° \cos 7.5°$. If a train takes 4 minutes to travel from A to B, find its average speed.

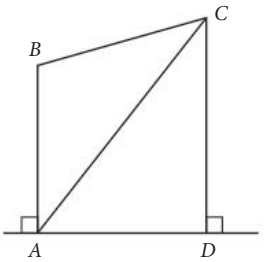
15 Two buildings of equal height are 40 m apart. At a point on the horizontal line joining the buildings' bases, the angles of elevation of the tops of the buildings are 47° and 28°. Show that the height h of the buildings is given by $h = \dfrac{40\tan 47°\tan 28°}{\tan 47° + \tan 28°}$.

16 An observer's eye is 2 m above the ground. A vertical pole fixed in the ground subtends an angle of 45° at the observer's eye. The angle of depression of the base of the flagpole from the eye is 15°. Show that the height of the flagpole is $2(1 + \tan 30°\tan 75°)$ metres.

17 From a point A, two points B and C are in line in a direction 049° T. From a point D that is 100 m from A in a direction 139° T, B is in a direction 352° T and C is in a direction 022° T. Calculate the distance between B and C.

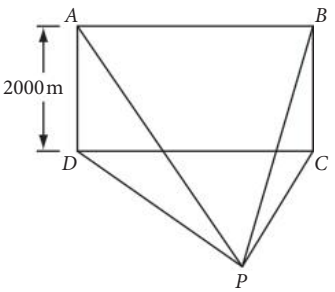
18 From a point A, the angle of elevation of the top of a tower due north of it is 20°. From point B, due east of the tower, the angle of elevation is 18°. A and B are 100 m apart. Show that the height h of the tower is given by $h = \dfrac{100}{\sqrt{\tan^2 72° + \tan^2 70°}}$.

19 In the diagram shown, $AB = 25$ cm, $BC = 27$ cm, $AC = 40$ cm. Calculate the length of AD.



20 $ABCD$ is a cyclic quadrilateral in which $AB = 5$ cm, $BC = 6$ cm, $CD = 7$ cm, $AD = 8$ cm. Show that $\cos(\angle ADC) = \dfrac{13}{43}$.

21 A and B represent successive positions of an aircraft flying horizontally at an altitude of 2000 metres. From a point P the angles of elevation of A and B are 15° and 25° respectively. If $\angle DPC = 60°$, calculate the distance AB.

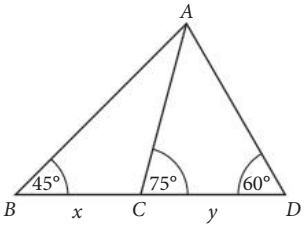


22 The elevation of a hill at a point P due east of it is 42° and at a point Q due south of it is 22°. If the distance from P to Q is 400 metres, find the height of the hill.

23 O, A, B are three points in order in a straight line. The bearings of A and B from O are both 020° T. From a point P that is 4 km from O in a NW direction, the bearings of A and B are 112° T and 064° T. Calculate the distance from A to B.

24 From a point P an observer finds that the angle of elevation of the top of a vertical tower is $\alpha°$. After walking x m horizontally towards the foot of the tower, the observer finds that the angle of elevation is $\beta°$. If the height of the tower is h m, prove that $h = \dfrac{x\sin\alpha°\sin\beta°}{\sin(\beta - \alpha)°}$.

25 Using the sine rule, show that $\dfrac{x}{y} = \dfrac{\sqrt{3}}{2}$ and also that $\dfrac{\text{Area } \Delta ABC}{\text{Area } \Delta ADC} = \dfrac{\sqrt{3}}{2}$.



26 A side of a hill can be regarded as a plane making an angle of 20° with the horizontal.

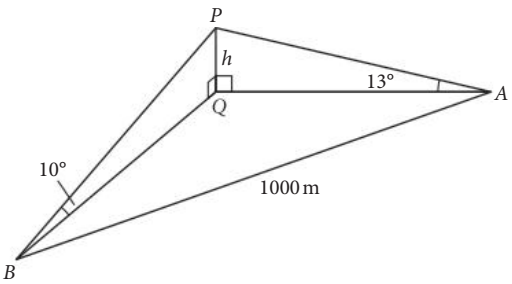
(a) From a point A at the base of the hill, I walk 100 m up the hill along a straight line of steepest slope to the point B. Find the vertical height of B above A.

(b) A tower 40 m tall is constructed at B. Find the angle subtended by the tower at my eyes, which are 1.5 m above the ground, when I stand at A.

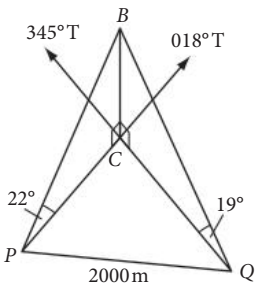
(c) A straight road is constructed on the hillside with a slope of 1 in 10, i.e. with a 1 m vertical rise for every 10 m horizontally forward. Find the angle this road must make with the line of steepest slope on the hillside.

27 A tower PQ of height h metres has an angle of elevation of 13° at a point A due east of it. From another point B, the bearing of the tower is 050° T and the angle of elevation is 10°. The points A and B are 1000 metres apart on the same level as the tower's base Q.

(a) Show that $\angle AQB = 140°$.
(b) Consider the triangle APQ and show that $AQ = h\tan 77°$.
(c) Find a similar expression for BQ.
(d) Use the cosine rule in the triangle AQB to calculate h to the nearest metre.

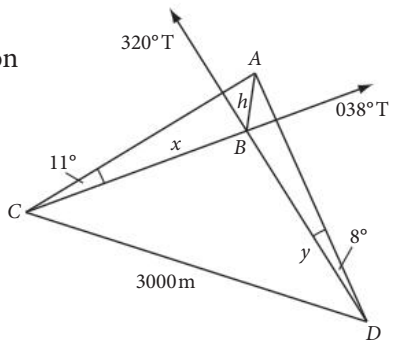


28 Two observers P and Q, who are 2000 metres apart, take the bearing and elevation of a balloon B at the same instant. The observer P finds the bearing to be 018° T and the angle of elevation to be 22°, while the observer Q finds the bearing to be 345° T and the angle of elevation to be 19°. Find the height BC of the balloon to the nearest metre.



29 AB is a tower. CD is a road. A person walking along the road observes the tower AB from C at a bearing of 038° T and with an angle of elevation of 11°. After travelling 3000 m along the road to D, the person observes the tower from D at a bearing of 320° T and an angle of elevation of 8°. By letting $CB = x$, $BD = y$ and $AB = h$:

(a) Show that $\angle CBD = 78°$.
(b) Express x in terms of h.
(c) Express y in terms of h.
(d) Using the cosine rule with ΔCBD, find the value of h correct to one decimal place.



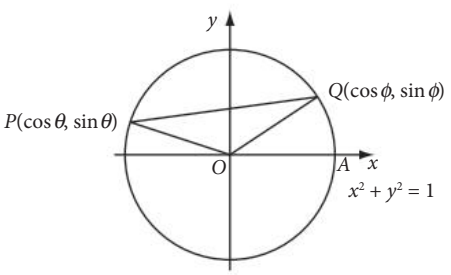
4.2 SPECIAL ANGLE FORMULAE

Sum and difference of two angles

We can derive formulae for the trigonometric functions of sums and differences of angles, i.e. formulae for $\cos(\theta \pm \phi)$, $\sin(\theta \pm \phi)$ and $\tan(\theta \pm \phi)$, in terms of trigonometric functions of the angles θ and ϕ.

We will first obtain a formula for $\cos(\theta - \phi)$, then deduce formulae for the others from it.

Starting from A, mark a unit circle to show $\angle AOP = \theta$ and $\angle AOQ = \phi$ (where $\theta > \phi$ for convenience) as shown. The coordinates of P and Q are thus $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ respectively, and $\angle POQ = \theta - \phi$.



Using the distance formula: $PQ^2 = (\cos\theta - \cos\phi)^2 + (\sin\theta - \sin\phi)^2$

$= 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$ [a]

Using the cosine rule in ΔPOQ: $PQ^2 = 1 + 1 - 2\cos(\angle POQ)$

$= 2 - 2\cos(\theta - \phi)$ [b]

Equating [a] and [b]: $2 - 2\cos(\theta - \phi) = 2 - 2(\cos\theta\cos\phi + \sin\theta\sin\phi)$

$\therefore$ $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ [1]

This formula [1] is true for all values of θ and ϕ.

By writing $(-\phi)$ in place of ϕ, remembering that $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, we also have:

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \qquad [2]$$

Similarly, with $(90° - \theta)$ in place of θ in [1]:

$$\cos[90° - (\theta + \phi)] = \cos(90° - \theta)\cos\phi + \sin(90° - \theta)\sin\phi$$

And using the complementary angles formulae, i.e. $\cos(90° - \alpha) = \sin\alpha$ and $\sin(90° - \alpha) = \cos\alpha$:

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \qquad [3]$$

Writing $(-\phi)$ in place of ϕ in [3]: $\sin(\theta - \phi) = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi \qquad [4]$$

Combining these for the tan functions:

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \qquad [5]$$

And for $(\theta - \phi)$:

$$\tan(\theta - \phi) = \frac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\cos\theta\cos\phi + \sin\theta\sin\phi}$$

Dividing numerator and denominator by $\cos\theta\cos\phi$ $(\cos\theta\cos\phi \neq 0)$:

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 - \tan\theta\tan\phi} \qquad [6]$$

Example 3

Find the expansion for each expression, simplifying where possible.

(a) $\sin(3x + 2y)$ **(b)** $\cos(2\alpha + \beta)$ **(c)** $\tan(A + 45°)$

Solution

(a) $\sin(3x + 2y) = \sin 3x\cos 2y + \cos 3x\sin 2y$

(b) $\cos(2\alpha + \beta) = \cos 2\alpha\cos\beta - \sin 2\alpha\sin\beta$

(c) $\tan(A + 45°) = \dfrac{\tan A + \tan 45°}{1 - \tan A\ \tan 45°} = \dfrac{1 + \tan A}{1 - \tan A}$

Example 4

Simplify each expression.

(a) $\sin(2\alpha + \beta)\cos\beta - \cos(2\alpha + \beta)\sin\beta$ **(b)** $\cos(2\theta - 3\alpha)\cos 2\theta + \sin(2\theta - 3\alpha)\sin 2\theta$

Solution

If you can recognise the form of the equation, you can use the two-angle expansion in reverse:

(a) $\sin(2\alpha + \beta)\cos\beta - \cos(2\alpha + \beta)\sin\beta = \sin[(2\alpha + \beta) - \beta] = \sin 2\alpha$

(b) $\cos(2\theta - 3\alpha)\cos 2\theta + \sin(2\theta - 3\alpha)\sin 2\theta = \cos[(2\theta - 3\alpha) - 2\theta]$
$= \cos(-3\alpha) = \cos 3\alpha$

If you didn't recognise the expansion, then you could expand the expressions on the left-hand side and collect like terms. For example, the solution to **(a)** becomes:

$$\begin{aligned}\sin(2\alpha+\beta)\cos\beta-\cos(2\alpha+\beta)\sin\beta &= [\sin 2\alpha\cos\beta+\cos 2\alpha\sin\beta]\cos\beta-[\cos 2\alpha\cos\beta-\sin 2\alpha\sin\beta]\sin\beta\\ &= \sin 2\alpha\cos^2\beta+\cos 2\alpha\sin\beta\cos\beta-\cos 2\alpha\cos\beta\sin\beta+\sin 2\alpha\sin^2\beta\\ &= \sin 2\alpha[\cos^2\beta+\sin^2\beta]=\sin 2\alpha\end{aligned}$$

Double-angle formulae

Using the sum-and-difference-of-two-angles formula [2] (see page 84), writing θ in place of ϕ:

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi \qquad [2]$$

$$\therefore\ \cos(\theta+\theta)=\cos\theta\cos\theta-\sin\theta\sin\theta$$

$$\cos 2\theta=\cos^2\theta-\sin^2\theta$$

From $\sin^2\theta+\cos^2\theta=1$:

$$\cos 2\theta=2\cos^2\theta-1 \qquad [7]$$

$$\cos 2\theta=1-2\sin^2\theta$$

Similarly, using formula [3] (from page 84), writing θ in place of ϕ:

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi \qquad [3]$$

$$\therefore\ \sin(\theta+\theta)=\sin\theta\cos\theta+\cos\theta\sin\theta$$

$$\sin 2\theta=2\sin\theta\cos\theta \qquad [8]$$

And using formula [5] (from page 84), writing θ in place of ϕ:

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi} \qquad [5]$$

$$\therefore\ \tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}$$

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta},\ \tan\theta\neq\pm1 \qquad [9]$$

Example 5

(a) Prove that $\dfrac{\cos 3\theta}{\sin\theta}+\dfrac{\sin 3\theta}{\cos\theta}=2\cot 2\theta$.

(b) Prove that $\dfrac{\sin 2\alpha+\sin\alpha}{1+\cos 2\alpha+\cos\alpha}=\tan\alpha$.

Solution

(a)

$$\begin{aligned}\text{LHS}&=\frac{\cos 3\theta}{\sin\theta}+\frac{\sin 3\theta}{\cos\theta}\\ &=\frac{\cos 3\theta\cos\theta+\sin 3\theta\sin\theta}{\sin\theta\cos\theta}\\ &=\frac{\cos(3\theta-\theta)}{\frac{1}{2}\sin 2\theta}\\ &=\frac{2\cos 2\theta}{\sin 2\theta}\\ &=2\cot 2\theta=\text{RHS}\end{aligned}$$

(b) **Method 1**

$$\begin{aligned}\text{LHS}&=\frac{\sin 2\alpha+\sin\alpha}{1+\cos 2\alpha+\cos\alpha}\\ &=\frac{2\sin\alpha\cos\alpha+\sin\alpha}{1+2\cos^2\alpha-1+\cos\alpha}\\ &=\frac{\sin\alpha(2\cos\alpha+1)}{\cos\alpha(2\cos\alpha+1)}\\ &=\frac{\sin\alpha}{\cos\alpha}\ \text{ if }\ 2\cos\alpha+1\neq0\\ &=\tan\alpha=\text{RHS}\end{aligned}$$

Method 2

In the denominator, use formula [7] to directly replace $1+\cos 2\alpha$ with $2\cos^2\alpha$.

Example 6

If $\tan A = -\frac{3}{4}$, $90° < A < 180°$, and $\cos B = \frac{5}{13}$, $0° < B < 90°$, write the exact value of the following.

(a) $\sin(A - B)$ **(b)** $\cos 2A$ **(c)** $\tan(A + B)$

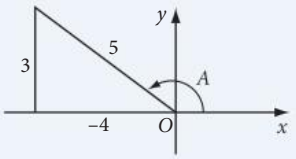
Solution

Draw diagrams to show the given ratio for each angle, then use the diagrams to find the other ratios for the angle.

$\sin A = \frac{3}{5}$

$\cos A = -\frac{4}{5}$

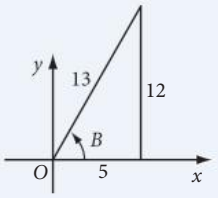
$\tan A = -\frac{3}{4}$



$\sin B = \frac{12}{13}$

$\cos B = \frac{5}{13}$

$\tan B = \frac{12}{5}$



(a) $\sin(A - B) = \sin A \cos B - \cos A \sin B$

$= \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{63}{65}$

(b) $\cos 2A = \cos^2 A - \sin^2 A$

$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$

(c) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$= \frac{-\frac{3}{4} + \frac{12}{5}}{1 + \frac{3}{4} \times \frac{12}{5}} = \frac{33}{56}$

EXERCISE 4.2 SPECIAL ANGLE FORMULAE

1 Expand:

(a) $\sin(A + 2B)$ **(b)** $\sin(2x - y)$ **(c)** $\cos(2x - 3y)$

(d) $\cos(2\theta + 60°)$ **(e)** $\tan(\theta + \alpha)$ **(f)** $\tan(A - 135°)$

2 Simplify:

(a) $\sin A \cos(A - B) + \cos A \sin(A - B)$ **(b)** $\cos(\theta + \alpha)\cos(\theta - \alpha) + \sin(\theta + \alpha)\sin(\theta - \alpha)$

(c) $\sin 2A \cos A - \cos 2A \sin A$ **(d)** $\cos 60° \cos 30° - \sin 60° \sin 30°$

(e) $\frac{\tan\theta - \tan 20°}{1 + \tan 20° \tan\theta}$ **(f)** $\frac{\tan 2\alpha + \tan\alpha}{1 - \tan 2\alpha \tan\alpha}$ **(g)** $\sin(2A + B)\cos(A + B) - \cos(2A + B)\sin(A + B)$

(h) $\cos(3\theta + \alpha)\cos(2\theta + \alpha) + \sin(3\theta + \alpha)\sin(\theta + \alpha)$ **(i)** $\frac{\tan 3x - \tan x}{1 + \tan 3x \tan x}$

3 The expression $\frac{\tan(A+B) + \tan C}{1 - \tan(A+B)\tan C}$ simplifies to:

A $\tan A + \tan B + \tan C$ **B** $\tan(A - B + C)$ **C** $\tan(A + B - C)$ **D** $\tan(A + B + C)$

4 **(a)** Find the exact value of $\sin 38° \cos 22° + \cos 38° \sin 22°$.

(b) Find the exact value of $\frac{\tan 119° + \tan 16°}{1 - \tan 119° \tan 16°}$.

(c) Find the exact value of $\cos 165°$. **(d)** Expand and simplify $\sin(x + 40°) + \sin(x - 40°)$.

5 Write the expansion of $\cos(A + B)$. From this deduce the expansion of $\cos(A - B)$.

6 Write the expansion of $\cos(\theta - \phi)$. Write $(90° - \theta)$ in place of θ to deduce the expansion of $\sin(\theta + \phi)$.

7 If θ and ϕ are angles between $0°$ and $90°$, $\sin\theta = \frac{3}{5}$, $\tan\phi = \frac{7}{24}$, find the following without using a calculator.

(a) $\sin(\theta - \phi)$ **(b)** $\cos(\theta + \phi)$ **(c)** $\tan(\theta - \phi)$

8 If $\tan A = 4$, $\tan B = \frac{3}{5}$, and A and B are acute angles, then $A - B = \ldots$

A $45°$ **B** $30°$ **C** $60°$ **D** $135°$

9 If $\tan\alpha=\frac{4}{3}$ and $\cos\beta=\frac{12}{13}$, where $0<\beta<\alpha<90°$, evaluate the following without using a calculator.

(a) $\sin 2\alpha$ **(b)** $\tan 2\alpha$ **(c)** $\cos(\alpha-\beta)$

10 **(a)** Using the expansion of $\sin(A+B)$, prove that $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) Using the expansion of $\tan(A+B)$, prove that $\tan 75° = 2+\sqrt{3}$.

11 Find the value (in simplest surd form) of:

(a) $\cos 75°$ **(b)** $\tan 15°$ **(c)** $\cos 15°$

12 If $\sin\theta=\frac{3}{4}$, $90°<\theta<180°$, evaluate (in surd form):

(a) $\sin 2\theta$ **(b)** $\cos 2\theta$ **(c)** $\tan 2\theta$ **(d)** In which quadrant is 2θ?

13 **(a)** By writing $\sin 3\theta$ as $\sin(2\theta+\theta)$, write $\sin 3\theta$ in terms of $\sin\theta$.

(b) Hence write $\cos 3\theta$ in terms of $\cos\theta$. **(c)** Hence write $\tan 3\theta$ in terms of θ.

14 Use the expansion of $\sin(A+B)$ to evaluate $\sin 195°$.

15 Simplify:

(a) $\frac{\sin 2A}{1+\cos 2A}$ **(b)** $\frac{1}{2}\sin 2\theta\tan\theta$ **(c)** $\cos^2 2\theta-\sin^2 2\theta$ **(d)** $\cos^2 30°-\sin^2 30°$

(e) $\sin 4x\cos 4x$ **(f)** $1+\cos(180°+2\theta)$ **(g)** $\sin x\cos x\cos 2x$ **(h)** $2\sin 2x\cos 2x$

(i) $(\sin\theta+\cos\theta)^2$ **(j)** $(\sin A-\cos A)^2$ **(k)** $\frac{2\tan\theta}{1-\tan^2\theta}$ for $\theta=22.5°$ **(l)** $\sin^2 50°+\sin^2 40°$

(m) $\sin(45°-x)\cos(45°-x)$ **(n)** $\frac{1-\cos 2\theta}{1+\cos 2\theta}$ **(o)** $2\cos^2 3x-1$

Prove the following identities (questions **16** to **36**):

16 $\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\cos 2\theta\cos\theta+\sin 2\theta\sin\theta}=\tan\theta$

17 $\frac{\sin A+\cos A\tan B}{\cos A-\sin A\tan B}=\tan(A+B)$

18 $\sin(\theta+\alpha)\sin(\theta-\alpha)=\sin^2\theta-\sin^2\alpha$

19 $\frac{\tan A-\tan B}{\tan A+\tan B}=\frac{\sin(A-B)}{\sin(A+B)}$

20 $\tan(45°+A)+\tan(45°-A)=\frac{2}{\cos 2A}$

21 $\frac{\sin 3A}{\sin A}-\frac{\cos 3A}{\cos A}=2$

22 $\frac{\sin 2\theta+1}{\cos 2\theta}=\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$

23 $\cos 4x=8\cos^4 x-8\cos^2 x+1$

24 $\frac{1-\cos x}{\sin x}=\tan\frac{x}{2}$

25 $\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=2\sec 2\theta$

26 $\frac{\sin x+1-\cos x}{\sin x-1+\cos x}=\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}$

27 $\sin(A+B+C)=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B-\sin A\sin B\sin C$

What is the resulting identity if C is replaced by $(90°-B)$?

28 $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin C\sin A-\cos C\sin A\sin B$

What is the resulting identity if B is replaced by $(90°-C)$?

29 $\frac{\sin A+\sin(90°-A)+1}{\sin A-\sin(90°-A)+1}=\cot\frac{A}{2}$

30 $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=1-\frac{1}{2}\sin 2\theta$

31 $\tan(\theta+\alpha)\tan(\theta-\alpha)=\frac{\tan^2\theta-\tan^2\alpha}{1-\tan^2\theta\tan^2\alpha}$

32 $\cot(x+y)=\frac{\cot x\cot y-1}{\cot x+\cot y}$

33 $\frac{2\cos\frac{\theta}{2}-1-\cos\theta}{2\cos\frac{\theta}{2}+1+\cos\theta}=\frac{1-\cos\frac{\theta}{2}}{1+\cos\frac{\theta}{2}}$

34 $\tan(\theta+45°)\tan(\theta-45°)=-1$

35 $\dfrac{\tan 2\theta - \tan\theta}{\tan 2\theta + \cot\theta} = \tan^2\theta$

36 $\dfrac{1-\tan\theta\tan 2\theta}{1+\tan\theta\tan 2\theta} = 4\cos^2\theta - 3$

37 If A, B and C are the angles of a triangle, prove that $\cos A\cos B - \sin A\sin B + \cos C = 0$.

38 If $\tan A = \dfrac{p}{q}$, express the following in terms of p and q.

(a) $q\sin A\cos A + p\sin^2 A$ **(b)** $p\sin 2A + q\cos 2A$

39 Find $\tan x$ in terms of $\tan\theta$ if $\tan\theta = \dfrac{\cos(\theta + x)}{\cos(\theta - x)}$.

40 Given that $\sin 18° = \frac{1}{4}\left(\sqrt{5}-1\right)$, find $\cos 36°$ in surd form.

41 Using the double-angle formula for $\tan\theta$, find $\tan 22.5°$ in simplest surd form.

42 Three points P, Q, R are in a horizontal plane. Angles RPQ and RQP are α and β respectively. If PQ is x units in length, show that the perpendicular distance y from R to PQ is given by $y = \dfrac{x\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}$.

4.3 HALF-ANGLE FORMULAE

We have established the double-angle formulae: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta},\ \tan\theta \neq \pm 1$$

Just as θ is the half-angle of 2θ, similarly $\dfrac{\theta}{2}$ is the half-angle of θ, so we can use these results to obtain:

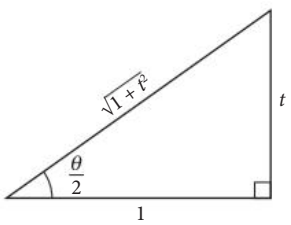
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}},\ \tan\frac{\theta}{2} \neq \pm 1$$

If we let $t = \tan\dfrac{\theta}{2}$, then: $\tan\theta = \dfrac{2t}{1-t^2}$

Using the right-angled triangle in which $\tan\dfrac{\theta}{2} = \dfrac{t}{1}$, we can obtain the ratios:



$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}} \quad \text{and} \quad \cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$

Hence: $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$$= 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}$$

$$= \frac{2t}{1+t^2}$$

and: $\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2}$

$$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$$

$$= \frac{1-t^2}{1+t^2}$$

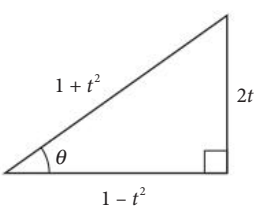
These results, known as the 't formulae', may be summarised as follows:

If $t = \tan\dfrac{\theta}{2}$, then:

$$\tan\theta = \frac{2t}{1-t^2}$$

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$



Example 7

Given that $\tan\theta = \frac{5}{12}$, find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

Solution

$\tan\theta = \frac{5}{12}$ so $t = \frac{5}{12}$: $\sin 2\theta = \frac{2t}{1+t^2}$ $\cos 2\theta = \frac{1-t^2}{1+t^2}$

$= \frac{2\times\frac{5}{12}}{1+\left(\frac{5}{12}\right)^2}$ $= \frac{1-\left(\frac{5}{12}\right)^2}{1+\left(\frac{5}{12}\right)^2}$

$= \frac{120}{169}$ $= \frac{119}{169}$

Note that if a calculator is used to find intermediate values then the answer will be only approximate. For example, $\tan\theta = \frac{5}{12}$ so $\theta = 22°\,37'$, $2\theta = 45°\,14'$, $\sin 2\theta \approx 0.709\,98$, while $\frac{120}{169} \approx 0.710\,06$ (both to five d.p.)—close, but not the same.

Example 8

Use the t formulae to prove that: $\frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1} = \tan\frac{\theta}{2}$

Solution

$$\text{LHS} = \frac{\cos\theta + \sin\theta - 1}{\cos\theta - \sin\theta + 1}$$

$$= \frac{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} + 1}$$

$$= \left(\frac{1-t^2+2t-(1+t^2)}{1+t^2}\right) \Big| \left(\frac{1-t^2-2t+(1+t^2)}{1+t^2}\right)$$

$$= \frac{1-t^2+2t-1-t^2}{1-t^2-2t+1+t^2}$$

$$= \frac{2t-2t^2}{2-2t}$$

$$= \frac{2t(1-t)}{2(1-t)}$$

$$= t = \tan\frac{\theta}{2} = \text{RHS}$$

EXERCISE 4.3 HALF-ANGLE FORMULAE

1 If $t = \tan\frac{\theta}{2}$, then $\sin\theta + \cos\theta = \ldots$

A $\frac{1+2t-t^2}{1+t^2}$ B $\frac{t^2-2t+1}{1+t^2}$ C $\frac{(1-t)^2}{1+t^2}$ D $\frac{(1+t)^2}{1+t^2}$

2 Simplify:

(a) $\frac{2\tan 9°}{1-\tan^2 9°}$ (b) $\frac{1-\tan^2 15°}{1+\tan^2 15°}$ (c) $\frac{1+\tan^2 22.5°}{2\tan 22.5°}$

3 If $t = \tan\frac{\theta}{2}$, express each of the following in terms of t:

(a) $\sin\theta - \cos\theta$ (b) $3\sin\theta + 4\cos\theta$ (c) $2\cos\theta - \sin\theta$ (d) $\cot\theta$

(e) $\cot\theta - \tan\theta$ (f) $\dfrac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta}$ (g) $1 - \frac{1}{2}\sin\theta\tan\frac{\theta}{2}$ (h) $1 + \tan\theta\tan\frac{\theta}{2}$

(i) $\dfrac{\tan\theta - \tan\frac{\theta}{2}}{\cot\frac{\theta}{2} + \tan\theta}$ (j) $\cot\frac{\theta}{2} - 2\cot\theta$ (k) $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$ (l) $\dfrac{\sin\theta+\sin\frac{\theta}{2}}{1+\cos\theta+\cos\frac{\theta}{2}}$

4 If $t = \tan\frac{\theta}{2}$, then $\dfrac{1-\cos\theta}{\sin\theta} = \ldots$

A 0 B $\dfrac{1+t^2}{t}$ C $\dfrac{1-2t+t^2}{2t}$ D t

5 If $t = \tan\frac{\theta}{2}$, solve for t the equation $\tan\theta = d$.

6 If $t = \tan\frac{\theta}{2}$, solve for t the equation $12\tan\theta = 5$, $180° < \theta < 270°$.

7 If $\tan\frac{\theta}{2} = \frac{a}{b}$, prove that $b\cos\theta + a\sin\theta = b$.

8 If $\sec\theta - \tan\theta = x$, prove that $x = \dfrac{1-t}{1+t}$.

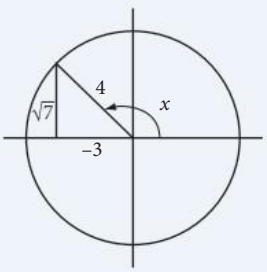
4.4 USING RADIANS

Example 9

If $\cos x = -\frac{3}{4}$ and $\frac{\pi}{2} \le x \le \pi$, find the value of: (a) $\sin x$ (b) $\sin 2x$

Solution

From the diagram (drawn to show the given ratio for the angle):



(a) $\sin x = \dfrac{\sqrt{7}}{4}$

(b) $\sin 2x = 2\sin x\cos x$

$= 2\times\dfrac{\sqrt{7}}{4}\times\left(-\frac{3}{4}\right)$

$= -\dfrac{3\sqrt{7}}{8}$

Example 10

Simplify:

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6}$ (b) $\sin\left(\frac{\pi}{2}-\theta\right) + \sin\left(\frac{\pi}{2}-\phi\right)$ (c) $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

Solution

(a) $\cos\frac{\pi}{3}\cos\frac{\pi}{6} - \sin\frac{\pi}{3}\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$ (using formula [2] from page 84)

$= \cos\frac{\pi}{2} = 0$

(b) $\sin\left(\frac{\pi}{2}-\theta\right) + \sin\left(\frac{\pi}{2}-\phi\right) = \cos\theta + \cos\phi$

(c) $\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right) = \frac{1}{2}\times 2\sin\left(\frac{\pi}{4}-x\right)\cos\left(\frac{\pi}{4}-x\right)$

$= \frac{1}{2}\sin 2\left(\frac{\pi}{4}-x\right)$ (using double-angle formula [8] from page 85)

$= \frac{1}{2}\sin\left(\frac{\pi}{2}-2x\right) = \frac{1}{2}\cos 2x$

Example 11

(a) By writing the expansions for $\cos(A+B)$ and $\cos(A-B)$, find a simplified expression for $\cos(A+B)+\cos(A-B)$.

(b) By writing $\theta = A+B$ and $\phi = A-B$, find an expression for $\cos\theta + \cos\phi$ as the product of two trigonometric functions.

Solution

(a) $\cos(A+B) = \cos A\cos B - \sin A\sin B$ [1]

$\cos(A-B) = \cos A\cos B + \sin A\sin B$ [2]

[1] + [2]: $\cos(A+B)+\cos(A-B) = \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B$

$= 2\cos A\cos B$

(b) $\cos(A+B)+\cos(A-B) = \cos\theta + \cos\phi$

$\theta = A+B$ [1]

$\phi = A-B$ [2]

[1] + [2]: $\theta+\phi = 2A$

$A = \frac{\theta+\phi}{2}$

[1] − [2]: $\theta-\phi = 2B$

$B = \frac{\theta-\phi}{2}$

Hence: $\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

EXERCISE 4.4 USING RADIANS

1 Simplify:

(a) $1+\tan^2\left(\frac{\pi}{2}-\alpha\right)$ **(b)** $1-\cos^2(\pi+\theta)$ **(c)** $\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)+\cos\theta\sin\left(\frac{\pi}{2}-\theta\right)$

(d) $2\cos^2\frac{\pi}{6}-1$ **(e)** $1-\sin\theta\cos\left(\frac{\pi}{2}-\theta\right)$ **(f)** $\sin(\pi-\theta)\cos\phi-\cos(\pi-\theta)\sin\phi$

2 $\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)$ simplifies to:

A $\cos 2x$ **B** $\sin 2x$ **C** $\cos x$ **D** $\sin x$

3 If $\tan\theta = \frac{3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, find the value of: **(a)** $\sin\theta$ **(b)** $\cos\theta$ **(c)** $\cos 2\theta$

4 If $\frac{\pi}{2} \le x \le \pi$ and $\cos x = -\frac{5}{6}$, find the value of: **(a)** $\sin x$ **(b)** $\sin 2x$ **(c)** $\tan 2x$

5 If $\operatorname{cosec}\alpha = -\frac{17}{8}$ and $\pi < \alpha < \frac{3\pi}{2}$, find the value of: **(a)** $\cot\alpha$ **(b)** $\tan 2\alpha$

6 If $\sin\theta = \frac{3}{5}$, $\frac{\pi}{2} \le \theta \le \pi$ and $\tan\phi = \frac{7}{24}$, $0 \le \phi \le \frac{\pi}{2}$, find the value of:

(a) $\sin(\theta-\phi)$ **(b)** $\cos(\theta-\phi)$ **(c)** $\tan(\theta-\phi)$

7 If $\tan x = \frac{5}{4}$, $\tan y = \frac{1}{9}$ and $0 < y < x < \frac{\pi}{2}$, prove that $x - y = \frac{\pi}{4}$.

8 Prove the following.

(a) $2\cos\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}-x\right) = \cos 2x$ **(b)** $\tan\left(\theta+\frac{\pi}{4}\right)\tan\left(\theta-\frac{\pi}{4}\right) = -1$

(c) $\left(\sec^2\theta - 1\right)\tan\left(\frac{\pi}{2}-\theta\right) = \tan\theta$ **(d)** $\tan\left(x+\frac{3\pi}{4}\right) = \frac{\tan x - 1}{\tan x + 1}$

(e) $\frac{1-\sin\left(\frac{\pi}{2}-2x\right)}{\sin 2x} = \tan x$ **(f)** $\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right) = 2\sec 2A$

9 By expanding each term on the left-hand side, prove that $\sin\left(\theta+\frac{\pi}{6}\right)\sin\left(\theta-\frac{\pi}{6}\right) = \sin^2\theta - \frac{1}{4}$.

10 Simplify: **(a)** $\dfrac{2\tan\theta}{1-\tan^2\theta}$ where $\theta=\dfrac{7\pi}{8}$ **(b)** $2\cos^2 3x-1$ where $x=\dfrac{2\pi}{9}$

11 **(a)** By writing expansions for $\sin(A+B)$ and $\sin(A-B)$, find a simplified expression for $\sin(A+B)+\sin(A-B)$.

(b) By writing $\theta=A+B$ and $\phi=A-B$, find an expression for $\sin\theta+\sin\phi$ as the product of two trigonometric functions.

12 If $0\le\theta\le\dfrac{\pi}{2}$, prove that $\tan\theta=\sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$. Hence show that the exact value of $\tan\dfrac{\pi}{8}$ is $\sqrt{2}-1$.

13 If $4\tan(\alpha-\beta)=3\tan\alpha$, prove that $\tan\beta=\dfrac{\sin 2\alpha}{7+\cos 2\alpha}$.

14 If $\sec\theta-\tan\theta=\dfrac{3}{5}$, show that $\sin\theta=\dfrac{8}{17}$. (Hint: Use t formulae.)

15 If $\tan\theta=t$, express $\sin 2\theta$ and $\cos 2\theta$ in terms of t. Find the values of t for which $(k+1)\sin 2\theta+(k-1)\cos 2\theta=k+1$.

16 Use the factors of x^3-y^3 to show that $\cos^6\theta-\sin^6\theta=\left(1-\dfrac{1}{4}\sin^2 2\theta\right)\cos 2\theta$.

17 If $\cos\theta=\dfrac{l^2-m^2}{l^2+m^2}$ and $0<\theta<\dfrac{\pi}{2}$, express $\tan\theta$ and $\sin 2\theta$ in terms of l and m.

18 If A, B and C are successive terms of an arithmetic series, prove that $\sin A+\sin C=2\sin B\cos(B-A)$.

19 Show that $4\sin\theta\sin\left(\theta-\dfrac{\pi}{3}\right)\sin\left(\theta-\dfrac{2\pi}{3}\right)=\sin 3\theta$.

20 If $\tan\alpha=k\tan\beta$, show that $(k-1)\sin(\alpha+\beta)=(k+1)\sin(\alpha-\beta)$.

4.5 SIMPLE TRIGONOMETRIC EQUATIONS

You have already solved simple equations where the variable is in degrees. With the variable in radians, we can now use the angle formulae we have found in this chapter to solve harder equations.

There is no general method for solving trigonometric equations and inequalities, but there are certain standard procedures and types, as illustrated in the examples below.

Example 12

Solve the equation $3\sin 2\theta=1.5$, $0\le\theta\le\pi$.

Solution

$3\sin 2\theta=1.5$

$\therefore\quad \sin 2\theta=0.5$

$0\le\theta\le\pi$ means that $0\le 2\theta\le 2\pi$:

$2\theta=\dfrac{\pi}{6},\dfrac{5\pi}{6}$

Hence: $\theta=\dfrac{\pi}{12},\dfrac{5\pi}{12}$

This solution can be checked graphically by using GeoGebra or other graphing software to find the intersection of $y=3\sin 2\theta$ and $y=1.5$. Over the domain $0\le\theta\le\pi$ these functions intersect only twice. If the domain is increased, then there will be two more intersections for each domain increase of π units.

Example 13

Solve the equation $\cos\left(2x-\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2}$, $0\le x\le 2\pi$.

Solution

Let $\theta=2x-\dfrac{\pi}{6}$: $\cos\theta=\dfrac{\sqrt{3}}{2}$

Hence: $\theta = \ldots -\frac{11\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi+\frac{\pi}{6}, 2\pi+\frac{11\pi}{6}, \ldots$

From the limits: $0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$

But $2x = \theta + \frac{\pi}{6}$: $0 \le \theta + \frac{\pi}{6} \le 4\pi$

$-\frac{\pi}{6} \le \theta \le 4\pi - \frac{\pi}{6}$

$\therefore \quad \theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi+\frac{\pi}{6}, 2\pi+\frac{11\pi}{6}$

$\therefore 2x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, 2\pi+\frac{\pi}{6}, 2\pi+\frac{11\pi}{6}$

$2x = 0, \frac{\pi}{3}, 2\pi, \frac{7\pi}{3}, 4\pi$

$x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

Example 14

Find the values of x for which $\cos 2x \le \frac{1}{\sqrt{2}}$, $0 \le x \le 2\pi$.

Solution

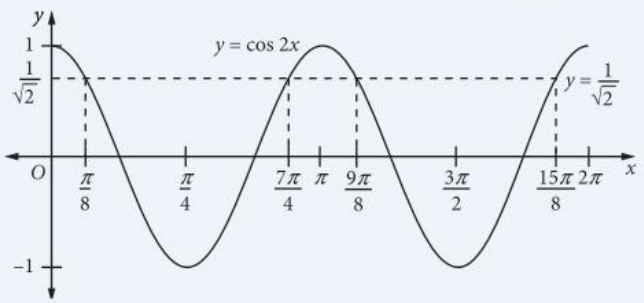
First solve the equation, then solve the inequality graphically.

Solve the equation: $\cos 2x = \frac{1}{\sqrt{2}} \qquad (0 \le x \le 2\pi)$

$2x = \frac{\pi}{4}, \frac{7\pi}{4}, 2\pi+\frac{\pi}{4}, 2\pi+\frac{7\pi}{4}$ (as $0 \le 2x \le 4\pi$, around the circle twice)

$x = \frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

Now sketch the graph of $y = \cos 2x$ for $0 \le x \le 2\pi$, showing the line $y = \frac{1}{\sqrt{2}}$ also:



Use the diagram to find where the graph of $y = \cos 2x$ is on or below the line $y = \frac{1}{\sqrt{2}}$.

Hence $\cos 2x \le \frac{1}{\sqrt{2}}$ for $\frac{\pi}{8} \le x \le \frac{7\pi}{8}$ and for $\frac{9\pi}{8} \le x \le \frac{15\pi}{8}$.

EXERCISE 4.5 SIMPLE TRIGONOMETRIC EQUATIONS

1 Solve for values of θ and x between 0 and 2π inclusive:

(a) $\sin\theta = \frac{\sqrt{3}}{2}$ **(b)** $\tan x = -1$ **(c)** $\cos x = -0.5$ **(d)** $\sqrt{3}\tan\theta = 1$

(e) $\sin 2\theta = -\frac{1}{2}$ **(f)** $\operatorname{cosec}\theta = -2$ **(g)** $\cot 2x = \sqrt{3}$ **(h)** $\sec 2\theta = \sqrt{2}$

2 Solve between 0° and 360°: **(a)** $\cos x = 0.4$ **(b)** $4\tan 2\theta + 3 = 0$

3 The solution to $\sqrt{2}\sin 2\theta + 1 = 0$ for $0 \le \theta \le 2\pi$ is:

A $\frac{5\pi}{4}, \frac{7\pi}{4}$ **B** $\frac{5\pi}{8}, \frac{7\pi}{8}$ **C** $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ **D** $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$

4 Solve for $-\pi \le x \le \pi$: **(a)** $2\cos 2x + 1 = 0$ **(b)** $\sqrt{2}\sin 2x - 1 = 0$

5 Solve between 0 and 2π inclusive:

(a) $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ **(b)** $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ **(c)** $\cos\left(2x + \frac{\pi}{3}\right) = \frac{1}{2}$ **(d)** $2\sin\left(2x - \frac{\pi}{6}\right) = 1$

(e) $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ **(f)** $2\cos\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$ **(g)** $\sin 2\theta = -\cos\frac{7\pi}{4}$ **(h)** $\sin x = \cos x$

6 Solve for $0° < x < 360°$: **(a)** $\cos(2x + 60°) = 0.7242$ **(b)** $5\sin(2x - 70°) + 4 = 0$

7 If $0 \le x \le 2\pi$, the solution to $\sin x \le \frac{\sqrt{3}}{2}$ is:

A $x \le \frac{\pi}{3}$ **B** $x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **C** $0 \le x \le \frac{\pi}{3}$ or $x \ge \frac{2\pi}{3}$ **D** $0 \le x \le \frac{\pi}{3}$ or $\frac{2\pi}{3} \le x \le 2\pi$

8 If $0 \le x \le 2\pi$, solve: **(a)** $\sin x \ge \frac{1}{2}$ **(b)** $\cos x < \frac{1}{2}$ **(c)** $\sin x > 0$ **(d)** $\cos x > \frac{\sqrt{3}}{2}$

(e) $\sin x < 1$ **(f)** $\cos x > 0$ **(g)** $\tan x > 1$ **(h)** $\sqrt{2}\cos x > -1$

9 Solve for $0 < x < \pi$: **(a)** $\sin 2x \ge \frac{1}{2}$ **(b)** $\cos 2x \le 0$

10 Solve for $-\pi < x < \pi$: **(a)** $2\sin 2x \le \sqrt{3}$ **(b)** $2\cos 2x > -1$

4.6 QUADRATIC TRIGONOMETRIC EQUATIONS

Example 15

(a) Solve the equation $\tan^2\theta + \tan\theta - 2 = 0$, $0° < \theta < 360°$.

(b) Solve the equation $\cos^2 x = 2\cos x$, $-\pi \le x \le \pi$.

Solution

(a) Factorise: $(\tan\theta - 1)(\tan\theta + 2) = 0$

$\tan\theta = 1$ or -2

$\theta = 45°, 225°$ or $116°34', 296°34'$

Solution is $\theta = 45°, 116°34', 225°, 296°34'$.

(b) Rearrange: $\cos^2 x - 2\cos x = 0$

Factorise: $\cos x(\cos x - 2) = 0$

$\therefore \cos x = 0$ or 2

Because $|\cos x| \le 1$, the only solution is $\cos x = 0$.

$\therefore x = -\frac{\pi}{2}, \frac{\pi}{2}$

Example 16

Solve the equation $\sec^2 x - 2\tan x = 4$ for $0 \le x \le 2\pi$.
(Trigonometric values rounded to 3 d.p. where necessary.)

Solution

The trigonometric functions are different, but they can be linked by the identity $\sec^2 x = 1 + \tan^2 x$:

$$\sec^2 x - 2\tan x = 4$$
$$1 + \tan^2 x - 2\tan x = 4$$
$$\tan^2 x - 2\tan x - 3 = 0$$
$$(\tan x - 3)(\tan x + 1) = 0$$
$$\tan x = -1 \text{ or } 3$$
$$x = \pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4}, 1.249, \pi + 1.249$$
$$x = \frac{3\pi}{4}, \frac{7\pi}{4}, 1.249, 4.391$$

EXERCISE 4.6 QUADRATIC TRIGONOMETRIC EQUATIONS

1 Solve for values between 0 and 2π inclusive:

(a) $\tan^2 x - 1 = 0$ **(b)** $\sin^2 x - \sin x = 0$ **(c)** $\cos^2\theta - 2\cos\theta + 1 = 0$

(d) $\sqrt{3}\tan^2 x + \tan x = 0$ **(e)** $4\sin^2\theta = 1$ **(f)** $\sin^2 x - \sin x \cos x = 0$

2 If $0 \le \theta \le 2\pi$, the solution to $2\cos^2\theta - 1 = 0$ is:

A $\frac{\pi}{4}, \frac{7\pi}{4}$ **B** $\frac{3\pi}{4}, \frac{5\pi}{4}$ **C** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **D** $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

3 Solve for values between 0 and 2π inclusive:

(a) $2\cos^2\theta - 3\cos\theta - 2 = 0$ **(b)** $2\cos^2\theta + \sin\theta = 1$

(c) $2\sin^2\theta - 3\cos\theta = 2$ **(d)** $(2\cos x + 1)(\sin x - 1) = 0$

4 Solve for $0 < x < 360°$: **(a)** $2\tan^2 x + \tan x = 15$ **(b)** $5\cos^2 x + 2\sin x = 2$

5 If $0 \le \theta \le 2\pi$, then the solution to $3\sin^2\theta - 4\cos\theta + 1 = 0$ is (approximately or exactly):

A $\theta = 0.841, 5.442$ **B** $\theta = 1.969, 4.315$ **C** $\theta = 2.301, 3.983$ **D** $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

6 Solve for $0 \le \theta \le 2\pi$: **(a)** $3\tan^3\theta - 3\tan^2\theta - \tan\theta + 1 = 0$ **(b)** $\cos^3\theta - 2\cos^2\theta + \cos\theta = 0$

7 If $t = \tan\frac{\theta}{2}$, solve for $0 \le \theta \le 2\pi$: **(a)** $t^2 - 3 = 0$ **(b)** $t^2 + 2t - 3 = 0$ **(c)** $t^2 - t - 6 = 0$

4.7 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

Example 17

Solve the equation $\sin 2x = 3\cos x$, $0 \le x \le 2\pi$.

Solution

As $\sin 2x = 2\sin x\cos x$: $2\sin x\cos x = 3\cos x$

$$\cos x(2\sin x - 3) = 0$$
$$\therefore \cos x = 0 \text{ or } \sin x = 1.5$$

Because $|\sin x| \le 1$, the only solution is $\cos x = 0$.

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Example 18

Solve the equation $\cos 2x \cos \alpha - \sin 2x \sin \alpha = -0.5$, $0 \le x \le 2\pi$, where $\alpha = \frac{\pi}{6}$.

Solution

Use the expansion of $\cos(A + B)$ to simplify the left-hand side:

$\cos 2x \cos \alpha - \sin 2x \sin \alpha = \cos(2x + \alpha)$

Hence, as $\alpha = \frac{\pi}{6}$: $\cos\left(2x + \frac{\pi}{6}\right) = -0.5$

$$2x + \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi + \frac{2\pi}{3}, 2\pi + \frac{4\pi}{3}$$

$$2x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{5\pi}{2}, \frac{19\pi}{6}$$

$$x = \frac{\pi}{4}, \frac{7\pi}{12}, \frac{5\pi}{4}, \frac{19\pi}{12}$$

EXERCISE 4.7 TRIGONOMETRIC EQUATIONS INVOLVING ANGLE FORMULAE

1 Solve: **(a)** $\cos 2\theta = \cos \theta$, $0 \le \theta \le 2\pi$ **(b)** $2\cos 2\theta = 4\cos \theta - 3$, $0 \le \theta \le 2\pi$

(c) $3\tan 2\theta = 2\tan \theta$, $0 \le \theta \le 2\pi$ **(d)** $\tan \theta + 2\cot \theta = 3$, $0° \le \theta \le 360°$

2 The solution to $5\sin x = 2\sec x$ for $0° \le x \le 180°$ is:

A $x = 11° 47'$ or $78° 13'$ **B** $x = 23° 34'$ or $156° 26'$

C $x = 26° 34'$ or $63° 26'$ **D** $x = 53° 8'$ or $126° 52'$

3 Solve: **(a)** $\cos 2x \cos \frac{\pi}{6} - \sin 2x \sin \frac{\pi}{6} = \frac{1}{2}$, $0 \le x \le 2\pi$ **(b)** $\sin 2x \cos \frac{\pi}{3} + \cos 2x \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $-\pi \le x \le \pi$

4 The solution to $\sin \theta = \cos 2\theta$ for $0 \le \theta \le 2\pi$ is:

A $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ **B** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **C** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **D** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

5 Solve $\tan \theta = \sin 2\theta$, $0 \le \theta \le 2\pi$.

6 Solve $\cos^2 \theta = 2\cos^2 \frac{\theta}{2}$, $0° \le \theta \le 360°$.

7 Solve $\sin 3x \cos x - \cos 3x \sin x = \frac{\sqrt{3}}{2}$, $0 \le x \le 2\pi$.

8 Solve $\tan 2\theta = 2\tan \theta$, $0 \le \theta \le 2\pi$.

4.8 AUXILIARY ANGLE METHOD

This method involves changing an equation of the form $a\sin x \pm b\cos x = c$ into the form $r\sin(x \pm \alpha) = c$. In this form, α is called the **auxiliary angle**.

This method can also be used to change $a\cos x \pm b\sin x = c$ into the form $r\cos(x \mp \alpha) = c$. The constants a, b, r and α are positive real numbers.

For example, to express $a\sin x + b\cos x$ in the form $r\sin(x + \alpha)$:

Let $a\sin x + b\cos x \equiv r\sin(x + \alpha)$

$$\equiv r(\sin x \cos \alpha + \cos x \sin \alpha)$$

$$\equiv r\sin x \cos \alpha + r\cos x \sin \alpha$$

This is an identity, so the coefficients of $\sin x$ and $\cos x$ on each side are the same.

i.e. $a = r\cos \alpha$

$b = r\sin \alpha$

$\therefore\ a^2 + b^2 = r^2(\cos^2 \alpha + \sin^2 \alpha)$

Hence: $r^2 = a^2 + b^2$

$r = \sqrt{a^2 + b^2}$ because r is a positive real number.

From the coefficients, we also have $\cos\alpha = \frac{a}{r}$ and $\sin\alpha = \frac{b}{r}$.

As a and b are positive constants, so $\cos\alpha$ and $\sin\alpha$ are also positive, hence α is in the first quadrant (i.e. it is an acute angle) such that $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{b}{a}$.

Hence, the auxiliary angle method gives us:

[1] $a\sin x + b\cos x \equiv r\sin(x+\alpha)$

[2] $a\sin x - b\cos x \equiv r\sin(x-\alpha)$

[3] $a\cos x + b\sin x \equiv r\cos(x-\alpha)$

[4] $a\cos x - b\sin x \equiv r\cos(x+\alpha)$

which then allows us to obtain

In each case $r = \sqrt{a^2+b^2}$ and α is in the first quadrant such that $\tan\alpha = \frac{b}{a}$.

Example 19

Express: **(a)** $\sqrt{3}\sin x - \cos x$ in the form $r\sin(x-\alpha)$ **(b)** $3\cos x - 4\sin x$ in the form $r\cos(x+\alpha)$.

Solution

(a) $\sqrt{3}\sin x - \cos x \equiv r\sin(x-\alpha)$

$\equiv r(\sin x\cos\alpha - \cos x\sin\alpha)$

$\equiv r\sin x\cos\alpha - r\cos x\sin\alpha$

Equate coefficients of $\sin x$ and $\cos x$: $r\cos\alpha = \sqrt{3}$ [1]

$r\sin\alpha = 1$ [2]

$[1]^2 + [2]^2$: $r^2(\cos^2\alpha + \sin^2\alpha) = 4$

$r^2 = 4$

$r = 2$ (as $r > 0$)

Hence, from [1] and [2]: $\cos\alpha = \frac{\sqrt{3}}{2}$ and $\sin\alpha = \frac{1}{2}$

As $\cos\alpha$ and $\sin\alpha$ are both positive, α is in the first quadrant such that $\tan\alpha = \frac{1}{\sqrt{3}}$, i.e. $\alpha = \frac{\pi}{6}$.

From the first equation: $\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right)$

(b) $3\cos x - 4\sin x = r\cos(x+\alpha)$

$a = 3, b = 4$: $r = \sqrt{3^2+4^2} = 5$

$\tan\alpha = \frac{4}{3}$: $\alpha = 53°\,8'$

$\therefore\ 3\cos x - 4\sin x = 5\cos(x + 53°\,8')$

Example 19 illustrates two different auxiliary angle methods that may be used. You should practise both.

Important uses of the auxiliary angle method

- Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y = a\sin x + b\cos x$ much easier.
- Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ allows you to solve equations of the type $a\sin x \pm b\cos x = c$.

Example 20

Sketch the graph of $y=\sqrt{3}\sin x-\cos x,\ 0\le x\le 2\pi$.

Solution

Example 19(a) (see page 97) has already shown that $\sqrt{3}\sin x-\cos x=2\sin\left(x-\frac{\pi}{6}\right)$. Hence: $y=2\sin\left(x-\frac{\pi}{6}\right)$

At the endpoints of the domain, $x=0$ and $x=2\pi$: $y=2\sin\left(-\frac{\pi}{6}\right)=2\sin\left(2\pi-\frac{\pi}{6}\right)=-1$

The greatest value of y is 2, where: $1=\sin\left(x-\frac{\pi}{6}\right)$

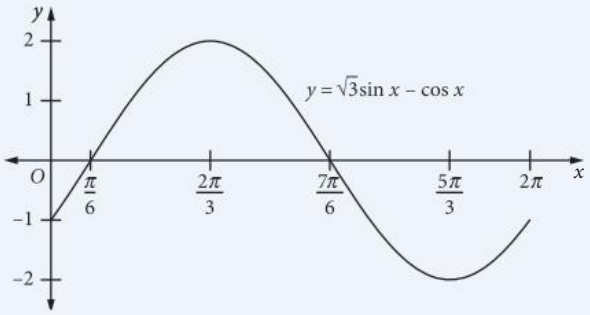
$$x-\frac{\pi}{6}=\frac{\pi}{2}$$
$$x=\frac{2\pi}{3}$$

The least value of y is -2, where: $-1=\sin\left(x-\frac{\pi}{6}\right)$

$$x-\frac{\pi}{6}=\frac{3\pi}{2}$$
$$x=\frac{5\pi}{3}$$

The graph crosses the x-axis where: $\sin\left(x-\frac{\pi}{6}\right)=0$

$$x-\frac{\pi}{6}=0,\ \pi$$
$$x=\frac{\pi}{6},\ \frac{7\pi}{6}$$



Example 21

Solve the following equations.

(a) $\sqrt{3}\sin x-\cos x=1,\ 0\le x\le 2\pi$

(b) $8\cos x+6\sin x=-3,\ 0^\circ\le x\le 360^\circ$

Solution

Method 1

(a) Example 19 (a) (see page 97) has already shown that $\sqrt{3}\sin x-\cos x=2\sin\left(x-\frac{\pi}{6}\right)$.

$$\therefore\quad 2\sin\left(x-\frac{\pi}{6}\right)=1$$
$$\sin\left(x-\frac{\pi}{6}\right)=\frac{1}{2}$$
$$x-\frac{\pi}{6}=\frac{\pi}{6},\ \frac{5\pi}{6}$$
$$x=\frac{\pi}{3},\ \pi$$

(b) $8\cos x+6\sin x=-3$

$a=8,\ b=6$: $\quad r=\sqrt{8^2+6^2}=10$

$\tan\alpha=\frac{6}{8}=0.75$ so $\alpha=36^\circ 52'$

$\therefore\ 10\cos(x-36^\circ 52')=-3$

$$\cos(x-36^\circ 52')=-0.3$$
$$x-36^\circ 52'=107^\circ 27',\ 252^\circ 33'$$
$$x=144^\circ 19',\ 289^\circ 25'$$

Method 2

We can express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ for all values of x except $x=\pm\pi,\pm 3\pi,\pm 5\pi,\ldots$ (because $\tan\frac{\pi}{2}$ is undefined).

The t formulae (see page 88) give $\sin x=\frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}$.

(a) $\sqrt{3}\sin x - \cos x = 1,\ 0 \le x \le 2\pi$: $\dfrac{2\sqrt{3}t}{1+t^2} - \dfrac{1-t^2}{1+t^2} = 1$ where $t = \tan\dfrac{x}{2}$

$$2\sqrt{3}t - (1-t^2) = 1+t^2$$
$$2\sqrt{3}t - 1 + t^2 = 1 + t^2$$
$$\sqrt{3}t = 1$$
$$t = \frac{1}{\sqrt{3}}$$
$$\therefore \quad \frac{x}{2} = \frac{\pi}{6} \text{ for } 0 \le \frac{x}{2} \le \pi$$
$$x = \frac{\pi}{3} \text{ for } 0 \le x \le 2\pi$$

Because $t = \tan\dfrac{x}{2}$ is undefined at $x = \pi$, we must now separately test whether $x = \pi$ is a solution.

$x = \pi$: LHS $= \sqrt{3}\sin\pi - \cos\pi = 0 - (-1) = 1 =$ RHS

Hence $x = \pi$ is also a solution. The complete solution is $x = \dfrac{\pi}{3}, \pi$.

(b) $8\cos x + 6\sin x = -3,\ 0° \le x \le 360°$: $\dfrac{8(1-t^2)}{1+t^2} + \dfrac{12t}{1+t^2} = -3$

$$8 - 8t^2 + 12t = -3 - 3t^2$$
$$5t^2 - 12t - 11 = 0$$
$$t = \frac{12 \pm \sqrt{144+220}}{10}$$
$$\tan\frac{x}{2} = 3.108, -0.708 \quad \text{(to 3 d.p.)}$$
$$\frac{x}{2} = 72°\,10',\ 144°\,42' \quad \text{for } 0° \le \frac{x}{2} \le 180°$$
$$x = 144°\,20',\ 289°\,24' \quad \text{for } 0° \le x \le 360°$$

(Note the slight difference in the answers due to the rounding error when solving the quadratic equation.)

Because $t = \tan\dfrac{x}{2}$ is undefined at $x = \pi$, we must now separately test whether $x = \pi$ is a solution.

$x = \pi$: LHS $= 8\cos\pi + 6\sin\pi = -8 + 0 = -8 \neq$ RHS

Hence $x = \pi$ is not a solution of the equation.

Important note:

If you use the t formulae substitution to solve equations of the type $a\cos x + b\sin x = c$, you must also test to see whether $x = \pm n\pi$ is a solution of the equation.

EXERCISE 4.8 AUXILIARY ANGLE METHOD

1 Express each of the following in the form $r\sin(x + \alpha)$.

(a) $\sin x + \cos x$
(b) $3\sin x + \sqrt{3}\cos x$
(c) $5\sin x + 12\cos x,\ 0° < \alpha < 90°$
(d) $2\sin x + \cos x,\ 0° < \alpha < 90°$

2 Express each of the following in the form $r\sin(x - \alpha)$.

(a) $\sin x - \sqrt{3}\cos x$
(b) $2\sin x - 3\cos x,\ 0° < \alpha < 90°$
(c) $2\sin x - \cos x,\ 0° < \alpha < 90°$
(d) $3\sin x - 3\cos x$

3 Express each of the following in the form $r\cos(x-\alpha)$.

(a) $\cos x+\sin x$ **(b)** $24\cos x+7\sin x,\ 0^\circ<\alpha<90^\circ$

(c) $2\cos x+2\sqrt{3}\sin x$ **(d)** $3\cos x+2\sin x,\ 0^\circ<\alpha<90^\circ$

4 Express each of the following in the form $r\cos(x+\alpha)$.

(a) $\cos x-\sin x$ **(b)** $\sqrt{3}\cos x-\sin x$

(c) $8\cos x-\sin x,\ 0^\circ<\alpha<90^\circ$ **(d)** $5\cos x-3\sin x,\ 0^\circ<\alpha<90^\circ$

5 Which expression is equivalent to $8\sin x-15\cos x$?

A $17\cos(x-61^\circ 56')$ **B** $17\sin(x-61^\circ 56')$

C $17\cos(x+61^\circ 56')$ **D** $17\sin(x+61^\circ 56')$

6 Find **(i)** the maximum and **(ii)** the minimum value of the following expressions. Also find the smallest positive value of x for which they have each value.

(a) $\sin x-\sqrt{3}\cos x$ **(b)** $\cos x+\sin x$ **(c)** $2\sqrt{3}\cos x-2\sin x$

(d) $5\sin x+12\cos x$ (answer in degrees)

7 Solve:

(a) $\cos x+\sin x=1,\ 0\le x\le 2\pi$ **(b)** $\cos x+\sqrt{3}\sin x=2,\ 0\le x\le 2\pi$

(c) $3\cos x+2\sin x=\sqrt{13},\ 0^\circ\le x\le 360^\circ$ **(d)** $3\sin x-\sqrt{3}\cos x=\sqrt{3},\ 0\le x\le 2\pi$

(e) $6\sin x+8\cos x=-5,\ 0^\circ\le x\le 360^\circ$ **(f)** $4\cos x+3\sin x=-1,\ 0^\circ\le x\le 360^\circ$

(g) $\cos x-\sqrt{3}\sin x=2,\ 0\le x\le 2\pi$ **(h)** $\cos x-\sin x=-1,\ -\frac{\pi}{2}\le x\le\frac{3\pi}{2}$

(i) $3\sin x+4\cos x=-2,\ -180^\circ\le x\le 180^\circ$ **(j)** $\sqrt{2}\sin x-\cos x=1.5,\ 0^\circ\le x\le 360^\circ$

8 Sketch the graph of $f(x)=\sqrt{3}\cos x-\sin x,\ 0\le x\le 2\pi$. Use your sketch to find the values of x for which:

(a) $f(x)=1$ **(b)** $f(x)>1$

9 Sketch the graph of $f(x)=\cos x+\sin x,\ -\pi\le x\le\pi$. Use your sketch to find the values of x for which:

(a) $f(x)=-1$ **(b)** $f(x)\ge -1$

4.9 GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

Equations of the form $\sin\theta=\sin\alpha$

Example 22

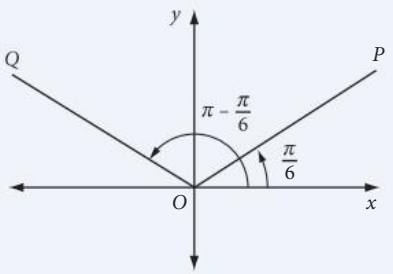
Find all values of θ for which $\sin\theta=\frac{1}{2}$.

Solution

$$\sin\theta=\frac{1}{2}$$

$$\therefore\ \sin\theta=\sin\frac{\pi}{6},\ \sin\left(\pi-\frac{\pi}{6}\right),\ \sin\left(2\pi+\frac{\pi}{6}\right),\ldots$$

Consider a coordinate diagram.



$$\angle XOP=\frac{\pi}{6}$$

$$\angle XOQ=\pi-\frac{\pi}{6}$$

The ray OP defines an infinite number of angles in the first quadrant. If we rotate OP about the origin (either clockwise or anticlockwise), then during each revolution it is along the original ray OP once.

Each full rotation increases the angle by 2π, so we find that OP is the terminal ray defining the angles:

- $\frac{\pi}{6}, 2\pi+\frac{\pi}{6}, 4\pi+\frac{\pi}{6}, \ldots$ for anticlockwise rotation
- $-2\pi+\frac{\pi}{6}, -4\pi+\frac{\pi}{6}, -6\pi+\frac{\pi}{6}, \ldots$ for clockwise rotation.

We can summarise this result as: $n\pi+\frac{\pi}{6}$ where $n=0, \pm2, \pm4, \ldots$ [1]

or: $n\times180°+30°$ (in degrees)

Similarly, the terminal ray OQ defines an infinite number of angles:

$\pi-\frac{\pi}{6}, 3\pi-\frac{\pi}{6}, 5\pi-\frac{\pi}{6}, \ldots$ for anticlockwise rotation

$-\pi-\frac{\pi}{6}, -3\pi-\frac{\pi}{6}, -5\pi-\frac{\pi}{6}, \ldots$ for clockwise rotation.

We can summarise this result as: $n\pi-\frac{\pi}{6}$ where $n=\pm1, \pm3, \pm5, \ldots$ [2]

or: $n\times180°-30°$ (in degrees)

Statements [1] and [2] can be written together as:

$$\theta=n\pi+(-1)^n\frac{\pi}{6}$$

or: $\theta=n\times180°+(-1)^n\times30°$ (in degrees)

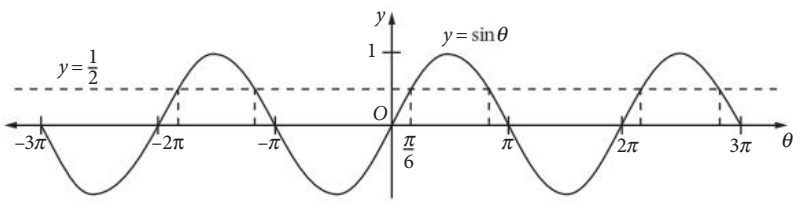
Note that $(-1)^n$ is 1 when n is zero or even, and is -1 when n is odd.

Following the example above, the general solution to the equation $\sin\theta=\sin\alpha$ can be written as:

$\theta=n\pi+(-1)^n\alpha$ (in radians)

$\theta=n\times180°+(-1)^n\alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y=\sin\theta$ and $y=\frac{1}{2}$ (from the example above), as shown in the following diagram.



From symmetry, we can observe that the line $y=\frac{1}{2}$ intersects the sine curve at values of θ that are $\frac{\pi}{6}$ units to the right of $n\pi$ where $n=0, \pm2, \pm4, \ldots$ and that are $\frac{\pi}{6}$ units to the left of $n\pi$ when $n=\pm1, \pm3, \ldots$

Both of these solutions are contained in the statement: $\theta=n\pi+(-1)^n\frac{\pi}{6}$

$$\text{or: } \theta=\begin{cases} n\pi+\frac{\pi}{6}, & n \text{ even or zero} \\ n\pi-\frac{\pi}{6}, & n \text{ odd} \end{cases}$$

Example 23

Find all angles θ for which $\sin\left(\theta+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

Solution

$$\sin\left(\theta+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

i.e. $$\sin\left(\theta+\frac{\pi}{4}\right)=\sin\frac{5\pi}{4}$$

$\therefore$ $$\theta+\frac{\pi}{4}=n\pi+(-1)^n\times\frac{5\pi}{4}$$

$\therefore$ $$\theta=n\pi+(-1)^n\times\frac{5\pi}{4}-\frac{\pi}{4}$$

$n=0$: $\theta=\frac{5\pi}{4}-\frac{\pi}{4}=\pi$;

$n=1$: $\theta=n\pi-\frac{5\pi}{4}-\frac{\pi}{4}=n\pi-\frac{3\pi}{2}=\left(n-\frac{3}{2}\right)\pi$;

$n=2$: $\theta=n\pi+\frac{5\pi}{4}-\frac{\pi}{4}=n\pi+\pi=(n+1;\quad\ldots$

$n=3$: $\theta=n\pi-\frac{5\pi}{4}-\frac{\pi}{4}=n\pi-\frac{3\pi}{2}=\left(n-\frac{3}{2}\right)\pi;\ldots$

$$\therefore\quad \theta=\begin{cases}(n+1)\pi, & n \text{ even or zero}\\ \left(n-\frac{3}{2}\right)\pi, & n \text{ odd}\end{cases}$$

Equations of the form $\cos\theta=\cos\alpha$

Example 24

Find all angles θ for which $\cos\theta=\frac{1}{2}$.

Solution

$$\cos\theta=\frac{1}{2}$$

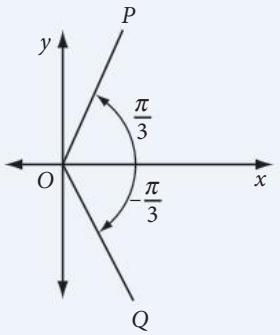
i.e. $\cos\theta=\cos\frac{\pi}{3}, \cos\left(2\pi-\frac{\pi}{3}\right), \cos\left(2\pi+\frac{\pi}{3}\right),\ldots$

In the diagram, OP defines the angles: $\frac{\pi}{3}, 2\pi+\frac{\pi}{3}, 4\pi+\frac{\pi}{3},\ldots$

and OQ defines the angles: $-\frac{\pi}{3}, 2\pi-\frac{\pi}{3}, 4\pi-\frac{\pi}{3},\ldots$

We can summarise these results as: $\theta=2n\pi\pm\frac{\pi}{3}$ where n is any integer

or: $\theta=n\times360^\circ\pm60^\circ$ (in degrees)

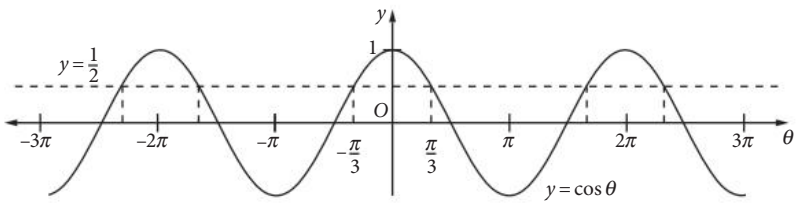


Following the example above, the general solution to the equation $\cos\theta=\cos\alpha$ can be written as:

$\theta=2n\pi\pm\alpha$ (in radians)

$\theta=n\times360^\circ\pm\alpha^\circ$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y=\cos\theta$ and $y=\frac{1}{2}$ (from the example above), as shown in the following diagram.



From symmetry, we can observe that the line $y=\frac{1}{2}$ intersects the cosine curve at values of θ that are $\frac{\pi}{3}$ units to the left and right of $2n\pi$ where $n=0,\pm2,\pm4,\ldots$ i.e. $\theta=2n\pi\pm\frac{\pi}{3}$.

Example 25

Find all values of x for which $2\cos(3x+30^\circ)+\sqrt{3}=0$.

Solution

$$\cos(3x+30^\circ)=-\frac{\sqrt{3}}{2}$$

$$=\cos150^\circ$$

$\therefore\quad 3x+30^\circ=n\times360^\circ\pm150^\circ$ (where n is an integer)

$3x+30^\circ=n\times360^\circ+150^\circ$ or $3x+30^\circ=n\times360^\circ-150^\circ$

$3x=n\times360^\circ+120^\circ$ $\qquad$ $3x=n\times360^\circ-180^\circ$

$x=n\times120^\circ+40^\circ$ $\qquad$ $x=n\times120^\circ-60^\circ$

Example 26

Find all angles θ for which $\sin 2\theta = \cos\theta$.

Solution

$\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ or $2\sin\theta = 1$

$= \cos\frac{\pi}{2}$ $\sin\theta = \frac{1}{2} = \sin\frac{\pi}{6}$

$\therefore \quad \theta = 2n\pi \pm \frac{\pi}{2}$ (in radians) $\therefore \quad \theta = n\pi + (-1)^n \times \frac{\pi}{6}$ (in radians)

$\theta = n \times 360° \pm 90°$ (in degrees) $\theta = n \times 180° + (-1)^n \times 30°$ (in degrees)

Hence the complete solution in radians is $\theta = 2n\pi \pm \frac{\pi}{2}$ or $\theta = n\pi + (-1)^n \times \frac{\pi}{6}$, while the complete solution in degrees is $\theta = n \times 360° \pm 90°$ or $\theta = n \times 180° + (-1)^n \times 30°$.

Equations of the form $\tan\theta = \tan\alpha$

Example 27

Give the general solution of the equation $\tan\theta = 1$.

Solution

$\tan\theta = 1$

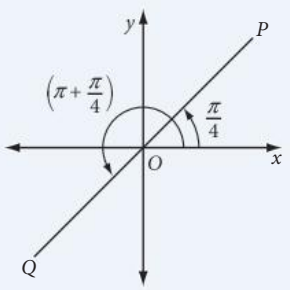
i.e. $\tan\theta = \tan\frac{\pi}{4}, \tan\left(\pi + \frac{\pi}{4}\right), \tan\left(2\pi + \frac{\pi}{4}\right), \ldots$

In the diagram, OP defines the angles: $\frac{\pi}{4}, 2\pi + \frac{\pi}{4}, 4\pi + \frac{\pi}{4}, \ldots$

and OQ defines the angles: $\pi + \frac{\pi}{4}, 3\pi + \frac{\pi}{4}, 5\pi + \frac{\pi}{4}, \ldots$

We can summarise these results as: $\theta = n\pi + \frac{\pi}{4}$ where n is any integer

or: $\theta = n \times 180° + 45°$ (in degrees)

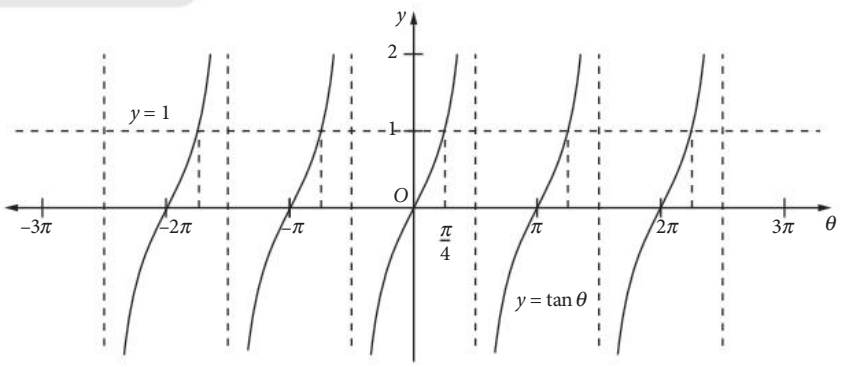


Following the example above, the general solution to the equation $\tan\theta = \tan\alpha$ can be written as:

$\theta = n\pi + \alpha$ (in radians)

$\theta = n \times 180° + \alpha°$ (in degrees) for any integer n.

The pattern for this general solution can also be seen by considering the value of θ at the points of intersection of the curves $y = \tan\theta$ and $y = 1$ (from the previous example), as shown in the diagram at right.



From symmetry, we can observe that the line $y = 1$ intersects the tangent curve at values of θ that are $\frac{\pi}{4}$ units to the right of $n\pi$ where $n = 0, \pm1, \pm2, \ldots$

Example 28

Find the values of x for which $\tan x = 3\cot x$.

Solution

$$\tan x = 3\cot x$$

$$\tan x = \frac{3}{\tan x}$$

$$\tan^2 x = 3$$

$$\tan x = \pm\sqrt{3}$$

i.e. $\tan x = \tan\frac{\pi}{3}$ or $\tan x = \tan\frac{2\pi}{3}$

$\therefore$ $x = n\pi + \frac{\pi}{3}$ (in radians) $\therefore$ $x = n\pi + \frac{2\pi}{3}$ (in radians)

$x = n \times 180^\circ + 60^\circ$ (in degrees) $x = n \times 180^\circ + 120^\circ$ (in degrees)

Example 29

Give the general solution to the equation $\sqrt{3}\cos x - \sin x = 1$.

Solution

$$\sqrt{3}\cos x - \sin x = 1$$

$$\sqrt{3}\cos x - \sin x = r\cos(x+\alpha) \qquad \text{where } r = \sqrt{3+1} = 2 \text{ and } \tan\alpha = \frac{1}{\sqrt{3}},\ \alpha = \frac{\pi}{6}$$

$\therefore$ $\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$

$\therefore$ $2\cos\left(x + \frac{\pi}{6}\right) = 1$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$$

$\therefore$ $x + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$x = 2n\pi + \frac{\pi}{6}$ or $2n\pi - \frac{\pi}{2}$

or: $x = n \times 360^\circ + 30^\circ$ or $n \times 360^\circ - 90^\circ$

EXERCISE 4.9 GENERAL SOLUTION OF TRIGONOMETRIC EQUATIONS

1 Find the general solution of each equation.

(a) $\sin x = 1$ **(b)** $\cos x = 0$ **(c)** $\tan x = -1$ **(d)** $\sqrt{3}\operatorname{cosec} x = 2$

(e) $\sec x = -2$ **(f)** $\cot x = \sqrt{3}$ **(g)** $2\sin\left(\theta - \frac{\pi}{6}\right) + 1 = 0$ **(h)** $\cos\frac{\theta}{2} = 1$

(i) $2\sin^2 x = 1$ **(j)** $\sin x = 0.3894$

2 The general solution to $4\cos^2 x - 1 = 0$ is:

A $x = \frac{\pi}{3}, \frac{2\pi}{3}$ **B** $x = 2n\pi \pm \frac{\pi}{3}$ **C** $x = 2n\pi \pm \frac{2\pi}{3}$ **D** $x = 2n\pi \pm \frac{\pi}{3}, x = 2n\pi \pm \frac{2\pi}{3}$

3 Find the general solution of each equation.

(a) $\cos^2 x - 2\cos x + 1 = 0$ **(b)** $\sin^2 x = \sin x$ **(c)** $\cos 2\theta = \sin\theta$

(d) $\sin^2 x = 1 - \cos x$ **(e)** $\cos 2\theta = 2 + \cos\theta$ **(f)** $\tan 2x = \cot x$

(g) $\cos 2x = \cos x$ **(h)** $2\sin x = \sec x$ **(i)** $\tan^2 x = \tan x$

4 The general solution (in degrees) to $2\cos\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$ is:

A $x = 30°, 330°$ **B** $x = 45°, 345°$

C $x = n \times 180° + 45°, x = n \times 180° + 15°$ **D** $x = n \times 360° + 90°, x = n \times 360° + 30°$

5 Find the general solution of each equation.

(a) $\sqrt{2}\sin 2\theta + 1 = 0$ **(b)** $\tan\left(\theta - \frac{\pi}{3}\right) = -\sqrt{3}$ **(c)** $\cos 2x \cos\frac{\pi}{6} - \sin 2x \sin\frac{\pi}{6} = 0.5$

(d) $\tan\theta = \sin 2\theta$ **(e)** $\tan\theta = \cot\theta$ **(f)** $\sqrt{3}\sin x - \cos x = 1$

6 Find the general solution (in degrees) of each equation.

(a) $\sin x + \cos x = 1$ **(b)** $\cos x - \sqrt{3}\sin x = 2$ **(c)** $4\cos x + 3\sin x = -1$

(d) $\cos x - \sin x = 1$ **(e)** $\sin 3\theta + \sin\theta = 0$ **(f)** $\sin 4x - \sin 2x = 0$

7 Find the general solution of each equation.

(a) $\cos 4x + \cos 2x = 0$ **(b)** $\cos 3\theta = \cos\theta$ **(c)** $2\cos 2\theta = 4\cos\theta - 3$

(d) $3\tan 2\theta = 2\tan\theta$ **(e)** $\tan\left(2\theta - \frac{\pi}{4}\right) + 1 = 0$ **(f)** $2\cos\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$

(g) $\cos\theta = \sin\theta - 1$ **(h)** $2\sin^2 x + \cos x = 1$ **(i)** $\cos x - \sqrt{3}\sin x = 1$

8 Find the general solution (in degrees) of each equation.

(a) $4 + \sin\theta = 6\cos^2\theta$ **(b)** $\sin x = \cos x$ **(c)** $1 + 2\cos^2 x = 5\sin x$

9 Find the general solution of each equation.

(a) $\tan^3\theta - \tan\theta = 0$ **(b)** $\tan x = \sin x$ **(c)** $\sec 2x = \operatorname{cosec} 2x$

(d) $\sin 2\theta = \tan\theta$ **(e)** $\sin 2x + \sin 4x = \sin 3x$ **(f)** $\sin 3x = \sin 2x$

CHAPTER REVIEW 4

1 Simplify: **(a)** $\dfrac{1-t^2}{1+t^2}$ where $t=\tan\dfrac{\theta}{2}$ **(b)** $\dfrac{\tan\theta-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{6}\tan\theta}$

2 Solve $2\tan 2x-1=0$ for $0°<x<360°$.

3 **(a)** Express $2\sqrt{3}\cos\left(\theta+\dfrac{\pi}{6}\right)-2\cos\theta$ in the form $R\cos(\theta+\alpha)$, where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$.

(b) Hence, or otherwise, solve $2\sqrt{3}\cos\left(\theta+\dfrac{\pi}{6}\right)-2\cos\theta=1$ for $0<\theta<2\pi$.

4 **(a)** Show that $\cos(A+B)=\cos A\cos B(1-\tan A\tan B)$.

(b) Suppose that $0<A<\dfrac{\pi}{2}$ and $0<B<\dfrac{\pi}{2}$. Show by deduction that if $\tan A\tan B=1$ then $A+B=\dfrac{\pi}{2}$.

5 Show that: **(a)** $\dfrac{\cos\theta}{1+\sin\theta}=\sec\theta-\tan\theta$ **(b)** $\tan^2\theta=\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$, given that $\cos 2\theta\neq -1$.

6 **(a)** Express $3\sin x+4\cos x$ in the form $r\sin(x+\alpha)$ where $0\leq\alpha\leq\dfrac{\pi}{2}$.

(b) Hence, or otherwise, solve $3\sin x+4\cos x=5$ for $0\leq x\leq 2\pi$. Give your answer(s) correct to two decimal places.

(c) Write the general solution for $3\sin x+4\cos x=5$.

7 Use $\tan^2\theta=\dfrac{1-\cos 2\theta}{1+\cos 2\theta}$ to find the exact value of $\tan\dfrac{\pi}{8}$.

8 The elevation of a hill at a place K due east of the hill is 38°; at a place L, due south of K, the elevation of the hill is 26°. If the distance from K to L is 500 metres, calculate the height of the hill to the nearest metre.

9 It can be shown that $\cos 3\theta=4\cos^3\theta-3\cos\theta$. Use this result to solve $\cos 3\theta+\cos 2\theta+\cos\theta=0$ for $0\leq\theta\leq 2\pi$.

10 Show that the cubic equation $8x^3-6x+1=0$ can be reduced to the form $\cos 3\theta=-\dfrac{1}{2}$ by substituting $x=\cos\theta$. From this, deduce the following:

(a) $\cos\dfrac{2\pi}{9}+\cos\dfrac{4\pi}{9}=\cos\dfrac{\pi}{9}$ **(b)** $\sec\dfrac{2\pi}{9}+\sec\dfrac{4\pi}{9}=6+\sec\dfrac{\pi}{9}w$

(c) $\sec\dfrac{\pi}{9}\sec\dfrac{2\pi}{9}\sec\dfrac{4\pi}{9}=8$ **(d)** $\tan^2\dfrac{\pi}{9}+\tan^2\dfrac{2\pi}{9}+\tan^2\dfrac{4\pi}{9}=33$

11 If $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are the roots of the equation $x^3-(a+1)x^2+(c-a)x-c=0$, show that $\alpha+\beta+\gamma=n\pi+\dfrac{\pi}{4}$.

12 **(a)** Expand $\cos(2A+B)$ and hence prove that $\dfrac{1}{4}\cos 3\theta=\cos^3\theta-\dfrac{3}{4}\cos\theta$.

(b) By writing $x=k\cos\theta$ and giving k a suitable value, use the formula proved in part **(a)** to find the three roots of the equation $27x^3-9x=1$. Hence write the value of the product $\cos\dfrac{\pi}{9}\cos\dfrac{3\pi}{9}\cos\dfrac{5\pi}{9}\cos\dfrac{7\pi}{9}$.

CHAPTER 5
MISCELLANEOUS

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

5.1 INEQUALITIES

Here are some of the main properties of inequalities. In particular, you should note the operations that reverse the inequality sign:

- Adding or subtracting the same number from both sides does not alter an inequality.
- Multiplying both sides of an inequality by a negative number reverses the direction of the inequality:
  e.g. $3 > 2$ but $-6 < -4$ after multiplying by -2.
- Multiplication by a positive number does not alter the direction of the inequality.
- Taking the reciprocal of both sides of an inequality reverses its direction when both sides have the same sign, but **not** if the signs are different:
  e.g. $5 < 7$ but $\frac{1}{5} > \frac{1}{7}$
  $3 > -4$ and $\frac{1}{3} > -\frac{1}{4}$
  It is interesting to prove this result, as follows:

  Let $a > b$ and a, b have the same sign.
  Divide both sides by ab, which must be positive as a, b have the same sign:
  $$\frac{a}{ab} > \frac{b}{ab}$$
  $$\frac{1}{b} > \frac{1}{a}$$
  $$\therefore \frac{1}{a} < \frac{1}{b}$$

  Let $a > b$ and a, b have different signs.
  Divide both sides by ab, which must be negative as a, b have different signs:
  $$\frac{a}{ab} < \frac{b}{ab} \quad \text{(inequality changes)}$$
  $$\frac{1}{b} < \frac{1}{a}$$
  $$\therefore \frac{1}{a} > \frac{1}{b}$$
- Squaring both sides of an inequality: this produces a positive number on both sides, so the direction of the inequality $a^2 \lessgtr b^2$ depends on which of $|a|$ and $|b|$ is larger.

  Thus, if a and b are positive and $a > b$ then $a^2 > b^2$ (i.e. you can square both sides of an inequality without changing direction if both sides are known to be positive).

  This can be proved as follows:

  Let $a > b$ and a, b both be positive.
  Multiply the original inequality by a, which is positive: $a^2 > ab$
  Multiply the original inequality by b, which is positive: $ab > b^2$
  Link the two results together: $a^2 > ab > b^2$
  $\therefore$ $a^2 > b^2$
- Square root of both sides of an inequality: this is only defined if both sides of the inequality are non-negative. The direction of the inequality does not change, i.e. if $a > b$ then $\sqrt{a} > \sqrt{b}$, provided a is positive and b is positive or zero.

We will now look at some methods for solving harder inequalities.

Quadratic inequalities

These are also covered in the Mathematics course (see *New Senior Mathematics for Years 11 & 12*).

Example 1

Solve $x^2 - 4x > 0$.

Solution

It is very common to think that to solve a quadratic inequality you should simply factorise it like a quadratic equation, but this is wrong: $x(x-4) > 0$

$x > 0,\ x - 4 > 0$

$x > 0,\ x > 4$

If you now substitute any negative number for x in the original inequality, you will see that it is a solution, so $x > 0$, $x > 4$ must not be the complete answer. Similarly, $x = 1$ is not a solution even though it is included in $x > 0$, $x > 4$, so this method must be wrong.

What has happened? Where is the error?

The answer is that for $x(x-4) > 0$ to be true (i.e. for the product of two factors to be positive), then both x and $x - 4$ must be positive **or** both factors must be negative.

The solution can proceed as follows:

$x(x-4) > 0$

$x > 0$	and	$x - 4 > 0$	**or**	$x < 0$	and	$x - 4 < 0$
$x > 0$	and	$x > 4$		$x < 0$	and	$x < 4$

For both $x > 0$ and $x > 4$ to be true: $x > 4$

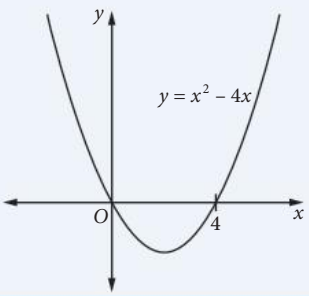
For both $x < 0$ and $x < 4$ to be true: $x < 0$

Hence the correct solution is $x < 0$, $x > 4$ (which is read as '$x < 0$ or $x > 4$').

Alternatively, a graphical method can be used:

- Sketch the parabola $y = x^2 - 4x$ (concave up, cuts the x-axis at 0 and 4).
- Identify the x values for which the parabola is above the x-axis (as we are looking for the places where $y > 0$). These values are the solution.

The graph shows that the solution is $x < 0$ or $x > 4$.



The graphical method in Example **1** has the advantage of providing a visual picture. It is also an easy method for higher-degree inequalities.

Polynomial inequalities

- Use the factored form of the polynomial to sketch the graph of the function.
- Use the graph to identify the x values for which the graph is above (or below) the x-axis (depending on the inequality). These values are the solution.
- If the inequality is $\leq$ or $\geq$, be careful to include the values where the graph cuts or touches the x-axis.

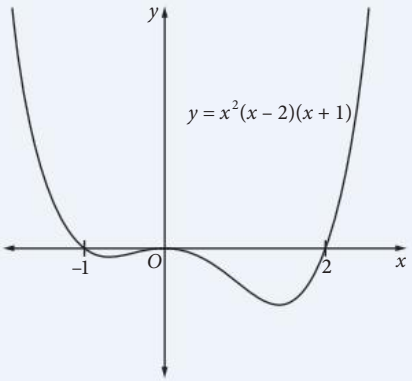
Example 2

Solve $x^2(x-2)(x+1) \geq 0$.

Solution

Sketch $y = x^2(x-2)(x+1)$, noting the double zero at $x = 0$ (a turning point on the x-axis), and zeros at $x = 2$ and $x = -1$.

Find where the graph is on or above the x-axis. The solution is $x \leq -1$, $x = 0$, $x \geq 2$.



Rational function inequalities (x in a denominator)

Example 3

Solve $\frac{1}{x-2} \geq -1$.

Solution

There are two important things to note here:

- $x \neq 2$, as the function $f(x) = \frac{1}{x-2}$ does not exist when $x = 2$
- if we multiply both sides by $(x-2)$, we don't know whether $(x-2)$ is positive or negative, so we don't know whether to reverse the inequality or not.

There are several possible methods to solve an inequality like this (see below). You should memorise the one or two methods that you find easiest and most useful, but you should be aware of the other methods too.

Method 1 (algebraic method requiring consideration of all possible cases)

$$\frac{1}{x-2} \geq -1$$

If $x > 2$, multiply by $(x-2)$ which is positive:

$1 \geq -(x-2)$

$1 \geq -x + 2$

$-1 \geq -x$

$1 \leq x$ (multiplying by -1)

$x \geq 1$

Both $x > 2$ and $x \geq 1$ must be true.

This requires $x > 2$.

If $x < 2$, multiply by $(x-2)$ which is negative:

$1 \leq -(x-2)$ (note change of inequality)

$1 \leq -x + 2$

$-1 \leq -x$

$1 \geq x$ (multiplying by -1)

$x \leq 1$

Both $x < 2$ and $x \leq 1$ must be true.

This requires $x \leq 1$.

Thus the complete solution is $x > 2$, $x \leq 1$.

Method 2 (algebraic method avoiding the need to consider different cases)

$\frac{1}{x-2} \geq -1$ Note that $x \neq 2$.

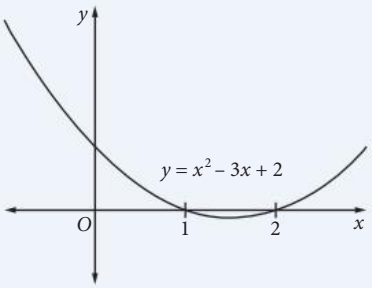
Multiply both sides by $(x-2)^2$, which is known to be positive (so the inequality does not change):

$x - 2 \geq -(x-2)^2$

$x - 2 \geq -x^2 + 4x - 4$

$x^2 - 3x + 2 \geq 0$

We can now solve this quadratic inequality using the graphical method (as shown in Example **1**).



The graph (at right) is on or above the x-axis for $x \leq 1$, $x \geq 2$. However, $x \neq 2$, so the solution is $x \leq 1$, $x > 2$.

Method 3 (variation on Method 2, producing an inequality in factorised form)

$\frac{1}{x-2} \geq -1$ Note that $x \neq 2$.

Make the right-hand side become zero, then simplify the resulting left-hand side:

$$\frac{1}{x-2} + 1 \geq 0$$

$$\frac{1+(x-2)}{x-2} \geq 0$$

$$\frac{x-1}{x-2} \geq 0$$

Multiply both sides by $(x-2)^2$:

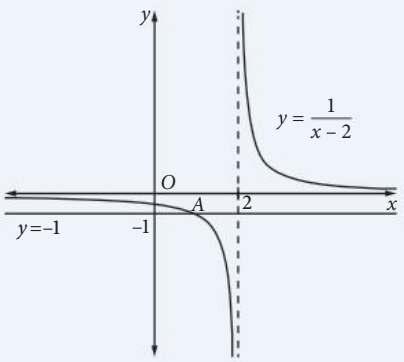
$$(x-1)(x-2) \geq 0$$

This is the factorised form of the quadratic inequality obtained in Method 2. From this point on, the solution is identical to Method 2.

Method 4 (graphical method)

$\frac{1}{x-2} \geq -1$

Sketch a graph of $y = \frac{1}{x-2}$: it is a hyperbola with vertical asymptote at $x = 2$ and horizontal asymptote at the x-axis.



Also graph the horizontal line $y = -1$.

Use the graph to find the x values for which the hyperbola is on or above the horizontal line $y = -1$. We can see that this happens for all x values from point A back to the left, including point A, and for all x values to the right of the asymptote.

To find the x-coordinate of the point A we solve $\frac{1}{x-2} = -1$: the solution is $x = 1$.
Thus the solution is $x \leq 1$, $x > 2$.

Method 5 (like the graphical method, but without drawing the graph)

$\frac{1}{x-2} \geq -1$

List all the x values that are excluded because they make the denominator become zero:

$x \neq 2$ (the vertical asymptote)

Solve the inequality's corresponding equation:

$$\frac{1}{x-2} = -1$$

$$1 = -x + 2$$

$x = 1$ This is the x-coordinate of the point of intersection of $y = \frac{1}{x-2}$ and $y = -1$.

The x values found by these two steps are 'critical values', to be marked on a number line. Use open circles to mark the excluded (asymptotic) values. If the inequality is $<$ or $>$, use open circles to mark the x values from solving the equation; if the inequality is $\leq$ or $\geq$, use filled circles.

Now choose representative x values from each section of the number line and test to see whether the x values satisfy the original inequality:

Test $x = 0$: Is $\frac{1}{(0-2)} \geq -1$? Yes

Test $x = 1\frac{1}{2}$: Is $\frac{1}{1\frac{1}{2}-2} \geq -1$? No

Test $x = 3$: Is $\frac{1}{(3-2)} \geq -1$? Yes

This finds where the graph of $y = \frac{1}{x-2}$ is above the line $y = -1$.
Thus the solution is $x \leq 1$, $x > 2$.

Example 4

Solve $\frac{1}{x^2 - x} > \frac{1}{x^2 - 1}$.

Solution

Using **Method 3** (from Example **3**):

$$\frac{1}{x(x-1)} > \frac{1}{(x-1)(x+1)} \quad \text{so } x \neq 0, 1, -1$$

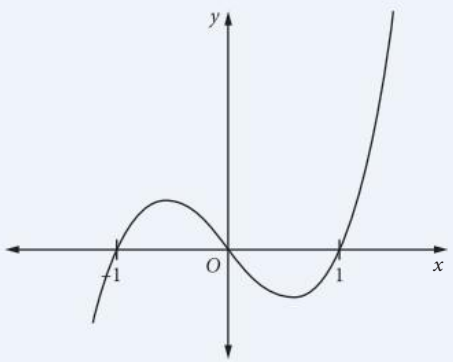
$$\frac{1}{x(x-1)} - \frac{1}{(x-1)(x+1)} > 0$$

$$\frac{(x+1)-x}{x(x-1)(x+1)} > 0$$

$$\frac{1}{x(x-1)(x+1)} > 0$$

Multiply by $x^2(x-1)^2(x+1)^2$:

$$x(x-1)(x+1) > 0$$



The graph is above the y-axis for $-1 < x < 0$, $x > 1$, which is the required solution.

Using **Method 5** (from Example **3**):

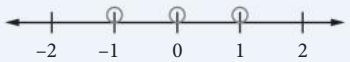
$x \neq 0, 1, -1$

Solve $\frac{1}{x^2 - x} = \frac{1}{x^2 - 1}$:

$$x^2 - 1 = x^2 - x$$

$$x = 1$$

Critical values are 0, 1, −1; all must be not included (i.e. open circles on number line):



Test $x = -2$:

Is $\frac{1}{\left((-2)^2 - (-2)\right)} > \frac{1}{\left((-2)^2 - 1\right)}$? No

Test $x = -\frac{1}{2}$:

Is $\frac{1}{\left((-\frac{1}{2})^2 - (-\frac{1}{2})\right)} > \frac{1}{\left((-\frac{1}{2})^2 - 1\right)}$? Yes

Test $x = \frac{1}{2}$:

Is $\frac{1}{\left((\frac{1}{2})^2 - (\frac{1}{2})\right)} > \frac{1}{\left((\frac{1}{2})^2 - 1\right)}$? No

Test $x = 2$:

Is $\frac{1}{\left((2)^2 - (2)\right)} > \frac{1}{\left((2)^2 - 1\right)}$? Yes

The solution is $-1 < x < 0$, $x > 1$.

Example 5

Solve $|x-2| < \frac{x}{2}$.

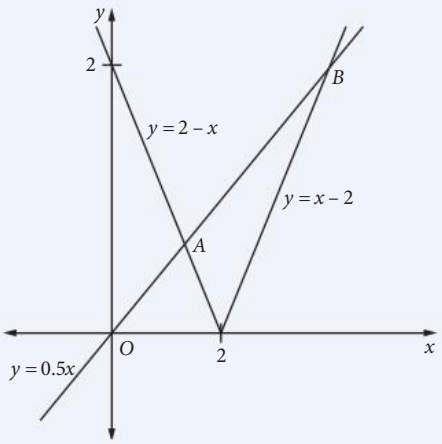
Solution

From the definition of absolute value, we know that $|x-2|$ is simply equal to $x-2$ when $x \geq 2$, but it is equal to $-(x-2)$ when $x < 2$.

Using graphical method (**Method 4** from Example **3**):

Graph $y = |x-2|$ and $y = \frac{x}{2}$.

Note that the 'V'-shaped absolute value graph consists of the ray $y = 2 - x$ for $x < 2$ and the ray $y = x - 2$ for $x \geq 2$.



The solution of the inequality $|x-2| < \frac{x}{2}$ is shown on the graph where the 'V'-shaped absolute value graph is below the straight line $y = \frac{x}{2}$. We need to find the x-coordinates of the points of intersection A and B.

- At A, $y = 2 - x$ and $y = \frac{x}{2}$ intersect. Solving these: $x = \frac{4}{3}$
- At B, $y = x - 2$ and $y = \frac{x}{2}$ intersect. Solving these: $x = 4$

Thus the solution is: $\frac{4}{3} < x < 4$

Using analytical method (**Method 1** from Example **3**):

For $x \geq 2$: $x - 2 < \frac{x}{2}$

$\frac{x}{2} < 2$

$x < 4$

Both $x \geq 2$ and $x < 4$ must be true.

$\therefore 2 \leq x < 4$

For $x < 2$: $2 - x < \frac{x}{2}$

$2 < \frac{3x}{2}$

$x > \frac{4}{3}$

Both $x < 2$ and $x > \frac{4}{3}$ must be true.

$\therefore \frac{4}{3} < x < 2$

Thus the complete solution is $\frac{4}{3} < x < 4$.

Example 6

Solve $|x-4| + |x+2| > 7$.

Solution

Although it is possible to solve this analytically, a graphical approach is easier.

- First sketch $y = |x-4|$ and $y = |x+2|$ on the same axes, writing the equations of their rays.
  Remember that $y = |x-4|$ is made up of the ray $y = x - 4$ (with positive gradient) and $y = 4 - x$.
  Similarly, $y = |x+2|$ is made up of $y = x + 2$ and $y = -x - 2$.

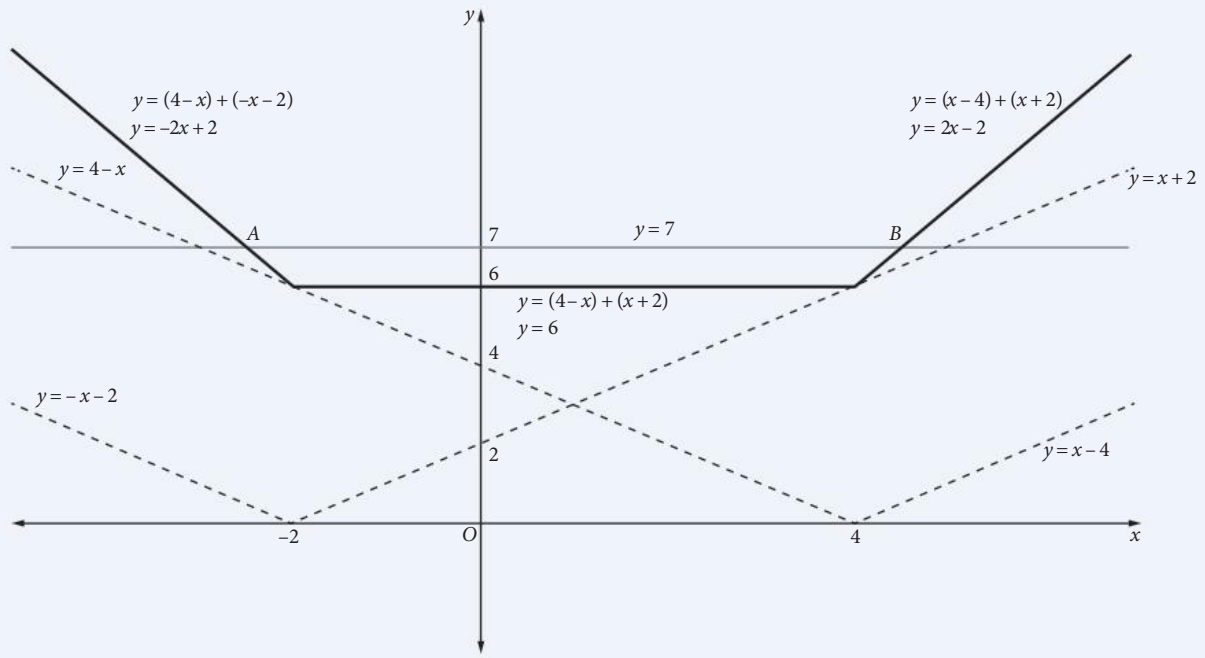
  Note that the graphs have the bases of their 'V'-shapes at $x = 4$ and $x = -2$.

Next, sketch $y = |x - 4| + |x + 2|$ by determining the equations of each of its three parts:

- For $x < -2$, we find the equation by adding the ordinates (y values) that apply. Over this domain the two rays that apply are $y = -x - 2$ and $y = 4 - x$, so the equation is $y = (-x - 2) + (4 - x)$, i.e. $y = -2x + 2$. This can easily be sketched. At $x = -2$, $y = -2(-2) + 2 = 6$, so the ray starts at $(-2, 6)$ and is drawn back to the left with a gradient of -2 (which is steeper than the existing rays on the diagram).
- For $-2 < x < 4$, the two rays that apply are $y = -x - 2$ and $y = x - 4$, so the equation is $y = (-x - 2) + (x - 4)$, i.e. $y = 6$. Add this horizontal interval to the graph.
- For $x > 4$, the two rays that apply are $y = x + 2$ and $y = x - 4$, so the equation is $y = (x + 2) + (x - 4)$, i.e. $y = 2x - 2$.
  Draw this ray on the graph. It starts at $(4, 6)$ and proceeds to the right with a gradient of 2.

We have now sketched the 'trough'-shaped graph of $y = |x - 4| + |x + 2|$ and found the equations of each of its three parts.

- Draw the horizontal line $y = 7$ on the graph and label points A and B where $y = |x - 4| + |x + 2|$ and $y = 7$ intersect.



To solve $|x - 4| + |x + 2| > 7$, we need to find the x values for which the graph is above the horizontal line $y = 7$, which means we need to find the x-coordinates of A and B.

- To find A, solve simultaneously $y = -2x + 2$ and $y = 7$: $x = -\frac{5}{2}$
- To find B, solve simultaneously $y = 2x - 2$ and $y = 7$: $x = \frac{9}{2}$

Hence the solution is $x < -\frac{5}{2}$, $x > \frac{9}{2}$.

Example 7

(a) Sketch $y = \sqrt{1 - x^2}$ and state the domain of this function.

(b) On the same diagram sketch $y = |x - 1|$.

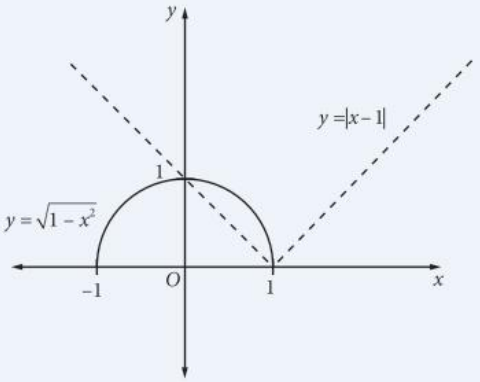
(c) Hence, or otherwise, solve $|x - 1| \geq \sqrt{1 - x^2}$.

(d) Solve $x - 1 \geq \sqrt{1 - x^2}$.

Solution

(a) The sketch is shown at right.
The domain is $-1 \leq x \leq 1$.

(b)



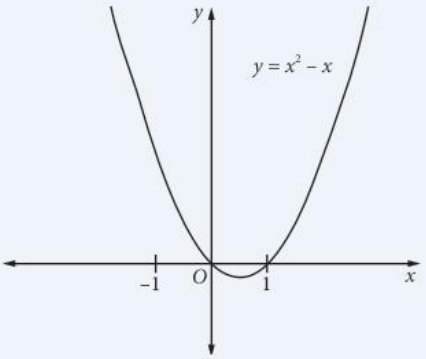
(c) To solve $|x - 1| \geq \sqrt{1 - x^2}$ we need to state the x values for which the 'V'-shaped absolute graph is on or above the semicircle. The graphs intersect at $x = 0$ and $x = 1$. We must also remember the domain from (a), i.e. that the only applicable x values are between -1 and 1. Thus the solution is $-1 \leq x \leq 0, x = 1$.

Alternatively:

In this inequality both sides are known to be non-negative (as $\sqrt{1 - x^2}$ means the positive square root), so we can square both sides and know that the inequality will not change.

$$\begin{aligned} (|x - 1|)^2 &\geq \left(\sqrt{1 - x^2}\right)^2 \\ x^2 - 2x + 1 &\geq 1 - x^2 \\ 2x^2 - 2x &\geq 0 \\ x^2 - x &\geq 0 \end{aligned}$$

Graphing $y = x^2 - x$:
The parabola is on or above the x-axis for $x \leq 0, x \geq 1$.
But the domain is restricted to $-1 \leq x \leq 1$, so the solution is $-1 \leq x \leq 0, x = 1$.

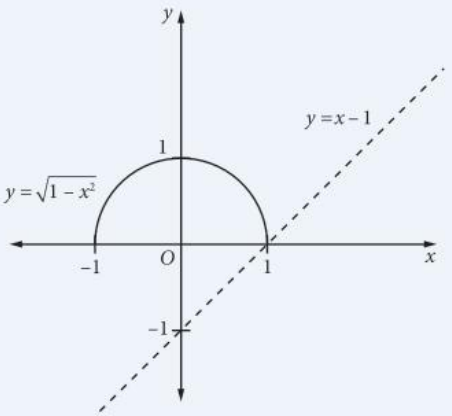


(d) Graphing $y = \sqrt{1 - x^2}$ and $y = x - 1$:

The only point of intersection is at $x = 1$.

We now find the x values for which the straight line is on or above the semicircle, keeping in mind that x values must be between -1 and 1.

Thus the solution is $x = 1$.



EXERCISE 5.1 INEQUALITIES

1 When asked to solve $\frac{2}{x-1} \leq \frac{1}{2}$, four students began their solutions as follows:

Student A: $4 \leq x-1$ Student B: $4(x-1) \leq (x-1)^2$

Student C: $\frac{x-1}{2} \geq 2$ Student D: $4(x-1) \leq (x-1)^2$ noting that $x \neq 1$

(a) Which student has started their solution correctly?
(b) Explain the errors made by the other students.

For questions **2** to **23**, solve the following inequalities.

2 $x^2 - 2x - 15 \leq 0$ **3** $x(x-1) \leq 0$ **4** $x(x-1) \leq 6$ **5** $4x^2 - 12x + 10 > 0$

6 $-3x^2 + 10x + 8 \leq 0$ **7** $(x-1)(x+3)(x-2) < 0$ **8** $(2-x)(x-5)(x+1) > 0$

9 $x^2(x-1) \leq 0$ **10** $\frac{1}{x} > 6$ **11** $\frac{1}{|x|} < 3$ (Hint: The denominator is known to be non-negative.)

12 $\frac{x-3}{x+1} > 0$ **13** $\frac{x-2}{x+3} > -2$ **14** $\frac{4x-3}{2x+1} \leq 3$ **15** $\frac{2}{1-x} > -1$

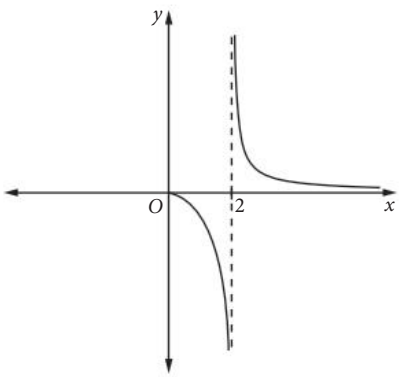
16 $\frac{2x-3}{4x-5} + 2 < 0$ **17** $\frac{1}{(x-1)(x-3)} \leq -1$ **18** $\frac{7}{(3-x)(x+3)} > -1$ **19** $\frac{x}{x^2-1} < 0$

20 $2^{2x} - 2(2^x) \leq -1$ **21** $2^{2x} - 5(2^x) + 4 \leq 0$ **22** $\frac{2x-4}{x+3} > \frac{x+2}{2x+6}$ **23** $1 - x < 2x + 1 < x + 4$

24 **(a)** Show that $f(x) = \frac{x}{x^2-4}$ is an odd function.

(b) Copy and complete the graph of $y = \frac{x}{x^2-4}$ (as shown at right).

(c) Hence solve $\frac{x}{x^2-4} < \frac{1}{3}$.



25 **(a)** On the same axes sketch $y = |x-2|$ and $y = 2x$. **(b)** Hence solve $|x-2| < 2x$.

26 **(a)** On the same axes sketch $y = 1 - |x|$ and $y = 3x$. **(b)** Hence solve $|x| + 3x > 1$.

27 **(a)** On the same axes sketch $y = |x-2|$ and $y = \frac{1}{x}$. **(b)** Hence solve $|x-2| > \frac{1}{x}$.

28 Solve $x^2 - |x| > 0$.

29 Solve $\left|\frac{1-x}{2x+1}\right| \geq 1$. (Hint: $|2x+1|$ is known to be non-negative.)

30 Solve $\frac{x}{x+1} < 1$. (*Note:* This looks like a standard problem but in fact requires some analysis, depending on which method you use.)

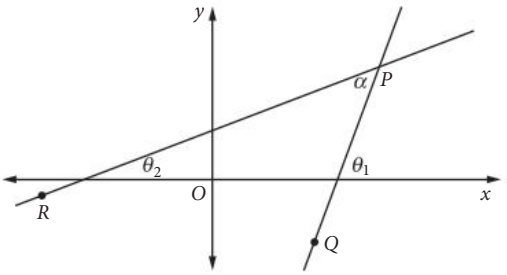
31 **(a)** On the same axes sketch $y = |x+1|$ and $y = |x-5|$. **(b)** Hence graph $y = |x+1| + |x-5|$.

(c) Solve $|x+1| + |x-5| > 7$. **(d)** Solve $|x+1| + |x-5| = 6$.

32 Solve $|x+1| - |x-2| \geq 2$.

5.2 ANGLE BETWEEN TWO INTERSECTING LINES

Let lines PQ and PR have gradients m_1 and m_2, and let them form angles θ_1 and θ_2 with the x-axis as shown.



Recall that $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

How do we find the size of α, the acute angle between the two lines at their point of intersection?

Method 1

We can use a calculator to find θ_1 and θ_2 (using $\theta_1 = \tan^{-1} m_1$).
Then $\alpha = \theta_1 - \theta_2$ since $\theta_1 = \theta_2 + \alpha$ (exterior angle of triangle).

Method 2

$$\alpha = \theta_1 - \theta_2$$
$$\tan\alpha = \tan(\theta_1 - \theta_2)$$
$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$
$$= \frac{m_1 - m_2}{1 + m_1 m_2}$$

Hence angle α can be found using the formula: $\tan\alpha = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

This formula will give the acute angle **or** the obtuse angle formed between the lines, depending on which lines are chosen for the gradients m_1 and m_2. To ensure that it finds the acute angle, we can use the formula:

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Note that when the lines are parallel, $m_1 = m_2$. The formula then gives $\tan\alpha = 0$.
When the lines are perpendicular, $m_1 m_2 = -1$, so the formula gives $\tan\alpha$ to be undefined (i.e. α is a right angle).

Example 8

Find the acute angle between the lines $x + 2y = 5$ and $x - 3y = -3$.

Solution

Method 1

The gradient of $x + 2y = 5$ is $-\frac{1}{2}$: $\tan\theta_1 = -\frac{1}{2}$

$\theta_1 = 180° - 26°34'$

$= 153°26'$

The gradient of $x - 2y = 5$ is $\frac{1}{3}$: $\tan\theta_2 = \frac{1}{3}$

$\theta_2 = 18°26'$

$\therefore\ \alpha = \theta_1 - \theta_2 = 135°$

The acute angle is 45°.

Method 2

$$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$
$$\tan\alpha = \left|\frac{-\frac{1}{2} - \frac{1}{3}}{1 + \left(-\frac{1}{2}\right)\frac{1}{3}}\right|$$
$$\tan\alpha = |-1|$$
$$\tan\alpha = 1$$

$\therefore\ \alpha = 45°$ and so the acute angle is 45°.

EXERCISE 5.2 ANGLE BETWEEN TWO INTERSECTING LINES

For questions **1** to **6** below, find the acute angle between each pair of straight lines with:

1 gradients 4 and $\frac{3}{5}$ **2** gradients -3 and 2 **3** gradients $\frac{3}{4}$ and $-\frac{4}{3}$

4 equations $3x - 4y = 3$ and $x - 2y = 11$ (to the nearest minute)

5 equations $y = 4x + 5$ and $6x + 3y = 7$ (to the nearest minute)

6 equations $x - 2y = 6$ and $y = 3x - 1$

7 The points $(0, 2)$, $(2, 3)$, $(4, 0)$ and $(2, -1)$ are the vertices of a quadrilateral. Find the tangent of the acute angle between its diagonals.

8 Prove that the points $(4, 0)$, $(6, 1)$, $(4, 3)$, $(3, 2)$ are the vertices of a cyclic quadrilateral by proving that its opposite angles are supplementary.

9 What is the gradient of a straight line that intersects the line $y = 2x + 5$ at an angle of 45°?

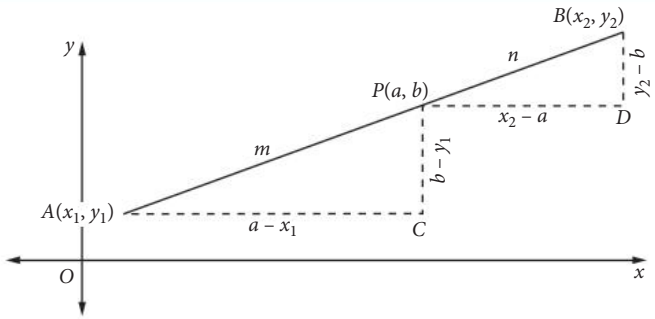
A -1 **B** -3 **C** $-\frac{1}{3}$ **D** -3 or $\frac{1}{3}$

5.3 DIVISION IN A RATIO

Internal division

If A and B are two points in the plane whose coordinates are (x_1, y_1) and (x_2, y_2) respectively, then the coordinates of the point that divides AB **internally** in the ratio $m : n$ are:

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$



Proof

Let $P(a, b)$ be the point in the interval joining $A(x_1, y_1)$ and $B(x_2, y_2)$ that divides AB internally in the ratio $m : n$:

$$\frac{AP}{PB} = \frac{m}{n}$$

The triangles APC and PBD are similar triangles. Hence:

$$\frac{AC}{PD} = \frac{AP}{PB} \quad \text{and} \quad \frac{CP}{DB} = \frac{AB}{PB}$$

$$\frac{a - x_1}{x_2 - a} = \frac{m}{n} \qquad \frac{b - y_1}{y_2 - b} = \frac{m}{n}$$

$$na - nx_1 = mx_2 - ma \qquad nb - ny_1 = my_2 - mb$$

$$a = \frac{mx_2 + nx_1}{m+n} \qquad b = \frac{my_2 + ny_1}{m+n}$$

Note that m, which corresponds to the interval AP, is multiplied by the coordinates of B while n, which corresponds to the interval PB, is multiplied by the coordinates of A.

Example 9

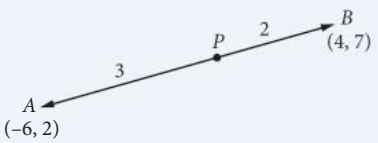
Find the coordinates of the point that divides the interval joining the points $(-6, 2)$ and $(4, 7)$ in the ratio $3 : 2$.

Solution

At P, using the formula:

$$x = \frac{3 \times 4 + 2 \times -6}{3+2} = 0$$

$$y = \frac{3 \times 7 + 2 \times 2}{3+2} = 5$$

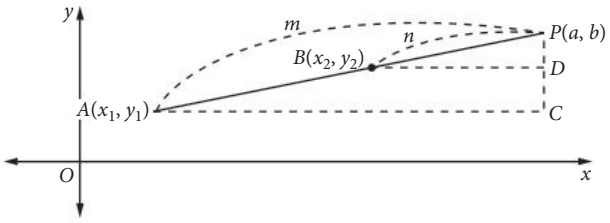


The coordinates of P are $(0, 5)$.

External division

If the point P divides the interval AB **externally** in the ratio $m:n$, then its coordinates are:

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$



Note that for **external** division, the point P is **outside** the interval AB, beyond B, but $\frac{AP}{PB} = \frac{m}{n}$ as for internal division.

Proof

The triangles APC and PBD are similar triangles. Hence:

$$\frac{AC}{BD} = \frac{AP}{PB} \qquad \text{and} \qquad \frac{CP}{DP} = \frac{AP}{PB}$$

$$\frac{a - x_1}{a - x_2} = \frac{m}{n} \qquad \frac{b - y_1}{b - y_2} = \frac{m}{n}$$

$$a = \frac{mx_2 - nx_1}{m-n} \qquad b = \frac{my_2 - ny_1}{m-n}$$

$$a = \frac{mx_2 + (-n)x_1}{m + (-n)} \qquad b = \frac{my_2 + (-n)y_1}{m + (-n)}$$

This is the same form as for internal division but with n replaced by $-n$.

P dividing AB **externally** in the ratio $3:2$ can be represented as $3:-2$, and then the same formula can be applied for both internal and external division.

Example 10

Find the coordinates of the point that divides the interval joining $(3, -2)$ and $(-5, 4)$ externally in the ratio $5:3$.

Solution

x-coordinate $= \frac{5 \times -5 - 3 \times 3}{5 - 3} = -17$ $\qquad$ y-coordinate $= \frac{5 \times 4 - 3 \times -2}{5 - 3} = 13$

The point is $(-17, 13)$.

Example 11

If A is the point $(3, -1)$, B is $(6, 4)$, and P is $(12, 14)$, find the ratio in which P divides the interval AB.

Solution

Let the ratio be $m:n$. Using the formula: $x_P = \frac{nx_1 + mx_2}{m+n}$ $\qquad$ $y_P = \frac{ny_1 + my_2}{m+n}$

$$12 = \frac{n(3) + m(6)}{m+n} \qquad 14 = \frac{n(-1) + m(4)}{m+n}$$

It seems as if we can find m and n by solving a pair of simultaneous equations:

$$12(m+n) = 3n + 6m \qquad 14(m+n) = -n + 4m$$
$$6m + 9n = 0 \qquad 10m + 15n = 0$$
$$2m + 3n = 0 \qquad 2m + 3n = 0$$

This is unexpected: the equations are identical, so we can't solve simultaneously to find the values of m and n. However, we can still use this equation to find $\frac{m}{n}$: $\quad 2m = -3n$

$$\frac{m}{n} = -\frac{3}{2}$$

We now know that P divides the interval AB externally in the ratio $3:2$.

EXERCISE 5.3 DIVISION IN A RATIO

1 On a number line, let point S be at 0 and point T at 100. Match the correct location of point P for each of the following instances.

(a) P divides ST in the ratio 3 : 2
(b) P divides TS in the ratio 3 : 2
(c) P divides ST externally in the ratio 3 : 2
(d) P divides ST in the ratio 2 : −3

A P is at −200
B P is at 40
C P is at 60
D P is at 300

2 Find the coordinates of the point that divides the interval from (−5, 6) to (4, −3) in the ratio:

(a) 1 : 2 **(b)** 2 : 1 **(c)** 3 : 1

3 Find the coordinates of the point that divides externally the interval joining the points:

(a) (−5, 6) and (−2, 3) in the ratio 3 : 2 **(b)** (−5, 6) and (1, 0) in the ratio 3 : 1.

4 Find the coordinates of the points of trisection of the interval joining points (1, 2) and (7, 8).

5 $A(-2,-4)$, $B(4,8)$ and $C(7,-5)$ are the vertices of a triangle. Find the coordinates of:

(a) D, the midpoint of AB
(b) G, the point on CD that divides CD in the ratio 2 : 1
(c) E, the midpoint of BC
(d) the point on AE that divides AE in the ratio 2 : 1
(e) F, the midpoint of AC
(f) the point on BF that divides BF in the ratio 2 : 1.
(g) A **median** is an interval drawn from a vertex of a triangle to the midpoint of the opposite side. Use your answers to **(b)**, **(d)** and **(f)** to complete this sentence: The medians of a triangle are …
(h) The point where the medians intersect is called the **centroid** of the triangle. In what ratio does the centroid divide each median?

6 The vertices of a triangle are the points $P(5,6)$, $Q(-4,3)$ and $R(2,-3)$. Find the coordinates of its centroid.

7 Show that the point $P(-2,0)$ is on the line joining $A(7,-3)$ and $B(-5,1)$. Find the ratio in which it divides the interval AB.

5.4 EQUATIONS REDUCIBLE TO QUADRATICS

This is an extension of work you have done earlier.

Example 12

Solve $\dfrac{x+5}{x-1} - 10\left(\dfrac{x-1}{x+5}\right) = 3$.

Solution

Note the presence of $\dfrac{x+5}{x-1}$ and its reciprocal $\dfrac{x-1}{x+5}$.

Let $a = \dfrac{x+5}{x-1}$. The equation becomes: $a - \dfrac{10}{a} = 3$

$$a^2 - 3a - 10 = 0$$

$$a = 5 \quad \text{or} \quad a = -2$$

$$\frac{x+5}{x-1} = 5 \qquad \frac{x+5}{x-1} = -2$$

$$x + 5 = 5x - 5 \qquad x + 5 = -2x + 2$$

$$x = 2.5 \quad \text{or} \quad x = -1$$

Hence the solutions are: $x = -1, 2.5$

Example 13

Solve $x^2 + x - 5 = \dfrac{10}{x^2 + x + 4}$.

Solution

Note the repeated presence of $x^2 + x$.

Let $a = x^2 + x$. The equation becomes: $a - 5 = \dfrac{10}{a + 4}$

$$a^2 - a - 30 = 0$$

$$a = 6 \quad \text{or} \quad a = -5$$

$$x^2 + x - 6 = 0 \qquad x^2 + x + 5 = 0$$

$$x = -3, 2 \quad \text{or} \quad \text{No real roots}$$

Hence the solutions are $x = -3, 2$.

Example 14

In this example, note the symmetry of the coefficients.

(a) If $a = x + \dfrac{1}{x}$, show that $a^2 = x^2 + \dfrac{1}{x^2} + 2$.

(b) Show that $x = 0$ is not a solution of $x^4 - 7x^3 - 28x^2 - 7x + 1 = 0$.

(c) Solve $x^4 - 7x^3 - 28x^2 - 7x + 1 = 0$ by first dividing by x^2, noting $x \neq 0$, and then using a suitable substitution.

Solution

(a) $a = x + \dfrac{1}{x}$

$$a^2 = \left(x + \frac{1}{x}\right)^2$$

$$a^2 = x^2 + \frac{1}{x^2} + 2$$

(b) At $x = 0$: LHS $= 0 - 0 - 0 - 0 + 1 \neq 0$

$\therefore$ $x = 0$ is not a solution.

(c) $x^4 - 7x^3 - 28x^2 - 7x + 1 = 0$

Divide by x^2, noting $x \neq 0$: $x^2 - 7x - 28 - \dfrac{7}{x} + \dfrac{1}{x^2} = 0$

$$x^2 + \frac{1}{x^2} - 7x - \frac{7}{x} - 28 = 0$$

$$x^2 + \frac{1}{x^2} - 7\left(x + \frac{1}{x}\right) - 28 = 0$$

Let $a = x + \dfrac{1}{x}$ and use the result of **(a)**: $a^2 - 2 - 7a - 28 = 0$

$$a^2 - 7a - 30 = 0$$

$$a = -3 \quad \text{or} \quad a = 10$$

$$x + \frac{1}{x} = -3 \qquad x + \frac{1}{x} = 10$$

$$x^2 + 3x + 1 = 0 \qquad x^2 - 10x + 1 = 0$$

$$x = \frac{-3 \pm \sqrt{5}}{2} \qquad x = \frac{10 \pm \sqrt{96}}{2}$$

$\therefore$ $x = \dfrac{-3 \pm \sqrt{5}}{2}$ or $x = 5 \pm 2\sqrt{6}$

Example 15

Solve $\sqrt{x^2-4} = 9 - 2x^2$.

But first, a note of warning!

Be very careful when solving equations with square roots. At some stage you will have to square both sides, and this can sometimes introduce spurious (i.e. false, fake, wrong) solutions.

For example, the only solution to the equation $x + 2 = 5$ is $x = 3$. But if you square both sides, you will get $x^2 + 4x + 4 = 25$, which has the solutions $x = 3$ **and** $x = -7$. The $x = -7$ is a spurious solution to the original equation.

Another example: $\sqrt{x-4} = -3$ clearly has no solutions, but squaring both sides gives $x - 4 = 9$, which produces the spurious solution $x = 13$.

Always check your solutions by substituting the values back into the original equation or by checking the domain restrictions of the original equation. For example, $\sqrt{x^2-4} = 9 - 2x^2$ implies the following restrictions on x:

- $x^2 - 4 \geq 0$ (square root of a negative is not defined)
- $9 - 2x^2 \geq 0$ (LHS ≥ 0 because $\sqrt{\ }$ is the positive square root $\therefore$ RHS ≥ 0)

Solution

We could square both sides, producing a quartic equation to solve. This would become very difficult.

Alternatively, let $a = \sqrt{x^2-4}$ so that $a^2 = x^2 - 4$ and so $x^2 = a^2 + 4$. The equation becomes:

$$a = 9 - 2(a^2 + 4)$$

$$2a^2 + a - 1 = 0$$

$$a = \frac{1}{2} \quad \text{or} \quad a = -1$$

$$\sqrt{x^2-4} = \frac{1}{2} \qquad \sqrt{x^2-4} = -1$$

$$x^2 - 4 = \frac{1}{4} \quad \text{or} \quad \text{No real solutions (if we squared both sides, we would get spurious solutions)}$$

$$x = \pm\frac{\sqrt{17}}{2}$$

Confirm by substitution that both $x = \frac{\sqrt{17}}{2}$ and $x = -\frac{\sqrt{17}}{2}$ are solutions.

EXERCISE 5.4 EQUATIONS REDUCIBLE TO QUADRATICS

1 Solve the following. **(a)** $\frac{2x}{x-5} - 18\left(\frac{x-5}{2x}\right) = 3$ **(b)** $2\left(\frac{3x-1}{x+5}\right) + 12\left(\frac{x+5}{3x-1}\right) = -11$

(c) $\frac{x+2}{2x-1} - 2 = \frac{1-2x}{x+2}$ **(d)** $\frac{6x+6}{7x} + 11 = \frac{70x}{x+1}$ **(e)** $\frac{x^2}{x-1} = 3 + \frac{10(x-1)}{x^2}$

2 Solve the following. **(a)** $x^2 + 8x + 5 = \frac{6}{x^2+8x+4}$ **(b)** $x^2 - 2x - 4 = \frac{-6}{x^2-2x+3}$

(c) $x^2 - x - 20 = \frac{40}{x^2-x-2}$ **(d)** $x^2 - 3x + 6 + \frac{160}{x^2-3x-20} = 0$

3 Note the symmetry of the coefficients and use the substitution $a = x + \frac{1}{x}$ to solve:

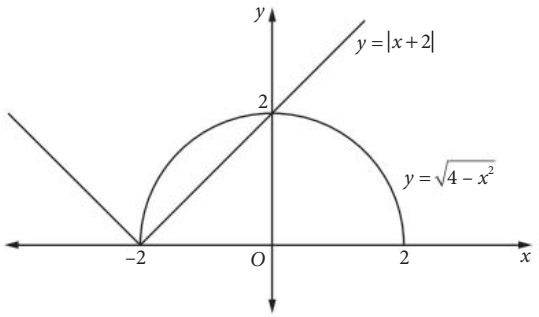
(a) $x^4 - 3x^3 - 26x^2 - 3x + 1 = 0$ **(b)** $x^4 + 12x^3 - 11x^2 + 12x + 1 = 0$

4 Solve the following. **(a)** $\sqrt{x+5} = 7 - x$ **(b)** $\sqrt{53 - x^2} = x - 5$ **(c)** $\frac{x+1}{2x+3} - \sqrt{\frac{x+1}{2x+3}} = 6$

(d) $\sqrt{x^2+28} - \frac{16}{\sqrt{x^2+28}} = 6$ **(e)** $2x^{\frac{3}{2}} - 5x + 2x^{\frac{1}{2}} = 0$ (Hint: Take out the common factor.)

CHAPTER REVIEW 5

1 The diagram shows the graphs of $y = |x+2|$ and $y = \sqrt{4-x^2}$.



The solution of $\sqrt{4-x^2} \le |x+2|$ is:

A $0 \le x \le 2$ **B** $-2 \le x \le 0$ **C** $x = -2, 0 \le x \le 2$ **D** $x \ge 0$

2 Solve the following inequalities.

(a) $\dfrac{2}{1-x} > 1$ **(b)** $\dfrac{1}{x+3} \le \dfrac{2}{x}$ **(c)** $\dfrac{1}{|x-1|} > \dfrac{1}{|x+1|}$ **(d)** $\dfrac{x}{x^2-1} \le 0$

(e) $\dfrac{4x}{x^2-x-6} \le 1$ **(f)** $\dfrac{x}{x-1} < 0$ **(g)** $|x-1| + |x+3| > 6$ **(h)** $x^2 \le |2x-1|$

3 Find the acute angle between each pair of lines. (Answer to the nearest minute.)

(a) $y = 2x - 1$ and $x + 3y + 6 = 0$ **(b)** $y = 3$ and $x + y = 1$ **(c)** $x = 4$ and $y = 2x$

4 Find the exact gradient(s) of lines that make an angle of 60° with the line $y = 2x + 1$.

5 P is the point $(-1, 5)$ and Q is $(2, -3)$. Find the coordinates of the point that divides the interval PQ:

(a) internally in the ratio 2 : 1 **(b)** externally in the ratio 3 : 1.

6 **(a)** Show that $A(1, -4)$, $B(-3, -16)$ and $C(3, 2)$ are collinear.

(b) Find the ratio in which B divides the interval AC.

7 Solve the following. **(a)** $\dfrac{x^2-1}{x} - \dfrac{18x}{x^2-1} = 3$ **(b)** $x^2 - 8x - 2 = \dfrac{12}{x^2-8x-1}$

(c) $x^2 + \sqrt{x^2-9} = 9$ **(d)** **(i)** If $a = x + \dfrac{1}{x}$, show that $x^2 + \dfrac{1}{x^2} = a^2 - 2$.

(ii) Hence solve $x^4 + 2x^3 - 33x^2 + 2x + 1 = 0$.

CHAPTER 6
PARAMETRIC EQUATIONS AND THE PARABOLA

Outcomes covered:

PE1 appreciates the role of mathematics in the solution of practical problems

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

PE4 uses the parametric representation together with differentiation to identify geometric properties of parabolas

PE6 makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations

6.1 PARAMETRIC REPRESENTATION

It is common in mathematics to express two related variables (e.g. x and y) in terms of a third variable (e.g. t or θ), so that, for example: $x = f(t)$, $y = g(t)$

$$x = f(\theta),\ y = g(\theta)$$

Equations like these are called **parametric equations** and the third variable (e.g. t or θ) is called the **parameter**.

Recall that the functions cosine and sine can be defined as the x- and y-coordinates respectively of a point on the unit circle $x^2 + y^2 = 1$. Thus the unit circle can be represented by the parametric equations:

$$x = \cos\theta,\ \ y = \sin\theta$$

where θ is the parameter. When the unit circle is described by the equation $x^2 + y^2 = 1$, it is said to be in **Cartesian** form.

Example 1

Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t$, $y = t^2$.

Solution

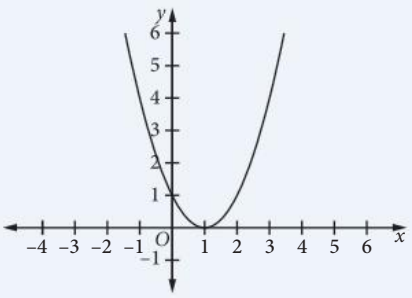
$x = 1 + t$ [1]

$y = t^2$ [2]

From [1]: $t = x - 1$

Substitute into [2]: $y = (x-1)^2$

Hence the Cartesian equation is $y = (x-1)^2$ and the graph is the parabola shown.



Example 2

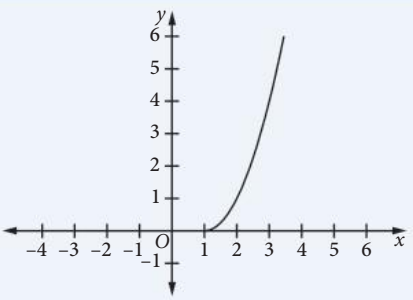
Find the Cartesian equation of the curve whose parametric equations are $x = 1 + t$, $y = t^2$, $t \geq 0$.

Solution

We proceed as with Example **1** to find $y = (x-1)^2$, but we now need to consider the additional condition $t \geq 0$.

$x = 1 + t$ and $t \geq 0$, so $x \geq 1$.

Hence the Cartesian equation is $y = (x-1)^2$ with its domain restricted to $x \geq 1$, as shown on the graph.



Example 3

Find the Cartesian equation of the curve whose parametric equations are $x = \sin\theta$, $y = \cos 2\theta$.

Solution

Rather than trying to make θ the subject of one of the parametric equations, use the identity $\cos 2\theta = 1 - 2\sin^2\theta$.

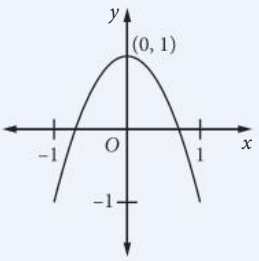
$x = \sin\theta$ [1]

$y = \cos 2\theta$ [2]

Substituting [1] and [2] into the identity, we find $y = 1 - 2x^2$, which is the Cartesian equation.

However, we must also consider the restriction on the domain. The sine function has a range from -1 to 1 (i.e. $-1 \le \sin\theta \le 1$), so $-1 \le x \le 1$.

Hence the complete statement of the Cartesian equation is $y = 1 - 2x^2$, $-1 \le x \le 1$ as shown on the graph.



EXERCISE 6.1 PARAMETRIC EQUATIONS

For questions **1** to **14**, find the Cartesian equation of the curves whose parametric equations are:

1 $x = 2t$, $y = t + 2$ **2** $x = t$, $y = t^2$ **3** $x = t$, $y = \frac{1}{t}$ **4** $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

5 $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le \pi$ **6** $x = t + 3$, $y = t^2 - 5$, $t \ge 0$ **7** $x = 2u - 2$, $y = 3u + 1$, $1 \le u \le 3$

8 $x = v^3$, $y = 1 - v^2$, $-1 \le v \le 1$ **9** $x = t + 2$, $y = t^2 - 1$ **10** $x = \cos 2t$, $y = \cos t$, $0 \le t \le 2\pi$

11 $x = 2t^2$, $y = 4t$ **12** $x = 2\cos\theta$, $y = \sqrt{3}\sin\theta$, $0 \le \theta \le 2\pi$

13 $x = 2\cos t$, $y = \sin t$, $0 \le t \le \pi$ **14** $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$

15 Two boats on a lake start sailing at the same time. Boat A moves on a course given by $x = \frac{t}{2}$, $y = t + 1$, while boat B moves on a course given by $x = t - 2$, $y = -2t + 9$, where t is the time elapsed in hours.

(a) Find the Cartesian equation for the course of each boat. Show that the courses intersect at the point $(1, 3)$.

(b) Do the boats collide? Justify your answer.

16 Which of the following gives the Cartesian equation for the parametric equations $x = \tan\frac{\theta}{2}$, $y = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$?

A $y = 2x$ **B** $y = \frac{2x}{1-x^2}$ **C** $y = 2x$, $-1 < x < 1$ **D** $y = \frac{2x}{1-x^2}$, $-1 < x < 1$

6.2 PARAMETRIC EQUATIONS OF THE PARABOLA

The parabola $x^2 = 4ay$ can be represented by the parametric equations: $x = 2at$, $y = at^2$

We can verify this by eliminating the parameter:

$x = 2at$ [1]

$y = at^2$ [2]

From [1]: $t = \frac{x}{2a}$

Substitute into [2]: $y = a\left(\frac{x}{2a}\right)^2$

$x^2 = 4ay$

The point $P(2at, at^2)$ on the parabola is the variable point that depends on the value of t, so it is frequently called 'the point t'.

You have already seen how differential calculus can be used to find the gradient of the tangent and normal to a curve at any point on the curve, and hence to find the equation of the tangent and normal. We will now apply those principles to the parabola, with particular reference to the parametric equations of the parabola. We will also consider some simple locus problems.

Parametric equation of the tangent

At P, $x = 2at$ and $y = at^2$, $\frac{dx}{dt} = 2a$ and $\frac{dy}{dt} = 2at$: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 2at \times \frac{1}{2a}$

$= t$

= gradient of tangent at any point t

or $y = \frac{1}{4a}x^2$

$\frac{dy}{dx} = \frac{1}{2a}x$

$= \frac{1}{2a} \times 2at$ where $x = 2at$

$= t$

Hence the parameter t must denote $\tan\theta$, where θ is the angle that the tangent makes with the positive direction of the x-axis.

The equation of the tangent at P is: $y - y_1 = m(x - x_1)$

$\therefore \quad y - at^2 = t(x - 2at)$

$$y = tx - at^2$$

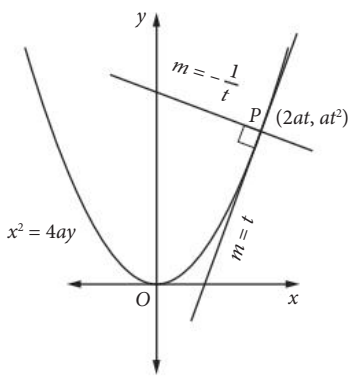
Parametric equation of the normal

The gradient of the normal at P is $-\frac{1}{t}$, $(t \neq 0)$, so the equation of the normal is:

$y - y_1 = m(x - x_1)$

$\therefore\ y - at^2 = -\frac{1}{t}(x - 2at)$

$$x + ty = at^3 + 2at$$



Cartesian equation of the tangent

From the equation of the parabola: $y = \frac{1}{4a}x^2$

$\frac{dy}{dx} = \frac{1}{2a}x$

$= \frac{1}{2a}x_1$ at any point (x_1, y_1)

Equation of the tangent is: $y - y_1 = m(x - x_1)$

$\therefore \quad y - y_1 = \frac{x_1}{2a}(x - x_1)$

$2ay - 2ay_1 = xx_1 - (x_1)^2$

$= xx_1 - 4ay_1$ because $(x_1)^2 = 4ay_1$

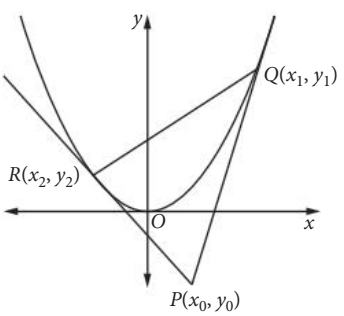
$\therefore \quad xx_1 = 2a(y + y_1)$

Compare this equation of the tangent with the Cartesian equation of the parabola. What do you notice?

Equation of the chord of contact

Let $P(x_0, y_0)$ be a point external to the parabola $x^2 = 4ay$. From P there are two tangents that can be drawn to touch the parabola at points $Q(x_1, y_1)$ and $R(x_2, y_2)$. The chord QR is called the **chord of contact** of the tangents to $x^2 = 4ay$ from P.

We can prove that the equation of the chord of contact from $P(x_0, y_0)$ is $xx_0 = 2a(y + y_0)$, but first we must establish a vital concept by way of a simple example.



- It is clear that the point $(2, 5)$ is on the line $3x + 4y = 26$, because $3(2) + 4(5) = 26$.
  Now note that $2(3) + 5(4) = 26$ also.
  Thus the point $(3, 4)$ is also on the line $2x + 5y = 26$.

Now, we already know that the Cartesian equation of the tangent at Q is $xx_1 = 2a(y + y_1)$ (as shown earlier).
But $P(x_0, y_0)$ is on this tangent, so $x_0x_1 = 2a(y_0 + y_1)$.
Using the concept just established, we note that the point $Q(x_1, y_1)$ is on the line $xx_0 = 2a(y + y_0)$.
Similarly, the equation of the tangent at R is $xx_2 = 2a(y + y_2)$.
But $P(x_0, y_0)$ is on this tangent, so $x_0x_2 = 2a(y_0 + y_2)$. Again using the established concept, we note that the point $R(x_2, y_2)$ is also on the line $xx_0 = 2a(y + y_0)$.
We now know that both Q and R are on the line $xx_0 = 2a(y + y_0)$. The line through Q and R is the chord of contact. Hence:

The equation of the chord of contact from $P(x_0, y_0)$ is $xx_0 = 2a(y + y_0)$.

Example 4

Find the equation of the tangent and the normal to the parabola $x = 6t$, $y = 3t^2$ at the point on the parabola where $t = 1$.

Solution

At $t = 1$, $x = 6$ and $y = 3$; also $\frac{dx}{dt} = 6$ and $\frac{dy}{dt} = 6t$: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= 6t \times \frac{1}{6}$

$= t$

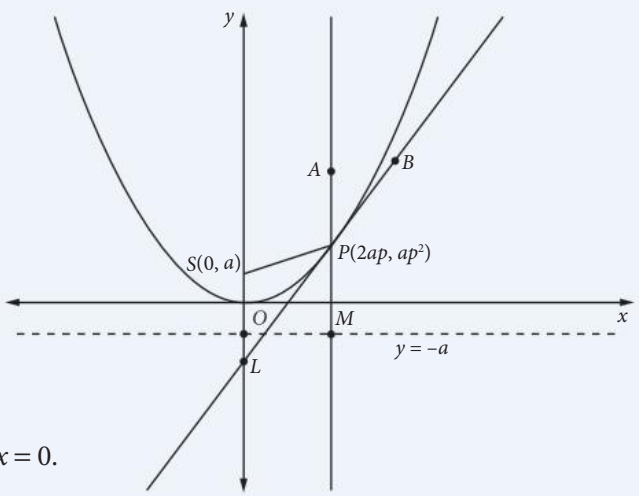
$= 1$ when $t = 1$

Equation of tangent: $y - 3 = 1(x - 6)$, i.e. $y = x - 3$

Equation of normal: $y - 3 = -1(x - 6)$, i.e. $x + y - 9 = 0$

Example 5

The diagram shows the parabola $x^2 = 4ay$ with focus S.
The point $P(2ap, ap^2)$, where $p \neq 0$, is on the parabola.
The tangent to the parabola at P, $y = px - ap^2$, meets the y-axis at L.
The line through P parallel to the axis of the parabola intersects the directrix at M.
A is on MP produced, B is on LP produced.



(a) Show that $SL = SP$.
(b) Hence show that $\angle APB = \angle SPL$.

Solution

(a) L is the point of intersection of $y = px - ap^2$ and $x = 0$.
i.e. L is $(0, -ap^2)$

Hence $SL = ap^2 + a$ (distance SL is $SO + OL$, where O is the origin)

To find SP, we do not need to use the distance formula: it is much easier to use the focus–directrix definition of the parabola. Thus $SP = PM = ap^2 + a$

$\therefore SL = SP$

(b) $\angle SLP = \angle SPL$ (base angles of isosceles triangle are equal)

$\angle APB = \angle SLP$ (corresponding angles, $AP \parallel SL$)

$\therefore \angle APB = \angle SPL$

The example above proves the **reflection property of the parabola**, namely:

> The tangent to a parabola at a given point is equally inclined to the axis of the parabola and to the focal chord through the point.

This property is used to concentrate incoming signals in communication reception dishes. It means that a parabolic mirror has the property that every ray (including electromagnetic rays such as television, radio and other communication signals) that enters parallel to the axis of the parabolic dish will pass through the focus after reflection from the mirror. This intensifies the signal at the focus.

Similarly, torches, headlights and so on emit their light from a globe placed at the focus in front of a parabolic mirror. As a result, every ray of light is reflected from the surface of the mirror along a direction parallel to the axis (i.e. straight out). In other words:

- A ray from the focus of a parabola is reflected as a ray parallel to the axis of the parabola.
- A ray parallel to the axis of a parabola is reflected through the focus.

Example 6

Find the equation of the chord of contact from $(6, 5)$ to the parabola $x^2 = 6y$.

Solution

From $x^2 = 4ay$: $4a = 6$, $a = \frac{3}{2}$

Equation of chord of contact: $xx_0 = 2a(y + y_0)$

i.e. $6x = 2 \times \frac{3}{2}(y + 5)$

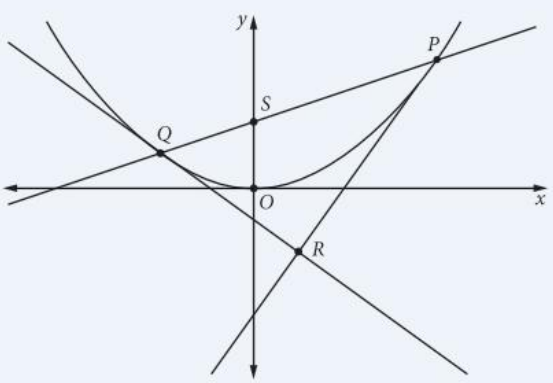
$2x - y - 5 = 0$

Example 7

Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $x^2 = 4ay$, with $p, q \neq 0$.

PQ is a focal chord. Tangents to the parabola are drawn from P and Q to intersect at R.

The equation of the tangent at P is $y = px - ap^2$.



(a) Use the fact that PQ is a focal chord to prove that $pq = -1$.

(b) Find the coordinates of R.

(c) Hence prove that the tangents at the endpoints of a focal chord intersect at right angles on the directrix.

Solution

(a) PQ is a focal chord so P, S, Q are collinear: $m_{PS} = m_{QS}$

$$\frac{ap^2 - a}{2ap - 0} = \frac{aq^2 - a}{2aq - 0}$$
$$\frac{a(p^2 - 1)}{2ap} = \frac{a(q^2 - 1)}{2aq}$$
$$q(p^2 - 1) = p(q^2 - 1)$$
$$qp^2 - q - pq^2 + p = 0$$
$$pq(p - q) + (p - q) = 0$$
$$(p - q)(pq + 1) = 0$$

So $p - q = 0$ or $pq + 1 = 0$

But $p \neq q$, so $pq = -1$.

(b) Tangent at P: $y = px - ap^2$ [1]
Tangent at Q: $y = qx - aq^2$ [2]

Find R by solving [1] and [2] simultaneously.

Substitute [1] into [2]: $px - ap^2 = qx - aq^2$

$$px - qx = ap^2 - aq^2$$
$$x(p - q) = a(p - q)(p + q)$$

Dividing by $(p - q)$: $x = a(p + q)$ (noting $p \neq q$)

Substitute into [2]: $y = pa(p + q) - ap^2$

$$y = apq$$

Hence R is the point $(a(p + q), apq)$.

(c) Gradient of tangent at P is p, and gradient of tangent at Q is q.

From **(a)**: $pq = -1$, so tangents at P, Q meet at right angles.

From **(b)**: at R, $y = apq$

i.e. R is restricted to be on the line $y = -a$, as $pq = -1$.

Thus the tangents at the endpoints of a focal chord will always intersect at right angles on the directrix.

TECHNOLOGY EXPLORATION

Tangents at the endpoints of a focal chord

We can use GeoGebra to demonstrate graphically that the tangents at the endpoints of a focal chord intersect at right angles on the directrix. We will do this as in Example **7** above, for the case where $a = 2$.

1. In GeoGebra, enter *(0,2)* into the input bar to create a new point as the focus. Right-click on this point and select 'Rename' to rename this as point S.
2. Next enter *y=–2* into the input bar to plot the line $y = -2$ as the directrix. GeoGebra will name this line a.
3. Using the **Parabola** tool , click on the focus point S and the directrix line a.

This will create a parabola with focus $(0, 2)$ and directrix $y = -2$. GeoGebra will name the parabola c.

4. Using the **New Point** tool , click on the parabola to plot a point on it. Rename this point P.
5. Use the **Line through Two Points** tool to construct a line through P and S.
6. Use the **Intersect Two Objects** tool to plot the points of intersection of the parabola and the line PS. Rename the second point (i.e. the point that is not the same as P) as Q.

If you now use the **Move** tool , you will be able to drag point P along the parabola and see the corresponding changes in the line PS and the point Q.

7 Use the **Tangents** tool to construct tangents from P and Q to the parabola.
(Click on P and then the parabola; next, click on Q and then the parabola.)

8 Use the **Intersect Two Objects** tool to plot the point of intersection of the tangents from P and Q. Rename this point R.

9 Use the **Angle** tool to name $\angle PRQ$ by clicking on P, then R, then Q.

GeoGebra will show the angle to be a right angle. (If it is shown as the reflex angle 270° instead, then right-click on the angle, select 'Object Properties > Basic' and un-tick the option 'Allow Reflex Angle'.)

If you now use the **Move** tool again, you will be able to drag point P along the parabola to see the corresponding changes in the point R as well as in the tangents from P and Q.

Notice that as P moves on the parabola, the tangents from the ends of the focal chord PQ always intersect at right angles on the directrix.

Example 8

The chord of contact from the point $(3,-1)$ to the parabola $x^2 = 4ay$ intersects the parabola at points A and B. Let M be the midpoint of AB. Find the coordinates of M.

Solution

Using $xx_0 = 2a(y + y_0)$, the equation of the chord of contact AB is $3x = 2a(y-1)$.

To find A and B, solve simultaneously: $x^2 = 4ay$ [1]

$3x = 2a(y-1)$ [2]

From [2]: $2ay = 3x + 2a$

Substitute into [1]: $x^2 = 2(3x + 2a)$

$x^2 - 6x - 4a = 0$ [3]

We now have a problem: we cannot solve this by factorisation. We could use the quadratic formula, but that is likely to get very messy when we have to substitute to find the y-coordinates and then use the midpoint formula.

However, there is a clever way to avoid this problem:

Let the roots of equation [3] be x_A and x_B. (The roots must be the x-coordinates of A and B.)

Now the x-coordinate of M is $x_M = \frac{x_A + x_B}{2}$ and $x_A + x_B$ is the sum of the roots of the quadratic equation [3], which we know is 6 (using the sum of roots formula $\alpha + \beta = -\frac{b}{a}$). Hence $x_M = 3$.

We can substitute into [2] to find the y-coordinate of M, as M is on the chord of contact: $3(3) = 2a(y_M - 1)$

$y_M = \frac{9}{2a} + 1$

Thus M is $(3, \frac{9}{2a} + 1)$.

EXERCISE 6.2 PARAMETRIC EQUATIONS OF THE PARABOLA

1 The equation of the tangent to the parabola $x^2 = 4y$ at the point $(-1, \frac{1}{4})$ is:

A $2x + 4y + 1 = 0$ **B** $2x + 2y + 1 = 0$ **C** $2x + 2y - 1 = 0$ **D** $2x - 4y + 1 = 0$

2 The equation of the normal to the parabola $x^2 = 12y$ at the point $(-6p, 3p^2)$ is:

A $-3py + x = 6p + 3p^3$ **B** $py + x = 6p + 3p^3$ **C** $py + x = -6p + 3p^3$ **D** $py - x = 6p + 3p^3$

3 Find the Cartesian equation of the tangent and normal to the following parabolas at the points indicated.

(a) $x = 2t, y = t^2$ at the point where $t = 1$

(b) $x = 12t, y = 6t^2$ at the point where $t = -2$

(c) $x = t + 2, y = t^2$ at the point where $t = 1$

(d) $x = 4t, y = -2t^2$ at the point where $t = -1$

4 The parabola $x^2 = 8y$ has focus S. The chord of contact from the point $T(2,-1)$ to the parabola cuts the parabola at A and B.

(a) Find the equation of the chord of contact.

(b) Show that the area of ΔABS is $\frac{1}{3}$ of the area of ΔABT. (Hint: You do not need to find the length AB.)

5 P is the point $(4,4)$ on the parabola $x^2 = 4y$. The tangent at P cuts the x-axis at A. The normal at P cuts the y-axis at B. Find the coordinates of C, the midpoint of AB.

6 On the parabola $x^2 = 4ay$, P is the point with parameter p and Q is the point with parameter $2p$.

(a) Given that the equation of the tangent at P is $y = px - ap^2$, show that the tangent at Q has the equation $y = 2px - 4ap^2$.

(b) The tangents at P and Q intersect at R. Show that R is the point $(3ap, 2ap^2)$.

7 From what external point are the tangents to the parabola $x^2 = 4y$ drawn so that $3x - 2y = 4$ is the equation of the chord of contact?

8 A chord of contact to the parabola $x^2 = 4y$ has the equation $2y - 3x + 2 = 0$. From what external point are the tangents drawn?

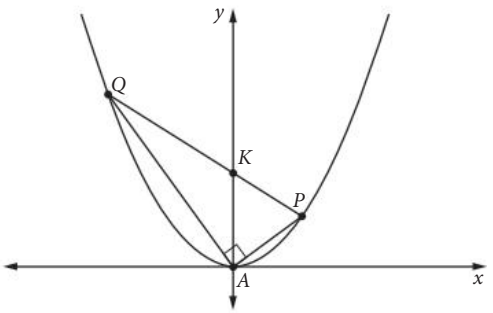
9 $P(2at, at^2)$ is any point on the parabola $x^2 = 4ay$. The normal to the parabola at P meets the y-axis at G and the x-axis at H. $Q(u, v)$ is the fourth vertex of the rectangle $GOHQ$. Show that $u = vt$.

10 **(a)** Factorise $p^3 - 13p + 12$ either by using the factor theorem or by noting that $p^3 - 13p + 12 = p^3 - p - 12p + 12$.

(b) Find the coordinates of three points on the parabola $x^2 = 4y$ such that the normals through these three points pass through the point $(-12, 15)$.
(Hint: Let $P(2p, p^2)$ be such a point. Write the equation of the normal at P, then show that p satisfies $p^3 - 13p + 12 = 0$.)

11 How many normals pass through $(0, ka)$, a point on the axis of the parabola $x^2 = 4ay$, for $k > 2$? For $k = 3$, find where the normal meets the parabola again.

12 Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $x^2 = 4ay$ with vertex A. The points P and Q move together so that the chords AP and AQ are mutually perpendicular.



(a) Show that $pq = -4$.

(b) Show that the equation of the chord PQ is $y - \frac{1}{2}(p+q)x + apq = 0$.

(c) Show that as P and Q move as described above, the chord PQ cuts the axis of the parabola at a fixed point K. You may want to visualise this by constructing the information using GeoGebra or other graphing software.

13 Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $x = 2at$, $y = at^2$. The tangents from P and Q meet at right angles at A. The respective normals intersect at B.

(a) Draw a diagram to show this information.

(b) Given that the tangent at P is $y - px + ap^2 = 0$ and the normal is $py + x = 2ap + ap^3$, find the coordinates of A and B.

(c) Explain why AB is parallel to the axis of the parabola.

14 **(a)** Show that $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2$.

(b) Show that the chord of contact of tangents from the point $(a, -a)$ to the parabola $x^2 = 4ay$ has length $5a$.

15 P and Q are points on the parabola $x^2 = 4ay$. Tangents TP and TQ are drawn from an external point T to cut the x-axis at A and B. Show that the line joining the focus to the midpoint of AB is perpendicular to the chord of contact PQ.

6.3 LOCUS PROBLEMS AND THE PARABOLA $x^2 = 4ay$

To find the locus of any point G:

1. Draw a diagram to show the information.
2. Use coordinate geometry to find the coordinates of G.
3. Write the coordinates as a pair of parametric equations.
4. Form the Cartesian equation of the locus of G by eliminating the parameter(s).

Sometimes there may be restrictions on the locus (the question should provide directions). If so, some thoughtful analysis will be required to identify the restrictions.

Problems with only one parameter

When there is only one parameter in the parametric equations, it is usually possible to:

1. Rewrite the simpler equation to make the parameter the subject.
2. Substitute this rewritten equation into the other to eliminate the parameter.

Example 9

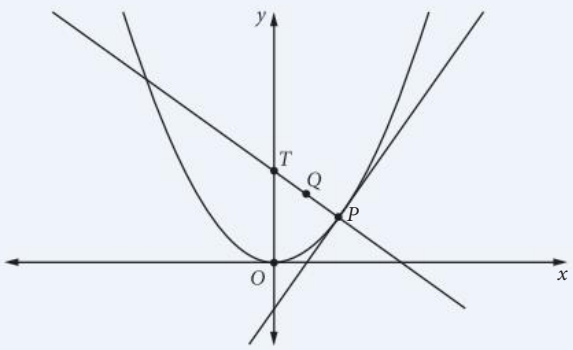
The normal at any point $P(2ap, ap^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at T.

(a) Find the Cartesian equation of the locus of the midpoint Q of PT, given that the normal at P is $py + x = 2ap + ap^3$.

(b) Show that as P moves on the parabola $x^2 = 4ay$, Q also moves along a parabola. State the vertex and focal length of the parabola on which Q moves.

Solution

(a) *Step 1* — Draw a diagram.



Step 2 — Find the coordinates of Q.

T is the point where $py + x = 2ap + ap^3$ intersects the y-axis ($x = 0$):

$py = 2ap + ap^3$

$y = 2a + ap^2$

T is $(0, 2a + ap^2)$

Q is the midpoint of $P(2ap, ap^2)$ and $T(0, 2a + ap^2)$

$\therefore$ Q is $(ap, a + ap^2)$

Step 3 — Write the coordinates of Q as parametric equations.

$x = ap$ [1]

$y = a + ap^2$ [2]

Step 4 — Form the Cartesian equation by eliminating the parameter.

From [1]: $p = \frac{x}{a}$

Substitute into [2]: $y = a + a\left(\frac{x}{a}\right)^2$

$y = a + \frac{x^2}{a}$

$ay = a^2 + x^2$

$x^2 = ay - a^2$ which is the locus of Q

(b) $x^2 = a(y-a)$

$x^2 = 4 \times \frac{a}{4} \times (y-a)$ which is the equation of a parabola with vertex $(0, a)$ and focal length $\frac{a}{4}$.

GeoGebra or other graphing software could also be used to demonstrate this.

Problems with two parameters

When there are two parameters in the parametric equations (e.g. p and q), it is necessary to:

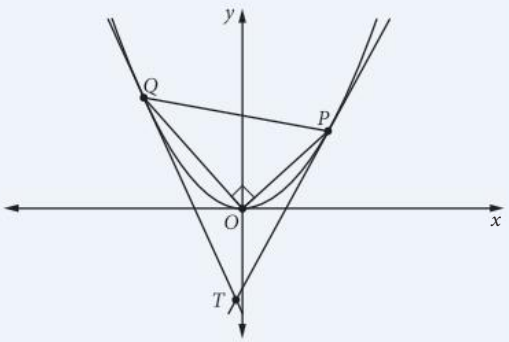
1. Find and use additional information from the question that will provide a link between p and q.
2. Use the parametric equations of the point plus the linking information (i.e. work with three equations) to form the Cartesian equation.

Often you will also need to use an identity, usually $(p+q)^2 = p^2 + q^2 + 2pq$, as an additional part of the process.

Example 10

$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$ such that the chord PQ subtends a right angle at the vertex (i.e. OP is perpendicular to OQ). The tangents from P and Q intersect at T. The equation of the tangent at P is $y - px + ap^2 = 0$. Find the Cartesian equation of the locus of T.

Solution



Step 1 — Draw a diagram.

Step 2 — Find the coordinates of T.

Tangent at P: $y - px + ap^2 = 0$ [1]

Tangent at Q: $y - qx + aq^2 = 0$ [2]

[1] − [2]: $-px + qx + ap^2 - aq^2 = 0$

$(q-p)x = a(q-p)(q+p)$

Dividing by $q-p$, noting $p \neq q$: $x = a(q+p)$

Substitute into [1]: $y - pa(q+p) + ap^2 = 0$

$y - apq - ap^2 + ap^2 = 0$

$y = apq$

Thus T is $(a(p+q), apq)$.

Step 3 — Write T as parametric equations. $x = a(p+q)$ [3]

$y = apq$ [4]

Now — Find the link between p and q.

We are told that OP is perpendicular to OQ, so the product of their gradients is −1.

$$m_{OP} = \frac{ap^2 - 0}{2ap - 0} = \frac{p}{2}$$

Similarly: $m_{OQ} = \frac{q}{2}$

Thus: $\frac{p}{2} \times \frac{q}{2} = -1$

$pq = -4$ [5] (this is the link)

Step 4 — Form the Cartesian equation by eliminating parameters (using equations [3], [4] and [5]).

Substitute [5] into [4] to obtain $y = -4a$.

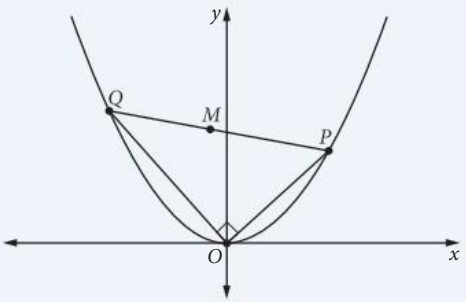
Note that a is a constant in this question, so the equation $y = -4a$ is the equation of a horizontal straight line. The locus of T has thus been found.

If you can substitute the link into either of the parametric equations to obtain $x = k$ or $y = k$, where k is a constant, then you have found the locus to be either a vertical line ($x = k$) or horizontal line ($y = k$).

Example 11

$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$ such that the chord PQ subtends a right angle at the vertex. Find the Cartesian equation of M, the midpoint of PQ.

Solution



Step 1 — Draw a diagram.

Step 2 — Find the coordinates of M.

Using midpoint formula: M is $\left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

i.e. M is $\left(a(p+q), \frac{a}{2}(p^2 + q^2)\right)$

Step 3 — Write M as parametric equations.

$x = a(p + q)$ [1]

$y = \frac{a}{2}(p^2 + q^2)$ [2]

Now — Find the link between p and q.

(As in Example **10**): $pq = -4$ [3]

Step 4 — Form the Cartesian equation by eliminating parameters.

Note the presence of $p + q$, $p^2 + q^2$ and pq in equations [1], [2], [3].

This indicates that we can use the identity $(p + q)^2 = p^2 + q^2 + 2pq$ [4]

From [1]: $p + q = \frac{x}{a}$ [1′]

From [2]: $p^2 + q^2 = \frac{2y}{a}$ [2′]

Substitute [1′], [2′], [3] into [4]:

$$\left(\frac{x}{a}\right)^2 = \frac{2y}{a} + 2(-4)$$

$$\frac{x^2}{a^2} = \frac{2y}{a} - 8$$

$$x^2 = 2ay - 8a^2$$

$$x^2 = 2a(y - 4a)$$

This is the Cartesian equation of M. The locus is a parabola with vertex $(0, 4a)$ and focal length $\frac{a}{2}$.

Locus problems with restrictions on the locus

In some situations, restrictions are implied for the locus. For example:

- Every tangent to a parabola is 'outside' the parabola (except at the point of contact), so the point of intersection of two tangents must be outside the parabola.
- The midpoint of every chord of a parabola must be 'inside' the parabola.

Example 12

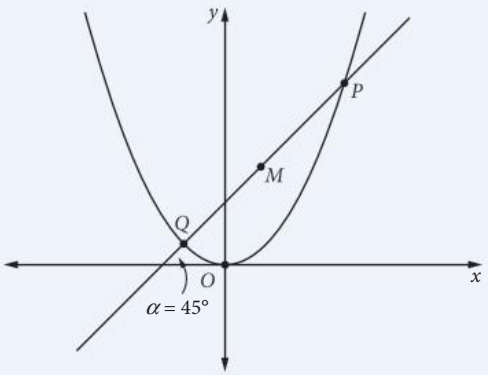
The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are on the parabola $x^2 = 4ay$ so that PQ is inclined at 45° to the positive direction of the x-axis. M is the midpoint of PQ.

(a) Show that $p + q = 2$. **(b)** Find the locus of M.

(c) Clearly state any restrictions on the locus of M.

Solution

(a) Angle of inclination of PQ is 45°, so PQ has gradient 1.



But $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{p+q}{2}$

So $\dfrac{p+q}{2} = 1$

$\therefore\ p + q = 2$ (This is the link.)

(b) M is $\left(a(p+q), \dfrac{a}{2}(p^2+q^2)\right)$

At M: $x = a(p+q)$ [1]

$y = \dfrac{a}{2}(p^2+q^2)$ [2]

Substituting $p + q = 2$ into [1], we find $x = 2a$ as the Cartesian equation of the locus of M.

(c) The locus of M is the vertical line $x = 2a$, but M must be inside the parabola. We can solve simultaneously to find the point where the line $x = 2a$ intersects the parabola.

Substitute $x = 2a$ into $x^2 = 4ay$: $4a^2 = 4ay$

$\therefore\quad y = a$

The part of the vertical line that is inside the parabola is that part where $y > a$.

Thus the locus of M is the line $x = 2a$ with its range restricted to $y > a$.

EXERCISE 6.3 LOCUS PROBLEMS AND THE PARABOLA $x^2 = 4ay$

1 If R is $(2p + 2q, 4p^2 + 4q^2)$ and it is known that $pq = 3$, the Cartesian equation of the locus of R is:

A $y = x^2$ **B** $y = x^2 - 24$ **C** $y = x^2 + 24$ **D** $y = 24x^2$

2 The tangent to the parabola $x^2 = 4ay$ at the variable point $P(2at, at^2)$ cuts the x-axis at Q and cuts the y-axis at R.

(a) Find the equation of the tangent at P. **(b)** Find the coordinates of Q and R.

(c) Find the Cartesian equation of the locus of M, the midpoint of QR.

3 P is a variable point on the parabola $x = 2t$, $y = t^2$ with focus S and vertex A. Q is the midpoint of SP and R is the midpoint of AQ.

(a) Draw a diagram to show this information. **(b)** Find the coordinates of R.

(c) Find the Cartesian equation of the locus of R.

(d) Show that the locus is a parabola and find its vertex and focus.

4 The normal at a point $P(4t, 2t^2)$ on the parabola $x^2 = 8y$ intersects the y-axis at Q. In terms of t, find the coordinates of Q and find the equation of the locus of the midpoint of PQ.

5 Let P be a variable point on the parabola $x^2 = 4ay$ whose focus is at S. Find the locus of the midpoint of PS. Show that it is a parabola and write the coordinates of its vertex and focus.

6 On the parabola $x^2 = 4ay$, P is the point with parameter p and Q is the point with parameter $2p$. The tangents from P and Q intersect at T.

(a) Write the coordinates of P and Q in terms of p.

(b) Given that the tangent at P has equation $y - px + ap^2 = 0$, show that the tangent at Q has equation $y - 2px + 4ap^2 = 0$.

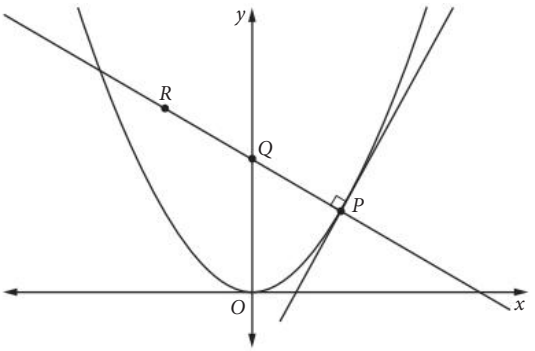
(c) Find the coordinates of T.

(d) Find the Cartesian equation of the locus of T.

7 P is a variable point on the parabola $x^2 = 4y$ so that P has coordinates $(2p, p^2)$. The tangent from P cuts the parabola $x^2 = -4y$ at Q and R. (*Note:* This is not the diagram below right.)

(a) Find the midpoint of QR.

(b) Show that the Cartesian equation of the locus of the midpoint of QR is $3x^2 = -4y$.



8 The normal at any point $P(2at, at^2)$ on the parabola $x^2 = 4ay$ cuts the y-axis at Q and is produced to R so that $PQ = QR$ (see diagram at right).

(a) Express the coordinates of R in terms of t.

(b) Find the Cartesian equation of the locus of R.

9 P is a variable point $(2at, at^2)$ on the parabola $x^2 = 4ay$ and S is the focus. The line joining P and S is produced to Q so that $PS = SQ$.

(a) Find the coordinates of Q in terms of t.

(b) Find the Cartesian equation of the locus of Q.

10 The chord of contact of the tangents to the parabola $x^2 = 4ay$ from the point $P(x_0, y_0)$ passes through the point $Q(0, 2a)$.

(a) Show that $y_0 = -2a$.

(b) Show that the locus of the midpoint of PQ is the x-axis.

11 $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$ such that the chord PQ subtends a right angle at the origin. M is the midpoint of PQ.

(a) Find the coordinates of M.

(b) Use the gradients of OP and OQ to show that $pq = -4$.

(c) Show that the Cartesian equation of the locus of M is $x^2 = 2(y-4)$.

12 The tangents to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at R.

(a) Show that the equation of the tangent at P is $y - px + ap^2 = 0$.

(b) Show that R is the point $(a(p+q), apq)$.

(c) Find the Cartesian equation of the locus of R given the following.

(i) $pq = -2$ **(ii)** PQ has gradient 3 **(iii)** $q = p + 2$

13 The normals to the parabola $x^2 = 4ay$ at the points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ meet at R.

(a) Show that the equation of the normal at P is $py + x = 2ap + ap^3$.

(b) Show that R is the point $(-apq(p+q), a(p^2 + q^2 + pq + 2))$.

(c) Find the Cartesian equation of the locus of R given the following.

(i) $pq = -2$ **(ii)** PQ has gradient 3

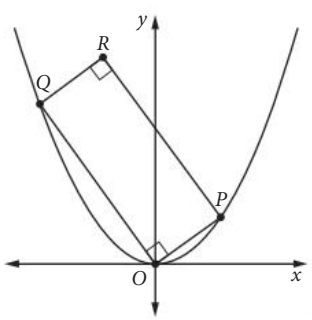
14 P and Q are the endpoints of a focal chord of the parabola $x^2 = 4ay$ with focus S. The coordinates of P and Q are $(2ap, ap^2)$ and $(2aq, aq^2)$ respectively.

(a) Find the gradients of PS and QS.

(b) Show that $pq = -1$.

(c) Find the coordinates of the midpoint R of PQ.

(d) Show that the equation of the locus of R is $x^2 = 2a(y - a)$.



15 $P(2at, at^2)$ and $Q(2as, as^2)$ are two points on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the origin O. If R is the fourth vertex of the rectangle $POQR$, show that:

(a) $st = -4$ **(b)** R is specified parametrically by $x = \dfrac{2a(t^2 - 4)}{t}$, $y = \dfrac{a(t^4 + 16)}{t^2}$

(c) the Cartesian equation of the locus of R is $x^2 = 4a(y - 8a)$. Describe the locus.

16 $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$ such that the tangents at P and Q intersect at an angle of 45°. The equation of the tangent at P is given as $y - px + ap^2 = 0$.

(a) Show that T, the point of intersection of the tangents at P and Q, has coordinates $(a(p+q), apq)$.

(b) Show that $|p-q| = |1+pq|$.

(c) Simplify $(p+q)^2 - (p-q)^2$.

(d) Hence show that the Cartesian equation of the locus of T is $(y+3a)^2 - x^2 = 8a^2$.

CHAPTER REVIEW 6

1 Match the parametric equations to their Cartesian equations. (Answers **A B C D** may be used more than once.)

(a) $x = 2ap, y = -ap^2$ **(b)** $x = ap^2, y = 2ap$ **(c)** $x = -2ap, y = -ap^2$ **(d)** $x = 2ap, y = ap^2$

A $y^2 = 4ax$ **B** $y^2 = -4ax$ **C** $x^2 = 4ay$ **D** $x^2 = -4ay$

2 The point $P(t+3, t^2 - 5)$ is a variable point on a parabola. Find the Cartesian equation of this parabola.

3 $x = t - 2, y = 2t^2 - 1$ are the parametric equations for a parabola.

(a) Find the Cartesian equation.

(b) Find the vertex, focus and directrix.

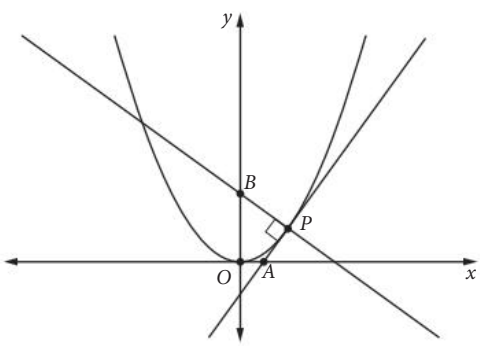
4 A tangent to the parabola $x^2 = 8y$ is parallel to the line $x - y - 3 = 0$. Find the coordinates of the point of contact of this tangent to the parabola.

5 Without using the distance formula, find the coordinates of the points on the parabola $x^2 = 4ay$ whose distance from $S(0, a)$ is $5a$.

6 $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$. The normal at P cuts the parabola again at Q. Show that $p^2 + pq + 2 = 0$.

7 The tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ meets the x-axis at A and the y-axis at B. The equation of the tangent at P is given as $y - px + ap^2 = 0$. O is the origin. C is the fourth vertex of the rectangle $OACB$. Show that the locus of C is the parabola $x^2 = -ay$.

8 The parabola $x^2 = 4y$ is shown. The tangent to the parabola at $P(2p, p^2)$, $p > 0$ cuts the x-axis at A.
The normal to the parabola at P cuts the y-axis at B.



(a) Show that the equation of the tangent at P is $y - px + p^2 = 0$.

(b) Find the coordinates of A.

(c) Show that B has coordinates $(0, p^2 + 2)$.
(Hint: The equation of the normal PB is given as $py + x = 2ap + ap^3$.)

(d) Let C be the midpoint of AB. Find the Cartesian equation of the locus of C, clearly stating any restrictions that apply to this locus.

9 $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two points on the parabola $x^2 = 8y$. The tangents at P and Q intersect at T, where T is $(2(p+q), 2pq)$. The chord PQ produced passes through $(0, -4)$.

(a) Show that the equation of the chord PQ is $y - \frac{1}{2}(p+q)x + 2pq = 0$.

(b) Show that $pq = 2$.

(c) Find the Cartesian equation of the locus of T, stating any restrictions on the locus.

This is an interesting question to illustrate with GeoGebra (or similar graphing software), using either the **Locus** tool or the **Trace** feature (right-click on the point T and select 'Trace On').

CHAPTER 7
MATHEMATICAL INDUCTION

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
HE2 uses inductive reasoning in the construction of proofs
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

The nature of proof by induction

We can often see that a particular pattern seems to apply to some situation, and we may then attempt to formulate a rule that applies to that situation. For example, we might note that 1 is a factor of 420, so is 2, so is 3, so is 4 and so is 5. From these observations we might make the statement that 'every positive integer is a factor of 420', or in other words 'n is a factor of 420 for all positive integers n'. This statement is clearly wrong: $n = 9$ is a **counter-example** which proves that the statement is false (because 9 is a positive integer and not a factor of 420).

Note: It only takes one counter-example to disprove a general statement.

For example, can you find counter-examples to disprove the following statements?

- $n^2 + n + 11$ is a prime number for all integers $n \geq 1$.
- $n^2 - n + 41$ is a prime number for all integers $n \geq 1$.

More importantly: *how do you prove that a general statement is true?* Even if you demonstrate that a statement is true for the first 1 000 000 cases, that would still not **prove** the statement to be true in all cases.

Mathematical induction is a method of proof that can be used in exactly these circumstances (for example, to prove that something is true for all positive integers n, or for all positive integers n greater than a specified starting value, or for all positive odd integers n).

A proof by induction consists of a two-step process followed by a conclusion.

1. Prove that the statement is true in the first case possible.
2. Prove that if the statement is true in any one case (i.e. for $n = k$), then it must also be true in the next case (i.e. for $n = k + 1$, or for $n = k + 2$ if the statement is true for even integers only, etc.).
3. Write a conclusion:

   The statement is true for $n = k + 1$ if it is true for $n = k$. (As proved in step 2.)
   The statement is true for $n = 1$ (i.e. for the first possible case). (As proved in step 1.)
   Thus the statement is true for all integers $n \geq 1$.

In step 2 we prove that there is the potential for a 'chain reaction' to occur: if it is true for any one case then it is also true for the next case, which means that it is also true for the case after that, and so on forever.

In step 1 we prove that it is true in the first case: this is like flicking the switch to start a chain reaction, or pushing over the first domino of an infinite line of dominoes. The infinite cascade of falling dominoes means that the statement is true for all appropriate values of n from the starting value to infinity.

We will now look at several different types of problems that require proof by induction.

7.1 PROVING SUMS OR PRODUCTS BY INDUCTION

Note the following pattern:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$

We observe that the sum of n consecutive odd numbers starting from 1 is always n^2.
Can we prove this? Yes, this can be proved by induction. (See the example below.)

Example 1

Prove that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all integers $n \geq 1$.

Solution

The two methods shown below are equivalent, but Method 2 introduces a symbolic representation of the statement being proved. When you are familiar with Method 2 you will find that it requires less writing than with Method 1.

No matter which method you use, you must always write all three steps and use the appropriate words emphasised below (or their equivalent) in each step.

Method 1

Step 1 Prove that the statement is true for $n = 1$.

When $n = 1$: LHS $= 1$ RHS $= 1^2 = 1$

LHS = RHS $\therefore$ the statement is true for $n = 1$

Step 2 Assume the statement is true for $n = k$, where k is any integer greater than or equal to 1.

i.e. assume that $\boxed{1 + 3 + 5 + \ldots + (2k - 1)} = k^2$ [a]

Now prove that the statement will be true for $n = k + 1$ if it is true for $n = k$.

i.e. prove that $1 + 3 + 5 + \ldots + (2[k + 1] - 1) = (k + 1)^2$

$$\text{LHS} = 1 + 3 + 5 + \ldots + (2[k + 1] - 1)$$
$$= \boxed{1 + 3 + 5 + \ldots + (2k - 1)} + (2[k + 1] - 1)$$

using [a]: $= k^2 + (2k + 1)$

(We can only prove this is true when $n = k + 1$ if it is true when $n = k$.)

$$= (k + 1)^2$$
$$= \text{RHS}$$

Step 3 Conclusion

The statement is true for $n = k + 1$ if it is true for $n = k$ (Step 2)

The statement is true for $n = 1$ (Step 1)

$\therefore$ by induction, the statement is true for all integers $n \geq 1$.

Method 2

Let $S(n)$ be the statement that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ where n is a positive integer.

Step 1 Prove that $S(1)$ is true.

$\text{LHS} = 1 \qquad \text{RHS} = 1^2 = 1$

$\text{LHS} = \text{RHS} \qquad \therefore S(1) \text{ is true}$

Step 2 Assume $S(k)$ is true.

i.e. assume that $\boxed{1 + 3 + 5 + \ldots + (2k - 1)} = k^2$ [a]

Now prove that $S(k + 1)$ is true if $S(k)$ is true.

i.e. prove that $1 + 3 + 5 + \ldots + (2[k + 1] - 1) = (k + 1)^2$

$$\begin{aligned}\text{LHS} &= 1 + 3 + 5 + \ldots + (2[k + 1] - 1)\\ &= \boxed{1 + 3 + 5 + \ldots + (2k - 1)} + (2[k + 1] - 1)\end{aligned}$$

using [a]: $= k^2 + (2k + 1)$

$= (k + 1)^2$

$= \text{RHS}$

Step 3 Conclusion

$S(k + 1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 2

Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all positive integers n,

i.e. prove that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Solution

Let $S(n)$ be the statement that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ where n is a positive integer.

Step 1 Prove that $S(1)$ is true.

$\text{LHS} = 1^2 = 1 \qquad \text{RHS} = \frac{1(1+1)(2\times1+1)}{6} = \frac{1(2)(3)}{6} = 1$

$\text{LHS} = \text{RHS} \qquad \therefore S(1) \text{ is true}$

Step 2 Assume $S(k)$ is true for a positive integer k.

i.e. assume that $\boxed{1^2 + 2^2 + 3^2 + \ldots + k^2} = \frac{k(k+1)(2k+1)}{6}$ [a]

Now prove that $S(k + 1)$ is true if $S(k)$ is true.

i.e. prove that $1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{(k+1)\big((k+1)+1\big)\big(2(k+1)+1\big)}{6}$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{LHS} = \boxed{1^2 + 2^2 + 3^2 + \ldots + k^2} + (k+1)^2$$

using [a]: $= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

Now look at the RHS: we want $\frac{(k+1)}{6}$ as a factor.

$$= \frac{(k+1)}{6}[k(2k+1)+6(k+1)]$$

$$= \frac{(k+1)}{6}[2k^2+7k+6]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} = \text{RHS}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 3

Prove by induction that $(2^2-1)(3^2-1)\ldots(n^2-1) = \frac{(n!)^2(n+1)}{2n}$ for all integers $n \geq 2$.

Solution

Note that this involves a product of terms rather than a sum of terms.

Let $S(n)$ be the statement that $(2^2-1)(3^2-1)\ldots(n^2-1) = \frac{(n!)^2(n+1)}{2n}$ for integers $n \geq 2$.

Step 1 Prove that $S(2)$ is true.

$\text{LHS} = (2^2-1) = 3$ $\quad$ $\text{RHS} = \frac{(2!)^2(2+1)}{2\times 2} = 3$

$\text{LHS} = \text{RHS}$ $\quad$ $\therefore$ $S(2)$ is true

Step 2 Assume $S(k)$ is true for a positive integer $k \geq 2$.

i.e. assume that $\boxed{(2^2-1)(3^2-1)\ldots(k^2-1)} = \frac{(k!)^2(k+1)}{2k}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $(2^2-1)(3^2-1)\ldots([k+1]^2-1) = \frac{((k+1)!)^2((k+1)+1)}{2(k+1)}$

$$= \frac{((k+1)!)^2(k+2)}{2(k+1)}$$

$$\text{LHS} = \boxed{(2^2-1)(3^2-1)\ldots(k^2-1)}([k+1]^2-1)$$

using [a]: $= \frac{(k!)^2(k+1)}{2k} \times (k^2+2k)$

$$= \frac{(k!)^2(k+1)}{2k} \times k(k+2)$$

$$= \frac{(k!)^2(k+1)(k+2)}{2}$$

$$= \frac{(k!)^2(k+1)(k+2)}{2} \times \frac{(k+1)}{(k+1)}$$

$$= \frac{((k+1)!)^2(k+2)}{2(k+1)} = \text{RHS}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 4

Prove by induction that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for all integers $n \geq 1$.

Solution

Let $S(n)$ be the statement that $n + (n+1) + (n+2) + \ldots + 2n = \frac{3n(n+1)}{2}$ for positive integer n.

Step 1 Prove that $S(1)$ is true.

$\text{LHS} = 1 + (2 \times 1)$ (Note that the sum on the LHS starts with n and finishes with $2n$.)
$= 3$

$\text{RHS} = \frac{3 \times 1(1+1)}{2} = 3$

LHS = RHS $\therefore$ $S(1)$ is true

Step 2 Assume $S(k)$ is true for an integer $k \geq 1$.

i.e. assume that $\boxed{k + (k+1) + (k+2) + \ldots + 2k} = \frac{3k(k+1)}{2}$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $(k+1) + (k+2) + \ldots + 2(k+1) = \frac{3(k+1)((k+1)+1)}{2}$

Note that when $n = k+1$, the sum on the LHS starts with $(k+1)$ and finishes with $2(k+1)$.

$$\text{LHS} = (k+1) + (k+2) + \ldots + 2k + (2k+1) + (2k+2)$$

Now we have a problem! At this point we need to use the substitution of line [a], but we are missing the first term (the k). In going from $S(k)$ to $S(k+1)$ we have lost the first term but gained two extra terms on the LHS. The solution is to break up the extra two terms $(2k+1) + (2k+2)$ as a k term and a $(3k+3)$ term.

$$\text{LHS} = \boxed{k + (k+1) + (k+2) + \ldots + 2k} + (3k+3)$$

using [a]: $= \frac{3k(k+1)}{2} + (3k+3)$

$= \frac{3(k+1)}{2}[k+2]$ taking the common factor of $\frac{3(k+1)}{2}$

= RHS

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

EXERCISE 7.1 PROVING SUMS OR PRODUCTS BY INDUCTION

1 If $S(n)$ is the statement that $n + 2n + 3n + \ldots + n^2 = \dfrac{n^2(n+1)}{2}$, then $S(5)$ represents the statement:

A $1 + 2 + 3 + \ldots + 5 = \dfrac{5 \times 6}{2}$ **B** $1 + 2 + 3 + \ldots + 25 = \dfrac{25 \times 26}{2}$

C $1 + 2 + 3 + \ldots + 25 = \dfrac{25 \times 6}{2}$ **D** $5 + 10 + 15 + 20 + 25 = \dfrac{25 \times 6}{2}$

Prove each of the following by induction for all positive integers n.

2 $1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$

3 $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

4 $\displaystyle\sum_{k=1}^{n} (3k-1) = \frac{n(3n+1)}{2}$

5 $1 + r + r^2 + r^3 + \ldots + r^{n-1} = \dfrac{1-r^n}{1-r}$

6 $\dfrac{1}{1\times 2} + \dfrac{1}{2\times 3} + \dfrac{1}{3\times 4} + \quad + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

7 $\displaystyle\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \frac{n}{2(n+2)}$

8 $\displaystyle\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$

9 $\dfrac{1^2}{1\times 3} + \dfrac{2^2}{3\times 5} + \dfrac{3^2}{5\times 7} + \quad + \dfrac{n^2}{(2n-1)(2n+1)} = \dfrac{n(n+1)}{2(2n+1)}$

10 $1^3 + 2^3 + 3^3 + \ldots + n^3 = \dfrac{n^2(n+1)^2}{4}$. Hence find $\displaystyle\lim_{n\to\infty}\left(\frac{1^3 + 2^3 + 3^3 + \quad + n^3}{n^4}\right)$

11 $\left(1 - \dfrac{1}{2^2}\right)\left(1 - \dfrac{1}{3^2}\right) \quad \left(1 - \dfrac{1}{n^2}\right) = \dfrac{n+1}{2n}$ for $n \geq 2$

12 $(n+1) + (n+2) + \ldots + 2n = \dfrac{n(3n+1)}{2}$

13 $1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$

14 $\displaystyle\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1}\left[\frac{n(n+1)}{2}\right]$

15 $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \quad + \dfrac{2^n - 1}{2^n} = n - 1 + \dfrac{1}{2^n}$

16 $\displaystyle\sum_{k=1}^{n} \log\left(\frac{k+1}{k}\right) = \log(n+1)$

7.2 PROVING DIVISIBILITY BY INDUCTION

The following points are essential for properly understanding this type of proof.

- If an integer N is divisible by an integer d then $N = d \times q$, where q is an integer.
  For example, 18 is divisible by 3 because $18 = 3 \times 6$ (where 6 is an integer). But 19 is not divisible by 3 even though $19 = 3 \times 6\frac{1}{3}$, because $6\frac{1}{3}$ is not an integer.
- The integers are **closed** under addition, subtraction and multiplication. This means that whenever we add, subtract or multiply integers, the answer will always be another integer. (This is not true for division: some divisions of integers give integer answers, but others do not.)

Example 5

Prove by induction that $4^n - 1$ is divisible by 3 for all integers $n \geq 1$.

Solution

Let $S(n)$ be the statement that $4^n - 1$ is divisible by 3 for integer n.

Step 1 Prove that $S(1)$ is true.

$4^1 - 1 = 3$, which is divisible by 3. (Note that we can't use the LHS–RHS style of proof here.)

$\therefore$ $S(1)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k \geq 1$.

i.e. assume that $4^k - 1 = 3M$ where M is an integer

$4^k = 3M + 1$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $4^{k+1} - 1$ is divisible by 3

$4^{k+1} - 1 = 4 \times 4^k - 1$

using [a]: $= 4(3M + 1) - 1$

$= 12M + 3$

$= 3(4M + 1)$ which is divisible by 3, because $4M + 1$ is an integer.

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 1$.

Example 6

Prove by induction that $3^n + 7^n$ is divisible by 10 for all positive odd integers n.

Solution

Let $S(n)$ be the statement that $3^n + 7^n$ is divisible by 10 for positive odd integer n.

Step 1 Prove that $S(1)$ is true.

$3^1 + 7^1 = 10$, which is divisible by 10.

$\therefore$ $S(1)$ is true.

Step 2 Assume $S(k)$ is true for an odd integer $k \geq 1$.

i.e. assume that $3^k + 7^k = 10M$ where M is an integer

$3^k = 10M - 7^k$ [a] (We could have made 7^k the subject instead of 3^k.)

Now prove that $S(k+2)$ is true if $S(k)$ is true.

(Note the $S(k+2)$ instead of $S(k+1)$, as we are working with odd integer values of n.)

i.e. prove that $3^{k+2} + 7^{k+2}$ is divisible by 10

$3^{k+2} + 7^{k+2} = 3^2 \times 3^k + 7^2 \times 7^k$

using [a]: $= 9(10M - 7^k) + 49 \times 7^k$

$= 90M - 9 \times 7^k + 49 \times 7^k$

$= 90M + 40 \times 7^k$

$= 10(9M + 4 \times 7^k)$

This is divisible by 10, because $9M + 4 \times 7^k$ is an integer.

Step 3 Conclusion

$S(k+2)$ is true if $S(k)$ is true (Step 2)

$S(1)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all odd integers $n \geq 1$.

EXERCISE 7.2 PROVING DIVISIBILITY BY INDUCTION

1 If k and M are integers, which of the following expressions does *not* always generate an integer?

A $9M + 4 \times 7k$ **B** $9M - 4 \times 7k$ **C** $9M \div 4 \times 7k$ **D** $9M \times 4 \times 7k$

Prove the following by induction.

2 $5^n + 3$ is divisible by 4 for all positive integers n.

3 $3^{2n} - 1$ is divisible by 8 for all positive integers n.

4 $3^n + 2^n$ is divisible by 5 for all odd integers $n \geq 1$.

5 $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

6 **(a)** Factorise $k(k+1)(k+2) + 3(k+1)(k+2)$.

(b) Hence prove that $n(n+1)(n+2)$ is divisible by 3 for all positive integers n.

7 $3^{3n} + 2^{n+2}$ is divisible by 5 for all positive integers n.

8 **(a)** Show that $(k+3)^3 = k^3 + 9k^2 + 27k + 27$.

(b) Hence prove that the sum of the cubes of three consecutive positive integers is divisible by 3.

9 Prove that the polynomial $(x-1)^{n+2} + x^{2n+1}$ is divisible by $x^2 - x + 1$ for all positive integers n.
(*Note:* In step 2, we can't say $(x-1)^{k+2} + x^{2k+1} = (x^2 - x + 1)M$ where M is an integer. We must say $(x-1)^{k+2} + x^{2k+1} = (x^2 - x + 1)M(x)$, where $M(x)$ is a polynomial, and continue this through the rest of the proof.)

10 Prove that $x^n - 1$ is divisible by $x - 1$ for all positive integers n.
(Use $x^{k+1} - 1 = x^{k+1} - x^k + x^k - 1$.)

7.3 PROVING INEQUALITIES BY INDUCTION

Example 7

Prove by induction that $3^n > 1 + 2n$ for all integers $n > 1$ (i.e. prove that $3^n - 1 - 2n > 0$).

Solution

Let $S(n)$ be the statement that $3^n - 1 - 2n > 0$ for integer n.

Step 1 Prove that $S(2)$ is true. (Note that $n = 2$ is the first case.)

$$\begin{aligned} \text{LHS} &= 3^2 - 1 - 4 \\ &= 4 \\ &> 0 \end{aligned}$$

$\therefore$ $S(2)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k \geq 2$.

i.e. assume that $3^k - 1 - 2k > 0$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $3^{k+1} - 1 - 2(k+1) > 0$

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 1 - 2k - 2 \\ &= 3 \times 3^k - 2k - 3 \end{aligned}$$

We need to link [a] to this, so we need to group $3^k - 1 - 2k$ together. However, the term in 3^k is multiplied by 3, so we need $3(3^k - 2k - 1)$. Form this group and 'pay back' the extra terms as required:

$$\begin{aligned} \text{LHS} &= 3 \times 3^k - 6k - 3 + 4k \\ &= 3(3^k - 2k - 1) + 4k \\ &> 0 \qquad \text{as } 3^k - 1 - 2k > 0 \quad \text{from [a] and } k > 0 \end{aligned}$$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

EXERCISE 7.3 PROVING INEQUALITIES BY INDUCTION

Prove the following by induction.

1 $n^2 - 11n + 30 \geq 0$ for all integers $n \geq 6$.

2 $n^2 > -5n + 14$ for all integers $n > 2$.

3 $12^n > 7^n + 5^n$ for all integers $n \geq 2$.

4 $10^n \geq 2^n + 3^n + 5^n$ for all positive integers n.

5 If $x > 0$, prove that $(1+x)^n \geq 1 + nx$ for all positive integers n.

6 It is given that $A > 0$, $B > 0$ and n is a positive integer.

(a) Factorise $A^{n+1} - A^nB + B^{n+1} - B^nA$

(b) Hence show that $A^{n+1} + B^{n+1} \geq A^nB + B^nA$

(c) Prove by induction that $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$ for all positive integers n.

7.4 OTHER PROOFS BY INDUCTION

You will need creative thinking and insight to build the three-step induction proof in unfamiliar situations.

The Fibonacci numbers are a well-known sequence in which each new term is formed by adding the two previous terms (starting with the first two terms 1, 1). Thus the sequence begins 1, 1, 2, 3, 5, 8, 13, 21, …

We will use the symbol F_n to represent the n-th Fibonacci number, so $F_3 = 2$, $F_8 = 21$ and so on.

Note that the Fibonacci numbers can be defined as $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 3$.

Example 8

Prove by induction that $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$ for all integers $n \geq 2$.

Solution

Let $S(n)$ be the statement that $F_1 + F_2 + F_3 + \ldots + F_n = F_{n+2} - 1$ for integer n.

Step 1 Prove that $S(2)$ is true (i.e. prove that $F_1 + F_2 = F_4 - 1$).

$\text{LHS} = 1 + 1 = 2$ $\quad$ $\text{RHS} = 3 - 1 = 2$

$\text{LHS} = \text{RHS}$ $\quad$ $\therefore$ $S(2)$ is true

Step 2 Assume $S(k)$ is true for an integer $k \geq 2$.

i.e. assume that $\boxed{F_1 + F_2 + F_3 + \ldots + F_k} = F_{k+2} - 1$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that $F_1 + F_2 + F_3 + \ldots + F_{k+1} = F_{k+3} - 1$

LHS $= \boxed{F_1 + F_2 + F_3 + \ldots + F_k} + F_{k+1}$

using [a]: $= F_{k+2} - 1 + F_{k+1}$

Fibonacci definition: $= F_{k+3} - 1 \qquad (F_n = F_{n-1} + F_{n-2})$

$= \text{RHS}$

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 2$.

Example 9

Construct a proof by induction of the geometrical property that 'the angle sum of an n-sided polygon is $(n-2) \times 180°$ for all integers $n \geq 3$'.

Solution

Let $S(n)$ be the statement that the angle sum of an n-sided polygon is $(n-2) \times 180°$ for integer n.

Step 1 Prove that $S(3)$ is true.

When $n = 3$, angle sum $= (3-2) \times 180° = 180°$, which is the angle sum of a triangle.

$\therefore$ $S(3)$ is true.

Step 2 Assume $S(k)$ is true for an integer $k > 3$.

i.e. assume that the angle sum of a k-sided polygon is $(k-2) \times 180°$ [a]

Now prove that $S(k+1)$ is true if $S(k)$ is true.

i.e. prove that the angle sum of a $(k+1)$-sided polygon is $([k+1]-2) \times 180°$

$= (k-1) \times 180°$

In the $(k+1)$-sided polygon, construct a diagonal that divides the polygon into a k-sided polygon and a triangle. (This can always be done.)

Angle sum of $(k+1)$-sided polygon = (angle sum of k-sided polygon) + (angle sum of triangle)

using [a]: $= (k-2) \times 180° + 180°$

$= (k-1) \times 180°$ (as required)

Step 3 Conclusion

$S(k+1)$ is true if $S(k)$ is true (Step 2)

$S(3)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for all integers $n \geq 3$.

EXERCISE 7.4 OTHER PROOFS BY INDUCTION

1 If F_n represents the n-th Fibonacci number, prove by induction that $F_1 + F_3 + F_5 + \ldots + F_{2n-1} = F_{2n}$ for all positive integers n.

2 If F_n represents the n-th Fibonacci number, prove by induction that F_{4n} is divisible by 3 for all positive integers n.

3 Consider the sequence of numbers defined by $T_1 = 3$, $T_n = 2 \times T_{n-1} + 3$ for all $n \geq 2$.

(a) List the first five terms of this sequence.

(b) Prove by induction that $T_n = 3(2^n - 1)$ for all integers $n \geq 1$.

4 Prove by induction that $\sin(x + n\pi) = (-1)^n \sin x$ for all integers $n \geq 1$.

5 Prove by induction that the greatest number of regions that a circle can be divided into by n straight lines is $\frac{1}{2}(n^2 + n + 2)$ for all positive integers n.

6 Suppose there is a country in which the only currency is \$3 coins and \$5 coins.

(a) Construct a table to show how the amounts of money from \$1 to \$15 (in whole dollars) can be made using only \$3 and \$5 coins. Note that some amounts are impossible to achieve.

(b) Complete the following statement, then prove it by induction:
'Using only \$3 and \$5 coins, it is possible to make \$$n$ for all integers $n \geq \ldots$'

CHAPTER REVIEW 7

1 Prove by induction that $\sum_{r=1}^{n}(4r-6) = 2n(n-2)$.

2 Prove by induction that $1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for all integers $n \geq 1$.

3 Prove by induction that $7^n - 1$ is divisible by 3 for all positive integers n.

4 Prove by induction that $4^{n+1} + 5^{2n-1}$ is divisible by 21 for all positive integers n.

5 Prove by induction that $n^2 \geq 3n - 2$ for all integers $n \geq 2$.

6 If $x > 0$ and $y > 0$, prove by induction that $(x + y)^n > x^n + y^n$ for all integers $n \geq 2$.

7 Prove by induction that $\sum_{k=1}^{n} \frac{k+2}{k(k+1)2^k} = 1 - \frac{1}{(n+1)2^n}$ for all integers $n \geq 1$.

8 Prove by induction that $5^n + 12n - 1$ is divisible by 16 for all positive integers n.

9 Prove by induction that $7^n + 6^n$ is divisible by 13 for all odd positive integers n.

10 Prove by induction that $2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n+1)!$ for all positive integers n.

11 **(a)** By writing $\cos([2k+1]x)$ as $\cos(2kx + x)$, and remembering that $\cos 2x = 1 - 2\sin^2 x$, show that:
$\frac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \frac{\sin(2(k+1)x)}{2\sin x}$

(b) Use the result of **(a)** to prove by induction that $\cos x + \cos 3x + \ldots + \cos([2n-1]x) = \frac{\sin(2nx)}{2\sin x}$ for all positive integers n.

CHAPTER 8
ITERATIVE METHODS

Outcomes covered:

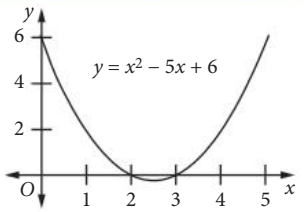
HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

8.1 APPROXIMATE SOLUTIONS OF EQUATIONS

In Chapter 2, we looked at finding the roots of polynomial equations by factoring the polynomial.

We associate the roots of an equation with the x values of its graph's points of intersection with the x-axis. (For example, the roots of the quadratic equation $x^2 - 5x + 6 = 0$ are the x values of the points of intersection of the parabola $y = x^2 - 5x + 6$ with the x-axis.)

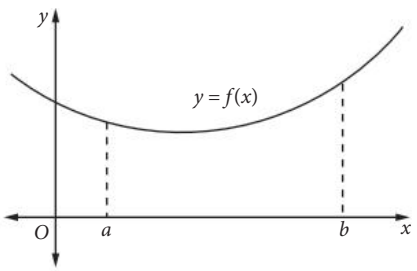


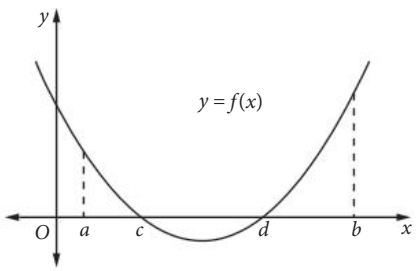
The graph of $y = x^2 - 5x + 6$ cuts the x-axis where $x = 2$ or 3.

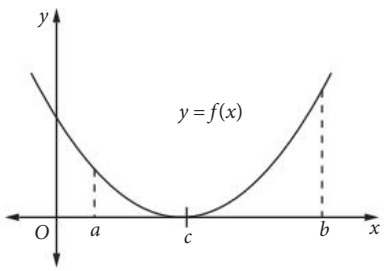
Our problem is finding where the curve crosses (or touches) the x-axis.

For a given function $y = f(x)$, if $f(a) > 0$ and $f(b) < 0$ then there is at least one root of the equation $f(x) = 0$ between $x = a$ and $x = b$, provided that the function is continuous in the interval $a \le x \le b$. This may seem obvious, but 'obviousness' is not proof! The actual proof of this is beyond the scope of this course.

However, we can still investigate this situation. Note that when $f(a)$ and $f(b)$ have the same sign, we cannot always be certain that there is no root between a and b. Consider the cases below where $f(a)$ and $f(b)$ have the same sign.

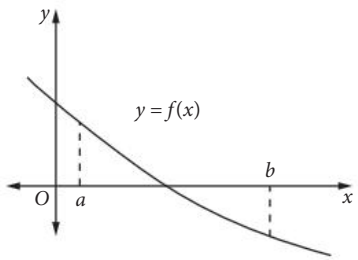


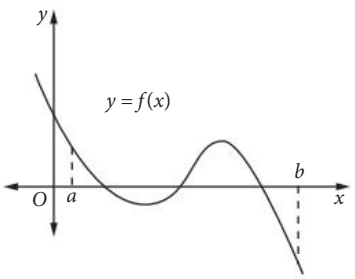


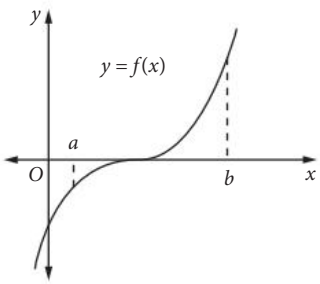


The left diagram shows no real root between a and b. The middle diagram shows two distinct roots between a and b. The right diagram shows one root of multiplicity 2: there are two coincident roots between a and b.

If $f(a)$ and $f(b)$ have opposite signs then there must be at least one real root between a and b, provided the curve is continuous for $a \le x \le b$.

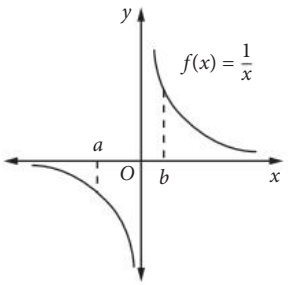






In the left diagram, $f(a)$ and $f(b)$ have opposite signs and there is one root between a and b. In the middle diagram, $f(a)$ and $f(b)$ have opposite signs and there are several roots between a and b. In the right diagram, $f(a)$ and $f(b)$ have opposite signs and there is a multiple (coincident) root between a and b.
(For example, $f(x) = x^3 - 6x^2 + 12x - 8 = (x - 2)^3$ has a triple root at $x = 2$.)

Of course, if a function is **not** continuous between $x = a$ and $x = b$, then the fact that $f(a)$ and $f(b)$ have opposite signs does not necessarily imply the presence of a root. (For example, see the diagram at right.)



Even if we cannot solve an equation, the principles above may tell us whether a solution exists or not. We can then attempt to find an approximate solution.

Approximate methods of solving equations usually involve starting with an estimate of the solution and then improving on the estimate by an **iterative process**. In an iterative process, a sequence of steps is performed repeatedly ('applied' or 'iterated') and at the end of each cycle an improved estimate is obtained. An iterative process may never give the exact solution, but it should improve as time goes on (even if it improves only slowly).

Example 1

Between which two consecutive integer values of x will a root of $f(x) = 0$ occur, if $f(x) = 8x^3 - 15x^2 - x + 6$?

Solution

$f(-1) = -8 - 15 + 1 + 6 = -16 < 0$
$f(0) = 6 > 0$
$f(1) = 8 - 15 - 1 + 6 = -2 < 0$
$f(2) = 64 - 60 - 2 + 6 = 8 > 0$

In this example there are three possible answers: $-1 < x < 0$, $0 < x < 1$ and $1 < x < 2$.

EXERCISE 8.1 APPROXIMATE SOLUTIONS OF EQUATIONS

1 Show that polynomial equation $P(x) = 0$ has at least one root in the interval given.

(a) $P(x) = x^2 + 5x - 3$, $0 < x < 1$ **(b)** $P(x) = x^2 + 2x - 4$, $1 < x < 2$
(c) $P(x) = x^3 + 5$, $-2 < x < -1$ **(d)** $P(x) = 2x^2 - 3x - 10$, $3 < x < 4$
(e) $P(x) = x^3 + 2x^2 - 2x - 4$, $1 < x < 2$ **(f)** $P(x) = x^4 - 3x^2 + 2x + 3$, $-1 < x < 0$

2 If $f(x) = x^3 - x^2 + 3$, then the equation $f(x) = 0$ has a root in which interval?

A $-2 < x < -1$ **B** $-1 < x < 0$ **C** $0 < x < 1$ **D** $1 < x < 2$

3 For each equation, use the remainder theorem (see Chapter 2 page 45) to find two regions that each contain a root. Define each region using a pair of consecutive integers.

(a) $4x^2 - 8x + 2 = 0$ **(b)** $x^2 - 6x + 6 = 0$ **(c)** $x^3 - 2x^2 - 3x + 2 = 0$
(d) $x^3 - 3x - 1 = 0$ **(e)** $x^3 + 2x^2 - 2x = 0$ **(f)** $x^4 - 3x^2 + 2x + 3 = 0$

4 Show that each equation has a root in the given interval.

(a) $8x - 2\sin x - 3 = 0$, $0 < x < 1$ **(b)** $\log_e x - \cos x + 1 = 0$, $0.1 < x < 1$
(c) $2\sin x - 3\cos x + 1 = 0$, $4 < x < 5$ **(d)** $\log_e(x + 2) + \sin x + 2 = 0$, $-1.9 < x < -1$

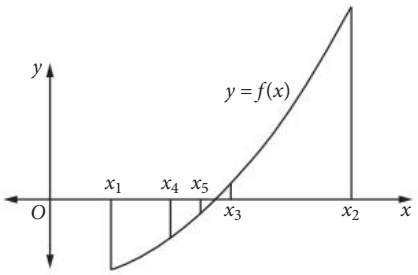
8.2 HALVING THE INTERVAL

In this iterative method, you first find two values of x for which $f(x)$ has different signs. You then take the average of these two values for x (this is 'halving the interval') to find a better approximation for the root of the equation $f(x) = 0$.

- First, obtain two estimates of the root: x_1 below and x_2 above the actual root.
  If $f(x_1)$ and $f(x_2)$ have opposite signs then there is a root of $f(x)$ in the interval $x_1 < x < x_2$.
- Next, compute the midpoint $x_3 = \dfrac{x_1 + x_2}{2}$ and find the value of $f(x_3)$.
  If $f(x_3) = 0$ then x_3 is the desired root. If $f(x_3) \neq 0$, go to the next step.

- Decide whether the root is in the half-interval from x_1 to x_3 or the half-interval from x_3 to x_2 by considering the sign of $f(x_3)$. Now return to the start, this time using the value of x_3 as the new value of one of the two estimates (i.e. so that the two new values of $f(x)$ are still different signs).

The diagram shows what 'halving the interval' looks like on a graph.



Example 2

Show that $x^3 + x^2 + x - 8 = 0$ has a root between $x = 1$ and $x = 2$. Apply the 'halving the interval' method twice to determine an estimate of the root.

Solution

$f(x) = x^3 + x^2 + x - 8$

$f(1) = 1 + 1 + 1 - 8 = -5$

$f(2) = 8 + 4 + 2 - 8 = 6$

$f(1)$and $f(2)$ have opposite signs, so there is a root between $x = 1$ and $x = 2$.
Call the iterative values x_L ('L' for 'lower') and x_U ('U' for 'upper') with a midpoint x_M.

$x_L = 1$ and $x_U = 2$: $x_M = \dfrac{1+2}{2} = 1.5$

$f(1.5) = (1.5)^3 + (1.5)^2 + 1.5 - 8 = -0.875 < 0$

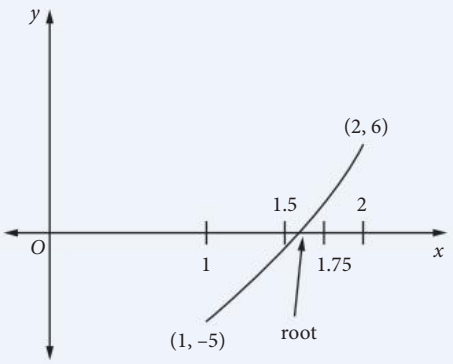
$f(1.5) < 0$, so the root is between $x = 1.5$ and $x = 2$.

$x_L = 1.5$ and $x_U = 2$: $x_M = \dfrac{1.5+2}{2} = 1.75$

$f(1.75) = (1.75)^3 + (1.75)^2 + 1.75 - 8 = 2.172 > 0$

$f(1.75) > 0$, so the root is between $x = 1.5$ and $x = 1.75$.

Because $f(1.5)$ is the closest to zero, the root is nearer to $x = 1.5$.



Continuing this iterative process will generate a better approximation for this root of the equation.

Note that this will be the only root of this cubic equation: $f'(x) = 3x^2 + 2x + 1$ is a positive definite function, so the curve $y = f(x)$ has no turning points and hence cannot cut the x-axis again.

Halving the interval—summary

Step 1 Obtain two estimates of the root, x_L (below the root) and x_U (above the root).

Step 2 Calculate the midpoint $x_M = \dfrac{x_L + x_U}{2}$.

Step 3 Calculate $f(x_M)$. If x_M is a good enough estimate of the root, stop the process. Otherwise, proceed to step 4.

Step 4 Examine the sign of $f(x_M)$ to determine whether the root is between x_L and x_M or between x_M and x_U.

Step 5 If the root is between x_L and x_M, make x_M the new x_U and return to step 2; if the root is between x_M and x_U, make x_M the new x_L and return to step 2.

EXERCISE 8.2 HALVING THE INTERVAL

1 Use the 'halving the interval' method to solve each equation correct to two decimal places, given that:

(a) $x^3 - 3x - 20 = 0$ has a root between 3 and 3.1
(b) $x^4 + x^2 - 80 = 0$ has a root between 2.9 and 3.0
(c) $10x = 2\sin x + 5$ has a root between 0.6 and 0.7
(d) $\log_e x = \cos x$ has a root between 1.2 and 1.4
(e) $\log_e (x + 2) = e^x - 2$ has a root between 1.1 and 1.2.

2 The 'halving the interval' method is applied three times to find the root of the equation $x^4 - x^2 - 2 = 0$ that is between $x = 1$ and $x = 2$. The answer obtained is:

A $x = 1.25$ **B** $x = 1.375$ **C** $x = 1.5$ **D** $x = 1.625$

3 Use the 'halving the interval' method to solve each equation correct to two decimal places, given that:

(a) $8x - 2\sin x - 3 = 0$ has a root between $x = 0$ and $x = 1$
(b) $\log_e x - \cos x + 1 = 0$ has a root between $x = 0.6$ and $x = 0.9$
(c) $2\sin x - 3\cos x + 1 = 0$ has a root between $x = 4$ and $x = 5$
(d) $\log_e (x + 2) + \sin x + 2 = 0$ has a root between $x = -1.7$ and $x = -1.5$.

4 The equation $x^3 - x^2 + 1 = 0$ has a root that is approximately -1. Use the 'halving the interval' method to find a better approximation to this root correct to two decimal places.

8.3 NEWTON'S METHOD

Suppose that x_1 is close to a root of the equation $f(x) = 0$. The equation of the tangent at $x = x_1$ is:

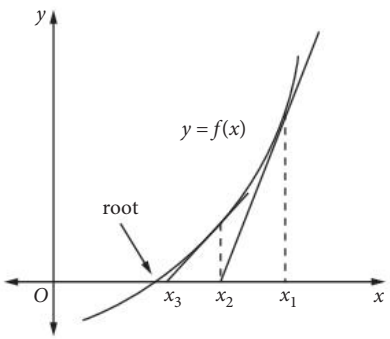
$$y - y_1 = m\,(x - x_1)$$

i.e. $$y - f(x_1) = f'(x_1)\,(x - x_1)$$

This tangent crosses the x-axis at a point where $x = x_2$ and $y = 0$.

Hence $$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

i.e. $$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$



This equation provides the basis for an iterative process known as Newton's method (after Isaac Newton, one of the method's early developers).
If x_1 is close to the desired root, then under certain conditions x_2 is closer. If we now replace x_1 by x_2, calling the new result x_3, then x_3 may be closer still. Newton's method usually requires fewer applications for a given accuracy than the 'halving the interval' method, but more care must be taken when applying it.

The diagram above shows that if x_1 is our first approximation, then x_2 is closer to the root than x_1, and x_3 is closer than x_2. If we continue the iterative process then we can find an x_4 closer to the root than x_3, and so on.

Newton's method is particularly useful in finding square roots, cubic roots etc.

Example **3** reconsiders Example **2** using Newton's method:

Example 3

The equation $x^3 + x^2 + x - 8 = 0$ has a root near $x = 2$. Use Newton's method with two applications to obtain a better approximation.

Solution

Calculations below are rounded to three decimal places where necessary.

$$f(x) = x^3 + x^2 + x - 8 \qquad \text{Make } x_1 = 2: \quad f(x_1) = 8 + 4 + 2 - 8 = 6$$
$$f'(x) = 3x^2 + 2x + 1 \qquad f'(x_1) = 12 + 4 + 1 = 17$$

Now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 2 - \dfrac{6}{17}$ $\qquad f(x_2) = (1.647)^3 + (1.647)^2 + 1.647 - 8 = 0.827$

$= 1.647$ $\qquad f'(x_2) = 3 \times 1.647^2 + 2 \times 1.647 + 1 = 12.43$

Now $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1.647 - \dfrac{0.827}{12.43}$

$= 1.58$

Because $f(1.58) = 0.021 \approx 0$, $x = 1.58$ is a close approximation to the root of the given equation.

Example 4

Use Newton's method with two applications to find $\sqrt{2}$ to two decimal places.

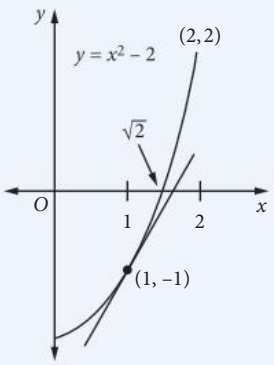
Solution

The positive solution of the quadratic equation $x^2 - 2 = 0$ gives the square root of 2.

The graph of $f(x) = x^2 - 2$ crosses the x-axis between $x = 1$ and $x = 2$, apparently closer to $x = 1$, so take $x = 1$ as the first approximation:

$f(x) = x^2 - 2$ $\qquad \therefore \quad f(1) = -1$

$f'(x) = 2x$ $\qquad \therefore \quad f'(1) = 2$



Now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 1 + \dfrac{1}{2}$

$= 1.5$

$f(x_2) = (1.5)^2 - 2 = 0.25$

$f'(x_2) = 2 \times 1.5 = 3$

Now $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1.5 - \dfrac{0.25}{3}$

$= 1.42$ (to 2 d.p.)

As Example **4** shows, Newton's method is applicable not only to polynomial equations but also to other types of equations, including transcendental equations. Newton's method will work as long as the function is continuous and differentiable near the root of the equation.

Example 5

Find the root of $2 \sin x - x = 0$ near $x = 2$. Use Newton's method with two applications, remembering that x is in radians.

Solution

Calculations below are rounded to four decimal places where necessary.

$f(x) = 2 \sin x - x$ $\qquad \therefore \quad f(2) = 2 \sin 2 - 2 = -0.1814$

$f'(x) = 2 \cos x - 1$ $\qquad \therefore \quad f'(2) = 2 \cos 2 - 1 = -1.8323$

Make $x_1 = 2$:

Now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 2 - \dfrac{-0.8114}{-1.8323}$

$= 1.9010$

$f(x_2) = f(1.9010) = -0.0090$

$f'(x_2) = f'(1.9010) = -1.6485$

Now $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 1.9010 - \dfrac{-0.0090}{-1.6485}$

$= 1.8955$

$f(1.8955) = 0.000\,009$

$\therefore$ $x = 1.8955$ is a good approximation.

In Newton's method, your initial estimate of the root is **very important**.

If the initial estimate is not close enough to the actual root, then the iterative process may not continue to get closer to the root.

Example 6

Use Newton's method to find the root of $2\sin x - x = 0$ near $x = 6$. After two applications of Newton's method, comment on your result.

Solution

Calculations below are rounded to four decimal places where necessary.

$f(x) = 2\sin x - x \quad \therefore \quad f(6) = 2\sin 6 - 6 = -6.5588$

$f'(x) = 2\cos x - 1 \quad \therefore \quad f'(6) = 2\cos 6 - 1 = 0.9203$

Make $x_1 = 6$:

Now $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$= 6 - \dfrac{-6.5588}{0.9203}$

$= 13.1268$

$f(x_2) = f(13.1268) = -12.0637$

which is further from zero than $f(x_1)$

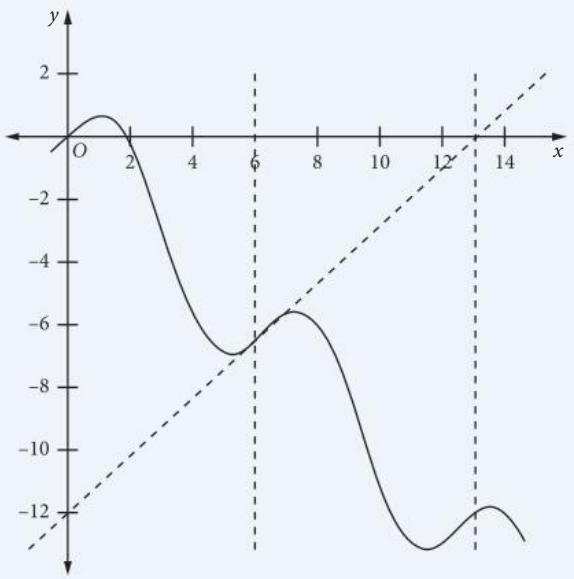
$f'(x_2) = f'(13.1268) = 0.6941$

Now $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

$= 13.1268 - \dfrac{-12.0637}{0.6941}$

$= 30.5071$

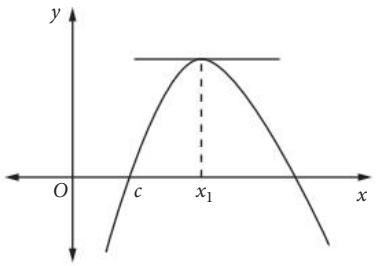
$f(x_3) = -32.0847$



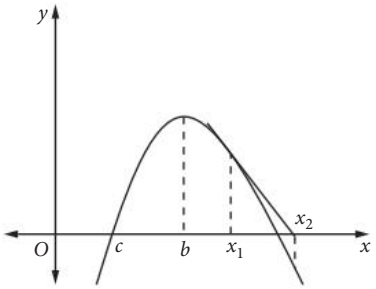
The function values are getting further away from zero, hence the x value is getting further from the root. The graph of $f(x)$ above shows why the root is not obtained when $x = 6$ is used as the initial estimate: the tangent at $x = 6$ cuts the x-axis at $x = 13.127$, but the tangent at $x = 13.127$ will cut the x-axis even further to the right. The first turning point for $x > 0$ is at $x = \dfrac{\pi}{3} \approx 1.05$, while the second turning point is at $x = \dfrac{5\pi}{3} \approx 5.24$. For Newton's method to work, the initial approximation for the root must be between these two values of x, i.e. $1.05 < x < 5.24$.

Failure of Newton's method

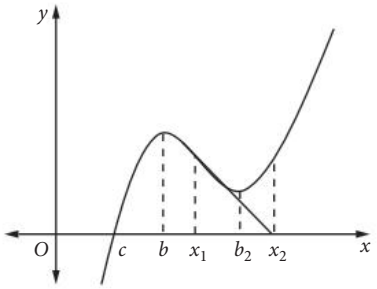
1 If $f'(x_1) = 0$ then the result becomes $x_2 = x_1 - \dfrac{f(x_1)}{0}$, which is undefined. This means that if the first estimate for the root is at a stationary point, then Newton's method cannot be used to find the root of the equation, because the tangent at that point is horizontal and will not cut the x-axis.



2 If $f'(b) = 0$, where b is between the estimate $x = x_1$ and the root at $x = c$, then Newton's method will not work:



In this situation, the new approximation x_2 will move closer to the other root instead of to the required root c.



In this situation, the new approximation will not be closer to the root c but will be on the other side of the turning point at $x = b_2$. Continued applications of Newton's method will see the approximations oscillate about b_2.

These points are demonstrated in Example **7** below.

Example 7

Use Newton's method to find a root of $x^3 - 3x = 0$ near: **(a)** $x = 1$ **(b)** $x = 0.5$ **(c)** $x = 1.5$

Solution

$f(x) = x^3 - 3x \qquad f'(x) = 3x^2 - 3$

$\therefore f'(x) = 0 \quad \text{when } x = \pm 1$

(a) $x_1 = 1, f(x_1) = -2, f'(x_1) = 0$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1 - \frac{-2}{0} = \text{undefined}$$

This estimate does not lead to a root of the equation.

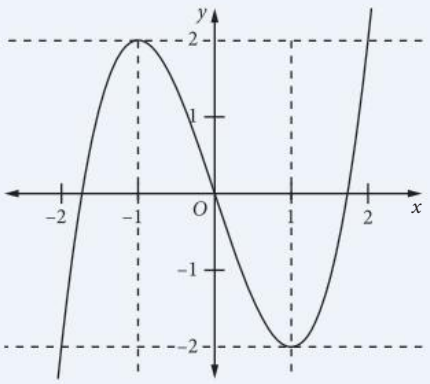
(b) $x_1 = 0.5, f(x_1) = -1.375, f'(x_1) = -2.25$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.5 - \frac{-1.375}{-2.25}$$

$$= -0.11$$

This initial estimate will lead to the root at $x = 0$, as shown in the diagram.



(c) $x_1 = 1.5, f(x_1) = -1.125, f'(x_1) = 3.75$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.5 - \frac{-1.125}{3.75}$$

$$= 1.8$$

$f(1.8) = 0.432$. This initial estimate will lead to the root near 2, as shown in the diagram.

Handy hint

If $|f'(x_1)| < 1$, then $\frac{f(x)}{f'(x)}$ will increase in size and Newton's method will move further away from the initial estimate of the root of the equation. If this happens, the initial estimate may not be a good one.

TECHNOLOGY EXPLORATION

Newton's method

If x_0 is an approximation to a root of the function $f(x)$, then the intersection of the tangent from $(x_0, f(x_0))$ with the x-axis is usually a better approximation.

For example, consider the function $f(x) = 0.3x^3 - 0.2x^2 - 2x + 2$:

1. In GeoGebra, enter *f(x)=0.3*x^3-0.2*x^2-2x+2* in the input bar. Notice that the three roots are between –3 and –1, 2 and 3, and 2 and 3.
2. Use the **Slider** tool to create a slider for the value of x_0. Enter the name as *x_0* and set the slider to have Min = –3, Max = 3 and Increment = 1.
3. Create the point $A(x_0, f(x_0))$ on the curve $f(x)$ by entering *A=(x_0,f(x_0))* in the input bar.
4. Create a tangent at the point A by entering *tangent[A,f(x)]* in the input bar.
5. Find the point of intersection of the tangent with the x-axis by using the **Intersect Two Objects** tool . Click on the tangent and then on the x-axis.
6. In the input bar, enter the formula for the approximation for the root, *x_1=x_0-f(x_0)/f'(x_0)*. What do you notice about the value x_1 and the point of intersection of the tangent with the x-axis?
7. Move the slider for x_0 to see how the approximation varies with a different initial value.

EXERCISE 8.3 NEWTON'S METHOD

1 Use two applications of Newton's method to solve each equation, given that:

(a) $x^3 - 3x - 20 = 0$ has a root near $x = 3$
(b) $x^4 + x^2 - 80 = 0$ has a root near $x = 3$
(c) $10x = 2\sin x + 5$ has a root near $x = 0.5$
(d) $\log_e x = \cos x$ has a root near $x = 1$
(e) $\log_e(x + 2) = e^x - 2$ has a root near $x = 1$.

2 Using Newton's method, a root of $x^3 + 3x - 7 = 0$ correct to one decimal place is:

A $x = 1.6$ **B** $x = 1.5$ **C** $x = 1.4$ **D** $x = 1.3$

3 Use Newton's method to find the approximate value of:

(a) $\sqrt[3]{10}$ **(b)** $\sqrt{5}$ **(c)** $\sqrt{15}$ **(d)** $\sqrt[3]{28}$ **(e)** $\sqrt[5]{34}$ **(f)** $4^{\frac{2}{3}}$

4 The equation $x^3 - x^2 + x - 2 = 0$ has a root that is approximately 1. Find a better approximation to two decimal places using the 'halving the interval' method and using Newton's method.

5 The equation $x^3 - 8x + 1 = 0$ has a root that is approximately –3. Find a better approximation to two decimal places using the 'halving the interval' method and using Newton's method.

6 **(a)** Show that there is a solution to the equation $x + e^{-x^2} = 0$ between $x = -0.7$ and $x = -0.6$.
(b) By halving the interval once, find the solution correct to two decimal places.
(c) Use one application of Newton's method to find a better approximation to the root of this equation.

7 The function $f(x) = x - \cos 2x$ has a zero near $x = 0.5$. Use one application of Newton's method to obtain a better approximation to this zero. Give your answer correct to two decimal places.

8 The function $f(x) = \sin x - \frac{x}{3}$ has a zero near $x = 2.2$. Taking $x = 2.2$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Give your answer correct to three decimal places.

CHAPTER REVIEW 8

1 **(a)** The graphs of $y = x$ and $y = \sqrt{\dfrac{1-x}{x}}$ meet at exactly one point, P. Let α be the x-coordinate of P. Explain why α is a root of the equation $x^3 + x - 1 = 0$.

(b) Take 0.6 as a first approximation for α. Use one application of Newton's method to find a second approximation for α.

2 **(a)** Show that the function $g(x) = x^2 - \log_e(x + 2)$ has a zero between 1 and 1.2.

(b) Use the 'halving the interval' method to find an approximation to this zero of $g(x)$ correct to one decimal place.

3 One solution of the equation $2\cos 2x = x - 1$ is close to $x = 0.9$. Use one application of Newton's method to find another approximation to this solution. Give your answer correct to three decimal places.

4 The function $f(x) = \tan x + \log_e x$ has a zero near $x = 2$. Use two applications of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places.

5 A first approximation to the solution of $3\theta - \tan\theta = 0$ is $\theta = 1.2$. Use one application of Newton's method to find a better approximation. Give your answer correct to four decimal places.

6 **(a)** By considering $f(x) = 3\log_e x - x$, show that the curve $y = 3\log_e x$ and the line $y = x$ meet at the points P and Q, where P has an x-coordinate between 1.5 and 2 and Q has an x-coordinate between 4 and 5.

(b) Starting at $x = 4.4$, use one application of Newton's method to find an approximation to the x-coordinate of Q. Give your answer correct to two decimal places.

(c) Use two applications of the 'halving the interval' method to find an approximation to the x-coordinate of P. Give your answer correct to two decimal places.

CHAPTER 9
INTEGRATION TECHNIQUES

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
HE4 uses the relationship between functions, inverse functions and their derivatives
HE6 determines integrals by reduction to a standard form through a given substitution
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

9.1 INDEFINITE INTEGRALS AND SUBSTITUTION

Some integrals can only be solved using particular substitutions for the variables. In this Mathematics Extension 1 course, any substitutions needed to find an integral are given. In the Mathematics Extension 2 course, you may need to decide on the substitution needed. (Of course, you can always use your own substitution if it helps you to find an integral, even if no substitution has been given.)

Integration using a substitution can be considered as the converse of the method of differentiating a composite function—it is like using the chain rule backwards.

The aim of a substitution is to transform an integral into one that involves a standard result, e.g. $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$. Variable substitution works as follows:

Let $y = \int f(u)du$ where $u = g(x)$

$\therefore \quad \frac{dy}{du} = f(u)$

But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= f(u) \times \frac{du}{dx}$

$\therefore \quad y = \int f(u) \times \frac{du}{dx}dx$

$$\int f(u) \times \frac{du}{dx}dx = \int f(u)du$$

This 'backwards' form of the chain rule is convenient when the substitution of $u = g(x)$ will let us express a function being integrated as the product of $\frac{du}{dx}$ and a function of u. For example:

- If $f(x) = 3x^2(x^3 - 1)^4$ then we can substitute $u = x^3 - 1$. As $\frac{du}{dx} = 3x^2$, we can write $2x^2$ as $\frac{2}{3} \times 3x^2$ so that $f(x)$ is written: $f(x) = \frac{2}{3}(x^3 - 1)^4 \times (3x^2)$
  $= \frac{2}{3}u^4 \frac{du}{dx}$ where $u = x^3 - 1$
- If $f(x) = x\sqrt{1 + x^2}$, we can see that $2x$ is the derivative of $1 + x^2$, so if we make the substitution $u = 1 + x^2$ and write x as $\frac{1}{2}(2x)$ then $f(x) = \frac{1}{2}u^{\frac{1}{2}} \frac{du}{dx}$ where $u = 1 + x^2$ and $\frac{du}{dx} = 2x$.
- If $f(x) = \frac{x+1}{(x^2 + 2x)^3}$, we can see that $2x + 2$ is the derivative of $x^2 + 2x$, so if we make the substitution $u = x^2 + 2x$ and write $x + 1$ as $\frac{1}{2}(2x + 2)$ then $f(x) = \frac{1}{2}u^{-3} \frac{du}{dx}$.

- If $f(x) = x\sqrt{1-x}$ then we can make the substitution $u = 1 - x$. As $x = 1 - u$, $\frac{du}{dx} = -1$, so:

$$\begin{aligned} f(x) &= -x\sqrt{1-x} \times (-1) \\ &= -(1-u)u^{\frac{1}{2}} \times \frac{du}{dx} \\ &= -\left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right) \times \frac{du}{dx} \\ &= \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) \times \frac{du}{dx} \\ &= f(u)\frac{du}{dx} \end{aligned}$$

Example 1

Find: **(a)** $\int 3x^2\left(x^3-1\right)^4 dx$ using the substitution $u = x^3 - 1$

(b) $\int x\sqrt{1+x^2}\, dx$ using the substitution $u = 1 + x^2$

(c) $\int \frac{x+1}{\left(x^2+2x\right)^3} dx$ using the substitution $u = x^2 + 2x$.

Solution

(a) $u = x^3 - 1$, $\frac{du}{dx} = 3x^2$

$$\begin{aligned} \int 3x^2\left(x^3-1\right)^4 dx &= \int u^4 \times \frac{du}{dx} dx \\ &= \int u^4\, du \\ &= \frac{1}{5}u^5 + C \\ &= \frac{1}{5}\left(x^3-1\right)^5 + C \end{aligned}$$

(b) $u = 1 + x^2$, $\frac{du}{dx} = 2x$

$$\begin{aligned} \int x\sqrt{1+x^2}\, dx &= \frac{1}{2}\int 2x\sqrt{1+x^2}\, dx \\ &= \frac{1}{2}\int u^{\frac{1}{2}} \times \frac{du}{dx} dx \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2} \times \tfrac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}} + C \end{aligned}$$

(c) $u = x^2 + 2x$, $\frac{du}{dx} = 2x + 2$

$$\begin{aligned} \int \frac{x+1}{\left(x^2+2x\right)^3} dx &= \frac{1}{2}\int (2x+2)\left(x^2+2x\right)^{-3} dx \\ &= \frac{1}{2}\int u^{-3} \times \frac{du}{dx} dx \\ &= \frac{1}{2}\int u^{-3}\, du \\ &= \frac{1}{2} \times \left(\frac{1}{-2}\right)u^{-2} + C \\ &= \frac{-1}{4\left(x^2+2x\right)^2} + C \end{aligned}$$

A quick way to check your answer is to differentiate it to see that it gives the integrand.

Example 2

Find: (a) $\int x\sqrt{1-x}\,dx$ using the substitution $u = 1 - x$ (b) $\int \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(c) $\int (3x-5)^4\,dx$ using the substitution $u = 3x - 5$.

Solution

(a) $u = 1 - x, \frac{du}{dx} = -1, x = 1 - u$

$$\begin{aligned}\int x\sqrt{1-x}\,dx &= \int (1-u)u^{\frac{1}{2}} \times \frac{du}{dx}\,dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)(-1)\,dx \\ &= -\int \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du \qquad \left(\text{note that } du = (-1)\,dx\right) \\ &= -\left(\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C \\ &= \frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + C\end{aligned}$$

(b) $u = 1 + t, \frac{du}{dt} = 1, t = u - 1$

$$\begin{aligned}\int \frac{t}{\sqrt{1+t}}\,dt &= \int \frac{u-1}{\sqrt{u}} \times \frac{du}{dt} \times dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)(1)\,dt \\ &= \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du \qquad \left(\text{note that } du = (1)\,dt\right) \\ &= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C\end{aligned}$$

(c) $u = 3x - 5, \frac{du}{dx} = 3$

$$\begin{aligned}\int (3x-5)^4\,dx &= \frac{1}{3}\int 3(3x-5)^4\,dx \\ &= \frac{1}{3}\int u^4 \times \frac{du}{dx}\,dx \\ &= \frac{1}{3}\int u^4\,du \\ &= \frac{1}{3} \times \frac{1}{5}u^5 + C \\ &= \frac{1}{15}(3x-5)^5 + C\end{aligned}$$

EXERCISE 9.1 INDEFINITE INTEGRALS AND SUBSTITUTION

1 Find: (a) $\int 2x\left(x^2-1\right)^4\,dx$ using the substitution $u = x^2 - 1$

(b) $\int 3x^2\left(x^3+4\right)^3\,dx$ using the substitution $u = x^3 + 4$

(c) $\int x^2\sqrt{x^3+1}\,dx$ using the substitution $u = x^3 + 1$.

2 Find: (a) $\int (2t+1)^3\,dt$ using the substitution $u = 2t + 1$

(b) $\int \frac{2x}{\sqrt{x^2-4}}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int (2x+1)\left(x^2+x+2\right)^5\,dx$ using the substitution $u = x^2 + x + 2$.

3 Using $u = 2x + 3$, $\int \frac{dx}{(2x+3)^3} = \ldots$

A $4(2x+3)^4 + C$ B $\frac{1}{4(2x+3)^4} + C$ C $-4(2x+3)^2 + C$ D $\frac{-1}{4(2x+3)^2} + C$

4 Find: **(a)** $\int (3-2x)^6\,dx$ using the substitution $u = 3-2x$

(b) $\int \dfrac{3x+1}{\left(3x^2+2x+5\right)^2}\,dx$ using the substitution $u = 3x^2+2x+5$

(c) $\int \left(x^2-2x\right)\left(x^3-3x^2+1\right)^4\,dx$ using the substitution $u = x^3-3x^2+1$.

5 Find: **(a)** $\int 3x^2\left(x^3+1\right)^4\,dx$ using the substitution $u = x^3+1$

(b) $\int \dfrac{t}{\sqrt{1-t^2}}\,dt$ using the substitution $u = 1-t^2$

(c) $\int (3x-5)^{\frac{2}{3}}\,dx$ using the substitution $u = 3x-5$.

6 Find: **(a)** $\int 2t\sqrt{1-t^2}\,dt$ using the substitution $u = 1-t^3$

(b) $\int x\sqrt{a^2-x^2}\,dx$ using the substitution $u = a^2-x^2$

(c) $\int z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2+1$.

7 Find: **(a)** $\int y\sqrt{y+1}\,dy$ using the substitution $u = y+1$

(b) $\int \dfrac{x}{(x-1)^3}\,dx$ using the substitution $u = x-1$

(c) $\int \dfrac{x}{\sqrt{2x-1}}\,dx$ using the substitution $u = 2x-1$.

8 **(a)** Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = 1+x^3$.

(b) Find $\int x^2\sqrt{1+x^3}\,dx$ using the substitution $u = x^3$.

(c) Why is the substitution in **(a)** easier to use than the substitution in **(b)**?

9 $\dfrac{dy}{dx} = x\sqrt{x^2-4}$ and $y = 2$ at $x = \sqrt{5}$. Use the substitution $u = x^2-4$ to find y in terms of x.

10 If $f'(x) = \dfrac{3x}{\sqrt{x^2+1}}$ for all x and $f(0) = 2$, use the substitution $u = x^2+1$ to find $f(x)$.

11 $\dfrac{dx}{dt} = \dfrac{t-1}{\sqrt{t^2-2t+4}}$ and $x = 10$ when $t = 0$. Use the substitution $u = t^2-2t+4$ to find x in terms of t.

12 Given that $\dfrac{dr}{d\theta} = \dfrac{3}{(1-r)^4}$ and $r(0) = 0$, use the substitution $u = 1-r$ to find r in terms of θ.

13 At any point where $x > \frac{1}{2}$, the gradient of a curve is given by $\dfrac{dy}{dx} = \sqrt{2x-1}$. If the point $(2.5, 9)$ is on the curve, use the substitution $u = 2x-1$ to find the equation of the curve.

14 Given that $\dfrac{dt}{dx} = \dfrac{1}{2(4-x)^2}$ and $x = 0$ when $t = 0$, use the substitution $u = 4-x$ to find x in terms of t.

9.2 DEFINITE INTEGRALS AND SUBSTITUTION

When using a substitution to evaluate a definite integral you must take care with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Example 3

Evaluate: (a) $\int_1^2 2x\sqrt{x^2-1}\,dx$ using the substitution $u = x^2 - 1$

(b) $\int_{-5}^{3} x\sqrt{4-x}\,dx$ using the substitution $u = 4 - x$

(c) $\int_0^1 x^2\left(x^3+1\right)^4 dx$ using the substitution $u = x^3 + 1$.

Solution

(a) $u = x^2 - 1, \frac{du}{dx} = 2x$

Limits: for $x = 1$, $u = 1^2 - 1 = 0$
for $x = 2$, $u = 2^2 - 1 = 3$

$$\int_1^2 2x\sqrt{x^2-1}\,dx = \int_0^3 \sqrt{u} \times \frac{du}{dx}\,dx$$
$$= \int_0^3 u^{\frac{1}{2}}\,du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^3$$
$$= \frac{2}{3}\left(3^{\frac{3}{2}} - 0\right) = 2\sqrt{3}$$

(b) $u = 4 - x, \frac{du}{dx} = -1$

$x = 4 - u$, so $x\sqrt{4-x} = (4-u)\sqrt{u}$
or $-x\sqrt{4-x} = (u-4)\sqrt{u}$

Limits: for $x = -5$, $u = 4 + 5 = 9$
for $x = 3$, $u = 4 - 3 = 1$

$$\int_{-5}^{3} x\sqrt{4-x}\,dx = \int_9^1 (u-4)\sqrt{u} \times \frac{du}{dx}\,dx$$
$$= \int_9^1 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$$
$$= \left[\frac{2}{5}u^{\frac{5}{2}} - 4 \times \frac{2}{3}u^{\frac{3}{2}}\right]_9^1$$
$$= \left(\frac{2}{5} - \frac{8}{3}\right) - \left(\frac{2}{5} \times 9^{\frac{5}{2}} - \frac{8}{3} \times 9^{\frac{3}{2}}\right)$$
$$= \frac{2}{5} - \frac{8}{3} - \frac{2}{5} \times 3^5 + \frac{8}{3} \times 3^3$$
$$= -\frac{412}{15}$$

(c) $u = x^3 + 1, \frac{du}{dx} = 3x^2$

Limits: for $x = 0$, $u = 1$
for $x = 1$, $u = 1^3 + 1 = 2$

$$\int_0^1 x^2\left(x^3+1\right)^4 dx = \frac{1}{3}\int_1^2 u^4 \times \frac{du}{dx}\,dx$$
$$= \frac{1}{3}\int_1^2 u^4\,du$$
$$= \frac{1}{3}\left[\frac{1}{5}u^5\right]_1^2$$
$$= \frac{1}{15}\left[u^5\right]_1^2$$
$$= \frac{1}{15}(32-1) = \frac{31}{15}$$

Useful result

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ If you reverse the limits of integration, you change the sign of the integral.

In Example 3(b) the integral $\int_9^1 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$ could have been written as $-\int_1^9 \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$.

Check that this integral gives the same answer.

EXERCISE 9.2 DEFINITE INTEGRALS AND SUBSTITUTION

1 Evaluate: (a) $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $u = 1 - x^2$

(b) $\int_{-1}^{2} x\sqrt{2-x}\,dx$ using the substitution $u = 2 - x$

(c) $\int_0^2 \frac{2x}{\sqrt{x^2+1}}\,dx$ using the substitution $u = x^2 + 1$.

2 Evaluate: **(a)** $\int_{0.5}^{1.5} \frac{2-2x}{\sqrt{2x-x^2}}\,dx$ using the substitution $u = 2x - x^2$

(b) $\int_{2}^{3} x\sqrt{x-1}\,dx$ using the substitution $u = x - 1$

(c) $\int_{-0.5}^{0.5} \frac{x}{\sqrt{1-x^2}}\,dx$ using the substitution $u = 1 - x^2$.

3 Using the substitution $u = 1 + x^2$, $\int_{0}^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx =$

A 0.5 **B** 1 **C** 2 **D** $\sqrt{3}-1$

4 Evaluate: **(a)** $\int_{3}^{4} (2x-3)\left(x^2-3x+2\right)^2\,dx$ using the substitution $u = x^2 - 3x + 2$

(b) $\int_{0}^{2} \frac{x}{\left(x^2+2\right)^2}\,dx$ using the substitution $u = x^2 + 2$

(c) $\int_{1.5}^{4} \sqrt{2t+1}\,dt$ using the substitution $u = 2t + 1$.

5 Evaluate: **(a)** $\int_{0}^{1} \frac{t}{\sqrt{1+t}}\,dt$ using the substitution $u = 1 + t$

(b) $\int_{0}^{1} 3x^2\left(x^3-1\right)^4\,dx$ using the substitution $u = x^3 - 1$

(c) $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$ using the substitution $u = a^2 - x^2$.

6 Evaluate: **(a)** $\int_{0}^{\sqrt{2}} z\sqrt[3]{z^2+1}\,dz$ using the substitution $u = z^2 + 1$

(b) $\int_{2}^{3} x\sqrt{x^2-4}\,dx$ using the substitution $u = x^2 - 4$

(c) $\int_{1}^{5} \sqrt{2x-1}\,dx$ using the substitution $u = 2x - 1$.

7 Evaluate: **(a)** $\int_{0}^{3} y\sqrt{y+1}\,dy$ using the substitution $u = y + 1$

(b) $\int_{0}^{4} x\sqrt{16-x^2}\,dx$ using the substitution $u = 16 - x^2$

(c) $\int_{-1}^{1} x(3x-1)^4\,dx$ using the substitution $u = 3x - 1$.

8 **(a)** Evaluate $\int_{1}^{2} \frac{2x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = 4 - x^2$.

(b) Evaluate $\int_{1}^{2} \frac{2x}{\sqrt{4-x^2}}\,dx$ using the substitution $u = x^2$.

(c) Why is the substitution in **(a)** easier to use than the substitution in **(b)**?

9 Using the substitution $u = 1 - x^2$, calculate the area bounded by the curve $y = x\sqrt{1-x^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

10 Find the area of the region bounded by the curve $y = \frac{x}{\sqrt{x^2-1}}$, the x-axis and the lines $x = \sqrt{2}$ and $x = \sqrt{5}$.

11 Sketch the curve defined by $y^2 = x^2(1-x)$ and find the area of the loop. Use the substitution $u = 1 - x$.

12 If $f(x) = x^2(x-3)$, calculate the area between the x-axis, the lines $x = 4$ and $x = 6$ and the curve $y = \sqrt{f(x)}$. Use the substitution $u = x - 3$.

13 Calculate the area of the region bounded by the curve $y = \dfrac{x}{\sqrt{x^2+4}}$, the x-axis and the lines $x = 0$ and $x = 2\sqrt{3}$. Use the substitution $u = x^2 + 4$.

Useful result

You may have noticed that for an integral of the form $\int f'(x)\big(f(x)\big)^n\,dx$, using the substitution $u = f(x)$ gives $\dfrac{du}{dx} = f'(x)$ so that the integral becomes: $\int f'(x)\big(f(x)\big)^n\,dx = \int u^n\,du$

$$= \frac{1}{n+1}u^{n+1} + C = \frac{1}{n+1}\big(f(x)\big)^{n+1} + C$$

If you can recognise that $\int f'(x)\big(f(x)\big)^n\,dx = \dfrac{1}{n+1}\big(f(x)\big)^{n+1} + C$, you can obtain your answer faster. This result will be especially useful in section 9.5 (page 167) when integrating powers of trigonometric functions.

9.3 INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

$\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ cannot be found without using a substitution. However, a useful substitution comes from the formulae for $\cos 2x$. This will generally not be given in a question, but you must be able to use it when necessary.

The important results are as follows: $\cos 2x = \cos^2 x - \sin^2 x$

Using $\cos^2 x + \sin^2 x = 1$: $\cos 2x = 2\cos^2 x - 1$

$\therefore$ $\cos^2 x = \dfrac{1+\cos 2x}{2}$

Thus: $\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$

$$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Similarly: $\cos 2x = 1 - 2\sin^2 x$

$\therefore$ $\sin^2 x = \dfrac{1-\cos 2x}{2}$

Thus: $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

Example 4

(a) Find $3\int \sin^2 x\,dx$. **(b)** Evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx$. **(c)** Find $\int \cos^2 3x\,dx$.

Solution

(a) $3\int \sin^2 x\,dx = 3 \times \frac{1}{2}\int (1-\cos 2x)\,dx$

$$= \frac{3}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{3x}{2} - \frac{3\sin 2x}{4} + C$$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos^2 x\,dx = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} (1+\cos 2x)\,dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} - \left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} + \frac{\pi}{2} - 0\right)$

$= \frac{1}{2}\left(\frac{3\pi}{4} + \frac{1}{2}\right) = \frac{1}{8}(3\pi + 2)$

(c) $\int \cos^2 3x\,dx = \frac{1}{2}\int (1+\cos 6x)\,dx$

$= \frac{1}{2}\left(x + \frac{\sin 6x}{6}\right) + C$

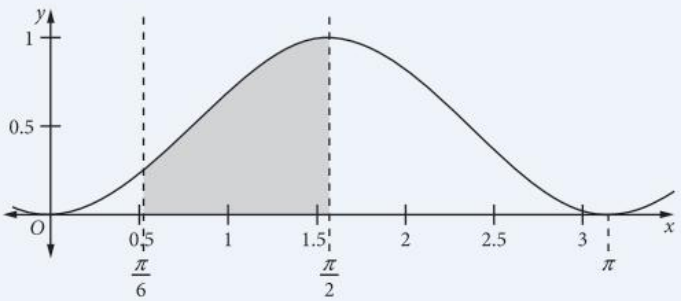
$= \frac{x}{2} + \frac{\sin 6x}{12} + C$

Example 5

Find the area bounded by the curve $y = \sin^2 x$, the x-axis and the ordinates at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Solution

Sketch the function.



$\text{Area} = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x\,dx$

$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{2}\sin\pi - \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0 - \frac{\pi}{6} + \frac{1}{2}\times\frac{\sqrt{3}}{2}\right)$

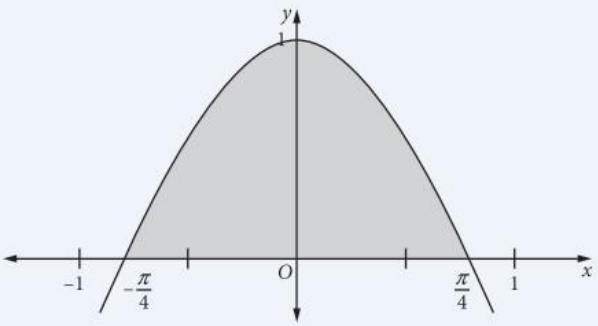
$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) = \frac{4\pi + 3\sqrt{3}}{24}$ square units

Example 6

The region under the curve $y = \cos 2x$ between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the volume of the solid of revolution formed.

Solution

Sketch the function.



$\text{Volume} = \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} y^2\,dx$

$= \pi\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2x\,dx$

$= \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 4x)\,dx$

$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$= \frac{\pi}{2}\left[\frac{\pi}{4} + 0 - \left(-\frac{\pi}{4} + 0\right)\right] = \frac{\pi^2}{4}$ cubic units

EXERCISE 9.3 INTEGRATION OF $\sin^2 x$ AND $\cos^2 x$

1 Find: **(a)** $\int 2\cos^2 x\,dx$ **(b)** $\int 2\sin^2 x\,dx$ **(c)** $\int \sin^2 \frac{x}{2}\,dx$

(d) $\int 2\cos^2 \frac{x}{2}\,dx$ **(e)** $\int \sin^2 3x\,dx$ **(f)** $\int \cos^2 4x\,dx$

2 Evaluate: **(a)** $\int_0^{\frac{\pi}{2}} 2\sin^2 x\,dx$ **(b)** $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx$ **(c)** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\,dx$

(d) $\int_{\frac{\pi}{2}}^{2\pi} \cos^2 \frac{x}{3}\,dx$ **(e)** $4\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \sin^2 4x\,dx$ **(f)** $2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 3x\,dx$

3 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. The volume of the solid of revolution formed is given by:

A $\int_0^{\pi} \sin^2 x\,dx$ **B** $\pi\int_0^{\pi} \sin x\,dx$ **C** $\pi\int_0^{\pi} \sin^2 x\,dx$ **D** $\pi\int_0^{\pi} \sin x^2\,dx$

4 The region under the curve $y = \cos x$ between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

5 The region under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis. Find the volume of the solid of revolution formed.

6 The gradient at the point (x, y) of the function $y = f(x)$ is given by $\frac{dy}{dx} = \sin^2 x$. The curve passes through the point $(0, -2)$. Find y where $x = \frac{\pi}{4}$.

7 If $\frac{d^2y}{dx^2} = 2\cos^2 x$ and when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 0$, $y = 0$, then find y in terms of x.

8 The region bounded by the curves $y = \cos 2x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

9 The region bounded by the curves $y = \sin 3x$, the x-axis and the ordinate $x = \frac{\pi}{12}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

10 The region bounded by the curves $y = \sin x$, $y = \cos x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

(a) Find the point of intersection of the two curves.

(b) Calculate the exact value of the volume of the solid of revolution formed.

11 The region bounded by $y = \sin x$, $y = \cos x$ and the y-axis between $x = 0$ and $x = \frac{\pi}{4}$ is rotated about the x-axis. Calculate the exact value of the volume of the solid of revolution formed.

12 **(a)** Sketch the region bounded by the curve $y = \sin x + \cos x$ and the coordinate axes in the first quadrant. Show the intercepts on the axes.

(b) Calculate the area of the region in part **(a)**.

(c) The region in part **(a)** is rotated about the x-axis to form a solid of revolution. Calculate the volume of the solid formed.

9.4 INTEGRALS OF THE TYPE $\int f'(x)(f(x))^n\,dx$

You have seen this type of integral before (e.g. with integrals like $\int 2x(x^2+1)^3\,dx = \frac{1}{4}(x^2+1)^4 + C$). We will now look at trigonometric integrals such as $\int \cos x \sin^2 x\,dx$, i.e. where $f(x) = \sin x$ and $f'(x) = \cos x$. In trigonometric integrals of this type the substitution will not always be given.

Example 7

(a) Find $\int \cos x \sin^2 x \, dx$. **(b)** Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$.

Solution

(a) $\int \cos x \sin^2 x \, dx$: Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$$\begin{aligned}\int \cos x \sin^2 x \, dx &= \int u^2 \times \frac{du}{dx} \times dx \\ &= \int u^2 \, du \\ &= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3 x + C\end{aligned}$$

If you can recognise that $\int \cos x \sin^2 x \, dx$ is of the form $\int f'(x)\big(f(x)\big)^2 \, dx = \frac{1}{3}\big(f(x)\big)^3 + C$, then you can write the answer immediately as $\int \cos x \sin^2 x \, dx = \frac{1}{3}\sin^3 x + C$.

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx$: Let $u = \cos x$, $\frac{du}{dx} = -\sin x$ For $x = \frac{\pi}{3}$, $u = \frac{1}{2}$; for $x = \frac{\pi}{2}$, $u = 0$

$$\begin{aligned}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx &= \int_{\frac{1}{2}}^{0} -u^2 \times \frac{du}{dx} \times du \\ &= \int_{\frac{1}{2}}^{0} -u^2 \, du \\ &= \left[-\tfrac{1}{3}u^3\right]_{\frac{1}{2}}^{0} = 0 + \tfrac{1}{3} \times \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{24}\end{aligned}$$

Note: The result $\int_{\frac{1}{2}}^{0} -u^2 \, du = \int_{0}^{\frac{1}{2}} u^2 \, du$ (reversing the limits of the integral and changing the sign of the integrand) could have been used: $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx = -\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (-\sin x)\cos^2 x \, dx$

$$\begin{aligned}&= -\left[\tfrac{1}{3}\cos^3 x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= -\tfrac{1}{3}\left(0^3 - \left(\tfrac{1}{2}\right)^3\right) = \tfrac{1}{24}\end{aligned}$$

Substitution simplified

If we treat $\frac{du}{dx}$ as a fraction then $\frac{du}{dx} \times dx = du$.

If $u = \sin x$ then $\frac{du}{dx} = \cos x$ can be written as $du = \cos x \, dx$.

We are really replacing $f'(x)\,dx$ by du. This makes the algebra involved much simpler.

Example 8

Find $\int \tan^2 x \sec^2 x \, dx$.

Solution

Let $u = \tan x$, $du = \sec^2 x \, dx$: $\int \tan^2 x \sec^2 x \, dx = \int u^2 \, du$

$$= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3 x + C$$

EXERCISE 9.4 INTEGRALS OF THE TYPE $\int f'(x)(f(x))^n dx$

1 Find: (a) $\int \sin x \cos^2 x\, dx$ (b) $\int \tan x \sec^2 x\, dx$ (c) $\int \sin x \cos^3 x\, dx$

(d) $\int \cos x \sin^4 x\, dx$ (e) $\int (1+\cos 2x)\sin x\, dx$ (f) $\int \sin x \cos x\, dx$

2 Evaluate: (a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos^3 x\, dx$ (b) $\int_{-\pi}^{\pi} \sin^3 x \cos x\, dx$ (c) $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

(d) $\int_{\pi}^{\frac{3\pi}{2}} \sin x \cos x\, dx$ (e) $\int_0^{\pi} 2\sin\theta \cos^2\theta\, d\theta$ (f) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(x-\frac{\pi}{4}\right) dx$

9.5 INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTION

If an expression of the form $a^2 - x^2$ occurs in the integrand, the standard trigonometric substitutions $x = a\sin\theta$ or $x = a\cos\theta$ will help to find the integral. If an expression of the form $a^2 + x^2$ occurs in the integrand then the standard substitution is $x = a\tan\theta$.

Note: In this course, if these substitutions are needed then they should be given in the question.

Example 9

Evaluate $\int_0^2 \sqrt{4-x^2}\, dx$ using the substitution $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution

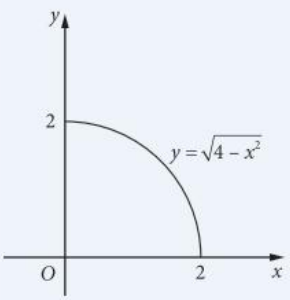
$x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $\frac{dx}{d\theta} = 2\cos\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = |2\cos\theta| = 2\cos\theta$ because $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\therefore \cos\theta \ge 0$

Limit for $x = 2$: $2 = 2\sin\theta$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

Limit for $x = 0$: $0 = 2\sin\theta$, $\sin\theta = 0$, $\theta = 0$

$$\int_0^2 \sqrt{4-x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{4-x^2} \times \frac{dx}{d\theta} \times d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\cos\theta \times 2\cos\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$
$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$
$$= 2\left[\frac{\pi}{2} + \frac{1}{2}\sin\pi - \left(0 + \frac{1}{2}\sin 0\right)\right]$$
$$= 2 \times \frac{\pi}{2} = \pi$$



Geometrically, the graph of $y = \sqrt{4-x^2}$ between $x = 0$ and $x = 2$ is the quarter of a circle with equation $x^2 + y^2 = 4$ (in the first quadrant). This integration thus proves the formula for the area of a circle: $A = \pi r^2$.

Area $= \frac{1}{4}\pi r^2 = \frac{1}{4} \times \pi \times 4 = \pi$ when $r = 2$.

Note: If the restriction on θ had been different, e.g. $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$, then $\cos\theta \ge 0$ would not be true and we would not be able to use $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$. Instead, we may have needed $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = -2\cos\theta$.

Remember: Always take care with trigonometric substitutions and check the sign of the function for the given domain.

Example 10

Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using the substitution $x = \tan\theta$.

Solution

Method 1

$x = \tan\theta$, $\frac{dx}{d\theta} = \sec^2\theta$: $\quad \frac{1}{1+x^2} = \frac{1}{1+\tan^2\theta} = \frac{1}{\sec^2\theta}$

Limit for $x = 1$: $\quad 1 = \tan\theta$, $\theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta$, $\theta = 0$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+x^2} \times \frac{dx}{d\theta} \times d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta$$

$$= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

Method 2

$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

Limit for $x = 1$: $\quad 1 = \tan\theta$, $\theta = \frac{\pi}{4}$

Limit for $x = 0$: $\quad 0 = \tan\theta$, $\theta = 0$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\tan^2\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} \times \sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta$$

$$= [\theta]_0^{\frac{\pi}{4}} = \frac{\pi}{4}$$

Example 11

Find $\int \frac{1-\sin x}{x+\cos x}\,dx$ using the substitution $u = x + \cos x$.

Solution

$u = x + \cos x$, $du = (1 - \sin x)\,dx$: $\quad \int \frac{1-\sin x}{x+\cos x}\,dx = \int \frac{du}{u}$

$$= \ln|u| + C$$

$$= \ln|x + \cos x| + C$$

This example involves an indefinite integral, so we must remember to re-substitute for u at the end to make the answer a function of x.

EXERCISE 9.5 INTEGRALS INVOLVING TRIGONOMETRIC SUBSTITUTION

1 Evaluate $\int_{-4}^{4} \sqrt{16-x^2}\,dx$ using the substitution $x = 4\sin\theta$. What shape have you just found the area of?

2 Evaluate $\int_0^1 x\sqrt{1-x^2}\,dx$ using the substitution $x = \sin\theta$.

3 Evaluate $\int_0^2 \frac{dx}{4+x^2}$ using the substitution $x = 2\tan\theta$.

4 Evaluate $\int_0^{0.5} \frac{dx}{(1-x^2)^{\frac{3}{2}}}$ using the substitution $x = \sin\theta$.

5 Evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin^2 x \cos^2 x\,dx$ using $\sin x \cos x = \frac{1}{2}\sin 2x$.

6 Evaluate $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$ using the substitution $x = \cos\theta$.

7 Evaluate $\int_{\frac{1}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}\,dx$ using the substitution $x = \cos\theta$.

8 Evaluate $\int_{-\frac{1}{2}}^{0} \frac{dx}{1+4x^2}$ using the substitution $x = \frac{1}{2}\tan\theta$.

9 Evaluate $\int_{0}^{4} x\sqrt{16-x^2}\,dx$ using the substitution $x = 4\sin\theta$.

10 Evaluate $\int_{0}^{\sqrt{3}} \frac{1}{9+x^2}\,dx$ using the substitution $x = 3\tan\theta$.

11 Evaluate $\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x}{(\tan x+3)^4}\,dx$ using the substitution $u = \tan x + 3$.

12 Use the substitution $u = \tan x$ to find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$.

13 Use the substitution $u = \cos x + \sin x$ to find $\int \frac{\sin x - \cos x}{\sin x + \cos x}\,dx$.

14 **(a)** By writing $\sec x = \frac{1}{\cos x}$, show that $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(b) Using the substitution $u = 1 + \sec x$, find $\int \frac{\sec x \tan x}{1+\sec x}\,dx$.

Summary of integration techniques

- $\int f(u) \times \frac{du}{dx} \times dx = \int f(u)\,du$
- $\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + C$
- $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$
- $\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$

CHAPTER REVIEW 9

1 Using the substitution $u = \sqrt{x}$, find $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

2 **(a)** Use the substitution $u = 1 - x$ to evaluate $\int_0^1 2x\sqrt{1-x}\,dx$.

(b) Use the substitution $u = x - 3$ to evaluate $\int_3^4 x\sqrt{x-3}\,dx$.

3 Using the substitution $u = x^3 + 1$, or otherwise, evaluate $\int_0^1 x^2 e^{x^3+1}\,dx$.

4 Evaluate $\int_0^{\frac{\pi}{4}} \sin\theta\cos^2\theta\,d\theta$.

5 Use the substitution $u = \log_e x$ to evaluate $\int_e^{e^3} \frac{1}{x(\log_e x)^2}\,dx$.

6 Use the substitution $u = 9 - x^2$ to evaluate $\int_1^2 \frac{2x}{\sqrt{9-x^2}}\,dx$.

7 Find: **(a)** $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx$ **(b)** $\int_0^{\frac{\pi}{6}} \cos^2 2x\,dx$

8 Using the substitution $u = 3x^2 + 1$, or otherwise, find $\int x(3x^2+1)^{\frac{3}{4}}\,dx$.

9 **(a)** Differentiate $e^{2x}(2\sin x - \cos x)$. **(b)** Hence, or otherwise, find $\int e^{2x}\sin x\,dx$.

10 Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$ and the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is rotated about the x-axis.

11 Use the substitution $u = 25 - x^2$ to evaluate $\int_2^3 \frac{2x}{\sqrt{25-x^2}}\,dx$.

12 **(a)** By expanding the left-hand side, show that $\sin(6x + 3x) + \sin(6x - 3x) = 2\sin 6x\cos 3x$.

(b) Hence find $\int \sin 6x\cos 3x\,dx$.

13 Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the x-axis and the line $x = \frac{\pi}{6}$ is rotated about the x-axis.

14 Use the list of standard integrals (see page 284) to evaluate $\int_0^{\frac{\pi}{9}} \sec 3x\tan 3x\,dx$.

15 **(a)** Starting from the identity $\sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$ and using the double-angle formulae, prove the identity $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(b) Hence find $\int (3\sin\theta - 4\sin^3\theta)\,d\theta$.

CHAPTER 10
INVERSE FUNCTIONS

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics
HE4 uses the relationship between functions, inverse functions and their derivatives
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

10.1 INVERSE FUNCTIONS

If you do not have GeoGebra, you can perform the following activities using other graphing software or by hand. Be sure to adjust the sizing of your graphs so that both axes have the same scale.

Reflection in the line $y = x$

TECHNOLOGY EXPLORATION

Reflecting points in the line $y = x$

1. In GeoGebra, make sure the scale is the same for both axes, then enter $y=x$ in the input bar to construct the line $y = x$. Enter $(1,3)$ and then enter $(-2,4)$ in the input bar to plot those points.
2. Using the **Reflect Object about Line** tool , click on each point and then on the line to reflect the points in the line $y = x$.
3. Complete the following sentence:
   When reflected in the line $y = x$, the corresponding points to $(1, 3)$ and $(-2, 4)$ are …

You should see that when a point is reflected in the line $y = x$, 'what was x becomes y and what was y becomes x'.

In general, the point (a, b) becomes the point (b, a) on reflection in $y = x$. The midpoint of these two points is $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$, which is on the line $y = x$.

4. Using the **Segment between Two Points** tool , click on each point and then on its reflected point to join them.
5. Complete the following sentence:
   The line $y = x$ is the … of the interval joining a point to its corresponding reflected point.

Reflecting straight lines in the line $y = x$

1. In a new GeoGebra window (using the same scale for both axes), enter $y=x$ in the input bar and then enter $y=2x$ to construct the lines $y = x$ and $y = 2x$.
2. Using the **Reflect Object about Line** tool , click on the $y = 2x$ line and then on the $y = x$ line to reflect $y = 2x$ in the line $y = x$. What is the equation of the reflected line?

The gradient of the line $y = 2x$ is 2 (i.e. for every 1 unit horizontally, the line rises by 2 units vertically). You should see that the corresponding reflected line rises by 1 unit vertically for every 2 units horizontally.

3. Complete the following sentence:
   When reflected in the line $y = x$, the gradient of the line $y = 2x$ and the gradient of its reflected image multiply to give …

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Note that the lines are **not** perpendicular, as that would require $m_1 \times m_2 = -1$.

4 Complete the following sentence:
When reflected in the line $y = x$, the line $y = 2x$ and its image intersect at …

5 Write the equation of a line that does not intersect its image when reflected in the line $y = x$.

6 Write the equation of a line that is its own image when reflected in the line $y = x$.

Reflecting curves in the line $y = x$

1 In a new GeoGebra window (using the same scale for both axes), enter *y=x* in the input bar and then enter *y=x^3* to construct the line $y = x$ and the curve $y = x^3$.

2 Using the **Reflect Object about Line** tool, click on the $y = x^3$ curve and then on the $y = x$ line to reflect $y = x^3$ in the line $y = x$.

3 Complete the following sentence:
The graph of any curve and its image when reflected in the line $y = x$ intersect at …

4 Using the **New Point** tool, click on the curve $y = x^3$ to create any point on the curve $y = x^3$ except (0,0).
Then, using the **Tangents** tool, click on the point and on the curve to construct the tangent to the curve at that point. You can find the tangent's gradient using its equation displayed in GeoGebra's **Algebra** view or by using the **Slope** tool.

5 Using the **Reflect Object about Line** tool, click on the point at which the tangent was constructed and then on the line $y = x$ to reflect the point in the line. Construct the tangent at this reflected point and note its gradient. Do the gradients at the reflected points multiply to give 1?

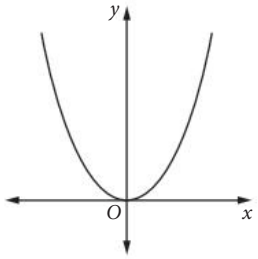
Note again that on reflection, the vertical rise per 1 unit of horizontal change becomes the horizontal change per 1 unit of vertical rise.

6 Repeat steps **4–5** for several different points, or use the **Move** tool to drag the point along the curve $y = x^3$. (You might also investigate the **Record to Spreadsheet** tool to produce a table of values of the gradients of both tangents.)

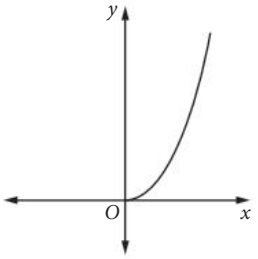
Because 'what was x becomes y and what was y becomes x' on reflection, the equation of the curve obtained when $y = x^3$ is reflected in the line $y = x$ should be $x = y^3$. Making y the subject of that equation, we obtain $y = x^{\frac{1}{3}}$.

7 In GeoGebra, enter *y=x^(1/3)* in the input bar to graph $y = x^{\frac{1}{3}}$. Confirm that this is the reflection of $y = x^3$ in the line $y = x$.

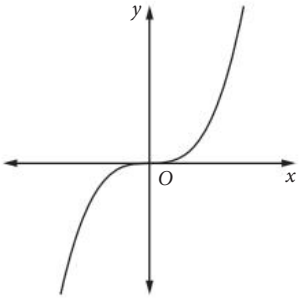
One-to-one functions



$f(x) = x^2$



$g(x) = x^2, x \geq 0$



$h(x) = x^3$

A **one-to-one** function is a function that has no ordered pairs with the same y-coordinate. Any line parallel to the x-axis will intersect the graph of a one-to-one function no more than once. We say that such graphs pass the **horizontal line test**. This means that in the diagrams above, $g(x)$ and $h(x)$ are examples of one-to-one functions.

Note that $f(x)$ is not one-to-one, because a straight line parallel to the x-axis could intersect the graph of $f(x)$ at two points, e.g. $(-a, a^2)$ and (a, a^2). This means that the graph of $f(x)$ is a **two-to-one** function.

An example of a **many-to-one** function is $s(x) = \sin x$. But a function $t(x) = \sin x$, $0 \leq x \leq \frac{\pi}{2}$ is a one-to-one restriction of $s(x)$.

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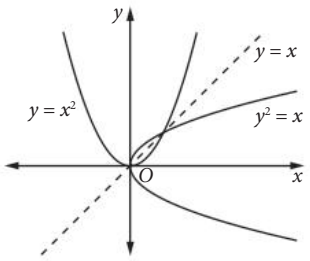
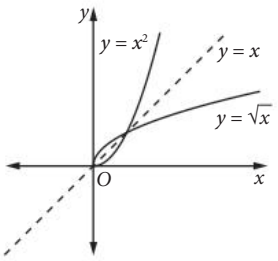
One-to-one functions are significant because of their properties when reflected in the line $y = x$. This is important in the development of inverse functions. For example, note that the reflection of the graph of $y = x^2$ in the line $y = x$ is the parabola $y^2 = x$, which is the graph of a relation and not a function (see the left diagram below).

The reflection of the graph of $g(x)$ in the line $y = x$ is the positive branch only of the parabola $y^2 = x$, i.e. $y = +\sqrt{x}$. This (the diagram on the right) is the graph of a function.

The graph of a function will pass the vertical line test (i.e. no vertical line can cut the graph more than once).

The graph of a one-to-one function will also pass the horizontal line test (i.e. no horizontal line can cut the graph more than once).





If $f(x)$ is a one-to-one function, then the reflection of its graph in the line $y = x$ is the graph of a function, i.e. if a graph passes the horizontal line test then its reflection in the line $y = x$ will pass the vertical line test.

Example 1

Define a restriction on the domain of f, where $f(x) = x^2 - 4x + 5$, so that f is one-to-one.

Solution

$f(x) = (x - 2)^2 + 1$: the graph of f is a parabola (concave up) with vertex at $(2, 1)$.

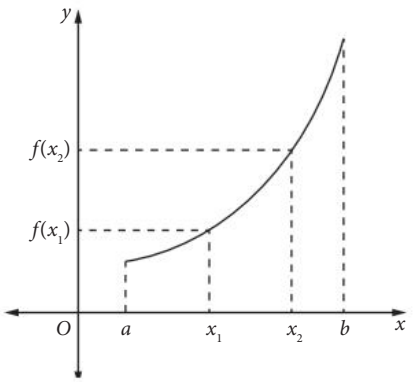
f is thus one-to-one for $x \geq 2$ or for $x \leq 2$.

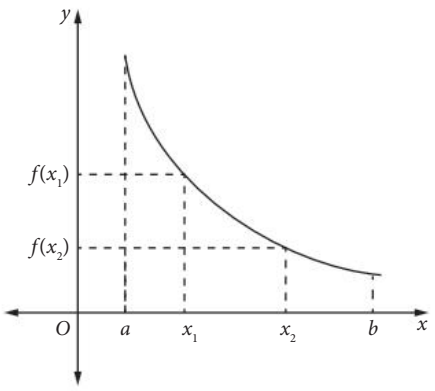
Thus: $f_1(x) = x^2 - 4x + 5, x \geq 2$ and $f_2(x) = x^2 - 4x + 5, x \leq 2$ are both one-to-one restrictions of f.

These restrictions give the largest possible domains for which f is one-to-one. (Of course, any narrower restriction such as $f_3(x) = x^2 - 4x + 5, x \geq 5$ would also give a one-to-one function.)

Increasing and decreasing functions

A function f defined on the interval $a \leq x \leq b$ is said to be **strictly monotonic increasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. This means that the curve always slopes upwards to the right, as shown in the left diagram below.





Similarly, a function is said to be **strictly monotonic decreasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. This means that the curve slopes downwards to the right, as shown in the right diagram above.

Functions that are strictly monotonic increasing or decreasing are clearly one-to-one.

Inverse functions

Consider two operations such that each is the 'reverse' of the other. For example, the **cube** of 2 is 8 and the **cube root** of 8 is 2. The cube function defined by $f(x) = x^3$, $x \in R$ (i.e. 'x is a member of the set of real numbers'), and the

NSM_SB_10.indd 173 18/04/13 10:46 AM

cube root function defined by $g(x)=\sqrt[3]{x}$, $x\in R$ are said to be a pair of **inverse** functions: $f(2)=8$ and $g(8)=2$.

If we cube any real number x and then take the cube root of the answer, the result is x again. Likewise, if we take the cube root of any number x and then cube the answer, the result is x.

$$f(x)=x^3 \qquad g(x)=\sqrt[3]{x}$$
$$f(g(x))=(g(x))^3 \qquad g(f(x))=\sqrt[3]{f(x)}$$
$$=\left(\sqrt[3]{x}\right)^3 \qquad =\sqrt[3]{x^3}$$
$$=x \qquad =x$$

Thus we can define, for any pair of inverse functions:

$$f(g(x))=g(f(x))=x$$

If we denote $g(x)$ as $f^{-1}(x)$ (meaning 'the inverse function of f'):

$$f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x$$

The **square** of 2 is 4, but in the reverse operation 4 has two **square roots**, ± 2. If we wish to return to 2 in the square root operation, we must take the positive square root.

This means that the square function defined by $f(x)=x^2$, $x\geq 0$ and the positive square root function defined by $g(x)=\sqrt{x}$, $x\geq 0$ are a pair of inverse functions. For the inverse to define a function, the functions must be one-to-one.

$$f(x)=x^2 \quad (x\geq 0) \qquad g(x)=\sqrt{x} \quad (x\geq 0)$$
$$f(g(x))=(g(x))^2 \qquad g(f(x))=\sqrt{f(x)}$$
$$=(\sqrt{x})^2 \qquad =\sqrt{x^2}$$
$$=x \qquad =x \text{ if } x\geq 0$$

Thus $f(g(x))=g(f(x))=x$

i.e. $f\left(f^{-1}(x)\right)=f^{-1}(f(x))=x$ where $g(x)=f^{-1}(x)$

Example 2

Find the inverse function f^{-1} for the function f where $f(x)=2x+4$.

Solution

We require f^{-1} such that: $f(f^{-1}(x))=x=f^{-1}(f(x))$

$$\therefore\ 2f^{-1}(x)+4=x$$

i.e. $$f^{-1}(x)=\frac{1}{2}(x-4)$$

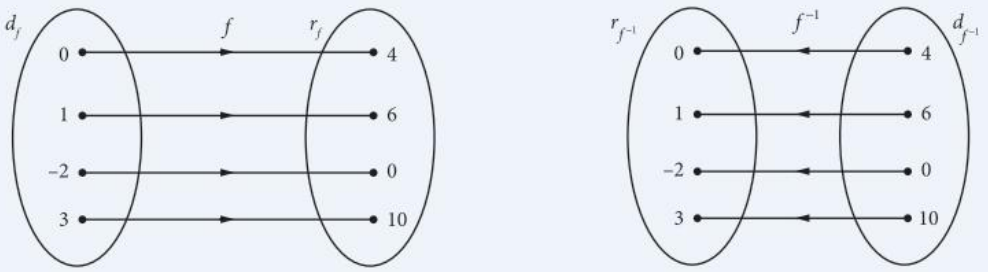
It is now easy to show that: $f^{-1}(f(x))=\frac{1}{2}[(2x+4)-4]$

$$=\frac{1}{2}(2x)$$
$$=x$$

Hence $f^{-1}(x)=\frac{1}{2}(x-4)$ is the rule defining this inverse function. We now need to find the domain and range of f^{-1}.

Let d_f and r_f denote the domain and range respectively of f, and $d_{f^{-1}}$ and $r_{f^{-1}}$ denote the domain and range respectively of f^{-1}.

Consider a few typical elements of d_f, e.g. 0, 1, −2, 3. Under the rule $f(x)=2x+4$:



NSM_SB_10.indd 174 18/04/13 10:46 AM

This correspondence can be represented by the ordered pairs $(0,4)$, $(1,6)$, $(-2,0)$, $(3,10)$.

Every real number in d_f has a corresponding real number image in r_f.

The inverse mapping is obtained by operating f^{-1} on the elements of r_f. This means that these elements can now be regarded as belonging to $d_{f^{-1}}$, i.e. $r_f = d_{f^{-1}}$.

For example, we have: $f^{-1}(4)=\frac{1}{2}(4-4)=0$ $\quad f^{-1}(6)=\frac{1}{2}(6-4)=1$

$f^{-1}(0)=-2$ $\quad f^{-1}(10)=3$

This inverse correspondence can be represented by $(4,0)$, $(6,1)$, $(0,-2)$, $(10,3)$.

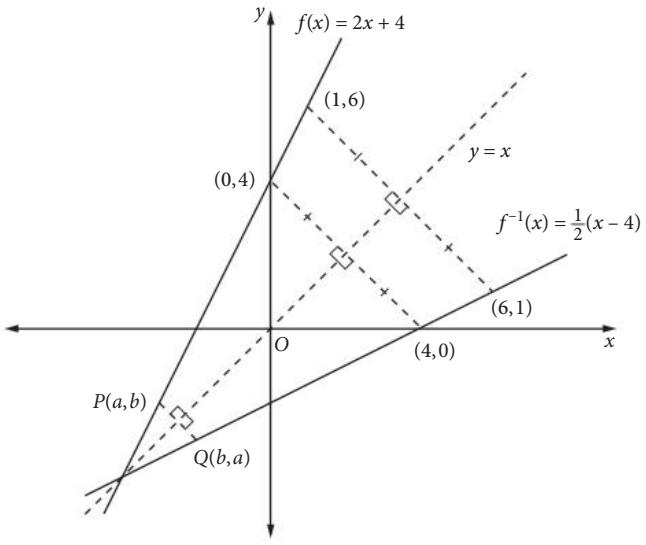
The elements 0, 1, −2, 3 that belong to the range of f^{-1} can be identified as the elements of the domain of f, and so $r_{f^{-1}} = d_f$.

The two diagrams in Example **2**, considered together, show that an element $x \in d_f$ is mapped back to itself by the composite function $f(f^{-1}(x))$. This is as required by the general defining property $f(f^{-1}(x))=x$.

Example **2** illustrates several important general properties of inverse functions.

1. $d_f = r_{f^{-1}}$ and $r_f = d_{f^{-1}}$, so the domain and range are interchanged between a function and its inverse.
2. If (a,b) is a point on the graph of f then (b,a) is a point on the graph of f^{-1}. These are called **corresponding points** on the graphs.
3. Representing $P(a,b)$ and $Q(b,a)$ as corresponding points on the graphs of f and f^{-1}, the coordinates of the midpoint of PQ are $\left[\frac{1}{2}(a+b),\frac{1}{2}(b+a)\right]$, which is a point on the graph of $y=x$ for any a and b. Hence $y=x$ bisects the line segment PQ so that P and Q are mirror images of each other in this line.
4. The graph of f^{-1} is thus seen to be a reflection of the graph of f in the line $y=x$. This is very useful when sketching graphs. It also follows that any intersection between the graphs of f and f^{-1} must be on $y=x$.

The diagram shows the graphs of f and f^{-1} for Example **2**.



The property that (x,y) and (y,x) are on the graphs of f and f^{-1} respectively provides an alternative means of finding f^{-1}. For example, the function f of Example **2** is defined by:

$y=f(x)$ for $x \in d_f$ and $y \in r_f$

i.e. $y=2x+4$ for $x \in R$ and $y \in R$

Hence f^{-1} is defined by:

$x=2y+4$ for $y \in R$ and $x \in R$

(interchanging x and y)

i.e. $y=\frac{1}{2}(x-4)$ for $x \in R$, $y \in R$

Thus: $f^{-1}(x)=\frac{1}{2}(x-4)$ as before.

Existence of inverse functions

Not every function has an inverse function. The graphs of a function and its inverse must be reflections of each other in the line $y=x$. When the graph of a one-to-one function is reflected in $y=x$, the reflection is itself the graph of a function. But when the graph of a many-to-one function is reflected in $y=x$, the reflection is not the graph of a function—it is the graph of a relation (which would fail the vertical line test), so it cannot be called f^{-1}.

A function f will have an inverse function f^{-1} if and only if f is a one-to-one function.

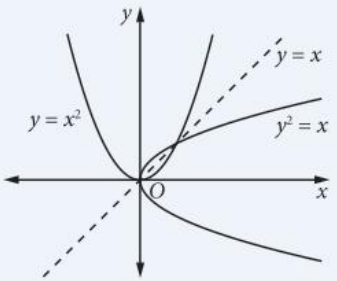
NSM_SB_10.indd 175 18/04/13 10:46 AM

Example 3

Investigate the existence of an inverse function when f is defined by $f(x) = x^2$.

Solution

f is not a one-to-one function. Its graph, the parabola $y = x^2$, produces the parabola with equation $y^2 = x$ when reflected in the line $y = x$. This latter parabola consists of an upper branch $y = \sqrt{x}$ and a lower branch $y = -\sqrt{x}$: it is the graph of a relation with two values of y for each value of x in the domain (except $x = 0$). Therefore f^{-1} does not exist.



Example 4

Investigate the existence of an inverse function when f is defined by $f(x) = x^2$, $x \geq 0$.

Solution

This is a restriction of the function in Example **3**. In this case we have only the right-hand branch of the parabola $y = x^2$, which is the graph of a one-to-one function. Hence f^{-1} exists and can be determined as follows.

Method 1 (using algebraic definition)

We require f^{-1} such that $f^{-1}(f(x)) = x$.

$\therefore \left(f^{-1}(x)\right)^2 = x$

$\therefore \quad f^{-1}(x) = \pm\sqrt{x}$ Which one?

The range of f^{-1} consists of non-negative numbers, so take the positive square root.

$\therefore \quad f^{-1}(x) = \sqrt{x}$

Method 2 (using reflection property)

Let $y = x^2$.

Interchange x and y: $x = y^2$

Make y the subject: $y = \pm\sqrt{x}$ Which one?

The range of f^{-1} consists of non-negative numbers, so take the positive square root.

$\therefore f^{-1}(x) = \sqrt{x}$

Example 5

Consider the function $y = -\dfrac{1}{\sqrt{x}}$.

(a) Explain why the largest possible domain of the function is $x > 0$.

(b) Use calculus to show that the function has no stationary points and is strictly monotonic increasing for all x in its domain.

(c) Show that the graph of the function does not meet the x-axis.

(d) Find y when x is a very small number and also when x is a very large number. Hence state the range of the function.

(e) Explain why an inverse function exists.

(f) Sketch the graph of $y = -\dfrac{1}{\sqrt{x}}$ and its inverse on the same axes.

(g) Find the equation of the inverse function, carefully specifying any restrictions on its domain.

NSM_SB_10.indd 176 18/04/13 10:46 AM

Solution

(a) $x > 0$ because you can't take the square root of a negative number and you can't divide by 0.

(b) $y = -x^{-\frac{1}{2}} \quad \therefore \frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{2}}$ i.e. $\frac{dy}{dx} = \frac{1}{2\sqrt{x^3}}$

$\frac{dy}{dx} \neq 0$, so there are no stationary points.

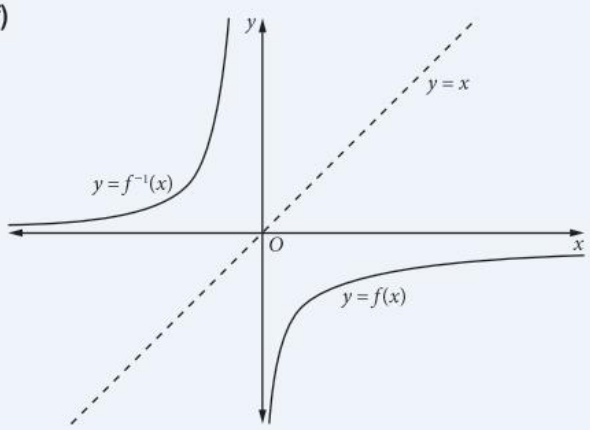
For all x in the domain, $\frac{dy}{dx} > 0$, so the function is increasing.

(c) There are no solutions to $-\frac{1}{\sqrt{x}} = 0$, hence no x-intercepts.

(d) As $x \to 0$, $y \to -\infty$; as $x \to \infty$, $y \to 0$; Range is $y < 0$

(e) Inverse function exists because f is strictly monotonic increasing (one-to-one).

(f)



(g) f is $y = -\frac{1}{\sqrt{x}}$ with domain $x > 0$ and range $y < 0$

$\therefore f^{-1}$ has domain $x < 0$ and range $y > 0$

To find the equation of f^{-1}, interchange x and y:

$x = -\frac{1}{\sqrt{y}}$

Make y the subject: $y = \frac{1}{x^2}$

Remember the restriction on the domain of f^{-1}, i.e. $x < 0$

Hence the inverse function is $f^{-1}(x) = \frac{1}{x^2}$, $x < 0$

Example 6

$f(x)$ is defined by $f(x) = x^2 - 2x - 1$, $x \leq 1$.

(a) Sketch $y = f(x)$ and the inverse function $y = f^{-1}(x)$ on the same diagram.

(b) State the domain and range of f^{-1}.

(c) Find the equation of the inverse function.

(d) Show that f and f^{-1} intersect where $x = \frac{3 - \sqrt{13}}{2}$.

(e) Show that the gradient of the tangent to $y = f(x)$ at the point $(-1, 2)$ is -4. Hence find the coordinates of the point on $y = f^{-1}(x)$ where the gradient is -0.25.

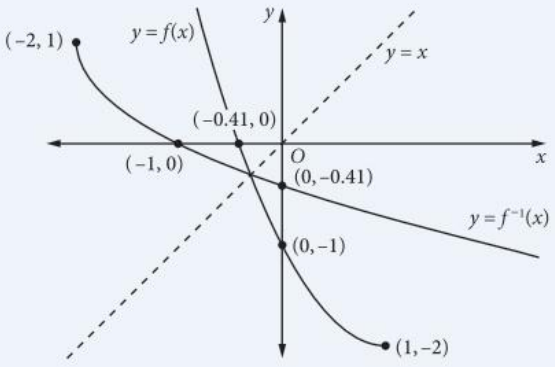
Solution

(a) $y = x^2 - 2x - 1$ is a concave-up parabola with vertex at $(1, -2)$.

It cuts the y-axis at $(0, -1)$ and cuts the x-axis at $(1 + \sqrt{2}, 0)$ and $(1 - \sqrt{2}, 0)$.

As the domain of f is restricted to $x \leq 1$, its graph is the left half of the parabola.

The graph of f^{-1} is then obtained by reflection in the line $y = x$.



(b) For f^{-1} the domain is $x \geq -2$ and the range is $y \leq 1$.

NSM_SB_10.indd 177 18/04/13 10:47 AM

(c) f is $y = x^2 - 2x - 1$

Interchange x and y: $x = y^2 - 2y - 1$

Make y the subject:

Method 1 (complete the square)

$$y^2 - 2y = x + 1$$
$$y^2 - 2y + 1 = x + 2$$
$$(y-1)^2 = x + 2$$
$$y - 1 = \pm\sqrt{x+2}$$
$$y = 1 \pm \sqrt{x+2}$$

Which one?

The range of f^{-1} is $y \leq 1$, so it must be:

$$y = 1 - \sqrt{x+2}$$

Method 2 (quadratic formula)

$y^2 - 2y - x - 1 = 0$

This is a quadratic in y with $a = 1$, $b = -2$ and $c = -(x+1)$.

$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-[x+1])}}{2}$$
$$y = \frac{2 \pm \sqrt{4 + 4x + 4}}{2}$$
$$y = \frac{2(1 \pm \sqrt{x+2})}{2}$$
$$y = 1 \pm \sqrt{x+2}$$

Which one?

The range of f^{-1} is $y \leq 1$, so it must be:

$$y = 1 - \sqrt{x+2}$$

(d) To find the point of intersection, we would usually solve simultaneously the equations of the two curves. However, for these two equations we need to solve $x^2 - 2x - 1 = 1 - \sqrt{x+2}$: that is a formidable task.

A better way is to remember that a function and its inverse intersect on the line $y = x$.

Now we can find the solution of $x^2 - 2x - 1 = x$, i.e. $x^2 - 3x - 1 = 0$.

The solution is $x = \dfrac{3 \pm \sqrt{13}}{2}$.

However, f and f^{-1} intersect at one point only (as shown in the diagram above). The second solution is a different point of intersection that would exist if the entire parabola were intersected with the line $y = x$ (instead of only half the parabola).

From the diagram, we can see that the required value in this case is the smaller of the two solutions, so the point of intersection is at $x = \dfrac{3 - \sqrt{13}}{2}$.

(e) $y = x^2 - 2x - 1 \quad \therefore \dfrac{dy}{dx} = 2x - 2$

At $x = -1$, the gradient of the tangent is -4.

At corresponding points in the reflection in $y = x$, the product of the gradients of a function and its inverse must be 1, so f^{-1} has a gradient of -0.25 at the point $(2, -1)$.

Exponential and logarithmic functions

Exponential and logarithmic functions, which you have met in previous work, provide a very good example of inverse functions.

An exponential function has the form $f(x) = a^x$. The base a is any positive real number except 1, the domain is R and the range is R^+ (the set of positive real numbers). It is a strictly monotonic increasing function, i.e. a one-to-one function, without having to restrict its domain.

Let $y = f(x) = a^x$

Interchanging x and y: $x = a^y$

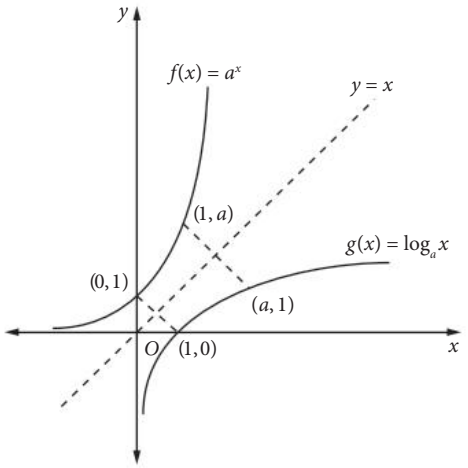
$\therefore \quad y = \log_a x$ (definition of a logarithm)

NSM_SB_10.indd 178 18/04/13 10:47 AM

The function g where $g(x) = \log_a x$ is called a logarithmic function. Its domain is R^+ (the range of f) and its range is R (the domain of f). It is also a strictly monotonic increasing function.

Hence $\log_a a^x = x$ for all real x (i.e. for all x in the domain of the exponential function), and $a^{\log_a x} = x$ for all $x > 0$ (i.e. for all x in the domain of the logarithmic function).

It should be noted that for any pair of inverse functions f and g, f is the inverse of g and g is the inverse of f. Neither has priority over the other.



EXERCISE 10.1 INVERSE FUNCTIONS

1 Which of the following are one-to-one functions?

A $f(x) = x - 2$ **B** $f(x) = x^2 - 2x + 1, x \geq 1$ **C** $f(x) = \sqrt{4 - x^2}$

D $f(x) = 9 - x$ **E** $f(x) = \dfrac{1}{9 - x}$ **F** $f(x) = |9 - x|$ **G** $f(x) = 9 - x^2$

H $f(x) = \cos x, 0 \leq x \leq \dfrac{\pi}{2}$ **I** $f(x) = \cos x$ **J** $f(x) = x^3 - 4x, -2 \leq x \leq 2$

2 Find the largest possible domain for which the following are one-to-one increasing functions.

(a) $f(x) = \sqrt{4 - x^2}$ **(b)** $f(x) = \sqrt{x^2 - 4}$ **(c)** $f(x) = -\dfrac{1}{x + 2}$

(d) $f(x) = 3x - x^2$ **(e)** $f(x) = x^2 + 6x + 8$

3 For each of the following, find the inverse function and state the domain and range of the inverse.

(a) $f(x) = 2x - 4$ **(b)** $f(x) = x^2 - 1, x \geq 0$ **(c)** $g(x) = \sqrt{x - 3}$ **(d)** $f(x) = \sqrt{9 - x^2}, -3 \leq x \leq 0$

(e) $f(x) = x^3$ **(f)** $f(x) = (x + 2)^2, x \leq -2$ **(g)** $f(x) = x^2 + 2x, x \geq 0$ **(h)** $f(x) = \log_e(x + 1)$

(i) $f(x) = 2 - \sqrt{x - 2}$ **(j)** $g(x) = \sqrt{(5 - x)} - 1$ **(k)** $f(x) = 2^{-x}, x > 0$ **(l)** $h(x) = \dfrac{1}{x + 1}, x > -1$

4 **(a)** Sketch the graph of $y = \dfrac{1}{1 + x^2}$.

(b) State two domains for which the function $f(x) = \dfrac{1}{1 + x^2}$ is one-to-one. (State the largest possible domain in each case.)

(c) Find and graph the two functions that are the inverse of the functions defined in **(b)**.

5 Explain why the following functions do not have an inverse function. Suggest suitable restrictions to their domain so that the restricted functions will have an inverse.

(a) $f(x) = \sqrt{a^2 - x^2}$ **(b)** $f(x) = 4 - x^2$ **(c)** $f(x) = \dfrac{1}{x^2}$

6 Show that the following pairs of functions are inverses by showing that $f(g(x)) = g(f(x)) = x$.

(a) $f(x) = 2x - 1$ and $g(x) = \frac{1}{2}(x + 1)$ **(b)** $f(x) = \dfrac{e^x}{2}$ and $g(x) = \log_e 2x$

(c) $f(x) = \sqrt{16 - x^2}, -4 \leq x \leq 0$ and $g(x) = -\sqrt{16 - x^2}, 0 \leq x \leq 4$

(d) $f(x) = 2x - x^2, x \geq 1$ and $g(x) = 1 + \sqrt{1 - x}, x \leq 1$ **(e)** $f(x) = \dfrac{1}{2x - 1}, x > \frac{1}{2}$ and $g(x) = \dfrac{x + 1}{2x}, x > 0$

NSM_SB_10.indd 179 18/04/13 10:47 AM

7 $f(x)=\sqrt{4-x^2}$, $-2 \le x \le 0$. The inverse function is given by:

A $x^2+y^2=4$ **B** $y=\sqrt{4-x^2}$, $-2 \le x \le 2$ **C** $y=\sqrt{4-x^2}$, $0 \le x \le 2$ **D** $y=-\sqrt{4-x^2}$, $0 \le x \le 2$

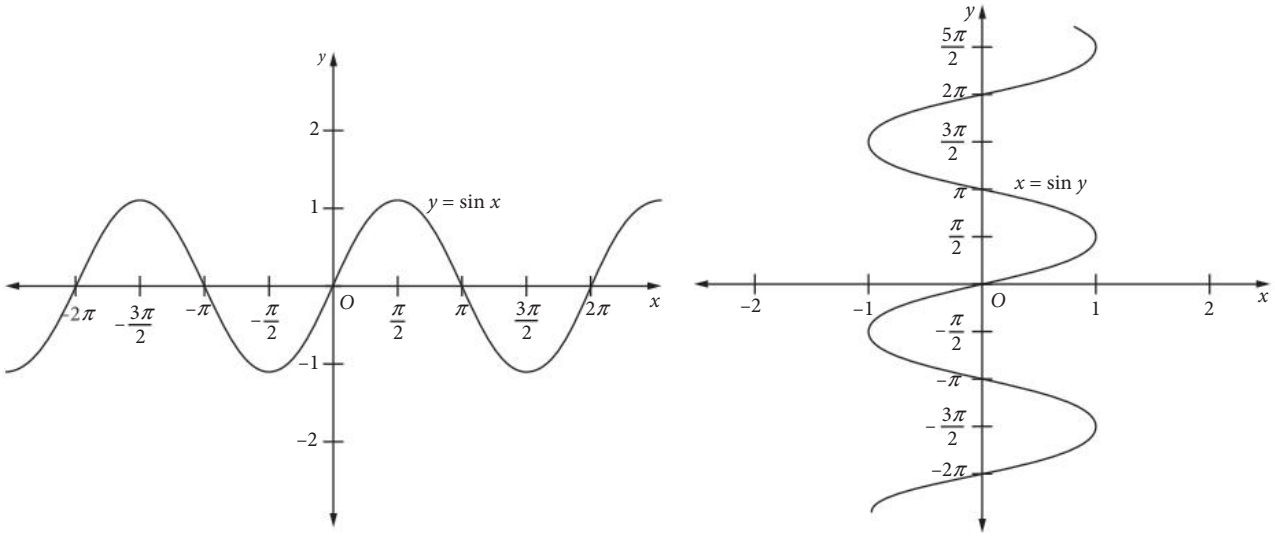
8 **(a)** Show that $y=x^3-9x+6\sqrt{3}$ has turning points at $(\sqrt{3},0)$ and $(-\sqrt{3}, 12\sqrt{3})$.
(b) Use the 'sum of roots' to find the other point where the graph cuts the x-axis.
(c) Let $f(x)=x^3-9x+6\sqrt{3}$ for the restricted domain $-\sqrt{3} \le x \le \sqrt{3}$. Sketch the graph of $y=f(x)$ and explain why the inverse function $f^{-1}(x)$ exists.
(d) State the domain and range of $f^{-1}(x)$.
(e) Sketch $y=f^{-1}(x)$
(f) Find the equation of the tangent to the curve $y=f^{-1}(x)$ at the point where the curve cuts the x-axis.

9 **(a)** Show that $y=e^x-1$ is a monotonic increasing function.
(b) Show that the graph of $y=e^x-1$ has a gradient of 1 at the point $(0,0)$.
(c) Carefully sketch $y=e^x-1$, $y=x$ and the inverse function on the same diagram.
(d) Show that the inverse function is $y=\log_e(x+1)$.
(e) Solve $\log_e(x+1) < x$.

10.2 INVERSE TRIGONOMETRIC FUNCTIONS

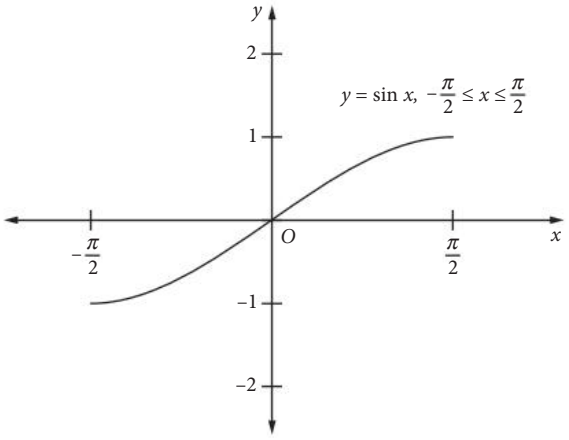
The inverse sine function

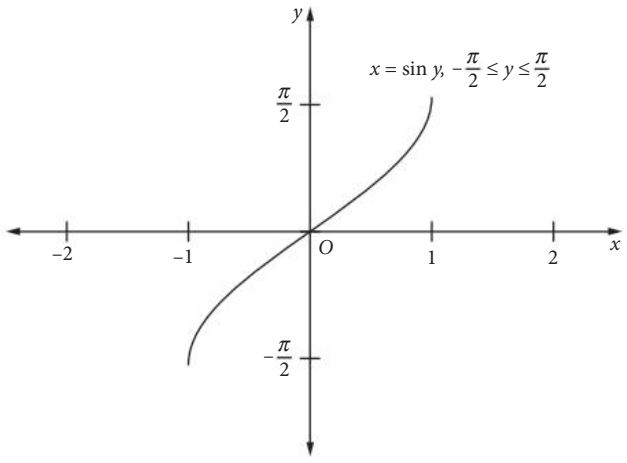
The trigonometric function $y=\sin x$ is a many-to-one function (see graph below left). The reflection of $y=\sin x$ in the line $y=x$ is shown below right. It has the equation $x=\sin y$, but clearly it is not a function.



There are infinitely many ways to restrict the domain of $y=\sin x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so we will now consider the function $f(x)=\sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The graph of this is shown below left, and its reflection in the line $y=x$ is shown below right (see next page).
Note that the reflection is that part of $x=\sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

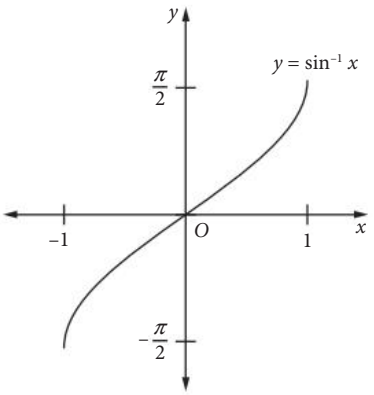
NSM_SB_10.indd 180 18/04/13 10:47 AM





By restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, we form a one-to-one function whose inverse is that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. Our final concern is how to make y the subject of this inverse function.
The answer is to say that $y = \sin^{-1} x$, where $y = \sin^{-1} x$ means 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$'.
Note that this raised number notation $^{-1}$ here is different to $y = \sin^2 x$, where the ^{2} means $y = (\sin x)^2$.
Here $y = \sin^{-1} x$ **does not mean** $y = (\sin x)^{-1}$, as that would be $y = \operatorname{cosec} x$.
We have instead a new function, the **inverse sine function**, $y = \sin^{-1} x$.

Note the following features of the inverse sine function:



- increasing function
- domain is $-1 \le x \le 1$
  Remember: 'you can only take $\sin^{-1}$ of values from −1 to 1 inclusive'
- range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
  Remember: '$\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive'
- vertical tangents at the endpoints
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function.
  Hence $\sin^{-1}(-x) = -\sin^{-1} x$.

Graph the functions $y = \sin(\sin^{-1} x)$ and $y = \sin^{-1}(\sin x)$ for $-2\pi \le x \le 2\pi$ using GeoGebra (or other graphing software) or by completing a table of values.

Note:

- $y = \sin(\sin^{-1} x)$ exists only for values of x between −1 and 1 inclusive;
  $y = \sin(\sin^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \sin^{-1}(\sin x)$ exists for all real x, but it is equivalent to $y = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
  Its graph has a 'sawtooth' shape.

Example 7

Find the exact values of the following. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (b) $\sin^{-1}\left(-\frac{1}{2}\right)$ (c) $\sin^{-1} 1.2$
(d) $\sin^{-1}(\sin 1.2)$ (e) $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$ (f) $\sin^{-1}(\sin\pi)$

NSM_SB_10.indd 181 18/04/13 10:47 AM

Solution

(a) As $-1 \le \frac{\sqrt{3}}{2} \le 1$, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ exists. It is the number y or angle y^c (i.e. in radians) such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and whose sine is $\frac{\sqrt{3}}{2}$. Hence $\sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$.

(b) Similarly, $\sin^{-1}\left(-\frac{1}{2}\right)$ can be evaluated as a number y or angle y^c, such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, whose sine is $-\frac{1}{2}$. Hence $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Alternatively: $\sin^{-1} x$ is an odd function, so $\sin^{-1}\left(-\frac{1}{2}\right) = -\sin^{-1}\left(\frac{1}{2}\right) = -\frac{\pi}{6}$.

(c) 1.2 is not within the domain $-1 \le x \le 1$, so $\sin^{-1} 1.2$ does not exist.

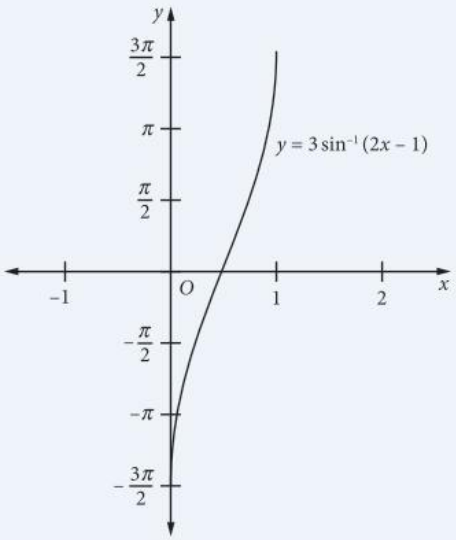
(d) 1.2 is within the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\sin^{-1}(\sin 1.2) = 1.2$.

(e) $\frac{\pi}{4}$ is within the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$.

(f) π is outside the domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, so $\sin^{-1}(\sin\pi) \ne \pi$. Instead, $\sin^{-1}(\sin\pi) = \sin^{-1} 0 = 0$.

Example 8

By first considering the domain and range, sketch $y = 3\sin^{-1}(2x - 1)$.



Solution

Remember, 'you can only take $\sin^{-1}$ of values from -1 to 1 inclusive', so $-1 \le 2x - 1 \le 1$. Solving this inequality:

$0 \le x \le 1$ is the required domain.

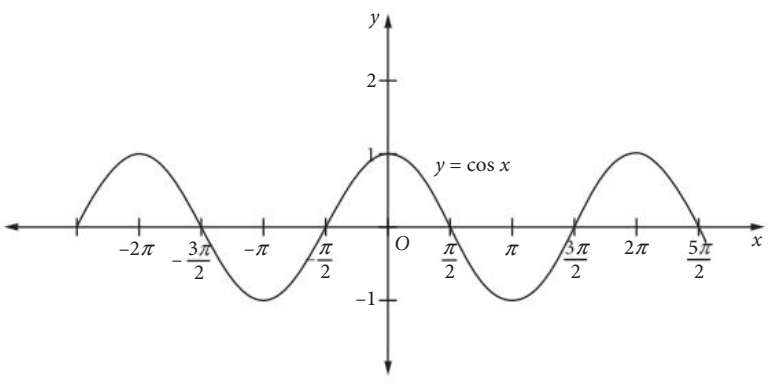
Similarly, '$\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive', so the range of $3\sin^{-1}(2x - 1)$ is $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ (i.e. 3 times the range of $y = \sin^{-1} x$).

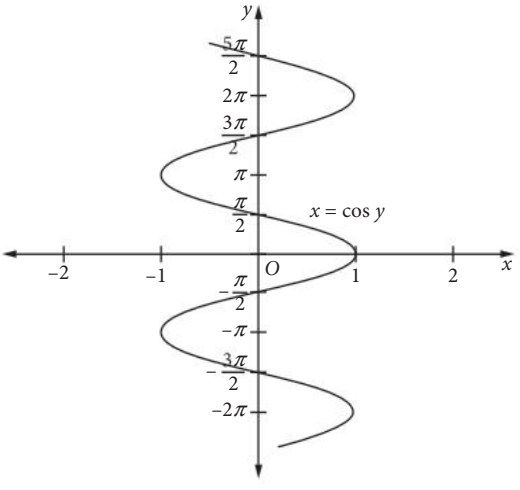
In other words: $-\frac{\pi}{2} \le \sin^{-1}(2x - 1) \le \frac{\pi}{2}$

Multiplying by 3: $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

The inverse cosine function

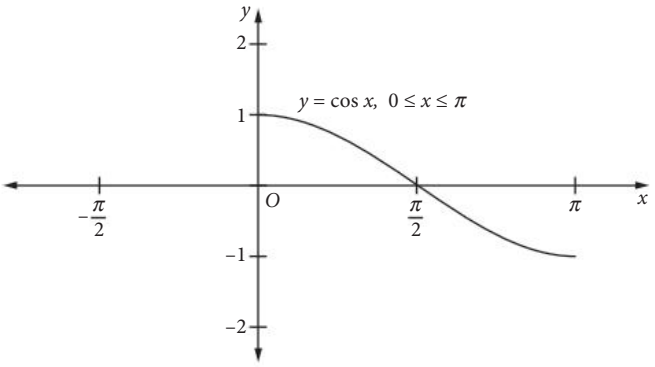
The trigonometric function $y = \cos x$ is a many-to-one function (see graph below left). The reflection of $y = \cos x$ in the line $y = x$ is shown below right. It has the equation $x = \cos y$, but clearly it is not a function.

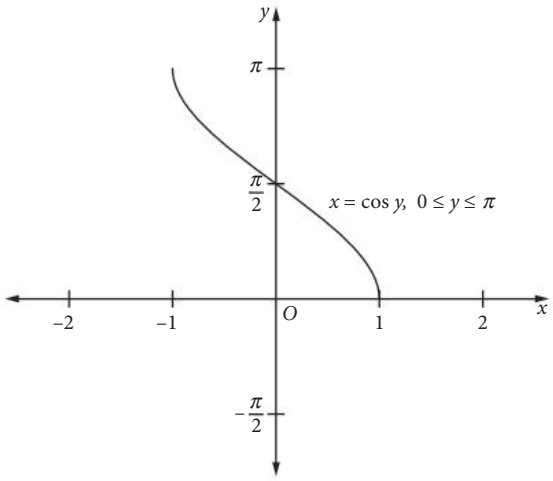




NSM_SB_10.indd 182 18/04/13 10:47 AM

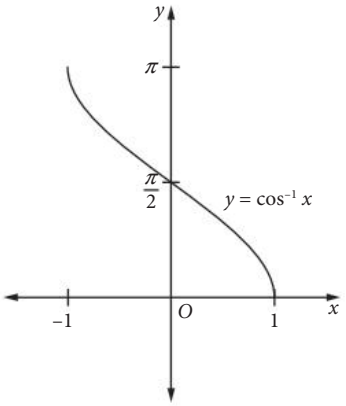
There are infinitely many ways to restrict the domain of $y = \cos x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $0 \leq x \leq \pi$, so we will now consider the function $f(x) = \cos x$, $0 \leq x \leq \pi$. The graph of this is shown below left, and its reflection in the line $y = x$ is shown below right.
Note that the reflection is that part of $x = \cos y$ for which $0 \leq y \leq \pi$.





By restricting the domain of $y = \cos x$ to $0 \leq x \leq \pi$, we form a one-to-one function whose inverse is that part of $x = \cos y$ for which $0 \leq y \leq \pi$.
This function is the **inverse cosine function**, $y = \cos^{-1} x$, where $y = \cos^{-1} x$ means 'that part of $x = \cos y$ for which $0 \leq y \leq \pi$'.

Note the following features of the inverse cosine function:



- decreasing function
- domain is $-1 \leq x \leq 1$
  Remember: 'you can only take $\cos^{-1}$ of values from −1 to 1 inclusive'
- range is $0 \leq y \leq \pi$
  Remember: '$\cos^{-1}$ gives values from 0 to π inclusive'
- vertical tangents at the endpoints
- gradient of −1 at $\left(0, \frac{\pi}{2}\right)$
- the function is neither even nor odd, but it does have rotational symmetry about its y-intercept.
- In particular: for any x in the domain, the sum of the function's heights at x and $-x$ will always be π:
  $\cos^{-1}(-x) + \cos^{-1} x = \pi$
  $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Graph the functions $y = \cos(\cos^{-1} x)$ and $y = \cos^{-1}(\cos x)$ for $-2\pi \leq x \leq 2\pi$ using GeoGebra (or other graphing software) or by completing a table of values.

Note:

- $y = \cos(\cos^{-1} x)$ exists only for values of x between −1 and 1 inclusive;
  $y = \cos(\cos^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \cos^{-1}(\cos x)$ exists for all real x, but it is equivalent to $y = x$ only for $0 \leq x \leq \pi$.
  Its graph has a 'sawtooth' shape.

NSM_SB_10.indd 183 18/04/13 10:47 AM

Example 9

Find the exact values of the following. (a) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (b) $\cos\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ (c) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

(d) $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)$ (e) $\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)$ (f) $\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$

Solution

(a) **Method 1**

Let $y=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Then $\cos y=-\frac{1}{\sqrt{2}}$ and $0\le y\le\pi$

$\therefore y=\frac{3\pi}{4}$

$\therefore \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\frac{3\pi}{4}$

Method 2

$$\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)=\pi-\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$=\pi-\frac{\pi}{4}$$
$$=\frac{3\pi}{4}$$

(b) $\cos(\cos^{-1}x)=x$ for $-1\le x\le 1$, so $\cos\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)=-\frac{1}{2}$

(c) $\cos^{-1}(\cos x)=x$ for $0\le x\le\pi$, so $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$

(d) $\frac{5\pi}{3}$ is not in the domain $0\le x\le\pi$, so $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)\ne\frac{5\pi}{3}$

The solution is: $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$

(e) **Method 1**

$$\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)=\sin\frac{2\pi}{3}$$
$$=\frac{\sqrt{3}}{2}$$

Method 2

$$\sin\left(\cos^{-1}\left(-\frac{1}{2}\right)\right)=\sin\left(\pi-\cos^{-1}\frac{1}{2}\right)$$
$$=\sin\left(\pi-\frac{\pi}{3}\right)$$
$$=\sin\frac{\pi}{3}$$
$$=\frac{\sqrt{3}}{2}$$

Method 2 here shows a good approach. When you have to take an inverse trigonometric function of a negative value, use the symmetry properties of the inverse trigonometric functions:

$\sin^{-1}(-x)=-\sin^{-1}x$ $\quad\cos^{-1}(-x)=\pi-\cos^{-1}x$ $\quad\tan^{-1}(-x)=-\tan^{-1}x$

This process ensures that the function is evaluated with a first quadrant angle.

(f) **Method 1**

Let $\cos^{-1}\left(-\frac{2}{3}\right)=\theta$

Then $\cos\theta=-\frac{2}{3}$ and $0\le\theta\le\pi$

So θ is a second quadrant angle.

We need to evaluate:

$$\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)=\tan\theta$$

Method 2

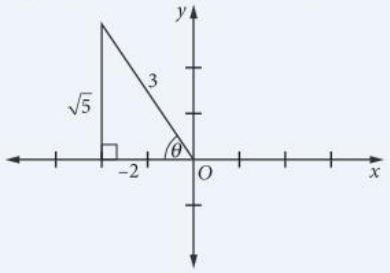
Use the symmetry properties:

$\cos^{-1}\left(-\frac{2}{3}\right)=\pi-\cos^{-1}\frac{2}{3}$ and $\tan(\pi-\theta)=-\tan\theta$

$$\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)=\tan\left(\pi-\cos^{-1}\frac{2}{3}\right)$$
$$=-\tan\left(\cos^{-1}\frac{2}{3}\right)$$

NSM_SB_10.indd 184 18/04/13 10:47 AM

The graph below shows this process:



θ is in the second quadrant, $\cos\theta = -\frac{2}{3}$

We want the value of $\tan\theta$.

$\therefore \tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right) = \tan\theta = -\frac{\sqrt{5}}{2}$

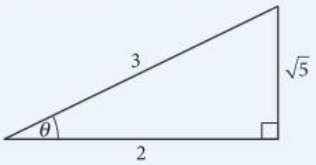
Now find the exact value of $\tan\left(\cos^{-1}\frac{2}{3}\right)$.

Let $\theta = \cos^{-1}\frac{2}{3}$

$\therefore \cos\theta = \frac{2}{3}$ (where θ is acute)

We evaluate $\tan\left(\cos^{-1}\frac{2}{3}\right) = \tan\theta$.

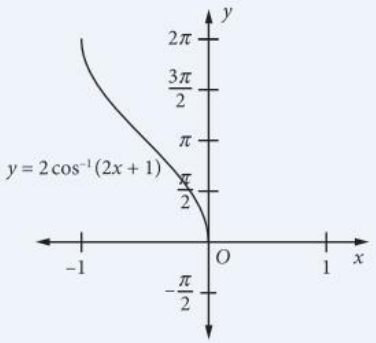
The diagram below shows that if $\cos\theta = \frac{2}{3}$ then $\tan\theta = \frac{\sqrt{5}}{2}$:



Hence $\tan\left(\cos^{-1}\left(-\frac{2}{3}\right)\right) = -\frac{\sqrt{5}}{2}$.

Example 10

State the domain and range of the function $y = 2\cos^{-1}(2x+1)$.
Hence sketch the graph of the function.



Solution

You can only take the inverse cosine of values between −1 and 1 inclusive, so $-1 \leq 2x+1 \leq 1$.

Solving this, the domain is $-1 \leq x \leq 0$.

As inverse cos gives values between 0 and π inclusive:

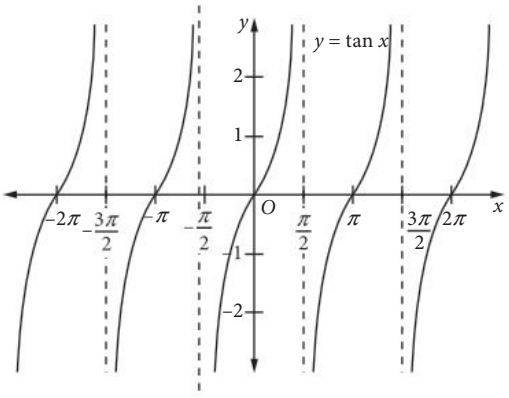
$$0 \leq \cos^{-1}(2x+1) \leq \pi$$

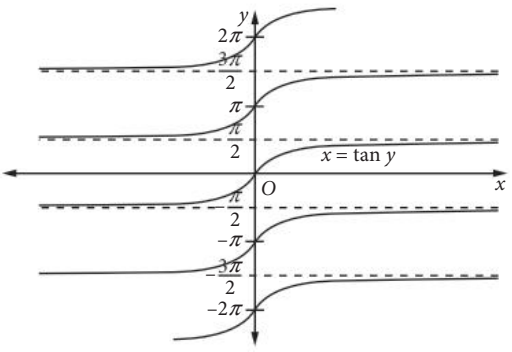
$$0 \leq 2\cos^{-1}(2x+1) \leq 2\pi$$

$\therefore$ the range is $0 \leq y \leq 2\pi$

The inverse tangent function

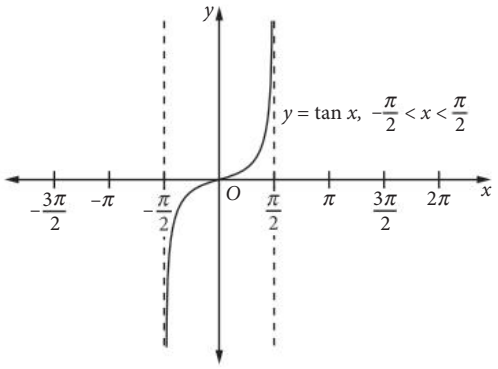
The trigonometric function $y = \tan x$ is a many-to-one function (see graph below left). The reflection of $y = \tan x$ in the line $y = x$ is shown below right. It has the equation $x = \tan y$, but clearly it is not a function.

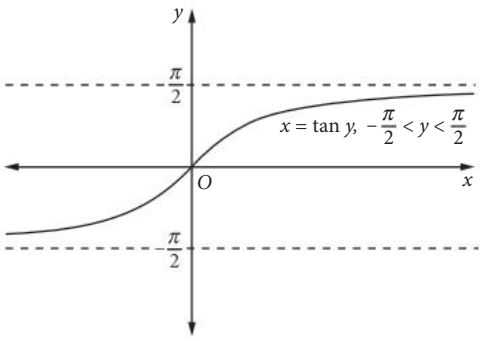




There are infinitely many ways to restrict the domain of $y = \tan x$ to make it one-to-one. Mathematicians have agreed that the most useful restriction is $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so we will now consider the function $f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
The graph of this is shown below left, and its reflection in the line $y = x$ is shown below right (see next page).
Note that the reflection is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

NSM_SB_10.indd 185 18/04/13 10:47 AM

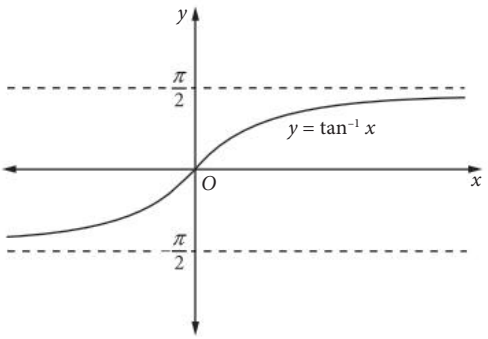




By restricting the domain of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we form a one-to-one function whose inverse is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

This function is the **inverse tangent function**, $y = \tan^{-1} x$, where $y = \tan^{-1} x$ means 'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

Note the following features of the inverse tangent function:



- increasing function
- domain is all real x
  Remember: 'you can take $\tan^{-1}$ of any real value'
- range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$
  Remember: '$\tan^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ not inclusive'
- horizontal asymptotes at $y = \pm\frac{\pi}{2}$
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function.
  Hence $\tan^{-1}(-x) = -\tan^{-1} x$.

Graph the functions $y = \tan(\tan^{-1} x)$ and $y = \tan^{-1}(\tan x)$ for $-2\pi \le x \le 2\pi$ using GeoGebra (or other graphing software) or by completing a table of values.

Note:

- $y = \tan(\tan^{-1} x)$ is equivalent to $y = x$ for all real x.
- $y = \tan^{-1}(\tan x)$ exists for all real x except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

  It is equivalent to $y = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Its graph looks like an infinite set of parallel intervals with open circles on each end.

Example 11

Find the exact values of the following.

(a) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **(b)** $\tan(\tan^{-1} 1)$ **(c)** $\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)$ **(d)** $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$

Solution

(a) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. It is the value of θ, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive), for which $\tan\theta = \frac{1}{\sqrt{3}}$.

(b) $\tan(\tan^{-1} 1) = 1$ **(c)** $\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, as $\frac{\pi}{3}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (not inclusive)

NSM_SB_10.indd 186 18/04/13 10:47 AM

(d) $\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)$ is not equal to $-\frac{4\pi}{3}$, because $-\frac{4\pi}{3}$ is not between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Using the symmetry properties that $\tan\theta$ and $\tan^{-1}x$ are odd functions, and that $\tan(\pi+\theta)=\tan\theta$:

$$\tan^{-1}\left(\tan\left(-\frac{4\pi}{3}\right)\right)=\tan^{-1}\left(-\tan\left(\frac{4\pi}{3}\right)\right)=-\tan^{-1}\left(\tan\left(\frac{4\pi}{3}\right)\right)=-\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)=-\frac{\pi}{3}$$

Example 12

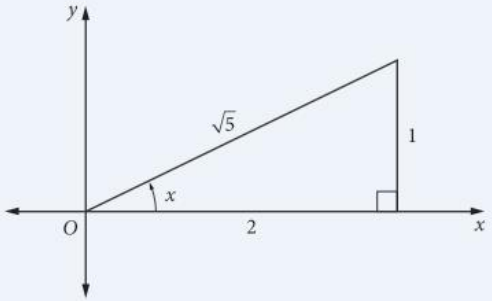
Find the exact value of $\sin\left(2\tan^{-1}\frac{1}{2}\right)$.

Solution

Let $\tan^{-1}\frac{1}{2}=x$

Thus $\tan x=\frac{1}{2}$ and $-\frac{\pi}{2}<x<\frac{\pi}{2}$

Hence x can be represented as a first quadrant angle.



Then: $\sin\left(2\tan^{-1}\frac{1}{2}\right)=\sin 2x$

$$=2\sin x\cos x$$

$$=2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$$

$$=\frac{4}{5}$$

Example 13

Find $\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]$.

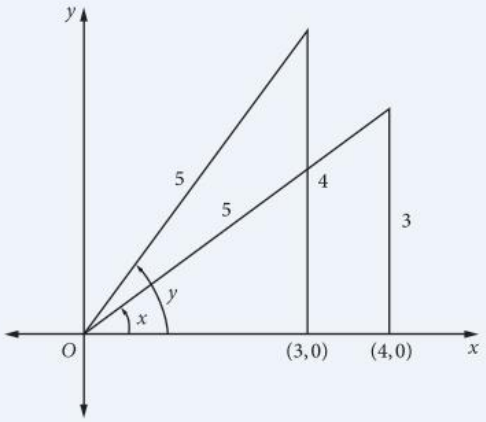
Solution

$$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\left(-\frac{4}{3}\right)\right]=\sin\left[\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{4}{3}\right]$$

$$=\sin(x-y)\quad\text{where } x=\cos^{-1}\frac{4}{5}\text{ and } y=\tan^{-1}\frac{4}{3}$$

$$=\sin\left(\cos^{-1}\frac{4}{5}\right)\cos\left(\tan^{-1}\frac{4}{3}\right)-\cos\left(\cos^{-1}\frac{4}{5}\right)\sin\left(\tan^{-1}\frac{4}{3}\right)$$

[Using expansion of $\sin(x-y)$]



$\cos^{-1}\frac{4}{5}=x$, so $\cos x=\frac{4}{5}$ and $0\le x\le\pi$

$\tan^{-1}\frac{4}{3}=y$, so $\tan y=\frac{4}{3}$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$

Hence both x and y can be represented as first quadrant angles:

$$\sin\left[\cos^{-1}\frac{4}{5}+\tan^{-1}-\frac{4}{3}\right]=\sin x\cos y-\cos x\sin y$$

$$=\left(\frac{3}{5}\right)\left(\frac{3}{5}\right)-\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)$$

$$=-\frac{7}{25}$$

NSM_SB_10.indd 187 18/04/13 10:47 AM

Example 14

Prove that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$.

Solution

Let $\alpha=\sin^{-1}x \quad \therefore \sin\alpha=x$ where $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$

Recall that $\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha$, so $\cos\left(\frac{\pi}{2}-\alpha\right)=x$

Also, as $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, thus $0\le\frac{\pi}{2}-\alpha\le\pi$

$\therefore \frac{\pi}{2}-\alpha=\cos^{-1}x$ (noting that $\theta=\cos^{-1}x$ only when $\cos\theta=x$ and $0\le\theta\le\pi$)

$\therefore \frac{\pi}{2}-\sin^{-1}x=\cos^{-1}x$ so $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

You should remember this result: $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ for $-1\le x\le 1$

We will prove this by an alternative method involving differentiation later in section 10.3 (page 190).

You could also verify it by graphing $y=\sin^{-1}x+\cos^{-1}x$.

Example 15

Solve $2\sin^{-1}x+\cos^{-1}x=\frac{\pi}{3}$.

Solution

Using the result of Example **14**, the equation becomes: $2\sin^{-1}x+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{3}$

$$\sin^{-1}x=-\frac{\pi}{6}$$

$$\therefore \quad x=\sin\left(-\frac{\pi}{6}\right)$$

$$\therefore \quad x=-\frac{1}{2}$$

EXERCISE 10.2 INVERSE TRIGONOMETRIC FUNCTIONS

1 $\cos^{-1}x-\cos^{-1}(-x)=\ldots$

A π **B** $2\cos^{-1}x-\pi$ **C** $2\cos^{-1}x$ **D** 0

2 Find the exact values of the following (if they exist).

(a) $\sin^{-1}1$ **(b)** $\sin^{-1}0$ **(c)** $\sin^{-1}(-1)$ **(d)** $\sin^{-1}\frac{\sqrt{3}}{2}$ **(e)** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **(f)** $\cos^{-1}1$

(g) $\cos^{-1}0$ **(h)** $\cos^{-1}(-1)$ **(i)** $\cos^{-1}\frac{1}{2}$ **(j)** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **(k)** $\tan^{-1}1$ **(l)** $\tan^{-1}(-1)$

(m) $\tan^{-1}\sqrt{3}$ **(n)** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ **(o)** $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$ **(p)** $\sin^{-1}(\sin 80°)$ **(q)** $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$

(r) $\sin^{-1}\left(\sin\left(-\frac{3\pi}{4}\right)\right)$ **(s)** $\cos^{-1}\left(\cos\frac{\pi}{3}\right)$ **(t)** $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ **(u)** $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$ **(v)** $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$

3 Evaluate the following.

(a) $\cos\left(\sin^{-1}\frac{1}{2}\right)$ **(b)** $\tan\left(\tan^{-1}\left(-\frac{5}{13}\right)\right)$ **(c)** $\tan^{-1}(\tan 245°)$ **(d)** $\cos^{-1}(\cos 540°)$

(e) $\cos\left(\tan^{-1}\left(-\sqrt{3}\right)\right)$ **(f)** $\cos\left(2\sin^{-1}\frac{\sqrt{3}}{2}\right)$ **(g)** $\cos\left(2\cos^{-1}\frac{5}{13}\right)$ **(h)** $\sec\left(\sin^{-1}\left(-\frac{1}{3}\right)\right)$

NSM_SB_10.indd 188 18/04/13 10:47 AM

4 Show that:

(a) $\tan^{-1}4-\tan^{-1}\frac{3}{5}=\frac{\pi}{4}$ **(b)** $\sin^{-1}\frac{3}{5}+\sin^{-1}\left(-\frac{3}{5}\right)=0$

(c) $\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{5}{13}=\frac{\pi}{2}$ **(d)** $\cos^{-1}\frac{3}{5}-\tan^{-1}\left(-\frac{3}{4}\right)=\frac{\pi}{2}$

(e) $2\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{24}{25}$ **(f)** $\sin^{-1}\frac{3}{5}+\tan^{-1}\frac{7}{24}=\cos^{-1}\frac{3}{5}$

(g) $\sin^{-1}\frac{5}{13}+\tan^{-1}\frac{16}{63}=\cos^{-1}\frac{4}{5}$

5 Find the exact values of the following.

(a) $\sin\left[\sin^{-1}\left(\frac{3}{5}\right)\right]+\sin\left[\sin^{-1}\left(-\frac{3}{5}\right)\right]$ **(b)** $\sin\left[\sin^{-1}\frac{3}{5}+\sin^{-1}\left(-\frac{3}{5}\right)\right]$

(c) $\cos\left[\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{4}{5}\right]$ **(d)** $\sin\left(2\tan^{-1}\frac{4}{3}\right)$

(e) $\cos\left[\tan^{-1}\frac{4}{3}-\cos^{-1}\frac{5}{13}\right]$ **(f)** $\sin\left[\cos^{-1}\frac{3}{5}+\tan^{-1}\left(-\frac{3}{4}\right)\right]$

(g) $\tan\left[\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{12}{13}\right]$

6 Prove that: **(a)** $\sin^{-1}(x)=\cos^{-1}\sqrt{1-x^2}$, $0\le x\le 1$ **(b)** $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}$, $0\le x\le 1$

7 Sketch graphs of each of the following, stating their domain and range.

(a) $y=2\sin^{-1}x$ **(b)** $y=2\sin^{-1}3x$ **(c)** $y=3\sin^{-1}(2x+1)$ **(d)** $y=3\sin^{-1}2x+1$

(e) $y=2\tan^{-1}x$ **(f)** $y=2\tan^{-1}(x-1)$ **(g)** $y=2\cos^{-1}\left(\frac{x}{2}\right)$

(h) $y=2\sin^{-1}(-x)$ **(i)** $y=\sin(\cos^{-1}x)$, by first simplifying $\sin(\cos^{-1}x)$

8 Consider the functions $f(x)=2\sin^{-1}(2x-1)$ and $g(x)=2\sin^{-1}(1-2x)$.

(a) Find the domain and range of each function.

(b) Show that $g(x)=-f(x)$.

(c) Sketch the graphs of the two functions.

9 Solve the following equations. For part **(c)**, answer correct to two decimal places and check your solution by substituting values back into the original equation.

(a) $\sin^{-1}x\cos^{-1}x=0$ **(b)** $\sin^{-1}(1-x)+2\cos^{-1}(x-1)=\frac{\pi}{2}$ **(c)** $\sin^{-1}x\cos^{-1}x=-1$

10 Solve the following pairs of simultaneous equations.

(a) $2\sin^{-1}x+\cos^{-1}y=-\frac{\pi}{12}$, $\sin^{-1}x-2\cos^{-1}y=-\frac{2\pi}{3}$ **(b)** $\sin^{-1}x+\cos^{-1}y=-\frac{\pi}{12}$, $\cos^{-1}x-\sin^{-1}y=\frac{5\pi}{12}$

11 **(a)** Show that $\sin(\sin^{-1}x-\cos^{-1}x)=2x^2-1$ for $-1\le x\le 1$.

(b) Show that if $-1\le x\le 1$ then $-1\le 2x^2-1\le 1$.

(c) Hence solve $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(5x-4)$.

12 Prove that $f(x)=\tan(\cos^{-1}x)$ is an odd function.

13 If $f(x)=3\cos^{-1}\left(\frac{x}{2}\right)$, determine the inverse function f^{-1} and specify the domain and range of f^{-1}.

14 **(a)** Sketch the graph of $y=\sin^{-1}x+\cos^{-1}x$. **(b)** Hence find $\int_{-1}^{1}(\sin^{-1}x+\cos^{-1}x)dx$.

NSM_SB_10.indd 189 18/04/13 10:47 AM

10.3 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

The result $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

Proof

Let $y = f(x)$. Differentiate both sides with respect to y:

$$\frac{d}{dy}(y) = \frac{d}{dy}(f(x))$$

$$\therefore \quad 1 = \frac{d}{dx}(f(x)) \times \frac{dx}{dy} \qquad \text{(using chain rule on RHS)}$$

$$\therefore \quad 1 = \frac{dy}{dx} \times \frac{dx}{dy}$$

Demonstration

Let $y = x^3 - 1 \quad \therefore \quad x = (y+1)^{\frac{1}{3}}$

$$\frac{dy}{dx} = 3x^2 \qquad \frac{dx}{dy} = \frac{1}{3}(y+1)^{-\frac{2}{3}}$$

$$\frac{dy}{dx} \times \frac{dx}{dy} = 3x^2 \times \frac{1}{3}(y+1)^{-\frac{2}{3}}$$

$$= \frac{x^2}{(y+1)^{\frac{2}{3}}}$$

$$= \frac{x^2}{(x^3)^{\frac{2}{3}}}$$

$$= \frac{x^2}{x^2} = 1$$

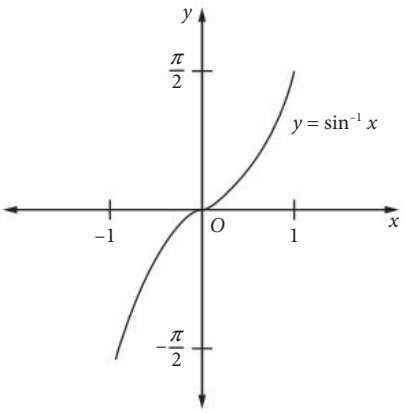
Derivative of $\sin^{-1} x$

$\sin^{-1} x$ is defined for $-1 \le x \le 1$.

Let $y = \sin^{-1} x \quad \therefore \quad x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\frac{dx}{dy} = \cos y$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} \text{ noting that } \cos y \neq 0 \ \therefore \ -\frac{\pi}{2} < y < \frac{\pi}{2}$$



Now, using $\cos^2 y + \sin^2 y = 1$: $\frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$ Which one?

As y is an angle in the first or fourth quadrants, $\cos y$ must be positive.

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \quad \text{for } -1 < x < 1$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1$$

Note that the derivative is not defined at $x = \pm 1$ (the graph of $\sin^{-1} x$ has vertical tangents at its endpoints).

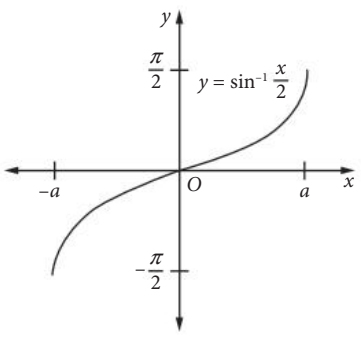
Also note that the derivative is positive for all x in its domain (as $\sin^{-1} x$ is an increasing function).

Derivative of $\sin^{-1} \frac{x}{a}$

$\sin^{-1} \frac{x}{a}$ is defined for $-a \le x \le a$.

Let $y = \sin^{-1} \frac{x}{a} \quad \therefore \quad x = a \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\frac{dx}{dy} = a \cos y$$



NSM_SB_10.indd 190 18/04/13 10:47 AM

$$\therefore \quad \frac{dy}{dx} = \frac{1}{a\cos y} \qquad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\text{i.e.} \quad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}} \qquad \text{for } -a < x < a$$

$$\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$$

Derivative of $\cos^{-1} x$

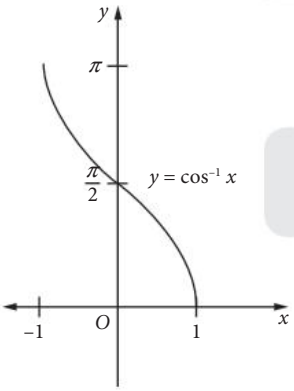
$\cos^{-1} x$ is defined for $-1 \leq x \leq 1$.

Let $y = \cos^{-1} x \quad \therefore \quad x = \cos y, \ 0 \leq y \leq \pi$

$$\frac{dx}{dy} = -\sin y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sin y} \qquad \text{for } 0 < y < \pi$$

$$\text{i.e.} \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \qquad \text{for } -1 < x < 1$$



$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

As for the derivative of $\sin^{-1} x$, the last step again uses $\cos^2 y + \sin^2 y = 1$.

Note that the gradient is negative for all x in the domain.

Note also that the domain of the derived function is different to the domain of $\cos^{-1} x$. Why?

Derivative of $\cos^{-1}\frac{x}{a}$

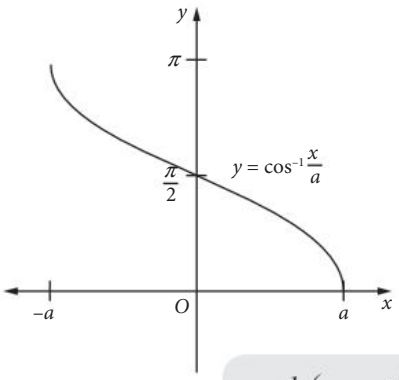
$\cos^{-1}\frac{x}{a}$ is defined for $-a \leq x \leq a$.

Let $y = \cos^{-1}\frac{x}{a} \quad \therefore \quad x = a\cos y, \ 0 \leq y \leq \pi$

$$\frac{dx}{dy} = -a\sin y$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{a\sin y} \qquad \text{for } 0 < y < \pi$$

$$\text{i.e.} \quad \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}} \qquad \text{for } -a < x < a$$



$$\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$$

Derivative of $\tan^{-1} x$

$\tan^{-1} x$ is defined for all x.

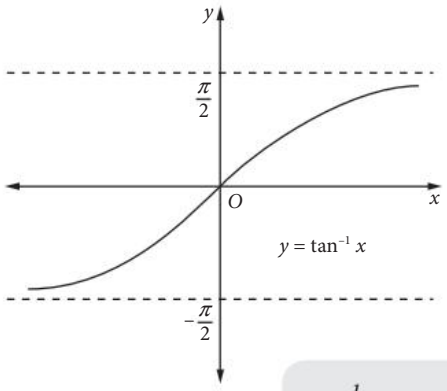
Let $y = \tan^{-1} x \quad \therefore \quad x = \tan y, \ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{dx}{dy} = \sec^2 y$$

$$= 1 + \tan^2 y$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1 + \tan^2 y} \qquad \text{for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$= \frac{1}{1 + x^2} \qquad \text{for all } x$$



$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2} \quad \text{for all } x$$

NSM_SB_10.indd 191 18/04/13 10:47 AM

Derivative of $\tan^{-1}\frac{x}{a}$

$\tan^{-1}\frac{x}{a}$ is defined for all x.

Let $y = \tan^{-1}\frac{x}{a}$ $\quad\therefore\quad x = a\tan y, -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\frac{dx}{dy} = a\sec^2 y$$
$$= a(1+\tan^2 y)$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{a(1+\tan^2 y)} \text{ for } -\frac{\pi}{2} < y < \frac{\pi}{2}$$
$$= \frac{a}{a^2+x^2} \qquad \text{for all } x$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right) = \frac{a}{a^2+x^2} \qquad \text{for all } x$$

Example 16

Find the derivative of $\cos^{-1}(2x+1)$, stating the values of x for which it is defined.

Solution

$\cos^{-1}(2x+1)$ is defined for $-1 \le 2x+1 \le 1$

i.e. for $-2 \le 2x \le 0$

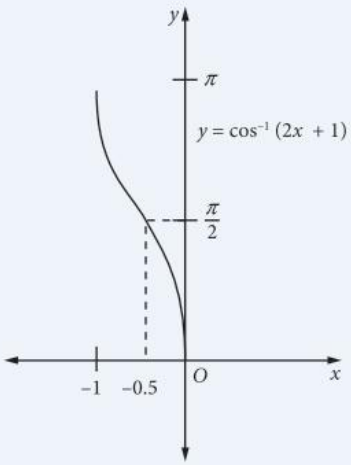
$-1 \le x \le 0$

Hence $\cos^{-1}(2x+1)$ is defined for $-1 \le x \le 0$.

Let $y = \cos^{-1}(2x+1)$

$= \cos^{-1} u$ where $u = 2x+1$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$
$$= -\frac{1}{\sqrt{1-u^2}}\times 2$$
$$= -\frac{2}{\sqrt{1-(2x+1)^2}} \quad \text{provided } -1 < x < 0$$
$$= -\frac{2}{\sqrt{-4x(x+1)}}$$



Example 17

Find the derivative of $x\cos^{-1}(2x+1)$, stating the values of x for which the derivative is defined.

Solution

Following the previous example and using the product rule:

$$\frac{d}{dx}\left(x\cos^{-1}(2x+1)\right) = x\times\frac{-2}{\sqrt{-4x(x+1)}} + \cos^{-1}(2x+1)$$
$$= \frac{-2x}{\sqrt{-4x(x+1)}} + \cos^{-1}(2x+1)$$

On the RHS, the first term is defined for $-1 < x < 0$ and the second term is defined for $-1 \le x \le 0$. Thus the complete RHS is defined for $-1 < x < 0$ and these are the values for x for which the derivative is defined.

NSM_SB_10.indd 192 18/04/13 10:47 AM

Example 18

Differentiate $\sin^{-1}(\cos x)$. Hence sketch the graph of $y = \sin^{-1}(\cos x)$ for $-\pi \le x \le \pi$.

Solution

Let $y = \sin^{-1}(\cos x)$

$= \sin^{-1} u$ where $u = \cos x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

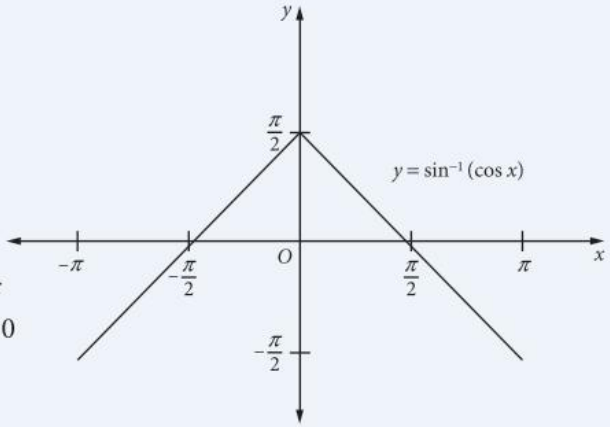
$$= \frac{1}{\sqrt{1-u^2}} \times -\sin x$$

$$= \frac{-\sin x}{\sqrt{1-\cos^2 x}}$$

$$= \frac{-\sin x}{\sqrt{\sin^2 x}}$$

$$= \frac{-\sin x}{|\sin x|}$$

x	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
y	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	0	$-\frac{\pi}{2}$



Now $|\sin x| = \sin x$ for $\sin x \ge 0$, i.e. for $0 \le x \le \pi$

$= -\sin x$ for $\sin x \le 0$, i.e. for $-\pi \le x \le 0$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{-\sin x}{\sin x} = -1 & \text{for } 0 < x < \pi \\ \frac{-\sin x}{-\sin x} = 1 & \text{for } -\pi < x < 0 \end{cases}$$

and $\frac{dy}{dx}$ is not defined when $x = -\pi, 0, \pi$. If there were no restrictions on the domain, the graph would repeat itself (i.e. it would be periodic with period 2π).

The range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and the derivative is not defined for any values of $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

Note that these sharp peaks are **not** turning points.

Example 19

(a) Differentiate $\sin^{-1} x + \cos^{-1} x$. **(b)** Hence show that $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Solution

(a) $$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

(b) $\sin^{-1} x + \cos^{-1} x$ is a constant as its derivative is 0.

The value of the constant can be found by evaluating the function at any x in its domain.

Where $x = 0$: $\sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$

$\therefore$ $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ (You may wish to verify this by substituting other values for x.)

NSM_SB_10.indd 193 18/04/13 10:47 AM

EXERCISE 10.3 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

1 The derivative of $\sin^{-1} 3x$ is:

A $\dfrac{1}{\sqrt{9-x^2}}$ B $\dfrac{3}{\sqrt{1-9x^2}}$ C $\dfrac{1}{\sqrt{1-9x^2}}$ D $\dfrac{3}{\sqrt{9-x^2}}$

2 Differentiate the following.

(a) $y = \tan^{-1} 5x$ **(b)** $y = 3\tan^{-1}(1-x)$ **(c)** $y = \tan^{-1} x^2$ **(d)** $y = (\tan^{-1} x)^2$

(e) $y = \tan^{-1}\left(\sqrt{x}\right)$ **(f)** $y = \tan^{-1} e^x$ **(g)** $y = e^{\tan^{-1} x}$ **(h)** $y = e^x \tan^{-1} x$

(i) $y = \sin^{-1}\left(\dfrac{x}{4}\right)$ **(j)** $y = 2\cos^{-1}\left(\dfrac{3x}{2}\right)$ **(k)** $y = \log_e(\sin^{-1} x)$ **(l)** $y = \log_e(\cos^{-1} 2x)^2$

(m) $y = \sin^{-1}\left(\dfrac{1}{x}\right)$ **(n)** $y = \dfrac{1}{\cos^{-1} x}$ **(o)** $y = \dfrac{1}{x}\sin^{-1} x$ **(p)** $y = x\sin^{-1}\left(\sqrt{x}\right)$

(q) $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ **(r)** $y = \cos^{-1} x + \cos^{-1}(-x)$ **(s)** $y = \tan x \tan^{-1} x$ **(t)** $y = \tan^{-1}\left(\sqrt{x^2-1}\right)$

3 Find the equation of the tangent to the curve $y = 2\tan^{-1}(2x+1)$ at the point on the curve where $x = -1$.

4 **(a)** If $y = \tan^{-1}(\tan 2x)$, show that $\dfrac{dy}{dx} = 2$ for all x in its domain.

(b) Explain why the function is undefined at $x = \pm\dfrac{\pi}{4}$.

(c) Sketch the graph of $y = \tan^{-1}(\tan 2x)$ for $-\dfrac{\pi}{4} < x < \dfrac{\pi}{4}$.

(d) Hence show that $\displaystyle\int_{-\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^{-1}(\tan 2x)\,dx = \frac{7\pi^2}{576}$.

5 If $y = \cos^{-1} x + \cos^{-1}(-x)$, find $\dfrac{dy}{dx}$ and show that $y = \pi$ for all x in the domain.

6 If $y = \sin^{-1} x + \sin^{-1}(-x)$, find $\dfrac{dy}{dx}$ and show that $y = 0$ for all x in the domain.

7 **(a)** Differentiate $x\tan^{-1} x$. **(b)** Hence find $\displaystyle\int \tan^{-1} x\,dx$.

(c) Use the substitution $u = \log_e x$ to evaluate $\displaystyle\int_1^e \frac{\tan^{-1}(\log_e x)}{x}\,dx$.

8 **(a)** State the domain of $f(x) = \tan^{-1} x + \tan^{-1}\left(\dfrac{1}{x}\right)$.

(b) Find $f'(x)$. **(c)** Find $f(1)$ and $f(-1)$. **(d)** Sketch the graph of $y = f(x)$.

9 Find the coordinates of any stationary points on the curve $y = 2\sin x + 3\cos x$, $0 \le x \le \dfrac{\pi}{2}$, and determine their nature.

10.4 INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

In the previous section we established the derivatives of the inverse trigonometric functions. From these, it follows that:

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C,\ -a < x < a \qquad \int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\frac{x}{a} + C,\ -a < x < a$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \text{ for all } x$$

NSM_SB_10.indd 194 18/04/13 10:47 AM

Example 20

Find: (a) $\int \frac{1}{\sqrt{4-x^2}}dx$ (b) $\int \frac{-1}{\sqrt{9-x^2}}dx$ (c) $\int \frac{1}{4+x^2}dx$ (d) $\int \frac{2}{\sqrt{3-x^2}}dx$

Solution

(a) $\int \frac{1}{\sqrt{4-x^2}}dx = \sin^{-1}\frac{x}{2}+C \text{ (as } a=2)$

(b) $\int \frac{-1}{\sqrt{9-x^2}}dx = \cos^{-1}\frac{x}{3}+C \text{ (as } a=3)$

(c) $\int \frac{1}{4+x^2}dx = \frac{1}{2}\tan^{-1}\frac{x}{2}+C \text{ (as } a=2)$

(d) $\int \frac{2}{\sqrt{3-x^2}}dx = 2\sin^{-1}\frac{x}{\sqrt{3}}+C \text{ (as } a=\sqrt{3})$

Sometimes a slight adjustment is necessary to make the coefficient of x^2 equal to 1.

Example 21

Find: (a) $\int \frac{1}{\sqrt{1-9x^2}}dx$ (b) $\int \frac{1}{1+4x^2}dx$ (c) $\int \frac{2}{\sqrt{4-25x^2}}dx$

Solution

(a) $\int \frac{1}{\sqrt{1-9x^2}}dx = \int \frac{1}{\sqrt{9\left(\frac{1}{9}-x^2\right)}}dx = \frac{1}{3}\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}dx = \frac{1}{3}\sin^{-1}3x+C \quad \left(\text{as } a=\frac{1}{3}\right)$

(b) $\int \frac{1}{1+4x^2}dx = \frac{1}{4}\int \frac{1}{\frac{1}{4}+x^2}dx = \frac{1}{4}\times 2\tan^{-1}2x+C = \frac{1}{2}\tan^{-1}2x+C \quad \left(\text{as } a=\frac{1}{2}\right)$

(c) $\int \frac{2}{\sqrt{4-25x^2}}dx = 2\int \frac{1}{\sqrt{25\left(\frac{4}{25}-x^2\right)}}dx = \frac{2}{5}\int \frac{1}{\sqrt{\frac{4}{25}-x^2}}dx = \frac{2}{5}\sin^{-1}\frac{5x}{2}+C \quad \left(\text{as } a=\frac{2}{5}\right)$

Example 22

Evaluate: (a) $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}}$ (b) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx$ (c) $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ (d) $\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$

Solution

(a) $\int_0^{2.5} \frac{dx}{\sqrt{25-x^2}} = \left[\sin^{-1}\frac{x}{5}\right]_0^{2.5}$

$= \sin^{-1}\frac{1}{2} - \sin^{-1}0$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

(b) $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}dx = \left[\cos^{-1}\frac{x}{2}\right]_{-\sqrt{3}}^{\sqrt{3}}$

$= \cos^{-1}\frac{\sqrt{3}}{2} - \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$= \frac{\pi}{6} - \frac{5\pi}{6}$

$= -\frac{2\pi}{3}$

NSM_SB_10.indd 195 18/04/13 10:47 AM

(c) $\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_1^{\sqrt{3}}$

$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$

$= \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$

(d) As $\sqrt{1-4x^2} = \sqrt{4\left(\frac{1}{4}-x^2\right)}$

$= 2\sqrt{\frac{1}{4}-x^2}$

$\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{dx}{\sqrt{\frac{1}{4}-x^2}}$

$= \frac{1}{2}\left[\sin^{-1} 2x\right]_{-\frac{1}{4}}^{\frac{1}{4}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) - \frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{12} + \frac{\pi}{12}$

$= \frac{\pi}{6}$

Example 23

Evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$.

Solution

There is no standard derivative that yields $\sin^{-1} x$, so finding its primitive is beyond the scope of this Mathematics Extension 1 course. (However, the primitive can be found using the method of 'integration by parts' as studied in the Mathematics Extension 2 course.)

It is possible to evaluate this integral by an alternative method. We can regard the integral as an area and use our knowledge of inverse functions to determine this area.

The graph of $f(x) = \sin^{-1} x$ is shown in the left diagram below. The shaded area A is the area bounded by the graph, the x-axis and the lines $x = 0$ and $x = \frac{1}{2}$. This area is equal to $\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$.

The graph of $g(x) = \sin x$ is shown at right, with the reflected area A now enclosed by this graph, $y = \frac{1}{2}$ and the y-axis. The shaded area B on the diagram can be determined by integration, and so:

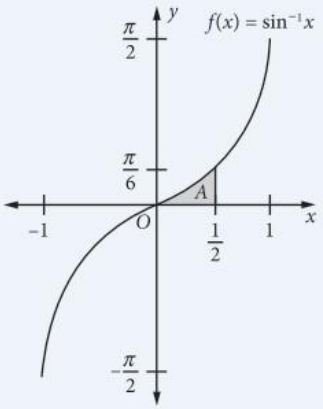
$\int_0^{\frac{1}{2}} \sin^{-1} x\, dx$ = Area of A

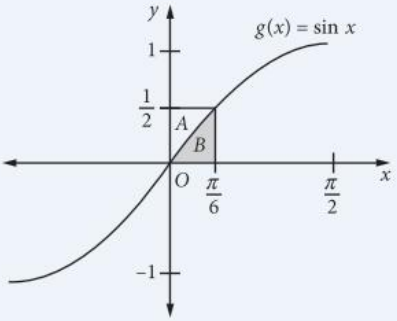
= Area of rectangle (i.e. $A + B$) − Area of B

$= \frac{\pi}{6} \times \frac{1}{2} - \int_0^{\frac{\pi}{6}} \sin x\, dx$

$= \frac{\pi}{12} - \left[-\cos x\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$





NSM_SB_10.indd 196 18/04/13 10:47 AM

EXERCISE 10.4 INTEGRATION INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

1 $\int \frac{dx}{\sqrt{16-(x+1)^2}} =$ (let $u = x+1$)

A $\sin^{-1} 4(x+1) + C$ **B** $\sin^{-1}\frac{x+1}{4} + C$ **C** $4\sin^{-1}\frac{x+1}{4} + C$ **D** $\frac{1}{4}\sin^{-1}\frac{x+1}{4} + C$

2 Find the following.

(a) $\int \frac{dx}{\sqrt{16-x^2}}$ (b) $\int \frac{3}{9+x^2}dx$ (c) $\int \frac{dx}{\sqrt{1-x^2}}$ (d) $\int \frac{-1}{\sqrt{5-x^2}}dx$

(e) $\int \frac{dx}{\sqrt{2-x^2}}$ (f) $\int \frac{5}{25+x^2}dx$ (g) $\int \frac{dx}{4+x^2}$ (h) $\int \frac{dx}{2+x^2}$

(i) $\int \frac{-1}{\sqrt{6-x^2}}dx$ (j) $\int \frac{dx}{\sqrt{1-4x^2}}$ (k) $\int \frac{dx}{1+9x^2}$ (l) $\int \frac{dx}{9+16x^2}$

(m) $\int \frac{dx}{x^2+16}$ (n) $\int \frac{dx}{\sqrt{1-3x^2}}$ (o) $\int \frac{-1}{\sqrt{16-9x^2}}dx$ (p) $\int \frac{dx}{\sqrt{5-4x^2}}$

(q) $\int \frac{dx}{4+(x+5)^2}$ (let $u = x+5$) (r) $\int \frac{dx}{\sqrt{2-(x-3)^2}}$ (let $u = x-3$) (s) $\int \frac{dx}{\sqrt{8-4(x+2)^2}}$ (let $u = x+2$)

3 Evaluate the following.

(a) $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ (b) $\int_0^{\sqrt{3}} \frac{dx}{x^2+9}$ (c) $\int_0^{\frac{1}{4}} \frac{dx}{\sqrt{1-4x^2}}$ (d) $\int_{-2}^{2} \frac{dx}{4+x^2}$

(e) $\int_{\frac{3}{5}}^{4} \frac{dx}{1+x^2}$ (f) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$ (g) $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$ (h) $\int_0^2 \frac{-1}{\sqrt{16-x^2}}dx$

(i) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$ (j) $\int_0^{\frac{1}{3}} \frac{dx}{1+9x^2}$ (k) $\int_0^1 \left(\frac{1}{1+x^2} + \frac{x}{1+x^2}\right)dx$ (l) $\int_0^{\sqrt{3}} \frac{dx}{\sqrt{3-x^2}}$

(m) $\int_{-4}^{4} \frac{dx}{x^2+16}$ (n) $\int_0^{\frac{\sqrt{3}}{6}} \frac{-1}{\sqrt{1-9x^2}}dx$ (o) $\int_0^{\frac{1}{2}} \frac{dx}{1+4x^2}$ (p) $\int_0^{\sqrt{2}} \frac{dx}{x^2+2}$

(q) $\int_{-1}^{1} \frac{dx}{\sqrt{2-x^2}}$ (r) $\int_{-5}^{5} \frac{dx}{\sqrt{100-x^2}}$ (s) $\int_0^{\frac{1}{2}} \cos^{-1} x\,dx$ (t) $\int_0^1 \tan^{-1} x\,dx$

4 Given that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ and $y = 1$ at $x = 0$, express y as a function of x.

5 Find the area bounded by the curve $y = \frac{1}{x^2+1}$, the x-axis and the ordinates $x = 1$ and $x = -1$.

6 On the same axes, sketch the graph of $y = 2\sin\frac{\pi x}{4}$ for $0 \le x \le 2$ and $x = 2\sin\frac{\pi y}{4}$ for $0 \le y \le 2$. Find the area of the region enclosed by the curves.

7 The curve $y = \frac{1}{\sqrt{1+x^2}}$ is rotated about the x-axis. Find the volume of the solid enclosed between $x = \frac{1}{\sqrt{3}}$ and $x = \sqrt{3}$.

8 Sketch the curve $y = \tan^{-1} x$ and find the area between this curve, the x-axis and the line $x = \sqrt{3}$.

9 Without evaluating the integral, explain why $\int_{-1}^{1} \tan^{-1} x\,dx$ is equal to zero.

10 Without using a calculator, prove that: (a) $\int_{\frac{3}{5}}^{4} \frac{dx}{1+x^2} = \frac{\pi}{4}$ (b) $\int_{-2}^{3} \frac{dx}{1+x^2} = \frac{3\pi}{4}$

NSM_SB_10.indd 197 18/04/13 10:47 AM

11 Find the volume of the solid of revolution formed by rotating the curve $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{2}$ about the x-axis. Hence find the volume of the solid formed by rotating the curve $y = \sin^{-1} x$ between $y = 0$ and $y = \frac{\pi}{2}$ about the y-axis.

12 Differentiate $x\cos^{-1} x - \sqrt{1-x^2}$ and use the result to evaluate $\int_0^1 \cos^{-1} x\,dx$.

13 **(a)** Sketch the graph of $f(x) = 1 - \cos x$, $0 \le x \le \pi$.

(b) Evaluate $\int_0^{\frac{\pi}{2}} f(x)dx$ and indicate the area represented by this integral on the graph of f.

(c) Find the inverse function $f^{-1}(x)$, stating its domain and range.

(d) Evaluate $\int_0^1 f^{-1}(x)dx$ and indicate the area represented by this integral on the graph of f.

14 **(a)** Prove that $\frac{d}{dx}(x\sin^{-1} x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$.

(b) Hence show that $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$. (You may use the substitution $u = 1 - x^2$.)

15 Use the substitution $x = \cos\theta$, where $0 \le \theta \le \pi$, to show that $\int \frac{\sqrt{1-x^2}}{x^2}dx = \cos^{-1} x - \frac{\sqrt{1-x^2}}{x} + C$.

16 Find $\int x\sqrt{9-x^2}\,dx$ using the substitution $x = 3\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Confirm your answer by using the alternative substitution $u = 9 - x^2$.

CHAPTER REVIEW 10

1 **(a)** Find the vertex and intercepts of the parabola $y = x^2 - 2x - 8$.
(b) State the restricted domain (containing $x = 0$) for which $f(x) = x^2 - 2x - 8$ has an inverse function.
(c) Find the equation of the inverse function $f^{-1}(x)$.
(d) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same graph.
(e) Show that $y = f(x)$ and $y = f^{-1}(x)$ intersect when $x^2 - 3x - 8 = 0$. Find the exact value of α, the x-coordinate of the point of intersection T.
(f) Find the gradient of $f^{-1}(x)$ at T.

2 Find the exact value of the following. **(a)** $\tan^{-1}\sqrt{3}$ **(b)** $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ **(c)** $\sin(\sin^{-1} 1)$

(d) $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$ **(e)** $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ **(f)** $\sin\left(2\tan^{-1}\frac{1}{3}\right)$ **(g)** $\tan(2\cos^{-1}(-0.4))$

3 Show that: **(a)** $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2} = \tan^{-1} 2$ **(b)** $\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{2\sqrt{5}}{25}$

4 For each of the following, state the domain and range and sketch the graph.

(a) $y = 2\cos^{-1}\frac{x}{2}$ **(b)** $y = \frac{1}{2}\sin^{-1}(2x-1)$ **(c)** $y = \frac{1}{2}\sin^{-1}(1-2x)$ (Hint: $1 - 2x = -[2x - 1]$)

(d) $y = \frac{\pi}{2} - \sin^{-1} 2x$ **(e)** $y = \cos(\sin^{-1} x)$

5 Find the equation of the normal to the curve $y = \tan^{-1}(2x + 1)$ at the point on the curve where $y = -\frac{\pi}{4}$.

6 Sketch the graph of the inverse function of $y = 3\cos^{-1}\frac{x}{2}$.

7 Differentiate the following. **(a)** $\sin^{-1} x$ **(b)** $\cos^{-1} 2x$ **(c)** $\tan^{-1}\frac{x}{2}$ **(d)** $e^x\sin^{-1} x$

(e) $e^{\sin^{-1} x}$ **(f)** $\tan^{-1}\frac{2}{x}$ **(g)** $x\cos^{-1} 2x$ **(h)** $\sqrt{\sin^{-1} x}$ **(i)** $\log_e(\tan^{-1} x)^3$

8 **(a)** Differentiate $x\cos^{-1} x$. **(b)** Hence find $\int \cos^{-1} x\,dx$. (The substitution $u = 1 - x^2$ will be helpful.)

NSM_SB_10.indd 198 18/04/13 10:47 AM

9 Find the following integrals. **(a)** $\int \frac{dx}{\sqrt{8-x^2}}$ **(b)** $\int \frac{dx}{64+x^2}$

(c) $\int \frac{4}{4+9x^2}dx$ **(d)** $\int \frac{dx}{\sqrt{4-8x^2}}$ **(e)** $\int \frac{dx}{\sqrt{16-(x+2)^2}}$ (using $u=x+2$)

10 Evaluate the following. **(a)** $\int_0^3 \frac{dx}{x^2+9}$ **(b)** $\int_0^{\sqrt{5}} \frac{dx}{\sqrt{20-x^2}}$ **(c)** $\int_0^{\frac{\sqrt{6}}{3}} \frac{dx}{9x^2+2}$

11 **(a)** Show that $\frac{x^3+x+2}{1+x^2} = x+\frac{2}{1+x^2}$. **(b)** Hence find $\int \frac{x^3+x+2}{1+x^2}dx$.

12 **(a)** Sketch $y=2\sin x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and show the area represented by $\int_0^1 \sin^{-1}\frac{x}{2}dx$.

(b) Hence evaluate $\int_0^1 \sin^{-1}\frac{x}{2}dx$.

13 Find the following integrals. These are a mix of different integral types that you have seen in this Mathematics Extension 1 course.

(a) $\int \frac{dx}{1+x}$ **(b)** $\int \frac{dx}{1+x^2}$ **(c)** $\int \frac{x}{1+x^2}dx$ **(d)** $\int \frac{1+x^2}{x}dx$

(e) $\int \frac{x^2}{1+x^2}dx$ (Hint: $\frac{x^2}{1+x^2} = 1-\frac{1}{1+x^2}$) **(f)** $\int \frac{x^3}{1+x^2}dx$ **(g)** $\int \frac{dx}{(1+x)^2}$

(h) $\int \frac{dx}{\sqrt{1+x}}$ **(i)** $\int \frac{dx}{\sqrt{1-x^2}}$ **(j)** $\int \frac{x}{\sqrt{1-x^2}}dx$ (using $u=1-x^2$)

(k) $\int \frac{e^x}{2+e^{2x}}dx$ (using $u=e^x$) **(l)** $\int \frac{\tan^{-1}3x}{1+9x^2}dx$ (using $u=\tan^{-1}3x$)

14 **(a)** Sketch both $y=\sin^{-1}x$ and $y=\cos^{-1}x$ on the same number plane.

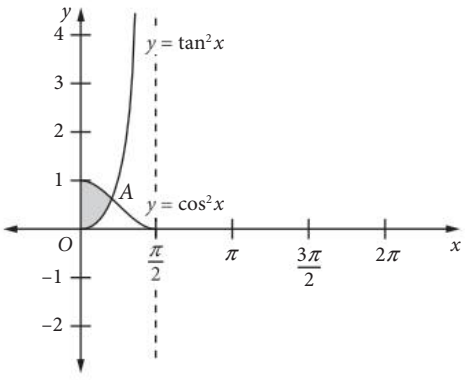
Hence determine the domain of the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

(b) Find the intercepts of the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ with the coordinate axes.

(c) Show that the function $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$ is increasing for all x in its domain.

(d) Sketch the graph of $y=\log_e\left(\frac{\sin^{-1}x}{\cos^{-1}x}\right)$.

15 The graph shows the curves $y=\cos^2 x$ and $y=\tan^2 x$ for $0 \le x \le \frac{\pi}{2}$.



(a) Show that the point of intersection A has the x-coordinate $\cos^{-1}\left(\sqrt{\frac{-1+\sqrt{5}}{2}}\right)$.

(b) Find the shaded area correct to one decimal place.

16 **(a)** Write the value of $\cos^{-1}x+\sin^{-1}x$ for $-1 \le x \le 1$.

(b) Use the substitution $u=\sin^{-1}x$ to show that: $\int_0^{\frac{1}{\sqrt{2}}} \left(\cos^{-1}x\right)\sqrt{\frac{\sin^{-1}x}{1-x^2}}dx = \frac{7\pi^{\frac{5}{2}}}{240}$

NSM_SB_10.indd 199 18/04/13 10:47 AM

CHAPTER 11
BINOMIAL THEOREM

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

11.1 EXPANSION OF $(1 + x)^n$, PASCAL'S TRIANGLE

By multiplying to expand the brackets, it can be shown that $(1 + x)^n$ is equal to the following:

$n = 0$: $(1 + x)^0 = 1$

$n = 1$: $(1 + x)^1 = 1 + x$

$n = 2$: $(1 + x)^2 = 1 + 2x + x^2$

$n = 3$: $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$

$n = 4$: $(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

$n = 5$: $(1 + x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$n = 6$: $(1 + x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

We can see that in each case, the expansion of $(1 + x)^n$ for $n = 1, 2, 3, \ldots$ is a polynomial of degree n in the variable x.

The coefficients of the successive powers of the expansion of $1 + x$ can be arranged in a triangular pattern called **Pascal's triangle**, as shown at right.

$(1 + x)^0$						1						
$(1 + x)^1$					1		1					
$(1 + x)^2$				1		2		1				
$(1 + x)^3$			1		3		3		1			
$(1 + x)^4$		1		4		6		4		1		
$(1 + x)^5$	1		5		10		10		5		1	
$(1 + x)^6$	1		6		15		20		15		6	1

Properties of Pascal's triangle

1. The first and last number in each row is 1.
2. Every number not at the end of a row is the sum of the two numbers in the row above it to the left and right (e.g. $10 = 4 + 6$, $15 = 5 + 10$, $20 = 10 + 10$).
3. The number of terms in each row is one more than the value of n.
4. The second and second-last terms in each row are the value of n.

These properties enable us to write the expansion of $(1 + x)^n$ for any integer value of n. For example:

$n = 7$: $(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$ (8 terms)

$n = 8$: $(1 + x)^8 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$ (9 terms)

Unfortunately, using these properties of Pascal's triangle to find the coefficients of any particular expansion requires all the previous rows. For example, to expand $(1 + x)^{12}$ in this way we would first need all the rows up to $n = 11$ to find the 12th row of Pascal's triangle.

However, there is a more efficient way to find the coefficients. We can establish a relationship for the coefficients for the expansion of the general case $(1 + x)^n$.

To do this, we can use the notation ${}^nC_r$ in place of the coefficient, where the right subscript r is the power of x and the left superscript n is the power n. ${}^nC_r$, previously defined as the number of combinations of n different objects taken r at a time (see Chapter 3 page 71), is also by definition the coefficient of x^r in $(1 + x)^n$.

Thus we can write the general expansion of $(1+x)^n$ as:

$$\begin{aligned}(1+x)^n &= {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n \\ &= \sum_{r=0}^{n} {}^nC_rx^r\end{aligned}$$

If we write $\frac{b}{a}$ in place of x and multiply both sides by a^n, we obtain:

$$\begin{aligned}(a+b)^n &= a^n\left(1+\frac{b}{a}\right)^n \\ &= a^n\left({}^nC_0 + {}^nC_1\frac{b}{a} + {}^nC_2\left(\frac{b}{a}\right)^2 + \quad + {}^nC_r\left(\frac{b}{a}\right)^r + \quad + {}^nC_n\left(\frac{b}{a}\right)^n\right) \\ &= {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \quad + {}^nC_ra^{n-r}b^r + \quad + {}^nC_nb^n \\ &= \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r\end{aligned}$$

This is the **binomial theorem**. Notice that in each term the sum of the powers in $a^{n-r}b^r$ is $n-r+r=n$. As the power of a decreases with each term, the power of b increases by the same amount.

Example 1

Write the expansion of $(1+2x)^4$, using Pascal's triangle to obtain the coefficients.

Solution

Write: $(1+2x)^4 = {}^4C_0 + {}^4C_1(2x)^1 + {}^4C_2(2x)^2 + {}^4C_3(2x)^3 + {}^4C_4(2x)^4$

From Pascal's triangle: ${}^4C_0 = 1,\ {}^4C_1 = 4,\ {}^4C_2 = 6,\ {}^4C_3 = 4,\ {}^4C_4 = 1$

$\therefore$ $(1+2x)^4 = 1 + 4\times 2x + 6\times 4x^2 + 4\times 8x^3 + 1\times 16x^4$

$= 1 + 8x + 24x^2 + 32x^3 + 16x^4$

Example 2

Write the expansion of $(1-x)^3$.

Solution

Write: $(1-x)^3 = {}^3C_0 + {}^3C_1(-x)^1 + {}^3C_2(-x)^2 + {}^3C_3(-x)^3$

From Pascal's triangle: ${}^3C_0 = 1,\ {}^3C_1 = 3,\ {}^3C_2 = 3,\ {}^3C_3 = 1$

$\therefore$ $(1-x)^3 = 1 + 3\times(-x) + 3\times x^2 + 1\times(-x^3)$

$= 1 - 3x + 3x^2 - x^3$

Example 3

Write the simplified expansion of $\left(1+\sqrt{3}\right)^6$.

Solution

$$\begin{aligned}\left(1+\sqrt{3}\right)^6 &= {}^6C_0 + {}^6C_1\times\sqrt{3} + {}^6C_2\times\left(\sqrt{3}\right)^2 + {}^6C_3\times\left(\sqrt{3}\right)^3 + {}^6C_4\times\left(\sqrt{3}\right)^4 + {}^6C_5\times\left(\sqrt{3}\right)^5 + {}^6C_6\times\left(\sqrt{3}\right)^6 \\ &= 1 + 6\times\sqrt{3} + 15\times\left(\sqrt{3}\right)^2 + 20\times\left(\sqrt{3}\right)^3 + 15\times\left(\sqrt{3}\right)^4 + 6\times\left(\sqrt{3}\right)^5 + \left(\sqrt{3}\right)^6 \\ &= 1 + 6\sqrt{3} + 45 + 60\sqrt{3} + 135 + 54\sqrt{3} + 27 \\ &= 208 + 120\sqrt{3}\end{aligned}$$

Example 4

Find the coefficient of a^4 in $(1+5a)^5$.

Solution

Method 1

$(1+5a)^5 = 1 + 5\times 5a + 10\times(5a)^2 + 10\times(5a)^3 + 5\times(5a)^4 + \ldots$

After the required term is found, there is no need to find succeeding terms.

$(1+5a)^5 = 1 + 25a + 250a^2 + 1250a^3 + 3125a^4 + \ldots$

The coefficient of a^4 is 3125.

Method 2

For the expansion we can write $T_1 = {}^5C_0$, $T_2 = {}^5C_1(5a)^1$ and so on, leading to the general term $T_{r+1} = {}^5C_r(5a)^r$.

For the term in a^4, $r = 4$: $T_5 = {}^5C_4(5a)^4 = 5\times 5^4 a^4 = 3125a^4$

Hence the coefficient of a^4 is 3125.

Example 5

Use the expansion of $\left(1-\frac{1}{x}\right)^5$ to find an approximation for $\left(\frac{99}{100}\right)^5$ correct to four decimal places.

Solution

We can write this as an expansion: $\left(\frac{99}{100}\right)^5 = \left(1-\frac{1}{100}\right)^5$

$$= 1 - 5\times\frac{1}{100} + 10\times\frac{1}{100^2} - 10\times\frac{1}{100^3} +$$

$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} +$$

$$= 1 - 0.05 + 0.001 - 0.00001 +$$

$$\approx 0.9510$$

As the accuracy is only to four decimal places, the terms smaller than 0.000 01 are insignificant and can be ignored.

$\therefore \left(\frac{99}{100}\right)^5 \approx 0.9510$ correct to four decimal places.

EXERCISE 11.1 EXPANSION OF $(1+x)^n$, PASCAL'S TRIANGLE

1 Use Pascal's triangle to find the expansion of each of the following.

(a) $(1+x)^6$ **(b)** $(1+x)^8$ **(c)** $(1+b)^7$ **(d)** $(1+2x)^3$ **(e)** $(1+4x)^5$

(f) $(1+3x)^6$ **(g)** $(1-x)^3$ **(h)** $(1-a)^8$ **(i)** $(1-2x)^4$ **(j)** $\left(1+\frac{x}{2}\right)^5$

(k) $\left(1-\frac{x}{3}\right)^6$ **(l)** $\left(1+\frac{2x}{5}\right)^5$ **(m)** $(1+x^2)^3$ **(n)** $(1-x^2)^4$ **(o)** $\left(x+\frac{1}{x}\right)^5$

2 Find the coefficient of x^2 in the expansion of:

(a) $(1+x)^4$ **(b)** $(1+x)^7$ **(c)** $(1-x)^5$ **(d)** $(1+3x)^3$ **(e)** $(1+2x)^5$ **(f)** $(1-3x)^4$

3 Find the coefficient of b^3 in the expansion of: **(a)** $(1+2b)^4$ **(b)** $(1-4b)^3$ **(c)** $\left(1+\frac{2b}{3}\right)^5$

4 The coefficient of x^4 in the expansion of $\left(x-\sqrt{2}\right)^6$ is:

A -30 **B** 30 **C** -60 **D** 60

5 Find the fourth term in the expansion of: **(a)** $(1+x)^5$ **(b)** $(1-2a)^7$ **(c)** $\left(1+\frac{3x}{4}\right)^6$

6 Write the simplified expansion for each of the following. **(a)** $\left(1+\sqrt{2}\right)^4$ **(b)** $\left(1-\sqrt{3}\right)^3$

(c) $\left(1+\sqrt{5}\right)^6$ **(d)** $\left(1+2\sqrt{3}\right)^3$ **(e)** $\left(1-3\sqrt{2}\right)^4$ **(f)** $\left(1+\frac{\sqrt{3}}{2}\right)^5$

7 Use the expansion of $\left(1-\frac{1}{x}\right)^n$ to find the value of each expression correct to four decimal places.

(a) $\left(\frac{9}{10}\right)^6$ **(b)** $\left(\frac{19}{20}\right)^5$ **(c)** $\left(\frac{99}{100}\right)^4$

11.2 MORE PASCAL'S TRIANGLE EXPANSIONS

We will now consider expansions of the type:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \quad + {}^nC_ra^{n-r}b^r + \quad + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

where the ${}^nC_r$ are the numbers in the $(n+1)$-th row of Pascal's triangle.

Example 6

Write the expansion of $(a+x)^5$.

Solution

$$(a+x)^5 = {}^5C_0a^5x^0 + {}^5C_1a^4x^1 + {}^5C_2a^3x^2 + {}^5C_3a^2x^3 + {}^5C_4a^1x^4 + {}^5C_5a^0x^5$$

where the ${}^5C_r$ are the numbers in the 6th row of Pascal's triangle.

$$(a+x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5a^1x^4 + x^5$$

After writing the expansion, you can check your result by adding the powers on the variables in each term to check that the sum is 5.

Example 7

Write the expansion of $(2+x)^4$.

Solution

Coefficients are the numbers in the fifth row of Pascal's triangle:

$$\begin{aligned}(2+x)^4 &= 1\times 2^4x^0 + 4\times 2^3x^1 + 6\times 2^2x^2 + 4\times 2^1x^3 + 1\times 2^0x^4\\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

Example 8

Write the expansion of $(2a - b)^5$.

Solution

Coefficients are the numbers in the sixth row of Pascal's triangle:

$$\begin{aligned}(2a-b)^5 &= (2a)^5 + 5\times(2a)^4\times(-b)^1 + 10\times(2a)^3\times(-b)^2 + 10\times(2a)^2\times(-b)^3 + 5\times(2a)\times(-b)^4 + (-b)^5\\ &= 32a^5 - 5\times 16a^4b + 10\times 8a^3b^2 - 10\times 4a^2b^3 + 10ab^4 - b^5\\ &= 32a^5 - 80a^4b + 80a^3b^2 - 40a^2b^3 + 10ab^4 - b^5\end{aligned}$$

Example 9

Write the expansion of $\left(x+\frac{1}{x}\right)^4$.

Solution

Coefficients are the numbers in the fifth row of Pascal's triangle:

$$\begin{aligned}\left(x+\frac{1}{x}\right)^4 &= x^4 + 4x^3\times\frac{1}{x} + 6x^2\times\frac{1}{x^2} + 4x\times\frac{1}{x^3} + \frac{1}{x^4}\\ &= x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\end{aligned}$$

In this example, two terms that were functions of x create a constant term in the expansion. We can say that 'the term independent of x' (i.e. the term without an x) is 6.

EXERCISE 11.2 MORE PASCAL'S TRIANGLE EXPANSIONS

1 Use Pascal's triangle to find the expansion of the following.

(a) $(2+x)^3$ **(b)** $(3+x)^5$ **(c)** $(b+2)^4$ **(d)** $(a+b)^7$ **(e)** $(x-y)^8$ **(f)** $(1+3x)^6$

(g) $(3-x)^3$ **(h)** $(2-a)^8$ **(i)** $(3y-2x)^4$ **(j)** $\left(x+\frac{1}{x}\right)^5$ **(k)** $\left(x-\frac{1}{x}\right)^4$ **(l)** $\left(x^2+\frac{1}{x}\right)^6$

2 Write the simplified expansion for each of the following. **(a)** $\left(2+\sqrt{2}\right)^4$ **(b)** $\left(4-\sqrt{3}\right)^3$

(c) $\left(\sqrt{2}+\sqrt{5}\right)^6$ **(d)** $\left(\sqrt{5}+\sqrt{3}\right)^5$ **(e)** $\left(2\sqrt{3}-3\sqrt{2}\right)^4$ **(f)** $\left(\sqrt{6}+2\sqrt{3}\right)^5$

3 Write the coefficient of x^5 in the expansion of: **(a)** $(3+x)^6$ **(b)** $(2-x)^7$ **(c)** $\left(x+\frac{1}{x}\right)^7$

4 The coefficient of x^5 in the expansion of $(2x+5)^8$ is:

A 32 000 **B** 4000 **C** 224 000 **D** 1792

5 Find the term independent of x in each expansion. **(a)** $(2x+3)^4$ **(b)** $\left(x+\frac{1}{x}\right)^6$ **(c)** $\left(x+\frac{1}{x^2}\right)^6$

11.3 PASCAL'S TRIANGLE RELATIONS AND THE BINOMIAL THEOREM

There are two separate Pascal's triangle relations: for the outer coefficients and for the other (inner) coefficients.

Pascal's triangle relation—outer coefficients

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \quad + {}^nC_ra^{n-r}b^r + \quad + {}^nC_nb^n$$

When $a = 1$ and $b = 0$: $(1+0)^n = {}^nC_0(1)^n + 0 + 0 + \ldots$

$\therefore\ {}^nC_0 = 1$

When $a = 0$ and $b = 1$: $(0+1)^n = 0 + 0 + \ldots + {}^nC_n(1)^n$

$\therefore\ {}^nC_n = 1$

Pascal's triangle relation—inner coefficients

$$(1+x)^{n-1} = {}^{n-1}C_0 + {}^{n-1}C_1x + {}^{n-1}C_2x^2 + \quad + {}^{n-1}C_rx^r + \quad + {}^{n-1}C_{n-1}x^{n-1}$$

$$x(1+x)^{n-1} = \quad {}^{n-1}C_0x + {}^{n-1}C_1x^2 + \ldots + {}^{n-1}C_{r-1}x^r + \ldots + {}^{n-1}C_{n-2}x^{n-1} + {}^{n-1}C_{n-1}x^n$$

Adding the above two lines, the left-hand side becomes: $(1+x)^{n-1} + x(1+x)^{n-1} = (1+x)(1+x)^{n-1} = (1+x)^n$

Hence:

$$(1+x)^n = {}^{n-1}C_0 + \left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \left({}^{n-1}C_1 + {}^{n-1}C_2\right)x^2 + \quad + \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)x^r + \quad + \left({}^{n-1}C_{n-2} + {}^{n-1}C_{n-1}\right)x^{n-1} + {}^{n-1}C_{n-1}x^n$$

Comparing these two expansions for $(1+x)^n$, we now have:

$$(1+x)^n = {}^nC_0 + {}^nC_1x + \quad + {}^nC_rx^r + \quad + {}^nC_nx^n \qquad [1]$$

$$(1+x)^n = {}^{n-1}C_0 + \left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \quad + \left({}^{n-1}C_{r-1} + {}^{n-1}C_r\right)x^r + \quad + {}^{n-1}C_{n-1}x^n \qquad [2]$$

Equating the coefficients of x^r: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 \le r \le n-1$

In terms of Pascal's triangle, this result is equivalent to saying that a term in one row is the sum of the two terms in the row above to the left and to the right (e.g. ${}^5C_3 = {}^4C_2 + {}^4C_3$).

The formula for ${}^nC_r$

The formula for ${}^nC_r$ can be proved by mathematical induction. (For a discussion of induction, see Chapter 7.)

When $n = 4$, $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

The ratios of successive coefficients are:

- $\frac{4}{1} = \frac{n}{1}$
- $\frac{6}{4} = \frac{3}{2} = \frac{n-1}{2}$
- $\frac{4}{6} = \frac{2}{3} = \frac{n-2}{3}$
- $\frac{1}{4} = \frac{n-3}{4}$

Thus, the coefficients in this expansion from left to right are:

- 1
- $\frac{n}{1}$
- $\frac{n}{1} \times \frac{n-1}{2}$
- $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$

This leads us to guess that perhaps ${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \times 2 \times 3 \times \ldots \times r}$, $1 \le r \le n$.

Call this statement $S(n)$. The coefficient of ${}^nC_0$ has already been shown to be 1.

Step 1 Prove that statement $S(n)$ is true for $n = 2$.

$(1+x)^2 = 1 + 2x + x^2 = {}^2C_0x^0 + {}^2C_1x^1 + {}^2C_2x^2$

$\therefore\ {}^2C_0 = 1,\ {}^2C_1 = \frac{2}{1},\ {}^2C_2 = \frac{2}{1} \times \frac{2-1}{2} = 1$

$\therefore\ S(2)$ is true.

Step 2 Assume $S(n-1)$ is true (to show that if $S(n-1)$ is true, then $S(n)$ is true).

i.e. assume that ${}^{n-1}C_r = \frac{(n-1)(n-2)(n-3)\ldots(n-r)}{1 \times 2 \times 3 \times \ldots \times r}$ is true

First, for $r = 1$: assume that ${}^{n-1}C_1 = \frac{n-1}{1}$ from the statement $S(n-1)$.

We have previously shown that ${}^{n-1}C_0 = 1$.

Adding these results using the Pascal's triangle relation (also shown previously):

$${}^{n}C_1 = {}^{n-1}C_0 + {}^{n-1}C_1$$

$${}^{n}C_1 = 1 + \frac{n-1}{1} = \frac{n}{1}$$, which agrees with the statement $S(n)$.

Now, for $r \geq 2$, i.e. $r - 1 \geq 1$: assume that ${}^{n-1}C_{r-1} = \dfrac{(n-1)(n-2)(n-3)\quad(n-r+1)}{1\times 2\times 3\times\quad\times(r-1)}$ from the statement $S(n-1)$. Again using the Pascal's triangle relation:

$$\begin{aligned}
{}^{n}C_r &= {}^{n-1}C_{r-1} + {}^{n-1}C_r \\
&= \frac{(n-1)(n-2)(n-3)\quad(n-r+1)}{1\times 2\times 3\times\quad\times(r-1)} + \frac{(n-1)(n-2)(n-3)\quad(n-r)}{1\times 2\times 3\times\quad\times r} \\
&= \frac{(n-1)(n-2)(n-3)\quad(n-r+1)}{1\times 2\times 3\times\quad\times(r-1)}\left[1 + \frac{n-r}{r}\right] \\
&= \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{1\times 2\times 3\times\ldots\times r} \quad \text{as } 1 + \frac{n-r}{r} = \frac{n}{r}
\end{aligned}$$

Assuming $S(n-1)$ is true, this now proves the statement $S(n)$ for $r \geq 2$ (which was also already proved for $r = 1$ above).

Step 3 Conclusion

$S(n)$ is true for $r \geq 1$ if $S(n-1)$ is true (Step 2)

$S(2)$ is true (Step 1)

$\therefore$ by induction, $S(n)$ is true for $n \geq 2$, $r \geq 1$

Note: This formula for ${}^{n}C_r$ is the same as the formula established for it in Chapter 3. This can now be used to obtain binomial expansions without having to construct the rows of Pascal's triangle.

Another useful form of this formula is: $${}^{n}C_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Binomial theorem—alternative approach

The expansion of $(1+x)^n$ can be thought of as the product of n factors, each of which is $(1+x)$. The term containing x^r must be found by selecting x out of any r of the n factors and 1 out of the remaining $(r-1)$ factors. The coefficient of x^r is therefore equal to the number of ways of selecting r lots of x from a total of n lots of x. Hence by definition the coefficient of x^r is ${}^{n}C_r$. By making $r = 0, 1, 2, 3, \ldots, n$ in succession we can obtain the coefficient of each term. Thus:

$$(1+x)^n = 1 + {}^{n}C_1 x + {}^{n}C_2 x^2 + \quad + {}^{n}C_r x^r + \quad + x^n = \sum_{r=0}^{n} {}^{n}C_r x^r$$

The coefficients ${}^{n}C_0$ and ${}^{n}C_n$ are both equal to 1.

Making $x = \dfrac{b}{a}$ and multiplying both sides by a^n gives the expansion for $(a+b)^n$:

$$(a+b)^n = a^n + {}^{n}C_1 a^{n-1} b + {}^{n}C_2 a^{n-2} b^2 + \quad + {}^{n}C_r a^{n-r} b^r + \quad + b^n = \sum_{r=0}^{n} {}^{n}C_r a^{n-r} b^r$$

Note:

- The expansion of $(a+b)^n$ contains $(n+1)$ terms. ${}^{n}C_r a^{n-r} b^r$ is the $(r+1)$-th term and is called the **general term**.

- If we write $-b$ in place of b, the expansion becomes:
  $(a-b)^n = a^n - {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 - \quad + {}^nC_ra^{n-r}(-b)^r + \quad + (-b)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_ra^{n-r}b^r$
  The terms are alternately positive and negative.
- Because $(a+b)^n = (b+a)^n$ and ${}^nC_r = {}^nC_{n-r}$, any term from one end of the expansion of $(a+b)^n$ is the same term counted from the opposite end of the expansion of $(b+a)^n$.

Example 10

Write the expansion of $(2x-y)^5$.

Solution

$$(2x-y)^5 = (2x)^5 - {}^5C_1(2x)^4y + {}^5C_2(2x)^3y^2 - {}^5C_3(2x)^2y^3 + {}^5C_4(2x)y^4 - y^5$$
$$= 32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$$

Example 11

Find the coefficient of x^4 in each expansion. **(a)** $(1+x)^6$ **(b)** $(3x-2)^7$

Solution

(The alternative notation $\binom{n}{r} = {}^nC_r$ is used below. Either notation may be used.)

Method 1

(a) $(1+x)^6 = 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 +$

The expansion does not need to be written out in full, only far enough to find the required term.

Coefficient of x^4: $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6\times5\times4!}{4!\times2} = \frac{6\times5}{2} = 15$

(Recall from Chapter 3 page 72 that $\binom{6}{4} = \binom{6}{6-4} = \binom{6}{2} = \frac{6\times5}{2\times1} = 15$, which can simplify the calculation.)

(b) $(3x-2)^7 = (3x)^7 - {}^7C_1(3x)^6\times2 + {}^7C_2(3x)^5\times2^2 - {}^7C_3(3x)^4\times2^3 +$

Term in x^4: $(-1)\times{}^7C_3\times3^4\times2^3x^4 = -\frac{7\times6\times5}{3\times2\times1}\times81\times8x^4$

Coefficient of x^4: $-22\,680$

Method 2

(a) In the expansion of $(1+x)^6$, $T_{r+1} = \binom{6}{r}x^r$

Term in x^4: $T_5 = \binom{6}{4}x^4 = \frac{6\times5}{2\times1}x^4 = 15x^4$

Coefficient of x^4: 15

(b) In the expansion of $(3x-2)^7$, $T_{r+1} = \binom{7}{r}(3x)^{7-r}(-2)^r$

The term for x^4 requires $7-r=4$, so $r=3$.

Term in x^4: $T_4 = \binom{7}{3}(3x)^4(-2)^3 = -\frac{7\times6\times5}{3\times2\times1}\times81\times8x^4 = -22\,680x^4$

Coefficient of x^4: $-22\,680$

Example 12

Write the sixth term in the expansion of $(2x-3y)^8$.

Solution

The general term is ${}^8C_r(2x)^r(-3y)^{8-r}$. This is the $(r+1)$-th term: $T_{r+1} = {}^8C_r(2x)^r(-3y)^{8-r}$

For the sixth term, $r+1=6$, so $r=5$: $T_6 = {}^8C_5(2x)^5(-3y)^3$ (Remember ${}^8C_5 = {}^8C_3$)

$$= (-1)^3 \frac{8\times7\times6}{3\times2\times1} \times 2^5 3^3 x^5 y^3$$

$$= -48384x^5y^3$$

EXERCISE 11.3 PASCAL'S TRIANGLE RELATIONS AND THE BINOMIAL THEOREM

1 Without using a calculator, find the value of:

(a) ${}^5C_2$ **(b)** ${}^8C_4$ **(c)** ${}^{10}C_6$ **(d)** $\binom{7}{3}$ **(e)** $\binom{9}{4}$ **(f)** $\binom{15}{12}$

2 Write the expansion of: **(a)** $(1+x)^6$ **(b)** $(2x+3y)^4$ **(c)** $(x^2+a)^5$ **(d)** $\left(a+\frac{1}{a}\right)^6$

(e) $(x^2-3y^2)^4$ **(f)** $\left(\frac{2a}{3}-\frac{3}{2b}\right)^5$ **(g)** $\left(2y-\frac{3x^2}{2}\right)^4$ **(h)** $(x^3-2x^{-2})^6$ **(i)** $\left(\frac{a^2}{2}-3b\right)^5$

3 Find the coefficient of x^3 for each expansion.

(a) $(x+2)^5$ **(b)** $(x-3)^4$ **(c)** $(4-x)^3$ **(d)** $(3+x)^6$ **(e)** $(2x+1)^7$ **(f)** $(3x-2)^8$

4 Write the fourth term in the expansion of:

(a) $(x+2)^5$ **(b)** $(x-3)^4$ **(c)** $(4-x)^3$ **(d)** $(3y+x)^6$ **(e)** $(2x+y)^7$ **(f)** $(3x-2z)^8$

5 The fourth term in the expansion of $\left(\frac{a}{b}-\frac{b}{a}\right)^6$ is:

A -20 **B** 20 **C** $\frac{15a^2}{b^2}$ **D** $-\frac{15b^2}{a^2}$

6 Write and simplify the required term for each.

(a) the fourth term of $\left(\frac{m}{2}+3n\right)^8$ **(b)** the seventh term of $\left(\frac{2x}{3}-\frac{3}{2x}\right)^{11}$

(c) the fifth term of $\left(a-\frac{2b}{3}\right)^7$ **(d)** the fourth term of $\left(x-\frac{1}{x}\right)^n$

7 Find the coefficient of:

(a) x^2 in $\left(x+\frac{1}{x}\right)^8$ **(b)** x^{17} in $(2x^2-3x)^{12}$ **(c)** a^5b^4 in $\left(3a-\frac{b}{3}\right)^9$ **(d)** a^9 in $\left(2a-\frac{3}{a}\right)^{13}$

8 Write and simplify the $(k+2)$-th term of $(a+b)^{2n}$.

11.4 HARDER BINOMIAL EXPANSIONS

Example 13

In the expansion of $\left(2x^3-\frac{1}{x}\right)^{12}$, calculate:

(a) the term independent of x (constant term) **(b)** the coefficient of x^8.

Solution

Method 1

$$\left(2x^3-\frac{1}{x}\right)^{12}=\left(2x^3\right)^{12}-{}^{12}C_1\left(2x^3\right)^{11}\times\frac{1}{x}+{}^{12}C_2\left(2x^3\right)^{10}\times\frac{1}{x^2}-$$

1st term contains x^{36}, 2nd term contains x^{32}, 3rd term contains x^{28}, …

Following this pattern, the 10th term contains x^0 (constant term) and the 8th term contains x^8.

(a) Constant term $= T_{10}$

$$={}^{12}C_9\left(2x^3\right)^3\left(-\frac{1}{x}\right)^9$$

$$=-\frac{12\times11\times10}{3\times2\times1}\times2^3$$

$$=-1760$$

(b) Coefficient of x^8 = coefficient of T_8

$$=-\binom{12}{7}\times2^5$$

$$=-\frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1}\times32$$

$$=-25344$$

Method 2

$$T_{r+1}={}^{12}C_r\left(2x^3\right)^{12-r}\times\left(-\frac{1}{x}\right)^r$$

$$=(-1)^r\times{}^{12}C_r\times2^{12-r}\times x^{36-3r}\times x^{-r}$$

$\therefore$ Power of $x = 36-3r-r$

$= 36-4r$

(a) For the constant term, the power of x is zero:

$36-4r=0$

$r=9$

Constant term: $(-1)^9\times{}^{12}C_9\times2^{12-9}\times x^{36-27}\times x^{-9}$

$=-{}^{12}C_3\times2^3\times x^0$

$=-1760$

(b) Coefficient of x^8:

$36-4r=8$

$r=7$

Coefficient of x^8: $(-1)^7\times{}^{12}C_7\times2^{12-7}$

$=-{}^{12}C_5\times2^5$

$=-25344$

Example 14

Find the coefficient of x in the expansion of $\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$.

Solution

Method 1

$$\left(x+\frac{1}{x}\right)^4=x^4+{}^4C_1x^3\left(\frac{1}{x}\right)+{}^4C_2x^2\left(\frac{1}{x}\right)^2+{}^4C_3x\left(\frac{1}{x}\right)^3+\left(\frac{1}{x}\right)^4$$

$$=x^4+{}^4C_1x^2+{}^4C_2+{}^4C_3\times\frac{1}{x^2}+\frac{1}{x^4}$$

$$\left(x-\frac{1}{x}\right)^3=x^3+{}^3C_1x^2\left(-\frac{1}{x}\right)+{}^3C_2x\left(-\frac{1}{x}\right)^2+\left(-\frac{1}{x}\right)^3$$

$$=x^3-{}^3C_1x+{}^3C_2\times\frac{1}{x}-\frac{1}{x^3}$$

$$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3=\left[x^4+{}^4C_1x^2+{}^4C_2+{}^4C_3\times\frac{1}{x^2}+\frac{1}{x^4}\right]\times\left[x^3-{}^3C_1x+{}^3C_2\times\frac{1}{x}-\frac{1}{x^3}\right]$$

Rather than expanding the whole expression, it is better to find only the products of terms that will generate a term in x. The terms that will do this are:

- $x^4 \times \frac{1}{x^3} = x$
- $x^2 \times \frac{1}{x} = x$
- $1 \times x = x$
- $\frac{1}{x^2} \times x^3 = x$

Thus the term in x is: $x^4 \times \left(-\frac{1}{x^3}\right) + {}^4C_1x^2 \times {}^3C_2 \times \frac{1}{x} + {}^4C_2 \times \left(-{}^3C_1x\right) + {}^4C_3 \times \frac{1}{x^2} \times x^3$

$= -x + 4 \times 3x - 6 \times 3x + 4x$

$= -3x$

$\therefore$ Coefficient of x: -3

Method 2

$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3 = \left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^3$ using the fact: $\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right) = \left(x^2-\frac{1}{x^2}\right)$

$$= \left(x+\frac{1}{x}\right)\left(\left(x^2\right)^3 - {}^3C_1\left(x^2\right)^2\left(\frac{1}{x^2}\right) + {}^3C_2\left(x^2\right)\left(\frac{1}{x^2}\right)^2 - \left(\frac{1}{x^2}\right)^3\right)$$

$$= \left(x+\frac{1}{x}\right)\left(x^6 - {}^3C_1x^2 + {}^3C_2\left(\frac{1}{x^2}\right) - \frac{1}{x^6}\right)$$

On expansion, the only term in x is $\frac{1}{x} \times \left(-{}^3C_1x^2\right)$

$\therefore$ Coefficient of x: -3

Example 15

Three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio $6:3:1$. Find the value of n and state which terms possess these coefficients.

Solution

Let the three consecutive coefficients be ${}^nC_r$, ${}^nC_{r+1}$, ${}^nC_{r+2}$.

Hence: $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{6}{3}$ and $\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = 3$

So: $\frac{n!}{r!(n-r)!} \times \frac{(r+1)!(n-r-1)!}{n!} = 2$

$$\frac{n!}{r!(n-r)(n-r-1)!} \times \frac{(r+1)r!(n-r-1)!}{n!} = 2$$

$$\frac{r+1}{n-r} = 2 \qquad [1]$$

Similarly: $\frac{n!}{(r+1)!(n-r-1)!} \times \frac{(r+2)!(n-r-2)!}{n!} = 3$

$$\frac{n!}{(r+1)!(n-r-1)(n-r-2)!} \times \frac{(r+2)(r+1)!(n-r-2)!}{n!} = 3$$

$$\frac{r+2}{n-r-1} = 3 \qquad [2]$$

Rewrite [1]: $2n - 3r = 1$ [3]

Rewrite [2]: $3n - 4r = 5$ [4]

$-4 \times [3]$: $-8n + 12r = -4$ [5]

$3 \times [4]$: $9n - 12r = 15$ [6]

[5] + [6]: $n = 11$

$\therefore \quad r = 7$

The terms possessing these coefficients are ${}^{11}C_7x^7$, ${}^{11}C_8x^8$ and ${}^{11}C_9x^9$.

Example 16

Find the ratio of the $(r+1)$-th term to the r-th term in the expansion of $(a+b)^n$.

Solution

$(r+1)$-th term: $T_{r+1} = {}^nC_r a^{n-r}b^r$

r-th term: $T_r = {}^nC_{r-1}a^{n-r+1}b^{r-1}$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r a^{n-r}b^r}{{}^nC_{r-1}a^{n-r+1}b^{r-1}}$$

$$= \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-r+1)!}{n!} \times \frac{a^{n-r}b^r}{a^{n-r+1}b^{r-1}}$$

$$= \frac{n-r+1}{r} \times \frac{b}{a}$$

The result from Example **16** provides the ratio of successive terms in the expansion of $(a+b)^n$:

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{b}{a}$$

This ratio also allows us to find the greatest term or the greatest coefficient in the binomial expansion, as shown in the following example.

Example 17

Find the greatest coefficient in the expansion of $(5+2x)^{12}$.

Solution

$T_{r+1} = {}^{12}C_r 5^{12-r}(2x)^r \qquad T_r = {}^{12}C_{r-1}5^{13-r}(2x)^{r-1}$

Let C_{r+1} and C_r be the coefficients of T_{r+1} and T_r respectively:

$$\frac{C_{r+1}}{C_r} = \frac{{}^{12}C_r 5^{12-r}2^r}{{}^{12}C_{r-1}5^{13-r}2^{r-1}}$$

$$= \frac{12!}{r!(12-r)!} \times \frac{(r-1)!(13-r)!}{12!} \times \frac{2}{5}$$

$$= \frac{13-r}{r} \times \frac{2}{5}$$

$$= \frac{26-2r}{5r}$$

So: $C_{r+1} \geq C_r$ if $26 - 2r \geq 5r$

$26 \geq 7r$

$r \leq 3\frac{5}{7}$

If $r = 3$, $C_4 > C_3$

If $r = 2$, $C_3 > C_2$ and so on

$\therefore C_4 > C_3 > C_2 \ldots$

If $r = 4$, $C_5 < C_4$

If $r = 5$, $C_6 < C_5$ and so on

$\therefore C_1 < C_2 < C_3 < C_4 > C_5 > C_6 \ldots$

Hence the greatest coefficient is C_4 (when $r = 3$): $C_4 = {}^{12}C_3 \times 5^9 \times 2^3$

$= 5^{10} \times 2^5 \times 11$

Example 18

Find the numerically greatest terms in the expansion of $(1-2x)^9$ when $x=\frac{1}{3}$.

Solution

We are only interested in the numerically greatest term, so we can consider the easier expansion of $(1+2x)^9$ instead.

Ratio of successive terms: $\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\times\frac{b}{a}$

Now $n=9$, $a=1$, $b=\frac{2}{3}$: $\frac{T_{r+1}}{T_r}=\frac{10-r}{r}\times\frac{\frac{2}{3}}{1}$

$$=\frac{20-2r}{3r}$$

$T_{r+1}\geq T_r$ if: $\frac{20-2r}{3r}\geq 1$

$$20-2r\geq 3r$$

$$r\leq 4$$

Hence $T_1<T_2<T_3<T_4=T_5>T_6>T_7\ldots$

The terms increase in magnitude until the fourth term, which is equal to the fifth term.
Terms after the fifth term decrease in magnitude.

$$T_4={}^9C_3\left(\frac{2}{3}\right)^3=\frac{224}{9}\qquad T_5={}^9C_4\left(\frac{2}{3}\right)^4=\frac{224}{9}$$

Thus in the expansion of $(1-2x)^9$ the numerically greatest terms are the fourth term $T_4=-\frac{224}{9}$ and the fifth term $T_5=\frac{224}{9}$.

EXERCISE 11.4 HARDER BINOMIAL EXPANSIONS

1 Find the constant term (i.e. the term independent of x) in each expansion.

(a) $\left(2x-\frac{1}{x^2}\right)^9$ **(b)** $\left(x^2-\frac{1}{x^4}\right)^{12}$ **(c)** $\left(\frac{3x^2}{2}+\frac{2}{3x}\right)^9$ **(d)** $\left(2x+\frac{1}{x}\right)^{2n}$

2 Find the middle term (or terms) in the expansion of: **(a)** $\left(x^2-\frac{1}{2x}\right)^{10}$ **(b)** $\left(3a+\frac{b^3}{6}\right)^9$

3 The coefficient of x^{-5} in the expansion of $\left(x^3-\frac{2}{x^2}\right)^{10}$ is:

A $2^6\binom{10}{6}$ **B** $-2^6\binom{10}{6}$ **C** $2^7\binom{10}{7}$ **D** $-2^7\binom{10}{7}$

4 Simplify $\left(\sqrt{3}+1\right)^6+\left(\sqrt{3}-1\right)^6$.

5 Simplify $\left(\sqrt{x-1}+1\right)^5-\left(\sqrt{x-1}-1\right)^5$.

6 Express each of the following in binomial form and then evaluate.

(a) 1.01^8 to five significant figures, where $1.01^8=(1+0.01)^8$
(b) 2.2^9 to the nearest integer
(c) 0.99^{10} to five decimal places, where $0.99^{10}=(1-0.01)^{10}$
(d) 0.998^{20} to six decimal places.

7 In the expansion of $\left(x-\frac{1}{x}\right)^6\left(x+\frac{1}{x}\right)^8$, find the coefficient of: **(a)** x^{10} **(b)** x^7 **(c)** x^{-8}

8 Find the coefficient of a^{-2} in the expansion of $\left(a+\frac{1}{a}\right)^3\left(a-\frac{1}{a}\right)^5$.

9 Using the factors of $x^2 - 3x + 2$, expand $(x^2 - 3x + 2)^6$ in ascending powers of x as far as the term in x^3.

10 Write the expansion of $(1 + 2x)(1 - x)^8$ in ascending powers of x as far as the term in x^4 and use it to evaluate 1.4×0.8^8 to four decimal places.

11 In the expansion of $(2 + 3x)^n$, the coefficients of x^3 and x^4 are in the ratio 8 : 15. Find n.

12 In the expansion of $(x + a)^3(x + b)^5$, the coefficients of x^7 and x^6 are 9 and 12 respectively. Find the values of a and b.

13 Find the value of r if the coefficients of the r-th term from the beginning and the r-th term from the end of $(2x + 3)^{15}$ are in the ratio 8 : 27.

14 Find n if the coefficients of the second, third and fourth terms in the expansion of $(1 + x)^n$ are successive terms of an arithmetic series.

15 In the expansion of $(1 + x)^n$, the ratios of three consecutive coefficients are 6 : 14 : 21. Find the value of n.

16 The second, third and fourth terms in the expansion of $(a + b)^n$ are 12, 60 and 160 respectively. Find the values of a, b and n.

17 Find the ratio of the $(r + 1)$-th term to the r-th term in the expansions of:

(a) $(2x + 3y)^{10}$ **(b)** $(3a - b)^8$

18 Find the numerically greatest term or terms in each of the following expansions.

(a) $(3 + 4x)^{12}$ if $x = \frac{1}{2}$ **(b)** $(1 - 2x)^9$ if $x = 3$ **(c)** $(2x + y)^{10}$ if $x = 4$, $y = -5$

(d) $(2x + y)^{12}$ if $x = 4$, $y = -5$ **(e)** $(1 - 3x)^7$ if $x = \frac{1}{4}$ **(f)** $(2x + 5y)^{10}$ if $x = 3$, $y = \frac{1}{2}$

11.5 RELATIONSHIPS BETWEEN BINOMIAL COEFFICIENTS

Many useful relationships can be developed from the binomial expansion of $(1 + x)^n$. Some may be as simple as substituting values for x, such as $x = 0$ or 1. Others are more complicated and may require both sides of the expression to be differentiated or integrated before substituting for x. In these situations it is more common to write $\binom{n}{r}$ instead of ${}^nC_r$.

Example 19

Given that $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad + \binom{n}{n}x^n$, investigate what happens when:

(a) $x = 1$ **(b)** $x = -1$

Solution

(a) $x = 1$: $(1+1)^n = \binom{n}{0} + \binom{n}{1} \times 1 + \binom{n}{2} \times 1^2 + \quad + \binom{n}{r} \times 1^r + \quad + \binom{n}{n} \times 1^n$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{r} + \quad + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$$

This result proves that the sum of the binomial coefficients in the expansion of $(1 + x)^n$ is 2^n.
This is the same as saying that the sum of the numbers in the n-th row of Pascal's triangle is 2^n.

Because $\binom{n}{0} = 1$, we also have: $2^n - 1 = \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{r} + \quad + \binom{n}{n} = \sum_{r=1}^{n} \binom{n}{r}$

This result means that the total number of combinations of n things, taking at least one thing at a time, is $2^n - 1$.

(b) $x = -1$: $(1-1)^n = \binom{n}{0} + \binom{n}{1} \times (-1) + \binom{n}{2} \times (-1)^2 + \binom{n}{3} \times (-1)^3 + \quad + \binom{n}{r} \times (-1)^r + \quad + \binom{n}{n} \times (-1)^n$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \quad + \binom{n}{r} \times (-1)^r + \quad + \binom{n}{n} \times (-1)^n$$

When r is even the term is positive; when r is odd the term is negative. Separating the odd and even values of r:

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \quad = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} +$$

$$= \frac{1}{2} \times 2^n$$

$$= 2^{n-1}$$

$\therefore$ In the expansion of $(1 + x)^n$:

Sum of coefficients of even terms = Sum of coefficients of odd terms $= 2^{n-1}$

Example 20

By expanding both sides of the identity $(1 + x)^5 = (1 + x)^3(1 + x)^2$, show that: $\binom{5}{2} = \binom{3}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{0}\binom{2}{2}$

Solution

$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$ $\quad\quad$ $\therefore$ Coefficient of x^2: $\binom{5}{2}$

This means that we need to find all the terms in x^2 in the expansion of $(1 + x)^3(1 + x)^2$.

$$(1+x)^3 = \binom{3}{0} + \binom{3}{1}x + \binom{3}{2}x^2 + \binom{3}{3}x^3 \qquad (1+x)^2 = \binom{2}{0} + \binom{2}{1}x + \binom{2}{2}x^2$$

The x^2 term must be: $\binom{3}{0}\binom{2}{2}x^2 + \binom{3}{1}x \times \binom{2}{1}x + \binom{3}{2}x^2$

i.e. $\left[\binom{3}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{0}\binom{2}{2}\right]x^2$

Equating coefficients: $\binom{5}{2} = \binom{3}{2} + \binom{3}{1}\binom{2}{1} + \binom{3}{0}\binom{2}{2}$

Example 21

By expanding both sides of the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, show that:

$$\binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \quad + \binom{m}{2}\binom{n}{r-2} + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}$$

Solution

$$(1+x)^{m+n} = 1 + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \quad + \binom{m+n}{r}x^r + \qquad \text{Here the coefficient of } x^r \text{ is: } \binom{m+n}{r}$$

$$(1+x)^m(1+x)^n = \left[1 + \binom{m}{1}x + \binom{m}{2}x^2 + \quad + \binom{m}{r}x^r + \quad\right] \times \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad\right]$$

Here the coefficient of x^r is: $\binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \quad + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}$

Equating coefficients of x^r: $\binom{m+n}{r} = \binom{m}{r} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \quad + \binom{m}{2}\binom{n}{r-2} + \binom{m}{1}\binom{n}{r-1} + \binom{n}{r}$

Example 22

(a) Differentiate both sides of the expansion of $(1+x)^n$.

(b) Put $x = 1$ to show that: $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \quad + n\binom{n}{n} = n \times 2^{n-1}$

(c) By choosing an appropriate value for x, show that: $\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \quad + (-1)^{n-1}n\binom{n}{n} = 0$

Solution

(a) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad + \binom{n}{n}x^n$

Differentiating: $n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \quad + r\binom{n}{r}x^{r-1} + \quad + n\binom{n}{n}x^{n-1}$

(b) Each term in the LHS of the result is multiplied by consecutive integers from 1 to n. This is the clue that we must need the derivative of the expansion of $(1+x)^n$.

Let $x = 1$: $n(1+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2} \times 1 + 3\binom{n}{3} \times 1^2 + \quad + r\binom{n}{r} \times 1^{r-1} + \quad + n\binom{n}{n} \times 1^{n-1}$

$$n \times 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \quad + r\binom{n}{r} + \quad + n\binom{n}{n}$$

(c) The value of the expression is zero. This suggests substituting $x = -1$.

Let $x = -1$: $n(1-1)^{n-1} = \binom{n}{1} + 2\binom{n}{2} \times (-1) + 3\binom{n}{3} \times (-1)^2 + \quad + r\binom{n}{r} \times (-1)^{r-1} + \quad + n\binom{n}{n} \times (-1)^{n-1}$

$$0 = \binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \quad + (-1)^{r-1}r\binom{n}{r} \times + \quad + (-1)^{n-1}n\binom{n}{n}$$

Example 23

(a) Integrate both sides of the expansion of $(1+x)^n$.

(b) Put $x=1$ in the result of **(a)** to show that: $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\quad+\frac{1}{r+1}\binom{n}{r}+\quad+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}-1}{n+1}$

Solution

(a) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\quad+\binom{n}{r}x^r+\quad+\binom{n}{n}x^n$

Integrating: $\frac{(1+x)^{n+1}}{n+1}=\binom{n}{0}x+\frac{1}{2}\binom{n}{1}x^2+\frac{1}{3}\binom{n}{2}x^3+\quad+\frac{1}{r+1}\binom{n}{r}x^{r+1}+\quad+\frac{1}{n+1}\binom{n}{n}x^{n+1}+K$

Remember, you **must** include a constant of integration (K above). Because it is a constant, it can be evaluated using any value of x.

Let $x=0$: $\frac{1^{n+1}}{n+1}=K \qquad \therefore\ K=\frac{1}{n+1}$

$\therefore\quad \frac{(1+x)^{n+1}}{n+1}=\frac{1}{n+1}+\binom{n}{0}x+\frac{1}{2}\binom{n}{1}x^2+\frac{1}{3}\binom{n}{2}x^3+\quad+\frac{1}{r+1}\binom{n}{r}x^{r+1}+\quad+\frac{1}{n+1}\binom{n}{n}x^{n+1}$

(b) Let $x=1$: $\frac{(1+1)^{n+1}}{n+1}=\frac{1}{n+1}+\binom{n}{0}\times 1+\frac{1}{2}\binom{n}{1}\times 1^2+\frac{1}{3}\binom{n}{2}\times 1^3+\quad+\frac{1}{r+1}\binom{n}{r}\times 1^{r+1}$

$$+\quad+\frac{1}{n+1}\binom{n}{n}\times 1^{n+1}$$

$$\frac{2^{n+1}}{n+1}=\frac{1}{n+1}+\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\quad+\frac{1}{r+1}\binom{n}{r}+\quad+\frac{1}{n+1}\binom{n}{n}$$

$$\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\quad+\frac{1}{r+1}\binom{n}{r}+\quad+\frac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}}{n+1}-\frac{1}{n+1}$$

$$=\frac{2^{n+1}-1}{n+1}$$

Hints for proving binomial coefficient relationships

1. If the result is a power of 2, try $x=1$.
2. If the result is zero, try $x=-1$.
3. If each term in the result is multiplied by consecutive integers from 1 to n, try differentiating the expansion of $(1+x)^n$.
4. If each term in the result is divided by consecutive integers from 1 to $n+1$, try integrating the expansion of $(1+x)^n$. Don't forget the constant of integration and the need to substitute a value (e.g. $x=0$) to find it.
5. Where necessary, use the symmetry relation $\binom{n}{r}=\binom{n}{r-1}$ to simplify expansions in which $x=1$.

EXERCISE 11.5 RELATIONSHIPS BETWEEN BINOMIAL COEFFICIENTS

1 By giving x appropriate numerical values in the expansion of $(1+x)^n$, prove that:

(a) $\sum_{k=1}^{n}\binom{n}{k}=2^n-1$ **(b)** $\sum_{k=0}^{n}(-2)^k\binom{n}{k}=(-1)^n$ **(c)** $\sum_{k=0}^{n}3^k\binom{n}{k}=2^{2n}$

2 By expanding both sides of the identity $(1+x)^{m+n} = (1+x)^m(1+x)^n$, prove that:

$$\binom{m+n}{3} = \binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}$$

3 By expanding both sides of the identity $(1+x)^{m+3} = (1+x)^m(1+x)^3$, prove that:

$$\binom{m+3}{r} = \binom{m}{r} + 3\binom{m}{r-1} + 3\binom{m}{r-2} + \binom{m}{r-3}$$

4 By expanding both sides of the identity $(1+x)^{n+2} = (1+x)^n(1+x)^2$, prove that:

$$\binom{n+2}{r} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

5 **(a)** By considering the coefficient of x^r in the geometric series $1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$, prove that: $\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \quad + \binom{r}{r} = \binom{n+1}{r+1}$

(b) Similarly, by considering $x^n + x^{n-1}(1+x) + x^{n-2}(1+x)^2 + \ldots + (1+x)^n$, find the sum:

$$\binom{n}{r} + \binom{n-1}{r-1} + \binom{n-2}{r-2} + \quad + \binom{n-r+1}{1}$$

6 **(a)** Write an expression for $r\binom{n}{r}$.

(b) Write an expression for $n\binom{n-1}{r-1}$. Hence show that: $r\binom{n}{r} = n\binom{n-1}{r-1}$

(c) If $P_r = \binom{n}{r}x^r(1-x)^{n-r}$, find the value of $P_1 + P_2 + P_3 + \ldots + P_n$.

Also show that $P_1 + 2P_2 + 3P_3 + \ldots + nP_n = nx$.

7 **(a)** Write the expansion of $(1-x^2)^n$.

(b) By considering the coefficients of x^{2r} (where $2r < n$) in the expansion of $(1+x)^n(1-x)^n$ and $(1-x^2)^n$, show that: $\binom{n}{r-1}\binom{n}{r+1} - \binom{n}{r-2}\binom{n}{r+2} + \quad + (-1)^r\binom{n}{1}\binom{n}{2r-1} - (-1)^r\binom{n}{2r} = \frac{1}{2}\binom{n}{r}\left[\binom{n}{r} - 1\right]$

8 **(a)** Write the expansion of $(1+x)^n$.

(b) By differentiating, show that: $\sum_{k=1}^{n} k\binom{n}{k} = n \times 2^{n-1}$

(c) By integrating, show that: $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \quad + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}$

9 By considering the value of $(1+x)^{2n}$ for $x = 1$, prove that: $\sum_{r=0}^{n} {}^{2n}C_r = 2^{2n-1} + \frac{(2n)!}{2(n!)^2}$

10 The binomial theorem states that $(1+x)^n = \sum_{r=0}^{n}\binom{n}{r}x^r$ for all integers $n \geq 1$.

(a) Show that: $\sum_{r=1}^{n}\binom{n}{r}rx^r = nx(1+x)^{n-1}$

(b) By differentiating the result from part **(a)**, or otherwise, show that: $\sum_{r=1}^{n}\binom{n}{r}r^2 = n(n+1) \times 2^{n-2}$

(c) Assume that n is even. Show that $\binom{n}{2}2^2 + \binom{n}{4}4^2 + \binom{n}{6}6^2 + \quad + \binom{n}{n}n^2 = n(n+1) \times 2^{n-3}$ for $n \geq 4$.

11 Use the expansion of $(1+x)^n = (1+x)^3(1+x)^{n-3}$ to prove that:

$$\binom{n}{r} = \binom{n-3}{r} + 3\binom{n-3}{r-1} + 3\binom{n-3}{r-2} + \binom{n-3}{r-3}$$

CHAPTER REVIEW 11

1 **(a)** By using the binomial theorem to expand $(1+x)^n$ and then differentiating the result, show that:

$$n(1+x)^n = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \quad + r\binom{n}{r}x^r + \quad + n\binom{n}{n}x^{n-1}$$

(b) Hence deduce that: $n \times 3^{n-1} = \binom{n}{1} + 2\binom{n}{2}2 + 3\binom{n}{3}2^2 + \quad + r\binom{n}{r}2^{r-1} + \quad + n\binom{n}{n}2^{n-1}$

2 Use the binomial theorem to find the term independent of x in the expansion of $\left(x - \frac{2}{x^2}\right)^{12}$.

3 **(a)** Sum the geometric series $1 + (1+x) + (1+x)^2 + \ldots + (1+x)^n$.

(b) Hence show that: $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \quad + \binom{n}{r} = \binom{n+1}{r+1}$

4 The binomial theorem states that: $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \quad + \binom{n}{n}x^n$

(a) Show that: $2^n = \sum_{k=0}^{n}\binom{n}{k}$

(b) Hence, or otherwise, find the value of: $\binom{50}{0} + \binom{50}{1} + \binom{50}{2} + \quad + \binom{50}{50}$

(c) Show that: $n \times 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$

5 Find the coefficient of x^n in the expansion of $(1 - x + x^2)(1+x)^{2n+1}$.

6 If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, prove that:

$$\frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{2a_2}{a_2 + a_3}$$

7 If X is the sum of all odd-numbered terms (i.e. first, third, fifth, …) in the expansion of $(a+b)^n$, and Y is the sum of all even-numbered terms (i.e. second, fourth, sixth, …), show that $X^2 - Y^2 = (a^2 - b^2)^n$.

8 **(a)** Use the binomial theorem to expand $(1+x)^n$.

(b) Put $x = 2$ in the binomial expansion to obtain an expression for 3^n.

(c) Put $x = -2$ in the binomial expansion to obtain an expression for $(-1)^n$.

9 Prove that: $\binom{2n+1}{1} + \binom{2n+1}{2} + \binom{2n+1}{3} + \quad + \binom{2n+1}{n} = 4^n - 1$

10 If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ form an arithmetic series, prove that $n + 2$ is a perfect square.

CHAPTER 12
FURTHER PROBABILITY

Outcomes covered:

PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

12.1 COUNTING TECHNIQUES IN PROBABILITY

This section builds on the counting techniques developed in Chapter 3. A brief summary of the necessary knowledge follows.

Factorial notation

$n! = n(n-1)(n-2)\ldots\times 3\times 2\times 1$

$0! = 1$

Permutations

A permutation is an *ordered* selection or arrangement of all or part of a set of objects.

- The number of arrangements of n different objects is $n!$
- The number of arrangements of r objects selected from n different objects is ${}^nP_r = \frac{n!}{(n-r)!}$
- The number of ways of arranging n objects when there are p identical objects of one kind, q identical objects of another kind, etc. is $\frac{n!}{p!q!}$

Combinations

A combination is an *unordered* selection of all or part of a set of objects.
A combination (or 'selection') is an unordered permutation (or 'arrangement').

- The number of combinations of r objects selected from n different objects is ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Useful results

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{and} \quad {}^nC_{n-r} = \binom{n}{n-r} = \frac{n!}{(n-r)!r!}$$

Thus: ${}^nC_r = {}^nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

Probability

$$P(A) = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}},\ 0 \le P(A) \le 1 \qquad P(\bar{A}) = 1 - P(A)$$

Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$

Not mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

Example 1

PINs (Personal Identification Numbers) are short number codes used as a security device.

(a) How many four-digit PINs can be made using the digits 0, 1, 2, … 9?

(b) What is the probability that someone can randomly guess the four-digit PIN?

(c) As a further security measure, some financial institutions require a four-symbol code that is made using the 26 letters of the alphabet and/or the digits 0–9. How many different four-symbol codes can be created?

(d) What is the probability of someone correctly guessing one of these four-symbol codes?

(e) What is the probability of someone correctly guessing the four-digit PIN and the correct four-symbol code?

Solution

(a) Number of PINs $= 10^4 = 10\,000$

(b) $P(\text{guessing PIN}) = \frac{1}{10^4} = \frac{1}{10\,000}$

(c) There are 26 letters of the alphabet and 10 digits. Number of codes $= 36^4 = 1\,679\,616$

(d) $P(\text{guessing symbol code}) = \frac{1}{36^4} = \frac{1}{1\,679\,616}$

(e) $P(\text{guessing both PIN and symbol code}) = \frac{1}{10^4} \times \frac{1}{36^4} = \frac{1}{16\,796\,160\,000}$

Example 2

In horse racing, a 'quinella' is when the first two horses are picked correctly (in any order). In a twelve-horse race, what is the probability of picking the quinella, assuming that each horse is equally likely to win?

Solution

12 horses total, 2 picked for a quinella: Number of quinella combinations $= {}^{12}C_2 = 66$

Only one of these combinations wins: $P(\text{winning quinella}) = \frac{1}{66}$

Example 3

Eight people in total, including two people A and B, randomly arrange themselves in a straight line. What is the probability that:

(a) A and B are next to each other

(b) A and B are not next to each other

(c) A and B occupy the end positions

(d) there are at least 3 people between A and B?

Solution

(a) 8 people can be arranged in a row in 8! ways $= 40\,320$
As A and B are next to each other, consider them as one unit.
This unit and the other 6 people can be arranged in 7! ways.
A and B can be arranged in 2 ways (AB or BA).
Total ways of arranging the people $= 2 \times 7!$

$$P(A \text{ and } B \text{ next to each other}) = \frac{2 \times 7!}{8!} = \frac{2 \times 7!}{8 \times 7!} = \frac{1}{4}$$

(b) $P(A \text{ and } B \text{ not next to each other}) = 1 - P(A \text{ and } B \text{ next to each other}) = 1 - \frac{1}{4} = \frac{3}{4}$

(c) A and B can be placed at the ends in 2 ways (A at the beginning and B at the end, or B at the beginning and A at the end). The remaining 6 places can be filled in 6! ways.

Total number of ways with A and B at the ends $= 2 \times 6!$

$$P(A \text{ and } B \text{ occupy end positions}) = \frac{2 \times 6!}{8!} = \frac{2 \times 6!}{8 \times 7 \times 6!} = \frac{1}{28}$$

(d) A diagram helps to visualise the possibilities:

$A _ _ _ B _ _ _$, $A _ _ _ _ B _ _$, $A _ _ _ _ _ B _$, $A _ _ _ _ _ _ B$.
$_ A _ _ _ B _ _$, $_ A _ _ _ _ B _$, $_ A _ _ _ _ _ B$.
$_ _ A _ _ _ B _$, $_ _ A _ _ _ _ B$.
$_ _ _ A _ _ _ B$.

A and B can be swapped, so this gives $10 \times 2 = 20$ arrangements for A and B with 6! ways of filling the remaining 6 places.

Total number of favourable arrangements $= 20 \times 6!$

$$P(\text{at least 3 people between } A \text{ and } B) = \frac{20 \times 6!}{8!} = \frac{20 \times 6!}{8 \times 7 \times 6!} = \frac{5}{14}$$

Example 4

A bag contains nine cubes: 3 that are white and 6 that are black. Two cubes are drawn at random without replacement. Calculate the probability that both cubes are black.

Solution

Method 1

Number of ways 2 cubes can be selected from 9 cubes without replacement $= {}^9C_2$
Number of ways the 2 cubes can be black $= {}^6C_2$

$$P(\text{both cubes are black}) = \frac{{}^6C_2}{{}^9C_2} = \frac{6 \times 5}{2 \times 1} \times \frac{2 \times 1}{9 \times 8} = \frac{5}{12}$$

Method 2

Let A be the event 'black cube in the first draw' and B be the event 'black cube in the second draw'.
Each cube is equally likely to be selected, so $P(A) = \frac{6}{9}$.
The outcome of the second draw is dependent on the outcome of the first draw:
if event A has happened then there are 8 cubes left, of which 5 are black.

Hence $P(B|A) = \frac{5}{8}$

Now $P(AB) = P(A) \times P(B|A) = \frac{6}{9} \times \frac{5}{8} = \frac{5}{12}$

Example 5

A group of 9 people contains 3 teachers and 6 students. A random sample of 5 people is selected. What is the probability that it contains: **(a)** exactly 2 teachers **(b)** not more than 2 teachers?

Solution

(a) Exactly 2 teachers means the sample contains 2 teachers and 3 students.

Number of ways of selecting 5 people from 9 people $= \binom{9}{5}$

Number of ways of selecting 2 teachers from 3 teachers $= \binom{3}{2}$

Number of ways of selecting 3 students from 6 students $= \binom{6}{3}$

Number of ways of selecting 2 teachers and 3 students $= \binom{3}{2} \times \binom{6}{3}$

Hence: $P(\text{2 teachers and 3 students}) = \dfrac{\binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}}$

Remember $\binom{9}{5} = \binom{9}{4}$: $P(\text{2 teachers and 3 students}) = \dfrac{\binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}} = 3 \times \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \dfrac{4 \times 3 \times 2 \times 1}{9 \times 8 \times 7 \times 6} = \dfrac{10}{21}$

(b) Not more than 2 teachers means either 0 teachers and 5 students, 1 teacher and 4 students or 2 teachers and 3 students.

0 teachers and 5 students can be selected in $\binom{6}{5}$ ways.

1 teacher and 4 students can be selected in $\binom{3}{1} \times \binom{6}{4}$ ways.

2 teachers and 3 students can be selected in $\binom{3}{2} \times \binom{6}{3}$ ways.

Hence: $P(\text{not more than 2 teachers}) = \dfrac{\binom{6}{5} + \binom{3}{1} \times \binom{6}{4} + \binom{3}{2} \times \binom{6}{3}}{\binom{9}{5}} = \dfrac{6 + 45 + 60}{126} = \dfrac{37}{42}$

Example 6

From a set of five cards numbered 1, 2, 3, 4, 5, two cards are selected at random without replacement.

(a) What is the probability that both cards are odd-numbered?

(b) If the cards are used to make a two-digit number, what is the probability that this number is odd?

(c) If the cards are used to make a two-digit number, what is the probability that this number has one odd digit and one even digit?

Solution

(a) Number of different two-card selections $= {}^5C_2 = 10$

Number of two-card selections where both are odd $= {}^3C_2 \times {}^2C_0 = 3$

$P(\text{both cards are odd}) = \dfrac{3}{10}$

(b) **Method 1**

The second digit must be odd, so the order is important.

Number of odd numbers formed $= {}^3P_1 \times {}^4P_1 = 12$

Number of two-digit numbers formed $= {}^5P_2 = 20$

$P(\text{odd two-digit number formed}) = \dfrac{12}{20} = \dfrac{3}{5}$

Method 2

To be an odd number, the cards must be odd–odd or even–odd.

$$P(\text{odd number}) = P(\text{odd, odd}) + P(\text{even, odd})$$

$$= \frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4}$$

$$= \frac{12}{20}$$

$$= \frac{3}{5}$$

(c) One even digit and one odd digit: number of ways $= 2 \times 3 = 6$

Number of different two-card selections $= {}^5C_2 = 10$

$$P(\text{one even digit and one odd digit}) = \frac{6}{10} = \frac{3}{5}$$

If we consider the order: $P(\text{one even digit and one odd digit}) = \frac{12}{20} = \frac{3}{5}$ (i.e. the same answer)

Example 7

A carton contains 10 transistors, two of which are defective. A sample of three transistors is drawn at random from the carton without replacement. Find the probability that not more than one transistor is defective.

Solution

Number of possible selections $= {}^{10}C_3 = 120$

Number of favourable selections = [0 defective] + [1 defective]

'0 defective' means that 0 (none) of the 2 defectives are selected and 3 of the 8 non-defectives are selected, i.e. ${}^2C_0 \times {}^8C_3$ ways.

'1 defective' means that 1 of the 2 defectives is selected and 2 of the 8 non-defectives are selected, i.e. ${}^2C_1 \times {}^8C_2$ ways.

Number of favourable selections $= {}^2C_0 \times {}^8C_3 + {}^2C_1 \times {}^8C_2 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} + 2 \times \frac{8 \times 7}{2 \times 1} = 56 + 56 = 112$

$$P(\text{not more than 1 defective}) = \frac{112}{120} = \frac{14}{15}$$

Example 8

A scientific study of penguins uses the 'capture–recapture' technique. In the first stage of the study, 36 penguins are caught, tagged and then released. Later, in the second stage of the study, some penguins are again captured from the same area. Of these penguins, 27 of them are found to be tagged, which is 30% of the total captured in this second stage.

(a) In the second stage of the study, how many penguins are captured in total?

(b) Calculate the estimate for the total population of penguins in this area.

Solution

(a) 27 is 30% of total captured, N: $27 = 0.3N$

$N = \frac{27}{0.3} = 90$ Hence 90 penguins were captured.

(b) P is the total population of penguins in the area.

Use the ratio: $\frac{27}{90}=\frac{36}{P}$

$$P=\frac{36\times 90}{27}=120$$

or: $0.3=\frac{36}{P}$ The penguin population is about 120.

EXERCISE 12.1 COUNTING TECHNIQUES IN PROBABILITY

1 Five cards are drawn at random from a standard pack of 52 playing cards. What is the probability that:
(a) they are all from the same suit **(b)** they include four Aces **(c)** they include three 10s and two 9s?

2 A bag contains five red balls and four white balls. Three balls are withdrawn without replacement. The probability of drawing at least two red balls is:
A $\frac{1}{21}$ **B** $\frac{5}{42}$ **C** $\frac{20}{42}$ **D** $\frac{25}{42}$

3 A student writes a random three-digit number using the digits 1–9. What is the probability that the three digits in the number are all the same?

4 In horse racing, a 'trifecta' is when the first three horses are picked in the correct order. In a ten-horse race, what is the probability of picking the trifecta, assuming that each horse is equally likely to win?

5 A bag contains three white balls, four red balls and five black balls. Three balls are drawn at random without replacement. What is the probability that they are: **(a)** different colours **(b)** the same colour?

6 From a group of seven teachers and five students, a random selection of seven people is made. What is the probability that the selection contains at least four teachers?

7 A committee of three judges and four lawyers is to be chosen from six judges and seven lawyers. What is the probability that it contains a particular judge and a particular lawyer?

8 Five cards are drawn from a standard pack of 52 playing cards (without replacement). What is the probability of drawing at least three Aces?

9 A person correctly picks the first and second horses in a race of ten horses. If each horse was equally likely to win, what is the probability of this?

10 The letters of the word PROMISE are arranged in a row. What is the probability that there are three letters between 'P' and 'R'?

11 Six giraffes randomly arrange themselves in a line. What is the probability that the tallest giraffe and the shortest giraffe are together?

12 Four different trucks and three different cars are parked in a row. What is the probability that the trucks and the cars occupy alternate positions?

13 A car's number plates contain three letters of the alphabet followed by three numerals. How many number plates like this can be made? What proportion of these would contain three letters the same and three numerals the same?

14 The letters of the word INDEPENDENCE are arranged in a row. What is the probability of all the letters 'E' being together? Indicate whether each statement below is a correct or incorrect step in solving this problem.

(a) Number of arrangements of the letters $=\frac{12!}{4!3!2!}$

(b) Number of ways that the letters 'E' are together $=\frac{9!}{4!3!2!}$

(c) Number of ways that the letters 'E' are together $=\frac{9!}{3!2!}$

(d) Probability of the letters 'E' being together $=\frac{1}{55}$

15 A carton contains a dozen eggs, of which three have a double yolk. If three eggs are taken to make a cake, find the probability that the three eggs all have double yolks.

16 A team of six is selected from ten people. What is the probability that the youngest and oldest people are on the team?

17 A box contains five red cubes and four white cubes. Three cubes are drawn in succession without replacement. What is the probability that:
(a) the first two cubes are red and the third cube is white
(b) any two cubes are red and one cube is white?

18 An angler has caught 15 fish, of which three are undersized. A random sample of three fish is drawn without replacement by an inspector. The angler is fined if one or more of the fish in the sample is undersized. What is the probability that the angler is fined?

19 Eight different jackets are divided into two piles. What is the probability that there will be four in each pile?

20 A party of twelve people, including A and B, are arranged at random in a straight line. What is the probability that A and B are not next to one another?

21 A carton contains 15 memory cards, of which 5 are defective. If a random sample of 6 memory cards is drawn from the carton (without replacement), determine the probability of there being 0, 1, 2, 3, 4, 5 defective memory chips in the sample.

22 An urn contains 12 distinguishable cubes of which 5 are red and the remainder black. If a random sample of 6 cubes is drawn without replacement, calculate the probabilities of 0, 1, 2, 3, 4, 5 red cubes in the sample.

23 Two boxes each contain eight balls. In box A there are three black and five white balls; in box B there are one black and seven white balls. For each box, find the probability that two balls chosen at random without replacement will both be white.

24 A sample of three coins is selected without replacement from a handful of eight coins that consists of four 10c coins and four 20c coins. What is the probability that the sample contains at least two 10c coins?

25 A hand of five cards is dealt from a standard pack of 52 playing cards. What is the probability that it contains at least one Ace?

26 From a group of 12 people, of whom eight are painters and four are carpenters, a sample of four is selected at random. What is the probability that the sample contains at least two carpenters?

27 Box A contains six white and four black balls. Box B contains two white and two black balls. From box A, two balls are selected at random and placed in box B. Two balls are then selected at random from box B. What is the probability that exactly one of these two balls is white?

28 From a set of ten cards numbered 1–10, two cards are drawn without replacement. What is the probability that:
(a) both numbers are even **(b)** one is even and the other is odd
(c) the sum of the two numbers is 12 **(d)** both numbers are even and the sum of the two numbers is 12?

29 The letters of the word TOMATO are arranged in a row. What is the probability that:
(a) the two letters 'O' are together **(b)** the two letters 'O' are not together?

30 The letters of the word TUESDAY are arranged at random in a row. What is the probability that:
(a) the vowels and consonants occupy alternate positions **(b)** the vowels are together
(c) the vowels are together and the letter 'T' occupies the first place?

31 Four girls and four boys arrange themselves at random in a row. What is the probability that the girls and the boys occupy alternate positions?

32 Six different robots, including A and B, arrange themselves at random in a row. What is the probability that:
(a) A and B occupy the end positions **(b)** A and B are not next to each other
(c) there are at least three robots between A and B?

33 The digits 1, 2, 3, 4, 5, 6 are used to form numbers that contain two or more digits. The same digit cannot be used more than once in a number. What proportion of the numbers formed are even numbers?

34 Five cards are selected with replacement from a standard pack of 52 playing cards. What is the probability that the selected cards include:

(a) exactly three hearts **(b)** four Aces **(c)** no hearts?

35 A game involves choosing six numbers from a set of 44 numbers.

(a) What is the probability of winning (i.e. choosing the six numbers correctly) in a single game?

(b) If you play four games, choosing a different set of numbers for each game, what is the probability that one of your games will win?

36 A scientific study uses the 'capture–recapture' technique. In the first stage of the study, 48 possums are caught, tagged and then released. Later, in the second stage of the study, some possums are again captured from the same area. Of these possums, 24 of them are found to be tagged, which is 40% of the total captured in this second stage.

(a) In the second stage of the study, how many possums are captured in total?

(b) Calculate the estimate for the total population of possums in this area.

An interesting problem—who shares your birthday?

Example 9

In a random group of people, what is the probability that at least two people in the group have the same birthday (i.e. the same day and month, but not necessarily the same year)? This example investigates this problem.

(*Note:* We will ignore 29 February and leap years, assuming instead 1 year = 365 days.)

(a) In a group of four people, what is the probability that there is at least one birthday shared?

(b) In a group of ten people, what is the probability that there is at least one birthday shared?

(c) In a group of n people, what is the probability that there is at least one birthday shared?

(d) For what value of n is there a 50% chance that at least two people in the group have the same birthday?

Solution

(a) If two people have their birthday on different days, then there are 364 days on which the second person could have been born.

In a group of 4 people, the probability that no two people are born on the same day $= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$

Probability of at least one birthday in common $= 1 - \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \approx 0.0164$

(b) In a group of 10 people:

Probability that no two people are born on the same day $= \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \quad \times \frac{355}{365} = \frac{{}^{364}P_{10}}{365^{10}}$

Probability of at least one birthday in common $= 1 - \frac{{}^{364}P_{10}}{365^{10}} \approx 0.1411$

(c) In a group of n people:

Probability that no two people are born on the same day $= \frac{{}^{364}P_{n}}{365^{n}}$

Probability of at least one birthday in common $= 1 - \frac{{}^{364}P_{n}}{365^{n}}$

(d) To find a 50% chance we must solve the equation: $1 - \frac{{}^{364}P_{n}}{365^{n}} = 0.5$

That is, solve for n the equation: $\frac{{}^{364}P_{n}}{365^{n}} = 0.5$

This cannot be done analytically, but by substituting values for n we can find the answer by trial and error.

From **(b)** we know $n > 10$. Trying $n = 20$: $\dfrac{{}^{364}P_{20}}{365^{20}} \approx 0.556$

Trying $n = 25$: $\dfrac{{}^{364}P_{25}}{365^{25}} \approx 0.4012$

Trying $n = 22$: $\dfrac{{}^{364}P_{22}}{365^{22}} \approx 0.5234$

Trying $n = 23$: $\dfrac{{}^{364}P_{23}}{365^{23}} \approx 0.4917$

Hence the closest value of n is 23.

An interesting problem similar to the birthday investigation above is the 'Hermit Problem', which you may wish to research.

12.2 BINOMIAL PROBABILITY

In probability we often find situations in which the same trial is repeated several times and there are only **two** possible outcomes in each trial. (When there are only two outcomes, they are often called 'success' and 'failure'.) For example:

Games of chance

If a die is rolled, there is a probability $p = \frac{1}{6}$ of a 4 being rolled on the die and a probability $q = \frac{5}{6}$ that a 4 is not rolled on the die. Any one trial (i.e. when a die is rolled once) can be seen to have only two possible outcomes, the occurrence of a 4 or the non-occurrence of a 4—hence the term 'binomial trials'.

If a die is rolled three times (or equivalently, if three dice are rolled once), then the three trials are **independent**, i.e. the outcome of one trial has absolutely no influence on the outcome of any other trial and the probability of a 4 in each of the three trials is the same, $\frac{1}{6}$. For three trials we would want to calculate the probability of rolling zero, one, two or three 4s.

Similarly, if a coin is tossed five times then the probability of 'heads' turning up in each toss is $p = \frac{1}{2}$. The five tosses are independent and there are two possible outcomes in each (binomial) trial, heads or tails.

Target shooting

An archer finds that a bullseye is scored on average 4 times out of 5. Thus the probability that a bullseye is scored in any one trial is $p = \frac{4}{5}$ and the probability of not scoring a bullseye in a trial is $q = \frac{1}{5}$.

At a tournament the archer shoots 8 arrows at a target. Assuming that the 8 trials (shots) are independent and that the chance of a bullseye is the same in each trial, we can calculate the probability of 0, 1, 2, 3, … , 8 bullseyes.

Sampling manufactured products

A manufacturer finds that 10% of a factory's products are defective. If a sample of 20 products is randomly selected, each product has a probability $p = \frac{1}{10}$ of being defective and a probability $q = \frac{9}{10}$ of not being defective. We can calculate the probability of 0, 1, 2, … , 20 defectives in the sample.

Clinical trials

Researchers find that 80% of patients suffering from disease X recover after taking drug Y. Ten patients (not specifically selected) suffering from this disease are treated with the drug. Each patient has a probability $p = 0.8$ of recovering and a probability $p = 0.2$ of not recovering, and each case is independent.

The examples above share some common properties:

- there are n independent trials
- each trial has two possible outcomes ('success' or 'failure'—e.g. rolling a 4, tossing heads, a bullseye, a defective product, or recovering from disease X could be called a 'success')
- in each example, the probability of a success is the same in each of the n trials.

Example 10

A die is rolled 3 times. What is the probability of zero, one, two or three 6s being rolled?

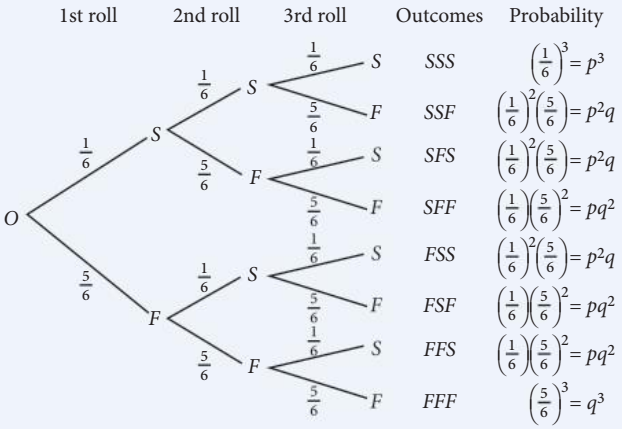
Solution

Probability of rolling a 6 in a single trial: $p = \frac{1}{6}$

Probability of not rolling a 6 in a single trial: $q = \frac{5}{6}$

Call rolling a 6 a 'success' (S) and not rolling a 6 a 'failure' (F).

The eight possible outcomes and their corresponding probabilities can be represented by a tree diagram, with three sets of branches from left to right representing the probability of 'success' or 'failure' in each of the three die rolls.



The possible outcomes of the three independent trials can also be listed in a table with their respective probabilities.

Number of sixes	Possible outcomes	Probability
0	FFF	$\left(\frac{5}{6}\right)^3 = q^3$
1	SFF	$\left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right)$
	FSF	$\left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right)$
	FFS	$\left(\frac{5}{6}\right)^2 \times \left(\frac{1}{6}\right)$
		$= 3q^2p$
2	SSF	$\left(\frac{5}{6}\right) \times \left(\frac{1}{6}\right)^2$
	SFS	$\left(\frac{5}{6}\right) \times \left(\frac{1}{6}\right)^2$
	FSS	$\left(\frac{5}{6}\right) \times \left(\frac{1}{6}\right)^2$
		$= 3qp^2$
3	SSS	$\left(\frac{1}{6}\right)^3 = p^3$

If X denotes the number of successes in the three trials, then X is a random variable that can have the values $r = 0, 1, 2, 3$. We can consider the sample space as consisting of 2^3 (i.e. 8) sample points, but the points are not equally weighted.

$P(X = 0) = P(0 \text{ sixes in any of the 3 trials})$

$$= \left(\frac{5}{6}\right)^3 = q^3$$

$$= \frac{125}{216}$$

$P(X = 1) = P(1 \text{ six in any one of the 3 trials, but 0 sixes in the other 2 trials})$

$$= \binom{3}{1}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) = \binom{3}{1}q^2p$$

$$= \frac{75}{216}$$

$P(X = 1) = P(1 \text{ six in any one of the 3 trials, but 0 sixes in the other 2 trials})$

$$= \binom{3}{1}\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right) = \binom{3}{1}q^2p$$

$$= \frac{75}{216}$$

$P(X = 2) = P(1 \text{ six in any 2 of the 3 trials, but 0 sixes in the other trial})$

$$= \binom{3}{2}\left(\frac{5}{6}\right)\left(\frac{1}{6}\right)^2 = \binom{3}{2}qp^2$$

$$= \frac{15}{216}$$

$P(X=3) = P(1 \text{ six in each of the 3 trials})$

$$= \left(\frac{1}{6}\right)^3 = p^3$$

$$= \frac{1}{216}$$

From this example we can see the pattern of the binomial trials. The table above lists all possible outcomes of the 3 trials, all of which are mutually exclusive, so the sum of their probabilities is unity:

$$\text{i.e. } q^3 + \binom{3}{1}q^2p + \binom{3}{2}qp^2 + p^3 = 1 \qquad [1]$$

Checking numerical values: $\frac{125}{216} + \frac{75}{216} + \frac{15}{216} + \frac{1}{216} = 1$

Also: $q + p = \frac{5}{6} + \frac{1}{6} = 1$

Notice that the left-hand side of [1] is the binomial expansion of $(q+p)^3$:

$$\text{i.e. } (q+p)^3 = q^3 + \binom{3}{1}q^2p + \binom{3}{2}qp^2 + p^3$$

$$= P(0 \text{ sixes}) + P(1 \text{ six}) + P(2 \text{ sixes}) + P(3 \text{ sixes})$$

Example 11

An archer finds that in the long run she scores a bullseye 3 times out of 4. At a tournament she shoots 5 arrows at a target. Assuming that each trial (shot) is independent, find the probability of:

(a) exactly 3 bullseyes **(b)** at least 4 bullseyes **(c)** a bullseye in the second trial only.

Solution

$P(\text{bullseye in one trial})$: $p = \frac{3}{4}$

$P(\text{not a bullseye in one trial})$: $q = \frac{1}{4}$

If X denotes the number of successes in the 5 trials then X is a binomial variable that can have the values 0, 1, 2, 3, 4, 5.

(a) $P(X=3) = P(1 \text{ bullseye in any 3 trials and 0 bullseyes in the other 2 trials})$

$$= \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$$

$$= \frac{270}{1024} = \frac{135}{512}$$

(b) 'At least 4 bullseyes' means exactly 4 bullseyes or exactly 5 bullseyes. These events are mutually exclusive, hence: $P(X \geq 4) = P(X=4) + P(X=5)$

$$= \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5$$

$$= \frac{405}{1024} + \frac{243}{1024}$$

$$= \frac{648}{1024} = \frac{81}{128}$$

(c) A bullseye only in the second trial means that we need a 'failure' in all the other trials. This gives the pattern *FSFFF*. $P(\text{bullseye only in second trial}) = \frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{1024}$

There is no binomial coefficient in the answer to part **(c)** because it can only happen in one way.

Example 12

For a certain species of bird, there is a chance of 3 in 5 that a fledgling will survive its first month after birth. From a brood of 10 fledglings, what is the probability that:

(a) none will survive **(b)** more than one will survive?

Solution

$p=\frac{3}{5},\ q=\frac{2}{5}$ If X denotes the number of fledglings that survive then X is a binomial variable that can have the values 0, 1, 2, 3 …, 10.

(a) $P(X=0)=\left(\frac{2}{5}\right)^{10}\approx 0.0001$

This is a very small answer. It seems reasonable to expect that some of the fledglings would survive. How many survivors would you expect?

(b) $P(X>1)=P(X=2)+P(X=3)+\ldots+P(X=10)$.

All outcomes are possible, from 0 to 10 survivors, hence: $P(X>1)=1-\left[P(X=0)+P(X=1)\right]$

$$=1-\left[\left(\frac{2}{5}\right)^{10}+\binom{10}{1}\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^{9}\right]$$

$$\approx 0.9983$$

From the examples above, we can generalise for n independent binomial trials.

$$(q+p)^n=q^n+\binom{n}{1}q^{n-1}p+\binom{n}{2}q^{n-2}p^2+\quad+\binom{n}{r}q^{n-r}p^r+\quad+p^n$$

$$=P(X=0)+P(X=1)+P(X=2)+\quad+P(X=r)+\quad+P(X=n)$$

If X denotes the number of successes in n independent trials, we can see that the probability of exactly r successes and $(n-r)$ failures is given by the $(r+1)$-th term in the expansion of $(q+p)^n$:

$$P(X=r)=\binom{n}{r}q^{n-r}p^r \quad \text{where } 0\le r\le n$$

$$=\binom{n}{r}(1-p)^{n-r}p^r \quad \text{as } p+q=1$$

Sampling with replacement from a small population

Example 13

A box contains 15 mobile phones, of which 5 are defective. A phone is sampled by being taken from the box, examined and then put back into the box. This process is repeated until four samplings have been made. What is the probability of:

(a) at least one defective phone being sampled

(b) not more than two defective phones being sampled?

Solution

Because each phone is replaced after being sampled, the probability p of a 'success' (i.e. a defective phone) is the same, i.e. $p=\frac{1}{3}$ for each of the four trials.

This is therefore a case of binomial probability. The binomial variable X can have the values 0, 1, 2, 3, 4.

$p=\frac{1}{3},\ q=\frac{2}{3},\ n=4$

(a) $P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= 1 - P(X=0)$

$= 1 - \left(\frac{2}{3}\right)^4 = \frac{65}{81}$

(b) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$= \left(\frac{2}{3}\right)^4 + \binom{4}{1}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right) + \binom{4}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2$

$= \frac{16}{81} + \frac{32}{81} + \frac{24}{81} = \frac{8}{9}$

Sampling without replacement from a large population

In Example **13** above we considered sampling with replacement from a small population of 15 phones. If instead the sampling is done without replacement, the probability that the first phone sampled is defective is $\frac{5}{15}$, but the probability that the second phone sampled is defective is $\frac{4}{14}$ if the first phone was defective or $\frac{5}{14}$ if the first phone was not defective, and so on. The probability p of a defective phone is now changing considerably with each sample, so we are not dealing with a binomial variable.

However, if we take a sample without replacement from a very large population, e.g. 1000 or more factory-produced phones of which 20% are defective, then the probability that the first phone selected is defective is $\frac{200}{1000}$, or 0.2. The probability that the second phone selected is defective given that the first phone was defective is then $\frac{199}{999}$, which is still approximately 0.2.

In general, the probability p of any article from a large population is practically unaltered by the sampling, even without replacement. The larger the population, the more this is true. In this case we can use the binomial distribution as an approximation.

Example 14

Three coins are tossed 400 times. How many occasions would we expect 0, 1, 2, or 3 heads?

Solution

Let X denote the number of heads in a single toss of 3 coins.

The number of occasions on which we would expect r heads $= 400P(X=r)$ for $r = 0, 1, 2, 3$.

This can also be called the **expected value** of the experiment, np, where n is the number of trials and p is the probability of the outcome.

$400P(X=0) = 400\left(\frac{1}{2}\right)^3 = 50$ $\qquad$ $400P(X=2) = 400\binom{3}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = 150$

$400P(X=1) = 400\binom{3}{1}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = 150$ $\qquad$ $400P(X=3) = 400\left(\frac{1}{2}\right)^3 = 50$

It should not be surprising that the sum of the expected values $50 + 150 + 150 + 50 = 400$, the total number of trials.

Example 15

A manufacturer finds that x% of a factory's products are defective. If a sample of 5 products is randomly selected, calculate the probability that the sample contains:

(a) 0 defectives **(b)** exactly 2 defectives **(c)** more than 1 defective.

Solution

The binomial approximation can be used with $p=\frac{x}{100}$, $q=\frac{100-x}{100}$, $n=5$.

The binomial variable X can have the values 0, 1, 2, 3, 4, 5.

(a) $P(X=0)=q^5=\left(\frac{100-x}{100}\right)^5$

$=10^{-10}(100-x)^5$

(b) $P(X=2)=\binom{5}{2}\left(\frac{100-x}{100}\right)^3\left(\frac{x}{100}\right)^2$

$=10^{-10}\times 10x^2(100-x)^3$

$=10^{-9}x^2(100-x)^3$

(c) $P(X>1)=P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$=1-[P(X=0)+P(X=1)]$

$=1-\left[\left(\frac{100-x}{100}\right)^5+\binom{5}{1}\left(\frac{100-x}{100}\right)^4\left(\frac{x}{100}\right)\right]$

$=1-10^{-10}[(100-x)^5+5x(100-x)^4]$

Example 16

A die is rolled 30 times. What is the most likely number of sixes rolled?

Solution

This is an application of Examples **17** and **18** in Chapter 11 (see pages 211–212).

The 'most likely number of sixes' is the number of sixes that would occur most frequently. This is the same as finding the greatest term in the binomial expansion of $(q+p)^n$, i.e. in this case $\left(\frac{5}{6}+\frac{1}{6}\right)^{30}$.

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\times\frac{p}{q}=\frac{31-r}{r}\times\frac{1}{5}=\frac{31-r}{5r}$$

$T_{r+1}\geq T_r$ if $\frac{31-r}{5r}\geq 1$

$31-r\geq 5r$

$31\geq 6r$

$r\leq 5\frac{1}{6}$

The greatest term must occur when $r=5$, so the most likely number of sixes is 5.

$$T_6=P(X=5)=\binom{30}{5}\left(\frac{5}{6}\right)^{25}\left(\frac{1}{6}\right)^5$$

This is also what we would find using the expected value result, $np=30\times\frac{1}{6}=5$.

EXERCISE 12.2 BINOMIAL PROBABILITY

1 In 12 rolls of a die, what is the probability that exactly 4 sixes are rolled?

2 Hospital records show that 75% of patients suffering from a certain medical condition do not completely recover. What is the probability that, of six randomly selected patients with the condition, all will completely recover?

3 An archer finds that he hits the target on average 4 times out of 5. If he shoots 4 arrows, what is the probability of: **(a)** more hits than misses **(b)** at least 3 misses?

4 If the probability of a child being a boy is $\frac{1}{2}$ (ignoring multiple births), then the proportion of families with exactly six children that have at least three boys is:

A $\frac{5}{16}$ **B** $\frac{11}{32}$ **C** $\frac{41}{64}$ **D** $\frac{21}{32}$

5 A manufacturer of metal pistons finds that on average 12% of the pistons are rejected because they are either oversize or undersize. What is the probability that a random batch of 10 pistons will contain:

(a) no more than 2 rejects **(b)** at least 2 rejects?

6 Three uniform tetrahedral dice (triangular pyramids) each have one face grey, another face blue, another face red and the final face green. When rolled onto a table, three faces of each dice can be seen. What is the probability that:

(a) no grey face can be seen **(b)** exactly two grey faces can be seen

(c) at least two red faces can be seen **(d)** three blue faces and only one green face can be seen?

7 A die is rolled ten times. What is the probability that at least two sixes are rolled?

8 If the probability of any child being a girl is 0.5, what is the probability that a family of four children contains four girls?

9 A dental inspector finds that about 20% of people in a certain town have tooth decay. What is the probability that out of six people examined, only the first, third and fifth will have tooth decay?

10 Hospital records show that 40% of patients with a certain disease fail to recover. Calculate the probability that in a sample of ten patients with the disease, only one fails to recover.

If these particular ten patients had been subjected to a new treatment, what might you infer about the effectiveness of this treatment?

11 It is found that seven out of every ten students from the Year 12 of a certain school go on to university. Assuming a binomial distribution, find the probability that out of a group of five Year 12 students chosen at random:

(a) exactly three enter university **(b)** any number up to and including three enter university

(c) only the first three students chosen enter university.

12 A study finds that 40% of patients with a particular disease are cured after treatment with the drug streptomycin. If a doctor uses streptomycin to treat ten randomly chosen patients with this disease, what is the probability of: **(a)** 9 cures **(b)** more than 8 cures?

(c) Another doctor treats ten patients with this disease using a new treatment and obtains more than 8 cures. Comment on this result.

13 An online poll finds that 60% of people are in favour of a certain proposal. If five of the people polled are selected again at random, what is the probability that a majority of them are against the proposal?

14 A die is biased so that in ten independent trials, the probability that an even number appears five times is twice the probability that an even number appears four times. What is the probability that an even number will appear in a single trial?

15 The probability that a person aged 60 or a person aged 65 will stay healthy during the next year is 0.97 or 0.96 respectively. From a group of five people including three people aged 60 and two aged 65, what is the probability that:

(a) not more than one will get sick within the next year

(b) the two 65-year-olds will stay healthy and not more than one 60-year-old will get sick within the next year?

16 How many times must a fair coin be tossed so that the probability of getting heads at least once exceeds 0.9?

17 In an exam consisting of ten true/false questions, a student guesses all the answers. What is the probability that the student gets 70% or more correct?

18 Assuming that the chance of death by accident is $\frac{1}{300}$, estimate the probability that from a random selection of 150 there will be at least one death by accident.

19 John estimates that the probability of his winning any one game of tennis against a particular opponent is $\frac{1}{3}$. How many games should be played so that the probability that John wins at least one game is greater than 0.9?

20 On average, a proofreader corrects one wrong word in every 800 words. Assuming that a page contains 200 words, find the probability of more than one correction per page.

21 Steel rods are manufactured to be 4 cm in diameter, but they are acceptable as long as the diameter is between 3.98 cm and 4.02 cm. At a certain factory, 3% are rejected as oversize and 3% are rejected as undersize. A random sample of ten rods is selected from the factory. What is the probability that the sample contains not more than one rod that has to be rejected?

22 Four families each have four children. All the children have either dark hair or light hair. If the probability of dark hair is $\frac{1}{2}$ and the probability of light hair is $\frac{1}{2}$, what is the probability that:

(a) at least one of these families has two children with dark hair and two children with light hair

(b) each family has at least one child with dark hair?

23 An archer finds that he hits the target on average nine times out of every ten and scores a bullseye on average once every five shots. He shoots four arrows. What is the probability that:

(a) he hits the target each time **(b)** he scores at least two bullseyes?

24 In a packet of mixed flower seeds, 40% are known to be for pink flowers and the remainder are for yellow flowers.

(a) Calculate the probability of the seeds producing 0, 1, 2, … , 6 pink flowers in a row of six plants.

If 2500 rows are planted of six plants each, how many rows would be expected to contain:

(b) all pink flowers **(c)** all yellow flowers?

25 If ten coins are tossed 50 times, in how many cases would you expect the number of heads to exceed the number of tails?

26 On average, a darts player scores a bullseye in two out of every three throws. She throws at a target in sets of five throws. Estimate how many sets of five throws with three bullseyes would occur in a thousand sets.

27 A factory has seven machines. Four 'Model A' machines are in use 80% of the time on average, while three 'Model B' machines are in use 60% of the time on average. If the manager walks into the factory at a randomly selected time, what is the probability that:

(a) one 'Model B' machine and two 'Model A' machines are in use **(b)** two machines in use?

28 If rain falls randomly on an average of 12 days out of every 30, find the probability that:

(a) the first three days of a week are fine and the rest of the week has rain

(b) rain falls on exactly three days of a week **(c)** at least three days of a week are fine?

29 A manufacturer of toothbrushes finds that on average one toothbrush in every 20 is faulty. The toothbrushes are sold in packets of five. In a batch of 200 packets, how many packets would be expected to have 0, 1, 2, 3, 4, 5 faulty toothbrushes?

30 On a particular morning, a machine in a factory workshop is active or idle according to the table at right.

Time (a.m.)	Machine state
9:00–9:30	Idle
9:30–10:20	Active
10:20–10:30	Idle
10:30–11:30	Active
11:30–12:00	Idle

(a) If an engineer walks through the workshop at a randomly selected time between 9:00 a.m. and 12:00 noon, determine the probability that the machine is idle.

If an engineer inspects the machine at six randomly selected times during the same period, determine the probability that the machine is idle: **(b)** at all six times **(c)** at all but one of the times.

31 To estimate the quality of mass-produced items, a customer uses the following double-sampling scheme. A random sample of ten is inspected: if no defectives are found, the customer accepts the lot; if four or more defectives are found, the customer rejects the lot. But if one, two or three defectives are found, the customer then takes a second sample of ten, and if the total of defectives in the two samples together is four or more, the customer rejects the lot, but otherwise the customer accepts the lot. Calculate the probability that a lot which is known to be 10% defective is accepted.

32 In a sporting challenge, the probability that a competitor will complete the course is p and the probability that a competitor will not complete the course is $q = 1 - p$. Teams consist of either two, four or six competitors. A team scores points if more than half its members complete the course.

(a) Show that the probability that a four-member team will have at least three of its members complete the course is $p^4 + 4p^3q$.

(b) Hence, or otherwise, find an expression for the probability that a four-member team will have at least three of its members not complete the course.

(c) Hence, or otherwise, find an expression in terms of q only for the probability that a four-member team will score points.

(d) Find an expression in terms of q only for the probability that a two-member team will score points.

(e) Find an expression in terms of q only for the probability that a six-member team will score points.

(f) Using your answers from **(c)** and **(e)**, find the range of values of q for which the six-member team is more likely to score points than the four-member team.

CHAPTER REVIEW 12

1 A box contains four marbles labelled 2, 3, 4, 6. If two marbles are drawn at random from the box, what is the probability that their sum is greater than 7?

2 **(a)** In how many ways can a committee of three parents and four teachers be chosen from six parents and eight teachers?

(b) The members of the committee are seated in a row of seven chairs on a stage. In how many different ways can they be seated?

(c) Find the number of ways of seating this committee if the parents refuse to sit at the ends of the row.

(d) What is the probability that for a chosen committee, the parents do not sit at the ends of the row?

3 Three-digit numbers are formed using the digits 1, 2, 3, 4, 5, 6, 7. If none of the digits are repeated, what is the probability of forming an even number?

4 The letters of the word CONSIDER are rearranged. What is the probability that the rearranged set of letters will begin with 'SI'?

5 Three cards are selected from a standard pack of 52 playing cards. What is the probability that at least one of the cards will be a King?

6 The letters of the word ARRANGEMENT are arranged in a row.

(a) In how many ways can this be done using all the letters?

(b) What is the probability that the vowels are all together?

7 Seven-digit numbers are formed using all the digits 2, 2, 3, 4, 6, 7, 7.

(a) How many different numbers can be formed?

(b) What is the probability that the number formed is odd?

(c) What is the probability that the number formed is greater than 4 000 000?

(d) What is the probability that the number formed is less than 7 000 000?

8 If a coin is tossed six times, what is the probability of obtaining exactly three heads?

9 Four ordinary six-sided dice are rolled. What is the probability that exactly three of the dice roll a 5? Leave your answer in unsimplified form.

10 A test consists of six multiple-choice questions. Each question has four possible answers: one that is correct and three that are incorrect. Amanda randomly selects an answer to each of the six questions.

(a) What is the probability that Amanda selects four correct and two incorrect answers?

(b) What is the probability that Amanda selects four or more correct answers?

(c) What is the probability that Amanda selects at least one incorrect answer?

11 Mr and Mrs Zeno and their four children go to the theatre. They are randomly allocated six adjacent seats in a single row. What is the probability that Mr and Mrs Zeno are allocated seats next to each other?

12 A four-person team is chosen at random from seven women and nine men.

(a) In how many ways can the team be chosen?

(b) What is the probability that the team will consist of four men?

13 There are seven matches in each round of a football season. Sanjiv takes part in a competition in which he earns one point if he picks more than half of the winning teams for a round, or zero points otherwise. The probability that Sanjiv picks the team that wins any given match is $\frac{3}{4}$.

(a) Show that the probability that Sanjiv earns one point for a given round is 0.9294 correct to four decimal places.

(b) Hence find the probability that Sanjiv earns one point in each round of a 13-round season. Give your answer correct to three decimal places.

(c) Find the probability that Sanjiv earns no more than 10 points during the 13-round season. Give your answer correct to two decimal places.

CHAPTER 13
APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay

HE4 uses the relationship between functions, inverse functions and their derivatives

HE5 applies the chain rule to problems including those involving velocity and acceleration as functions of displacement

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

13.1 RELATED RATES

Rates of change were introduced in Chapter 17 of *New Senior Mathematics for Years 11 & 12*. We will now look at more sophisticated applications of rates of change.

Problems with **related rates** arise when there is a function that relates two variables, e.g. x and y, where both variables are also functions of another variable, e.g. time t. For example, we may need to determine $\frac{dy}{dt}$ when $\frac{dx}{dt}$ is known. In such cases it is necessary to use the chain rule, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$.

Remember that in a context like this, 'increasing' means a positive rate of change while 'decreasing' means a negative rate of change.

Example 1

A spherical balloon is being inflated so that its radius increases at the constant rate of 3 cm/min. At what rate is its volume increasing when the radius of the balloon is 5 cm?

Solution

If r is the radius of the balloon, its volume is $V = \frac{4}{3}\pi r^3$.

We are given $\frac{dr}{dt} = 3$ and asked to find $\frac{dV}{dt}$ when $r = 5$.

By the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

But: $V = \frac{4}{3}\pi r^3$

So: $\frac{dV}{dr} = 4\pi r^2$

Thus: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 3$

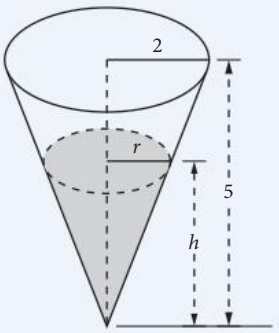
When $r = 5$: $\frac{dV}{dt} = 4\pi \times 5^2 \times 3$

$= 300\pi \text{ cm}^3/\text{min}$

Example 2

A vessel containing water has the shape of an inverted right circular cone with base radius 2 m and height 5 m. The water flows from the apex of the cone at a constant rate of $0.2\ \text{m}^3/\text{min}$. Find the rate at which the water level is dropping when the depth of the water is 4 m.

Solution

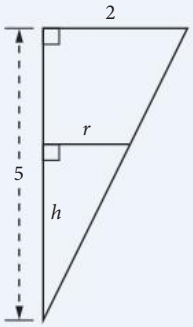


Let the depth of the water be h m, the radius of the cone at the water level be r m and the volume of the water be $V\ \text{m}^3$ at time t minutes.
The volume of the water at any time t is $V = \frac{1}{3}\pi r^2 h$.

We are given $\frac{dV}{dt} = -0.2$ and asked to find $\frac{dh}{dt}$ when $h = 4$.

$\frac{dV}{dt}$ is negative: the volume is decreasing, because the water is flowing out of the vessel.

To find the link between r and h we can use similar triangles.



From proportional sides: $\frac{r}{h} = \frac{2}{5}$

$$r = \frac{2h}{5}$$

Volume: $V = \frac{1}{3}\pi r^2 h$

Substitute $r = \frac{2h}{5}$: $V = \frac{1}{3}\pi \times \left(\frac{2h}{5}\right)^2 \times h$

$$V = \frac{4\pi h^3}{75}$$

Hence: $\frac{dV}{dh} = \frac{4\pi h^2}{25}$

Chain rule: $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$\therefore$ $-0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$

$$\frac{dh}{dt} = -\frac{5}{4\pi h^2}$$

When $h = 4$: $\frac{dh}{dt} = -\frac{5}{4\pi \times 16} = -\frac{5}{64\pi}$

$$\approx -0.0249\ \text{m/min}$$

Thus the water level is decreasing at a rate of 0.0249 m/min.

Alternatively: V and h are both dependent on time, so we can differentiate both sides of the volume equation with respect to time.

$$V = \frac{4\pi h^3}{75}$$

$$\frac{dV}{dt} = \frac{4\pi}{75} \times \frac{d}{dt}\left(h^3\right)$$

Chain rule: $= \frac{4\pi}{75} \times \frac{d}{dh}\left(h^3\right) \times \frac{dh}{dt}$

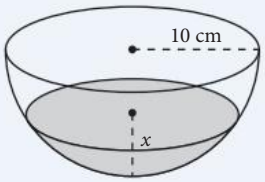
$$= \frac{4\pi}{75} \times 3h^2 \times \frac{dh}{dt}$$

$\frac{dV}{dt} = -0.2$, so: $-0.2 = \frac{4\pi h^2}{25} \times \frac{dh}{dt}$ as before.

Example 3

The volume of water in a hemispherical bowl of radius 10 cm is $V = \frac{1}{3}\pi x^2(30-x)$, where x cm is the depth of water at time t.

The bowl is being filled at a constant rate of 2π cm^3/min. At what rate is the depth increasing when the depth is 2 cm?



Solution

We are given $\frac{dV}{dt} = 2\pi$ and asked to find $\frac{dx}{dt}$ when $x = 2$.

The related variables are V, x and t.

$$V = \frac{1}{3}\pi x^2(30-x)$$
$$= 10\pi x^2 - \frac{1}{3}\pi x^3$$
$$\frac{dV}{dx} = 20\pi x - \pi x^2 = \pi x(20-x)$$

Method 1

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$
$$2\pi = \pi x(20-x) \times \frac{dx}{dt}$$
$$\frac{dx}{dt} = \frac{2}{x(20-x)}$$

Method 2

$$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$$
$$\frac{dx}{dt} = \frac{1}{\pi x(20-x)} \times 2\pi$$
$$\frac{dx}{dt} = \frac{2}{x(20-x)}$$

When $x = 2$: $\frac{dx}{dt} = \frac{1}{18}$ cm/s

As an extension to this question, we might ask: 'At what depth is the depth increasing at a minimum rate?'

This is the same as asking: 'For what value of x is $\frac{dx}{dt}$ a minimum?'

Note that the formula for the volume of water in the hemispherical bowl at any depth a can be calculated by finding the volume generated by rotating the circle with equation $x^2 + y^2 = 100$ between $y = 10 - a$ and $y = 10$.

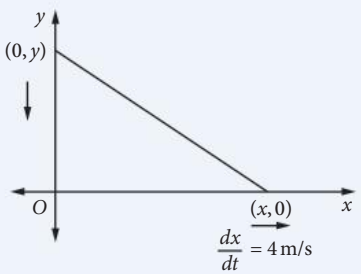
Example 4

A ladder 10 m long has its upper end against a vertical wall and its lower end on a horizontal floor. The lower end is slipping away from the wall at a constant speed of 4 m/s. Find the rate at which the upper end of the ladder is slipping down the wall when the lower end is 6 m from the wall. What is this rate when the upper end is very close to the ground?

Solution

At any time t, the lower end of the ladder is x m from the wall and the upper end is y m above the ground.

We are given $\frac{dx}{dt} = 4$ and asked to find $\frac{dy}{dt}$ when $x = 6$.



By Pythagoras' theorem: $x^2 + y^2 = 100$

$$y = \sqrt{100 - x^2},\ 0 \le x \le 10$$
$$\frac{dy}{dx} = \frac{1}{2}\left(100 - x^2\right)^{-\frac{1}{2}} \times (-2x)$$
$$= \frac{-x}{\sqrt{100 - x^2}},\ 0 \le x < 10$$

For $\frac{dx}{dt} = 4$:
$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= \frac{-x}{\sqrt{100 - x^2}} \times 4$$

At $x = 6$:
$$\frac{dy}{dt} = \frac{-4 \times 6}{\sqrt{100 - 36}} = -3$$

The top of the ladder is descending at the rate of 3 m/s when $x = 6$.

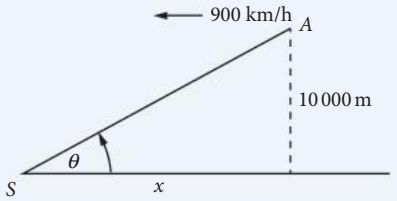
From the equations, we can see that as $y \to 0$, $x \to 10$ and $\frac{dy}{dt} \to -\infty$. This means that the speed of the upper end of the ladder is increasing as it falls (until it suddenly stops when it reaches the ground).

Example 5

A radar tracking station is located at ground level, vertically below the path of an approaching aircraft flying at 900 km/h at a constant height of 10 000 m. The tracking station transmits a radar beam that turns to follow the aircraft perfectly as it approaches. Find the rate in degrees per second at which the radar beam is turning when the aircraft is at a horizontal distance of 3 km from the station.

Solution

S represents the tracking station and x is the horizontal displacement of the aircraft at time t.



Find $\frac{d\theta}{dt}$ when $x = 3000$ m.

Converting units: $900 \text{ km/h} = 900 \times \frac{1000}{3600} \text{ m/s} = 250 \text{ m/s}$

Hence: $\frac{dx}{dt} = -250$

Right-angled triangle: $x = \frac{10000}{\tan\theta} = 10\,000(\tan\theta)^{-1}$

Chain rule differentiation:
$$\frac{dx}{d\theta} = 10000 \times (-1)(\tan\theta)^{-2} \times \sec^2\theta$$
$$= -10000 \times \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos^2\theta}$$
$$= -10000 \operatorname{cosec}^2\theta$$

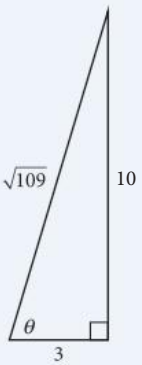
Reciprocal of both sides: $\frac{d\theta}{dx} = -\frac{\sin^2\theta}{10000}$

Chain rule:
$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$$
$$= -\frac{\sin^2\theta}{10\,000} \times \frac{dx}{dt}$$

For $\frac{dx}{dt} = -250$: $\frac{d\theta}{dt} = -\frac{\sin^2\theta}{10\,000} \times (-250) = \frac{\sin^2\theta}{40}$

From the triangle, with $x = 3000$ m: $\tan\theta = \frac{10000}{3000} = \frac{10}{3}$

From the triangle: $\sin\theta = \frac{10}{\sqrt{109}}$



Hence
$$\frac{d\theta}{dt} = \frac{1}{40} \times \frac{100}{109} = \frac{5}{218} \text{ radians per second}$$
$$= \frac{5}{218} \times \frac{180}{\pi} \text{ degrees per second}$$
$$\approx 1.3 \text{ degrees per second}$$

EXERCISE 13.1 RELATED RATES

1 A spherical balloon is being filled with air at the rate of $100\,\text{cm}^3/\text{min}$. At what rate is the radius of the balloon increasing when the radius is 5 cm?

2 Water is being poured at a constant rate of $3\,\text{cm}^3/\text{s}$ into an inverted right conical vessel whose apex angle is 90°. At what rate is the water level rising when the depth is π cm?

3 A lamp is 6 m directly above a straight footpath. A person 2 m tall walks along the footpath away from the light at a constant speed of 1 m/s. At what speed is the end of the person's shadow moving along the path? At what speed is the length of the shadow increasing?

4 Given $A = x\sqrt{100 - x^2}$ and $\frac{dx}{dt} = 0.3$, the value of $\frac{dA}{dt}$ when $x = 8$ is:

A $-\frac{72}{5}$ **B** $\frac{72}{5}$ **C** $-\frac{7}{5}$ **D** $\frac{7}{5}$

5 A particle moves on the circumference of the semicircle $y = \sqrt{4 - x^2}$. If $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ at $x = 1$.

6 A melting snowball is decreasing in volume at a constant rate of $8\,\text{cm}^3/\text{min}$. If the melting snowball is always a perfect spherical shape, find the rate at which its radius is changing when the radius is 4 cm.

7 A particle is moving along a curve with equation $y = x^2$. If $\frac{dx}{dt} = 2$ at $x = 3$, find $\frac{dy}{dt}$ at that instant.

8 A circular oil slick floats on the surface of still water. Its area is increasing at a rate of $10\,\text{m}^2/\text{min}$. At what rate is the radius (r metres) increasing?

9 A point P moves on the curve $y = x^3$ so that its x-coordinate increases at a constant rate of 5 units per second. When $x = 1$:
(a) at what rate is the y-coordinate of P increasing
(b) at what rate is the gradient of the curve increasing?

10 The perimeter of a circular sector is 20 cm. The radius is increasing at a rate of 5 cm/s.
(a) At what rate is the angle of the sector changing when the radius length is 10 cm?
(b) At what rate is the area changing when the radius is 10 cm?

11 A ladder 5 metres long is standing vertically, flat against a vertical wall, while its lower end is on the horizontal floor. The lower end moves horizontally away from the wall at a constant speed of 1 m/s while the upper end stays in contact with the wall. Find the speed at which the upper end is moving down the wall four seconds after the lower end has left the wall.

12 A vessel is shaped so that when the depth of water in it is x cm, the volume of the water is $V\,\text{cm}^3$ where $V = 108x + x^3$. Water is poured into the vessel at a constant rate of $30\,\text{cm}^3/\text{s}$. At what rate is the water level rising when its depth is 8 cm?

13 A trough l metres long has a cross-section in the shape of an isosceles triangle with base length $2a$ metres and height b metres. Water leaks from the trough at a constant rate of $c\,\text{m}^3/\text{min}$. Find the rate at which the water level is falling when the depth of the water is $\frac{b}{2}$ metres.

14 Sand is poured into a heap in the shape of a right circular cone whose semi-vertex angle is α, where $\tan\alpha = \frac{3}{4}$. When the height of the cone is 16 cm, the height is increasing at a rate of 2 cm/min. At that instant, at what rate is the volume increasing?

15 When a certain gas expands at constant temperature, its pressure P and volume V are given by the relation $PV^{1.4} = k$, a constant. At a certain instant the pressure is $25\,\text{g/cm}^2$ and the volume is $32\,\text{cm}^3$. If the volume is increasing at the rate of $5\,\text{cm}^3/\text{s}$, at what rate is the pressure changing at that instant?

16 A kite 50 metres above the ground is being carried horizontally by the wind at a rate of 4 m/s. If the kite's string is always perfectly straight, how fast must the length of the string be increasing when the string is 100 metres long?

17 Sand is poured into a heap at a constant rate of $4\,\text{cm}^3/\text{min}$ so that the heap is in the shape of a right circular cone whose height is always equal to the radius of the base. When the heap is 10 cm high:

(a) how fast is the height increasing **(b)** how fast is the area of the base increasing?

18 A boat is pulled in to a wharf by a rope at a speed of 20 m/min. If the rope is attached to a point on the boat 7 m vertically below the wharf, at what rate is the rope being drawn in when the boat is 24 m from the wharf?

19 A straight railway track and a straight road intersect at right angles. At a given instant a car travelling at 40 km/h and a train travelling at 50 km/h are moving away from the intersection and are 40 km and 30 km from the intersection respectively.

(a) The car and train continue moving in straight lines without changing their speed. One hour later, at what rate is the distance between the car and the train changing?

(b) At what rate would the distance between the car and train be changing if they were both travelling towards the intersection?

20 Two straight roads meet at an angle of 60°. A starts from this intersection and travels along one road at 40 km/h. One hour later, B starts from the intersection and travels along the other road at 50 km/h. Three hours after A starts, at what rate is the distance between A and B changing?

21 A conical tank with a vertical axis has a semi-vertical angle of 45°. Water, initially at a depth of 5 metres, leaks out through a hole at the bottom of the tank at a rate of $0.2\sqrt{h}\ \text{m}^3/\text{min}$ when the depth is h metres. Find the rate at which the depth is decreasing when the depth is 4 metres.

22 The altitude of a right-angled triangle is 6 cm and its base is increasing at a constant rate of 2 cm/s. At what rate is the hypotenuse increasing when its length is 10 cm?

23 A loading chute is in the shape of a square pyramid with base length 10 m and depth 8 m. Liquid is poured in at the top at a rate of $4\,\text{m}^3/\text{min}$. At what rate is the level rising when the depth is 4 m?

24 A metal sphere is dissolving in acid. It remains spherical and the rate at which its volume decreases is proportional to its surface area. Show that the radius of the sphere decreases at a constant rate.

25 A circular cylinder with height 6 cm and base radius 4 cm sits on a table with its axis vertical. A point source of light moves vertically up at a speed of 3 cm/s above the central axis of the cylinder, thus casting a circular shadow on the table. Find the rate at which the radius of the shadow is decreasing when the light is 4 cm above the top of the cylinder.

26 Grain is ejected from a chute at a rate of $0.1\,\text{m}^3/\text{min}$ to form a heap on a flat horizontal floor. The heap is in the form of a circular cone of semi-vertical angle 45°. Find the rate (in metres per minute) at which the height of the cone is increasing at the instant 3 minutes after the opening of the chute.

27 In triangle ABC, $AB = 10$ cm, $AC = 12$ cm and angle A is increasing at the rate of 0.1 radians per second. At what rate is:

(a) the area of ΔABC increasing **(b)** the length of BC increasing, when angle A is $\frac{\pi}{3}$ radians?

28 A spherical mothball evaporates at a rate proportional to its surface area so that its volume $V\,\text{cm}^3$ and radius r after t weeks are related by the equation $\frac{dV}{dt} = -4k\pi r^2$, where k is a positive constant.

(a) Show that $\frac{dr}{dt} = -k$.

(b) If the initial radius of the mothball is 1 cm and the radius after 10 weeks is 0.5 cm, express r in terms of t.

29 The volume V and surface area S of a sphere of radius r are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$ respectively.

(a) Show that $\frac{dV}{dr} = S$ and $\frac{dV}{dt} = S\frac{dr}{dt}$.

(b) A spherical ball of radius 24 mm is immersed in an acid bath so that its volume decreases at a changing rate equal to three times its surface area (while remaining constantly spherical). Find the time taken for each of: **(i)** the radius **(ii)** the volume to reduce to one-eighth of their original size.

30 **(a)** Show that the formula for the volume V of a right circular cone of base radius r and height h can be expressed as $V = \frac{1}{3}\pi h^3 \tan^2\alpha$, where α is the semi-vertex angle.

(b) Water flows out through a hole at the vertex angle of an inverted cone, whose angle is 60°, at a rate equal to π times the square root of the depth of the water at any time. At what rate (in cm/s) would the water level be dropping when the depth is 9 cm?

13.2 EXPONENTIAL CHANGE

This is an extension of the work on exponential growth and decay that you have studied before, in which we considered the equation $\frac{dN}{dt} = kN$. That solution was found to be $N = Ae^{kt}$, where A is the initial value of N. It is a useful exercise to derive this result by integration.

Example 6

Show by integration that the solution to $\frac{dN}{dt} = kN$ is $N = Ae^{kt}$, where A is the value of N when $t = 0$.

Solution

$$\frac{dN}{dt} = kN$$

Reciprocal of both sides: $$\frac{dt}{dN} = \frac{1}{kN}$$

Integrate with respect to N: $$t = \int \frac{1}{kN}\,dN$$

$$t = \frac{1}{k}\log_e N + C$$

When $t = 0$, $N = A$: $$0 = \frac{1}{k}\log_e A + C$$

$$C = -\frac{1}{k}\log_e A$$

$\therefore$ $$t = \frac{1}{k}\log_e N - \frac{1}{k}\log_e A$$

$$t = \frac{1}{k}\log_e \frac{N}{A}$$

$$kt = \log_e \frac{N}{A}$$

$$\frac{N}{A} = e^{kt}$$

$$N = Ae^{kt}$$

Solution of $\frac{dN}{dt} = k(N - P)$ where k and P are constants

$\frac{dN}{dt} = k(N - P)$ means that the rate of change of N is proportional to the excess of N over a fixed quantity P.

This can be applied to several real-life physical processes.

Given $\frac{dN}{dt} = k(N - P)$, where k and P are constants, it is easy to show that a solution of this equation is $N = P$.

If $N = P$, then $\frac{dN}{dt} = k(P - P) = 0$; because P is a constant, differentiating the equation $N = P$ with respect to t also gives $\frac{dN}{dt} = 0$.

It can also be shown by substitution that $N = P + Ae^{kt}$ is a solution to $\frac{dN}{dt} = k(N-P)$:

$$\text{LHS} = \frac{dN}{dt} \qquad \text{RHS} = k(N-P)$$
$$= Ake^{kt} \qquad = k(P + Ae^{kt} - P)$$
$$= Ake^{kt} = \text{LHS}$$

It is important to be able to derive this result by integration.

$$\frac{dN}{dt} = k(N-P)$$

Reciprocal of both sides: $\frac{dt}{dN} = \frac{1}{k(N-P)}, \quad N \neq P$

Integrate with respect to N: $t = \frac{1}{k}\int \frac{1}{N-P}\,dN$

$$t = \frac{1}{k}\log_e(N-P) + C, \quad N > P$$

$$k(t-C) = \log_e(N-P)$$
$$N - P = e^{k(t-C)}$$
$$= e^{kt} \times e^{-kC}$$

Let $A = e^{-kC}$ (a constant): $N - P = Ae^{kt}$

$$N = P + Ae^{kt}$$

If $\frac{dN}{dt} = k(N-P)$ then $N = P + Ae^{kt}$

If $k < 0$, $Ae^{kt} \to 0$ as $t \to \infty$ and hence $N \to P$ from above.

Example 7

N is increasing according to the equation $\frac{dN}{dt} = 0.4(N-30)$. If $N = 60$ when $t = 0$:

(a) show that $N = 50 + Ae^{0.4t}$ is a solution to this equation, where A is a constant

(b) calculate the value of N when $t = 20$.

Solution

(a) Differentiate $N = 50 + Ae^{0.4t}$: $\frac{dN}{dt} = 0.4Ae^{0.4t}$

Rewrite $N = 50 + Ae^{0.4t}$: $Ae^{0.4t} = N - 50$

Substitute into $\frac{dN}{dt} = 0.4Ae^{0.4t}$: $\frac{dN}{dt} = 0.4(N-50)$

Hence $N = 50 + Ae^{0.4t}$ is a solution to the equation $\frac{dN}{dt} = 0.4(N-30)$.

(b) At $t = 0$, $N = 60$: $60 = 50 + A$

$$A = 10$$
$$\therefore \quad N = 50 + 10e^{0.4t}$$

At $t = 20$: $N = 50 + 10e^{8}$

$$N \approx 29\,860$$

Example 8

The mass M of a particular southern right whale is modelled as $M = 55 - 54e^{-kt}$, where M is measured in tonnes, t is the age of the whale in years and k is a positive constant.

(a) Show that the rate of growth of the whale's mass is given by the differential equation $\frac{dM}{dt} = k(55 - M)$.

(b) What is the birth mass of the whale? (i.e. at $t = 0$)

(c) When the whale is one year old, its mass is 10 tonnes. Show that $k = \ln\left(\frac{6}{5}\right)$.

(d) What is the mass of the whale when it is 10 years old (to the nearest tonne)?

(e) If male southern right whales grow to about 55 tonnes and females grow to about 85 tonnes, determine the gender of this whale, giving reasons for your answer.

Solution

(a) $M = 55 - 54e^{-kt}$

Differentiate with respect to t: $\frac{dM}{dt} = 54ke^{-kt}$

Rewrite $M = 55 - 54e^{-kt}$: $54e^{-kt} = 55 - M$

Substitute into $\frac{dM}{dt} = 54ke^{-kt}$: $\frac{dM}{dt} = k(55 - M)$

Hence the rate of growth of the whale's mass is $\frac{dM}{dt} = k(55 - M)$.

(b) At $t = 0$: $M = 55 - 54e^{0}$

$M = 55 - 54 = 1$

The birth mass of the whale is 1 tonne.

(c) At $t = 1$, $M = 10$: $10 = 55 - 54e^{-k}$

$54e^{-k} = 45$

$e^{-k} = \frac{45}{54} = \frac{5}{6}$

$e^{k} = \frac{6}{5}$

$k = \ln\left(\frac{6}{5}\right)$

(d) At $t = 10$: $M = 55 - 54e^{-10\ln\left(\frac{6}{5}\right)}$

$M = 55 - 54e^{10\ln\left(\frac{5}{6}\right)}$

$M \approx 46.3$

The mass of the whale to the nearest tonne is 46 tonnes.

(e) As $t \to \infty$: $M \to 55 - 54e^{-\infty}$

$M \to 55$

The limiting mass of the whale is 55 tonnes, so it is most likely to be a male.

Newton's Law of Cooling

Newton's Law of Cooling states that the cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium:

$$\frac{dT}{dt} = -k(T - M)$$

where T is the temperature at any time t
and M is the temperature of the surrounding medium (a constant).

Example 9

The original temperature of a body is 100°C, the temperature of its surroundings is 20°C and the body cools to 70°C in 10 minutes. Assuming Newton's Law of Cooling, i.e. $\frac{dT}{dt} = -k(T - 20)$ where T is the temperature of the body at time t, find:

(a) the temperature after 20 minutes **(b)** the time taken to cool from 100°C to 60°C.

Solution

$$\frac{dT}{dt} = -k(T-20)$$

Reciprocal of both sides: $\frac{dt}{dT} = \frac{-1}{k(T-20)}, \quad T \neq 20$

Integrate with respect to T: $t = -\frac{1}{k}\int \frac{1}{T-20}\,dT$

$$t = -\frac{1}{k}\log_e(T-20) + C, \quad T > 20$$

$$-k(t-C) = \log_e(T-20)$$

$$T - 20 = e^{-k(t-C)}$$

Let $A = e^{-kC}$ (a constant): $T = 20 + Ae^{-kt}$

When $t = 0$, $T = 100$: $100 = 20 + A$

$$A = 80$$

When $t = 10$, $T = 70$: $70 = 20 + 80e^{-10k}$

$$e^{-10k} = \frac{50}{80} = 0.625$$

$$-10k = \log_e 0.625$$

$$-10k \approx -0.47$$

$$k = 0.047 \quad \text{(using the approximate logarithm value)}$$

$\therefore \quad T = 20 + 80e^{-0.047t}$

(a) When $t = 20$: $T = 20 + 80e^{-0.94}$

$$T \approx 51.25$$

After 20 minutes the temperature is approximately 51.25°C.

(b) When $T = 60$: $60 = 20 + 80e^{-0.047t}$

$$e^{-0.047t} = \frac{40}{80} = 0.5$$

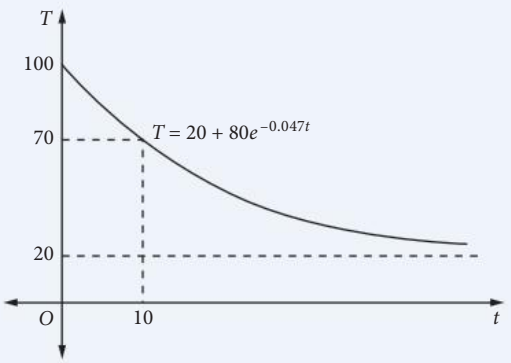
$$-0.047t = \log_e 0.5$$

$$-0.047t \approx -0.6931$$

$$t \approx 14.7$$

The temperature reaches 60°C after approximately 14.7 minutes.

The graph of $T = 20 + 80e^{-0.047t}$ shows that the temperature of the body never falls below the temperature of the surroundings. As $t \to \infty$, $T \to 20$ from above.



Wilhelmy's Law

Many chemical reactions follow the law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time:

$$\frac{dx}{dt} = k(a-x), \quad 0 \leq x \leq a$$

where a is the initial concentration and x is the amount transformed at time t.

Example 10

A chemical reaction follows the rule $\frac{dx}{dt} = k(a-x)$, where a is the initial concentration and x is the amount of the reagent transformed at time t. Thus when $t = 0$, $x = 0$. If $a = 10$ and after two minutes $x = 4$, find the concentration of the reagent after five minutes.

Solution

$$\frac{dx}{dt} = k(10-x), \quad 0 \le x \le 10$$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{k(10-x)}, \quad 0 \le x < 10$

Integrate with respect to x: $t = \frac{1}{k}\int \frac{1}{10-x}\,dx$

$$kt = -\log_e(10-x) + C$$

When $t = 0$, $x = 0$: $0 = -\log_e 10 + C$

$$C = \log_e 10$$

$\therefore$ $kt = \log_e 10 - \log_e(10-x)$

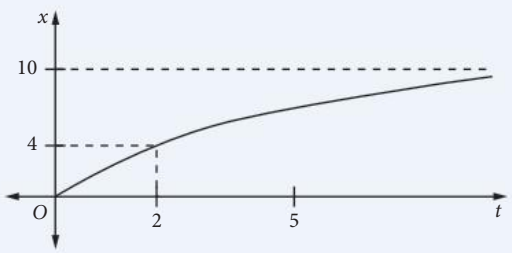
$$kt = \log_e \frac{10}{10-x}$$

Use inverse functions: $\frac{10}{10-x} = e^{kt}$

$$\frac{10-x}{10} = e^{-kt}$$

$$10 - x = 10e^{-kt}$$

$$x = 10(1 - e^{-kt}) \text{ for } t \ge 0$$



Note that the asymptote of the graph is $x = 10$, approached from below. This is consistent with the restriction $0 \le x < 10$ for $t \ge 0$.

When $t = 2$, $x = 4$: $4 = 10(1 - e^{-2k})$

$$0.4 = 1 - e^{-2k}$$

$$e^{-2k} = 0.6$$

Reciprocal of both sides: $e^{2k} = \frac{5}{3}$

$$2k = \log_e\left(\frac{5}{3}\right)$$

$$k = \frac{1}{2}\log_e\left(\frac{5}{3}\right)$$

When $t = 5$: $x = 10(1 - e^{-5k})$

But: $e^{-2k} = 0.6$

$$e^{-5k} = \left(e^{-2k}\right)^{\frac{5}{2}} = \left(\frac{3}{5}\right)^{\frac{5}{2}} \approx 0.279$$

$\therefore$ $x = 10(1 - 0.279)$

$$x = 10 - 2.79$$

or inverse functions: $-2k = \log_e 0.6$

$$-2k = -0.51$$

$$k = 0.255$$

i.e. $x = 10(1 - e^{-5k})$

$$x = 10(1 - e^{-1.275})$$

$$x = 10(1 - 0.279)$$

$$x = 10 - 2.79$$

Hence the concentration is 2.79 units after five minutes. (Remember that $(10 - x)$ is the concentration remaining.)

EXERCISE 13.2 EXPONENTIAL CHANGE

1 N is decreasing according to the equation $\frac{dN}{dt} = -0.4(N-30)$. If $N = 60$ when $t = 0$:

(a) show that $N = 30 + Ae^{-0.4t}$ is a solution of this equation, where A is a constant

(b) calculate the value of N when $t = 5$.

2 N is increasing according to the equation $\frac{dN}{dt} = 0.2(N-40)$. If $N = 50$ when $t = 0$:

(a) show that $N = 40 + Ae^{0.2t}$ is a solution to this equation, where A is a constant

(b) calculate the value of N when $t = 10$.

3 The original temperature of a body is 120°C, the temperature of its surroundings is 50°C and the body cools to 70°C in ten minutes. Assuming Newton's Law of Cooling, i.e. $\frac{dT}{dt} = -k(T-50)$ where T is the temperature of the body at time t, find:

(a) the temperature after 20 minutes **(b)** the time taken to cool to 60°C.

4 If $N = 70$ when $t = 0$, which expression is the correct solution to $\frac{dN}{dt} = -0.5(N-20)$?

A $N = 20 + 50e^{0.5t}$ **B** $N = 20 + 50e^{-0.5t}$ **C** $N = 20 - 50e^{0.5t}$ **D** $N = 20 - 50e^{-0.5t}$

5 A metal bar has a temperature of 1230°C and cools to 1030°C in ten minutes when the surrounding temperature is 30°C. Assume Newton's Law of Cooling, i.e. $\frac{dT}{dt} = -k(T-30)$ where T is the temperature of the body at time t.

(a) Show that $T = 30 + 1200e^{-kt}$ satisfies both Newton's Law of Cooling and the initial conditions.

(b) Find the temperature after 20 minutes. **(c)** Find the time taken to cool from 1230°C to 80°C.

6 Water at 20°C is placed in a freezer where the air is at a constant temperature of −10°C. The temperature of the water falls to 15°C in five minutes. Assume Newton's Law of Cooling, i.e. $\frac{dT}{dt} = -k(T+10)$ where T is the temperature of the body at time t.

(a) Show that $T = -10 + 30e^{-kt}$ satisfies both Newton's Law of Cooling and the initial conditions.

(b) Find the temperature of the water after another five minutes (when $t = 10$).

7 A body whose temperature is 180°C is immersed in a liquid that is at 60°C. In one minute the temperature of the body has fallen to 120°C. Assume Newton's Law of Cooling, i.e. $\frac{dT}{dt} = -k(T-60)$ where T is the temperature of the body at time t.

(a) Show that $T = 60 + 120e^{-kt}$ satisfies both Newton's Law of Cooling and the initial conditions.

(b) At what time would the temperature of the body have fallen to 90°C?

8 A current of i amperes (or 'amps') flows through a coil of inductance L henrys and resistance R ohms. The current at any time is given by $i = \frac{E}{R}\left(1 - e^{\frac{-Rt}{L}}\right)$ where E is the electromotive force (i.e. the voltage) in volts. Show that $L\frac{di}{dt} + Ri = E$.

9 A vessel is filled at a variable rate so that the volume of liquid in the vessel at any time t is given by $V = A(1 - e^{-kt})$.

(a) Show that $\frac{dV}{dt} = k(A-V)$.

(b) If a quarter of the vessel is filled in the first five minutes, what fraction is filled in the next five minutes?

(c) Show that $\lim_{t\to\infty} V = A$.

10 A rectangular vessel is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 20 cm, flows into the other compartment, initially empty, at a rate proportional to the difference between the levels in each compartment. The differential equation for this process is $\frac{dx}{dt} = k(20-2x)$, where x cm is the depth of the liquid in one of the vessels at any time t minutes.

(a) Show that $x = 10(1 - e^{-2kt})$.

(b) If the level in the second compartment rises 2 cm in the first five minutes, at what time will the difference in levels be 2 cm?

11 In a certain chemical process, the amount y grams of a certain substance at time t hours is given by the formula $y = 3 + e^{-kt}$.

(a) Show that $\frac{dy}{dt} = -k(y-3)$.

(b) If initially y decreases at a rate of 0.08 grams per hour, find the value of k.

(c) Find the rate of change when $y = 3.5$.

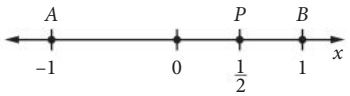
(d) What values can y take?

13.3 VELOCITY AND ACCELERATION AS FUNCTIONS OF x

The expressions $x(t)$, $v(t) = \dot{x}(t)$ and $a(t) = \ddot{x}(t) = \dot{v}(t)$, which denote displacement, velocity and acceleration respectively, clearly define functions of time t. This makes sense on practical grounds, because an ordinary particle cannot be in two different places at the same time: for each t there corresponds only one value of $x(t)$ that satisfies the function. Similarly $(t, v(t))$ and $(t, a(t))$ define functions because a particle cannot have more than one velocity and one acceleration at any given time.

However, sometimes instead of expressing acceleration in terms of t it is useful to express it in terms of the displacement x or in terms of the velocity v. In this case we must be careful, because equations relating a, v and x will not necessarily define functions.

Consider a particle that moves so that its displacement at time $t \geq 0$ is given by $x(t) = \sin t$. This means that the particle is moving back and forth between the limits $x = \pm 1$. (This is an example of simple harmonic motion, an important kind of motion that is explored in more detail later in this chapter.)



The particle is at $x = \frac{1}{2}$ whenever $\sin t = \frac{1}{2}$, i.e. when $t = \frac{\pi}{6}, \pi - \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, \ldots$ so that $x(t)$ defines a many-to-one function of time. Similarly, $v(t) = \dot{x}(t) = \cos t$ defines a many-to-one function of time.

Note that $v^2 = \cos^2 t = 1 - \sin^2 t = 1 - x^2$. Hence the equation $v^2 = 1 - x^2$ expresses v in terms of x. However, this equation does not express v as a function of x, because (for example) at $x = \frac{1}{2}$ we may have $v = \pm\frac{\sqrt{3}}{2}$. The point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ corresponds to $t = \frac{\pi}{6}$ while $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ corresponds to $t = \frac{5\pi}{6}$. There are an infinite number of values of t for each value of x. As the particle moves to and fro between A and B, its velocity has two possible values at any point P in between, depending on whether the particle is moving towards A or towards B at the time. At the extremes A and B the velocity is zero, so at these points we can say the particle is instantaneously at rest.

Remember that differential calculus only makes sense when applied to functions. In this context $\frac{dv}{dx}$ can only have meaning if $v = v(x)$ defines a function. Hence if we want to differentiate the equation $v^2 = 1 - x^2$ we must restrict v to be either positive or negative. This means that in the equation $v^2 = 1 - x^2$ we consider v to apply only to velocities as the particle moves from A to B, i.e. $v > 0$, or only from B to A, i.e. $v < 0$.

If $v(x)$ specifies a velocity function of x according to one of these restrictions, then we can use the chain rule of differentiation:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= v\frac{dv}{dx} \quad \text{as } v = \frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

- Given $a = h(v)$, use $\begin{cases} \dfrac{dv}{dt} & \text{if initial conditions are values for } t \text{ and } v \\ v\dfrac{dv}{dx} & \text{if initial conditions are values for } x \text{ and } v \end{cases}$

It is customary to write derivatives with respect to time using dots above the dependent variable,
e.g. $\dot{x} = \dfrac{dx}{dt}$, $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\dot{v} = \dfrac{dv}{dt}$.

Example 11

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If the acceleration is $3 - 2x$, find v in terms of x given that $v = 2$ when $x = 1$.

Solution

$\ddot{x} = 2 - 2x$ gives the acceleration as a function of x and the initial conditions are v and x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 3 - x$

Integrate with respect to x: $\quad \frac{1}{2}v^2 = \int (3 - 2x)\,dx$

$$\frac{1}{2}v^2 = 3x - x^2 + C$$

At $x = 1$, $v = 2$: $\quad 2 = 6 - 4 + C$

$$C = 0$$

$\therefore \quad \frac{1}{2}v^2 = 3x - x^2$

$$v^2 = 6x - 2x^2$$

$$v = \pm\sqrt{6x - 2x^2}$$

The conditions include $x = 1$, $v = 2$, which means that v is positive when $x = 1$.
To satisfy the initial conditions we must take the positive square root.

$\therefore \quad v = \sqrt{6x - 2x^2}$

It is worth noting the domain of the velocity function. For v to exist, $6x - 2x^2 \geq 0$, so $0 \leq x \leq 3$: this means that the motion exists only for $0 \leq x \leq 3$.

Example 12

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = 3x^2$ and $v = -\sqrt{2}$, $x = 1$ when $t = 0$, find x as a function of t.

Solution

$\ddot{x} = 3x^2$ gives acceleration as a function of x, so to find v as a function of x:

Use $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\quad \dfrac{d}{dx}\left(\frac{1}{2}v^2\right) = 3x^2$

Integrate with respect to x: $\frac{1}{2}v^2 = \int 3x^2\,dx$

$\frac{1}{2}v^2 = x^3 + C_1$

At $x = 1$, $v = -\sqrt{2}$: $1 = 1 + C_1$

$C_1 = 0$

$\therefore$ $\frac{1}{2}v^2 = x^3$

$\frac{1}{2}v^2 = x^3$

$v^2 = 2x^3$

$v = \pm\sqrt{2x^3}$

The conditions include $x = 1$, $v = -\sqrt{2}$, which means that v is negative when $x = 1$.
To satisfy the initial conditions we must take the negative square root.

$\therefore$ $v = -\sqrt{2}x^{\frac{3}{2}}$

Now $v = \frac{dx}{dt}$: $\frac{dx}{dt} = -\sqrt{2}x^{\frac{3}{2}}$

Reciprocal of both sides: $\frac{dt}{dx} = -\frac{1}{\sqrt{2}}x^{-\frac{3}{2}}$

Integrate with respect to t: $t = -\frac{1}{\sqrt{2}}\int x^{-\frac{3}{2}}\,dx$

$t = -\frac{1}{\sqrt{2}} \times \left(-\frac{2}{1}\right)x^{-\frac{1}{2}} + C_2$

$t = \sqrt{2}x^{-\frac{1}{2}} + C_2$

At $x = 1$, $t = 0$: $0 = \sqrt{2} + C_2$

$C_2 = -\sqrt{2}$

$\therefore$ $t = \sqrt{2}x^{-\frac{1}{2}} - \sqrt{2}$

Find x in terms of t: $\sqrt{2}x^{-\frac{1}{2}} = t + \sqrt{2}$

$x^{-\frac{1}{2}} = \frac{t+\sqrt{2}}{\sqrt{2}}$

$x^{\frac{1}{2}} = \frac{\sqrt{2}}{t+\sqrt{2}}$

$x = \frac{2}{\left(t+\sqrt{2}\right)^2}$

Example 13

The velocity of a particle is $v = 3x + 7$ m s^{-1}.

(a) Find an expression for the acceleration.

(b) If the initial displacement is 1 m to the right of the origin, find the displacement as a function of time.

Solution

(a) $v = 3x + 7$

$$\frac{dv}{dx} = 3$$

Use $\ddot{x} = v\frac{dv}{dx}$: $\ddot{x} = (3x+7)\times 3$

$= 3(3x+7)\ \text{m s}^{-2}$

(b) $\frac{dx}{dt} = 3x + 7$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{3x+7}$

Integrate with respect to x: $t = \int \frac{dx}{3x+7}$

$$t = \tfrac{1}{3}\log_e(3x+7) + C$$

At $t = 0$, $x = 1$: $C = -\tfrac{1}{3}\log_e 10$

$\therefore$ $t = \tfrac{1}{3}\log_e(3x+7) - \tfrac{1}{3}\log_e 10$

$$t = \tfrac{1}{3}\log_e\left(\frac{3x+7}{10}\right)$$

$$3t = \log_e\left(\frac{3x+7}{10}\right)$$

$$\left(\frac{3x+7}{10}\right) = e^{3t}$$

$$3x + 7 = 10e^{3t}$$

$$x = \tfrac{1}{3}\left(10e^{3t} - 7\right)$$

Example 14

A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If $\ddot{x} = -e^{-\frac{x}{2}}$ and $v = 2$, $x = 0$ when $t = 0$, find x as a function of t.

Solution

Use $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -e^{-\frac{x}{2}}$

Integrate with respect to x: $\frac{1}{2}v^2 = -\int e^{-\frac{x}{2}}\,dx$

$$\frac{1}{2}v^2 = 2e^{-\frac{x}{2}} + C_1$$

At $x = 0$, $v = 2$: $2 = 2 + C_1$

$C_1 = 0$

$\therefore$ $\frac{1}{2}v^2 = 2e^{-\frac{x}{2}}$

$$v^2 = 4e^{-\frac{x}{2}}$$

$$v = \pm 2e^{-\frac{x}{4}}$$

The conditions include $x = 0$, $v = 2$, which means that v is positive when $x = 0$; also $e^{-\frac{x}{4}} > 0$ for all x. To satisfy the initial conditions we must take the positive square root.

$\therefore$ $v = 2e^{-\frac{x}{4}}$

Now $v = \frac{dx}{dt}$: $\frac{dx}{dt} = 2e^{-\frac{x}{4}}$

Reciprocal of both sides: $\frac{dt}{dx} = \frac{1}{2}e^{\frac{x}{4}}$

Integrate with respect to t: $t = \frac{1}{2}\int e^{\frac{x}{4}}\,dx$

$t = \frac{1}{2} \times 4e^{\frac{x}{4}} + C_2$

$t = 2e^{\frac{x}{4}} + C_2$

At $t = 0$, $x = 0$: $0 = 2 + C_2$

$C_2 = -2$

$\therefore$ $t = 2e^{\frac{x}{4}} - 2$

Find x in terms of t: $e^{\frac{x}{4}} = \frac{t+2}{2}$

$\frac{x}{4} = \log_e\left(\frac{t+2}{2}\right)$

$x = 4\log_e\left(\frac{t+2}{2}\right)$

EXERCISE 13.3 VELOCITY AND ACCELERATION AS FUNCTIONS OF x

1 A particle moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. If its acceleration is given by $\ddot{x} = 4 + x$ and $v = 1$ when $x = 0$, find v when $x = 1$.

2 The acceleration of a particle moving in a straight line is given by $\frac{d^2x}{dt^2} = 7 - 2x$ and the particle starts from rest at the point where $x = 0$. Find the speed in terms of x. What values can x take?

3 At time t, the displacement of a particle moving in a straight line is x. If the acceleration is given by $\frac{d^2x}{dt^2} = 3 - 4x$ and the particle starts from rest at $x = 1$, find its velocity at any position. At what other point, if any, does the particle come to rest?

4 The acceleration of a particle moving in a straight line is given by $\ddot{x} = kx$ where $k = 1$. If $\dot{x} = 10\ \text{m s}^{-1}$ when $x = 6\,\text{m}$, the velocity of the particle when it is 15 m from the origin is:

A $-17\,\text{m s}^{-1}$ **B** $\sqrt{79}\,\text{m s}^{-1}$ **C** $17\,\text{m s}^{-1}$ **D** $23\,\text{m s}^{-1}$

5 A particle leaves O with velocity $2\,\text{m s}^{-1}$. Its acceleration is $-\frac{1}{6}\sqrt[3]{x}\ \text{m s}^{-2}$ when its displacement from O is x metres. Find its displacement when it first comes to rest.

6 A particle moves in a straight line and its acceleration at any time t is $\cos x$. If $v = 0$ and $x = 0$ when $t = 0$, express v in terms of x.

7 A particle moves in a straight line and its acceleration at any time is given by $\frac{d^2x}{dt^2} = \sin^2 x$. Find $\frac{dx}{dt}$ given that $\frac{dx}{dt} = 1$ when $x = 0$.

8 If $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - 3x^2$ and $v = 2$ when $x = 0$, find v in terms of x.

9 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\ddot{x} = x - 3$ and $\dot{x} = -2$ when $x = 1$, find $\dot{x}$ when $x = 0$.

10 The velocity of a particle is given by $v = 4 + x^2\ \text{m s}^{-1}$.

(a) Find the acceleration as a function of x.

(b) If initially $x = -2\,\text{m}$, what is the displacement after $\frac{\pi}{4}$ seconds?

11 If $\frac{dx}{dt}=\frac{1}{x+4}$ and $x=0$ when $t=0$, find t when $x=2$.

12 If $\frac{dx}{dt}=x+4$ and $x=-3$ when $t=0$, express x in terms of t.

13 The velocity of a particle at any time t is $2(4-x)^2$. Find its position $x(t)$ at any time t given $x(0)=0$.

14 If $\frac{dx}{dt}=(3-x)^2$ and $x=2$ when $t=0$, find: **(a)** x as a function of t **(b)** $\frac{d^2x}{dt^2}$ as a function of x.

15 A particle moves in a straight line. At time t its displacement from a fixed origin is x. If $\dot{x}=x+3$:

(a) express $\ddot{x}$ in terms of x **(b)** find x when $t=1$, given that $x=-2$ when $t=0$.

16 The acceleration of a body moving under gravitational attraction towards a planet varies inversely as the square of its distance from the centre of the planet. This can be written as $\frac{d^2x}{dt^2}=-\frac{k}{x^2}$ where x is the distance from the centre of the planet and k is a constant. If the body starts from rest at a distance a from the centre of the planet, show that its speed at x (before it hits the planet) is given by $\frac{dx}{dt}=\sqrt{\frac{2k(a-x)}{ax}}$.

17 A particle is moving in a straight line with its acceleration as a function of x given by $\ddot{x}=-e^{-2x}$. It is initially at the origin and travelling with a velocity of 1 metre per second.

(a) Show that $\dot{x}=e^{-x}$. **(b)** Hence show that $x=\log_e(t+1)$.

18 A particle is moving so that $\ddot{x}=32x^3+48x^2+16x$. Initially $x=-2$ and the velocity v is -8.

(a) Show that $v^2=16x^2(1+x)^2$. **(b)** Hence, or otherwise, show that $-4t=\int\frac{1}{x(1+x)}dx$.

(c) It can be shown that for some constant C, $\log_e\left(1+\frac{1}{x}\right)=4t+C$. (Mathematics Extension 2 students should derive this result.) Using this equation and the initial conditions, find x as a function of t.

19 The acceleration of a particle P is given by the equation $\frac{d^2x}{dt^2}=32x(x^2+9)$, where x metres is the displacement of P from a fixed point O after t seconds. Initially the particle is at O and has a velocity $36\,\text{m s}^{-1}$ in the positive direction.

(a) Show that the speed at any position x is given by $4(x^2+9)\,\text{m s}^{-1}$.

(b) Hence find the time taken for the particle to travel 3 metres from O.

20 A body falls from rest so that its velocity v metres per second after t seconds is $v=80(1-e^{-0.4t})$.

(a) Show that the acceleration is proportional to $(80-v)$.

(b) Calculate the distance fallen in the first five seconds.

(c) Calculate the distance fallen when $v=60$.

21 A particle is brought to top speed by an acceleration that varies linearly with the distance travelled, i.e. $\ddot{x}=kx+C$ where k and C constants. It starts from rest with an acceleration of $3\,\text{m s}^{-2}$ and reaches top speed in a distance of 160 metres. Find:

(a) the top speed **(b)** the speed when the particle has moved 80 metres.

13.4 SIMPLE HARMONIC MOTION (SHM)

We will now consider the motion of a particle (a point) moving along a straight line so that its displacement from a fixed point at time t is given by a sine or cosine function.

Example 15

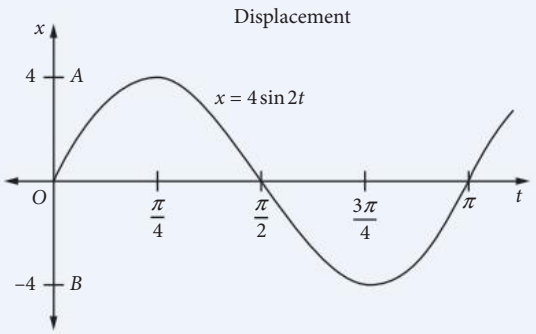
A particle moves in a straight line so that its displacement x m from a fixed point O at time t seconds is defined by $x = 4\sin 2t$. After considering properties of the graph of $x = 4\sin 2t$ to analyse the motion of the particle, find expressions for:

(a) the velocity **(b)** the acceleration. **(c)** Discuss the motion of the particle.

Solution

Look at the properties of the sine curve to help determine the nature of the movement of the particle.

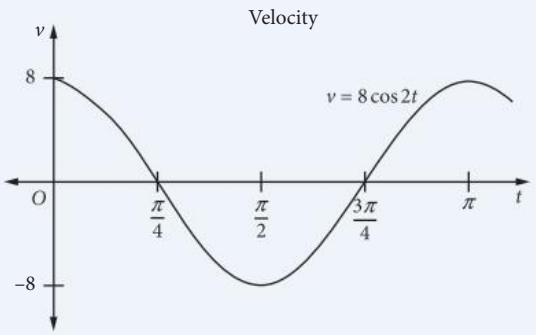
- When $t = 0$, $x = 0$
- When $t = \frac{\pi}{4}$, $x = 4$
- When $t = \frac{3\pi}{4}$, $x = -4$
- When $t = \pi$, $x = 0$



Remember that the particle is moving in a straight line along the x-axis. It starts at $x = 0$ and takes $\frac{\pi}{4}$ seconds (approximately 0.8 s) to move to A, another $\frac{\pi}{4}$ seconds to return to O, another $\frac{\pi}{4}$ seconds to move to B and then another $\frac{\pi}{4}$ seconds to return to O again. This pattern of movement repeats every π seconds. We say that the particle **oscillates** from A to B about the point O, the centre of the motion, with a **period** of π. The distance between O and the extreme positions A and B (in this case 4 m) is called the **amplitude**.

(a) $v = \frac{dx}{dt} = 8\cos 2t$ [1]

- When $t = 0$, $v = 8$ at $x = 0$
- When $t = \frac{\pi}{4}$, $v = 0$ at $x = 4$
- When $t = \frac{\pi}{2}$, $v = -8$ at $x = 0$
- When $t = \frac{3\pi}{4}$, $v = 0$ at $x = -4$
- When $t = \pi$, $v = 8$ at $x = 0$



In the original displacement diagram above, the particle is at rest at A and B because $v = 0$ when $t = \frac{\pi}{4}$ at $x = 4$ and when $t = \frac{3\pi}{4}$ at $x = -4$.

The velocity diagram above right shows that $v = 0$ when $t = \frac{\pi}{4}$ and when $t = \frac{3\pi}{4}$.

When $t = 0$, $x = 0$ and $v = 8$. When $t = \frac{\pi}{2}$, $x = 0$ and $v = -8$. This means that at O, $v = 8$ when the particle is travelling in the positive direction (towards A) but $v = -8$ when the particle is travelling in the negative direction (towards B).

Squaring [1]: $v^2 = 64\cos^2 2t$

$$v^2 = 64(1 - \sin^2 2t)$$

$$v^2 = 64\left(1 - \frac{x^2}{16}\right)$$

$$v^2 = 4(16 - x^2) \qquad [2]$$

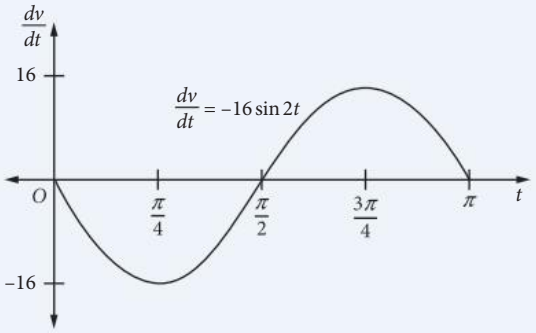
Equation [2] gives the velocity in terms of x.

(b) $\ddot{x} = \dfrac{dv}{dt} = -16\sin 2t$ [3]

$\ddot{x} = -4x$ [4]

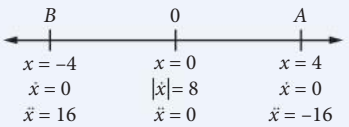
Equation [3] gives the acceleration in terms of t. Equation [4] gives the acceleration in terms of x. $\ddot{x} = -4x$ tells us that when $x = 0$, $\ddot{x} = 0$; when $x = 4$ (at A), $\ddot{x} = -16$; and when $x = -4$ (at B), $\ddot{x} = 16$. $\ddot{x}$ and x always have the opposite sign.

- When $t = 0$, $\ddot{x} = 0$ at $x = 0$
- When $t = \dfrac{\pi}{4}$, $\ddot{x} = -16$ at $x = 4$
- When $t = \dfrac{\pi}{2}$, $\ddot{x} = 0$ at $x = 0$
- When $t = \dfrac{3\pi}{4}$, $\ddot{x} = 16$ at $x = -4$
- When $t = \pi$, $\ddot{x} = 0$ at $x = 0$



(c) In this type of motion, the particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and the magnitude of this acceleration is proportional to its distance from the fixed point.

Taking O as the fixed point, the above description tells us that when the particle is at A $(x > 0)$ its acceleration is towards O $(\ddot{x} < 0)$, while when the particle is at B $(x < 0)$ its acceleration is again towards O $(\ddot{x} > 0)$.



Note that the equation of motion $x = 4\sin 2t$ in this example could also be written as $x = 4\cos\left(2t - \dfrac{\pi}{2}\right)$.

In a situation such as Example **15**, $\ddot{x}$ and x are always opposite in sign and the magnitude of the acceleration is always proportional to the distance from O. This means we can take the differential equation $\ddot{x} = -4x$ from this example and write it as a general equation $\ddot{x} = -kx$ to define all motion of this type.

This type of motion is called **simple harmonic motion** (SHM) and it can be applied to many physical situations. Because k is a positive constant, it is usually replaced by n^2 so that we have $\ddot{x} = -n^2x$ as the basic equation of SHM.

In general, for simple harmonic motion we have:

- displacement x: $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$
- velocity $\dot{x}$: $\dot{x} = -an\sin(nt + \alpha)$
- acceleration $\ddot{x}$: $\ddot{x} = -an^2\cos(nt + \alpha)$

  or $\ddot{x} = -n^2x$

Important results

1 When $\dot{x} = 0$ (e.g. at A and B in Example **15**) the acceleration is greatest.

2 When $\ddot{x} = 0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. the velocity has its greatest or least value).

The general equation $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$

If when $t = 0$, $x = a$, the particle is initially at A (the extreme point) and $a = a\cos\alpha$, then $\cos\alpha = 1$ and $\alpha = 0$. The equation of motion can then be written $x = a\cos nt$.

$x = a\cos nt$ [1]

$v = \dot{x} = -na\sin nt$ [2]

$v^2 = n^2a^2\sin^2 nt$

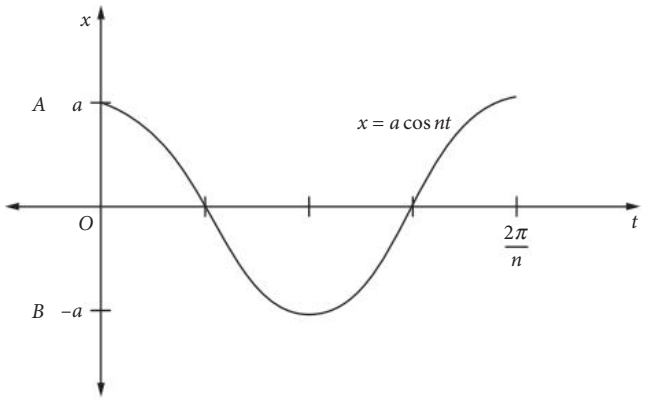
$= n^2a^2(1 - \cos^2 nt)$

$= n^2a^2\left(1 - \frac{x^2}{a^2}\right)$

$v^2 = n^2(a^2 - x^2)$ [3]

$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$

$\ddot{x} = -n^2x$ [4]



- The period T is the time for one complete oscillation: $T = \frac{2\pi}{n}$
- The frequency f is the number of oscillations per unit time: $f = \frac{1}{T} = \frac{n}{2\pi}$
- The amplitude a is the distance from the centre of motion O to either of the extreme points A or B.

Simple harmonic motion problems are usually solved using either $x = a\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(a^2 - x^2)$. The starting point will depend on the information given.

Simple harmonic motion (SHM)—summary

$x = a\cos(nt + \alpha)$
$= a\cos nt$ if $x(0) = a$
$v = \dot{x} = -na\sin nt$

$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$
$\ddot{x} = -n^2x$

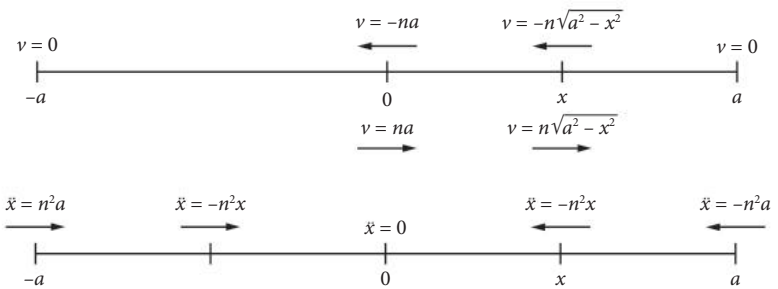
$v^2 = n^2(a^2 - x^2)$ $-a \le x \le a$

$T = \frac{2\pi}{n} = \frac{1}{f}$

Note:

- $x = a\sin(nt + \alpha)$ can also describe the displacement function, but it is more conventional to use $x = a\cos(nt + \alpha)$.
- In problems involving a pendulum, the motion usually starts at the maximum displacement, so $\alpha = 0$ and the equation of motion becomes $x = a\cos nt$.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.



Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob (approximately SHM)
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

Example 16

The displacement x m of a particle moving in a straight line is given by $x = 4\cos 6t$. Discuss the motion of the particle.

Solution

$x = 4\cos 6t$ is of the form $x = a\cos(nt + \alpha)$, so the motion is simple harmonic about the origin.

Amplitude $a = 4$ $\quad n = 6$ $\quad$ Period $= \dfrac{2\pi}{n} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$

When $t = 0$: $x = 4\cos 0 = 4$

$\therefore$ The particle starts 4 cm to the right of O.

Velocity: $\dot{x} = -24\sin 6t$

When $t = 0$: $\dot{x} = -24\sin 0 = 0$

$\therefore$ The particle is initially at rest.

Acceleration: $\ddot{x} = -144\cos 6t$

$\ddot{x} = -36x$

When $t = 0$: $\ddot{x} = -144$

$\therefore$ The initial acceleration is 144 m s^{-2} towards O.

The motion is simple harmonic with amplitude 4 m, period $\frac{\pi}{3}$ seconds, initially at rest 4 cm to the right of O with an acceleration of 144 m s^{-2} towards O.

Example 17

The motion of a particle moving along a straight line is given by the equation $\dfrac{d^2x}{dt^2} = -16x$.

If $x = 0$ and $v = 4$ when $t = 0$, find its displacement at any time t and state the period and amplitude.

Solution

Because $\dfrac{d^2x}{dt^2} = -16x = -n^2x$ where $n = 4$, the motion is simple harmonic.

$\therefore \quad x = a\cos(nt + \alpha)$

For $n = 4$: $\quad x = a\cos(4t + \alpha)$ $\quad [1]$

$v = \dot{x} = -4a\sin(4t + \alpha)$ $\quad [2]$

When $t = 0$, $x = 0$ in [1]: $\quad 0 = a\cos\alpha$

$$\alpha = \frac{\pi}{2}$$

When $t = 0$, $v = 4$ in [2]: $\quad 4 = -4a\sin\dfrac{\pi}{2}$

$$a = -1$$

$$\therefore \quad x = -\cos\left(4t + \frac{\pi}{2}\right)$$

We can use the identity $-\cos\theta = \cos(\pi - \theta)$ to remove the negative sign and write the answer in a more familiar form:

$$-\cos\left(4t + \frac{\pi}{2}\right) = \cos\left(\pi - 4t - \frac{\pi}{2}\right)$$

$$= \cos\left(\frac{\pi}{2} - 4t\right)$$

$$= \cos\left(4t - \frac{\pi}{2}\right)$$

$$\therefore \quad x = \cos\left(4t - \frac{\pi}{2}\right) \quad \text{Period} = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2} \quad \text{Amplitude} = a = 1$$

It is worth noticing that $\cos\left(\frac{\pi}{2}-4t\right)=\sin 4t$, so the equation of motion could be written as $x=\sin 4t$. This would make it easier to sketch the function.

Example 18

A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x}=-4x$. Find its period, amplitude and displacement at time t given that at $t=0$, $x=3$ and $v=-6\sqrt{3}$.

Solution

$\ddot{x}=-4x=-n^2x$ where $n=2$, so the motion is simple harmonic.

$$x=a\cos(nt+\alpha)$$

For $n=2$: $x=a\cos(2t+\alpha)$

When $t=0$, $x=3$: $3=a\cos\alpha$ [1]

Velocity: $\dot{x}=-2a\sin(2t+\alpha)$

When $t=0$, $v=-6\sqrt{3}$: $-6\sqrt{3}=-2a\sin\alpha$

$3\sqrt{3}=a\sin\alpha$ [2]

[2] ÷ [1]: $\frac{a\sin\alpha}{a\cos\alpha}=\frac{3\sqrt{3}}{3}$

$\tan\alpha=\sqrt{3}$

As $a>0$: $\sin\alpha>0$ (from [2]) and $\cos\alpha>0$ (from [1]), so α is in the first quadrant and is an acute angle.

Hence: $\alpha=\frac{\pi}{3}$

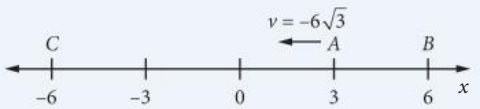
From [1]: $3=a\cos\frac{\pi}{3}$

$a=6$

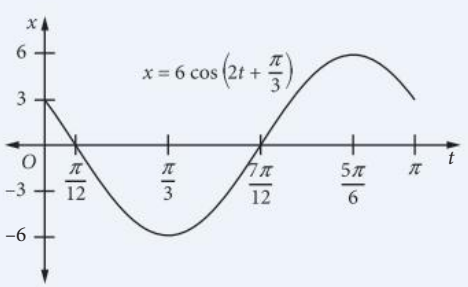
$\therefore\quad x=6\cos\left(2t+\frac{\pi}{3}\right)$

Hence the period $=\frac{2\pi}{2}=\pi$, amplitude $=6$ and displacement is given by $x=6\cos\left(2t+\frac{\pi}{3}\right)$.

This diagram shows the motion of the particle at $t=0$.
It starts at A and moves towards the centre of the motion.
C and B are the extreme points of the motion.



This diagram shows the displacement x for any time t in the domain $0\le t\le\pi$.



Example 19

The speed v m s^{-1} of a particle moving in a straight line is given by $v^2 = 6 + 4x - 2x^2$, where the magnitude of its displacement from a fixed point O is x m. Show that the motion is simple harmonic and find:

(a) the centre of the motion **(b)** the period **(c)** the amplitude.

Solution

$v^2 = f(x)$, so it seems likely that using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ might help.

$$v^2 = 6 + 4x - 2x^2$$

$$\frac{1}{2}v^2 = 3 + 2x - x^2$$

Differentiate with respect to x: $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2 - 2x$

Hence: $\ddot{x} = -2(x-1)$

With $y = x - 1$: $\ddot{y} = -2y$ (as $\ddot{y} = \ddot{x}$)

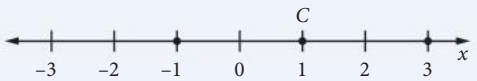
This is simple harmonic motion about $y = 0$.

(a) With $y = x - 1$, SHM about $y = 0$ is the same as SHM about $x = 1$.

(b) $n^2 = 2$, so $n = \sqrt{2}$ and the period $= \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$.

(c) The extreme positions of the particle are found where $v = 0$.

$$6 + 4x - 2x^2 = 0$$
$$-2(x^2 - 2x - 3) = 0$$
$$(x+1)(x-3) = 0$$
$$x = -1 \text{ or } 3$$



The particle oscillates between $x = -1$ and $x = 3$ about the centre $x = 1$, so the amplitude is 2 m.

Example 20

A particle moves in a straight line so that its position at any time t is given by $x = 3\cos 2t + 4\sin 2t$.

(a) Show that $3\cos 2t + 4\sin 2t = 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$.

(b) Show that the motion is simple harmonic and find its greatest speed in metres per second.

Solution

(a) Expression $= 3\cos 2t + 4\sin 2t$

$\sqrt{3^2 + 4^2} = 5$: $= 5\left(\frac{3}{5}\cos 2t + \frac{4}{5}\sin 2t\right)$

$\cos\alpha = \frac{3}{5}$, $\sin\alpha = \frac{4}{5}$: $= 5(\cos 2t \cos\alpha + \sin 2t \sin\alpha)$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{4}{3}$: $= 5\cos(2t - \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $\tan\alpha = \frac{4}{3}$

(b) $x = 5\cos(2t - \alpha)$

Velocity: $\dot{x} = -10\sin(2t - \alpha)$

Acceleration: $\ddot{x} = -20\cos(2t - \alpha)$

$= -4x$

$\ddot{x} = -n^2 x$ with $n = 2$, so the motion is simple harmonic.

The greatest speed occurs when $\ddot{x} = 0$, i.e. when $\cos(2t - \alpha) = 0$.

Hence: $2t - \alpha = \frac{\pi}{2}$

Thus: $\dot{x} = -10\sin\frac{\pi}{2}$

$= -10$ The greatest speed is $10\,\text{m s}^{-1}$.

Example 21

A particle moving in a straight line with SHM has a speed of 15 m/s when passing through its mean position (the centre of the motion). Find the amplitude of the motion and the acceleration in the extreme positions, given that the period of the motion is 2 seconds.

Solution

$x = 0 \qquad |v| = 15 \qquad T = 2 = \frac{2\pi}{n}$ hence $n = \pi$.

We know the value of x, v and n. Use the result $v^2 = n^2(a^2 - x^2)$ to find a.

$$225 = \pi^2(a^2 - 0)$$

$$a^2 = \frac{225}{\pi^2}$$

$$a = \frac{15}{\pi} \quad \text{as } a > 0$$

Motion is SHM, so: $\ddot{x} = -n^2 x$

Hence: $\ddot{x} = -\pi^2 x$

At the extreme position, $x = a = \frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \frac{15}{\pi} = -15\pi$

At the other extreme position, $x = -\frac{15}{\pi}$: $\ddot{x} = -\pi^2 \times \left(-\frac{15}{\pi}\right) = 15\pi$

Thus the amplitude of the motion is $\frac{15}{\pi}$ metres and the magnitude of the acceleration is $15\pi\,\text{m s}^{-2}$.

Example 22

A particle is moving in a straight line with SHM. The velocity of the particle is respectively $\sqrt{5}\,\text{m s}^{-1}$ and $2\,\text{m s}^{-1}$ at distances of 1 m and 2 m from the centre of motion. Find:

(a) the length of the path **(b)** the period of the motion.

Solution

(a) Given $x = 1$, $v = \sqrt{5}$; $x = 2$, $v = 2$.

Hence use: $v^2 = n^2(a^2 - x^2)$

At $x = 1$, $v = \sqrt{5}$: $5 = n^2(a^2 - 1)$ [1]

At $x = 2$, $v = 2$: $4 = n^2(a^2 - 4)$ [2]

[1] ÷ [2]: $\dfrac{5}{4} = \dfrac{n^2(a^2-1)}{n^2(a^2-4)}$

$$5a^2 - 20 = 4a^2 - 4$$
$$a^2 = 16$$
$$a = 4 \quad \text{as } a > 0$$

The amplitude of the motion is 4 m, so the length of the path ($2a$) is 8 m.

(b) Substitute $a = 4$ into [1]: $5 = 15n^2$

$$\therefore \quad n = \frac{1}{\sqrt{3}} \quad \text{as } n > 0$$

Thus the period of the motion is $T = \dfrac{2\pi}{n} = 2\pi\sqrt{3}$ seconds.

Example 23

Assume that the tides of the ocean rise and fall in SHM. The depth of water y metres in a harbour channel is given by $y = 10 - 2\cos\dfrac{\pi t}{6}$ where t is the number of hours after low tide. On a particular day, low tide occurs at 7 a.m. when the channel is 8 metres deep. On the same day high tide occurs at 1 p.m. when the channel is 12 metres deep.

(a) Show that these values satisfy the equation $y = 10 - 2\cos\dfrac{\pi t}{6}$.

(b) What is the depth of the channel at 10 a.m.?

(c) State the amplitude and period of the motion.

(d) Draw the graph of $y = 10 - 2\cos\dfrac{\pi t}{6}$ for $0 \le t \le 12$.

(e) A ship needs needs at least 9 metres of water in the channel to be able to pass through safely. Use your graph to find when the ship can safely pass through the channel.

(f) Find the answer to part **(e)** algebraically.

Solution

(a) At 7 a.m., $t = 0$:

$y = 10 - 2\cos(0) = 10 - 2 = 8$ m

The depth at 7 a.m. is 8 metres.

At 1 p.m., $t = 6$:

$y = 10 - 2\cos(\pi) = 10 - 2 \times (-1) = 10 + 2 = 12$ m

The depth at 1 p.m. is 12 metres.

(b) 10 a.m., $t = 3$:

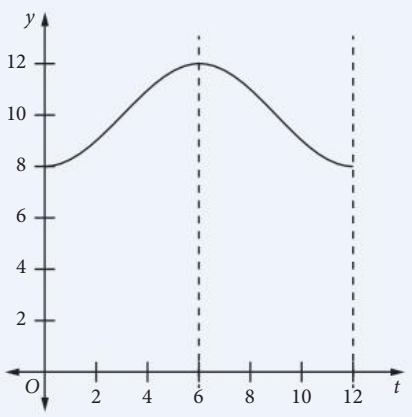
$$y = 10 - 2\cos\left(\frac{\pi}{2}\right) = 10 - 0 = 10 \text{ m}$$

The depth at 10 a.m. is 10 metres.

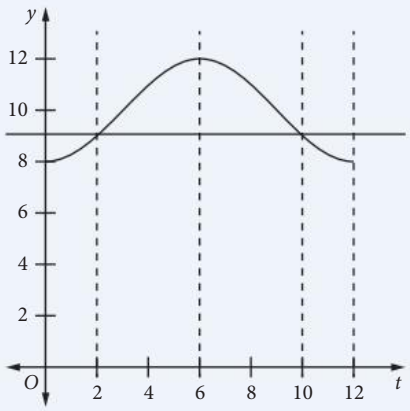
(This is half-way between low tide and high tide.)

(c) Amplitude = 2 m; period = 6 hours

(d)



(e) Draw the line $y = 9$ on the graph. When the curve is above the line it is safe to pass through the channel.



It is safe to pass through the channel from 2 hours after low tide (i.e. after 9 a.m.) until 10 hours after low tide (i.e. before 5 p.m.).

(f) To find the answer algebraically, solve $10 - 2\cos\left(\frac{\pi t}{6}\right) \geq 9$ as an equation and interpret the result:

$$2\cos\left(\frac{\pi t}{6}\right) = 1$$

$$\cos\left(\frac{\pi t}{6}\right) = \frac{1}{2}$$

$$\frac{\pi t}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\therefore \quad t = 2, 10$$

EXERCISE 13.4 SIMPLE HARMONIC MOTION (SHM)

1 The displacement x m of a particle moving in a straight line is given by $x = 6\cos 4t$. Discuss the motion of the particle.

2 The equation of motion of a particle moving with simple harmonic motion is $\ddot{x} = -9x$. Find its period, amplitude and greatest speed if: **(a)** $x = 0, \dot{x} = 2$ when $t = 0$ **(b)** $x = 2, \dot{x} = 2$ when $t = 0$.

3 A particle is moving in a straight line. If x metres is its displacement at time t seconds and $\left(\frac{dx}{dt}\right)^2 = 5\left(4 - x^2\right)$, find the acceleration in terms of x only. Show that the motion is simple harmonic and find its period and amplitude.

4 The velocity v metres per second of a particle moving in simple harmonic motion along the x-axis is given by the equation $v^2 = 100 - 4x^2$. The amplitude in metres of the motion of the particle is:

A 2 **B** 5 **C** 10 **D** 50

5 A particle moves in a straight line. At time t seconds, its displacement x cm from a fixed point O in the line is given by $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$. Express the acceleration in terms of x only and hence show that the motion is simple harmonic. Find:

(a) the period **(b)** the amplitude **(c)** the speed when $x = -2.5$ **(d)** the acceleration when $x = -2.5$.

6 The speed of a point moving along the x-axis is v m s^{-1} and the displacement of the point from the origin is x m. Prove in each case below that the motion is simple harmonic, finding **(i)** the centre of motion and **(ii)** the period and amplitude, when v is given by:

(a) $v^2 = 300 + 100x - 25x^2$ **(b)** $v^2 = 128 - 32x - 16x^2$ **(c)** $v^2 = 6 + 4x - 2x^2$

7 A particle is moving in SHM along the x-axis. Its velocity v at x is given by $v^2 = 30 - 4x - 2x^2$.

(a) Find all values of x for which the particle is at rest.
(b) Find an expression for the acceleration of the particle in terms of x.
(c) Find the maximum speed of the particle.

8 The displacement x metres at time t seconds of a point moving in a straight line is given by $x = a\cos(nt + \varepsilon)$. Find the form that this expression takes if initially:

(a) $\dot{x} = 0$ and $x = -5$ **(b)** $x = 0$ and the velocity is negative.

9 The velocity of a particle moving in a straight line is $3\sqrt{16 - x^2}$ m s^{-1} where x m is the displacement of the particle from a fixed point in the line. Is the motion simple harmonic? Find the acceleration when $x = 4$. What is the maximum speed?

10 Solve the differential equation $\frac{d^2x}{dt^2}+16x=0$ subject to the conditions $x=3$ and $\frac{dx}{dt}=16$ when $t=0$. Find the maximum displacement and the maximum speed if x metres is the displacement of the particle moving in a straight line at time t seconds.

11 A particle moves with simple harmonic motion. If the particle starts from the equilibrium position (centre of motion) with velocity $4\,\text{m s}^{-1}$ towards the origin and the period is $\frac{\pi}{2}$ seconds, find:

(a) the displacement at time t **(b)** the amplitude.

12 A particle moves with simple harmonic motion. When it is 2 m from its equilibrium position, its velocity is $6\,\text{m s}^{-1}$; when it is 3 m from equilibrium, its velocity is $4\,\text{m s}^{-1}$. Find the period of this motion and the amplitude.

13 If $x=a\sin nt+b\cos nt$, find the velocity and acceleration of a particle whose displacement from a fixed point O is x at time t. Show that $\ddot{x}=-n^2x$ and find the amplitude and maximum speed.

14 The position x m of a particle relative to a fixed point O at any time t seconds is $x=5-2\cos^2 t$. By finding the acceleration in terms of x, show that the motion is simple harmonic. Find:

(a) the centre of motion **(b)** the period **(c)** the amplitude.

15 A particle executes simple harmonic motion of period 8 seconds and amplitude 10 m. Calculate the velocity and acceleration when its displacement is 6 m from the centre of motion. Find also its maximum acceleration.

16 A particle moves in a straight line so that its position x metres from a fixed point O at time t seconds is given by $x=10+8\sin 2t+6\cos 2t$. Prove that the motion is simple harmonic. Find the period and amplitude.

17 A particle moves in a straight line. At time t seconds its distance from a point O on the line is x metres. The following is an incomplete table of observations:

t	0	7	9	11	18
x	0			0.5	0

Complete the table using two different assumptions, namely:

(a) that the particle moves with uniform acceleration

(b) that the particle performs simple harmonic motion with a period of 12 seconds.

18 A floating buoy oscillates up and down with the waves, rising and falling 2 metres about its mean position. Find its greatest velocity and acceleration if the period of the motion is 3 seconds.

19 A particle moving with SHM starts from rest at $x=5$ and after two seconds reaches $x=2.5$. Find:

(a) an expression for the displacement at time $t\geq 0$ **(b)** the speed at $x=0$

(c) the amplitude, frequency and period **(d)** the maximum speed **(e)** the maximum acceleration.

20 The amplitude of a particle moving with SHM is 5 m and the acceleration when 2 m from the mean position is $4\,\text{m s}^{-2}$. Find the speed of the particle when at the mean position and when 4 m from the mean position.

21 A particle is moving with SHM of period π seconds and a maximum velocity of 8 metres per second. Find the amplitude and the velocity at a distance of 3 metres from the central position.

22 A particle executes SHM of period $\frac{2\pi}{\sqrt{5}}$ seconds. If it starts from rest with displacement 10 metres, find:

(a) the frequency **(b)** the amplitude.

23 A point moving with SHM has a speed of 5 m/s when passing through its mean position O. Find the speed and acceleration when it is 1.5 m from O, given that the period is π seconds.

24 A particle is moving with SHM of amplitude 10 metres. Find how much time it takes to travel 6 metres from its mean position if the period is 10 seconds.

25 A point moves with SHM in such a way that its speed is 8 and 6 m/s respectively at distances 3 and 4 m from the mean position. Calculate the period of the motion and the magnitude of the greatest acceleration.

26 Assume that the tides rise and fall in SHM. At low tide the channel in a harbour is 9 m deep and at high tide it is 12 m deep. Low tide is at 9 a.m. and high tide is at 4 p.m.

(a) What is the depth of the channel at 12:30 p.m.? **(b)** What is the amplitude of the motion?

(c) What is the period of the motion?

(d) Show that the depth y m of the water in the channel is given by $y = 10.5 - 1.5\cos\frac{\pi t}{7}$ where t is the number of hours after low tide.

(e) A ship needs at least 10 m of water to pass through the channel safely. Between what times can the ship safely navigate the channel?

27 Assume that over several days of constant weather, the cycle of changing temperature each day is simple harmonic between 13°C at 4 a.m. and 23°C at 4 p.m.

(a) What is the temperature at 10 a.m.?

(b) Show that the pattern of temperature can be represented by $T = 18 - 5\cos\frac{\pi t}{12}$ where T is the temperature in °C and t is the time in hours after 4 a.m.

(c) At what time of day would the temperature be: **(i)** 18°C **(ii)** 15°C **(iii)** 21°C?

28 A particle is moving in simple harmonic motion in a straight line. Its maximum speed is $3\,\text{m s}^{-1}$ and its maximum acceleration is $8\,\text{m s}^{-2}$. Find the amplitude and period of the motion.

13.5 PROJECTILE MOTION

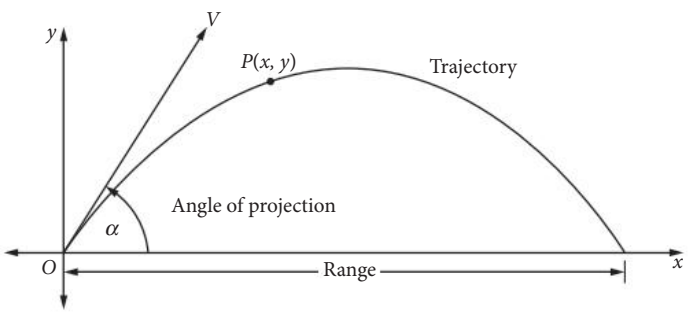
In this section we will investigate the motion of a particle that is projected in any direction and subject only to gravitational acceleration (i.e. any effect of air resistance is ignored).

The motion is considered entirely in the vertical plane containing the direction of the initial velocity. The **angle of projection** (or angle of elevation) is the angle at which the particle is projected, measured from the horizontal.

The **range** on the horizontal plane is the distance between the point of projection and the point where the particle meets the horizontal plane through the point of projection (see diagram below). The **trajectory** is the path of the particle.

Note: Under gravity, any particle fired at an angle α to the horizontal (where $0 < \alpha < \pi$) will travel on a parabolic path (trajectory). This will be proved later in this section.

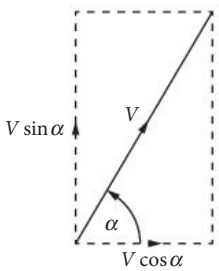
To obtain the equations of motion for this kind of projectile motion, we first consider typical projectile motion as illustrated in the diagram below.



The point of projection is chosen as the origin. The initial velocity of projection is V.

As the angle of projection is α, from the diagram we find that V has two components:

$V\cos\alpha$ (horizontal)
$V\sin\alpha$ (vertical)



The moving particle is affected only by the acceleration due to gravity, which has a magnitude $g\,\text{m s}^{-2}$ directed vertically down.

Thus $\ddot{x} = 0$ and $\ddot{y} = -g$ define the motion in the horizontal and vertical directions respectively.

The initial conditions of the motion are: $t = 0$, $x = 0$, $y = 0$, $\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha$

Horizontal projectile motion

As $\ddot{x} = 0$, we have $\dot{x} = C_1$ and initially $\dot{x} = V\cos\alpha$, so we have:

$\dot{x} = V\cos\alpha$ for $0 \le t \le T$, where T is the time of flight. [1]

Integrate with respect to t: $x = \int V\cos\alpha\, dt = Vt\cos\alpha + C_2$

When $t = 0$, $x = 0$: $C_2 = 0$

$\therefore\ x = Vt\cos\alpha$ for $0 \le t \le T$ [2]

Vertical projectile motion

$\ddot{y} = -g$

Integrate with respect to t: $\dot{y} = \int -g\, dt = -gt + C_3$

When $t = 0$, $\dot{y} = V\sin\alpha$: $C_3 = V\sin\alpha$

$\therefore\ \dot{y} = V\sin\alpha - gt$ for $0 \le t \le T$ [3]

Integrate with respect to t: $y = \int (V\sin\alpha - gt)\, dt = Vt\sin\alpha - \frac{1}{2}gt^2 + C_4$

When $t = 0$, $y = 0$: $C_4 = 0$

$\therefore\ y = Vt\sin\alpha - \frac{1}{2}gt^2$ for $0 \le t \le T$ [4]

Equation [3] produces the formula $v = u + at$ and equation [4] produces $s = ut + \frac{1}{2}at^2$, which you may know from Physics.

Equations [2] and [4] are the **parametric** equations (see also Chapter 6 page 123) that define any point on the trajectory in terms of the time t after projection (for $0 \le t \le T$ where T is the time of flight).

Equations of motion for a projectile

These equations can be memorised, but you must also be able to derive them and evaluate the constant of integration.

Horizontal	Vertical
$\ddot{x} = 0$	$\ddot{y} = -g$
$\dot{x} = V\cos\alpha$	$\dot{y} = V\sin\alpha - gt$
$x = Vt\cos\alpha$	$y = Vt\sin\alpha - \frac{1}{2}gt^2$

Equation of the path

The **Cartesian equation** (i.e. an equation in terms of x and y, not in terms of t) of the projectile's trajectory is obtained by eliminating t from equations [2] and [4]:

$y = Vt\sin\alpha - \frac{1}{2}gt^2$

Substitute $t = \frac{x}{V\cos\alpha}$: $y = V\sin\alpha \times \frac{x}{V\cos\alpha} - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$

Hence: $y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$ is the equation of the trajectory. [5]

The equation of the trajectory is particularly useful when finding the angle of projection. It is a quadratic function in x (because g, V and α are all constants). Thus the motion is parabolic, described by a concave-down parabola.

Note:
The parametric equations of the trajectory can be obtained using the equations for uniformly accelerated motion. If $\dot{x}$ and $\dot{y}$ denote the velocity components in the x and y directions respectively:

$\dot{x} = V\cos\alpha$ (constant velocity) $\qquad$ $\dot{y} = V\sin\alpha - gt$ (using $v = u + at$)

$x = Vt\cos\alpha$ $\qquad$ $y = Vt\sin\alpha - \frac{1}{2}gt^2$ (using $s = ut + \frac{1}{2}at^2$)

You must take care if using these results.

Maximum height of a projectile

The particle reaches its highest point when $\dot{y} = 0$. From [3], this is when $t = \dfrac{V\sin\alpha}{g}$.

Substitute into [4]: $y_{\max} = \dfrac{V^2\sin^2\alpha}{g} - \dfrac{1}{2}\dfrac{V^2\sin^2\alpha}{g} = \dfrac{V^2\sin^2\alpha}{2g}$

Alternatively: starting with the equation of the trajectory [5], the highest point is reached when $\dfrac{dy}{dx} = 0$.

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \qquad [5]$$

Differentiate with respect to x: $\dfrac{dy}{dx} = \tan\alpha - \dfrac{gx}{V^2\cos^2\alpha}$

At $\dfrac{dy}{dx} = 0$: $\tan\alpha - \dfrac{gx}{V^2\cos^2\alpha} = 0$

$$\frac{gx}{V^2\cos^2\alpha} = \frac{\sin\alpha}{\cos\alpha}$$

$$x = \frac{V^2\sin\alpha\cos\alpha}{g}$$

Substitute into [5]: $y = \dfrac{V^2\sin\alpha\cos\alpha}{g} \times \dfrac{\sin\alpha}{\cos\alpha} - \dfrac{g}{2V^2\cos^2\alpha} \times \left(\dfrac{V^2\sin\alpha\cos\alpha}{g}\right)^2$

$$= \frac{V^2\sin^2\alpha}{g} - \frac{g}{2V^2\cos^2\alpha} \times \frac{V^4\sin^2\alpha\cos^2\alpha}{g^2}$$

$$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g} = \frac{V^2\sin^2\alpha}{2g}$$

In this case, using the parametric forms of the equations is easiest.

The maximum height of the projectile is: $y_{\max} = \dfrac{V^2\sin^2\alpha}{2g}$

Range on a horizontal plane through the point of projection

When the projectile strikes the horizontal plane, $y = 0$. Thus from [4]:

$$Vt\sin\alpha - \frac{1}{2}gt^2 = 0$$

$$t\left(V\sin\alpha - \frac{1}{2}gt\right) = 0$$

$$t = 0 \quad \text{or} \quad \frac{2V\sin\alpha}{g}$$

As $t = 0$ is when the particle was first projected, the required time is $t = \dfrac{2V\sin\alpha}{g}$.
Thus the time of flight T is given by $T = \dfrac{2V\sin\alpha}{g}$.

Substitute $t = \dfrac{2V\sin\alpha}{g}$ into [2]: $x = V\cos\alpha \times \dfrac{2V\sin\alpha}{g}$

$$= \frac{2V^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{V^2\sin 2\alpha}{g}$$

i.e. $R = \dfrac{V^2\sin 2\alpha}{g}$ where R denotes the range.

Alternatively: if only the range is required, without needing to know the time of flight, then the trajectory equation [5] can be used.

Substitute $y = 0$ into [5]: $0 = x\left(\tan\alpha - \dfrac{gx}{2V^2\cos^2\alpha}\right)$

$$x = 0 \quad \text{or} \quad x = \frac{2V^2\cos^2\alpha\tan\alpha}{g} = \frac{2V^2\cos\alpha\sin\alpha}{g} = \frac{V^2\sin 2\alpha}{g}$$

$x = 0$ is the initial position and $x = \dfrac{V^2\sin 2\alpha}{g}$ is where it meets the horizontal plane again, which gives the range.

Maximum range

For a given speed of projection V, the equation $R = \dfrac{V^2\sin 2\alpha}{g}$ defines the range as a function of α. This function of α has its greatest value when $\sin 2\alpha = 1$, i.e. when $\alpha = 45°$. Thus the greatest range of the particle is achieved when it is projected at an angle of 45° to the horizontal.

The greatest range of a projectile is: $R_{\max} = \dfrac{V^2}{g}$ when $\alpha = 45°$

It is important for you to be able to derive projectile formulae when necessary, rather than to memorise the formulae.

Example 24

A particle is projected from a horizontal plane at an angle of 30° with a speed of 100 m/s.

(a) Given that $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$, show that the equation of the trajectory is $y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{1500}$ if $g = 10\,\text{m s}^{-2}$.

(b) Find the range of the projectile and the time of flight.

(c) Find the maximum height reached by the projectile.

Solution

When $t = 0$: $x = 0$, $y = 0$, $V = 100$, $\alpha = 30°$, $g = 10$

(a)

	$\dot{x} = V\cos\alpha$	$\dot{y} = V\sin\alpha - gt$
$V = 100$, $\alpha = 30°$, $g = 10$:	$\dot{x} = 100\cos 30°$	$\dot{y} = 100\sin 30° - 10t$
	$= 100\dfrac{\sqrt{3}}{2} = 50\sqrt{3}$	$= 50 - 10t$
Integrate with respect to t:	$x = 50\sqrt{3}t + C_1$	$y = 50t - 5t^2 + C_2$
$t = 0$, $x = 0$, $y = 0$:	$C_1 = 0$	$C_2 = 0$
$\therefore$	$x = 50\sqrt{3}t$	$y = 50t - 5t^2$
Find t in terms of x:	$t = \dfrac{x}{50\sqrt{3}}$	

Substitute into y:

$$y = 50\times\frac{x}{50\sqrt{3}} - 5\times\left(\frac{x}{50\sqrt{3}}\right)^2$$

$$= \frac{x}{\sqrt{3}} - \frac{x^2}{1500}$$

The equation of the trajectory is $y = \dfrac{x}{\sqrt{3}} - \dfrac{x^2}{1500}$.

(b) For range, $y = 0$: $50t - 5t^2 = 0$

$5t(10 - t) = 0$

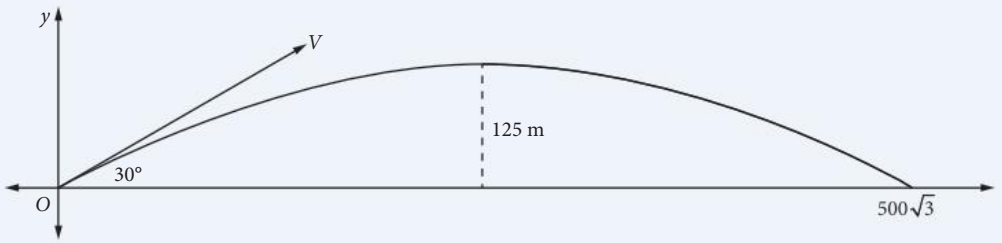
$t = 0$ or 10 Time of flight is 10 seconds.

When $t = 10$: $x = 500\sqrt{3} \approx 866\,\text{m}$ The range is $500\sqrt{3}$ metres.

(c) By symmetry, the maximum height occurs when $t = 5$.

When $t = 5$: $y_{max} = 50 \times 5 - 5 \times 25 = 125$ The maximum height is 125 metres.

The diagram below shows the graph of the projectile's path from the information found.



Example 25

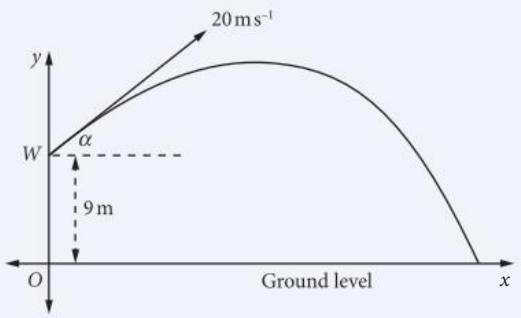
A particle is projected from a window 9 metres above horizontal ground at an angle α to the horizontal, where $\tan\alpha = \frac{3}{4}$, with an initial velocity of 20 m/s. Ignoring air resistance, using $x = Vt\cos\alpha$ and $y = 9 + Vt\sin\alpha - \frac{1}{2}gt^2$, find:

(a) the Cartesian equation of the path, using $g = 10\,\text{m s}^{-2}$

(b) the maximum height above the ground

(c) the time taken for the particle to strike the ground

(d) the horizontal distance from the window to the point where the particle strikes the ground.

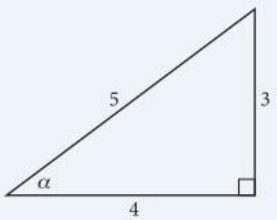


Solution

(a) $\tan\alpha = \frac{3}{4}$, so $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$:

$\therefore \quad V\cos\alpha = 20 \times \frac{4}{5} = 16$

and $V\sin\alpha = 20 \times \frac{3}{5} = 12$



Horizontally: $x = Vt\cos\alpha$

$x = 16t$

Hence: $t = \frac{x}{16}$

Vertically: $y = 9 + Vt\sin\alpha - \frac{1}{2}gt^2$

$y = 9 + 12t - 5t^2$

Substitute for t: $y = 9 + 12 \times \frac{x}{16} - 5\left(\frac{x}{16}\right)^2$

$y = 9 + \frac{3x}{4} - \frac{5x^2}{256}$

The Cartesian equation of the path is $y = 9 + \frac{3x}{4} - \frac{5x^2}{256}$.

(b) **Method 1**, using $y = f(x)$

At maximum height, $\frac{dy}{dx} = 0$.

$$y = 9 + \frac{3x}{4} - \frac{5x^2}{256}$$

Differentiate y: $\frac{dy}{dx} = \frac{3}{4} - \frac{5x}{128}$

Solve: $0 = \frac{3}{4} - \frac{5x}{128}$

$$x = \frac{96}{5}$$

Substitute to find y: $y = 9 + \frac{3}{4} \times \frac{96}{5} - \frac{5}{256} \times \left(\frac{96}{5}\right)^2$

$$= 9 + \frac{72}{5} - \frac{36}{5} = 16.2$$

Method 2, using $y = g(t)$

At maximum height, $\dot{y} = 0$.

$$y = 9 + 12t - 5t^2$$

$$\dot{y} = 12 - 10t$$

$$0 = 12 - 10t$$

$$t = \frac{6}{5}$$

$$y = 9 + 12 \times \frac{6}{5} - 5 \times \left(\frac{6}{5}\right)^2$$

$$= 9 + \frac{72}{5} - \frac{36}{5} = 16.2$$

The maximum height above the ground is 16.2 metres.

(c) When the particle strikes the ground, $y = 0$.
To find t, use $y = 9 + 12t - 5t^2$:

$$0 = 9 + 12t - 5t^2$$
$$(3 + 5t)(3 - t) = 0$$
$$\therefore\ t = 3 \quad (\text{as } t \geq 0)$$

It takes 3 seconds for the particle to reach the ground.

(d) When $t = 3$: $x = 16 \times 3 = 48$
The particle strikes the ground 48 metres from the base of the building.

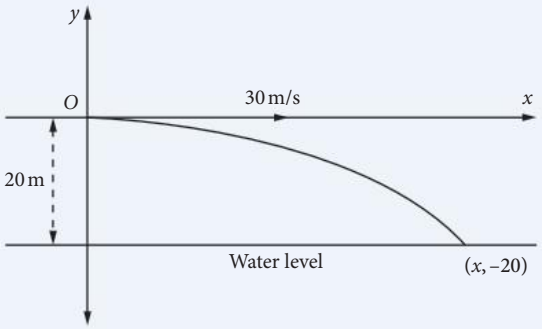
Note that the choice of the origin O is arbitrary. It is usually chosen so as to make the calculations as easy as possible. Try repeating the previous example with the origin at the point of projection, i.e. at the window, so that the ground is at $y = -9$. All answers should be the same, although the greatest height will not be simply the value of y.

Example 26

A body is projected horizontally from the top of a cliff 20 m above water with a velocity of 30 m s^{-1}. Find:

(a) the time taken to reach the water

(b) the horizontal distance travelled.



Solution

Take the origin O to be the point of projection at the top of the cliff.

When $t = 0$: $x = 0$, $y = 0$, $\dot{x} = 30$, $\dot{y} = 0$
(because the particle is projected horizontally)

Horizontally: $\ddot{x} = 0$

$t = 0$, $\dot{x} = 30$: $\dot{x} = C_1 = 30$ [1]

$$x = 30t + C_2$$

$t = 0$, $x = 0$: $C_2 = 0$

$$\therefore\ x = 30t \quad [2]$$

Vertically: $\ddot{y} = -g$

$\dot{y} = -gt + C_3$

$t = 0,\ \dot{y} = 0$: $C_3 = 0$

$\dot{y} = -gt$ [3]

$y = -\frac{1}{2}gt^2 + C_4$

$t = 0,\ y = 0$: $C_4 = 0$

$\therefore\ y = -\frac{1}{2}gt^2$ [4]

(a) At water level: $y = -20$

$\therefore$ from [4]: $-20 = -\frac{1}{2}gt^2$

$g = 10$: $t^2 = 4$

$t = 2$ (as $t > 0$)

It takes 2 seconds to reach the water.

(b) From [2]: $x = 30t$

When $t = 2$: $x = 60$

The horizontal distance travelled is 60 metres.

Note:
If there is no vertical component of projection then the velocity of projection does not affect the time taken to reach the water. If you drop an object and at the same time throw another object horizontally, you will notice that both objects reach the ground at the same time. In both cases the vertical motion is governed by $y = -\frac{1}{2}gt^2$, independent of any horizontal motion, just as in the previous example.

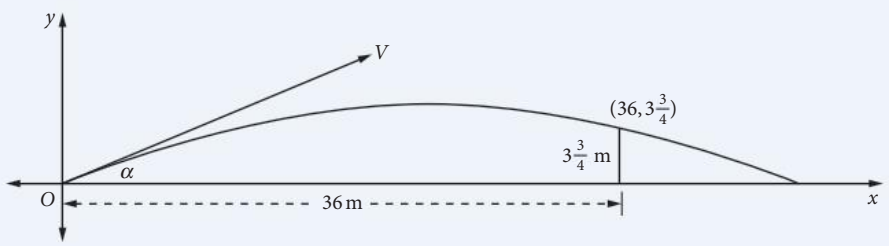
Example 27

A particle is projected from ground level and 1.5 seconds later it just clears a wall 3.75 metres high at a horizontal distance of 36 metres. If the initial velocity was V m s^{-1} and the angle of projection from the origin was α, the equation of the trajectory is given by $x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$. Calculate:

(a) the velocity and angle of projection **(b)** the range.

Solution

$x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$; when $t = 0$: $x = 0$, $y = 0$; when $t = 1.5$: $x = 36$, $y = 3.75$



(a) $x = Vt\cos\alpha$

$t = 1.5,\ x = 36$: $36 = 1.5V\cos\alpha$

$V\cos\alpha = 24$ [1]

$y = Vt\sin\alpha - 5t^2$

$t = 1.5,\ y = 3.75$: $3.75 = 1.5V\sin\alpha - 5 \times 2.25$

$15 = 6V\sin\alpha - 45$

$6V\sin\alpha = 60$

$V\sin\alpha = 10$ [2]

[2] ÷ [1]: $\frac{V\sin\alpha}{V\cos\alpha} = \frac{10}{24}$

$\tan\alpha = \frac{5}{12}$

$\alpha = 22° 37'$

As $\tan\alpha = \frac{5}{12}$: $\sin\alpha = \frac{5}{13}$, $\cos\alpha = \frac{12}{13}$ from the right-angled triangle

From [1]: $V\cos\alpha = 24$

$$V \times \frac{12}{13} = 24$$

$$V = 26$$

The velocity of projection is $26\,\text{m s}^{-1}$ at an angle of $22° 37'$.

(b) $y = Vt\sin\alpha - 5t^2$

At $y = 0$: $0 = 26t \times \frac{5}{13} - 5t^2$

$$5t^2 - 10t = 0$$

$$5t(t-2) = 0$$

$$t = 0 \text{ or } 2$$

$x = Vt\cos\alpha$

When $t = 2$: $x = 26 \times 2 \times \frac{12}{13} = 48$ The range is 48 metres.

Alternatively: use the result $R = \frac{V^2\sin 2\alpha}{g}$, which does not require you to first find t.

You have to realise that $g = 10$ was used in the given equations of the trajectory:

$$R = \frac{V^2\sin 2\alpha}{g} = \frac{V^2 \times 2\sin\alpha\cos\alpha}{10}$$

$$= \frac{26 \times 26 \times 2 \times 5 \times 12}{10 \times 13 \times 13}$$

Example 28

A stone is thrown to hit a small object on top of a wall that is 20 metres horizontally from the point of projection and 5 metres high. If the stone is thrown from ground level with a speed of $40\,\text{m s}^{-1}$, show that there are two angles of projection that will enable the object to be hit.

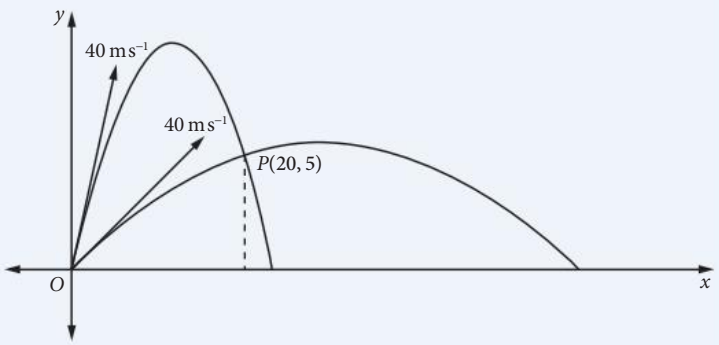
The equations of motion of the projectile are $x = Vt\cos\theta$, $y = Vt\sin\theta - 5t^2$; V is the initial velocity and θ is the angle of projection.

Solution

$x = Vt\cos\theta$, $y = Vt\sin\theta - 5t^2$

When $t = 0$: $x = 0$, $y = 0$, $V = 40$

At $x = 20$, $y = 5$



Horizontally: $x = Vt\cos\theta$

At $x = 20$, $V = 40$: $20 = 40t\cos\theta$

$$t = \frac{1}{2\cos\theta} \qquad [1]$$

Vertically: $y = Vt\sin\theta - 5t^2$

At $y = 5$, $V = 40$: $5 = 40t\sin\theta - 5t^2$ [2]

Substitute [1] into [2]: $5 = \dfrac{40\sin\theta}{2\cos\theta} - \dfrac{5}{4\cos^2\theta}$

$20 = 80\tan\theta - 5\sec^2\theta$

$\sec^2\theta = 1 + \tan^2\theta$: $20 = 80\tan\theta - 5 - 5\tan^2\theta$ [3]

$5\tan^2\theta - 80\tan\theta + 25 = 0$

$\tan^2\theta - 16\tan\theta + 5 = 0$

Quadratic formula: $\tan\theta = \dfrac{16 \pm \sqrt{256 - 20}}{2}$

$= \dfrac{16 \pm \sqrt{236}}{2}$

$= 8 \pm \sqrt{59}$

$\tan\theta \approx 15.68$ or 0.3189

$\theta \approx 86°\,21'$ or $17°\,41'$

The angles of projection that allow the object to be hit are $17°\,41'$ and $86°\,21'$.

Alternatively: obtain the Cartesian equation of the trajectory from the parametric equations, then substitute $x = 20$, $y = 5$, $V = 40$ to obtain equation [3] above.

TECHNOLOGY EXPLORATION

Projectile motion

A projectile is projected at a speed between 10 and 20 m/s at an angle between 30 and 80 degrees. Investigate the possible trajectories of the projectile.

1. In GeoGebra, right-click in the **Graphics** view and select 'Graphics… > Basic'. Change the settings to x Min = -5, x Max = 50, y Min = -5, y Max = 25.
2. Use the **Slider** tool to create three sliders as follows:
   v with Min = 10, Max = 20, Increment = 1
   a with Min = 30, Max = 80, Increment = 1
   t with Min = 0, Max = 3, Increment = 0.01
3. In GeoGebra's input bar, enter *P=(v*t*cos(a),-0.5*(9.81)*t^2v*t*sin(a))* to create a new point *P*. This represents the parametric equations of the horizontal and vertical displacements of a point undergoing projectile motion. Note that 9.81 is a more accurate value for gravitational acceleration (which we often round to 10).
4. Choose an initial velocity and an initial angle of projection by adjusting the sliders for *v* and *a*. Set the slider *t* for time to 0. Right-click on the point *P* and set 'Trace On'.
5. Move the slider for *t* and notice the trace of the point *P*. What shape does it make?
6. Set the slider for *t* to zero again and adjust the value of *a*. Notice the change in the trajectory of the point *P*. Repeat this several times for different values of *a.*
7. Repeat the previous step, but this time set a particular value for *a* and repeat for different values of *t*.

EXERCISE 13.5 PROJECTILE MOTION

1. The equation of the trajectory of a particle projected from the origin is given by $y = \dfrac{x}{2} - \dfrac{x^2}{800}$. Find:
   (a) where it hits the ground **(b)** the greatest height reached.

2 A particle is projected with a speed of 100 metres per second from a horizontal plane at an angle of 60°. Given $\dot{x} = 100\cos 60°$ and $\dot{y} = 100\sin 60° - 10t$, find:

(a) when the particle reaches its greatest height **(b)** the time of flight
(c) the equation of the trajectory **(d)** the horizontal range.

3 A particle is projected with a speed of 80 m/s from a horizontal plane at an angle of 30°.

(a) Given $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$, show that the equation of the trajectory is $y = \frac{x}{\sqrt{3}} - \frac{x^2}{960}$ if $g = 10\,\text{m s}^{-2}$.
(b) Find the range of the projectile and the time of flight.
(c) Find the maximum height reached by the projectile.

4 An object is projected horizontally from the top of a building 125 m high at a speed of $10\,\text{m s}^{-1}$. If $g = 10\,\text{m s}^{-2}$, find: **(a)** the time taken to reach the ground **(b)** the horizontal distance travelled.

5 A cricket ball is hit with a velocity of $12.5\,\text{m s}^{-1}$ at an angle with the horizontal whose tangent is $\frac{4}{3}$.

(a) Given that $\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$, where $\tan\alpha = \frac{4}{3}$, show that the equation of the trajectory is $y = \frac{4x}{3} - \frac{4x^2}{45}$ if $g = 10\,\text{m s}^{-2}$.

(b) Find the greatest height reached. **(c)** Find the time of flight and the horizontal distance travelled.

6 A particle is projected so that at any time t, its position (x, y) is given by $x = 36t$, $y = 15t - \frac{1}{2}gt^2$, where distances are in metres and time is in seconds. If θ is the angle of projection and V is the initial velocity, which of the following statements is correct?

A $V = 36\,\text{m s}^{-1}$ and $\theta = \sin^{-1}\left(\frac{5}{13}\right)$ **B** $V = 15\,\text{m s}^{-1}$ and $\theta = \cos^{-1}\left(\frac{12}{13}\right)$

C $V = 39\,\text{m s}^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{13}\right)$ **D** $V = 39\,\text{m s}^{-1}$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$

7 At what angle should a particle be projected from the origin with a velocity of $50\,\text{m s}^{-1}$ in order to pass through a point with the coordinates (200, 25), where distance is measured in metres and $g = 10\,\text{m s}^{-2}$?

8 A ball is projected so that its horizontal range is 45 metres. It passes through a point 11.25 metres vertically above and 22.5 metres horizontally from the point of projection. Find the angle of projection and the speed of projection. (Assume $g = 10\,\text{m s}^{-2}$)

9 A stone is projected from the edge of a cliff with a speed of 30 m/s to hit an object 120 m horizontally from the edge and 35 m vertically below. Find the angle of projection. (Assume $g = 10\,\text{m s}^{-2}$)

10 A ball thrown horizontally with speed V from a point H metres above the ground lands at a horizontal distance D metres from the point of release. If $g = 10\,\text{m s}^{-2}$, find:

(a) V if $D = 3$ and $H = 1.25$ **(b)** H if $V = 10$ and $D = 20$.

11 A particle is projected from a point whose coordinates are (0, 0) with a velocity of $20\,\text{m s}^{-1}$ horizontally and $100\,\text{m s}^{-1}$ vertically. Find the time when the projectile passes through the point whose distance from O is 120 m and find also its vertical height at this time. (Assume $g = 10\,\text{m s}^{-2}$)

12 A particle is projected so that at any time t its position (x, y) is given by $x = 36t$, $y = 15t - 5t^2$, where the distance components are measured in metres and the time is in seconds. Find:

(a) the velocity of projection **(b)** the angle of projection
(c) the greatest height reached **(d)** the range.

13 A particle is projected so that at any time t the equation of the trajectory is $x = 15t$, $y = 20t - 5t^2$, where the components are measured in metres. Find:

(a) the point of projection **(b)** the speed of projection
(c) the coordinates of the highest point of the path.

14 A shell is fired at 200 metres per second to strike a target 2 km away. At what angle should it be fired? (Use $g = 10\,\text{m s}^{-2}$.)

15 A plane travelling at 630 km/h drops a package from a height of 1000 m. Use $g = 10\,\text{m s}^{-2}$.

(a) How long does the package take to reach the ground?
(b) What horizontal distance does the package travel?

16 A particle is projected to just clear two walls that are 7 m tall and that are 7 m and 14 m respectively from the point of projection. It is given that $x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$.

(a) If α is the angle of projection, prove that $\tan\alpha = 1.5$.
(b) Show that if the walls are h metres high and are respectively b metres and c metres distant from the point of projection, then $\tan\alpha = \dfrac{h(b+c)}{bc}$.

17 A particle is projected with a speed of $25\,\text{m s}^{-1}$ in the direction of a point P that is 7 m vertically above and 24 m horizontally from the point of projection. (Use $g = 10\,\text{m s}^{-2}$.)

(a) Find the coordinates of the point on the trajectory directly below P.
(b) At the instant of the particle's projection, a second particle is dropped from P. Prove that the two particles will collide (i.e. will be in the same place at the same time).

18 A particle projected at an angle of 15° has a horizontal range of 80 m. (Use $g = 10\,\text{m s}^{-2}$.)

(a) What is the velocity of projection?
(b) What would the range be if the angle were increased to 45° with the same velocity of projection?

19 A stone is projected horizontally with a velocity of $15\,\text{m s}^{-1}$ from the top of a building 30 m high. At the same instant another stone is projected from the foot of the building with a velocity of $30\,\text{m s}^{-1}$ at an angle of 60° to the horizontal. The two stones collide. If the coordinates of the foot of the building are (0, 0), find the coordinates of the point of collision. (Use $g = 10\,\text{m s}^{-2}$.)

20 A golfer hits a ball from a point on a level fairway. Two seconds later, it strikes the fairway 50 m away. Find (if $g = 10\,\text{m s}^{-2}$):

(a) the velocity and angle of projection **(b)** the maximum height of the ball above the fairway.

21 A particle is projected with velocity $20\,\text{m s}^{-1}$ to hit a target at a horizontal distance 20 m from the point of projection and at a vertical height of 10 m. Find two possible angles of projection. (Use $g = 10\,\text{m s}^{-2}$.)

22 A particle is projected so that at any time t the coordinates of its position relative to a fixed point O are given by the parametric equations $x = 30t$ and $y = 40t - 5t^2$, where the components are measured in metres and time in seconds. Find:

(a) the velocity of projection **(b)** the angle of projection
(c) the Cartesian coordinates of the highest point of the path.

23 A particle projected from a point meets the horizontal plane through the point of projection after travelling a horizontal distance a. In the course of its trajectory it attains a greatest height b above the point of projection.

(a) Find the horizontal and vertical components of the velocity of projection in terms of a and b.
(b) Show that when the particle has travelled a horizontal distance x, it has reached a height of $\dfrac{4bx(a-x)}{a^2}$.

24 A football, kicked at $16\,\text{m s}^{-1}$, just passes over a crossbar 4 m high and 16 m away. Show that if α is the angle of projection, $5\tan^2\alpha - 16\tan\alpha + 9 = 0$ (using $g = 10\,\text{m s}^{-2}$.)

CHAPTER REVIEW 13

1 Rain is falling and collects in an inverted cone so that the volume collected increases at a constant rate of 4π cm^3 per hour. If the radius r of the cone is half its height h, find the rate (in cm per hour) at which the height is increasing when $h = 3$.

2 The radius of a cylinder increases at a constant rate of 0.1 cm per minute while its height remains constant at 10 cm. At what rate is the volume of the cylinder increasing when the radius is 2 cm?

3 A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 2$. Its acceleration is given by $\ddot{x} = x + 6$. Using $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, find the speed of the particle at $x = 3$.

4 A particle moves along the x-axis, starting from a position 2 metres to the right of the origin (i.e. $x = 2$ when $t = 0$), with an initial velocity of $\frac{5\sqrt{2}}{2}$ m s^{-1} and an acceleration $\ddot{x} = x^3 + x$.

(a) Show that $\dot{x} = \frac{x^2 + 1}{\sqrt{2}}$. **(b)** Hence find an expression for x in terms of t.

5 The equation of motion for a particle moving in simple harmonic motion is given by $\frac{d^2x}{dt^2} = -n^2x$ where n is a positive constant and x is the displacement of the particle at time t.

(a) Show that the square of the velocity of the particle is $v^2 = n^2(a^2 - x^2)$, where $v = \frac{dx}{dt}$ and a is the amplitude of the motion.

(b) Find the maximum speed of the particle. **(c)** Find the maximum acceleration of the particle.

(d) The particle is initially at the origin. Write a formula for x as a function of t. Hence find the first time that the particle's speed is a quarter of its maximum speed.

6 A particle moves with simple harmonic motion on the x-axis about the origin. It is initially at its extreme negative position. The amplitude of the motion is 16 and the particle returns to its initial position every 5 seconds.

(a) Write an equation for the position of the particle at time t seconds.

(b) How much time does the particle take to move from a rest position to the point half-way between the rest position and the equilibrium position?

7 A particle moves in a straight line. Its displacement x metres after t seconds is $x = \sin 2t - \sqrt{3}\cos 2t + 3$.

(a) Prove that the particle is moving in simple harmonic motion about $x = 3$ by showing that $\ddot{x} = -4(x - 3)$.

(b) What is the period of the motion?

(c) Express the velocity of the particle in the form $\dot{x} = A\cos(2t - \alpha)$, where α is in radians.

(d) Hence, or otherwise, find all times within the first π seconds when the particle is moving at 2 metres per second in either direction.

8 A cup of hot coffee at a temperature T°C loses heat in a cooler environment. It cools according to the law $\frac{dT}{dt} = -k(T - T_0)$, where t is the time elapsed in minutes, T_0 is the temperature of the environment in degrees Celsius and k is a constant.

(a) At recess, Mr Masters makes his cup of coffee with water at 100°C. The temperature in the staff room is 25°C when he places his coffee on his desk. Three minutes later his coffee is just the temperature he likes it, 75°C. Find the value of k.

(b) Before Mr Masters gets a chance to drink his coffee, he leaves the staff room to help a student with a maths problem about exponential growth and decay. What is the temperature of the coffee when he returns, five minutes later?

9 A particle is projected with a velocity whose horizontal and vertical components are 6 m s^{-1} and 4 m s^{-1} respectively. Find the horizontal range.

10 A cricketer hits a cricket ball towards a fielder who is 65 m away. The ball reaches a maximum height of 4.9 m and the horizontal component of its velocity is 28 m s^{-1}. Find the constant speed with which the fielder must run forward, starting at the instant the ball is hit, in order to catch the ball at a height of 1.3 m above the ground. (Use $g = 9.8$ m s^{-2}.)

11 Find the speed and direction of a particle, projected from a point 15 m above the horizontal ground, that at its highest point just clears the top of a wall 26.25 m high and 30 m away. (Use $g = 10$ m s^{-2}.)

CHAPTER 14
HARDER APPLICATIONS

Outcomes covered:

HE1 appreciates interrelationships between ideas drawn from different areas of mathematics

HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

14.1 GRAPHING RATIONAL ALGEBRAIC FUNCTIONS

Rational algebraic functions are functions such as $y=\frac{2x}{x^2+1}$, $y=\frac{3x^2}{x+1}$, $y=x+\frac{2}{x}$.

When graphing rational functions, we use the typical approach to curve sketching:

- identify restrictions on the domain and/or the range
- find intercepts on the coordinate axes, where possible
  (i.e. we can always find the y-intercept, if it exists, but it is sometimes very difficult or impossible to find the x-intercept)
- identify whether the function is odd or even (to use the associated symmetry properties)
- find stationary points and identify their nature
- find asymptotes and use them to guide the shape of the curve.

In the following examples, pay particular attention to the determination of the asymptotes. You have previously worked with vertical and horizontal asymptotes, but we will now encounter oblique (sloping) asymptotes and curved asymptotes.

Example 1

Sketch the curve with equation $y=\frac{1}{x^2+4}$.

Solution

- Because $x^2+4\neq 0$ for any value of x, the denominator is never zero, so y is defined for all x.
  There are no vertical asymptotes.
- The curve does not cross the x-axis and crosses the y-axis at $y=\frac{1}{4}$
- The function is clearly even (because the only term in x is x^2). The curve is symmetrical about the y-axis.
- Dividing the numerator and denominator by x^2 we have: $y=\frac{\frac{1}{x^2}}{1+\frac{4}{x^2}}$
  As $x\to\infty$, $\frac{1}{x^2}\to 0$ and $\frac{4}{x^2}\to 0$, $\therefore\ y\to 0^+$
  Similarly, as $x\to-\infty$, $y\to 0^+$
  $\therefore\ y=0$ is a horizontal asymptote.

  The graph of every function of the form $y=\frac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials with the degree of P_1 < degree of P_2, has $y=0$ as a horizontal asymptote.

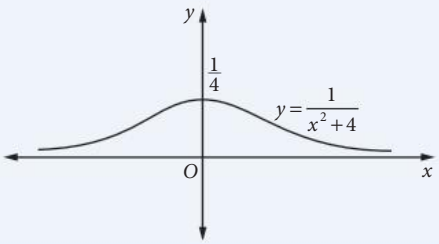
- $\frac{dy}{dx}=\frac{-2x}{(x^2+4)^2}=0$ at $x=0$

The denominator $(x^2+4)^2$ is always positive, so the sign of $\frac{dy}{dx}$ is determined by the numerator $-2x$.

$\frac{dy}{dx} < 0$ for $x > 0$ and $\frac{dy}{dx} > 0$ for $x < 0$

$\therefore$ Maximum turning point at $(0, \frac{1}{4})$

- The range is $0 < y \leq \frac{1}{4}$



Example 2

Sketch the curve with equation $y = \frac{x^2}{x^2-4}$.

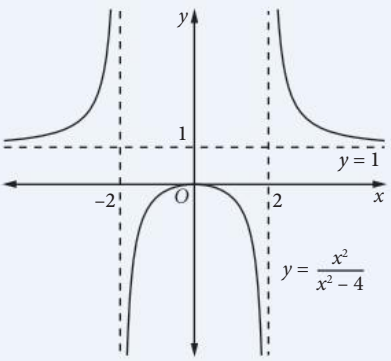
Solution

By inspection:

- The function is undefined at $x = \pm 2$: As $x \to 2^+$, $y \to \infty$ As $x \to 2^-$, $y \to -\infty$
  As $x \to -2^+$, $y \to \infty$ As $x \to -2^-$, $y \to -\infty$
  The lines $x = \pm 2$ are vertical asymptotes.
- The curve passes through the origin.
- It is an even function (because the only terms in x have even powers).
- By division: $y = 1 + \frac{4}{x^2-4}$
  As $x \to \pm\infty$, $\frac{4}{x^2-4} \to 0$ but remains positive.
  $\therefore$ $y \to 1^+$ (i.e. y approaches 1 from above, so it is always slightly more than 1)
  $\therefore$ $y = 1$ is a horizontal asymptote.

The graph of every function of the form $y = \frac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials of the same degree, will have a horizontal asymptote that is not the x-axis.
It is the line $y = k$, where $k =$ (leading coefficient of P_1) $\div$ (leading coefficient of P_2).

- $\frac{dy}{dx} = \frac{-8x}{(x^2-4)^2}$, stationary point(s) where $\frac{-8x}{(x^2-4)^2} = 0$ i.e. at $x = 0$.
  The denominator $(x^2-4)^2$ is always positive, so the sign of $\frac{dy}{dx}$ is determined by the numerator $-8x$.
  $\frac{dy}{dx} < 0$ for $x > 0$ and $\frac{dy}{dx} > 0$ for $x < 0$
  $\therefore$ Maximum turning point at $(0, 0)$



Example 3

Sketch the curve with equation $y = \frac{x^3}{x^2+1}$.

Solution

Note that the degree of the numerator is greater than the degree of the denominator.

The graph of every function of the form $y = \frac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials with degree of P_1 > degree of P_2, has an asymptote that is either oblique (i.e. an inclined straight line) or that is another curve.

- The domain is all real x $\therefore$ no vertical asymptotes.
- The graph passes through $(0, 0)$ and this is the only contact with the x-axis.
- The function is odd $\therefore$ the graph is symmetric with reflection in the origin.
- By division: $y = x - \frac{x}{x^2+1}$

  As $x \to \infty$, $\frac{x}{x^2+1} \to 0$ but remains positive.

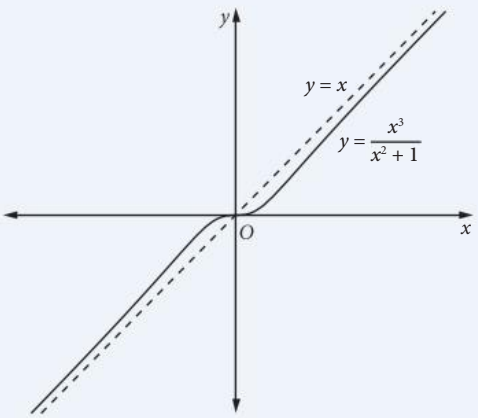
  $\therefore$ $y \to x^-$ (i.e. approaching from below, $y = x$ minus a small positive amount)

  $\therefore$ as $x \to \infty$, the graph of $y = \frac{x^3}{x^2+1}$ becomes asymptotic to the line $y = x$ but remains below this asymptote.

  As $x \to -\infty$, $\frac{x}{x^2+1} \to 0$ but remains negative.

  $\therefore$ $y \to x^+$ (i.e. approaching from above, $y = x$ minus a small negative amount)

  $\therefore$ as $x \to -\infty$, the graph of $y = \frac{x^3}{x^2+1}$ becomes asymptotic to the line $y = x$ but remains above this asymptote.



- $\frac{dy}{dx} = \frac{x^2(x^2+3)}{(x^2+1)^2}$

  We note that $\frac{dy}{dx} > 0$ for all x except at $x = 0$.

  $\therefore$ Horizontal point of inflection at $(0, 0)$

Example 4

Sketch the curve with equation $y = x^2 + \frac{1}{x^2}$.

Solution

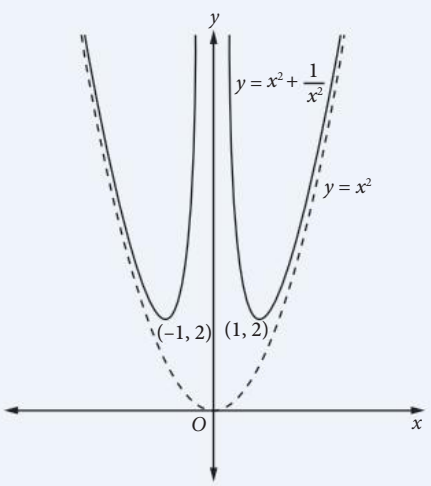
By inspection:

- vertical asymptote at $x = 0$
- no intercepts on the axes
- even function
- as $x \to \pm\infty$, $\frac{1}{x^2} \to 0$ but remains positive.

  $\therefore$ graph is asymptotic to the parabola $y = x^2$ (and remains above it)
- $\frac{dy}{dx} = 2x - \frac{2}{x^3}$, stationary points where $2x - \frac{2}{x^3} = 0$, i.e. at $x = \pm 1$

  $\frac{d^2y}{dx^2} = 2 + \frac{6}{x^4}$, so $\frac{d^2y}{dx^2} > 0$ at $x = \pm 1$

  $\therefore$ Minimum turning points at $(1, 2)$ and $(-1, 2)$

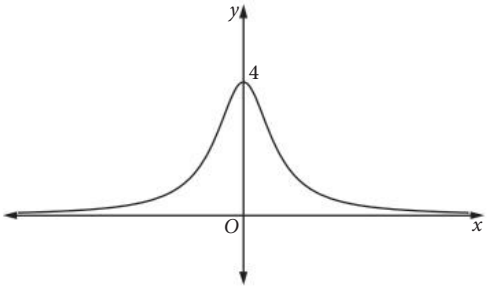


EXERCISE 14.1 GRAPHING RATIONAL ALGEBRAIC FUNCTIONS

1 The function whose graph is shown is:

A $y=\dfrac{4}{x^2-1}$ **B** $y=\dfrac{4}{x^2+1}$

C $y=\dfrac{4x}{x^2+1}$ **D** $y=\dfrac{4x^2}{x^2+1}$



For questions **2**–**15** sketch the graph of the function, showing any stationary points and asymptotes. State the range in each case.

2 $y=\dfrac{2}{x^2-1}$ **3** $y=\dfrac{2x}{x^2-1}$ **4** $y=\dfrac{2x^2}{x^2-1}$ **5** $y=\dfrac{-2}{x^2+1}$ **6** $y=\dfrac{4x^2}{x^2+1}$

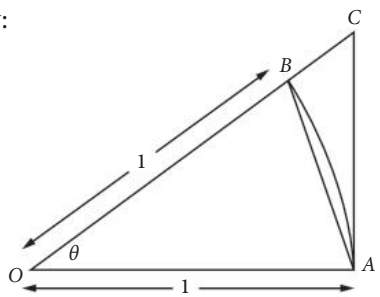
7 $y=\dfrac{x^2+1}{x}$ **8** $y=x^2+\dfrac{1}{x}$ **9** $y=\dfrac{x^2+3}{x(x-3)}$ **10** $y=\dfrac{x+1}{(x-1)^2}$ **11** $y=1+\dfrac{x}{x^2+1}$

12 $y=\dfrac{1-x^2}{1+x^2}$ **13** $y=x^2-\dfrac{1}{x}$ **14** $y=\dfrac{2x^3+1}{x}$ **15** $y=\dfrac{x+2}{x^2-3}$

14.2 THE LIMIT $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

You have seen this limit earlier in the course. It can be investigated geometrically:

Consider a sector OAB of a circle of unit radius.
The tangent at A meets OB produced at C.
Let the size of $\angle AOB$ be θ radians.



$\therefore$ Area $\Delta AOB <$ Area of sector $AOB <$ Area ΔAOC

For $0<\theta<\dfrac{\pi}{2}$: $\dfrac{1}{2}\sin\theta<\dfrac{\theta}{2}<\dfrac{1}{2}\tan\theta$

$\therefore$ $\sin\theta<\theta<\tan\theta$

Taking reciprocals: $\dfrac{1}{\sin\theta}>\dfrac{1}{\theta}>\dfrac{1}{\tan\theta}$

$\dfrac{1}{\sin\theta}>\dfrac{1}{\theta}>\dfrac{\cos\theta}{\sin\theta}$

$\therefore$ $1>\dfrac{\sin\theta}{\theta}>\cos\theta$

As $\theta\to 0$, $\cos\theta\to 1$; because the value of $\dfrac{\sin\theta}{\theta}$ is between 1 and $\cos\theta$: $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

Here θ is in radians, so this fundamental limit is only true when the variable is in radians; it does not work in degrees.

You can verify this limit using a calculator and small values for θ.

Note that this limit effectively means that when θ is small, $\sin\theta\approx\theta$ and $\cos\theta\approx 1$.

Example 5

Find: (a) $\lim_{x\to 0}\frac{\sin 4x}{2x}$ (b) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

Solution

(a) $\lim_{x\to 0}\frac{\sin 4x}{2x}=\lim_{x\to 0}\frac{\sin 4x}{4x}\times 2$

$=2\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ where $4x=\theta$

$=2\times 1$

$=2$

(b) $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\times\cos\theta\right)$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{\theta\to 0}\cos\theta$

$=1\times 1$

$=1$

EXERCISE 14.2 THE LIMIT $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$

1 Evaluate: (a) $\lim_{h\to 0}\frac{\sin 2h}{h}$ (b) $\lim_{x\to 0}\frac{\sin x}{3x}$ (c) $\lim_{x\to 0}\frac{\sin 5x}{2x}$ (d) $\lim_{\theta\to 0}\frac{\theta}{\sin\theta}$

2 For the expression $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}$, indicate whether each statement below is correct or incorrect.

(a) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\lim_{\theta\to 0}\left\{\left(\frac{\sin\theta}{\theta}\right)^2\times\theta\right\}$ (b) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\theta\times\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2$

(c) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\lim_{\theta\to 0}\theta$ (d) $\lim_{\theta\to 0}\frac{\sin^2\theta}{\theta}=0$

3 Evaluate: (a) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ (b) $\lim_{x\to 0}\frac{\sin(\pi-x)}{x}$ (c) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$ (d) $\lim_{h\to 0}\frac{\tan 2h}{3h}$

(e) $\lim_{x\to 0}\frac{\sin 2x}{\sin x}$ (f) $\lim_{x\to 0}\frac{\sin 3x}{2x}$ (g) $\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{x}$ (h) $\lim_{x\to 0}\frac{5x}{\sin 2x}$

4 (a) Express $\sin x-\sqrt{3}\cos x$ in the form $A\sin(x-\alpha)$ where $A>0$ and $0<\alpha<\frac{\pi}{2}$.

(b) Hence find $\lim_{x\to\frac{\pi}{3}}\left(\frac{\sin x-\sqrt{3}\cos x}{x-\frac{\pi}{3}}\right)$

14.3 HARDER APPLICATIONS OF MATHEMATICS COURSE TOPICS

In the following exercise you will need to apply your understanding of the Mathematics course to more complex and difficult problems. In many cases you will need to apply your skills and knowledge to new situations.

EXERCISE 14.3 HARDER APPLICATIONS OF MATHEMATICS COURSE TOPICS

1 Use differentiation from first principles, i.e. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, to find the derivative of each of the following:

(a) $y=\frac{1}{x}$ (b) $y=\sqrt{x}$ (Hint: Rationalise the numerator.)

2 (a) Factorise $a-b$ as a difference of two cubes, i.e. $a-b=\left(a^{\frac{1}{3}}\right)^3-\left(b^{\frac{1}{3}}\right)^3$

(b) Hence differentiate $y=\sqrt[3]{x}$ from first principles.

3 (a) Write the value of $\sec^2 x-\tan^2 x$ and state the values of x for which it is true.

(b) Sketch the graph of $y=\sec^2 x-\tan^2 x$ for $-\pi\le x\le 2\pi$.

4 **(a)** Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

(b) Hence find: **(i)** $\sum_{k=10}^{25} k^2$ **(ii)** $\sum_{k=1}^{30}\left(\frac{1}{2}k^2+7k-5\right)$

5 **(a)** Find the exact value of a if $5^x = e^{ax}$. **(b)** Find the derivative of 5^x.

(c) Hence find the exact value of $\int_0^2 5^x\,dx$.

(d) Use the trapezoidal rule with five function values to approximate $\int_0^2 5^x\,dx$ correct to one decimal place.

(e) Use the trapezoidal rule with 21 function values to approximate $\int_0^2 5^x\,dx$ correct to one decimal place. (Hint: Apply your knowledge of series to simplify the calculation.)

6 **(a)** Use Simpson's rule with five function values to approximate $\int_{-2}^{6}(3x-x^2)dx$ correct to one decimal place.

(b) Use Simpson's rule with five function values to find the area enclosed by the curve $y = 3x - x^2$, the x-axis and the ordinates at $x = -2$ and $x = 6$.

(c) Use Simpson's rule with five function values to find the volume formed when the area enclosed by the curve $y = 3x - x^2$, the x-axis and the ordinates at $x = -2$ and $x = 6$ is rotated about the x-axis. Answer correct to one decimal place.

7 For the function $f(x) = \log_e x$:

(a) Use differentiation from first principles to show that $\lim_{h\to 0}\left(\frac{\log_e\left(1+\frac{h}{x}\right)}{h}\right) = \frac{1}{x}$

(b) Hence show that $\lim_{h\to 0}\left(\log_e(1+h)^{\frac{1}{h}}\right) = 1$

(c) Hence, by letting $n = \frac{1}{h}$, show that $\lim_{n\to\infty}\left[n\log_e\left(\frac{n+1}{n}\right)\right] = 1$

8 The series $18 + 11 + 6\frac{5}{6} + \ldots$ has its general term given by $T_n = A\times\left(\frac{1}{2}\right)^n + B\times\left(\frac{2}{3}\right)^n$.

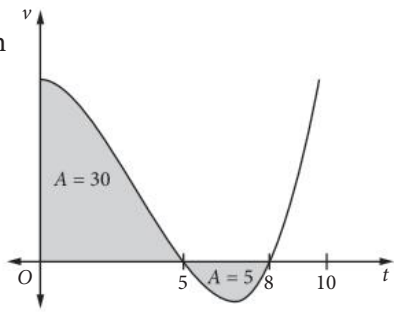
(a) Find A and B. **(b)** Find the limiting sum of the series.

9 The velocity–time graph at right shows the first 10 seconds of motion of an object moving in a straight line. The graph also shows the areas between the curve and the t-axis from $t = 0$ to $t = 5$ and from $t = 5$ to $t = 8$.

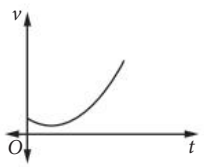
(a) Find the distance travelled by the object before it first comes to rest.

(b) Find the total distance travelled by the object before it comes to rest for the second time.

(c) If the initial displacement is 20 metres, find the displacement when the object comes to rest for the second time.



10 Each graph below shows the velocity–time relationship for an object moving in a straight line. In which case does the object change its direction only once?

A


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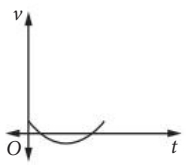
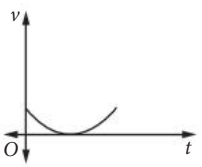
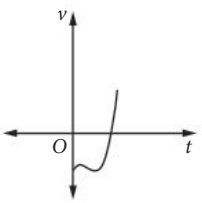


C


D


11 Show that the line $y = x - 1$ is a tangent to the curve $y = x^3 - 4x^2 + 5x - 1$.

12 For the series $1 + \left(1+\frac{1}{2}\right) + \left(1+\frac{1}{2}+\frac{1}{4}\right) + \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\right) + \ldots$

(a) Find the n-th term. **(b)** Find the sum of the first n terms.

13 For the function $y=\dfrac{1}{e^x-1}$:

(a) State the vertical asymptote. **(b)** Explain why the curve does not touch the x-axis.

(c) Find $\lim\limits_{x\to\infty}\left(\dfrac{1}{e^x-1}\right)$ and $\lim\limits_{x\to-\infty}\left(\dfrac{1}{e^x-1}\right)$. In each case state whether the limit is approached from above or below.

(d) Find $\lim\limits_{x\to0^+}\left(\dfrac{1}{e^x-1}\right)$ and $\lim\limits_{x\to0^-}\left(\dfrac{1}{e^x-1}\right)$.

(e) Find $\dfrac{dy}{dx}$ and explain why the curve has no stationary points and is decreasing for all x in its domain.

(f) Hence sketch the graph of the function.

14 **(a)** Show that $\dfrac{2e^x}{e^x-1}=2+\dfrac{2}{e^x-1}$.

(b) Use a similar approach to that of question **13** to sketch the graph of $y=\dfrac{2e^x}{e^x-1}$.

15 **(a)** Sketch $y=e^x$ and $y=\dfrac{1}{x}$ on the same axes.

(b) On the same axes, locate the x-intercept of the curve $y=e^x-\dfrac{1}{x}$.

(c) State the vertical asymptote of the curve $y=e^x-\dfrac{1}{x}$. State the equation of the curve to which $y=e^x-\dfrac{1}{x}$ is asymptotic as $x\to\pm\infty$.

(d) Find the derivative of $e^x-\dfrac{1}{x}$. Show that the curve $y=e^x-\dfrac{1}{x}$ has no stationary points and is increasing for all x in its domain.

(e) Hence sketch the graph of $y=e^x-\dfrac{1}{x}$.

CHAPTER REVIEW 14

1 Sketch the following, showing any intercepts on the coordinate axes, stationary points and asymptotes.

(a) $y=\dfrac{4x^2}{2x^2+1}$ **(b)** $y=\dfrac{4x^2}{x^2-1}$ **(c)** $y=\dfrac{2x^3+2x+1}{x^2+1}$ **(d)** $y=\dfrac{x^3+1}{x}$ **(e)** $y=\dfrac{x^3-1}{x}$

(f) $y=\dfrac{x\log_e x+1}{x}$ **(g)** $y=\dfrac{1}{e^x+1}$ **(h)** $y=\dfrac{-e^x}{e^x+1}$ **(i)** $y=e^x+\dfrac{1}{x}$

2 A function $y=f(x)$ is defined for all real x. An incomplete definition of the function is $f(x)=\begin{cases}|x-1| & \text{if } x>0\\ 0 & \text{if } x=0\\ \ldots\ldots & \text{if } x<0\end{cases}$
Complete the definition so that:

(a) $f(x)$ is even **(b)** $f(x)$ is odd.

3 Match each parabola with its number of points of intersection with the circle $x^2+y^2=4$.

(a) $y=x^2$ **(b)** $y=2(x+1)(x-1)$ **(c)** $y=3(x+1)(x-1)$ **(d)** $y=x^2+2$

A 1 **B** 2 **C** 3 **D** 4

4 **(a)** On the same axes sketch $x^2+y^2=8$ and $y=\dfrac{4}{x}$, showing the coordinates of the points of intersection.

(b) On the same diagram sketch $y=\dfrac{1}{x}$ and $y=x$, showing the coordinates of the points where they intersect.

(c) Show that every hyperbola of the form $y=\dfrac{k}{x}$ (where $k>0$) has a gradient of -1 at the points where it intersects $y=x$.

(d) Hence find the equations of the four hyperbolae that have vertical and horizontal asymptotes and that have exactly three points of intersection with the circle $x^2+y^2=8$.

5 Evaluate: $\sum\limits_{k=-3}^{10}\left(6k-15+2^k\right)$

6 Find: **(a)** $\lim\limits_{x\to0}\dfrac{\sin3x}{x}$ **(b)** $\lim\limits_{x\to0}\dfrac{\sin4x}{3x}$ **(c)** $\lim\limits_{x\to0}\dfrac{\sin\left(\frac{x}{3}\right)}{3x}$ **(d)** $\lim\limits_{x\to0}\dfrac{4x}{\sin3x}$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

SUMMARY—PRELIMINARY

1 CIRCLE GEOMETRY

Tests for congruent triangles

SSS Two triangles are congruent if the three sides of one triangle are respectively equal to the three sides of the other triangle.

SAS Two triangles are congruent if two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other triangle.

AAS Two triangles are congruent if two angles and a side of one triangle are respectively equal to two angles and the corresponding side of the other triangle.

RHS Two right-angled triangles are congruent if the hypotenuse and one side of one triangle are equal to the hypotenuse and the corresponding side of the other triangle.

Tests for similar triangles

- Two triangles are similar if the three angles of one triangle are respectively equal to the three angles of the other triangle. (**AAA**)
- Two triangles are similar if the three pairs of corresponding sides are proportional.
- Two triangles are similar if two pairs of corresponding sides are proportional and the angles included by these sides are equal.

Pythagoras' theorem

In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

Simple properties of a circle

- A **tangent** to a circle is perpendicular to the radius at the point of contact.
- If two circles touch, then they have a common tangent at the point of contact. If one circle is inside the other, the circles touch internally; otherwise, they touch externally.
- Two circles that do not touch also have common tangents. **Direct** common tangents do not pass between the circles; **transverse** common tangents do.
- A circle can be drawn to pass through any three non-collinear points.

Assumptions (for circle geometry)

1 Equal arcs on circles of equal radii subtend equal angles at the centre.

2 If two arcs of circles of equal radii subtend equal angles at their centres, then the arcs are equal.

Theorems (for circle geometry)

You do not need to reproduce proofs for theorems 1–5.

1 Equal angles at the centre of a circle stand on equal chords.

2 Equal chords subtend equal angles at the centre of a circle.

3 The angle at the centre of a circle is twice the angle at the circumference that is subtended by the same arc.

4 The tangent to a circle is perpendicular to the radius drawn to the point of contact.

5 A straight line drawn through a point on a circle, perpendicular to the radius at that point, is a tangent to the circle.

6 The perpendicular from the centre of a circle to a chord bisects the chord.

7 The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.

8 Equal chords in equal circles are equidistant from the centres.

9 Chords in a circle that are equidistant from the centre are equal.

10 Any three non-collinear points are on a unique circle whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.

11 Angles in the same segment are equal.

12 The angle in a semicircle is a right angle.

13 The opposite angles of a cyclic quadrilateral are supplementary.

14 The exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.

15 If the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic. (This is also a test for four points to be concyclic.)

16 If an interval subtends equal angles at two points on the same side of the interval, then the two points and the endpoints of the interval are concyclic. (This is also a test for a cyclic quadrilateral.)

17 The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

18 Tangents to a circle from an external point are equal.

19 The products of the intercepts of two intersecting chords are equal.

20 The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.

21 When two circles touch, the line through both centres passes through the point of contact.

Note:

- Theorem 19 also holds when the two chords intersect outside the circle. It can be proved as an application of theorem 20.
- The converse of theorem 19 gives another test for concyclic points: if two straight lines AB and CD are divided both internally or both externally at the same point P such that $PA \times PB = PC \times PD$, then the four points A, B, C and D are concyclic. (This result is not included in this course.)

2 POLYNOMIALS

- A real polynomial $P(x)$ is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, where n, $n-1, \ldots$ are all positive integers and a_n, $a_{n-1}, \ldots$ are the coefficients, which for convenience will usually be chosen as integers.
- The term $a_n x^n$ is the **leading term**. If $a_n \neq 0$, the polynomial is said to be of the n-th degree.
- $P(x)$ is defined for all real x and is a continuous and differentiable function of x.
- $P(x) = 0$ is a polynomial equation of degree n. Real numbers x that satisfy this polynomial equation are called the real **roots** of the equation or the real **zeros** of the corresponding polynomial.
- The polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ will have at most n real roots.
- Expressions are **not** polynomials if they include powers of x that are not positive integers, e.g. $x^{\frac{1}{2}} + 2 + x$, $\frac{1}{x} + 2x^{-3}$.

Division of polynomials

- If $P(x) = 3x^2 + 7x + 4$, then $P(1)$ is the value of $P(x)$ when $x = 1$, which is found by substituting $x = 1$ into the polynomial: $P(1) = 3(1)^2 + 7(1) + 4 = 14$ i.e. $P(a) = 3a^2 + 7a + 4$
- $\frac{x^2+7x+12}{x+3}$, $x \neq -3$ or $(x^2 + 7x + 12) \div (x + 3)$, $x \neq -3$ are two ways of writing a long division of polynomials.
- Dividend = Divisor × Quotient + Remainder
  Degree of the remainder < degree of the divisor.
  Degree of the quotient < degree of the dividend.
- When the divisor is a linear function (first degree), the remainder will be a constant.

The remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$ until the remainder R does not contain x, then $R = P(a)$.

For any polynomial: $P(x) = (x - a)Q(x) + R$ where $Q(x)$ is another polynomial.

Note:

- If $P(x)$ is divided by $x + a$, as $x + a = x - (-a)$, then $R = P(-a)$.
- If $P(x)$ is divided by $ax - b$, as $ax - b = a\left(x - \frac{b}{a}\right)$, then $R = P\left(\frac{b}{a}\right)$.

The factor theorem

When the remainder of a division is zero, the divisor is a factor of the expression being divided:

- For a polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.
- Conversely: if $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

By finding the zeros of a polynomial (i.e. values of a such that $P(a) = 0$), we can **factorise** the polynomial. Zeros can be found by trial and error and then by long division.

Polynomial functions

The **general polynomial function** is the function f where $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ $(a_n \neq 0)$. This function is defined for all real values of x and is continuous and differentiable.

For $a \neq 0$, we have:

- $f(x) = ax + b$ (general linear function)
- $f(x) = ax^2 + bx + c$ (general quadratic function)
- $f(x) = ax^3 + bx^2 + cx + d$ (general cubic function)

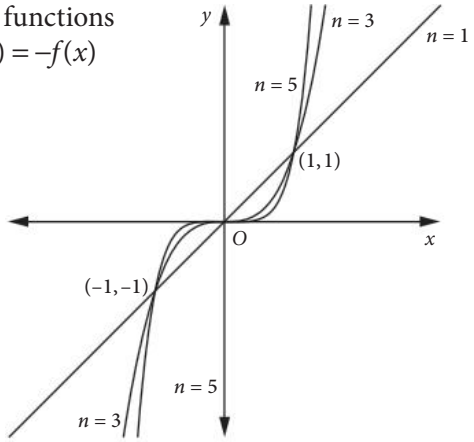
If the coefficient of the highest power of x is unity (1), the polynomial is said to be **monic**.

Graphs of polynomial functions

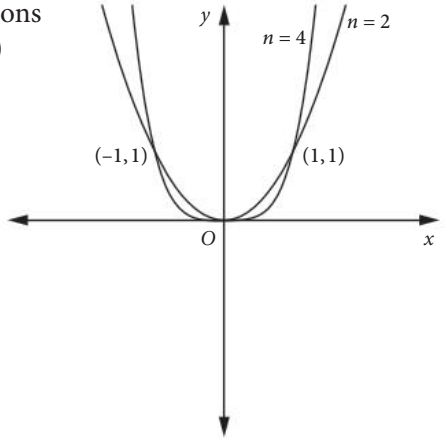
The simplest polynomial function of degree n is $f(x) = x^n$.

Graphs of these basic polynomials for $n = 1, 3, 5$ on the left, for $n = 2, 4$ on the right:

Odd functions
$f(-x) = -f(x)$



Even functions
$f(-x) = f(x)$



Important features of $f(x) = x^n$

1 The x-axis is a tangent to each graph at the origin ($n \neq 1$).

2 $f(x) = x^n$ for even values of n defines **even functions** (see above right).
For even functions, $f(-x) = f(x)$, so their graphs are symmetrical about the y-axis.

3 $f(x) = x^n$ for odd values of n defines **odd functions** (see above left).
For odd functions, $f(-x) = -f(x)$. Because $f(-x)$ and $f(x)$ are opposite in sign, the graph of f for $x \leq 0$ can be obtained by rotating the graph of f for $x \geq 0$ through 180° about the origin.

4 Recognising that a function is odd or even means that you only need to draw half of the graph in detail. The other half can then be drawn using the symmetry properties.

5 Odd functions will have an inverse function, although some may require a restriction on the domain.

6 Even functions will not have a single inverse function, but can be split into two parts (by restricting the domain) so that each part has an inverse function.
(Inverse functions are covered in Chapter 10.)

7 Note that most functions are neither even nor odd, e.g. $f(x) = x^2 + x$, $f(x) = e^x$.

Cubic functions

A general cubic function is a polynomial function f of the third degree, defined by $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are constants and $a \neq 0$.

Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$, where α is a real number.

Summary of polynomial functions

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, $a_n \neq 0$, then:

1 For very large $|x|$, $P(x) \approx a_n x^n$.

2 A polynomial of odd degree always has at least one real zero (i.e. its graph cuts the x-axis at least once).

3 At least one maximum or minimum value of P occurs between any two distinct real zeros.

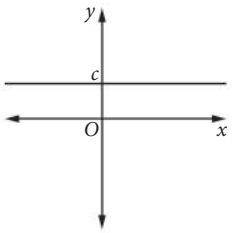
4 For a polynomial of odd degree, the ends of the graph go in opposite directions.

5 For a polynomial of even degree, the ends of the graph go in the same direction.

6 When the graph of a polynomial function meets the x-axis, it may cut it (single zero), touch it (double zero) or cut it at a point of inflexion (triple zero).

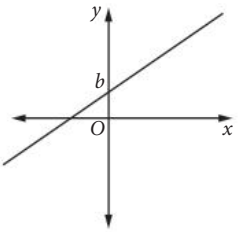
Polynomial graphs, standard forms

Degree 0 (constant)
$P(x) = c$



Does not cut
No zeros

Degree 1 (linear)
$P(x) = mx + b$

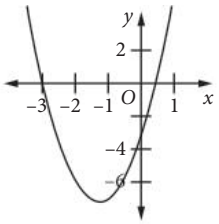


Cuts once
One zero

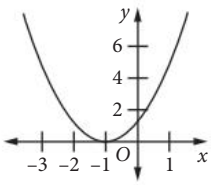
Degree 2 (quadratic)

$P(x) = ax^2 + bx + c$

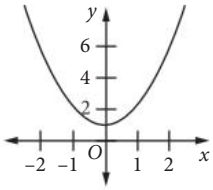
$a > 0$:



Cuts twice
$\Delta > 0$
Two zeros

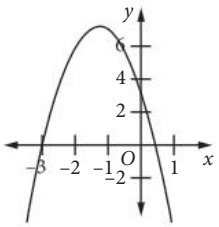


Touches
$\Delta = 0$
Double zero

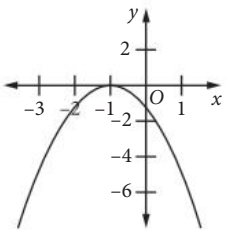


Does not cut
$\Delta < 0$
No real zeros

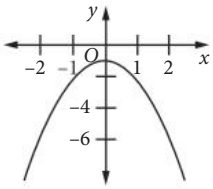
$a < 0$:



Cuts twice
$\Delta > 0$
Two zeros



Touches
$\Delta = 0$
Double zero



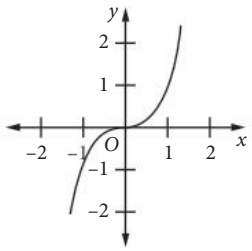
Does not cut
$\Delta < 0$
No real zeros

A quadratic polynomial may have two, one or no real zeros.

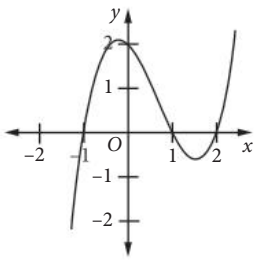
Degree 3 (cubic)

$P(x) = ax^3 + bx^2 + cx + d$

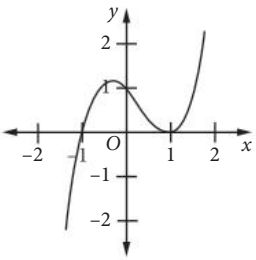
$a > 0$:



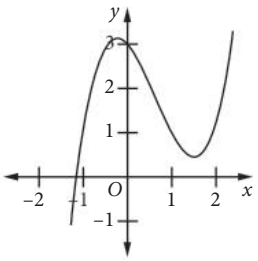
Cuts once
Triple zero



Cuts three times
Three zeros

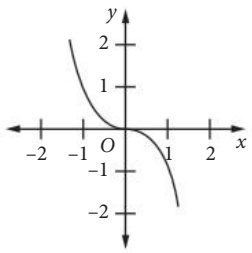


Cuts once and touches
One zero, one double zero

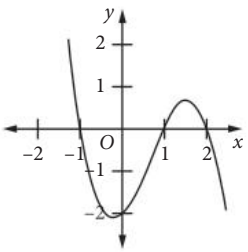


Cuts once
One zero

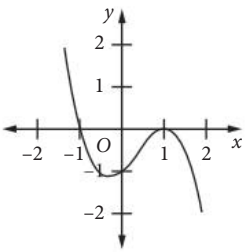
$a < 0$:



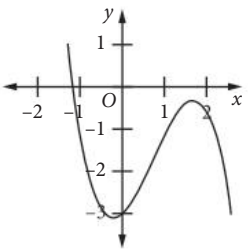
Cuts once
Triple zero



Cuts three times
Three zeros



Cuts once and touches
One zero, one double zero



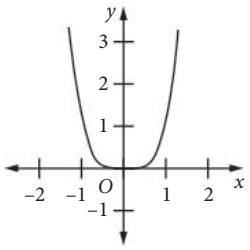
Cuts once
One zero

A cubic polynomial may have one, two or three real zeros.
A cubic polynomial always has at least one real zero.

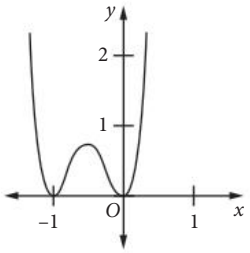
Degree 4 (quartic)

$P(x) = ax^4 + bx^3 + cx^2 + dx + e$

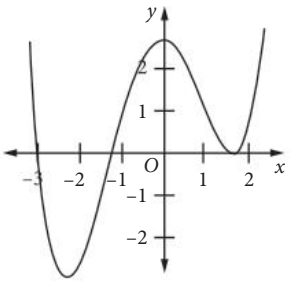
$a > 0$:



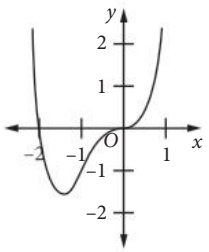
Touches
Quadruple zero



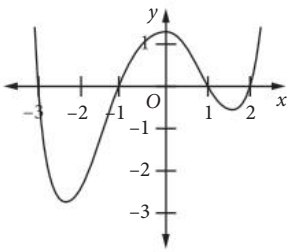
Touches twice
Two double zeros



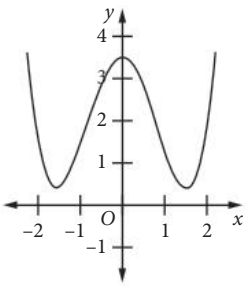
Touches once, cuts twice
Two zeros, one double zero



Cuts twice
One zero, one triple zero



Cuts four times
Four zeros



Does not cut
No real zeros

The addition of an appropriate constant to each equation can create a polynomial with no real zeros.

$a < 0$: A negative a inverts each of the six graphs above so that they open downwards, with the properties of their zeros the same.

Summary of quartic polynomials

1 A quartic polynomial may have four, three, two, one or no real zeros.

2 If a quartic polynomial has only one real zero then it must be a quadruple zero.

3 If a quartic polynomial has only two distinct real zeros, then they are either a triple zero and a single zero or they are both double zeros.

4 If a quartic polynomial has only three distinct real zeros, then they are a double zero and two single zeros.

5 If a quartic polynomial has four distinct real zeros, then it can be factorised into four real linear factors.

6 If a quartic polynomial has no real zeros, then it cannot be factorised into any real linear factors.

Relationship between roots and coefficients

Quadratic equations

The general quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.
The roots of this equation are α and β, so that $(x - \alpha)(x - \beta) = 0$.

- Sum of roots: $\alpha + \beta = -\frac{b}{a}$
- Product of roots: $\alpha\beta = \frac{c}{a}$

Identities:

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

Cubic equations

The general cubic equation is $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$.
The roots of this equation are α, β and γ so that $(x - \alpha)(x - \beta)(x - \gamma) = 0$.

- Sum of roots: $\alpha + \beta + \gamma = -\frac{b}{a}$
- Sum of products of pairs of roots: $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$
- Product of roots: $\alpha\beta\gamma = -\frac{d}{a}$

Identity:

- $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

Note: These relationships between roots and coefficients are not enough to find the roots of an equation without some additional information.

Quartic equations (4th degree)

The general quartic equation is
$ax^4 + bx^3 + cx^2 + dx + e = 0,\ a \neq 0.$

The roots of this equation are α, β, γ and δ so that
$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0.$

- Sum of roots: $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$
- Sum of products of pairs of roots:
  $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
- Sum of products of triplets of roots:
  $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
- Product of roots: $\alpha\beta\gamma\delta = \frac{e}{a}$

3 PERMUTATIONS AND COMBINATIONS

Fundamental counting principle

If an outcome can happen in m different ways, and a second outcome can happen in n different ways, then the total number of ways in which the two outcomes can happen together is $m \times n$.

Permutations

A **permutation** is an ordered selection or arrangement of all or part of a set of objects.

Number of permutations ${}^nP_r$

The symbol ${}^nP_r$ means the number of arrangements (i.e. ordered subsets, permutations) of any n different objects taken r at a time.

This denotes the number of ways of filling r places with the n different objects that are available ($r \leq n$).

- ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$

Arrangements in a circle

- The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, is $\frac{n!}{n} = (n-1)!$.
- In a circular arrangement there is no start or finish, so to count the arrangements we need to fix one object as our 'starting' object.

Mutually exclusive operations—the addition principle

Mutually exclusive arrangements cannot both be true at the same time.

For example, 'A is first and B is fifth' excludes the possibility that 'A is first and B is sixth'—these two descriptions cannot both be true at the same time.

Arrangement of n objects when some are identical

The number of ways of arranging n objects in a row when p of the objects are identical, and q of the objects are identical (but different to the others), and so on, is $\frac{n!}{p!q!}$.

Combinations

A **combination** (or 'selection') is an unordered permutation of all or part of a set of objects. The order is not important.

Number of combinations ${}^nC_r$ or $\binom{n}{r}$

The symbol ${}^nC_r$ or $\binom{n}{r}$ denotes the number of combinations of n different objects taken r at a time.

An unordered sample can be ordered in $r!$ ways:

- ${}^nC_r = \binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!}$
- An ordered sample of r objects taken from n different objects can be chosen in ${}^nP_r$ ways.
- An unordered sample of r objects taken from n different objects can be chosen in ${}^nC_r$ ways.

Important result:

- ${}^nC_{n-r} = \frac{n!}{(n-r)!r!}$
  so ${}^nC_r = {}^nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$

This means that the number of combinations of n objects taken r at a time is equal to the number of combinations of n objects taken $(n-r)$ at a time.

For each set of r objects selected, there is left behind a set containing $(n-r)$ objects. Thus there must be the same number of sets containing $(n-r)$ objects as there are containing r objects.

Combinations of any number of objects

The symbol ${}^nC_r$ also represents the number of r-subsets in a given set, where $0 \leq r \leq n$.

If no specific value of r is stated, r can assume any value between 0 and n inclusive. Hence the number of subsets that each contain at least one element, i.e. the number of non-empty subsets in an n-set, is:

${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r$

From a set of n objects, each object can be dealt with in two ways: it can be included or it can be excluded. The number of ways of dealing with the n objects is thus $2 \times 2 \times 2 \times \ldots$ to n factors, or 2^n. But this includes the case when all n objects are excluded, so the number of combinations of at least one object is $2^n - 1$.

$\therefore\ {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_r = 2^n - 1$

4 FURTHER TRIGONOMETRY

Sum and difference of two angles

- $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$
- $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$
- $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$
- $\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$
- $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$
- $\tan(\theta-\phi)=\dfrac{\tan\theta-\tan\phi}{1-\tan\theta\tan\phi}$

Double-angle formulae

- $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=1-2\sin^2\theta$
- $\sin 2\theta=2\sin\theta\cos\theta$
- $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$, $\tan\theta\neq\pm1$

Half-angle formulae

- $\cos\theta=\cos^2\dfrac{\theta}{2}-\sin^2\dfrac{\theta}{2}$
- $\sin\theta=2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$
- $\tan\theta=\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$, $\tan\dfrac{\theta}{2}\neq\pm1$

t formulae $\left(\text{where } t=\tan\dfrac{\theta}{2}\right)$

- $\tan\theta=\dfrac{2t}{1-t^2}$
- $\sin\theta=\dfrac{2t}{1+t^2}$
- $\cos\theta=\dfrac{1-t^2}{1+t^2}$

Auxiliary angle method

In each case $r=\sqrt{a^2+b^2}$ and α is in the first quadrant such that $\tan\alpha=\dfrac{b}{a}$:

- $a\sin x+b\cos x\equiv r\sin(x+\alpha)$
- $a\sin x-b\cos x\equiv r\sin(x-\alpha)$
- $a\cos x+b\sin x\equiv r\cos(x-\alpha)$
- $a\cos x-b\sin x\equiv r\cos(x+\alpha)$

Note:

(a) Writing $a\sin x+b\cos x$ in the form $r\sin(x+\alpha)$ tells you that the greatest and least values of the function are r and $-r$ respectively. This makes sketching functions like $y=a\sin x+b\cos x$ a lot easier.

(b) Writing $a\sin x+b\cos x$ in the form $r\sin(x+\alpha)$ allows you to solve equations of the type $a\sin x\pm b\cos x=c$.

(c) If you use t formulae substitution to solve equations of the type $a\cos x+b\sin x=c$, you must also test to see whether $x=\pm n\pi$ is a solution of the equation.

General solution of trigonometric equations

- For $\sin\theta=\sin\alpha$:
  $\theta=n\pi+(-1)^n\alpha$ (in radians)
  $\theta=n\times180^\circ+(-1)^n\alpha^\circ$ (in degrees) for any integer n.
- For $\cos\theta=\cos\alpha$:
  $\theta=2n\pi\pm\alpha$ (in radians)
  $\theta=n\times360^\circ\pm\alpha^\circ$ (in degrees) for any integer n.
- For $\tan\theta=\tan\alpha$:
  $\theta=n\pi+\alpha$ (in radians)
  $\theta=n\times360^\circ+\alpha^\circ$ (in degrees) for any integer n.

5 MISCELLANEOUS

Inequalities—properties

- Adding or subtracting the same number from both sides does not alter an inequality.
- Multiplying both sides of an inequality by a negative number reverses the direction of the inequality.
- Multiplication by a positive number does not alter the direction of the inequality.
- Taking the reciprocal of both sides of an inequality reverses its direction when both sides have the same sign, but not if the signs are different.
- Squaring both sides of an inequality: this produces a positive number on both sides, so the direction of the inequality $a^2\lessgtr b^2$ depends on which of $|a|$ and $|b|$ is larger.
- Square root of both sides of an inequality: this is only defined if both sides of the inequality are positive. The direction of the inequality does not change, i.e. if $a>b$ then $\sqrt{a}>\sqrt{b}$, provided a is positive and b is positive or zero.

Angle between two intersecting lines

The acute angle α between the two lines at their point of intersection is given by $\tan\alpha=\dfrac{m_1-m_2}{1+m_1m_2}$.

- If the lines are parallel, $\alpha=0$ and so $\tan\alpha=0$, $m_1=m_2$.
- If the lines are perpendicular, $\alpha=90^\circ$ and $\tan 90^\circ$ is not defined, $m_1m_2=-1$.

Division in a ratio

If A and B are two points in the plane whose coordinates are (x_1,y_1) and (x_2,y_2) respectively, then the coordinates of the point that divides AB **internally** in the ratio $m:n$ are $\left(\dfrac{mx_2+nx_1}{m+n},\dfrac{my_2+ny_1}{m+n}\right)$.

If the point P divides the interval AB **externally** in the ratio $m:n$, then the ratio can be rewritten as $m:-n$ to use the same formula.

6 PARAMETRIC EQUATIONS AND THE PARABOLA

Parametric representation

Parametric representation is when two related variables (e.g. x and y) are expressed in terms of a third variable (e.g. t or θ), so that, for example:

$x = f(t), y = g(t)$
$x = f(\theta), y = g(\theta)$

Equations like these are called **parametric equations** and the third variable (e.g. t or θ) is called the **parameter**.

- Recall that cosine and sine can be defined as x- and y-coordinates respectively of a point on the unit circle $x^2 + y^2 = 1$. The unit circle can be represented by the parametric equations $x = \cos\theta$, $y = \sin\theta$, where θ is the parameter.
- When the unit circle is described by the equation $x^2 + y^2 = 1$ (i.e. without the parameter), it is said to be in **Cartesian** form.

Parametric equation of the parabola

- The parabola $x^2 = 4ay$ can be represented by the parametric equations $x = 2at$, $y = at^2$.
- The point $(2at, at^2)$ on the parabola is frequently called 'the point t'.

Parametric equation of the tangent

- As $x = 2at$ and $y = at^2$, $\frac{dx}{dt} = 2a$ and $\frac{dy}{dt} = 2at$.
- $\frac{dy}{dx} = t$, which is the gradient of the tangent at the point t.
- The equation of the tangent is $y = tx - at^2$.

Parametric equation of the normal

- The gradient of the normal is $-\frac{1}{t}$, $(t \neq 0)$, so the equation of the normal is $x + ty = at^3 + 2at$.

Cartesian equation of the tangent

- $y = \frac{1}{4a}x^2$ so $\frac{dy}{dx} = \frac{x}{2a}$
- The Cartesian equation of the tangent is $xx_1 = 2a(y + y_1)$.

Equation of the chord of contact

Let $P(x_0, y_0)$ be a point external to the parabola $x^2 = 4ay$. From P there are two tangents that can be drawn to touch the parabola at points $Q(x_1, y_1)$ and $R(x_2, y_2)$. The chord QR is called the **chord of contact** of the tangents to $x^2 = 4ay$ from P.

The equation of the chord of contact from $P(x_0, y_0)$ is $xx_0 = 2a(y + y_0)$.

Locus problems and the parabola $x^2 = 4ay$

To find the locus of any point G:

1. Draw a diagram to show the information.
2. Use coordinate geometry to find the coordinates of G.
3. Write the coordinates as a pair of parametric equations.
4. Form the Cartesian equation of the locus of G by eliminating the parameter(s).

Sometimes there may be restrictions on the locus (the question should provide directions). If so, some thoughtful analysis will be required to identify the restrictions.

Locus problems with only one parameter

When there is only one parameter in the parametric equations, it is usually possible to:

1. Rewrite the simpler equation to make the parameter the subject.
2. Substitute this rewritten equation into the other to eliminate the parameter.

Locus problems with two parameters

When there are two parameters in the parametric equations (e.g. p and q), it is necessary to:

1. Find and use additional information from the question that will provide a link between p and q.
2. Use the parametric equations of the point plus the linking information (i.e. work with three different equations) to form the Cartesian equation.

Often you will also need to use an identity, usually $(p + q)^2 = p^2 + q^2 + 2pq$, as an additional part of the process.

Locus problems with restrictions on the locus

In some situations, restrictions are implied for the locus. For example:

- Because every tangent to a parabola is 'outside' the parabola (except at the point of contact), then the point of intersection of two tangents must be outside the parabola.
- The midpoint of every chord of a parabola must be 'inside' the parabola.

SUMMARY—HSC

7 MATHEMATICAL INDUCTION

Note: It only takes one counter-example to disprove a general statement.

Mathematical induction is a method of proof that can be used, for example, to prove that something is true for all positive integers n, or for all positive integers n greater than a specified starting value, or for all positive odd integers n.

A proof by induction consists of a two-step process followed by a conclusion:

1 Prove that the statement is true in the first case possible.

2 Prove that if the statement is true in any one case (i.e. for $n = k$), then it must also be true in the next case (i.e. for $n = k + 1$, or for $n = k + 2$ if the statement is true for even integers only, etc.).

3 Write a conclusion:

The statement is true for $n = k + 1$ if it is true for $n = k$.	(As proved in step 2.)
The statement is true for $n = 1$ (i.e. for the first possible case).	(As proved in step 1.)

Thus the statement is true for all integers $n \geq 1$.

8 ITERATIVE METHODS

- For a given function $y = f(x)$, if $f(a)$ and $f(b)$ have opposite signs then there is at least one root of the equation $f(x) = 0$ between $x = a$ and $x = b$, provided that the function is continuous in the interval $a \leq x \leq b$.
- Approximate methods of solving equations usually involve starting with an estimate of the solution and then improving on the estimate by an **iterative process**: a sequence of steps is performed repeatedly ('applied' or 'iterated') and at the end of each cycle an improved estimate is obtained. An iterative process may never give the exact solution, but it should always improve as time goes on (even if it improves only slowly).

Halving the interval

First find two values of x for which $f(x)$ has different signs. Then take the average of these two values for x to find a better approximation for the root of the equation $f(x) = 0$.

- First, obtain two estimates of the root: x_1 below and x_2 above the actual root.
  If $f(x_1)$ and $f(x_2)$ have opposite signs then there is a root of $f(x)$ in the interval $x_1 < x < x_2$.
- Next, compute the midpoint $x_3 = \frac{x_1 + x_2}{2}$ and find the value of $f(x_3)$.
  If $f(x_3) = 0$ then x_3 is the desired root. If $f(x_3) \neq 0$, go to the next step.
- Decide whether the root is in the half-interval from x_1 to x_3 or the half-interval from x_3 to x_2 by considering the sign of $f(x_3)$.
  Now return to the start, this time using the value of x_3 as the new value of one of the two estimates (i.e. so that the two new values of $f(x)$ are still different signs).

Newton's method

Suppose that x_1 is close to a root of the equation $f(x) = 0$. The equation of the tangent at $x = x_1$ is $y - y_1 = m(x - x_1)$, i.e. $y - f(x_1) = f'(x_1)(x - x_1)$.
This tangent crosses the x-axis at a point where $x = x_2$ and $y = 0$. Hence $0 - f(x_1) = f'(x_1)(x_2 - x_1)$,
i.e. $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.
Substituting x_2 for x_1 and repeatedly calculating this will usually be faster and more accurate than the 'halving the interval' method, but under certain conditions it will fail to approach the correct value.

Failure of Newton's method

1 If $f'(x_1) = 0$ then the result becomes $x_2 = x_1 - \frac{f(x_1)}{0}$, which is undefined. If the first estimate for the root is at a stationary point, then Newton's method cannot be used to find the root of the equation, because the tangent at that point is horizontal and will not cut the x-axis.

2 If $f'(b) = 0$, where b is between the estimate $x = x_1$ and the root at $x = c$, then Newton's method will not work.

Note: If $|f'(x_1)| < 1$, then $\frac{f(x)}{f'(x)}$ will increase in size and Newton's method will move further away from the initial estimate of the root of the equation. If this happens, the initial estimate may not be a good one.

9 INTEGRATION TECHNIQUES

Indefinite integrals and substitution

- $\int u^n\,du = \frac{1}{n+1}u^{n+1} + C$
- Let $y = \int f(u)\,du$ where $u = g(x)$, so that $\frac{dy}{du} = f(u)$

  But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = f(u) \times \frac{du}{dx}$

  so $y = \int f(u) \times \frac{du}{dx}\,dx = \int f(u)\,du$

  This 'backwards' form of the chain rule is convenient when the substitution of $u = g(x)$ will let us express a function being integrated as the product of $\frac{du}{dx}$ and a function of u.

Definite integrals and substitution

When using a substitution to evaluate a definite integral you must take care with the limits of integration. The original limits are for values for x, but after substitution the variable will become u (or some other new variable), so the limits similarly need to become values for u (or the other new variable). To do this, substitute the limits into the change-of-variable equation to find the limits for the new variable.

Useful results:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$: if you reverse the limits of integration, you change the sign of the integral.
- For an integral of the form $\int f'(x)\big(f(x)\big)^n\,dx$,

  the substitution $u = f(x)$ gives $\frac{du}{dx} = f'(x)$

  so that the integral becomes:

  $\int f'(x)\big(f(x)\big)^n\,dx = \int u^n\,du = \frac{1}{n+1}u^{n+1} + C$

  $= \frac{1}{n+1}\big(f(x)\big)^{n+1} + C$
- If you can recognise that

  $\int f'(x)\big(f(x)\big)^n\,dx = \frac{1}{n+1}\big(f(x)\big)^{n+1} + C$,

  you can obtain your answer faster.

Integration of $\sin^2 x$ and $\cos^2 x$

$\int \sin^2 x\,dx$ and $\int \cos^2 x\,dx$ can be found using substitutions from the formulae for $\cos 2x$. This will generally not be given in a question, but you must be able to use it when necessary.

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
- $\cos^2 x = \frac{1 + \cos 2x}{2}$,

  $\int \cos^2 x\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
- $\sin^2 x = \frac{1 - \cos 2x}{2}$,

  $\int \sin^2 x\,dx = \frac{1}{2}\int (1 - \cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

Integrals of the type $\int f'(x)(f(x))^n\,dx$

In trigonometric integrals of this type such as $\int \cos x \sin^2 x\,dx$, i.e. where $f(x) = \sin x$ and $f'(x) = \cos x$, the substitution will not always be given.

- $\int \cos x \sin^2 x\,dx = \frac{1}{3}\sin^3 x + C$

Substitution simplified

If we treat $\frac{du}{dx}$ as a fraction then $\frac{du}{dx} \times dx = du$.

If $u = \sin x$ then $\frac{du}{dx} = \cos x$ can be written as $du = \cos x\,dx$.

We are really replacing $f'(x)\,dx$ by du.

Integrals involving trigonometric substitution

If an expression of the form $a^2 - x^2$ occurs in the integrand, the standard trigonometric substitutions are $x = a\sin\theta$ or $x = a\cos\theta$.

If an expression of the form $a^2 + x^2$ occurs in the integrand, the standard substitution is $x = a\tan\theta$.

Note: If these substitutions are needed then they should be given in the question.

10 INVERSE FUNCTIONS

- If any line drawn parallel to the y-axis intersects the graph of $f(x)$ no more than once, then there are no ordered pairs with the same y-coordinate. This is the **vertical line test**.
- If any line drawn parallel to the x-axis intersects the graph of $f(x)$ no more than once, then there are no ordered pairs with the same x-coordinate. This is the **horizontal line test**.
- If the graph of $f(x)$ satisfies both the vertical line test and the horizontal line test then it is a one-to-one function. This means that for each value of x there is only one value of y, and for each value of y there is only one value of x.
- The necessary and sufficient condition for the reflection of the graph of $f(x)$ in the line $y = x$ to be the graph of a function is that $f(x)$ is one-to-one.

Increasing and decreasing functions

- A function f defined on the interval $a \le x \le b$ is said to be **strictly monotonic increasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. This means that the curve always slopes up to the right.

- A function f defined on the interval $a \le x \le b$ is said to be **strictly monotonic decreasing** when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. This means that the curve always slopes down to the right.

Strictly monotonic increasing or decreasing functions are one-to-one.

Inverse functions

If $f(g(x)) = g(f(x)) = x$ then $g(x) = f^{-1}(x)$ is the inverse function of f.

- If d is the domain and r is the range of f, then r is the domain and d is the range of f^{-1}. Thus if (a, b) is a point on f then (b, a) is a point on f^{-1}.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, when the inverse exists. This means that if $y = f(x)$ and $y = f^{-1}(x)$ intersect, then they do so on $y = x$.

Existence of inverse functions

A function f will have an inverse function f^{-1} if and only if f is a one-to-one function.

Exponential and logarithmic functions

$\log_a a^x = x$ for all real x (i.e. for all x in the domain of the exponential function), and $a^{\log_a x} = x$ for all $x > 0$ (i.e. for all x in the domain of the logarithmic function). Hence $y = \log_a x$ and $y = a^x$ are inverse functions.

The inverse sine function

By restricting the domain of $y = \sin x$ to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, we form a one-to-one function whose inverse is that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

We write $y = \sin^{-1} x$ to mean 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$'.

Note that $y = \sin^{-1} x$ **does not mean** $y = (\sin x)^{-1}$, as that would be $y = \operatorname{cosec} x$.

Features of $\sin^{-1} x$:

- increasing function
- domain is $-1 \le x \le 1$ (you can only take $\sin^{-1}$ of values from -1 to 1 inclusive)
- range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
  ($\sin^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ inclusive)
- vertical tangents at the endpoints
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function. Hence $\sin^{-1}(-x) = -\sin^{-1} x$.
- $y = \sin(\sin^{-1} x)$ exists only for values of x between -1 and 1 inclusive; $y = \sin(\sin^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \sin^{-1}(\sin x)$ exists for all real x, but it is equivalent to $y = x$ only for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.
  Its graph has a 'sawtooth' shape.

The inverse cosine function

By restricting the domain of $y = \cos x$ to $0 \le x \le \pi$, we form a one-to-one function whose inverse is that part of $x = \cos y$ for which $0 \le y \le \pi$.

We write $y = \cos^{-1} x$ to mean 'that part of $x = \cos y$ for which $0 \le y \le \pi$'.

Features of $\cos^{-1} x$:

- decreasing function
- domain is $-1 \le x \le 1$ (you can only take $\cos^{-1}$ of values from -1 to 1 inclusive)
- range is $0 \le y \le \pi$
  ($\cos^{-1}$ gives values from 0 to π inclusive)
- vertical tangents at the endpoints
- gradient of -1 at $\left(0, \frac{\pi}{2}\right)$
- the function is neither even nor odd, but it does have rotational symmetry about its y-intercept.
- In particular: for any x in the domain, the sum of the function's heights at x and $-x$ will always be π:
  $\cos^{-1}(-x) + \cos^{-1} x = \pi$
  $\cos^{-1}(-x) = \pi - \cos^{-1} x$
- $y = \cos(\cos^{-1} x)$ exists only for values of x between -1 and 1 inclusive;
  $y = \cos(\cos^{-1} x)$ is equivalent to $y = x$ for this entire domain.
- $y = \cos^{-1}(\cos x)$ exists for all real x, but it is equivalent to $y = x$ only for $0 \le x \le \pi$.
  Its graph has a 'sawtooth' shape.

The inverse tangent function

By restricting the domain of $y = \tan x$ to $-\frac{\pi}{2} < x < \frac{\pi}{2}$, we form a one-to-one function whose inverse is that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

We write $y = \tan^{-1} x$ to mean 'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

Features of $\tan^{-1} x$:

- increasing function
- domain is all real x
  (you can take $\tan^{-1}$ of any real value)
- range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$
  ($\tan^{-1}$ gives values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ not inclusive)
- horizontal asymptotes at $y = \pm\frac{\pi}{2}$
- gradient of 1 at $(0, 0)$
- for all x in the domain, $f(-x) = -f(x)$, i.e. it is an odd function. Hence $\tan^{-1}(-x) = -\tan^{-1} x$.

- $y = \tan(\tan^{-1} x)$ is equivalent to $y = x$ for all real x.
- $y = \tan^{-1}(\tan x)$ exists for all real x except $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
  It is equivalent to $y = x$ only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
  Its graph is like an infinite set of parallel intervals 'with open circles on each end' (i.e. missing the values that are multiples of $\pi \pm \frac{\pi}{2}$).

Differentiation of inverse trigonometric functions

Note: $\frac{dy}{dx} \times \frac{dx}{dy} = 1$

Derivative of $\sin^{-1} x$

Let $y = \sin^{-1} x$ for $-1 \le x \le 1$

$\therefore x = \sin y$ where $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\frac{dx}{dy} = \cos y \quad \therefore \frac{dy}{dx} = \frac{1}{\cos y}$ noting that $\cos y \neq 0$

$\therefore -\frac{\pi}{2} < y < \frac{\pi}{2}$

Using $\cos^2 y + \sin^2 y = 1$: $\frac{dy}{dx} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$

As y is an angle in the first or fourth quadrants, $\cos y$ must be positive.

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$

$\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

Note that the derivative is not defined at $x = \pm 1$ (the graph of $\sin^{-1} x$ has vertical tangents at its endpoints). Also note that the derivative is positive for all x in its domain (as $\sin^{-1} x$ is an increasing function).

Derivative of $\sin^{-1} \frac{x}{a}$

Let $y = \sin^{-1} \frac{x}{a}$ for $-a \le x \le a$

$\therefore x = a \sin y, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\frac{dx}{dy} = a\cos y \quad \therefore \frac{dy}{dx} = \frac{1}{a\cos y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

As for the derivative of $\sin^{-1} x$:

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - a^2\sin^2 y}}$ for $-a < x < a$

$\therefore \frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$

Derivative of $\cos^{-1} x$

Let $y = \cos^{-1} x$ for $-1 \le x \le 1$

$\therefore x = \cos y$ where $0 \le y \le \pi$

$\frac{dx}{dy} = -\sin y \quad \therefore \frac{dy}{dx} = -\frac{1}{\sin y}$ for $0 < y < \pi$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-\cos^2 y}}$ for $-1 < x < 1$

$\therefore \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1$

Derivative of $\cos^{-1} \frac{x}{a}$

Let $\cos^{-1} \frac{x}{a}$ for $-a \le x \le a \quad \therefore x = a\cos y, \quad 0 \le y \le \pi$

$\frac{dx}{dy} = -a\sin y \quad \therefore \frac{dy}{dx} = -\frac{1}{a\sin y}$ for $0 < y < \pi$

$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - a^2\cos^2 y}}$ for $-a < x < a$

$\therefore \frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}, \quad -a < x < a$

Derivative of $\tan^{-1} x$

Let $y = \tan^{-1} x$ for all $x \quad \therefore x = \tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y$

$\therefore \frac{dy}{dx} = \frac{1}{1+\tan^2 y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$ for all x

$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ for all x

Derivative of $\tan^{-1} \frac{x}{a}$

Let $y = \tan^{-1} \frac{x}{a}$ for all x

$\therefore x = a\tan y$ where $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\frac{dx}{dy} = a\sec^2 y = a(1+\tan^2 y)$

$\therefore \frac{dy}{dx} = \frac{a}{a^2 + a^2\tan^2 y}$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$

$\therefore \frac{dy}{dx} = \frac{a}{a^2 + x^2}$ for all x

$\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{a}{a^2 + x^2}$ for all x

Integration involving inverse trigonometric functions

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad -a < x < a$$

$$\int \frac{-1}{\sqrt{a^2 - x^2}}\,dx = \cos^{-1}\frac{x}{a} + C, \quad -a < x < a$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad \text{for all } x$$

Expansion of $(1+x)^n$, Pascal's triangle

The coefficients of the successive powers of the $(1+x)^n$ expansion can be arranged in a triangular pattern called **Pascal's triangle**, as shown at right.

$(1+x)^0$ 1
$(1+x)^1$ 1 1
$(1+x)^2$ 1 2 1
$(1+x)^3$ 1 3 3 1
$(1+x)^4$ 1 4 6 4 1
$(1+x)^5$ 1 5 10 10 5 1
$(1+x)^6$ 1 6 15 20 15 6 1

Properties of Pascal's triangle

1 The first and last number in each row is 1.
2 Every number not at the end of a row is the sum of the two numbers in the row above it to the left and right (e.g. $10 = 4 + 6, 15 = 5 + 10, 20 = 10 + 10$).
3 The number of terms in each row is one more than the value of n.
4 The second and second-last terms in each row are the value of n.

Binomial expansion

(a) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \quad + {}^nC_rx^r + \quad + {}^nC_nx^n = \sum_{r=0}^{n} {}^nC_rx^r$

(b) $(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \quad + {}^nC_ra^{n-r}b^r + \quad + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$

(c) ${}^nC_0 = 1, {}^nC_n = 1, {}^nC_r = {}^nC_{n-r}$

(d) ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \quad$ for $1 \le r \le n-1$

(e) The general term in the expansion of $(a+b)^n$ is $T_{r+1} = {}^nC_ra^{n-r}b^r$

(f) The ratio of successive terms of the binomial expansion of $(a+b)^n$ is $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times \frac{b}{a}$

Alternate notation:

- ${}^nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$
- $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad + \binom{n}{n}x^n$

Useful results:

(a) $x = 1$:

$$(1+1)^n = \binom{n}{0} + \binom{n}{1} \times 1 + \binom{n}{2} \times 1^2 + \quad + \binom{n}{r} \times 1^r + \quad + \binom{n}{n} \times 1^n$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{r} + \quad + \binom{n}{n} = \sum_{r=0}^{n} \binom{n}{r}$$

$$2^n - 1 = \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{r} + \quad + \binom{n}{n} = \sum_{r=1}^{n} \binom{n}{r} \quad \text{as} \binom{n}{0} = 1$$

(b) $x = -1$:

$$(1-1)^n = \binom{n}{0} + \binom{n}{1} \times (-1) + \binom{n}{2} \times (-1)^2 + \binom{n}{3} \times (-1)^3 + \quad + \binom{n}{r} \times (-1)^r + \quad + \binom{n}{n} \times (-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \quad + \binom{n}{r} \times (-1)^r + \quad + \binom{n}{n} \times (-1)^n$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \quad = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \quad = \frac{1}{2} \times 2^n = 2^{n-1}$$

In the expansion of $(1+x)^n$: Sum of coefficients of even terms = Sum of coefficients of odd terms $= 2^{n-1}$

12 FURTHER PROBABILITY

- $P(A)=\dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}, 0\le P(A)\le 1$
- $P(\bar{A})=1-P(A)$
- Mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B)$
- Not mutually exclusive events: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- Independent events: $P(A \text{ and } B) = P(A) \times P(B)$

Binomial probability

In probability we often find situations in which the same trial is repeated several times and there are only **two** possible outcomes in each trial (often called 'success' and 'failure'). This is binomial probability. Properties:

- there are n independent trials
- each trial has two possible outcomes ('success' or 'failure'—e.g. rolling a 4, tossing heads, a bullseye, a defective product, or recovering from disease X could be called a 'success')
- in each example, the probability of a success is the same in each of the n trials.

For n independent binomial trials: $(q+p)^n = q^n + \binom{n}{1}q^{n-1}p + \binom{n}{2}q^{n-2}p^2 + \quad + \binom{n}{r}q^{n-r}p^r + \quad + p^n$

$= P(X=0) + P(X=1) + P(X=2) + \quad + P(X=r) + \quad + P(X=n)$

If X denotes the number of successes in n independent trials, we can see that the probability of exactly r successes and $(n-r)$ failures is given by the $(r+1)$-th term in the expansion of $(q+p)^n$.

$P(X=r)=\binom{n}{r}q^{n-r}p^r=\binom{n}{r}(1-p)^{n-r}p^r$ where $0\le r\le n,\ p+q=1$

13 APPLICATIONS OF CALCULUS TO THE PHYSICAL WORLD

Related rates

Problems with **related rates** arise when there is a function that relates two variables, e.g. x and y, where both x and y are also functions of another variable, e.g. time t. For example, we may need to determine $\dfrac{dy}{dt}$ when $\dfrac{dx}{dt}$ is known. In such cases it is necessary to use the chain rule, $\dfrac{dy}{dt}=\dfrac{dy}{dx}\times\dfrac{dx}{dt}$.

The equation $\dfrac{dN}{dt} = k(N - P)$ where k, P are constants

- $N = Ae^{kt}$ is the solution of the equation $\dfrac{dN}{dt}=kN$, where A is the initial value of N.
- $N = P + Ae^{kt}$ is a solution of the equation $\dfrac{dN}{dt}=k(N-P)$, where A is a constant.
- If $k<0$, $Ae^{kt}\to 0$ as $t\to\infty$ and hence $N\to P$ from above.

Newton's Law of Cooling

The cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium:

$\dfrac{dT}{dt}=-k(T-M)$ where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant).

Wilhelmy's Law

Many chemical reactions follow the law that states that the rate of the reaction is proportional to the difference between the initial concentration of the reagent (i.e. the chemical reacting) and the amount transformed at any time: $\dfrac{dx}{dt}=k(a-x), 0\le x\le a$ where a is the initial concentration and x is the amount transformed at time t.

Velocity and acceleration as functions of x

The expressions $x(t)$, $v(t) = \dot{x}(t)$ and $a(t) = \ddot{x}(t) = \dot{v}(t)$, which denote displacement, velocity and acceleration respectively, define functions of time t. For each t there corresponds only one value of $x(t)$ that satisfies the function, because the particle can only be in one place at one time. Similarly $(t, v(t))$ and $(t, a(t))$ define functions because a particle cannot have more than one velocity and one acceleration at any given time.

However, sometimes it is useful to express acceleration in terms of the displacement x or the velocity v. In this case we must be careful, because equations relating a, v and x will not necessarily define functions.

Remember that calculus only makes sense when applied to functions, so $\frac{dv}{dx}$ can only have meaning if $v = v(x)$ defines a function. Hence if we want to differentiate the equation $v^2 = 1 - x^2$ we must restrict v to be either positive or negative, so we consider v to apply only to velocities as the particle moves from A to B, i.e. $v > 0$, or only from B to A, i.e. $v < 0$.

If $v(x)$ specifies a function according to one of these restrictions, then we can use the chain rule of differentiation:

$$\begin{aligned}\frac{dv}{dt} &= \frac{dv}{dx} \times \frac{dx}{dt} \\ &= v\frac{dv}{dx} \quad \text{as } v = \frac{dx}{dt} \\ &= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx} \\ &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\end{aligned}$$

Hence acceleration may be expressed in any of the forms $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, $v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

The form to use in a particular problem will depend on the form of the equation that defines acceleration:

- Given $a = f(t)$, use $\frac{dv}{dt}$ or $\frac{d^2x}{dt^2}$
- Given $a = g(x)$, use $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$
- Given $a = h(v)$, use $\frac{dv}{dt}$ if initial conditions are values for t and v; or $v\frac{dv}{dx}$ if initial conditions are values for x and v

It is customary to write derivatives with respect to time using dots above the dependent variable,

e.g. $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, $\dot{v} = \frac{dv}{dt}$.

Simple harmonic motion (SHM)

SHM is the motion of a point particle moving along a straight line so that its displacement x from a fixed point at time t is given by a sine or cosine function:

- displacement x: $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$
- velocity $\dot{x}$: $\dot{x} = -an\sin(nt + \alpha)$
- acceleration $\ddot{x}$: $\ddot{x} = -an^2\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$

Important results:

1 When $\dot{x} = 0$ (i.e. at the endpoints of the motion), the acceleration is greatest.

2 When $\ddot{x} = 0$ (i.e. at O, the centre of the motion), the speed is greatest (i.e. velocity has its greatest or least value).

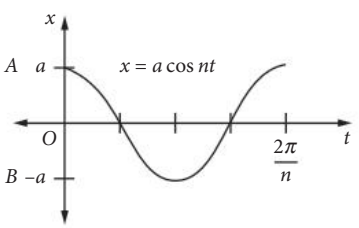
The general equation $x = a\cos(nt + \alpha)$, $\alpha > 0$, $n > 0$

If when $t = 0$, $x = a$, the particle is initially at the extreme point A and $a = a\cos\alpha$, then $\cos\alpha = 1$ and $\alpha = 0$. The equation of motion can then be written $x = a\cos nt$.

$$\begin{aligned}x &= a\cos(nt + \alpha) \\ &= a\cos nt \qquad \text{if } x(0) = a \\ v = \dot{x} &= -na\sin nt\end{aligned}$$

$$\frac{dv}{dt} = \ddot{x} = -n^2a\cos nt$$

$$\ddot{x} = -n^2x$$

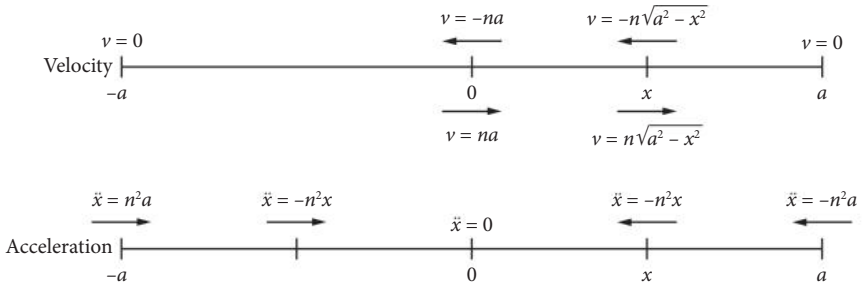


$v^2 = n^2(a^2 - x^2)$, $-a \le x \le a$ $\qquad T = \frac{2\pi}{n} = \frac{1}{f}$

The amplitude a is the distance from the centre of motion O to either of the extreme points A or B.

SHM problems are usually solved using either $x = a\cos(nt + \alpha)$ or $\ddot{x} = -n^2x$, or occasionally $v^2 = n^2(a^2 - x^2)$. The starting point will depend on the information given.

The following diagrams illustrate the velocity and acceleration of a particle undergoing SHM, including the extreme values.



Note that to the left of O, x is negative, so $-n^2x$ (the acceleration) is positive.

The equation $\ddot{x} = -n^2x$ describes the motion of a particle under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The force (and hence the acceleration) is zero at O, where the speed is greatest. The force (and hence the acceleration) is greatest at the extreme points, where the speed is zero.

This type of motion occurs in real physical situations (either approximately or exactly) where a particle oscillates about an equilibrium position. For example:

- the to-and-fro motion of a pendulum bob (approximately SHM)
- the up-and-down motion of a mass attached to a spring
- the bobbing motion of a buoy floating on water.

Projectile motion

This is the motion of a particle that is projected in any direction and subject only to gravitational acceleration (i.e. any effect of air resistance is ignored).

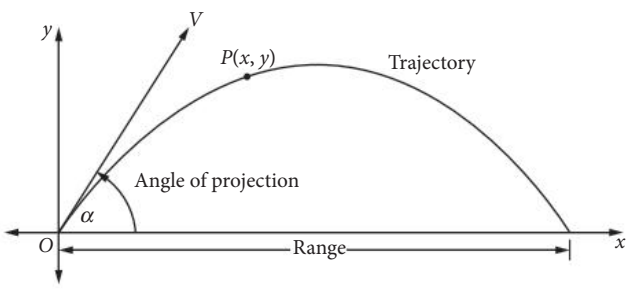
The motion is considered entirely in the vertical plane containing the direction of the initial velocity. The **angle of projection** (or angle of elevation) is the angle at which the particle is projected, measured from the horizontal.

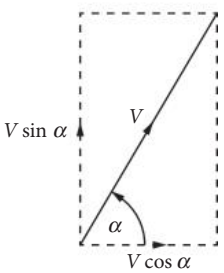
The **range** on the horizontal plane is the distance between the point of projection and the point where the particle meets the horizontal plane through the point of projection (see diagram below). The **trajectory** is the path of the particle.

Typical projectile motion:

The point of projection is chosen as the origin. The initial velocity of projection is V. The angle of projection is α. From the diagram we find that V has two components:

$V\cos\alpha$ (horizontal)
$V\sin\alpha$ (vertical)





The moving particle is affected only by the acceleration due to gravity, which has a magnitude g m s^{-2} directed vertically down.

Thus $\ddot{x} = 0$ and $\ddot{y} = -g$ define the motion in the horizontal and vertical directions respectively.

The initial conditions of the motion are: $t = 0$, $x = 0$, $y = 0$, $\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha$

Horizontal projectile motion

$\ddot{x} = 0$, we have $\dot{x} = C_1$ and initially $\dot{x} = V\cos\alpha$, so we have (where T is the time of flight):

$$\dot{x} = V\cos\alpha \qquad \text{for } 0 \le t \le T \qquad [1]$$

Integrate with respect to t: $x = \int V\cos\alpha\, dt = Vt\cos\alpha + C_2$

When $t = 0$, $x = 0$: $C_2 = 0$ $\therefore$ $x = Vt\cos\alpha \qquad \text{for } 0 \le t \le T \qquad [2]$

Vertical projectile motion

$$\ddot{y} = -g$$

Integrate with respect to t: $\dot{y} = \int -g\, dt = -gt + C_3$

When $t = 0$, $\dot{y} = V\sin\alpha$: $C_3 = V\sin\alpha$

$\therefore$ $\dot{y} = V\sin\alpha - gt \qquad \text{for } 0 \le t \le T \qquad [3]$

Integrate with respect to t: $y = \int (V\sin\alpha - gt)dt = Vt\sin\alpha - \frac{1}{2}gt^2 + C_4$

When $t = 0$, $y = 0$: $C_4 = 0$ $\therefore$ $y = Vt\sin\alpha - \frac{1}{2}gt^2 \qquad \text{for } 0 \le t \le T \qquad [4]$

Equation [3] produces the formula $v = u + at$ and equation [4] produces $s = ut + \frac{1}{2}at^2$, which you may know from Physics.

Equations [2] and [4] are the parametric equations that define any point on the trajectory in terms of the time t after projection ($0 \le t \le T$ where T is the time of flight).

Equation of the path

The Cartesian equation of the projectile's trajectory is obtained by eliminating t from equations [2] and [4] above:

$$y = Vt\sin\alpha - \frac{1}{2}gt^2$$

Substitute $t = \dfrac{x}{V\cos\alpha}$: $\quad y = V\sin\alpha \times \dfrac{x}{V\cos\alpha} - \dfrac{1}{2}g\left(\dfrac{x}{V\cos\alpha}\right)^2$

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \qquad [5]$$

This equation is particularly useful when finding the angle of projection.

Note:

The parametric equations of the trajectory can be obtained using the equations for uniformly accelerated motion. If $\dot{x}$ and $\dot{y}$ denote the velocity components in the x and y directions respectively:

$\dot{x} = V\cos\alpha$ (constant velocity) $\qquad$ $\dot{y} = V\sin\alpha - gt$ (using $v = u + at$)

$x = Vt\cos\alpha$ $\qquad$ $y = Vt\sin\alpha - \frac{1}{2}gt^2$ (using $s = ut + \frac{1}{2}at^2$)

You must take care if using these results.

Maximum height of a projectile

The particle reaches its highest point when $\dot{y} = 0$. From [3] above, this is when $t = \dfrac{V\sin\alpha}{g}$:

Substitute into [4]: $\quad y_{\max} = \dfrac{V^2\sin^2\alpha}{g} - \dfrac{1}{2}\dfrac{V^2\sin^2\alpha}{g} = \dfrac{V^2\sin^2\alpha}{2g}$

Alternatively: Starting with the equation of the trajectory [5], the highest point is reached when $\frac{dy}{dx}=0$:

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \qquad [5]$$

Differentiate with respect to x: $\frac{dy}{dx} = \tan\alpha - \frac{gx}{V^2\cos^2\alpha}$

At $\frac{dy}{dx}=0$: $\tan\alpha - \frac{gx}{V^2\cos^2\alpha} = 0$

$$x = \frac{V^2\sin\alpha\cos\alpha}{g}$$

Substitute into [5]: $y = \frac{V^2\sin\alpha\cos\alpha}{g}\times\frac{\sin\alpha}{\cos\alpha} - \frac{g}{2V^2\cos^2\alpha}\times\left(\frac{V^2\sin\alpha\cos\alpha}{g}\right)^2$

$$= \frac{V^2\sin^2\alpha}{2g}$$

In this case, using the parametric forms of the equations is easiest.

The maximum height of the projectile is: $y_{\max} = \frac{V^2\sin^2\alpha}{2g}$

Range on a horizontal plane through the point of projection

When the projectile strikes the horizontal plane, $y=0$. Thus from [4]: $Vt\sin\alpha - \frac{1}{2}gt^2 = 0$

$$t\left(V\sin\alpha - \frac{1}{2}gt\right) = 0$$

$$t = 0 \quad \text{or} \quad \frac{2V\sin\alpha}{g}$$

As $t=0$ is when the particle was first projected, the required time is $t = \frac{2V\sin\alpha}{g}$, so the time of flight T is given by $T = \frac{2V\sin\alpha}{g}$.

Substitute $t = \frac{2V\sin\alpha}{g}$ into [2]: $x = V\cos\alpha\times\frac{2V\sin\alpha}{g}$

$$= \frac{V^2\sin 2\alpha}{g}$$

i.e. $R = \frac{V^2\sin 2\alpha}{g}$ where R denotes the range.

Alternatively: if only the range is required, without needing to know the time of flight, then the trajectory equation [5] can be used.

Substitute $y=0$ into [5]: $0 = x\left(\tan\alpha - \frac{gx}{2V^2\cos^2\alpha}\right)$

$$x = 0 \quad \text{or} \quad x = \frac{V^2\sin 2\alpha}{g}$$

$x=0$ is the initial position and $x = \frac{V^2\sin 2\alpha}{g}$ is where it meets the horizontal plane again, which gives the range.

Maximum range

For a given speed of projection V, the equation $R = \frac{V^2\sin 2\alpha}{g}$ defines the range as a function of α. This function of α has its greatest value when $\sin 2\alpha = 1$, i.e. when $\alpha = 45°$.

Thus the greatest range of the particle is achieved when it is projected at an angle of 45° to the horizontal:

$R_{\max} = \dfrac{V^2}{g}$ when $\alpha = 45°$

It is important for you to be able to derive projectile formulae when necessary, rather than for the formulae to be memorised.

14 HARDER APPLICATIONS

Graphing rational algebraic functions

Examples of rational algebraic functions are: $y = \dfrac{2x}{x^2+1}$, $y = \dfrac{3x^2}{x+1}$, $y = x + \dfrac{2}{x}$

When graphing rational functions, we use the typical approach to curve sketching:

- identify restrictions on the domain and/or the range
- find intercepts on the coordinate axes, where possible
  (i.e. we can always find the y-intercept, if it exists, but it is sometimes very difficult or impossible to find the x-intercept)
- identify whether the function is odd or even (to use the associated symmetry properties)
- find stationary points and identify their nature
- find asymptotes and use them to guide the shape of the curve.

(a) The graph of every function of the form $y = \dfrac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials such that degree of P_1 < degree of P_2, has $y = 0$ as a horizontal asymptote.

(b) The graph of every function of the form $y = \dfrac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials of the same degree, has a horizontal asymptote that is not the x-axis. It is the line $y = k$, where: $k = \dfrac{\text{leading coefficient of } P_1}{\text{leading coefficient of } P_2}$

(c) The graph of every function of the form $y = \dfrac{P_1(x)}{P_2(x)}$, where P_1 and P_2 are polynomials such that degree of P_1 > degree of P_2, has an asymptote that is either oblique (i.e. an inclined straight line) or that is another curve.

The limit $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Note that this limit effectively means that when θ is small, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$.

MATHEMATICS EXTENSION 1 COURSE OUTCOMES

Objectives	Preliminary outcomes	HSC outcomes
Students will develop:	A student:	A student:
• appreciation of the scope, usefulness, beauty and elegance of mathematics	PE1 • appreciates the role of mathematics in the solution of practical problems	HE1 • appreciates interrelationships between ideas drawn from different areas of mathematics
• the ability to reason in a broad range of mathematical contexts	PE2 • uses multi-step deductive reasoning in a variety of contexts	HE2 • uses inductive reasoning in the construction of proofs
• skills in applying mathematical techniques to the solution of practical problems	PE3 • solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations	HE3 • uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay
• understanding of the key concepts of calculus and the ability to differentiate and integrate a range of functions	PE4 • uses the parametric representation together with differentiation to identify geometric properties of parabolas PE5 • determines derivatives which require the application of more than one rule of differentiation	HE4 • uses the relationship between functions, inverse functions and their derivatives HE5 • applies the chain rule to problems including those involving velocity and acceleration as functions of displacement HE6 • determines integrals by reduction to a standard form through a given substitution
• the ability to interpret and communicate mathematics in a variety of forms	PE6 • makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations	HE7 • evaluates mathematical solutions to problems and communicates them in an appropriate form

MATHEMATICS EXTENSION 1 PERFORMANCE BANDS

The typical performance in this band:

BAND E4

- Exhibits extensive knowledge and skills appropriate to the Mathematics and Mathematics Extension 1 courses
- Synthesises mathematical techniques, results and ideas creatively across the Mathematics and Mathematics Extension 1 courses to solve difficult problems
- Uses sophisticated multi-step mathematical reasoning
- Interprets, explains, justifies and evaluates solutions to problems
- Translates efficiently between practical problems and their mathematical model
- Communicates complex ideas and arguments effectively using appropriate mathematical language, notation, diagrams and graphs

BAND E3

- Exhibits knowledge and skills appropriate to the Mathematics and Mathematics Extension 1 courses
- Synthesises mathematical techniques, results and ideas from across the Mathematics and Mathematics Extension 1 courses to solve problems in areas such as geometry, calculus and probability
- Uses multi-step mathematical reasoning such as mathematical induction
- Translates between practical problems and their mathematical model in areas such as projectile motion
- Communicates effectively using appropriate mathematical language, notation, diagrams and graphs

BAND E2

- Exhibits knowledge of the techniques of the Mathematics and Mathematics Extension 1 courses
- Uses techniques of integration from the Mathematics Extension 1 course such as integration by substitution
- Uses logical reasoning in numerical contexts such as problems in algebra and geometry
- Applies calculus to solve practical problems
- Communicates using mathematical language, notation, diagrams and graphs

MATHEMATICS EXTENSION 1 PERFORMANCE BANDS

This table shows approximate relative skill levels in the performance bands.

Skill level	Band E2	Band E3	Band E4
1	Exhibits knowledge of the techniques of the Mathematics and Mathematics Extension 1 courses	Exhibits knowledge and skills appropriate to the Mathematics and Mathematics Extension 1 courses	Exhibits extensive knowledge and skills appropriate to the Mathematics and Mathematics Extension 1 courses
2	Uses techniques of integration from the Mathematics Extension 1 course such as integration by substitution	Synthesises mathematical techniques, results and ideas from across the Mathematics and Mathematics Extension 1 courses to solve problems in areas such as geometry, calculus and probability	Synthesises mathematical techniques, results and ideas creatively across the Mathematics and Mathematics Extension 1 courses to solve difficult problems
3	Uses logical reasoning in numerical contexts such as problems in algebra and geometry	Uses multi-step mathematical reasoning such as mathematical induction	Uses sophisticated multi-step mathematical reasoning
4			Interprets, explains, justifies and evaluates solutions to problems
5	Applies calculus to solve practical problems	Translates between practical problems and their mathematical model in areas such as projectile motion	Translates efficiently between practical problems and their mathematical model
6	Communicates using mathematical language, notation, diagrams and graphs	Communicates effectively using appropriate mathematical language, notation, diagrams and graphs	Communicates complex ideas and arguments effectively using appropriate mathematical language, notation, diagrams and graphs

ANSWERS

Trigonometric and logarithmic values may be approximate.

CHAPTER 1

EXERCISE 1.1

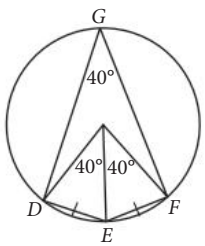
1 (a) $x = 70$ (angle at centre on equal chords)
$y = 55$ (ΔOCD is isosceles, $OC = OD$, radii)
(b) $\angle FOE = 110°$ (ΔOFE is isosceles, $OF = OE$, radii)
$y = 25$ (angles at centre equal so chords equal)
(c) $\angle OLM = 32°$ (ΔOLM is isosceles, $OL = OM$, radii)
$x = 64$ (exterior angle is sum of interior opposite angles)
$y = 16$ (angles at centre equal so chords equal)
(d) $x = 60$ (angle at centre twice angle at circumference)
(e) $y = 160$ (angle at centre twice angle at circumference)
(f) $\angle COE = 140°$ (angle at centre twice angle at circumference)
reflex $\angle COE = 220°$ (angles at a point)
$z = 110$ (angle at centre twice angle at circumference)

2 C

3 (a) $\angle DCB = 115°$ (angle sum of triangle)
reflex $\angle DOB = 230°$ (angle at centre twice angle at circumference)
$\angle DOB = 130°$ (angles at a point)
$x = 65$ (angle at centre twice angle at circumference)
(b) $\angle COE = 2y°$ (angle at centre twice angle at circumference)
reflex $\angle COE = 4y°$ (angle at centre twice angle at circumference)
$6y° = 360°$ (angles at a point), $y = 60$
(c) $OHPG$ is a rhombus so $\angle HOG = 56°$ (opposite angles of a rhombus are equal)
$\angle OHK = z°$ (ΔOHK is isosceles)
$2z° = 56°$ (exterior angle is sum of interior opposite angles)
$z = 28$

4 (a) correct (b) correct (c) correct (d) correct

5 $\angle DGF = 40°$



6 $\angle DOE = 2x°$ (angle at centre twice angle at circumference)
$OD = OE$ (radii)
$\therefore \Delta DOE$ is isosceles
$\therefore \angle DEO = y°$ (base angles of isosceles triangle)
$\therefore 2x° + 2y° = 180°$ (angle sum of triangle)
$\therefore x + y = 90$

7 ΔOCB is isosceles ($OC = OB$, radii)
$\angle OCB = \angle OBC$ (base angles of isosceles triangle OCB are equal)
$\angle OCB = \angle OBC = 38°$ (angle sum of ΔOCB is 180°)
$\therefore \angle ACO = 71° - 38° = 33°$
ΔOCA is isosceles ($OC = OA$, radii)
$\angle OCA = \angle OAC$ (base angles of isosceles triangle OCA are equal)
$\therefore \angle OAC = 33°$

8 $\angle KOM = 2x°$ (angle at centre twice angle at circumference)
$\therefore$ reflex $\angle KOM = (360 - 2x)°$ (angles at a point)
$\angle KLM = (180 - x)°$ (angle at centre twice angle at circumference)
$\therefore \angle KNM + \angle KLM = x° + (180 - x)° = 180°$

9 Let $\angle OCA = x°$
ΔAOC is isosceles ($OC = OA$, radii)
$\therefore \angle OCA = \angle OAC = x°$ (base angles of isosceles triangle OCA are equal)
$\therefore \angle AOC = (180 - 2x)°$ (angle sum of triangle)
$\therefore \angle AOB = (180 - 2x)°$ (equal chords subtend equal angles at the centre, $AC = AB$)
But ΔAOB is isosceles ($OB = OA$, radii)
$\therefore \angle OAB = \angle OBA = x°$ (angle sum of triangle)
$\therefore \angle CAB = 2x°$
$\therefore \angle XAC = (180 - 2x)°$ (XAB a straight line)
$\therefore \angle XAC = \angle AOC$ (both equal $(180 - 2x)°$)

EXERCISE 1.2

1 6 mm **2** $PM = 10$ mm, $PQ = 20$ mm **3** C

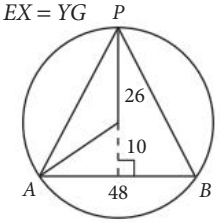
4 In ΔXYZ and ΔOYW:
$XY = 2OY$ (diameter is twice radius)
$ZY = 2WY$ (W is midpoint of YZ)
$\therefore \frac{XY}{OY} = \frac{ZY}{WY} = \frac{2}{1}$
$\angle XYZ = \angle OYW$ (common angle)
$\therefore \Delta XYZ \,|||\, \Delta OYW$ (2 sides proportional and included angle equal)
$\therefore \angle YZX = \angle YWO$ (corresponding angles of similar triangles)
$\therefore XZ \parallel OW$ (pair of corresponding angles equal when ZY cuts XZ and OW)
But $\angle YWO = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)
$\therefore \angle YZX = 90°$
Alternatively: the result 'the line through the midpoint of two sides of a triangle is parallel to the third side' could have been used.

5 Let N be the midpoint of ML and join ON.
$\angle ONL = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)
Similarly $\angle OAL = 90°$
But $\angle ALM = 90°$ (given)
$\therefore \angle AON = 90°$ (angle sum of a quadrilateral is 360°)
$\therefore AONL$ is a rectangle (each angle 90°)
$\therefore LN = OA$ (opposite sides of a rectangle)
But $ML = 2LN$ (N is the midpoint of ML)
$\therefore ML = 2OA$

6 $\angle BSO = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)
Similarly $\angle BTX = 90°$
$\therefore \angle SOX + \angle OXT + 90° + 90° = 360°$ (angle sum of quadrilateral)
$\therefore \angle SOX + \angle OXT = 180°$

7 $XF = FY$ (perpendicular from centre of circle to chord bisects chord)
$\angle CEF = \angle OFG = \angle DGF = 90°$ (given)
$\therefore CE \parallel OF \parallel DG$ (pairs of cointerior angles supplementary)
But $CO = OD$ (equal radii)
$\therefore EF = FG$ (if parallel lines cut off equal intercepts on one transversal, then intercepts on other transversals are also equal)
$EX = EF - XF$ and $YG = FG - FY$ so $EX = YG$

8 $AP^2 = 24^2 + 36^2$,
$AP = 12\sqrt{13}$ mm



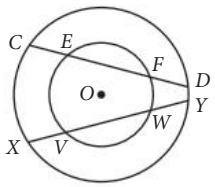
EXERCISE 1.3

1 Join OP and OQ.
In ΔOAQ and ΔOAP:
OA is a common side
$OQ = OP$ (radii)
$QA = PA$ (given)
$\therefore \Delta OAQ \equiv \Delta OAP$ (SSS)
$\therefore \angle OAQ = \angle OAP$ (corresponding angles in congruent triangles)
$\therefore BA$ bisects PAQ

2 B

3 O and P are equidistant from KN (equal chords in equal circles are equidistant from the centre)
Hence $OP \parallel KN$

4



$CD = XY$ (given)
$\therefore O$ is equidistant from CD and XY
$\therefore O$ is equidistant from EF and VW (parts of CD and XY)
$\therefore EF = VW$ (chords of a circle equidistant from the centre are equal)

5 $\angle OBM = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)
Similarly $\angle OAK = 90°$
But $\angle BOA = 90°$ (angle sum of quadrilateral $AOBX$ is 360°)
$\therefore AOBX$ is a rectangle.
But $OA = OB$ (equal chords are equidistant from the centre)
$AOBX$ is a square (rectangle with a pair of adjacent sides equal)

6 Draw the perpendicular OA from O to DE and the perpendicular PB from P to FC.
$\therefore OA = PB$ (equal chords in equal circles are equidistant from the centre)
In ΔOAC and ΔPBC:
$\angle OAC = \angle PBC = 90°$ (by construction)
$\angle OCA = \angle PCB$ (vertically opposite angles)
$OA = PB$ (proven above)
$\therefore \Delta OAC \equiv \Delta PBC$ (AAS)
$\therefore OC = CP$ (corresponding sides of congruent triangles)
$\therefore DG$ bisects OP

7 In ΔAOB and ΔAPB:
$OA = PA$ (radii of equal circles)
Similarly $OB = PB$
AB is a common side
$\therefore \Delta AOB \equiv \Delta APB$ (SSS)
$\therefore \angle OAB = \angle PAB$ (corresponding angles of congruent triangles)
In ΔAOC and ΔAPC:
$OA = PA$ (radii of equal circles)
$\angle OAB = \angle PAB$
AC is a common side
$\therefore \Delta AOC \equiv \Delta APC$ (SAS)
$\therefore OC = CP$ (corresponding sides of congruent triangles)

8 $OP \parallel KN$ so O and P are equidistant from KN
$\therefore$ the distance of O from KM is the same as the distance of P from LN.
Hence, as the circles have the same radius,
$KM = LN$ (chords of equal circles equidistant from the centre are equal)
But $KL = KM - LM$ and $MN = LN - LM$ so $KL = MN$

9 Let A be the midpoint of CD and B the midpoint of FG.
$\therefore \angle OAD = 90°$ (line joining midpoint of chord to centre of circle $\perp$ chord)
Similarly $\angle OBG = 90°$.
In ΔOAE and ΔOBE:
$\angle OAE = \angle OBE = 90°$
OE is a common side
$OA = OB$ (radii)
$\therefore \Delta OAE \equiv \Delta OBE$ (RHS)
$\therefore AE = BE$ (corresponding sides of congruent triangles)
But $DE = AE - AD$ and $GE = BE - BG$.
$AD = BG$ (half of equal chords)
$\therefore DE = GE$

10 Draw OF perpendicular to AB and OG perpendicular to CD.
In ΔOFE and ΔOGE:
$\angle OFE = \angle OGE = 90°$ (by construction)
OE is a common side
$OF = OG$ (equal chords are equidistant from the centre)
$\therefore \Delta OFE \equiv \Delta OGE$ (RHS)
$\therefore FE = GE$ (corresponding sides of congruent triangles)
$AF = FB$ (line from centre of circle perpendicular to chord bisects chord)
Similarly $CG = GD$
$\therefore AF = CG$ (halves of equal chords are equal)
But $AE = AF + FG$ and $CE = CG + GE$
$\therefore AE = CE$

EXERCISE 1.4

1 (a) $x = 40$ (angle at centre twice angle at circumference)
(b) $x = 100$ (angle at centre twice angle at circumference)
$y = 50$ (isosceles triangle)
(c) $z + 100 = 130$ (angle at centre twice angle at circumference)
$z = 30$
(d) $x = 230$ (angle at centre twice angle at circumference)
(e) $x = 60$ (angles in the same segment)
$y = 40$ (angle sum of triangle)
$z = 40$ (angles in the same segment)
(f) $x = 90$ (angle in a semicircle)
(g) $c = 15$ (angles in the same segment)
$b + 15 = 90$ (exterior angle is sum of interior opposite angles)
$b = 75$, $a = 75$ (angles in the same segment)
(h) $a = 68$ (base angles of isosceles triangle, equal radii)
$b = 22$ (angle in a semicircle)
$c = 22$ (base angles of isosceles triangle, equal radii)
(i) $a = 50$ (angle in a semicircle, angle sum of triangle)
$b = 50$ (angles in the same segment)
(j) $a = 50$ (angles in the same segment)
$b = 25$ (angle sum of triangle)
$c = 25$ (angles in the same segment)
(k) $y = 40$ (angle in a semicircle)
$a = 28$ (angle sum of triangle)
$b = 28$ (angles in the same segment)
$x = 40$ (angles in the same segment)
(l) $a = 42$ (angle in a semicircle)
$z = 55$ (angle sum of triangle)
$y = 42$ (angles in the same segment)
$x = 35$ (angles in the same segment)
(m) $e = 50$ (angle in a semicircle)
$f = 50$ (angles in the same segment)
$a = 40$ (base angles of isosceles triangle)
$d = 40$ (angles in the same segment)
$c = 40$ (angles in the same segment)
$b = 50$ (angle in a semicircle)
(n) $x = 28$ (alternate angles)
$y = 28$ (angles in the same segment)
$z = 28$ (angles in the same segment)
(o) $x = 47$ (angles in the same segment)
$y = 22$ (exterior angle is sum of interior opposite angles)

2 D

3 $\angle CAB = \angle CDB$ (angles in the same segment)
$\angle AEC = \angle DEB$ (vertically opposite angles)
$\angle ACD = \angle ABD$ (angles in the same segment)
$\therefore \Delta AEC$ and ΔDEB are equiangular

4 $\angle ABX = 90°$ (angle in a semicircle)
$\angle ABY = 90°$ (angle in a semicircle)
$\therefore \angle XBY = 90° + 90° = 180°$
$\therefore \angle XBY$ is a straight angle, hence X, B, Y are collinear

5 (a) (angle in a semicircle) (b) (both 90°)
(c) (angles in the same segment)
(d) (sum of angles in a triangle is 180°)

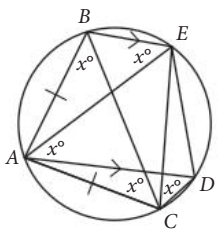
6 $\angle OAD = \angle DAB$ (AD bisects $\angle OAB$)
$OA = OD$ (radii)
$\therefore \Delta OAD$ is isosceles
$\therefore \angle ODA = \angle OAD$ (base angles of isosceles triangle)
$\angle ODA = \angle DAB$ (both equal $\angle OAD$)
But $\angle ODA$ and $\angle DAB$ are a pair of equal alternate angles formed when DA cuts OD and AB.
$\therefore OD \parallel AB$

7 $\angle DAC = \angle DBC$ (angles in the same segment)
$\angle DAC = \angle ACB$ (alternate angles equal, $DA \parallel CB$)
$\therefore \angle DBC = \angle ACB$
Hence ΔEBC is isosceles (two angles equal)
$\angle ADB = \angle ACB$ (angles in the same segment)
$\therefore \angle ADB = \angle DAC$
Hence ΔEDA is isosceles (two angles equal)

8 (a) (base angles of isosceles triangle)
(b) (angles in the same segment) (c) (sum of adjacent angles)
(d) (angle in a semicircle) (e) (angle sum of triangle)

9 ΔOCP is isosceles ($OC = CP$, given)
$\therefore \angle COP = \angle CPO = a°$
$\angle BCO = 2a°$ (exterior angle is sum of interior opposite angles)
ΔOCB is isosceles ($OC = OB$, radii)
$\therefore \angle OBC = \angle OCB = 2a°$
$\therefore \angle BOC = (180 - 4a)°$ (angle sum of triangle)
But AOP is a straight line so $\angle AOB = 180° - (180 - 4a)° - a°$
$\therefore \angle AOB = 3a°$

10 Let $\angle ABC = x°$
$\therefore \angle ACB = x°$ (base angles of isosceles triangle)
$\angle AEB = x°$ (angles in the same segment standing on AB)
$\angle EAD = \angle BEA = x°$
(alternate angles equal, $BE \parallel AD$)
$\angle ADC = \angle ABC = x°$
(angles in the same segment standing on AC)
$\therefore \angle EAD = \angle ADC$ (both equal $x°$)
$\therefore AE \parallel CD$ (transversal AD makes pair of equal alternate angles)

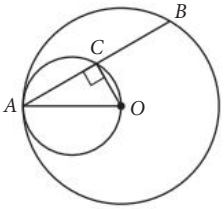


11 Let $\angle ABC = y°$
$\therefore \angle ADC = y°$ (opposite angles of a parallelogram are equal)
$\therefore \angle AEC = \angle ADC = y°$ (angles in the same segment standing on AC)
and $\angle DAE = \angle AEC = y°$ (alternate angles equal, $DA \parallel EB$)
$\therefore \angle AEB = \angle ABE = y°$
$\therefore \Delta ABE$ is isosceles

12 $\angle APD = \angle CPB$ (same angle)
$\angle DAP = \angle BCP$ (angles in the same segment)
$\therefore \Delta APD \,|||\, \Delta CPB$ (AAA)

13 (a) In ΔCDE and ΔBAE:
$\angle DCA = \angle DBA$ (angles in the same segment)
$\angle DEC = \angle AEB$ (vertically opposite angles)
$DC = AB$ (given)
$\therefore \Delta CDE \equiv \Delta BAE$ (AAS)
(b) $\therefore DE = AE$ (corresponding sides of congruent triangles)
$\therefore \Delta EDA$ is isosceles
(c) $\angle CDE = \angle BAE$ (corresponding angles of congruent triangles)
But $\angle EDA = \angle EAD$ (base angles of isosceles triangle)
$\therefore \angle CDA = \angle BAD$ (sum of equal parts)
In ΔADC and ΔDAB:
$\angle CDA = \angle BAD$ (from above)
DA is a common side
$DC = AB$ (given)
$\therefore \Delta ADC \equiv \Delta DAB$ (AAS)
(d) $\therefore AC = DB$ (corresponding sides of congruent triangles)

14 $\angle ACO = 90°$
(angle in a semicircle, diameter AO)
$\therefore OC \perp AB$
$\therefore C$ is the midpoint of AB
(perpendicular from centre of circle to chord bisects chord)

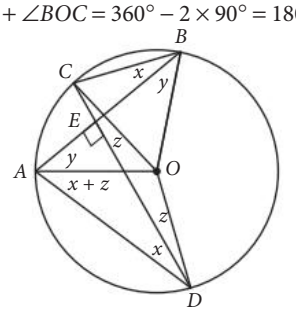


15 There is only one circle that passes through A, B and all the positions of P.
Or: A, B and all the positions of P are concyclic.

16 AB is a diameter of the circle through A, B and P.

17 $\angle OAC = 25°$, $\angle ACB = 37°$, $\angle ADC = 65°$

18 (a) $\angle BOC = 180° - 2(x + y)°$ (ΔBOC is isosceles, angle sum of triangle is 180°)
(b) $\angle AOD = 180° - 2(x + z)°$ (ΔAOD is isosceles, angle sum of triangle is 180°)
(c) $\angle AOD + \angle BOC = 360° - 2(2x + y + z)°$
In ΔAED, $x + (x + z) + y = 90$ ($AB \perp CD$)
$\therefore 2x + y + z = 90$
$\therefore \angle AOD + \angle BOC = 360° - 2 \times 90° = 180°$



EXERCISE 1.5

1 (a) $y = 82$ (opposite angles of a cyclic quadrilateral are supplementary)
Similarly $y = 102$
(b) $x = 65$ (angle in a semicircle, angle sum of triangle)
$y = 115$ (opposite angles of a cyclic quadrilateral are supplementary)
(c) $x = 54$ (opposite angles of a cyclic quadrilateral are supplementary, base angles of an isosceles triangle equal)
(d) $x = 65$ (opposite angles of a cyclic quadrilateral are supplementary)
(e) $x = 76$ (opposite angles of a cyclic quadrilateral are supplementary, straight angle equals 180°)
(f) $a = 105$ (base angles of an isosceles triangle equal, straight angle)
$b = 100$ (opposite angles of a cyclic quadrilateral are supplementary)
$c = 55$ (sum of angles of triangle)
(g) $a = 155$ (straight angle)
$b = 25$ (opposite angles of a cyclic quadrilateral are supplementary)
(h) $a = 122$ (opposite angles of a cyclic quadrilateral are supplementary)
$b = 18$ (sum of angles of triangle)
(i) $a = 74$ (angle at centre is twice the angle at the circumference)
$b = 106$ (opposite angles of a cyclic quadrilateral are supplementary)

$c = 53$ (opposite angles of kite equal)
$d = 42$ (angles at a point, sum of angles of quadrilateral)

(j) $x = 109$ (cointerior angles supplementary, reflex angle at centre twice angle at circumference)

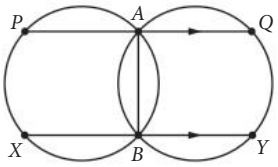
(k) $x = 52$ (reflex angle at centre twice angle at circumference, straight angle)

(l) $b = c = 109$ (base angles of isosceles triangle, exterior angle of a triangle)
$a = d = 71$ (opposite angles of a cyclic quadrilateral are supplementary)

2 C **3** (a) correct (b) correct (c) incorrect (d) correct

4 (a) (base angles of isosceles triangle equal)
(b) (straight angle is 180°)
(c) (opposite angles of a cyclic quadrilateral are supplementary)
(d) (pair of corresponding angles equal where AD cuts BC and ED)

5



$\angle QAB = \angle PXB$ (exterior angle of quadrilateral equals interior opposite angle)
$\angle XBA = \angle QAB$ (alternate angles, $PQ \parallel XY$)
$\angle AQY = \angle XBA$ (exterior angle of cyclic quadrilateral equals interior opposite angle)
$\therefore \angle PXY = \angle PQY$

6 $\angle ADB = x°$ (base angles of isosceles triangle equal)
$\angle BAD = (180 - 2x)°$ (angle sum of triangle)
$\angle BCD = 2x°$ (opposite angles of a cyclic quadrilateral are supplementary)

7 (a) Let $\angle DAC = x°$
$\therefore \angle DCA = \angle DAC = x°$ (base angles of isosceles triangle equal)
$\angle DBC = \angle DAC = x°$ (angles in the same segment)
$\therefore \angle BDC = \angle DBC = x°$ (base angles of isosceles triangle equal)
$\angle BAC = \angle BDC = x°$ (angles in the same segment)
and $\angle DBA = \angle DCA = x°$ (angles in the same segment)
$\therefore \angle DAB = 2x°$ ($\angle DAC + \angle CAB$)
Hence $\angle DCB = (180 - 2x°)$ (opposite angles of a cyclic quadrilateral are supplementary)
$\angle ABC = 2x°$ ($\angle ABD + \angle DBC$)
Hence $\angle ADC = (180 - 2x°)$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore \angle ADC = \angle DCB$

(b) $\angle ADC + \angle DAB = (180 - 2x°) + 2x° = 180°$
These are a pair of supplementary cointerior angles so $DC \parallel AB$

8 Let $\angle ABC = x°$, $\angle BAC = y°$
$\angle ACB = 90°$ (angle in a semicircle, diameter AB)
$\therefore \angle ABC + \angle BAC = 90°$ (angle sum of triangle)
$\therefore x + y = 90$
Now $\angle APC = (180 - x)°$ (opposite angles of cyclic quadrilateral $APCB$ are supplementary)
and $\angle CQB = (180 - y)°$ (opposite angles of cyclic quadrilateral $ACQB$ are supplementary)
$\therefore \angle APC + \angle CQB = (180 - x)° + (180 - y)°$
$= 360° - (x + y)°$
$= 360° - 90° = 3$ right angles

9 Let $\angle ABC = x°$
$\therefore \angle ADC = (180 - x)°$ (opposite angles of a cyclic quadrilateral are supplementary)
and $\angle DAB = x°$ (cointerior angles supplementary, $AB \parallel DC$)
$\therefore \angle DAB = \angle ABC$

10 (a) Let $\angle ADC = x°$
$\therefore \angle BCD = x°$ (given)
and $\angle BAD = (180 - x)°$ (opposite angles of a cyclic quadrilateral are supplementary)
But $\angle ADC + \angle BAD = x° + (180 - x)° = 180°$
Hence $\angle ADC$ and $\angle BAD$ are a pair of supplementary cointerior angles formed when DA cuts CD and BA.
$\therefore DC \parallel AB$

(b) $\angle DBA = \angle BDC$ (alternate angles, $DC \parallel AB$)
and $\angle BDC = \angle BAC$ (angles in the same segment)
$\therefore \angle DBA = \angle BAC$

11 In ΔABC and ΔCDA:
$\angle BAC = \angle ACD$ (alternate angles, $BA \parallel CD$)
$\angle ACB = \angle CAD$ (alternate angles, $BC \parallel AD$)
AC is a common side
$\therefore \Delta ABC \equiv \Delta CDA$ (AAS)
$\therefore \angle ABC = \angle CDA$ (matching angles in congruent triangles)
ΔABC is acute-angled so $\angle ABC < 90°$, thus $\angle CDA < 90°$.
If D is on the circle then $ADCB$ would be a cyclic quadrilateral and $\angle CDA = 180° - \angle ABC$ (opposite angles of a cyclic quadrilateral are supplementary).
This would only be possible if $\angle ABC = \angle CDA = 90°$
$\therefore D$ is not on the circle if ΔABC is acute-angled.

EXERCISE 1.6

1 $\angle CBD = 25°$ (angle sum of ΔBCD)
$\therefore \angle CAD = \angle CBD$ (both 25°)
These are a pair of equal angles subtended on the same side of CD, hence A, B, C and D are concyclic.
$\therefore \angle BAC = \angle BDC = 55°$ (angles in the same segment standing on BC)
$\angle BEA = 65°$ ($\angle CEA$ is a straight angle)
$\therefore \angle ABD = 60°$ (angle sum of ΔABE)

2 A **3** (a) correct (b) incorrect (c) correct (d) incorrect

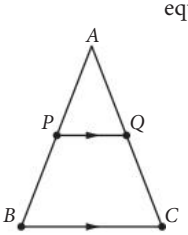
4 $\angle QPR = 75°$ (angle sum of ΔPQR)
$\therefore \angle QPR = \angle QSR = 75°$
These are a pair of equal angles subtended on the same side of QR, hence P, Q, R, S are concyclic.
$\angle PSQ = 35°$ (angles in the same segment)

5 $\angle ADB = \angle ABD = x°$ (base angles of isosceles triangle ABD)
$\therefore \angle DAB = (180 - 2x)°$ (angle sum of ΔABD)
and $\angle DCB = \angle DBC = 2x°$ (base angles of isosceles ΔCBD)
$\angle DAB + \angle DCB = (180 - 2x)° + 2x° = 180°$
$\therefore ABCD$ is a cyclic quadrilateral (pair of opposite angles supplementary)

6 $\angle LPM = \angle LNM = 90°$ (LM is hypotenuse of the right-angled triangles)
These are a pair of equal angles subtended on the same side of LM, hence L, M, N, P are concyclic.

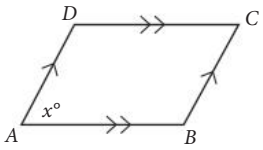
7 $\angle ABC = \angle BCA = \angle CAB = 60°$ (ΔABC is equilateral)
$\angle BAX = \angle CAX = 30°$ (AX bisects $\angle BAC$)
$\angle ABX = \angle XBC = 30°$ (BX bisects $\angle ABC$)
$\angle BDC = \angle DBC = 30°$ (base angles of isosceles triangle BDC)
$\angle DCE = 60°$ (angle sum of ΔBCD)
$\angle DEC = 90°$ (angle sum of ΔDCE)
$\angle AEB = 90°$ (vertically opposite angles)
$\angle AXE = 60°$ (angle sum of ΔAXE)
$\angle AXB = 120°$ (angle sum of ΔAXB)
A, D, C, X are concyclic ($\angle XAC = \angle XDC = 30°$ are a pair of equal angles standing on the same side of XC)

8



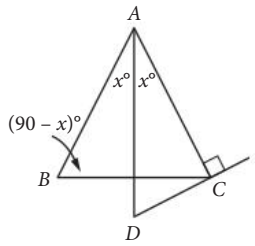
Let $\angle ABC = x°$
$\therefore \angle BPQ = (180 - x)°$ (cointerior angles are supplementary, $PQ \parallel BC$)
$\angle ACB = \angle ABC = x°$ (base angles of isosceles triangle)
$\therefore \angle BPQ + \angle ACB = (180 - x)° + x° = 180°$
$\therefore PBCQ$ is a cyclic quadrilateral (pair of opposite angles supplementary)

9



$ABCD$ is a parallelogram that is also cyclic.
Let $\angle DAB = x°$
$\angle DCB = \angle DAB = x°$ (opposite angles of a parallelogram are equal)
But $\angle DCB = (180 - x)°$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore x = 180 - x$
$x = 90$
$\therefore ABCD$ is a rectangle (a parallelogram with one angle 90°)

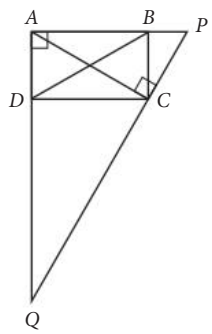
10



Let $\angle BAC = 2x°$
$\therefore \angle BAD = \angle DAC = x°$ (DA bisects $\angle BAC$)
$\angle ACD = 90°$ ($DC \perp AC$)
$\therefore \angle ADC = (90 - x)°$ (angle sum of ΔADC)
$\angle ABC = \angle ACB = (90 - x)°$ (base angles of an isosceles triangle)
$\therefore \angle ABC = \angle ACB = (90 - x)°$ (angle sum of triangle)
Hence $\angle ABC = \angle ADC = (90 - x)°$
$\therefore A, B, D, C$ are concyclic ($\angle ABC$ and $\angle ADC$ are a pair of equal angles standing in the same side of AB)

11 $\angle OQA = 90°$ (line joining centre of circle to midpoint of chord $\perp$ chord)
Similarly $\angle OPA = 90°$
$\therefore \angle OQA + \angle OPA = 180°$
$\therefore A, P, O, Q$ are concyclic (opposite angles of the quadrilateral are supplementary)

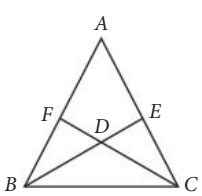
12



Let $\angle BPC = x°$
$\therefore \angle DCQ = x°$ (corresponding angle equal, $AB \parallel DC$, opposite sides of a rectangle)
But $\angle ACP = 90°$ (given)
$\therefore \angle DCA = (90 - x)°$ (QCP is a straight angle)
$\Delta ADC \equiv \Delta BCD$ (RHS)
$\therefore \angle ACD = \angle BDC$ (corresponding angles of congruent triangles)
$\therefore \angle BDC = (90 - x)°$
Now $\angle QDB = \angle QDC + \angle BDC$
$= 90° + (90 - x)° = (180 - x)°$
and $\angle BPQ = x°$
$\therefore P, B, D, Q$ are concyclic (opposite angles of the quadrilateral are supplementary)

13 Let $\angle BAD = x°$
$\therefore \angle DCE = \angle BAD = x°$ (exterior angle of a cyclic quadrilateral equals interior opposite angle)
and $\angle DCE = \angle DFE = x°$ (angles in the same segment)
$\therefore \angle GFD = (180 - x)°$ (GFE is a straight angle)
But $\angle GFD + \angle BAD = (180 - x)° + x° = 180°$
$\therefore GFDA$ is a cyclic quadrilateral (opposite angles of the quadrilateral are supplementary)

14 (a)



$\angle ABC = \angle ACB$ (base angles of isosceles triangle ABC)
$\angle DBC = \angle DCB$ (base angles of isosceles triangle DBC)
$\therefore \angle FBD = \angle ECD$
$\angle FDB = \angle EDC$ (vertically opposite angles)
$\Delta FDB \,|||\, \Delta EDC$ (equiangular)

(b) $\angle FBD$ and $\angle ECD$ are a pair of equal angles standing on the same side of FE.
$\therefore BFEC$ is a cyclic quadrilateral

15 $\angle EFD = \angle ECD$ (angles in the same segment)
$\angle BAD = \angle ECD$ (exterior angle of cyclic quadrilateral equals interior opposite angle)
$\therefore \angle GAD = \angle EFD$
$\therefore GFDA$ is a cyclic quadrilateral (exterior angle of quadrilateral equals interior opposite angle)

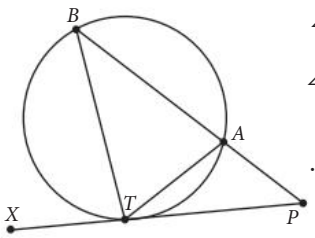
EXERCISE 1.7

1 (a) $a = 34$ (tangent is perpendicular to the radius drawn to the point of contact)
$b = 90$ (angle in a semicircle)
$c = 56$ (angle in the alternate segment)
(b) $2a = 46$, $a = 23$ (isosceles triangle, exterior angle of triangle, radius $\perp$ tangent, angle sum of triangle)
$b = 67$ (radius $\perp$ tangent)
(c) $a = 36$ (radius $\perp$ tangent)
$b = 36$ (isosceles triangle)
$c = 54$ (angle in the alternate segment)
(d) $c = 68$ (angle in the alternate segment)
$b = 68$ (isosceles triangle, angle in the alternate segment)
$a = 44$ (straight angle)
(e) $a = 22$ (radius $\perp$ tangent)
$b = 22$ (isosceles triangle)
$c = 44$ (exterior angle of triangle is sum of interior opposite angles)
$d = 68$ (angle in the alternate segment)
(f) $a = 25$ (radius $\perp$ tangent)
$b = 65$ (angle in the alternate segment)
$c = 25$ (alternate angles)
$d = 65$ (angle in a semicircle, angle sum of triangle)
(g) $x = y = 50$ (angle in the alternate segment)
(h) $y = 100$ (exterior angle of cyclic quadrilateral equals interior opposite angle)
$x = 100$ (angle in the alternate segment)

2 B **3** (a) incorrect (b) correct (c) incorrect (d) correct

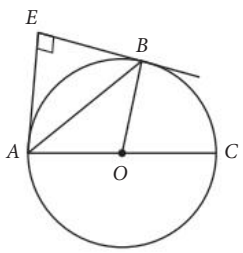
4 (a) $\angle ABD = 50°$ (angle in the alternate segment)
$\angle BAF = 50°$ (alternate angles, $BD \parallel FE$)
(b) $\angle ACD = 50°$ (angle in the alternate segment)
$\angle BCA = 50°$ (angle in the alternate segment)
$\angle BCD = 100°$

5



$\angle BTX = \angle BAT$ (angle in the alternate segment)
$\angle BAT = \angle APT + \angle PTA$ (exterior angle of triangle is sum of interior opposite angles)
$\therefore \angle BTX = \angle APT + \angle PTA$

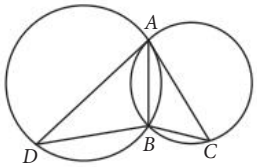
6



$OB \perp EB$ (tangent $\perp$ radius drawn to point of contact)
$\therefore \angle AEB = \angle OBE = 90°$ ($AE \perp EB$, given)
$\therefore AE \parallel OB$ (cointerior angles supplementary when EB cuts AE and OB)
$\therefore \angle EAB = \angle ABO$ (alternate angles, $AE \parallel OB$)
ΔOAB is isosceles ($OA = OB$, radii of circle)
$\therefore \angle OAB = \angle OBA$ (base angles of isosceles triangle)
$\therefore \angle OAB = \angle EAB$
$\therefore BA$ bisects $\angle CAE$

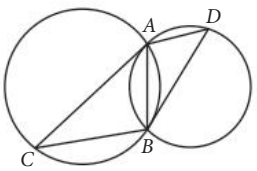
7 $\angle CAB = \angle ADB$ (angle in the alternate segment)
$\angle ACB = \angle DAB$ (angle in the alternate segment)
$\therefore \Delta ABC \,|||\, \Delta DBA$ (equiangular)

$\therefore \angle ABC = \angle ABD$ (matching angles in similar triangles)
If D, B, C are collinear then $\angle ABC + \angle ABD = 180°$ (straight angle)
$\therefore \angle ABC = \angle ABD = 90°$
Hence AD and AC are the diameters of the respective circles (angle in a semicircle)



8 $\angle PQS = \angle QRS$ (angle in the alternate segment)
$\angle PSQ = \angle SQR$ (alternate angles, $PS \parallel QR$)
$\therefore \Delta PSQ \,|||\, \Delta SQR$ (equiangular)
$\therefore \angle SPQ = \angle QSR$ (matching angles in similar triangles)

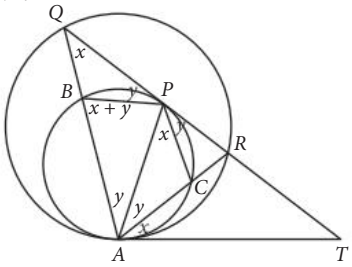
9 $\angle DBA = \angle BCA$ (angle in the alternate segment)
$\angle CAB = \angle ADB$ (angle in the alternate segment)
$\therefore \Delta DAB \,|||\, \Delta ACB$ (equiangular)
$\therefore \angle DAB = \angle CBA$ (matching angles in similar triangles)



But these are a pair of equal alternate angles formed when AB cuts AD and CB.
$\therefore AD \parallel CB$

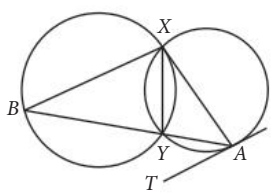
10 $\angle TPC = \angle PAC$ (angle in the alternate segment)
$\angle TPD = \angle PBD$ (angle in the alternate segment)
$\therefore \angle PAC = \angle PBD$
But these are a pair of equal corresponding angles formed when PAB cuts CA and DB.
$\therefore AC \parallel BD$

11 Let QR produced meet the tangent at A at T. Join BP and PC as shown.



(a) $\angle AQR = \angle TAR = x°$ (angle in the alternate segment)
and $\angle APC = \angle TAC = x°$ (angle in the alternate segment)
(b) $\angle PAC = \angle RPC = y°$ (angle in the alternate segment)
(c) Now $\angle RPA = (x + y)°$
$\therefore \angle ABP = \angle RPA = (x + y)°$ (angle in the alternate segment)
$\angle QPB = y°$ (exterior angle of ΔQPB is sum of interior opposite angles)
Hence $\angle PAB = \angle QPB = y°$ (angle in the alternate segment)
$\therefore \angle QAP = \angle PAR = y°$
$\therefore PA$ bisects $\angle RAQ$.

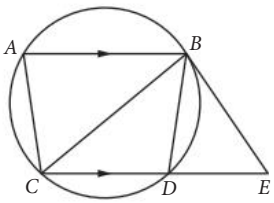
12 Let TA be the tangent at A to the second circle.



$\angle YXA = \angle TAY$ (angle in the alternate segment)
$\angle XBT = \angle YXA$ (angle in the alternate segment)
$\therefore \angle XBT = \angle BAT$
But these are a pair of equal alternate angles formed when BA cuts BX and TA.
$\therefore XB \parallel AT$.

13 Let $\angle PCB = x°$
$\therefore \angle CAB = \angle PCB = x°$ (angle in the alternate segment)
Let $\angle ACQ = \angle QCB = y°$ (QC bisects $\angle ACB$)
$\therefore \angle PCQ = (x + y)°$
and $\angle PQC = (x + y)°$ (exterior angle of ΔQAC is sum of interior opposite angles)
$\therefore \angle PCQ = \angle PQC$
$\therefore \Delta PCQ$ is isosceles
$\therefore PC = PQ$ (opposite equal angles in isosceles triangle)

14 $\angle EBD = \angle DCB$ (angle in the alternate segment)
$\angle DCB = \angle CBA$ (alternate angles, $AB \parallel CD$)
$\therefore \angle EBD = \angle CBA$
$\angle BDE = \angle BAC$ (exterior angle of a cyclic quadrilateral equals the interior opposite angle)
$\therefore \Delta EBD \,|||\, \Delta CBA$



15 Join PB, AB and BR.
Let $\angle TPQ = x°$, $\angle TRB = y°$
$\therefore \angle PBA = \angle TPA = x°$ (angle in the alternate segment)
and $\angle PAB = \angle QRB = y°$ (exterior angle of a cyclic quadrilateral equals the interior opposite angle)
$\therefore \angle APB = 180° - (x + y)°$ (angle sum of triangle)
$\therefore \angle TPB = x° + 180° - (x + y)°$
$= (180 - y)°$
$\therefore \angle TPB + \angle TRB = 180°$
$\therefore P, T, R, B$ are concyclic as the opposite angles of the $PTRB$ are supplementary.

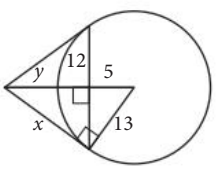
EXERCISE 1.8

1 (a) $x = 62$ (radius $\perp$ tangent, angle sum of triangle)
(b) $x = 35$ (radius $\perp$ tangent, exterior angle of a triangle)
(c) $b = 28$ (radius $\perp$ tangent, exterior angle of a triangle)
$a = 62$ (complementary angles)
(d) $x = 36$ (base angles of isosceles triangle, radius $\perp$ tangent)
(e) $b = 62$ (base angles of isosceles triangle)
$c = 56$ (angle sum of triangle)
$a = 28$ (radius $\perp$ tangent)
$d = 34$ (angle sum of triangle)
(f) $a = 90$ (radius $\perp$ tangent)
$c = 30$ (base angles of isosceles triangle)
$b = 120$ (angle sum of triangle)
(g) $y = 52$ (radius $\perp$ tangent)
$x = 36$ ($180 + x = 216$, reflex angle at centre)
$z = 56$ (angle sum of quadrilateral)
(h) $a = 40$ (angle in the alternate segment, angle sum of triangle)

2 C **3** 20 cm (tangents drawn to a circle from an external point are equal)

4 $(x + 21)^2 = (x + 3)^2 + 24^2$, $x = 4$

5 (a) 5 cm
(b) $x^2 = y^2 + 144$. $(y + 5)^2 = x^2 + 169$
$y^2 + 10y + 25 = y^2 + 144 + 169$
$10y = 288$
$y = 28.8$
$x^2 = 28.8^2 + 144$
$x = 31.2$ cm



6 $\angle ABC = 103°$, $\angle BCD = 92°$, $\angle CDA = 77°$, $\angle DAB = 88°$

7 $\left(4 + 2\sqrt{3}\right)$ cm

8 (a) $ZX = ZA$ (tangents to a circle from an external point are equal)

Similarly, $ZA = ZY$

$\therefore ZX = ZY$

(b) $\angle XZO = \angle OZA$ (tangents to a circle from an external point are equally inclined to the line joining the point to the centre of the circle)

Similarly $\angle YZP = \angle PZA$

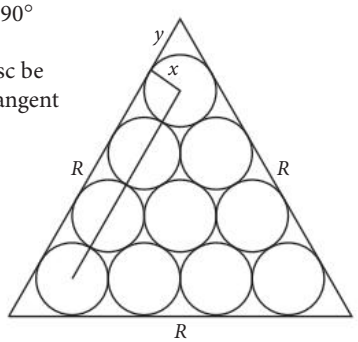
Now $\angle XZA + \angle AZY = 180°$ (XZY a straight angle)

$\therefore 2\angle OZA + 2\angle AZP = 180°$

$\therefore \angle OZA + \angle AZP = 90°$

$\therefore \angle OZP = 90°$

9 Let the radius of each disc be x and the length of the tangent from the corner be y.



$R = 6x + 2y$

$\tan 30° = \frac{x}{y}$ so $y = \sqrt{3}x$

$R = \left(6 + 2\sqrt{3}\right)x$

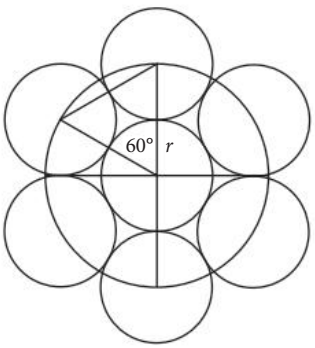
so $x = \dfrac{R}{2\left(3+\sqrt{3}\right)}$

Perimeter of discs $= 2\pi x + 3 \times 1.5 \times 2\pi x = 11\pi x$

$= \dfrac{11\pi R}{2\left(3+\sqrt{3}\right)} = \dfrac{11\pi R\left(3-\sqrt{3}\right)}{2(9-3)} = \frac{11}{12}\left(3-\sqrt{3}\right)\pi R$

10 (a) $\frac{r}{2}, \frac{3r}{2}$ (where 'touches' means 'shares a common tangent')

(b) Perimeter $= 6 \times \frac{2}{3} \times 2\pi \times \frac{r}{2} = 4\pi r$



11 $KD = KG$ (tangents to a circle from an external point are equal)

Similarly $LD = LE$, $ME = MF$, $NF = NG$.

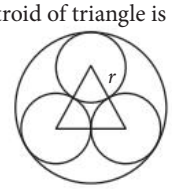
Now $KN + ML = KG + GN + LE + EM$

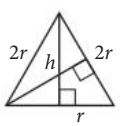
and $KL + NM = KD + DL + NF + FM$

Hence $KN + ML = KL + NM$

12 Height of equilateral triangle $= \sqrt{3}r$

Centroid of triangle is one-third of the way up the altitude.





$h^2 = 4r^2 - r^2$, $h = \sqrt{3}r$, $R = r + \dfrac{2\sqrt{3}r}{3}$

13 (a) Perimeter $= 3 \times 2\pi r - \pi r = 5\pi$

(b) Area of playground

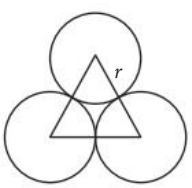
= Area of equilateral triangle + Area of 2.5 circles

$= \sqrt{3}r^2 + \dfrac{5\pi r^2}{2} = \dfrac{r^2}{2}\left(2\sqrt{3} + 5\pi\right)$

(c) Distance

$= r$ + two-thirds of the altitude of the equilateral triangle

$= r + \frac{2}{3} \times \sqrt{3}r = \frac{r}{3}\left(3 + 2\sqrt{3}\right)$

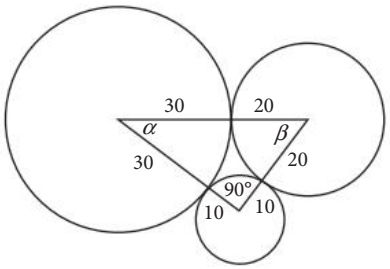


14 (a) $\sin\alpha = 0.6$, $\alpha = 0.6435$; $\sin\alpha = 0.8$, $\alpha = 0.9273$

Perimeter

$= (2\pi - 0.6435) \times 30 + (2\pi - 0.9273) \times 20 + \frac{3\pi}{2} \times 10$

$= 323.4\,\text{cm}$

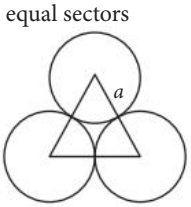


(b) Area $= 600 - \frac{1}{2}\left(30^2 \times 0.6435 + 20^2 \times 0.9273 + 10^2 \times \frac{\pi}{2}\right)$

$= 46.4\,\text{cm}^2$

15 (a) Area = Area of triangle − Area of three equal sectors

$= \sqrt{3}a^2 - \dfrac{\pi a^2}{2} = \dfrac{a^2}{2}\left(2\sqrt{3} - \pi\right)$



(b) Radius $= \dfrac{(2\sqrt{3}-3)a}{3}$

EXERCISE 1.9

1 (a) $CE = \dfrac{5 \times 4}{2.5} = 8\,\text{cm}$ (b) $CD = (10 + 3) = 13\,\text{cm}$ (c) $AB = 19\,\text{cm}$

(d) $CD = 12\,\text{cm}$

2 B **3** $AX \times XB = 4 \times 14 = 56$

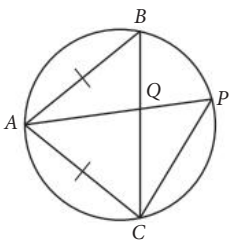
4 Produce MX to N, where N is on the circle

$MX = XN$ (perpendicular from centre to chord bisects chord)

$DX \times XD = MX \times XN$ (product of intercepts of chords are equal)

$\therefore DX \times XD = MX \times MX = MX^2$

5 (a)



$AB = AC$ so ΔABC is isosceles

$\therefore \angle ABC = \angle ACB$ (base angles of isosceles triangle)

$\angle ABC = \angle APC$ (angles in the same segment)

In ΔAQC and ΔAPC:

$\angle ACQ = \angle APC$ (both equal $\angle ABC$)

$\angle QAC = \angle PAC$ (common angle)

$\therefore \Delta AQC \,|||\, \Delta APC$

(b) $\dfrac{AC}{AP} = \dfrac{AQ}{AC}$ (corresponding sides of similar triangles are proportional)

$\therefore AC^2 = AQ \times AP$

$\therefore AB^2 = AQ \times AP$ ($AB = AC$, given)

6 (a) $\angle AQH = 90°$ (angle in a semicircle, diameter AH)

$\angle APK = 90°$ (angle in a semicircle, diameter AK)

$\therefore \angle AQH = \angle APK = 90°$ These are a pair of equal angles standing on the same side of HK

$\therefore H, Q, P, K$ are concyclic

(b) $\therefore PA \times AH = QA \times AK$ (product of intercepts of chords are equal)

7 (a) $6^2 - (5 - x)^2 = 4^2 - x^2$ (use Pythagoras' theorem twice)

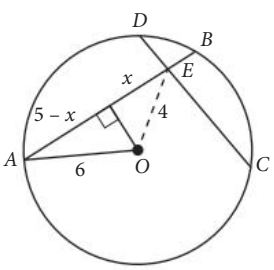
$36 - (25 - 10x + x^2) = 16 - x^2$

$x = 0.5$

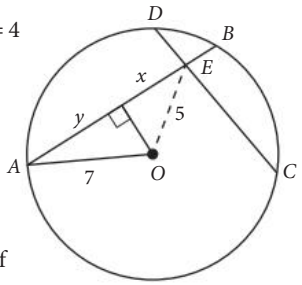
$\therefore$ half $AB = 5 - 0.5 = 4.5\,\text{cm}$ (perpendicular from centre to chord bisects chord)

$\therefore AB = 9\,\text{cm}$

$\therefore BE = 4$



Or: producing OE to meet the circle, its intercepts at E are $(6+4)$ and $(6-4)$
Hence $5 \times EB = 10 \times 2$, $BE = 4$



(b) $49 - y^2 = 25 - x^2$
$y^2 - x^2 = 24$
$AB = 2y$, so $BE = y - x$ and $AE = y + x$
$AE \times EB = (y + x)(y - x)$
$= y^2 - x^2 = 24$
Or: produce OE to meet the circle and use the product of intercepts.

EXERCISE 1.10

1 (a) $TA(TA + 7) = 12^2$, $TA = 9$ cm (b) $TX = 6$ cm
(c) $AB = 12$ cm (d) $AB = 5\sqrt{2}$ cm

2 (a) correct (b) incorrect (c) incorrect (d) correct

3 $PH^2 = PB \times PA$ (square on tangent is product of intercepts on secant)
Similarly $PK^2 = PB \times PA$
$\therefore PH^2 = PK^2$
$\therefore PH = PK$
Hence tangents to the circles from P are equal.

4 $TP^2 = TB \times TA$ (square on tangent is product of intercepts on secant)
But $TA = 2TP$
$\therefore TP^2 = TB \times 2TP$
$\therefore TP = 2TB$
$\therefore TA = 4TB$
$AB = TA - TB = 3TB$

5 (a) $\angle APB = 90°$ (angle in a semicircle)
$\angle KHB = 90°$ (given)
These are a pair of supplementary opposite angles in the quadrilateral $KPBH$
$\therefore K, P, B, H$ are concyclic
(b) Draw the tangent from A to touch the circle through K, P, B and H at Q.
$\therefore AQ^2 = AH \times AB$ (square on tangent is product of intercepts on secant)
Similarly $AQ^2 = AK \times AP$
$\therefore AH \times AB = AK \times AP$

6 $CE^2 = CA \times CD$ (square on tangent is product of intercepts on secant)
Similarly $DF^2 = DA \times DC$
$\therefore CE^2 + DF^2 = CA \times CD + DA \times DC$
$= CD(CA + AD)$
$= CD^2$

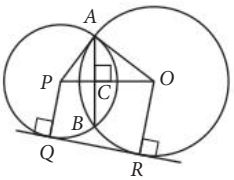
7 (a) $\angle KPA = 90°$ (angle in a semicircle, AK is diameter)
$\angle HQA = 90°$ (angle in a semicircle, AH is diameter)
$\therefore \angle KPA = \angle HQA$
These are a pair of equal angles standing on the same side of KH
$\therefore K, P, Q, H$ are concyclic
(b) $PA \times AH = QA \times AK$ (product of intercepts of chords are equal)

8 (a) $YB^2 = YQ \times YA$ (square on tangent is product of intercepts on secant)
$= (AY - AQ) \times AY$
$= AY^2 - AQ \times AY$
(b) $XB^2 = XP \times XA$ (square on tangent is product of intercepts on secant)
$= (AX - AP) \times AX$
$= AX^2 - AP \times AX$
(c) $\angle XBA = 90°$ (tangent is perpendicular to the radius drawn to the point of contact)
$AB^2 = AY^2 - YB^2$ (Pythagoras' theorem)
$\therefore AB^2 = AQ \times AY$ from (a)
$AB^2 = AX^2 - XB^2$ (Pythagoras' theorem)
$\therefore AB^2 = AP \times AX$ from (b)
$\therefore AP \times AX = AQ \times AY = AB^2$

EXERCISE 1.11

1 12 cm

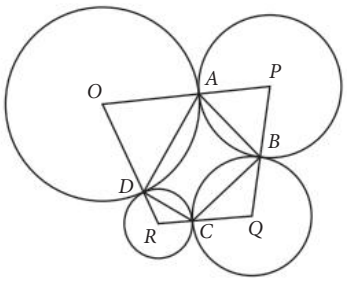
2 $AP = 10$ cm, $AO = 17$ cm, $AC = 8$ cm (OP bisects AB)
(a) $CO^2 = 17^2 - 8^2 = 225$, $CO = 15$ cm
$CP^2 = 10^2 - 8^2 = 36$, $CP = 6$ cm
$\therefore PO = 6 + 15 = 21$ cm
(b) $QR^2 = 21^2 - 7^2 = 392$
$QR = 14\sqrt{2} = 19.8$ cm



3 (a) $70 + 45 = 115$ cm
(b) $\sqrt{115^2 - 23^2} = \sqrt{12696} = 46\sqrt{6} = 112.7$ cm

4 Let the circles touch at Q
$\therefore O, Q, P$ are collinear (lines of centres passes through point of contact)
As O is the midpoint of DE and P is the midpoint of GF, then $DG \parallel OP \parallel EF$ (set of three lines cut off equal intercepts on two transversals and $DG \parallel EF$, opposite sides of a parallelogram)
$\therefore Q$ is the point of intersection of the diagonals DF and EG.
Now $\angle DQE = 90°$ (angle in a semicircle)
$\therefore \angle EQF = 90°$ (DQF is a straight line)
In ΔDQE and ΔEQF:
$DQ = QF$ (diagonals of a parallelogram bisect each other)
$\angle DQE = \angle EQF = 90°$
QE is a common side
$\therefore \Delta DQE \equiv \Delta EQF$ (SAS)
$\therefore DE = EF$ (corresponding sides of congruent triangles)

5 Let O, P, Q, R be the centres of the respective circles and A, B, C, D be their points of contact, as shown.



Let $\angle OAD = \angle ODA = x°$ (base angles of isosceles triangle)
$\therefore \angle AOD = (180 - 2x)°$
Similarly if $\angle PAB = \angle PBA = y°$, $\angle APB = (180 - 2y)°$
and if $\angle QBC = \angle QCB = z°$, $\angle BQC = (180 - 2z)°$
and if $\angle RCD = \angle RDC = w°$, $\angle CRD = (180 - 2w)°$
Now $(180 - 2x)° + (180 - 2y)° + (180 - 2z)° + (180 - 2w)° = 360°$
so $x + y + z + w = 180$
$\angle DAB = (180 - (x + y))°$ (OAP a straight line, line joining centres passes through point of contact)
Similarly $\angle BCD = (180 - (z + w))°$
$\therefore \angle DAB + \angle BCD = (180 - (x + y))° + (180 - (z + w))°$
$= 360° - (x + y + z + w)°$
$= 360° - 180° = 180°$
These are a pair of supplementary opposite angles in the quadrilateral $ABCD$.
$\therefore ABCD$ is a cyclic quadrilateral
$\therefore$ The four points of contact are concyclic.

6 (a) $\angle OAD = 90°$ (tangent $\perp$ radius drawn to the point of contact)
$\angle OCD = 90°$ (tangent $\perp$ radius drawn to the point of contact)

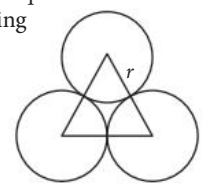
$\therefore \angle OAD + \angle OCD = 180°$
$\therefore ADCO$ is a cyclic quadrilateral (pair of opposite angles supplementary)
Similarly $DBPC$ is a cyclic quadrilateral.

(b) $\Delta ADO \equiv \Delta CDO$ (RHS)
$\therefore \angle ADO = \angle CDO$ (corresponding angles in congruent triangles)
$\therefore \angle ADC = 2\angle ODC$
Similarly $\angle BDC = 2\angle PDC$
$\therefore \angle ADC + \angle BDC = 2\angle ODC + 2\angle PDC = 2\angle ODP$
But $\angle ADC + \angle BDC = 180°$ (ADB a straight line)
$\therefore \angle ODP = 90°$

7 (a) correct (b) correct (c) correct (d) correct

8 (a) If the radius of each circle is r units, the length of the line joining the pairs of centres is $2r$ units (line joining centres passes through point of contact)
$\therefore$ triangle has each side $2r$ units, so is equilateral

(b) Each side of the triangle is a line joining the midpoints of the sides of an equilateral triangle, so it must also be equilateral.



9 $\angle WXP = \angle XRP$ (angle in the alternate segment)
$\angle WXP = \angle YXQ$ (vertically opposite angles)
$\angle YXQ = \angle XSQ$ (angle in the alternate segment)
$\therefore \angle PRS = \angle RSQ$
But these are a pair of equal alternate angles formed when RS cuts PR and SQ
$\therefore PR \parallel SQ$

10 Let $\angle LKC = y°$
$\therefore \angle CDK = \angle LKC = y°$ (angle in the alternate segment)
Let $\angle LKA = x°$
$\therefore \angle ABK = \angle LKA = x°$ (angle in the alternate segment)
$\angle CKA = \angle CKL - \angle AKL$
$\therefore \angle CKA = (y - x)°$
$\angle CDK = \angle BCD + \angle CDB$ (exterior angle of triangle is sum of interior opposite angles)
$\angle BCD = (y - x)°$
$\angle CKD = \angle BCD = (y - x)°$ (angle in the alternate segment)
$\therefore \angle CKA = \angle CKB$

11 Let $\angle DPB = x°$
$\therefore \angle PCB = \angle DPB = x°$ (angle in the alternate segment)
Let $\angle DPA = y°$
ΔDAP is isosceles ($DP = DA$, tangents from an external point)
$\therefore \angle DAP = \angle DPA = y°$ (base angles of isosceles triangle)
Now $\angle APB = (x + y)°$
and $\angle APC = 180° - (x + y)°$ (angle sum of ΔAPC)
$\therefore \angle APB + \angle APC = (x + y)° + 180° - (x + y)° = 180°$

CHAPTER REVIEW 1

1 $\angle EFD = 90°$ (angle in a semicircle)
$\angle EDG = 90°$ (tangent $\perp$ radius drawn to the point of contact)
$\therefore \angle FDG = 53°$
$\therefore \angle EGD = 37°$ (angle sum of ΔDGF)

2 $\angle ABC = 55°$ (angle at centre twice angle at circumference)
$\therefore \angle ABD = 85°$
$\angle ABD = \angle BDE = 85°$ (alternate angles, $AB \parallel CD$)
$\therefore x = 85$

3 Let $\angle KLN = x°$, $\angle LKM = y°$, $\angle KQN = w°$
$\therefore \angle KMN = \angle KLN = x°$ (angles in the same segment)
and $\angle LNM = \angle LKM = y°$ (angles in the same segment)
But $\angle LPM = (x + y)°$ (exterior angle of ΔLKP)
and $x = y + w$ (exterior angle of ΔKMQ)
Now $\angle LPM + \angle LQM = (x + y + w)°$
But $x = y + w$, so
$\angle LPM + \angle LQM = 2x° = 2\angle KLN$

4 $x(x + 5) = 36$, $(x + 9)(x - 4) = 0$, 4 cm

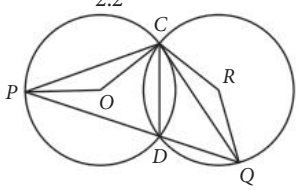
5 Let D be a point on the circle such that DA bisects $\angle BAC$.
In ΔABD and ΔACD:
$AB = AC$ (given)
$\angle BAD = \angle CAD$ (DA bisect $\angle BAC$)
AD is a common side
$\therefore \Delta ABD \equiv \Delta ACD$ (SAS)
$\therefore \angle ABD = \angle ACD$ (corresponding angles in congruent triangles)
But $\angle ABD + \angle ACD = 180°$ (opposite angles of a cyclic quadrilateral are supplementary)
$\therefore \angle ABD = \angle ACD = 90°$
$\therefore AD$ is a diameter of the circle as it subtends a right angle at the circumference
$\therefore O$ is on AD
$\therefore$ bisector of $\angle BAC$ passes through O

6 $BM \times MA = CM \times MD$ (product of intercepts of chords are equal)
But $CM = MD$ (perpendicular from centre to chord bisects chord)
$\therefore BM \times MA = MD^2$
Now $BD^2 = MD^2 + MB^2$ (Pythagoras' theorem)
so $BD^2 = MB \times MA + MB^2$
$= MB(MA + MB)$
$= MB \times AB$

7 Let $\angle PCD = x°$
$\therefore \angle CPB = \angle PCD = x°$ (alternate angles, $AB \parallel DC$)
and $\angle BEQ = \angle BPQ = x°$ (angles in the same segment)
$\angle DRE = \angle BEQ = x°$ (corresponding angles equal, $AR \parallel BE$)
$\therefore \angle PCD = \angle DRQ$ (both $x°$)

8 $AO^2 = 4^2 + (AO - 1.1)^2$
$\therefore AO = \frac{17.21}{2.2} \approx 7.82$ m

9



Let O and R be the centres of the circles such that $OP = RQ$.
Let $\angle PDC = x°$
$\therefore \angle CDQ = (180 - x)°$ (PDQ a straight angle)
Now $\angle POC = 2x°$ (angle at the centre is twice the angle at the circumference)
and reflex $\angle CRQ = (360 - 2x)°$ (angle at the centre is twice the angle at the circumference)
$\therefore \angle CRQ = 2x°$ (angles at a point)
$\therefore \angle POC = \angle CRQ = 2x°$
Hence the chords PC and CQ subtend equal angles at the centre of equal circles
$\therefore PC = CQ$

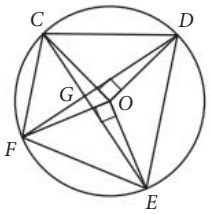
10 $\angle OWX = x°$ (OXW)
$\angle WOY = 2x°$ (angle at the centre is twice the angle at the circumference)
$\angle WOX = (180 - 2x)°$ (angle sum of triangle)
$\angle XWY = 90°$ (angle in a semicircle)
$\angle ZWY = \angle WXY = x°$ (angle in the alternate segment)
$\therefore \angle OWY = (90 - x)°$
$\angle OYW = \angle OWY = (90 - x)°$ (base angles of isosceles ΔOYW)
$\angle WYZ = (90 + x)°$ (XYZ a straight angle)
$\angle WZY = (90 - 2x)°$ (angle sum of triangle)

11 $\angle OQT = 90°$ (tangent $\perp$ radius drawn to the point of contact)
$\angle CEQ = \angle EQT = 90°$ (alternate angles, $CD \parallel QT$)
$\therefore OE \perp CD$

$\therefore$ E is the midpoint of CD (radius perpendicular to chord bisects chord)

$\therefore$ QE bisects CD

12 Let CE and DF intersect at G.



$\angle COD = \angle EOF = 90°$ (given)

$\therefore$ $\angle CED = 45°$ (angle at centre is twice angle at circumference)

Similarly $\angle EDF = 45°$

$\therefore$ $\angle DGE = 90°$ (angle sum of ΔDGE)

$\therefore$ $CE \perp DF$

13 $(x+32)^2 = x^2 + 1600$, $x = 9$ cm

14 $22°\,30'$

15 (a) $\angle EFC = 90°$ (angle in a semicircle)

$\therefore$ $\angle EBD = \angle EFC = 90°$ (exterior angle of cyclic quadrilateral equals interior opposite angle)

$\therefore$ $DB \perp AB$

(b) $\angle BED = \angle BFD$ (angles in the same segment)

$\angle AEC = \angle AFC$ (angles in the same segment)

Now $\angle AFB = 180° - (\angle BFD + \angle AFC)$

and $\angle CED = 180° - (\angle BED + \angle AEC)$

$= 180° - (\angle BFD + \angle AFC)$

$= \angle AFB$

16 (a) $\angle LPK = 90°$ (angle in a semicircle)

In ΔLKP and ΔLMP:

$\angle LPK = \angle LPM = 90°$

$LK = LM$ (given)

LP is a common side

$\therefore$ $\Delta LKP \equiv \Delta LMP$ (RHS)

$\therefore$ $\angle KLP = \angle MLP$ (corresponding angles in congruent triangles)

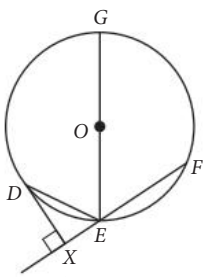
$\therefore$ PL bisects $\angle KLM$.

(b) $\angle LKM = \angle LMK = 40°$ (corresponding angles in congruent triangles)

$\angle NLK = 80°$ (exterior angle of triangle is sum of interior opposite angles)

$\therefore$ $\angle NOK$ 160° (angle at centre twice angle at circumference)

17 (a) $\angle GFE = 90°$ (angle in a semicircle)



$\angle DXE = 90°$ ($DX \perp XF$, given)

$\therefore$ $\angle GFE + \angle DXE = 180°$

These are a pair of supplementary cointerior angles formed when XF cuts GF and DX.

$\therefore$ $GF \parallel DX$

(b) $\angle DEX = 90° - \angle EDX$ (angle sum of triangle)

$\angle GEF = 90° - \angle EGF$ (angle sum of triangle)

$\angle DEG = 180° - (\angle DEX + \angle GEF)$ (straight angle)

$\angle DEG = 180° - (90° - \angle EDX + 90° - \angle EGF)$

$\therefore$ $\angle DEG = \angle EDX + \angle EGF$

18 Let $\angle CBD = y°$

$\angle CAD = \angle CBD = y°$ (angles in the same segment)

$\angle CAB = \angle CDB = x°$ (angles in the same segment)

$\angle DCB = 180° - (x+y)°$ (opposite angles of cyclic quadrilateral supplementary)

$\therefore$ $\angle CDQ + x° + 180° - (x+y)° = 180°$ (angle sum of ΔCDQ)

$\therefore$ $\angle CDQ = y°$

$\therefore$ $\angle CDA = \angle CAD = y°$

$\therefore$ ΔCAD is isosceles (two angles equal)

$\therefore$ $CD = CA$ (opposite equal angles in isosceles triangle)

19 Let $\angle QTK = x°$, $\angle QTL = y°$

$\therefore$ $\angle KNT = \angle QTK = x°$ (angle in the alternate segment)

and $\angle LMT = \angle QTL = y°$ (angle in the alternate segment)

$\angle LTK = (y - x)°$

and $\angle MTN = (y - x)°$ (exterior angle of triangle is sum of interior opposite angles)

$\therefore$ $\angle KTL = \angle MTN$

CHAPTER 2

EXERCISE 2.1

1 C

2 $-x^3 + x^2 + 6x$

(a) 3 (b) 0 (c) 1 (d) $-x^3$ (e) 3 (f) $x = -2, 0, 3$

3 (a) $5x^3 + x^2 - 6x + 7$ (i) 3 (ii) 7 (iii) 1 (iv) not monic (v) 3

(b) $-x^3 + 27$ (i) 3 (ii) 27 (iii) 0 (iv) not monic (v) 3

(c) $ax^3 + cx^2 + bx - d$ (i) 3 (ii) $-d$ (iii) c (iv) not monic (v) 3

(d) $5x^4 - 3x^3 + 2x$ (i) 4 (ii) 0 (iii) 0 (iv) not monic (v) 4

(e) $6x^4 + 2x^3 - 15x - 5$ (i) 4 (ii) -5 (iii) 0 (iv) not monic (v) 4

(f) $9x^4 - 12x^2 + 4$ (i) 4 (ii) 4 (iii) -12 (iv) not monic (v) 4

4 (a) polynomial (b) polynomial (c) polynomial

(d) polynomial (e) not polynomial, x^{-1} term (f) polynomial

(g) not polynomial, fractional and negative powers of x

(h) not polynomial, rational function

(i) not polynomial, x in exponent

5 D

6 (a) $2x^4 - 2x^2 + x + 5$ (b) $10x^6 - 4x^5 - 20x^3 + 8x^2 - 15x + 6$

(c) $2x^5 - 2x^4 + x^2 - 3x - 7$ (d) $15x^3 - 16x^2 + 9x - 2$

(e) $-6x^4 + 18x^2 - x - 15$

7 (a) $3x^3 + 2x^2 - 9x - 6$ (b) $3x^3 + 8x^2 + 7x + 2$ (c) $-x^2 + 18x - 5$

(d) $x^3 - x^2 - 5x - 3$ (e) $9x^2 + 12x + 4$ (f) $x^4 + 5x^3 - 9x^2 - 6x + 1$

EXERCISE 2.2

1 (a) $3x + 4$, remainder 13 (b) $x^2 + 1$, reminder 0

(c) $2x^2 - 5x + 6.5$, remainder -7.5

(d) $x^3 + 3x + 2$, remainder 2 (e) $x^2 + 3x + 9$, remainder 0

(f) $x^3 + x^2 + 3x - 12$, remainder 16

(g) $-x^2 + 5x - 11$, remainder 34

(h) $x^3 + x^2 - 2x - 2$, remainder 0

2 $-1, -97, 2a^3 - 4a^2 + 2a - 1, -37$ **3** 0, 0, $a^4 - 2a^3 + a^2 - 4$

4 A **5** $p = 2, q = 3$ **6** $a = 3, b = 4, c = 12$

7 (a) -7 (b) 74 (c) 4 (d) 142 (e) $-30\frac{15}{16}$ (f) $1\frac{3}{16}$

8 14 **9** $a = 3, b = 8$ **10** $m = 2, n = 1$ **11** 11

12 $a = 6, b = -2$ **13** $a = 4, b = -9$ **14** $a = 3$, remainder $= -5$

EXERCISE 2.3

1 (a) correct (b) incorrect (c) correct (d) incorrect

2 (a) $(x+1)(x-2)(x+5)$ (b) $(x+1)(x+7)(x-6)$

(c) $(x-2)(x+4)(2x-3)$ (d) $(x-2)(3x+2)(2x+1)$

(e) $(x+1)(3x+1)(x+2)$ (f) $(x+4)(2x+3)(x-2)$

3 (a) $(x-5)(2x+1)(3x-1)$ (b) $(x+3)(2x+1)(2x-1)$

(c) $(x+4)(3x-2)(2x+5)$ (d) $(2x+1)(3x+2)(x-6)$

(e) $x(x-2)(x-3)$ (f) $(x+2)(x-2)(x+2)$

4 (a) $(x-1)(3x-1)(2x+1)$ (b) $x(3-2x)(3x+1)$

(c) $(x-1)(2x-1)(x-2)$ (d) $x(2x-3)(x+3)$

(e) $(1+x)(2-x)(3+x)$ (f) $(x+1)(x+2)(x-4)$

5 D **6** $a = 7, b = 12$ **7** $p = -1, q = 30$

8 $a = 8, b = -4$, or $a = -\frac{1568}{25}$, $b = 2.8$

9 $a = -1, b = 16$, $(2x-1)(3x-2)$ **10** $a = -1, b = -6$

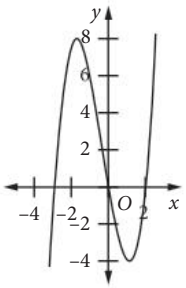
11 $(x+2)\left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)$

12 (a) $a = 2, b = 6$ (b) $2x + 6$ **13** $a = 2, c = -2; \beta = -5$ **14** 2

EXERCISE 2.4

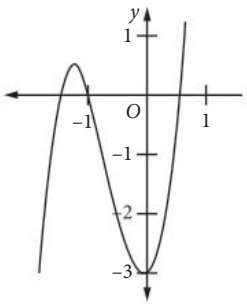
1 (a) (i) 0, 2, −3

(ii) $-3 < x < 0, x > 2$

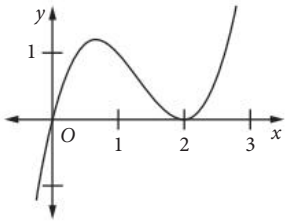


(b) (i) −1.5, −1, 0.5

(ii) $-1.5 < x < -1, x > 0.5$



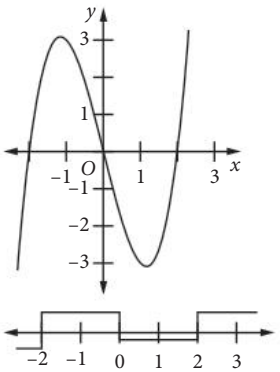
(c)



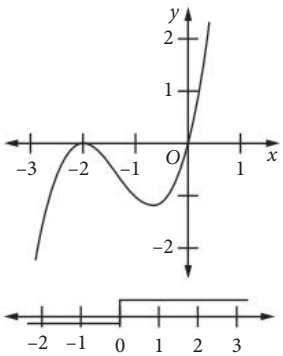
(i) 0, 2

(ii) $0 < x < 2, x > 2$

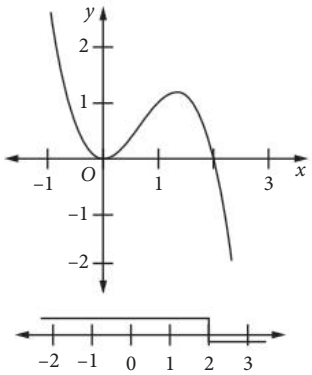
2 (a) $-2 \le x \le 0, x \ge 2$



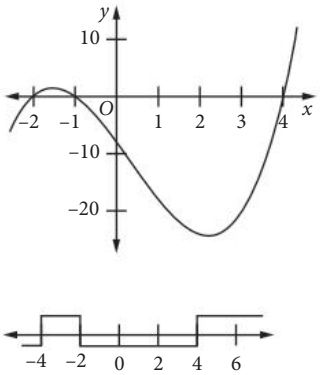
(b) $x \ge 0, x = -2$



(c) $x \le 2$



(d) $-2 \le x \le -1, x \ge 4$

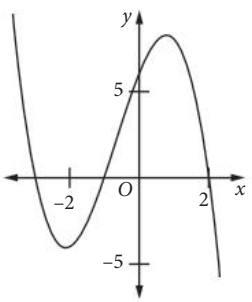


3 (a) $(x+1)(x+3)(2-x)$

(b) (i) −1, −3, 2

(ii) $x < -3, -1 < x < 2$

(c)

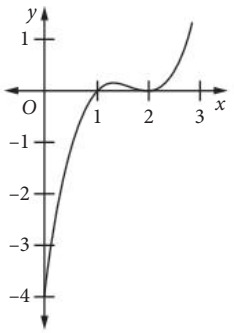


4 (a) $(x-1)(x-2)(x-2)$

(b) (i) 1, 2

(ii) $1 < x < 2, x > 2$

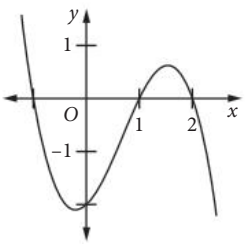
(c)



5 (a) $(1-x)(x+1)(x-2)$

(b) $x < -1, 1 < x < 2$

(c)



6 D

7 (a) Shift the graph horizontally to the left 3 units (parallel to the x-axis) so that it cuts the x-axis at $x = -3$.

(b) Shift the graph horizontally to the left 3 units (parallel to the x-axis), then move it down 2 units (parallel to the y-axis). The graph now cuts the y-axis at $y = 25$.

8 $x^3 - 8 = (x-2)(x^2+2x+4) = 0$ gives $x = 2$.

For $x^2 + 2x + 4 = 0$, $\Delta = 4 - 16 < 0$, no real roots.

Hence graph of f only cuts x-axis at $x = 2$.

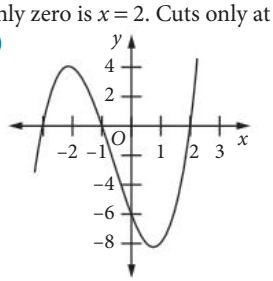
9 $f(x) = (x+3)(x-2)^2$

Cuts x-axis at $x = -3$, touches at $x = 2$.

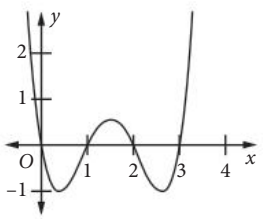
10 $f(x) = (x-2)(x^2-2x+4)$. $\Delta = -12$.

Only zero is $x = 2$. Cuts only at $x = 2$.

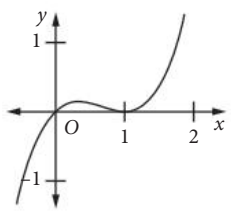
11 (a)



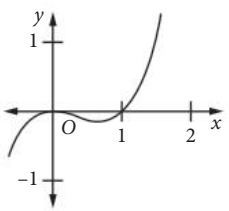
(b)



(c)



(d)



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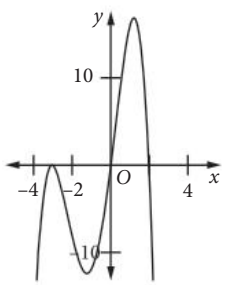

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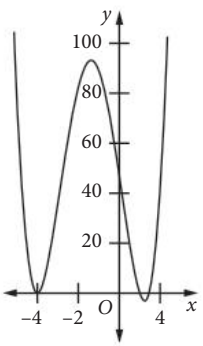
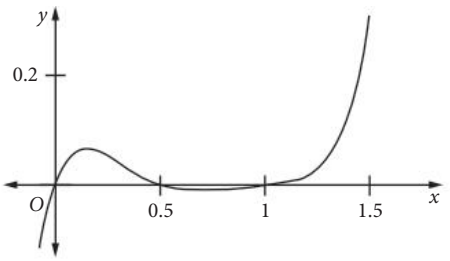
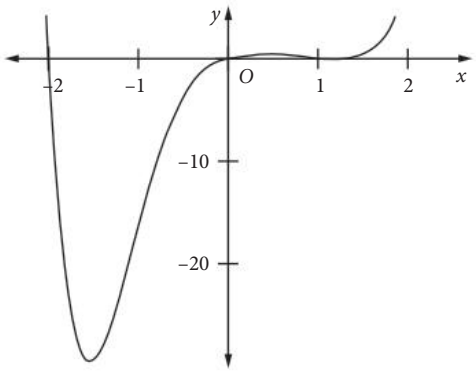


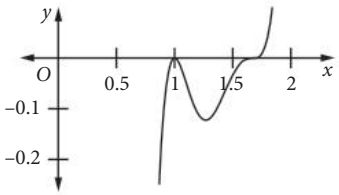
(g)



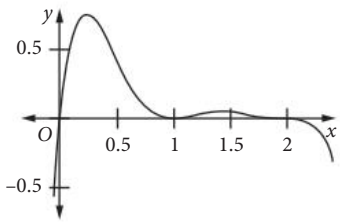
(h)



(i)

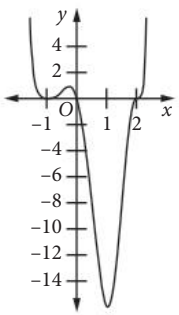


(j)

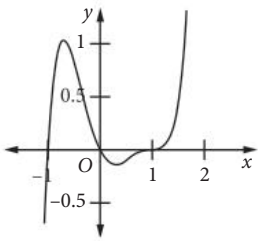


12 B

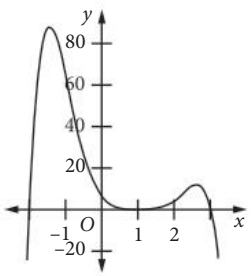
13 (a)



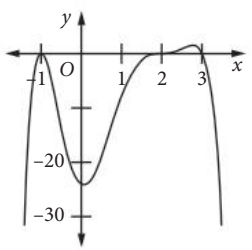
(b)



(c)



(d)



EXERCISE 2.5

1 $5x^2 + 74x + 5 = 0$

2 (a) −4 (b) 1 (c) 14 (d) −52

3 $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ **4** (a) $-\frac{q}{r}$ (b) $\frac{q^2 - 2pr}{r^2}$ **5** C

6 $x = -2, 2, 3$ **7** $x = 0.5, 2, 8$ **8** $x = -1.5, -2.5, -4$

9 $x = 6, \frac{2}{3}, -1$ **10** $x = \frac{5}{2}, \frac{5(2 \pm \sqrt{3})}{2}$

11 $m = -12, -20$; $1 \pm \sqrt{5}$, $-3 \pm \sqrt{5}$ **12** $\frac{1}{2}, \frac{2}{3}$

13 Let the roots be α, α, β.

$2\alpha + \beta = 0$ so $\beta = -2\alpha$

$\alpha^2 + 2\alpha\beta = q$ so $\alpha^2 - 4\alpha^2 = q$, $-3\alpha^2 = q$, $\alpha^2 = -\frac{q}{3}$

$\alpha^2\beta = -r$ so $-2\alpha^3 = -r$, $\alpha^3 = \frac{r}{2}$

Hence $(\alpha^2)^3 = -\frac{q^3}{27}$ so $\alpha^6 = -\frac{q^3}{27}$

and $(\alpha^3)^2 = \frac{r^2}{4}$ so $\alpha^6 = \frac{r^2}{4}$

so $-\frac{q^3}{27} = \frac{r^2}{4}$ or $4q^3 + 27r^2 = 0$

14 (a) 9 (b) −2

15 $x = 0.5, 0.5, 2$ **16** $\frac{4}{3}, -\frac{3}{2}, 1 \pm \sqrt{3}$

17 $\frac{4}{3}, \frac{3}{2}, 1 \pm \sqrt{2}$ **18** $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$

19 $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$ **20** $\frac{2}{3}$

21 $\sqrt{2}, -\sqrt{2}, 1+\sqrt{2}, 1-\sqrt{2}$ **22** $1+\sqrt{2}, 1-\sqrt{2}$

CHAPTER REVIEW 2

1 (a) $x^2 + 3x + \frac{4}{x-1}$ (b) $4x^3 - 12x^2 + 30x - 80 + \frac{200}{x+3}$

2 (a) −7 (b) −11

3 (a) $(x-1)(x+2)(x-3)$ (b) $(x+1)(x+2)(x+4)$

(c) $(x-1)(x+1)(x+5)$ (d) $(x-2)(x+2)(x+3)(x-4)$

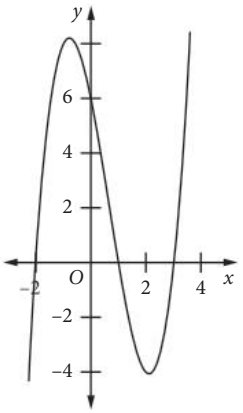
4 (a) $a = 2, b = 4$ (b) $2x + 4$ **5** $a = 2$, $P(-3) = -5$

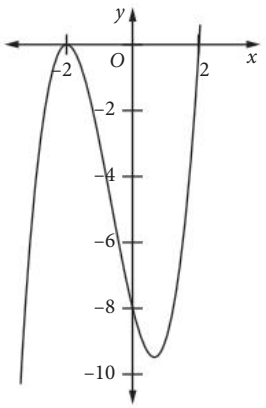
6 $a = -4, b = -2$; $P(3) = 17$ **7** $a = 3, c = -6$; $\alpha = -4$

8 (a) 1 (b) 0

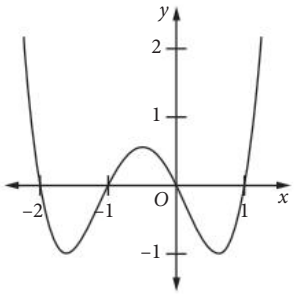
9 (a) 4 (b) −12 (c) −3

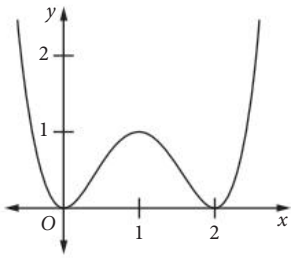
10 (a) $-2 < x < 1, x > 3$ (b) $x > 2$





(c) $x < -2, -1 < x < 0, x > 1$ (d) all $x, x \neq 0, 2$





CHAPTER 3

EXERCISE 3.1

1 $5 \times 2 = 10$ **2** 72 **3** 5040 **4** 4^{20}

5 45 **6** A **7** $26^3 \times 10^2 \times 26 = 45\,697\,600$

8 $6! = 720$ **9** (a) 60 (b) 120

10 (a) 45 697 600
(b) Answers will vary. It creates about 28 million more possible number plates.
11 40 320 **12**(a) $9 \times 9! = 3\,265\,920$ (b) 9×10^9

EXERCISE 3.2

1 (a) 60 (b) 12 (c) 720 (d) 1680 (e) 9 (f) 3 628 800
(g) $n(n-1)$ (h) $n(n-1)(n-2)(n-3)$
2 (a) 120 (b) 24 (c) 40 320 (d) 5040 (e) 60 (f) 7
(g) 90 (h) $n(n-1)$
3 120 **4** C **5** 336
6 (a) $2 \times 4! = 48$ (b) ${}^3P_2 \times 3! = 36$
7 (a) $4! \times 5! = 2880$ (b) ${}^4P_2 \times 6! = 8640$ (c) $2 \times 4! \times 4! = 1152$
8 (a) 720 (b) $4! \times 3! = 144$ **9** 720, 360
10 (a) incorrect (b) correct (c) incorrect (d) correct
11 $2 \times 5! \times 5! = 28\,800$ **12** (a) 12 (b) 128 **13** 216
14 (a) 120 (b) 2880 **15** $5! \times 4! \times 2! \times 3! = 34\,560$
16 30 240 **17** (a) 1440 (b) 2520 (c) 720
18 (a) 625 (b) 1250 (c) 1875
(d) 625 (number ends in 12, 24, 32, 44, 52)
19 288 **20** $120 = 6 \times 5 \times 4$, $r = 3$ **21** 720
22 (a) 48 (b) 72 **23** 3600 **24** 144
25 3 **26** 4
27 $\text{RHS} = \dfrac{n!}{(n-r)!} + r \times \dfrac{n!}{(n-(r-1))!}$

$= \dfrac{n!}{(n-r)!} + \dfrac{r \times n!}{(n+1-r)!}$

$= \dfrac{n!}{(n+1-r)!}[n+1-r+r]$

$= \dfrac{n!}{(n+1-r)!} \times (n+1) = \dfrac{(n+1)!}{(n+1-r)!} = {}^{n+1}P_r = \text{LHS}$

28 $\text{RHS} = \dfrac{(n-2)!}{(n-2-r)!} + 2r \times \dfrac{(n-2)!}{(n-2-(r-1))!} + \dfrac{r(r-1)(n-2)!}{(n-2-(r-2))!}$

$= (n-2)!\left[\dfrac{1}{(n-r-2)!} + \dfrac{2r}{(n-r-1)!} + \dfrac{r(r-1)}{(n-r)!}\right]$

$= \dfrac{(n-2)!}{(n-r)!}[(n-r)(n-r-1) + 2r(n-r) + r(r-1)]$

$= \dfrac{(n-2)!}{(n-r)!}[n^2 - 2nr - n + r^2 + r + 2nr - 2r^2 + r^2 - r]$

$= \dfrac{(n-2)!}{(n-r)!} \times n(n-1) = \dfrac{n!}{(n-r)!} = {}^nP_r = \text{LHS}$

29 2880; 864

EXERCISE 3.3

1 60 **2** D **3** (a) $\dfrac{9!}{2! \times 2!} = 90\,720$ (b) 20 160
4 (a) $\dfrac{{}^4P_2 \times 5!}{2!} = 720$ (b) 72 **5** (a) 181 440 (b) 1440
6 (a) $\dfrac{{}^7P_2 \times 8!}{3! \times 2!} = 141\,120$ (b) 20 160
7 (a) incorrect (b) incorrect (c) correct (d) correct
8 37 800 **9** 35 **10** (a) 1680 (b) 140
11 840 **12** $2(n-1)!$ **13** (a) 10! (b) $10! - 2 \times 8!$
14 $\text{RHS} = \dfrac{n!}{(n-r)!} + r \times \dfrac{n!}{(n-(r-1))!}$

$= \dfrac{n!}{(n+1-r)!}[n+1-r+r]$

$= \dfrac{n!}{(n+1-r)!} \times (n+1) = \dfrac{(n+1)!}{(n+1-r)!} = {}^{n+1}P_r = \text{LHS}$

15 4^4 or $4 + 48 + 36 + 144 + 24 = 256$

EXERCISE 3.4

1 (a) 21 (b) 9 (c) 70 (d) 66 (e) 495 (f) 15 (g) 1
(h) 15 (i) $\dfrac{n(n-1)}{2}$ (j) $\dfrac{n(n-1)(n-2)(n-3)}{24}$ (k) 1
(l) $\dfrac{n(n-1)(n-2)\quad(n-k+1)}{k!}$
2 (a) 252 (b) 167 960 (c) 3 268 760
(d) $\dfrac{n(n-1)(n-2)(n-3)(n-4)}{120}$ (e) 3003 (f) 31 824
(g) 12 870 (h) 17 296
3 ${}^8C_3 = 56$ **4** 2 598 960 **5** 125 970
6 A **7** ${}^5C_2 \times {}^4C_3 = 40$ **8** 20 **9** 1470
10 (a) incorrect (b) incorrect (c) correct (d) correct
11 ${}^8C_5 \times 1 = 56$ **12** (a) ${}^8C_4 = 70$ (b) ${}^8C_6 + {}^8C_5 + {}^8C_5 = 140$
13 (a) $\binom{52}{3} - \binom{48}{3} = 4804$ (b) $\binom{48}{3} + 4\binom{48}{2} = 21\,808$
14 700 **15** $5 \times \binom{10}{3}\binom{8}{3} + 5 \times \binom{10}{4}\binom{8}{2} + 5 \times \binom{10}{5}\binom{8}{1} = 73\,080$
16 (a) 280 (b) 742 **17** 595 **18** 98
19 (a) 720 (b) 45 **20** 1946 **21** 466
22 375 **23** (a) 336 (b) $336 \times 5! = 40\,320$
24 ${}^6C_4 \times {}^5C_3 \times 7! = 756\,000$, ${}^6C_4 \times {}^5C_3 \times 4! \times 4! = 86\,400$
25 7200, ${}^5C_3 \times {}^4C_2 \times 3 \times 4! = 4320$ **26** 1440
27 $(n-4)(n-5) = 30$, $n = 10$ **28** ${}^9C_4 \times {}^5C_3 \times 1 = 1260$
29 ${}^8C_3 + {}^8C_2 + {}^8C_1 = 92$
30 ${}^8C_2 \times {}^6C_2 \times {}^4C_2 \times 1 \div 4! = 105$ **31** $2^n - 2$
32 (a) $\text{LHS} = \dfrac{n!}{(n-r)!r!} + \dfrac{n!}{(n-r-1)!(r+1)!}$

$= \dfrac{n!}{(n-r-1)!r!}\left[\dfrac{1}{n-r} + \dfrac{1}{r+1}\right]$

$= \dfrac{n!}{(n-r-1)!r!}\left[\dfrac{r+1+n-r}{(n-r)(r+1)}\right]$

$= \dfrac{(n+1) \times n!}{(n-r) \times (n-r-1)! \times (r+1) \times r!}$

$= \dfrac{(n+1)!}{(n-r)! \times (r+1)!} = {}^{n+1}C_{r+1} = \text{RHS}$

(b)
$\text{LHS} = \dfrac{n!}{(n-k)!k!} + 2 \times \dfrac{n!}{(n-k+1)!(k-1)!} + \dfrac{n!}{(n-k+2)!(k-2)!}$

$= \dfrac{n!}{(n-k+2)!k!}[(n-k+2)(n-k+1) + 2k(n-k+2) + k(k-1)]$

$= \dfrac{n!}{(n+2-k)!k!}[n^2 + 3n + 2]$

$= \dfrac{(n+2)(n+1) \times n!}{(n+2-k)!k!} = \dfrac{(n+2)!}{(n+2-k)!k!} = {}^{n+2}C_k = \text{RHS}$

(c) $\text{RHS} = \dfrac{n-r+1}{r} \times \dfrac{n!}{(n-r+1)!(r-1)!} = \dfrac{n!}{(n-r)!r!} = {}^nC_r = \text{LHS}$

(d)
$\text{RHS} = \dfrac{m!}{(m-3)!3!} + \dfrac{m!}{(m-2)!2!} \times n + m \times \dfrac{n!}{(n-2)!2!} + \dfrac{n!}{(n-3)!3!}$

$= \dfrac{m(m-1)(m-2)}{6} + \dfrac{m(m-1) \times n}{2} + \dfrac{m \times n(n-1)}{2} + \dfrac{n(n-1)(n-2)}{6}$

$= \dfrac{1}{3!}[m^3 - 3m^2 + 2m + 3m^2n - 3mn + 3mn^2 - 3mn + n^3 - 3n^2 + 2n]$

$= \dfrac{1}{3!}[m^3 + n^3 - 3m^2 - 6mn - 3n^2 + 3m^2n + 3mn^2 + 2m + 2n]$

$= \dfrac{1}{3!}[m^3 + n^3 - 3(m+n)^2 + 3mn(m+n) + 2(m+n)]$

$= \dfrac{1}{3!}[(m+n)(m^2 - mn + n^2) - 3(m+n)^2 + 3mn(m+n) + 2(m+n)]$

$= \dfrac{(m+n)}{3!}[(m+n)^2 - 3(m+n) + 2]$

$= \dfrac{(m+n)(m+n-1)(m+n-2)}{3!} = {}^{m+n}C_3 = \text{LHS}$

(e) $\text{RHS} = \frac{n!}{(n-r)!r!} + 3\times\frac{n!}{(n-r+1)!(r-1)!} + 3\times\frac{n!}{(n-r+2)!(r-2)!} + \frac{n!}{(n-r+3)!(r-3)!}$

$= \frac{n!}{(n-r+1)!r!}[n-r+1+3r] + \frac{n!}{(n-r+3)!(r-2)!}[3(n-r+3)+r-2]$

$= \frac{n!}{(n-r+3)!r!}[(n+2r+1)(n-r+3)(n-r+2)+(3n-2r+7)(r)(r-1)]$

$= \frac{n!}{(n-r+3)!r!}[n^3+6n^2+11n+6]$

$= \frac{n!(n+1)(n+2)(n+3)}{(n-r+3)!r!} = \frac{(n+3)!}{(n+3-r)!r!} = {}^{n+3}C_r = \text{LHS}$

33 5

CHAPTER REVIEW 3

1 tree diagram **2** (a) 24 (b) 256 **3** 120
4 (a) ${}^7P_4 = 840$ (b) $7^4 = 2401$
5 (a) $3\times{}^4P_3 + 4\times 4! = 168$
(b) $3\times 5^3 + 4\times 5^4 - 1 = 2874$ (c) 108
6 (a) 30 240 (b) 240 **7** 210 **8** 3003
9 ${}^8C_4\times{}^9C_4 = 8820$ **10** (a) 142 506 (b) 20 **11** 48

CHAPTER 4

EXERCISE 4.1

1 20.06 cm, 122.5 cm^2 **2** 39.97 m, depression 0° 56′
3 212.3 m **4** B, C
5 (a) 30 m, 40 m (b) $10\sqrt{7}$ m
6 $\frac{h}{y} = \tan\beta$ so $y = h\cot\beta$

$\frac{h}{x+y} = \tan\alpha$ so $h = (x+y)\tan\alpha$

$h = x\tan\alpha + h\tan\alpha\cot\beta$

$x = h(\cot\alpha - \cot\beta)$

7 $h = y\tan 62°$, $h+5 = y\tan 68°$

$h+5 = \frac{h\tan 68°}{\tan 62°}$

$h = \frac{5\tan 62°}{\tan 68° - \tan 62°}$

8 Area of triangular cross-section $= \frac{1}{2}a\times a\cot\theta = \frac{1}{2}a^2\cot\theta$

Depth of water $= \frac{\text{Area}}{\text{Base}} = \frac{a^2\cot\theta}{2b}$

9 $AC = OC\tan\alpha$, $CG = \frac{1}{3}OC$

$\tan\angle AGC = \frac{AC}{CG} = 3\tan\alpha$

$\angle AGC = \alpha + \beta$

$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = 3\tan\alpha$

$\tan\alpha + \tan\beta = 3\tan\alpha - 3\tan^2\alpha\tan\beta$

$\tan\beta = \frac{2\tan\alpha}{1+3\tan^2\beta}$

10 A **11** 0.9067 km **12** 70° 32′
13 16 m **14** 71 km/h
15 $x = \frac{h}{\tan 47°}$, $40 - x = \frac{h}{\tan 28°}$

$40 - \frac{h}{\tan 47°} = \frac{h}{\tan 28°}$

$h = \frac{40\tan 47°\tan 28°}{\tan 47° + \tan 28°}$

16 $\frac{h-2}{2\tan 75°} = \tan 30°$

$h = 2(1+\tan 30°\tan 75°)$ metres

17 131 m
18 C is foot of tower.
$AC = h\tan 70°$, $BC = h\tan 72°$
$AC^2 + BC^2 = 100^2$
$h^2(\tan^2 70° + \tan^2 72°) = 100^2$

$h = \frac{100}{\sqrt{\tan^2 70° + \tan^2 72°}}$

19 26.55 cm
20 $AC^2 = 8^2 + 7^2 - 2\times 8\times 7\cos\theta$
$AC^2 = 5^2 + 6^2 - 2\times 5\times 6\cos(180° - \theta)$
$113 - 112\cos\theta = 61 + 60\cos\theta$
$\cos\theta = \frac{52}{172} = \frac{13}{43}$

21 6488 m **22** 147.4 m
23 3.881 km
24 $\frac{x}{\sin(\beta-\alpha)°} = \frac{h}{\sin\beta°}\times\frac{1}{\sin\alpha°}$

$h = \frac{x\sin\alpha°\sin\beta°}{\sin(\beta-\alpha)°}$

25 $\frac{x}{\sin 30°} = \frac{AC}{\sin 45°}$, $AC = \sqrt{2}x$

$\frac{AC}{\sin 60°} = \frac{y}{\sin 45°}$, $y = \frac{2AC}{\sqrt{6}}$

$y = \frac{2}{\sqrt{6}}\times\sqrt{2}x$, $\frac{x}{y} = \frac{\sqrt{3}}{2}$

Area $\Delta ABC = \frac{1}{2}x\times AC\sin 105° = \frac{1}{2}x\times AC\sin 75°$

Area $\Delta ADC = \frac{1}{2}y\times AC\sin 75°$

$\frac{\text{Area }\Delta ABC}{\text{Area }\Delta ADC} = \frac{x}{y} = \frac{\sqrt{3}}{2}$

26 (a) 34.2 m (b) 18° 32′ (c) 73°
27 (a) $\angle NBQ = 50°$, $\angle AQB = 140°$ (exterior angle of triangle)
(b) $\frac{AQ}{h} = \tan 77°$, $AQ = h\tan 77°$
(c) $\frac{BQ}{h} = \tan 80°$, $BQ = h\tan 80°$
(d) $1000^2 = h^2\tan^2 77° + h^2\tan^2 80° - 2h^2\tan 77°\tan 80°\cos 140°$

$h = \frac{1000}{\sqrt{\tan^2 77° + \tan^2 80° - 2\tan 77°\tan 80°\cos 140°}}$

$h = 106.3$ m

28 $\angle PCQ = 33°$, $PC = BC\tan 68°$, $QC = BC\tan 71°$
$2000^2 = BC^2(\tan^2 68° + \tan^2 71° - 2\tan 68°\tan 71°\cos 33°)$

$BC = \frac{2000}{\sqrt{\tan^2 68° + \tan^2 71° - 2\tan 68°\tan 71°\cos 33°}}$

$BC = 1264$ m
29 (a) Angle between directions is 78° (b) $x = h\tan 79°$
(c) $y = h\tan 82°$ (d) $h = 381.4$ m

EXERCISE 4.2

1 (a) $\sin A\cos 2B+\cos A\sin 2B$ (b) $\sin 2x\cos y-\cos 2x\sin y$
(c) $\cos 2x\cos 3y+\sin 2x\sin 3y$ (d) $\frac{1}{2}\cos 2\theta-\frac{\sqrt{3}}{2}\sin 2\theta$
(e) $\dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$ (f) $\dfrac{\tan A+1}{1-\tan A}$

2 (a) $\sin(2A-B)$ (b) $\cos 2\alpha$ (c) $\sin A$ (d) 0
(e) $\tan(\theta-20°)$ (f) $\tan 3\alpha$ (g) $\sin A$ (h) $\cos\theta$ (i) $\tan 2x$

3 D **4** (a) $\sin 60°=\frac{\sqrt{3}}{2}$ (b) $\tan 135°=-1$
(c) $\dfrac{-1-\sqrt{3}}{2\sqrt{2}}$ (d) $2\sin x\cos 40°$

5 $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\cos(\alpha-\beta)=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)$
$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

6 $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$
$\cos(90°-\theta-\phi)=\cos(90°-\theta)\cos\phi+\sin(90°-\theta)\sin\phi$
$\cos(90°-(\theta+\phi))=\sin\theta\cos\phi+\cos\theta\sin\phi$
$\sin(\theta+\phi))=\sin\theta\cos\phi+\cos\theta\sin\phi$

7 (a) $\frac{44}{125}$ (b) $\frac{3}{5}$ (c) $\frac{44}{117}$ **8** A **9** (a) $\frac{24}{25}$ (b) $-\frac{24}{7}$ (c) $\frac{56}{65}$

10 (a) $\sin(45°+30°)=\sin 45°\cos 30°+\cos 45°\sin 30°$
$\sin 75°=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}$
(b) $\tan(45°+30°)=\dfrac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$
$\tan 75°=\dfrac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}=\dfrac{(\sqrt{3}+1)^2}{3-1}=\dfrac{4+2\sqrt{3}}{2}=2+\sqrt{3}$

11 (a) $\dfrac{\sqrt{6}-\sqrt{2}}{4}$ (b) $2-\sqrt{3}$ (c) $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

12 (a) $-\frac{3\sqrt{7}}{8}$ (b) $-\frac{1}{8}$ (c) $3\sqrt{7}$ (d) 3rd quadrant

13 (a) $3\sin\theta-4\sin^3\theta$ (b) $4\cos^3\theta-3\cos\theta$ (c) $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

14 $\dfrac{\sqrt{2}\left(1-\sqrt{3}\right)}{4}$

15 (a) $\tan A$ (b) $\sin^2\theta$ (c) $\cos 4\theta$ (d) $\frac{1}{2}$ (e) $\frac{1}{2}\sin 8x$
(f) $2\sin^2\theta$ (g) $\frac{1}{4}\sin 4x$ (h) $\sin 4x$ (i) $1+\sin 2\theta$ (j) $1-\sin 2A$
(k) 1 (l) 1 (m) $\frac{1}{2}\cos 2x$ (n) $\tan^2\theta$ (o) $\cos 6x$

16 LHS $=\dfrac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\cos 2\theta\cos\theta+\sin 2\theta\sin\theta}$
$=\dfrac{\sin(2\theta-\theta)}{\cos(2\theta-\theta)}=\dfrac{\sin\theta}{\cos\theta}=\tan\theta=$ RHS

17 LHS $=\dfrac{\sin A+\cos A\tan B}{\cos A-\sin A\tan B}=\dfrac{\frac{\sin A}{\cos A}+\tan B}{1-\frac{\sin A}{\cos A}\tan B}$
$=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\tan(A+B)=$ RHS

18 LHS $=\sin(\theta+\alpha)\sin(\theta-\alpha)$
$=(\sin\theta\cos\alpha+\cos\theta\sin\alpha)(\sin\theta\cos\alpha-\cos\theta\sin\alpha)$
$=\sin^2\theta\cos^2\alpha-\cos^2\theta\sin^2\alpha$
$=\sin^2\theta(1-\sin^2\alpha)-(1-\sin^2\theta)\sin^2\alpha$
$=\sin^2\theta-\sin^2\theta\sin^2\alpha-\sin^2\alpha+\sin^2\theta\sin^2\alpha$
$=\sin^2\theta-\sin^2\alpha=$ RHS

19 LHS $=\dfrac{\tan A-\tan B}{\tan A+\tan B}=\dfrac{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}$
$=\dfrac{\sin A\cos B-\cos A\sin B}{\sin A\cos B+\cos A\sin B}=\dfrac{\sin(A-B)}{\sin(A+B)}=$ RHS

20 LHS $=\tan(45°+A)+\tan(45°-A)$
$=\dfrac{1+\tan A}{1-\tan A}+\dfrac{1-\tan A}{1+\tan A}$
$=\dfrac{(1+\tan A)^2+(1-\tan A)^2}{(1-\tan A)(1+\tan A)}$
$=\dfrac{1+2\tan A+\tan^2 A+1-2\tan A+\tan^2 A}{1-\tan^2 A}$
$=\dfrac{2+2\tan^2 A}{1-\tan^2 A}=\dfrac{2\sec^2 A}{1-\tan^2 A}$
$=\dfrac{2}{\cos^2 A-\sin^2 A}=\dfrac{2}{\cos 2A}=$ RHS

21 LHS $=\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}$
$=\dfrac{\sin 3A\cos A-\cos 3A\sin A}{\sin A\cos A}$
$=\dfrac{\sin(3A-A)}{\sin A\cos A}$
$=\dfrac{\sin 2A}{\sin A\cos A}=\dfrac{2\sin A\cos A}{\sin A\cos A}=2=$ RHS

22 RHS $=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}$
$=\dfrac{(\cos\theta+\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$
$=\dfrac{2\sin\theta\cos\theta+\sin^2\theta+\cos^2\theta}{\cos^2\theta-\sin^2\theta}=\dfrac{\sin 2\theta+1}{\cos 2\theta}=$ LHS

23 LHS $=\cos 4x$
$=2\cos^2 2x-1$
$=2(2\cos^2 x-1)^2-1$
$=8\cos^4 x-8\cos^2 x+1=$ RHS

24 LHS $=\dfrac{1-\cos x}{\sin x}$
$=\dfrac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$
$=\dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}=$ RHS

25 LHS $=\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\dfrac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$
$=\dfrac{(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$
$=\dfrac{2(\cos^2\theta+\sin^2\theta)}{\cos 2\theta}=\dfrac{2}{\cos 2\theta}=2\sec 2\theta=$ RHS

26 LHS $=\dfrac{\sin x+1-\cos x}{\sin x-1+\cos x}$
$=\dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}+2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}-2\sin^2\frac{x}{2}}$
$=\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}=\dfrac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}=$ RHS

27 LHS $=\sin[(A+B)+C]$
$=\sin(A+B)\cos C+\cos(A+B)\sin C$
$=(\sin A\cos B+\cos A\sin B)\cos C+(\cos A\cos B$
$-\sin A\sin B)\sin C$
$=\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B$
$-\sin A\sin B\sin C=$ RHS
$\sin(A+90°)=\cos A$

28 LHS $=\cos[(A+B)+C]$
$=\cos(A+B)\cos C-\sin(A+B)\sin C$
$=(\cos A\cos B-\sin A\sin B)\cos C-(\sin A\cos B$
$+\cos A\sin B)\sin C$
$=\cos A\cos B\cos C-\cos A\sin B\sin C-\cos B\sin C\sin A$
$-\cos C\sin A\sin B.$
$\cos(A+90°)=-\sin A$

29 LHS $=\dfrac{\sin A+\sin(90°-A)+1}{\sin A-\sin(90°-A)+1}$
$=\dfrac{\sin A+\cos A+1}{\sin A-\cos A+1}$
$=\dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}+2\cos^2\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin^2\frac{A}{2}}$
$=\dfrac{2\cos\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}=\dfrac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}=$ RHS

30 $\text{LHS} = \dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$

$= \dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$

$= 1-\sin\theta\cos\theta = 1-\frac{1}{2}\sin 2\theta = \text{RHS}$

31 $\text{LHS} = \tan(\theta+\alpha)\tan(\theta-\alpha)$

$= \dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}\times\dfrac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$

$= \dfrac{\tan^2\theta-\tan^2\alpha}{1-\tan^2\theta\tan^2\alpha} = \text{RHS}$

32 $\text{LHS} = \cot(x+y)$

$= \dfrac{1}{\tan(x+y)} = \dfrac{1-\tan x\tan y}{\tan x+\tan y}$

$= \dfrac{\frac{1}{\tan x\tan y}-1}{\frac{\tan x}{\tan x\tan y}+\frac{\tan y}{\tan x\tan y}} = \dfrac{\cot x\cot y-1}{\cot x+\cot y} = \text{RHS}$

33 $\text{LHS} = \dfrac{2\cos\frac{\theta}{2}-1-\cos\theta}{2\cos\frac{\theta}{2}+1+\cos\theta}$

$= \dfrac{2\cos\frac{\theta}{2}-2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}+2\cos^2\frac{\theta}{2}}$

$= \dfrac{2\cos\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(1+\cos\frac{\theta}{2}\right)} = \dfrac{1-\cos\frac{\theta}{2}}{1+\cos\frac{\theta}{2}} = \text{RHS}$

34 $\text{LHS} = \tan(\theta+45°)\tan(\theta-45°)$

$= \dfrac{\tan\theta+1}{1-\tan\theta}\times\dfrac{\tan\theta-1}{1+\tan\theta} = -1 = \text{RHS}$

35 $\text{LHS} = \dfrac{\tan 2\theta-\tan\theta}{\tan 2\theta+\cot\theta}$

$= \left(\dfrac{2\tan\theta}{1-\tan^2\theta}-\tan\theta\right)\times\dfrac{1}{\frac{2\tan\theta}{1-\tan^2\theta}+\frac{1}{\tan\theta}}$

$= \tan\theta\left(\dfrac{2-1+\tan^2\theta}{1-\tan^2\theta}\right)\times\dfrac{\tan\theta\left(1-\tan^2\theta\right)}{2\tan^2\theta+1-\tan^2\theta}$

$= \dfrac{\tan^2\theta\left(1+\tan^2\theta\right)}{1+\tan^2\theta} = \tan^2\theta = \text{RHS}$

36 $\text{LHS} = \dfrac{1-\tan\theta\tan 2\theta}{1+\tan\theta\tan 2\theta}$

$= \dfrac{1-\frac{\tan\theta\times 2\tan\theta}{1-\tan^2\theta}}{1+\frac{\tan\theta\times 2\tan\theta}{1-\tan^2\theta}}$

$= \dfrac{1-\tan^2\theta-2\tan^2\theta}{1-\tan^2\theta+2\tan^2\theta}$

$= \dfrac{1-3\tan^2\theta}{1+\tan^2\theta}$

$= \left(1-\dfrac{3\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta$

$= \cos^2\theta-3\sin^2\theta = 4\cos^2\theta-3 = \text{RHS}$

37 $A+B+C = 180°$

Hence $\sin(A+B+C) = \sin 180° = 0$

From question **28** we have:

$\sin(A+B+C) = \sin A\cos B\cos C+\sin B\cos C\cos A$
$+\sin C\cos A\cos B-\sin A\sin B\sin C$

So: $\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B$
$-\sin A\sin B\sin C = 0$

$\sin C\cos A\cos B-\sin A\sin B\sin C = -(\sin A\cos B\cos C$
$+\sin B\cos C\cos A)$

$\sin C(\cos A\cos B-\sin A\sin B) = -\cos C(\sin A\cos B$
$+\sin B\cos A)$

$\sin(180°-(A+B))(\cos A\cos B-\sin A\sin B) = -\cos C\sin(A+B)$

$\sin(A+B)(\cos A\cos B-\sin A\sin B) = -\cos C\sin(A+B)$

Hence: $\cos A\cos B-\sin A\sin B+\cos C = 0$

38 (a) p (b) q

39 $\dfrac{1-\tan\theta}{\tan\theta(1+\tan\theta)}$ **40** $\dfrac{1+\sqrt{5}}{4}$

41 $\sqrt{2}-1$

42 $PT+TQ = PQ = x$

$PT = \dfrac{y}{\tan\alpha}$, $TQ = \dfrac{y}{\tan\beta}$

$\dfrac{y}{\tan\alpha}+\dfrac{y}{\tan\beta} = x$

$y\left(\dfrac{\tan\beta+\tan\alpha}{\tan\alpha\tan\beta}\right) = x$

$y = \dfrac{x\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$

EXERCISE 4.3

1 A **2** (a) $\tan 18°$ (b) $\cos 30° = \frac{\sqrt{3}}{2}$ (c) $\dfrac{1}{\sin^2 45°} = 2$

3 (a) $\dfrac{t^2+2t-1}{1+t^2}$ (b) $\dfrac{4+6t-4t^2}{1+t^2}$ (c) $\dfrac{2-2t-2t^2}{1+t^2}$ (d) $\dfrac{1-t^2}{2t}$

(e) $\dfrac{1-6t^2+t^4}{2t\left(1-t^2\right)}$ (f) $\dfrac{1-6t^2+t^4}{1+2t^2+t^4}$ (g) $\dfrac{1}{1+t^2}$ (h) $\dfrac{1+t^2}{1-t^2}$

(i) t^2 (j) t (k) $\dfrac{1}{t}$ (l) t

4 D **5** $t = \dfrac{-1\pm\sqrt{1+d^2}}{d}$ **6** -5

7 $t = \frac{a}{b}$, $\text{LHS} = b\cos\theta+a\sin\theta$

$= \dfrac{b(1-t^2)}{1+t^2}+\dfrac{a\times 2t}{1+t^2}$

$= \dfrac{b\left(1-\frac{a^2}{b^2}\right)+\frac{2a^2}{b}}{1+\frac{a^2}{b^2}}$

$= \dfrac{b^3-a^2b+2a^2b}{b^2+a^2}$

$= \dfrac{b(b^2+a^2)}{b^2+a^2} = b = \text{RHS}$

8 $x = \sec\theta-\tan\theta$

$= \dfrac{1+t^2}{1-t^2}-\dfrac{2t}{1-t^2} = \dfrac{(1-t)^2}{1-t^2} = \dfrac{1-t}{1+t}$

EXERCISE 4.4

1 (a) $\operatorname{cosec}^2\alpha$ (b) $\sin^2\theta$ (c) 1 (d) $\frac{1}{2}$ (e) $\cos^2\theta$ (f) $\sin(\theta+\phi)$

2 B **3** (a) $-\frac{3}{\sqrt{34}}$ (b) $-\frac{5}{\sqrt{34}}$ (c) $\frac{8}{17}$

4 (a) $\frac{\sqrt{11}}{6}$ (b) $-\frac{5\sqrt{11}}{18}$ (c) $-\frac{5\sqrt{11}}{7}$

5 (a) $\frac{15}{8}$ (b) $\frac{240}{161}$ **6** (a) $\frac{4}{5}$ (b) $-\frac{3}{5}$ (c) $-\frac{4}{3}$

7 $\tan(x-y) = \dfrac{\tan x-\tan y}{1+\tan x\tan y} = \dfrac{\frac{5}{4}-\frac{1}{9}}{1+\frac{5}{4}\times\frac{1}{9}} = \dfrac{45-4}{36+5} = 1$

$x-y = \dfrac{\pi}{4}$

8 (a) $\text{LHS} = 2\cos\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}-x\right)$

$= 2\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)\left(\cos\frac{\pi}{4}\cos x+\sin\frac{\pi}{4}\sin x\right)$

$= 2\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)\left(\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x\right)$

$= \cos^2 x-\sin^2 x = \cos 2x = \text{RHS}$

(b) $\text{LHS} = \tan\left(\theta+\frac{\pi}{4}\right)\tan\left(\theta-\frac{\pi}{4}\right)$

$= \dfrac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\tan\frac{\pi}{4}}\times\dfrac{\tan\theta-\tan\frac{\pi}{4}}{1+\tan\theta\tan\frac{\pi}{4}}$

$= \dfrac{\tan\theta+1}{1-\tan\theta}\times\dfrac{\tan\theta-1}{1+\tan\theta} = -1 = \text{RHS}$

(c) $\text{LHS} = (\sec^2\theta-1)\tan\left(\frac{\pi}{2}-\theta\right)$

$= \tan^2\theta\cot\theta = \tan\theta = \text{RHS}$

(d) LHS $= \tan\left(x+\frac{3\pi}{4}\right)$

$= \dfrac{\tan x+\tan\frac{3\pi}{4}}{1-\tan x\tan\frac{3\pi}{4}} = \dfrac{\tan x-1}{\tan x+1} = \text{RHS}$

(e) LHS $= \dfrac{1-\sin\left(\frac{\pi}{2}-2x\right)}{\sin 2x}$

$= \dfrac{1-\cos 2x}{\sin 2x}$

$= \dfrac{2\sin^2 x}{2\sin x\cos x} = \dfrac{\sin x}{\cos x} = \tan x = \text{RHS}$

(f) LHS $= \tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)$

$= \dfrac{1+\tan A}{1-\tan A}+\dfrac{1-\tan A}{1+\tan A}$

$= \dfrac{(1+\tan A)^2+(1-\tan A)^2}{1-\tan^2 A}$

$= \dfrac{2(1+\tan^2 A)}{1-\tan^2 A}$

$= \dfrac{2}{\cos 2A}$ (adapting t formula)

$= 2\sec 2A = RHS$

9 LHS $= \sin\left(\theta+\frac{\pi}{6}\right)\sin\left(\theta-\frac{\pi}{6}\right)$

$= \left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)\left(\sin\theta\cos\frac{\pi}{6}-\cos\theta\sin\frac{\pi}{6}\right)$

$= \left(\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta\right)\left(\frac{\sqrt{3}}{2}\sin\theta-\frac{1}{2}\cos\theta\right) = \frac{3}{4}\sin^2\theta-\frac{1}{4}\cos^2\theta$

$= \sin^2\theta-\frac{1}{4}\left(\sin^2\theta+\cos^2\theta\right) = \sin^2\theta-\frac{1}{4} = \text{RHS}$

10 (a) -1 (b) $-\frac{1}{2}$

11 (a) $2\sin A\cos B$ (b) $2\sin\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$

12 RHS $= \sqrt{\dfrac{1-\cos 2\theta}{1+\cos 2\theta}}$

$= \sqrt{\dfrac{2\sin^2\theta}{2\cos^2\theta}} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta = \text{LHS}$

$\tan\frac{\pi}{8} = \sqrt{\dfrac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}} = \sqrt{\dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}} = \sqrt{\dfrac{\left(\sqrt{2}-1\right)^2}{2-1}} = \sqrt{2}-1$

13 $4\tan(\alpha-\beta) = 3\tan\alpha$

$\dfrac{4(\tan\alpha-\tan\beta)}{1+\tan\alpha\tan\beta} = 3\tan\alpha$

$4\tan\alpha-4\tan\beta = 3\tan\alpha+3\tan^2\alpha\tan\beta$

$\left(3\tan^2\alpha+4\right)\tan\beta = \tan\alpha$

$\tan\beta = \dfrac{\tan\alpha}{3\tan^2\alpha+4} = \dfrac{\frac{\sin\alpha}{\cos\alpha}}{\frac{3\sin^2\alpha}{\cos^2\alpha}+4}$

$= \dfrac{\sin\alpha\cos\alpha}{3\sin^2\alpha+4\cos^2\alpha} = \dfrac{\sin 2\alpha}{6+2\cos^2\alpha} = \dfrac{\sin 2\alpha}{7+\cos 2\alpha}$

14 $\sec\theta-\tan\theta = \frac{3}{5}$

$\dfrac{1+t^2}{1-t^2}-\dfrac{2t}{1-t^2} = \dfrac{3}{5}$

$\dfrac{(1-t)^2}{1-t^2} = \dfrac{3}{5}$

$\dfrac{1-t}{1+t} = \dfrac{3}{5}$

$5-5t = 3+3t$

$t = \dfrac{1}{4}$

$\sin\theta = \dfrac{2t}{1+t^2} = \dfrac{\frac{1}{2}}{1+\frac{1}{16}} = \dfrac{8}{17}$

15 1 or $\dfrac{1}{k}$

16 LHS $= \cos^6\theta-\sin^6\theta$

$= \left(\cos^3\theta-\sin^3\theta\right)\left(\cos^3\theta+\sin^3\theta\right)$

$= (\cos\theta-\sin\theta)(\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta)(\cos\theta+\sin\theta)$

$\times(\cos^2\theta-\sin\theta\cos\theta+\sin^2\theta)$

$= (\cos^2\theta-\sin^2\theta)(1+\sin\theta\cos\theta)(1-\sin\theta\cos\theta)$

$= \cos 2\theta(1-\sin^2\theta\cos^2\theta) = \left(1-\frac{1}{4}\sin^2 2\theta\right)\cos 2\theta = \text{RHS}$

17 $\dfrac{2lm}{l^2-m^2}, \dfrac{4lm(l^2-m^2)}{(l^2+m^2)^2}$

18 $B-A = C-B$ and $A = 2B-C$

$\cos(B-A) = \cos(C-B)$

$\cos(B-A) = \cos C\cos B+\sin C\sin B$

$2\sin B\cos(B-A) = 2\sin B\cos C\cos B+2\sin B\sin C\sin B$

$2\sin B\cos(B-A) = \sin 2B\cos C+2\sin^2 B\sin C$

$2\sin B\cos(B-A) = \sin 2B\cos C+(1-\cos 2B)\sin C$

$2\sin B\cos(B-A) = \sin 2B\cos C-\cos 2B\sin C+\sin C$

$2\sin B\cos(B-A) = \sin(2B-C)+\sin C$

$2\sin B\cos(B-A) = \sin A+\sin C$

19 LHS $= 4\sin\theta\sin\left(\theta-\frac{\pi}{3}\right)\sin\left(\theta-\frac{2\pi}{3}\right)$

$= 4\sin\theta\left(\sin\theta\cos\frac{\pi}{3}-\cos\theta\sin\frac{\pi}{3}\right)\left(\sin\theta\cos\frac{2\pi}{3}-\cos\theta\sin\frac{2\pi}{3}\right)$

$= 4\sin\theta\left(\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)\left(-\frac{1}{2}\sin\theta-\frac{\sqrt{3}}{2}\cos\theta\right)$

$= 4\sin\theta\left(-\frac{1}{4}\sin^2\theta+\frac{3}{4}\cos^2\theta\right) = \sin\theta\left(3\cos^2\theta-\sin^2\theta\right)$

RHS $= \sin 3\theta = \sin(2\theta+\theta)$

$= \sin 2\theta\cos\theta+\cos 2\theta\sin\theta$

$= 2\sin\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)\sin\theta$

$= \sin\theta(3\cos^2\theta-\sin^2\theta) = \text{LHS}$

20 LHS $= (k-1)\sin(\alpha+\beta)$

$= k(\sin\alpha\cos\beta+\cos\alpha\sin\beta)-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$

$= k(\sin\alpha\cos\beta-\cos\alpha\sin\beta+2\cos\alpha\sin\beta)+(\sin\alpha\cos\beta$

$-\cos\alpha\sin\beta-2\sin\alpha\cos\beta)$

$= k\sin(\alpha-\beta)+2k\cos\alpha\sin\beta+\sin(\alpha-\beta)-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta)+\dfrac{2\tan\alpha}{\tan\beta}\cos\alpha\sin\beta-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta)+2\sin\alpha\cos\beta-2\sin\alpha\cos\beta$

$= (k+1)\sin(\alpha-\beta) = \text{RHS}$

EXERCISE 4.5

1 (a) $\frac{\pi}{3}, \frac{2\pi}{3}$ (b) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (c) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (d) $\frac{\pi}{6}, \frac{7\pi}{6}$

(e) $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$ (f) $\frac{7\pi}{6}, \frac{11\pi}{6}$

(g) $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ (h) $\frac{\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{15\pi}{8}$

2 (a) 66° 25′, 293° 35′

(b) 71° 34′, 161° 34′, 251° 34′, 341° 34′

3 C **4** (a) $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$ (b) $-\frac{7\pi}{8}, -\frac{5\pi}{8}, \frac{\pi}{8}, \frac{3\pi}{8}$

5 (a) $0, \frac{\pi}{2}, 2\pi$ (b) $0, \pi, 2\pi$ (c) $0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

(d) $\frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$ (e) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ (f) $\frac{\pi}{12}, \frac{\pi}{4}, \frac{13\pi}{12}, \frac{5\pi}{4}$

(g) $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$ (h) $\frac{\pi}{4}, \frac{5\pi}{4}$

6 (a) 128° 12′, 171° 48′, 308° 12′, 351° 48′

(b) 8° 26′, 151° 34′, 188° 26′, 331° 34′

7 D

8 (a) $\frac{\pi}{6} \le x \le \frac{5\pi}{6}$ (b) $\frac{\pi}{3} < x < \frac{5\pi}{3}$ (c) $0 < x < \pi$

(d) $0 \le x \le \frac{\pi}{6}, \frac{11\pi}{6} < x \le 2\pi$ (e) $0 \le x \le 2\pi, x \ne \frac{\pi}{2}$

(f) $0 \le x < \frac{\pi}{2}, \frac{3\pi}{2} < x \le 2\pi$ (g) $\frac{\pi}{4} < x < \frac{\pi}{2}, \frac{5\pi}{4} < x < \frac{3\pi}{2}$

(h) $0 \le x < \frac{3\pi}{4}, \frac{5\pi}{4} < x \le 2\pi$

9 (a) $\frac{\pi}{12} \le x \le \frac{5\pi}{12}$ (b) $\frac{\pi}{4} \le x \le \frac{3\pi}{4}$

10 (a) $-\pi < x \le -\frac{5\pi}{6}, -\frac{2\pi}{3} \le x \le \frac{\pi}{6}, \frac{\pi}{3} \le x < \pi$

(b) $-\pi < x < -\frac{2\pi}{3}, -\frac{\pi}{3} < x < \frac{\pi}{3}, \frac{2\pi}{3} < x < \pi$

EXERCISE 4.6

1 (a) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (b) $0, \frac{\pi}{2}, \pi, 2\pi$ (c) $0, 2\pi$

(d) $0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi$ (e) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (f) $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

2 C **3** (a) $\frac{2\pi}{3}, \frac{4\pi}{3}$ (b) $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ (c) $\frac{\pi}{2}, \frac{3\pi}{2}$ (d) $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}$

4 (a) $68°12', 108°26', 248°12', 288°26'$

(b) $90°, 216°52', 323°8'$

5 A **6** (a) $\frac{\pi}{6}, \frac{\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}$ (b) $0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$

7 (a) $t = \pm\sqrt{3}$; $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ (b) $t = 1, -3$; $\theta = \frac{\pi}{2}, 3.79$

(c) $t = 3, -2$; $\theta = 2.50, 4.07$

EXERCISE 4.7

1 (a) $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ (b) $\frac{\pi}{3}, \frac{5\pi}{3}$ (c) $0, \pi, 2\pi$

(d) $45°, 63°26', 225°, 243°26'$

2 C **3** (a) $\frac{\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{7\pi}{4}$ (b) $-\pi, -\frac{5\pi}{6}, 0, \frac{\pi}{6}, \pi$

4 C **5** $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$

6 $128°10', 231°50'$ **7** $\frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$ **8** $0, \pi, 2\pi$

EXERCISE 4.8

1 (a) $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ (b) $2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$

(c) $13\sin(x+67°23')$ (d) $\sqrt{5}\sin\left(x+26°34'\right)$

2 (a) $2\sin\left(x-\frac{\pi}{3}\right)$ (b) $\sqrt{13}\sin\left(x-56°19'\right)$

(c) $\sqrt{5}\sin\left(x-26°34'\right)$ (d) $3\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

3 (a) $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ (b) $25\cos(x-16°16')$

(c) $4\cos\left(x-\frac{\pi}{3}\right)$ (d) $\sqrt{13}\cos\left(x-33°41'\right)$

4 (a) $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$ (b) $2\cos\left(x+\frac{\pi}{6}\right)$

(c) $\sqrt{65}\cos\left(x+7°8'\right)$ (d) $\sqrt{34}\cos\left(x+30°58'\right)$

5 B

6 (a) (i) $2, \frac{5\pi}{6}$ (ii) $-2, \frac{11\pi}{6}$ (b) (i) $\sqrt{2}, \frac{\pi}{4}$ (ii) $-\sqrt{2}, \frac{5\pi}{4}$

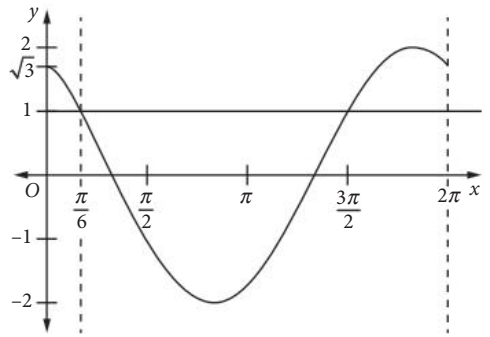
(c) (i) $4, \frac{11\pi}{6}$ (ii) $-4, \frac{5\pi}{6}$ (d) (i) $13, 22°37'$ (ii) $-13, 202°37'$

7 (a) $0, \frac{\pi}{2}, 2\pi$ (b) $\frac{\pi}{3}$ (c) $33°41'$ (d) $\frac{\pi}{3}, \pi$

(e) $156°52', 276°52'$ (f) $138°24', 295°20'$ (g) $\frac{5\pi}{3}$

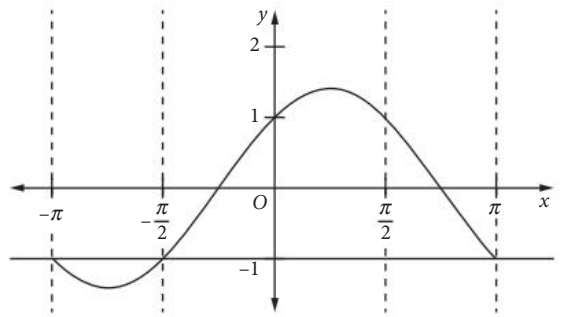
(h) $\frac{\pi}{2}, \pi$ (i) $-76°43', 150°27'$ (j) $95°16', 155°16'$

8



(a) $\frac{\pi}{6}, \frac{3\pi}{2}$ (b) $0 \le x < \frac{\pi}{6}, \frac{3\pi}{2} < x \le 2\pi$

9



(a) $-\pi, -\frac{\pi}{2}, \pi$ (b) $-\frac{\pi}{2} \le x \le \pi, x = -\pi$

EXERCISE 4.9

1 (a) $n\pi + (-1)^n \times \frac{\pi}{2}$ (b) $2n\pi \pm \frac{\pi}{2}$ (c) $n\pi - \frac{\pi}{4}$

(d) $n\pi + (-1)^n \times \frac{\pi}{3}$ (e) $2n\pi \pm \frac{2\pi}{3}$ (f) $n\pi + \frac{\pi}{6}$

(g) $n\pi + (-1)^n \times \left(-\frac{\pi}{6}\right) + \frac{\pi}{6}$ (h) $4n\pi$ (i) $n\pi \pm \frac{\pi}{4}$

(j) $n\pi + (-1)^n \times 0.4$

2 D

3 (a) $2n\pi$ (b) $n\pi$ or $n\pi + (-1)^n \times \frac{\pi}{2}$

(c) $n\pi + (-1)^n \times \frac{\pi}{6}$ or $n\pi + (-1)^n \times \left(-\frac{\pi}{2}\right)$

(d) $2n\pi \pm \frac{\pi}{2}, 2n\pi$ (e) $2n\pi \pm \pi$ (f) $n\pi \pm \frac{\pi}{6}$

(g) $2n\pi$ or $2n\pi \pm \frac{2\pi}{3}$ (h) $\frac{n\pi}{2} + (-1)^n \times \frac{\pi}{4}$ (i) $n\pi$ or $n\pi + \frac{\pi}{4}$

4 C

5 (a) $\frac{n\pi}{2} + (-1)^n \times \left(\frac{5\pi}{8}\right)$ (b) $n\pi$ (c) $n\pi + \frac{\pi}{12}$ or $n\pi - \frac{\pi}{4}$

(d) $n\pi$ or $2n\pi \pm \frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$ (e) $n\pi \pm \frac{\pi}{4}$

(f) $n\pi + (-1)^n \times \frac{\pi}{6} + \frac{\pi}{6}$

6 (a) $n \times 180° + (-1)^n \times 45° - 45°$ (b) $n \times 360° \pm 60°$

(c) $n \times 360° + 138°24'$ or $n \times 360° - 64°40'$

(d) $n \times 360°$ or $n \times 360° - 90°$ (e) $n \times 90°$

(f) $n\pi \pm \frac{\pi}{6}$ or $\frac{n\pi}{2}$

7 (a) $n\pi \pm \frac{\pi}{6}$ or $n\pi \pm \frac{\pi}{2}$ (b) $\frac{n\pi}{2}$ (c) $2n\pi \pm \frac{\pi}{3}$ (d) $n\pi$ (e) $\frac{n\pi}{2}$

(f) $n\pi \pm \frac{\pi}{12} + \frac{\pi}{6}$ (g) $n\pi + (-1)^n \times \frac{\pi}{4} + \frac{\pi}{4}$ (h) $2n\pi$ or $2n\pi \pm \frac{2\pi}{3}$

(i) $2n\pi \pm \frac{\pi}{3} - \frac{\pi}{3}$

8 (a) $n \times 180° + (-1)^n \times (-41°49')$ or $n \times 180° + (-1)^n \times 30°$

(b) $n \times 180° + 45°$ (c) $n \times 180° + (-1)^n \times 30°$

9 (a) $n\pi$ or $n\pi \pm \frac{\pi}{4}$ (b) $n\pi$ (c) $\frac{n\pi}{2} + \frac{\pi}{8}$

(d) $n\pi$ or $2n\pi \pm \frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$ (e) $\frac{n\pi}{3}$ or $2n\pi \pm \frac{\pi}{3}$

(f) $2n\pi$ or $\frac{4n\pi}{5} \pm \frac{\pi}{5}$

CHAPTER REVIEW 4

1 (a) $\cos\theta$ (b) $\tan\left(\theta - \frac{\pi}{6}\right)$

2 $13°17', 103°17', 193°17', 283°17'$

3 (a) $\cos\theta - \sqrt{3}\sin\theta = 2\cos\left(\theta + \frac{\pi}{3}\right)$ (b) $2\cos\left(\theta + \frac{\pi}{3}\right) = 1$, $\theta = \frac{4\pi}{3}$

4 (a) $\text{RHS} = \cos A\cos B\left(1 - \frac{\sin A}{\cos A} \times \frac{\sin B}{\cos B}\right)$

$= \cos A\cos B\left(\frac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}\right)$

$= \cos(A+B) = \text{LHS}$

(b) $\cos(A+B) = 0$ so $A + B = \frac{\pi}{2}$

5 (a) $\text{LHS} = \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$

$= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta}$

$= \frac{1-\sin\theta}{\cos\theta} = \sec\theta - \tan\theta = \text{RHS}$

(b) $\text{RHS} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta = \text{LHS}$

6 (a) $5\sin(x+0.927)$ (b) 0.64 (c) $n\pi + (-1)^n \times \frac{\pi}{2} - 0.927$

7 $\tan^2\frac{\pi}{8} = \frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{\left(\sqrt{2}-1\right)^2}{1}$

$\tan\frac{\pi}{8} = \sqrt{2}-1$

8 312 m **9** $\theta = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{3}, \frac{5\pi}{4}, \frac{7\pi}{4}$

10 $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

Now $8\cos^3\theta - 6\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$

Hence $\cos 3\theta = -\frac{1}{2}$, $3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, $\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

$x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$, or $x = -\cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}$

(a) sum of roots: $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$,

$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = -\cos\frac{8\pi}{9} = \cos\frac{\pi}{9}$

(b) sum of products of pairs of roots:

$-\cos\frac{\pi}{9}\cos\frac{2\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{4\pi}{9} - \cos\frac{\pi}{9}\cos\frac{4\pi}{9} = -\frac{3}{4}$

$\cos\frac{\pi}{9}\cos\frac{2\pi}{9} + \cos\frac{\pi}{9}\cos\frac{4\pi}{9} = \frac{3}{4} + \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = \dfrac{3}{4\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}} + \sec\frac{\pi}{9}$

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = \dfrac{3}{4\times\frac{1}{8}} + \sec\frac{\pi}{9}$ from product of roots,

$\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} = 6 + \sec\frac{\pi}{9}$

(c) product of roots: $\left(-\cos\frac{\pi}{9}\right)\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = -\frac{1}{8}$

So $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$, $\sec\frac{\pi}{9}\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} = 8$

(d) $\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} - \sec\frac{\pi}{9} = 6$

$\left(\sec\frac{4\pi}{9} + \sec\frac{2\pi}{9} - \sec\frac{\pi}{9}\right)^2 = 6^2$

$\sec^2\frac{4\pi}{9} + \sec^2\frac{2\pi}{9} + \sec^2\frac{\pi}{9}$

$+ 2\left(\sec\frac{2\pi}{9}\sec\frac{4\pi}{9} - \sec\frac{\pi}{9}\sec\frac{2\pi}{9} - \sec\frac{\pi}{9}\sec\frac{4\pi}{9}\right) = 36$

$1 + \tan^2\frac{4\pi}{9} + 1 + \tan^2\frac{2\pi}{9} + 1 + \tan^2\frac{\pi}{9}$

$+ 2\left(\dfrac{\cos\frac{\pi}{9} - \cos\frac{4\pi}{9} - \cos\frac{2\pi}{9}}{\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}\right) = 36$

$\tan^2\frac{\pi}{9} + \tan^2\frac{2\pi}{9} + \tan^2\frac{4\pi}{9} = 33$

11 $\tan\alpha + \tan\beta + \tan\gamma = a + 1$

$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = c - a$

$\tan\alpha\tan\beta\tan\gamma = c$

$\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha)}$

$= \dfrac{a+1-c}{1-(c-a)} = 1$

$\alpha + \beta + \gamma = n\pi + \frac{\pi}{4}$

12 (a) $\cos(2A + B) = \cos 2A\cos B - \sin 2A\sin B$

$= (2\cos^2 A - 1)\cos B - 2\sin A\cos A\sin B$

Let $A = B = \theta$:

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$

(b) $27x^3 - 9x = 1$ can be written as $\frac{27}{8}x^3 - \frac{9}{8}x = \frac{1}{8}$ or $\left(\frac{3x}{2}\right)^3 - \frac{3}{4}\times\frac{3x}{2} = \frac{1}{8}$,

so let $\frac{3x}{2} = \cos\theta$ or $x = \frac{2}{3}\cos\theta$

$\cos^3\theta - \frac{3}{4}\cos\theta = \frac{1}{8}$, $\frac{1}{4}\cos 3\theta = \frac{1}{8}$, $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$, $\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

Roots are $x = \frac{2}{3}\cos\frac{\pi}{9}, \frac{2}{3}\cos\frac{5\pi}{9}, \frac{2}{3}\cos\frac{7\pi}{9}$

Product of roots: $\frac{2}{3}\cos\frac{\pi}{9}\times\frac{2}{3}\cos\frac{5\pi}{9}\times\frac{2}{3}\cos\frac{7\pi}{9} = \frac{1}{27}$

$\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}$

Now $\cos\frac{3\pi}{9} = \cos\frac{\pi}{3} = \frac{1}{2}$ so

$\cos\frac{\pi}{9}\cos\frac{3\pi}{9}\cos\frac{5\pi}{9}\cos\frac{7\pi}{9} = \frac{1}{8}\times\frac{1}{2} = \frac{1}{16}$

CHAPTER 5

EXERCISE 5.1

1 (a) Student D

(b) Student A multiplied by $2(x-1)$ which could be positive or negative, B multiplied by $(x-1)^2$ but did not note $x \neq 1$, C took reciprocals but can't be sure that both sides are positive.

2 $-3 \le x \le 5$ **3** $0 \le x \le 1$ **4** $-2 \le x \le 3$

5 all real x **6** $x \le -\frac{2}{3}, x \ge 4$ **7** $x < -3, 1 < x < 2$

8 $x < -1, 2 < x < 5$ **9** $x \le 1$ **10** $0 < x < \frac{1}{6}$

11 $x < -\frac{1}{3}, x > \frac{1}{3}$ **12** $x < -1, x > 3$

13 $x < -3, x > -\frac{4}{3}$ **14** $x \le -3, x > -\frac{1}{2}$

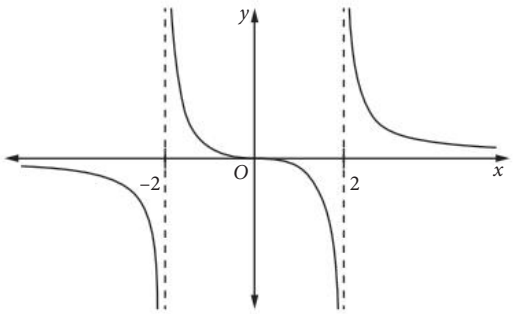
15 $x < 1, x > 3$ **16** $\frac{5}{4} < x < \frac{13}{10}$ **17** $1 < x < 3$

18 $x < -4, -3 < x < 3, x > 4$ **19** $x < -1, 0 < x < 1$

20 $x = 0$ **21** $0 \le x \le 2$ **22** $x < -3, x > \frac{10}{3}$ **23** $0 < x < 3$

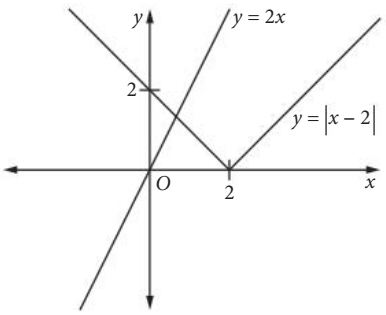
24 (a) $f(x) = \dfrac{x}{x^2-4}$, $f(-x) = \dfrac{-x}{(-x)^2-4} = -f(x)$

(b)



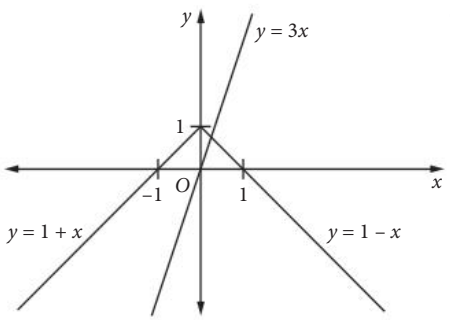
(c) $x < -2, -1 < x < 2, x > 4$

25 (a)



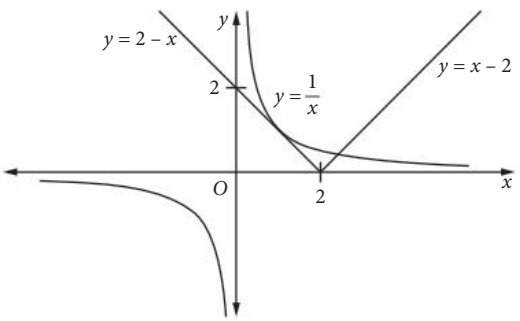
(b) $x > \frac{2}{3}$

26 (a)



(b) $x > \frac{1}{4}$

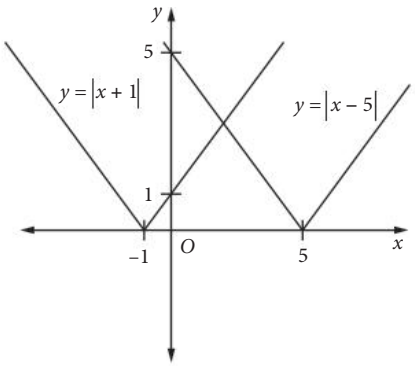
27 (a)



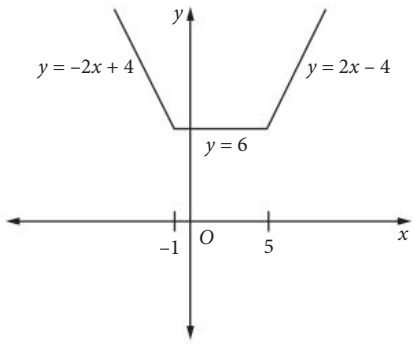
(b) $x < 0, x > 1 + \sqrt{2}$

28 $x < -1, x > 1$ **29** $-2 \le x \le 0$ **30** $x > -1$

31 (a)



(b)



(c) $x < -\frac{3}{2}, x > \frac{11}{2}$ (d) $-1 \le x \le 5$

32 $x \ge \frac{3}{2}$

EXERCISE 5.2

1 45° **2** 45° **3** 90°
4 10° 18′ **5** 40° 36′ **6** 45° **7** 2
8 $P(4,0), Q(6,1), R(4,3), S(3,2)$

$m_{PQ} = \frac{1}{2}$, $m_{QR} = -1$, $m_{RS} = 1$, $m_{SP} = -2$

$m_{QR} \times m_{RS} = -1 \times 1 = -1$ QR RS

$m_{PQ} \times m_{SP} = \frac{1}{2} \times (-2) = -1$ PQ SP

$\therefore \angle QRS + \angle QPS = 180°$

$\therefore$ $PQRS$ is a cyclic quadrilateral (pair of opposite angles supplementary)

9 D

EXERCISE 5.3

1 (a) C (b) B (c) D (d) A
2 (a) $(-2,3)$ (b) $(1,0)$ (c) $\left(\frac{7}{4}, -\frac{3}{4}\right)$
3 (a) $(4,-3)$ (b) $(4,-3)$ **4** $(3,4), (5,6)$
5 (a) $(1,2)$ (b) $\left(3, -\frac{1}{3}\right)$ (c) $\left(\frac{11}{2}, \frac{3}{2}\right)$ (d) $\left(3, -\frac{1}{3}\right)$
(e) $\left(\frac{5}{2}, -\frac{9}{2}\right)$ (f) $\left(3, -\frac{1}{3}\right)$ (g) concurrent (h) 2 : 1
6 $(1,2)$ **7** 3 : 1

EXERCISE 5.4

1 (a) $3, 7\frac{1}{2}$ (b) $-\frac{19}{7}, -\frac{13}{9}$ (c) 3 (d) $-\frac{2}{37}, \frac{3}{11}$
(e) $\frac{5 \pm \sqrt{5}}{2}$, $-1 \pm \sqrt{3}$
2 (a) $-7, -1, -4 \pm \sqrt{14}$ (b) $-1, 3$ (c) $0, 1, \frac{1 \pm \sqrt{89}}{2}$
(d) $-2, -1, 4, 5$
3 (a) $-2 \pm \sqrt{3}, \frac{7 \pm 3\sqrt{5}}{2}$ (b) $\frac{-13 \pm \sqrt{165}}{2}$
4 (a) 4 (b) 7 (c) $-\frac{26}{17}$ (d) ± 6 (e) $0, \frac{1}{4}, 4$

CHAPTER REVIEW 5

1 C
2 (a) $-1 < x < 1$ (b) $-6 \le x < -3, x > 0$ (c) $x > 0, x \ne 1$
(d) $x < -1, 0 \le x < 1$ (e) $x < -2, -1 \le x < 3, x \ge 6$ (f) $0 < x < 1$
(g) $x < -4, x > 2$ (h) $-1 - \sqrt{2} \le x \le -1 + \sqrt{2}, x = 1$
3 (a) 81° 52′ (b) 45° (c) 26° 34′
4 $\frac{-8 \pm 5\sqrt{3}}{11}$ **5** (a) $(1, -\frac{1}{3})$ (b) $(3\frac{1}{2}, -7)$
6 (a) $m_{AB} = \frac{12}{4} = 3$, $m_{BC} = \frac{-18}{-6} = 3$, $m_{AB} = m_{BC}$

Gradients equal, intervals have common point, so A, B, C are collinear.

(b) externally in ratio 2 : 3

7 (a) $3 \pm \sqrt{10}$ or $\frac{-3 \pm \sqrt{13}}{2}$ (b) $4 \pm \sqrt{21}$ or $4 \pm \sqrt{14}$ (c) ± 3
(d) (i) $a = x + \frac{1}{x}$, $a^2 = \left(x + \frac{1}{x}\right)^2$, $a^2 = x^2 + 2 + \frac{1}{x^2}$, $x^2 + \frac{1}{x^2} = a^2 - 2$
(ii) $\frac{-7 \pm 3\sqrt{5}}{2}$ or $\frac{5 \pm \sqrt{21}}{2}$

CHAPTER 6

EXERCISE 6.1

1 $2y = x + 4$ **2** $y = x^2$
3 $y = \frac{1}{x}, x \ne 0$ **4** $x^2 + y^2 = 4, -2 \le x \le 2$
5 $y = \sqrt{4 - x^2}, -2 \le x \le 2$ **6** $y = x^2 - 6x + 4, x \ge 3$
7 $2y = 3x + 8, 0 \le x \le 4$ **8** $y = 1 - x^{\frac{2}{3}}, -1 \le x \le 1$
9 $y = x^2 - 4x + 3$ **10** $2y^2 = x + 1, -1 \le x \le 1$
11 $y^2 = 8x, x \ge 0$ **12** $3x^2 + 4y^2 = 12, -2 \le x \le 2$
13 $y = \frac{1}{2}\sqrt{4 - x^2}, -2 \le x \le 2$ **14** $x^2 + y^2 = 1, -1 \le x \le 1$
15 (a) $y = 2x + 1, y = -2x + 5$
(b) No. Boat A is at $(1,3)$ when $t = 2$,
Boat B is at $(1,3)$ when $t = 3$.
16 D

EXERCISE 6.2

1 A **2** D
3 (a) $y = x - 1, x + y = 3$
(b) $2x + y + 24 = 0, x - 2y + 72 = 0$
(c) $y = 2x - 5, x + 2y - 5 = 0$
(d) $y = x + 2, x + y + 6 = 0$
4 (a) $x - 2y + 2 = 0$
(b) Distance from S to chord $= \left|\frac{0 - 4 + 2}{\sqrt{5}}\right| = \frac{2}{\sqrt{5}}$

Distance from T to chord $= \left|\frac{2 + 2 + 2}{\sqrt{5}}\right| = \frac{6}{\sqrt{5}}$

Perpendicular height of ΔABS is $\frac{1}{3}$ of perpendicular height of ΔABT, so Area $\Delta ABS = \frac{1}{3}$ Area ΔABT

5 $(1,3)$
6 (a) Tangent at Q is $y = (2p)x - a(2p)^2$, i.e. $y = 2px - 4ap^2$
(b) $px - ap^2 = 2px - 4ap^2$, $px = 3ap^2$, $x = 3ap$
$y = 3ap^2 - ap^2 = 2ap^2$, $R(3ap, 2ap^2)$
7 $(3,2)$ **8** $(3,1)$
9 Normal at P is $ty + x = 2at + at^3$
$\therefore$ G is $(0, 2a + at^2)$ and H is $(2at + at^3, 0)$
$\therefore$ $Q(u,v)$ is $(2at + at^3, 2a + at^2)$ $\therefore$ $u = vt$
10 (a) $(p-1)(p-3)(p+4)$ (b) $(2,1), (6,9), (-8,16)$
11 $3, (0,0), (\pm 2a, a)$
12 (a) $m_{AP} = \frac{ap^2}{2ap} = \frac{p}{2}$, $m_{AQ} = \frac{q}{2}$,

AP AQ so $\frac{p}{2} \times \frac{q}{2} = -1$, $pq = -4$

(b) $PQ: \frac{y - aq^2}{x - 2aq} = \frac{ap^2 - aq^2}{2ap - 2aq}$

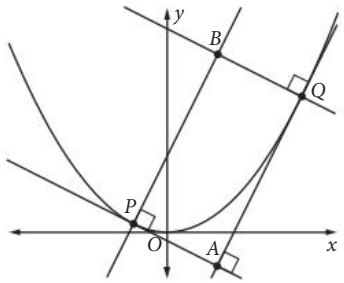
$y - aq^2 = \frac{p+q}{2}(x - 2aq)$

$y - \frac{p+q}{2}x = aq^2 - apq - aq^2$

$y - \frac{p+q}{2}x + apq = 0$

(c) $x = 0$: $y + apq = 0$, but $pq = -4$ so $y = 4a$
$\therefore$ K is $(0, 4a)$, a fixed point (as a is a constant)

13 (a)



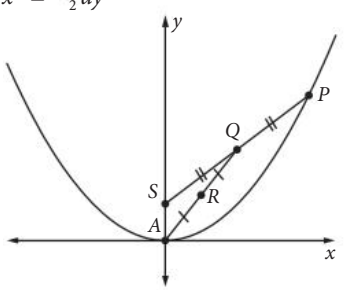
(b) $A(a(p+q), -a)$, $B(a(p+q), a(p^2+q^2+1))$
(c) x-coordinates are equal $\therefore$ $AB \parallel y$-axis

14 (a) LHS $= x_1^2 - 2x_1x_2 + x_2^2$
RHS $= x_1^2 + 2x_1x_2 + x_2^2 - 4x_1x_2 = x_1^2 - 2x_1x_2 + x_2^2 =$ LHS
(b) Chord of contact is $ax = 2a(y-a)$ or $2ay = ax + 2a^2$.
Solve with $x^2 = 4ay$ to get $x^2 = 2(ax + 2a^2)$
or $x^2 - 2ax - 4a^2 = 0$
This has roots x_1 and x_2 where $x_1 + x_2 = 2a$ (using sum of roots) and $x_1x_2 = -4a^2$ (using product of roots).
From chord of contact, make x the subject: $x = 2(y-a)$
Solve with $x^2 = 4ay$ to get $4(y^2 - 2ay + a^2) = 4ay$
or $y^2 - 3ay + a^2 = 0$
This has roots y_1 and y_2 where $y_1 + y_2 = 3a$ and $y_1y_2 = a^2$
Length of chord of contact $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
$= \sqrt{(x_1 + x_2)^2 - 4x_1x_2 + (y_1 + y_2)^2 - 4y_1y_2}$
$= \sqrt{(2a)^2 - 4(-4a^2) + (3a)^2 - 4(a^2)} = \sqrt{25a^2} = 5a$

15 Let P be $(2ap, ap^2)$ and Q be $(2aq, aq^2)$
$\therefore$ gradient of PQ is $\frac{p+q}{2}$
Focus S is $(0, a)$, A is $(ap, 0)$, B is $(aq, 0)$
Let M be midpoint of AB. M is $\left(\frac{a(p+q)}{2}, 0\right)$,
gradient of SM is $\frac{-2}{p+q}$
$\therefore$ $SM \perp PQ$ (product of gradients is -1)

EXERCISE 6.3

1 B
2 (a) $y - tx + at^2 = 0$ (b) $Q(at, 0)$ $R(0, -at^2)$
(c) $x^2 = -\frac{1}{2}ay$
3 (a)



(b) $\left(\frac{p}{2}, \frac{p^2+1}{4}\right)$ (c) $y = x^2 + \frac{1}{4}$
(d) $x^2 = y - \frac{1}{4}$: parabola, vertex $(0, 0.25)$, focus $(0, 0.5)$
4 $Q(0, 4 + 2t^2)$, locus of midpoint of PQ: $x^2 = 2(y-2)$
5 locus: $x^2 = 2a(y - \frac{a}{2})$, vertex $(0, \frac{a}{2})$, focus $(0, a)$
6 (a) $P(2ap, ap^2)$, $Q(4ap, 4ap^2)$
(b) Tangent at Q: $y - (2p)x + a(2p)^2 = 0$, $y - 2px + 4ap^2 = 0$
(c) $T(3ap, 2ap^2)$ (d) $x^2 = \frac{9}{2}ay$
7 (a) $(-2p, -3p^2)$
(b) Locus: $x = -2p$, $y = -3p^2$; $y = -3\left(\frac{-x}{2}\right)^2$, $4y = -3x^2$
8 (a) $(-2at, 4a + at^2)$ (b) $x^2 = 4a(y-4a)$
9 (a) $(-2at, 2a - at^2)$ (b) $x^2 = -4a(y-2a)$
10 (a) Chord of contact: $xx_0 = 2a(y + y_0)$, through $Q(0, 2a)$
$0 = 2a(y_0 + 2a)$, $y_0 = -2a$
(b) Midpoint of PQ: $\left(\frac{x_0}{2}, \frac{y_0 + 2a}{2}\right)$
But $y_0 = -2a$
$\therefore$ midpoint is $\left(\frac{x_0}{2}, 0\right)$
$\therefore$ locus of the midpoint of PQ is the x-axis

11 (a) $\left(p+q, \frac{p^2+q^2}{2}\right)$
(b) $m_{OP} = \frac{p}{2}$, $m_{OQ} = \frac{q}{2}$, $OP \perp OQ$, $\frac{p}{2} \times \frac{q}{2} = -1$, $pq = -4$
(c) $M\left(p+q, \frac{p^2+q^2}{2}\right)$
At M: $x = p + q$ [1]
$y = \frac{p^2+q^2}{2}$ [2]
$pq = -4$ [3]
Substitute [1], [2], [3] into $(p+q)^2 = p^2 + q^2 + 2pq$:
$x^2 = 2y - 8$
$x^2 = 2(y-4)$

12 (a) At P: $x = 2ap$, $y = ap^2$. So $\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = 2ap \times \frac{1}{2a} = p$
Equation of tangent: $y - ap^2 = p(x - 2ap)$
i.e. $y - px + ap^2 = 0$
(b) Tangent at Q: $y - qx + aq^2 = 0$
Solve simultaneously to get R:
$px - ap^2 = qx - aq^2$, $(p-q)x = a(p^2 - q^2)$, $x = a(p+q)$
$y = ap^2 + apq - ap^2 = apq$
$R(a(p+q), apq)$
(c) (i) $y = -2a$ (ii) $x = 6a$ (iii) $x^2 = 4a(y+a)$

13 (a) At P: $x = 2ap$, $y = ap^2$. So $\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = 2ap \times \frac{1}{2a} = p$
Gradient of normal: $\frac{-1}{p}$
Equation of normal: $y - ap^2 = -\frac{1}{p}(x - 2ap)$
i.e. $py + x = 2ap + ap^3$
(b) Normal at Q: $qy + x = 2aq + aq^3$
Solve simultaneously to get R:
$2ap + ap^3 - py = 2aq + aq^3 - qy$
$py - qy = 2ap - 2aq + ap^3 - aq^3$
$(p-q)y = 2a(p-q) + a(p-q)(p^2 + pq + q^2)$
$y = a(p^2 + pq + q^2 + 2)$
$x = 2ap + ap^3 - ap(p^2 + pq + q^2 + 2) = -apq(p+q)$
$R(-apq(p+q), a(p^2 + pq + q^2 + 2))$
(c) (i) $x^2 = 4a(y-4a)$ (ii) $x - 6y + 228a = 0$

14 (a) $\frac{p^2-1}{2p}, \frac{q^2-1}{2q}$
(b) $\frac{p^2-1}{2p} = \frac{q^2-1}{2q}$, $qp^2 - q = pq^2 - p$, $pq(p-q) = q - p$, $pq = -1$
(c) $\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$
(d) At R: $x = a(p+q)$ so $p + q = \frac{x}{a}$ [1]
At R: $y = \frac{a(p^2+q^2)}{2}$ $p^2 + q^2 = \frac{2y}{a}$ [2]
Also $pq = -1$ [3]
Substitute [1], [2], [3] into $(p+q)^2 = p^2 + q^2 + 2pq$:
$\left(\frac{x}{a}\right)^2 = \frac{2y}{a} - 2$
$x^2 = 2a(y-a)$

15 (a) $m_{OP} = \frac{at^2}{2at} = \frac{t}{2}$, $m_{OQ} = \frac{s}{2}$, $m_{OP} \times m_{OQ} = -1$,

$\frac{t}{2} \times \frac{s}{2} = -1$, $st = -4$

(b) $PR \parallel OQ$. Equation of PR:

$y - at^2 = \frac{s}{2}(x - 2at)$, $y - at^2 = \frac{-2}{t}(x - 2at)$

$y = \frac{-2x}{t} + 4a + at^2$

$QR \parallel OP$. Equation of QR:

$y - as^2 = \frac{t}{2}(x - 2as)$, $y - a\left(\frac{-4}{t}\right)^2 = \frac{t}{2}\left(x + \frac{8a}{t}\right)$

$y - \frac{16a}{t^2} = \frac{tx}{2} + 4a$, $y = \frac{tx}{2} + 4a + \frac{16a}{t^2}$

For R, solve: $\frac{-2x}{t} + 4a + at^2 = \frac{tx}{2} + 4a + \frac{16a}{t^2}$,

$\frac{tx}{2} + \frac{2x}{t} = at^2 - \frac{16a}{t^2}$

$x\left(\frac{t^2+4}{2t}\right) = a\left(\frac{t^4-16}{t^2}\right)$, $x = \frac{2a(t^2-4)}{t}$

$y = \frac{-2}{t} \times \frac{2a(t^2-4)}{t} + 4a + at^2$

$= -4a + \frac{16a}{t^2} + 4a + at^2 = \frac{a(t^4+16)}{t^2}$

(c) At R: $x = \frac{2a(t^2-4)}{t}$ $\frac{t^2-4}{t} = \frac{x}{2a}$

At R: $y = \frac{a(t^4+16)}{t^2}$ $\frac{t^4+16}{t^2} = \frac{y}{a}$

But $\left(\frac{t^2-4}{t}\right)^2 = \frac{t^4+16-8t^2}{t^2} = \frac{t^4+16}{t^2} - \frac{8t^2}{t^2} = \frac{t^4+16}{t^2} - 8$

$\left(\frac{x}{2a}\right)^2 = \frac{y}{a} - 8$ i.e. $x^2 = 4a(y - 8a)$, so the result is true.

The locus of R is a parabola, vertex $(0, 8a)$, focal length a, focus $(0, 9a)$.

16 (a) $y - px + ap^2 = 0$, $y - qx + aq^2 = 0$

Solve $px - ap^2 = qx - aq^2$:

$(p - q)x = a(p^2 - q^2)$, $x = a(p + q)$; $y = pa(p + q) - ap^2 = apq$

$T(a(p + q), apq)$

(b) $\tan(\angle PTQ) = 1$

gradient of tangent at $P = p$, gradient of tangent at $Q = q$

Using formula for angle between two lines:

$\tan \angle PTQ = \left|\frac{p-q}{1+pq}\right| = 1$ hence $|p - q| = |1 + pq|$

(c) $4pq$

(d) At T: $p + q = \frac{x}{a}$ [1]

Also $pq = \frac{y}{a}$ [2]

From (c): $(p - q)^2 = (p + q)^2 - 4pq$ [3]

From (b): $(p - q)^2 = (1 + pq)^2$ [4]

Substitute [4] into [3]:

$(1 + pq)^2 = (p + q)^2 - 4pq$ [5]

Substitute [1], [2] into [5]:

$\left(1 + \frac{y}{a}\right)^2 = \left(\frac{x}{a}\right)^2 - 4 \times \frac{y}{a}$

$1 + \frac{2y}{a} + \frac{y^2}{a^2} = \frac{x^2}{a^2} - \frac{4y}{a}$

$y^2 + 6ay + a^2 - x^2 = 0$

$y^2 + 6ay + 9a^2 - x^2 = 8a^2$

$\therefore (y + 3a)^2 - x^2 = 8a^2$

CHAPTER REVIEW 6

1 (a) D (b) A (c) D (d) C **2** $y = x^2 - 6x + 4$

3 (a) $y = 2x^2 + 8x + 7$

(b) vertex $(-2, -1)$, focus $\left(-2, -\frac{7}{8}\right)$, directrix $y = -\frac{9}{8}$

4 $(4, 2)$ **5** $(\pm 4a, 4a)$

6 Normal at P: $x + py = ap^3 + 2ap$

Q satisfies so $2aq + apq^2 = ap^3 + 2ap$, $p^3 - pq^2 + 2(p - q) = 0$,

$p(p^2 - q^2) + 2(p - q) = 0$

$(p - q)(p(p + q) + 2) = 0$, $p \neq q$ so $p^2 + pq + 2 = 0$

7 $A(ap, 0)$, $B(0, -ap^2)$, $C(ap, -ap^2)$

At C: $x = ap$, $p = \frac{x}{a}$, and $y = -ap^2$

$\therefore y = -a \times \left(\frac{x}{a}\right)^2$ $\therefore x^2 = -ay$

8 (a) See **Parametric equation of the tangent**, Chapter 6 page 125.

$y - p^2 = p(x - 2p)$, $y - p^2 = px - 2p^2$, $y - px + p^2 = 0$

(b) $(p, 0)$

(c) Normal is: $x + py = p^3 + 2p$

At B: $x = 0$, $y = p^2 + 2$: $B(0, p^2 + 2)$

(d) $y = 2x^2 + 1$, $x > 0$

9 (a) $\frac{y - 2p^2}{x - 4p} = \frac{2q^2 - 2p^2}{4q - 4p}$ $\frac{y - 2p^2}{x - 4p} = \frac{p + q}{2}$

(b) $(0, -4)$: $-4 - 0 + 2pq = 0$, $pq = 2$

(c) $y = 4$, with domain $x < -4\sqrt{2}$, $x > 4\sqrt{2}$

CHAPTER 7

Many of the proofs below are not complete: in most cases only the outline of the proof of $S(k + 1)$ is given. For more complete solutions, see the Student Worked Solutions book.

EXERCISE 7.1

1 D

2 For $S(k + 1)$:

$\text{LHS} = 1 + 2 + 3 + \ldots + k + k + 1$

$= \frac{k(k+1)}{2} + k + 1$

$= \frac{(k+1)}{2} \times (k + 2) = \text{RHS}$

3 For $S(k + 1)$:

$\text{LHS} = 1 + 2 + 4 + \ldots + 2^{k-1} + 2^k$

$= 2^k - 1 + 2^k = 2 \times 2^k - 1$

$= 2^{k+1} - 1 = \text{RHS}$

4 For $S(k + 1)$:

$\text{LHS} = 2 + 5 + 8 + \ldots + (3k - 1) + (3k + 2)$

$= \frac{k(3k+1)}{2} + 3k + 2$

$= \frac{(k+1)(3k+4)}{2} = \text{RHS}$

5 For $S(k + 1)$:

$\text{LHS} = 1 + r + r^2 + r^3 + \ldots + r^{k-1} + r^k$

$= \frac{1 - r^k}{1 - r} + r^k$

$= \frac{1 - r^k + r^k - r^{k+1}}{1 - r}$

$= \frac{1 - r^{k+1}}{1 - r} = \text{RHS}$

6 For $S(k + 1)$:

$\text{LHS} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \quad + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k+1}{k+2} = \text{RHS}$

7 For $S(k + 1)$:

$\text{LHS} = \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \quad + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+2)(k+3)}$

$= \frac{k}{2(k+2)} + \frac{1}{(k+2)(k+3)} = \frac{k(k+3)+2}{2(k+2)(k+3)} = \frac{k+1}{2(k+3)}$

$= \text{RHS}$

8 For $S(k+1)$:

$$\text{LHS} = \frac{1}{1\times3}+\frac{1}{3\times5}+\quad+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$
$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$
$$=\text{RHS}$$

9 For $S(k+1)$:

$$\text{LHS}=\frac{1^2}{1\times3}+\frac{2^2}{3\times5}+\frac{3^2}{5\times7}+\quad+\frac{k^2}{(2k-1)(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}$$
$$=\frac{k(k+1)}{2(2k+1)}+\frac{(k+1)^2}{(2k+1)(2k+3)}$$
$$=\frac{k(k+1)(2k+3)+2(k+1)^2}{2(2k+1)(2k+3)}$$
$$=\frac{(k+1)[k(2k+3)+2(k+1)]}{2(2k+1)(2k+3)}$$
$$=\frac{(k+1)(k+2)(2k+1)}{2(2k+1)(2k+3)}=\frac{(k+1)(k+2)}{2(2k+3)}=\text{RHS}$$

10 For $S(k+1)$:

$$\text{LHS}=1^3+2^3+3^3+\ldots+k^3+(k+1)^3$$
$$=\frac{k^2(k+1)^2}{4}+(k+1)^3=\frac{(k+1)^2(k^2+4k+4)}{4}$$
$$=\frac{(k+1)^2(k+2)^2}{4}=\text{RHS, limit}=\tfrac{1}{4}$$

11 For $S(k+1)$:

$$\text{LHS}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\quad\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$
$$=\frac{(k+1)}{2k}\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)}{2k}\times\frac{k(k+2)}{(k+1)^2}$$
$$=\frac{k+2}{2(k+1)}=\text{RHS}$$

12 For $S(k)$: $(k+1)+(k+2)+\ldots+2k=\dfrac{k(3k+1)}{2}$

For $S(k+1)$:

$$\text{LHS}=(k+2)+\ldots+2k+(2k+1)+(2k+2)$$
$$=(k+1)+(k+2)+\ldots+2k+(3k+2)$$
$$=\frac{k(3k+1)}{2}+(3k+2)=\frac{3k^2+7k+4}{2}=\frac{(k+1)(3k+4)}{2}$$
$$=\text{RHS}$$

13 For $S(k+1)$:

$$\text{LHS}=1\times1!+2\times2!+3\times3!+\ldots+k\times k!+(k+1)\times(k+1)!$$
$$=(k+1)!-1+(k+1)\times(k+1)!$$
$$=(k+1)!\times[1+(k+1)]-1$$
$$=(k+1)!\times(k+2)-1=(k+2)!-1=\text{RHS}$$

14 For $S(k+1)$:

$$\text{LHS}=1^2-2^2+3^2-4^2+\ldots+(-1)^{k-1}k^2+(-1)^k(k+1)^2$$
$$=(-1)^{k-1}\left[\frac{k(k+1)}{2}\right]+(-1)^k(k+1)^2$$
$$=(-1)^k\times\frac{(k+1)}{2}\times[-k+2(k+1)]$$
$$=(-1)^k\times\frac{(k+1)(k+2)}{2}=\text{RHS}$$

15 For $S(k+1)$:

$$\text{LHS}=\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\quad+\frac{2^k-1}{2^k}+\frac{2^{k+1}-1}{2^{k+1}}$$
$$=k-1+\frac{1}{2^k}+\frac{2^{k+1}-1}{2^{k+1}}=k-1+\frac{1}{2^k}+1-\frac{1}{2^{k+1}}$$
$$=k+\frac{2-1}{2^{k+1}}=k+\frac{1}{2^{k+1}}=\text{RHS}$$

16 For $S(k+1)$:

$$\text{LHS}=\log\left(\tfrac{2}{1}\right)+\log\left(\tfrac{3}{2}\right)+\log\left(\tfrac{4}{3}\right)+\quad+\log\left(\frac{k+1}{k}\right)+\log\left(\frac{k+2}{k+1}\right)$$
$$=\log(k+1)+\log\left(\frac{k+2}{k+1}\right)$$
$$=\log\left((k+1)\times\frac{(k+2)}{(k+1)}\right)=\log(k+2)=\text{RHS}$$

EXERCISE 7.2

1 C

2 $S(k)$ is $5^k=4M-3$

For $S(k+1)$: $5^{k+1}+3=5\times5^k+3=5(4M-3)+3$

$=20M-12=4(5M-3)$

which is divisible by 4 as $(5M-3)$ is an integer

3 $S(k)$ is $3^{2k}=8M+1$

For $S(k+1)$: $3^{2k+2}-1=9\times3^{2k}-1=9(8M+1)-1$

$=72M+8=8(9M+1)$

which is divisible by 8 as $(9M+1)$ is an integer

4 $S(k)$ is $3^k=5M-2^k$

For $S(k+2)$: $3^{k+2}+2^{k+2}=9\times3^k+4\times2^k$

$=9(5M-2^k)+4\times2^k=45M-5\times2^k=5(9M-2^k)$

which is divisible by 5 as $(9M-2^k)$ is an integer

5 $S(k)$ is $5^k=3M-2\times11^k$

For $S(k+1)$: $5^{k+1}+2\times11^{k+1}$

$=5\times5^k+22\times11^k=5(3M-2\times11^k)+22\times11^k$

$=15M+12\times11^k=3(5M+4\times11^k)$

which is divisible by 3 as $(5M+4\times11^k)$ is an integer

6 (a) $(k+1)(k+2)(k+3)$

(b) $S(k)$ is $k(k+1)(k+2)=3M$

For $S(k+1)$: $(k+1)(k+2)(k+3)$

$=k(k+1)(k+2)+3(k+1)(k+2)$

$=3M+3(k+1)(k+2)=3(M+(k+1)(k+2))$

which is divisible by 3 as $(M+(k+1)(k+2))$ is an integer

7 $S(k)$ is $3^{3k}=5M-2^{k+2}$

For $S(k+1)$: $3^{3k+3}+2^{k+3}$

$=27\times3^{3k}+2\times2^{k+2}=27(5M-2^{k+2})+2\times2^{k+2}$

$=135M-25\times2^{k+2}=5(27M-5\times2^{k+2})$

which is divisible by 5 as $(27M-5\times2^{k+2})$ is an integer

8 (a) $(k+3)^3=(k+3)(k+3)^2=k^3+9k^2+27k+27$

(b) Must prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 3 for all integers $n\geq1$.

$S(k)$ is $k^3+(k+1)^3+(k+2)^3=3M$

For $S(k+1)$: $(k+1)^3+(k+2)^3+(k+3)^3$

$=(k+1)^3+(k+2)^3+k^3+9k^2+27k+27$

$=3M+9k^2+27k+27=3(M+3k^2+9k+9)$

which is divisible by 3 as $(M+3k^2+9k+9)$ is an integer

9 Prove $S(1)$ is true: $(x-1)^3+x^3=2x^3-3x^2+3x-1$

$=(x^2-x+1)(2x-1)$

which is divisible by x^2-x+1

Assume $S(k)$ is true, i.e. $(x-1)^{k+2}=(x^2-x+1)M(x)-x^{2k+1}$

Prove $S(k+1)$ is true: $(x-1)^{k+3}+x^{2k+3}$

$=(x-1)(x-1)^{k+2}+x^2\times x^{2k+1}$

$=(x-1)[(x^2-x+1)M(x)-x^{2k+1}]+x^2\times x^{2k+1}$

$=(x-1)(x^2-x+1)M(x)-(x-1)\times x^{2k+1}+x^2\times x^{2k+1}$

$=(x-1)(x^2-x+1)M(x)+x^{2k+1}(x^2-x+1)$

$=(x^2-x+1)[(x-1)M(x)+x^{2k+1}]$

which is divisible by x^2-x+1

10 $S(k)$ is $x^k-1=(x-1)M(x)$

For $S(k+1)$:

$x^{k+1}-1=x^{k+1}-x^k+x^k-1$

$=x^k(x-1)+(x-1)M(x)=(x-1)[x^k+M(x)]$

which is divisible by $x-1$

EXERCISE 7.3

1 $S(k)$ is $k^2 - 11k + 30 \geq 0$

For $S(k+1)$: $(k+1)^2 - 11(k+1) + 30$

$= k^2 + 2k + 1 - 11k - 11 + 30$

$= (k^2 - 11k + 30) + (2k - 10) \geq 0$

as $k^2 - 11k + 30 \geq 0$ and $k \geq 6$

2 $S(k)$ is $k^2 + 5k - 14 > 0$

For $S(k+1)$: $(k+1)^2 + 5(k+1) - 14$

$= k^2 + 2k + 1 + 5k + 5 - 14$

$= (k^2 + 5k - 14) + (2k + 6) > 0$

as $k^2 + 5k - 14 > 0$ and $k > 0$

3 $S(k)$ is $12^k - 7^k - 5^k > 0$

For $S(k+1)$: $12^{k+1} - 7^{k+1} - 5^{k+1}$

$= 12 \times (12^k - 7^k - 5^k) + 5 \times 7^k + 7 \times 5^k > 0$

4 $S(k)$ is $10^k - 2^k - 3^k - 5^k \geq 0$

For $S(k+1)$: $10^{k+1} - 2^{k+1} - 3^{k+1} - 5^{k+1}$

$= 10 \times (10^k - 2^k - 3^k - 5^k) + 8 \times 2^k + 7 \times 3^k + 5 \times 5^k \geq 0$

5 $S(k)$ is $(1+x)^k - 1 - kx \geq 0$

For $S(k+1)$: $(1+x)^{k+1} - 1 - (k+1)x$

$= (1+x)[(1+x)^k - 1 - kx] + kx^2 \geq 0$

6 (a) $(A^n - B^n)(A - B)$

(b) If $A > B$ then $A^n > B^n$ as A, B both positive

$\therefore (A^n - B^n)(A - B) > 0$

If $A < B$ then $A^n < B^n$ as A, B both positive

$\therefore (A^n - B^n)(A - B) > 0$

If $A = B$ then $(A^n - B^n)(A - B) = 0$

$\therefore (A^n - B^n)(A - B) \geq 0$

$\therefore A^{n+1} - A^nB + B^{n+1} - B^nA \geq 0$

$\therefore A^{n+1} + B^{n+1} \geq A^nB + B^nA$

(c) $S(k)$ is $\left(\frac{A+B}{2}\right)^k - \frac{A^k + B^k}{2} \leq 0$

For $S(k+1)$:

$$\left(\frac{A+B}{2}\right)^{k+1} - \frac{A^{k+1} + B^{k+1}}{2} = \left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)^k - \frac{A^{k+1} + B^{k+1}}{2}$$

$$= \left(\frac{A+B}{2}\right)\left[\left(\frac{A+B}{2}\right)^k - \frac{A^k + B^k}{2}\right] + \frac{AB^k + BA^k - A^{k+1} - B^{k+1}}{4} \leq 0$$

using $S(k)$ and the result from (b)

EXERCISE 7.4

1 $S(k)$ is $F_1 + F_3 + F_5 + \ldots + F_{2k-1} = F_{2k}$

For $S(k+1)$: $F_1 + F_3 + F_5 + \ldots + F_{2k+1}$

$= F_1 + F_3 + F_5 + \ldots + F_{2k-1} + F_{2k+1}$

$= F_{2k} + F_{2k+1} = F_{2k+2}$

2 $S(k)$ is $F_{4k} = 3M$

For $S(k+1)$: $F_{4k+4} = F_{4k+2} + F_{4k+3}$

$= F_{4k} + F_{4k+1} + F_{4k+1} + F_{4k+2}$

$= F_{4k} + F_{4k+1} + F_{4k+1} + F_{4k} + F_{4k+1}$

$= 2 \times F_{4k} + 3 \times F_{4k+1} = 3(2M + F_{4k+1})$

which is divisible by 3 as $2M + F_{4k+1}$ is an integer

3 (a) 3, 9, 21, 45, 93

(b) $S(k)$ is $T_k = 3(2^k - 1)$

For $S(k+1)$: LHS $= T_{k+1}$

$= 2 \times T_k + 3 = 6(2^k - 1) + 3$

$= 6 \times 2^k - 3 = 3(2^{k+1} - 1) = \text{RHS}$

4 $S(k)$ is $\sin(x + k\pi) = (-1)^k \sin x$

For $S(k+1)$: LHS $= \sin(x + (k+1) \times \pi) = \sin((x + k\pi) + \pi)$

$= -\sin(x + k\pi)$ as $\sin(a + \pi) = -\sin\alpha$

$\therefore$ LHS $= -1 \times (-1)^k \sin x = (-1)^{k+1} \sin x = \text{RHS}$

5 To generate the greatest number of regions, each new line has to intersect each of the existing lines inside the circle. So, if there are 2 existing lines (giving 4 regions), then the 3rd line would cut the other 2 lines inside the circle, producing 3 extra regions for a total of 7 regions.

$S(k)$ is 'k straight lines gives a maximum of $\frac{1}{2}(k^2 + k + 2)$ regions'.

For $S(k+1)$: next line produces maximum of $(k+1)$ new regions

$\therefore$ number of regions $= \frac{1}{2}(k^2 + k + 2) + k + 1 = \frac{1}{2}(k^2 + 3k + 4)$

$= \frac{1}{2}((k+1)^2 + (k+1) + 2)$

6 (a)

\$1	(impossible)
\$2	(impossible)
\$3	$1 \times$ \$3
\$4	(impossible)
\$5	$1 \times$ \$5
\$6	$2 \times$ \$3
\$7	(impossible)
\$8	$1 \times$ \$3, $1 \times$ \$5
\$9	$3 \times$ \$3
\$10	$2 \times$ \$5
\$11	$2 \times$ \$3, $1 \times$ \$5
\$12	$4 \times$ \$3
\$13	$1 \times$ \$3, $2 \times$ \$5
\$14	$3 \times$ \$3, $1 \times$ \$5
\$15	$3 \times$ \$5

(b) 'Using only \$3 and \$5 coins, it is possible to make \$$n$ for all integers $n \geq 8$.'

For $S(k+1)$, there are 3 cases to consider.

Case 1: k is a multiple of 3, e.g. 9 or 12.

To generate $k+1$, replace $3 \times$ \$3 by $2 \times$ \$5

Case 2: k is a multiple of 5, e.g. 10.

To generate $k+1$, replace $1 \times$ \$5 by $2 \times$ \$3

Case 3: $k = 3p + 5q$ where p, q are positive integers.

To generate $k+1$, replace by $3(p+2) + 5(q-1)$

CHAPTER REVIEW 7

1 For $S(k+1)$:

LHS $= 2k(k-2) + 4(k+1) - 6$

$= 2k^2 - 2 = \text{RHS}$

2 For $S(k+1)$:

$$\text{LHS} = \frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

$$= \frac{(k+1)(k+2)}{3} \times [k+3] = \text{RHS}$$

3 $S(k)$ is $7^k = 3M + 1$

For $S(k+1)$: $7^{k+1} - 1 = 7(3M+1) - 1 = 3(7M+2)$

which is divisible by 3 as $7M + 2$ is an integer

4 $S(k)$ is $4^{k+1} = 21M - 5^{2k-1}$

For $S(k+1)$: $4^{k+2} + 5^{2k+1} = 4 \times (21M - 5^{2k-1}) + 25 \times 5^{2k-1}$

$= 84M + 21 \times 5^{2k-1} = 21(4M + 5^{2k-1})$

which is divisible by 21 as $4M + 5^{2k-1}$ is an integer

5 $S(k)$ is $k^2 - 3k + 2 \geq 0$

For $S(k+1)$: $(k+1)^2 - 3(k+1) + 2$

$= (k^2 - 3k + 2) + (2k - 2) \geq 0$

using $S(k)$ and $k \geq 2$

6 $S(k)$ is $(x+y)^k - x^k - y^k > 0$

For $S(k+1)$: $(x+y)^{k+1} - x^{k+1} - y^{k+1}$

$= (x+y)(x+y)^k - x \times x^k - y \times y^k$

$= (x+y)[(x+y)^k - x^k - y^k] + y \times x^k + x \times y^k > 0$

using $S(k)$ and $x > 0$, $y > 0$

7 For $S(k+1)$: LHS $= 1 - \frac{1}{(k+1)2^k} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$

$$= 1 - \frac{2(k+2)}{(k+1)(k+2)2^{k+1}} + \frac{k+3}{(k+1)(k+2)2^{k+1}}$$

$$= 1 - \frac{k+1}{(k+1)(k+2)2^{k+1}} = 1 - \frac{1}{(k+2)2^{k+1}} = \text{RHS}$$

8 $S(k)$ is $5^k = 16M - 12k + 1$
For $S(k+1)$: $5^{k+1} + 12(k+1) - 1$
$= 5(16M - 12k + 1) + 12k + 11$
$= 80M - 48k + 16 = 16(5M - 3k + 1)$
which is divisible by 16 as $5M - 3k + 1$ is an integer

9 $S(k)$ is $7^k = 13M - 6^k$
For $S(k+2)$: $7^{k+2} + 6^{k+2}$
$= 49(13M - 6^k) + 36 \times 6^k$
$= 637M - 13 \times 6^k = 13(49M - 6^k)$
which is divisible by 13 as $49M - 6^k$ is an integer

10 For $S(k+1)$:
LHS $= k(k+1)! + ((k+1)^2 + 1)(k+1)!$
$= (k+1)![k + k^2 + 2k + 2]$
$= (k+1)!(k+2)(k+1) = (k+1)(k+2)! =$ RHS

11 (a) LHS $= \dfrac{\sin 2kx}{2\sin x} + \cos(2k+1)x$

$= \dfrac{\sin 2kx}{2\sin x} + \cos 2kx \cos x - \sin 2kx \sin x$

$= \dfrac{\sin 2kx + 2\sin x(\cos 2kx \cos x - \sin 2kx \sin x)}{2\sin x}$

$= \dfrac{\sin 2kx + \sin 2x \cos 2kx - 2\sin 2kx \sin^2 x}{2\sin x}$

$= \dfrac{\sin 2x \cos 2kx - \sin 2kx(2\sin^2 x - 1)}{2\sin x}$

$= \dfrac{\sin 2x \cos 2kx + \sin 2kx \cos 2kx}{2\sin x}$

$= \dfrac{\sin(2(k+1)x)}{2\sin x} =$ RHS

(b) For $S(k+1)$:
LHS $= \dfrac{\sin 2kx}{2\sin x} + \cos(2k+1)x = \dfrac{\sin(2(k+1)x)}{2\sin x} =$ RHS
using the result of (a)

CHAPTER 8

EXERCISE 8.1

1 (a) $P(0) = -3$, $P(1) = 3$ (b) $P(1) = -1$, $P(2) = 4$
(c) $P(-2) = -3$, $P(-1) = 4$ (d) $P(3) = -1$, $P(4) = 10$
(e) $P(1) = -3$, $P(2) = 8$ (f) $P(-1) = -1$, $P(0) = 3$

2 A

3 (a) $0 < x < 1$, $1 < x < 2$ (b) $1 < x < 2$, $4 < x < 5$
(c) $-2 < x < -1$, $0 < x < 1$, $2 < x < 3$
(d) $-2 < x < -1$, $-1 < x < 0$, $1 < x < 2$
(e) $-3 < x < -2$, $0 < x < 1$ (f) $-2 < x < -1$, $-1 < x < 0$

4 (a) $f(0) = -3$, $f(1) = 3.3$ (b) $f(0.1) = -2.3$, $f(1) = 0.5$
(c) $f(4) = 1.4$, $f(5) = -1.8$ (d) $f(-1.9) = -1.2$, $f(-1) = 1.159$

EXERCISE 8.2

1 (a) $f(3) = -2$, $f(3.1) = 0.5$, $x = 3.08$
(b) $f(2.9) = -0.9$, $f(3) = 10$, $x = 2.91$
(c) $f(0.6) = -0.1$, $f(0.7) = 0.7$, $x = 0.62$
(d) $f(1.2) = -0.2$, $f(1.4) = 0.2$, $x = 1.30$
(e) $f(1.1) = 0.1$, $f(1.2) = -0.2$, $x = 1.15$

2 B

3 (a) $f(0) = -3$, $f(1) = 3.3$, $f(0.5) = 0.04$, $f(0.25) = 1.5$,
$f(0.375) = -0.7$, $f(0.4375) = -0.35$, $f(0.495) = 0.01$, $x = 0.50$
(b) $f(0.6) = -0.3$, $f(0.9) = 0.3$, $f(0.75) = -0.02$,
$f(0.825) = 0.13$, $x = 0.75$
Further substitution of values would show that $f(0.76) = 0.0007$, so $x = 0.76$ is a better answer. This answer would not come very quickly from halving the interval.
(c) $f(4) = 1.4$, $f(5) = -1.8$, $f(4.5) = -0.3$, $f(4.25) = 0.5$,
$f(4.375) = 0.1$, $f(4.4375) = -0.1$, $f(4.406\,25) = -0.003$, $x = 4.41$
(d) $f(-1.7) = -0.2$, $f(-1.5) = 0.3$, $f(-1.6) = 0.08$,
$f(-1.65) = -0.05$, $f(-1.63) = 0.007$, $x = -1.63$

4 $f(-1) = -1$, $f(0) = 1$, $f(-0.5) = 0.625$, $f(-0.75) = 0.02$, $x = -0.75$

EXERCISE 8.3

1 (a) $f(x) = x^3 - 3x - 20$, $f'(x) = 3x^2 - 3$
$x_2 = 3 - \frac{-2}{24} = 3.083$, $x_3 = 3.083 - \frac{0.0546}{25.51} = 3.081$
(b) $f(x) = x^4 + x^2 - 80$, $f'(x) = 4x^3 + 2x$
$x_2 = 3 - \frac{10}{114} = 2.912$, $x_3 = 2.912 - \frac{0.3858}{104.6} = 2.908$
(c) $f(x) = 10x - 2\sin x - 5$, $f'(x) = 10 - 2\cos x$
$x_2 = 0.5 - \frac{-0.9589}{8.245} = 0.616$, $x_3 = 0.616 - \frac{0.0044}{8.368} = 0.615$
(d) $f(x) = \log_e x - \cos x$, $f'(x) = \frac{1}{x} + \sin x$
$x_2 = 1 - \frac{-0.5403}{1.841} = 1.293$, $x_3 = 1.293 - \frac{-0.0173}{1.735} = 1.303$
(e) $f(x) = \log_e(x+2) - e^x + 2$, $f'(x) = \frac{1}{x+2} - e^x$
$x_2 = 1 - \frac{0.3803}{-2.385} = 1.159$, $x_3 = 1.159 - \frac{-0.0365}{-2.870} = 1.146$

2 C

3 (a) $f(x) = x^3 - 10$, $f'(x) = 3x^2$. $x_1 = 2$
$x_2 = 2 - \frac{-2}{12} = 2.167$, $x_3 = 2.167 - \frac{0.1760}{14.09} = 2.15$
(b) $f(x) = x^2 - 5$, $f'(x) = 2x$. $x_1 = 2$
$x_2 = 2 - \frac{-1}{4} = 2.25$, $x_3 = 2.25 - \frac{0.0625}{4.5} = 2.236$
(c) $f(x) = x^2 - 15$, $f'(x) = 2x$. $x_1 = 4$
$x_2 = 4 - \frac{1}{8} = 3.875$, $x_3 = 3.875 - \frac{0.0156}{7.75} = 3.873$
(d) 3.037 (e) 2.024 (f) 2.52

4 1.35, 1.35 **5** −2.89, −2.89

6 (a) $f(-0.7) = -0.087$, $f(-0.6) = 0.098$
(b) $x = -0.65$ (c) $x_1 = -0.7$, $x_2 = -0.653$

7 $x_2 = 0.5 - \frac{-0.0403}{2.683} = 0.52$

8 $x_2 = 2.2 - \frac{0.0752}{-0.9218} = 2.282$

CHAPTER REVIEW 8

1 (a) $x = \sqrt{\dfrac{1-x}{x}}$, $x^2 = \dfrac{1-x}{x}$, $x^3 = 1 - x$, $x^3 + x - 1 = 0$
(b) $x_2 = 0.6 - \frac{-0.184}{2.08} = 0.688$

2 (a) $g(1) = -0.0986$, $g(1.2) = 0.2768$
(b) $x_1 = 1.1$, $g(1.1) = 0.0786$

3 $x_2 = 0.9 - \frac{-0.3544}{-4.895} = 0.828$

4 $x_2 = 2 - \frac{-1.492}{6.274} = 2.24$, $x_3 = 2.24 - \frac{-0.4578}{3.045} = 2.39$

5 $x_2 = 1.2 - \frac{1.028}{-4.616} = 1.4227$

6 (a) $f(1.5) = -0.28$, $f(2) = 0.08$
Abscissa of P is between 1.5 and 2.
$f(4) = 0.16$, $f(5) = -0.17$
Abscissa of Q is between 4 and 5.
(b) $x_2 = 4.4 - \frac{0.0448}{-0.3182} = 4.54$
(c) $x_1 = 1.75$, $f(1.75) = -0.071$
$x_2 = 1.875$, $f(1.875) = 0.01$
$f(1.87) = 0.0078$, $f(1.88) = 0.0138$
Root is $x = 1.87$

CHAPTER 9

EXERCISE 9.1

1 (a) $\frac{1}{5}(x^2 - 1)^5 + C$ (b) $\frac{1}{4}(x^3 + 4)^4 + C$ (c) $\frac{2}{9}(x^3 + 1)^{\frac{3}{2}} + C$

2 (a) $\frac{1}{8}(2t + 1)^4 + C$ (b) $2\sqrt{x^2 - 4} + C$ (c) $\frac{1}{6}(x^2 + x + 1)^6 + C$

3 D

4 (a) $-\frac{1}{14}(3 - 2x)^7 + C$ (b) $\dfrac{-1}{2(3x^2 + 2x + 5)} + C$
(c) $\frac{1}{15}(x^3 - 3x^2 + 1)^5 + C$

5 (a) $\frac{1}{5}(x^3+1)^5+C$ (b) $-\sqrt{1-t^2}+C$ (c) $\frac{1}{5}(3x-5)^{\frac{5}{3}}+C$

6 (a) $-\frac{2}{3}(1-t^2)^{\frac{3}{2}}+C$ (b) $-\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}+C$

(c) $\frac{3}{8}(z^2+1)^{\frac{4}{3}}+C$

7 (a) $\frac{2}{5}(y+1)^{\frac{5}{2}}-\frac{2}{3}(y+1)^{\frac{3}{2}}+C$ (b) $\frac{-1}{x-1}-\frac{1}{2(x-1)^2}+C$

(c) $\frac{1}{6}(2x-1)^{\frac{3}{2}}+\frac{1}{2}(2x-1)^{\frac{1}{2}}+C$

8 (a) $\frac{du}{dx}=3x^2$, $\frac{1}{3}\int\sqrt{u}\,du=\frac{1}{3}\times\frac{2}{3}u^{\frac{3}{2}}+C=\frac{2}{9}(1+x^3)^{\frac{3}{2}}+C$

(b) $\frac{du}{dx}=3x^2$, $\frac{1}{3}\int\sqrt{1+u}\,du=\frac{1}{3}\times\frac{2}{3}(1+u)^{\frac{3}{2}}+C=\frac{2}{9}(1+x^3)^{\frac{3}{2}}+C$

(c) The integration of $\sqrt{u}$ is easier than the integration of $\sqrt{1+u}$.

9 $y=\frac{1}{3}(x^2-4)^{\frac{3}{2}}+\frac{5}{3}$ **10** $f(x)=3\sqrt{x^2+1}-1$

11 $x=\sqrt{t^2-2t+4}+8$ **12** $r=1-\sqrt[5]{1-15\theta}$

13 $y=\frac{1}{3}(2x-1)^{\frac{3}{2}}+\frac{19}{3}$ **14** $x=4-\frac{4}{8t+1}$

EXERCISE 9.2

1 (a) $\frac{1}{3}$ (b) $\frac{2\sqrt{3}}{5}$ (c) $2\left(\sqrt{5}-1\right)$ **2** (a) 0 (b) $\frac{44\sqrt{2}-16}{15}$ (c) 0

3 B **4** (a) $69\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $6\frac{1}{3}$ **5** (a) $\frac{4-2\sqrt{2}}{3}$ (b) $\frac{1}{5}$ (c) 0

6 (a) $\frac{9\sqrt[3]{3}-3}{8}$ (b) $\frac{5\sqrt{5}}{3}$ (c) $8\frac{2}{3}$ **7** (a) $7\frac{11}{15}$ (b) $21\frac{1}{3}$ (c) $-51\frac{1}{5}$

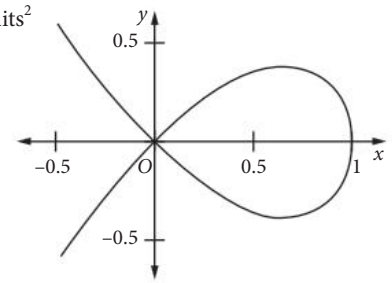
8 (a) $\int_3^0\frac{-du}{\sqrt{u}}=\left[-2\sqrt{u}\right]_3^0=2\sqrt{3}$

(b) $\int_1^4\frac{du}{\sqrt{4-u}}=\left[-2\sqrt{4-u}\right]_1^4=-0+2\sqrt{3}=2\sqrt{3}$

(c) Integrating $\sqrt{4-u}$ means that the negative sign needs to be taken into account. It is easily missed.

9 $\frac{1}{3}$ units^2 **10** 1 unit^2

11 Area $=\frac{8}{15}$ units^2



12 Area $=\int_4^6 x\sqrt{x-3}\,dx=\int_1^3(u+3)\sqrt{u}\,du=\int_1^3\left(u^{\frac{3}{2}}+3u^{\frac{1}{2}}\right)du$

$=\left[\frac{2}{5}u^{\frac{5}{2}}+2u^{\frac{3}{2}}\right]_1^3=\frac{12(4\sqrt{3}-1)}{5}$ units^2

13 Area $=\int_0^{2\sqrt{3}}\frac{x}{\sqrt{x^2+4}}dx=\frac{1}{2}\int_4^{16}\frac{du}{\sqrt{u}}=2$ units^2

EXERCISE 9.3

1 (a) $x+\frac{1}{2}\sin 2x+C$ (b) $x-\frac{1}{2}\sin 2x+C$

(c) $\frac{x}{2}-\frac{1}{2}\sin x+C$ (d) $x+\sin x+C$

(e) $\frac{x}{2}-\frac{1}{12}\sin 6x+C$ (f) $\frac{x}{2}+\frac{1}{16}\sin 8x+C$

2 (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}-\frac{1}{4}$ (c) $\frac{\pi}{12}$ (d) $\frac{3(\pi-\sqrt{3})}{4}$ (e) $\frac{\pi}{3}+\frac{\sqrt{3}}{8}$ (f) $\frac{\pi}{6}$

3 C **4** $\frac{\pi^2}{12}$ units^3 **5** $\frac{\pi^2}{2}$ units^3

6 $\frac{dy}{dx}=\frac{1}{2}(1-\cos 2x)$, $y=\frac{1}{2}\int(1-\cos 2x)\,dx=\frac{x}{2}-\frac{1}{4}\sin 2x+C$,

$y=\frac{x}{2}-\frac{1}{4}\sin 2x-2$, $y=\frac{\pi-18}{8}$

7 $\frac{d^2y}{dx^2}=1+\cos 2x$, $\frac{dy}{dx}=x+\frac{1}{2}\sin 2x-\frac{\pi}{2}$,

$y=\frac{x^2}{2}-\frac{\pi x}{2}-\frac{1}{4}\cos 2x+\frac{\pi^2-2}{8}$

8 $V=\pi\int_0^{\frac{\pi}{4}}\cos^2 2x\,dx=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}(1+\cos 4x)\,dx=\frac{\pi^2}{8}$ units^3

9 $V=\pi\int_0^{\frac{\pi}{12}}\sin^2 3x\,dx=\frac{\pi}{2}\int_0^{\frac{\pi}{12}}(1-\cos 6x)\,dx=\frac{\pi}{24}(\pi-2)$ units^3

10 (a) $\sin x=\cos x$, $x=\frac{\pi}{4}$

(b) $V=\pi\int_0^{\frac{\pi}{4}}\sin^2 x\,dx+\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2 x\,dx$

$=\frac{\pi}{2}\left\{\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}+\left[x+\frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right\}$

$=\frac{1}{4}(\pi^2-2\pi)$ units^3

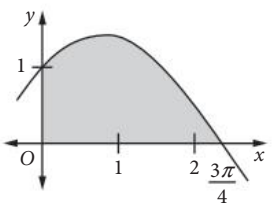
11 $V=\pi\int_0^{\frac{\pi}{4}}(\cos^2 x-\sin^2 x)\,dx$

$=\pi\int_0^{\frac{\pi}{4}}\cos 2x\,dx$

$=\pi\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$=\frac{\pi}{2}$ units^3

12 (a)



(b) $A=\int_0^{\frac{3\pi}{4}}(\sin x+\cos x)\,dx$

$=\left[-\cos x+\sin x\right]_0^{\frac{3\pi}{4}}=1+\sqrt{2}$ units^2

(c) $V=\int_0^{\frac{3\pi}{4}}(\sin x+\cos x)^2\,dx=\pi\int_0^{\frac{3\pi}{4}}(1+\sin 2x)\,dx$

$=\left[x-\frac{1}{2}\cos 2x\right]_0^{\frac{3\pi}{4}}=\frac{3\pi+2}{4}$ units^3

EXERCISE 9.4

1 (a) $-\frac{1}{3}\cos^3 x+C$ (b) $\frac{1}{2}\tan^2 x+C$ (c) $-\frac{1}{4}\cos^4 x+C$

(d) $\frac{1}{5}\sin^5 x+C$ (e) $-\frac{2}{3}\cos^3 x+C$ (f) $\frac{1}{2}\sin^2 x+C$

2 (a) $\frac{1}{16}$ (b) 0 (c) 0.5 (d) 0.5 (e) $\frac{4}{3}$ (f) $\frac{\pi}{2}$

EXERCISE 9.5

1 8π, a semicircle of radius 4 **2** $\frac{1}{3}$ **3** $\frac{\pi}{8}$ **4** $\frac{\sqrt{3}}{3}$

5 $\frac{\pi+2}{64}$ **6** 0 **7** $\sqrt{3}-\frac{\pi}{3}$ **8** $\frac{\pi}{8}$ **9** $\frac{64}{3}$ **10** $\frac{\pi}{18}$

11 $du=\sec^2 x\,dx$, $\int_3^4\frac{du}{u^4}=\left[\frac{-1}{3u^3}\right]_3^4=\frac{37}{5184}$

12 $du=\sec^2 x\,dx$, $\int e^u\,du=e^u+C=e^{\tan x}+C$

13 $du=\cos x-\sin x$, $\int\frac{-du}{u}=-\ln|u|+C=-\ln|\cos x+\sin x|+C$

14 (a) $\frac{d}{dx}\left(\frac{1}{\cos x}\right)=-1\times\frac{1}{\cos^2 x}\times(-\sin x)$

$=\frac{1}{\cos x}\times\frac{\sin x}{\cos x}=\sec x\tan x$

(b) $du = \sec x \tan x\, dx$, $\int \frac{du}{u} = \ln u + C = \ln|1 + \sec x| + C$

CHAPTER REVIEW 9

1 $2\,du = \frac{dx}{\sqrt{x}}$, $\int 2e^u\, du = 2e^u + C = 2e^{\sqrt{x}} + C$

2 (a) $2\int_1^0 \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right) du = \frac{8}{15}$ (b) $\int_0^1 (u+3)\sqrt{u}\, du = \frac{12}{5}$

3 $du = 3x^2\, dx$, $\frac{1}{3}\int_1^2 e^u\, du = \frac{1}{3}(e^2 - e)$

4 $\frac{4-\sqrt{2}}{12}$ 5 $du = \frac{dx}{x}$, $\int_1^3 \frac{du}{u^2} = \frac{2}{3}$

6 $-du = 2x\, dx$, $\int_8^5 \frac{-du}{\sqrt{u}} = \int_5^8 \frac{du}{\sqrt{u}} = \left[2\sqrt{u}\right]_5^8 = 4\sqrt{2} - 2\sqrt{5}$

7 (a) $\frac{2\pi - 3\sqrt{3}}{24}$ (b) $\frac{4\pi + 3\sqrt{3}}{48}$

8 $\frac{1}{6}du = x\, dx$, $\int \frac{1}{6}u^{\frac{3}{4}}\, du = \frac{2}{21}(3x^2+1)^{\frac{7}{4}} + C$

9 (a) $\frac{d}{dx}\left[e^{2x}(2\sin x - \cos x)\right] = 5e^{2x}\sin x$

(b) $\int e^{2x}\sin x\, dx = \frac{1}{5}\int \frac{d}{dx}\left\{e^{2x}(2\sin x - \cos x)\right\} dx$

$= \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$

10 $V = \pi\int_0^{\frac{\pi}{2}} \sin^2 2x\, dx = \frac{\pi}{2}\left[x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4}$ units^3

11 $\int_{16}^{21} \frac{du}{\sqrt{u}} = 2\sqrt{21} - 8$

12 (a) LHS $= \sin 6x\cos 3x + \cos 6x\sin 3x + \sin 6x\cos 3x$
$- \cos 6x\sin 3x = 2\sin 6x\cos 3x$

(b) $\frac{1}{2}\int (\sin 9x + \sin 3x)\, dx = -\frac{1}{18}\cos 9x - \frac{1}{6}\cos 3x + C$

13 $\frac{4\pi^2 - 3\sqrt{3}\pi}{48}$ units^3

14 $\frac{1}{3}\left[\sec 3x\right]_0^{\frac{\pi}{9}} = \frac{1}{3}\left(\sec\frac{\pi}{3} - \sec 0\right) = \frac{1}{3}$

15 (a) $\sin 3\theta = \sin\theta(1 - 2\sin^2\theta) + \cos\theta \times 2\sin\theta\cos\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta\cos^2\theta$
$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1 - 2\sin^2\theta)$
$= 3\sin\theta - 4\sin^3\theta$

(b) $\int \sin 3\theta\, d\theta = -\frac{1}{3}\cos 3\theta + C$

CHAPTER 10

TECHNOLOGY EXPLORATION (10.1)

Reflecting points in the line $y = x$

3 (3, 1) and (4, –2) 5 perpendicular bisector

Reflecting straight lines in the line $y = x$

2 $y = \frac{1}{2}x$ 3 1 4 the line $y = x$

5 (any line parallel to $y = x$) 6 (any line parallel to $y = -x$)

Reflecting curves in the line $y = x$

3 the line $y = x$ 5 yes

EXERCISE 10.1

1 A, B, D, E, H

2 (a) $-2 \le x \le 0$ (b) $x \ge 2$ (c) $x < -2, x > -2$ (d) $x \le 1.5$
(e) $x \ge -3$

3 (a) $f^{-1}(x) = \frac{1}{2}x + 2$, all real x, all real y

(b) $f^{-1}(x) = \sqrt{x+1}$, $x \ge -1$, $y \ge 0$

(c) $g^{-1}(x) = x^2 + 3$, $x \ge 0$, $y \ge 3$

(d) $f^{-1}(x) = -\sqrt{9 - x^2}$, $0 \le x \le 3$, $-3 \le y \le 0$

(e) $f^{-1}(x) = x^{\frac{1}{3}}$, all real x, all real y

(f) $f^{-1}(x) = -\sqrt{x} - 2$, $x \ge 0$, $y \le -2$

(g) $f^{-1}(x) = -1 + \sqrt{x+1}$, $x \ge 0$, $y \ge 0$

(h) $f^{-1}(x) = e^x - 1$, all real x, $y > -1$

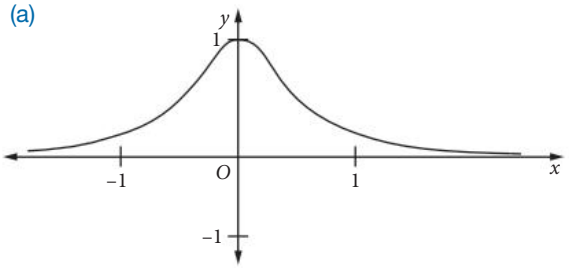
(i) $f^{-1}(x) = x^2 - 4x + 6$, $x \le 2$, $y \ge 2$

(j) $g^{-1}(x) = -x^2 - 2x + 4$, $x \ge -1$, $y \le 5$

(k) $f^{-1}(x) = -\log_2 x$, $0 < x < 1$, $y > 0$

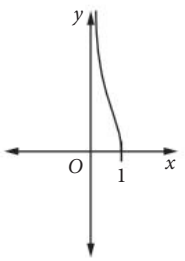
(l) $h^{-1}(x) = \frac{1}{x} - 1$, $x > 0$, $y > -1$

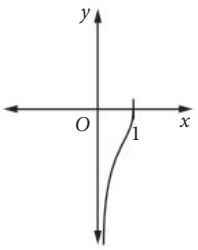
4 (a)



(b) $x \ge 0$, $x \le 0$

(c) $y = \sqrt{\frac{1}{x} - 1}$, $0 < x \le 1$, $y \ge 0$ or $y = -\sqrt{\frac{1}{x} - 1}$, $0 < x \le 1$, $y \le 0$





5 (a) not one-to-one, $-a \le x \le 0$, $0 \le x \le a$
(b) not one-to-one, $x \le 0$, $x \ge 0$
(c) not one-to-one, $x < 0$, $x > 0$

6 (a) $f(g(x)) = f\left(\frac{1}{2}(x+1)\right) = 2 \times \frac{1}{2}(x+1) - 1 = x$,
$g(f(x)) = g(2x-1) = \frac{1}{2}((2x-1)+1) = x$

(b) $f(g(x)) = f(\log_e 2x) = \frac{1}{2}e^{\log_e 2x} = \frac{1}{2} \times 2x = x$,
$g(f(x)) = g\left(\frac{1}{2}e^x\right) = \log_e\left(2 \times \frac{1}{2}e^x\right) = x$

(c) $f(g(x)) = f\left(-\sqrt{16 - x^2}\right) = \sqrt{16 - \left(-\sqrt{16 - x^2}\right)^2}$
$= \sqrt{x^2} = x$ as $0 \le x \le 4$
$g(f(x)) = g\left(\sqrt{16 - x^2}\right) = -\sqrt{16 - \left(\sqrt{16 - x^2}\right)^2}$
$= -\sqrt{x^2} = -(-x) = x$ as $-4 \le x \le 0$

(d) $f(g(x)) = f\left(1 + \sqrt{1 - x}\right)$
$= 2 \times \left(1 + \sqrt{1 - x}\right) - \left(1 + \sqrt{1 - x}\right)^2$
$= 2 + 2\sqrt{1 - x} - \left(1 + 2\sqrt{1 - x} + 1 - x\right) = x$,
$g(f(x)) = g\left(2x - x^2\right) = 1 + \sqrt{1 - \left(2x - x^2\right)}$
$= 1 + \sqrt{(x-1)^2} = x$ as $x \ge 1$

(e) $f(g(x)) = f\left(\frac{x+1}{2x}\right) = \frac{1}{2\times\frac{x+1}{2x}-1}$

$= \frac{1}{\frac{x+1-x}{x}} = x,$

$g(f(x)) = g\left(\frac{1}{2x-1}\right) = \frac{\frac{1}{2x-1}+1}{2\times\left(\frac{1}{2x-1}\right)} = \frac{\frac{1+2x-1}{2x-1}}{\frac{2}{2x-1}} = x$

7 D

8 (a) $\frac{dy}{dx} = 3x^2 - 9$ $\therefore$ stationary points at $x = \pm\sqrt{3}$

$\frac{d^2y}{dx^2} = 6x$:

Where $x = \sqrt{3}$, $\frac{d^2y}{dx^2} = 6\sqrt{3}$ which is positive

$\therefore$ max turning point at $(\sqrt{3}, 0)$

Where $x = -\sqrt{3}$, $\frac{d^2y}{dx^2} = -6\sqrt{3}$ which is negative

$\therefore$ min turning point at $\left(-\sqrt{3}, 12\sqrt{3}\right)$

(b) As there is a turning point at $x = \sqrt{3}$ on the x-axis, $x = \sqrt{3}$ is a double zero of the polynomial.

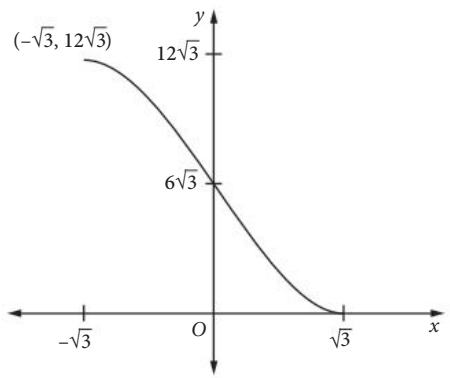
Let the zeros of the polynomial be $\sqrt{3}, \sqrt{3}$ and α

Using sum of roots $= -\frac{b}{a}$: $2\sqrt{3} + \alpha = 0$

$\therefore$ $\alpha = -2\sqrt{3}$

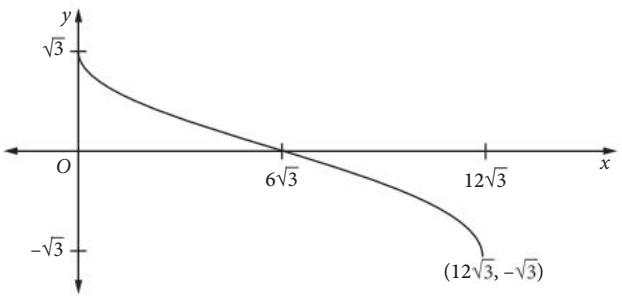
$\therefore$ other x-intercept is at $(-2\sqrt{3}, 0)$

(c)



(d) $0 \le x \le 12\sqrt{3}$, $-\sqrt{3} \le y \le \sqrt{3}$

(e)



(f) For f^{-1} the equation is $x = y^3 - 9y + 6\sqrt{3}$, but we can't make y the subject so we can't find $\frac{dy}{dx}$. However, we can find $\frac{dx}{dy} = 3y^2 - 9$, so $\frac{dy}{dx} = \frac{1}{3\times y^2 - 9}$.

At the point $(6\sqrt{3}, 0)$: $\frac{dy}{dx} = \frac{1}{3\times 0^2 - 9} = -\frac{1}{9}$

Thus, equation of tangent: $y - 0 = -\frac{1}{9}(x - 6\sqrt{3})$

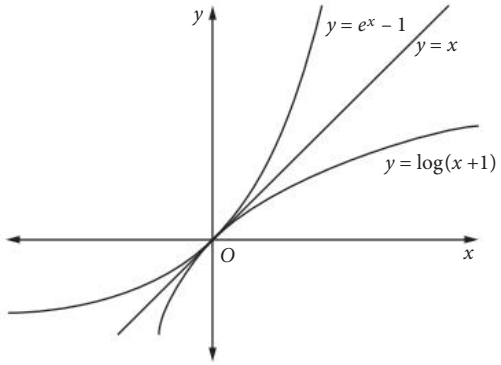
or $x + 9y = 6\sqrt{3}$

9 (a) $\frac{dy}{dx} = e^x$ which is positive for all x

$\therefore$ the function is increasing for all x

(b) Where $x = 0$, $\frac{dy}{dx} = e^0 = 1$

(c) From (b) we see that $y = x$ is a tangent to the curve at $(0, 0)$:



(d) Equation of f^{-1} is $x = e^y - 1$, which becomes $y = \log_e(x + 1)$

(e) From the graph of (c), $\log_e(x + 1) < x$ for $-1 < x < 0$, $x > 0$

EXERCISE 10.2

1 B

2 (a) $\frac{\pi}{2}$ (b) 0 (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{3}$ (e) $-\frac{\pi}{3}$ (f) 0 (g) $\frac{\pi}{2}$ (h) π
(i) $\frac{\pi}{3}$ (j) $\frac{5\pi}{6}$ (k) $\frac{\pi}{4}$ (l) $-\frac{\pi}{4}$ (m) $\frac{\pi}{3}$ (n) $-\frac{\pi}{6}$ (o) $\frac{\pi}{4}$ (p) 80°
(q) $\frac{\pi}{3}$ (r) $-\frac{\pi}{4}$ (s) $\frac{\pi}{3}$ (t) $\frac{2\pi}{3}$ (u) $\frac{\pi}{3}$ (v) $\frac{\pi}{6}$

3 (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{5}{13}$ (c) 65° (d) 180° (e) $\frac{1}{2}$ (f) $-\frac{1}{2}$
(g) $-\frac{119}{169}$ (h) $\frac{3\sqrt{2}}{4}$

4 (a) $0 < \tan^{-1}\frac{3}{5} < \tan^{-1}4 < \frac{\pi}{2}$, so $0 < \tan^{-1}4 - \tan^{-1}\frac{3}{5} < \frac{\pi}{2}$

Now $\tan(\tan^{-1}4 - \tan^{-1}\frac{3}{5})$

$= \frac{\tan(\tan^{-1}4) - \tan(\tan^{-1}\frac{3}{5})}{1 + \tan(\tan^{-1}4)\tan(\tan^{-1}\frac{3}{5})} = \frac{4 - \frac{3}{5}}{1 + 4\times\frac{3}{5}} = 1$

$\therefore$ $\tan^{-1}4 - \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$

(b) $\sin^{-1}\frac{3}{5} + \sin^{-1}\left(-\frac{3}{5}\right) = \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{3}{5} = 0$

(because $\sin^{-1}x$ is an odd function)

(c) $0 < \tan^{-1}\frac{5}{12} < \frac{\pi}{2}$ and $0 < \cos^{-1}\frac{5}{13} < \frac{\pi}{2}$,

so $0 < \tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13} < \pi$

Now $\cos(\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13})$

$= \cos(\tan^{-1}\frac{5}{12})\cos(\cos^{-1}\frac{5}{13}) - \sin(\tan^{-1}\frac{5}{12})\sin(\cos^{-1}\frac{5}{13}) = 0$

$\therefore$ $\tan^{-1}\frac{5}{12} + \cos^{-1}\frac{5}{13} = \frac{\pi}{2}$

(d) $0 < \cos^{-1}\frac{3}{5} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{3}{4} < \frac{\pi}{2}$,

so $0 < \cos^{-1}\frac{3}{5} + \tan^{-1}\frac{3}{4} < \pi$

Now $\cos\left(\cos^{-1}\frac{3}{5} - \tan^{-1}\left(-\frac{3}{4}\right)\right)$

$= \cos\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{3}{4}\right)$

$= \cos(\cos^{-1}\frac{3}{5})\cos(\tan^{-1}\frac{3}{4}) - \sin(\cos^{-1}\frac{3}{5})\sin(\tan^{-1}\frac{3}{4}) = 0$

$\therefore$ $\cos^{-1}\frac{3}{5} - \tan^{-1}\left(-\frac{3}{4}\right) = \frac{\pi}{2}$

(e) $0 < \sin^{-1}\frac{3}{5} < \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, so $0 < 2\sin^{-1}\frac{3}{5} < \frac{\pi}{2}$

Now $\sin(2\sin^{-1}\frac{3}{5}) = 2\sin(\sin^{-1}\frac{3}{5})\cos(\sin^{-1}\frac{3}{5})$

$= 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

$\therefore$ $2\sin^{-1}\frac{3}{5} = \sin^{-1}\frac{24}{25}$

(f) $0 < \sin^{-1}\frac{3}{5} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{7}{24} < \frac{\pi}{2}$,

so $0 < \sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24} < \pi$

Now $\cos(\sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24}) = \cos(\sin^{-1}\frac{3}{5})$

$\cos(\tan^{-1}\frac{7}{24}) - \sin(\sin^{-1}\frac{3}{5})\sin(\tan^{-1}\frac{7}{24}) = \frac{3}{5}$

$\therefore \sin^{-1}\frac{3}{5} + \tan^{-1}\frac{7}{24} = \cos^{-1}\frac{3}{5}$

(g) $0 < \sin^{-1}\frac{5}{13} < \frac{\pi}{2}$ and $0 < \tan^{-1}\frac{16}{63} < \frac{\pi}{2}$,

so $0 < \sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63} < \pi$

Now $\cos(\sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63})$

$= \cos(\sin^{-1}\frac{5}{13})\cos(\tan^{-1}\frac{16}{63}) - \sin(\sin^{-1}\frac{5}{13})\sin(\tan^{-1}\frac{16}{63}) = \frac{4}{5}$

$\therefore \sin^{-1}\frac{5}{13} + \tan^{-1}\frac{16}{63} = \cos^{-1}\frac{4}{5}$

5 (a) 0 (b) 0 (c) $\frac{16}{65}$ (d) $\frac{24}{25}$ (e) $\frac{63}{65}$ (f) $\frac{7}{25}$ (g) $-\frac{88}{9}$

6 (a) Let $\alpha = \sin^{-1}x$ where $0 \le x \le 1$

$\therefore 0 \le \alpha \le \frac{\pi}{2}$ and $\sin\alpha = x = \frac{x}{1}$

By Pythagoras' theorem: $\cos\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore \alpha = \cos^{-1}\sqrt{1-x^2}$ and so $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$

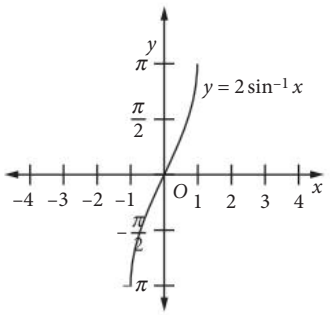
(b) Let $\alpha = \cos^{-1}x$ where $0 \le x \le 1$

$\therefore 0 \le \alpha \le \frac{\pi}{2}$ and $\cos\alpha = x = \frac{x}{1}$

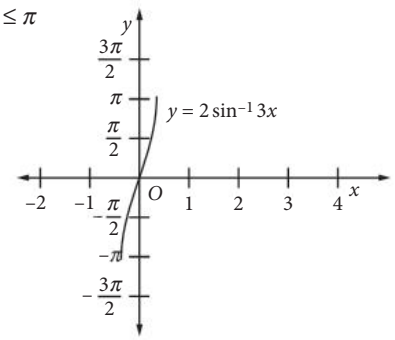
By Pythagoras' theorem: $\sin\alpha = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

$\therefore \alpha = \sin^{-1}\sqrt{1-x^2}$ and so $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$

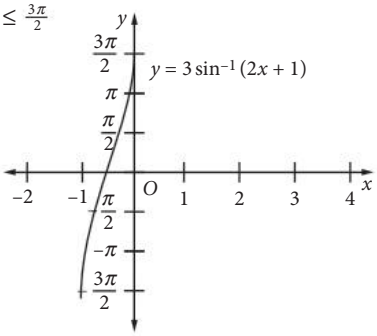
7 (a) $-1 \le x \le 1, -\pi \le y \le \pi$



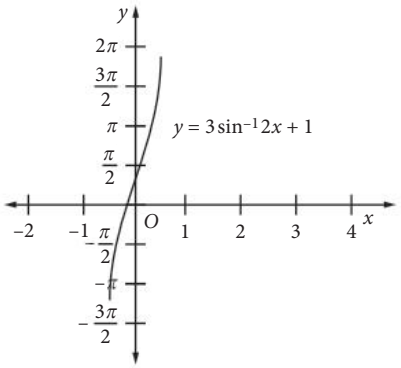
(b) $-\frac{1}{3} \le x \le \frac{1}{3}, -\pi \le y \le \pi$



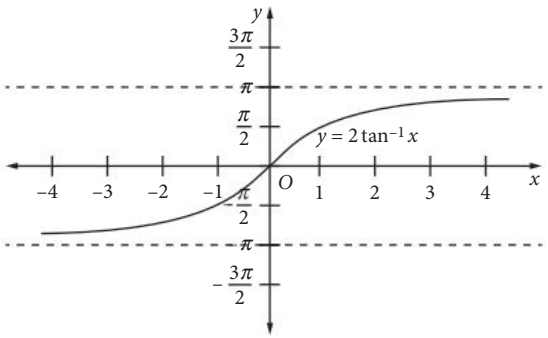
(c) $-1 \le x \le 0, -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



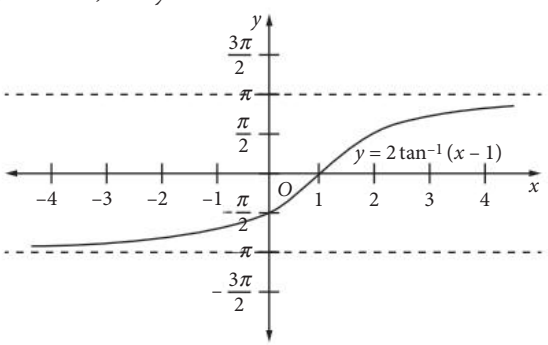
(d) $-\frac{1}{2} \le x \le \frac{1}{2}, -\frac{3\pi}{2} + 1 \le y \le \frac{3\pi}{2} + 1$



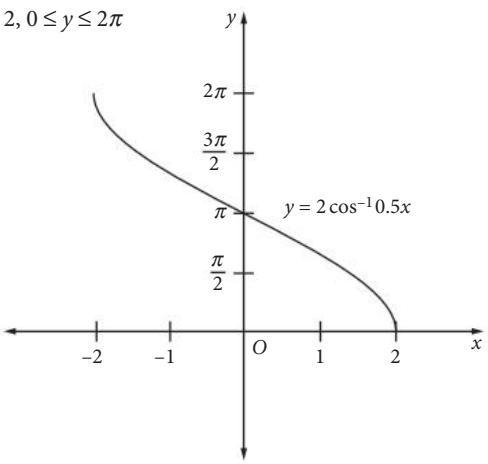
(e) all real $x, -\pi \le y \le \pi$



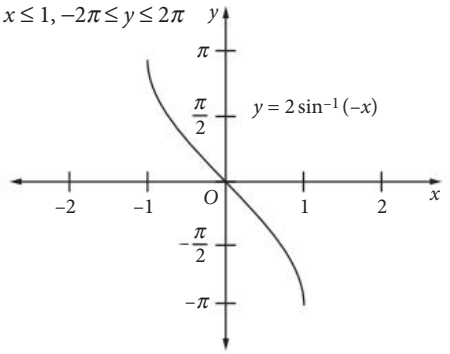
(f) all real $x, -\pi \le y \le \pi$



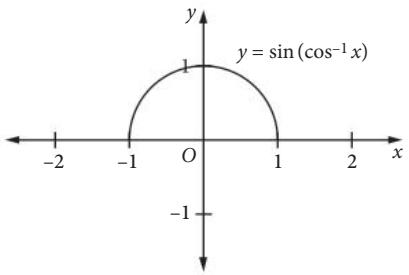
(g) $-2 \le x \le 2, 0 \le y \le 2\pi$



(h) $-1 \le x \le 1, -2\pi \le y \le 2\pi$

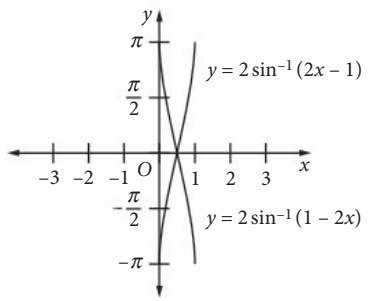


(i) Let $\alpha = \cos^{-1} x$ where $0 \le \alpha \le \pi$, i.e. $\cos\alpha = x$
From $\sin^2\alpha + \cos^2\alpha = 1$: $\sin\alpha = \pm\sqrt{1-x^2}$
Which one? $0 \le \alpha \le \pi$ so $\sin\alpha = +\sqrt{1-x^2}$
Hence $y = \sin(\cos^{-1} x)$ becomes $y = \sin\alpha$,
which becomes $y = \sqrt{1-x^2}$ i.e. a semicircle
Domain $-1 \le x \le 1$, Range $0 \le y \le 1$



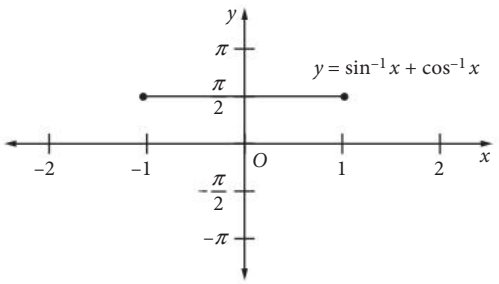
8 (a) Both have domain $0 \le x \le 1$, range $-\pi \le y \le \pi$
(b) $g(x) = 2\sin^{-1}(1-2x) = 2\sin^{-1}(-[-1+2x])$
$= -2\ \sin^{-1}(-1+2x) = -f(x)$

(c)



9 (a) $x = 0, 1$ (b) $x = 1.5$ (c) $x = -0.47$
10 (a) $x = -\frac{1}{2}, y = \frac{1}{\sqrt{2}}$ (b) $x = -\frac{1}{\sqrt{2}}, y = \frac{\sqrt{3}}{2}$
11 (a) LHS $= \sin(\sin^{-1} x)\cos(\cos^{-1} x) - \cos(\sin^{-1} x)\sin(\cos^{-1} x)$
$= x \times x - \sqrt{1-x^2} \times \sqrt{1-x^2} = x^2 - (1-x^2) = 2x^2 - 1$
(b) $-1 \le x \le 1$
$\therefore 0 \le x^2 \le 1, 0 \le 2x^2 \le 2, -1 \le 2x^2 - 1 \le 1$
(c) From (a) and (b), $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(2x^2 - 1)$
So the equation becomes $\sin^{-1}(2x^2 - 1) = \sin^{-1}(5x - 4)$
$\therefore 2x^2 - 1 = 5x - 4$
This has solutions $x = 1, \frac{3}{2}$ but the domain is $-1 \le x \le 1$
Hence the only solution is $x = 1$.

12 $f(-x) = \tan(\cos^{-1}(-x)) = \tan(\pi - \cos^{-1} x) = -\tan(\cos^{-1} x)$
$= -f(x)$ $\therefore$ odd function
13 $y = 2\cos\frac{x}{3}, 0 \le x \le 3\pi, -2 \le y \le 2$
14 (a) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ for $-1 \le x \le 1$, so the graph is:

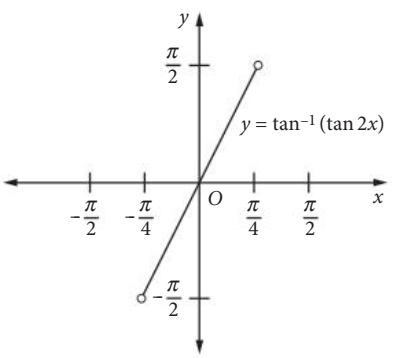


(b) $\int_{-1}^{1} (\sin^{-1} x + \cos^{-1} x)\,dx$
= Area of rectangle between interval and x-axis $= \pi$

EXERCISE 10.3

1 B
2 (a) $\frac{5}{1+25x^2}$ (b) $\frac{-3}{x^2-2x+2}$ (c) $\frac{2x}{1+x^4}$
(d) $\frac{2\tan^{-1} x}{1+x^2}$ (e) $\frac{1}{2\sqrt{x}(1+x)}$ (f) $\frac{e^x}{1+e^{2x}}$
(g) $\frac{e^{\tan^{-1} x}}{1+x^2}$ (h) $\frac{e^x}{1+x^2} + e^x \tan^{-1} x$ (i) $\frac{1}{\sqrt{16-x^2}}$
(j) $\frac{-6}{\sqrt{4-9x^2}}$ (k) $\frac{1}{\sqrt{1-x^2}\sin^{-1} x}$ (l) $\frac{-4}{\sqrt{1-4x^2}\cos^{-1} 2x}$
(m) $\frac{-1}{x\sqrt{x^2-1}}$ (n) $\frac{1}{(\cos^{-1} x)^2\sqrt{1-x^2}}$
(o) $\frac{1}{x\sqrt{1-x^2}} - \frac{\sin^{-1} x}{x^2}$ (p) $\sin^{-1}\sqrt{x} + \frac{\sqrt{x}}{2\sqrt{1-x}}$
(q) $\frac{2}{1+x^2}$ if $x>0$, $\frac{-2}{1+x^2}$ if $x<0$, undefined at $x=0$ (r) 0
(s) $\frac{\tan x}{1+x^2} + \tan^{-1} x \sec^2 x$ (t) $\frac{1}{x\sqrt{x^2-1}}$
3 $y = 2x + 2 - \frac{\pi}{2}$
4 (a) $\frac{dy}{dx} = \frac{2\sec^2 2x}{1+\tan^2 2x} = 2$ (b) $\tan\left(\pm\frac{\pi}{2}\right)$ is undefined

(c)



(d) $\int_{-\frac{\pi}{8}}^{\frac{\pi}{6}} \tan^{-1}(\tan 2x)\,dx$

= area of triangle above x-axis − area of triangle below x-axis

$= \frac{1}{2}\times\frac{\pi}{6}\times\frac{\pi}{3} - \frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{4} = \frac{\pi^2}{36} - \frac{\pi^2}{64} = \frac{7\pi^2}{576}$

5 $\frac{dy}{dx} = 0 \quad \therefore y$ is a constant for all x in the domain.

Where $x = 0$, $y = \cos^{-1}0 + \cos^{-1}0 = \pi$

6 $\frac{dy}{dx} = 0 \quad \therefore y$ is a constant for all x in the domain.

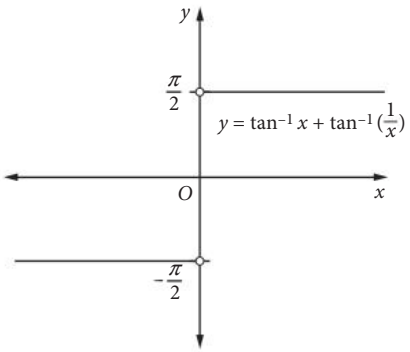
Where $x = 0$, $y = \sin^{-1}0 + \sin^{-1}0 = 0$

7 (a) $\frac{x}{1+x^2} + \tan^{-1}x$

(b) $x\tan^{-1}x - \frac{1}{2}\log_e(1+x^2) + C$ (c) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$

8 (a) all real x except 0 (b) 0 (c) $\frac{\pi}{2}, -\frac{\pi}{2}$

(d)



9 Max turning point at $\left(\tan^{-1}\frac{2}{3}, \sqrt{13}\right)$

EXERCISE 10.4

1 B

2 (a) $\sin^{-1}\frac{x}{4} + C$ (b) $\tan^{-1}\frac{x}{3} + C$ (c) $\sin^{-1}x + C$

(d) $\cos^{-1}\frac{x}{\sqrt{5}} + C$ (e) $\sin^{-1}\frac{x}{\sqrt{2}} + C$ (f) $\tan^{-1}\frac{x}{5} + C$

(g) $\frac{1}{2}\tan^{-1}\frac{x}{2} + C$ (h) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$ (i) $\cos^{-1}\frac{x}{\sqrt{6}} + C$

(j) $\frac{1}{2}\sin^{-1}2x + C$ (k) $\frac{1}{3}\tan^{-1}3x + C$

(l) $\frac{1}{12}\tan^{-1}\frac{4x}{3} + C$ (m) $\frac{1}{4}\tan^{-1}\frac{x}{4} + C$

(n) $\frac{1}{\sqrt{3}}\sin^{-1}\sqrt{3}x + C$ (o) $\frac{1}{3}\cos^{-1}\frac{3x}{4} + C$ (p) $\frac{1}{2}\sin^{-1}\frac{2x}{\sqrt{5}} + C$

(q) $\frac{1}{2}\tan^{-1}\frac{x+5}{2} + C$ (r) $\sin^{-1}\frac{x-3}{\sqrt{2}} + C$ (s) $\frac{1}{2}\sin^{-1}\frac{x+2}{\sqrt{2}} + C$

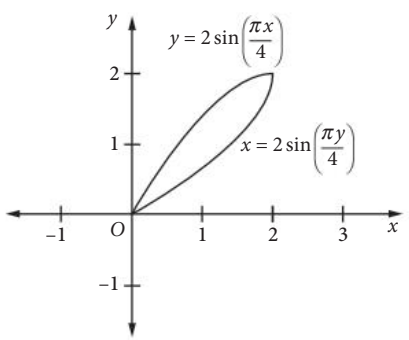
3 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{18}$ (c) $\frac{\pi}{12}$ (d) $\frac{\pi}{4}$ (e) $\frac{\pi}{4}$ (f) $\frac{\pi}{3}$ (g) $\frac{\pi}{4}$ (h) $-\frac{\pi}{6}$

(i) $\frac{\pi}{6}$ (j) $\frac{\pi}{12}$ (k) $\frac{\pi}{4} + \frac{1}{2}\log_e 2$ (l) $\frac{\pi}{2}$ (m) $\frac{\pi}{8}$ (n) $-\frac{\pi}{9}$

(o) $\frac{\pi}{8}$ (p) $\frac{\pi\sqrt{2}}{8}$ (q) $\frac{\pi}{2}$ (r) $\frac{\pi}{3}$ (s) $\frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$ (t) $\frac{\pi}{4} - \frac{1}{2}\log_e 2$

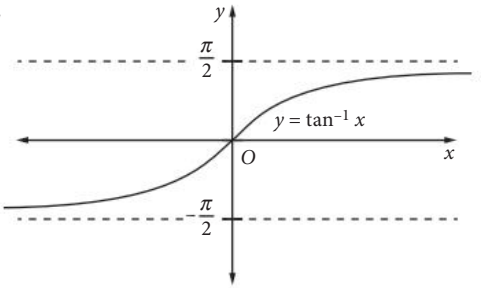
4 $y = 1 + \sin^{-1}x$ **5** $\frac{\pi}{2}$

6 $\frac{16}{\pi} - 4$



7 $\frac{\pi^2}{6}$

8 $\frac{\pi\sqrt{3}}{3} - \log_e 2$



9 When graphed, the area above the x-axis is equal to the area below the x-axis by symmetry. In general, $\int_{-a}^{a} f(x)\,dx = 0$ for every odd function $f(x)$.

10 (a) $\int_{\frac{3}{5}}^{4} \frac{dx}{x^2+1} = \tan^{-1}4 - \tan^{-1}\frac{3}{5}$

Both $\tan^{-1}4$ and $\tan^{-1}\frac{3}{5}$ are values between 0 and $\frac{\pi}{2}$,

and $\tan^{-1}4 > \tan^{-1}\frac{3}{5}$, so $0 < \tan^{-1}4 - \tan^{-1}\frac{3}{5} < \frac{\pi}{2}$.

Further, $\tan(\tan^{-1}4 - \tan^{-1}\frac{3}{5}) = 1$

when evaluated using $\tan(\alpha - \beta)$ expansion.

$\therefore \tan^{-1}4 - \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$ (i.e. the first quadrant angle)

(b) $\int_{-2}^{3} \frac{dx}{x^2+1} = \tan^{-1}3 - \tan^{-1}(-2) = \tan^{-1}3 + \tan^{-1}2$

Both $\tan^{-1}3$ and $\tan^{-1}2$ are values between $\frac{\pi}{4}$ and $\frac{\pi}{2}$

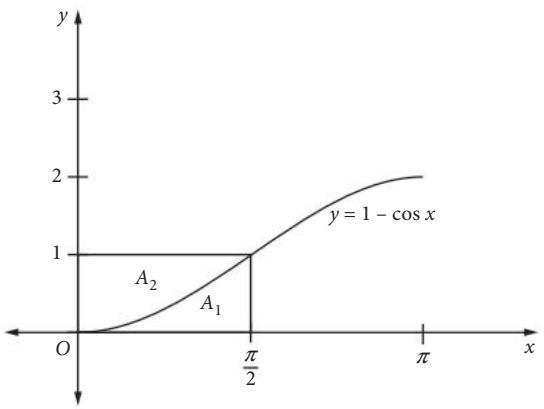
So $\frac{\pi}{2} < \tan^{-1}3 + \tan^{-1}2 < \pi$

Further, $\tan(\tan^{-1}3 + \tan^{-1}2) = -1$ when evaluated using $\tan(\alpha - \beta)$ expansion.

$\therefore \tan^{-1}3 + \tan^{-1}2 = \frac{3\pi}{4}$ (i.e. the scond quadrant angle)

11 $\frac{\pi^2}{4}, \frac{\pi^2}{4}$ **12** $\cos^{-1}x$, 1

13 (a)



(b) $\frac{\pi}{2} - 1$ (c) $f^{-1}(x) = \cos^{-1}(1-x)$, $0 \le x \le 2$, $0 \le y \le \pi$

(d) 1

14 (a) Product rule: $\frac{d}{dx}(x\sin^{-1}x) = 1\times\sin^{-1}x + x\times\frac{1}{\sqrt{1-x^2}}$

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

(b) From (a): $\int \sin^{-1}x\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x$

Hence $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \frac{1}{2}\sin^{-1}\frac{1}{2} + \sqrt{\frac{3}{4}} - (0 + \sqrt{1})$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

15 $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \int \frac{\sqrt{1-\cos^2\theta}}{\cos^2\theta}\times(-\sin\theta)\,d\theta$

$\displaystyle= \int \frac{-\sin^2\theta}{\cos^2\theta}\,d\theta = \int -\tan^2\theta\,d\theta$

$\displaystyle= \int (1-\sec^2\theta)\,d\theta$

$= \theta - \tan\theta + C = \cos^{-1}x - \tan(\cos^{-1}x) + C$

$\displaystyle= \cos^{-1}x - \frac{\sqrt{1-x^2}}{x} + C$

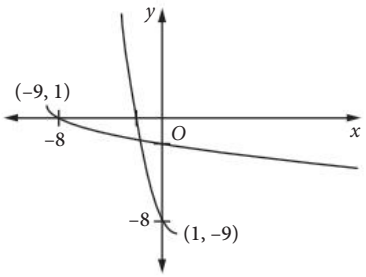
16 $-\frac{1}{3}(9-x^2)^{\frac{3}{2}} + C$

CHAPTER REVIEW 10

1 (a) vertex $(1,-9)$, intercepts $(0,-8)$, $(4,0)$, $(-2,0)$

(b) $x \le 1$ (c) $y = 1 - \sqrt{x+9}$

(d)



(e) Intersect on line $y = x$. Solving simultaneously $y = f(x)$ and $y = x$ to find the point of intersection:

$x^2 - 3x - 8 = 0$, $x = \dfrac{3 \pm \sqrt{41}}{2}$

But domain of f is $x \le 1$ $\quad \therefore \alpha = \dfrac{3-\sqrt{41}}{2}$

(f) For f^{-1}, $\dfrac{dy}{dx} = -\dfrac{1}{2\sqrt{x+9}}$.

Evaluating this for $x = \dfrac{3-\sqrt{41}}{2}$ is very difficult.

Better method: use the fact that at points of intersection of f and f^{-1}, product of their gradients is 1.

For f, $\dfrac{dy}{dx} = 2x - 2$. At T, $x = \dfrac{3-\sqrt{41}}{2}$, gradient of

$f = 2\left(\dfrac{3-\sqrt{41}}{2}\right) - 2 = 1 - \sqrt{41}$

$\therefore$ gradient of f^{-1} at $T = \dfrac{1}{1-\sqrt{41}} = -\dfrac{1+\sqrt{41}}{40}$

2 (a) $\frac{\pi}{3}$ (b) $\frac{3\pi}{4}$ (c) 1 (d) $\frac{\pi}{6}$ (e) $\frac{\pi}{6}$ (f) $\frac{3}{5}$ (g) $\frac{4\sqrt{21}}{17}$

3 (a) Both $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{1}{2}$ are between 0 and $\frac{\pi}{4}$

$\therefore 0 < \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2} < \frac{\pi}{2}$

i.e. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}$ is a first quadrant angle.

Now, $\tan(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2}) = 2$

using $\tan(\alpha+\beta)$ expansion

$\therefore \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{2} = \tan^{-1}2$

(b) Both $\tan^{-1}\frac{3}{4}$ and $\tan^{-1}\frac{1}{2}$ are between 0 and $\frac{\pi}{4}$,

and $\tan^{-1}\frac{3}{4} > \tan^{-1}\frac{1}{2}$

$\therefore 0 < \tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2} < \frac{\pi}{2}$

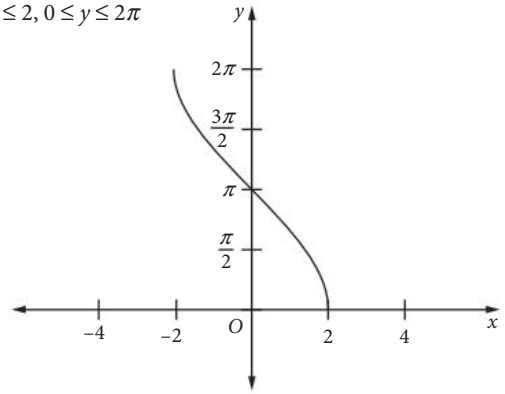
i.e. $\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2}$ is a first quadrant angle.

Now, $\sin(\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2}) = \frac{2\sqrt{5}}{25}$

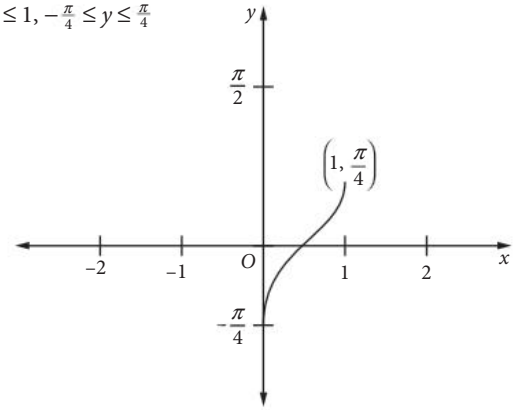
using $\sin(\alpha-\beta)$ expansion

$\therefore \tan^{-1}\frac{3}{4} - \tan^{-1}\frac{1}{2} = \sin^{-1}\frac{2\sqrt{5}}{25}$

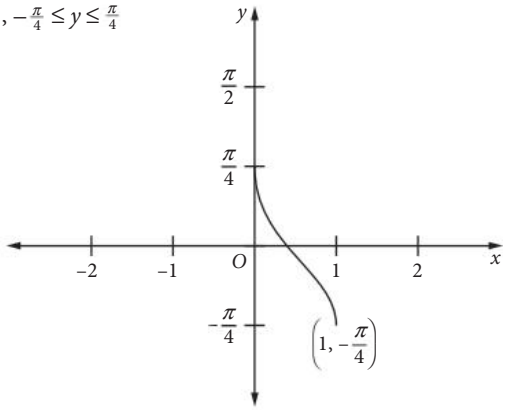
4 (a) $-2 \le x \le 2$, $0 \le y \le 2\pi$



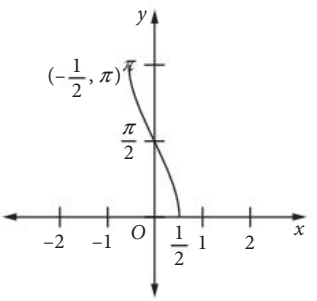
(b) $0 \le x \le 1$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$



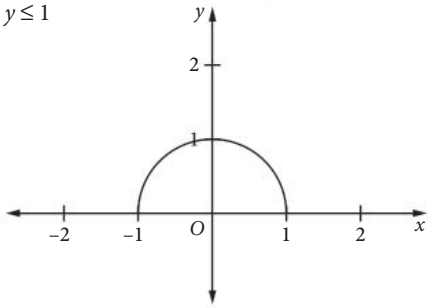
(c) $0 \le x \le 1$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$



(d) $-\frac{1}{2} \le x \le \frac{1}{2}$, $0 \le y \le \pi$

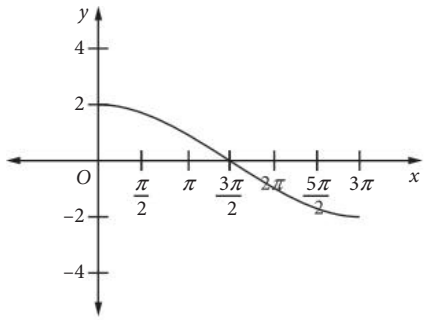


(e) $-1 \le x \le 1$, $0 \le y \le 1$



5 $y = -x - 1 - \frac{\pi}{4}$

6



7 (a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{-2}{\sqrt{1-4x^2}}$ (c) $\frac{2}{4+x^2}$

(d) $e^x \sin^{-1} x + \frac{e^x}{\sqrt{1-x^2}}$ (e) $\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$ (f) $\frac{-2}{4+x^2}$

(g) $\frac{-2x}{\sqrt{1-4x^2}} + \cos^{-1} 2x$ (h) $\frac{1}{2\sqrt{\sin^{-1} x}\sqrt{1-x^2}}$

(i) $\frac{3}{\tan^{-1} x(1+x^2)}$

8 (a) $\frac{-x}{\sqrt{1-x^2}} + \cos^{-1} x$ (b) $x\cos^{-1} x - \sqrt{1-x^2} + C$

9 (a) $\sin^{-1} \frac{x}{2\sqrt{2}} + C$ (b) $\frac{1}{8}\tan^{-1} \frac{x}{8} + C$ (c) $\frac{2}{3}\tan^{-1} \frac{3x}{2} + C$

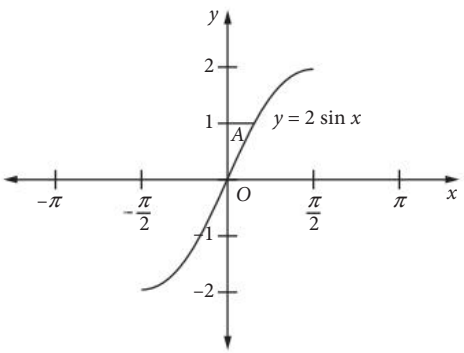
(d) $\frac{1}{2\sqrt{2}}\sin^{-1} \sqrt{2}x + C$ (e) $\sin^{-1} \frac{x+2}{4} + C$

10 (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9\sqrt{2}}$

11 (a) RHS $= \frac{x(1+x^2)}{1+x^2} + \frac{2}{1+x^2} = \frac{x^3+x+2}{1+x^2}$ = LHS

(b) $\frac{1}{2}x^2 + 2\tan^{-1} x + C$

12 (a)



(b) $\frac{\pi}{6} + \sqrt{3} - 2$

13 (a) $\log_e(1+x) + C$ (b) $\tan^{-1} x + C$ (c) $\frac{1}{2}\log_e(1+x^2) + C$

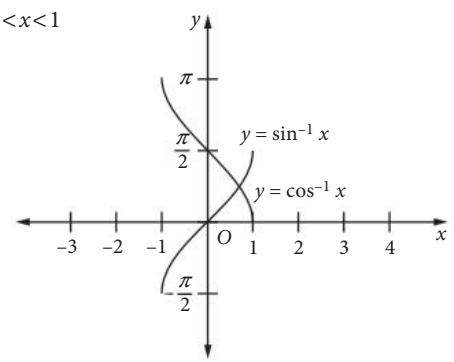
(d) $\log_e x + \frac{1}{2}x^2 + C$ (e) $x - \tan^{-1} x + C$

(f) $\frac{1}{2}x^2 - \frac{1}{2}\log_e(x^2+1) + C$ (g) $\frac{-1}{1+x} + C$ (h) $2\sqrt{1+x} + C$

(i) $\sin^{-1} x + C$ (j) $-\sqrt{1-x^2} + C$ (k) $\frac{1}{\sqrt{2}}\tan^{-1} \frac{e^x}{\sqrt{2}} + C$

(l) $\frac{1}{6}(\tan^{-1} 3x)^2 + C$

14 (a) $0 < x < 1$



(b) $\left(\frac{\pi}{4}, 0\right)$

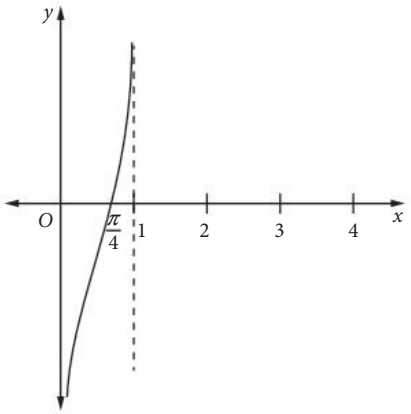
(c) $y = \log_e(\sin^{-1} x) - \log_e(\cos^{-1} x)$

$\therefore \frac{dy}{dx} = \frac{1}{\sin^{-1} x\sqrt{1-x^2}} + \frac{1}{\cos^{-1} x\sqrt{1-x^2}}$

Over the domain $0 < x < 1$, every factor in the expression is positive $\therefore \frac{dy}{dx} > 0$ for all x in the domain

Hence the function is increasing for all x in the domain.

(d)



15 (a) At A, $\cos^2 x = \tan^2 x$

$\therefore \cos^2 x = \sec^2 x - 1$

$\cos^4 x = 1 - \cos^2 x$

$\cos^4 x + \cos^2 x - 1 = 0$

$\therefore \cos^2 x = \frac{-1 \pm \sqrt{5}}{2}$

But $\frac{-1-\sqrt{5}}{2}$ is negative

$\therefore \cos^2 x = \frac{-1+\sqrt{5}}{2}$

$\cos x = \pm\sqrt{\frac{-1+\sqrt{5}}{2}}$

But $0 \le x \le \frac{\pi}{2}$

$\therefore \cos x > 0$

$\therefore \cos x = \sqrt{\frac{-1+\sqrt{5}}{2}}$

$x = \cos^{-1}\sqrt{\frac{-1+\sqrt{5}}{2}}$

(b) 0.5 units^2

16 (a) $\frac{\pi}{2}$

(b) $u = \sin^{-1} x \quad \therefore du = \frac{1}{\sqrt{1-x^2}}dx$

Where $x = 0$, $u = 0$; where $x = \frac{1}{\sqrt{2}}$, $u = \frac{\pi}{4}$

$$\int_0^{\frac{1}{\sqrt{2}}} \cos^{-1} x\sqrt{\frac{\sin^{-1} x}{1-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \left(\frac{\pi}{2} - u\right)\sqrt{u}\,du$$

$$= \left[\frac{\pi}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^{\frac{\pi}{4}} = \frac{\pi^{\frac{5}{2}}}{24} - \frac{\pi^{\frac{5}{2}}}{80} = \frac{7\pi^{\frac{5}{2}}}{240}$$

CHAPTER 11

EXERCISE 11.1

1 (a) $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

(b) $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

(c) $1 + 7b + 21b^2 + 35b^3 + 35b^4 + 21b^5 + 7b^6 + b^7$

(d) $1 + 6x + 12x^2 + 8x^3$

(e) $1 + 20x + 160x^2 + 640x^3 + 1280x^4 + 1024x^5$

(f) $1 + 18x + 135x^2 + 540x^3 + 1215x^4 + 1458x^5 + 729x^6$

(g) $1 - 3x + 3x^2 - x^3$

(h) $1 - 8a + 28a^2 - 56a^3 + 70a^4 - 56a^5 + 28a^6 - 8a^7 + a^8$

(i) $1 - 8x + 24x^2 - 32x^3 + 16x^4$

(j) $1 + \frac{5x}{2} + \frac{5x^2}{2} + \frac{5x^3}{4} + \frac{5x^4}{16} + \frac{x^5}{32}$

(k) $1 - 2x + \frac{5x^2}{3} - \frac{20x^3}{27} + \frac{5x^4}{27} - \frac{2x^5}{81} + \frac{x^6}{729}$

(l) $1+2x+\frac{8x^2}{5}+\frac{16x^3}{25}+\frac{16x^4}{125}+\frac{32x^5}{3025}$

(m) $1+3x^2+3x^4+x^6$ (n) $1-4x^2+6x^4-4x^6+x^8$

(o) $x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$

2 (a) 6 (b) 21 (c) 10 (d) 27 (e) 40 (f) 54

3 (a) 32 (b) −64 (c) $\frac{80}{27}$

4 B **5** (a) $10x^3$ (b) $-280a^3$ (c) $\frac{135x^3}{16}$

6 (a) $1+4\sqrt{2}+6\times2+4\times2\sqrt{2}+4=17+12\sqrt{2}$

(b) $1-3\sqrt{3}+3\times3-3\sqrt{3}=10-6\sqrt{3}$

(c) $1+6\sqrt{5}+15\times5+20\times5\sqrt{5}+15\times25+6\times25\sqrt{5}+125$
$=576+256\sqrt{5}$

(d) $1+6\sqrt{3}+36+24\sqrt{3}=37+30\sqrt{3}$

(e) $1-12\sqrt{2}+108-216\sqrt{2}+324=433-228\sqrt{2}$

(f) $1+\frac{5\sqrt{3}}{2}+\frac{15}{2}+\frac{15\sqrt{3}}{4}+\frac{45}{16}+\frac{9\sqrt{3}}{32}=\frac{362+209\sqrt{3}}{32}$

7 (a) $\left(1-\frac{1}{10}\right)^6$
$\approx1-0.6+0.15-0.02+0.0015-0.00006+$
≈0.5314

(b) $\left(1-\frac{1}{20}\right)^5=1-0.25+0.025-0.00125+0.00003+\quad\approx0.7738$

(c) $\left(1-\frac{1}{100}\right)^4=1-0.04+0.0006-\quad\approx0.9606$

EXERCISE 11.2

1 (a) $8+12x+6x^2+x^3$

(b) $243+405x+270x^2+90x^3+15x^4+x^5$

(c) $b^4+8b^3+24b^2+32b+16$

(d) $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$

(e) $x^8-8x^7y+28x^6y^2-56x^5y^3+70x^4y^4-56x^3y^5+28x^2y^6-8xy^7+y^8$

(f) $1+18x+135x^2+540x^3+1215x^4+1458x^5+729x^6$

(g) $27-27x+9x^2-x^3$

(h) $256-1024a+1792a^2-1792a^3+1120a^4-448a^5+112a^6-16a^7+a^8$

(i) $81y^4-216y^3x+216y^2x^2-96yx^3+16x^4$

(j) $x^5+5x^3+10x+\frac{10}{x}+\frac{5}{x^3}+\frac{1}{x^5}$

(k) $x^4-4x^2+6-\frac{4}{x^2}+\frac{1}{x^4}$

(l) $x^{12}+6x^9+15x^6+20x^3+15+\frac{6}{x^3}+\frac{1}{x^6}$

2 (a) $16+32\sqrt{2}+24\times2+8\times2\sqrt{2}+4=68+48\sqrt{2}$

(b) $64-48\sqrt{3}+36-3\sqrt{3}=100-51\sqrt{3}$

(c) $8+24\sqrt{10}+300+200\sqrt{10}+750+150\sqrt{10}+125$
$=1183+374\sqrt{10}$

(d) $25\sqrt{5}+125\sqrt{3}+150\sqrt{5}+150\sqrt{3}+45\sqrt{5}+9\sqrt{3}$
$=220\sqrt{5}+284\sqrt{3}$

(e) $144-288\sqrt{6}+1296-432\sqrt{6}+324=1764-720\sqrt{6}$

(f) $36\sqrt{6}+360\sqrt{3}+720\sqrt{6}+1440\sqrt{3}+720\sqrt{6}+288\sqrt{3}$
$=1476\sqrt{6}+2088\sqrt{3}$

3 (a) 18 (b) −84 (c) 7

4 C **5** (a) 81 (b) 20 (c) 15

EXERCISE 11.3

1 (a) 10 (b) 70 (c) 210 (d) 35 (e) 126 (f) 455

2 (a) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$

(b) $16x^4+96x^3y+216x^2y^2+216xy^3+81y^4$

(c) $x^{10}+5x^8a+10x^6a^2+10x^4a^3+5x^2a^4+a^5$

(d) $a^6+6a^4+15a^2+20+\frac{15}{a^2}+\frac{6}{a^4}+\frac{1}{a^6}$

(e) $x^8-12x^6y^2+54x^4y^4-108x^2y^6+81y^8$

(f) $\frac{32a^5}{243}-\frac{40a^4}{27b}+\frac{20a^3}{3b^2}-\frac{15a^2}{b^3}+\frac{135a}{8b^4}-\frac{243}{32b^5}$

(g) $16y^4-48y^3x^2+54y^2x^4-27yx^6+\frac{81x^8}{16}$

(h) $x^{18}-12x^{13}+60x^8-160x^3+\frac{240}{x^2}-\frac{192}{x^7}+\frac{64}{x^{12}}$

(i) $\frac{a^{10}}{32}-\frac{15a^8b}{16}+\frac{45a^6b^2}{4}-\frac{135a^4b^3}{2}+\frac{405a^2b^4}{2}-243b^5$

3 (a) 40 (b) −12 (c) −1 (d) 540 (e) 280 (f) −48 384

4 (a) $80x^2$ (b) $-108x$ (c) $-x^3$ (d) $540y^3x^3$ (e) $560x^4y^3$ (f) $-108\,864x^5z^3$

5 A

6 (a) $\binom{8}{3}\left(\frac{m}{2}\right)^5(3n)^3=\frac{189m^5n^3}{4}$ (b) $\binom{11}{6}\left(\frac{2x}{3}\right)^5\left(\frac{-3}{2x}\right)^6=\frac{693}{x}$

(c) $\binom{7}{4}a^3\left(\frac{-2b}{3}\right)^4=\frac{560}{81}a^3b^4$

(d) $\binom{n}{3}x^{n-3}\left(\frac{-1}{x}\right)^3=-\frac{n(n-1)(n-2)x^{n-6}}{6}$

7 (a) $\binom{8}{r}x^{8-r}\left(\frac{1}{x}\right)^r$, $r=3$, coefficient $=56$

(b) $\binom{12}{r}(2x^2)^{12-r}(-3x)^r$, $r=7$,
coefficient $=-11\times2^8\times3^9=-55\,427\,328$

(c) $\binom{9}{r}(3a)^{9-r}\left(-\frac{b}{3}\right)^r$, $r=4$, coefficient $=378$

(d) $\binom{13}{r}(2a)^{13-r}\left(\frac{-3}{a}\right)^r$, $r=2$,
coefficient $=13\times2^{12}\times3^3=1\,437\,696$

8 $\binom{2n}{k+1}a^{2n-(k+1)}b^{k+1}=\frac{(2n)!}{(k+1)!(2n-k-1)!}a^{2n-k-1}b^{k+1}$

EXERCISE 11.4

1 (a) −5376 (b) 495 (c) $\frac{224}{9}$ (d) $\frac{(2n)!}{(n!)^2}\times2^n$

2 (a) $-\frac{63}{8}x^5$ (b) $\frac{189}{8}a^5b^{12}$; $\frac{21}{16}a^4b^{15}$

3 D **4** 416 **5** $10x^2-8$

6 (a) 1.0829 (b) 1207 (c) 0.904 38 (d) 0.960 751

7 $\left(x^2-\frac{1}{x^2}\right)^6\left(x+\frac{1}{x}\right)^2$
$=\left(x^{12}-6x^8+15x^4-20+\frac{15}{x^4}-\frac{6}{x^8}+\frac{1}{x^{12}}\right)\left(x^2+2+\frac{1}{x^2}\right)$

(a) −5 (b) 0 (c) −12

8 $\left(a^2-\frac{1}{a^2}\right)^3\left(a-\frac{1}{a}\right)^2=\left(a^6-3a^2+\frac{3}{a^2}-\frac{1}{a^6}\right)\left(a^2-2+\frac{1}{a^2}\right)$; −6

9 $x^2-3x+2=(x-1)(x-2)$; $64-576x+2352x^2-5706x^3$

10 $1-6x+12x^2-42x^4$; 0.2128

11 $\binom{n}{3}2^{n-3}3^3x^3$, $\binom{n}{4}2^{n-4}3^4x^4$; $n=8$

12 $5b+3a=9$, $10b^2+15ab+3a^2=12$; $4a^2-9a-34=0$;
$a=-2$, $b=3$; $a=4.25$, $b=-0.75$

13 $\binom{15}{r-1}2^{16-r}3^{r-1}x^{16-r}$, $\binom{15}{15-r+1}2^{r-1}3^{16-r}x^{r-1}$; $r=7$

14 $1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{6}x^3+\quad$; $n=7$
($n=2$ is not a solution)

15 $\binom{n}{r-1}x^{r-1}$, $\binom{n}{r}x^r$, $\binom{n}{r+1}x^{r+1}$; $n=9$

16 $a=1$, $b=2$, $n=6$

17 (a) $\frac{3(11-r)y}{2rx}$ (b) $\frac{(r-9)b}{3ra}$

18 (a) T_6 (b) T_9 (c) T_5 (d) T_5 and T_6 (e) T_4 (f) T_4

EXERCISE 11.5

1 $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{n}x^n$

(a) Let $x = 1$:

$(1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{n}$

$2^n = 1 + \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{n}$

$2^n - 1 = \binom{n}{1} + \binom{n}{2} + \quad + \binom{n}{n}$

$\therefore \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1$

(b) Let $x = -2$:

$(1-2)^n = \binom{n}{0} + \binom{n}{1}(-2) + \binom{n}{2}(-2)^2 + \quad + \binom{n}{n}(-2)^n$

$(-1)^n = \binom{n}{0} - 2\binom{n}{1} + 4\binom{n}{2} + \quad + (-2)^n\binom{n}{n}$

$\therefore \sum_{k=0}^{n} (-2)^k \binom{n}{k} = (-1)^n$

(c) Let $x = 3$:

$(1+3)^n = \binom{n}{0} + \binom{n}{1} \times 3 + \binom{n}{2} \times 3^2 + \quad + \binom{n}{n} \times 3^n$

$4^n = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \quad + 3^n\binom{n}{n}$

$2^{2n} = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + \quad + 3^n\binom{n}{n}$

$\therefore \sum_{k=0}^{n} 3^k \binom{n}{k} = 2^{2n}$

2 $(1+x)^{m+n} = 1 + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \quad + \binom{m+n}{r}x^r +$

The coefficient of x^3 is $\binom{m+n}{3}$

$(1+x)^m(1+x)^n = \left[1 + \binom{m}{1}x + \binom{m}{2}x^2 + \quad + \binom{m}{r}x^r + \quad\right] \times \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad\right]$

The coefficient of x^3 is $\binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}$

Equating the coefficients of x^3: $\binom{m+n}{3} = \binom{m}{3} + \binom{m}{2}\binom{n}{1} + \binom{m}{1}\binom{n}{2} + \binom{n}{3}$

3 $(1+x)^{m+3} = 1 + \binom{m+3}{1}x + \binom{m+3}{2}x^2 + \quad + \binom{m+3}{r}x^r +$

The coefficient of x^r is $\binom{m+3}{r}$

$(1+x)^m(1+x)^3 = \left[1 + \binom{m}{1}x + \binom{m}{2}x^2 + \quad + \binom{m}{r}x^r + \quad\right] \times \left[1 + \binom{3}{1}x + \binom{3}{2}x^2 + x^3\right]$

The coefficient of x^r is $\binom{m}{r} + \binom{m}{r-1}\binom{3}{1} + \binom{m}{r-2}\binom{3}{2} + \binom{m}{r-3} = \binom{m}{r} + 3\binom{m}{r-1} + 3\binom{m}{r-2} + \binom{m}{r-3}$

Equating the coefficients of x^r: $\binom{m+3}{r} = \binom{m}{r} + 3\binom{m}{r-1} + 3\binom{m}{r-2} + \binom{m}{r-3}$

4 $(1+x)^{n+2} = 1 + \binom{n+2}{1}x + \binom{n+2}{2}x^2 + \quad + \binom{n+2}{r}x^r +$

The coefficient of x^r is $\binom{n+2}{r}$

$(1+x)^n(1+x)^2 = \left[1 + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad\right] \times \left[1 + \binom{2}{1}x + x^2\right]$

The coefficient of x^r is $\binom{n}{r} + \binom{n}{r-1}\binom{2}{1} + \binom{n}{r-2} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

Equating the coefficients of x^r: $\binom{n+2}{r} = \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

5 (a) The series has $n+1$ terms, so: $S_{n+1} = \dfrac{(1+x)^{n+1} - 1}{x} = \dfrac{(1+x)^{n+1}}{x} - \dfrac{1}{x}$

Coefficient of x^r comes from term in $x^{r+1} \div x$, i.e. $\binom{n+1}{r+1}$

Starting with term $(1+x)^n$ and working backwards to term $(1+x)^r$, coefficient of x^r is: $\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \quad + \binom{r}{r}$

Equating coefficients: $\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \quad + \binom{r}{r} = \binom{n+1}{r+1}$

(b) Common ratio $\frac{1+x}{x}$, $S_{n+1} = (1+x)^{n+1} - x^{n+1}$

Coefficient of x^r is $\binom{n+1}{r}$

Starting with terms $(1+x)^n$, $x(1+x)^{n-1}$ and working backwards to $x^{r-1}(1+x)^{n-r+1}$, $x^r(1+x)^{n-r}$, coefficient of x^r is:

$\binom{n}{r} + \binom{n-1}{r-1} + \binom{n-2}{r-2} + \quad + \binom{n-r+1}{1} + \binom{n-r}{0}$

Equating coefficients: $\binom{n+1}{r} = \binom{n}{r} + \binom{n-1}{r-1} + \binom{n-2}{r-2} + \quad + \binom{n-r+1}{1} + \binom{n-r}{0}$

$\binom{n+1}{r} = \binom{n}{r} + \binom{n-1}{r-1} + \binom{n-2}{r-2} + \quad + \binom{n-r+1}{1} + 1$

Hence $\binom{n}{r} + \binom{n-1}{r-1} + \binom{n-2}{r-2} + \quad + \binom{n-r+1}{1} = \binom{n+1}{r} - 1$

6 (a) $r\binom{n}{r} = \frac{r \times n!}{r!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}$ (b) $n\binom{n-1}{r-1} = \frac{n \times (n-1)!}{(r-1)!(n-r)!} = \frac{n!}{(r-1)!(n-r)!}$ $\therefore\ r\binom{n}{r} = n\binom{n-1}{r-1}$

(c) $P_1 = \binom{n}{1}x(1-x)^{n-1},\ P_2 = \binom{n}{2}x^2(1-x)^{n-2},\ P_3 = \binom{n}{3}x^3(1-x)^{n-3}, \ldots\ P_n = \binom{n}{n}x^n$

$$P_1 + P_2 + P_3 + \quad + P_n = \binom{n}{1}x(1-x)^{n-1} + \binom{n}{2}x^2(1-x)^{n-2} + \binom{n}{3}x^3(1-x)^{n-3} + \quad + \binom{n}{n}x^n$$

$$= \binom{n}{0}(1-x)^n + \binom{n}{1}x(1-x)^{n-1} + \binom{n}{2}x^2(1-x)^{n-2} + \binom{n}{3}x^3(1-x)^{n-3} + \quad + \binom{n}{n}x^n - \binom{n}{0}(1-x)^n$$

$$= \left((1-x)+x\right)^n - (1-x)^n = 1-(1-x)^n$$

Also:

Now $P_1 + P_2 + P_3 + \ldots + P_n = 1 - (1-x)^n$

Differentiate both sides: $\frac{d}{dx}(P_1) = \frac{d}{dx}\left[\binom{n}{1}x(1-x)^{n-1}\right] = \binom{n}{1}\left[(1-x)^{n-1} - (n-1)x(1-x)^{n-2}\right]$

$$= \binom{n}{1}(1-x)^{n-2}(1-nx)$$

$$\frac{d}{dx}(P_2) = \frac{d}{dx}\left[\binom{n}{2}x^2(1-x)^{n-2}\right]$$

$$= \binom{n}{2}\left[2x(1-x)^{n-2} - (n-2)x^2(1-x)^{n-3}\right]$$

$$= \binom{n}{2}x(1-x)^{n-3}(2-nx)$$

$$\frac{d}{dx}(P_3) = \frac{d}{dx}\left[\binom{n}{3}x^3(1-x)^{n-3}\right]$$

$$= \binom{n}{3}\left[3x^2(1-x)^{n-3} - (n-3)x^3(1-x)^{n-4}\right]$$

$$= \binom{n}{3}x^2(1-x)^{n-4}(3-nx)$$

$$\frac{d}{dx}(P_n) = \frac{d}{dx}\left[\binom{n}{n}x^n\right] = \binom{n}{n}nx^{n-1}$$

$$\frac{d}{dx}\left(1-(1-x)^n\right) = 0 + n(1-x)^{n-1} = n(1-x)^{n-1}$$

Hence: $\binom{n}{1}(1-x)^{n-2}(1-nx) + \binom{n}{2}x(1-x)^{n-3}(2-nx) + \binom{n}{3}x^2(1-x)^{n-4}(3-nx) + \quad + \binom{n}{n}nx^{n-1} = n(1-x)^{n-1}$

Multiply both sides by $(1-x)$:

$$\binom{n}{1}(1-x)^{n-1}(1-nx) + \binom{n}{2}x(1-x)^{n-2}(2-nx) + \binom{n}{3}x^2(1-x)^{n-3}(3-nx) + \quad + \binom{n}{n}nx^{n-1}(1-x) = n(1-x)^n$$

$$\text{LHS} = \binom{n}{1}(1-x)^{n-1} + 2\binom{n}{2}x(1-x)^{n-2} + 3\binom{n}{3}x^2(1-x)^{n-3} + \quad + n\binom{n}{n}x^{n-1} - n\binom{n}{1}x(1-x)^{n-1} - n\binom{n}{2}x^2(1-x)^{n-2} - n\binom{n}{3}x^3(1-x)^{n-3} + \quad - n\binom{n}{n}x^n$$

$$= \binom{n}{1}(1-x)^{n-1} + 2\binom{n}{2}x(1-x)^{n-2} + 3\binom{n}{3}x^2(1-x)^{n-3} + \quad + n\binom{n}{n}x^{n-1} - n\left(P_1 + P_2 + P_3 + \quad + P_n\right)$$

Hence:

$$\binom{n}{1}(1-x)^{n-1} + 2\binom{n}{2}x(1-x)^{n-2} + 3\binom{n}{3}x^2(1-x)^{n-3} + \quad + n\binom{n}{n}x^{n-1} - n\left(P_1 + P_2 + P_3 + \quad + P_n\right) = n(1-x)^n$$

$$\binom{n}{1}(1-x)^{n-1} + 2\binom{n}{2}x(1-x)^{n-2} + 3\binom{n}{3}x^2(1-x)^{n-3} + \quad + n\binom{n}{n}x^{n-1} - n\left(1-(1-x)^n\right) = n(1-x)^n$$

$$\binom{n}{1}(1-x)^{n-1} + 2\binom{n}{2}x(1-x)^{n-2} + 3\binom{n}{3}x^2(1-x)^{n-3} + \quad + n\binom{n}{n}x^{n-1} = n$$

Multiply both sides by x:

$$\binom{n}{1}x(1-x)^{n-1} + 2\binom{n}{2}x^2(1-x)^{n-2} + 3\binom{n}{3}x^3(1-x)^{n-3} + \quad + n\binom{n}{n}x^n = nx$$

$\therefore P_1 + 2P_2 + 3P_3 + \ldots + nPn = nx$

7 (a) $(1-x^2)^n = 1 - \binom{n}{1}x^2 + \binom{n}{2}x^4 - \binom{n}{3}x^6 + \quad + (-1)^r\binom{n}{r}x^{2r} + \quad + (-1)^n x^{2n}$

(b) Coefficient of x^{2r} in expansion of $(1-x^2)^n$ is $(-1)^r\binom{n}{r}$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad + \binom{n}{n}x^n$$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 + \quad + (-1)^r\binom{n}{r}x^r + \quad + (-1)^n\binom{n}{n}x^n$$

Term in x^{2r} in expansion of $(1+x)^n(1-x)^n$ is:

$$+\binom{n}{r+2}x^{r+2}(-1)^{r-2}\binom{n}{r-2}x^{r-2} + \binom{n}{r+1}x^{r+1}(-1)^{r-1}\binom{n}{r-1}x^{r-1} + \binom{n}{r}x^r(-1)^r\binom{n}{r}x^r + \binom{n}{r-1}x^{r-1}(-1)^{r+1}\binom{n}{r+1}x^{r+1} + \binom{n}{r-2}x^{r-2}(-1)^{r+2}\binom{n}{r+2}x^{r+2}$$

$$+ \quad + (-1)^{r-2}\binom{n}{r+2}\binom{n}{r-2}x^{2r} + (-1)^{r-1}\binom{n}{r+1}\binom{n}{r-1}x^{2r} + (-1)^r\binom{n}{r}\binom{n}{r}x^{2r} + (-1)^{r+1}\binom{n}{r-1}\binom{n}{r+1}x^{2r} + (-1)^{r+2}\binom{n}{r-2}\binom{n}{r+2}x^{2r} +$$

$$= (-1)^r\binom{n}{r}\binom{n}{r}x^{2r} + 2(-1)^{r-1}\binom{n}{r-1}\binom{n}{r+1}x^{2r} + 2(-1)^r\binom{n}{r-2}\binom{n}{r+2}x^{2r} + 2(-1)^{r-1}\binom{n}{r-3}\binom{n}{r+3}x^{2r} + \quad + 2(-1)^{2r-1}\binom{n}{1}\binom{n}{2r-1}x^{2r} + 2(-1)^{2r}\binom{n}{0}\binom{n}{2r}x^{2r}$$

Equating coefficients, dividing by $(-1)^r$:

$$\binom{n}{r}=\binom{n}{r}\binom{n}{r}-2\binom{n}{r-1}\binom{n}{r+1}+2\binom{n}{r-2}\binom{n}{r+2}-2\binom{n}{r-3}\binom{n}{r+3}+\quad+2(-1)^{r-1}\binom{n}{1}\binom{n}{2r-1}-2(-1)^r\binom{n}{2r}$$

$$2\binom{n}{r-1}\binom{n}{r+1}-2\binom{n}{r-2}\binom{n}{r+2}+2\binom{n}{r-3}\binom{n}{r+3}+\quad-2(-1)^{r-1}\binom{n}{1}\binom{n}{2r-1}-2(-1)^r\binom{n}{2r}=\binom{n}{r}\binom{n}{r}-\binom{n}{r}$$

$$\binom{n}{r-1}\binom{n}{r+1}-\binom{n}{r-2}\binom{n}{r+2}+\binom{n}{r-3}\binom{n}{r+3}+\quad+(-1)^r\binom{n}{1}\binom{n}{2r-1}-(-1)^r\binom{n}{2r}=\tfrac{1}{2}\binom{n}{r}\left[\binom{n}{r}-1\right]$$

8 (a) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\quad+\binom{n}{r}x^r+\quad+\binom{n}{n}x^n$

(b) Differentiating:

$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\quad+r\binom{n}{r}x^{r-1}+\quad+n\binom{n}{n}x^{n-1}$$

Let $x=1$:

$$n(1+1)^{n-1}=\binom{n}{1}+2\binom{n}{2}\times1+3\binom{n}{2}\times1^2+\quad+r\binom{n}{r}\times1^{r-1}+\quad+n\binom{n}{n}\times1^{n-1}$$

$$n\times2^{n-1}=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\quad+r\binom{n}{r}+\quad+n\binom{n}{n}$$

$$\therefore\ \sum_{k=1}^{n}k\binom{n}{k}=n\times2^{n-1}$$

(c) Integrating:

$$\frac{(1+x)^{n+1}}{n+1}=\binom{n}{0}x+\tfrac{1}{2}\binom{n}{1}x^2+\tfrac{1}{3}\binom{n}{2}x^3+\quad+\tfrac{1}{r+1}\binom{n}{r}x^{r+1}+\quad+\tfrac{1}{n+1}\binom{n}{n}x^{n+1}+K$$

When $x=0,\ K=\frac{1}{n+1}$

$$\therefore\ \frac{(1+x)^{n+1}}{n+1}=\tfrac{1}{n+1}+\binom{n}{0}x+\tfrac{1}{2}\binom{n}{1}x^2+\tfrac{1}{3}\binom{n}{2}x^3+\quad+\tfrac{1}{r+1}\binom{n}{r}x^{r+1}+\quad+\tfrac{1}{n+1}\binom{n}{n}x^{n+1}$$

Let $x=1$:

$$\frac{(1+1)^{n+1}}{n+1}=\tfrac{1}{n+1}+\binom{n}{0}\times1+\tfrac{1}{2}\binom{n}{1}\times1^2+\tfrac{1}{3}\binom{n}{2}\times1^3+\quad+\tfrac{1}{r+1}\binom{n}{r}\times1^{r+1}+\quad+\tfrac{1}{n+1}\binom{n}{n}\times1^{n+1}$$

$$\frac{2^{n+1}}{n+1}=\tfrac{1}{n+1}+\binom{n}{0}+\tfrac{1}{2}\binom{n}{1}+\tfrac{1}{3}\binom{n}{2}+\quad+\tfrac{1}{r+1}\binom{n}{r}+\quad+\tfrac{1}{n+1}\binom{n}{n}$$

$$\binom{n}{0}+\tfrac{1}{2}\binom{n}{1}+\tfrac{1}{3}\binom{n}{2}+\quad+\tfrac{1}{r+1}\binom{n}{r}+\quad+\tfrac{1}{n+1}\binom{n}{n}=\frac{2^{n+1}}{n+1}-\frac{1}{n+1}$$

$$=\frac{2^{n+1}-1}{n+1}$$

9 $(1+x)^{2n}=\binom{2n}{0}+\binom{2n}{1}x+\binom{2n}{2}x^2+\quad+\binom{2n}{r}x^r+\quad+\binom{2n}{2n}x^{2n}$

$x=1$:

$$2^{2n}=\binom{2n}{0}+\binom{2n}{1}+\binom{2n}{2}+\quad+\binom{2n}{r}+\quad+\binom{2n}{n}+\binom{2n}{n+1}+\binom{2n}{n+2}+\quad+\binom{2n}{2n}$$

By symmetry and as $2n$ is even: $\binom{2n}{0}=\binom{2n}{2n}$, $\binom{2n}{1}=\binom{2n}{2n-1}$ up to $\binom{2n}{n-1}=\binom{2n}{n+1}$, with $\binom{2n}{n}$ unpaired

Hence $2^{2n}=2\sum_{r=0}^{n}\binom{2n}{r}-\binom{2n}{n}$

$$2\sum_{r=0}^{n}\binom{2n}{r}=2^{2n}+\binom{2n}{n}$$

$$\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\tfrac{1}{2}\times\frac{(2n)!}{n!n!}$$

$$\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{(2n)!}{2(n!)^2}$$

10 (a) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\quad+\binom{n}{r}x^r+\quad+\binom{n}{n}x^n$

Differentiating:

$$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\quad+r\binom{n}{r}x^{r-1}+\quad+n\binom{n}{n}x^{n-1}$$

Hence $nx(1+x)^{n-1}=\binom{n}{1}x+2\binom{n}{2}x^2+3\binom{n}{3}x^3+\quad+r\binom{n}{r}x^r+\quad+n\binom{n}{n}x^n$

$$=\sum_{r=1}^{n}\binom{n}{r}rx^r$$

(b) Differentiating:

$n\left[(1+x)^{n-1}+x(n-1)(1+x)^{n-2}\right]=\binom{n}{1}+4\binom{n}{2}x+9\binom{n}{3}x^2+\quad+r^2\binom{n}{r}x^{r-1}+\quad+n^2\binom{n}{n}x^{n-1}$

$x=1$:

$n\left[2^{n-1}+(n-1)2^{n-2}\right]=\binom{n}{1}+4\binom{n}{2}+9\binom{n}{3}+\quad+r^2\binom{n}{r}+\quad+n^2\binom{n}{n}$

$n\times 2^{n-2}[2+n-1]=\sum_{r=1}^{n}\binom{n}{r}r^2$

$\sum_{r=1}^{n}\binom{n}{r}r^2=n(n+1)\times 2^{n-2}$

(c) For $n\geq 4$: $\sum_{r=1}^{n}\binom{n}{r}r^2=n(n+1)\times 2^{n-2}$

$\binom{n}{2}2^2+\binom{n}{4}4^2+\binom{n}{6}6^2+\quad+\binom{n}{n}n^2+\binom{n}{1}1^2+\binom{n}{3}3^2+\binom{n}{5}5^2+\quad+\binom{n}{n-1}(n-1)^2=n(n+1)\times 2^{n-2}$

Consider $\binom{n}{1}1^2+\binom{n}{3}3^2+\binom{n}{5}5^2+\quad+\binom{n}{n-1}(n-1)^2=\sum_{r=2}^{n}\binom{n}{r-1}(r-1)^2$ where n is even.

Let $k=r-1$: $\binom{n}{1}1^2+\binom{n}{3}3^2+\binom{n}{5}5^2+\quad+\binom{n}{n-1}(n-1)^2=\sum_{k=1}^{n}\binom{n}{k}k^2$

Hence $2\left[\binom{n}{2}2^2+\binom{n}{4}4^2+\binom{n}{6}6^2+\quad+\binom{n}{n}n^2\right]=n(n+1)\times 2^{n-2}$

So $\binom{n}{2}2^2+\binom{n}{4}4^2+\binom{n}{6}6^2+\quad+\binom{n}{n}n^2=n(n+1)\times 2^{n-3}$ for $n\geq 4$

11 $(1+x)^n=1+\binom{n}{1}x+\binom{n}{2}x^2+\quad+\binom{n}{r}x^r+$

Coefficient of x^r is: $\binom{n}{r}$

$(1+x)^3(1+x)^{n-3}=\left[1+\binom{3}{1}x+\binom{3}{2}x^2+x^3\right]\times\left[1+\binom{n-3}{1}x+\binom{n-3}{2}x^2+\quad+\binom{n-3}{r}x^r+\quad\right]$

Coefficient of x^r is: $\binom{n-3}{r}+\binom{n-3}{r-1}\binom{3}{1}+\binom{n-3}{r-2}\binom{3}{2}+\binom{n-3}{r-3}=\binom{n-3}{r}+3\binom{n-3}{r-1}+3\binom{n-3}{r-2}+\binom{n-3}{r-3}$

Equating coefficients of x^r: $\binom{n}{r}=\binom{n-3}{r}+3\binom{n-3}{r-1}+3\binom{n-3}{r-2}+\binom{n-3}{r-3}$

CHAPTER REVIEW 11

1 (a) $(1+x)^n=\binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\quad+\binom{n}{r}x^r+\quad+\binom{n}{n}x^n$

Differentiating:

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\quad+r\binom{n}{r}x^{r-1}+\quad+n\binom{n}{n}x^{n-1}$

(b) Let $x=2$:

$n(1+2)^{n-1}=\binom{n}{1}+2\binom{n}{2}\times 2+3\binom{n}{3}\times 2^2+\quad+r\binom{n}{r}\times 2^{r-1}+\quad+n\binom{n}{n}\times 2^{n-1}$

$n\times 3^{n-1}=\binom{n}{1}+2\binom{n}{2}\times 2+3\binom{n}{3}\times 2^2+\quad+r\binom{n}{r}\times 2^{r-1}+\quad+n\binom{n}{n}\times 2^{n-1}$

2 $\binom{12}{r}x^{12-r}\left(\frac{-2}{x^2}\right)^r$; $r=4$: $\binom{12}{4}x^8\left(\frac{-2}{x^2}\right)^4=7920$

3 (a) $S_{n+1}=\dfrac{(1+x)^{n+1}-1}{x}$

(b) Coefficient of x^r comes from term in $x^{r+1}\div x$, i.e. $\binom{n+1}{r+1}$

Starting with term $(1+x)^r$ and working forwards to term $(1+x)^n$, coefficient of x^r is: $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\quad+\binom{n}{r}$

Equating coefficients: $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\quad+\binom{n}{r}=\binom{n+1}{r+1}$

4 (a) $x=1$:

$(1+1)^n=\binom{n}{0}+\binom{n}{1}\times 1+\binom{n}{2}\times 1^2+\quad+\binom{n}{r}\times 1^r+\quad+\binom{n}{n}\times 1^n$

$2^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\quad+\binom{n}{r}+\quad+\binom{n}{n}=\sum_{k=0}^{n}\binom{n}{k}$

(b) $\binom{50}{0}+\binom{50}{1}+\binom{50}{2}+\quad+\binom{50}{50}=\sum_{k=0}^{50}\binom{n}{k}=2^{50}$

(c) Differentiating:

$n(1+x)^{n-1}=\binom{n}{1}+2\binom{n}{2}x+3\binom{n}{3}x^2+\quad+r\binom{n}{r}x^{r-1}+\quad+n\binom{n}{n}x^{n-1}$

Let $x = 1$:

$$n(1+1)^{n-1} = \binom{n}{1} + 2\binom{n}{2}\times 1 + 3\binom{n}{3}\times 1^2 + \quad + r\binom{n}{r}\times 1^{r-1} + \quad + n\binom{n}{n}\times 1^{n-1}$$

$$n\times 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \quad + r\binom{n}{r} + \quad + n\binom{n}{n}$$

$$\therefore\ n\times 2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$$

5 $\left(1-x+x^2\right)(1+x)^{2n+1} = \left(1-x+x^2\right)\left(\binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + \quad + \binom{2n+1}{n}x^n + \quad\right)$

Term in x^n: $\binom{2n+1}{n}x^n - x\binom{2n+1}{n-1}x^{n-1} + x^2\binom{2n+1}{n-2}x^{n-2}$

Coefficient: $\binom{2n+1}{n} - \binom{2n+1}{n-1} + \binom{2n+1}{n-2} = \frac{(2n+1)!}{n!(n+1)!} - \frac{(2n+1)!}{(n-1)!(n+2)!} + \frac{(2n+1)!}{(n-2)!(n+3)!}$

$$= (2n+1)!\left(\frac{(n+3)(n+2)(n+1)(n+2) - (n+3)(n+2)(n+1)n + (n+2)(n+1)n(n-1)}{(n+3)!(n+2)!}\right)$$

$$= (2n+1)!\left(\frac{(n+3)(n+2) - (n+3)n + n(n-1)}{(n+3)!n!}\right)$$

$$= \frac{(n^2+n+6)(2n+1)!}{n!(n+3)!}$$

6 Let $a_1 = \binom{n}{r-1}$, $a_2 = \binom{n}{r}$, $a_3 = \binom{n}{r+1}$, $a_4 = \binom{n}{r+2}$.

$$\frac{a_1}{a_1+a_2} = \frac{\binom{n}{r-1}}{\binom{n}{r-1}+\binom{n}{r}} = \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}}$$

$$= \frac{1}{1+\frac{(n-r+1)}{r}} = \frac{r}{r+(n-r+1)} = \frac{r}{n+1}$$

$$\frac{a_3}{a_3+a_4} = \frac{\binom{n}{r+1}}{\binom{n}{r+1}+\binom{n}{r+2}} = \frac{\frac{n!}{(r+1)!(n-r-1)!}}{\frac{n!}{(r+1)!(n-r-1)!} + \frac{n!}{(r+2)!(n-r-2)!}}$$

$$= \frac{1}{1+\frac{(n-r-1)}{r+2}} = \frac{r+2}{r+2+(n-r-1)} = \frac{r+2}{n+1}$$

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{r}{n+1} + \frac{r+2}{n+1} = \frac{2(r+1)}{n+1}$$

$$\frac{2a_2}{a_2+a_3} = \frac{2\binom{n}{r}}{\binom{n}{r}+\binom{n}{r+1}} = \frac{\frac{2\times n!}{r!(n-r)!}}{\frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!}} = \frac{2}{\frac{1+(n-r)}{r+1}} = \frac{2(r+1)}{n+1}$$

$$\therefore\ \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

7 $X = a^n + \binom{n}{2}a^{n-2}b^2 + \binom{n}{4}a^{n-4}b^4 +$

$Y = \binom{n}{1}a^{n-1}b + \binom{n}{3}a^{n-3}b^3 + \binom{n}{5}a^{n-5}b^5 +$

$X+Y = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \binom{n}{5}a^{n-5}b^5 +$

$= (a+b)^n$

$X-Y = a^n - \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 - \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 - \binom{n}{5}a^{n-5}b^5 +$

$= (a-b)^n$

$(X+Y)(X-Y) = (a+b)^n(a-b)^n \quad \therefore X^2 - Y^2 = (a^2-b^2)^n$

8 (a) $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \quad + \binom{n}{n}x^n$

(b) $3^n = \binom{n}{0} + 3\binom{n}{1} + 9\binom{n}{2} + \quad + 3^r\binom{n}{r} + \quad + 3^n\binom{n}{n}$

(c) $(-1)^n = \binom{n}{0} - 2\binom{n}{1} + 4\binom{n}{2} + \quad + (-2)^r\binom{n}{r} + \quad + (-2)^n\binom{n}{n}$

9 $(1+x)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + \quad + \binom{2n+1}{n}x^n + \binom{2n+1}{n+1}x^{n+1} + \quad + \binom{2n+1}{2n+1}x^{2n+1}$

As $(2n+1)$ is odd, there are an even number of terms, so:

By symmetry $\binom{2n+1}{0} = \binom{2n+1}{2n+1}$, $\binom{2n+1}{1} = \binom{2n+1}{2n}$ etc.

Hence: $(1+x)^{2n+1} = 2\left[\binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + \quad + \binom{2n+1}{n}x^n\right]$

Let $x=1$: $2^{2n+1}=2\left[\binom{2n+1}{0}+\binom{2n+1}{1}+\binom{2n+1}{2}+\quad+\binom{2n+1}{n}\right]$

$$2^{2n}=\binom{2n+1}{0}+\binom{2n+1}{1}+\binom{2n+1}{2}+\quad+\binom{2n+1}{n}$$
$$4^{n}=1+\binom{2n+1}{1}+\binom{2n+1}{2}+\binom{2n+1}{3}+\quad+\binom{2n+1}{n}$$
$$4^{n}-1=\binom{2n+1}{1}+\binom{2n+1}{2}+\binom{2n+1}{3}+\quad+\binom{2n+1}{n}$$

10 Let $C_r=\binom{n}{r-1}, C_{r+1}=\binom{n}{r}, C_{r+2}=\binom{n}{r+1}$.

$$\binom{n}{r}-\binom{n}{r-1}=\binom{n}{r+1}-\binom{n}{r}$$
$$\frac{n!}{r!(n-r)!}-\frac{n!}{(r-1)!(n-r+1)!}=\frac{n!}{(r+1)!(n-r-1)!}-\frac{n!}{r!(n-r)!}$$
$$\frac{(n-r+1)!(r+1)!}{r!(n-r)!}-\frac{(n-r+1)!(r+1)!}{(r-1)!(n-r+1)!}=\frac{(n-r+1)!(r+1)!}{(r+1)!(n-r-1)!}-\frac{(n-r+1)!(r+1)!}{r!(n-r)!}$$
$$(n-r+1)(r+1)-r(r+1)=(n-r+1)(n-r)-(n-r+1)(r+1)$$
$$nr+n-r^2-r+r+1-r^2-r=n^2-nr-nr+r^2+n-r-nr-n+r^2+r-r-1$$
$$4nr-4r^2+2=n^2-n$$
$$n+2=n^2-4nr+4r^2$$
$$n+2=(n-2r)^2$$

Hence $n+2$ is a perfect square.

CHAPTER 12

EXERCISE 12.1

1 (a) $\frac{4\times{}^{13}C_5}{{}^{52}C_5}=\frac{33}{16660}$ (b) $\frac{1}{54145}$ (c) $\frac{{}^4C_3\times{}^4C_2}{{}^{52}C_5}=\frac{1}{108290}$

2 D **3** $\frac{9}{9^3}=\frac{1}{81}$ **4** $\frac{1}{720}$ **5** (a) $\frac{3}{11}$ (b) $\frac{3}{44}$

6 $\frac{149}{198}$ **7** $\frac{{}^5C_2\times{}^6C_3}{{}^6C_3\times{}^7C_4}=\frac{2}{7}$ **8** $\frac{19}{10829}$ **9** $\frac{1}{90}$

10 $\frac{1}{7}$ **11** $\frac{1}{3}$ **12** $\frac{1}{35}$ **13** 17 576 000, $\frac{1}{67600}$

14 (a) correct (b) incorrect (c) correct (d) correct

15 $\frac{1}{220}$ **16** $\frac{1}{3}$

17 (a) $\frac{5}{9}\times\frac{4}{8}\times\frac{4}{7}=\frac{10}{63}$ (b) $\frac{10}{21}$ **18** $1-\frac{12}{15}\times\frac{11}{14}\times\frac{10}{13}=\frac{47}{91}$

19 $\frac{{}^8C_4}{{}^8C_1+{}^8C_2+{}^8C_3+{}^8C_4}=\frac{35}{81}$ **20** $\frac{5}{6}$

21 0.041 96, 0.2517, 0.4196, 0.2398, 0.0450, 0.0020

22 $\frac{1}{132}, \frac{5}{44}, \frac{25}{66}, \frac{25}{66}, \frac{5}{44}, \frac{1}{132}$ **23** $\frac{5}{14}, \frac{3}{4}$

24 $\frac{1}{2}$ **25** 0.3412 **26** $\frac{67}{165}$ **27** $\frac{128}{225}$

28 (a) $\frac{1}{2}\times\frac{4}{9}=\frac{2}{9}$ (b) $\frac{5}{9}$ (c) $\frac{8}{10}\times\frac{1}{9}=\frac{4}{45}$ (d) $\frac{2}{45}$

29 (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ **30** (a) $\frac{1}{35}$ (b) $\frac{1}{7}$ (c) $\frac{1}{35}$

31 $\frac{1}{35}$ **32** (a) $\frac{1}{15}$ (b) $\frac{2}{3}$ (c) $\frac{1}{5}$

33 $\frac{1}{2}$ **34** (a) $\frac{2717}{33320}$ (b) $\frac{1}{54145}$ (c) $\frac{2109}{9520}$

35 (a) $\frac{1}{{}^{44}C_6}=\frac{1}{7059052}$ (b) $\frac{4}{{}^{44}C_6}=\frac{1}{1764763}$

36 (a) 60 (b) 120

EXERCISE 12.2

1 $\binom{12}{4}\left(\frac{5}{6}\right)^8\left(\frac{1}{6}\right)^4=0.08883$ **2** $\left(\frac{1}{4}\right)^6=\frac{1}{4096}=0.0002441$

3 (a) $\binom{4}{3}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^3+\left(\frac{4}{5}\right)^4=0.8192$

(b) $\binom{4}{1}\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)+\left(\frac{1}{5}\right)^4=\frac{17}{625}=0.0272$

4 D

5 (a) $0.88^{10}+\binom{10}{1}0.88^9\times0.12+\binom{10}{2}0.88^8\times0.12^2=0.8913$

(b) $1-\left[0.88^{10}+\binom{10}{1}0.88^9\times0.12\right]=0.3417$

6 (a) $\left(\frac{1}{4}\right)^3=\frac{1}{64}$ (b) $\binom{3}{1}\times\left(\frac{3}{4}\right)^2\times\frac{1}{4}=\frac{27}{64}$

(c) $3\times\left(\frac{3}{4}\right)^2\times\frac{1}{4}+\left(\frac{3}{4}\right)^3=\frac{27}{32}$ (d) $\binom{3}{2}\times\left(\frac{1}{4}\right)^2\times\frac{1}{2}=\frac{3}{32}$

7 $1-\left[\left(\frac{5}{6}\right)^{10}+\binom{10}{1}\times\frac{1}{6}\times\left(\frac{5}{6}\right)^9\right]=0.5155$

8 $\left(\frac{1}{2}\right)^4=\frac{1}{16}$ **9** $\frac{64}{15625}$

10 $\binom{10}{1}\times0.6^9\times0.4=0.04031$ The new treatment is effective.

11 (a) $\binom{5}{3}\times0.3^2\times0.7^3=0.3087$

(b) $\binom{5}{3}\times0.3^2\times0.7^3+\binom{5}{2}\times0.3^3\times0.7^2+\binom{5}{1}\times0.3^4\times0.7=0.46935$

(c) $0.7^3\times0.3^2=0.03087$

12 (a) $\binom{10}{9}\times0.6\times0.4^9=0.001573$

(b) $\binom{10}{9}\times0.6\times0.4^9+0.4^{10}=0.001678$

(c) The new treatment is more successful.

13 $\binom{5}{3}\times0.6^2\times0.4^3+\binom{5}{4}\times0.6\times0.4^4+0.4^5=0.3174$

14 $\binom{10}{5}(1-p)^5p^5=2\binom{10}{6}(1-p)^6p^4, 3p=5-5p, p=\frac{5}{8}$

15 (a) $0.97^3\times0.96^2+0.97^3\times\binom{2}{1}\times0.96\times0.04$
$+\binom{3}{1}\times0.97^2\times0.03\times0.96^2=0.9893$

(b) $0.96^2\times\left[0.97^3+\binom{3}{1}\times0.97^2\times0.03\right]=0.9192$

16 $1-\frac{1}{2^n}>0.9, n=4$

17 $\left[\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{10}\right]\times\frac{1}{2^{10}}=\frac{11}{64}$

18 $1-\left(\frac{299}{300}\right)^{150}=0.394$

19 $1-\left(\frac{2}{3}\right)^n>0.9, n=\frac{-1}{\log\frac{2}{3}}=5.7, n=6$

20 $1-\left[\left(\frac{799}{800}\right)^{200}+\binom{200}{1}\times\left(\frac{799}{800}\right)^{199}\times\frac{1}{800}\right]=0.0264$

21 $0.94^{10}+\binom{10}{1}\times0.94^9\times0.06=0.8824$

22 (a) $p=\frac{3}{8}, 1-\left(\frac{5}{8}\right)^4=\frac{3471}{4096}$ (b) $\left(\frac{15}{16}\right)^4$

23 (a) $0.9^4=0.6561$ (b) $1-\left[\left(\frac{4}{5}\right)^4+\binom{4}{1}\times\frac{1}{5}\times\left(\frac{4}{5}\right)^3\right]=\frac{113}{625}$

24 (a) 0.0467, 0.1866, 0.3110, 0.2765, 0.1382, 0.0369, 0.0041
(b) 10 (c) 117

25 $P(5H,5T) = \binom{10}{5} \times \frac{1}{2^{10}} = \frac{63}{256}$,

$P(\text{more H than T}) = \left[1 - \frac{63}{256}\right] \div 2 = \frac{193}{512}$

In 50 tosses: $50 \times \frac{193}{512} = 19$

26 $P(X=3) = \binom{5}{3} \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^3 = \frac{80}{243}$, $1000 \times \frac{80}{243} = 329$

27 (a) $\binom{4}{2} \times 0.2^2 \times 0.8^2 \times \binom{3}{1} \times 0.4^2 \times 0.6 = 0.044\,24$

(b) $\binom{4}{2} \times 0.2^2 \times 0.8^2 \times 0.4^3 + \binom{4}{1} \times 0.2^3 \times 0.8 \times \binom{3}{1} \times 0.4^2 \times 0.6 + 0.2^4 \times \binom{3}{2} \times 0.4 \times 0.6^2 = 0.01789$

28 (a) $0.6^3 \times 0.4^4 = 0.005\,530$ (b) $\binom{7}{3} \times 0.6^4 \times 0.4^3 = 0.2903$

(c) $1 - \left[0.4^7 + \binom{7}{1} \times 0.6 \times 0.4^6 + \binom{7}{2} \times 0.6^2 \times 0.4^5\right] = 0.9037$

29 $\left(\frac{19}{20}\right)^5 \times 200 \approx 155$, $\binom{5}{1} \times \frac{1}{20} \times \left(\frac{19}{20}\right)^4 \times 200 \approx 41$,

$\binom{5}{2} \times \left(\frac{1}{20}\right)^2 \times \left(\frac{19}{20}\right)^3 \times 200 \approx 4$, $\binom{5}{3} \times \left(\frac{1}{20}\right)^3 \times \left(\frac{19}{20}\right)^2 \times 200 \approx 0$,

$\binom{5}{4} \times \left(\frac{1}{20}\right)^4 \times \left(\frac{19}{20}\right) \times 200 \approx 0$, $\left(\frac{1}{20}\right)^5 \times 200 \approx 0$

30 (a) $p = \frac{7}{18}$ (b) $\left(\frac{7}{18}\right)^6 = 0.003\,459$ (c) $\binom{6}{1} \times \left(\frac{7}{18}\right)^5 \times \frac{11}{18} = 0.032\,61$

31 $0.9^{10} + \binom{10}{1} \times 0.9^9 \times 0.1 \times \left[0.9^{10} + \binom{10}{1} \times 0.9^9 \times 0.1 + \binom{10}{2} \times 0.9^8 \times 0.1^2\right] + \binom{10}{2} \times 0.9^8 \times 0.1^2 \times \left[0.9^{10} + \binom{10}{1} \times 0.9^9 \times 0.1\right] + \binom{10}{3} \times 0.9^7 \times 0.1^3 \times 0.9^{10} = 0.8715$

32 (a) $P(X=3) + P(X=4) = \binom{4}{3} qp^3 + p^4 = p^4 + 4p^3q$

(b) $P(X=0) + P(X=1) = q^4 + \binom{4}{1} q^3 p = q^4 + 4pq^3$

(c) $P(X=3) + P(X=4) = p^4 + 4p^3q$
$= (1-q)^4 + 4q(1-q)^3 = (1+3q)(1-q)^3$

(d) $P(Y=2) = p^2 = (1-q)^2$

(e) $P(Z=4) + P(Z=5) + P(Z=6) = \binom{6}{4} q^2 p^4 + \binom{6}{5} qp^5 + p^6$
$= 15q^2(1-q)^4 + 6q(1-q)^5 + (1-q)^6$
$= (1-q)^4[10q^2 + 4q + 1]$

(f) $(1-q)^4[10q^2 + 4q + 1] - (1+3q)(1-q)^3 > 0$,
$2q^2(3-5q) > 0$, $0 < q < \frac{3}{5}$

CHAPTER REVIEW 12

1 0.5

2 (a) $\binom{6}{3} \times \binom{8}{4} = 1400$ (b) $7! = 5040$

(c) ${}^4P_2 \times 5! = 1440$ (d) $\frac{1440}{5040} = \frac{2}{7}$

3 $\frac{{}^3P_1 \times {}^6P_2}{{}^7P_3} = \frac{3}{7}$ **4** $\frac{1 \times 6!}{8!} = \frac{1}{56}$ **5** $1 - \frac{{}^{48}C_3}{{}^{52}C_3} = 0.2174$

6 (a) $\frac{11!}{2!2!2!2!} = 2\,494\,800$ (b) $\frac{\frac{4!}{2!2!} \times \frac{8!}{2!2!}}{\frac{11!}{2!2!2!2!}} = \frac{4}{165}$

7 (a) $\frac{7!}{2!2!} = 1260$ (b) $\frac{\frac{6!}{2!2!} + \frac{6!}{2!}}{\frac{7!}{2!2!}} = \frac{3}{7}$ (c) $\frac{\frac{2 \times 6!}{2!2!} + \frac{6!}{2!}}{\frac{7!}{2!2!}} = \frac{4}{7}$

(d) $1 - \frac{\frac{1 \times 6!}{2!}}{\frac{7!}{2!2!}} = \frac{5}{7}$

8 $\binom{6}{3} \times \left(\frac{1}{2}\right)^6 = \frac{5}{16}$ **9** ${}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 = \frac{5}{324}$

10 (a) $\binom{6}{4} \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^4 = \frac{135}{4096}$

(b) $\binom{6}{4} \times \left(\frac{3}{4}\right)^2 \times \left(\frac{1}{4}\right)^4 + \binom{6}{5} \times \left(\frac{3}{4}\right) \times \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 = \frac{77}{2048}$

(c) $1 - \left(\frac{1}{4}\right)^6 = \frac{4095}{4096}$

11 $\frac{2 \times 5!}{6!} = \frac{1}{3}$ **12** (a) $\binom{16}{4} = 1820$ (b) $\frac{{}^9C_4}{{}^{16}C_4} = \frac{9}{130}$

13 (a) $\binom{7}{4} \times \frac{1}{4^3} \times \left(\frac{3}{4}\right)^4 + \binom{7}{5} \times \frac{1}{4^2} \times \left(\frac{3}{4}\right)^5 + \binom{7}{6} \times \frac{1}{4} \times \left(\frac{3}{4}\right)^6 + \left(\frac{3}{4}\right)^7$
$= \frac{47 \times 3^4}{4^6} \approx 0.9294$

(b) $0.9294^{13} = 0.386$

(c) $1 - [{}^{13}C_{11} \times 0.0706^2 \times 0.9294^{11} + {}^{13}C_{12} \times 0.0706 \times 0.9294^{12} + 0.9294^{13}] = 0.06$

CHAPTER 13

EXERCISE 13.1

1 $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = \frac{100}{4\pi \times 25} = \frac{1}{\pi}$ cm/min

2 $r = h$, $\frac{dV}{dt} = 3$; $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dh} = \pi h^2$

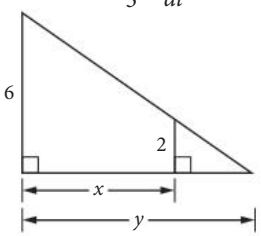
$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$, $\frac{dh}{dt} = \frac{3}{\pi h^2}$

$h = \pi$: $\frac{dh}{dt} = \frac{3}{\pi^3}$ cm s^{-1}

3 $\frac{dx}{dt} = 1$, $\frac{y}{y-x} = \frac{6}{2}$, $y = \frac{3x}{2}$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{3}{2} \times 1 = 1.5$ m s^{-1}

Length of shadow $= y - x$

But $y - x = \frac{y}{3}$, $\frac{d}{dt}(y-x) = \frac{1}{3} \times \frac{dy}{dt} = \frac{1}{3} \times 1.5 = 0.5$ m s^{-1}



4 C

5 $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}}$, $\frac{dy}{dt} = \frac{-x}{\sqrt{4-x^2}} \times 4$

$x = 1$: $\frac{dy}{dt} = \frac{-4}{\sqrt{3}} = -\frac{4\sqrt{3}}{3}$

6 $\frac{1}{8\pi}$ cm/min **7** 12

8 $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$, $\frac{dr}{dt} = \frac{5}{\pi r}$ m/min

9 (a) 15 units/s (b) 3 units/s

10 (a) $\frac{dr}{dt} = 5$; $2r + r\theta = 20$, $\theta = \frac{20}{r} - 2$

$\frac{d\theta}{dr} = -\frac{20}{r^2}$; $\frac{d\theta}{dt} = -\frac{20}{r^2} \times 5$

$r = 10$: $\frac{d\theta}{dt} = -1$ radians per second

(b) $A = \frac{1}{2}r^2\theta = 10r - r^2$, $\frac{dA}{dr} = 10 - 2r$,

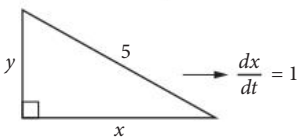
$\frac{dA}{dt} = (10 - 2r) \times 5$

$r = 10$: $\frac{dA}{dt} = -50$ cm^2 s^{-1}

11 $x^2 + y^2 = 25$, $y = \sqrt{25 - x^2}$, $\frac{dy}{dx} = \frac{-x}{\sqrt{25-x^2}}$,

$\frac{dx}{dt} = 1$, $x = t$; at $t = 4$, $x = 4$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = \frac{-x}{\sqrt{25-x^2}} \times 1 = \frac{-4}{\sqrt{25-16}} = -\frac{4}{3}$ m s^{-1}



12 $\frac{dV}{dt} = 30$, $\frac{dV}{dx} = 108 + 3x^2$, $\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$30 = (108 + 3x^2) \times \frac{dx}{dt}$, $\frac{dx}{dt} = \frac{30}{108 + 3x^2}$

At $x = 8$: $\frac{dx}{dt} = 0.1 \text{ cm s}^{-1}$

13 base $= 2y$, depth $= x$, $\frac{y}{x} = \frac{a}{b}$ (similar triangles), $y = \frac{ax}{b}$

$V = \frac{1}{2} \times 2y \times x \times l = \frac{alx^2}{b}$, $\frac{dV}{dt} = -c$, $\frac{dV}{dx} = \frac{2alx}{b}$

$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$, $-c = \frac{2alx}{b} \times \frac{dx}{dt}$

At $x = \frac{b}{2}$, $\frac{dx}{dt} = -\frac{c}{al}$

Level is falling at $\frac{c}{al}$ m/min.

14 base radius $= x$, height $= h$, $x = \frac{3h}{4}$

$V = \frac{1}{3}\pi x^2 h = \frac{3\pi h^3}{16}$, $\frac{dV}{dh} = \frac{9\pi h^2}{16}$, $\frac{dh}{dt} = 2$

At $h = 16$: $\frac{dV}{dt} = \frac{9\pi h^2}{16} \times 2 = 288\pi \text{ cm}^3/\text{min}$

15 $k = 3200$, $P = \frac{3200}{V^{1.4}}$, $\frac{dP}{dV} = \frac{3200 \times (-1.4)}{V^{2.4}} = -\frac{4480}{V^{2.4}}$

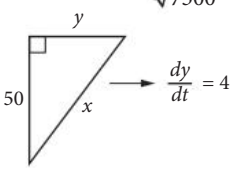
$V = 32$, $\frac{dV}{dt} = 5$, $\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$

$\frac{dP}{dt} = -\frac{4480}{V^{2.4}} \times 5$, $\frac{dP}{dt} = -\frac{4480}{32^{2.4}} \times 5 = -5\frac{15}{32}$ g/cm^2

16 $x^2 = y^2 + 50^2$, $\frac{dy}{dt} = 4$

$y = \sqrt{x^2 - 2500}$, $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 - 2500}}$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

At $x = 100$: $4 = \frac{100}{\sqrt{7500}} \times \frac{dx}{dt}$, $\frac{dx}{dt} = 2\sqrt{3} \text{ m s}^{-1}$

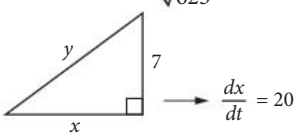


17 $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dt} = 4$, $\frac{dV}{dh} = \pi h^2$, $h = 10$

(a) $\frac{dh}{dt} = \frac{1}{25\pi}$ cm/min (b) $A = \pi h^2$, $\frac{dA}{dt} = 0.8 \text{ cm}^2/\text{min}$

18 $y = \sqrt{x^2 + 49}$, $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 49}}$, $\frac{dx}{dt} = 20$, $\frac{dy}{dt} = \frac{20x}{\sqrt{x^2 + 49}}$

At $x = 24$: $\frac{dy}{dt} = \frac{480}{\sqrt{625}} = 19.2$ m/min



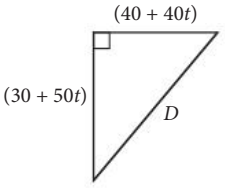
19 (a) $D = 10\sqrt{25 + 62t + 41t^2}$, $\frac{dD}{dt} = \frac{5(82t + 62)}{\sqrt{25 + 62t + 41t^2}}$

At $t = 1$: $\frac{dD}{dt} = 45\sqrt{2}$ km/h

(b) $D = 10\sqrt{(4 - 4t)^2 + (3 - 5t)^2} = 10\sqrt{25 - 62t + 42t^2}$,

$\frac{dD}{dt} = \frac{5(82t - 62)}{\sqrt{25 - 62t + 41t^2}}$

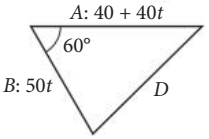
At $t = 1$: $\frac{dD}{dt} = 50$ km/h



20 $t = 0$, $A = 40$, $B = 0$; finding $\frac{dD}{dt}$ at $t = 2$ using cosine rule:

$D = 10\sqrt{16 + 12t + 21t^2}$, $\frac{dD}{dt} = \frac{5(12 + 42t)}{\sqrt{16 + 12t + 21t^2}}$

At $t = 2$: $\frac{dD}{dt} = \frac{5 \times 96}{\sqrt{124}} \approx 43.1$ km/h

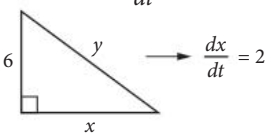


21 $V = \frac{1}{3}\pi h^3$, $\frac{dV}{dh} = \pi h^2$, $\frac{dV}{dt} = 0.2\sqrt{h}$

At $h = 4$: $\frac{dh}{dt} = \frac{1}{40\pi}$ m/min

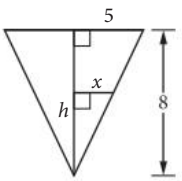
22 $y = \sqrt{x^2 + 36}$, $\frac{dx}{dt} = 2$, $\frac{dy}{dx} = \frac{x}{\sqrt{x^2 + 36}}$

$y = 10$, $x = 8$: $\frac{dy}{dt} = 1.6 \text{ cm s}^{-1}$



23 $\frac{x}{h} = \frac{5}{8}$, $V = \frac{1}{3}(2x)^2 \times h = \frac{25h^3}{48}$, $\frac{dV}{dh} = \frac{25h^2}{16}$

At $h = 4$: $\frac{dh}{dt} = \frac{4}{25} = 0.16$ m/min



24 $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = -4k\pi r^2$ where k is a constant

$\frac{dV}{dr} = 4\pi r^2$, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

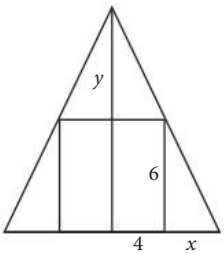
$-4k\pi r^2 = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = -k$

Radius decreases at a constant rate.

25 Similar triangles: $\frac{y}{6} = \frac{4}{x}$, $y = \frac{24}{x}$, $\frac{dy}{dx} = -\frac{24}{x^2}$

$\frac{dy}{dt} = 3$, $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$, $3 = -\frac{24}{x^2} \times \frac{dx}{dt}$

$y = 4$, $x = 6$: $\frac{dx}{dt} = -4.5 \text{ cm s}^{-1}$



26 0.0732 m/min

27 (a) Area $= 60\sin\theta$

$\frac{d(\text{Area})}{d\theta} = 60\cos\theta$, $\frac{d\theta}{dt} = 0.1$, $\frac{d(\text{Area})}{dt} = 6\cos\theta$

$\theta = \frac{\pi}{3}$: $\frac{d(\text{Area})}{dt} = 3\ \text{cm}^2\,\text{s}^{-1}$

(b) Cosine rule: $BC = \sqrt{244 - 240\cos\theta}$,

$\frac{d}{d\theta}(BC) = \frac{120\sin\theta}{\sqrt{244 - 240\cos\theta}}$

$\frac{d}{dt}(BC) = \frac{120\sin\theta}{\sqrt{244 - 240\cos\theta}} \times 0.1$

$= \frac{12\sin\theta}{\sqrt{244 - 240\cos\theta}}$

$\theta = \frac{\pi}{3}$: $\frac{d}{dt}(BC) = \frac{6\sqrt{3}}{\sqrt{244 - 120}} = \frac{3\sqrt{3}}{\sqrt{31}}\ \text{cm}\,\text{s}^{-1}$

28 (a) $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dr} = 4\pi r^2$, $\frac{dV}{dt} = -4k\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, $-4k\pi r^2 = 4\pi r^2 \times \frac{dr}{dt}$, $\frac{dr}{dt} = -k$

(b) $t = 0$, $r = 1$ cm; $t = 10$, $r = 0.5$ cm

$\frac{dr}{dt} = \frac{0.5 - 1}{10} = -0.05$, $r = -0.05t + C$

$1 = C$, $r = 1 - 0.05t$

29 (a) $\frac{dV}{dr} = 4\pi r^2 = S$, $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, $\frac{dV}{dt} = S \times \frac{dr}{dt}$

(b) $\frac{dr}{dt} = -3$, $r = 24$

(i) $r = -3t + C$; $t = 0$, $r = 24$; $C = 24$, $r = 24 - 3t$

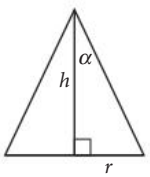
$r = 3$, $t = 7$

(ii) $V = 2304\pi$; $r = 12$, $12 = 24 - 3t$, $t = 4$

30 (a) $V = \frac{1}{3}\pi r^2 h$; $r = h\tan\alpha$:

$V = \frac{1}{3}\pi(h\tan\alpha)^2 h$

$= \frac{1}{3}\pi h^3 \tan^2\alpha$



(b) $\alpha = 30°$, $\frac{dV}{dt} = \pi\sqrt{h}$, $\frac{dV}{dh} = \pi h^2 \tan^2 30° = \frac{1}{3}\pi h^2$,

$\pi\sqrt{h} = \frac{1}{3}\pi h^2 \times \frac{dh}{dt}$

At $h = 9$: $\frac{dh}{dt} = \frac{3\sqrt{h}}{h^2} = \frac{1}{9}\ \text{cm}\,\text{s}^{-1}$

EXERCISE 13.2

1 (a) $N = 30 + Ae^{-0.4t}$, $\frac{dN}{dt} = -0.4Ae^{-0.4t}$, but $Ae^{-0.4t} = N - 30$,

so $\frac{dN}{dt} = -0.4(N - 30)$

(b) 34

2 (a) $N = 40 + Ae^{0.2t}$, $\frac{dN}{dt} = 0.2Ae^{0.2t}$, but $Ae^{0.2t} = N - 40$,

so $\frac{dN}{dt} = 0.2(N - 40)$

(b) 113.9

3 $\frac{dT}{dt} = -k(T - 50)$, $\frac{dt}{dT} = \frac{-1}{k(T - 50)}$, $T \neq 20$

$t = -\frac{1}{k}\int \frac{1}{T - 50}\,dt = -\frac{1}{k}\log_e(T - 50) + C$, $T > 50$

$-k(t - C) = \log_e(T - 50)$, $T - 50 = e^{-k(t-C)}$,

$T = 50 + Ae^{-kt}$ where $A = e^{-kC}$

$t = 0$, $T = 120$: $A = 70$

$t = 10$, $T = 70$: $70 = 50 + 70e^{-10k}$, $k = \frac{1}{10}\log_e 3.5 = 0.125$,

$T = 50 + 70e^{-0.125t}$

(a) $t = 20$: $T = 50 + 70e^{-2.5} = 55.7°\text{C}$

(b) $T = 60$: $60 = 50 + 70e^{-0.125t}$, $e^{-0.125t} = \frac{1}{7}$,

$0.125t = \ln 7$, $t = 15.6$ min

4 B

5 (a) $t = 0$, $T = 1230$: RHS $= 30 + 1200 \times 1 = 1230 =$ LHS

$\frac{dT}{dt} = -1200ke^{-kt} = -k(T - 30)$

(b) $t = 10$, $T = 1030$: $1030 = 30 + 1200e^{-10k}$, $k = \frac{1}{10}\ln\left(\frac{6}{5}\right)$

$t = 20$: $T = 30 + 1200e^{-2\ln\left(\frac{6}{5}\right)} = 30 + 1200 \times \left(\frac{5}{6}\right)^2 = 863.3°\text{C}$

(c) $80 = 30 + 1200e^{-kt}$, $e^{kt} = 24$, $kt = \ln 24$, $t = \frac{10\ln 24}{\ln 1.2} = 174$ min

$= 2\text{h}\,54$ min

6 (a) $t = 0$, $T = 20$: RHS $= -10 + 30 \times 1 = 20 =$ LHS

$\frac{dT}{dt} = -30ke^{-kt} = -k(T + 10)$

(b) $t = 5$, $T = 15$: $15 = -10 + 30e^{-5k}$, $k = \frac{1}{5}\ln\left(\frac{6}{5}\right)$

$t = 10$: $T = -10 + 30e^{-2\ln\left(\frac{6}{5}\right)} = -10 + 30 \times \left(\frac{5}{6}\right)^2$

$= 10\frac{5}{6} \approx 11°\text{C}$

7 (a) $t = 0$, $T = 180$: RHS $= 60 + 120 \times 1 = 180 =$ LHS

$\frac{dT}{dt} = -120ke^{-kt} = -k(T - 60)$

(b) $t = 1$, $T = 120$: $120 = 60 + 120e^{-k}$, $k = \ln 2$

$T = 90$: $90 = 60 + 120e^{-kt}$, $e^{-kt} = \frac{1}{4}$, $t = \frac{\ln 4}{\ln 2} = 2$ min

8 $\frac{di}{dt} = \frac{E}{R} \times \left(0 - \left(-\frac{R}{L}e^{-\frac{Rt}{L}}\right)\right) = \frac{E}{L}e^{-\frac{Rt}{L}}$

But $e^{-\frac{Rt}{L}} = 1 - \frac{Ri}{E}$ so $\frac{di}{dt} = \frac{E}{L}\left(1 - \frac{Ri}{E}\right)$

Hence $L\frac{di}{dt} = E - Ri$ or $L\frac{di}{dt} + Ri = E$

9 (a) $\frac{dV}{dt} = A(0 + ke^{-kt}) = kAe^{-kt}$

But $A - V = Ae^{-kt}$ so $\frac{dV}{dt} = k(A - V)$

(b) If total volume $= B$, then at $t = 5$: $V = \frac{B}{4}$

$\frac{B}{4} = A(1 - e^{-5k})$, $e^{-5k} = \frac{4A - B}{4A}$, $e^{5k} = \frac{4A}{4A - B}$,

$k = \frac{1}{5}\ln\left(\frac{4A}{4A - B}\right)$

At $t = 10$:

$V = A(1 - e^{-10k}) = A\left(1 - e^{-2\ln\left(\frac{4A}{4A-B}\right)}\right)$

$= A\left(1 - e^{\ln\left(\frac{4A-B}{4A}\right)^2}\right) = A\left(1 - \left(\frac{4A - B}{4A}\right)^2\right) = \frac{B}{2} - \frac{B}{16} \times \frac{B}{A}$

Now $B = \lim_{t\to\infty}\left(A(1 - e^{-kt})\right) = A$

so $\frac{B}{A} = 1$ and $V = \frac{B}{2} - \frac{B}{16} = \frac{7B}{16}$

Increase in volume $= \frac{7B}{16} - \frac{B}{4} = \frac{3B}{16}$,

fraction filled in next 5 minutes $= \frac{3}{16}$

(c) $\lim_{t\to\infty} V = \lim_{t\to\infty}\left(A(1 - e^{-kt})\right) = A\lim_{t\to\infty}\left(1 - e^{-kt}\right) = A$

10 (a) $\frac{dx}{dt} = k(20 - 2x)$, $\frac{dt}{dx} = \frac{1}{k(20 - 2x)}$

$t = \frac{1}{2k}\int \frac{1}{10 - x}\,dx = -\frac{1}{2k}\log_e(10 - x) + C$

$t = 0$, $x = 0$: $0 = -\frac{1}{2k}\log_e 10 + C$, $C = \frac{1}{2k}\log_e 10$

$t = -\frac{1}{2k}\log_e(10 - x) + \frac{1}{2k}\log_e 10$, $-2kt = \log_e\left(\frac{10 - x}{10}\right)$,

$\frac{10 - x}{10} = e^{-2kt}$, $x = 10(1 - e^{-2kt})$

(b) $t=5, x=2$: $2=10(1-e^{-10k})$, $e^{-10k}=0.8$, $k=\frac{1}{10}\log_e 1.25$

$x=10\left(1-e^{-\frac{t}{5}\log_e 1.25}\right)$

Where $x=9$: $9=10\left(1-e^{-\frac{t}{5}\log_e 1.25}\right)$, $e^{-\frac{t}{5}\log_e 1.25}=\frac{1}{10}$,

$e^{\frac{t}{5}\log_e 1.25}=10$, $\frac{t}{5}\log_e 1.25=\log_e 10$, $t=\dfrac{5\log_e 10}{\log_e 1.25}=51.6$ min

11 (a) $\dfrac{dy}{dt}=-ke^{-kt}$ but $e^{-kt}=y-3$ so $\dfrac{dy}{dt}=-k(y-3)$

(b) $k=0.08$ (c) $y=3.5$: $\dfrac{dy}{dt}=-0.08(3.5-3)=-0.04$ g/h

(d) $t=0, y=4$, $\lim\limits_{t\to\infty} y=\lim\limits_{t\to\infty}\left(3+e^{-0.08t}\right)=3$,

hence $3<y\le 4$

EXERCISE 13.3

1 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=4+x$, $\frac{1}{2}v^2=4x+\frac{1}{2}x^2+C$

$x=0, v=1$: $C=\frac{1}{2}$, $v^2=8x+x^2+1$

$x=1$: $v=\sqrt{10}$

2 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=7-2x$, $\frac{1}{2}v^2=7x-x^2+C$

$x=0, v=0$: $C=0$, $v^2=14x-2x^2$, $v=\sqrt{14x-2x^2}$, $0\le x\le 7$

3 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=3-4x$, $\frac{1}{2}v^2=3x-2x^2+C$

$x=1, v=0$: $C=-1$, $v^2=6x-4x^2-2$

At rest: $-2(2x^2-3x+1)=0$, $(2x-1)(x-1)=0$, $x=\frac{1}{2}$

4 C

5 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=-\frac{1}{6}\sqrt[3]{x}$, $\frac{1}{2}v^2=-\frac{1}{8}x^{\frac{4}{3}}+C$

$x=0, v=2$: $C=2$, $v^2=4-\frac{1}{4}x^{\frac{4}{3}}$

$v=0$: $x=8$ m

6 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=\cos x$, $\frac{1}{2}v^2=\sin x+C$

$x=0, v=0$: $C=0$, $v^2=2\sin x$, $v=\pm\sqrt{2\sin x}$

7 $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=\sin^2 x=\frac{1}{2}(1-\cos 2x)$,

$\frac{1}{2}v^2=\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C$

$x=0, v=1$: $C=\frac{1}{2}$, $v^2=x-\frac{1}{2}\sin 2x+1$,

$\dfrac{dx}{dt}=\sqrt{x-\frac{1}{2}\sin 2x+1}$

8 $\frac{1}{2}v^2=x^2-x^3+C$

$x=0, v=2$: $C=2$, $v^2=4+2x^2-2x^3$, $v=\sqrt{4+2x^2-2x^3}$,

$v>0$ from initial conditions

9 $\frac{1}{2}v^2=\frac{1}{2}x^2-3x+C$

$x=1, v=-2$: $C=4.5$, $v^2=x^2-6x+9$

$x=0$: $v^2=9$, $v=-3$ $(v<0)$

10 (a) $\ddot{x}=v\dfrac{dv}{dx}=(4+x^2)\times 2x=2x^3+8x$

(b) $\dfrac{dt}{dx}=\dfrac{1}{4+x^2}$, $t=\displaystyle\int\frac{1}{4+x^2}dx=\frac{1}{2}\tan^{-1}\frac{x}{2}+C$

$t=0, x=-2$: $C=\frac{\pi}{8}$, $t=\frac{1}{2}\tan^{-1}\frac{x}{2}+\frac{\pi}{8}$, $\tan^{-1}\frac{x}{2}=2\left(t-\frac{\pi}{8}\right)$,

$x=2\tan 2\left(t-\frac{\pi}{8}\right)$

$t=\frac{\pi}{4}$: $x=2$ m

11 $t=\frac{1}{2}x^2+4x+C$

$t=0, x=0$: $C=0$, $t=\frac{1}{2}x^2+4x$

$x=2$: $t=10$

12 $t=\displaystyle\int\frac{1}{x+4}dx$, $t=\ln(x+4)+C$

$t=0, x=-3$: $C=0$, $t=\ln(x+4)$, $x=e^t-4$

13 $v=2(4-x)^2$, $t=\frac{1}{2}\displaystyle\int\frac{1}{(4-x)^2}dx=\frac{1}{2(4-x)}+C$

$t=0, x=0$: $C=-\frac{1}{8}$, $t=\dfrac{1}{2(4-x)}-\frac{1}{8}$, $x=\dfrac{32t}{8t+1}$

14 $t=\displaystyle\int\frac{1}{(3-x)^2}dx=\frac{1}{3-x}+C$

$t=0, x=2$: $C=-1$, $t=\dfrac{1}{3-x}-1$

(a) $x=\dfrac{3t+2}{t+1}$ (b) $\dot{x}=\dfrac{1}{(t+1)^2}$, $\ddot{x}=\dfrac{-2}{(t+1)^3}=-2(3-x)^3$

15 (a) $\ddot{x}=\dot{x}+0=x+3$

(b) $t=\displaystyle\int\frac{1}{x+3}dx=\ln(x+3)+C$

$t=0, x=-2$: $C=0$, $x=e^t-3$

$t=1$: $x=e-3$

16 $\frac{1}{2}v^2=\dfrac{k}{x}+C$, $t=0, v=0, x=a$:

$C=-\dfrac{k}{a}$, $\dfrac{1}{2}v^2=\dfrac{k}{x}-\dfrac{k}{a}=\dfrac{k(a-x)}{ax}$,

$v^2=\dfrac{2k(a-x)}{ax}$, $|v|=\sqrt{\dfrac{2k(a-x)}{ax}}$

17 (a) $\frac{1}{2}v^2=\frac{1}{2}e^{-2x}+C$, $x=0, v=1$: $C=0$, $v^2=e^{-2x}$, $v=\pm e^{-x}$

$x=0, v>0$: $v=e^{-x}$

(b) $\dfrac{dx}{dt}=e^{-x}$, $\dfrac{dt}{dx}=e^x$, $t=e^x+C$

$t=0, x=0$: $C=-1$, $t=e^x-1$, $x=\ln(t+1)$

18 (a) $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=32x^3+48x^2+16x$, $\frac{1}{2}v^2=8x^4+16x^3+8x^2+C$

$x=-2, v=-8$: $C=0$, $v^2=16x^4+32x^3+16x^2$,

$v^2=16x^2(x^2+2x+1)=16x^2(x+1)^2$

(b) $v=\pm 4x(x+1)$; $x=-2, v<0$: $v=-4x(x+1)$

$\dfrac{dx}{dt}=-4x(1+x)$, $\dfrac{dt}{dx}=-\dfrac{1}{4x(1+x)}$, $-4t=\displaystyle\int\frac{1}{x(1+x)}dx$

(c) $\log_e\left(1-\frac{1}{2}\right)=0+C$, $t=0, x=-2$: $C=-\log_e 2$,

$4t=\log_e\left(1+\frac{1}{x}\right)+\log_e 2=\log_e 2\left(1+\frac{1}{x}\right)$, $2\left(1+\frac{1}{x}\right)=e^{4t}$,

$1+\frac{1}{x}=\frac{1}{2}e^{4t}$, $x=\dfrac{2}{e^{4t}-2}$

19 (a) $\ddot{x}=\dfrac{dv}{dt}=\dfrac{dv}{dx}\times\dfrac{dx}{dt}=8x\times 4(x^2+9)=32x(x^2+9)$

(b) $\dfrac{dx}{dt}=4(x^2+9)$, $\dfrac{dt}{dx}=\dfrac{1}{4}\times\dfrac{1}{9+x^2}$, $t=\frac{1}{4}\times\frac{1}{3}\tan^{-1}\frac{x}{3}+C$

$t=0, x=0$: $C=0$, $t=\frac{1}{12}\tan^{-1}\frac{x}{3}$

$x=3$: $t=\frac{1}{12}\tan^{-1}1=\frac{\pi}{48}$ s

20 (a) $\dfrac{d^2x}{dt^2}=80(0+0.4e^{-0.4t})=0.4\times 80e^{-0.4t}$,

$80e^{-0.4t}=80-v$, $\dfrac{d^2x}{dt^2}=0.4(80-v)\propto(80-v)$

(b) $\dfrac{dx}{dt}=80(1-e^{-0.4t})$, $x=80(t+2.5e^{-0.4t})+C$

$t=0, x=0, v=0$: $C=-200$, $x=80(t+2.5e^{-0.4t})-200$

$t=5$: $x=80(5+2.5e^{-2})-200=227$ m

(c) $v=60$: $e^{-0.4t}=\dfrac{80-60}{80}=0.25$, $t=2.5\log_e 4$

$x=80\left(2.5\log_e 4+2.5\times\frac{1}{4}\right)-200=127$ m

21 $x=0, \ddot{x}=3$: $C=3$, $\dfrac{d}{dx}\left(\frac{1}{2}v^2\right)=kx+3$, $\frac{1}{2}v^2=\frac{1}{2}kx^2+3x$,

$v^2=kx^2+6x$

$\ddot{x}=0$ when $x=160$: $k=-\dfrac{3}{160}$, $v^2=6x-\dfrac{3x^2}{160}$

(a) $x=160$, $v^2=3\times 160$, $v=4\sqrt{30}$ m s^{-1}

(b) $x=80$: $v=6\sqrt{10}$ m s^{-1}

EXERCISE 13.4

1 $x = 6\cos 4t$ is of the form $x = a\cos(nt + \alpha)$: motion is simple harmonic about the origin, $a = 6$, $n = 4$; period $= \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$.
Particle starts 6 cm to the right of O.
$\dot{x} = -24\sin 4t$, particle is initially at rest.
$\ddot{x} = -96\cos 4t$, $\ddot{x} = -16x$, initial acceleration is 96 cm s^{-2} towards O.

2 (a) $\ddot{x} = -9x$, $n = 3$, $T = \frac{2\pi}{3}$, $v = \sqrt{4 - 9x^2}$, $x = \frac{2}{3}\sin 3t$,
$a = \frac{2}{3}$, $v_{\max} = 2$

(b) $\ddot{x} = -9x$, $n = 3$, $T = \frac{2\pi}{3}$, $v^2 = 9\left(\frac{40}{9} - x^2\right)$, $a = \frac{2\sqrt{10}}{3}$,
$v_{\max} = 2\sqrt{10}$

3 $v^2 = 5(4 - x^2)$, $\frac{1}{2}v^2 = \frac{5}{2}(4 - x^2)$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{5}{2}(4 - x^2)\right]$,
$\ddot{x} = \frac{5}{2} \times (-2x) = -5x$: SHM, $n = \sqrt{5}$, $T = \frac{2\pi}{\sqrt{5}}$, $a = 2$

4 B

5 $x = 5\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$, $\dot{x} = -\frac{5\pi}{2}\sin\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$,
$\ddot{x} = -\frac{5\pi^2}{4}\cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right) = -\frac{\pi^2}{4}x$: SHM

(a) $T = 2\pi \div \frac{\pi}{2} = 4\text{ s}$ (b) $a = 5\text{ m}$

(c) $\frac{1}{2} = \cos\left(\frac{\pi}{2}t - \frac{\pi}{3}\right)$, $t = \frac{4}{3}$,
$\dot{x} = -\frac{5\pi}{2}\sin\left(\frac{2\pi}{3} - \frac{\pi}{3}\right) = -\frac{5\pi\sqrt{3}}{4}\text{ m s}^{-1}$

(d) $\ddot{x} = -\frac{\pi^2}{4} \times \left(-\frac{5}{2}\right) = \frac{5\pi^2}{8}\text{ m s}^{-2}$

6 (a) $\frac{1}{2}v^2 = 150 + 50x - \frac{25}{2}x^2$, $\ddot{x} = 50 - 25x = -25(x - 2)$:
SHM around $x = 2$
(i) $x = 2\text{ m}$
(ii) $T = \frac{2\pi}{5}$, $v^2 = 25(6 - x)(2 + x) = 25(16 - y^2)$
where $y = 2 - x$; $a = 4\text{ m}$

(b) $\frac{1}{2}v^2 = 64 - 16x - 8x^2$, $\ddot{x} = -16 - 16x = -16(x + 1)$:
SHM around $x = -1$
(i) $x = -1$
(ii) $T = \frac{\pi}{2}$, $v^2 = 16(2 - x)(4 + x) = 16(9 - y^2)$
where $y = x + 1$; $a = 3\text{ m}$

(c) $\frac{1}{2}v^2 = 3 + 2x - x^2$, $\ddot{x} = 2 - 2x = -2(x - 1)$:
SHM around $x = 1$
(i) $x = 1$
(ii) $T = \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$, $v^2 = 2(3 - x)(1 + x) = 2(4 - y^2)$
where $y = 1 - x$; $a = 2\text{ m}$

7 (a) $v^2 = 2(3 - x)(5 + x)$, $x = -5, 3$

(b) $\frac{1}{2}v^2 = 15 - 2x - x^2$, $\ddot{x} = -2 - 2x = -2(x + 1)$

(c) $x = -1$, $v^2 = 32$, $v_{\max} = 4\sqrt{2}$

8 (a) $\dot{x} = -an\sin(nt + \varepsilon)$
$t = 0$, $\dot{x} = 0$: $\varepsilon = 0$, $x = a\cos nt$
$t = 0$, $x = -5$: $-5 = a$, $x = -5\cos nt = 5\cos(\pi - nt)$
or $x = 5\sin\left(nt - \frac{\pi}{2}\right)$

(b) $t = 0$, $x = 0$: $0 = a\cos\varepsilon$
$\varepsilon = \frac{\pi}{2}$: $x = a\cos\left(nt + \frac{\pi}{2}\right)$, $\dot{x} = -an\sin\left(nt + \frac{\pi}{2}\right)$
$t = 0$, $v < 0$, $a > 0$: $x = a\cos\left(nt + \frac{\pi}{2}\right)$

9 $v^2 = 9(16 - x^2)$, $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{9}{2}(16 - x^2)\right]$, $\ddot{x} = -9x$:
SHM $x = 4$: $\ddot{x} = -36\text{ m s}^{-2}$, max speed $= 12\text{ m s}^{-1}$

10 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -16x$, $v^2 = 16(25 - x^2)$, $x = 5\sin(4t - C)$
where $\sin C = -\frac{3}{5}$
$\therefore \cos C = \frac{4}{5}$, $x = 4\sin 4t + 3\cos 4t$
max displacement $= 5\text{ m}$, max speed $= 20\text{ m s}^{-1}$

11 (a) $t = 0$, $x = 0$, $\dot{x} = -4$, $\frac{2\pi}{n} = \frac{\pi}{2}$, $n = 4$
$x = a\sin 4t$, $\dot{x} = -4a\cos 4t$, $-4 = -4a$, $a = 1$: $x = \sin 4t$
(b) $a = 1\text{ m}$

12 $36 = n^2(a^2 - 4)$, $16 = n^2(a^2 - 9)$, $\frac{36}{16} = \frac{a^2 - 4}{a^2 - 9}$, $a = \sqrt{13}$, $n = 2$
period $= \pi\text{ s}$, amplitude $= \sqrt{13}\text{ m}$

13 $\dot{x} = an\cos nt - bn\sin nt$,
$\ddot{x} = -an^2\sin nt - bn^2\cos nt = -n^2(a\sin nt + b\cos nt) = -n^2x$,
amplitude $= \sqrt{a^2 + b^2}$
$\dot{x} = n(a\cos nt - b\sin nt)$, max speed $= n\sqrt{a^2 + b^2}$

14 $x = 4 - \cos 2t$, $\dot{x} = 2\sin 2t$,
$\ddot{x} = 4\cos 2t = 4(4 - x) = -4(x - 4)$: SHM
(a) $x = 4\text{ m}$ (b) $n = 2$, $T = \pi\text{ s}$ (c) $a = 1\text{ m}$

15 $n = \frac{\pi}{4}$, $a = 10$: $v^2 = \frac{\pi^2}{16}\left(100 - x^2\right)$; $x = 6$: $v = 2\pi\text{ cm s}^{-1}$
$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left[\frac{\pi^2}{32}\left(100 - x^2\right)\right] = -\frac{\pi^2}{16}x$, $\ddot{x} = \frac{3\pi^2}{8}\text{ m s}^{-2}$
max acceleration when $x = 10$: $\ddot{x}_{\max} = \frac{5\pi^2}{8}\text{ m s}^{-2}$

16 $\dot{x} = 16\cos 2t - 12\sin 2t$,
$\ddot{x} = -32\sin 2t - 24\cos 2t = -4(8\sin 2t + 6\cos 2t) = -4(x - 10)$:
SHM about $x = 10$, $n = 2$, $T = \pi\text{ s}$, $a = 10\text{ m}$

17 (a) $x = ut + \frac{1}{2}at^2$, $0.5 = 11u + \frac{1}{2} \times 121$, $1 = 22u + 121a$
$0 = 18u + \frac{1}{2} \times 324a$, $0 = u + 9a$
$\therefore\ x = \frac{9t}{77} - \frac{t^2}{154}$: $t = 7$, $x = 0.5$; $t = 9$, $x = \frac{81}{154}$

(b) $n = \frac{\pi}{6}$, $x = -\sin\frac{\pi t}{6}$: $t = 7$, $x = \frac{1}{2}$; $t = 9$, $x = 1$

18 $a = 2$, $n = \frac{2\pi}{3}$, $v^2 = \frac{4\pi^2}{9}(4 - x^2)$, $v_{\max} = \frac{4\pi}{3}\text{m s}^{-1}$,
$\ddot{x}_{\max} = \frac{8\pi^2}{9}\text{ m s}^{-2}$

19 (a) $x = 5\cos\frac{\pi t}{6}$ (b) $\frac{5\pi}{6}$ (c) $a = 5$, $f = \frac{1}{12}$, $T = 12\text{ s}$
(d) $\frac{5\pi}{6}$ (e) $\frac{5\pi^2}{36}$

20 $v^2 = 2(25 - x^2)$; $x = 0$: speed $= 5\sqrt{2}\text{ m s}^{-1}$
$x = 4$: speed $= 3\sqrt{2}\text{ m s}^{-1}$

21 $v^2 = 4(a^2 - x^2)$; $a = 4\text{ m}$, $v = 2\sqrt{7}\text{ m s}^{-1}$

22 (a) $f = \frac{\sqrt{5}}{2\pi}$ Hz (vibrations per second)
(b) $n = \sqrt{5}$, $a = 10\text{ m}$

23 $n = 2$, $a = 2.5$, $v^2 = 4(6.25 - x^2)$; $x = 1.5$: $v = 4\text{ m s}^{-1}$, $\ddot{x} = 6\text{ m s}^{-2}$

24 $a = 10$, $n = \frac{\pi}{5}$, $x = 10\sin\frac{\pi t}{5}$; $x = 6$: $t = 1.02\text{ s}$

25 $64 = n^2(a^2 - 9)$, $36 = n^2(a^2 - 16)$: $a = 5$, $n = 2$, $T = \pi\text{ s}$,
$\ddot{x}_{\max} = 20\text{ m s}^{-2}$

26 (a) 10.5 m (b) $a = 1.5\text{ m}$ (c) $T = 14\text{ h}$
(d) $n = \frac{\pi}{7}$:
$t = 0$ (9 a.m.): $y = 10.5 - 1.5 = 9\text{ m}$
$t = 3.5$ (12:30 p.m.): $y = 10.5 - 1.5\cos\frac{\pi}{2} = 10.5\text{ m}$
$t = 7$ (4 p.m.): $y = 10.5 - 1.5\cos\pi = 10.5 + 1.5 = 12\text{ m}$.
(e) $y = 10$, $10 = 10.5 - 1.5\cos\frac{\pi t}{7}$, $1.5\cos\frac{\pi t}{7} = 0.5$, $\cos\frac{\pi t}{7} = \frac{1}{3}$,
$\frac{\pi t}{7} = 1.231, 5.052$; $t = 2.743\text{ h}, 11.257\text{ h}$
Between 9 a.m. + 2 h 45 min = 11:45 a.m.
and 9 a.m. + 11 h 15 min = 8:15 p.m.

27 (a) 18°C
(b) $t = 0$ (4 a.m.), $T = 13$: $T = 18 - 5\cos 0 = 13$
$t = 12$ (4 p.m.), $T = 23$: $T = 18 - 5\cos\pi = 18 + 5 = 23$
$\therefore$ formula satisfies given conditions

(c) (i) 10 a.m and 10 p.m. (from part (a))

(ii) $T = 15$: $15 = 18 - 5\cos\frac{\pi t}{12}$, $5\cos\frac{\pi t}{12} = 3$, $\cos\frac{\pi t}{12} = 0.6$,

$\frac{\pi t}{12} = 0.9273$, $t = 3$ h 33 min = at 7:33 a.m and 12:27 a.m.

(iii) $T = 21$: $21 = 18 - 5\cos\frac{\pi t}{12}$, $5\cos\frac{\pi t}{12} = -3$,

$\cos\frac{\pi t}{12} = -0.6$, $\frac{\pi t}{12} = 2.214$, $t = 8$ h 27 min
= at 12:27 p.m. and 7:33 p.m.

28 $na = 3$, $n^2a = 8$: $n = \frac{8}{3}$, $a = \frac{9}{8}$ m, $T = \frac{3\pi}{4}$ s

EXERCISE 13.5

1 (a) $y = 0$, $800x(400 - x) = 0$, $x = 400$ (b) $x = 200$, $y = 50$

2 (a) $t = 5\sqrt{3}$ s (b) $10\sqrt{3}$ s

(c) $x = 50t$, $y = 50\sqrt{3}t - 5t^2$, $t = \frac{x}{50}$: $y = \sqrt{3}x - \frac{x^2}{500}$

(d) $500\sqrt{3}$ m

3 (a) $V = 80$, $\alpha = 30°$ from $(0,0)$, $\dot{x} = 40\sqrt{3}$, $x = 40\sqrt{3}t$,
$\dot{y} = 40 - 10t$, $y = 40t - 5t^2$

$t = \frac{x}{40\sqrt{3}}$: $y = 40 \times \frac{x}{40\sqrt{3}} - 5 \times \left(\frac{x}{40\sqrt{3}}\right)^2$, $y = \frac{x}{\sqrt{3}} - \frac{x^2}{960}$

(b) $y = 0$: $0 = \frac{x}{\sqrt{3}} - \frac{x^2}{960}$, $x = 320\sqrt{3}$ m, $t = \frac{320\sqrt{3}}{40\sqrt{3}} = 8$ s

(c) $t = 4$, $y = 160 - 80 = 80$ m

4 (a) $t = 0$, $x = 0$, $y = 125$, $V = 10$, $\alpha = 0°$:
$x = 10t$, $\dot{y} = -10t$, $y = 125 - 5t^2$
$y = 0$: $t = 5$ s

(b) $x = 50$ m

5 (a) $V = 12.5$, $\tan\alpha = \frac{4}{3}$: $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$, $\dot{x} = 7.5$,
$\dot{y} = 10 - 10t$, $x = 7.5t$, $y = 10t - 5t^2$

$t = \frac{2x}{15}$: $y = 10 \times \frac{2x}{15} - 5 \times \left(\frac{2x}{15}\right)^2$, $y = \frac{4}{3}x - \frac{4}{45}x^2$

(b) $\dot{y} = 0$, $t = 1$; greatest height $y = 5$ m

(c) $y = 0$, $t = 2$ s: $x = 15$ m

6 D

7 $V = 50$: $x = 50t\cos\alpha$, $y = 50t\sin\alpha - 5t^2$: $t = \frac{200}{50\cos\alpha} = \frac{4}{\cos\alpha}$

$25 = 50\sin\alpha \times \frac{4}{\cos\alpha} - \frac{80}{\cos^2\alpha}$

$80\sec^2\alpha - 200\tan\alpha + 25 = 0$

$16(1 + \tan^2\alpha) - 40\tan\alpha + 5 = 0$

$16\tan^2\alpha - 40\tan\alpha + 21 = 0$

$(4\tan\alpha - 3)(4\tan\alpha - 7) = 0$

$\tan\alpha = \frac{3}{4}, \frac{7}{4}$; $\alpha = 36°\,52', 60°\,15'$

8 $x = Vt\cos\alpha$, $y = Vt\sin\alpha - 5t^2$, $y = x\tan\alpha - \frac{5x^2}{V^2\cos^2\alpha}$

$x = 45$, $y = 0$: $45\tan\alpha - \frac{10125}{V^2\cos^2\alpha} = 0$, $V^2 = \frac{225}{\sin\alpha\cos\alpha}$

$x = 22.5$, $y = 11.25$: $11.25 = 22.5\tan\alpha - \frac{5 \times 22.5^2}{V^2\cos^2\alpha}$

$11.25 = 22.5\tan\alpha - \frac{5 \times 22.5^2}{\cos^2\alpha} \times \frac{\sin\alpha\cos\alpha}{225}$

$45 = 90\tan\alpha - \frac{45\sin\alpha}{\cos\alpha}$

$\tan\alpha = 1$, $\alpha = 45°$: $V^2 = 225 \times 2 = 450$, $V = 15\sqrt{2}$ m s^{-1}

9 $V = 30$ to $(120, -35)$: $y = x\tan\alpha - \frac{x^2}{180\cos^2\alpha}$,

$-35 = 120\tan\alpha - \frac{120^2}{180\cos^2\alpha}$, $16\sec^2\alpha - 24\tan\alpha - 7 = 0$,

$16\tan^2\alpha - 24\tan\alpha + 9 = 0$

$\tan\alpha = \frac{3}{4}$, $\alpha = 36°\,52'$

10 (a) $D = Vt$, $H = 5t^2$: $V = 6$ m s^{-1} (b) $H = 20$ m

11 $t = 0$, $\dot{x} = 20$, $\dot{y} = 100$: $x = 20t$, $y = 100t - 5t^2$
$x = 120$: $t = 6$ s, $y = 420$ m

12 (a) $V\cos\alpha = 36$, $V\sin\alpha = 15$: $\frac{15^2}{V^2} + \frac{36^2}{V^2} = 1$, $V = 39$ m s^{-1}

(b) $\tan\alpha = \frac{5}{12}$, $\alpha = 22°\,37'$

(c) greatest height $= \frac{39^2}{20} \times \left(\frac{5}{13}\right)^2 = 11.25$ m

(d) range $= \frac{39^2}{10} \times 2 \times \frac{5}{13} \times \frac{12}{13} = 108$ m

13 (a) $(0,0)$

(b) $V\cos\alpha = 15$, $V\sin\alpha = 20$: $\frac{20^2}{V^2} + \frac{15^2}{V^2} = 1$, $V = 25$ m s^{-1}

(c) greatest height $= \frac{25^2}{20} \times \left(\frac{4}{5}\right)^2 = 20$ m

$20 = 20t - 5t^2$, $t^2 - 4t + 4 = 0$, $t = 2$:
$x = 15 \times 2 = 30$, point is $(30, 20)$

14 $V = 200$, $R = 2000$ m; $R = \frac{V^2\sin 2\alpha}{g}$, $2000 = \frac{200^2\sin 2\alpha}{10}$,
$\sin 2\alpha = 0.5$, $\alpha = 15°, 75°$

15 (a) $y = 1000$: $1000 = 5t^2$, $t = 10\sqrt{2}$ s

(b) $V_x = 630$ km/h $= \frac{7}{40}$ km/s: $x = \frac{7}{40} \times 10\sqrt{2} = \frac{7\sqrt{2}}{4}$ km

16 (a) $(7,7)$: $7 = Vt\cos\alpha$, $7 = Vt\sin\alpha - 5t^2$, $t = \frac{7}{V\cos\alpha}$,

$7 = 7\tan\alpha - 5 \times \left(\frac{7}{V\cos\alpha}\right)^2$, $V^2 = \frac{35}{\cos^2\alpha(\tan\alpha - 1)}$

$(14,7)$: $14 = Vt\cos\alpha$, $7 = Vt\sin\alpha - 5t^2$, $t = \frac{14}{V\cos\alpha}$,

$7 = 14\tan\alpha - 5 \times \left(\frac{14}{V\cos\alpha}\right)^2$, $V^2 = \frac{140}{\cos^2\alpha(2\tan\alpha - 1)}$

$\frac{35}{\cos^2\alpha(\tan\alpha - 1)} = \frac{140}{\cos^2\alpha(2\tan\alpha - 1)}$,

$2\tan\alpha - 1 = 4(\tan\alpha - 1)$, $\tan\alpha = 1.5$

(b) (b,h): $b = Vt\cos\alpha$, $h = Vt\sin\alpha - 5t^2$, $t = \frac{b}{V\cos\alpha}$,

$h = b\tan\alpha - 5 \times \left(\frac{b}{V\cos\alpha}\right)^2$, $V^2 = \frac{5b^2}{\cos^2\alpha(b\tan\alpha - h)}$

(c,h): $c = Vt\cos\alpha$, $h = Vt\sin\alpha - 5t^2$, $t = \frac{c}{V\cos\alpha}$,

$h = c\tan\alpha - 5 \times \left(\frac{c}{V\cos\alpha}\right)^2$, $V^2 = \frac{5c^2}{\cos^2\alpha(c\tan\alpha - h)}$

$\frac{5b^2}{\cos^2\alpha(b\tan\alpha - h)} = \frac{5c^2}{\cos^2\alpha(c\tan\alpha - h)}$,

$b^2(c\tan\alpha - h) = c^2(b\tan\alpha - h)$,

$bc(b - c)\tan\alpha = h(b^2 - c^2)$, $\tan\alpha = \frac{h(b + c)}{bc}$

17 (a) $V = 25$, $P(24,7)$: $\tan\alpha = \frac{7}{24}$, $\sin\alpha = \frac{7}{25}$, $\cos\alpha = \frac{24}{25}$

$x = 25 \times \frac{24}{25}t$, $x = 24t$; $y = 25 \times \frac{7}{25}t - 5t^2$, $y = 7t - 5t^2$

$x = 24$, $t = 1$, $y = 2$: point is $(24,2)$

(b) $y_2 = 7 - 5t^2$, $t = 1$: $y_2 = 7 - 5 = 2$
Both particles at $(24,2)$ at $t = 1$ second, so particles collide.

18 (a) $\alpha = 15°$, $R = 80$: $80 = \frac{V^2\sin 30°}{10}$, $V = 40$ m s^{-1}

(b) $\alpha = 45°$, $V = 40$: $R = \frac{40^2 \times \sin 90°}{10} = 160$ m

19 $t = 0$, $x_1 = 0$, $y_1 = 30$, horizontal $V_1 = 15$: $x_1 = 15t$, $y_1 = 30 - 5t^2$
$t = 0$, $x_2 = 0$, $y_2 = 0$, $V = 30$, $\alpha = 60°$: $x_2 = 15t$, $y_2 = 15t\sqrt{3} - 5t^2$
Horizontal position is the same for each as $x_1 = x_2$.
To collide, vertical position must be the same:

$30 - 5t^2 = 15t\sqrt{3} - 5t^2$, $t = \frac{2\sqrt{3}}{3}$, $x = 10\sqrt{3}$,

$y = 30 - 5 \times \frac{4}{3} = 23\frac{1}{3}$, collision at $\left(10\sqrt{3}, 23\frac{1}{3}\right)$

20 (a) $t = 2$, $x = 50$: $50 = V\cos\alpha \times 2$, $V\cos\alpha = 25$,
$0 = V\sin\alpha \times 2 - 20$, $V\sin\alpha = 10$

$\tan\alpha = \frac{2}{5}$, $\alpha = 21°\,48'$, $V = 10 \mid \frac{2}{\sqrt{29}} = 5\sqrt{29}$ m s^{-1}

(b) greatest height $= \frac{25\times 29}{20}\times\frac{4}{29} = 5$ m

21 $V = 20$ to $(20, 10)$: $10 = 20\tan\alpha - 5\sec^2\alpha$,
$\tan^2\alpha - 4\tan\alpha + 3 = 0$, $\tan\alpha = 1, 3$
$\alpha = 45°, 71°34'$

22 (a) $V\cos\alpha = 30$, $V\sin\alpha = 40$, $V^2 = 30^2 + 40^2 = 50^2$, $V = 50\,\text{m s}^{-1}$
(b) $\tan\alpha = \frac{4}{3}$, $\alpha = 53°8'$ (c) $(120, 80)$

23 (a) $R = a$, $y_{\max} = b$: $\frac{V^2\sin 2\alpha}{g} = a$, $\frac{V^2\sin^2\alpha}{2g} = b$

$\frac{4\sin\alpha\cos\alpha}{\sin^2\alpha} = \frac{a}{b}$, $\tan\alpha = \frac{4b}{a}$, $\sin\alpha = \frac{4b}{\sqrt{a^2+16b^2}}$,
$\cos\alpha = \frac{a}{\sqrt{a^2+16b^2}}$

$V^2 = \frac{2bg(a^2+16b^2)}{16b^2} = \frac{g(a^2+16b^2)}{8b}$, $V = \sqrt{\frac{g(a^2+16b^2)}{8b}}$

$V\cos\alpha = \sqrt{\frac{g(a^2+16b^2)}{8b}}\times\frac{a}{\sqrt{a^2+16b^2}} = \frac{a}{2}\sqrt{\frac{g}{2b}}$

$V\sin\alpha = \sqrt{\frac{g(a^2+16b^2)}{8b}}\times\frac{4b}{\sqrt{a^2+16b^2}} = \sqrt{2bg}$

(b) $x = \frac{at}{2}\sqrt{\frac{g}{2b}}$, $y = t\sqrt{2bg} - \frac{1}{2}gt^2$, $t = \frac{2x}{a}\sqrt{\frac{2b}{g}}$:

$y = \frac{2x}{a}\sqrt{\frac{2b}{g}}\times\sqrt{2bg} - \frac{1}{2}g\left(\frac{2x}{a}\sqrt{\frac{2b}{g}}\right)^2$

$= \frac{4bx}{a} - \frac{4bx^2}{a^2} = \frac{4bx(a-x)}{a^2}$

24 $V = 16$ to $(16, 4)$: $4 = 16\tan\alpha - \frac{5\times 16^2}{16^2\cos^2\alpha}$,
$4 = 16\tan\alpha - 5\sec^2\alpha$, $5 + 5\tan^2\alpha - 16\tan\alpha + 4 = 0$,
$5\tan^2\alpha - 16\tan\alpha + 9 = 0$

CHAPTER REVIEW 13

1 $r = \frac{h}{2}$, $V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3$, $\frac{dV}{dh} = \frac{1}{4}\pi h^2$, $\frac{dV}{dt} = 4\pi$
$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$: $4\pi = \frac{1}{4}\pi h^2\times\frac{dh}{dt}$, $\frac{dh}{dt} = \frac{16}{h^2}$
$h = 3$: $\frac{dh}{dt} = \frac{16}{9}$ cm/h

2 $\frac{dr}{dt} = 0.1$, $V = \pi r^2 h$, h constant: $\frac{dV}{dr} = 2\pi rh$
$\frac{dV}{dt} = \frac{dV}{dr}\times\frac{dr}{dt}$: $\frac{dV}{dt} = 2\pi rh\times 0.1 = \frac{\pi rh}{5}$
$h = 10$, $r = 2$: $\frac{dV}{dt} = 4\pi$ cm/min

3 $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + 6$, $\frac{1}{2}v^2 = \frac{1}{2}x^2 + 6x + C$
$t = 0$, $v = 0$, $x = 2$: $C = -14$, $v^2 = x^2 + 12x - 28$
$x = 3$: $v = \sqrt{17}$

4 (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x^3 + x$, $\frac{1}{2}v^2 = \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$
$x = 2$, $v = \frac{5\sqrt{2}}{2}$: $C = \frac{1}{4}$, $v^2 = \frac{1}{2}(x^4 + 2x^2 + 1)$,
$v = \pm\frac{1}{\sqrt{2}}\left(x^2+1\right)$
$v > 0$: $v = \frac{1}{\sqrt{2}}\left(x^2+1\right)$

(b) $\frac{dx}{dt} = \frac{x^2+1}{\sqrt{2}}$, $\frac{dt}{dx} = \frac{\sqrt{2}}{x^2+1}$: $t = \sqrt{2}\tan^{-1}x + C$
$t = 0$, $x = 2$: $C = -\sqrt{2}\tan^{-1}2$
$t = \sqrt{2}\tan^{-1}x - \sqrt{2}\tan^{-1}2$
$\frac{t}{\sqrt{2}} = \tan^{-1}x - \tan^{-1}2$
$\tan\left(\frac{t}{\sqrt{2}}\right) = \tan\left(\tan^{-1}x - \tan^{-1}2\right) = \frac{x-2}{1+2x}$
$x = \frac{2+\tan\left(\frac{t}{\sqrt{2}}\right)}{1-2\tan\left(\frac{t}{\sqrt{2}}\right)}$

5 (a) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$, $\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + C$
$x = a$, $v = 0$: $C = \frac{1}{2}a^2x^2$, $\frac{1}{2}v^2 = \frac{1}{2}a^2x^2 - \frac{1}{2}n^2x^2$,
$v^2 = n^2(a^2 - x^2)$
(b) $x = 0$: $v^2 = n^2a^2$, $v = na$
(c) $x = -a$: $\ddot{x} = n^2a$
(d) $t = 0$, $x = 0$, $x = a\sin nt$, $\dot{x} = an\cos nt$, $\dot{x} = \frac{an}{4}$:
$\frac{an}{4} = an\cos nt$, $\cos nt = \frac{1}{4}$, $nt = \cos^{-1}\left(\frac{1}{4}\right)$, $t = \frac{1}{n}\cos^{-1}\left(\frac{1}{4}\right)$

6 (a) $t = 0$, $x = -a$; $a = 16$, $T = 5$;
$n = \frac{2\pi}{5}$, $x = 16\sin\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right)$ or $x = 16\cos\left(\frac{2\pi t}{5} + \pi\right)$
(b) $x = -8$: $-8 = 16\sin\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right)$, $\sin\left(\frac{2\pi t}{5} - \frac{\pi}{2}\right) = -\frac{1}{2}$,
$\frac{2\pi t}{5} - \frac{\pi}{2} = -\frac{\pi}{6}$, $\frac{2\pi t}{5} = \frac{\pi}{3}$, $t = \frac{5}{6}$ s

7 (a) $\dot{x} = 2\cos 2t + 2\sqrt{3}\sin 2t$,
$\ddot{x} = -4\sin 2t + 4\sqrt{3}\cos 2t = -4(\sin 2t - \sqrt{3}\cos 2t)$
$= -4(x - 3)$
(b) $n = 2$, $T = \pi$ s
(c) $\dot{x} = 2(\cos 2t + \sqrt{3}\sin 2t) = 4\left(\frac{1}{2}\cos 2t + \frac{\sqrt{3}}{2}\sin 2t\right)$
$= 4\cos\left(2t - \frac{\pi}{3}\right)$
(d) $\dot{x} = 2$, $4\cos\left(2t - \frac{\pi}{3}\right) = 2$, $\cos\left(2t - \frac{\pi}{3}\right) = \frac{1}{2}$,
$2t - \frac{\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$;
$2t = 0, \frac{2\pi}{3}, 2\pi$; $t = 0, \frac{\pi}{3}, \pi$ s
$\dot{x} = -2$, $4\cos\left(2t - \frac{\pi}{3}\right) = -2$, $\cos\left(2t - \frac{\pi}{3}\right) = -\frac{1}{2}$,
$2t - \frac{\pi}{3} = \frac{2\pi}{3}, \frac{4\pi}{3}$; $2t = \pi, \frac{5\pi}{3}$; $t = \frac{\pi}{2}, \frac{5\pi}{6}$ s
Hence $t = 0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi$ s

8 (a) $\frac{dT}{dt} = -k(T - 25)$, $T = 25 + Ae^{-kt}$
$t = 0$, $T = 100$: $A = 75$, $T = 25 + 75e^{-kt}$
$t = 3$, $T = 75$: $50 = 75e^{-3k}$, $k = \frac{1}{3}\ln\frac{3}{2} = 0.135$
(b) $t = 8$: $T = 25 + 75e^{-8k}$,
$T = 25 + 75e^{\ln\left(\frac{2}{3}\right)^8} = 25 + 75\times\left(\frac{2}{3}\right)^8 = 50.4°\text{C}$

9 $V_x = 6$, $V_y = 4$: $V = 2\sqrt{13}$, $\tan\alpha = \frac{2}{3}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{3}{\sqrt{13}}$
$R = \frac{V^2\sin 2\alpha}{g} = \frac{52}{10}\times 2\times\frac{2}{\sqrt{13}}\times\frac{3}{\sqrt{13}} = 4.8$ m

10 $y_{\max} = 4.9$, $V_x = 28$: $\frac{V^2\sin^2\alpha}{2\times 9.8} = 4.9$, $V\sin\alpha = 9.8$,
$y = 9.8t - 4.9t^2$; $x = 28t$
$y = 1.3$: $1.3 = 9.8t - 4.9t^2$
$(7t-1)(7t-13) = 0$, $t = \frac{1}{7}, \frac{13}{7}$
$t = \frac{13}{7}$ on the way down; $x = 52$. Fielder runs $65 - 52 = 13$ metres in $\frac{13}{7}$ seconds: running speed $= 7\,\text{m s}^{-1}$

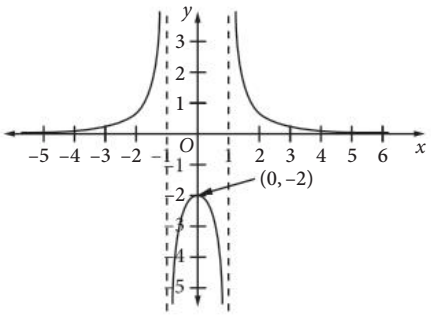
11 Origin at point of projection.
greatest height $\frac{45}{4} = \frac{V^2\sin^2\alpha}{20}$, $V^2 = \frac{225}{\sin^2\alpha}$
range $60 = \frac{V^2\sin 2\alpha}{10}$, $V^2 = \frac{600}{\sin 2\alpha}$:
$\frac{600}{2\sin\alpha\cos\alpha} = \frac{225}{\sin^2\alpha}$, $\sin\alpha(4\sin\alpha - 3\cos\alpha) = 0$
$\tan\alpha = \frac{3}{4}$, $\alpha = 36°52'$
$\sin\alpha = \frac{3}{5}$: $V^2 = 225\times\frac{25}{9} = 625$, $V = 25\,\text{m s}^{-1}$

CHAPTER 14

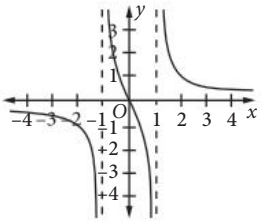
EXERCISE 14.1

1 B

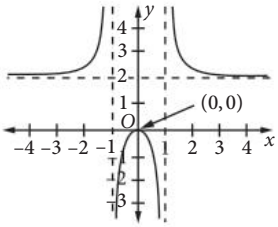
2 $y > 0, y \leq -2$



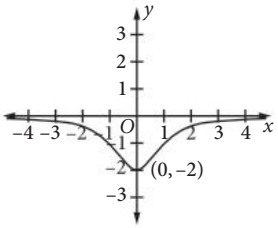
3 all real y



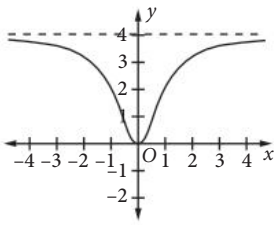
4 $y > 2, y \leq 0$



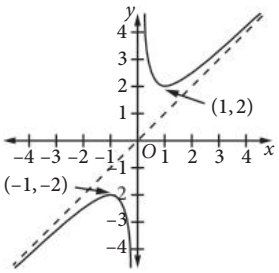
5 $-2 \leq y < 0$



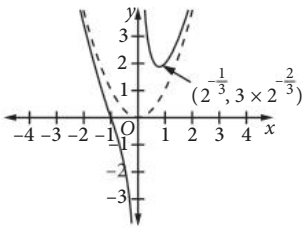
6 $0 \leq y < 4$



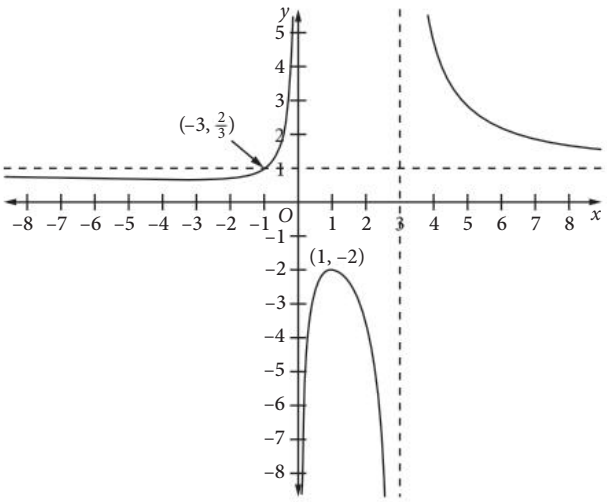
7 $y \leq -2, y \geq 2$



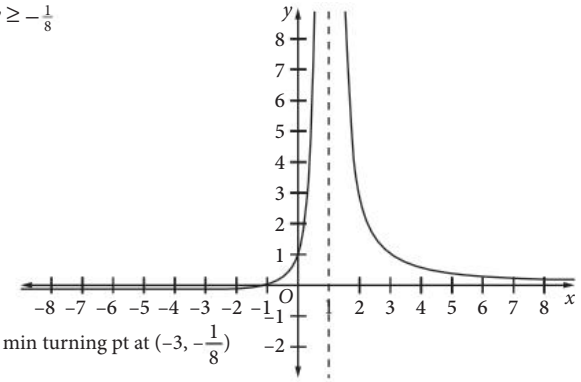
8 all real y



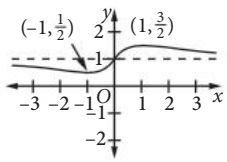
9 $y \leq -2, y \geq \frac{2}{3}$



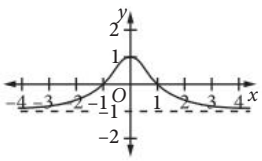
10 $y \geq -\frac{1}{8}$



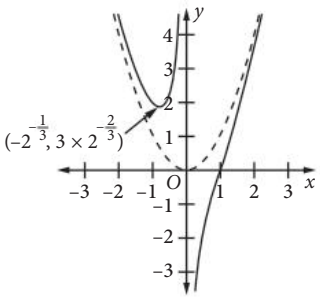
11 $\frac{1}{2} \leq y \leq \frac{3}{2}$



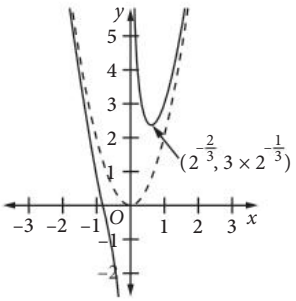
12 $-1 < y \leq 1$



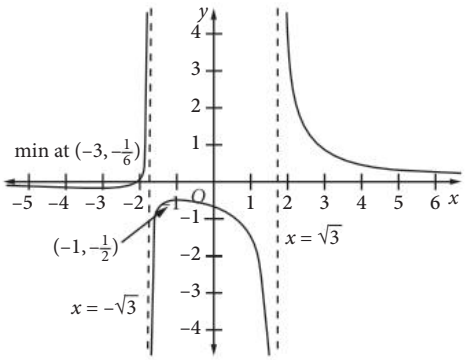
13 all real y



14 all real y



15 $y \leq -\frac{1}{2}, y \geq -\frac{1}{6}$



EXERCISE 14.2

1 (a) 2 (b) $\frac{1}{3}$ (c) 2.5 (d) 1

2 (a) correct (b) incorrect (c) correct (d) correct

3 (a) 2 (b) 1 (c) 0.5 (d) $\frac{2}{3}$ (e) 2 (f) 1.5 (g) 0.5 (h) 2.5

4 (a) $2\sin(x - \frac{\pi}{3})$ (b) 2

EXERCISE 14.3

1 (a) $f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{1}{h} \times \frac{x-(x+h)}{x(x+h)}$

$= \lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$

(b) $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

$= \lim_{h\to 0} \frac{1}{h} \times \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})}$

$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$

2 (a) $a-b=\left(a^{\frac{1}{3}}\right)^3-\left(b^{\frac{1}{3}}\right)^3=\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}\right)$

(b) $f'(x)=\lim_{h\to 0}\dfrac{(x+h)^{\frac{1}{3}}-x^{\frac{1}{3}}}{h}$

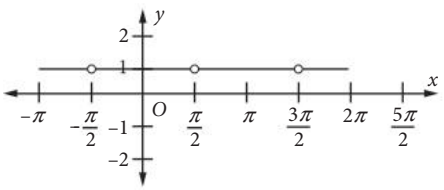
$=\lim_{h\to 0}\frac{1}{h}\times\dfrac{\left((x+h)^{\frac{1}{3}}-x^{\frac{1}{3}}\right)\left((x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}{\left((x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}$

$=\lim_{h\to 0}\dfrac{x+h-x}{h\left((x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}$

$=\lim_{h\to 0}\dfrac{1}{\left((x+h)^{\frac{2}{3}}+(x+h)^{\frac{1}{3}}x^{\frac{1}{3}}+x^{\frac{2}{3}}\right)}=\dfrac{1}{3\sqrt[3]{x}}$

3 (a) 1; all x except $\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\ldots$

(b)



4 (a) Prove true for $n=1$:

LHS $=1^2=1$

RHS $=\frac{1}{6}\times 1\times(1+1)\times(2+1)=1=$ LHS

Assume true for $n=k$:

i.e. assume $1^2+2^2+\ldots+k^2=\frac{1}{6}k(k+1)(2k+1)$

Now prove $n=k+1$ is true if $n=k$ is true:

LHS $=1^2+2^2+\ldots+k^2+(k+1)^2=\frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$=\frac{1}{6}(k+1)[k(2k+1)+6(k+1)]=\frac{1}{6}(k+1)(k+2)(2k+3)$

= RHS

Conclusion: true for $n=k+1$ if true for $n=k$;

true for $n=1$;

$\therefore$ by induction, true for all integer $n\geq 1$

(b) (i) $\sum_{k=10}^{25}k^2=\sum_{k=1}^{25}k^2-\sum_{k=1}^{9}k^2$

$=\frac{1}{6}\times 25\times 26\times 51-\frac{1}{6}\times 9\times 10\times 19=5240$

(ii) $\sum_{k=1}^{30}\left(\frac{1}{2}k^2+7k-5\right)=\sum_{k=1}^{30}\left(\frac{1}{2}k^2\right)+\sum_{k=1}^{30}(7k-5)$

$=\frac{1}{2}\times\frac{1}{6}\times 30\times 31\times 61$

$+$ [sum of arithmetic series, $a=2$, $d=7$, $n=30$]

$=4727.5+3105=7832.5$

5 (a) $a=\log_e 5$ (b) $5^x\times\log_e 5$ (c) $\dfrac{24}{\log_e 5}$ (d) 15.7 (e) 14.9

6 (a) −26.7 (b) 32 units^2 (c) 1038.9 units^3

7 (a) $f'(x)=\lim_{h\to 0}\dfrac{\log(x+h)-\log x}{h}$

$=\lim_{h\to 0}\left(\dfrac{\log\left(\frac{x+h}{x}\right)}{h}\right)=\lim_{h\to 0}\left(\dfrac{\log\left(1+\frac{h}{x}\right)}{h}\right)$

But $f'(x)=\frac{1}{x}$ $\therefore\lim_{h\to 0}\left(\dfrac{\log\left(1+\frac{h}{x}\right)}{h}\right)=\dfrac{1}{x}$

(b) The result of (a) is true for all $x>0$.

For $x=1$, $\lim_{h\to 0}\left(\dfrac{\log(1+h)}{h}\right)=1$

$\therefore\lim_{h\to 0}\left(\frac{1}{h}\log(1+h)\right)=1$ $\therefore\lim_{h\to 0}\left(\log(1+h)^{\frac{1}{h}}\right)=1$

(c) Let $n=\frac{1}{h}$, so $h=\frac{1}{n}$; as $h\to 0$, $n\to\infty$

$\therefore\lim_{n\to\infty}\left(\log\left(1+\frac{1}{n}\right)^n\right)=1$ $\therefore\lim_{n\to\infty}\left(n\log\left(\frac{n+1}{n}\right)\right)=1$

8 (a) $A=12$, $B=18$ (b) 48

9 (a) 30 m (b) 35 m (c) 45 m **10** D

11 Points of intersection where $x^3-4x^2+5x-1=x-1$,

$x^3-4x^2+4x=0$, $x(x-2)^2=0$

Double root at $x=2$ means that the line must be a tangent to the curve at the point on the curve where $x=2$.

(*Note:* The gradient of the curve at $x=2$ it is equal to 1, which is the gradient of the line.)

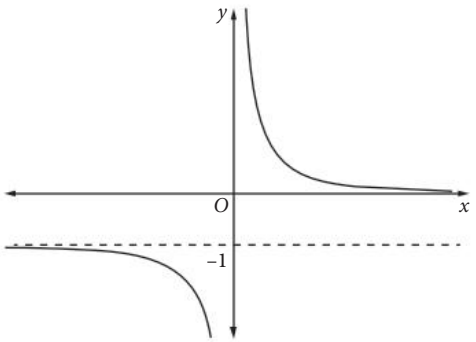
12 (a) $2-2^{1-n}$ (b) $2n-2+2^{1-n}$

13 (a) $x=0$ (b) $\frac{1}{e^x-1}\neq 0$

(c) 0 from above, −1 from below (d) $+\infty,-\infty$

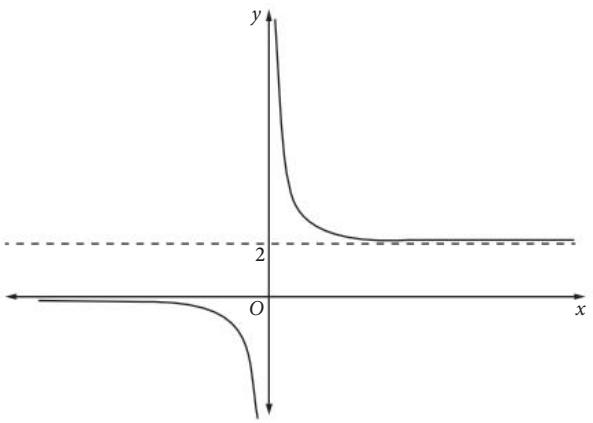
(e) $\dfrac{-e^x}{(e^x-1)^2}$, $\dfrac{dy}{dx}\neq 0$; $\dfrac{dy}{dx}<0$ for all x in the domain

(f)

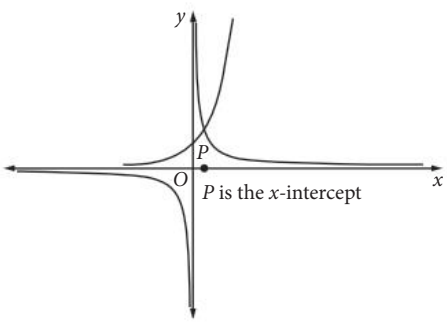


14 (a) RHS $=\dfrac{2(e^x-1)+2}{e^x-1}=\dfrac{2e^x}{e^x-1}=$ LHS

(b)



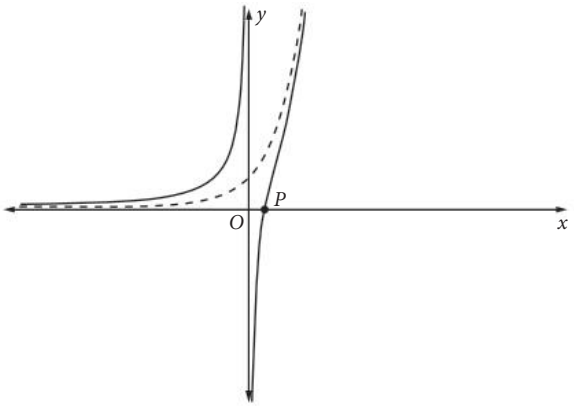
15 (a)–(b)



(c) $x=0$, $y=e^x$

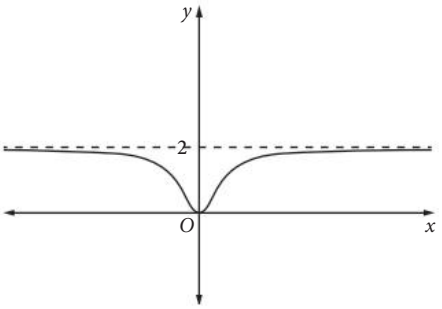
(d) $e^x+\dfrac{1}{x^2}$, $\dfrac{dy}{dx}\neq 0$; $\dfrac{dy}{dx}>0$ for all x in the domain

(e)

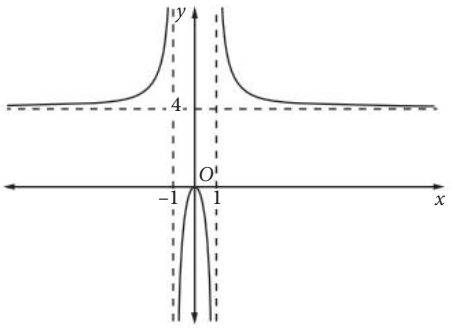


CHAPTER REVIEW 14

1 (a)



(b)



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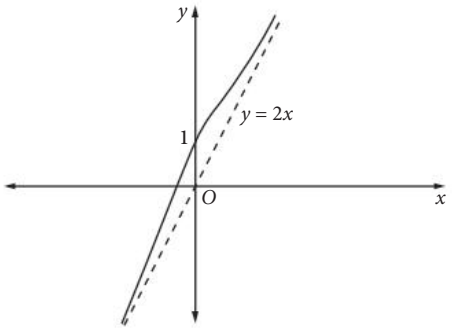
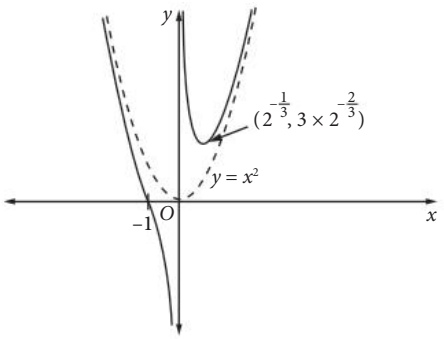
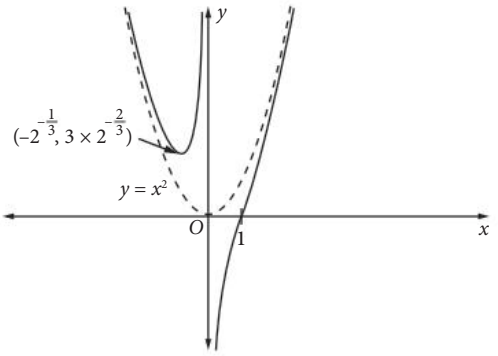


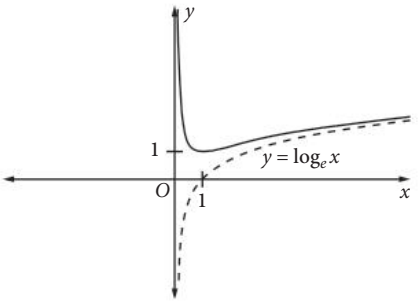
(d)



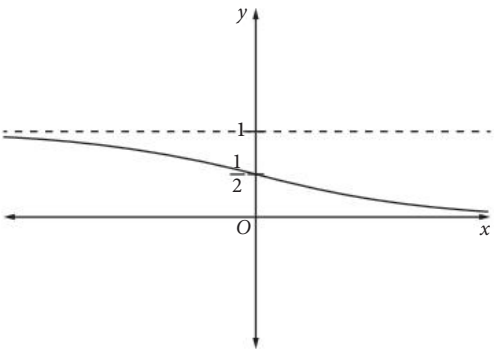
(e)



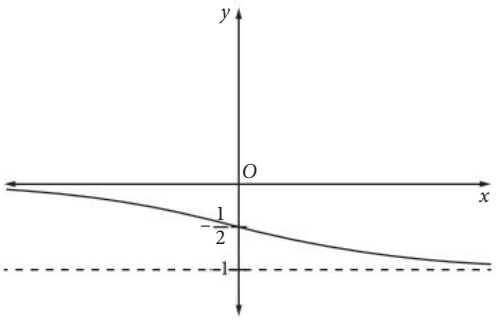
(f)



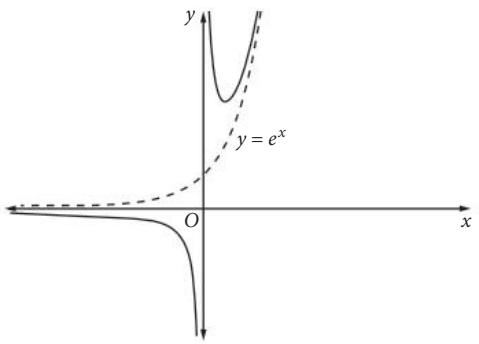
(g)



(h)

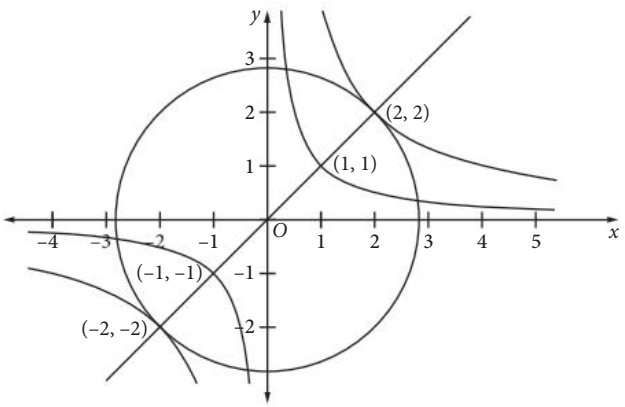


(i)



2 (a) $|x+1|$ (b) $-|x+1|$ **3** (a) B (b) C (c) D (d) A

4 (a)–(b)



(c) $\frac{dy}{dx} = -\frac{k}{x^2}$; the line $y = x$ cuts the hyperbola at $\left(\sqrt{k}, \sqrt{k}\right)$

At this point, $\frac{dy}{dx} = -\frac{k}{\left(\sqrt{k}\right)^2} = -1$ for all values of k $(k \neq 0)$

$\therefore$ All hyperbolae have tangents perpendicular to $y = x$ at their points of intersection with $y = x$.

(d) Translate $y = \frac{1}{x}$ 1 unit to the right and 1 unit up:

$y = \frac{1}{x-1} + 1 \quad \therefore y = \frac{x}{x-1}$;

Translate $y = \frac{1}{x}$ 1 unit to the left and 1 unit down:

$y = \frac{1}{x+1} - 1 \quad \therefore y = -\frac{x}{x+1}$;

Translate $y = -\frac{1}{x}$ 1 unit to the right and 1 unit down:

$y = -\frac{1}{x-1} - 1 \quad \therefore y = -\frac{x}{x-1}$;

Translate $y = -\frac{1}{x}$ 1 unit to the left and 1 unit up:

$y = -\frac{1}{x+1} + 1 \quad \therefore y = \frac{x}{x+1}$

5 $\sum_{k=-3}^{10} \left(6k - 15 + 2^k\right) = \sum_{k=-3}^{10} (6k - 15) + \sum_{k=-3}^{10} \left(2^k\right)$

This is the sum of 14 terms of an arithmetic series plus the sum of 14 terms of a geometric series.

$= 84 + 2047.875 = 2131.875$

6 (a) 3 (b) $\frac{4}{3}$ (c) $\frac{1}{9}$ (d) $\frac{4}{3}$

GLOSSARY

A

acceleration

The rate of change of velocity with respect to time: $\ddot{x}, \dot{v}(t), \frac{dv}{dt}, \frac{d^2x}{dt^2}, v\frac{dv}{dx}$ or $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$, standard units m s^{-2}.

amplitude

(1) The amplitude of a sine or cosine function is half the difference between the greatest and least values of the function. The amplitude of $y = a\sin nx$ is a.

(2) In simple harmonic motion (SHM), the distance a from the centre of motion O to either of the extreme points.

angle at the centre

The angle at the centre of a circle standing on an arc.

angle at the circumference

The angle at the circumference of a circle standing on an arc.

angle between two lines

The acute angle α between two lines at their point of intersection. The size of this angle is given by $\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_2 m_2}\right|$.

angle in a semicircle

The angle at the circumference of a circle standing on a diameter.

angle in the alternate segment

When there is an angle between a tangent to a circle and a chord drawn to the point of contact, the angle in the alternate segment is the angle at the circumference on the opposite side of a chord to that angle.

angles in the same segment

The angles at the circumference of a circle on the same side of a chord, standing on that chord.

angle of projection

The angle, measured from the horizontal, at which a particle is projected.

arc

(1) Part of the circumference of a circle.

(2) Part of a curve.

arrangements in a circle

The number of ways of arranging n different objects in a circle, regarding clockwise and anticlockwise arrangements as different, is $\frac{n!}{n} = (n-1)!$. As there is no start or finish to a circle, one object needs to be fixed as the 'starting object' for counting purposes.

asymptote

A line or curve that a function approaches but never reaches.

auxiliary angle method

Writing $a\sin x + b\cos x$ in the form $r\sin(x+\alpha)$ or $r\cos(x-\alpha)$, where $r = \sqrt{a^2+b^2}$ and α is in the first quadrant such that $\tan\alpha = \frac{b}{a}$.

B

binomial probability

When the same trial is repeated several times and there are only two possible outcomes in each trial (often called 'success' and 'failure'), the probability of r successes is given by $P(X = r) = \binom{n}{r} q^{n-r}p^r$, where $0 \le r \le n$, where p is the probability of a 'success' and $q = 1 - p$ is the probability of a 'failure'. This expression is the corresponding term of the binomial expansion of $(p+q)^n$.

binomial theorem

The sum of two terms raised to the nth power can be expanded as a series of n terms according to the following formula:

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \\ + {}^nC_ra^{n-r}b^r + \quad + {}^nC_nb^n = \sum_{r=0}^{n} {}^nC_ra^{n-r}b^r$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \quad + \binom{n}{r}x^r + \\ + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$$

C

chord of a circle

An interval (i.e. straight line) joining two points on a circle.

chord of contact

From an external point $P(x_0, y_0)$ to a parabola $x^2 = 4ay$, two tangents to the parabola can be drawn. The line connecting the points where these tangents touch the parabola is the chord of contact and its equation is $xx_0 = 2a(y + y_0)$.

circle

The set of all points in a plane that are equidistant from a fixed point in the plane.

circumcircle of a triangle

A circle that passes through the vertices of a triangle and whose centre is the point of intersection of the perpendicular bisectors of the sides of the triangle.

combination

An unordered selection of all or part of a set of objects (i.e. for which the order is not important). The number of combinations of r objects from a set of n objects total is: ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

Also note: ${}^nC_r = {}^nC_{n-r}$ or $\binom{n}{r} = \binom{n}{n-r}$

See also *permutation*.

concentric circles

Circles that have the same centre but different radii.

concyclic points

Any three or more non-collinear points that are on the same circle.

cubic function *or* cubic polynomial

A polynomial function f of the third degree, defined by $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are constants, $a \neq 0$. Every cubic polynomial has at least one linear factor of the form $(x + \alpha)$ where α is a real number.

cyclic quadrilateral

A quadrilateral whose vertices are on a circle (i.e. the vertices are concyclic).

D

decreasing function

A function f defined on the interval $a \leq x \leq b$ is said to be strictly monotonic decreasing when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) < f(x_1)$. In other words, the curve always slopes downwards to the right so the function never has a positive gradient.

degree of a polynomial

A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is of the nth degree if $a_n \neq 0$.

diameter of a circle

A chord through the centre of a circle.

displacement

The signed (positive or negative) distance of a particle from the origin; $x(t)$.

division algorithm

For any polynomial $P(x) = (x - a)Q(x) + R$ where $Q(x)$ is a polynomial.

division of an interval

The coordinates of the point P that divides the interval AB internally in the ratio $m:n$, given $A(x_1, y_1)$ and $B(x_2, y_2)$, are: $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

For external division, the ratio is written as $m:-n$.

E

even function

A function is even if $f(-x) = f(x)$ for all values of x in the domain. The function is also symmetrical about the y-axis.

F

factorial notation (!)

The notation $n!$ is called 'n factorial' and means $n \times (n-1)(n-2) \times \ldots \times 3 \times 2 \times 1$.

factor theorem

(1) For any polynomial $P(x)$, if $P(a) = 0$ then $(x - a)$ is a factor of $P(x)$.

(2) For any polynomial $P(x)$, if $(x - a)$ is a factor of $P(x)$ then $P(a) = 0$.

fundamental counting principle

If one event can occur in m ways and another event can occur in n ways, then the two-stage event (both events together) can occur in $m \times n$ ways. Also called 'multiplication principle'.

H

halving the interval

An iterative method for approximating the root of a function (see also *Newton's method*). First, find two values of x for which $f(x)$ has different signs; then, take the average ('halving the interval') of the two x values to find a better approximation for the root of the equation $f(x) = 0$.

horizontal line test

To check that any horizontal line that can be drawn will only cut a curve once.

I

increasing function

A function f defined on the interval $a \leq x \leq b$ is said to be strictly monotonic increasing when, for all x_1 and x_2 in the domain, if $x_2 > x_1$ then $f(x_2) > f(x_1)$. In other words, the curve always slopes upwards to the right so the function never has a negative gradient.

indefinite integral

The primitive of a function, written $\int f(x)dx$.

independent events

Events are independent if the occurrence or non-occurrence of one event cannot change the probability of the occurrence of another event, i.e. the events have no effect on each other.

induction (mathematical)

A method used to prove that something is true for all positive integers n (or e.g. for all positive integers n greater than a specified starting value, or for a subset such as all positive odd integers n). First, the proposition is proved true for a starting value of n; next, it is proved that if the proposition is true for $n = k$ then it is true for $n = k + 1$; by induction, it is therefore true for all values of n greater than the starting value.

initially

When time $t = 0$ (at the beginning).

integration by substitution
Changing the variable in an integral to make it easier to determine the integral, e.g. by substituting a new variable $u = g(x)$.

inverse cosine function
$y = \cos^{-1} x$ is the inverse cosine function; it means 'that part of $x = \cos y$ for which $0 \le y \le \pi$'.

inverse function
If $f(g(x)) = g(f(x)) = x$ then $g(x) = f^{-1}(x)$ is the inverse function of $f(x)$. The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, when the inverse function exists.

inverse sine function
$y = \sin^{-1} x$ is the inverse sine function; it means 'that part of $x = \sin y$ for which $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$'.

inverse tangent function
$y = \tan^{-1} x$ is the inverse tangent function; it means 'that part of $x = \tan y$ for which $-\frac{\pi}{2} < y < \frac{\pi}{2}$'.

iterative process
A process in which a sequence of steps is performed repeatedly ('applied' or 'iterated'), obtaining an improved value at the end of each sequence ('application' or 'iteration').

L

leading term
The term $a_n x^n$ of a polynomial where $a_n \neq 0$.

locus problem
A problem to be solved by eliminating the parameter from some parametric equations (usually $x(t)$, $y(t)$) to form a new Cartesian equation ($y = f(x)$).

M

major arc
An arc that is more than half the circumference of a circle.

major sector
A circular sector that is larger than a semicircle.

major segment
A circular segment that is larger than a semicircle.

minor arc
An arc that is less than half the circumference of a circle.

minor sector
A circular sector that is smaller than a semicircle.

minor segment
A circular segment that is smaller than a semicircle.

multiplication principle
See *fundamental counting principle*.

mutually exclusive events
Two events are mutually exclusive (or 'disjoint')) if membership of one event excludes membership of the other, so that they cannot occur simultaneously.

N

Newton's Law of Cooling
The cooling rate of a body is proportional to the difference between the temperature of the body and the temperature of the surrounding medium: $\frac{dT}{dt} = -k(T - M)$ where T is the temperature at any time t and M is the temperature of the surrounding medium (a constant). A solution of this differential equation is $T = M + Ae^{-kt}$ where A is a constant.

Newton's method
An iterative method for approximating the root of a function (see also *halving the interval*). If x_1 is close to a root of the equation $f(x) = 0$, then $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ should be a better approximation to that root.

normal to a parabola
The line perpendicular to the tangent at a point of contact on a parabola. The equation of the normal at the point t on $x^2 = 4ay$ is $x + ty = at^3 + 2at$.

O

odd function
A function is odd if $f(-x) = -f(x)$ for all values of x in the domain. The function has rotational symmetry about the origin.

one-to-one function
A function for which any vertical line can only cut it once and any horizontal line can only cut it once; every one-to-one function has an inverse function.

P

parametric equations
When two related variables (e.g. x and y) are expressed in terms of a third variable, the 'parameter' (e.g. t or θ), so that $x = f(t)$, $y = g(t)$ or $x = f(\theta)$, $y = g(\theta)$.

Pascal's triangle
An arrangement of numbers that gives the coefficients of the terms in the binomial expansion of $(1 + x)^n$.

permutation
An ordered selection or arrangement of all or part of a set of objects. The number of permutations of r objects from a set of n objects is ${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2) \quad (n-r+1)$.
See also *combination*.

point of contact
A point where two curves touch.

polynomial
An algebraic expression of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where n, $n-1, \ldots$ are all positive integers and a_n, $a_{n-1}, \ldots$ are the coefficients (for convenience, usually chosen as integers).

projectile motion

The motion of a particle when it is projected in any direction and subject only to gravitational acceleration, i.e. ignoring any air resistance.

Q

quadratic function *or* quadratic polynomial

A polynomial function f of the second degree, defined by $f(x) = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$. A quadratic polynomial with no real zeros cannot be factorised into linear factors of the form $(x + \alpha)$ where α is a real number.

quartic function *or* quartic polynomial

A polynomial function f of the fourth degree, defined by $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d, e are constants, $a \neq 0$. A quartic polynomial with no real zeros cannot be factorised into linear factors of the form $(x + \alpha)$ where α is a real number.

quinella

In horse racing, when the first two horses are picked correctly (in any order).

R

range on the horizontal plane

The horizontal distance between a particle's point of projection and the point where the particle meets the horizontal plane through the point of projection.

remainder theorem

If a polynomial $P(x)$ is divided by $(x - a)$ until a remainder R does not contain x, then $R = P(a)$.

rest

When a particle is 'at rest', its velocity is zero.

roots (of an equation)

(1) The values of x for which $f(x) = 0$.

(2) The abscissae of the points at which the graph of $y = f(x)$ cuts or touches the x-axis.

S

secant

(1) A straight line joining two points on a curve.

(2) The reciprocal of the cosine function. (Abbreviated 'sec'.)

sector

A portion of a plane bounded by a circle's arc and two radii.

segment

A portion of a plane bounded by a circle's arc and a chord.

semicircle

A half of a circle bounded by a diameter and the circumference.

simple harmonic motion (SHM)

A kind of motion where a particle oscillates about an equilibrium position. It is the motion of a point moving in a straight line so that its displacement x from a fixed point O at time t is given by a sine or cosine function.

SHM occurs when a particle is under the influence of a force that is directed towards the origin O and is proportional to the distance of the particle from O. The equation of motion is $\ddot{x} = -n^2 x$.

subtend

To 'extend under', to 'be opposite to', or to 'be held by'.

T

t formulae

Expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of t where $t = \tan\frac{\theta}{2}$.

tangent

The limiting position of a secant is called a tangent to a curve. It touches the curve and has the same gradient as the curve at that point.

tangents from an external point

Two tangents drawn to a circle from an external point.

tangents to a parabola

(1) The equation of the tangent at a point t on $x^2 = 4ay$ is $y = tx - at^2$.

(2) The equation of the tangent at a point (x_1, y_1) on $x^2 = 4ay$ is $xx_1 = 2a(y + y_1)$.

trajectory

The path followed by a particle.

trifecta

In horse racing, when the first three horses are picked in the correct order.

V

velocity

The rate of change of displacement with respect to time: $\dot{x}(t)$, $v(t)$ or $\frac{dx}{dt}$, standard units m s^{-1}.

vertical line test

To check that any vertical line that can be drawn will only cut a curve once.

If a vertical line cannot pass through a curve more than once, then the curve represents a function.

Z

zero factorial (0!)

By definition, $0! = 1$.

zeros (of a function)

The values of x for which a function $f(x) = 0$.