Mathematical Methods Units 3 & 4

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Introduction

Insight's *VCE Revision Questions: Mathematical Methods Units 3 & 4* contains questions, worked solutions, mark allocations and tips to help you develop skills for your assessment tasks. The questions cover all areas of study in Units 3 and 4 of VCE Mathematical Methods. A good habit to implement is to test yourself by working through this resource. The process of actively recalling information assists with deeper learning, and you will be able to identify any errors or omissions in your working by comparing your answers with the provided solutions.

Questions are grouped by area of study, and then under the headings 'Exam 1' and 'Exam 2' to clearly signal the questions for which a CAS calculator can be used. As extended response questions in Exam 2 often draw on knowledge from multiple areas of study, these questions are all collected in one section of this book.

Calculator screenshots are included in the worked solutions for Exam 2 questions. For reasons of space, these are from a TI-Nspire CAS calculator.

By using this resource as part of your study regime throughout the year, you will be prepared for questions you may encounter in your end-of-year VCE exams.

We wish you well with your studies.

The Insight Team

Questions

Area of Study 1 Functions, relations and graphs

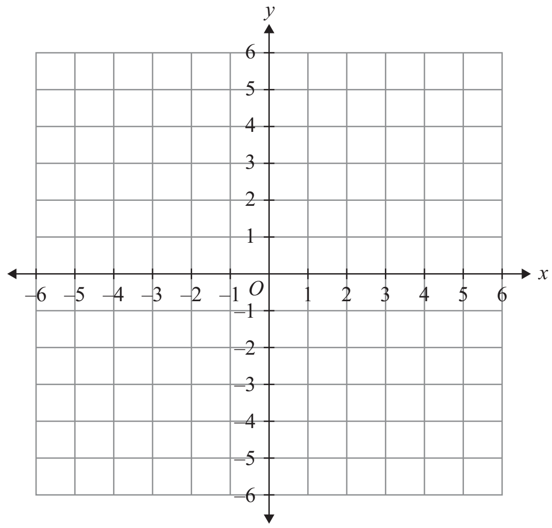
EXAM 1

Question 1 (5 marks)

Let $f: R\backslash\left\{-\frac{1}{2}\right\}, f(x) = 3 - \frac{4}{2x+1}$.

a. Sketch the graph of $y = f(x)$.

Label the axis intercepts with their coordinates and label any asymptotes with their equation. 3 marks



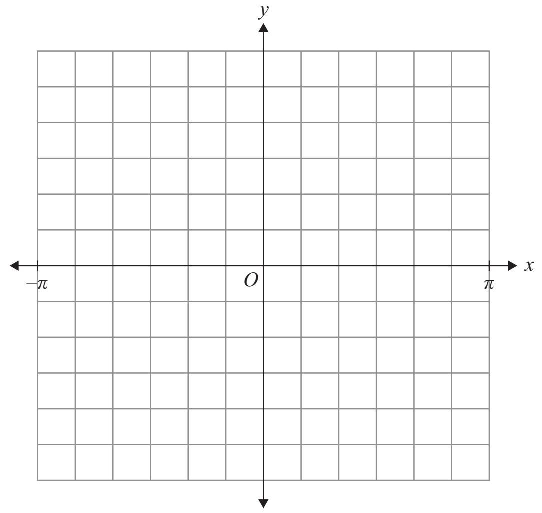
b. Hence, solve $3 - \frac{4}{2x+1} \geq 0$. 2 marks

Question 2 (3 marks)

Let $f: [-\pi, \pi] \to R$, $f(x) = 2\cos(2x) - 1$.

Sketch the graph of $y = f(x)$.

Label all axis intercepts and end points with their coordinates.



Question 3 (1 mark)

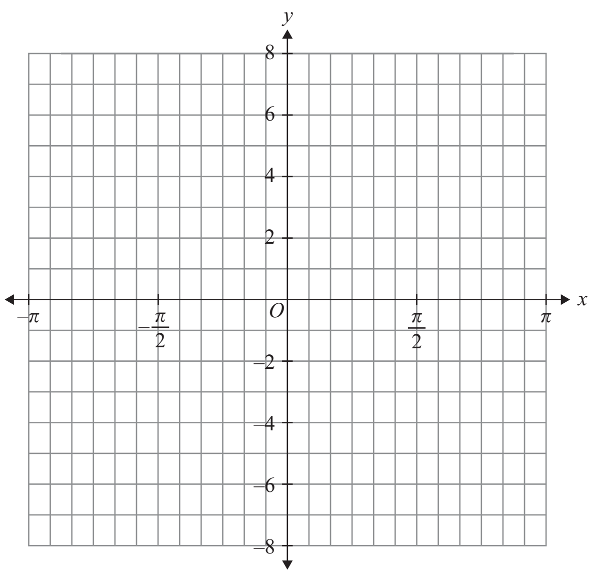
Let $f(x) = \log_e(x) + \log_e(5 - 2x)$.

State the maximal domain of f.

Question 4 (4 marks)

Let $f: \left[-\frac{7\pi}{12}, \frac{\pi}{4}\right] \to R$, $f(x) = 2 - 4\sin(2x)$.

Sketch the graph of $y = f(x)$ on the axes below. Label all end points and axis intercepts with their coordinates.



Question 5 (1 mark)

Let $f: \left(-\frac{1}{2}, 1\right] \to R$, $f(x) = 2(x-1)^2 - 3$.

State the range of f.

Question 6 (1 mark)

Let $f(x) = \log_2(x+1) - \log_2(4-x)$.

State the domain of f.

Question 7 (2 marks)

Let $f: [-1, 2] \to R$, $f(x) = \frac{1}{x+2} + \frac{1}{4-x}$.

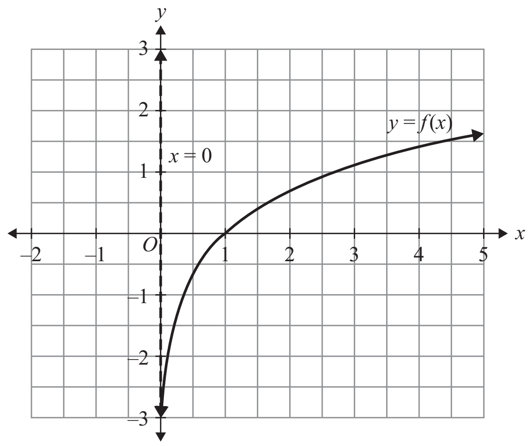
Find the range of $f(x)$.

Question 8 (4 marks)

a. Solve $1 - 2\log_e(x) = 0$. 1 mark

b. Solve $\log_e(x) = 1 - 2\log_e(x)$. 1 mark

c. The function $f: (0, \infty) \rightarrow R, f(x) = \log_e(x)$ is shown below.

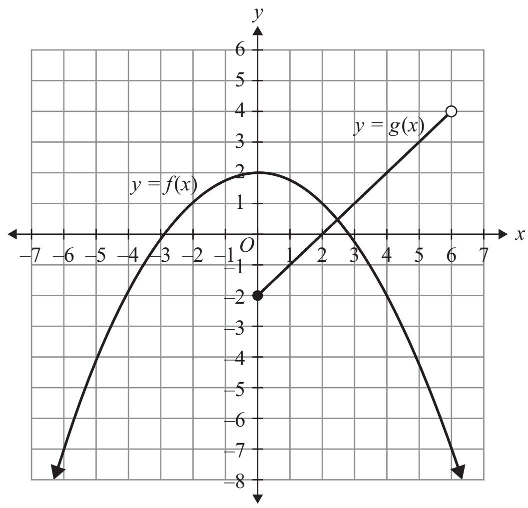


Let $g: (0, \infty) \rightarrow R, g(x) = 1 - 2f(x)$.

Sketch the graph of $y = g(x)$ on the axes above. Label any points of intersection between the graphs of f and g, and any axis intercepts, with their coordinates. 2 marks

Question 9 (2 marks)

The graphs of $y = f(x)$ and $y = g(x)$ are shown below. On the same axes, sketch the graph of $y = f(x) + g(x)$.



Question 10 (2 marks)

Determine the set of values of k for which the graph of $f(x) = ke^x$ intersects the graph of $g(x) = 3 - e^{-x}$ twice.

Question 11 (2 marks)

Let $f: \left(-\infty, \frac{5}{4}\right], f(x) = \sqrt{5 - 4x}$.

The transformation $T: R^2 \rightarrow R^2\ T(x, y) = (ax + c, y + d)$ maps the graph of $y = f(x)$ onto the graph of $y = \sqrt{x}, x \in R$. Find the values of a, c and d.

Question 12 (3 marks)

Let $f: R \rightarrow R, f(x) = \sin(ax)$, where $k \in R\backslash\{0\}$ and $g: R \rightarrow R, g(x) = x^2$.

a. Determine the range of f. 1 mark

b. Hence, or otherwise, determine the range of $g(f(x))$. 1 mark

c. If the function f has the property $f(x + h) = f(x)$ for all $h \in Z$, determine the value of a.

1 mark

EXAM 2: MULTIPLE CHOICE

Question 1

Let $f: R \to R, f(x) = -2\sin\left(\frac{\pi x}{2}\right) + 3$.

The period and range of this function are, respectively

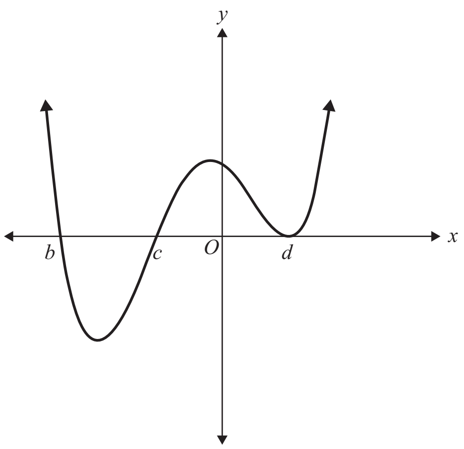
A. 4 and $[-5, 1]$

B. 4 and $[1, 5]$

C. $\frac{\pi}{2}$ and $[-5, 1]$

D. $\frac{\pi}{2}$ and $[1, 5]$

Question 2



The rule for a function with the graph shown above could be

A. $y = (x + b)(x + c)(x - d)^2$

B. $y = (x - b)(x - c)(x + d)^2$

C. $y = (x - b)(x - c)(x - d)^2$

D. $y = (x + b)(x + c)(x - d)$

Question 3

The range of the function $f: (-3, 5] \to R, f(x) = -x^2 - 3x - 5$ is

A. $[-45, -5)$

B. R

C. $[-45, -5]$

D. $[-45, -2.75]$

Question 4

The transformation $T: R^2 \to R^2$, which maps the curve with equation $y = e^x$ to the curve with equation $y = e^{2x+4} - 3$, could have the rule

A. $T(x, y) = (0.5x - 4, y + 3)$

B. $T(x, y) = (x + 4, 2y - 3)$

C. $T(x, y) = (x - 2, 2y - 3)$

D. $T(x, y) = (0.5x - 2, y - 3)$

Question 5

The range of the function $f: [-1, 7) \to R, f(x) = -x^2 + 4x - b$ is

A. $[-21 - b, -5 - b]$

B. $(-21 - b, -5 - b]$

C. $[-21 - b, 4 - b]$

D. $(-5 - b, 4 - b]$

Question 6

The function with the rule

$f(x) = -2\tan\left(-\frac{x}{3} + \frac{2\pi}{5}\right) + 4$ has a period of

A. 6π

B. 3π

C. $\frac{\pi}{3}$

D. $\frac{\pi}{6}$

Question 7

The graph of the function

$f: (0, \infty) \to R, f(x) = \frac{1}{\sqrt{x}}$ is reflected in the y-axis and then translated 4 units left and 3 units down.

Which one of the following is the rule of the transformed graph?

A. $y = \frac{1}{\sqrt{-x-4}} - 3$

B. $y = \frac{1}{\sqrt{x-4}} - 3$

C. $y = \frac{-1}{\sqrt{x-4}} + 3$

D. $y = \frac{-1}{\sqrt{x+4}} - 3$

Question 8

Let $f: R \to R, f(x) = 3\cos\left(\frac{2\pi x}{5}\right) - 2$.

The period and range of this function, respectively, are

A. $\frac{1}{5}$ and $[-3, 3]$

B. $\frac{1}{5}$ and $[-5, 1]$

C. 5 and $[-3, 3]$

D. 5 and $[-5, 1]$

Question 9

Let $f: [-2, 3) \to R, f(x) = 4 - 2x$.

The range of f is

A. $(-2, 8]$

B. $[-2, 8]$

C. $[-2, 8)$

D. $(-2, 8)$

Question 10

Let $f: D \to R, f(x) = \frac{9-2x}{x-3}$, where D is the maximal domain of f.

The graph of f has asymptotes

A. $x = -3, y = 2$

B. $x = -3, y = -2$

C. $x = 3, y = 2$

D. $x = 3, y = -2$

Question 11

The length of the line segment that joins the points with coordinates (1, 4) and (3, a) is

A. $\sqrt{a^2 + 8a + 20}$

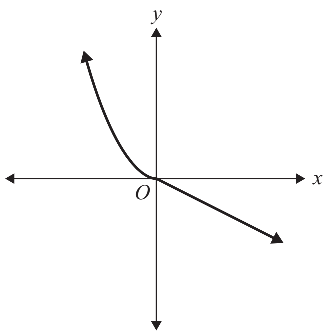
B. $\sqrt{a^2 - 8a + 20}$

C. $\sqrt{a^2 - 2a + 10}$

D. $\sqrt{a^2 + 2a + 10}$

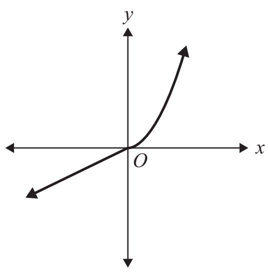
Question 12

Part of the graph of $y = f(x)$ is shown below.

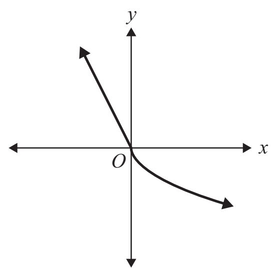


Which one of the following could be the graph of $y = f^{-1}(x)$, where f^{-1} is the inverse of f?

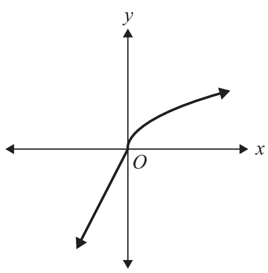
A.



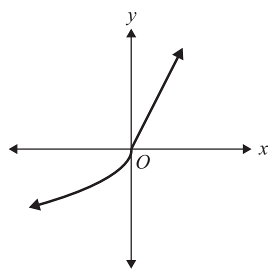
B.



C.



D.



Question 13

Consider the transformation T, defined as

- reflection in the y-axis, then
- dilation from the y-axis by a factor of 2, then
- dilation from the x-axis by a factor of 3, then
- translation by 1 unit in the positive direction of the x-axis.

The transformation T maps the graph of $y = f(x)$ onto the graph of $y = g(x)$.

If $g(x) = 3\sqrt{4 - x}$, then $f(x)$ is equal to

A. $\sqrt{3 - 2x}$

B. $\sqrt{5 - 4x}$

C. $\sqrt{3 + x}$

D. $\sqrt{5 + 4x}$

Question 14

Let $f: R \to R, f(x) = 3 - 2\sin\left(\frac{4\pi}{5}x\right)$.

The period and amplitude of f respectively, are

A. $\frac{2}{5}$ and 2

B. $\frac{5}{2}$ and -2

C. $\frac{2}{5}$ and -2

D. $\frac{5}{2}$ and 2

Question 15

A transformation $T: R^2 \to R^2$, which maps the graph of $y = \frac{-2}{\sqrt{3x - 1}} + 1$ onto the graph of $y = \frac{1}{\sqrt{x}}$, is

A. $(x, y) \to \left(3x - 1, -\frac{1}{2}y + \frac{1}{2}\right)$

B. $(x, y) \to \left(3x - 1, -\frac{1}{2}y - \frac{1}{2}\right)$

C. $(x, y) \to \left(3x + 1, \frac{1}{2}y + \frac{1}{2}\right)$

D. $(x, y) \to \left(-\frac{1}{3}x - 1, 2y + \frac{1}{2}\right)$

Question 16

If the graph of $g(x) = f(2x) + 3$ passes through the point $(2, -4)$, then the graph of f must pass through the point

A. $(1, -7)$

B. $(1, -1)$

C. $(2, -7)$

D. $(4, -1)$

Question 17

The sum of the solutions to the equation $2\cos(2x - 1) = 0$ for $x \in [-n\pi, (n + 1)\pi]$, where $n \in Z$, is

A. 2π

B. $2n\pi$

C. $(n + 1)\pi$

D. $(2n + 1)\pi$

Area of Study 2 Algebra, number and structure

EXAM 1

Question 1 (2 marks)

Solve the equation $2\cos(2x) - 1 = 0$ for $x \in [-\pi, \pi]$.

Question 2 (4 marks)

Let $f(x) = 2x(x - 1)$ and $g(x) = \log_2(x + 1)$.

a. Find $f\left(g\left(-\frac{1}{2}\right)\right)$. 2 marks

b. Find the minimum value of $g \circ f$. 2 marks

Question 3 (4 marks)

Let $f(x) = 4\sin^3(x) - 4\sin^2(x) - 3\sin(x) + 3$.

a. Show that $f(x) = (4\sin^2(x) - 3)(\sin(x) - 1)$. 1 mark

b. Hence, solve $4\sin^3(x) - 4\sin^2(x) - 3\sin(x) + 3 = 0$ for $x \in [0, 2\pi]$. 3 marks

Question 4 (2 marks)

Let $f: D \to R, f(x) = \log_e(x) + \log_e(5 - 2x)$.

Solve the equation $f(x) = 0$.

Question 5 (2 marks)

Let $f: \left(-\frac{1}{2}, 1\right] \to R, f(x) = 2(x-1)^2 - 3$.

Find the rule for f^{-1}.

Question 6 (2 marks)

Let $f(x) = \log_2(x+1) - \log_2(4-x)$.

Solve the equation $\log_2(x+1) - \log_2(4-x) = 0$.

Question 7 (3 marks)

Let $f: \left(-\infty, \frac{1}{3}\right) \to R, f(x) = \sqrt{1-3x}$.

a. Find the rule for f^{-1}. 2 marks

b. State the domain of f^{-1}. 1 mark

Question 8 (6 marks)

Consider the functions $f: R \rightarrow R, f(x) = \sin\left(\frac{x}{2}\right)$ and $g: R \rightarrow R, g(x) = a\sin(\pi x) + a$, where a is a real number.

a. i. For which values of a is $g(f(x)) \geq 0$ for all values of x? 1 mark

ii. For which value of x is $g(f(x))$ at its maximum over the interval $[0, \pi]$ when $g(f(x)) \geq 0$? 3 marks

b. For which values of a is $f(g(x)) \geq 0$ for all values of x? 2 marks

Question 9 (6 marks)

Let $f: R\backslash\left\{\frac{1}{2}\right\} \to R, f(x) = \frac{3}{4x-2} + \frac{1}{2}$.

a. State the range of f. 1 mark

b. Show that $f \circ f$ exists. 1 mark

c. Find the rule for $f \circ f$. 2 marks

d. Hence, or otherwise, define the inverse function of f. 2 marks

Question 10 (2 marks)

Solve the system of linear equations below, giving the general solution in terms of a parameter.

$2x - z = -3$

$-4y + z = 1$

Question 11 (5 marks)

Consider the functions $f(x) = 4x^2$ and $g(x) = ax^2 + 5x$, where $a \in R$.

a. The functions f and g have the property that $f(x) = g(x)$ when $x = 0$.

Show that the only other possible value of x for which $f(x) = g(x)$ is $x = \frac{5}{4-a}$. 2 marks

b. Hence, or otherwise, find the set of values of a for which there are two solutions to the equation $f(x) = g(x)$. 1 mark

c. Consider the functions $m: [-3, 0] \to R$, $m(x) = 4x^2$ and $n: [-3, 0] \to R$, $n(x) = ax^2 + 5x$.

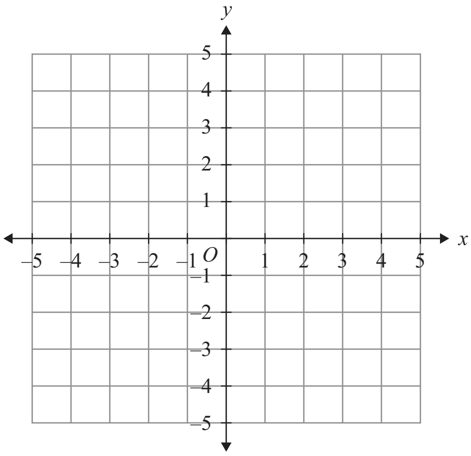
Find the set of values of a for which there are two solutions to the equation $m(x) = n(x)$. 2 marks

Question 12 (4 marks)

Let $f: [-2, \infty) \to R, f(x) = \sqrt{x+2} - 1$.

a. State the rule and domain of the inverse function f^{-1}. 2 marks

b. Sketch the graph of $y = f^{-1}(f(x))$. Label any axis intercepts with their coordinates. 2 marks



EXAM 2: MULTIPLE CHOICE

Question 1

The inverse function of $f: (3, \infty) \to R$, $f(x) = \frac{1}{\sqrt{x-3}}$ is

A. $f^{-1}: R^+ \to R \quad f(x) = \frac{1}{x^2} - 3$

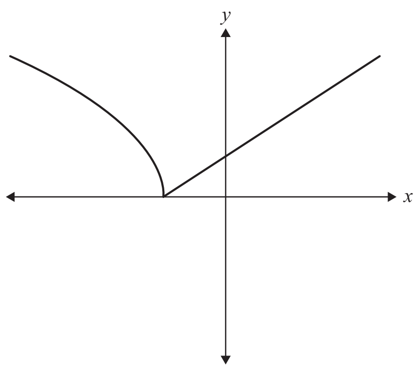
B. $f^{-1}: R^+ \to R \quad f(x) = \frac{1}{x^2} + 3$

C. $f^{-1}: R\backslash\{0\} \to R \quad f(x) = \frac{1}{x^2} + 3$

D. $f^{-1}: R^+ \to R \quad f(x) = x^2 + 3$

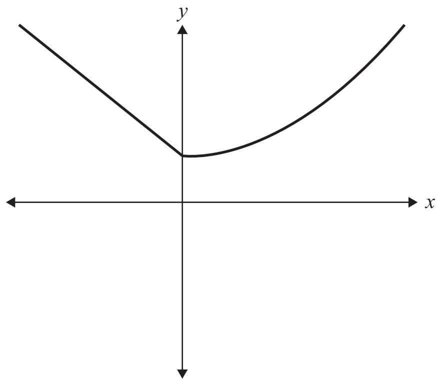
Question 2

The graph of the function with the equation $y = f(x)$ is shown below.

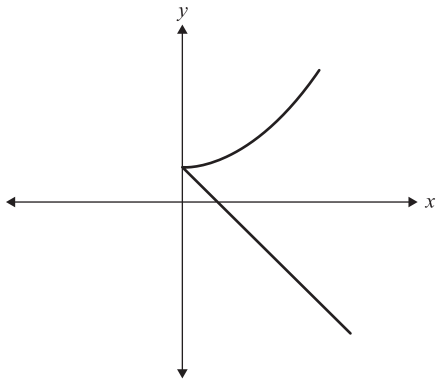


Which of the following is most likely to be the graph of $y = -f^{-1}(x)$?

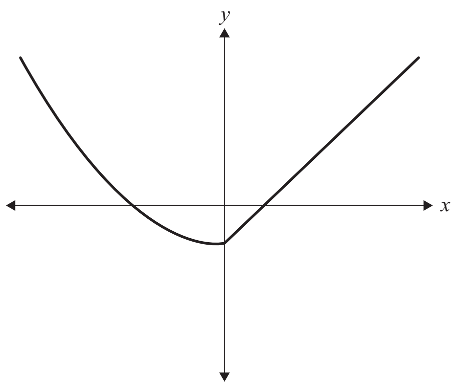
A.



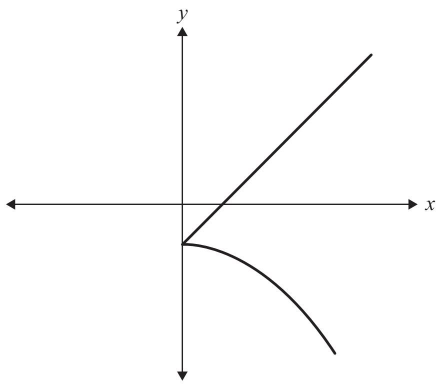
B.



C.



D.



Question 3

For the polynomial $P(x) = x^3 - ax^2 + 6x - 7$ for which $P(2) = -3$, the value of a is

A. 4

B. –4

C. –3

D. 3

Question 4

Which one of the following is the inverse function of the function $f: [-1, \infty) \rightarrow R, f(x) = 1 - 2\sqrt{x+1}$?

A. $f^{-1}: (-\infty, 1] \rightarrow R, f^{-1}(x) = \frac{1}{4}(x-1)^2 - 1$

B. $f^{-1}: [1, \infty) \rightarrow R, f^{-1}(x) = \frac{1}{4}(x-1)^2 - 1$

C. $f^{-1}: (-\infty, 1] \rightarrow R, f^{-1}(x) = -\frac{1}{4}(x-1)^2 + 1$

D. $f^{-1}: (-\infty, 1] \rightarrow R, f^{-1}(x) = \frac{1}{4}(x-1)^2 + 1$

Question 5

The largest value of a for which the inverse function of $g: (-a, a) \rightarrow R, g(x) = x\cos(x)$ exists is closest to

A. $a = 0.83$

B. $a = 0.84$

C. $a = 0.85$

D. $a = 0.86$

Question 6

Two functions, f and g, are defined as follows:

$f: \left(-\infty, -\frac{2}{3}\right) \rightarrow R, f(x) = \log_e(2 - 3x)$

$g: B \rightarrow R, g(x) = 3x^2 - 4\sqrt{2}x + 1$

where B is the maximal domain of g for which the function with rule $f(g(x))$ is defined. B is equal to

A. $\left(\frac{-2\sqrt{2} - \sqrt{17}}{3}, \frac{-2\sqrt{2} + \sqrt{17}}{3}\right)$

B. $\left(\frac{2\sqrt{2} - \sqrt{7}}{3}, \frac{2\sqrt{2} + \sqrt{7}}{3}\right)$

C. $\left(\frac{2\sqrt{2} - \sqrt{3}}{3}, \frac{2\sqrt{2} + \sqrt{3}}{3}\right)$

D. $\left(-\infty, \frac{2\sqrt{2} - \sqrt{7}}{3}\right)$

Question 7

Let $f: R \to R, f(x) = x(x+2)$ and
$g: \left[-6, -\frac{3}{2}\right] \to R, g(x) = 3x - 2$.

If the function h has the rule $h = f - g$, then the domain of the inverse function of h is

A. $\left[-\frac{4}{3}, \frac{1}{6}\right]$

B. $\left[-\frac{3}{4}, 24\right]$

C. $\left[\frac{23}{4}, 44\right]$

D. $\left[\frac{7}{4}, 40\right]$

Question 8

Suppose that $P(x) = ax^3 + bx^2 + x - 5$, $P(1) = -3$ and $P(3) = -83$. The sum of a and b equals

A. –1

B. 0

C. 1

D. 2

Question 9

The simultaneous linear equations $2x + ky = 5$ and $kx + 8y = -10$ have no solution when

A. $k = -4$

B. $k = 4$

C. $k \in \{-4, 4\}$

D. $k \in R\backslash\{4\}$

Question 10

If $x - a$ is a factor of $x^3 - 2x^2 - (3a+1)x + a$, where $a \in R\backslash\{0\}$, then the value of a is

A. 0

B. 4

C. –5

D. 5

Question 11

The graph of $f(x) = x^3 - 6x^2 + b$ has exactly three x-intercepts for

A. $b > 0$

B. $b < 32$

C. $b \in R\backslash[0, 32]$

D. $b \in (0, 32)$

Question 12

The following pseudocode implements the bisection method for finding roots, that is, the algorithm terminates when the difference between the left and right bounds is less than the specified tolerance.

```
Inputs:
  f(x), the function to
  approximately solve: f(x) = 0
  l, the left bound
  r, the right bound
  tolerance, the tolerance
  required

Define bisection(f(x), l, r,
tolerance)

If (f(l)<0 And f(r)<0) Or
(f(l)>0 And f(r)>0) Then
  Return "Illegal bounds: both
  on same side of x-axis."
End If
While r - l > tolerance Do
  m ← (l + r) ÷ 2
  If f(m)=0 Then
    Return m
  Else If (f(m)<0 And f(l)<0)
  Or (f(m)>0 And f(l)>0) Then
    [          ]
  Else
    r ← m
  End If
  End While
Return (l + r) ÷ 2
```

Which of the following lines of pseudocode should be placed in the empty box?

A. `Return m`

B. `r ← m`

C. `m ← r`

D. `l ← m`

Area of Study 3 Calculus

EXAM 1

Question 1 (4 marks)

a. Let $y = \dfrac{\sin(x)}{x^3 - 3x}$.

Find $\dfrac{dy}{dx}$. 2 marks

b. Let $f(x) = x^3 e^{1-4x}$.

Evaluate $f'(-1)$. 2 marks

Question 2 (3 marks)

Let $f: \left(-\infty, \frac{5}{4}\right] \to R$, where $f(x) = 2 + \sqrt{5 - 4x}$.

a. Find $f'(x)$. 1 mark

b. Find the equation of the tangent to $y = f(x)$ at $x = -1$.

Give your answer in the form $ax + by = c$. 2 marks

Question 3 (6 marks)

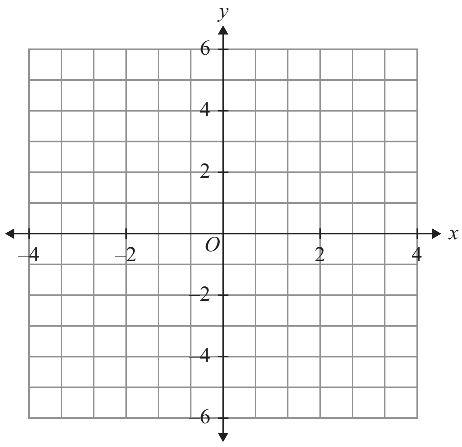
Let $f: [0, \infty) \to R, f(x) = \frac{4}{2x-1} - 1$.

a. Evaluate $f\left(\frac{3}{2}\right)$. 1 mark

b. Find the equation of the tangent to the graph of f at $x = \frac{3}{2}$. Express your answer in the form $y = ax + b$, where $a, b \in R$. 2 marks

c. Sketch the graph of $y = f(x)$ on the axes below.

Label each axis intercept with its coordinates and label each asymptote with its equation. Sketch the tangent to the graph of f at $x = \frac{3}{2}$ and label its point of intersection with $f(x)$. 3 marks



Question 4 (3 marks)

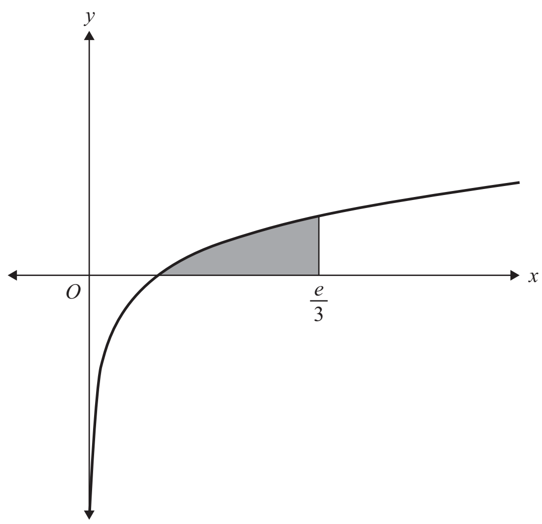
Let $f: [-\pi, \pi] \rightarrow R$, where $f(x) = 2\cos(2x) - 1$.

Calculate the average value of f over the interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{6}$.

Question 5 (4 marks)

a. Show that $\frac{d}{dx}(x\log_e(3x) - x) = \log_e(3x)$. 1 mark

The graph of the function $f: R^+ \to R$, where $f(x) = \log_e(3x)$, is shown below.



b. Hence find the exact area of the shaded region shown above, which is bounded by the graph of $f(x) = \log_e(3x)$, the x-axis and the line $x = \frac{e}{3}$. 3 marks

Question 6 (4 marks)

a. Let $y = \cos(x)\cos(4x)$.

Find $\frac{dy}{dx}$. 2 marks

b. Let $f(x) = 2e^{2x}$.

Evaluate $f'(\log_e(2))$. 2 marks

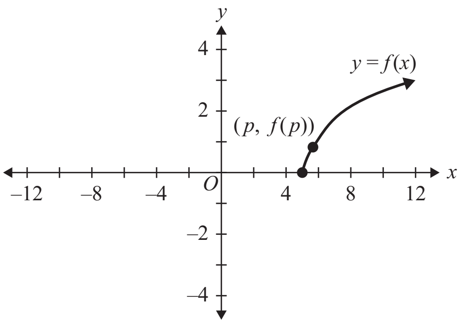
Question 7 (4 marks)

a. Show that $\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right) = \frac{1}{x^2} - \frac{\log_e(x)}{x^2}$. 1 mark

b. Hence calculate $\int_1^2 \frac{\log_e(x)}{x^2}\,dx$. 3 marks

Question 8 (3 marks)

Let $f\colon [5, \infty) \to R$, where $f(x) = \sqrt{x-5}$. Part of the graph of $y = f(x)$ is shown below.



Point P, with coordinates $(p, f(p))$, lies on the graph of $y = f(x)$.

a. Find the gradient of the tangent to the curve at point P in terms of p. 1 mark

b. Find the value of p such that the tangent to the curve at P passes through the point $(0, 0)$. 2 marks

Question 9 (4 marks)

a. If $y = (2x + 1)\log_e(2x + 1)$, find $\frac{dy}{dx}$. 2 marks

b. Let $f(x) = \cos(4x^2)$.

Evaluate $f'\left(\frac{\sqrt{\pi}}{4}\right)$. 2 marks

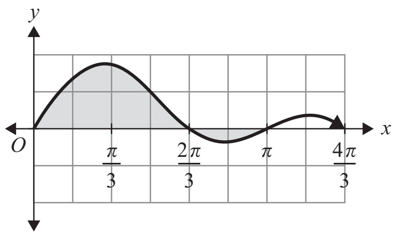
Question 10 (2 marks)

Let $\frac{dy}{dx} = 2 - e^{-x}$.

Given that $y = 4 - \frac{1}{e^2}$ when $x = 2$, find y in terms of x.

Question 11 (3 marks)

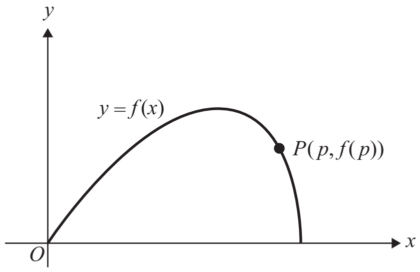
The graph of $f(x) = \sin(x) + \sin(2x)$ is shown below.



Calculate the area of the region bounded by the graph of $y = f(x)$ and the x-axis between $x = 0$ and $x = \pi$.

Question 12 (6 marks)

Let $f: [0, a) \to R$, $f(x) = \sqrt{-x^3 + kx^2}$, where $k \in R^+$. The graph of $y = f(x)$ is shown below.



a. Find the maximal value of a in terms of k. 2 marks

b. P is a point on the graph $y = f(x)$ with coordinates $(p, f(p))$.

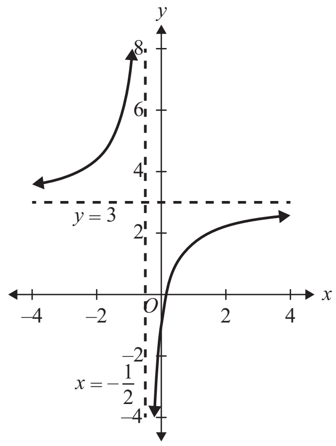
i. Show that the length, L, of the chord OP is given by $L = \sqrt{(k+1)p^2 - p^3}$. 1 mark

ii. For $0 < k \leq 2$, the chord OP is longest when $p = k$.

Determine the value of p in terms of k for which the length of the chord OP is longest when $k > 2$. 3 marks

Question 13 (3 marks)

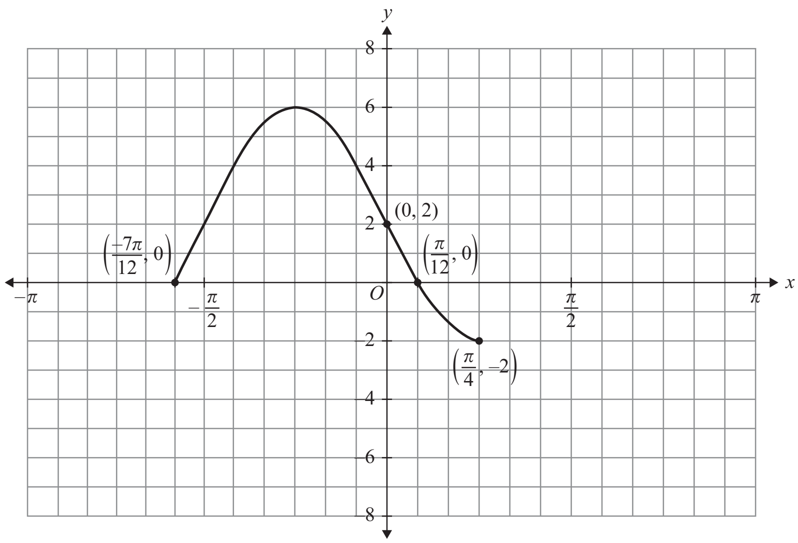
Let $f\colon R\backslash\left\{-\frac{1}{2}\right\},\ f(x) = 3 - \frac{4}{2x+1}$. Part of the graph of $y = f(x)$ is shown below.



The area enclosed by the graph of $y = f(x)$, the line $x = 1$ and the line $x = 3$ can be expressed in the form $a + \log_e\left(\frac{b}{c}\right)$. Find the values of a, b and c.

Question 14 (3 marks)

Let $f\colon \left[-\frac{7\pi}{12}, \frac{\pi}{4}\right] \to R,\ f(x) = 2 - 4\sin(2x)$. The graph of $y = f(x)$ is shown below.



Find the area enclosed by the graph of $y = f(x)$ and the x-axis over the interval $x \in \left[-\frac{7\pi}{12}, \frac{\pi}{12}\right]$.

Question 15 (2 marks)

Let $f: [-1, 2] \rightarrow R,\ f(x) = \frac{1}{x+2} + \frac{1}{4-x}$.

Solve $f'(x) = 0$.

Question 16 (3 marks)

Let $f: R \rightarrow R,\ f(x) = (x - k)e^{-x}$, where $k \in R$.

a. Show that $f'(x) = -(x - 1 - k)e^{-x}$. 1 mark

b. The graph of f has a stationary point at P. Find the coordinates of P in terms of k. 2 marks

Question 17 (4 marks)

Let $f: \left(-\infty, \frac{1}{3}\right) \to R,\ f(x) = \sqrt{1-3x}$.

a. **i.** Find $f'(x)$. 1 mark

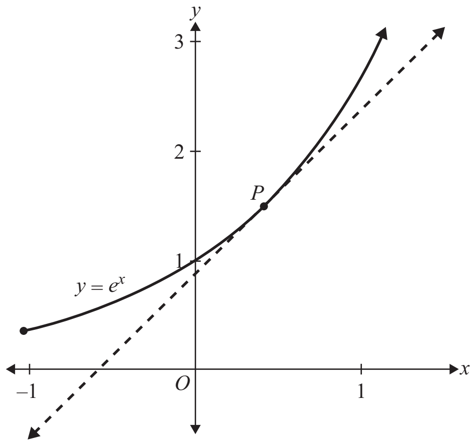
ii. Find an antiderivative of $f(x)$. 1 mark

b. Let $g: (-\pi, \pi) \to R,\ g(x) = \dfrac{\sin(x)}{\cos(x)+1}$.

Evaluate $g'\left(\frac{\pi}{2}\right)$. 2 marks

Question 18 (4 marks)

The graph of the relation $y = e^x$ is shown on the axes below. P is a point on the graph with coordinates (p, e^p). The tangent at P is also shown.



a. Show that the equation of the tangent at P is $y = e^p x + e^p(1 - p)$. 1 mark

b. Find the value of p for which the area bounded by $y = e^x$, its tangent at $x = p$, and the lines $x = 0$ and $x = 1$ is minimised. 3 marks

Question 19 (3 marks)

Let f: $[0, 2] \to R$, $f(x) = x^3 - 4x^2 + 2$.

Newton's method is to be used to approximate the value of an x-intercept of this function.

Starting with an initial estimate of $x_0 = 2$, use two iterations of Newton's method to find the values of x_1 and x_2.

Question 20 (8 marks)

Consider the functions f: $[-1, a] \to R$, where $f(x) = \frac{1}{x+2} + \frac{1}{4-x}$, and g: $[-1, a] \to R$, $g(x) = \frac{-6}{x^2 - 2x - 8}$.

a. Verify that $f(x) = g(x)$. 2 marks

b. Show that $g'(x) = \dfrac{12(x-1)}{(x^2-2x-8)^2}$. 1 mark

c. Find the x-coordinate of the stationary point of g and state the nature of the stationary point. 2 marks

d. State the largest value of a such that the inverse function, g^{-1}, exists. 1 mark

e. Find the rule for $g^{-1}(x)$. 2 marks

Question 21 (7 marks)

Let $f: (0, \infty) \to R, f(x) = \dfrac{\log_e(x)}{x^2}$.

a. Find $\dfrac{d}{dx}\left(\dfrac{\log_e(x)}{x}\right)$. 1 mark

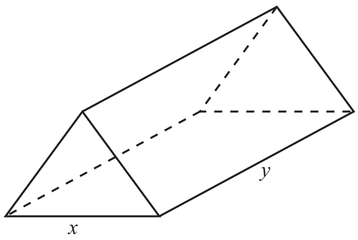
b. Hence determine an antiderivative of $f(x)$. 2 marks

c. Use your result from **part b.** to determine $\int_1^{e^2} f(x)\,dx$. 2 marks

d. Hence find the average value of f over the interval $[1, e^2]$. 2 marks

Question 22 (5 marks)

A solid triangular prism is shown below.



The cross-section of the prism is an equilateral triangle, and the sum of the lengths of the edges of the prism is E cm.

a. Show that the volume of the prism is given by $V = \dfrac{\sqrt{3}\,Ex^2 - 6\sqrt{3}\,x^3}{12}$. 3 marks

b. Find the side length, x, of the triangle that maximises the volume of the prism. 2 marks

EXAM 2: MULTIPLE CHOICE

Question 1

Consider the tangent to the graph of $y = x^3$ at the point $(-2, -8)$.

Which one of the following points lies on this tangent?

A. $(-1, -4)$

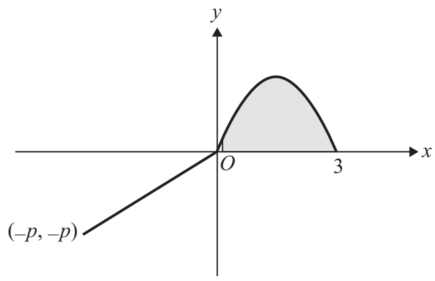
B. $(-1, 4)$

C. $(2, -40)$

D. $(1, 8)$

Question 2

The graph of a function f is shown below for $x \in [-p, 3]$. The graph is a straight line for $x \in [-p, 0]$.



The average value of f over the interval $[-p, 3]$ is zero and the area of the shaded region is $\frac{81}{8}$ square units. The value of p is

A. 3

B. 9

C. $\frac{81}{2}$

D. $\frac{9}{2}$

Question 3

If $y = \log_e(\sqrt{f(2x)})$, then $\frac{dy}{dx}$ is equal to

A. $\frac{1}{\sqrt{f(2x)}}$

B. $\frac{1}{2\sqrt{f(2x)}}$

C. $\frac{f'(2x)}{2\sqrt{f(2x)}}$

D. $\frac{f'(2x)}{f(2x)}$

Question 4

If $k = \int_2^5 \frac{2}{x}\,dx$, then $e^{\frac{k}{2}}$ is equal to

A. $\frac{5}{2}$

B. $e^{\frac{5}{2}} - e^1$

C. 5

D. $e^5 - e^2$

Question 5

If $\int_{-1}^{2} f(x)\,dx = 4$, then $\int_{-1}^{2} (3 - f(x))\,dx$ is equal to

A. -1

B. 5

C. 1

D. -4

Question 6

The area of the region bounded by the x-axis, the y-axis and the curve $f(x) = e^{2x} - 2e^x - 3$ is

A. $4\log_e(3)$

B. 3

C. $3\log_e(3)$

D. $4 - \log_e(3)$

Question 7

If $y = 2e(e^x - 1)$, then the rate of change of y with respect to x when $x = 0$ is

A. $2e^2$

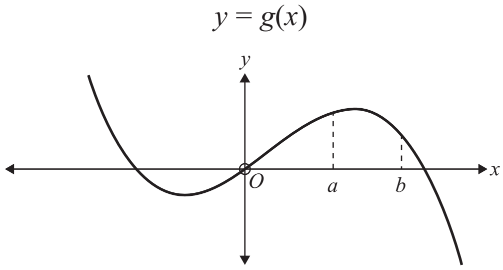
B. $2e - 1$

C. 0

D. $2e$

Question 8

The graph of the function with equation $y = f(x)$ is shown below.



Let g be a function such that $g'(x) = f(x)$.

Over the interval (a, b), the graph of g will have a

A. minimum turning point

B. maximum turning point

C. positive gradient

D. negative gradient

Question 9

The gradient of a line perpendicular to the line that passes through $(-3, 2)$ and $(1, 5)$ is

A. $\frac{3}{4}$

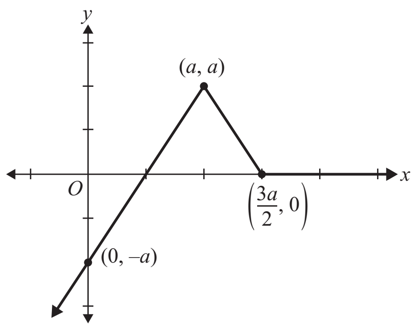
B. $\frac{4}{3}$

C. $-\frac{3}{4}$

D. $-\frac{4}{3}$

Question 10

Part of the graph of a function f is shown below.



The average value of f over the interval $[0, 2a]$ is

A. $\frac{3a}{8}$

B. $\frac{a}{6}$

C. $\frac{a}{8}$

D. $\frac{a}{2}$

Question 11

The equation of the line perpendicular to the tangent to the graph of $y = \frac{1}{2}\sin(3x) + 1$ at the point where $x = -\frac{\pi}{18}$ is

A. $y = \frac{4\sqrt{3}}{9}x + \frac{2\sqrt{3}\pi - 162}{81}$

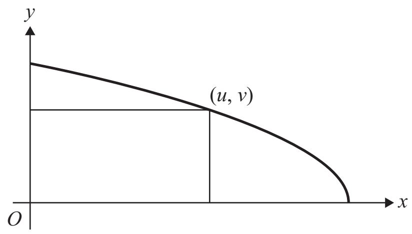
B. $y = \frac{4\sqrt{3}}{9}x + \frac{2\sqrt{3}\pi + 162}{81}$

C. $y = -\frac{4\sqrt{3}}{9}x - \frac{162 + 2\sqrt{3}\pi}{81}$

D. $y = -\frac{4\sqrt{3}}{9}x - \frac{8\pi\sqrt{3} - 243}{324}$

Question 12

A rectangle is formed as shown below, using the coordinate axes to form two sides and with a vertex at point (u, v), where $u > 0$. The vertex lies on the graph of $y = \sqrt{9 - x}$.



What is the maximum area of the rectangle?

A. $\frac{13\sqrt{5}}{2\sqrt{2}}$

B. $6\sqrt{3}$

C. 6

D. 10

Use the following information to answer Questions 13 and 14.

Let f and g be two functions such that $f(0) = 3$, $f'(0) = -6$, $g(0) = 4$ and $g'(0) = 2$.

Question 13

If $h(x) = \frac{g(x)}{f(x)}$, then the value of $h'(0)$ is

A. $-\frac{1}{3}$

B. $\frac{10}{3}$

C. $\frac{8}{3}$

D. $\frac{10}{9}$

Question 14

If $k(x) = f(x)\sqrt{g(x)}$, then the value of $k'(0)$ is

A. $-6\sqrt{2}$

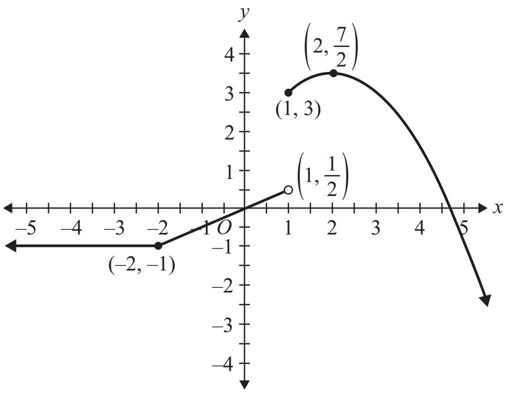
B. $3\sqrt{2} - 12$

C. $\frac{3\sqrt{2} - 24}{2}$

D. $-\frac{21}{2}$

Question 15

The graph of a function f is shown below.



The maximal set of values of x for which f is a strictly increasing function is

A. $[-2, 2]$

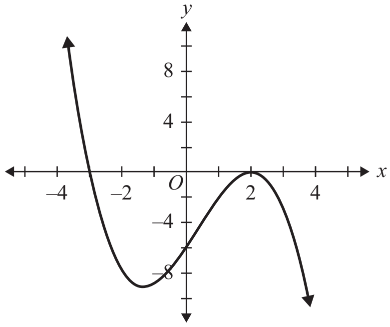
B. $(-2, 2)$

C. $(-2, 1) \cup (1, 2)$

D. $[-2, 1) \cup (1, 2)$

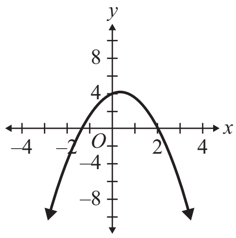
Question 16

Part of the graph of $y = f(x)$ is shown below.

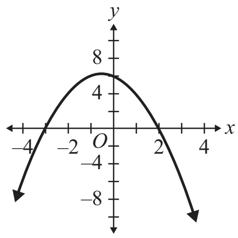


Which of the following could be the graph of $y = f'(x)$?

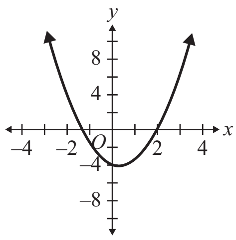
A.



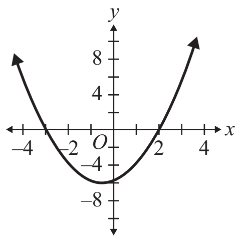
B.



C.



D.



Question 17

Consider the line perpendicular to the tangent to the graph of $y = x^2$ at the point $(-3, 9)$.

This line passes through the x-axis where

A. $x = 57$

B. $x = 12$

C. $x = -57$

D. $x = -55$

Question 18

The average value of the function with rule $f(x) = -6(x-1)^2(x-a)$ over the interval $[1, a]$, where a is a real number and $a > 1$, is 4.

The value of a is

A. 7

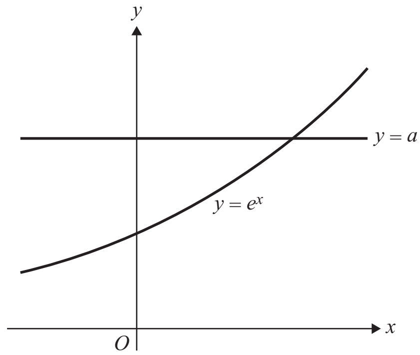
B. 6

C. 5

D. 3

Question 19

Parts of the graphs with equations $y = e^x$ and $y = a$, where $a > 1$, are shown below.



The total area of the region bounded by the y-axis, the line $y = a$, where $a > 1$, and the curve with equation $y = e^x$ is equal to

A. $1 + a^2 - e^a$

B. $a - 1 - a\log_e(a)$

C. $a\log_e(a) + 1 - a$

D. $\dfrac{e^a - 1}{\log_e(a)}$

Question 20

If $\int_0^2 f(x)\,dx = 5$, then $\int_{-4}^0 3f\left(-\frac{x}{2}\right)dx$ is

A. 30

B. 10

C. 15

D. 20

Question 21

The graph of the function $f(x) = a^2x - a\sqrt{x}$, where $a > 0$, has a stationary point when $x = 1$.

The value of a is

A. $\dfrac{1}{\sqrt{2}}$

B. $\dfrac{1}{2}$

C. $\dfrac{1}{2\sqrt{2}}$

D. $\dfrac{1}{4}$

Question 22

The average value of $1 + 2x^2$ over the interval $[1, b]$, where $b > 1$, is 15.

The value of b is

A. 3

B. 4

C. 5

D. 6

Question 23

Let $f: R \to R, f(x) = ax^2(3 - x)$, where $a \in R$. The y-intercept of the tangent to f at $x = 3$ is 9.

The value of a is

A. 9

B. $\frac{1}{27}$

C. $\frac{1}{9}$

D. $\frac{1}{3}$

Question 24

If $\int_2^6 f(x)\,dx = 5$, then $\int_1^3 (f(2x) + x)\,dx$ is

A. 9

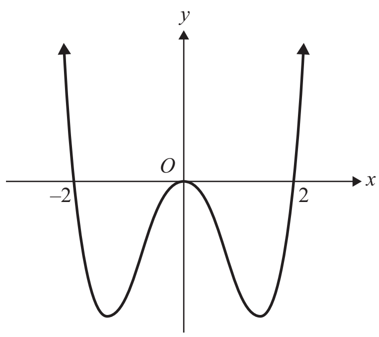
B. $\frac{9}{2}$

C. $\frac{13}{2}$

D. $\frac{15}{2}$

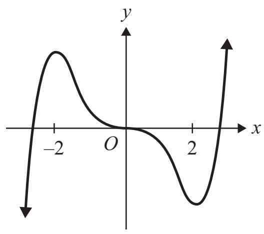
Question 25

The graph of $y = f'(x)$ is shown below.

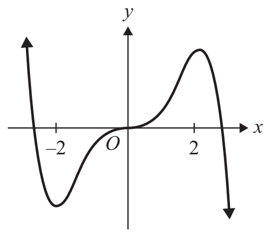


The corresponding part of the graph of $y = f(x)$ is best represented by

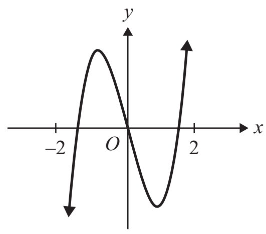
A.



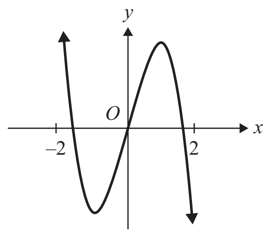
B.



C.



D.



Question 26

If $\int_k^{2k} \frac{1}{\sqrt{2x+1}}\,dx = 2$ and $k > -\frac{1}{2}$, then k is

A. $2\sqrt{3} + 3$

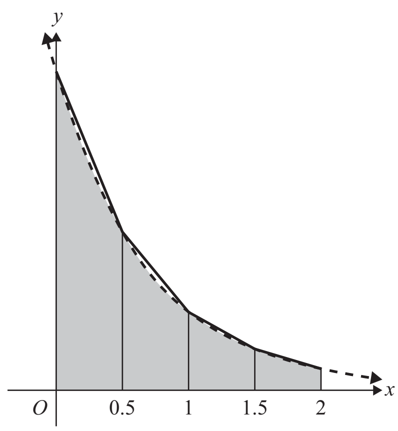
B. 12

C. 10

D. 8

Question 27

The area between the x-axis and the graph of $y = 2^{-x}$ over the interval $[0, 2]$ is approximated using four trapeziums of equal width, as shown below.



The error in this approximation, when compared with the actual area, is closest to

A. 0.00

B. 0.01

C. 0.04

D. 0.18

Use the following information for Questions 28 and 29.

An implementation of Newton's method that uses a fixed number of iterations is shown below.

```
Inputs:
f(x), the function to
approximately solve f(x)=0
guess, the initial guess x0
n, the number of iterations

Define newtons(f(x), guess, n)
  i ← 0
  deriv(x) ← The derivative of
  f(x)
While i < n Do
  guess ← guess - f(guess) ÷
  deriv(guess)
  i ← i + 1
End While
Return guess
```

This algorithm is being used to solve the equation $x^3 - x^2 + 4x + 1 = 0$, using an initial estimate of $x_0 = 0$.

Question 28

If the algorithm is run for three iterations, the output is

A. $x_3 = -\frac{14\,578}{62\,505}$

B. $x_3 = -\frac{1\,902\,917}{8\,158\,995}$

C. $x_3 = -\frac{7}{30}$

D. $x_3 = \frac{1\,501\,199}{875\,796}$

Question 29

The minimum number of iterations required for a tolerance of 0.00001 or less is

A. 1

B. 2

C. 3

D. 4

Question 30

The following pseudocode describes the trapezium method for approximating the area under a curve.

```
Inputs:
  f(x), the function we are
  approximating the area under
  a, the left terminal
  b, the right terminal
  width, the width of each
  trapezium
Define trapezium(f(x), a, b,
width)
  left ← a
  totalArea ← 0
While left < b Do
  right ← left + width
  leftHeight ← f(left)
  rightHeight ← f(right)
  area ← width × (leftHeight +
  rightHeight) ÷ 2
  totalArea ← totalArea + area
  left ← right
End While
Return totalArea
```

The algorithm above will be used to approximate the area under the graph of $f(x) = \frac{12}{x}$ between $x = 1$ and $x = 5$, using trapeziums of width 1 unit.

After three iterations, the variable `totalArea` is

A. 14

B. 17.5

C. 18.85

D. 19.3133

Question 31

Let $f: [0, \infty) \to R, f(x) = -x^4 + 3x^2 - x$.

The coordinates of the point of inflection of the graph of $y = f(x)$, correct to three decimal places, are

A. $(-0.707, 1.957)$

B. $(0, 0)$

C. $(0.170, -0.084)$

D. $(0.707, 0.543)$

Question 32

Let f be a one-to-one differentiable function such that $f(5) = 3$ and $f'(5) = -2$. The function g is differentiable and $g(x) = f^{-1}(x)$ for $x \in R$.

The equation of the tangent to the graph of g, where $x = 3$, is

A. $y = 2x - 1$

B. $y = \frac{1}{2}x + \frac{1}{2}$

C. $y = -\frac{1}{2}x + \frac{11}{2}$

D. $y = -\frac{1}{2}x + \frac{13}{2}$

Area of Study 4 Data analysis, probability and statistics

EXAM 1

Question 1 (4 marks)

For a particular plant species the probability that any one seed will germinate is $\frac{1}{4}$. Assume that one seed germinating is independent of any other seed germinating and of the pot that the seed is planted in.

a. Marie plants three seeds in a pot.

i. Find the probability that none of the three seeds germinate. 1 mark

ii. Show that the probability that exactly two seeds germinate is $\frac{9}{64}$. 1 mark

b. Marie plants a second pot with three seeds. Find the probability that exactly two seeds germinate in only one of the two pots she has planted. Express your answer in the form $\frac{a \times 3^b}{4^c}$, where a, b and c are positive integers. 2 marks

Question 2 (2 marks)

The events A and B from a sample space are independent.

If $\Pr(A' \cap B) = 0.2$ and $\Pr(A) = 0.4$, find $\Pr(A \cap B)$.

Question 3 (3 marks)

A set of 10 cards comprises two red cards and eight black cards. Two cards are randomly drawn from the set without replacement.

a. Find the probability that at least one red card is drawn. 1 mark

b. Find the probability that the first card drawn is red, given that at least one red card is drawn. 2 marks

Question 4 (2 marks)

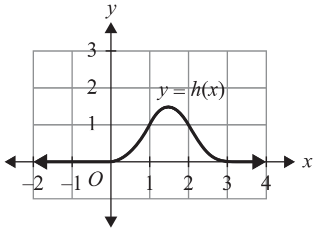
A board game uses a customised eight-sided die. Three sides of the die have a victory symbol. During a game the die is rolled 10 times. Let $\hat{P}$ be the proportion of victory symbols rolled.

a. Find $\mathrm{E}(\hat{P})$. 1 mark

b. Find the variance of $\hat{P}$. 1 mark

Question 5 (8 marks)

The graph of the function $h: R \to R$, $h(x) = \begin{cases} x^2 & 0 \le x < 1 \\ -2\left(x - \frac{3}{2}\right)^2 + \frac{3}{2} & 1 \le x < 2 \\ (x-3)^2 & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$ is shown below.



Let X be a continuous random variable with the probability density function $f(x) = ah(x)$, where a is a positive real number.

a. Show that $a = \frac{1}{2}$. 4 marks

b. i. Let Y be a continuous random variable with probability density function $f(y) = h(by)$, where b is a positive real number.

Find the value of b. 1 mark

ii. For which values of y is $f(y) > 0$? 1 mark

iii. Both X and Y are sampled.

If X and Y are independent, find $\Pr(X < 1 \cup Y < 1)$. 2 marks

Question 6 (2 marks)

Eggs can be either brown or white. A random sample of eggs, sample A, laid on a particular day on a large egg farm was taken in order to estimate the proportion of brown eggs that the farm produces.

Sample A contained a brown eggs, and, based on this sample, a 95% confidence interval for the proportion of brown eggs laid on the farm was calculated.

In order to reduce the width of the confidence interval, a second, larger sample, sample B, was taken. Sample B was found to contain the same proportion of brown eggs as sample A, but the 95% confidence interval calculated from sample B was only a quarter as wide as the 95% confidence interval calculated from sample A.

Determine the number of eggs in sample B, expressing your answer in terms of a.

Question 7 (8 marks)

The time, in minutes, it takes to produce a component on an assembly line is a continuous random variable X with the probability density function

$$g(x) = \begin{cases} \dfrac{x}{150} & 0 \le x \le 15 \\ \dfrac{20 - x}{50} & 15 \le x \le 20. \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the probability that it takes less than 15 minutes to produce a component. 1 mark

b. 80% of components are produced in less than k minutes. Find the value of k. Express your answer in the form $a + b\sqrt{c}$ where a, b and c are integers. 3 marks

Changes are made to the assembly line so that the new time to produce the same component is given by the probability density function h, where $h(x) = pg\left(\frac{x}{q}\right)$.

c. Describe the dilations, in terms of p and q, that map the graph of $y = g(x)$ onto the graph of $y = h(x)$. 2 marks

d. After the changes to the assembly line, 80% of components are produced in less than 16 minutes.

Use your answer to **part c.** to find the values of p and q. 2 marks

EXAM 2: MULTIPLE CHOICE

Question 1

A binomial random variable, X, has $\mathrm{E}(X) = 3$ and $\mathrm{Var}(X) = \frac{9}{4}$.

$\Pr(X = 1)$ is

A. $\left(\frac{1}{4}\right)^{12}$

B. $\left(\frac{3}{4}\right)^{12}$

C. $\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{11}$

D. $12\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{11}$

Question 2

A continuous random variable, X, has a normal distribution with mean 60 and standard deviation 7.

If the random variable Z has the standard normal distribution, then the probability that X is less than 46 is

A. $\Pr(Z > 2)$

B. $\Pr(Z > -2)$

C. $\Pr(Z > 74)$

D. $\Pr(X > 2)$

Question 3

The binomial random variable X has $\mathrm{E}(X) = \frac{40}{7}$ and $\Pr(X = 1) = \frac{5\,120}{5\,764\,801}$.

The value of $\Pr(X = 4)$ is closest to

A. 0.2371

B. 0.2549

C. 0.1937

D. 0.1214

Question 4

The random variable X has a normal distribution with mean 1.5 and standard deviation 0.4.

If $\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$, where the random variable Z has the standard normal distribution, then the value of a is

A. $\frac{89}{67}$

B. $\frac{33}{25}$

C. $\frac{46}{35}$

D. $\frac{45}{34}$

Question 5

X is a normally distributed random variable with mean 100.

If $\Pr(X > 120) = 0.2$, then the standard deviation of X is closest to

A. 21

B. 22

C. 23

D. 24

Question 6

A and B are independent events from a sample space. If $\Pr(A) = 0.4$ and $\Pr(B) = 0.3$, then $\Pr(A \cup B')$ is

A. 0.42

B. 0.82

C. 0.36

D. 0.92

Question 7

A darts player has a 20% chance of hitting the bullseye with any one dart. They throw 50 darts at the bullseye. The outcome of each throw is independent of the previous throw. The probability that n or more throws are successful is less than 0.3.

The smallest value of n, where n is an integer, is

A. 11

B. 12

C. 13

D. 14

Question 8

The function f is a probability density function with rule

$$f(x) = \begin{cases} \dfrac{ax(5-x)}{50}e^{-\frac{x}{10}} & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

The value of a is

A. $\dfrac{50}{\sqrt{e}} - 30$

B. $\dfrac{\sqrt{e}}{-50 + 30\sqrt{e}}$

C. $\dfrac{\sqrt{e}}{50 - 30\sqrt{e}}$

D. $\dfrac{\sqrt{e}}{50\sqrt{e} - 30}$

Question 9

The time Nitesh takes to walk to school in the morning is normally distributed, with a mean of μ minutes and a standard deviation of 3 minutes.

It takes Nitesh 10 minutes or less to walk to school on 90% of occasions.

The value of μ, correct to one decimal place, is

A. 6.1

B. 6.2

C. 6.5

D. 13.8

Question 10

A bag contains six red marbles and four blue marbles. Three marbles are randomly drawn from the bag without replacement.

Given that at least two of the marbles are red, the probability that all three are red is closest to

A. $\frac{1}{8}$

B. $\frac{1}{4}$

C. $\frac{2}{3}$

D. $\frac{1}{6}$

Question 11

A random sample of students are asked if they are studying Chemistry. From this sample, an approximate 95% confidence interval for the proportion of students who are studying Chemistry is calculated to be (0.3828, 0.5283).

The number of students in the sample who are studying Chemistry is closest to

A. 180

B. 82

C. 90

D. 112

Question 12

The organising committee for a Year 12 formal asks a random sample of students if they plan to attend the event. From the data collected, the committee calculates a 95% confidence interval of (0.35, 0.85) for the proportion of students planning to attend the event.

Which of the following statements is **not** a correct interpretation of this confidence interval?

A. Less than 35% of the student population may be planning to attend the formal.

B. There is a 2.5% chance that more than 85% of students plan to attend the formal.

C. 60% of the students may be planning to attend the formal.

D. The committee can expect a 95% chance that the true proportion of students planning to attend the formal would be inside the 95% confidence interval.

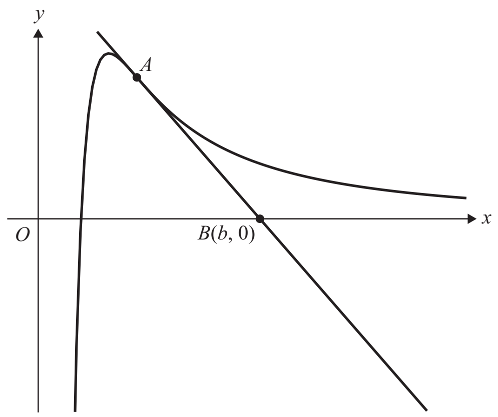
Extended response Exam **2**

Question 1 (6 marks)

Let $f: (0, \infty) \rightarrow R, f(x) = \dfrac{\log_e(x)}{x^2}$.

a. Find the rule for $f'(x)$. 1 mark

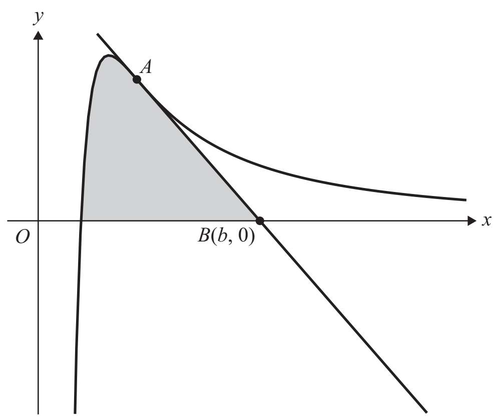
Shown on the axes below is the graph of f and the tangent to f at its inflection point, A.



b. **i.** Write down the coordinates of A. 2 marks

ii. The tangent passes through $B(b, 0)$. Determine the value of b. 1 mark

The area bounded by the graph of f, the x-axis and the tangent to f at point A is shaded below.



c. Determine the area of the shaded region, giving your answer in the form $a + be^c$ where a, b and c are rational numbers. 2 marks

Question 2 (6 marks)

Let $f: R \to R, f(x) = a(x - 6)(x - 1)(x + 2)$.

a. The coordinates of the local minimum of f are $(4, -108)$. Verify that the value of a is 3. 1 mark

b. Find $f'(x)$. 1 mark

c. Find the coordinates of the local maximum of f. 1 mark

d. Let g be the function $g: R \to R, g(x) = 4x^3 - 4x^2 - 15x + 3$.

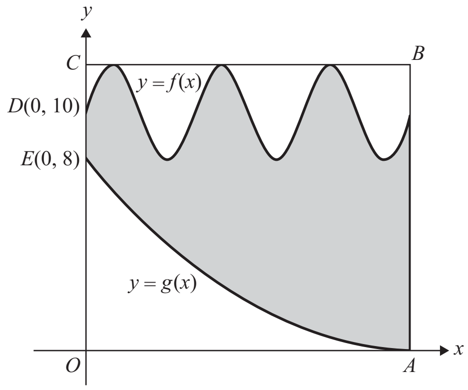
A sequence of four transformations is applied to g.

Describe a sequence of two dilations followed by two translations for which the image of g would have the same coordinates as those of f. 3 marks

Question 3 (7 marks)

A component for a machine is to be cut from a rectangular sheet of plastic, $OABC$. The length of OA is 15 cm and the length of OC is 12 cm.

The component is shown by the shaded region below.



The upper curved edge of the component can be modelled by the function $f(x) = 10 + 2\sin\left(\frac{2\pi x}{5}\right)$.

a. Determine the area of the rectangular sheet of plastic $OABC$, in square centimetres. 1 mark

b. The lower curved edge of the component, which runs from point E to point A, can be modelled by the function $g(x) = a(x - 15)^2$.

Show that the value of a is $\frac{8}{225}$. 1 mark

c. Write down the rule for a function, $d(x)$, that gives the vertical distance between the upper and lower edges of the component. 1 mark

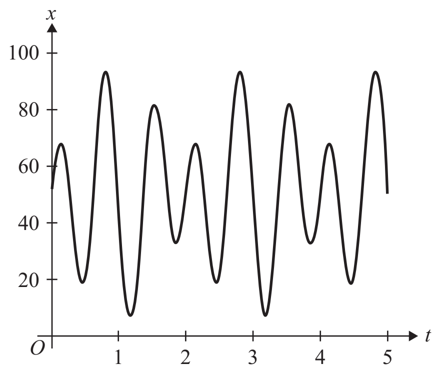
d. Find the value of x for which this distance calculated in **part c.** is a maximum. Give your answer in centimetres, correct to three decimal places. 1 mark

e. Find all values of x for which the distance between the upper and lower curved edges of the component is equal to the average distance between the upper and lower parts of the component. Give your answer correct to three decimal places. 3 marks

Question 4 (7 marks)

Let $x(t) = 50 + 30\sin(3\pi t) - 15\sin(2\pi t)$, where $t \geq 0$.

Part of the graph of $x(t)$ is shown below.



a. State the period of $x(t)$. 1 mark

b. Find the derivative $x'(t)$. 1 mark

c. Find the maximum value of $x(t)$, and the value of t for which $x(t)$ first reaches its maximum. Give values correct to two decimal places. 2 marks

d. Find the area bounded by the graph of $x(t)$, the horizontal axis, the line $t = 0$ and the line $t = 1$. 1 mark

e. The integral $\int_1^2 (p \cdot x(t) + q)\, dt$, where p and q are integers, has the same value as the area found in **part d.**

Find the values of p and q. 2 marks

Question 5 (9 marks)

Consider the function $f: R \to R, f(x) = x(x-4)^3 + 1$.

a. Determine the coordinates of the stationary points of the graph of $y = f(x)$. 2 marks

b. Find the values of x for which $f'(x) \leq 0$ and $f(x) \leq 0$. Round values to three decimal places where necessary. 1 mark

c. Write an integral that gives the area bounded by the graph of $y = f(x) - 1$ and the x-axis. 1 mark

d. **i.** Describe a sequence of transformations that maps the graph of $y = f(x)$ on to the graph of $y = f(2x) - 1$. 2 marks

ii. Find the coordinates of the x-intercepts of the graph of $y = f(2x) - 1$. 1 mark

e. The graph of $g(x) = f(px + q)$, where p and q are real numbers and $p > 0$, has stationary points where $x = 3\sqrt{2}$ and $x = 5\sqrt{2}$. Determine the values of p and q. 2 marks

Question 6 (14 marks)

When a drug is given to the body, it is absorbed and mixes with the blood. The amount of drug in the bloodstream first rises and then falls. An equation that models this process for the drug Spuriosa-A is $x = 14t^{0.3}e^{-0.2t}$, where x is the concentration of the drug in the bloodstream, in milligrams per litre, t hours after the drug is first administered.

John is given a dose of Spuriosa-A at 9 am on a particular day.

The time it takes for the concentration of Spuriosa-A in John's bloodstream to reach its maximum is 1.5 hours, that is $t = \frac{3}{2}$.

a. Find the time it takes, from when John was given the dose of Spuriosa-A, for the concentration in his bloodstream to fall to half its maximum value. Give your answer in minutes, correct to the nearest minute. 2 marks

b. The average rate of change of the concentration of Spuriosa-A in John's bloodstream over the first a hours is 2 milligrams per litre per hour.

Find the value of a, in hours, correct to two decimal places. 2 marks

c. Find the average concentration of Spuriosa-A in John's bloodstream between 9 am and 1 pm on this day. Express the answer in milligrams per litre, correct to four decimal places. 2 marks

d. John is given a second dose of Spuriosa-A at 4:30 pm on the same day as the first dose.

Find the concentration of Spuriosa-A in John's bloodstream at 6 pm on the same day. Express the answer in milligrams per litre, correct to four decimal places. 1 mark

e. The concentration, g milligrams per litre, of another drug, Spuriosa-B, in the bloodstream t hours after one dose is first given can be modelled by the function $g(t) = at^b e^{-ct}$, where a, b and c are positive constants.

It is known that Spuriosa-B reaches a maximum concentration of 12 milligrams per litre after 90 minutes and that the concentration of Spuriosa-B after 7.5 hours is half the maximum value.

i. Show that $g(t)$ has a stationary point when $t = \frac{b}{c}$. 2 marks

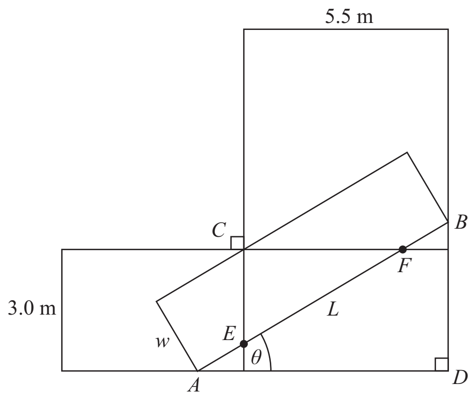
ii. Hence, show that $c = \frac{2b}{3}$. 1 mark

iii. Hence, or otherwise, find the value of b. Express the answer in the form $\frac{\log_e(p)}{p^2 - \log_e(q)}$, where p and q are positive integers. 3 marks

iv. State the values of a and c, correct to four decimal places. 1 mark

Question 7 (10 marks)

Consider the problem of positioning a rectangle of width w units and length L units inside the polygon with the dimensions shown below. The remaining dimensions of the polygon are unspecified.



The rectangle makes contact with the inside corner of the polygon at C and with the sides at points A and B. The angle θ is measured in radians. Points E and F show where the continuation of two sides of the polygon would intersect with the rectangle.

a. i. State, in terms of w and θ, an expression for the length of EC and the length of CF. 2 marks

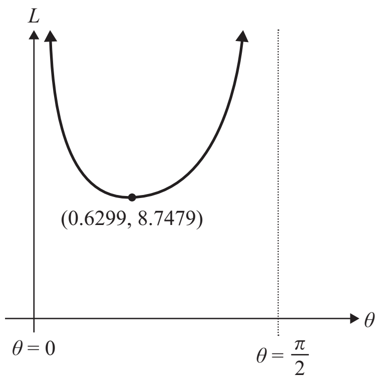
ii. Hence show that the length of $EF = \dfrac{w}{\sin(\theta)\cos(\theta)}$. 1 mark

b. **i.** Show that the length of $AE = \dfrac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$. 2 marks

ii. Find, in terms of w and θ, an expression for length FB. 2 marks

c. Hence show that $L = \dfrac{3\cos(\theta) + 5.5\sin(\theta) - w}{\cos(\theta)\sin(\theta)}$. 1 mark

d. Consider another rectangle, of width 1.5 metres and length L metres, also positioned inside the polygon. A graph of L as a function of θ for this rectangle is shown below.

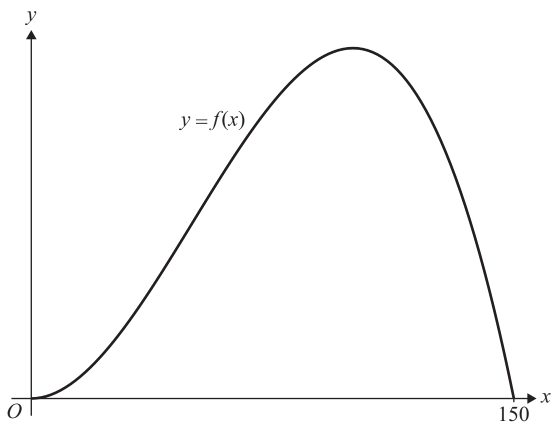


i. What is the widest rectangle that can fit within the polygon? 1 mark

ii. Briefly explain whether or not a rectangle of width 1.5 units and length 4.4 units can be rotated 90° within the polygon. 1 mark

Question 8 (13 marks)

One side of the stabiliser of a vintage aircraft needs to be painted. A cross-section of the stabiliser is shown below sketched on a set of coordinate axes. The curved edge of the cross-section can be modelled by the rule $f(x) = \dfrac{x^2(150 - x)}{2500}$, $x \in [0, 150]$.

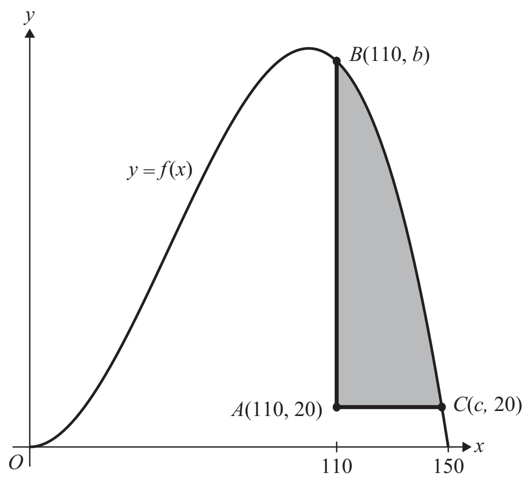


a. Find $f'(x)$. 1 mark

b. State the set of values for which f is strictly decreasing. 1 mark

c. Find the area of the cross-section shown above, that is, the area between $y = f(x)$ and the x-axis. 1 mark

A rudder, which enables the pilot to move the aircraft from side to side, forms part of the stabiliser.



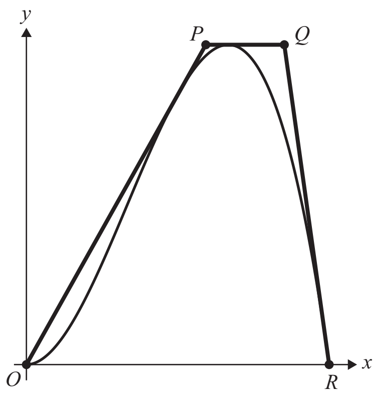
The rudder is shown by the shaded region in the diagram above. The bottom left corner of the rudder is located at $A(110, 20)$.

d. **i.** State the value of c, correct to three decimal places. 1 mark

ii. State the value of b. 1 mark

e. Find the area of the rudder, correct to three decimal places. 2 marks

A trapezium, $OPQR$, is used to determine an approximation of the area of the stabiliser, as shown in the diagram below. The tangent to f that passes through the origin $O(0, 0)$ is used for the segment OP, the horizontal tangent to f is used for the line segment PQ, and the tangent to $y = \frac{x^2(150 - x)}{2500}$ at the point $R(150, 0)$ is used for the segment QR.



f. **i.** Find the equation of the line segment OP. 1 mark

ii. Find the equation of the line segment QR. 1 mark

iii. Show that the x-coordinates of P and Q are $\frac{800}{9}$ and $\frac{1150}{9}$, respectively. 2 marks

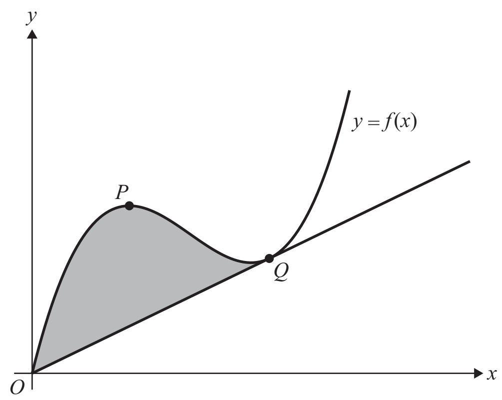
iv. Find the area of the trapezium $OPQR$. 1 mark

v. Find the error in the approximation obtained in **part f.iv.** as a percentage of the actual area, correct to two decimal places. 1 mark

Question 9 (12 marks)

Let $f: [0, \infty) \rightarrow R, f(x) = (ax - 1)(x - 1)^2 + 1$, where $a > \frac{1}{4}$.

The shaded region in the diagram below shows the area bounded by the graph of f and the tangent to the graph of f that passes through the origin.



Point P is the local maximum of f, and point Q is the point of intersection between the graph of f and the tangent to the graph of f that passes through the origin.

a. Find the coordinates of P in terms of a. 1 mark

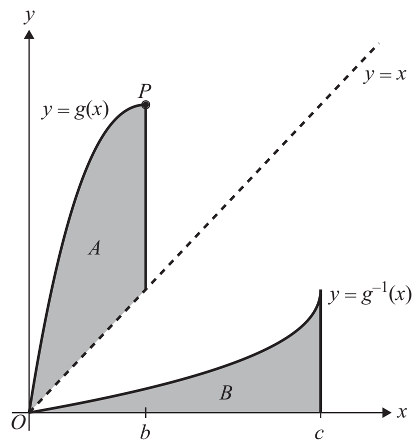
b. Find the x-coordinate of Q in terms of a. 1 mark

c. Find the area of the shaded region in terms of a. 3 marks

d. Find the value of a, where $a > \frac{1}{4}$, for which the area of the shaded region is a minimum. 2 marks

Let $g: [0, b] \to R$, $g(x) = (ax - 1)(x - 1)^2 + 1$, where $a > \frac{1}{4}$, and let b be the x-coordinate of P, where P is the local maximum of f.

The diagram below shows the graphs of g, g^{-1} and $y = x$. The region denoted by A is bounded by the graph of g, the line $y = x$ and $x = b$. The region denoted by B is bounded by g^{-1}, the x-axis and $x = c$, where c is the maximum value in the domain of g^{-1}.



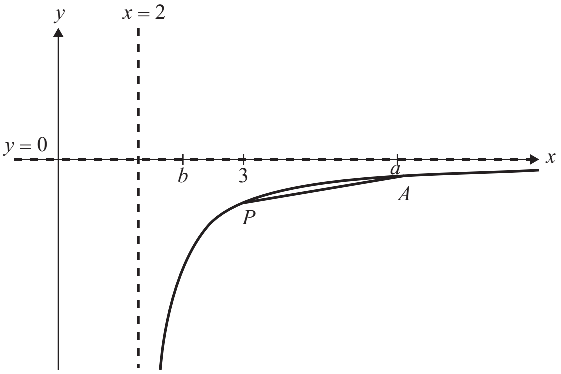
e. Find the area of B. 3 marks

f. Find the value of a for which the areas of A and B are equal. 2 marks

Question 10 (8 marks)

Consider the function $f: (2, \infty) \to R,\ f(x) = \dfrac{6}{2 - x}$.

Part of the graph of $y = f(x)$ is shown below. Points P and A lie on the graph of $y = f(x)$ with x-coordinates 3 and a, respectively.



a. **i.** Find the gradient of the line segment PA in terms of a. 1 mark

ii. Find the value of x over the interval $(2, a]$ such that the gradient of PA is equal to the gradient of the tangent to the graph of $y = f(x)$. 2 marks

b. i. Evaluate $\int_3^{e+2} \frac{6}{2 - x}\, dx$. 1 mark

ii. Find the value of b for $b \in (2, 3)$ such that $\int_b^3 \frac{6}{2 - x}\, dx = 6$. 1 mark

c. i. Find, in terms of a, the area bounded by the line segment PA, the x-axis, the line $x = 3$ and the line $x = a$. 2 marks

ii. For what value of a does the area calculated in **part c.i.** equal 6? 1 mark

Question 11 (9 marks)

Let $f: R \to R, f(x) = ax(x - k)^2$ and $g: R \to R, g(x) = \frac{4ak^2}{9}x$, where a and k are positive real numbers.

a. Find the coordinates of the local maximum of f. Give your answer in terms of a and k. 2 marks

b. Show that the graphs of f and g meet at the local maximum of f. 1 mark

Let $h:\left[0, \frac{k}{3}\right] \to R$, $h(x) = ax(x - k)^2$, where a and k are positive real numbers.

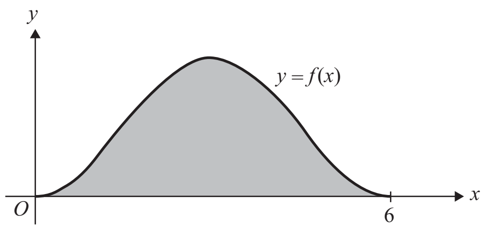
c. Determine the area bounded by the graphs of h and g. 2 marks

d. Show that the maximum value of a for h and h^{-1} to meet exactly twice is $\frac{9}{4k^2}$. 2 marks

e. If $a = \frac{9}{4k^2}$, determine the value of k when the area bounded by h and h^{-1} equals $\frac{1}{12}$. 2 marks

Question 12 (4 marks)

Jordan has designed a BMX bike ramp. A cross-section of the ramp is shown below sketched on a set of coordinate axes.



The curved edge of the cross-section has the equation $f(x) = a - a\cos\left(\frac{\pi x}{3}\right)$, where $0 \le x \le 6$, and the maximum gradient of $f(x)$ is $\frac{2\pi}{3}$.

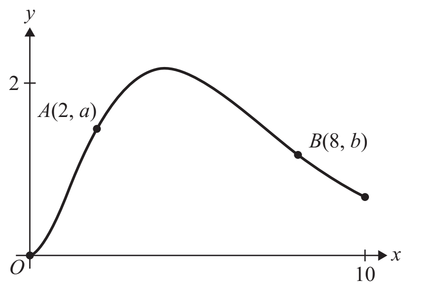
a. Show that $a = \frac{\pi}{3}$. 1 mark

b. Find the area of the cross-section, which is shaded in the diagram above. 1 mark

c. The area of the cross-section that falls between the lines $x = 3 - c$ and $x = 3 + c$ has an area of 8 square units. Find the value of c. 2 marks

Question 13 (6 marks)

Let $f : [0, 10] \to R, f(x) = x^2 e^{-\frac{x}{2}}$. The graph of $y = f(x)$ is shown below.



a. State the interval for which f is strictly increasing. 1 mark

b. Points $A(2, a)$ and $B(8, b)$ lie on the graph of $y = f(x)$. Determine the value of x for which $f'(x)$ is equal to the gradient of the line that joins A and B. Give your answer correct to three decimal places. 2 marks

c. Let $g: D \to R, g(x) = x^2$ and $h: D \to R, h(x) = f(g(x))$, where D is the largest domain for which h is defined.

i. Determine the domain of $h'(x)$, where $h'(x)$ is the derivative of $h(x)$. 2 marks

ii. Find $h'(x)$. 1 mark

Question 14 (12 marks)

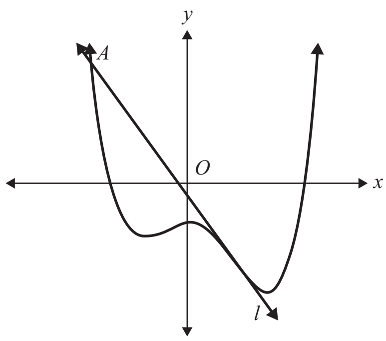
a. Consider the function $g(x) = \frac{4}{3}x^3 - x^2 - 2x$.

i. Given that $f'(x) = g(x)$ and $f(0) = -1$, show that $f(x) = \frac{1}{3}x^4 - \frac{1}{3}x^3 - x^2 - 1$. 1 mark

The graph of $y = f(x)$ has stationary points at $x = 0$ and at two other points.

ii. Find the other values of x for which the graph of $y = f(x)$ has stationary points. Express your answer in the form $\frac{a \pm \sqrt{b}}{c}$, where a, b and c are integers. 2 marks

b. The diagram below shows part of the graph of $f(x)$ and the tangent l to the graph at $x = 1$. The tangent crosses the graph at A.

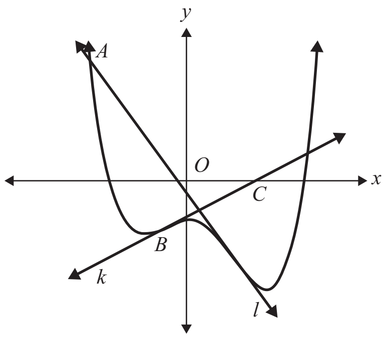


i. Find the equation of the tangent l. 1 mark

ii. State the coordinates of A. 1 mark

iii. Find the area bounded by the graphs of the tangent l and $y = f(x)$. 2 marks

c. The tangent line k, to $f(x)$ at $B(u, v)$, where $u, v < 0$, has a gradient of $\frac{7}{12}$ and crosses the x-axis at C.



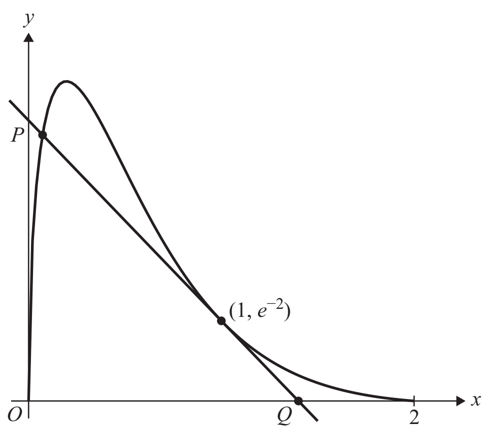
Find the coordinates of B and C. 3 marks

d. Determine the acute angle between the tangents k and l. Give your answer in degrees, correct to two decimal places. 2 marks

Question 15 (10 marks)

Let $f: [0, 2] \to R$, $f(x) = \sqrt{x}\,(2 - x)\,e^{-2x}$.

The diagram below shows the graph of f and its tangent at $x = 1$. The tangent intersects the graph of f at P and the horizontal axis at Q.



a. Find $f'(x)$. Express your answer in the form $\frac{(ax^2 + bx + c)e^{-2x}}{2\sqrt{x}}$, where a, b and c are integers. 1 mark

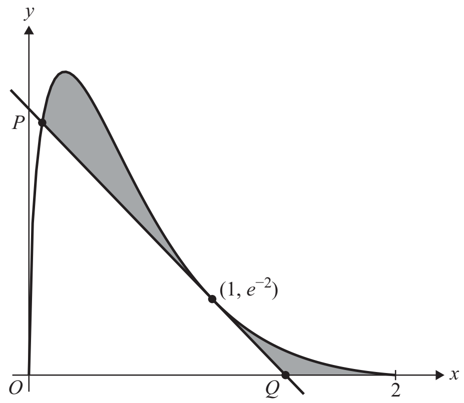
b. **i.** Find the value of x at which the maximum value of f occurs. 1 mark

ii. Find the maximum value of f, correct to four decimal places. 1 mark

c. **i.** Find the equation of the tangent to the graph of f at $x = 1$. Express the equation in the form $y = \frac{mx + n}{2e^2}$, where m and n are integers. 1 mark

ii. Find the x-coordinate of P, correct to four decimal places. 1 mark

iii. Consider the shaded region in the diagram below.



Find the area of the shaded region, correct to four decimal places. 2 marks

d. Let $R(r, s)$ be the point on the tangent to f at $x = 1$ that is closest to the origin, O.

i. Find the coordinates of R. 2 marks

ii. Find the distance from R to the origin, O. 1 mark

Question 16 (6 marks)

Susie is gathering data on two species of ants, the jumping jack ant and the red fire ant. The two species are very difficult to tell apart and both species are equally likely to be caught.

One technique for distinguishing between the two types of ant is to measure the length of their bodies.

It is known that the body length of red fire ants is normally distributed with a mean of 25 mm and a standard deviation of 5 mm.

a. Find the probability that a randomly chosen red fire ant has a body length that is shorter than 18 mm. Give your answer correct to four decimal places. 1 mark

It is also known that the body length of jumping jack ants is normally distributed, and that 10% of them have a body length shorter than 20 mm and 10% have a body length longer than 28 mm.

b. Find the mean and standard deviation of the body length of a jumping jack ant. Give the standard deviation correct to two decimal places. 3 marks

c. During her studies, Susie finds a site near the coast that has these two types of ants in abundance. 70% of the ants on the site are jumping jack ants and 30% are red fire ants.

Susie examines a single ant from this site and finds its body to be shorter than 18 mm.

What is the probability that it is a jumping jack ant? Give your answer correct to three decimal places. 2 marks

Question 17 (14 marks)

The amount of time spent on homework on any given school night by a Year 9 student is a normally distributed random variable with a mean of 60 minutes and a standard deviation of 25 minutes.

a. **i.** Find the probability that a randomly selected Year 9 student will undertake less than 68 minutes of homework on a school night. Give your answer correct to four decimal places. 1 mark

ii. Find the probability that exactly three students in a random group of nine Year 9 students will undertake less than 68 minutes of homework on a given school night. Give your answer correct to four decimal places. 2 marks

b. The probability that a Year 9 student will undertake more than a minutes of homework on a school night is 0.65.

Find a, correct to one decimal place. 1 mark

A typical Year 9 student undertakes between 20 minutes and b minutes of homework on a school night. The probability that a student is typical is 0.9.

c. **i.** Find the value of b, correct to one decimal place. 2 marks

ii. Find the probability that a typical Year 9 student undertakes less than 60 minutes of homework. Give your answer correct to four decimal places. 2 marks

d. A student is considered to have done an insufficient amount of homework if less than a certain number of minutes of homework is undertaken on a school night. It has been found that 5.5% of students undertake an insufficient amount of homework.

The probability that more than one student undertakes an insufficient amount of homework in a randomly selected group of Year 9 students is greater than 0.4.

Find the smallest possible size of the group. 2 marks

The amount of time required for a Year 9 student to complete all their homework is a normally distributed random variable with a mean of 85 minutes and a standard deviation of 20 minutes.

A Year 9 student who spends more than 2 hours completing homework is classified as spending an excessive amount of time doing homework.

e. Find the percentage of Year 9 students who will be able to complete all their homework without spending an excessive amount of time on it. Give your answer correct to two decimal places. 2 marks

The amount of time required for a Year 7 student to complete their homework is a normally distributed random variable with a mean of 30 minutes. The probability that more than 25 minutes is required is equal to 0.83.

f. Find the standard deviation of the amount of time required for a Year 7 student to complete their homework. Give your answer in minutes, correct to two decimal places. 2 marks

Question 18 (10 marks)

Apples grown in Victoria have weights that are normally distributed with a mean of 165 grams and a standard deviation of 23 grams.

a. Determine the probability that a randomly selected Victorian apple has a mass between 140 grams and 160 grams. Give your answer correct to four decimal places. 1 mark

Nationwide standards classify apples as large, average or small according to their weight. The lightest 5% of apples grown in Victoria have a mass that is classified as small.

b. Determine the largest mass that would result in an apple being classified as small. Give your answer correct to the nearest gram. 1 mark

c. A randomly selected Victorian apple is found to be small. What is the probability that it weighs more than 120 grams? Give your answer correct to four decimal places. 2 marks

d. Find the probability that more than two apples from a random sample of 20 Victorian apples are small apples. Give your answer correct to three decimal places. 2 marks

e. How many Victorian apples would need to be sampled to ensure that the probability that the sample contained at least five small apples is more than 0.8? 2 marks

f. 8% of a random sample of 400 apples grown in New South Wales were found to be small.

i. Find the 95% confidence interval for the proportion of apples grown in New South Wales that have a mass that would classify them as small. 1 mark

ii. Explain why this confidence interval suggests that the proportion of small apples grown in New South Wales could be different to the proportion of small apples grown in Victoria. 1 mark

Question 19 (13 marks)

A certain temperate rainforest contains many species of trees. Scientists are interested in two types in particular, which they have denoted as type A and type B.

The mature type A tree has a diameter that is normally distributed with a mean of 24 cm and a standard deviation of 4 cm.

a. Find the probability that a randomly selected type A tree has a diameter greater than 25 cm, given that it has a diameter greater than 22 cm. Give your answer correct to four decimal places. 2 marks

Type B trees have a diameter, in cm, described by the probability density function

$$f(x) = \begin{cases} \dfrac{3x}{39\,304}(34 - x)\left(e^{\frac{x}{17}} - 1\right) & 0 \le x \le 34 \\ 0 & \text{otherwise} \end{cases}.$$

b. Determine the expected diameter of a randomly selected type B tree, in cm. Give your answer correct to four decimal places. 1 mark

c. Determine the standard deviation of a randomly selected type B tree, in cm. Give your answer correct to four decimal places. 2 marks

d. 30% of type B trees have a diameter greater than d cm. Determine the value of d. Give your answer correct to two decimal places. 2 marks

A type B tree is classified as large if it has a diameter greater than 30 cm. The probability that a randomly selected type B tree is large is 0.1021, correct to four decimal places.

e. Find the probability that in a random sample of 70 type B trees, four or more are large. Give your answer correct to four decimal places. 2 marks

f. Let $\hat{P}$ be the random variable that represents the proportion of type B trees that are large in a random sample of 70 type B trees.

Find $\Pr(\hat{P} > 0.1 \mid \hat{P} < 0.2)$, correct to four decimal places. Do not use a normal approximation. 2 marks

g. In a sample of 70 type B trees, an approximate $c\%$ confidence interval for the proportion of trees that are large is calculated to be $(0.08265, 0.203065)$.

Determine the value of c, to the nearest integer. 2 marks

Worked solutions

Area of Study 1 Functions, relations and graphs

EXAM 1

Question 1a.

Worked solution

To find the x-intercept, let $f(x) = 0$:

$$0 = 3 - \frac{4}{2x+1}$$

$$3 = \frac{4}{2x+1}$$

$$3(2x+1) = 4$$

$$6x + 3 = 4$$

$$6x = 1$$

$$x = \frac{1}{6}$$

To find the y-intercept, let $x = 0$:

$$y = 3 - \frac{4}{0+1} = 3 - 4$$

$$y = -1$$

There are two methods for finding the asymptotes.

<u>Method 1</u>

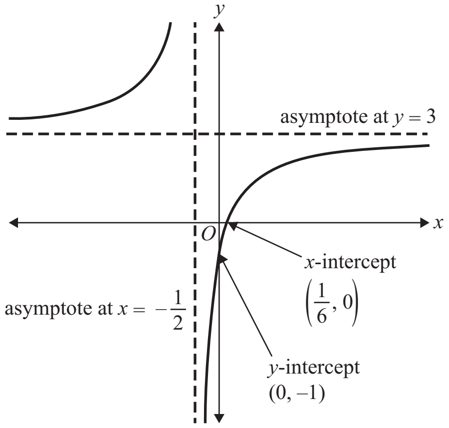
Rewrite the equation as $f(x) = \dfrac{-4}{2\left(x+\frac{1}{2}\right)} + 3$.

By inspection, the equations of the asymptotes are $y = 3$ and $x = -\frac{1}{2}$.

<u>Method 2</u>

Because $x \to +\infty$, $f(x) \to 3$ and $x \to -\infty$, $f(x) \to 3$, the horizontal asymptote must be at $y = 3$.

A function is undefined when the denominator is equal to zero, that is, when $2x + 1 = 0$. It follows that $x = -\frac{1}{2}$. Therefore the vertical asymptote is at $x = -\frac{1}{2}$.



Mark allocation: 3 marks

- 1 mark for the correct shape of the graph. It must be a negative hyperbola with asymptotic behaviour.
- 1 mark for correctly labelling both intercepts.
- 1 mark for correct equations for both asymptotes.

TIP

» Make sure you sketch the asymptotes with dotted or dashed lines.

Question 1b.

Worked solution

From the graph in **part a.** we can see that the function is positive to the left of the vertical asymptote and to the right of the x-intercept. At the x-intercept the function is zero, so this value should also be included.

$x \in \left(-\infty, -\frac{1}{2}\right) \cup \left[\frac{1}{6}, \infty\right)$

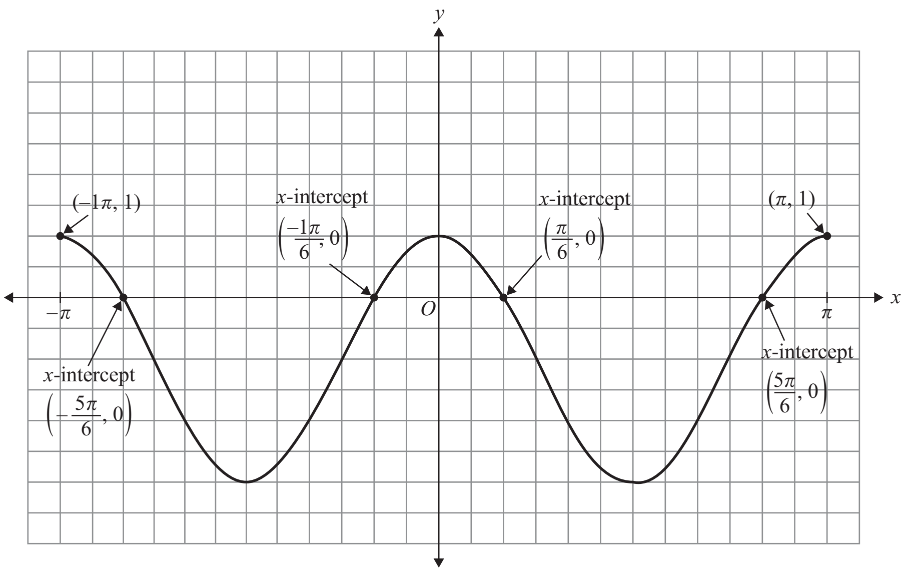
Mark allocation: 2 marks

- 1 answer mark for the left interval.
- 1 answer mark for the right interval.

Note: both intervals must be correctly combined for full marks.

Question 2

Worked solution



Mark allocation: 3 marks

- 1 mark for a correct cosine graph showing two cycles and an amplitude of 2.
- 1 mark for end points labelled correctly.
- 1 mark for intercepts labelled correctly.

TIP

» **Remember to label the required points with their coordinates.**

Question 3

Worked solution

$\log_e(x)$ implies that $x > 0$.

$\log_e(5 - 2x)$ implies that $5 - 2x > 0$ and hence that $x < \frac{5}{2}$.

The domain is $x > 0 \cap x < \frac{5}{2}$.

Hence $0 < x < \frac{5}{2}$.

Mark allocation: 1 mark

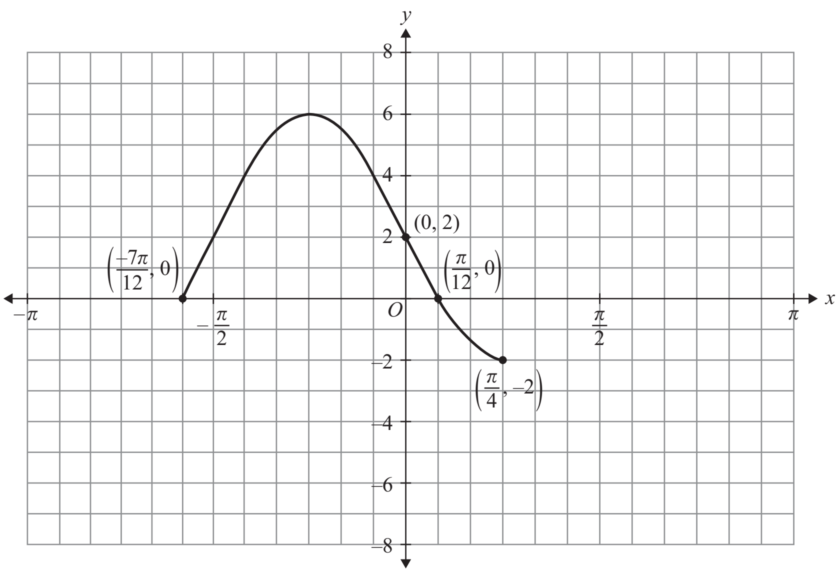
- 1 answer mark for $0 < x < \frac{5}{2}$ or the interval $\left(0, \frac{5}{2}\right)$.

Question 4

Worked solution

From the given equation, the period is $\frac{2\pi}{2} = \pi$ and the amplitude is 4. If there was not a domain restriction, the median would be 2.

The final graph should include the details shown below.



End points can be found by substitution:

$$f\left(-\frac{7\pi}{12}\right) = 2 - 4\sin\left(2 \times -\frac{\pi}{12}\right)$$
$$= 2 - 4\sin\left(-\frac{\pi}{6}\right)$$
$$= 2 - 4 \times -\frac{1}{2}$$
$$= 0$$

$\therefore$ The left end point is at $\left(-\frac{7\pi}{12}, 0\right)$.

$$f\left(\frac{\pi}{4}\right) = 2 - 4\sin\left(2 \times \frac{\pi}{4}\right)$$
$$= 2 - 4\sin\left(\frac{\pi}{2}\right)$$
$$= 2 - 4$$
$$= -2$$

$\therefore$ The right end point is at $\left(\frac{\pi}{4}, -2\right)$.

x-intercepts can be found by solving $f(x) = 0$:

$$2 - 4\sin(2x) = 0$$
$$\sin(2x) = \frac{1}{2}$$
$$2x = \sin^{-1}\left(\frac{1}{2}\right), \text{ where } -\frac{7\pi}{12} \le x \le \frac{\pi}{4}$$
$$\therefore -\frac{7\pi}{6} \le 2x \le \frac{\pi}{2}$$

Note: The exact value of $\sin\left(\frac{\pi}{6}\right)$ should be recognised as $\frac{1}{2}$.

Given that we have an expression where $\sin(\theta)$ is positive, solutions will be in the first and second quadrants.

$$2x = -\pi - \frac{\pi}{6}, \frac{\pi}{6}$$
$$= -\frac{7\pi}{6}, \frac{\pi}{6}$$
$$x = -\frac{7\pi}{12}, \frac{\pi}{12}$$

Alternatively, once it has been identified that the left end point at $x = -\frac{7\pi}{12}$ is an x-intercept, the other x-intercept, $x = \frac{\pi}{12}$, can be calculated using symmetry about the local maximum at $x = -\frac{\pi}{4}$.

Mark allocation: 4 marks

- 4 marks: A correctly drawn sketch, with all end points and axis intercepts labelled correctly. The local maximum at $\left(-\frac{\pi}{4}, 6\right)$ is also correctly located.
- 2–3 marks: A sinusoidal-shaped curve with the correct amplitude, median and period. Some axis intercepts or end points may not be labelled. The curve may be drawn for the full horizontal length of the provided axes. The curve may be reflected across the median.
- 1 mark: A sinusoidal-shaped curve with a range of $[-2, 6]$ or a non-sinusoidal shaped curve with correct end points.

TIP

» **When sketching sine and cosine functions, your first step should always be to determine the period, amplitude and median value of the function.**

Question 5

Worked solution

The turning point is at $(1, -3)$, hence the quadratic function is strictly decreasing over the domain $\left(-\frac{1}{2}, 1\right)$. Thus the range can be found by evaluating the function at the end points of the domain.

$$f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2} - 1\right)^2 - 3$$
$$= 2\left(-\frac{3}{2}\right)^2 - 3$$
$$= 2 \times \frac{9}{4} - 3 = \frac{9}{2} - 3 = \frac{3}{2}$$

and $f(1) = -3$.

The range is $\left[-3, \frac{3}{2}\right)$.

Mark allocation: 1 mark

- 1 mark for the correct range: $\left[-3, \frac{3}{2}\right)$.

TIPS

- » A quick sketch to visualise the end points and turning points of a function is helpful.
- » The y value of the end points do not always give the range of the function. In this case, the end point is the turning point (hence, the smallest y value), but this is not always the case.

Question 6

Worked solution

$f(x)$ is defined when both $\log_2(x + 1)$ and $\log_2(4 - x)$ are defined.

$\log_2(x + 1)$ implies that $x > -1$ and $\log_2(4 - x)$ implies that $x < 4$, hence $-1 < x < 4$. The domain of f is $(-1, 4)$.

Mark allocation: 1 mark

- 1 answer mark for $(-1, 4)$ or $-1 < x < 4$.

TIP

- » Remember that the logarithm of zero or a negative number is undefined.

Question 7

Worked solution

The domain is $[-1, 2]$, so evaluate $f(-1)$ and $f(2)$ and check for any stationary points within the domain.

Check for stationary points by solving $f'(x) = 0$:

$$f(x) = (x+1)^{-1} + (-x+4)^{-1}$$

$$f'(x) = -(x+1)^{-2} - (-(-x+4))^{-2}$$

$$= -\frac{1}{(x+1)^2} + \frac{1}{(4-x)^2} = 0$$

$$\frac{1}{(x+2)^2} = \frac{1}{(4-x)^2}$$

$$x^2 + 4x + 4 = 16 - 8x + x^2$$

$$12x = 12$$

$$x = 1$$

$$f(1) = \frac{1}{1+2} + \frac{1}{4-1} = \frac{2}{3}$$

Evaluate the end points:

$$f(-1) = \frac{1}{-1+2} + \frac{1}{4-(-1)} \text{ and } f(2) = \frac{1}{2+2} + \frac{1}{4-2}$$

$$= \frac{6}{5} \qquad\qquad = \frac{3}{4}$$

$\therefore$ For $x \in [-1, 2]$, min $= \frac{2}{3}$, max $= \frac{6}{5}$.

$\therefore$ Range $= \left[\frac{2}{3}, \frac{6}{5}\right]$

Mark allocation: 2 marks

- 1 method mark for evaluating $f(-1)$ and $f(2)$, or for finding the derivative: $f'(x) = -\frac{1}{(x+1)^2} + \frac{1}{(4-x)^2}$.
- 1 answer mark for the correct range: $\left[\frac{2}{3}, \frac{6}{5}\right]$.

TIP

» Always check the end points and the values of any stationary points when finding the range of a function with a restricted domain.

Question 8a.

Worked solution

$$1 - 2\log_e(x) = 0$$

$$1 = 2\log_e(x)$$

$$\log_e(x) = \frac{1}{2}$$

$$x = e^{\frac{1}{2}}$$

Mark allocation: 1 mark

- 1 answer mark for $x = e^{\frac{1}{2}}$ or the equivalent expression $\sqrt{e}$.

Question 8b.

Worked solution

$$\log_e(x) = 1 - 2\log_e(x)$$
$$3\log_e(x) = 1$$
$$\log_e(x) = \frac{1}{3}$$
$$x = e^{\frac{1}{3}}$$

Mark allocation: 1 mark

- 1 answer mark for $x = e^{\frac{1}{3}}$ or the equivalent expression $\sqrt[3]{e}$.

Question 8c.

Worked solution

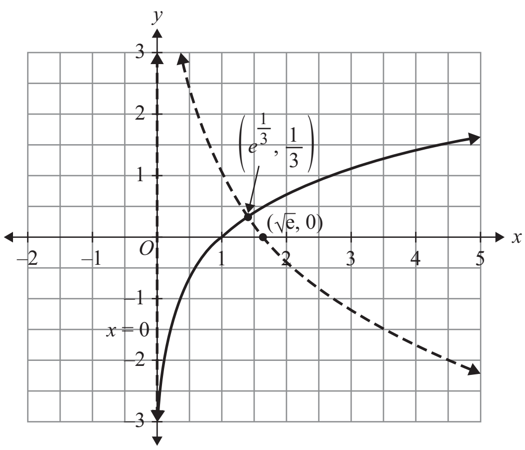
$g(x) = 1 - 2\log_e(x)$

There is a vertical asymptote at $x = 0$.

From **part a.** the x-intercept is $x = e^{\frac{1}{2}}$.

From **part b.** the point of intersection occurs at $x = e^{\frac{1}{3}}$.

$$g\left(e^{\frac{1}{3}}\right) = \log_e\left(e^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}\log_e(e) = \frac{1}{3}$$



Mark allocation: 2 marks

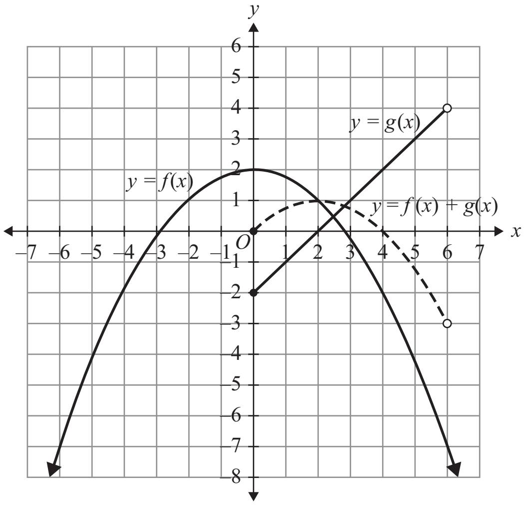
- 1 method mark for a strictly decreasing curve that does not cross the y-axis.
- 1 answer mark for a correctly shaped curve with the point of intersection and x-intercept correctly labelled.

TIP

» Writing the function in the form $g(x) = -2 \times \log_e(x) + 1$ makes it easier to recognise the transformations that have been performed on the graph of $f(x) = \log_e(x)$. $f(x)$ has been reflected in the x-axis, dilated by a factor of 2 from the x-axis and translated up 1 unit. You can use these transformations to help you sketch the correct graph.

Question 9

Worked solution



Mark allocation: 2 marks

- 1 answer mark for the correct shape of the graph and passing through $(2, 1)$.
- 1 answer mark for a graph drawn over the correct domain, with a closed circle at $(0, 0)$ and an open circle at $(6, -3)$.

Question 10

Worked solution

Begin by equating the functions and then making x the subject:

$ke^x = 3 - e^{-x}$

$ke^x = 3 - \frac{1}{e^x}$

Now multiply everything by e^x:

$ke^{2x} = 3e^x - 1$

$ke^{2x} - 3e^x + 1 = 0$

This is a quadratic in e^x, which can be more easily seen if we apply substitution.

If we let $w = e^x$, the substitution results in $kw^2 - 3w + 1 = 0$.

Thus only two solutions are possible if the discriminant is positive:

$\Delta = 9 - 4k > 0$

$k < \frac{9}{4}$

We need to consider whether all values of k, where $k < \frac{9}{4}$, will produce two solutions.

Solving the equation $kw^2 - 3w + 1 = 0$ using the quadratic formula gives

$$w = \frac{3 \pm \sqrt{9 - 4k}}{2k}$$

$$\therefore e^x = \frac{3 \pm \sqrt{9 - 4k}}{2k}$$

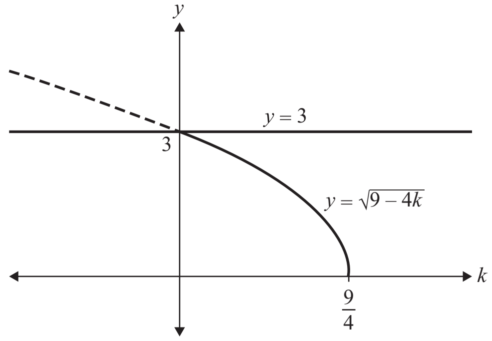
$$x = \log_e\left(\frac{3 \pm \sqrt{9 - 4k}}{2k}\right)$$

$x = \log_e\left(\frac{3 \pm \sqrt{9-4k}}{2k}\right)$ is only defined if $\frac{3 \pm \sqrt{9-4k}}{2k} > 0$, thus we require $\sqrt{9-4k} < 3$.

To solve this inequality, first solve $\sqrt{9-4k} = 3$:

$$\sqrt{9-4k} = 3$$
$$9 - 4k = 9$$
$$-4k = 0$$
$$k = 0$$

Then consider where the graph of $y = \sqrt{9-4k}$ is below the line $y = 3$:



Hence there will be two solutions if $0 < k < \frac{9}{4}$.

Mark allocation: 2 marks

- 1 method mark for finding a k value of $\frac{9}{4}$ (stated as either $k = \frac{9}{4}$, $k > \frac{9}{4}$ or $k < \frac{9}{4}$).
- 1 answer mark for $0 < k < \frac{9}{4}$.

Question 11

Worked solution

Method 1

Let $y = \sqrt{5-4x}$ and $y' = \sqrt{x'}6$.

Therefore, by equating equivalent parts, we get:

$x' = 5 - 4x$ $\qquad\qquad$ $y' = y$

$x' = -4x + 5$

Since $x' = ax + b$ and $y' = y + d$, we can equate coefficients to get $a = -4$, $c = 5$ and $d = 0$.

Method 2

Identify the transformations, and their order, that map $y = \sqrt{5-4x} = \sqrt{-4x+5}$ to $y = \sqrt{x}$.

Reflect in y-axis: $f(x) \to f(-x) = \sqrt{4x+5} = g(x)$.

Dilate by a factor of 4 from the y-axis: $g(x) \to g\left(\frac{x}{4}\right) = \sqrt{x+5} = h(x)$.

Translate 5 units right: $h(x) \to h(x-5) = \sqrt{x}$.

Hence $T(x,y) = (-4x+5, y)$.

Equating the coefficients of $T(x,y)$ with $(ax+c, y+d)$ gives $a = -4$, $c = 5$ and $d = 0$.

Answer: $a = -4$, $c = 5$ and $d = 0$.

Note: The translation must be applied *last* in the sequence of transformations, since the order of operations in the expression $ax + c$ has the addition completed after the multiplication.

Mark allocation: 2 marks

- 1 method mark for at least one correct value.
- 1 answer mark for all three correct values.

Question 12a.

Worked solution

$\text{ran } f = [-1, 1]$

Mark allocation: 1 mark

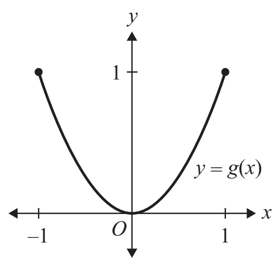
- 1 mark for the correct answer.

Question 12b.

Worked solution

Since $\text{ran } f = [-1, 1]$, we need to consider what values $g(x)$ produces when the input values are $[-1, 1]$.

It is helpful to sketch a graph of $g(x)$ over the domain $[-1, 1]$.



From the graph we can see that the range is $[0, 1]$.

Thus $\text{ran } g(f(x)) = [0, 1]$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIPS

» Sketching a relevant graph is often helpful when answering questions regarding composite functions.

» Many questions involving composite functions, such as the one above, do not require you to actually find the rule for the composite function.

Question 12c.

Worked solution

We are told that $f(x+h) = f(x)$, where $h \in Z$.

Hence $f(x) = f(x \pm 1) = f(x \pm 2) = \ldots$

It is helpful to think what this means graphically, noting that h represents a horizontal translation (left or right) and both $f(x+h)$ and $f(x)$ have the same period.

The graphs of $y = f(x)$, $y = f(x \pm 1)$, $y = f(x \pm 2) \ldots$ will be all identical. This is possible only if the horizontal translation, h, is one full period, or an integer multiple of the period.

Thus, since $h = \pm1, \pm2, \pm3, \ldots$, the possible values of the period are $1, 2, 3 \ldots$.

Hence for all possible values of h to be equal to the period, or an integer multiple of the period, the period must be 1.

The period of f is $\frac{2\pi}{a}$. Hence $\frac{2\pi}{a} = 1 \Rightarrow a = 2\pi$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIPS

» Recognising that if $f(x+h) = f(x)$ then the graphs of $f(x+h)$ and $f(x)$ are identical is useful when thinking about this question.

EXAM 2: MULTIPLE CHOICE

Question 1

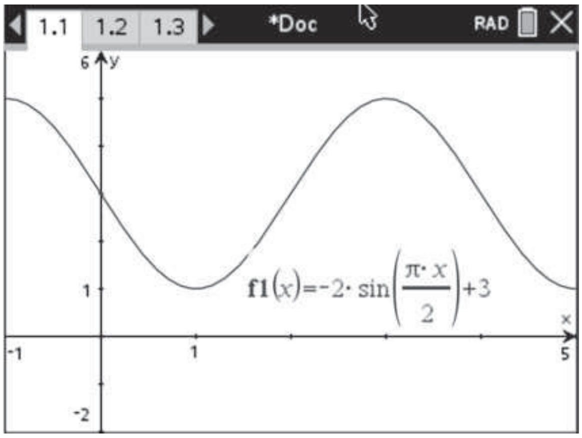
Answer: ***B***

Explanatory notes

The period is $\frac{2\pi}{\frac{\pi}{2}} = 4$.

The median is 3 and the amplitude is 2.
Hence the range is $3 - 2$ to $3 + 2$, that is, $[1, 5]$.

This can be checked by sketching the graph using a CAS.



Question 2

Answer: ***C***

Explanatory notes

The x-intercepts (roots) are at $x = b$, $x = c$ and $x = d$, with a repeated root at $x = d$.

So $y = a(x - b), (x - c)(x - d)^2$.
Considering each given option shows that $y = (x - b)(x - c)(x - d)^2$ is the only possibility.

TIP

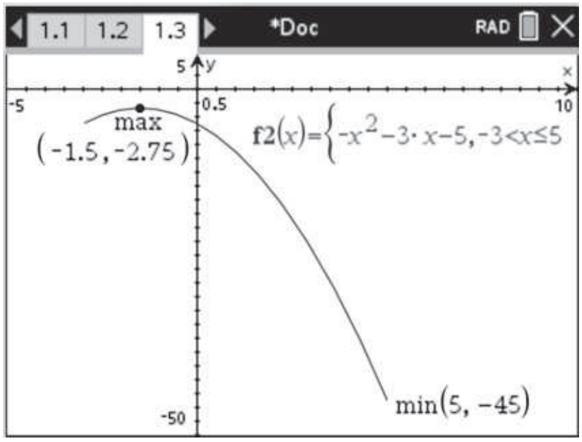
» Don't be confused by the fact that the roots at b and c are negative.

Question 3

Answer: ***D***

Explanatory notes

Sketching the graph using a CAS shows that the range is from the right end point, where $y = -45$, to the maximum turning point, where $y = -2.75$.



Question 4

Answer: ***D***

Explanatory notes

$y = e^{2x+4} - 3$ can be written as $y + 3 = e^{2x+4}$, and therefore $y' = y + 3$ and $x' = 2x + 4$.

Making x and y the subject, we get $y = y' - 3$ and $x = \frac{1}{2}(x' - 4) = \frac{1}{2}x' - 2$.

The transformation rule that corresponds to these equations is option D.

Question 5

Answer: ***C***

Explanatory notes

Method 1

First check if the turning point is within the domain. Do this either by putting the equation in turning-point form or using calculus. Completing the square with a CAS is efficient: $f(x) = -(x-2)^2 + 4 - b$.

$\therefore$ There is a maximum turning point at $(2, 4-b)$.

Considering symmetry, $f(7) < f(-1) < f(2)$, so the range is $(f(7), f(2)]$.

Since $f(7) = -21 - b$, the range is $(-21 - b, 4 - b]$.

Note: The end point $x = 7$ of the domain is not included, therefore the end point $f(7) = -21 - b$ of the range is excluded.

Method 2

Substitute any convenient value for b and draw the graph for $x \in [-1, 7)$ using a CAS. The range can be seen by inspection. Test each given option by substituting the chosen value of b and reject the options that give the wrong range.

For example, choose $b = 0$ so that $f(x) = -x^2 + 4x$.

There is a maximum turning point at (2, 4).

By inspection, the range associated with the domain $x \in [-1, 7)$ is $(-21, 4]$.

Substitute $b = 0$ into each option and reject the options that do not give a range of $(-21, 4]$.

Answer: $(-21 - b, 4 - b]$

Question 6

Answer: ***B***

Explanatory notes

The function is in the form $y = a\tan(bx + c) + d$, $\therefore$ period $= \frac{\pi}{|b|}$.

$$f(x) = -2\tan\left(-\frac{x}{3} + \frac{2\pi}{5}\right) + 4$$

$$\text{Period} = \frac{\pi}{\frac{1}{3}} = 3\pi$$

Question 7

Answer: ***A***

Explanatory notes

Method 1: By recognising transformations

Reflect the function in the y-axis: $y = \frac{1}{\sqrt{-x}}$.

Now translate the result 4 units left and 3 units down: $y = \frac{1}{\sqrt{-x-4}} - 3$.

Method 2: By substitution

$x' = -x - 4 \qquad y' = y - 3$

$x = -x' - 4 \qquad y = y' + 3$

Substituting these into the original equation and making y' the subject gives

$$y' + 3 = \frac{1}{\sqrt{-x'-4}}$$

$$y' = \frac{1}{\sqrt{-x'-4}} - 3$$

Question 8

Answer: ***D***

Explanatory notes

The period is $\frac{2\pi}{\frac{2\pi}{5}} = 5$.

Range: $\min = -2 - 3 = -5, \ \max = -2 + 3 = 1$

$\therefore$ range $= [-5, 1]$

Question 9

Answer: ***A***

Explanatory notes

Since $f(-2) = 8$ is included and $f(3) = -2$ is excluded, the range of f is $(-2, 8]$.

TIP

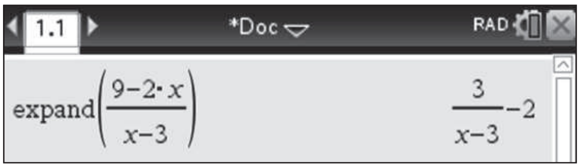
» It may be helpful to draw a sketch of the function for the given domain.

Question 10

Answer: ***D***

Explanatory notes

Express f in the form $a + \frac{b}{x-3}$ to find the vertical and horizontal asymptotes.



Hence the graph of f has a vertical asymptote of $x = 3$ and a horizontal asymptote of $y = -2$.

TIP

» Although you could do this by hand, having good knowledge of a CAS calculator allows the computation to be performed more easily.

Question 11

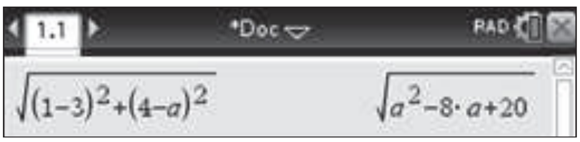
Answer: ***B***

Explanatory notes

Use the formula for the distance between two points:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(1-3)^2 + (4-a)^2}$$

$$= \sqrt{a^2 - 8a + 20}$$

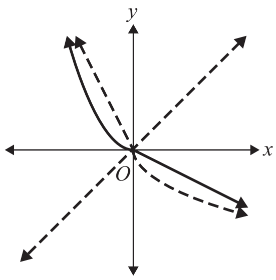


Question 12

Answer: ***B***

Explanatory notes

The inverse can be found by reflecting the graph of function f in the line $y = x$, as shown below.



Note: The function with the dotted line is the inverse function.

TIP

» You could eliminate some options in this question by drawing on the fact that points in the 2nd quadrant are always reflected to/from the 4th quadrant, whereas points in the 1st and 3rd quadrants don't change quadrants.

Question 13

Answer: ***B***

Explanatory notes

$x' = -2x + 1, \quad y' = 3y$

Since $g(x)$ is the rule for the transformed function, we have $g(x)$: $y' = 3\sqrt{4 - x'}$.

Substitute the expressions for x' and y' into the equation $y' = 3\sqrt{4 - x'}$ to obtain

$$3y = 3\sqrt{4 - (-2x + 1)}$$

$$y = \sqrt{3 + 2x}$$

Question 14

Answer: ***D***

Explanatory notes

Period:

$$\frac{2\pi}{\frac{4\pi}{5}} = \frac{5}{2}$$

Amplitude: 2

TIP

» The amplitude is always positive.

Question 15

Answer: ***A***

Explanatory notes

Rearrange $y = \frac{-2}{\sqrt{3x-1}} + 1$ into the form $\frac{y-1}{-2} = \frac{1}{\sqrt{3x-1}}$.

The image is $y' = \frac{1}{\sqrt{x'}}$, so set

$x' = 3x - 1$ and $y' = \frac{y-1}{-2} = -\frac{1}{2}y + \frac{1}{2}$.

Question 16

Answer: ***D***

Explanatory notes

We have $g(2) = f(4) + 3 = -4$ and so $f(4) = -7$. Therefore the graph of f passes through the point $(4, -7)$.

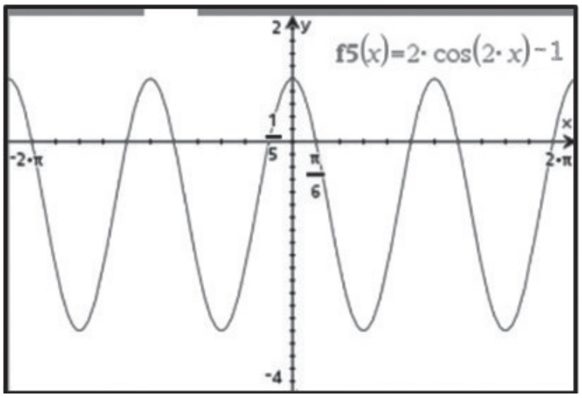
Question 17

Answer: ***D***

Explanatory notes

Method 1

Sketch the graph of $y = 2\cos(2x) - 1$ and consider its symmetry:



Note that the sum of the solutions (i.e. the x-intercepts of the graph) for $x \in [-n\pi, n\pi]$ is zero. This is because the positive solutions are $x = \frac{\pi}{6}, \frac{5\pi}{6}, \ldots$ and the negative solutions are $x = -\frac{\pi}{6}, -\frac{5\pi}{6}, \ldots$

Hence for $x \in [-n\pi, (n+1)\pi]$ we can just find the sum of the solutions for $x \in [n\pi, (n+1)\pi]$.

The period of $y = 2\cos(2x) - 1$ is π and it is helpful to consider the sum of the solutions for each successive period:

For $x \in [0, \pi]$, solutions are $\frac{\pi}{6}, \frac{5\pi}{6}$.

$\therefore$ Sum $= \pi$

For $x \in [\pi, 2\pi]$, solutions are $\frac{\pi}{6} + \pi, \frac{5\pi}{6} + \pi$.

$\therefore$ Sum $= \pi + 2\pi$

For $x \in [2\pi, 3\pi]$, solutions are $\frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi$.

$\therefore$ Sum $= \pi + 4\pi$

For $x \in [n\pi, (n+1)\pi]$, solutions are $\frac{\pi}{6} + n\pi$, $\frac{5\pi}{6} + n\pi$.

$\therefore$ Sum $= \pi + 2n\pi = (2n+1)\pi$

Method 2

Systematically finding the sum of solutions for several values of n with a CAS shows that the sum is equal to $(2n+1)\pi$.

Area of Study 2 Algebra, number and structure

EXAM 1

Question 1

Worked solution

$2\cos(2x) - 1 = 0,\ -2\pi \le 2x \le 2\pi$

$\cos(2x) = \frac{1}{2}$, reference angle is $\frac{\pi}{3}$

$2x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$

$x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$

Mark allocation: 2 marks

- 1 method mark for the reference angle: $\frac{\pi}{3}$.
- 1 answer mark for all four angles correct.

TIP

» Remember to change the domain when solving for $2x$ rather than x.

Question 2a.

Worked solution

$g\left(-\frac{1}{2}\right) = \log_2\left(-\frac{1}{2} + 1\right) = \log_2\left(\frac{1}{2}\right) = \log_2(2^{-1}) = -\log_2(2) = -1$

$f\left(g\left(-\frac{1}{2}\right)\right) = f(-1) = 2(-1)(-1 - 1) = 2(-1)(-2) = 4$

Mark allocation: 2 marks

- 1 answer mark for evaluating $g\left(-\frac{1}{2}\right) = -1$.
- 1 answer mark for the correct answer: $f\left(g\left(-\frac{1}{2}\right)\right) = 4$.

Question 2b.

Worked solution

$g(x)$ is strictly increasing, hence the minimum value of $g(f(x))$ will occur when $f(x)$ is at its minimum.

The minimum value of $f(x)$ occurs halfway between the two x-axis intercepts, that is, at $x = \frac{1}{2}$.

$\min f = f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right) = -\frac{1}{2}$

$\min g = g\left(-\frac{1}{2}\right) = \log_2\left(-\frac{1}{2} + 1\right) = \log_2\left(\frac{1}{2}\right) = -1$

The minimum value of $g(f(x))$ is -1.

Mark allocation: 2 marks

- 1 answer mark for the correct x value.
- 1 answer mark for calculating the minimum value.

Question 3a.

Worked solution

$$\begin{aligned}\text{RHS} &= (4\sin^2(x) - 3)(\sin(x) - 1)\\ &= 4\sin^2(x)\times\sin(x) + 4\sin^2(x)\times(-1) + (-3)\times\sin(x) + (-3)\times(-1)\\ &= 4\sin^3(x) - 4\sin^2(x) - 3\sin(x) + 3\\ &= f(x)\\ &= \text{LHS}\end{aligned}$$

Alternatively:

Let $p = \sin(x)$.

$$\begin{aligned}f(p) &= 4p^3 - 4p^2 - 3p + 3\\ &= 4p^2(p-1) - 3(p-1)\\ &= (4p^2 - 3)(p-1)\end{aligned}$$

$$f(x) = (4\sin^2(x) - 3)(\sin(x) - 1)$$

Alternatively:

Let $p = \sin(x)$.

$$f(p) = 4p^3 - 4p^2 - 3p + 3$$

$$f(1) = 0$$

Hence $(p-1)$ is a factor.

$$\begin{aligned}f(p) &= 4p^3 - 4p^2 - 3p + 3\\ &= (a\cdot p^2 + b\cdot p + c)(p-1)\end{aligned}$$

Equating coefficients gives $a = 4$, $c = -3$, and $-3p - b\cdot p = -3p$, $b = 0$.

Hence $f(p) = (4p^2 - 3)(p-1)$ and $f(x) = (4\sin^2(x) - 3)(\sin(x) - 1)$.

Mark allocation: 1 mark

- 1 method mark for the correct answer and for clearly shown reasoning.

TIPS

- You may choose to simplify the expression by equating the common function, $\sin(x)$, to a variable. This will make factorising the expression easier.
- This type of operation, where an expression is turned into a polynomial by substitution, is also used to solve questions involving exponential equations.
- The results of a 'show that' question are often useful for answering a subsequent question.

Question 3b.

Worked solution

From **part a.**, $(4\sin^2(x) - 3)(\sin(x) - 1) = 0$.

Hence $4\sin^2(x) - 3 = 0$ or $\sin(x) = 1$, where $x \in [0, 2\pi]$.

If $4\sin^2(x) - 3 = 0$

$$\sin^2(x) = \frac{3}{4}$$

$$\sin(x) = \pm\frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

If $\sin(x) = 1$, $x = \frac{\pi}{2}$.

$\therefore x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Mark allocation: 3 marks

- 1 method mark for $\sin(x) = \pm\frac{\sqrt{3}}{2}$ or 1.
- 1 answer mark for at least one of the correct solutions.
- 1 answer mark for all correct solutions.

TIP

- For Exam 1 you are expected to know the exact values of circular functions and the symmetry properties of the unit circle.

Question 4

Worked solution

$$\begin{aligned}\log_e(x) + \log_e(5-2x) &= 0\\ \log_e(x(5-2x)) &= 0\\ x(5-2x) &= 1\\ -2x^2 + 5x - 1 &= 0\\ 2x^2 - 5x + 1 &= 0\\ x &= \frac{5 \pm \sqrt{25-8}}{4}\\ &= \frac{5 \pm \sqrt{17}}{4}\end{aligned}$$

Note: For $\log_e(x) + \log_e(5-2x)$ to be defined, we require $x > 0$ and $5 - 2x > 0$, thus $0 < x < \frac{5}{2}$.

Since $4 < \sqrt{17} < 5$, $\frac{5-\sqrt{17}}{4} > 0$ and $\frac{5+\sqrt{17}}{4} < \frac{5}{2}$, both solutions are valid.

Mark allocation: 2 marks

- 1 answer mark for the correct quadratic equation, that is, $2x^2 - 5x + 1 = 0$, or equivalent.
- 1 answer mark for the correct answer, that is, $x = \frac{5+\sqrt{17}}{4}, \frac{5-\sqrt{17}}{4}$, or equivalent.

Question 5

Worked solution

Let $y = -2(x-1)^2 - 3$.

To find the inverse, create a new equation by interchanging x and y.

$x = -2(y-1)^2 - 3$

Rearrange this equation to solve for y:

$$\begin{aligned}2(y-1)^2 - 3 &= x\\ 2(y-1)^2 &= x + 3\\ (y-1)^2 &= \frac{x+3}{2}\\ y - 1 &= \pm\sqrt{\frac{x+3}{2}}\\ y &= \pm\sqrt{\frac{x+3}{2}} + 1\end{aligned}$$

The range of f^{-1} is $\left(-\frac{1}{2}, 1\right]$, therefore the negative root is required.

This gives $f^{-1}(x) = -\sqrt{\frac{x+3}{2}} + 1$.

Mark allocation: 2 marks

- 1 method mark for interchanging variables and rearranging the equation.
- 1 answer mark for the correct answer.

TIPS

- » When determining the correct root, consider the range of f^{-1}, which is equal to the domain of f.
- » Ensure that the final answer is written in terms of $f^{-1}(x)$ and not left in terms of y. If the answer is left in terms of y, you will not be awarded the answer mark.

Question 6

Worked solution

$$\log_2(x+1) - \log_2(4-x) = 0$$

$$\log_2\left(\frac{x+1}{4-x}\right) = 0$$

$$\frac{x+1}{4-x} = 2^0$$

$$\frac{x+1}{4-x} = 1$$

$$x + 1 = 4 - x$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Mark allocation: 2 marks

- 1 method mark for applying log laws to create the expression $\log_2\left(\frac{x+1}{4-x}\right)$ or equivalent; or for rearranging the equation to $\log 2(x+1) = \log 2(4-x)$ and equating the arguments of the log functions.
- 1 answer mark for the correct answer.

Question 7a.

Worked solution

Let $y = \sqrt{1-3x}$.

To find the rule for the inverse, write a new equation with x and y interchanged.

$$x = \sqrt{1-3y}$$

$$1 - 3y = x^2$$

$$y = -\frac{1}{3}(x^2 - 1) \text{ or } y = -\frac{1}{3}x^2 + \frac{1}{3}$$

Therefore the rule for the inverse is

$$f^{-1}(x) = -\frac{1}{3}(x^2 - 1) \text{ or } f^{-1}(x) = -\frac{1}{3}x^2 + \frac{1}{3}.$$

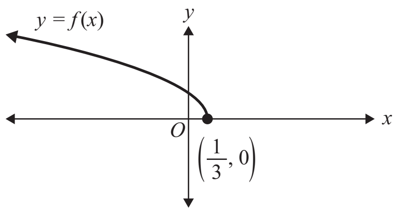
Mark allocation: 2 marks

- 1 method mark for writing an equation with x and y interchanged.
- 1 answer mark for $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{1}{3}$, or equivalent.

Question 7b.

Worked solution

dom f^{-1} = ran f. A graph can be used to find the range of f.



The range of f is $(0, \infty)$.

Therefore the domain of f^{-1} is $(0, \infty)$.

Mark allocation: 1 mark

- 1 answer mark for $(0, \infty)$.

TIP

» When asked for the domain of an inverse function, you may find it helpful to sketch the original function so that you can determine its range.

Question 8a.i.

Worked solution

The limits of the range of g are $a \times -1 + a = 0$ and $a \times 1 + a = 2a$.

Hence, when $a \geq 0$, the range of g is $[0, 2a]$.

Therefore, if $a \geq 0$, $g(f(x)) \geq 0$.

Mark allocation: 1 mark

- 1 mark for $a \geq 0$.

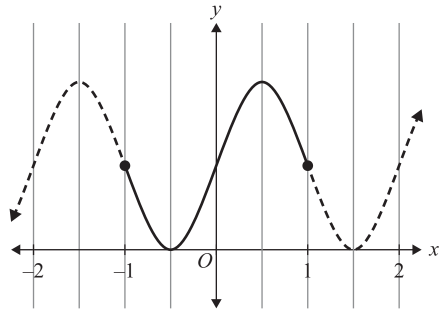
TIP

» As no graphs for the functions are provided, you could sketch a relevant portion of each function to help you visualise the problem.

Question 8a.ii.

Worked solution

$f(x)$ has a range of $[-1, 1]$, so first consider when $g(x)$ is at its maximum over this interval.



$g(x)$ is at its maximum when $x = \frac{1}{2}$.

Therefore we need to solve $f(x) = \frac{1}{2}$ on the interval $x \in [0, \pi]$.

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\frac{x}{2} = \frac{\pi}{6} \left(\text{solutions on the interval } \frac{x}{2} \in \left[0, \frac{\pi}{2}\right]\right)$$

$$x = \frac{\pi}{3}$$

Alternatively, the maximum value of $g(f(x))$ is $2a$. Therefore solve $g(f(x)) = 2a$:

$$g(f(x)) = 2a$$

$$a\sin(\pi f(x)) + a = 2a$$

$$\sin(\pi f(x)) = 1$$

$$\pi f(x) = \frac{\pi}{2}$$

$$f(x) = \frac{1}{2}$$

$$\sin\left(\frac{x}{2}\right) = \frac{1}{2}$$

$$\frac{x}{2} = \frac{\pi}{6} \left(\text{solutions on the interval } \frac{x}{2} \in \left[0, \frac{\pi}{2}\right]\right)$$

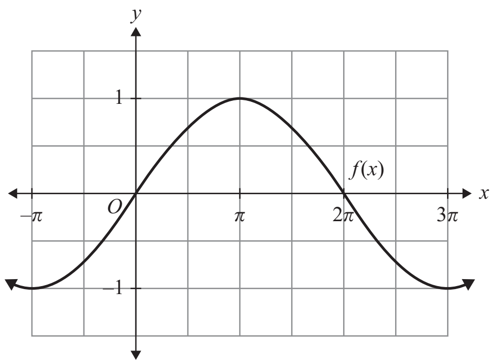
$$x = \frac{\pi}{3}$$

Mark allocation: 3 marks

- 1 method mark for identifying that $g(x)$ is at its maximum at $x = \frac{1}{2}$ or that $g(f(x)) = 2a$.
- 1 method mark for $\sin\left(\frac{x}{2}\right) = \frac{1}{2}$.
- 1 answer mark for $x = \frac{\pi}{3}$.

Question 8b.

Worked solution



$f(x) \geq 0$ when $x \in [0, 2\pi]$. Hence the range of $g(x)$ must fall within this interval.

The range of g is $[0, 2a]$ when $a \geq 0$, and $[2a, 0]$ when $a < 0$.

Therefore we need $2a = 2\pi$ and hence $a = \pi$.

Therefore, for $f(g(x)) \geq 0$, we must have $a \in [0, \pi]$.

Alternatively, $f(g(x)) = \sin\left(\frac{1}{2}(a\sin(\pi x) + a)\right)$.

For $f(g(x)) \geq 0$, we must have $0 \leq \frac{1}{2}(a\sin(\pi x) + a) \leq \pi$.

$0 \leq \frac{1}{2}(a\sin(\pi x) + a) \leq \pi$

$0 \leq a\sin(\pi x) + a \leq 2\pi$

$a\sin(\pi x) + a$ has values on the interval $[0, 2a]$, therefore $2a \leq 2\pi$ and $a \geq 0$.

Hence $a \in [0, \pi]$.

Alternatively, $f(x) \geq 0$ when $x \in [0, 2\pi]$. Hence the range of $g(x)$ must fall within this interval. $g(x)$ is a sinusoidal function, so for its range to be in the interval $[0, 2\pi]$, its median position must be greater than zero and less than π. From the equation for $g(x)$, its median position is at a, hence $a \in [0, \pi]$.

Mark allocation: 2 marks

- 1 method mark for identifying that $2a = 2\pi$, or that $f(x) \geq 0$ when $x \in [0, 2\pi]$, or that $0 \leq \frac{1}{2}(a\sin(\pi x) + a) \leq \pi$.
- 1 answer mark for $a \in [0, \pi]$, or equivalent.

Question 9a.

Worked solution

$\text{ran } f = R\backslash\left\{\frac{1}{2}\right\}$

Mark allocation: 1 mark

- 1 answer mark for the correct range.

Question 9b.

Worked solution

$\text{ran } f = R \backslash \left\{\frac{1}{2}\right\} = \text{dom } f$

Hence $\text{ran } f \subseteq \text{dom } f$, so $f \circ f$ exists.

Mark allocation: 1 mark

- 1 answer mark for justifying that $\text{ran } f \subseteq \text{dom } f$, followed by the correct conclusion.

Question 9c.

Worked solution

$$
\begin{aligned}
f \circ f(x) &= \frac{3}{4\left(\frac{3}{4x-2} + \frac{1}{2}\right) - 2} + \frac{1}{2} \\
&= \frac{3}{\frac{12}{4x-2} + 2 - 2} + \frac{1}{2} \\
&= \frac{3}{\frac{6}{2x-1}} + \frac{1}{2} \\
&= 3 \times \frac{2x-1}{6} + \frac{1}{2} \\
&= \frac{2x-1}{2} + \frac{1}{2} = x - \frac{1}{2} + \frac{1}{2} \\
&= x
\end{aligned}
$$

Mark allocation: 2 marks

- 1 answer mark for the substitution $f \circ f(x) = \frac{3}{4\left(\frac{3}{4x-2} + \frac{1}{2}\right) - 2} + \frac{1}{2}$, or any simplified version of this.
- 1 answer mark for $f \circ f(x) = x$.

Question 9d.

Worked solution

From **part c.** we have $f \circ f(x) = x$. It follows that $f^{-1}(x) = f(x)$ and hence $f^{-1}(x) = \frac{3}{4x-2} + \frac{1}{2}$.

The question asks us to 'define' f^{-1}, so we must include the domain, where

$\text{dom } f^{-1} = \text{ran } f : f^{-1} : R \backslash \left\{\frac{1}{2}\right\} \to R,\ f^{-1}(x) = \frac{3}{4x-2} + \frac{1}{2}$.

Mark allocation: 2 marks

- 1 answer mark for the correct rule: $f^{-1}(x) = \frac{3}{4x-2} + \frac{1}{2}$.
- 1 answer mark for fully defining the inverse, including the domain.

Question 10

Worked solution

By adding the two equations we can eliminate z.

$2x - 4y = -2$

Let $x = k$.

$$\begin{aligned} 2k - 4y &= -2 \\ -4y &= -2 - 2k \\ y &= \frac{1+k}{2} \end{aligned}$$

Substituting this into the first simultaneous equation we get

$$\begin{aligned} 2k - z &= -3 \\ -z &= -3 - 2k \\ z &= 3 + 2k \end{aligned}$$

Thus the general solution is $x = k,\ y = \frac{1+k}{2},\ z = 3 + 2k,\ k \in R$.

Alternative but equivalent answers can be found if a different pronumeral is set to k.
For example:

$x = 2k - 1,\ y = k,\ z = 4k + 1,\ k \in R$

$x = \frac{k-3}{2},\ y = \frac{k-1}{4},\ z = k,\ k \in R$

Mark allocation: 2 marks

- 1 method mark for setting any variable to a parameter and attempting to write at least one other variable in terms of that parameter (i.e. for using the substitution method, elimination method or by transposing an equation).
- 1 answer mark for the correct answer, or equivalent.

Question 11a.

Worked solution

Let $f(x) = g(x)$.

$$4x^2 = ax^2 + 5x$$

$$4x^2 - ax^2 - 5x = 0$$

$$(4 - a)x^2 - 5x = 0$$

$$x((4 - a)x - 5) = 0$$

$$x = 0,\ (4 - a)x = 5$$

$x = \frac{5}{4-a}$, as required

Mark allocation: 2 marks

- 1 method mark for setting $f(x) = g(x)$.
- 1 answer mark for the correct answer.

TIP

» You should not substitute $x = \frac{5}{4-a}$ into $f(x)$ and $g(x)$ for this 'show that' question, because you are required to show that this is the **only** other value of x for which $f(x) = g(x)$. Substitution would only show that this is a solution, not the **only** solution (apart from $x = 0$).

Question 11b.

Worked solution

From **part a.**, $f(x) = g(x)$ when $x = 0$ or $x = \frac{5}{4-a}$.

$x = 0$ is always a solution, so there will be a second solution if $x = \frac{5}{4-a}$ is defined.

$x = \frac{5}{4-a}$ gives a second solution if $4 - a \neq 0 \Rightarrow a \neq 4$.

Hence there are two solutions if $a \in R \backslash \{4\}$.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 11c.

Worked solution

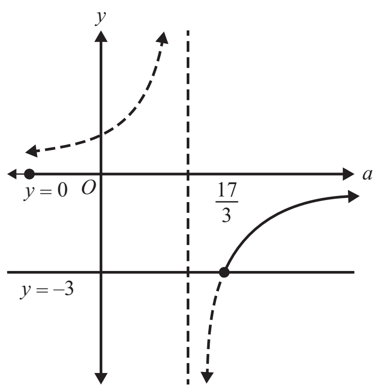
There will be two solutions if $x = \frac{5}{4-a}$ is within the domain $[-3, 0]$.

Noticing that $\frac{5}{4-a} \neq 0$, or that we want a second solution where $x \neq 0$, we can reduce the domain under consideration to $[-3, 0)$.

Thus we require $-3 \leq \frac{5}{4-a} < 0$.

Method 1: Graphical approach

Sketch the graph of $y = \frac{5}{4-a}$ and determine when $-3 \leq y < 0$:



Solve $\frac{5}{4-a} = 3 \Rightarrow 5 = 12 - 3a \Rightarrow a = \frac{17}{3}$.

We can see from the graph that $-3 \leq y < 0$ when $a \geq \frac{17}{3}$.

Hence there are two solutions if $a \geq \frac{17}{3}$.

You could also write this conclusion as $a \in \left[\frac{17}{3}, \infty\right)$.

Method 2: Algebraic approach

Since $\frac{5}{4-a} \neq 0$, we can solve $-3 \leq \frac{5}{4-a} < 0$.

$\frac{5}{4-a} < 0$ provided that $4 - a < 0 \Rightarrow a > 4$.

We also need to find the values of a for which $\frac{5}{4-a} \geq -3$. Since this involves solving an inequality, we need to consider the sign of the denominator, $4 - a$.

We have established that $4 - a < 0$, so

$$\begin{aligned} \frac{5}{4-a} &\geq -3 \\ 5 &\leq -3(4-a) \\ 5 &\leq -12 + 3a \\ 3a &\geq 17 \\ a &\geq \frac{17}{3} \end{aligned}$$

You could also write this conclusion as $a \in \left[\frac{17}{3}, \infty\right)$.

Mark allocation: 2 marks

- 1 method mark for finding the value $\frac{17}{3}$.
- 1 answer mark for the correct answer.

TIP

» A graphical method of solving inequalities is often the most straightforward.

Question 12a.

Worked solution

If $y = f(x)$, the inverse is

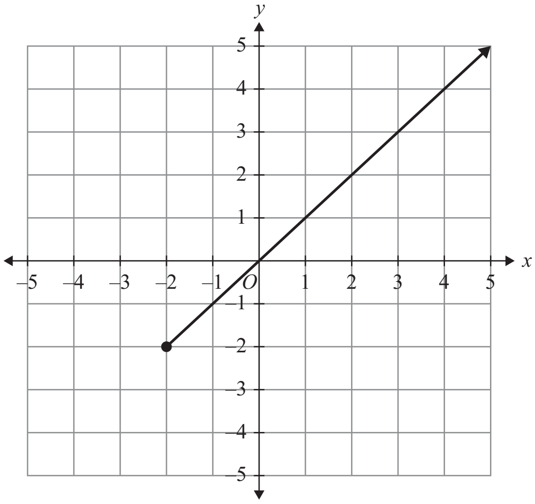
$$\begin{aligned} x &= \sqrt{y+2} - 1 \\ x + 1 &= y + 2 \\ (x+1)^2 &= y + 2 \\ y &= (x+1)^2 - 2 \\ \therefore f^{-1}(x) &= (x+1)^2 - 2 \\ \text{dom } f^{-1} &= \text{ran } f \\ &= [-1, \infty) \end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for the correct rule for the inverse. This must be expressed using $f^{-1}(x)$.
- 1 answer mark for the correct domain.

Question 12b.

Worked solution



Mark allocation: 2 marks

- 1 method mark for a sketch of $y = x$.
- 1 answer mark if the graph is sketched over the correct domain and with the axial intercept, $(0, 0)$, labelled. The end point at $(-2, -2)$ does not have to be labelled but it must be in the correct location.

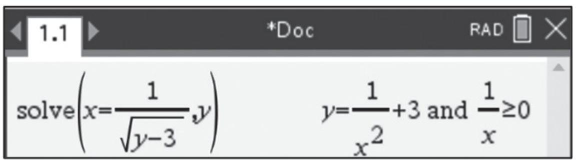
EXAM 2: MULTIPLE CHOICE

Question 1

Answer: ***B***

Explanatory notes

Using a CAS gives the inverse function as $y = \frac{1}{x^2} + 3$.



Since the range of the original function becomes the domain of its inverse, the domain is R^+.

So the answer is option B.

TIP

» You could also use the ***draw inverse*** function on a calculator to sketch the inverse.

Question 2

Answer: ***B***

Explanatory notes

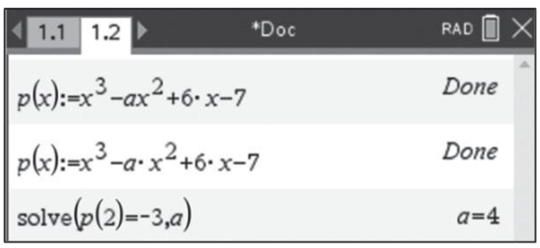
Reflect the graph in the line $y = x$ to get the graph of the inverse and then reflect the inverse in the x-axis to get the graph of $y = -f^{-1}(x)$.

Question 3

Answer: ***A***

Explanatory notes

A CAS can be used:



Question 4

Answer: ***A***

Explanatory notes

Find the inverse function f^{-1}. Let $y = 1 - 2\sqrt{x+1}$, and swap the variables: $x = 1 - 2\sqrt{y+1}$.

This gives $y = \frac{1}{4}(x-1)^2 - 1$.

Since $\text{dom } f^{-1} = \text{ran } f = (-\infty, 1]$, $f^{-1}: (-\infty, 1] \to R, f^{-1}(x) = \frac{1}{4}(x-1)^2 - 1$.

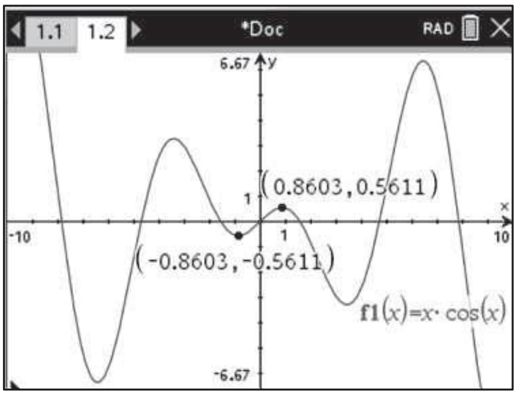
Question 5

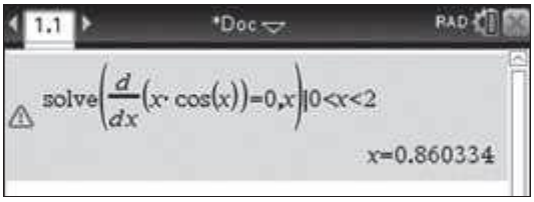
Answer: ***D***

Explanatory notes

g needs to be a one-to-one function in order to have an inverse function. Hence the domain cannot include both sides of a turning point.

Use a CAS to draw a graph of g and determine the turning point. This will show that the largest value of a is closest to $a = 0.86$.





Question 6

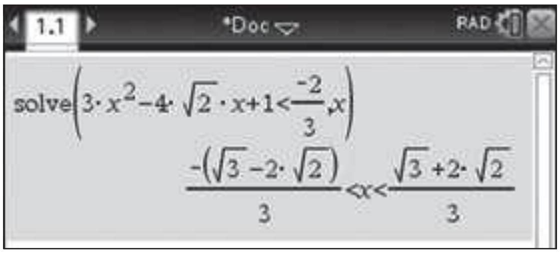
Answer: ***C***

Explanatory notes

We require $\text{ran } g \subseteq \text{dom } f$.

Therefore $g(x) = 3x^2 - 4\sqrt{2}x + 1 < -\frac{2}{3}$.

Solve $3x^2 - 4\sqrt{2}x + 1 < -\frac{2}{3}$ using a CAS:



So $B = \left\{x: \frac{2\sqrt{2} - \sqrt{3}}{3} < x < \frac{2\sqrt{2} + \sqrt{3}}{3}\right\}$.

Note: The given domain of f is $\left(-\infty, -\frac{2}{3}\right)$, which is not its maximal domain. Hence you cannot just find $f(g(x))$ and determine its maximal domain. (This will give a larger and incorrect domain.)

Question 7

Answer: ***C***

Explanatory notes

$h(x) = x(x+2) - (3x-2)$

$\text{dom}(h) = \left[-6, -\frac{3}{2}\right]$

From a simple graph it can be seen that there is no turning point in the interval $\left[-6, -\frac{3}{2}\right]$ and that $\text{ran}(h) = \left[\frac{23}{4}, 44\right]$.

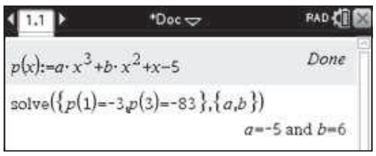
So $\text{dom}(h^{-1}) = \text{ran}(h) = \left[\frac{23}{4}, 44\right]$.

Question 8

Answer: ***C***

Explanatory notes

Use a CAS to determine the values of a and b by first defining the function, substituting in the two points and then solving for the parameters.



The values of a and b are –5 and 6, respectively, and their sum is 1.

Question 9

Answer: ***B***

Explanatory notes

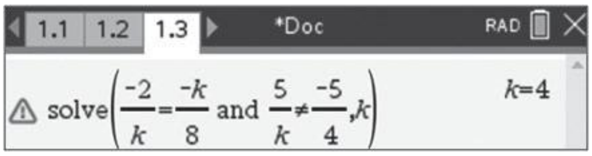
For the equations to have no solution, the corresponding graphs will have the same gradient but different y-intercepts.

Convert both equations to the form $y = mx + c$ so that the gradient (m) and y-intercept (c) can be easily seen:

$$y = \frac{5 - 2x}{k} = -\frac{2}{k}x + \frac{5}{k}$$

$$y = \frac{-10 - kx}{8} = -\frac{k}{8}x - \frac{5}{4}$$

Thus no solution exists if $-\frac{2}{k} = -\frac{k}{8}$ and $\frac{5}{k} \neq -\frac{5}{4}$. These equations may be solved simultaneously with a CAS:



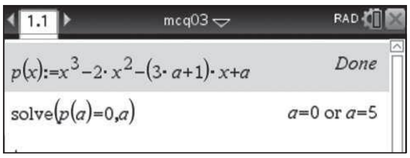
Question 10

Answer: ***D***

Explanatory notes

If $x - a$ is a factor of $p(x) = x^3 - 2x^2 - (3a + 1)x + a$, then $p(a) = 0$.

Use a CAS to solve $p(a) = 0$:



Since $a \neq 0$, $a = 5$.

Question 11

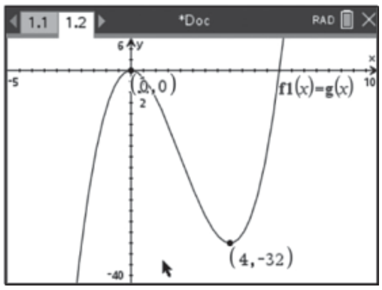
Answer: ***D***

Explanatory notes

Consider $g(x) = x^3 - 6x^2$ and $f(x) = x^3 - 6x^2 + b$. The graph of $f(x)$ is the image of the graph of $g(x)$ after a vertical translation of b units. This translation is up if $b > 0$ and down if $b < 0$.

Consider the graph of g.

$g'(x) = 3x^2 - 12x$ and the turning points of g are $(0, 0)$ and $(4, -32)$:



If we translate the graph of g up more than zero units but less than 32 units, there will be three x-intercepts.

Therefore there are exactly three x-intercepts when $b \in (0, 32)$.

Question 12

Answer: ***D***

Explanatory notes

This line is executed only when the mid value is on the same side of the x-intercept as the left bound. So the mid value also gives a left (lower) bound on the x-intercept, and the left bound (`l`) should be updated to the mid value (`m`).

Area of Study 3 Calculus

EXAM 1

Question 1a.

Worked solution

Using the quotient rule gives

$$\frac{dy}{dx} = \frac{(x^3 - 3x)\cos(x) - (3x^2 - 3)\sin(x)}{(x^3 - 3x)^2}$$

Mark allocation: 2 marks

- 1 method mark for attempting to use the quotient rule.
- 1 answer mark for the correct solution.

TIPS

» You could also use the product rule to find the derivative.

» You do not need to simplify the answer further.

Question 1b.

Worked solution

Using the product and chain rule gives

$$f'(x) = 3x^2 e^{1-4x} - 4x^3 e^{1-4x}$$

Evaluating the derivative with $x = -1$ gives $f'(-1) = 3e^5 + 4e^5 = 7e^5$.

Mark allocation: 2 marks

- 1 answer mark for the correct derivative.
- 1 answer mark for stating $f'(-1) = 7e^5$.

TIPS

» Don't forget to evaluate the derivative.

» Ensure that your final answer is simplified to get full marks.

Question 2a.

Worked solution

Using the chain rule gives

$$\begin{aligned} f'(x) &= \frac{1}{2}(5 - 4x)^{-\frac{1}{2}} \times -4 \\ &= -\frac{2}{\sqrt{5 - 4x}} \end{aligned}$$

Mark allocation: 1 mark

- 1 answer mark for the correct solution in simplified form.

Question 2b.

Worked solution

Since $f'(-1) = -\frac{2}{3}$ and $f(1) = 5$, the equation of the tangent is

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -\frac{2}{3}(x + 1)$$

$$y = -\frac{2x}{3} + \frac{13}{3}$$

$3y + 2x = 13$ or $2x + 3y = 13$

Mark allocation: 2 marks

- 1 method mark for finding $m = -\frac{2}{3}$ or $y = 5$ when $x = -1$.
- 1 answer mark for the correct equation in the correct form.

TIP

» The tangent equation could also be found using $y = mx + c$.

Question 3a.

Worked solution

$$f\left(\frac{3}{2}\right) = \frac{4}{\left(2 \times \frac{3}{2}\right) - 1} - 1$$

$$= \frac{4}{3 - 1} - 1$$

$$= \frac{4}{2} - 1$$

$$= 2 - 1$$

$$= 1$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3b.

Worked solution

First find $f'(x)$ and evaluate it at $x = \frac{3}{2}$ to find the gradient of the tangent:

$$f(x) = 4(2x - 1)^{-1} - 1$$

$$f'(x) = 4 \times \left(-1 \times 2 \times (2x - 1)^{-2}\right) = -\frac{8}{(2x - 1)^2}$$

$$f'\left(\frac{3}{2}\right) = -\frac{8}{\left(2\left(\frac{3}{2}\right) - 1\right)^2} = -\frac{8}{2^2} = -2$$

Therefore the equation of the tangent is

$$y - 1 = -2\left(x - \frac{3}{2}\right)$$

$$y = -2x + 4$$

Mark allocation: 2 marks

- 1 method mark for $f'\left(\frac{3}{2}\right) = -2$.
- 1 answer mark for $y = -2x + 4$.

Question 3c.

Worked solution

Find the end point at $x = 0$ (which is also a y-intercept):

$$f(0) = \frac{4}{2 \times 0 - 1} - 1$$

$$= \frac{4}{-1} - 1$$

$$= -5$$

To find the x-intercept, let $y = 0$ and solve for x:

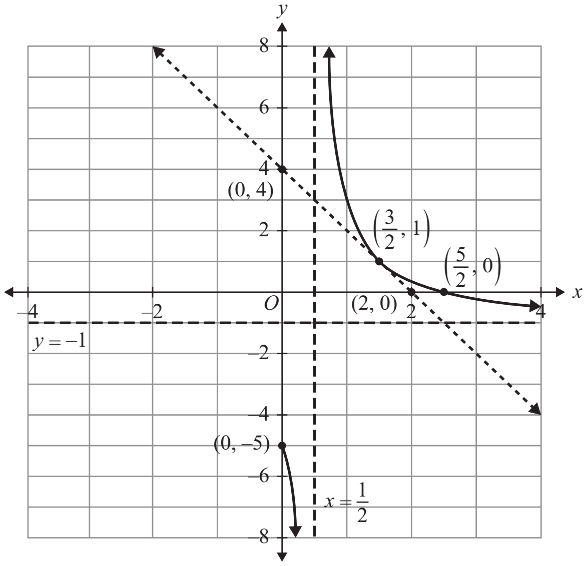
$$0 = \frac{4}{2x - 1} - 1$$

$$\frac{4}{2x - 1} = 1$$

$$4 = 2x - 1$$

$$2x = 5$$

$$x = \frac{5}{2}$$



Mark allocation: 3 marks

- 1 method mark for the tangent drawn through $(0, 4)$, $\left(\frac{3}{2}, 1\right)$ and $(2, 0)$. (The axis intercepts of the tangent do not need to be labelled but should be correctly located.)
- 1 method mark for sketching $f(x)$ with the correct hyperbola shape (even if drawn over a larger domain), showing asymptotic behaviour.
- 1 answer mark if everything is correct. This must include $f(x)$ sketched over the correct domain, both asymptotes labelled, the axial intercepts of $f(x)$ at $(0, -5)$, $\left(\frac{5}{2}, 0\right)$ labelled and the point of intersection $\left(\frac{3}{2}, 1\right)$ also labelled.

TIP

» When sketching hyperbolas, you should first draw the asymptotes.

Question 4

Worked solution

$$\text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\frac{\pi}{6}+\frac{\pi}{4}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} (2\cos(2x) - 1)\,dx$$

$$= \frac{24}{10\pi}\left[\sin(2x) - x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{6}}$$

$$= \frac{12}{5\pi}\left[\left(\sin\left(\frac{\pi}{3}\right) - \frac{\pi}{6}\right) - \left(\sin\left(-\frac{\pi}{2}\right) + \frac{\pi}{4}\right)\right]$$

$$= \frac{12}{5\pi}\left[\frac{\sqrt{3}}{2} - \frac{\pi}{6} + 1 - \frac{\pi}{4}\right]$$

$$= \frac{12}{5\pi}\left[\frac{\sqrt{3}}{2} + 1 - \frac{5\pi}{12}\right]$$

Mark allocation: 3 marks

- 1 method mark for setting up the integral for the average value.
- 1 method mark for correctly applying antidifferentiation.
- 1 answer mark for the correct answer.

Question 5a.

Worked solution

$$\frac{d}{dx}(x\log_e(3x) - x) = x\frac{1}{x} + \log_e(3x) - 1$$

$$= 1 + \log_e(3x) - 1$$

$$= \log_e(3x)$$

Mark allocation: 1 mark

- 1 answer mark for working that leads to the required derivative.

Question 5b.

Worked solution

Find the x-intercept:

$$\begin{aligned}\log_e(3x) &= 0\\ 3x &= e^0\\ 3x &= 1\\ x &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\text{Area} &= \int_{\frac{1}{3}}^{\frac{e}{3}} \log_e(3x)\,dx\\ &= \left[x\log_e(3x) - x\right]_{\frac{1}{3}}^{\frac{e}{3}}\\ &= \frac{e}{3}\log_e(e) - \frac{e}{3} - \log_e(1) + \frac{1}{3}\\ &= \frac{1}{3}\end{aligned}$$

Mark allocation: 3 marks

- 1 mark for finding the x-intercept.
- 1 method mark for $\int \log_e(3x)\,dx = x\log_e(3x) - x$.
- 1 answer mark for the correct answer.

Question 6a.

Worked solution

$\frac{d}{dx}(\cos(x)) = -\sin(x)$ and $\frac{d}{dx}(\cos(4x)) = -4\sin(4x)$.

$$\begin{aligned}\frac{dy}{dx} &= \cos(x) \times -4\sin(4x) + -\sin(x) \times \cos(4x)\\ &= -4\cos(x)\sin(4x) - \sin(x)\cos(4x)\end{aligned}$$

Mark allocation: 2 marks

- 1 method mark for applying the product rule.
- 1 answer mark for the correct answer.

Question 6b.

Worked solution

$f'(x) = 4e^{2x}$

$$\begin{aligned}f'(\log_e(2)) &= 4e^{2\log_e(2)}\\ &= 4e^{\log_e(4)}\\ &= 4 \times 4\\ &= 16\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for the correct derivative: $f'(x) = 4e^{2x}$.
- 1 answer mark for the correct answer: $f'(\log_e(2)) = 16$.

Question 7a.

Worked solution

The expression is of the form $\frac{u}{v}$, hence the quotient rule can be used to find the derivative:

$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$ and $\frac{d}{dx}(x) = 1$

$$\frac{dy}{dx} = \frac{\frac{1}{x} \times x - \log_e(x) \times 1}{x^2}$$

$$= \frac{1 - \log_e(x)}{x^2}$$

$$= \frac{1}{x^2} - \frac{\log_e(x)}{x^2}$$

Mark allocation: 1 mark

- 1 method mark for applying the quotient rule to obtain the derivative.

Note: You can also use the product rule to obtain the derivative.

Question 7b.

Worked solution

From **part a.** we know that $\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right) = \frac{1}{x^2} - \frac{\log_e(x)}{x^2}$.

Hence $\int\left(\frac{1}{x^2} - \frac{\log_e(x)}{x^2}\right)dx = \frac{\log_e(x)}{x}$.

This can be rearranged so that just $\int\frac{\log_e(x)}{x^2}\,dx$ is on the left-hand side:

$$\int\left(\frac{1}{x^2} - \frac{\log_e(x)}{x^2}\right)dx = \frac{\log_e(x)}{x}$$

$$-\int\frac{\log_e(x)}{x^2}dx = -\int\frac{1}{x^2}dx + \frac{\log_e(x)}{x}$$

$$\int\frac{\log_e(x)}{x^2}dx = \int\frac{1}{x^2}dx - \frac{\log_e(x)}{x}$$

$$= \int x^{-2}\,dx - \frac{\log_e(x)}{x}$$

$$= -x^{-1} - \frac{\log_e(x)}{x} \quad \text{(this is an antiderivative)}$$

Then integrate both sides over the required interval:

$$\int_1^2\frac{\log_e(x)}{x^2}dx = \left[-\frac{1}{x} - \frac{\log_e(x)}{x}\right]_1^2$$

$$= \left[-\frac{1}{2} - \frac{\log_e(2)}{2}\right] - \left[-\frac{1}{1} - \frac{\log_e(1)}{1}\right]$$

$$= \left[-\frac{1}{2} - \frac{\log_e(2)}{2}\right] - [-1]$$

$$= \frac{1}{2} - \frac{\log_e(2)}{2}$$

Mark allocation: 3 marks

- 1 method mark for rearranging the result from **part a.**, either before or after integrating both sides, to construct an appropriate equation with $\int \frac{\log_e(x)}{x^2}dx$ on the left side.
- 1 mark for obtaining the antiderivative: $-\frac{1}{x} - \frac{\log_e(x)}{x}$.
- 1 mark for the correct answer: $\int_1^2 \frac{\log_e(x)}{x^2}dx = \frac{1}{2} - \frac{\log_e(2)}{2}$.

TIPS

» The word 'hence' tells you that you should use the answer to the previous question to respond to the current question. These sorts of 'integration by recognition' problems are common in Exam 1.

» Because you are ultimately calculating a definite integral, you can use an antiderivative that omits the constant of integration, c, during the working.

Question 8a.

Worked solution

The gradient of the tangent is given by $f'(p)$:

$$f(x) = \sqrt{x-5} = (x-5)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(x-5)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x-5}}$$

$$f'(p) = \frac{1}{2\sqrt{p-5}}$$

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 8b.

Worked solution

The tangent to f passes through $(p, \sqrt{p-5})$ and has the equation

$$y - f(p) = \frac{1}{2\sqrt{p-5}}(x-p)$$

$$y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(x-p)$$

Substituting $x = 0$ and $y = 0$ gives

$$-\sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(-p)$$

$$-\sqrt{p-5} = -\frac{p}{2\sqrt{p-5}}$$

$$2(p-5) = p$$

$$p = 10$$

Alternatively, if the tangent passes through $(0, 0)$, then it has the equation $y = \frac{1}{2\sqrt{p-5}}x$.

Using the gradient from **part a.** and substituting $x = p$ and $y = f(p) = \sqrt{p-5}$ gives

$$\sqrt{p-5} = \frac{1}{2\sqrt{p-5}} \times p$$

$$p - 5 = \frac{1}{2}p$$

$$\frac{1}{2}p = 5$$

$$p = 10$$

Alternatively, a line through the origin has the equation $y = mx$, with some unknown gradient, m.

If this line is a tangent, then the equation $mx = \sqrt{x-5}$ has a single solution:

$$m^2x^2 = x - 5$$

$$m^2x^2 - x + 5 = 0$$

For a single solution, $\Delta = 0$.

$$\Delta = (-1)^2 - 4 \times m^2 \times 5$$

$$= 1 - 20m^2$$

$$1 - 20m^2 = 0$$

$$m^2 = \frac{1}{20}$$

$$m = \pm\frac{1}{2\sqrt{5}}$$

Since $m > 0$, $m = \frac{1}{2\sqrt{5}}$.

This gradient can be equated to the gradient found in **part a.**

$$\frac{1}{2\sqrt{p-5}} = \frac{1}{2\sqrt{5}}$$

$$p - 5 = 5$$

$$p = 10$$

Or the gradient can be substituted into the earlier equation to solve for x:

$$m^2x^2 - x + 5 = 0$$

$$\frac{1}{20}x^2 - x + 5 = 0$$

$$x^2 - 20x + 100 = 0$$

$$(x - 10)^2 = 0$$

$$x = 10$$

Hence $p = 10$.

Mark allocation: 2 marks

- 1 method mark for determining the equation of the tangent: $y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}(x - p)$ or $y - \sqrt{p-5} = \frac{1}{2\sqrt{p-5}}x + \frac{p-10}{2\sqrt{p-5}}$, or for stating the equation of the tangent as $y = \frac{1}{2\sqrt{p-5}}x$.
- 1 answer mark for $p = 10$.

Question 9a.

Worked solution

$\frac{d}{dx}(2x+1) = 2$ and $\frac{d}{dx}\left(\log_e(2x+1)\right) = \frac{2}{2x+1}$

$$\frac{dy}{dx} = (2x+1) \times \frac{2}{(2x+1)} + (2) \times \log_e(2x+1)$$
$$= 2 + 2\log_e(2x+1)$$

Mark allocation: 2 marks

- 1 method mark for applying the product rule.
- 1 answer mark for $\frac{dy}{dx} = 2 + 2\log_e(2x+1)$, or an equivalent answer (such as a factorised expression).

Question 9b.

Worked solution

Apply the chain rule to obtain $f'(x) = -8x\sin(4x^2)$:

$$f'\left(\frac{\sqrt{\pi}}{4}\right) = -8 \times \left(\frac{\sqrt{\pi}}{4}\right) \times \sin\left(4\left(\frac{\sqrt{\pi}}{4}\right)^2\right)$$
$$= -2\sqrt{\pi}\sin\left(\frac{\pi}{4}\right)$$
$$= -2\sqrt{\pi} \times \frac{\sqrt{2}}{2}$$
$$= -\sqrt{2\pi}$$

Mark allocation: 2 marks

- 1 answer mark for calculating the correct derivative: $f'(x) = -8x\sin(4x^2)$.
- 1 answer mark for the correct answer: $f'\left(\frac{\sqrt{\pi}}{4}\right) = -\sqrt{2\pi}$.

Question 10

Worked solution

$$y = \int 2 - e^{-x}dx$$
$$= 2x + e^{-x} + c$$

Substitute $x = 2$ and $y = 4 - \frac{1}{e^2}$, and solve for c:

$$4 - \frac{1}{e^2} = 2 \times 2 + e^{-2} + c$$
$$= 4 + \frac{1}{e^2} + c$$
$$c = -\frac{2}{e^2}$$

$y = 2x + e^{-x} - 2e^{-2}$ or $y = 2x + \frac{1}{e^x} - \frac{2}{e^2}$

Mark allocation: 2 marks

- 1 answer mark for antidifferentiating $\frac{dy}{dx}$ to give $y = 2x + e^{-x} + c$, or for $y = 2x + e^{-x}$ if given as the final answer.
- 1 answer mark for the equation $y = 2x + e^{-x} - 2e^{-2}$, or an equivalent with positive indices.

Question 11

Worked solution

$$\text{Area} = \int_0^{\frac{2\pi}{3}} \sin(x) + \sin(2x)\,dx - \int_{\frac{2\pi}{3}}^{\pi} \sin(x) + \sin(2x)\,dx$$

$$= \left[-\cos(x) - \frac{1}{2}\cos(2x)\right]_0^{\frac{2\pi}{3}} - \left[-\cos(x) - \frac{1}{2}\cos(2x)\right]_{\frac{2\pi}{3}}^{\pi}$$

$$= \left[\left(-\cos\left(\frac{2\pi}{3}\right) - \frac{1}{2}\cos\left(\frac{4\pi}{3}\right)\right) - \left(-\cos(0) - \frac{1}{2}\cos(0)\right)\right]$$

$$-\left[\left(-\cos(\pi) - \frac{1}{2}\cos(2\pi)\right) - \left(-\cos\left(\frac{2\pi}{3}\right) - \frac{1}{2}\cos\left(\frac{4\pi}{3}\right)\right)\right]$$

$$= \left[\left(\frac{1}{2} + \frac{1}{4}\right) - \left(-1 - \frac{1}{2}\right)\right] - \left[\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{4}\right)\right]$$

$$= \left[\frac{9}{4}\right] - \left[-\frac{1}{4}\right]$$

$$= \frac{10}{4} = \frac{5}{2} \text{ square units}$$

Mark allocation: 3 marks

- 1 method mark for an appropriate expression for calculating the area, such as $\int_0^{\frac{2\pi}{3}} f(x)\,dx - \int_{\frac{2\pi}{3}}^{\pi} f(x)\,dx$.
- 1 method mark for calculating the antiderivative of $\sin(x) + \sin(2x)$ as $-\cos(x) - \frac{1}{2}\cos(2x)$.
- 1 answer mark for the correct answer: $\frac{5}{2}$ square units.

Note: Units are not required.

TIP

» Even if you cannot solve a definite integral, you should still write down the expression that you will solve, as you might get a method mark.

Question 12a.

Worked solution

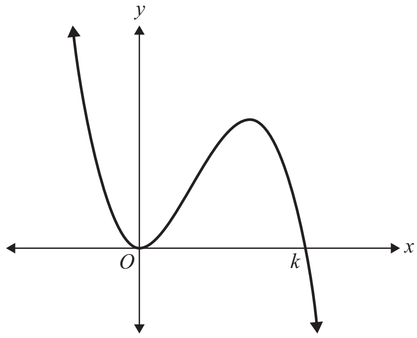
Method 1

The expression under the square root sign must be greater than or equal to zero, hence

$$-x^3 + kx^2 \geq 0$$

$$-x^2(x - k) \geq 0$$

Consider the graph of $y = -x^2(x-k)$:



The graph is a negative cubic function with x-intercepts at $x = 0$ and $x = k$.

The graph is on or above the x-axis for $0 \le x \le k$.

Since the domain is given as $[0, a)$, the maximal value is $a = k$.

Method 2

Let $f(x) = 0$.

$$\sqrt{-x^3 + kx^2} = 0$$
$$-x^3 + kx^2 = 0$$
$$-x^2(x-k) = 0$$
$$x = 0,\ k$$
$$a = k$$

Mark allocation: 2 marks

- 1 method mark for a suitable method, such as equating $f(x)$ to zero or by considering when $-x^3 + kx^2 \ge 0$.
- 1 answer mark for $a = k$.

TIP

» **Remember to read the question carefully, as a common mistake is to think that a question of this type is asking for the maximum value of the function.**

Question 12b.i.

Worked solution

P is at $\left(p, \sqrt{-p^3 + kp^2}\right)$.

The length of the chord is given by

$$L = \sqrt{(p-0)^2 + \left(\sqrt{-p^3 + kp^2} - 0\right)^2}$$
$$= \sqrt{p^2 - p^3 + kp^2}$$
$$= \sqrt{(k+1)p^2 - p^3}$$

Mark allocation: 1 mark

- 1 mark for clear reasoning and correct working, such as using the distance formula.

TIP

» You must know the formula for the distance between two points in the Cartesian plane. It is not given on the VCAA Mathematical Methods formula sheet.

Question 12b.ii.

Worked solution

$$L = \sqrt{-p^3 + (k+1)p^2}$$

$$= (-p^3 + (k+1)p^2)^{\frac{1}{2}}$$

$$\frac{dL}{dp} = (-3p^2 + 2(k+1)p) \times \frac{1}{2} \times (-p^3 + (k+1)p^2)^{-\frac{1}{2}}$$

$$= \frac{-3p^2 + 2(k+1)p}{2\sqrt{-p^3 + (k+1)p^2}}$$

Maximum occurs when $\frac{dL}{dp} = 0$.

$$0 = \frac{-3p^2 + 2(k+1)p}{2\sqrt{-p^3 + (k+1)p^2}}$$

$$0 = -3p^2 + 2(k+1)p$$

$$0 = -p(3p - 2(k+1))$$

$$p = 0, \frac{2(k+1)}{3}, \text{ but } p > 0$$

$$p = \frac{2(k+1)}{3}$$

Alternatively, minimise the squared distance $-x^3 + (k+1)x^2$ instead of minimising the distance.

Mark allocation: 3 marks

- 1 answer mark for calculating the derivative of $\sqrt{-x^3 + (k+1)x^2}$ or $-x^3 + (k+1)x^2$.
- 1 method mark for equating the derivative to zero.
- 1 answer mark for $p = \frac{2(k+1)}{3}$.

TIPS

» When maximising or minimising distance functions, it is often easier to maximise or minimise the squared distance.

» When omitting a solution, such as $p = 0$, you should provide a reason for the omission.

Question 13

Worked solution

$$\text{Area} = \int_1^3 3 - \frac{4}{2x+1}\,dx$$
$$= \left[3x - 2\log_e(2x+1)\right]_1^3$$
$$= \left(9 - 2\log_e(7)\right) - \left(3 - 2\log_e(3)\right)$$
$$= 6 - 2\log_e\left(\frac{7}{3}\right)$$
$$= 6 + 2\log_e\left(\frac{3}{7}\right)$$
$$= 6 + \log_e\left(\frac{9}{49}\right)$$

So $a = 6$, $b = 9$ and $c = 49$.

Mark allocation: 3 marks

- 1 method mark for setting up the integral.
- 1 method mark for correctly applying antidifferentiation.
- 1 answer mark for the correct area in the correct form and for stating the values of a, b and c.

TIP

» **Make sure you write your answer for the area in the given form so that you can correctly state the values of a, b and c.**

Question 14

Worked solution

$$\int_{-\frac{7\pi}{12}}^{\frac{\pi}{12}} 2 - 4\sin(2x)\,dx$$
$$= \left[2x + 2\cos(2x)\right]_{-\frac{7\pi}{12}}^{\frac{\pi}{12}}$$
$$= \left[2 \times \frac{\pi}{12} + 2\cos\left(2 \times \frac{\pi}{12}\right)\right] - \left[2 \times -\frac{7\pi}{12} + 2\cos\left(2 \times -\frac{7\pi}{12}\right)\right]$$
$$= \left[\frac{\pi}{6} + \sqrt{3}\right] - \left[-\frac{7\pi}{6} - \sqrt{3}\right]$$
$$= \frac{4\pi}{3} + 2\sqrt{3}$$

Mark allocation: 3 marks

- 1 method mark for a correctly composed integral expression: $\int_{-\frac{7\pi}{12}}^{\frac{\pi}{12}} 2 - 4\sin(2x)\,dx$.
- 1 method mark for evaluating the integral and substituting the end points of the interval, for example, obtaining $\left[2 \times \frac{\pi}{12} + 2\cos\left(2 \times \frac{\pi}{12}\right)\right] - \left[2 \times -\frac{7\pi}{12} + 2\cos\left(2 \times -\frac{7\pi}{12}\right)\right]$.
- 1 answer mark for the correct answer: $\frac{4\pi}{3} + 2\sqrt{3}$ or $\frac{4\pi + 6\sqrt{3}}{3}$.

Question 15

Worked solution

$$f'(x) = \frac{-1}{(x+2)^2} + \frac{1}{(4-x)^2} = 0$$

$$\frac{-(x+2)^2 + (4-x)^2}{(x+2)^2(4-x)^2} = 0$$

$$-(x+2)^2 + (4-x)^2 = 0$$

$$(4-x)^2 = (x+2)^2$$

$$4 - x = x + 2$$

$$2x = 2$$

$$x = 1$$

Mark allocation: 2 marks

- 1 answer mark for the correct derivative.
- 1 answer mark for the correct value: $x = 1$.

Question 16a.

Worked solution

$$f'(x) = 1 \times e^{-x} + (x-k) \times -e^{-x}$$

$$= e^{-x} - (x-k)e^{-x}$$

$$= (1 - x + k)e^{-x}$$

$$= -(x - 1 - k)e^{-x}$$

Mark allocation: 1 mark

- 1 method mark for applying the product rule and obtaining the correct answer.

TIPS

» If you have trouble applying the product rule for differentiation, break the problem into smaller steps. Begin by identifying parts of the function and finding their derivatives.

» In 'show that' questions, you should write your final answer in the same format as that shown in the question.

Question 16b.

Worked solution

At the stationary point, $f'(x) = 0$.

$$-(x - 1 - k)e^{-x} = 0$$
$$x - 1 - k = 0,\ e^{x} \neq 0$$
$$x = k + 1$$
$$f(k+1) = (k + 1 - k)e^{-(k+1)}$$
$$= e^{-(k+1)}$$

P has coordinates $\left(k + 1,\ e^{-(k+1)}\right)$.

Mark allocation: 2 marks

- 1 answer mark for the x-coordinate: $x = k + 1$.
- 1 answer mark for the y-coordinate: $y = e^{-(k+1)}$.

TIP

» Pay attention to the format in which answers are to be provided. A question like this may ask for just the x value where the stationary point occurs, for both coordinates or for the maximum value.

Question 17a.i.

Worked solution

$$f(x) = (1 - 3x)^{\frac{1}{2}} = (-3x + 1)^{\frac{1}{2}}$$
$$f'(x) = (-3) \times \frac{1}{2}(-3x + 1)^{-\frac{1}{2}}$$
$$= -\frac{3}{2}(-3x + 1)^{-\frac{1}{2}}$$
$$= -\frac{3}{2\sqrt{1 - 3x}}$$

Mark allocation: 1 mark

- 1 mark for the correct answer, or equivalent.

TIPS

» The formula for differentiating $(ax + b)^n$ is on the VCAA Mathematical Methods formula sheet.

» It can be helpful to write $(1 - 3x)^{\frac{1}{2}}$ as $(-3x + 1)^{\frac{1}{2}}$ in order to use the formula provided.

Question 17a.ii.

Worked solution

$f(x) = (1-3x)^{\frac{1}{2}} = (-3x+1)^{\frac{1}{2}}$

$$\int(-3x+1)^{\frac{1}{2}}\,dx = \frac{(-3x+1)^{\frac{3}{2}}}{(-3)\times\frac{3}{2}} + c$$

$$= -\frac{2}{9}(1-3x)^{\frac{3}{2}} + c \quad \text{or} \quad -\frac{2}{9}(1-3x)^{\frac{3}{2}} \text{ (an antiderivative)}$$

Note: The question asks for an antiderivative, so the case where the constant of integration, c, is equal to zero may be given. Alternatively, the answer may include '$+\,c$'.

Mark allocation: 1 mark

- 1 answer mark for $-\frac{2}{9}(1-3x)^{\frac{3}{2}}$.

TIP

» When a question asks for **an** antiderivative, you can use any value for the constant of integration, including zero. When a question asks for **the** antiderivative, you must provide an expression describing a family of functions and leave the constant of integration as a parameter (usually c is used for this).

Question 17b.

Worked solution

$\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}(\cos(x)+1) = -\sin(x)$.

$$g'(x) = \frac{\cos(x)\times(\cos(x)+1) - \sin(x)\times(-\sin(x))}{(\cos(x)+1)^2}$$

$$= \frac{\cos^2(x)+\cos(x)+\sin^2(x)}{(\cos(x)+1)^2}$$

$$= \frac{\cos(x)+1}{(\cos(x)+1)^2}$$

$$= \frac{1}{\cos(x)+1}$$

$$g'\left(\frac{\pi}{2}\right) = \frac{1}{\cos\left(\frac{\pi}{2}\right)+1}$$

$$= \frac{1}{0+1} = 1$$

Alternatively, $x = \frac{\pi}{2}$ may be substituted earlier, as in

$$g'(x) = \frac{\cos(x) \times (\cos(x) + 1) - \sin(x) \times (-\sin(x))}{(\cos(x) + 1)^2}$$

$$g'\left(\frac{\pi}{2}\right) = \frac{\cos\left(\frac{\pi}{2}\right) \times (\cos\left(\frac{\pi}{2}\right) + 1) - \sin\left(\frac{\pi}{2}\right) \times (-\sin\left(\frac{\pi}{2}\right))}{(\cos\left(\frac{\pi}{2}\right) + 1)^2}$$

$$= \frac{0 \times (0 + 1) - 1 \times (-1)}{(0 + 1)^2}$$

$$= \frac{1}{1} = 1$$

Mark allocation: 2 marks

- 1 answer mark for calculating the correct derivative: $g'(x) = \frac{1}{\cos(x) + 1}$, or equivalent.
- 1 answer mark for the correct answer: $g'\left(\frac{\pi}{2}\right) = 1$.

TIP

» The values of $\sin\left(\frac{\pi}{2}\right) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$ are easy to work with, so you may find this problem faster to solve if you don't simplify the expression you get from applying the quotient rule. Instead evaluate the $\sin$ and $\cos$ functions immediately after finding an expression for $g'(x)$.

Question 18a.

Worked solution

The gradient of the tangent is given by $\frac{dy}{dx}$.

$$\frac{dy}{dx} = e^x$$

At P, $\frac{dy}{dx}\Big|_{x=p} = e^p$

The tangent passes through point (p, e^p).

Therefore the equation of the tangent is

$$y - e^p = e^p(x - p)$$

$$y = e^p(x - p) + e^p$$

$$y = e^p x + e^p(1 - p)$$

Mark allocation: 1 mark

- 1 method mark for evaluating the derivative and showing relevant working to determine the equation of the tangent. The final equation must be in the given form.

Question 18b.

Worked solution

The enclosed area is given by

$$A = \int_0^1 e^x - (e^p x + e^p(1-p))\,dx$$

$$= \int_0^1 e^x - e^p x + e^p(p-1)\,dx$$

$$= \left[e^x - \frac{e^p}{2}x^2 + e^p(p-1)x\right]_0^1$$

$$= \left[e - \frac{e^p}{2} + e^p(p-1)\right] - [1]$$

$$= e^p\left(p - \frac{3}{2}\right) + e - 1$$

The area is minimised when $\frac{dA}{dp} = 0$.

$$\frac{dA}{dp} = e^p \times \frac{d}{dp}\left(p - \frac{3}{2}\right) + \frac{d}{dp}(e^p) \times \left(p - \frac{3}{2}\right)$$

$$= e^p \times 1 + e^p \times \left(p - \frac{3}{2}\right)$$

$$= e^p\left(p - \frac{1}{2}\right)$$

$$\frac{dA}{dp} = 0 = e^p\left(p - \frac{1}{2}\right)$$

Therefore

$$p - \frac{1}{2} = 0\,,\ e^p \neq 0$$

$$p = \frac{1}{2}$$

Mark allocation: 3 marks

- 1 method mark for calculating the rule for the area in terms of p: $A = e^p\left(p - \frac{3}{2}\right) + e - 1$.
- 1 method mark for calculating the derivative of the rule for the area in terms of p: $\frac{dA}{dp} = e^p\left(p - \frac{1}{2}\right)$.
- 1 answer mark for $p = \frac{1}{2}$.

TIP

» In each step in this question, you need to be careful about which variable you are integrating or differentiating with respect to.

Question 19

Worked solution

The formula for Newton's method is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Each iteration of Newton's method requires us to find the y value and the derivative at the current estimate of x.

$f'(x) = 3x^2 - 8x$

For $x_0 = 2$: $x_1 = 2 - \frac{f(2)}{f'(2)}$

$f(2) = (2)^3 - 4(2)^2 + 2 = -6,\ f'(2) = 3(2)^2 - 8(2) = -4$

Hence

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 2 - \frac{-6}{-4} \\ &= \frac{1}{2} \end{aligned}$$

Repeating for the second iteration gives

For $x_1 = \frac{1}{2}$: $x_2 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}$

$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 2 = \frac{1}{8} - 1 + 2 = \frac{9}{8}$

$f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) = \frac{3}{4} - 4 = -\frac{13}{4}$

Hence

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= \frac{1}{2} - \frac{\left(\frac{9}{8}\right)}{\left(-\frac{13}{4}\right)} \\ &= \frac{1}{2} - \frac{9}{8} \times \left(-\frac{4}{13}\right) \\ &= \frac{1}{2} + \frac{9}{26} \\ &= \frac{22}{26} \\ &= \frac{11}{13} \end{aligned}$$

Mark allocation: 3 marks

- 1 method mark for any substitution of values into the formula for Newton's method.
- 1 answer mark for the correct value of x_1.
- 1 answer mark for the correct value of x_2.

Question 20a.

Worked solution

$$f(x)=\frac{1}{x+2}+\frac{1}{4-x}$$
$$=\frac{4-x}{(x+2)(4-x)}+\frac{x+2}{(x+2)(4-x)}$$
$$=\frac{4-x+x+2}{(x+2)(4-x)}$$
$$=\frac{6}{(x+2)(4-x)}$$
$$=\frac{6}{-x^2+2x+8}$$
$$=\frac{-6}{x^2-2x-8}$$
$$=g(x)$$

Mark allocation: 2 marks

- 1 method mark for deriving $f(x)=\frac{6}{(x+2)(4-x)}$.
- 1 answer mark for noting that $f(x)$ is also equal to $\frac{-6}{x^2-2x-8}$ and that this is also $g(x)$.

Question 20b.

Worked solution

$$g(x)=-6(x^2-2x-8)^{-1}$$
$$g'(x)=-6\times-1\times(2x-2)(x^2-2x-8)^{-2}$$
$$=\frac{6(2x-2)}{(x^2-2x-8)^2}$$
$$=\frac{12(x-1)}{(x^2-2x-8)^2}$$

Mark allocation: 1 mark

- 1 answer mark for correct working leading to $g'(x)=\frac{12(x-1)}{(x^2-2x-8)^2}$.

Question 20c.

Worked solution

Solve $g'(x)=0$ to determine if the function has any stationary points.

$$g'(x)=\frac{12(x-1)}{(x^2-2x-8)^2}=0$$
$$12(x-1)=0$$
$$x=1$$

There is one stationary point at $x=1$. To determine the nature of the stationary point, consider the sign of $g'(x)$ either side of $x=1$.

Since $g'(x) = \dfrac{12(x-1)}{(x^2-2x-8)^2}$ and $(x^2-2x-8)^2 > 0$, we only need to consider the sign of $12(x-1)$:

- when $x < 1,\ 12(x-1) < 0 \therefore g'(x) < 0$ and
- when $x > 1,\ 12(x-1) > 0 \therefore g'(x) > 0$.

Hence $g(x)$ has a local minimum at $x = 1$.

Mark allocation: 2 marks

- 1 answer mark for deriving $x = 1$.
- 1 answer mark for the correct nature of the stationary point: local minimum.

Question 20d.

Worked solution

For the inverse function to exist, $f(x)$ needs to be a one-to-one function.

Since $f(x) = g(x)$, we can consider what we know about the graph of $g(x)$ from **part c.**

$g(x)$, and hence $f(x)$, has a minimum turning point at $x = 1$, so the domain cannot include both sides of $x = 1$. Since the domain is $[-1, a]$, the largest value of a is $a = 1$.

Hence $a = 1$.

Mark allocation: 1 mark

- 1 consequential answer mark for $a = 1$. (The answer should agree with value found in **part c.**)

Question 20e.

Worked solution

Let $x = \dfrac{-6}{y^2-2y-8}$, where $y = g^{-1}(x)$.

Hence

$$\begin{aligned}
y^2-2y-8 &= -\frac{6}{x}\\
(y-1)^2-9 &= -\frac{6}{x}\\
(y-1)^2 &= -\frac{6}{x}+9\\
y-1 &= \pm\sqrt{-\frac{6}{x}+9}\\
y &= \pm\sqrt{-\frac{6}{x}+9}+1\\
&= 1 \pm \sqrt{9-\frac{6}{x}}
\end{aligned}$$

Since $\text{dom } g = [-1, 1]$ and $\text{dom } g = \text{ran } g^{-1}$, only the negative square root is required:
$g^{-1}(x) = 1 - \sqrt{9-\frac{6}{x}}$

Mark allocation: 2 marks

- 1 method mark for swapping x and y and obtaining $y^2-2y-8 = -\frac{6}{x}$, or equivalent.
- 1 answer mark for the correct inverse: $g^{-1}(x) = 1 - \sqrt{9-\frac{6}{x}}$.

TIPS

- » Ensure that you use the correct notation, g^{-1}, and give consideration to the domain.
- » Completing the square may be required when finding the inverse of a function containing a quadratic expression.

Question 21a.

Worked solution

Using the quotient rule gives $\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right) = \frac{1 - \log_e(x)}{x^2}$.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 21b.

Worked solution

From **part a.** we have

$$\frac{d}{dx}\left(\frac{\log_e(x)}{x}\right) = \frac{1}{x^2} - \frac{\log_e(x)}{x^2}$$

Thus $\frac{\log_e(x)}{x^2} = \frac{1}{x^2} - \frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)$.

Therefore

$$\int \frac{\log_e(x)}{x^2}\,dx = \int \left(\frac{1}{x^2} - \frac{d}{dx}\left(\frac{\log_e(x)}{x}\right)\right)dx$$

$$= -\frac{1}{x} - \frac{\log_e(x)}{x} + c$$

Since an antiderivative is required, $-\frac{1}{x} - \frac{\log_e(x)}{x}$ is also acceptable.

Mark allocation: 2 marks

- 1 mark for rearranging the result from **part a.** and attempting to integrate both sides.
- 1 mark for the correct answer. The constant of integration, c, may be omitted since a question that asks for an antiderivative does not require it. (When it is omitted, we are giving the case where $c = 0$.)

Question 21c.

Worked solution

Using the result from **part b.** we have

$$\int_1^{e^2} (x)\,dx = \left[-\frac{1}{x} - \frac{\log_e(x)}{x}\right]_1^{e^2}$$

$$= \left(-\frac{1}{e^2} - \frac{\log_e(e^2)}{e^2}\right) - \left(-\frac{1}{1} - \frac{\log_e(1)}{1}\right)$$

$$= \left(-\frac{1}{e^2} - \frac{2}{e^2}\right) - (-1)$$

$$= 1 - \frac{3}{e^2}$$

Mark allocation: 2 marks

- 1 method mark for substituting the bounds after integrating the previous result.
- 1 answer mark for the correct answer.

Question 21d.

Worked solution

The average value of f over the interval $[1, e^2]$ is

$$\frac{1}{e^2-1}\int_1^{e^2} (x)\,dx = \frac{1}{e^2-1}\left(1 - \frac{3}{e^2}\right)$$

$$= \frac{1}{e^2-1}\left(\frac{e^2-3}{e^2}\right)$$

$$= \frac{e^2-3}{e^2(e^2-1)}$$

Mark allocation: 2 marks

- 1 method mark for setting up the expression for the average value: $\frac{1}{e^2-1}\int_1^{e^2} f(x)\,dx$.
- 1 answer mark for the correct answer.

Question 22a.

Worked solution

The sum of the edges is equal to E cm, therefore $6x + 3y = E$ and $y = \frac{E}{3} - 2x$.

Using Pythagoras' theorem, the vertical height, h, of the triangle is

$$h^2 = x^2 - \left(\frac{x}{2}\right)^2 = \frac{3x^2}{4}$$

$$h = \sqrt{\frac{3x^2}{4}} = \frac{\sqrt{3}x}{2}$$

The area of a triangular end is $\frac{1}{2}bh = \frac{1}{2}x\frac{\sqrt{3}x}{2} = \frac{\sqrt{3}x^2}{4}$.

The volume of the prism is

$$V = A_{\text{triangle}} \times y$$
$$= \frac{\sqrt{3}x^2}{4}\left(\frac{E}{3} - 2x\right)$$
$$= \frac{\sqrt{3}\,Ex^2}{12} - \frac{\sqrt{3}x^3}{2}$$
$$= \frac{\sqrt{3}\,Ex^2 - 6\sqrt{3}x^3}{12}$$

Mark allocation: 3 marks

- 1 method mark for finding $y = \frac{E}{3} - 2x$.
- 1 method mark for finding the vertical height of a triangular end: $h = \frac{\sqrt{3}x}{2}$.
- 1 answer mark for finding the area of a triangular end and using it to determine the volume of the prism.

Question 22b.

Worked solution

The maximum volume occurs when $\frac{dV}{dx} = 0$.

$$V = \frac{\sqrt{3}\,Ex^2 - 6\sqrt{3}x^3}{12}$$
$$= \frac{1}{12}(\sqrt{3}\,Ex^2 - 6\sqrt{3}x^3)$$

$$\frac{dV}{dx} = \frac{1}{12}(2\sqrt{3}\,Ex - 18\sqrt{3}x^2)$$
$$= \frac{1}{6}(\sqrt{3}\,Ex - 9\sqrt{3}x^2) = 0$$
$$= \frac{1}{6}x(\sqrt{3}E - 9\sqrt{3}x) = 0$$

$$x = 0,\ 9\sqrt{3x} = \sqrt{3}E$$
$$x = \frac{\sqrt{3}E}{9\sqrt{3}}$$
$$= \frac{E}{9}$$

Consideration of the general shape of the graph of V (a negative cubic function) confirms that $x = \frac{E}{9}$ corresponds to a maximum turning point.

So the maximum volume occurs when $x = \frac{E}{9}$ cm.

Mark allocation: 2 marks

- 1 answer mark for finding $\frac{dV}{dx} = \frac{1}{12}(2\sqrt{3}\,Ex - 18\sqrt{3}x^2)$, or equivalent.
- 1 answer mark for the correct answer: $x = \frac{E}{9}$.

EXAM 2: MULTIPLE CHOICE

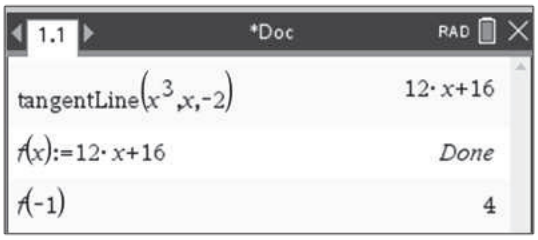
Question 1

Answer: ***B***

Explanatory notes

Use a CAS to find the equation of the tangent when $x = -2$: $y = 12x + 16$.

Check which of the given points satisfies this equation.



Only $(-1, 4)$ satisfies the equation.

Question 2

Answer: ***D***

Explanatory notes

For the average value of the function to be zero, the triangular area bounded by f and the x-axis over the interval $[-p, 0]$ has to equal $\frac{81}{8}$. Therefore

$$\text{Area of triangle} = \frac{1}{2}p^2 = \frac{81}{8}$$

$$p^2 = \frac{81}{4},\ p > 0$$

$$p = \frac{9}{2}$$

Question 3

Answer: ***D***

Explanatory notes

Using the chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{f(2x)}} \times \frac{1}{2\sqrt{f(2x)}} \times f'(2x) \times 2$$

$$= \frac{f'(2x)}{f(2x)}$$

Or, using logarithmic laws, the equation for y can be written as

$$y = \log_e(f(2x)^{\frac{1}{2}}) = \frac{1}{2}\log_e(f(2x)).$$

So $$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{f(2x)} \times f'(2x) \times 2$$

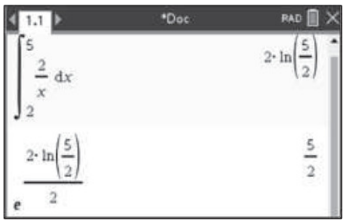
$$= \frac{f'(2x)}{f(2x)}$$

Question 4

Answer: ***A***

Explanatory notes

Using a CAS gives $\frac{5}{2}$.



Question 5

Answer: ***B***

Explanatory notes

$$\int_{-1}^{2}(3 - f(x))\,dx = \int_{-1}^{2}3\,dx - \int_{-1}^{2}f(x)\,dx$$

$$= [3x]_{-1}^{2} - 4$$

$$= 6 + 3 - 4$$

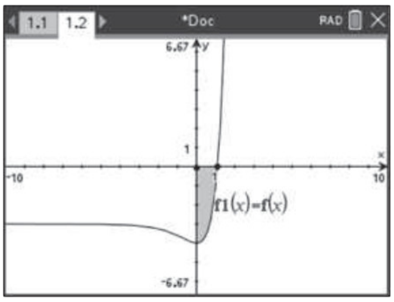
$$= 5$$

Question 6

Answer: ***C***

Explanatory notes

From a sketch drawn on a CAS, the area required can be seen as the shaded region shown below.



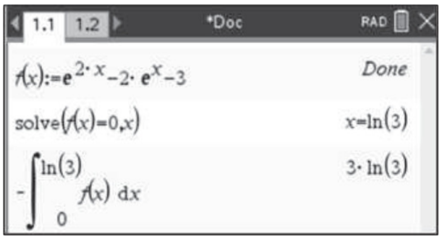
Calculate the exact value of the x-intercept with a CAS by solving

$f(x) = 0$

$x = \log_e 3$

The area is below the x-axis, so

$$A = -\int_0^{\log_e 3} f(x)\,dx = 3\log_e 3$$



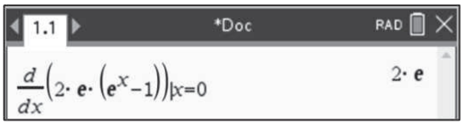
Question 7

Answer: ***D***

Explanatory notes

The rate of change is given by $\frac{dy}{dx}$.

Using a CAS gives $2e$.



Question 8

Answer: ***C***

Explanatory notes

The graph shown is the graph of the derivative of g, that is, the graph of the gradient function. The graph is positive over the interval (a, b), so the graph of g has a positive gradient.

Note: The graph shown does not have an x-intercept or a change of sign over the interval (a, b), therefore there are no turning points or stationary points of inflection over this interval.

Question 9

Answer: ***D***

Explanatory notes

The gradient of the line that passes through $(-3, 2)$ and $(1, 5)$ is $\frac{5-2}{1-(-3)} = \frac{3}{4}$.

Therefore the gradient of the line perpendicular to it is $-\frac{4}{3}$.

Question 10

Answer: ***C***

Explanatory notes

From the coordinates of the given points and the scale of the x-axis you can determine that the coordinates of the other x-intercept must be $\left(\frac{a}{2}, 0\right)$.

$$\begin{aligned}
\text{Average value} &= \frac{1}{2a-0}\int_0^{2a} f(x)\,dx \\
&= \frac{1}{2a}\left(\int_0^{\frac{a}{2}} f(x)\,dx + \int_{\frac{a}{2}}^{\frac{3a}{2}} f(x)\,dx\right) \\
&= \frac{1}{2a}(-\text{area of }\Delta_1 + \text{area of }\Delta_2) \\
&= \frac{1}{2a}\left(-\frac{1}{2}\times\frac{a}{2}\times a + \frac{1}{2}\times a\times a\right) \\
&= \frac{a}{8}
\end{aligned}$$

Note: The integral $\int_0^{\frac{a}{2}} f(x)\,dx$ will produce a negative value, hence we multiply the area of the corresponding triangle by -1.

Alternatively, this question can be answered geometrically by sectioning the specified area into known shapes and calculating their signed areas. (If the area is under the x-axis, the signed area is negative.)

The triangle between $x = 0$ and $x = \frac{a}{2}$ is below the x-axis and hence the signed area $= -\frac{1}{2}\times\frac{a}{2}\times a = -\frac{a^2}{4}$.

The triangle between $x = \frac{a}{2}$ and $x = \frac{3a}{2}$ is double the previously calculated area, so is $\frac{a^2}{2}$.

The total area is $-\frac{a^2}{4} + \frac{a^2}{2} = \frac{a^2}{4}$.

$$\text{Average value} \times 2a = \frac{a^2}{4}$$

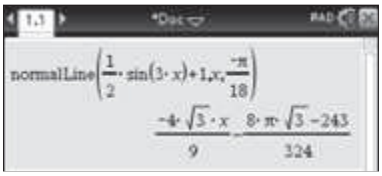
$$\Rightarrow \text{Average value} = \frac{a}{8}$$

Question 11

Answer: ***D***

Explanatory notes

Using a CAS gives $y = -\frac{4\sqrt{3}}{9}x - \frac{8\pi\sqrt{3} - 243}{324}$.



Question 12

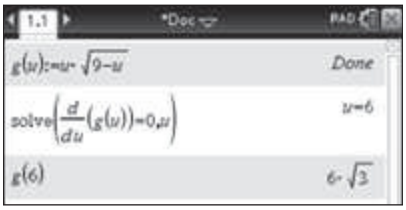
Answer: ***B***

Explanatory notes

The height of the rectangle is $v = \sqrt{9-u}$, hence the area is $A = u\sqrt{9-u}$.

The area is maximised when $\frac{dA}{du} = 0$. Using a CAS shows that this occurs when $u = 6$.

The maximum area is therefore $6\sqrt{3}$.



Question 13

Answer: ***B***

Explanatory notes

From the quotient rule

$$h'(x) = \frac{g'(x)f(x) - f'(x)g(x)}{(f(x))^2}$$

$$\Rightarrow h'(0) = \frac{g'(0)f(0) - f'(0)g(0)}{(f(0))^2}$$

Substitute the given values:

$$h'(0) = \frac{(2)(3) - (-6)(4)}{(3)^2} = \frac{10}{3}$$

Question 14

Answer: ***D***

Explanatory notes

Let $j(x) = \sqrt{g(x)} = (g(x))^{\frac{1}{2}}$ so that $k(x) = f(x) \cdot j(x)$.

From the product rule

$$k'(x) = f'(x)j(x) + f(x)j'(x)$$

$$\Rightarrow k'(0) = f'(0)j(0) + f(0)j'(0)$$

From the chain rule

$$j'(x) = \frac{g'(x)}{2\sqrt{g(x)}} \Rightarrow j'(0) = \frac{g'(0)}{2\sqrt{g(0)}}$$

Therefore

$$k'(0) = f'(0)j(0) + f(0)\frac{g'(0)}{2\sqrt{g(0)}}$$

$$= f'(0)\sqrt{g(0)} + f(0)\frac{g'(0)}{2\sqrt{g(0)}}$$

Substitute the given values:

$$k'(0) = (-6)\sqrt{4} + (3)\frac{2}{2\sqrt{4}} = -\frac{21}{2}$$

Question 15

Answer: ***A***

Explanatory notes

Definition: f is a strictly increasing function over an interval, I, when $f(x_1) > f(x_2)$ for all $x_2 > x_1$, where $x_1, x_2 \in I$.

Answer: $[-2, 2]$

Note: If $f'(x) > 0$ for an interval, then $f(x)$ is strictly increasing over that interval. But the converse is *not* always true.

If a function is strictly increasing over an interval, it is *not* always true that $f'(x) > 0$ for that *entire* interval. $f'(x)$ may equal to zero or may not even exist at some point in the interval.

Still, solving $f'(x) > 0$ will usually be an important step when trying to find the set of points for which $f(x)$ is strictly increasing, but it does not always give the maximal set.

Question 16

Answer: ***A***

Explanatory notes

The graph of $y = f(x)$ has a turning point at $x = 2$ and another within the interval $x \in (-2, -1)$.

Therefore the graph of $y = f'(x)$ has x-intercepts at $x = 2$ and when $x \in (-2, -1)$. This eliminates options B and D.

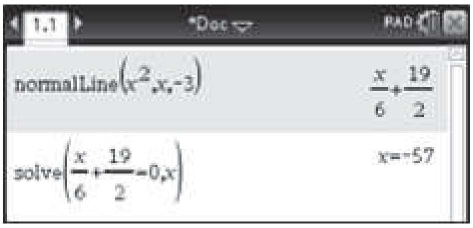
The gradient of f changes from positive to negative at $x = 2$, so $f'(x) > 0$ to the left of $x = 2$ and $f'(x) < 0$ to the right of $x = 2$. In option A the graph of $f'(x)$ is above the x-axis immediately to the left of $x = 2$ and below the x-axis immediately to the right of $x = 2$.

Question 17

Answer: ***C***

Explanatory notes

A CAS may be used. The equation of the line that is perpendicular to $y = x^2$ at point $(-3, 9)$ is $y = \frac{x}{6} + \frac{19}{2}$. This linear function passes through the x-axis when $x = -57$.



Alternatively, $\frac{dy}{dx} = 2x$, so when $x = -3$, $\frac{dy}{dx} = -6$.
Hence the gradient at $x = -3$ is –6 and the gradient of the perpendicular line is $\frac{1}{6}$.

The equation of the perpendicular line is then $y - 9 = \frac{1}{6}(x + 3)$.

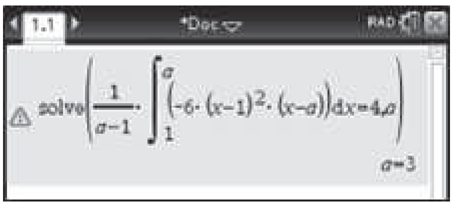
When $y = 0$, $\frac{1}{6}(x + 3) + 9 = 0 \Rightarrow x = -57$,
so $-54 = x + 3 \Rightarrow x = -57$.

Question 18

Answer: ***D***

Explanatory notes

Solve the equation
$\frac{1}{a-1}\int_1^a -6(x-1)^2(x-a)\,dx = 4$ for a.
Using a CAS, we find that $a = 3$.



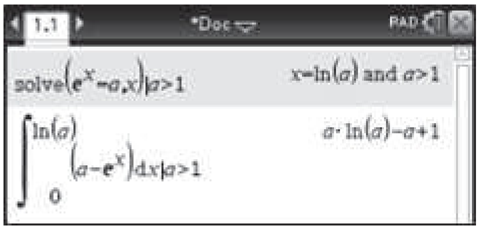
Question 19

Answer: ***C***

Explanatory notes

Find the point of intersection of the graphs $e^x = a$ and $x = \log_e(a)$.

The required area is
$\int_0^{\log_e a} (a - e^x)\,dx = a\log_e(a) - a + 1$.



Question 20

Answer: ***A***

Explanatory notes

<u>Method 1</u>

The average value of f over the interval $[0, 2]$ is $\frac{1}{2}\int_0^2 f(x)\,dx = \frac{5}{2}$. Therefore it must be the case that $\frac{1}{4}\int_{-4}^0 f\left(-\frac{x}{2}\right)\,dx = \frac{5}{2}$, since dilation from or reflection in the y-axis does not change a function's average value.

Therefore $\int_{-4}^0 3f\left(-\frac{x}{2}\right)\,dx = 4 \times 3 \times \frac{5}{2} = 30$.

<u>Method 2</u>

$f\left(-\frac{x}{2}\right)$ is the image of $f(x)$ after reflection in the x-axis and dilation by a factor of 2 from the y-axis (in either order).

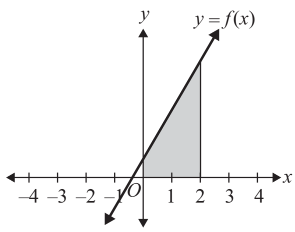
Determine the effect of the transformation on an example area and the corresponding integral.

We can consider *any* function for $f(x)$ to determine the effect of these transformations on an area given by $\int_0^2 f(x)\,dx$.

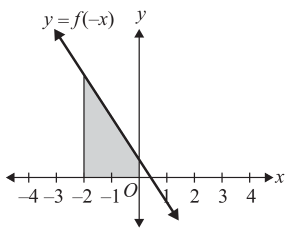
Consider a simple function $f(x)$ where $f(x) > 0$ for $x \in [0, 2]$, as shown below.

The graphs of $y = f(x)$, $y = f(-x)$ and $y = f\left(-\frac{x}{2}\right)$ are shown. The area for $x \in [0, 2]$ is shaded in the first graph and then the altered area is shaded after the relevant transformations.

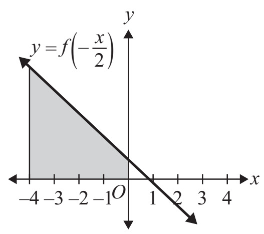
Graph A



Graph B



Graph C



The shaded area shown in graphs A and B is equal. Thus $\int_{-2}^{0} f(-x)\,dx = \int_{0}^{2} f(x)\,dx$.

In graph C the width of the shaded area is doubled (due to the dilation) but the height is unchanged. Thus it is double the original area. Hence $\int_{-4}^{0} f\left(-\frac{x}{2}\right)dx = 2\int_{0}^{2} f(x)\,dx$.

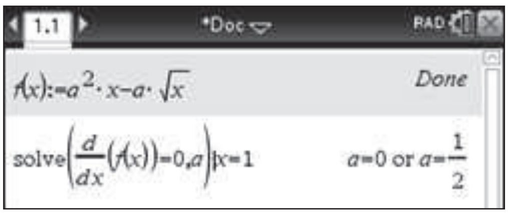
$$\begin{aligned}\int_{-4}^{0} 3f\left(-\frac{x}{2}\right)dx &= 3\int_{-4}^{0} f\left(-\frac{x}{2}\right)dx\\ &= 3\left(2\int_{0}^{2} f(x)\,dx\right)\\ &= 6 \times 4\\ &= 30\end{aligned}$$

Question 21

Answer: ***B***

Explanatory notes

Use a CAS to solve $f'(x) = 0$ for a when $x = 1$.



Since $a > 0$, $a = \frac{1}{2}$.

Question 22

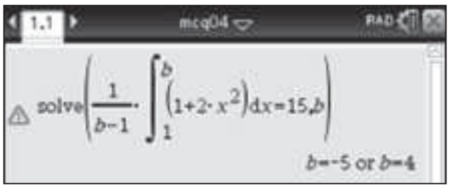
Answer: ***B***

Explanatory notes

The average value of $1 + 2x^2$ over the interval $[1, b]$ is $\frac{1}{b-1}\int_{1}^{b}(1 + 2x^2)\,dx$.

Use a CAS to determine the value of b.

Since $b > 1$, $b = 4$.



Question 23

Answer: ***D***

Explanatory notes

Using the tangent line function on a CAS, we see that the tangent to f at $x = 3$ has the equation $y = 27a - 9ax$. Since this is in gradient–intercept form, the y-intercept is $y = 27a$. Therefore $27a = 9 \Rightarrow a = \frac{1}{3}$.

Question 24

Answer: ***C***

Explanatory notes

$f(2x)$ is the image of $f(x)$ after dilation by a factor of $\frac{1}{2}$ from the y-axis.

<u>Method 1</u>

$$\int_{1}^{3}(f(2x) + x)\,dx = \int_{1}^{3} f(2x)\,dx + \int_{1}^{3} x\,dx$$

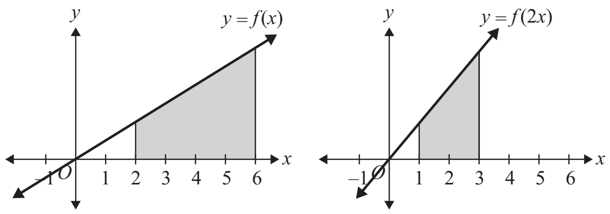
$\int_{1}^{3} x\,dx$ can be easily calculated, so consider $\int_{1}^{3} f(2x)\,dx$.

Determine the effect of the dilation on an example area and the corresponding integral.

We can consider any function for $f(x)$ to determine the effect of the dilation on an area given by $\int_{2}^{6} f(x)\,dx$.

Consider a simple function $f(x)$ where $f(x) > 0$ for $x \in [2, 6]$, as shown below.

The graphs of $y = f(x)$ and $y = f(2x)$ are shown.



The shaded area in the second graph has the same height as the first area, but is half as wide. Thus the second area is half that of the first area.

Hence $\int_1^3 f(2x)\,dx = \frac{1}{2}\int_2^6 f(x)\,dx$ and thus

$$\int_1^3 (f(2x) + x)\,dx = \int_1^3 f(2x)\,dx + \int_1^3 x\,dx$$

$$= \frac{1}{2}\int_2^6 f(x)\,dx + \left[\frac{1}{2}x^2\right]_1^3$$

$$= \frac{1}{2} \times 5 + 4$$

$$= \frac{13}{2}$$

Method 2

Since $\int_2^6 f(x)\,dx = 5$, the average value of $f(x)$ over $[2, 6]$ is $\frac{5}{4}$ and the average value of $f(2x)$ over $[1, 3]$ is $\frac{5}{4}$.

Therefore $\int_1^3 f(2x)\,dx = 2 \times \frac{5}{4} = \frac{5}{2}$ and

$$\int_1^3 (f(2x) + x)\,dx = \frac{5}{2} + \int_1^3 x\,dx$$

$$= \frac{5}{2} + \left[\frac{1}{2}x^2\right]_1^3$$

$$= \frac{5}{2} + \frac{9}{2} - \frac{1}{2}$$

$$= \frac{13}{2}$$

Question 25

Answer: ***A***

Explanatory notes

The x-intercepts of the graph of $f'(x)$ correspond to the stationary points of $f(x)$. Thus the graph of $f(x)$ has stationary points when $x = -2$, $x = 0$ and $x = 2$.

The nature of each stationary point can be determined by looking at the sign of the gradient function on each side of the corresponding x-intercept:

- The stationary point at $x = -2$ is a local maximum (as $f'(x)$ changes from positive to negative).
- The stationary point at $x = 0$ is a point of inflection (as $f'(x)$ is negative on both sides).
- The stationary point at $x = 2$ is a local minimum (as $f'(x)$ changes from negative to positive).

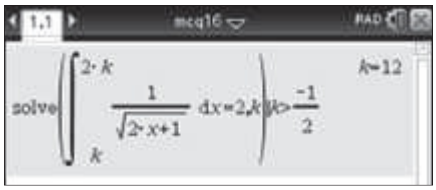
Therefore option A is correct.

Question 26

Answer: ***B***

Explanatory notes

Use a CAS to evaluate the integral and solve the equation. The result is $k = 12$.



Question 27

Answer: ***B***

Explanatory notes

The total area of the trapeziums is

$$\text{Area} \approx \frac{2-0}{2(4)}[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$$

$$= \frac{3\sqrt{2}}{8} + \frac{9}{16}$$

The area between the x-axis and the graph of $y = 2^{-x}$ over the interval $[0, 2]$ is

$$\int_0^2 2^{-x}\,dx = \frac{3}{4\log_e 2}$$

The error is

$$\left(\frac{3\sqrt{2}}{8} + \frac{9}{16}\right) - \frac{3}{4\log_e 2} \approx 0.0108088052232$$

Question 28

Answer: ***A***

Explanatory notes

$f'(x) = 3x^2 - 2x + 4$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{1}{4} = -\frac{1}{4}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -\frac{1}{4} - \frac{\left(\frac{-5}{64}\right)}{\left(\frac{75}{16}\right)} = -\frac{7}{30}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -\frac{7}{30} - \frac{\left(\frac{-13}{27\,000}\right)}{\left(\frac{463}{100}\right)} = -\frac{14\,578}{62\,505}$$

Question 29

Answer: ***D***

Explanatory notes

Use values from **Question 28** (converted to decimals):

$x_1 = -0.25$

$x_2 \approx -0.233\ 333$

$x_3 \approx -0.233\ 229$

These approximations differ by more than 0.00001, so we still have not reached the tolerance required. Continuing to the fourth iteration gives $x_4 \approx -0.233\,229$.

This is the same number as x_3 when rounded to six decimal places, so the estimate has changed by less than the tolerance of 0.00001. (In fact it has changed by less than $0.000\,001$.)

Question 30

Answer: ***B***

Explanatory notes

We are using the algorithm with inputs $f(x) = \frac{12}{x}$, $a = 1$, $b = 5$ and $\text{width} = 1$.

Each iteration of the algorithm calculates the area of a single trapezium, starting on the left, and adds it to the total. So after three iterations we have calculated the areas of three of the four total trapeziums.

$$A_1 = \frac{1 \times (f(1) + f(2))}{2}$$

$$= \frac{12 + 6}{2}$$

$$= 9$$

$$A_2 = \frac{1 \times (f(2) + f(3))}{2}$$

$$= \frac{6 + 4}{2}$$

$$= 5$$

$$A_3 = \frac{1 \times (f(3) + f(4))}{2}$$

$$= \frac{4 + 3}{2}$$

$$= 3.5$$

Therefore, total area is $9 + 5 + 3.5 = 17.5$.

TIP

» This algorithm for the trapezium method calculates the area of each trapezium separately. You may find that other algorithms use the formula given on the VCAA Mathematical Methods formula sheet, which makes use of the fact that most y values are shared by two adjacent trapeziums.

Question 31

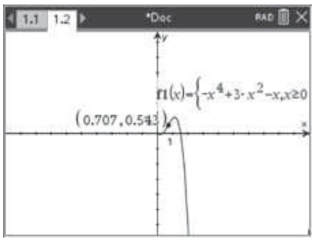
Answer: ***D***

Explanatory notes

Make sure you notice the domain of f is restricted.

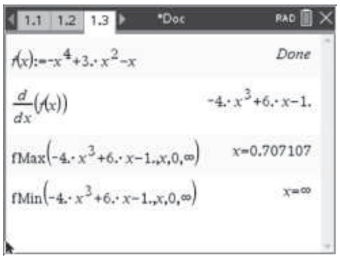
Method 1

A quick approach to finding the point of inflection is to sketch $f(x)$ with a CAS and use the inflection point tool.



Method 2

Find the minimum or maximum of the derivative and check which of the given options has a matching coordinate.



Question 32

Answer: ***D***

Explanatory notes

The derivative of the inverse, $g(x)$, is given by $g'(x) = \frac{1}{f'(g(x))}$.

The gradient of the graph of the tangent to the function g at $x = 3$ is $g'(3)$.

$$g'(3) = \frac{1}{f'(g(3))}, \text{ since } f(5) = 3, g(3) = 5$$

$$= \frac{1}{f'(5)}$$

$$= -\frac{1}{2}$$

$f(x)$ goes through $(5, 3)$, so $g(x)$ also goes through $(3, 5)$. Thus the tangent to g at $g = 3$ passes through $(3, 5)$. The equation of the tangent is

$$y - 5 = -\frac{1}{2}(x - 3)$$

$$y = -\frac{1}{2}x + \frac{13}{2}$$

TIPS

- Remember that the x values of the inverse function are the y values of the original function, and vice versa.
- It is worth including the rule for the derivative of an inverse function in the bound reference you are allowed to take into the examination room.

Area of Study 4 Data analysis, probability and statistics

EXAM 1

Question 1a.i.

Worked solution

Pr(germinating) = $\frac{1}{4}$, therefore Pr(not germinating) = $\frac{3}{4}$.

Assuming that each seed germinates or doesn't germinate independently of any other, then

Pr(3 seeds fail to germinate) = $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 1a.ii.

Worked solution

Method 1

Let X be the random variable '*Number of seeds that germinate*'.

$X \sim \text{Binomial}\left(p = \frac{1}{4}, n = 3\right)$

$$\begin{aligned}\Pr(X = 2) &= \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 \\ &= 3 \times \frac{3}{64} \\ &= \frac{9}{64}\end{aligned}$$

Method 2

List the possible options for two seeds germinating: GGG', $GG'G$, $G'GG$.

Hence there are three ways that exactly two seeds germinate, so $\Pr(X = 2) = 3\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^1 = \frac{9}{64}$.

Mark allocation: 1 mark

- 1 answer mark for correct working that leads to $\frac{9}{64}$.

Question 1b.

Worked solution

For exactly two seeds to germinate in only one of the pots we need exactly two to germinate in the first pot and not the second pot, or vice versa.

$$\begin{aligned}\Pr(2, 2') + \Pr(2', 2) &= 2\left(\frac{9}{64} \times \frac{55}{64}\right) \\ &= \frac{2 \times 55 \times 3^2}{4^3 \times 4^3} \\ &= \frac{110 \times 3^2}{4^6}\end{aligned}$$

Mark allocation: 2 marks

- 1 answer mark for stating $2\left(\frac{9}{64} \times \frac{55}{64}\right)$, or equivalent.
- 1 answer mark for the correct answer expressed in the specified form.

TIP

» **Always consider the order in which things can happen when answering probability questions.**

Question 2

Worked solution

Method 1

$$\begin{aligned}\Pr(A' \cap B) = 0.2 &= \Pr(B) - \Pr(A \cap B)\\ &= \Pr(B) - \Pr(A) \times \Pr(B)\\ &= \Pr(B) - 0.4 \times \Pr(B)\\ &= 0.6 \times \Pr(B)\end{aligned}$$

$$\Pr(B) = \frac{0.2}{0.6} = \frac{1}{3}$$

Since $\Pr(A) = 0.4$, $\Pr(A') = 0.6$.

As A and B are independent, $\Pr(A' \cap B) = 0.2 = \Pr(A') \times \Pr(B) = 0.6 \times \Pr(B)$, giving $\Pr(B) = \frac{0.2}{0.6} = \frac{1}{3}$.

Using $\Pr(B) = \frac{1}{3}$, and noting that A and B are independent, $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$, which gives $\Pr(A \cap B) = 0.4 \times \frac{1}{3} = \frac{2}{15}$.

Method 2

Let x denote $\Pr(A \cap B)$.

$\Pr(B) = \Pr(A' \cap B) + \Pr(A \cap B) = 0.2 + x$

$$\begin{aligned}x &= \Pr(A) \times \Pr(B) = 0.4 \times (0.2 + x)\\ x &= 0.08 + 0.4x\\ 0.6x &= 0.08\\ x &= \frac{0.08}{0.6} = \frac{2}{15}\end{aligned}$$

Therefore $\Pr(A \cap B) = \frac{2}{15}$.

Method 3

A Karnaugh map for this problem is shown below.

	A	A'	
B	m	0.2	n
B'			
	0.4		

This can be used to develop the equations $n = m + 0.2$ and $m = 0.4 \times n$, giving the resulting equation $\Pr(A \cap B) = m = 0.4 \times (m + 0.2)$, which is identical to the equation solved using Method 2.

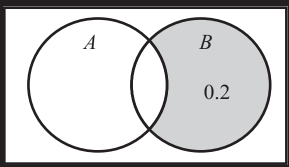
Mark allocation: 2 marks

- 1 method mark for recognising the independence of A and B, such as stating $\Pr(A \cap B) = 0.4 \times (0.2 + \Pr(A \cap B))$, or for calculating $\Pr(B) = \frac{1}{3}$.
- 1 answer mark for the solution: $\Pr(A \cap B) = \frac{2}{15}$.

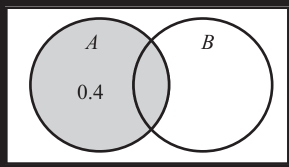
TIP

» Venn diagrams or a Karnaugh map can be used to help visualise the probabilities given in this problem. Venn diagrams would look as shown below.

$\Pr(A' \cap B) = 0.2$



$\Pr(A) = 0.4$



Question 3a.

Worked solution

Method 1

$\Pr(R \geq 1) = 1 - \Pr(BB) = 1 - \frac{4}{5} \times \frac{7}{9} = \frac{17}{45}$

Method 2

$\Pr(RB) + \Pr(BR) + \Pr(RR) = \frac{1}{5} \times \frac{8}{9} + \frac{4}{5} \times \frac{2}{9} + \frac{1}{5} \times \frac{1}{9} = \frac{17}{45}$

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

TIP

» A tree diagram could also be used to help solve this problem.

Question 3b.

Worked solution

$\Pr(\text{first card drawn is red} \mid \text{at least one red card is drawn})$

$= \dfrac{\Pr(\text{first card drawn is red} \cap \text{at least one red card is drawn})}{\Pr(\text{at least one red card is drawn})}$

$= \dfrac{\Pr(\text{first card drawn is red})}{\Pr(\text{at least one red card is drawn})}$

$= \dfrac{\frac{1}{5}}{\left(\frac{17}{45}\right)}$

$= \frac{1}{5} \times \frac{45}{17}$

$= \frac{9}{17}$

Mark allocation: 2 marks

- 1 method mark for simplifying the conditional probability expression to $\frac{\Pr(\text{first card drawn is red})}{\Pr(\text{at least one red card is drawn})}$ or for giving either the numerator, $\frac{1}{5}$, or the denominator, $\frac{17}{45}$.
- 1 answer mark for the correct answer.

Question 4a.

Worked solution

$\mathrm{E}(\hat{P}) = p$, the probability that any single die roll shows a victory symbol. (This formula is included in the VCAA Mathematical Methods formula sheet.)

Thus $\mathrm{E}(\hat{P}) = \frac{3}{8}$ (or 0.375).

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 4b.

Worked solution

The variance of $\hat{P}$ is given by $\frac{p(1-p)}{n}$, where $p = \frac{3}{8}$, $n = 10$.

$$E(\hat{p}) = \frac{p(1-p)}{n} = \frac{\frac{3}{8}\left(1-\frac{3}{8}\right)}{10}$$

$$= \frac{\frac{3}{8} \times \frac{5}{8}}{10}$$

$$= \frac{3}{128}$$

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

TIP

» To help you with this question, you could use the formula for the standard deviation of the sample proportion provided on the VCAA Mathematical Methods formula sheet, since $\mathrm{sd}(\hat{P}) = \sqrt{\mathrm{var}(\hat{P})}$.

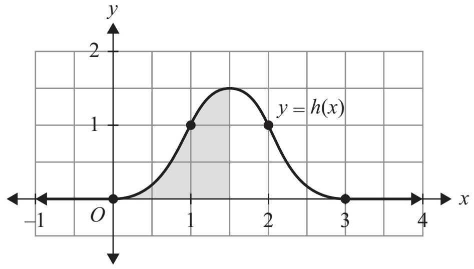
Question 5a.

Worked solution

For $f(x)$ to be a probability distribution function, $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Hence $a\left[\int_0^1 x^2\,dx + \int_1^2 \left(-2\left(x-\frac{3}{2}\right)^2 + \frac{3}{2}\right)dx + \int_2^3 (x-3)^2\,dx\right] = 1$.

From symmetry, the problem can be simplified by recognising that the area over the interval $\left[0, \frac{3}{2}\right]$ is equal to $\frac{1}{2}$ the area over the interval $[0, 3]$.



Thus $a\left[\int_0^1 x^2\,dx + \int_1^{\frac{3}{2}} -2\left(x-\frac{3}{2}\right)^2 + \frac{3}{2}\,dx\right] = \frac{1}{2}.$

$$a\left(\left[\frac{1}{3}x^3\right]_0^1 + \left[-\frac{2}{3}\left(x-\frac{3}{2}\right)^3 + \frac{3}{2}x\right]_1^{\frac{3}{2}}\right) = \frac{1}{2}$$

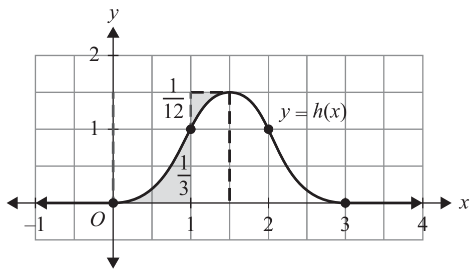
$$a\left(\frac{1}{3} + \left[0+\frac{9}{4}\right] - \left[-\frac{2}{3}\left(-\frac{1}{2}\right)^3 + \frac{3}{2}\right]\right) = \frac{1}{2}$$

$$a\left(\frac{1}{3} + \frac{9}{4} - \frac{1}{12} - \frac{3}{2}\right) = \frac{1}{2}$$

$$a = \frac{1}{2}$$

Alternatively, the area enclosed by $f(x)$ over the interval $[0, 1]$, as well as the area enclosed between the line $y = \frac{3}{2}$ and $f(x)$ over the interval $\left[1, \frac{3}{2}\right]$, are similar to each other, with an area dilation factor of $\frac{1}{4}$ (i.e. a factor of $\frac{1}{2}$ in each direction).

Hence, from $\int_0^1 x^2\,dx = \frac{1}{3}$, this area is $\frac{1}{12}$.



The area of the rectangle enclosed by the line $y = \frac{3}{2}$ and the x-axis on the interval $\left[1, \frac{3}{2}\right]$ is $\frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$.

Hence the area under the curve over the interval $\left[0, \frac{3}{2}\right]$ is given by $\frac{1}{3} + \frac{3}{4} - \frac{1}{12} = 1$ and $a = \frac{1}{2}$.

Mark allocation: 4 marks

- 1 method mark for recognising that the total area under f equals 1, for example, by setting the integral equal to 1.
- 1 method mark for at least one section of correct integral antidifferentiation.
- 1 answer mark for $\int_0^1 x^2\,dx = \frac{1}{3}$.
- 1 answer mark the correct answer.

TIP

» It can save time to look for ways to simplify complex integral expressions by using symmetry or transformations of known areas.

Question 5b.i.

Worked solution

The graph of $f(y)$ is the image of the graph of $h(x)$ after dilation by a factor of $\frac{1}{b}$ from the vertical axis.

From **part a.**, the total area under the graph of h is 2 square units and so will represent a probability density function only if it is dilated by a factor of $\frac{1}{2}$ from either the horizontal or the vertical axis.

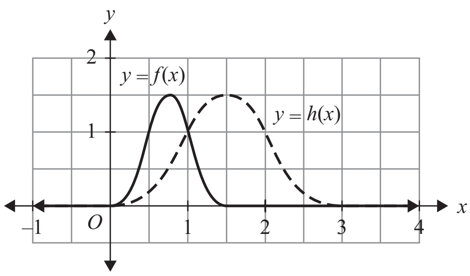
To dilate the function from the vertical axis by a factor of $\frac{1}{2}$, the required value of b is 2.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 5b.ii.

Worked solution



$h(x) > 0$ for $x \in (0, 3)$, so the function $f(x)$ is non-zero when $x \in \left(0, \frac{3}{2}\right)$.

Mark allocation: 1 mark

- 1 answer mark for $\left(0, \frac{3}{2}\right)$, or equivalent.

Question 5b.iii.

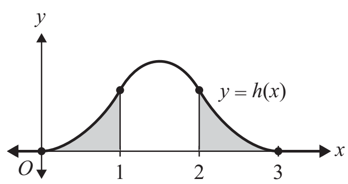
Worked solution

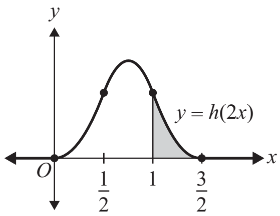
$\Pr(X < 1 \cup Y < 1) = \Pr(X < 1) + \Pr(Y < 1) - \Pr(X < 1) \times \Pr(Y < 1)$, so begin by finding $\Pr(X < 1)$ and $\Pr(Y < 1)$.

Method 1

From earlier, or by calculation, $\int_0^1 h(x)\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$

and by symmetry $\int_2^3 h(x)\,dx = \int_2^3 (x-3)^2\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$.





$$\Pr(X < 1) = \int_0^1 \frac{1}{2}h(x)\,dx = \frac{1}{2}\int_0^1 h(x)\,dx$$

$$= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$\Pr(Y < 1) = \int_0^1 h(2y)\,dy = \int_0^1 h(2x)\,dx = 1 - \int_1^{\frac{3}{2}} h(2x)\,dx$$

By symmetry, $\int_1^{\frac{3}{2}} h(2x)\,dx = \int_0^{\frac{1}{2}} h(2x)\,dx$.

And since $h(2x)$ is the image of $h(x)$ a dilation by a factor of $\frac{1}{2}$ from the vertical axis, the area represented by $\int_0^{\frac{1}{2}} h(2x)\,dx$ is equal to $\frac{1}{2}\int_0^1 h(x)\,dx$.

Hence $\int_0^{\frac{1}{2}} h(2x)\,dx = \frac{1}{2}\int_0^1 h(x)\,dx = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

Hence $\Pr(Y < 1) = 1 - \int_1^{\frac{3}{2}} h(2x)\,dx = 1 - \frac{1}{6} = \frac{5}{6}$.

We can now calculate $\Pr(X < 1 \cup Y < 1)$:

$$\Pr(X < 1 \cup Y < 1) = \Pr(X < 1) + \Pr(Y < 1) - \Pr(X < 1) \times \Pr(Y < 1)$$

$$= \frac{1}{6} + \frac{5}{6} - \frac{1}{6} \times \frac{5}{6}$$

$$= \frac{31}{36}$$

Method 2

The two probabilities can be calculated using integration:

$$\Pr(X < 1) = \int_0^1 \frac{1}{2}x^2\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^1$$

$$= \left[\frac{1}{6}\right] - [0]$$

$$= \frac{1}{6}$$

$$\Pr(Y < 1) = 1 - \int_1^{1.5} (2x-3)^2\,dx$$

$$= 1 - \left[\frac{(2x-3)^3}{6}\right]_1^{1.5}$$

$$= 1 - \left([0] - \left[\frac{(-1)^3}{6}\right]\right)$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

We can now calculate $\Pr(X < 1 \cup Y < 1)$:

$$\Pr(X < 1 \cup Y < 1) = \Pr(X < 1) + \Pr(Y < 1) - \Pr(X < 1) \times \Pr(Y < 1)$$

$$= \frac{1}{6} + \frac{5}{6} - \frac{1}{6} \times \frac{5}{6}$$

$$= \frac{31}{36}$$

Mark allocation: 2 marks

- 1 method mark for stating $\Pr(X < 1) = \frac{1}{6}$ and $\Pr(Y < 1) = \frac{5}{6}$, or for $\Pr(X < 1 \cup Y < 1) = \Pr(X < 1) + \Pr(Y < 1) - \Pr(X < 1) \times \Pr(Y < 1)$.
- 1 answer mark for stating $\Pr(X < 1 \cup Y < 1) = \frac{31}{36}$.

Question 6

Worked solution

$$\text{CI} = \left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$$\text{Width of CI} = \hat{p} + z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} - \left(\hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

$$= 2z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Both samples have the same sample proportion, so $\hat{p}$ is the same for both samples.

Both intervals are 95% confidence intervals, so the z value is the same for both.

Sample A size $= a$ and let sample B size $= b$.

$$\text{Width of CI}_\text{A} = 2z\sqrt{\frac{\hat{p}(1-\hat{p})}{a}}$$

$$\text{Width of CI}_\text{B} = 2z\sqrt{\frac{\hat{p}(1-\hat{p})}{b}}$$

We want CI_B to be a quarter as wide as CI_A, hence

$$2z\sqrt{\frac{\hat{p}(1-\hat{p})}{b}} = \frac{1}{4}\left(2z\sqrt{\frac{\hat{p}(1-\hat{p})}{a}}\right)$$

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{b}} = \frac{1}{4}\sqrt{\frac{\hat{p}(1-\hat{p})}{a}}$$

$$\frac{\hat{p}(1-\hat{p})}{b} = \frac{1}{16}\left(\frac{\hat{p}(1-\hat{p})}{a}\right)$$

$$b = 16a$$

Mark allocation: 2 marks

- 1 method mark for relevant working, such as setting the width of $\text{CI} = 2z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ or $b = (4)^2 a$.
- 1 answer mark for the correct answer.

Question 7a.

Worked solution

$$\Pr(X < 15) = \int_0^{15} \frac{x}{150}\,dx$$

$$= \left[\frac{1}{150}x^2\right]_0^{15}$$

$$= \frac{1}{500} \times 225$$

$$= \frac{1}{2}$$

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 7b.

Worked solution

$\Pr(X < k) = \int_0^k g(x)\,dx = 0.8$ or use $\Pr(X > k) = \int_k^{20} g(x)\,dx = 0.2$

We know from **part a.** that $k > 15$, so it is easiest to use $\Pr(X > k) = \int_k^{20} g(x)\,dx = 0.2$.

$$\int_k^{20} \frac{20 - x}{50}\,dx = 0.2$$

$$\frac{1}{50}\int_k^{20} (20 - x)dx = 0.2$$

$$\frac{1}{50}\left[20x - \frac{1}{2}x^2\right]_k^{20} = 0.2$$

$$400 - 200 - \left(20k - \frac{1}{2}k^2\right) = 10$$

$$\frac{1}{2}k^2 - 20k + 190 = 0$$

$$k^2 - 40k + 380 = 0$$

$$k = \frac{40 \pm \sqrt{1600 - 1520}}{2}$$

$$= \frac{40 \pm \sqrt{80}}{2}$$

$$= \frac{40 \pm 4\sqrt{5}}{2}$$

$$= 20 \pm 2\sqrt{5}$$

Since $k < 20$, $k = 20 - 2\sqrt{5}$.

Mark allocation: 3 marks

- 1 answer mark for $\Pr(X < k) = \int_0^k g(x)\,dx = 0.8$ or $\Pr(X > k) = \int_k^{20} g(x)\,dx = 0.2$.
- 1 method mark for an integral evaluated to produce a quadratic equation and an equation set to zero, such as $\frac{1}{2}k^2 - 20k + 190 = 0$. Allow any small error.
- 1 answer mark for the correct answer.

Question 7c.

Worked solution

$h(x) = pg\left(\frac{x}{q}\right)$, hence the graph of $y = g(x)$ is the image of $y = f(x)$ after dilation by a factor of p from the horizontal axis and a factor of q from the vertical axis.

Mark allocation: 2 marks

- 1 answer mark for one dilation factor correctly described.
- 1 answer mark for the other dilation factor correctly described.

Question 7d.

Worked solution

Only the dilation from the vertical axis changes the location of k.

The image of $k = 20 - 2\sqrt{5}$ after dilation by factor of q is $k_{new} = q(20 - 2\sqrt{5})$, thus

$q(20 - 2\sqrt{5}) = 16$

$$q = \frac{16}{20 - 2\sqrt{5}}$$

$$= \frac{8}{10 - \sqrt{5}} \qquad \left(\text{or } \frac{8(10 - \sqrt{5})}{95}\right)$$

Since $h(x)$ is a probability density function, the total area under the graph of $h(x)$ must equal 1. Since the area under the graph of $y = g(x)$ has been dilated from the vertical axis by a factor of q, to keep the area equal to 1 we need to dilate by a factor of $\frac{1}{q}$ from the horizontal axis.

Hence $p = \frac{1}{q} = \frac{10 - \sqrt{5}}{8}$.

Mark allocation: 2 marks

- 1 method mark for $p = \frac{1}{q}$, $q = \frac{1}{p}$, or one correct value of p or q.
- 1 answer mark if both p and q are correct.

EXAM 2: MULTIPLE CHOICE

Question 1

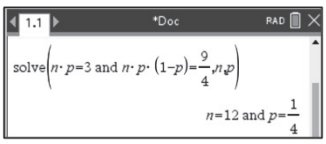
Answer: ***D***

Explanatory notes

$E(X) = np = 3$ and $Var(X) = np(1 - p) = \frac{9}{4}$

These simultaneous equations can be solved with a CAS or by hand.

$n = 12, p = \frac{1}{4}$



So

$$\Pr(X = 1) = \binom{12}{1}(p)^1(1 - p)^{11} = 12\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^{11}$$

Note: BinomPdf on a CAS does not give the answer in the required form.

TIP

» **The formulas for $E(X)$ and $Var(X)$ are on the VCAA Mathematical Methods formula sheet.**

Question 2

Answer: ***A***

Explanatory notes

$$\Pr(X < x) = \Pr\left(Z < \frac{x - \mu}{\sigma}\right)$$

$$\Pr(X < 46) = \Pr\left(Z < \frac{46 - 60}{7}\right)$$

$$= \Pr(Z < -2)$$

$$= \Pr(Z > 2), \text{ using symmetry}$$

Question 3

Answer: ***D***

Explanatory notes

$\mathrm{E}(X) = \frac{40}{7}$, hence

$np = \frac{40}{7}$ Equation (1)

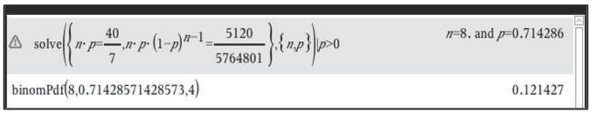
$\Pr(X = 1) = \frac{5120}{5\ 764\ 801}$

$^{n}C_{1}p^{1}(1-p)^{n-1} = \frac{5120}{5\ 764\ 801}$

$\Rightarrow np(1-p)^{n-1} = \frac{5120}{5\ 764\ 801}$ Equation (2)

Using a CAS to solve equations (1) and (2) simultaneously gives $n = 8$ and $p = \frac{5}{7}$.

Now use the CAS to find $\Pr(X = 4)$:

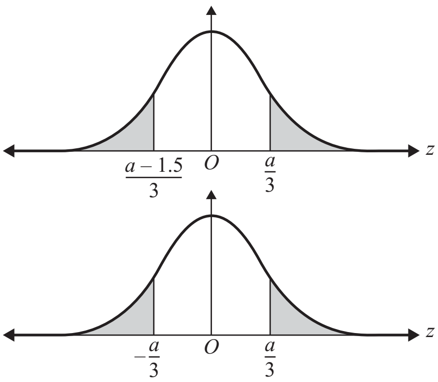


Answer: 0.1214

Question 4

Answer: ***D***

Explanatory notes



Method 1

$X = a \Rightarrow Z = \frac{a-1.5}{0.4}$, therefore

$\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$

$\Rightarrow \Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z > \frac{a}{3}\right)$

By considering symmetry,

$\frac{a-1.5}{0.4} = -\frac{a}{3} \Rightarrow a = \frac{45}{34}$.

Alternatively, express both inequalities in the same direction:

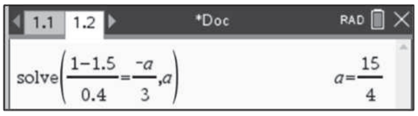
$\Pr\left(Z < \frac{a-1.5}{0.4}\right) = \Pr\left(Z < -\frac{a}{3}\right)$

Therefore $\frac{a-1.5}{0.4} = -\frac{a}{3}$.

Solving with a CAS gives $a = \frac{45}{34}$.

Method 2

Use a CAS to test the equality of $\Pr(X < a) = \Pr\left(Z > \frac{a}{3}\right)$ for each option.



This method is inefficient.

Question 5

Answer: ***D***

Explanatory notes

Use the standard normal $Z \sim N(0, 1)$:

$$\Pr(Z > 120) = 0.2$$

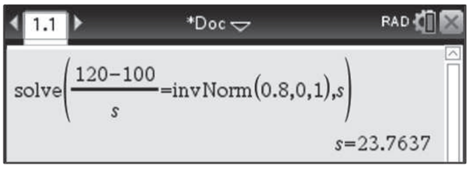
$$\Pr\left(Z > \frac{120-100}{\sigma}\right) = 0.2$$

$$\frac{120-100}{\sigma} = 0.8416$$

$$\sigma = 23.7637$$

So σ is closest to 24.

This can be done in a single line using a CAS:



Question 6

Answer: ***B***

Explanatory notes

Since $\Pr(B) = 0.3$, it follows that $\Pr(B') = 1 - 0.3 = 0.7$. Further, since A and B are independent events,

$$\Pr(A \cap B') = \Pr(A) \times \Pr(B')$$
$$= 0.4 \times 0.7 = 0.28$$

So $\Pr(A \cup B') = \Pr(A) + \Pr(B') - \Pr(A \cap B')$

$$= 0.4 + 0.7 - 0.28$$
$$= 0.82$$

Question 7

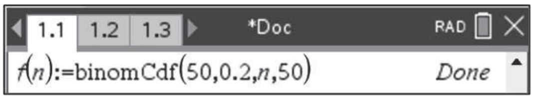
Answer: ***B***

Explanatory notes

Let $X \sim \text{Bi}(50,\ 0.20)$. We want to find the smallest value of n such that $\Pr(X \geq n) = 0.3$.

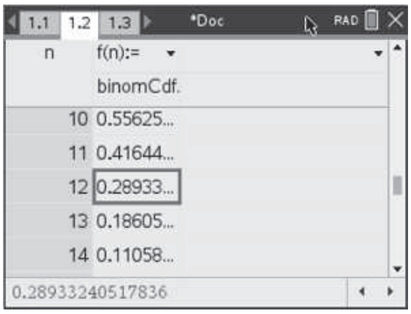
Method 1: Create a table of values (recommended method).

Define $f(n)$ on a CAS as shown, using n as the lower bound.



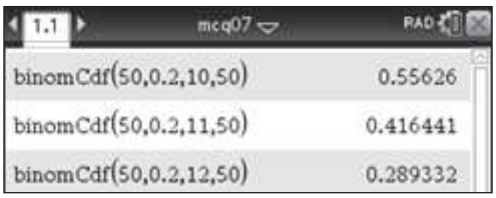
Open a **Lists** and **Spreadsheets** page and paste in $f(n)$ using Ctrl T. Find the lowest value n that produces a probability (i.e. a value of $f(n)$) that is less than 30%.

The answer is $n = 12$.



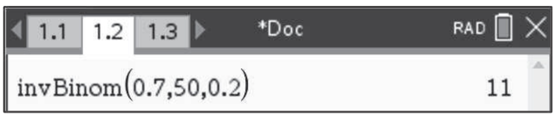
Method 2: Use trial and error to find that $n = 12$.

Method 3: Use the inverse binomial function on a CAS.



In order to use this function, we must first express the equation in the form $\Pr(X \leq x)$.

$\Pr(X \geq n) = 0.3 \Rightarrow \Pr(X \leq n - 1) = 0.7$. (Remember that X is discrete.)



Thus $n - 1 = 11 \Rightarrow n = 12$.

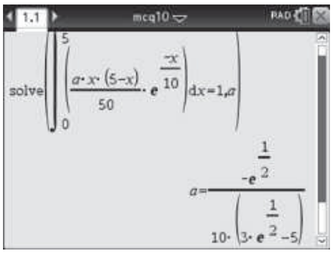
Question 8

Answer: ***C***

Explanatory notes

If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^5 \frac{ax(5-x)}{50} e^{-x/10}\,dx = 1$.

Use a CAS to determine the value of a.



Thus $a = \frac{\sqrt{e}}{50 - 30\sqrt{e}}$.

Question 9

Answer: ***B***

Explanatory notes

Let $X \sim \text{N}(\mu, 3^2)$ and $Z \sim \text{N}(0, 1)$.

$\Pr(X < 10) = 0.9$, so $\Pr\left(Z < \frac{10-\mu}{3}\right) = 0.9$.

Use a CAS to solve for μ.

Thus $\mu = 6.2$, correct to one decimal place.

Question 10

Answer: ***B***

Explanatory notes

$$\Pr(3 \text{ red} \,|\, \text{at least 2 red}) = \frac{\Pr(3 \text{ red} \cap \text{at least 2 red})}{\Pr(\text{at least 2 red})}$$

$$= \frac{\Pr(3 \text{ red})}{\Pr(\text{at least 2 red})}$$

$$= \frac{\Pr(3 \text{ red})}{\Pr(2 \text{ red}) + \Pr(3 \text{ red})}$$

Since the sampling is without replacement,

$$\frac{\Pr(3 \text{ red})}{\Pr(2 \text{ red}) + \Pr(3 \text{ red})} = \frac{{}^6C_3}{{}^6C_2\,{}^4C_1 + {}^6C_3\,{}^4C_0}$$

Alternatively, consider all the possibilities:

$$\frac{\Pr(3\text{ red})}{\Pr(\text{at least 2 red})} = \frac{\Pr(RRR)}{\Pr(RRB, RBR, BRR, RRR)}$$

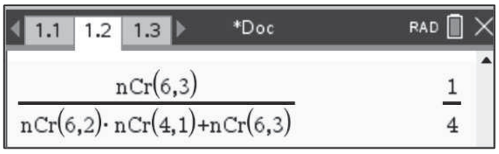
$$\Pr(RRR) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$$

$$\Pr(RRB) = \frac{6}{10} \times \frac{5}{9} \times \frac{4}{8} = \frac{1}{6}$$

$$\Pr(RBR) = \frac{6}{10} \times \frac{4}{9} \times \frac{5}{8} = \frac{1}{6}$$

$$\Pr(BRR) = \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8} = \frac{1}{6}$$

Thus $\frac{\Pr(RRR)}{\Pr(RRB, RBR, BRR, RRR)} = \frac{\frac{1}{6}}{4 \times \frac{1}{6}} = \frac{1}{4}$.



Alternatively, consider that once two red marbles are removed, there is an even number of red and blue marbles remaining. Since all four possible outcomes with at least two red marbles must have the same probability, the answer is $\frac{1}{4}$.

Question 11

Answer: ***B***

Explanatory notes

First find the values of $\hat{p}$ and n.

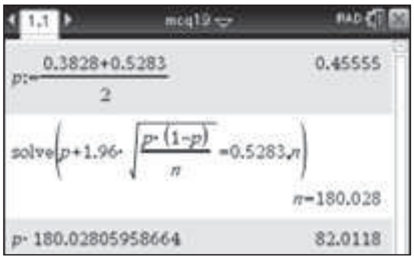
$\hat{p}$ lies in the middle of the confidence interval, so $\hat{p} = \frac{0.3828 + 0.5283}{2} = 0.45555$.

Then solve the equation

$\hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.5283$ for n (the sample size).

Thus $n = 180$ (to the nearest integer).

The number of students who study Chemistry in this random sample is $180 \times 0.4555 = 82$ (to the nearest integer).



TIP

» **Read the question carefully. The question asks for the number of students studying Chemistry, not the number of students in the sample.**

Question 12

Answer: ***B***

Explanatory notes

Option A is reasonable. The true proportion may lie outside the confidence intervals.

Option B is incorrect. Confidence intervals after the sample has been taken do not give the probability of the true value being in the interval (or above it). The 95% confidence level is the probability that a confidence interval calculated from a random sample contains the true proportion. However, once the random sample has been taken, its confidence interval either contains the true value or does not.

Option C is reasonable. 0.6 is the mid point of the confidence interval and so is the sample proportion.

Option D is reasonable. This is the definition of a 95% confidence interval.

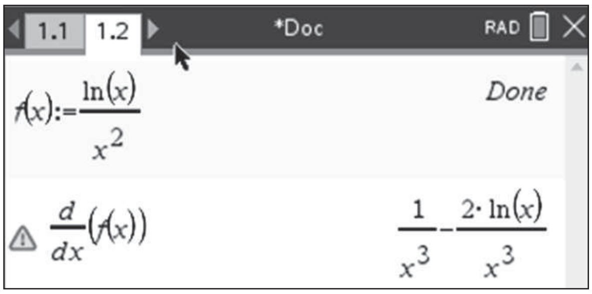
Extended response Exam 2

Question 1a.

Worked solution

$$f'(x) = \frac{1}{x^3} - \frac{2\log_e(x)}{x^3}$$

This can be found using a CAS:



Mark allocation: 1 mark

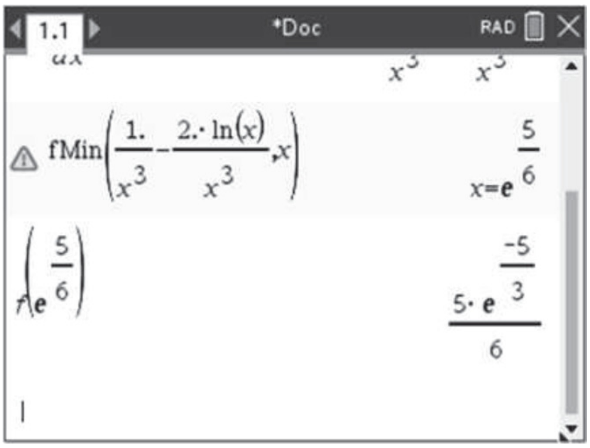
- 1 mark for the correct answer.

Question 1b.i.

Worked solution

Method 1

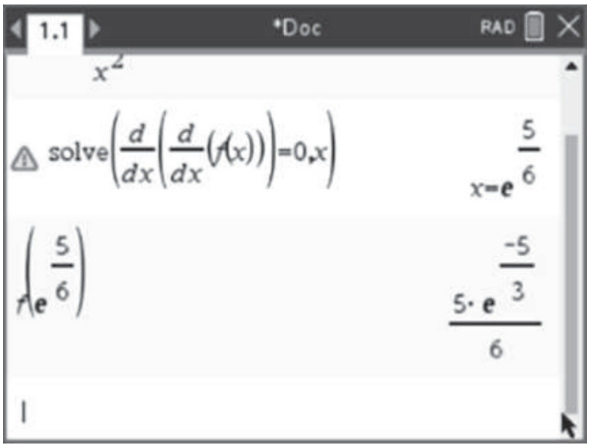
Notice that the inflection point occurs at the point with the minimum gradient. So a CAS can be used to find the x value for which the derivative is minimum, and then substitute this into the original function to find the y value.



The coordinates of A are $\left(e^{\frac{5}{6}}, \frac{5}{6}e^{-\frac{5}{3}}\right)$.

Method 2

Use the fact that all points of inflection have a second derivative value of zero: $f''(x) = 0$.



Mark allocation: 2 marks

- 1 mark for the correct x value.
- 1 mark for the correct y value.

TIPS

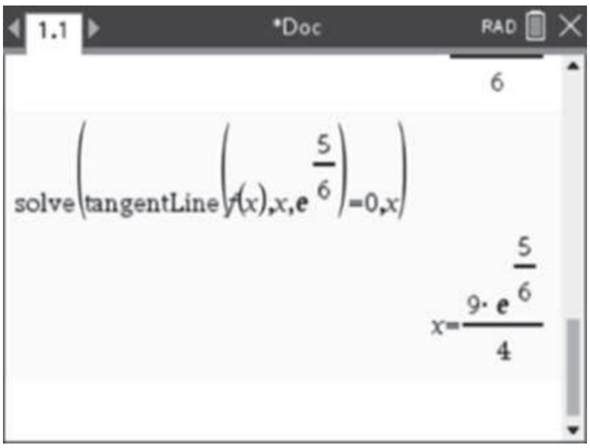
» The CAS graphing screen can find inflection points (as decimal values), but for exact values you will need to find the maximum or minimum of the derivative or use the second derivative.

» All inflection points must have $f''(x) = 0$, but not all points with $f''(x) = 0$ are inflection points.

Question 1b.ii.

Worked solution

Use a CAS to find the point where the tangent to f at A intersects the x-axis.



Therefore $b = \frac{9}{4}e^{\frac{5}{6}}$.

Mark allocation: 1 mark

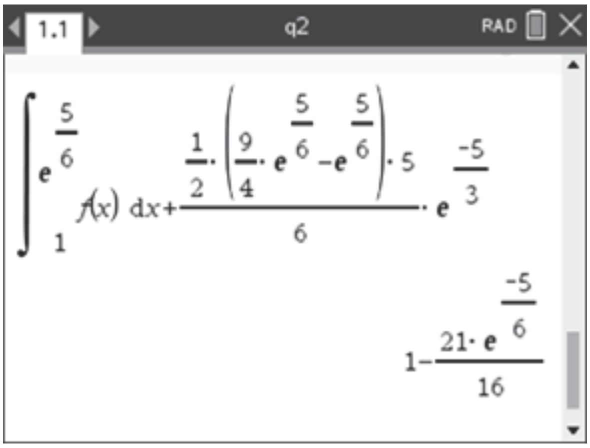
- 1 mark for the correct answer.

Question 1c.

Worked solution

The area of the shaded region is made up of an area under the curve and a triangle under the tangent, that is:

$$\int_1^{e^{\frac{5}{6}}} f(x)\,dx + \frac{1}{2}\left(\frac{9}{4}e^{\frac{5}{6}} - e^{\frac{5}{6}}\right) \times \frac{5}{6}e^{-\frac{5}{3}} = 1 - \frac{21}{16}e^{-\frac{5}{6}}$$



Alternatively, the area can be calculated as the area under the curve and the area under the tangent:

$$\int_1^{\frac{9}{4}e^{\frac{5}{6}}} f(x)\,dx - \int_{e^{\frac{5}{6}}}^{\frac{9}{4}e^{\frac{5}{6}}} \left(\frac{3}{2}e^{-\frac{5}{3}} - \frac{2e^{\frac{-5}{2}}x}{3}\right) dx = 1 - \frac{21}{16}e^{-\frac{5}{6}}$$

Mark allocation: 2 marks

- 1 mark for determining an integral expression with correct terminals.
- 1 mark for the correct answer.

Question 2a.

Worked solution

Substitute $(4, -108)$ into the equation and solve for a.

$-108 = a(4-6)(4-1)(4+2)$

$-108 = -36a$

$a = \frac{-108}{-36} = 3$

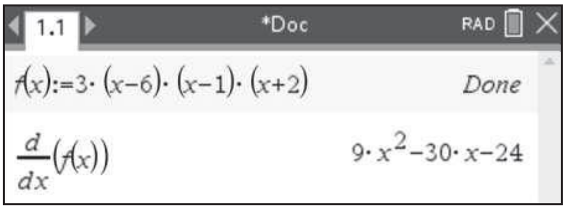
Mark allocation: 1 mark

- 1 answer mark for appropriate working to verify that $a = 3$.

Question 2b.

Worked solution

Use CAS to obtain $f'(x)$.



$f'(x) = 9x^2 - 30x - 24$

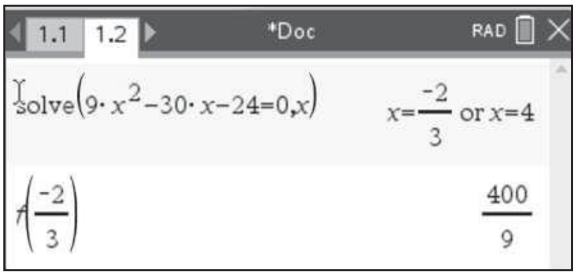
Mark allocation: 1 mark

- 1 answer mark for the correct rule for the derivative.

Question 2c.

Worked solution

Solve $f'(x) = 0$ to find the x-coordinate of the local maximum and then use substitution to find the y-coordinate.



Local max is at $\left(-\frac{2}{3}, \frac{400}{9}\right)$

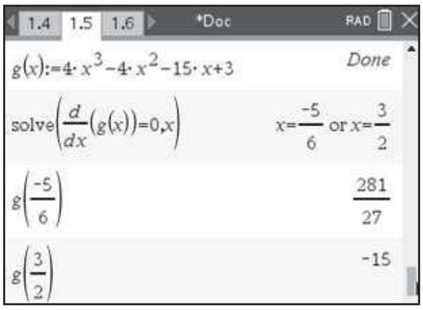
Mark allocation: 1 mark

- 1 answer mark for the correct coordinates.

Question 2d.

Worked solution

Use CAS to find the stationary points of g.



Local max of g: $\left(-\frac{5}{6}, \frac{281}{27}\right)$, local min of $g\left(\frac{3}{2}, -15\right)$.

The required transformations map the stationary points from g to the corresponding points from f.

i.e. $\left(-\frac{5}{6}, \frac{281}{27}\right) \rightarrow \left(-\frac{2}{3}, \frac{400}{9}\right)$ and $\left(\frac{3}{2}, -15\right) \rightarrow (4, -108)$

Let the transformation be $T(x, y) = (ax + b, cy + d)$.

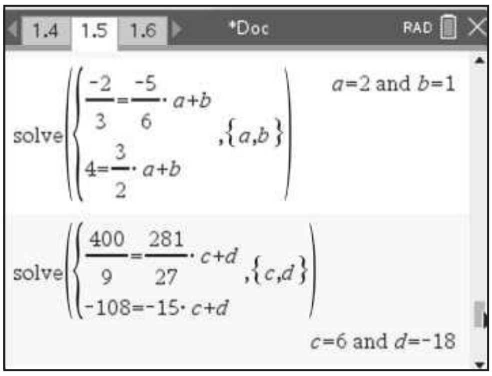
Hence, $x' = ax + b \qquad y' = cy + d$

Substituting in each pair of corresponding x values and each pair of y values produces the following pairs of simultaneous equations.

$-\frac{2}{3} = -\frac{5}{6}a + b$ and $4 = \frac{3}{2}a + b$

$\frac{400}{9} = \frac{281}{27}c + d$ and $-108 = -15c + d$

Use CAS to solve the above pairs of simultaneous equations.



Hence, $a = 2, b = 1, c = 6, d = -18$. Thus, $x' = 2x + 1$ and $y' = 6y - 18$.

The transformations are:

Dilation from the vertical axis by a factor of 2

Dilation from the horizontal axis by a factor of 6

Translation 1 unit right

Translation 18 units down.

Mark allocation: 3 marks

- 1 answer mark for the coordinates of the stationary points of g.
- 1 method mark for simultaneous equations formed, or other suitable method, or 2 correct transformations given.
- 1 answer mark for 4 correct transformations described.

Question 3a.

Worked solution

The area $OABC$ is $15 \times 12 = 180\text{ cm}^2$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3b.

Worked solution

Note that the y-intercept of the parabola is $(0, 8)$. Therefore $g(0) = 225a = 8 \Rightarrow a = \frac{8}{225}$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3c.

Worked solution

The rule for the distance between the two functions is

$$d(x) = 10 + 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}(x - 15)^2$$

or

$$\begin{aligned} d(x) &= 10 + 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}(x - 15)^2 \\ &= 2\sin\left(\frac{2\pi x}{5}\right) - \frac{8}{225}x^2 + \frac{16}{15}x + 2 \end{aligned}$$

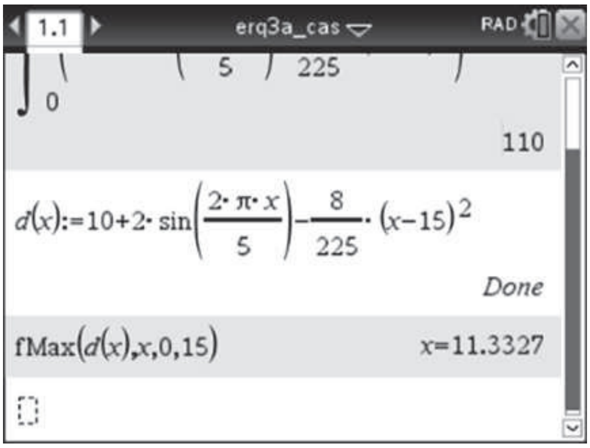
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3d.

Worked solution

Use a CAS to determine when the maximum value of $d(x)$ occurs over the interval $[0, 15]$.



Answer: $x = 11.333$

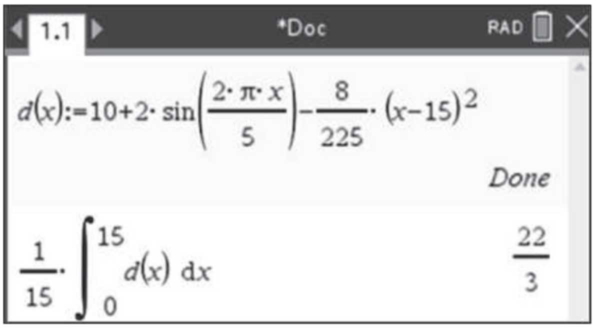
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 3e.

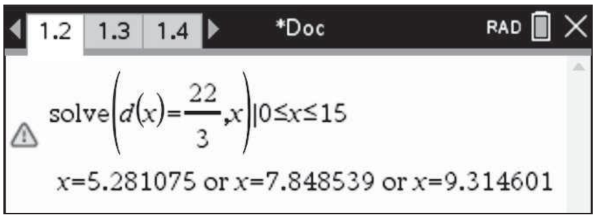
Worked solution

The average distance between components is given by $\frac{1}{15-0}\int_0^{15} d(x)\,dx$.
Use a CAS to evaluate the expression.



Average distance $= \frac{22}{3}$

Use a CAS to solve the equation for x.



Answer: $x = 5.281$, $x = 7.849$ and $x = 9.135$

Mark allocation: 3 marks

- 1 mark for finding $d(x) = \frac{22}{3}$.
- 2 marks for all three correct answers.

Question 4a.

Worked solution

The period of $y = 30\sin(3\pi t)$ is $\frac{2\pi}{3\pi} = \frac{2}{3}$ and the period of $y = 15\sin(2\pi t)$ is 1.

The lowest common multiple of $\frac{2}{3}$ and 1 is 2.

Therefore the period of $x(t)$ is 2.

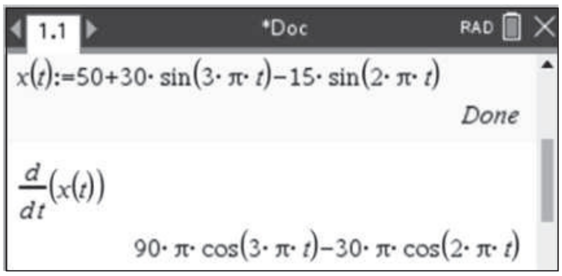
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 4b.

Worked solution

A CAS can be used:



$x'(t) = 90\pi \cos(3\pi t) - 30\pi \cos(2\pi t)$

Mark allocation: 1 mark

- 1 mark for the correct derivative.

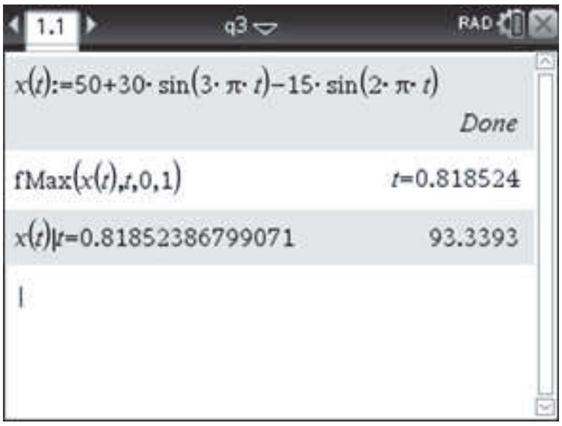
TIP

» It is not necessary to factorise the result.

Question 4c.

Worked solution

Use the maximum function in a CAS.



Alternatively, solve $x'(t) = 0$.

The maximum value of $x(t)$ is 93.34 and this first occurs at $t = 0.82$.

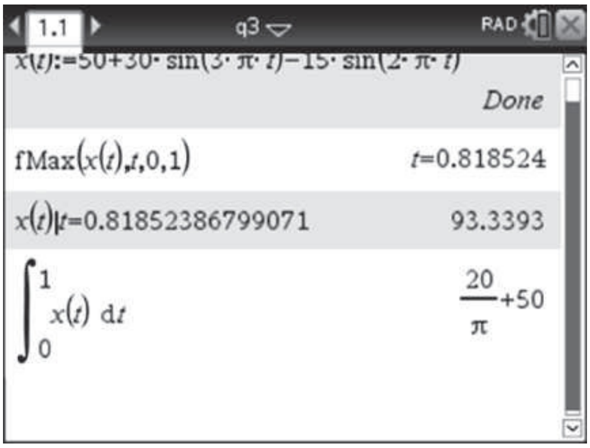
Mark allocation: 2 marks

- 1 mark for correctly calculating the maximum value.
- 1 mark for correctly calculating the value of t.

Question 4d.

Worked solution

The area is found by evaluating the integral $\int_0^1 x(t)\,dt = \frac{20}{\pi} + 50$.



Mark allocation: 1 mark

- 1 mark for the correct answer.

TIP

» **Unless answering to a number of decimal places specified in the question, always give exact answers.**

Question 4e.

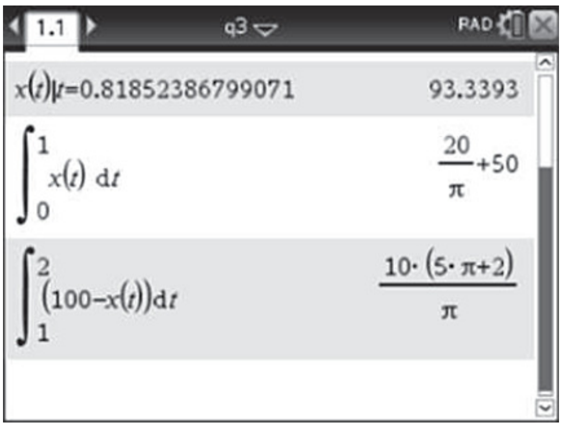
Worked solution

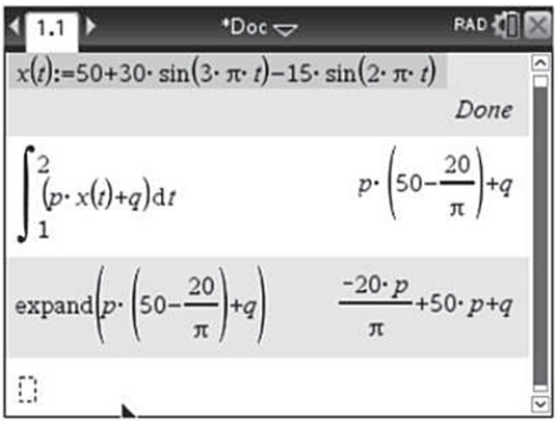
By inspecting the graph, it can be seen that a reflection of the function $x(t)$ in the t-axis followed by a shift upwards of 100 units produces a function with the required property.

$\int_1^2 (100 - x(t))\,dt = \frac{20}{\pi} + 50$

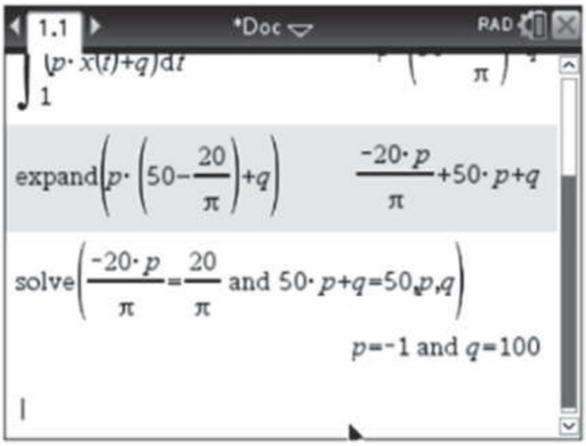
Therefore $p = -1$ and $q = 100$.

Alternatively, calculate the integral using a CAS.





Then equate the terms containing π and separately equate the remaining terms (since the pronumerals cannot be irrational).



Mark allocation: 2 marks

- 1 mark for correctly calculating p.
- 1 mark for correctly calculating q.

Question 5a.

Worked solution

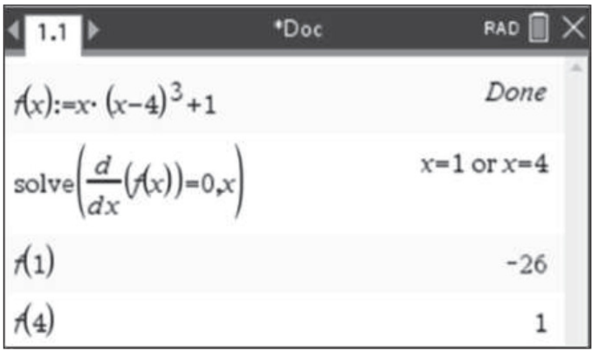
To determine the stationary points, solve

$f'(x) = 0 \Rightarrow 4(x-4)^2(x-1) = 0$

$\Rightarrow x = 1, x = 4$

$f(1) = -26$ and $f(4) = 1$.

So $(1, -26)$ and $(4, 1)$ are the stationary points.



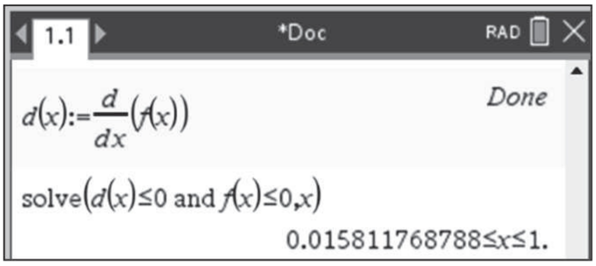
Mark allocation: 2 marks

- 1 mark for the correct x values.
- 1 mark for the correct y values.

Question 5b.

Worked solution

Use a CAS to solve $f'(x) \le 0$ and $f(x) \le 0$ simultaneously.



Answer: $0.016 \le x \le 1$

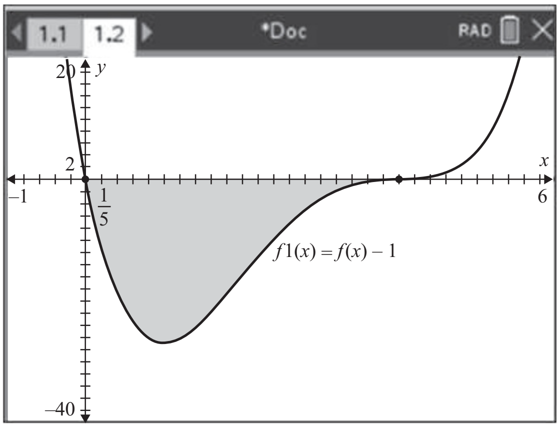
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 5c.

Worked solution

Begin by looking at the graph of $y = f(x) - 1$ to determine where the bounded area is located.



The graph of $y = f(x) - 1$ has x-intercepts at $x = 0$ and $x = 4$, so the area can be calculated as

$$\text{Area} = -\int_0^4 f(x)\,dx$$

Alternatively, the area can be calculated using

$$\text{Area} = \int_4^0 f(x)\,dx \quad \text{or} \quad = \left|\int_0^4 f(x)\,dx\right|$$

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIPS

- » When calculating areas, always remember to look at the graph to determine whether any part of the area is below the x-axis.
- » Rather than writing the rule for the function in the integral, it is more efficient to use $f(x)$.

Question 5d.i.

Worked solution

Dilation by a factor of $\frac{1}{2}$ in the x direction (or from the y-axis), followed by a translation of 1 unit down.

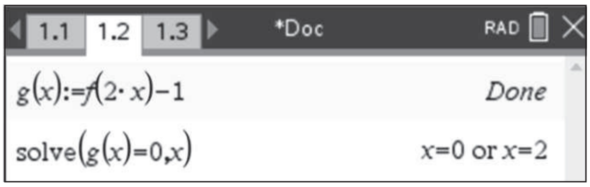
Mark allocation: 2 marks

- 1 mark for the dilation specified correctly.
- 1 mark for the translation specified correctly.

Question 5d.ii.

Worked solution

Method 1: Solve $f(2x) - 1$ with a CAS.



Method 2: Halve the x-intercepts of $y = f(x) - 1$.

Answer: $(0, 0)$ and $(2, 0)$

Mark allocation: 1 mark

- 1 mark for stating both sets of coordinates correctly.

TIP

» The question asks for coordinates, so ensure you answer in this way.

Question 5e.

Worked solution

Stationary points of $f(x)$ occur when $x = 1$ and $x = 4$.

Stationary points of $g(x) = f(px + q)$ occur when $x = 3\sqrt{2}$ and $x = 5\sqrt{2}$.

We are told that $p > 0$, so there is no reflection in the y-axis. Hence the stationary point at $x = 3\sqrt{2}$ is the image of the point at $x = 1$ and the stationary point at $x = 5\sqrt{2}$ is the image of the point at $x = 4$.

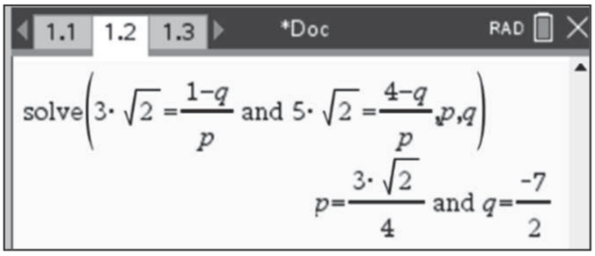
Since $g(x) = f(px + q)$ is the image equation, we can write $y' = f(px' + q)$.

Equating equivalent parts with $y = f(x)$, we have $x = px' + q$.

Hence $x' = \dfrac{x - q}{p}$.

Substituting $x = 1, x' = 3\sqrt{2}$ and $x = 4, x' = 5\sqrt{2}$, we get $3\sqrt{2} = \dfrac{1-q}{p}$ and $5\sqrt{2} = \dfrac{4-q}{p}$.

Solve these simultaneous equations for p and q with a CAS.



Mark allocation: 2 marks

- 1 mark for an appropriate process.
- 1 mark for the correct answer.

Question 6a.

Worked solution

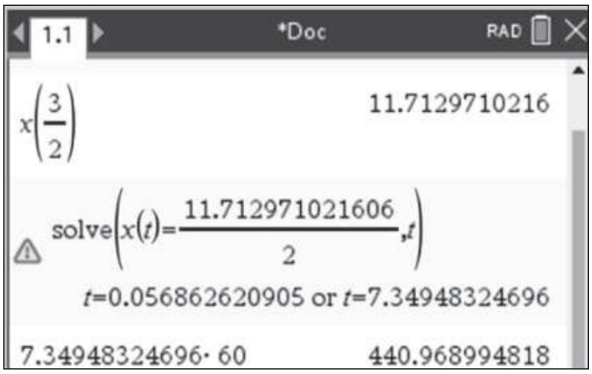
$x_{max} = f\left(\dfrac{3}{2}\right)$, and half the maximum is $\dfrac{f\left(\frac{3}{2}\right)}{2} = \dfrac{11.71\ldots}{2}$

Use a CAS to solve $x(t) = \dfrac{f\left(\frac{3}{2}\right)}{2} = \dfrac{11.71\ldots}{2}$.

$t = 0.056\ldots$ or $t = 7.349\ldots$

We want to determine when the concentration is falling, therefore the larger value of t is required: $t = 7.349\ldots$

Convert this to minutes by multiplying by 60.



Answer: 441 minutes

Mark allocation: 2 marks

- 1 mark for finding half the maximum: $\dfrac{f\left(\frac{3}{2}\right)}{2}$ or $11.71\ldots$
- 1 mark for the correct answer.

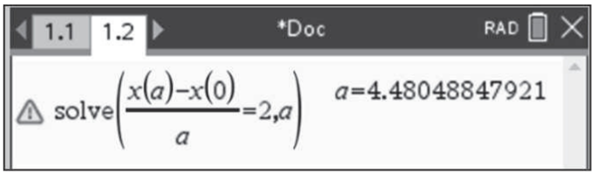
TIP

» Graphing the function allows you to visualise how the blood concentration changes over time. It also provides a way to check your answer.

Question 6b.

Worked solution

Use a CAS to solve $\text{Average rate of change} = \dfrac{x(a) - x(0)}{a - 0} = 2.$



Answer: $a = 4.48$ hours

Mark allocation: 2 marks

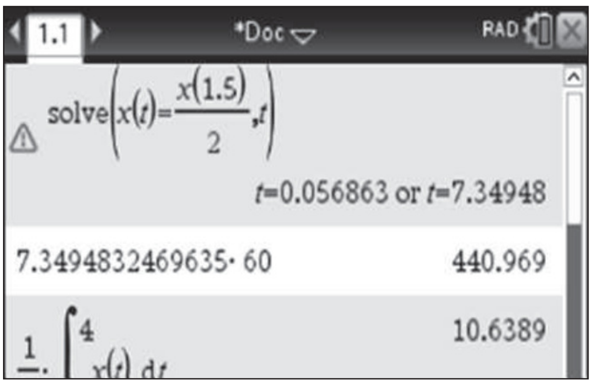
- 1 mark for $\dfrac{x(a) - x(0)}{a} = 2$ (or with a denominator of $a = 0$).
- 1 mark for the correct answer.

Question 6c.

Worked solution

At 9 am $t = 0$ and at 1 pm $t = 4$.

Average concentration $= \dfrac{1}{4 - 0}\int_0^4 x(t)\, dt = 10.6389.$



Answer: 10.6389 mg/L

Mark allocation: 2 marks

- 1 method mark for an appropriate integral to calculate the average value.
- 1 answer mark for the correct answer.

Question 6d.

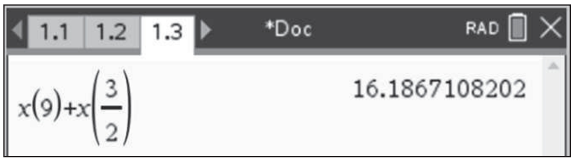
Worked solution

Method 1

The amount of the first dose remaining at 6 pm is $x(9)$.

The second dose is given at 4:30 pm, and 6 pm is 1.5 hours after this. So the amount of the second dose in John's bloodstream at 6 pm can be found using $x(1.5)$.

The total amount at 6 pm from both doses is given by $x(9) + x(1.5) = 16.1867\text{ mg/L}$.



Answer: 16.1867 mg

Method 2

The second dose is given at 4:30 pm when $t = 7.5$.

Define $C(t)$ to be the total concentration of the drug in the bloodstream t hours after the first dose:

$C(t) = x(t) + x(t - 7.5)$, where t is the number of hours after the first dose.

At 6 pm, $t = 9$.

$C(9) = x(9) + x(9 - 7.5) = x(9) + x(1.5) = 16.1867$

Answer: 16.1867 mg/L

Mark allocation: 1 mark

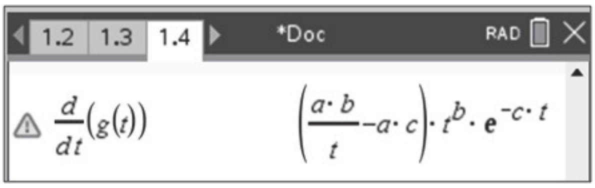
- 1 mark for the correct answer.

Question 6e.i.

Worked solution

Method 1

Stationary points occur when $g'(t) = \left(\frac{ab}{t} - ac\right) t^b e^{-ct} = 0$.



Using the null factor law:

$\frac{ab}{t} - ac = 0, \ t^b \neq 0, \ e^{-ct} \neq 0$

$\frac{ab}{t} = ac$

$t = \frac{ab}{ac}, a \neq 0$

$t = \frac{b}{c}$

Therefore $g(t)$ has a stationary point when $t = \frac{b}{c}$.

Method 2

$$g'(t) = \left(\frac{ab}{t} - ac\right) t^b e^{-ct}$$

$$g'\left(\frac{b}{c}\right) = \left(\frac{ab}{\left(\frac{b}{c}\right)} - ac\right) \left(\frac{b}{c}\right)^b e^{-c\left(\frac{b}{c}\right)}$$

$$= (ac - ac)\left(\frac{b}{c}\right)^b e^{-b}$$

$$= 0\left(\frac{b}{c}\right)^b e^{-b}$$

$$= 0$$

Therefore $g(t)$ has a stationary point when $t = \frac{b}{c}$.

Mark allocation: 2 marks

- 1 mark for the correct expression for $g'(x)$.
- 1 mark for $g'(t) = 0$ when $t = \frac{b}{c}$, derived with suitable steps.

Question 6e.ii.

Worked solution

Maximum occurs at $t = 1.5$. Substitute this into $t = \frac{b}{c}$.

$$1.5 = \frac{b}{c}$$

$$c = \frac{b}{1.5} = \frac{b}{\frac{3}{2}}$$

$$= \frac{2b}{3}$$

Mark allocation: 1 mark

- 1 mark for working leading to $c = \frac{2b}{3}$.

Question 6e.iii.

Worked solution

When $t = 1.5$, $g(t) = 12$.

$g(1.5) = 12 \Rightarrow g(1.5) = a(1.5)^b e^{-1.5c} = 12$ Equation (1)

x has half its maximum value when $t = 7.5$. Hence $g(7.5) = 6$.

$g(7.5) = a(7.5)^b e^{-7.5c} = 6$ Equation (2)

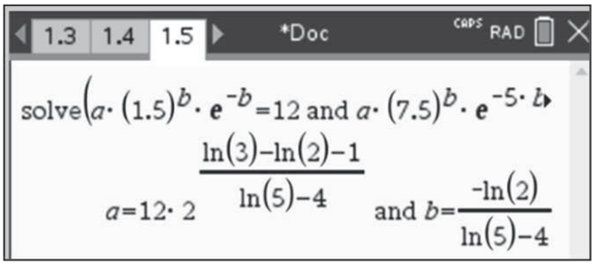
From **part 6.e.ii.**, $c = \frac{2b}{3}$.

Substitute $c = \frac{b}{1.5}$ into equations (1) and (2):

$$a(1.5)^b e^{-b} = 12$$

$$a(7.5)^b e^{-5b} = 6$$

Use a CAS calculator to solve for b.



Thus $b = \dfrac{-\log_e 2}{\log_e 5 - 4} = \dfrac{\log_e 2}{4 - \log_e 5} = \dfrac{\log_e 2}{2^2 - \log_e 5}$.

Answer: $b = \dfrac{\log_e(2)}{2^2 - \log_e(5)}$

Mark allocation: 3 marks

- 1 mark for equation (1), or equivalent.
- 1 mark for equation (2), or equivalent.
- 1 mark for the correct answer.

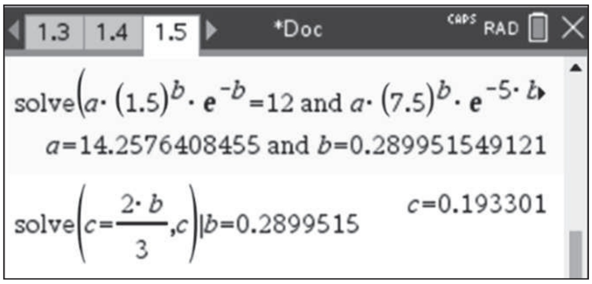
TIP

» Make sure that your final answer is written in the required form: $\dfrac{\log_e(p)}{p^2 - \log_e(q)}$, where p and q are positive integers.

Question 6e.iv.

Worked solution

Use a CAS to solve the equations derived above simultaneously,



Answers: $a = 14.2576$ and $c = 0.1933$.

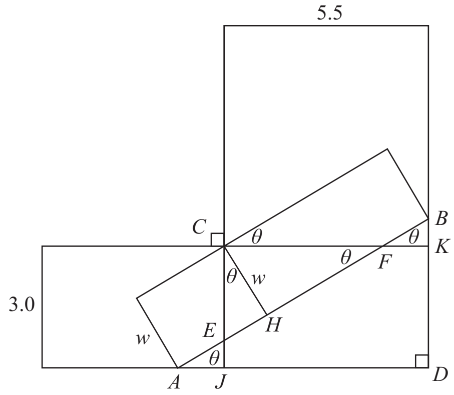
Mark allocation: 1 mark

- 1 mark for both correct answers.

Question 7a.i.

Worked solution

The following annotated diagram will be useful for defining notation to use when writing solutions. It will also help identify right-angled triangles useful in answering this question.



From the right-angled triangle ECH:

$\cos(\theta) = \frac{w}{EC}$

$\Rightarrow EC = \frac{w}{\cos(\theta)}$

From the right-angled triangle CFH:

$\sin(\theta) = \frac{w}{CF}$

$\Rightarrow CF = \frac{w}{\sin(\theta)}$

Mark allocation: 2 marks

- 1 mark for recognising and using an appropriate right-angled triangle.
- 1 mark for correct answers.

Question 7a.ii.

Worked solution

Method 1

From the right-angled triangle ECF:

$$
\begin{aligned}
(EF)^2 &= (EC)^2 + (CF)^2 \\
&= \left(\frac{w}{\cos(\theta)}\right)^2 + \left(\frac{w}{\sin(\theta)}\right)^2 \\
&= \frac{w^2}{\cos^2(\theta)} + \frac{w^2}{\sin^2(\theta)} \\
&= \frac{w^2\sin^2(\theta) + w^2\cos^2(\theta)}{\cos^2(\theta)\sin^2(\theta)} \\
&= \frac{w^2(\sin^2(\theta) + \cos^2(\theta))}{\cos^2(\theta)\sin^2(\theta)} \\
&= \frac{w^2}{\cos^2(\theta)\sin^2(\theta)} \\
EF &= \frac{w}{\cos(\theta)\sin(\theta)}
\end{aligned}
$$

Method 2

For the right-angled triangle ECF, where the angle at vertex F is θ,

$\cos(\theta) = \dfrac{\frac{w}{\sin(\theta)}}{EF}$ and $\sin(\theta) = \dfrac{\frac{w}{\cos(\theta)}}{EF}$.

Rearranging either ratio to make EF the subject will give the required result:

$$EF = \frac{w}{\sin(\theta)\cos(\theta)}$$

Mark allocation: 1 mark

- 1 mark for recognising an appropriate right-angled triangle, applying Pythagoras' theorem and substituting the result from **part a.i.**

OR

- 1 mark for using appropriate trigonometric ratios with results from **part a.i.** to derive the length.

TIPS

- This is a 'show that' question, therefore extra care must be taken to give appropriate working. One way is to follow these steps:
  1. Ignore the given result and instead treat the question as asking 'Find, in terms of w and θ, an expression for length EC'.
  2. Write out a solution.
  3. Compare your answer with the given expression.
- The result of a 'show that' question will usually be needed in the solution to a later question.
- The word 'hence' means that the answer from the previous part must be used as part of the solution.

Question 7b.i.

Worked solution

From the right-angled triangle AEJ:

$$\sin(\theta) = \frac{JE}{AE} = \frac{JC - EC}{AE}$$

$$\Rightarrow AE = \frac{JC - EC}{\sin(\theta)}$$

Substitute $EC = \dfrac{w}{\cos(\theta)}$ (from **part a.i.**) and $JC = 3$:

$$AE = \frac{3 - \frac{w}{\cos(\theta)}}{\sin(\theta)}$$

$$= \frac{\frac{3\cos(\theta) - w}{\cos(\theta)}}{\sin(\theta)}$$

$$= \frac{3\cos(\theta) - w}{\cos(\theta)} \div \sin(\theta)$$

$$= \frac{3\cos(\theta) - w}{\cos(\theta)\sin(\theta)}$$

Alternatively, from the diagram it can be seen that $AF = \frac{3}{\sin(\theta)}$.

Hence $AE = AF - EF = \frac{3}{\sin(\theta)} - \frac{w}{\sin(\theta)\cos(\theta)}$.

Simplifying the above equation gives the result $AE = \frac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$, as required.

Mark allocation: 2 marks

- 1 mark for recognising and using an appropriate right-angled triangle, using $EC = \frac{w}{\cos(\theta)}$ or deriving $AF = \frac{3}{\sin(\theta)}$.
- 1 mark for sufficient steps to establish the required result.

Question 7b.ii.

Worked solution

From the right-angled triangle FKB:

$$\cos(\theta) = \frac{FK}{FB} = \frac{CK - CF}{FB}$$

$$\Rightarrow FB = \frac{CK - CF}{\cos(\theta)}$$

Substitute $CF = \frac{w}{\sin(\theta)}$ and $CK = 5.5$:

$$FB = \frac{5.5 - \frac{w}{\sin(\theta)}}{\cos(\theta)}$$

Simplify this expression. (A CAS can be used.)

Answer: $FB = \frac{5.5\sin(\theta) - w}{\sin(\theta)\cos(\theta)}$

Mark allocation: 2 marks

- 1 mark for recognising and using an appropriate right-angled triangle.
- 1 mark for the correct answer.

Question 7c.

Worked solution

$L = AE + EF + FB$, substitute $AE = \frac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)}$, $EF = \frac{w}{\sin(\theta)\cos(\theta)}$, $FB = \frac{5.5\sin(\theta) - w}{\sin(\theta)\cos(\theta)}$:

$$L = \frac{3\cos(\theta) - w}{\sin(\theta)\cos(\theta)} + \frac{w}{\sin(\theta)\cos(\theta)} + \frac{5.5\sin(\theta) - w}{\sin(\theta)\cos(\theta)}$$

$$= \frac{3\cos(\theta) + 5.5\sin(\theta) - w}{\cos(\theta)\sin(\theta)}$$

Mark allocation: 1 mark

- 1 mark for recognising that L is the sum of three lengths, and substituting correct expressions for each of those lengths.

Question 7d.i.

Worked solution

The width of the widest possible rectangle is equal to the width of the narrowest part of polygon.

Answer: $w = 3.0$ metres

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 7d.ii.

Worked solution

For a given width, the longest rectangle that can be rotated within the polygon will be the shortest one that remains in contact with the inside corner and walls. Therefore a rectangle of width 1.5 units and length 4.4 units can be rotated within the polygon because $L_{rectangle} = 4.4 < L_{min} = 8.7479$.

Mark allocation: 1 mark

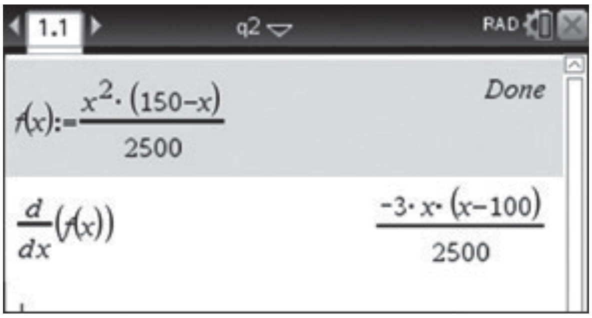
- 1 mark for the correct conclusion and justification: $L_{rectangle} = 4.4 < L_{min} = 8.7479$.

Question 8a.

Worked solution

$$\frac{dy}{dx} = -\frac{3x(x-100)}{2500}$$

Use a CAS to answer this question.



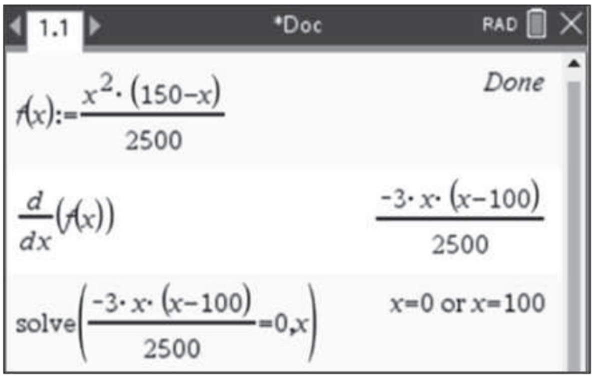
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8b.

Worked solution

Solve $\frac{dy}{dx} = 0$ to find the turning points.



The turning points are at $x = 0$ and $x = 100$.

Now consider the diagram of the stabiliser. There is a maximum at $x = 100$, so the function is strictly decreasing for $x \in [100, 150]$.

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIP

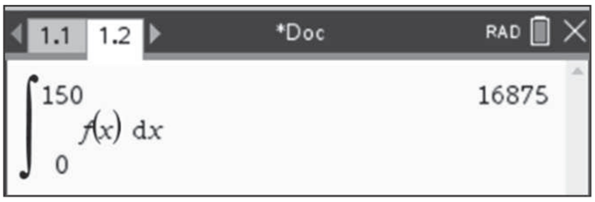
» **The set of values for which this function is strictly decreasing includes the end point and the turning point.**

Question 8c.

Worked solution

$$\text{Area} = \int_0^{150} f(x)\,dx$$
$$= 16\,875$$

Use a CAS to answer this question.



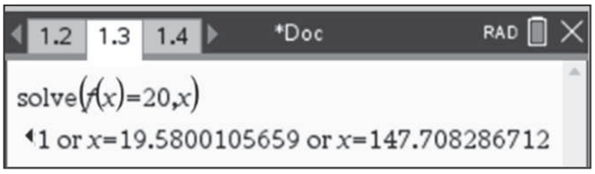
Mark allocation: 1 mark

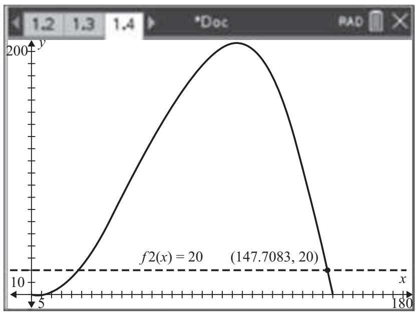
- 1 mark for the correct answer.

Question 8d.i.

Worked solution

The value of c can be found by solving $f(x) = 20$ for x, using the solve function on a CAS or using a CAS to find the point of intersection between $y = f(x)$ and $y = 20$.





The value of c is 147.708.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8d.ii.

Worked solution

The value of b is found by evaluating $f(110)$.

So $b = \frac{968}{5}$ or 193.6.

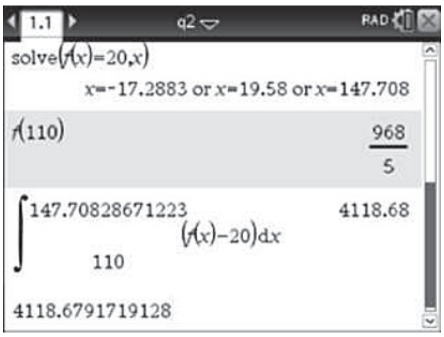
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8e.

Worked solution

The area of the rudder is found by evaluating the integral $\int_{110}^{147.708...} (f(x) - 20)\,dx = 4118.679$.



Mark allocation: 2 marks

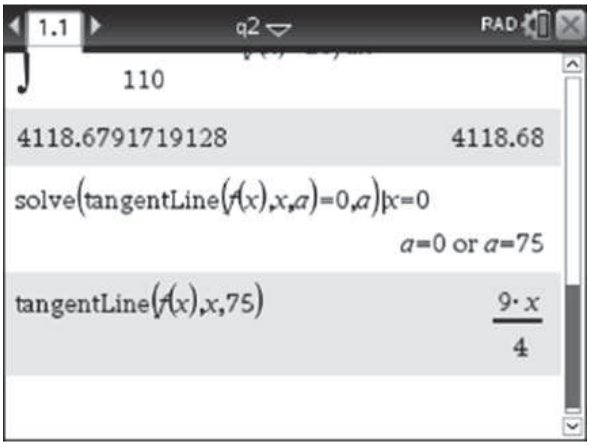
- 1 mark for formulating the integral.
- 1 mark for the correct answer, to three decimal places.

Question 8f.i.

Worked solution

The line segment OP lies on the tangent to the curve that passes through the origin. To find this tangent, first find the general equation for the tangent to the curve (e.g. at a point $x = a$) and then determine the value of a when the tangent passes through the origin (i.e. when $x = 0$).

We require the tangent to the curve when $x = 75$. This can be found quickly using a CAS.



The equation of the tangent is $y = \frac{9x}{4}$.

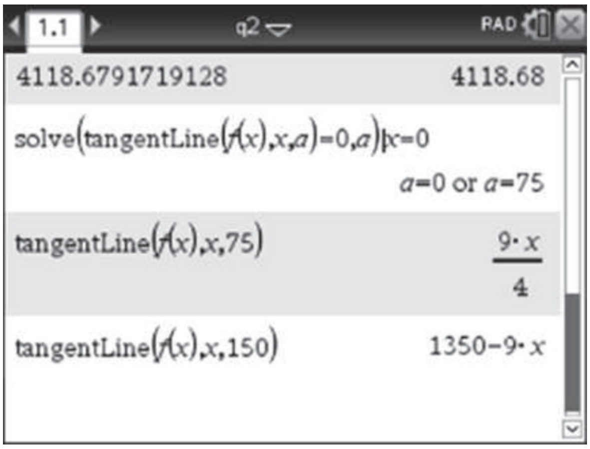
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 8f.ii.

Worked solution

The tangent to the curve when $x = 150$ is $y = 1350 - 9x$. Again, a CAS can be used.



Mark allocation: 1 mark

- 1 mark for the correct answer.

TIP

» If the function is defined on a restricted domain, then the tangent does not exist at the end points. To avoid this problem, do not define the function on a restricted domain.

Question 8f.iii.

Worked solution

In a previous question we found that the stationary point of f occurs when $x = 100$. Hence the equation of PQ is $y = f(100) = 200$.

P is the point of intersection between $y = \frac{9}{4}x$ and $y = 200$.

At P, $\frac{9}{4}x = 200$

$x = \frac{800}{9}$, as required

Q is the point of intersection between $y = 1350 - 9x$ and $y = 200$.

At Q, $1350 - 9x = 200$

$-9x = 1150$

$x = \frac{1150}{9}$, as required

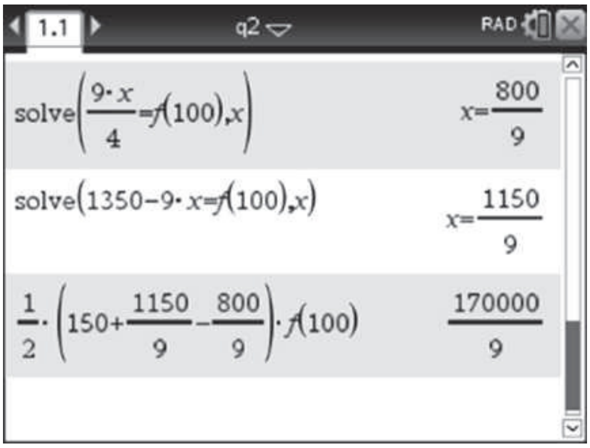
Mark allocation: 2 marks

- 1 mark for each equation, followed by the required solution.

Question 8f.iv.

Worked solution

The area of the trapezium $OPQR$ is $\frac{1}{2}\left(150 + \left(\frac{1150}{9} - \frac{800}{9}\right)\right) \times f(100) = \frac{170\,000}{9}$.



Mark allocation: 1 mark

- 1 mark for the correct answer.

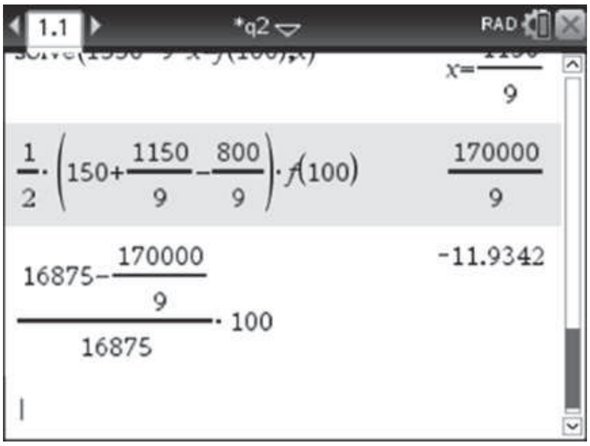
Question 8f.v.

Worked solution

The error is the true value minus the approximate value. To find this as a percentage of the actual area, we divide the error by the actual area and multiply by 100:

$$\frac{16\,875 - \frac{170\,000}{9}}{16\,875} \times 100 = -11.93$$

Therefore the error in the approximation is 11.93%.



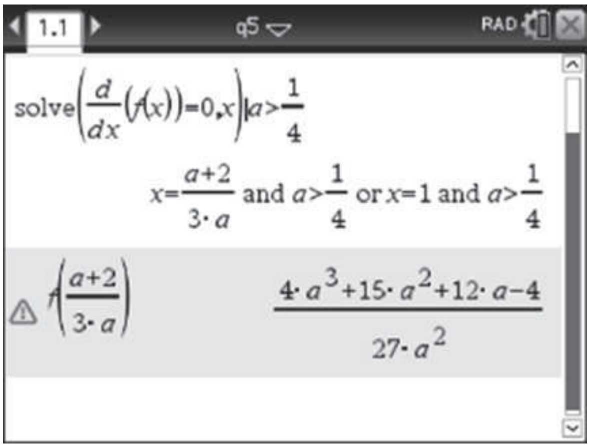
Mark allocation: 1 mark

- 1 mark for the correct answer, expressed as a positive value.

Question 9a.

Worked solution

The turning point at P occurs when $x = \frac{a+2}{3a}$. Therefore the coordinates of P are $\left(\frac{a+2}{3a}, \frac{4a^3+15a^2+12a-4}{27a^2}\right)$.



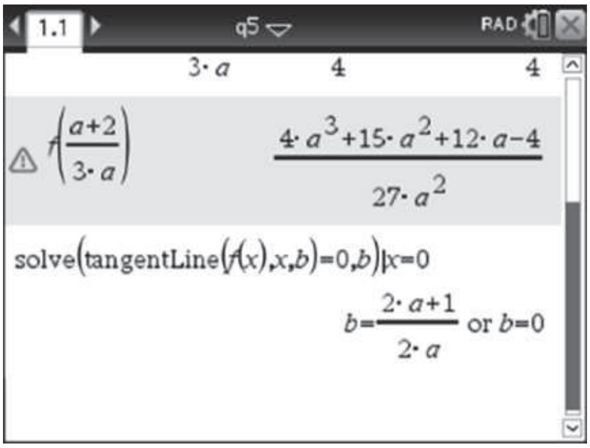
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 9b.

Worked solution

Find the tangent to f at any point on the curve (e.g. $x = b$). The tangent passes through the origin for certain values of b. These values can be obtained quickly using a CAS.



The x-coordinate of Q cannot be zero, so it must be $\frac{2a+1}{2a}$.

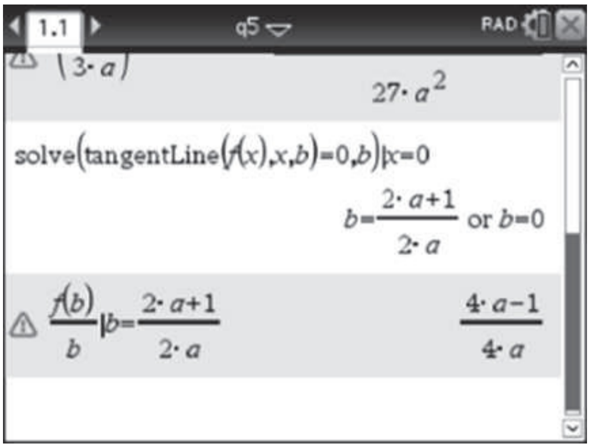
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 9c.

Worked solution

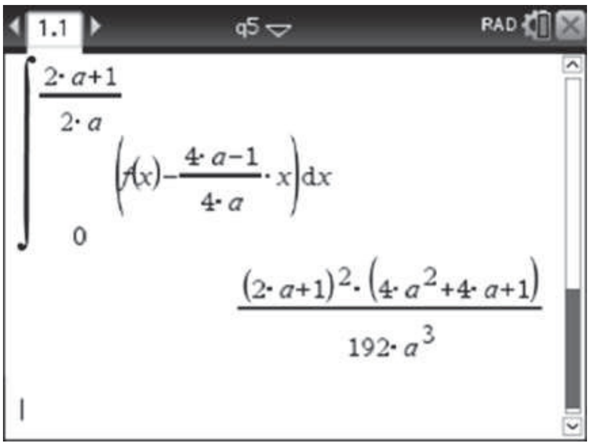
The gradient of the tangent is $\frac{f(b)}{b}$, where $b = \frac{2a+1}{2a}$. This evaluates to $\frac{4a-1}{4a}$.



Therefore the equation of the tangent to f that passes through the origin is $y = \left(\frac{4a-1}{4a}\right)x$.

The area of the shaded region is

$$\int_0^{\frac{2a+1}{2a}} \left(f(x) - \left(\frac{4a-1}{4a}\right)x\right) dx = \frac{(2a+1)^2(4a^2+4a+1)}{192a^3} = \frac{(2a+1)^4}{192a^3}$$



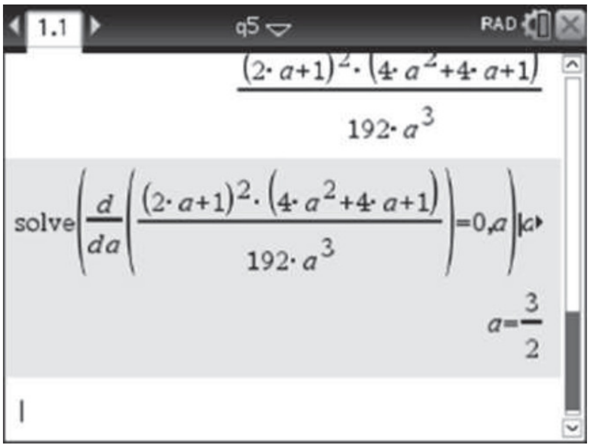
Mark allocation: 3 marks

- 1 mark for the equation of the tangent.
- 1 mark for the integral with correct terminals.
- 1 mark for the correct answer.

Question 9d.

Worked solution

Solve $\frac{d}{da}\left(\frac{(1+2a)^4}{192a^3}\right) = 0$, where $a > \frac{1}{4}$.



So $a = \frac{3}{2}$.

Mark allocation: 2 marks

- 1 mark for differentiating the result from **part c.** and setting it to zero.
- 1 mark for the correct value of a.

Question 9e.

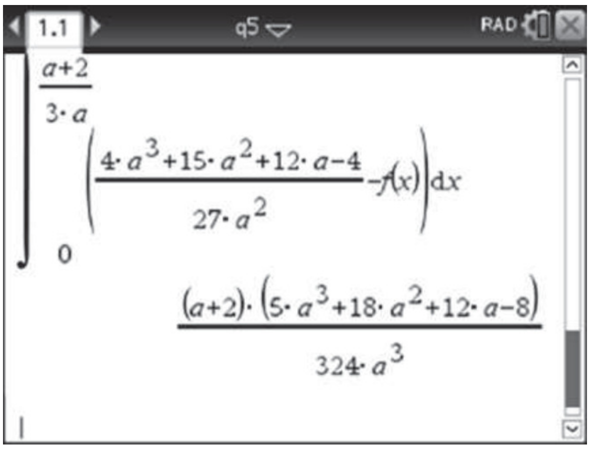
Worked solution

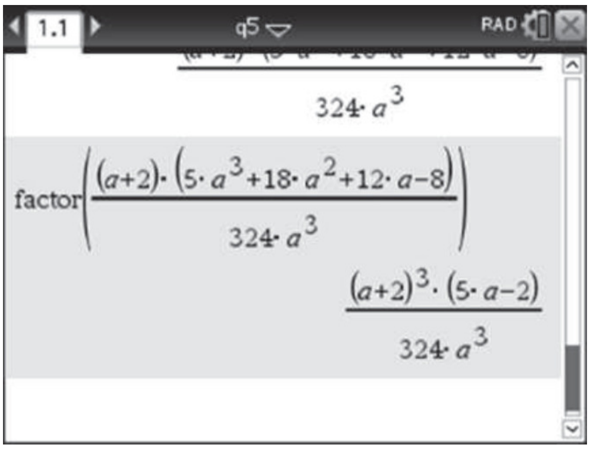
The area of B is equal to

$$\int_0^{\frac{a+2}{3a}}\left(\frac{4a^3+15a^2+12a-4}{27a^2} - g(x)\right)dx$$

$$= \frac{(a+2)(5a^3+18a^2+12a-8)}{324a^3}$$

$$= \frac{(a+2)^3(5a-2)}{324a^3}$$





Mark allocation: 3 marks

- 2 marks for the integral of the difference between the functions, and for the correct terminals.
- 1 mark for the correct answer.

Question 9f.

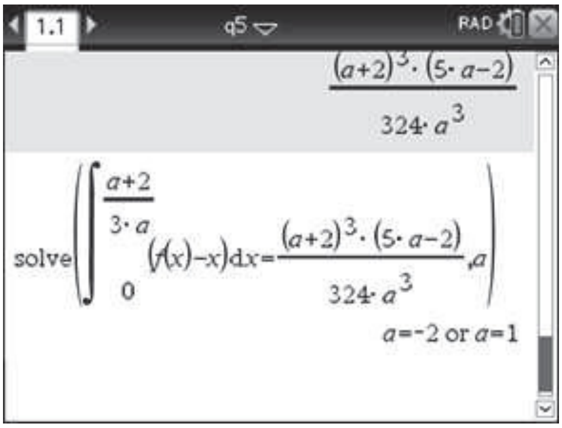
Worked solution

The area of A is $\int_0^{\frac{a+2}{3a}} (g(x) - x)\,dx$.

Equating this area to the area found in **part e.** gives

$\int_0^{\frac{a+2}{3a}} (g(x) - x)\,dx = \frac{(a+2)^3(5a-2)}{324a^3}$, $a > \frac{1}{4}$

Solving with a CAS gives $a = 1$.



Mark allocation: 2 marks

- 1 mark for equating the integral for area A with the area found in **part e.**
- 1 mark for the correct answer.

Question 10a.i.

Worked solution

Using a CAS, the gradient of PA is $\frac{6}{a-2}$.

Mark allocation: 1 mark

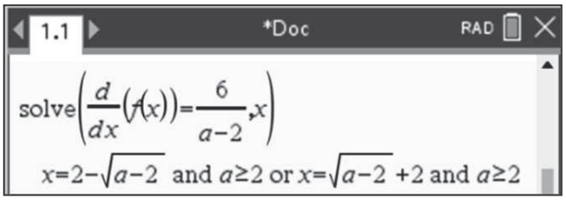
- 1 mark for the correct answer in simplified form.

Question 10a.ii.

Worked solution

The gradient of the tangent is given by $f'(x)$, so we need to solve $f'(x) = \frac{6}{a-2}$ for x.

Using a CAS gives $x = \pm\sqrt{a-2} + 2$.



Since $x > 2$, $x = \sqrt{a-2} + 2$.

Mark allocation: 2 marks

- 1 method mark for the equation $f'(x) = \dfrac{6}{a-2}$ (i.e. $f'(x)$ equated with the answer from **part a.i.**).
- 1 answer mark for the correct answer. (The negative solution must be discarded.)

TIP

» Always consider the domain when assessing solutions.

Question 10b.i.

Worked solution

Evaluate using a CAS, giving an answer of 6.

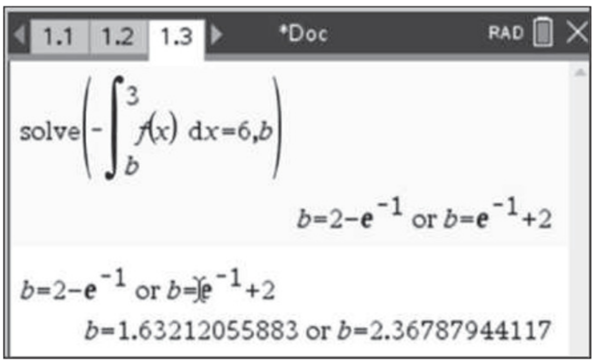
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 10b.ii.

Worked solution

Set up an integral equation and solve it using a CAS.



Note that since $b > 2$, $b = e^{-1} + 2$.

Mark allocation: 1 mark

- 1 mark for the answer $b = e^{-1} + 2$. (The negative solution must be discarded.)

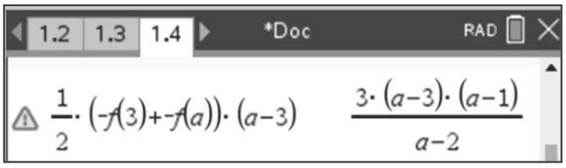
Question 10c.i.

Worked solution

The shape of the region is a trapezium. The relevant area formula is $A = \frac{1}{2}(a + b)h$.

The length of the parallel sides of this trapezium is given by $-f(3)$ and $-f(a)$.

Therefore $A = \frac{1}{2}(-f(3) + -f(a))(a - 3) = \frac{3(a-1)(a-3)}{a-2}$.



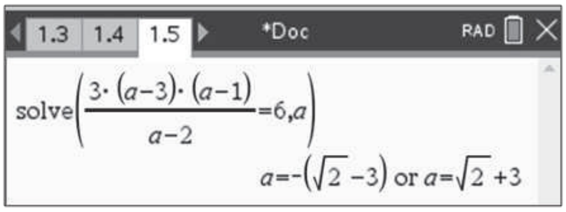
Mark allocation: 2 marks

- 1 method mark for recognising the area as a trapezium and for using the trapezium area formula (which must be shown with the correct expressions substituted).
- 1 answer mark for the correct answer.

Question 10c.ii.

Worked solution

Use a CAS to solve $\frac{3(a-1)(a-3)}{(a-2)} = 6$ for a.



Since $a > 3$, $a = \sqrt{2} + 3$.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 11a.

Worked solution

Use a CAS to solve $f'(x) = 0$.

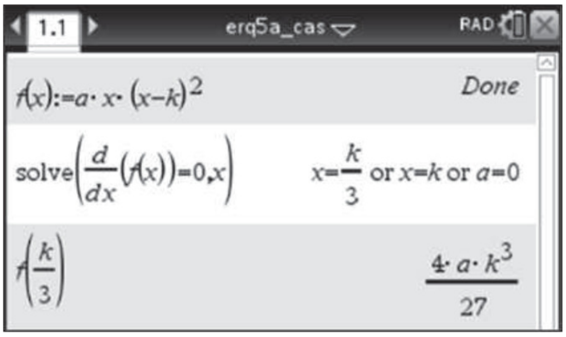
$f'(x) = 0$

$x = \frac{k}{3}$ or $x = k$

Since $a > 0$, f is a positive cubic function. Furthermore, since $k > 0$, $\frac{k}{3} < k$.

Check the graph of the function on your calculator. The shape of the graph shows that the turning point on the left at $x = \frac{k}{3}$ is the local maximum.

Use a CAS to find $f\left(\frac{k}{3}\right) = \frac{4ak^3}{27}$.



Answer: $\left(\frac{k}{3}, \frac{4ak^3}{27}\right)$

Mark allocation: 2 marks

- 1 answer mark for stating the equation $f'(x) = 0$.
- 1 answer mark for the correct coordinates.

Question 11b.

Worked solution

$$g\left(\frac{k}{3}\right) = \frac{4ak^2}{9}\left(\frac{k}{3}\right)$$
$$= \frac{4ak^3}{27}$$

Hence $g(x)$ also passes through $\left(\frac{k}{3}, \frac{4ak^2}{27}\right)$, so both graphs meet at the local maximum of f.

Mark allocation: 1 mark

- 1 mark for confirming that $g\left(\frac{k}{3}\right) = \frac{4ak^3}{27}$.

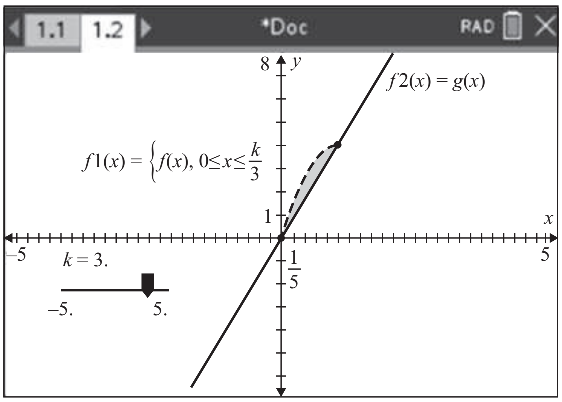
Question 11c.

Worked solution

Notice that $h(x)$ has the same rule as $f(x)$ but a different domain.

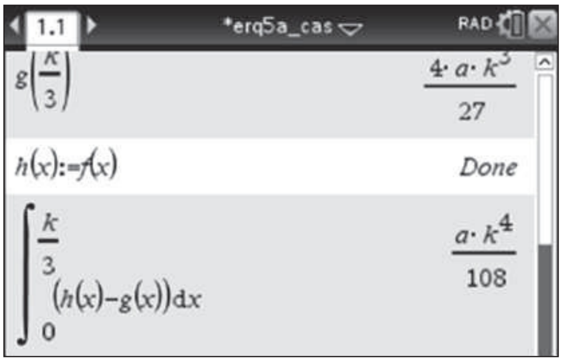
It is helpful to consider the graphs of h and g to determine where the bounded area lies. This can be done most easily by letting a equal a positive constant, such as $a = 1$.

We know from **part b.** that the graphs of h and g meet at the local maximum of h.



Using a CAS to evaluate the integral, the area bounded by the graphs of h and g is

$$\text{Area} = \int_0^{\frac{k}{3}} (h(x) - g(x))\, dx$$
$$= \frac{ak^4}{108}$$



Answer: $\frac{ak^4}{108}$

Mark allocation: 2 marks

- 1 answer mark for the integral $\int_0^{\frac{k}{3}}(h(x)-g(x))\,dx$.
- 1 answer mark for the correct answer.

Question 11d.

Worked solution

Method 1

For h and h^{-1} to meet twice, the graph of h must intersect the line $y = x$ twice.

There is always a point of intersection at $(0, 0)$, and there will be a second point of intersection if the right end point of h lies on or below the line $y = x$.

The right end point of h is at $\left(\frac{k}{3}, \frac{4ak^3}{27}\right)$. This point will lie on or below the line $y = x$ if $\frac{k}{3} \le \frac{4ak^3}{27}$ and $a \le \frac{9}{4k^2}$.

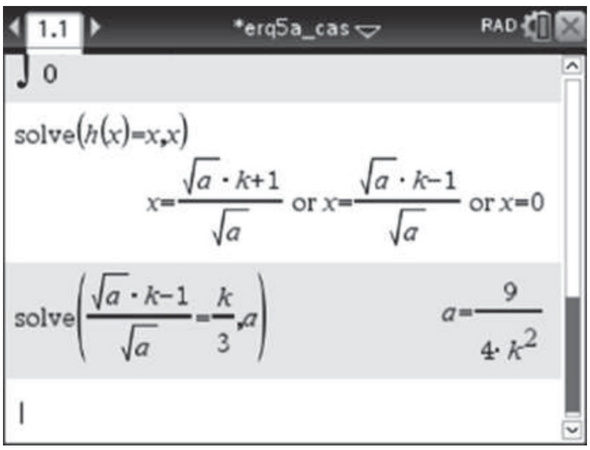
Hence the maximum value of a is $a = \frac{9}{4k^2}$.

Method 2

The graphs of $y = h(x)$ and $y = x$ meet when $x = 0$ and when $x = \frac{\sqrt{a}k \pm 1}{\sqrt{a}}$.

For the graphs of $y = h(x)$ and $y = x$ to meet exactly twice over the interval $\left[0, \frac{k}{3}\right]$ with the value of a as large as possible, it must be the case that $\frac{\sqrt{a}k - 1}{\sqrt{a}} = \frac{k}{3}$.

Therefore $a = \frac{9}{4k^2}$.



Mark allocation: 2 marks

- 1 mark for finding $x = 0$ and $x = \frac{\sqrt{a}k \pm 1}{\sqrt{a}}$, or for determining that the end point of f lies on $y = x$ when $a = \frac{9}{4k^2}$.
- 1 mark for solving $\frac{\sqrt{a}k - 1}{\sqrt{a}} = \frac{k}{3}$, or for justifying that $a \le \frac{9}{4k^2}$, followed by $a_{\max} = \frac{9}{4k^2}$.

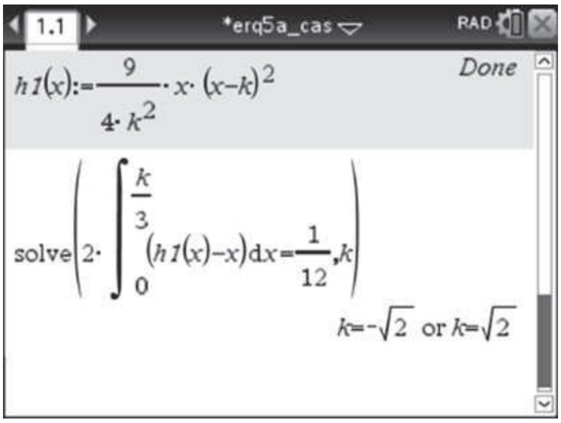
Question 11e.

Worked solution

h and h^{-1} meet at $x = 0$ and $x = \frac{k}{3}$, so the area bounded by the graphs is

$\int_0^{\frac{k}{3}} (h(x) - h^{-1}(x))\,dx = 2\int_0^{\frac{k}{3}} (h(x) - x)\,dx$

By solving $2\int_0^{\frac{k}{3}} (h(x) - x)\,dx = \frac{1}{12}$, where $a = \frac{9}{4k^2}$, we find that $k = \sqrt{2}$.



Answer: $k = \sqrt{2}$

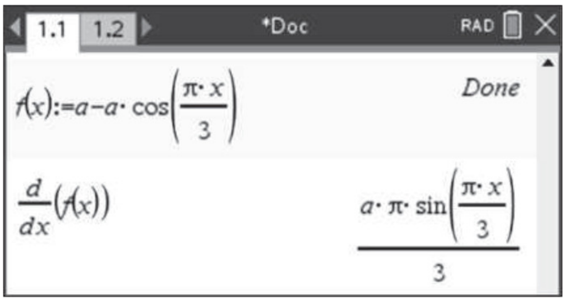
Mark allocation: 2 marks

- 1 answer mark for setting up a suitable integral with correct terminals, such as $\int_0^{\frac{k}{3}} (h(x) - h^{-1}(x))\,dx$ or $2\int_0^{\frac{k}{3}} (h(x) - x)\,dx$.
- 1 answer mark for $k = \sqrt{2}$.

Question 12a.

Worked solution

The gradient is given by $f'(x) = \frac{a\pi}{3}\sin\left(\frac{\pi x}{3}\right)$.



The maximum value of $f'(x)$ is $\frac{a\pi}{3}$.

Hence $\frac{a\pi}{3} = \frac{2\pi}{3}$ and $a = 2$.

Mark allocation: 1 mark

- 1 mark for using the maximum of $f'(x)$ to justify that $a = 2$.

Question 12b.

Worked solution

Area $= \int_0^6 f(x)\,dx = 12$

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIP

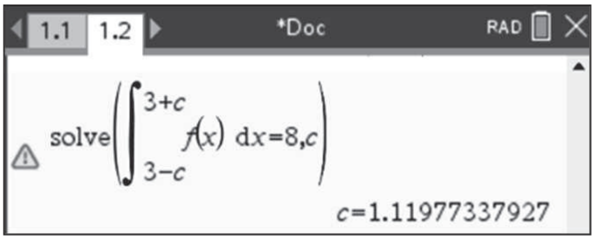
» This question earns only 1 mark, so there is no requirement to show any working, although you may do so if you wish.

Question 12c.

Worked solution

The required area is given by $\int_{3-c}^{3+c} f(x)\,dx = 8$ or $\int_{3}^{3+c} f(x)\,dx = 4$.

Use a CAS to find c.



$c = 1.12$

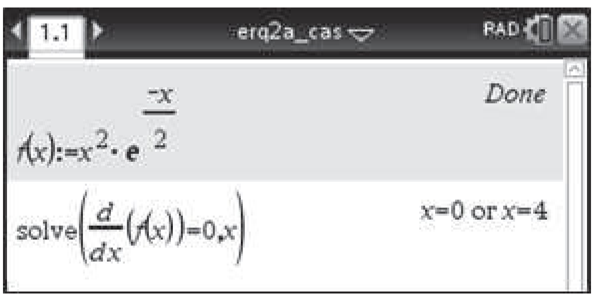
Mark allocation: 2 marks

- 1 method mark for setting a suitable integral equal to 8, or for using symmetry with an integral equal to 4.
- 1 answer mark for the correct answer.

Question 13a.

Worked solution

Use a CAS to solve $f'(x) = 0$ and find the turning points: $x = 0$ and $x = 4$.



Note from the graph that f is strictly increasing between these turning points, so the interval is $[0, 4]$.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

TIP

» The function is strictly increasing at the end points of the interval where $f(x) > 0$. The only time you don't include the end points of an interval over which a function is strictly increasing is when the function is not defined at those points.

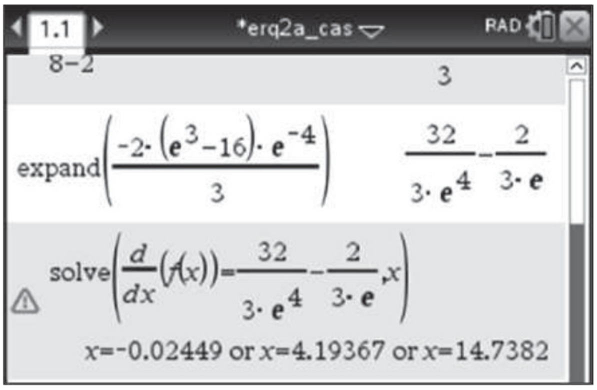
Question 13b.

Worked solution

Use a CAS to find the gradient of the line joining A and B.

$$\text{Gradient} = \frac{f(8) - f(2)}{8 - 2} = \frac{32}{3e^4} - \frac{2}{3e}$$

Use a CAS to solve $f'(x) = \frac{32}{3e^4} - \frac{2}{3e}$, where $0 \le x \le 10$.



Answer: $x = 4.194$

Mark allocation: 2 marks

- 1 answer mark for finding the gradient of AB.
- 1 answer mark for the correct answer.

Question 13c.i.

Worked solution

In order for $f(g(x))$ to be defined, we require $\text{ran } g \subseteq \text{dom } f$.

Since $\text{dom } f = [0, 10]$, we require $\text{ran } g \subseteq [0, 10]$.

$$0 \le g(x) \le 10$$

$$0 \le x^2 \le 10$$

$$-\sqrt{10} \le x \le \sqrt{10}$$

$$D = \left[-\sqrt{10}, \sqrt{10}\right]$$

Therefore the domain of h' is $(-\sqrt{10}, \sqrt{10})$, as the derivative is not defined at the end points.

Answer: $(-\sqrt{10}, \sqrt{10})$

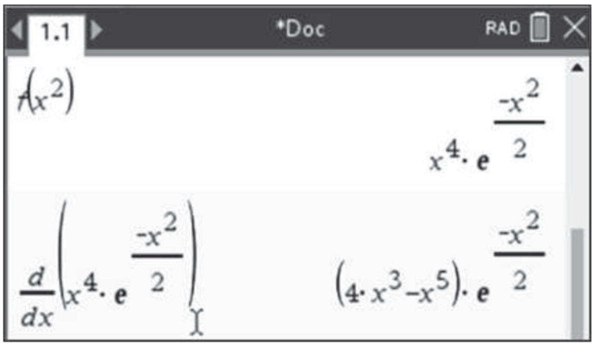
Mark allocation: 2 marks

- 1 mark for finding $D = [-\sqrt{10}, \sqrt{10}]$.
- 1 mark for the correct answer.

Question 13c.ii.

Worked solution

Use a CAS:



Answer: $h'(x) = (4x^3 - x^5)e^{-\frac{x^2}{2}}$

Mark allocation: 1 mark

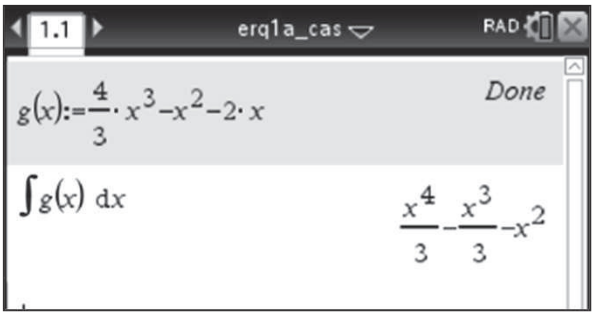
- 1 answer mark for the correct answer.

Question 14a.i.

Worked solution

Integrate by hand, or use a CAS, to obtain

$$f(x) = \int \frac{4}{3}x^3 - x^2 - 2x$$
$$= \frac{1}{3}x^4 - \frac{1}{3}x^3 - x^2 + c$$



Then apply the condition that $f(0) = -1$ to find the value of c:

$f(0) = c = -1$

Answer: $\frac{1}{3}x^4 - \frac{1}{3}x^3 - x^2 - 1$

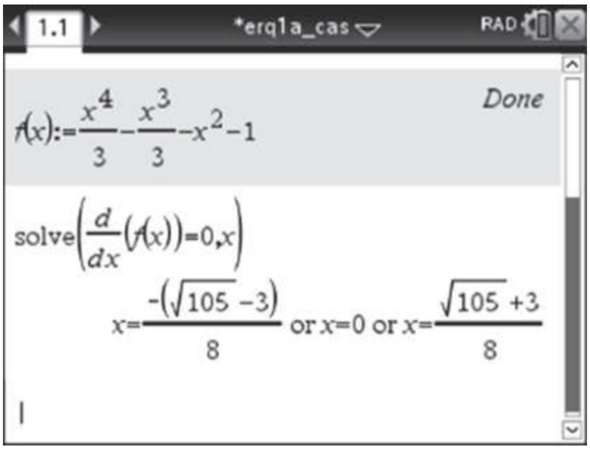
Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 14a.ii.

Worked solution

Use a CAS to solve $f'(x) = 0$ and find the stationary points.



Answer: $x = \frac{3 \pm \sqrt{105}}{8}$

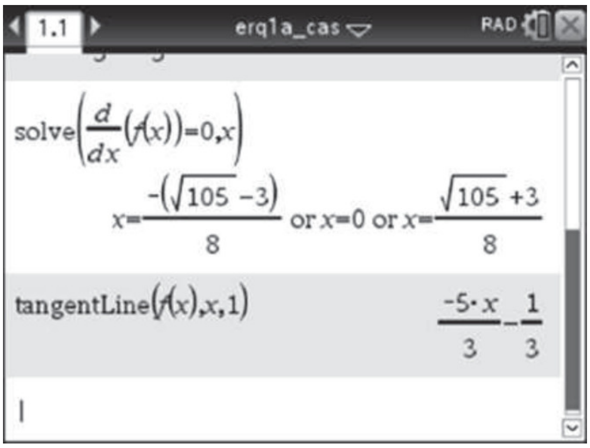
Mark allocation: 2 marks

- 1 answer mark for solving $f'(x) = g(x) = \frac{4}{3}x^3 - x^2 - 2x = 0$.
- 1 answer mark for correct answer.

Question 14b.i.

Worked solution

Use a CAS to find the equation of the tangent to $f(x)$ at $x = 1$.



Answer: $y = -\frac{5}{3}x - \frac{1}{3}$

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

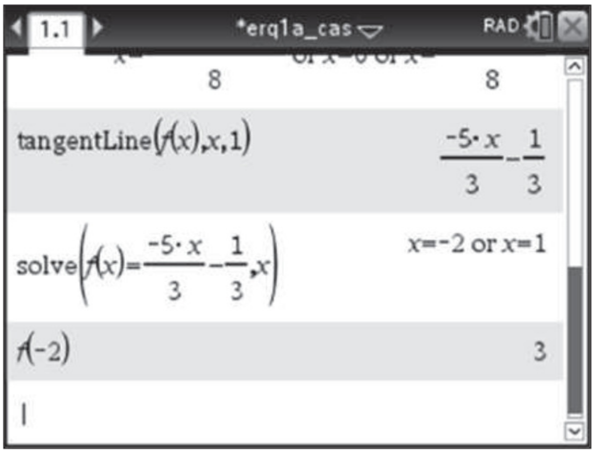
Question 14b.ii.

Worked solution

Use a CAS to solve $f(x) = -\frac{5}{3}x - \frac{1}{3}$ for x, giving $x = -2$ and $x = 1$.

Hence at A, $x = -2$ and $f(-2) = 3$.

The coordinates of A are therefore $(-2, 3)$.



Answer: $(-2, 3)$

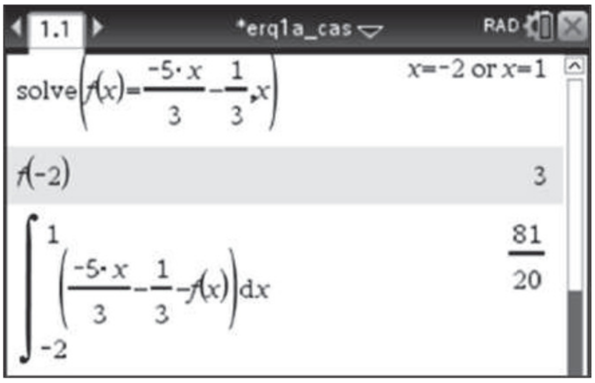
Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 14b.iii.

Worked solution

Use a CAS to establish that the area bounded by the tangent l and the graph of $f(x)$ is equal to $\int_{-2}^{1}\left(-\frac{5}{3}x-\frac{1}{3}-f(x)\right)dx=\frac{81}{20}$.



Mark allocation: 2 marks

- 1 method mark for setting up an appropriate integral, using results found earlier for l and the intersection points.
- 1 answer mark for the correct answer.

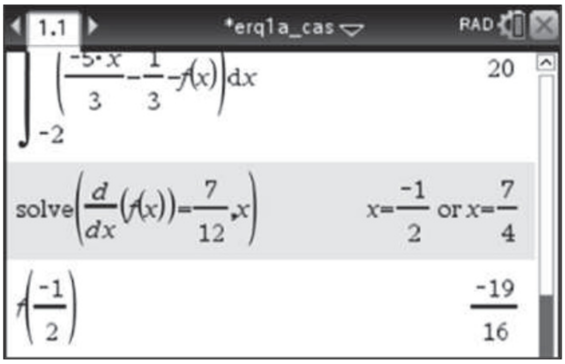
Question 14c.

Worked solution

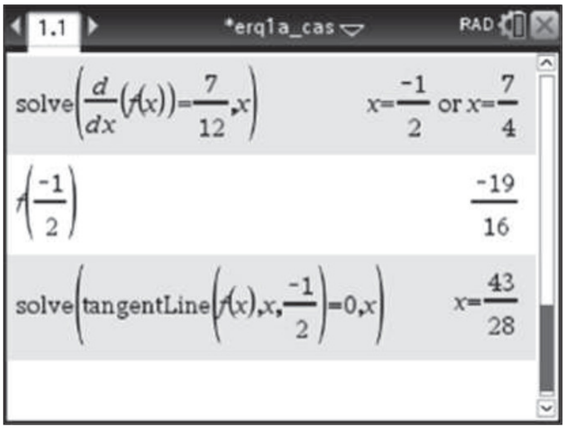
Using a CAS to solve $f'(x) = \frac{7}{12}$ gives $x = -\frac{1}{2}$ and $\frac{7}{4}$, since $u < 0,\ u = -\frac{1}{2}$.

$f\left(-\frac{1}{2}\right) = -\frac{19}{16}$

Thus the coordinates of B are $\left(-\frac{1}{2}, -\frac{19}{16}\right)$.



Now use a CAS to determine when the tangent to the graph of f crosses the x-axis. The coordinates of C are $\left(\frac{43}{28}, 0\right)$.



Answer: $B\left(-\frac{1}{2}, -\frac{19}{16}\right)$ and $C\left(\frac{43}{28}, 0\right)$.

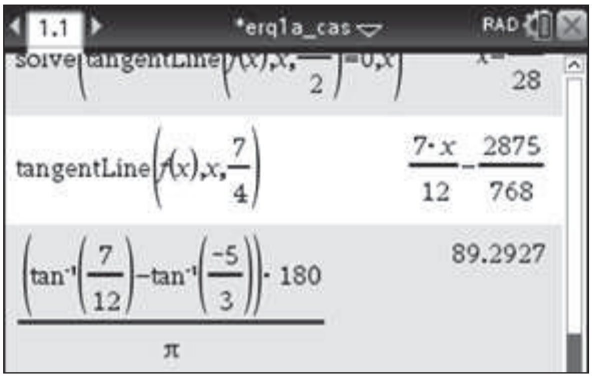
Mark allocation: 3 marks

- 1 answer mark for solving $f'(x) = \frac{7}{12}$ to find $x = -\frac{1}{2}$.
- 1 answer mark for the correct coordinates of B.
- 1 answer mark for the correct coordinates of C.

Question 14d.

Worked solution

From earlier results we know that $\text{grad}_l = -\frac{5}{3}$ and, $\text{grad}_k = \frac{7}{12}$. The angle between the two tangents is $\theta = \tan^{-1}\left(\frac{7}{12}\right) - \tan^{-1}\left(-\frac{5}{3}\right) = 89.29°$.



Answer: 89.29°

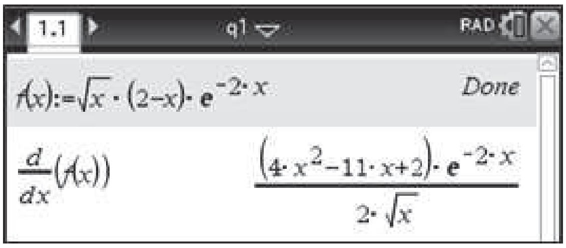
Mark allocation: 2 marks

- 1 method mark if either $\tan^{-1}\left(\frac{7}{12}\right)$ or $\tan^{-1}\left(-\frac{5}{3}\right)$ is used to find angle.
- 1 answer mark for the correct answer.

Question 15a.

Worked solution

Use a CAS:



Answer: $f'(x) = \dfrac{(4x^2 - 11x + 2)e^{-2x}}{2\sqrt{x}}$

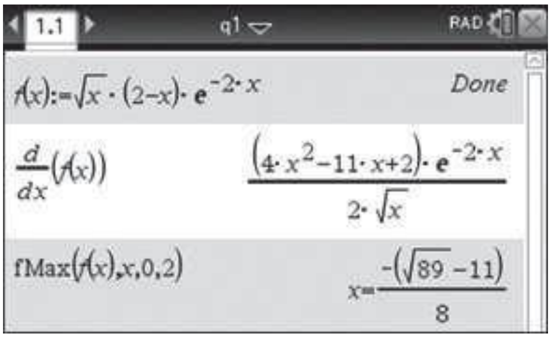
Mark allocation: 1 mark

- 1 answer mark for the correct answer in the required form.

Question 15b.i.

Worked solution

A CAS can be used to find the maximum value of f.



The maximum occurs when $x = \dfrac{11 - \sqrt{89}}{8}$.

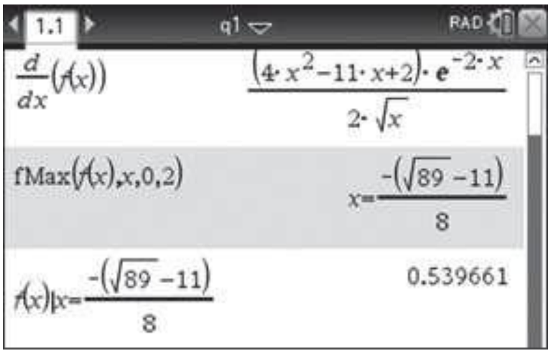
Mark allocation: 1 mark

- 1 answer mark for the correct answer.

Question 15b.ii.

Worked solution

Use a CAS to evaluate the function when $x = \frac{11 - \sqrt{89}}{8}$.



$f\left(\frac{11 - \sqrt{89}}{8}\right) = 0.5397$

The function maximum is 0.5397, correct to four decimal places.

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

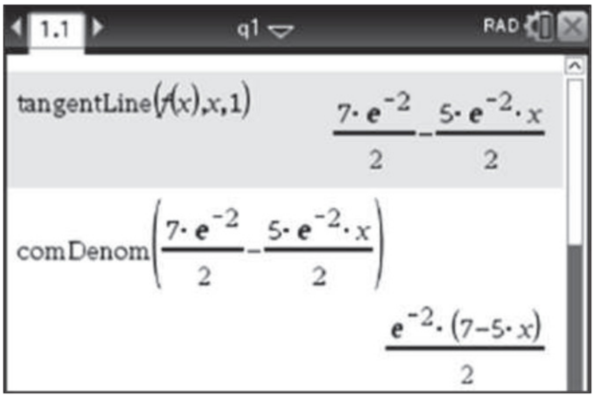
TIP

» The value of *f* must be stated for this question, not the coordinates of the maximum.

Question 15c.i.

Worked solution

The **tangentLine** function on a CAS can be used.



The common denominator command (**comDenom**) can be used to help express the answer in the required form, or this can be done by hand.

The equation of the tangent is $y = \frac{7 - 5x}{2e^2}$.

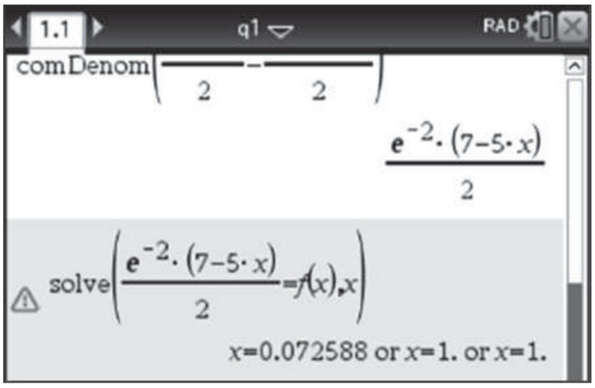
Mark allocation: 1 mark

- 1 answer mark for the correct answer in the required form.

Question 15c.ii.

Worked solution

P is the point of intersection between the tangent and $y = f(x)$. Use a CAS to solve $\frac{7-5x}{2e^2} = f(x)$.



Note that at P, $x < 1$.

The x-coordinate of P is $x = 0.0726$, correct to four decimal places.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 15c.iii.

Worked solution

Point Q is the x-intercept of the tangent. At this point:

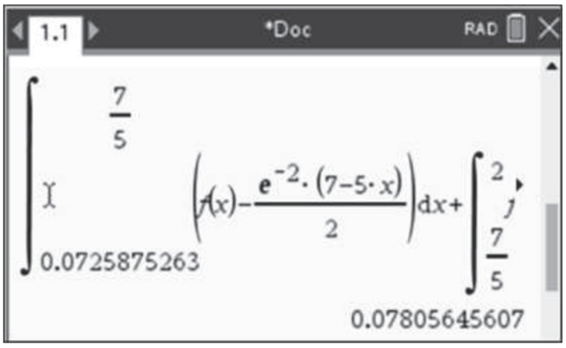
$$\frac{7-5x}{2e^2} = 0$$

$$x = \frac{7}{5}$$

The area of the shaded region is therefore

$$\text{Area} = \int_{0.072...}^{\frac{7}{5}} \left(f(x) - \frac{7-5x}{2e^2}\right) dx + \int_{\frac{7}{5}}^{2} f(x)\, dx$$

$$= 0.0781$$



Mark allocation: 2 marks

- 1 answer mark for setting up appropriate integrals to calculate the area.
- 1 answer mark for the correct answer.

Question 15d.i.

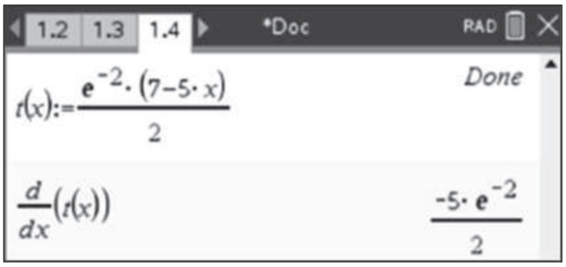
Worked solution

Method 1

The shortest distance is along the line perpendicular to the tangent passing through the origin. Since we know the gradient of the tangent, we can calculate the slope of the perpendicular line. Furthermore, since it passes through the origin, its y-intercept must be zero.

It is helpful to define the equation of the tangent through $x = 1$ on a CAS.

Let $t(x) = \frac{e^{-2}(7 - 5x)}{2}$, so $t'(x) = \frac{-5e^{-2}}{2}$.



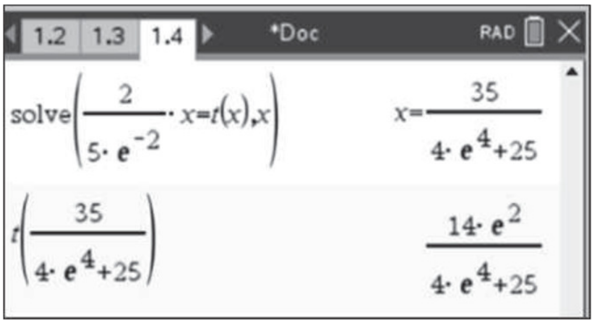
So the gradient of a line perpendicular to the tangent is $\frac{-1}{\frac{-5e^{-2}}{2}} = \frac{2}{5e^{-2}}$.

Since the perpendicular passes through the origin, its y-coordinate is $y = 0$. Thus the equation of the perpendicular is $y = \frac{2}{5e^{-2}}x$.

This line intersects the tangent when $t(x) = \frac{2}{5e^{-2}}x$.

Using a CAS to find the coordinates of the point of intersection gives

$$x = \frac{35}{4e^4 + 25}, f\left(\frac{35}{4e^4 + 25}\right) = \frac{14e^2}{4e^4 + 25}$$



So the coordinates of R are $\left(\frac{35}{4e^4 + 25}, \frac{14e^2}{4e^4 + 25}\right)$.

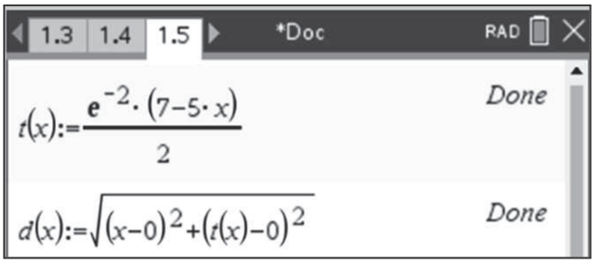
Method 2

Use the distance formula to form an expression for the distance from the origin to a point on the tangent. R is a general point on the tangent.

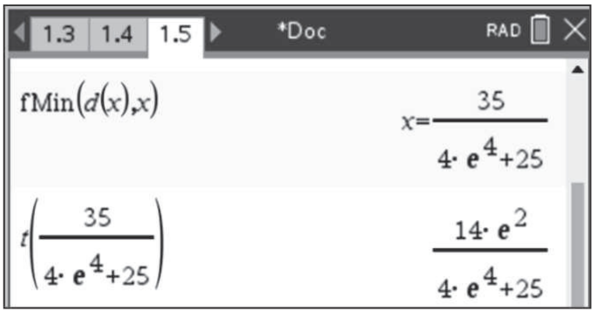
It is helpful to define the equation of the tangent through $x = 1$ on a CAS.

Let $t(x) = \frac{e^{-2}(7 - 5x)}{2}$. Point R is a general point on this line, so let $R(x, t(x))$.

The distance between R and $(0, 0)$ is $\text{dist} = \sqrt{(x - 0)^2 + (t(x) - 0)^2}$.



Then use the **fMin** command on a CAS to find the coordinates of the point which minimises this distance.



Therefore the coordinates of R are $\left(\frac{35}{4e^4+25}, \frac{14e^2}{4e^4+25}\right)$.

Mark allocation: 2 marks

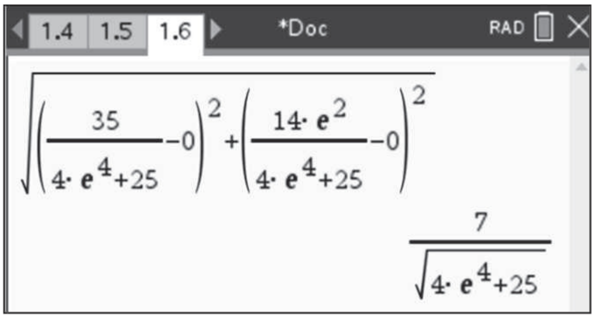
- 1 answer mark for the equation of the perpendicular or a rule for the distance between R and the origin, O, in terms of x.
- 1 answer mark for the correct coordinates.

Question 15d.ii.

Worked solution

Use the distance formula on a CAS to find the required distance:

$$\text{dist} = \sqrt{\left(\frac{35}{4e^4+25} - 0\right)^2 + \left(\frac{14e^2}{4e^4+25} - 0\right)^2} = \frac{7}{\sqrt{4e^4+25}}$$



The distance from R to the origin is $\frac{7}{\sqrt{4e^4+25}}$.

Mark allocation: 1 mark

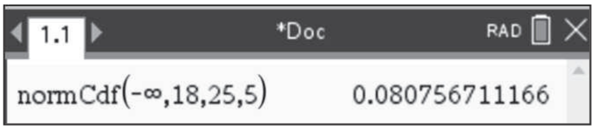
- 1 answer mark for the correct answer.

Question 16a.

Worked solution

$X_{RF} \sim N(25, 5^2)$

Using a CAS, $\Pr(X_{RF} < 18) = 0.0808$.



Answer: 0.0808

Mark allocation: 1 mark

- 1 answer mark for the correct answer.

TIPS

» For a 1 mark question it is sufficient just to state the answer. There is no requirement to show any working, or define the names of any probability distributions used, but you may do so if you wish. The exception to this is if it were a 'show that' question, in which case detailed working must be included.

» Use obvious and unambiguous notation, such as X_{RF} and X_{JJ}, to represent the length of a red fire ant and that of a jumping jack ant, respectively. Other notation could be used, such as X and Y, but the same pronumeral should not be used to represent different things.

Question 16b.

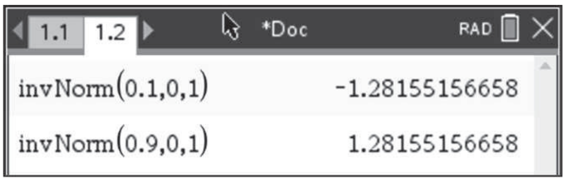
Worked solution

$X_{JJ} \sim N(24, \sigma^2)$ and $Z \sim N(0, 1)$

$\Pr(X_{JJ} < 20) = 0.1$

$\Pr(Z < z_1) = 0.1$, where $z_1 = \dfrac{20 - \mu}{\sigma}$.

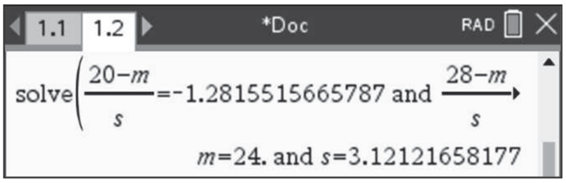
Use **InvNorm** on a CAS to find z_1,



$z_1 = -1.281\ldots$

Since $\Pr(X_{JJ} > 28) = \Pr(X_{JJ} < 20)$, we know that 20 and 28, and their corresponding z values, are symmetrically placed about the mean. Hence $z_2 = 1.281\ldots$ and $z_2 = \dfrac{28 - \mu}{\sigma}$.

Use a CAS to solve the simultaneous equations $\dfrac{20 - \mu}{\sigma} = -1.281\ldots$ and $\dfrac{28 - \mu}{\sigma} = 1.281\ldots$



$\mu = 24$, $\sigma = 3.12$

The mean can also be found using symmetry: $\mu = \dfrac{20 + 28}{2} = 24$.

Mark allocation: 3 marks

- 1 method mark for setting up two equations to find the z values, or using another suitable method.
- 1 answer mark for the correct mean.
- 1 answer mark for correct standard deviation.

TIPS

» Remember that in order to use Inverse Normal on a CAS, the probability must be expressed as $\Pr(X < x)$.

» It is helpful to remember that when finding the mean and/or the standard deviation of a normally distributed random variable, z values need to be used, where $z = \dfrac{z - \mu}{\sigma}$.

Question 16c.

Worked solution

$$\Pr(\text{JJ} \mid \text{length}) < 18 = \frac{\Pr(\text{JJ} \cap X < 18)}{\Pr(\text{length} < 18)}$$

$$= \frac{\Pr(X_{\text{JJ}} < 18)}{\Pr(\text{JJ}) \times \Pr(X_{\text{JJ}} < 18) + \Pr(\text{RF}) \times \Pr(X_{\text{RF}} < 18)}$$

$$= \frac{0.7 \times 0.027\ldots}{0.7 \times 0.027\ldots + 0.3 \times 0.080\ldots}$$

$$= 0.440$$

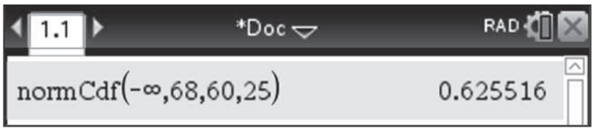
Mark allocation: 2 marks

- 1 method mark for recognising that conditional probability is involved, with either: (a) the numerator given as an expression equivalent to $\Pr(\text{JJ length} < 18)$; or (b) the denominator containing a correct probability expression or a correct value.
- 1 answer mark for the correct answer.

Question 17a.i.

Worked solution

Let X be the amount of time spent on homework, with $X \sim N(60, 25)$. Use a CAS to calculate $\Pr(X < 68) = 0.6255$.



Answer: 0.6255

Mark allocation: 1 mark

- 1 mark for the correct answer.

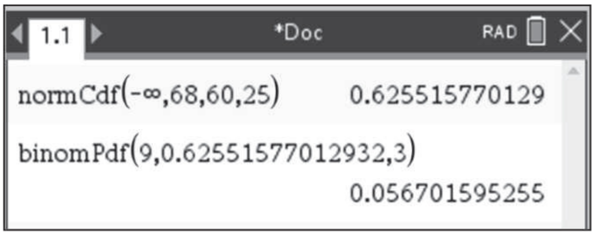
Question 17a.ii.

Worked solution

Let Y be the number of Year 9 students who will undertake less than 68 minutes of homework on a given school night.

$Y \sim \text{Bi}(n = 9, 0.6255\ldots)$

Use the **BinomialPdf** command on a CAS to show that $\Pr(Y = 3) = 0.0567$.



Mark allocation: 2 marks

- 1 mark for recognising that a binomial distribution is involved, with correct values for its parameters.
- 1 mark for the correct answer.

TIPS

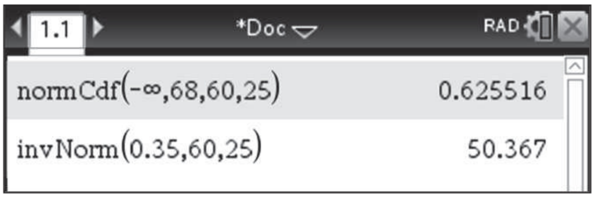
» Switching from a continuous random variable (such as normal) to a discrete random variable (such as binomial) is a common question type.

» Don't use the same random variable name in parts **i.** and **ii.** Rather than using X in both parts, use X in one and Y in the other.

Question 17b.

Worked solution

Use the **InvNorm** command on a CAS.



$\Pr(X > a) = 0.65$

$\Rightarrow \Pr(X < a) = 0.35$

$a = 50.4$

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIPS

» Where possible and when time permits, it is useful to check an answer for its reasonableness. Since $\Pr(X > a) > 0.5$, the answer must be less than the mean of 60, so check that your answer agrees with this.

» In order to use the InvNorm command, the probability needs to be rewritten in the form $\Pr(X < x)$.

Question 17c.i.

Worked solution

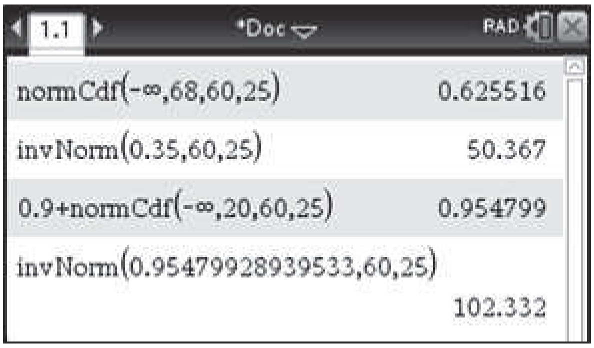
$\Pr(20 < X < b) = 0.9$

$\Rightarrow \Pr(X < b) - \Pr(X < 20) = 0.9$

$\Rightarrow \Pr(X < b) = 0.9 + \Pr(X < 20)$

$\Rightarrow \Pr(X < b) = 0.954799$

Using the **InvNorm** command on a CAS gives $b = 102.3$.



Answer: 102.3

Mark allocation: 2 marks

- 1 answer mark for the equation $\Pr(X < b) - \Pr(X < 20) = 0.9$, or equivalent.
- 1 answer mark for the correct answer.

Question 17c.ii.

Worked solution

This is a conditional probability question. The given condition (i.e. the event that is known to have occurred) is that the student is typical.

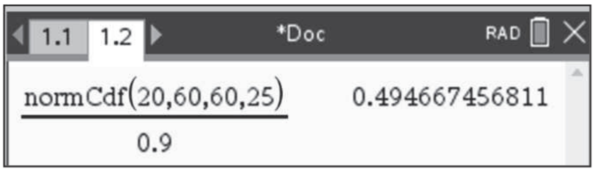
$$\Pr(20 < X < 60 | \text{typical}) = \frac{\Pr(20 < X < 60 \cap \text{typical})}{\Pr(\text{typical})}$$

$$= \frac{\Pr(20 < X < 60 \cap 20 < X < 102.3)}{\Pr(\text{typical})}$$

Note: We know from **part b.i.** that a typical student falls within the range $20 < X < 102.3$, and that $\Pr(\text{typical}) = 0.9$.

$$\Pr(20 < X < 60 | \text{typical student}) = \frac{\Pr(20 < X < 60)}{0.9}$$

$$= 0.4947$$



Answer: 0.4947

Mark allocation: 2 marks

- 1 method mark for recognising that conditional probability is involved, with either the numerator expressed as $\Pr(20 < X < 60)$ or a correct value for this, or the denominator is expressed as 0.9.
- 1 answer mark for the correct answer.

Question 17d.

Worked solution

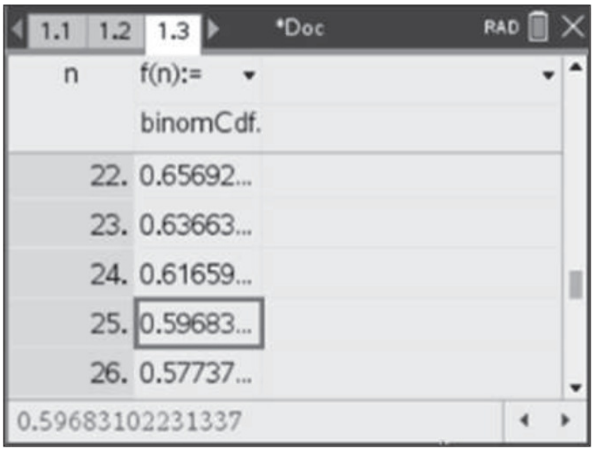
Let W be the random variable for the number of Year 9 students who do an insufficient amount of homework.

$W \sim \text{Bi}(n, 0.055)$

The smallest value of n such that $\Pr(W > 1) > 0.4$ is required.

$\Pr(W > 1) > 0.4 \Rightarrow \Pr(W \leq 1) < 0.6$. ($\Pr(W \geq 2) > 0.4$ can also be used.)

On a CAS, define the function $f(n) = \text{binomCdf}(n, 0.054799, 0, 1)$ and use a table (or trial and error) to find the value of n.



Answer: $n = 25$

Mark allocation: 2 marks

- 1 method mark for recognising that a binomial distribution is involved and for using the correct value of $p = 0.055$.
- 1 answer mark for the correct answer.

TIP

» **Finding a minimum sample size is a common question type. Make sure you can use your CAS calculator in the way described in the solution.**

Question 17e.

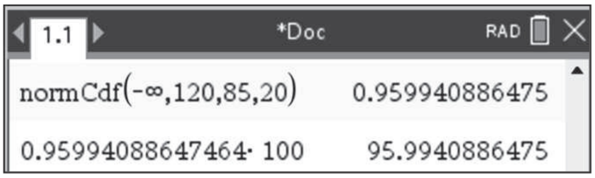
Worked solution

Let M be the random variable that is the amount of time required for a Year 9 student to complete their homework.

$M \sim \text{N}(85, 20^2)$

$\Pr(M < 120) = 0.9599\ldots$

$0.9599\ldots \times 100 = 95.99\%$



Answer: 95.99%

Mark allocation: 2 marks

- 1 answer mark for finding $\Pr(M < 120) = 0.9599\ldots$ or $\Pr(M > 120) = 0.0400\ldots$
- 1 answer mark for the correct answer expressed as a percentage.

Question 17f.

Worked solution

Let C be the random variable that is the amount of time required for a Year 7 student to complete their homework.

$C \sim N(30, \sigma^2)$

$\Pr(C > 25) = 0.83$

$\Pr(Z > z^*) = 0.83$

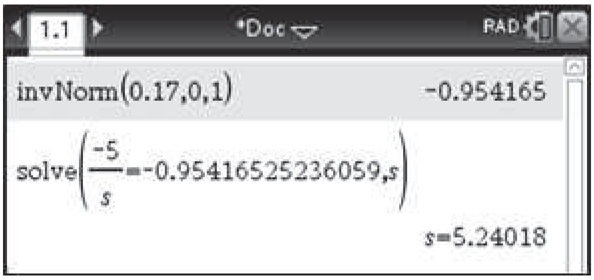
$\Rightarrow \Pr(Z < z^*) = 0.17$

Use the **InvNorm** command on a CAS to find that $z^* = -0.9541652$.

$z = \dfrac{c - \mu}{\sigma} \Rightarrow z^* = \dfrac{25 - 30}{\sigma} = -\dfrac{5}{\sigma}$

$\Rightarrow -0.954\ldots = \dfrac{5}{\sigma}$

Then solve for σ.



Answer: 5.24

Mark allocation: 2 marks

- 1 answer mark for finding $z^* = -0.954\ldots$
- 1 mark for the correct answer.

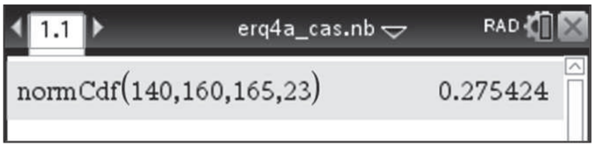
Question 18a.

Worked solution

$X \sim \text{N}(165, 23^2)$

$\Pr(140 < X < 160) = 0.2754$

Use the **NormCdf** command on a CAS to find the required probability.



Answer: 0.2754

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 18b.

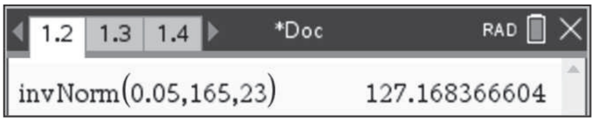
Worked solution

$\Pr(X < x) = 0.05$

$x = 127.168\ldots$

$\quad = 127\text{ g}$ (correct to nearest gram)

Use the **InvNorm** command on a CAS.



Answer: 127 grams

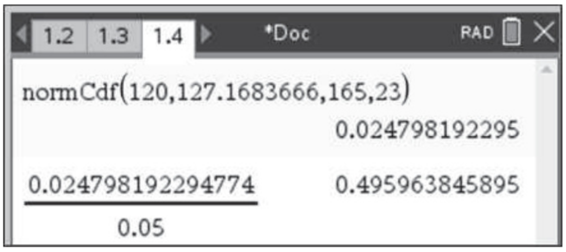
Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 18c.

Worked solution

$$\begin{aligned}\Pr(X > 120 \mid \text{small}) &= \frac{\Pr(X > 120 \cap \text{small})}{\Pr(\text{small})} \\ &= \frac{\Pr(X > 120 \cap X < 127.168\ldots)}{\Pr(X < 127.168\ldots)} \\ &= \frac{\Pr(120 < X < 127.168\ldots)}{0.05} \\ &= 0.4959\ldots \\ &= 0.4960 \text{ (correct to four decimal places)}\end{aligned}$$



Answer: 0.4960

Mark allocation: 2 marks

- 1 method mark for recognising that conditional probability is involved, with either the numerator expressed as $\Pr(120 < X < 127.168\ldots)$ or as a correct value, or the denominator is expressed as 0.05.
- 1 answer mark for the correct answer.

TIP

» Use unrounded values in calculations to ensure that the final answer is accurate, even when a rounded value has been given as the answer to an earlier question.

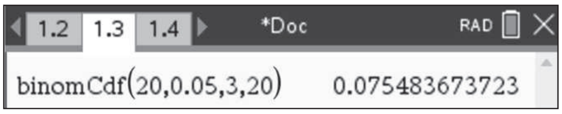
Question 18d.

Worked solution

Let $Y \sim \text{Bi}(20, 0.05)$

$\Pr(Y > 2) = \Pr(Y \geq 3)$

$= 0.075$



Answer: 0.075

Mark allocation: 2 marks

- 1 method mark for $Y \sim \text{Bi}(20, 0.1)$ or $\Pr(Y \geq 3)$, or an equivalent probability expression using $\geq$ or $\leq$. Note that just stating $\Pr(Y > 2)$ is not sufficient for this mark.
- 1 answer mark for the correct answer.

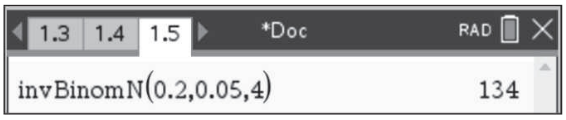
Question 18e.

Worked solution

Let $W \sim \text{Bi}(n, 0.05)$. We want to find n such that $\Pr(W \geq 5) > 0.8$.

This is equivalent to $\Pr(W < 5) < 0.2 \Rightarrow \Pr(W \leq 4) < 0.2$.

Use the Inverse Binomial function on a CAS to find that $n = 134$.



Answer: 134

Mark allocation: 2 marks

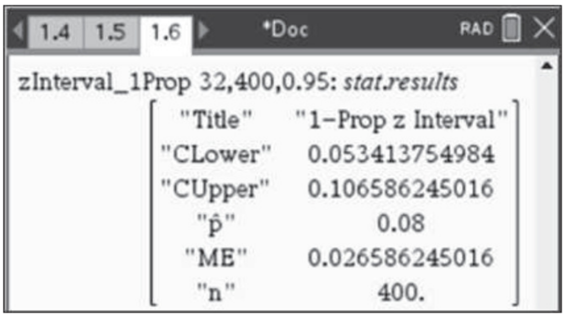
- 1 method mark for stating $W \sim \text{Bi}(n, 0.05)$ or for relevant working (such as $\Pr(W \leq 4) < 0.2$).
- 1 mark for the correct answer.

Question 18f.i.

Worked solution

8% of 400 is 32.

Use CAS to determine the confidence interval: $95\%\ \text{CI} = (0.053, 0.107)$.



Answer: $(0.053, 0.107)$

Mark allocation: 1 mark

- 1 mark for the correct answer.

TIP

» **Remember to place round brackets around a confidence interval.**

Question 18f.ii.

Worked solution

The proportion of apples that are small in Victoria is 0.05. This value lies outside the NSW confidence interval, so the population proportions could be different.

Mark allocation: 1 mark

- 1 mark for the correct answer.

Question 19a.

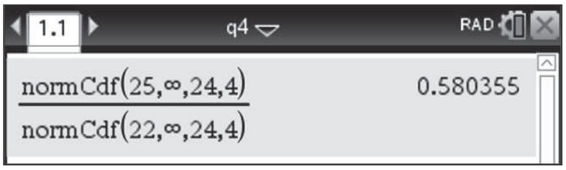
Worked solution

$X \sim \mathrm{N}(24, 4^2)$

$$\Pr(X > 25 \mid X > 22) = \frac{\Pr(X > 25 \cap X > 22)}{\Pr(X > 22)}$$

$$= \frac{\Pr(X > 25)}{\Pr(X > 22)}$$

$$= 0.5804$$



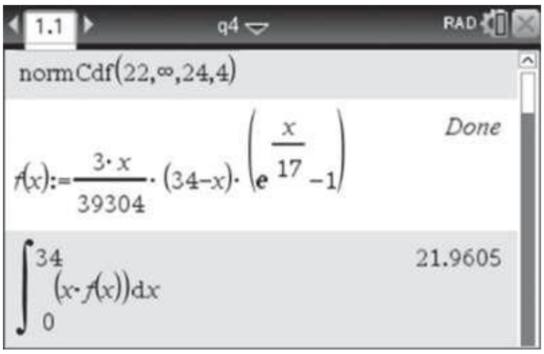
Mark allocation: 2 marks

- 1 method mark for the conditional probability expressed as $\frac{\Pr(X > 25)}{\Pr(X > 22)}$, or equivalent, with either the numerator or denominator correct.
- 1 answer mark for the correct answer.

Question 19b.

Worked solution

Use a CAS to calculate $\mathrm{E}(X) = \int_0^{34} x f(x)\,dx = 21.9605$ cm.



Mark allocation: 1 mark

- 1 answer mark for the correct answer.

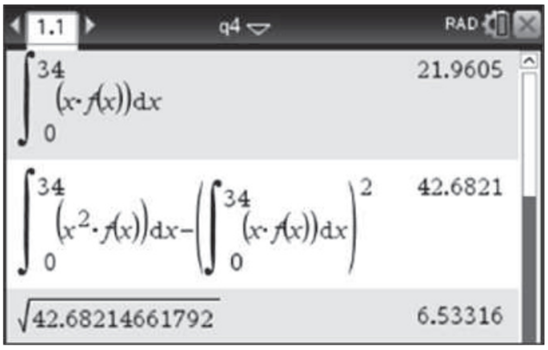
Question 19c.

Worked solution

Use the formula for the variance:

$$\begin{aligned}\text{var}(X) &= \text{E}(X^2) - (\text{E}(X))^2 \\ &= \int_0^{34} x^2 f(x)dx - \left(\int_0^{34} xf(x)\right)^2 \\ &= 42.682\ldots\end{aligned}$$

The standard deviation is $\text{sd}(X) = \sqrt{\text{var}(X)} = 6.5332$ cm.



Mark allocation: 2 marks

- 1 method mark for a suitable integral expression, using the variance formula.
- 1 answer mark for the correct answer.

TIP

» Do not round the variance before taking the square root.

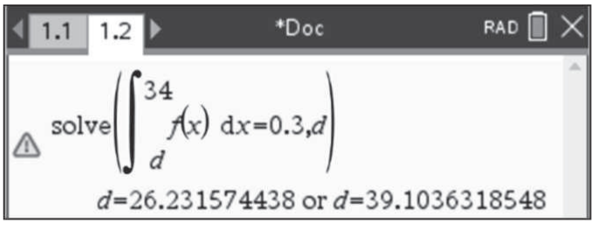
Question 19d.

Worked solution

$\Pr(X > d) = 0.3$

$\int_d^{34} f(x) = 0.3$

Solving for d, where $0 \le d \le 34$, gives $d = 26.23$ cm.



Alternatively, $\int_0^d f(x) = 0.7$ can be used.

Mark allocation: 2 marks

- 1 answer mark for correctly formulating an integral equation.
- 1 answer mark for the correct answer.

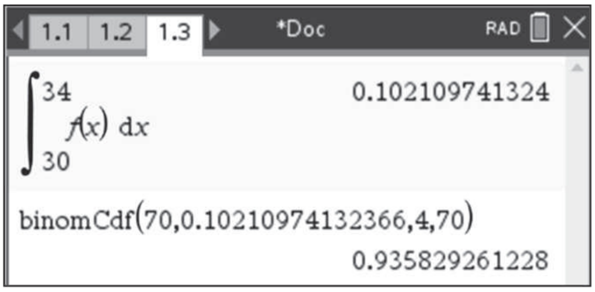
Question 19e.

Worked solution

$\Pr(X > 30) = \int_{30}^{34} f(x)dx = 0.102\ldots$

Let $W \sim \text{Bi}(70, 0.1021)$.

$\Pr(W \geq 4) = 0.9358$



Mark allocation: 2 marks

- 1 answer mark for $\Pr(X > 30) = 0.102\ldots$
- 1 answer mark for the correct answer.

Question 19f.

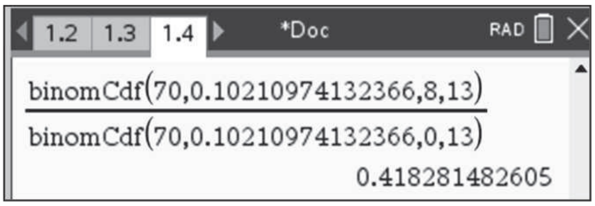
Worked solution

We need to find $\Pr(\hat{P} > 0.1 \mid \hat{P} < 0.2)$. Begin by converting the values of $\hat{p}$ to values of W:

$\hat{p} = 0.1 \Rightarrow W = 0.1 \times 70 = 7$

$\hat{p} = 0.2 \Rightarrow W = 0.2 \times 70 = 14$

$$\begin{aligned}\Pr(\hat{P} > 0.1 \mid \hat{P} < 0.2) &= \Pr(W > 7 \mid W < 14) \\ &= \frac{\Pr(W > 7 \cap W < 14)}{\Pr(W < 14)} \\ &= \frac{\Pr(7 < W < 14)}{\Pr(W < 14)} \\ &= \frac{\Pr(8 \leq W \leq 13)}{\Pr(W \leq 13)} \\ &= 0.4183\end{aligned}$$



Mark allocation: 2 marks

- 1 method mark for a suitable probability expression expressed in terms of the integer values of the binomial random variable.
- 1 answer mark for the correct answer.

Question 19g.

Worked solution

First determine the value of $\hat{p}$:

$$\hat{p} = \frac{0.203065 + 0.08265}{2} = 0.142858$$

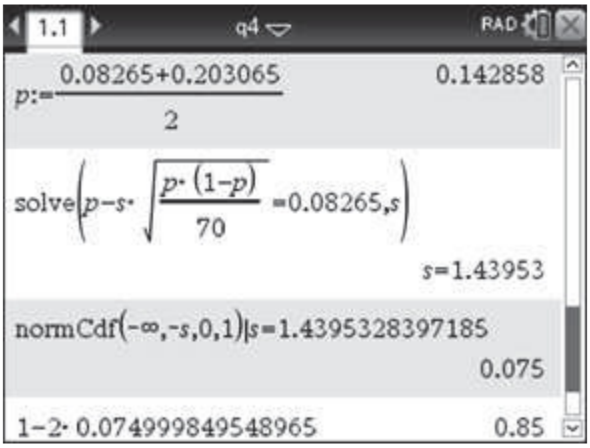
Therefore

$$0.08265 = \hat{p} - s\sqrt{\frac{\hat{p}(1-\hat{p})}{70}}$$

$$\Rightarrow s = 1.43953$$

Let $Z \sim \text{N}(0, 1)$. Hence $\Pr(Z < -s) = 0.075$.

Therefore $c = 1 - 2 \times 0.075 = 0.85$.



Mark allocation: 2 marks

- 1 answer mark for finding $\hat{p}$.
- 1 answer mark for the correct answer.

