Essential Insight Exam Guide

Mathematics Specialist

Year 12 WACE
Western Australian Curriculum

2025 Edition

Jeremy Chen

Essential Insight Exam Guide
Mathematics Specialist
Year 12 WACE

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Acknowledgements
School Curriculum and Standards Authority. (2019-2023). ATAR Examinations and Marking Keys. The School Curriculum and Standards Authority does not endorse this publication or product.

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Unit 3

Unit 3.1 – Complex numbers

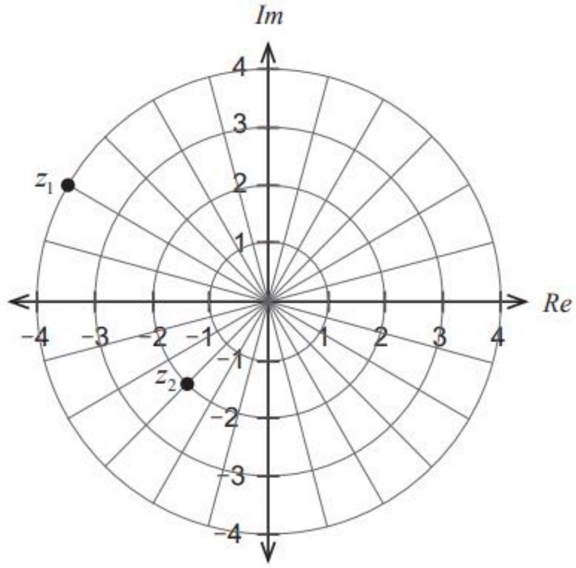
Section 1

2023 Section 1 Question 2 **Complex numbers**	$P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + 14$ is a fifth order polynomial with real coefficients. It is known that $P(z) = (z - z_0)\,Q(z)$ where z_0 is real and $Q(z)$ is a fourth order polynomial. Two roots of $P(z)$ are $z_1 = 1 + i$ and $z_2 = 2 + \sqrt{3}i$. (a) Determine $Q(z)$ in expanded form. (3 marks) (b) Determine the values of the coefficients a, b, c and d. (2 marks)

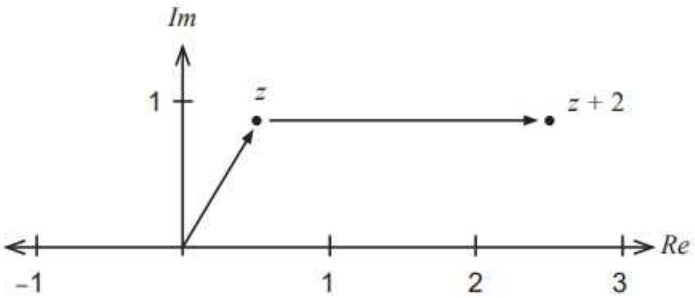
2023 Section 1 Question 6 **Complex numbers**	Solve the complex equation $z^4 = 2 - 2\sqrt{3}i$ giving solutions in the form $rcis\theta$ where $-\pi < \theta \le \pi$. (5 marks)

2023 Section 1 Question 8 **Complex numbers**	In the following simultaneous equations, a and b are real numbers. $a^3 = 3ab^2 + 14$ $b^3 = 3a^2b + 2\sqrt{5}$ In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a + bi)^3$. Hence, or otherwise, determine the exact value of $a^2 + b^2$. (4 marks)

2022 Section 1 Question 6 **Complex numbers**	Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below. (a) Determine the exact polar form for z_2. (2 marks) (b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)



	(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p. Justify your answer. (3 marks)

2021 Section 1 Question 1 **Complex numbers**	The Argand diagram below shows the complex numbers z and $z+2$ where $z = cis\left(\frac{\pi}{3}\right)$.  Determine the exact value for: (a) $Arg\,(-z)$. (1 mark)

	(b) $\|z+2\|$. (3 marks)

2021 Section 1 Question 6 **Complex numbers**	Consider the quartic polynomial $P(z) = z^4 - 6z^3 + 31z^2 - 52z + 60$. (a) Given that $P(2+4i) = 0$, determine a quadratic factor of $P(z)$. (2 marks) (b) Hence solve the equation $z^4 - 6z^3 + 31z^2 - 52z + 60 = 0$. (3 marks)

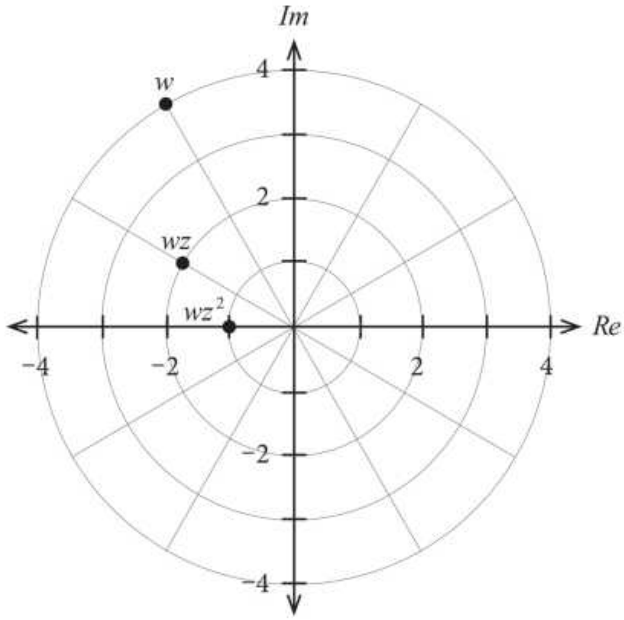
2021 Section 1 Question 7 **Complex numbers**	The number 2021 can be expressed as a product of two consecutive prime numbers: 43 × 47 = 2021. Consider the complex equation $z^{43} = 1$. (a) Write an expression for the roots of $z^{43} = 1$. (2 marks) Let w be any one of the roots of the equation $z^{43} = 1$. (b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. (3 marks)

2020 Section 1 Question 8 **Complex numbers**	Consider the complex sum: $\sum_{n=1}^{2020} n i^n = 1i^1 + 2i^2 + 3i^3 + ... + 2020i^{2020}$ Express the value of this sum in the form $r cis\theta$ where $-\pi < \theta \leq \pi$. (3 marks)

2019 Section 1 Question 2 **Complex numbers**	Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers. (a) Show that $(z - 2i)$ is a factor of $P(z)$. (2 marks) (b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$. (4 marks)

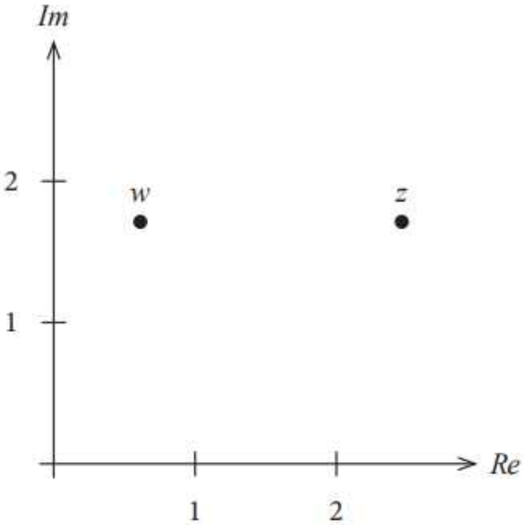
2019 Section 1 Question 9 **Complex numbers**	Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$. If the roots are designated as $z_0, z_1, z_2, \ldots, z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times \ldots \times z_{n-1}$. (4 marks)

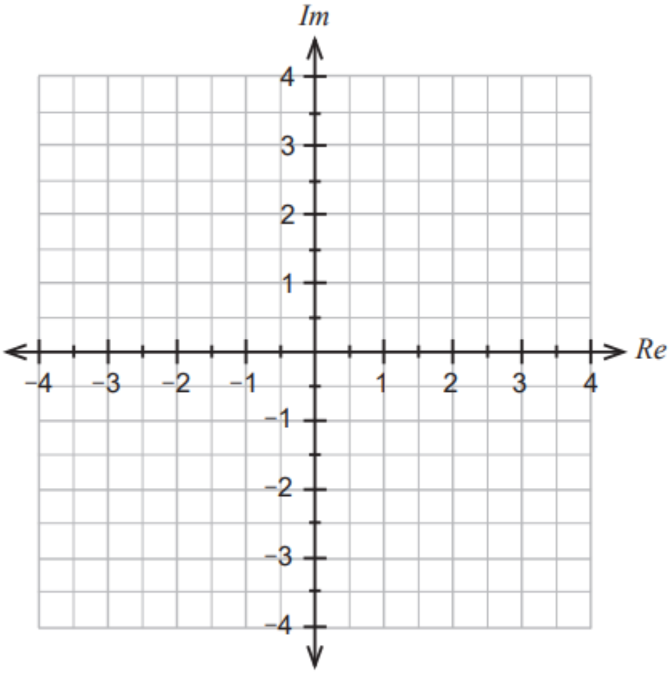
[Section 2](#)

2023 Section 2 Question 10 **Complex numbers**	The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.  (a) Express wz and wz^2 in exact polar form. (2 marks) Consider the geometric transformation(s) applied to transform $w \to wz \to wz^2$ etc. (b) Describe the geometric transformation(s) performed by successive multiplication by z. (2 marks)

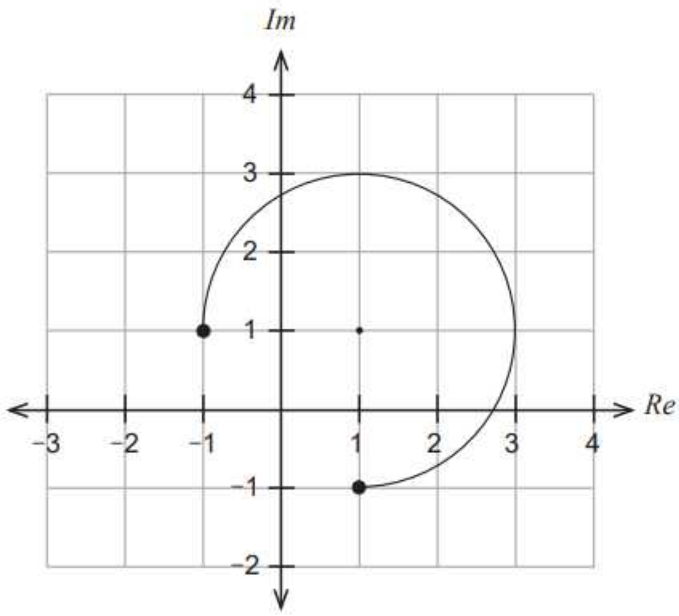
(c) Determine z in exact polar form. (1 mark)

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}.
(2 marks)

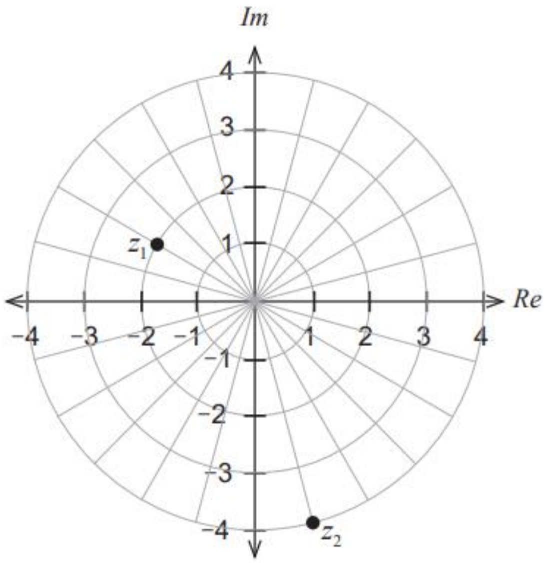
2023 Section 2 Question 12 **Complex numbers**	Complex numbers z and w are shown in the Argand diagram below. It is known that: $\lvert z\rvert = 3,\ Arg\,(z) = \theta$ where $0 < \theta < \frac{\pi}{4}$ $w = z - k$ such that $Arg\,(w) = 2\theta$ where $Im\,(k) = 0,\ k > 0.$  (a) Represent the given information on the Argand diagram. (3 marks) (b) Determine a simplified expression for k in terms of θ. Justify your answer. (3 marks)

2022 Section 2 Question 9 **Complex numbers**	(a) Sketch the locus of a complex number z satisfying the condition: $\lvert z-2i\rvert + \lvert z-(3-2i)\rvert = 5$ (2 marks)  (b) Describe the locus of the equation $(z+i)(\overline{z+i})=2$. (3 marks)

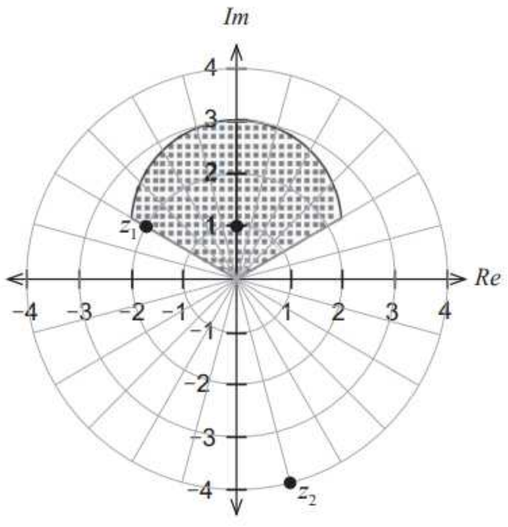
(c) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (3 marks)

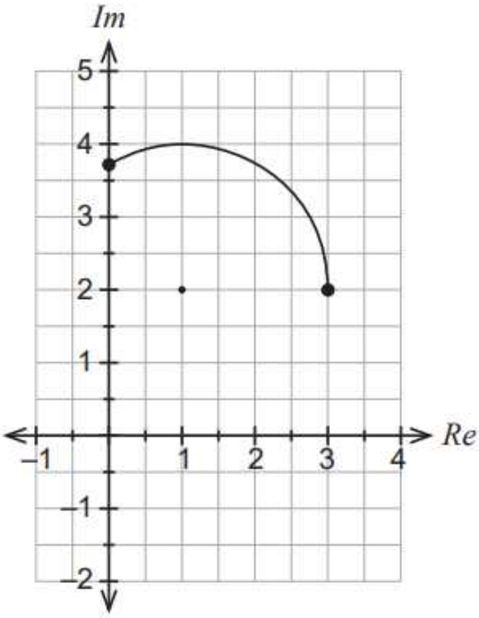
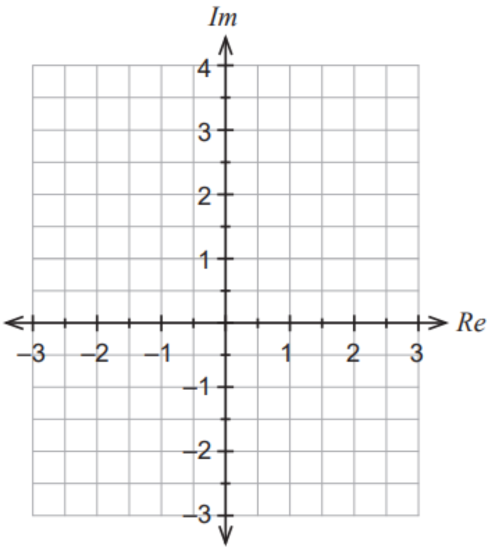


2022 Section 2 Question 18 **Complex numbers**	(a) Show that for all positive integers n and complex numbers z where $0 \le \theta \le \frac{\pi}{2}$, $(z^n - cis\,(\theta))(z^n + cis\,(-\theta)) = z^{2n} - (2i\sin\theta)\,z^n - 1$. (3 marks) (b) Hence, using the result from part (a), obtain all the solutions to the equation $z^6 - (i)z^3 - 1 = 0$ in exact polar form. (4 marks)

2021 Section 2 Question 11 **Complex numbers**	Two complex numbers z_1 and z_2 are shown in the Argand plane below.  (a) Write the expression for z_1 in exact polar form. (2 marks) (b) Write the expression for z_1 in exact Cartesian form. (1 mark) (c) Plot the complex number iz_1 on the Argand diagram above. (2 marks)

(d) A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)

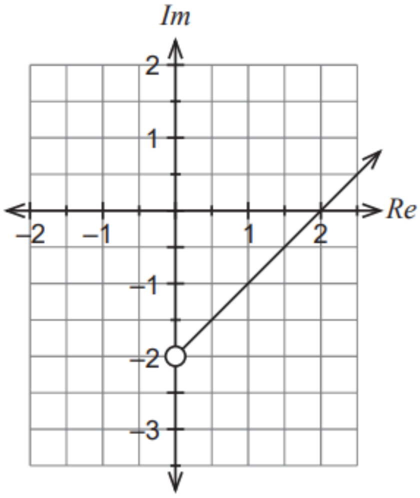


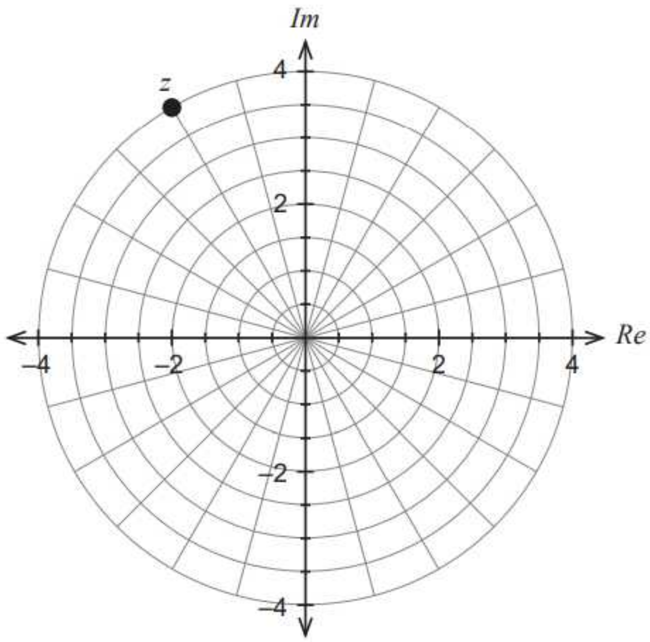
2020 Section 2 Question 10 **Complex numbers**	(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)  (b) Sketch the locus of the equation $\lvert z+2\rvert = \lvert z-i\rvert + \sqrt{5}$ in the Argand diagram below. (3 marks) 

2020 Section 2 Question 11 **Complex numbers**	Let z, w and u be complex numbers where: $w = (1+i)\bar{z}$ $\quad Arg(w) = \frac{\pi}{3}$ $\quad \lvert w \rvert = 2$ $u = \frac{z}{2-2i}$ (a) Determine $Arg(u)$ exactly. (3 marks) (b) Determine $\lvert u \rvert$ exactly. (2 marks)

2020 Section 2 Question 13 **Complex numbers**	Solve the equation $z^4 = 8\sqrt{3} + 8i$ giving exact solutions in the form $r\,cis\,\theta$ where $-\pi < \theta \leq \pi$. (4 marks)

2020 Section 2 Question 15 **Complex numbers**	Let $z = r cis\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$. (a) Express in terms of r and θ the complex number $\frac{\bar{z}}{-\sqrt{2}(i+1)}$. (3 marks) (b) Express $\alpha = Arg(z - ri)$ in terms of θ where $0 < \alpha < 2\pi$. (3 marks)

2019 Section 2 Question 10 **Complex numbers**	The sketch of the locus of a complex number $z = x + iy$ is shown below.  (a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. (2 marks) (b) Determine the minimum value for $\|z - i\|$ as an exact value. (3 marks)

2019 Section 2 Question 12 **Complex numbers**	Let $w = \frac{1-i}{2\sqrt{2}}$. (a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \le \pi$. (2 marks) The complex number z is represented in the Argand diagram below.  (b) Express z exactly in the form $z = a + bi$. (2 marks)

(c) Determine the exact polar form for wz and $w^2 z$. (2 marks)

(d) On the Argand diagram on page 6, plot the position for wz and $w^2 z$. Ensure that each position is labelled clearly. (2 marks)

Consider the geometric transformation(s) applied to transform $z \to wz \to w^2 z \to w^3 z$ etc

(e) Describe the geometric transformation(s) performed by the successive multiplication by w.
(2 marks)

Marking Guide – Section 1

<table>
<tr><td>2023
Section 1
Question 2

Complex numbers</td><td>

$P(z) = z^5 + az^4 + bz^3 + cz^2 + dz + 14$ is a fifth order polynomial with real coefficients. It is known that $P(z) = (z - z_0)\,Q(z)$ where z_0 is real and $Q(z)$ is a fourth order polynomial. Two roots of $P(z)$ are $z_1 = 1 + i$ and $z_2 = 2 + \sqrt{3}i$.

(a) Determine $Q(z)$ in expanded form. (3 marks)

Solution

If two roots are $1+i$ and $2+\sqrt{3}i$, then so are the conjugates $1-i$ and $2-\sqrt{3}i$.

$$\begin{aligned} \therefore\ Q(z) &= \left(z-(1+i)\right)\left(z-(1-i)\right)\left(z-\left(2+\sqrt{3}i\right)\right)\left(z-\left(2-\sqrt{3}i\right)\right) \\ &= \left((z-1)-i\right)\left((z-1)+i\right)\left((z-2)-\sqrt{3}i\right)\left((z-2)+\sqrt{3}i\right) \\ &= \left((z-1)^2+1\right)\left((z-2)^2+3\right) \\ &= \left(z^2-2z+2\right)\left(z^2-4z+7\right) \quad \ldots (1) \\ &= z^4-6z^3+17z^2-22z+14 \quad \ldots (2) \end{aligned}$$

Specific behaviours

✓ states that are $1-i$ and $2-\sqrt{3}i$ are also roots of $P(z)$
✓ expresses $Q(z)$ as a product of four linear factors correctly
✓ determines $Q(z)$ as either expression (1) or (2)

(b) Determine the values of the coefficients a, b, c and d. (2 marks)

Solution

Since $P(z) = z^5+az^4+bz^3+cz^2+dz+14$

$= (z-z_0)\left(z^4-6z^3+17z^2-22z+14\right)$

Then the constant term $14 = (-z_0)(14)$ $\quad \therefore\ z_0 = -1$

$\therefore\ P(z) = (z+1)\left(z^4-6z^3+17z^2-22z+14\right)$

$= z^5-5z^4+11z^3-5z^2-8z+14$

Hence $a=-5,\ b=11,\ c=-5,\ d=-8$

Specific behaviours

✓ determines the value for z_0
✓ states the values for a, b, c and d correctly

</td></tr>
</table>

2023 Section 1 Question 6 Complex numbers	Solve the complex equation $z^4 = 2 - 2\sqrt{3}i$ giving solutions in the form $rcis\theta$ where $-\pi < \theta \le \pi$. (5 marks)

Solution

$$|z^4| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4 \qquad Arg(z^4) = \tan^{-1}\left(\frac{-2\sqrt{3}}{2}\right) = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$\therefore$ Solve $z^4 = 4cis\left(-\frac{\pi}{3}\right)$

$$\therefore\ z = 4^{\frac{1}{4}}cis\left(\frac{-\pi}{12} + k\left(\frac{\pi}{2}\right)\right) \quad k = 0,1,2,3 \quad \dots (1)$$

Roots are: $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$

$z_1 = \sqrt{2}\,cis\left(\frac{5\pi}{12}\right)$

$z_2 = \sqrt{2}\,cis\left(\frac{11\pi}{12}\right)$

$z_3 = \sqrt{2}\,cis\left(-\frac{7\pi}{12}\right)$ Note: $z_3 = \sqrt{2}\,cis\left(\frac{17\pi}{12}\right)$ does not satisfy $-\pi < \theta \le \pi$.

Specific behaviours

✓ states the value for $|z^4|$ correctly

✓ states the value for $Arg(z^4)$ correctly

✓ states the principal solution $z_0 = \sqrt{2}\,cis\left(-\frac{\pi}{12}\right)$

✓ indicates a separation of $\frac{\pi}{2}$ between solution arguments

✓ states all solutions correctly using the condition $-\pi < \theta \le \pi$

2023 Section 1 Question 8 Complex numbers	In the following simultaneous equations, a and b are real numbers. $a^3 = 3ab^2 + 14$ $b^3 = 3a^2b + 2\sqrt{5}$ In order to determine the value of $a^2 + b^2$ from these equations, it is useful to consider the complex expansion for $(a + bi)^3$. Hence, or otherwise, determine the exact value of $a^2 + b^2$. (4 marks)

Solution

$$(a+bi)^3 = a^3 + 3a^2(bi) + 3a(bi)^2 + (bi)^3$$
$$= a^3 + (3a^2b)i - 3ab^2 - (b^3)i$$
$$= (a^3 - 3ab^2) + (3a^2b - b^3)i$$

From equation (1) we have: $a^3 - 3ab^2 = 14$

From equation (2) : $3a^2b - b^3 = -2\sqrt{5}$

Hence $(a+bi)^3 = 14 - 2\sqrt{5}\,i$.

$\therefore \left|(a+bi)^3\right| = \sqrt{14^2 + (2\sqrt{5})^2} = \sqrt{196+20} = \sqrt{216}$

$\therefore \left|(a+bi)\right|^3 = \sqrt{216}$ since $|z^3| = |z|^3$

$\therefore |a+bi| = (216)^{\frac{1}{6}}$

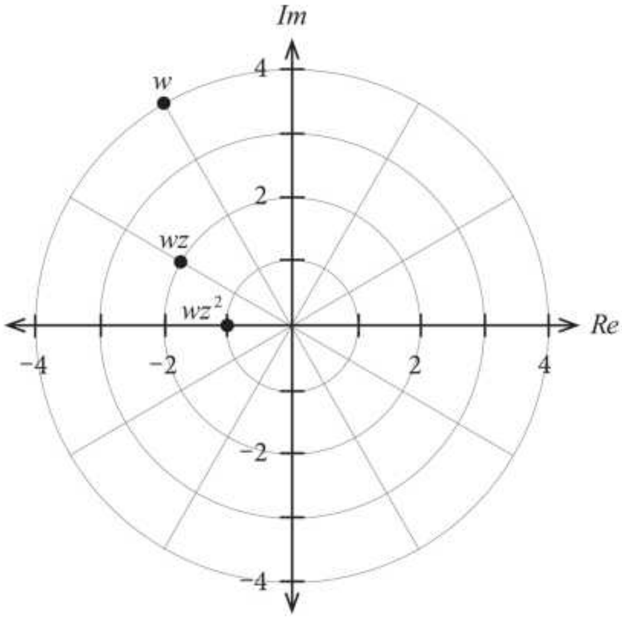
$\therefore |a+bi|^2 = (216)^{\frac{1}{3}}$

i.e. $a^2 + b^2 = \sqrt[3]{216} = 6$ Note: accept $\sqrt[3]{216}$ as the final answer.

Specific behaviours

- ✓ obtains $(a+bi)^3$ correctly as $(a^3 - 3ab^2) + (3a^2b - b^3)i$ or its equivalent
- ✓ deduces $(a+bi)^3 = 14 - 2\sqrt{5}\,i$
- ✓ obtains the value for $|a+bi| = (216)^{\frac{1}{6}}$ or its equivalent
- ✓ deduces the value of $a^2 + b^2$

2023 Section 2 Question 10 **Complex numbers**	The complex number $w = 4cis\left(\frac{2\pi}{3}\right)$ is shown in the Argand diagram, along with the complex numbers wz and wz^2.



(a) Express wz and wz^2 in exact polar form. (2 marks)

Solution
$wz = 2cis\left(\frac{5\pi}{6}\right)$ $\quad wz^2 = cis(\pi)$
Specific behaviours
✓ writes the correct modulus for each complex number ✓ writes the correct argument for each complex number or ✓ writes the correct modulus and argument for one complex number ✓ writes the correct modulus and argument for the other complex number

Consider the geometric transformation(s) applied to transform $w \to wz \to wz^2$ etc.

(b) Describe the geometric transformation(s) performed by successive multiplication by z.
(2 marks)

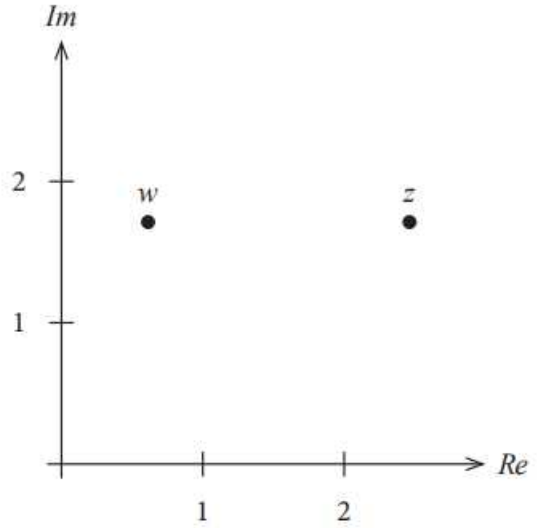
Solution
Successive multiplication by z results in the modulus changing by a factor of $\frac{1}{2}$ and the argument increasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 0.5. Rotation anti-clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as a dilation by factor 0.5 ✓ describes the change in the argument as an anti-clockwise rotation by 30°

(c) Determine z in exact polar form. (1 mark)

Solution
$z = \frac{1}{2} cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct polar form for z

(d) Describe the geometric transformation(s) performed by successive multiplication by z^{-1}.
(2 marks)

Solution
$z^{-1} = \left(\frac{1}{2} cis\left(\frac{\pi}{6}\right)\right)^{-1} = 2\,cis\left(-\frac{\pi}{6}\right)$ Successive multiplication by z^{-1} results in the modulus changing by a factor of 2 and the argument decreasing by $\frac{\pi}{6}$ i.e. 30°. Geometric description: Each vector is scaled by a factor of 2. Rotation clockwise (about origin) by 30°.
Specific behaviours
✓ describes the change in the modulus as an enlargement/dilation by factor 2 ✓ describes the change in the argument as a clockwise rotation by 30°

2023 Section 2 Question 12 **Complex numbers**	Complex numbers z and w are shown in the Argand diagram below. It is known that: $\lvert z\rvert = 3,\ Arg\,(z) = \theta$ where $0 < \theta < \frac{\pi}{4}$ $w = z - k$ such that $Arg\,(w) = 2\theta$ where $Im\,(k) = 0,\ k > 0.$  (a) Represent the given information on the Argand diagram. (3 marks) **Solution** Shown on diagram. **Specific behaviours** ✓ indicates $\lvert z\rvert = 3$ and $Arg(z) = \theta$ using line segments or vectors ✓ indicates horizontal length equal to k ✓ indicates $Arg(w) = 2\theta$ (b) Determine a simplified expression for k in terms of θ. Justify your answer. (3 marks) **Solution** Applying the Cosine rule in ΔOAB: $3^2 = k^2 + k^2 - 2(k)(k)\cos(\pi - 2\theta)$ i.e. $9 = 2k^2 - 2k^2\cos(-2\theta)$ i.e. $9 = 2k^2 + 2k^2\cos(2\theta)$ i.e. $9 = 2k^2(1 + \cos(2\theta))$ $\therefore\ 9 = 2k^2(2\cos^2\theta) = 4k^2\cos^2\theta$ This yields $k = \frac{3}{2\cos\theta}$. **Specific behaviours** ✓ states that $\lvert w\rvert = k$ or refers to $s\angle OAB = \theta = s\angle AOB$ ✓ forms an equation relating k, θ using appropriate trigonometry ✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution

Applying the Sine rule in ΔOAB :

$$\frac{k}{\sin\theta} = \frac{3}{\sin(\pi - 2\theta)}$$

i.e. $k = \dfrac{3\sin\theta}{\sin(2\theta)} = \dfrac{3\sin\theta}{2\sin\theta\cos\theta}$

This yields $k = \dfrac{3}{2\cos\theta}$.

Specific behaviours

- ✓ states that $|w| = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
- ✓ forms an equation relating k, θ using appropriate trigonometry
- ✓ obtains the correct simplified expression for k in terms of θ

Alternative Solution

Let $z = 3\cos\theta + (3\sin\theta)i$

$\therefore \quad w = (3\cos\theta - k) + (3\sin\theta)i$

But we also have $w = k\cos 2\theta + (k\sin 2\theta)i$

Equating imaginary parts: $k\sin 2\theta = 3\sin\theta$

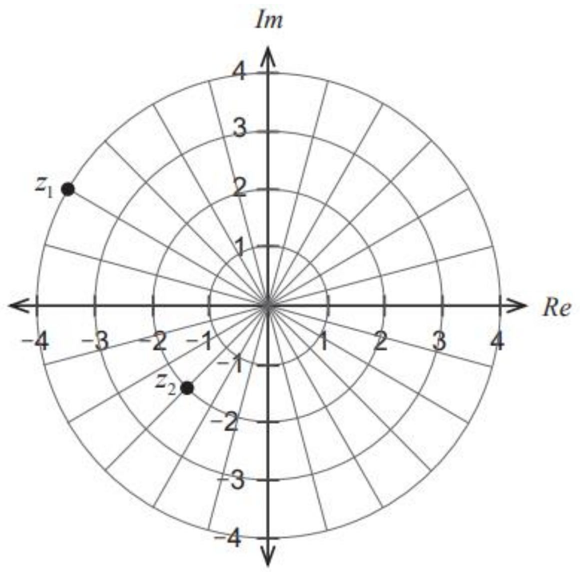
$$\therefore \quad k = \frac{3\sin\theta}{\sin 2\theta}$$

This yields $k = \dfrac{3}{2\cos\theta}$.

Specific behaviours

- ✓ states that $|w| = k$ or refers to $s\angle OAB = \theta = s\angle AOB$
- ✓ forms an equation relating k, θ using the real or imaginary parts
- ✓ obtains the correct simplified expression for k in terms of θ

2022 Section 1 Question 6 **Complex numbers**	Two complex numbers $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ and z_2 are shown in the Argand plane below.



(a) Determine the exact polar form for z_2. (2 marks)

Solution
$z_2 = 2cis\left(-\frac{3\pi}{4}\right)$ Accept also $2cis\left(\frac{5\pi}{4}\right)$.
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

(b) Plot the complex number $w = z_1 \times (z_2)^{-1}$ on the Argand diagram above. (3 marks)

Solution
$w = \left(4cis\left(\frac{5\pi}{6}\right)\right)\times\left(2cis\left(-\frac{3\pi}{4}\right)\right)^{-1} = 4cis\left(\frac{5\pi}{6}\right)\times\frac{1}{2}cis\left(\frac{3\pi}{4}\right)$ $= 2cis\left(\frac{5\pi}{6}+\frac{3\pi}{4}\right)$ $= 2cis\left(\frac{19\pi}{12}\right)$ or $2cis\left(-\frac{5\pi}{12}\right)$ w shown on the Argand diagram above.
Specific behaviours
✓ applies DeMoivre's Theorem correctly to determine z_2^{-1} ✓ determines the correct polar form for w ✓ plots the correct position for w

(c) If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution of the equation $z^n = r$ where r is a positive real number and n is a positive integer, determine the smallest possible value for r in the form 2^p.

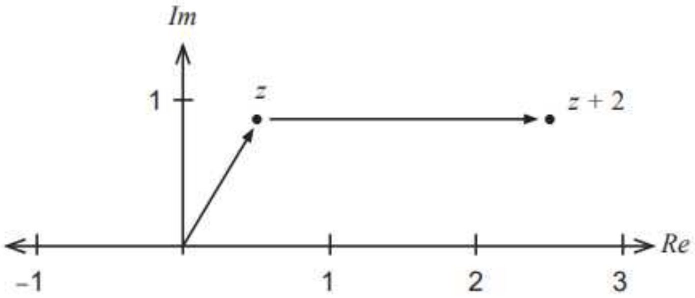
Justify your answer. (3 marks)

Solution
If $z_1 = 4cis\left(\frac{5\pi}{6}\right)$ is a solution then $\left(4cis\left(\frac{5\pi}{6}\right)\right)^n = r\,cis(2\pi k)$ i.e. $2^{2n}cis\left(\frac{5n\pi}{6}\right) = r\,cis(2\pi k)$ where $k = 0, 1, 2, \ldots, n-1$ i.e. $\frac{5n}{6} = 2k$ or $n = \frac{12k}{5}$ Hence the smallest possible value of $n = 12$ $(\text{when } k = 5)$ so that $n \in \mathbb{Z}^+$. $\therefore\ r = 2^{2\times12} = 2^{24}$ is the smallest value
Specific behaviours
✓ forms the equation that determines the relationship between n and integer k ✓ deduces the smallest value for n or k ✓ states the smallest value for r as a power of 2

2021 Section 1 Question 1

Complex numbers

The Argand diagram below shows the complex numbers z and $z+2$ where $z = cis\left(\frac{\pi}{3}\right)$.



Determine the exact value for:

(a) $Arg\,(-z)$. (1 mark)

Solution
$Arg(-z) = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$. Also accept $Arg(-z) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
Specific behaviours
✓ states the correct value

	(b) $\|z+2\|$. (3 marks)

Solution
Applying the cosine rule $\quad \|z+2\|^2 = 1^2+2^2-2(1)(2)\cos\left(\frac{2\pi}{3}\right)$
$= 1+4-4\left(-\frac{1}{2}\right) = 5+2 = 7$
$\therefore \quad \|z+2\| = \sqrt{7}$
Specific behaviours
✓ determines an angle of 120° between the vectors representing z and $z+2$
✓ applies the cosine rule correctly
✓ determines the value for $\|z+2\|$ correctly

Alternative Solution
$z = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \quad \therefore \; z+2 = \left(\frac{5}{2}\right) + \frac{\sqrt{3}}{2}i$
$\|z+2\|^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{25}{4} + \frac{3}{4} = 7$
$\therefore \quad \|z+2\| = \sqrt{7}$
Specific behaviours
✓ determines $z+2$ in Cartesian form correctly
✓ forms the expression for $\|z+2\|^2$ correctly
✓ determines the value for $\|z+2\|$ correctly

2021 Section 1 Question 6 Complex numbers	Consider the quartic polynomial $P(z)=z^4-6z^3+31z^2-52z+60$. (a) Given that $P(2+4i)=0$, determine a quadratic factor of $P(z)$. (2 marks)

Solution
Since $P(2+4i)=0$ then we also have $P(2-4i)=0$ as all coefficients are real. $Q(z) = (z-(2+4i))(z-(2-4i))$ $= (z^2-4z+20)$
Specific behaviours
✓ states that $P(2-4i)=0$ or states that $z-(2-4i)$ is a factor ✓ determines the quadratic factor $Q(z)$ correctly

(b) Hence solve the equation $z^4-6z^3+31z^2-52z+60=0$. (3 marks)

Solution
$P(z) = (z^2-4z+20)(z^2-2z+3)$ i.e. $T(z)=z^2-2z+3$ i.e. $P(z) = (z^2-4z+20)((z-1)^2+2)$ $= (z-(2+4i))(z-(2-4i))(z-(1+\sqrt{2}i))(z-(1-\sqrt{2}i))$ Solving $T(z)=0$ gives $z = 1 \pm \sqrt{2}\,i$ Solutions are $z = 2 \pm 4i,\ 1 \pm \sqrt{2}\,i$
Specific behaviours
✓ determines the quadratic factor $T(z)$ correctly ✓ states that $z = 1+\sqrt{2}\,i$ is a solution ✓ states that $z = 1-\sqrt{2}\,i$ is a solution

2021 Section 1 Question 7 Complex numbers	The number 2021 can be expressed as a product of two consecutive prime numbers: 43 × 47 = 2021. Consider the complex equation $z^{43} = 1$. (a) Write an expression for the roots of $z^{43} = 1$. (2 marks)

Solution
The equation $z^{43}=1$ has 43 roots where any root is of the form given by: $w = cis\left(\frac{2\pi k}{43}\right)$ where $k=0,1,2,\ \ldots\ ,42$.
Specific behaviours
✓ writes the correct form $cis\left(\frac{2\pi k}{43}\right)$ ✓ states that the integer $k=0,1,2,\ \ldots\ ,42$.

	Let w be any one of the roots of the equation $z^{43} = 1$. (b) How many of these roots will also be a solution of the equation $z^{47} = 1$? Justify your answer. (3 marks)

Solution
If w is also a root of $z^{47} = 1$ then we must show that $w^{47} = 1$. Examining the expression $w^{47} = \left(cis\left(\frac{2\pi k}{43}\right)\right)^{47}$ $\quad k = 0,1,2,...,42$ $= cis\left(\frac{47 \times 2\pi k}{43}\right) = cis\left(\frac{47 \times k \times 2\pi}{43}\right)$ This will be equal to ONE if and only if $\frac{47 \times k}{43}$ is an integer. If this occurs then the argument for w^{47} will be a multiple of 2π and hence $w^{47} = 1$. Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into k when $k = 1, 2, \ldots, 42$. Hence $\frac{47 \times k}{43}$ can never be an integer where $k = 1, 2, \ldots, 42$. When $k = 0$, $w = 1$ is a solution of BOTH $z^{43} = 1$ and $z^{47} = 1$. $\therefore$ *Only ONE* of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$.
Specific behaviours
✓ forms the expression for w^{47} correctly in terms of the integer k ✓ states that only ONE of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$ ✓ justifies the answer using the fact that the argument for w^{47} is never an even multiple of π (for $k \neq 0$)

	Alternative Solution The equation $z^{43} = 1$ has 43 roots where any root is of the form given by: $w = cis\left(\frac{2\pi k}{43}\right)$ where $k = 0,1,2, \ ... \ ,42.$ If w is also a root of $z^{47} = 1$ then $w = cis\left(\frac{2\pi m}{47}\right)$ where $m = 0,1,2, \ ... \ ,46.$ Hence we require : $w = cis\left(\frac{2\pi k}{43}\right) = cis\left(\frac{2\pi m}{47}\right)$ Hence $\left(\frac{2\pi k}{43}\right) = \left(\frac{2\pi m}{47}\right)$ where $k = 0,1,2, \ ... \ ,42$ and $m = 0,1,2, \ ... \ ,46.$ i.e. $\left(\frac{k}{43}\right) = \left(\frac{m}{47}\right)$ i.e. $m = \frac{47 \times k}{43}$ must be an integer. Since 43 and 47 are both prime numbers, then 43 does not divide into 47 and that 43 will not divide into k when $k = 1,2, \ ... \ ,42$. Hence $\frac{47 \times k}{43}$ can never be an integer where $k = 1,2, \ ... \ ,42$. When $k = 0, \ m = 0$, then $w = 1$ is a solution of BOTH $z^{43} = 1$ and $z^{47} = 1$. $\therefore$ *Only ONE* of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$. **Specific behaviours** ✓ forms the expression for the roots of $z^{47} = 1$ correctly in terms of the integer m (a different parameter to k) ✓ states that only ONE of the roots $(w = 1)$ of $z^{43} = 1$ is also a root of $z^{47} = 1$ ✓ justifies the answer using the fact $m = \frac{47 \times k}{43}$ cannot be an integer (unless both $k = 0, \ m = 0$)

2020 Section 1 Question 8 **Complex numbers**	Consider the complex sum: $\sum_{n=1}^{2020} n\,i^n = 1i^1 + 2i^2 + 3i^3 + \ldots + 2020i^{2020}$ Express the value of this sum in the form $r\,cis\,\theta$ where $-\pi < \theta \le \pi$. (3 marks)

Solution

$$\sum_{n=1}^{4} n\,i^n = 1(i)^1 + 2(i)^2 + 3(i)^3 + 4(i)^4$$
$$= i - 2 - 3i + 4 = 2 - 2i$$

$$\sum_{n=5}^{8} n\,i^n = 5(i)^5 + 6(i)^6 + 7(i)^7 + 8(i)^8$$
$$= 5i - 6 - 7i + 8 = 2 - 2i$$

$$\sum_{n=9}^{12} n\,i^n = 9(i)^9 + 10(i)^{10} + 11(i)^{11} + 12(i)^{12}$$
$$= 9i - 10 - 11i + 12 = 2 - 2i$$

Hence $\sum_{n=1}^{2020} n\,i^n = (2-2i)+(2-2i)+(2-2i)+\ldots$ 505 terms

$$= 505(2-2i)$$
$$= 1010 - 1010i$$
$$= 1010\sqrt{2}\,cis\left(-\frac{\pi}{4}\right) \quad or \quad \frac{2020}{\sqrt{2}}cis\left(-\frac{\pi}{4}\right)$$

Specific behaviours

✓ evaluates the sum of the first 4 terms correctly
✓ generalises that the sum of the first 4 terms repeats 505 times
✓ simplifies correctly in the form $r\,cis\,\theta$

2019 Section 1 Question 2 **Complex numbers**	Consider the function $P(z) = z^4 - 2z^3 + 14z^2 - 8z + 40$, defined over the complex numbers. (a) Show that $(z - 2i)$ is a factor of $P(z)$. (2 marks)

Solution

$$P(2i) = (2i)^4 - 2(2i)^3 + 14(2i)^2 - 8(2i) + 40$$
$$= 16(1) - 16(-1)(i) + 14(4)(-1) - 16i + 40$$
$$= 16 + 16i - 56 - 16i + 40 \quad \ldots (1)$$
$$= 0$$

Hence $(z-2i)$ is a factor of $P(z)$.

Specific behaviours

✓ substitutes $z = 2i$ correctly into $P(z)$

✓ obtains the 5 terms in expression (1) to deduce $P(2i) = 0$

(b) Hence or otherwise, solve the equation $P(z) = 0$, giving solutions in the form $a + bi$. (4 marks)

Solution
Since $(z-2i)$ is a factor then so is $(z+2i)$.
Hence $(z+2i)(z-2i)=(z^2+4)$ is also a factor of $P(z)$.
$\therefore\ P(z) = (z^2+4)Q(z)$ where $Q(z)=z^2-2z+10$
i.e.
Solving $Q(z)=0$ $\quad z^2-2z+10 = 0$ $\quad$ OR $\quad \therefore\ (z^2+4)=0$
$\therefore\ (z-1)^2+9=0$ $\quad \therefore\ z = \pm 2i$
$\therefore\ (z-1)^2=-9$
i.e. $z = 1 \pm 3i$
Specific behaviours
✓ deduces $(z+2i)$ is a factor of $P(z)$ or states $z=-2i$ is a solution
✓ deduces (z^2+4) is a factor of $P(z)$
✓ factorises $P(z)$ as $(z^2+4)(z^2-2z+10)$
✓ states $z=1\pm3i$ as solutions to $P(z)=0$

2019 Section 1 Question 9

Complex numbers

Consider the complex equation $z^n - 1 = 0$, where n is any positive integer $n \geq 3$.

If the roots are designated as $z_0, z_1, z_2, \ldots, z_{n-1}$, then determine the exact value for the product of the roots $p = z_0 \times z_1 \times z_2 \times \ldots \times z_{n-1}$.

(4 marks)

Solution

$z^n = 1 = cis(0) \quad \therefore \quad z = cis\left(\frac{0+2\pi k}{n}\right) = cis\left(\frac{2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n-1$

$\therefore \quad z_0 = cis(0) = 1,\ z_1 = cis\left(\frac{2\pi}{n}\right),\ z_2 = cis\left(\frac{4\pi}{n}\right),\ z_3 = cis\left(\frac{6\pi}{n}\right),\ z_4 = cis\left(\frac{8\pi}{n}\right)$

$z_{n-1} = cis\left(\frac{2(n-1)\pi}{n}\right)$

$$\begin{aligned} p &= cis(0)cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)cis\left(\frac{6\pi}{n}\right) \ldots cis\left(\frac{2(n-1)\pi}{n}\right) \\ &= cis\left(0+\frac{2\pi}{n}+\frac{4\pi}{n}+\frac{6\pi}{n}+\ldots\frac{2(n-1)\pi}{n}\right) \\ &= cis\left(\frac{2\pi}{n}(1+2+3+\ldots(n-1))\right) \\ &= cis\left(\frac{2\pi}{n}\times\frac{(n-1)(n)}{2}\right) \\ &= cis((n-1)\pi) = \cos(n-1)\pi + i\sin(n-1)\pi \end{aligned}$$

Since $\sin(n-1)\pi = 0$ for all integer values of n and $\cos(n-1)\pi = \pm 1$, then

Product $p = 1$ if n is ODD

$p = -1$ if n is EVEN.

Specific behaviours

✓ expresses the roots in the form $cis\left(\frac{2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n-1$

✓ forms the product $p = cis\left(\frac{2\pi}{n}\right)cis\left(\frac{4\pi}{n}\right)\ldots cis\left(\frac{2(n-1)\pi}{n}\right)$ correctly

✓ uses DeMoivres Theorem to obtain $cis((n-1)\pi)$ correctly

✓ states the two possible values correctly for n even and odd

	Alternative Solution
	Equation is $z^n - 1 = 0$ Given that the roots are: $z_0,\ z_1,\ z_2,\ \dots,\ z_{n-1}$ means that the equation can be written in the form $(z-z_0)(z-z_1)(z-z_2)\dots(z-z_{n-1}) = 0$ i.e. $(z-z_0)(z-z_1)(z-z_2)\dots(z-z_{n-1}) = z^n - 1$ Hence the LHS constants $(-z_0)(-z_1)(-z_2)\dots(-z_{n-1}) = -1$ (equating constants) Since there are n factors : IF n is EVEN then we have $(z_0)(z_1)(z_2)\dots(z_{n-1}) = -1$ i.e. $p = -1$ IF n is ODD then we have $-(z_0)(z_1)(z_2)\dots(z_{n-1}) = -1$ i.e. $p = 1$
	Specific behaviours
	✓ expresses the LHS in the form $(z-z_0)(z-z_1)(z-z_2)\dots(z-z_{n-1})$ ✓ states that the product of the constant terms $(-z_0)(-z_1)(-z_2)\dots(-z_{n-1}) = -1$ ✓ states that the product depends on whether n is even or odd ✓ states the correct value for the product for each case

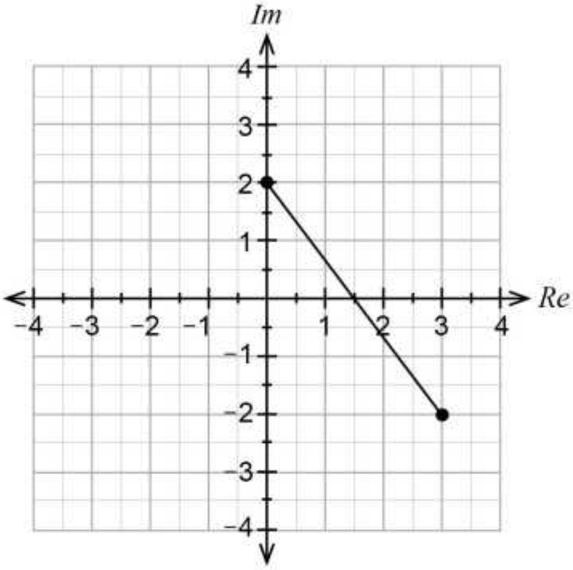
2022 Section 2 Question 9 Complex numbers	

2022 Section 2 Question 9

Complex numbers

(a) Sketch the locus of a complex number z satisfying the condition:

$|z-2i|+|z-(3-2i)|=5$ (2 marks)

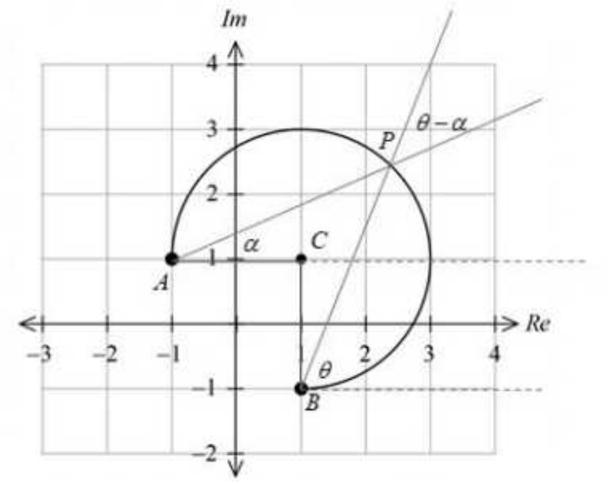


Solution
Locus can be interpreted as: "the distance of z from $2i$ added to the distance of z from $3-2i$ is equal to 5 units." Since the distance from $(0,2)$ and $(3,-2)$ is 5 units, then the locus is a line segment of the points connecting the end points $z=2i$ and $z=3-2i$.
Specific behaviours
✓ indicates a locus that contains $(0,2)$ and $(3,-2)$ ✓ indicates a line segment with correct end points

(b) Describe the locus of the equation $(z+i)(\overline{z+i})=2$. (3 marks)

Solution
$(z+i)(\overline{z+i})=2$ i.e. $\lvert z+i\rvert^2=2$ OR $x^2+(y+1)^2=2$ where $x=\mathrm{Re}(z),\ y=\mathrm{Im}(z)$ i.e. $\lvert z-(-i)\rvert=\sqrt{2}$ Description: Locus is a CIRCLE with radius $\sqrt{2}$ and centre $z=-i$.
Specific behaviours
✓ uses appropriate complex number properties to re-write the equation correctly ✓ states that the locus is a CIRCLE with a radius of $\sqrt{2}$ units ✓ states that the centre is $z=-i$ or $(0,-1)$

(c) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (3 marks)



Solution
Locus is given by $\lvert z-(1+i)\rvert = 2$ and $-\frac{\pi}{2} \le Arg(z-(1+i)) \le \pi$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ states an inequality about the argument from $1+i$ ✓ states the correct limits for the argument from $1+i$

Alternative Solution 1
Locus is given by $\lvert z-(1+i)\rvert = 2$ and $-\frac{\pi}{4} \le Arg(z) \le \frac{3\pi}{4}$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ states an inequality about the argument from the origin ✓ states the correct limits for the argument from the origin

Alternative Solution 2
Locus is given by $\lvert z-(1+i)\rvert = 2$ Using the Central Angle Theorem: $s\angle ACB = \frac{\pi}{2} \quad \therefore \ s\angle APB = \frac{\pi}{4}$ *i.e.* $\theta - \alpha = \frac{\pi}{4} \quad \therefore \ Arg(z-(1-i)) - Arg(z-(-1+i)) = \frac{\pi}{4}$ $\therefore \ Arg\left(\frac{z-(1-i)}{z-(-1+i)}\right) = \frac{\pi}{4}$
Specific behaviours
✓ states the equation $\lvert z-(1+i)\rvert = 2$ ✓ writes a difference of arguments equal to $\frac{\pi}{4}$ OR its equivalent ✓ writes the correct expression for each of the two arguments

2022 Section 2 Question 18 **Complex numbers**	

2022 Section 2 Question 18

Complex numbers

(a) Show that for all positive integers n and complex numbers z where $0 \le \theta \le \frac{\pi}{2}$,

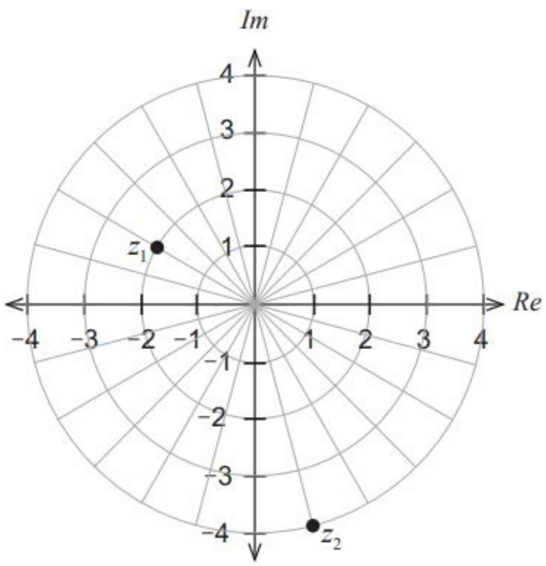
$(z^n - cis\,(\theta))(z^n + cis\,(-\theta)) = z^{2n} - (2i\sin\theta)\,z^n - 1$. (3 marks)

Solution
$\left(z^n - cis(\theta)\right)\left(z^n + cis(-\theta)\right)$
$= z^{2n} - cis(\theta)z^n + cis(-\theta)z^n - cis(\theta)cis(-\theta)$
$= z^{2n} - \left(cis(\theta) - cis(-\theta)\right)z^n - cis(0)$
$= z^{2n} - \left(2i\sin\theta\right)z^n - 1$
Specific behaviours
✓ expands the binomial products correctly to obtain 4 terms
✓ uses the property $cis(\theta)cis(-\theta) = cis(0) = 1$
✓ uses the property $cis(\theta) - cis(-\theta) = 2i\sin\theta$

(b) Hence, using the result from part (a), obtain all the solutions to the equation $z^6 - (i)z^3 - 1 = 0$ in exact polar form. (4 marks)

Solution
To solve $z^6 - (i)z^3 - 1 = 0$
i.e. $\left(z^3\right)^2 - \left(2\left(\frac{1}{2}\right)i\right)z^3 - 1 = 0$
$\therefore\ n = 3$ and $\sin\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{6}$ using $0 \le \theta \le \frac{\pi}{2}$ from part (a).
Hence solve $\left(z^3 - cis\left(\frac{\pi}{6}\right)\right)\left(z^3 + cis\left(-\frac{\pi}{6}\right)\right) = 0$
$\therefore\ z^3 = cis\left(\frac{\pi}{6}\right) \quad \therefore\ z = cis\left(\frac{\pi}{18}\right), cis\left(\frac{13\pi}{18}\right), cis\left(\frac{25\pi}{18}\right)$ or $cis\left(-\frac{11\pi}{18}\right)$
OR $z^3 = -cis\left(-\frac{\pi}{6}\right) = cis(\pi)cis\left(-\frac{\pi}{6}\right) = cis\left(\frac{5\pi}{6}\right)$
$\therefore\ z = cis\left(\frac{5\pi}{18}\right), cis\left(\frac{17\pi}{18}\right), cis\left(\frac{29\pi}{18}\right)$ or $cis\left(-\frac{7\pi}{18}\right)$
Specific behaviours
✓ determines the value of n and θ correctly
✓ deduces that $z^3 = cis\left(\frac{\pi}{6}\right)$ and $z^3 = cis\left(\frac{5\pi}{6}\right)$
✓ gives ALL solutions for $z^3 = cis\left(\frac{\pi}{6}\right)$ correctly
✓ gives ALL solutions for $z^3 = cis\left(\frac{5\pi}{6}\right)$ correctly

2021 Section 2 Question 11 Complex numbers	Two complex numbers z_1 and z_2 are shown in the Argand plane below.



(a) Write the expression for z_1 in exact polar form. (2 marks)

Solution
$z_1 = 2cis\left(\frac{5\pi}{6}\right)$
Specific behaviours
✓ states the correct modulus ✓ states the correct argument

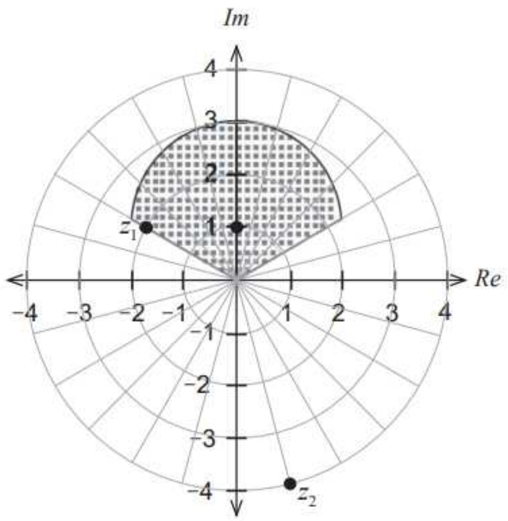
(b) Write the expression for z_1 in exact Cartesian form. (1 mark)

Solution
$z_1 = 2\cos\frac{5\pi}{6} + 2i\sin\frac{5\pi}{6} = 2\left(-\frac{\sqrt{3}}{2}\right) + 2i\left(\frac{1}{2}\right) = -\sqrt{3} + i$
Specific behaviours
✓ states the correct Cartesian form

(c) Plot the complex number iz_1 on the Argand diagram above. (2 marks)

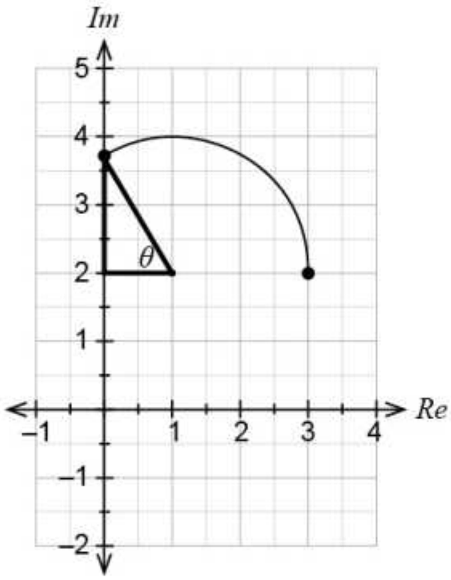
Solution
$iz_2 = cis\left(\frac{\pi}{2}\right).4cis(a) = 4cis\left(\frac{\pi}{2}+a\right)$ i.e. multiplying by i is to rotate z_2 90° anti-clockwise about the origin
Specific behaviours
✓ determines the correct value for iz_2 ✓ plots the correct position for iz_2

(d) A sketch of the locus of a complex number z is shown below. The upper boundary of the locus is part of a circle, centred at $z = i$. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)



Solution
The locus is part of the interior of the circle with centre $z = i$ and radius 2. $\lvert z-i \rvert \le 2$ with $\frac{\pi}{6} \le Arg(z) \le \frac{5\pi}{6}$
Specific behaviours
✓ uses the form $\lvert z-c \rvert \le r$ ✓ states $c = i$ and $r = 2$ ✓ uses the form $\theta_1 \le Arg(z) \le \theta_2$ ✓ states $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{5\pi}{6}$

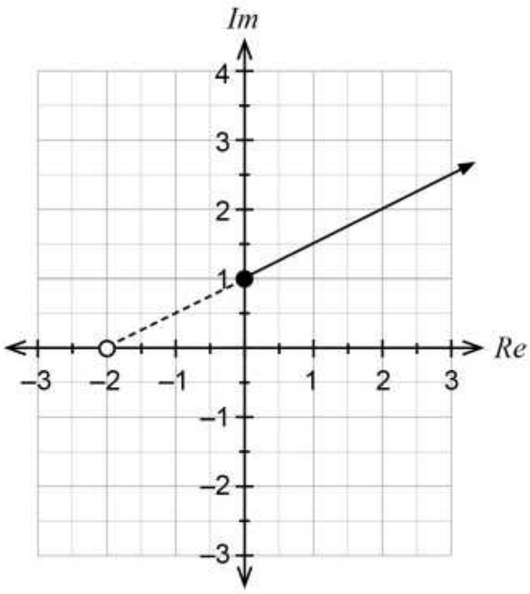
2020 Section 2 Question 10 **Complex numbers**	(a) The sketch of the locus of a complex number z has been shown below. Write equations or inequalities in terms of z (without using $x = Re(z)$ or $y = Im(z)$) for the indicated locus. (4 marks)



Solution
Arc is part of the circle $\left\|z-(1+2i)\right\| = 2$ such that $0 \le Arg\left(z-(1+2i)\right) \le \pi-\theta$ where $\tan\theta = \sqrt{3}$ i.e. $\theta = \frac{\pi}{3}$ $\therefore$ Locus is $\left\|z-(1+2i)\right\| = 2$, $0 \le Arg\left(z-(1+2i)\right) \le \frac{2\pi}{3}$
Specific behaviours
✓ writes the equation of the form $\left\|z-(1+2i)\right\| = r$ correctly ✓ states $r = 2$ ✓ writes the inequality of the form $c \le Arg\left(z-(1+2i)\right) \le k$ ✓ uses correct trigonometry to determine the limits c, k

(b) Sketch the locus of the equation $|z+2| = |z-i| + \sqrt{5}$ in the Argand diagram below.

(3 marks)



Solution
Shown above. This equation can be interpreted as 'the distance from $z=-2$ is equal to $\sqrt{5}$ more than the distance from $z=i$'.
Specific behaviours
✓ indicates the locus as a ray (part of a line) ✓ indicates $(0,1)$ i.e. from $z=i$, is an element of the locus ✓ indicates the correct ray (correct slope from $z=-2$ to $z=i$)

2020 Section 2 Question 11 **Complex numbers**	Let z, w and u be complex numbers where: $w = (1+i)\bar{z}$ $\quad Arg(w) = \frac{\pi}{3}$ $\quad \lvert w \rvert = 2$ $u = \frac{z}{2-2i}$ (a) Determine $Arg(u)$ exactly. (3 marks) **Solution** $Arg(w) = Arg(1+i) + Arg(\bar{z})$ $\frac{\pi}{3} = \frac{\pi}{4} - Arg(z)$ $\therefore \quad Arg(z) = -\frac{\pi}{12}$ $Arg(u) = Arg(z) - Arg(2-2i)$ $= -\frac{\pi}{12} - \left(-\frac{\pi}{4}\right)$ $= \frac{\pi}{6}$ **Specific behaviours** ✓ expresses relationships between arguments correctly ✓ determines $Arg(z)$ correctly ✓ determines $Arg(u)$ correctly (b) Determine $\lvert u \rvert$ exactly. (2 marks) **Solution** $\lvert w \rvert = \lvert 1+i \rvert \times \lvert \bar{z} \rvert$ $2 = \sqrt{2} \times \lvert z \rvert$ $\therefore \quad \lvert z \rvert = \sqrt{2}$ $\lvert u \rvert = \frac{\lvert z \rvert}{\lvert 2-2i \rvert} = \frac{\sqrt{2}}{2\sqrt{2}} = \frac{1}{2}$ **Specific behaviours** ✓ expresses relationship between the modulus of numbers correctly ✓ determines $\lvert u \rvert$ correctly

2020 Section 2 Question 13 **Complex numbers**	Solve the equation $z^4 = 8\sqrt{3} + 8i$ giving exact solutions in the form $r\,cis\,\theta$ where $-\pi < \theta \le \pi$. (4 marks)

Solution

$z^4 = 8\sqrt{3} + 8i \qquad r^2 = \left(8\sqrt{3}\right)^2 + 8^2$

$= 8^2(3) + 8^2 = 4\left(8^2\right) \quad \therefore \; r = 2(8) = 16$

$\tan\theta = \dfrac{8}{8\sqrt{3}} \quad \therefore \; \theta = \dfrac{\pi}{6}$

Hence solve $z^4 = 16cis\left(\dfrac{\pi}{6}\right)$

Solutions are given by $z = 16^{\frac{1}{4}}cis\left(\dfrac{\frac{\pi}{6} + 2\pi k}{4}\right)$ where $k = 0,1,2,3$.

Solutions are: $z_0 = 2cis\left(\dfrac{\pi}{24}\right)$

$z_1 = 2cis\left(\dfrac{13\pi}{24}\right)$

$z_2 = 2cis\left(\dfrac{13\pi}{24} + \dfrac{12\pi}{24}\right) = 2cis\left(-\dfrac{23\pi}{24}\right)$

$z_3 = 2cis\left(-\dfrac{11\pi}{24}\right)$

Specific behaviours

✓ determines the modulus correctly for $8\sqrt{3} + 8i$

✓ determines the argument correctly for $8\sqrt{3} + 8i$

✓ states one solution as $z = 2cis\left(\dfrac{\pi}{24}\right)$

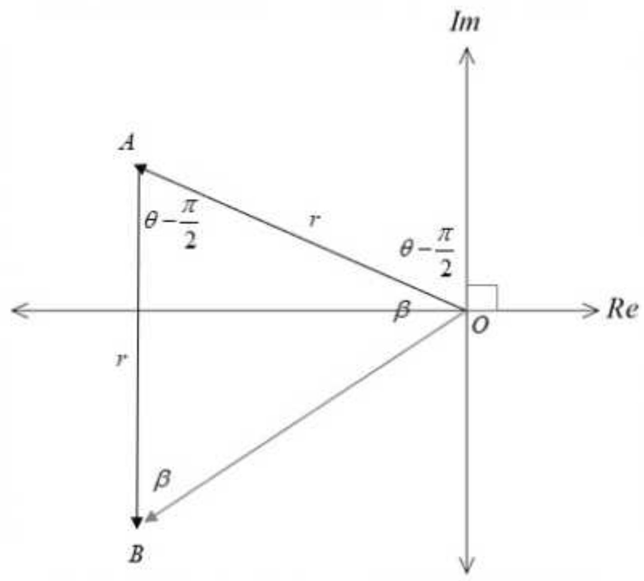
✓ states the correct arguments for the other 3 solutions (using $-\pi < \theta \le \pi$)

2020 Section 2 Question 15 **Complex numbers**	Let $z = r cis\theta$ be a complex number such that $\frac{\pi}{2} < \theta < \pi$.

(a) Express in terms of r and θ the complex number $\frac{\bar{z}}{-\sqrt{2}\,(i+1)}$. (3 marks)

Solution
$\frac{\bar{z}}{-\sqrt{2}(i+1)} = \frac{rcis(-\theta)}{2cis\left(-\frac{3\pi}{4}\right)} = \frac{r}{2}cis\left(\frac{3\pi}{4}-\theta\right)$
Specific behaviours
✓ writes the correct expression for $\bar{z}$ ✓ converts $-\sqrt{2}(i+1)$ into polar form correctly ✓ simplifies expression correctly in polar form

(b) Express $\alpha = Arg(z - ri)$ in terms of θ where $0 < \alpha < 2\pi$. (3 marks)

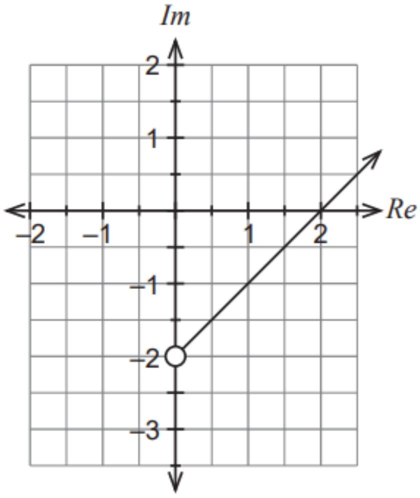


Solution
Vector $\overline{OB}$ represents $z-ri$. $s\angle OAB = \theta-\frac{\pi}{2}$ In isosceles ΔOAB, $OA = AB$ Hence $s\angle AOB = s\angle ABO = \beta$ We have $2\beta + \left(\theta-\frac{\pi}{2}\right) = \pi$ (Angle sum in ΔOAB) $\beta = \frac{3\pi}{4}-\frac{\theta}{2}$ Hence $\alpha = Arg(z-ri) = \theta+\beta = \frac{3\pi}{4} + \frac{\theta}{2}$
Specific behaviours
✓ draws the correct representation for $z-ri$ ✓ uses that $OA = AB$ to deduce congruent angles in ΔOAB ✓ deduces $\alpha = \frac{3\pi}{4}+\frac{\theta}{2}$

2019 Section 2 Question 10

Complex numbers

The sketch of the locus of a complex number $z = x + iy$ is shown below.



(a) Given that the equation for the above locus is written as $Arg(z - z_0) = k\pi$, determine the value of the constants z_0 and k. (2 marks)

Solution
The equation can be read as the argument of z from z_0 is equal to $k\pi$. i.e. $Arg\left(z-(-2i)\right) = \dfrac{\pi}{4}$ i.e. $z_0 = -2i,\ \ k = \dfrac{1}{4}$
Specific behaviours
✓ states the correct value for z_0 ✓ states the correct value for k

(b) Determine the minimum value for $|z - i|$ as an exact value. (3 marks)

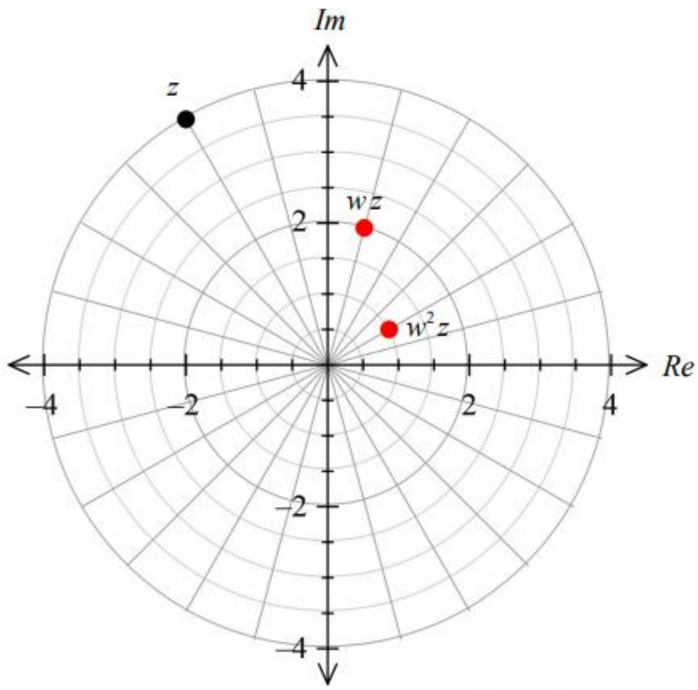
Solution
We require the minimum distance of a point in the locus from $z = i$ (point A). This will be the perpendicular distance AB to the locus. Point B will be the point $(1.5, -0.5i)$. Hence $AB = \sqrt{\left(\dfrac{3}{2}\right)^2 + \left(\dfrac{3}{2}\right)^2} = \sqrt{\dfrac{18}{4}} = \dfrac{3\sqrt{2}}{2}$ Hence the minimum value for $\lvert z-i \rvert = \dfrac{3\sqrt{2}}{2}$.
Specific behaviours
✓ indicates how the minimum value $\lvert z-i \rvert$ is found ✓ determines coordinates for point B correctly ✓ determines the minimum value $\lvert z-i \rvert$ correctly

2019 Section 2 Question 12 **Complex numbers**	Let $w = \frac{1-i}{2\sqrt{2}}$.

(a) Express w in the form $w = r\,cis\theta$, where $-\pi < \theta \le \pi$. (2 marks)

Solution
$w = \frac{1-i}{2\sqrt{2}} = \frac{\sqrt{2}cis\left(-\frac{\pi}{4}\right)}{2\sqrt{2}} = \frac{1}{2}cis\left(-\frac{\pi}{4}\right)$
Specific behaviours
✓ determines the correct modulus r ✓ determines the correct argument θ

The complex number z is represented in the Argand diagram below.



(b) Express z exactly in the form $z = a + bi$. (2 marks)

Solution
From the Argand diagram $z = 4cis\left(\frac{2\pi}{3}\right)$ Hence $z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + 2\sqrt{3}i$
Specific behaviours
✓ determines the polar form for z correctly (interprets the Argand diagram) ✓ determines the correct exact values for a, b

(c) Determine the exact polar form for wz and $w^2 z$. (2 marks)

Solution
Given $w = \frac{1}{2}cis\left(-\frac{\pi}{4}\right)$ and $z = 4cis\left(\frac{2\pi}{3}\right)$ Then $wz = \frac{1}{2}\times 4cis\left(-\frac{\pi}{4}+\frac{2\pi}{3}\right) = 2cis\left(\frac{5\pi}{12}\right)$ Also $w^2z = \left(\frac{1}{2}\right)^2 \times 4\times cis\left(-\frac{\pi}{2}+\frac{2\pi}{3}\right) = 1cis\left(\frac{\pi}{6}\right)$
Specific behaviours
✓ determines the correct modulus for both wz and w^2z ✓ determines the correct argument for both wz and w^2z

(d) On the Argand diagram on page 6, plot the position for wz and $w^2 z$. Ensure that each position is labelled clearly. (2 marks)

Solution
Indicated on the Argand diagram.
Specific behaviours
✓ indicates the correct modulus for both wz and w^2z (distance from origin) ✓ indicates the correct argument for both wz and w^2z (angle to real axis)

Consider the geometric transformation(s) applied to transform $z \rightarrow wz \rightarrow w^2 z \rightarrow w^3 z$ etc

(e) Describe the geometric transformation(s) performed by the successive multiplication by w. (2 marks)

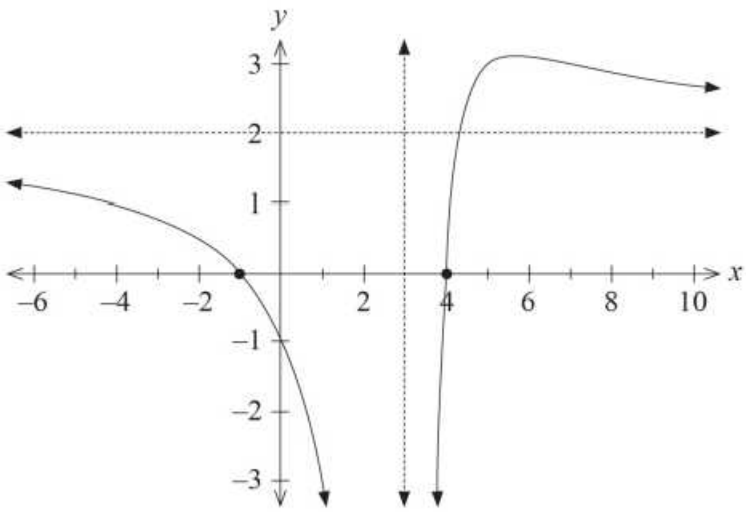
Solution
Successive multilplication by w results in the modulus changing by a factor of $\frac{1}{2}$ (successive points becoming twice as close to the origin) and the argument decreasing by 45° or $\frac{\pi}{4}$. Geometric description: Each vector is REDUCED by a factor of 0.5. Each vector is ROTATED clockwise (about origin) by 45°
Specific behaviours
✓ describes the change in the modulus a dilation by factor 0.5 ✓ describes the change in the argument as a clockwise rotation by 45° or $\frac{\pi}{4}$

Unit 3.2 – Functions and sketching graphs

Section 1

<table>
<tr>
<td>2023
Section 1
Question 1

Functions and sketching graphs</td>
<td>The graph of the function $f(x)=\dfrac{k(x+a)(x-b)}{(x-c)^2}$ is shown below. The constants a, b, c and k are positive.



Complete the table below by determining the values for a, b, c and k.

<table>
<tr><th>a</th><th>b</th><th>c</th><th>k</th></tr>
<tr><td></td><td></td><td></td><td></td></tr>
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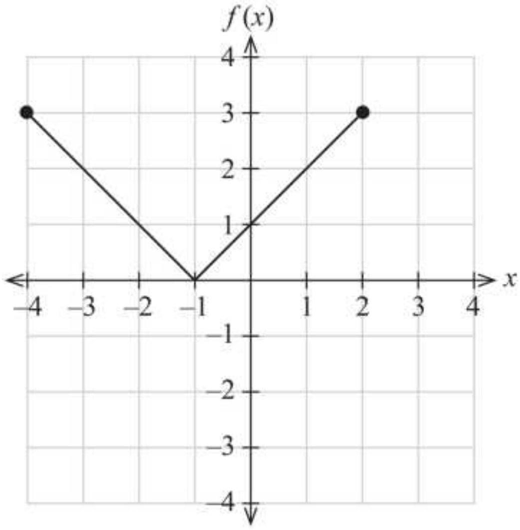
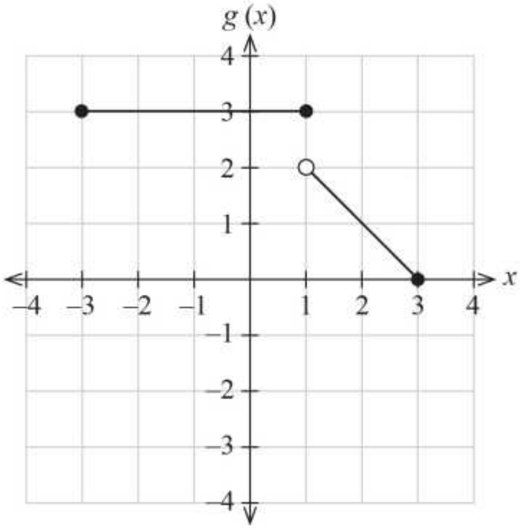
2023
Section 1
Question 4

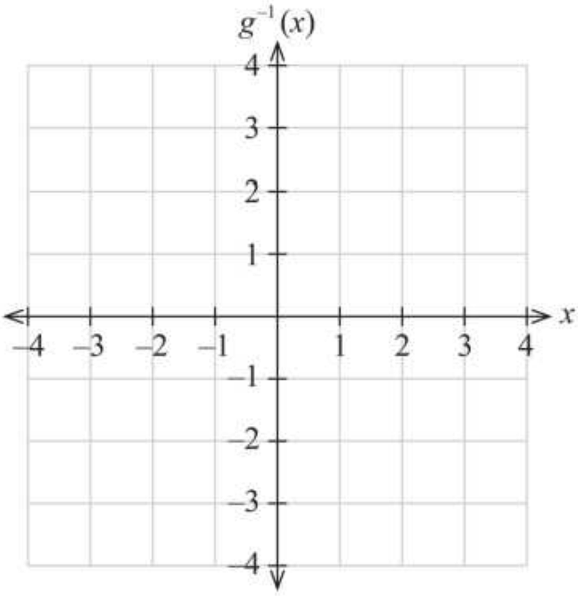
Functions and sketching graphs

The graphs of functions $f(x)$ and $g(x)$ are shown.





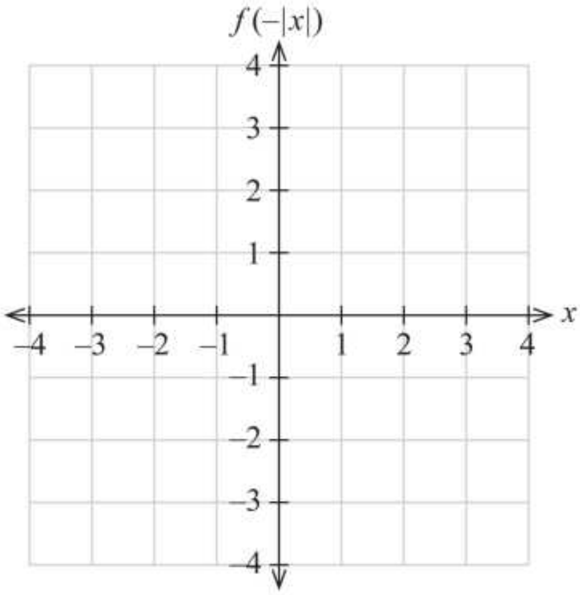
(a) Sketch the graph of $y = g^{-1}(x)$ on the axes below. (2 marks)



(b) State the value for $g\left(f^{-1}(0)\right)$. (2 marks)

(c) Determine the set of values of x such that $f(g(x))$ is defined. (2 marks)

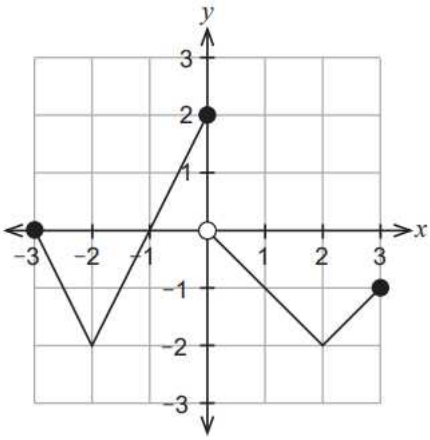
(d) Sketch the graph of $y = f(-|x|)$ on the axes below. (2 marks)



(e) The equation $|x+1| = k - |x+a|$ has an infinite number of solutions, with the solution set being $-3 \le x \le -1$. Determine the values of the constants a and k. (3 marks)

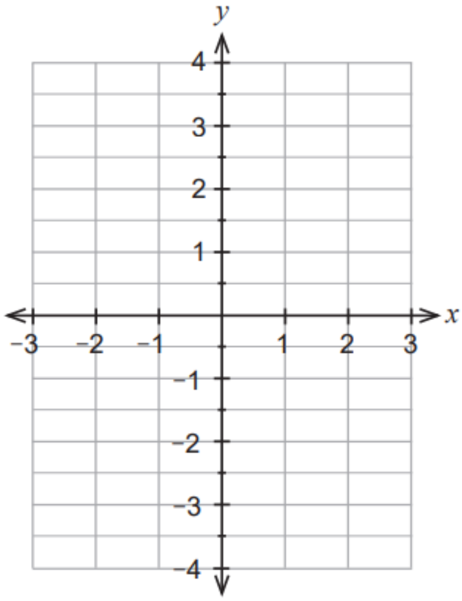
2022 Section 1 Question 1 **Functions and sketching graphs**	Consider functions $f(x) = \sqrt{4-x}$ and $g(x) = \frac{1}{x^2}$. (a) Determine the exact value of $g(f(-5))$. (2 marks) (b) Determine the domain for $f(g(x))$. (3 marks) (c) Explain why function g is not a one-to-one function. (1 mark)

2022 Section 1 Question 2 **Functions and sketching graphs**	The graph of $y = f(x)$ is shown below.

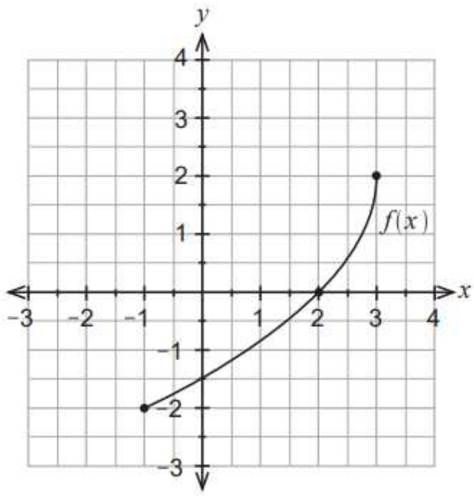


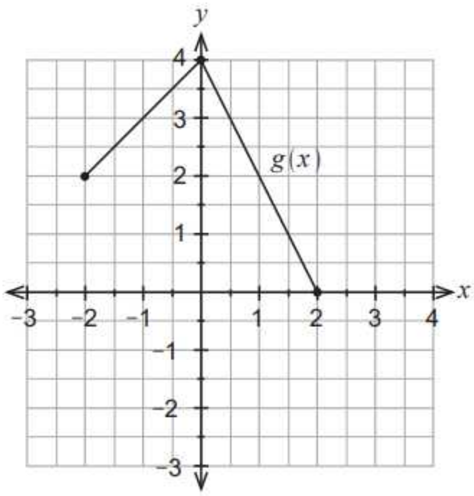
(a) Solve the equation $|f(x)| = x$. (2 marks)

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)



2021 Section 1 Question 2 **Functions and sketching graphs**	The graphs of functions f and g are shown below.





(a) Sketch the graph of function f^{-1} on the same axes used for function f. (2 marks)

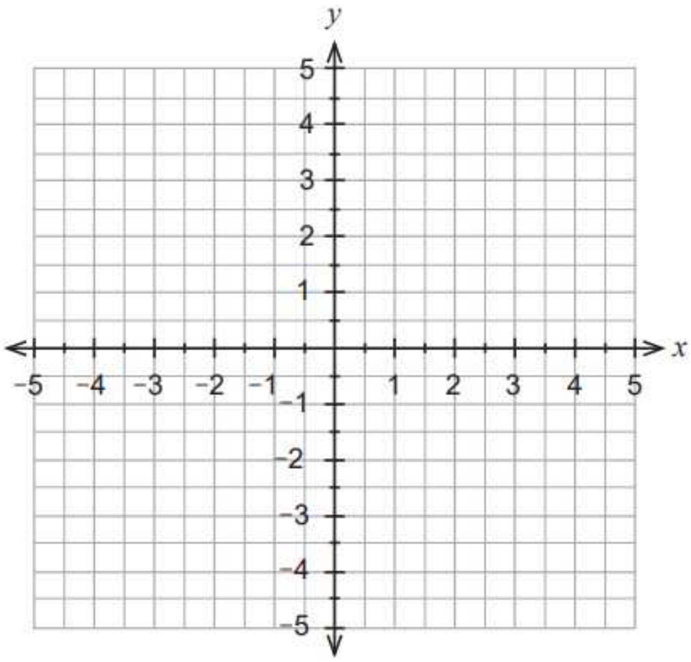
(b) Explain why the inverse of g is not a function. (1 mark)

The defining rule for function f is $f(x) = 2 - 2\sqrt{3-x}$ where $-1 \le x \le 3$.

(c) Determine the rule for $y = f^{-1}(x)$. (3 marks)

(d) Determine the exact value for $g\,(\,f(0))$. (2 marks)

	(e) Determine the domain for the function $y = f(g(x))$. Justify your answer. (3 marks)

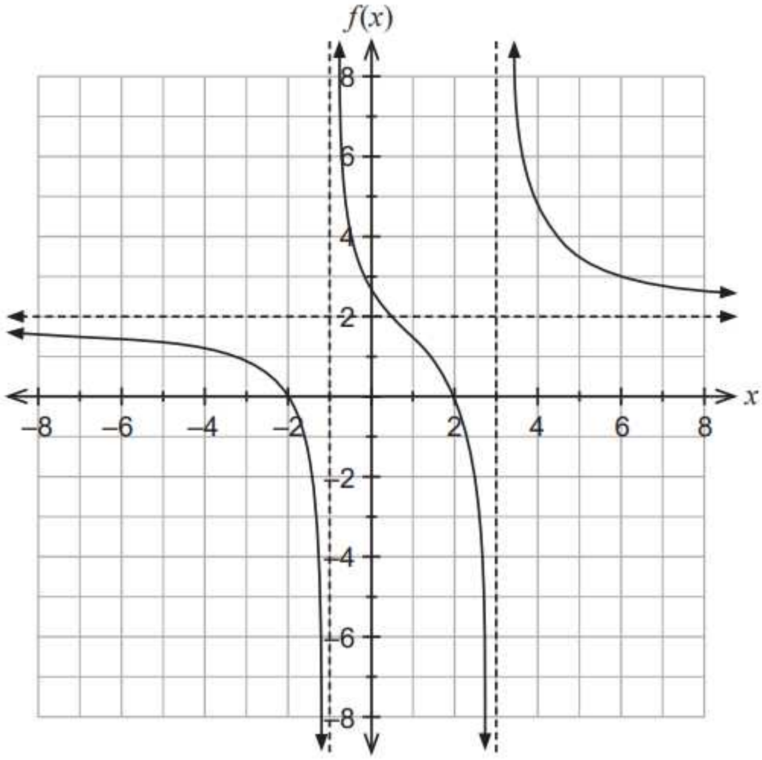
2021 Section 1 Question 4 **Functions and sketching graphs**	Consider the function $f(x) = \frac{x^2 - 4}{x + 1} = x - 1 - \frac{3}{x+1}$. Sketch the graph of the function $y = f(x)$ on the axes below. Indicate clearly the x and y intercepts and any asymptotes. 

2020
Section 1
Question 3

Functions and sketching graphs

The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by

$f(x) = \dfrac{a(x^2 - b)}{(x + c)(x - d)}$ where a, b, c and d are positive constants.

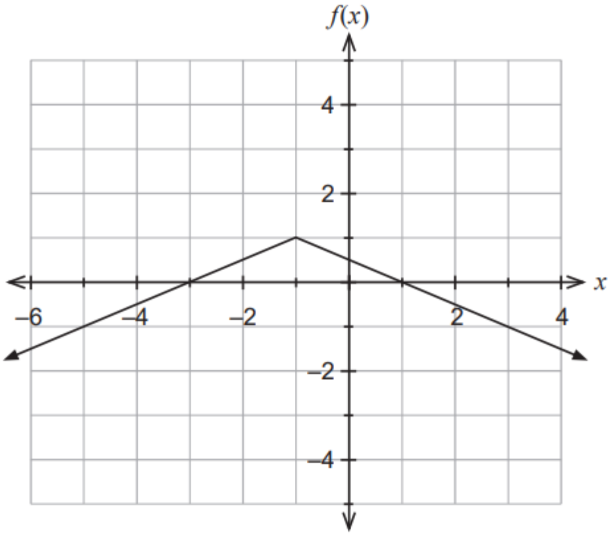
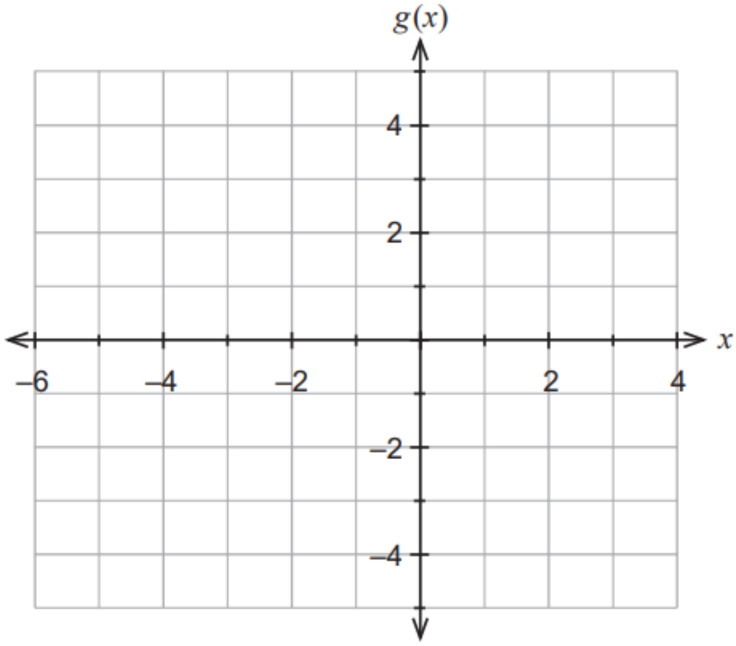


Determine the value of the constants a, b, c and d. Justify your answers. (6 marks)

a	b	c	d

2020
Section 1

The graph of $f(x) = 1 - \dfrac{|x+1|}{2}$ is shown below.

Question 5 Functions and sketching graphs	 (a) Sketch the graph of $g(x) = \frac{1}{f(x)}$ on the axes below. (4 marks) 

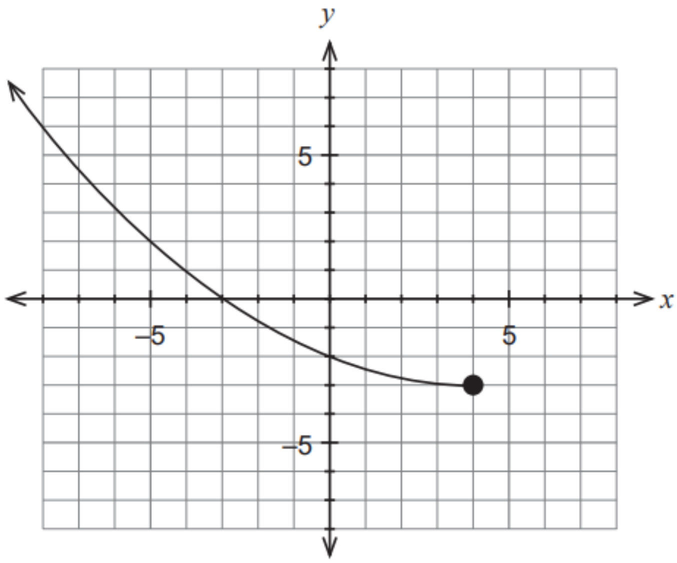
	(b) Hence give the domain and range for $h(x) = \dfrac{4}{2 - \lvert x + 1 \rvert}$. (3 marks)

2020 Section 1 Question 6 **Functions and sketching graphs**	Consider $f(x) = 2\tan(x)$ where $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$. Let $g(x) = f^{-1}(x)$ be the inverse of function f. (a) Determine the defining rule for $y = g(x)$. (2 marks) (b) By using implicit differentiation show that $g'(x)$ can be written in the form $\dfrac{a}{x^2 + b}$. (4 marks)

(c) Show that $\frac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\frac{q}{x^2+4}+\frac{r}{x-3}$ and hence determine the values for q and r. (3 marks)

(d) Hence determine $\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. (4 marks)

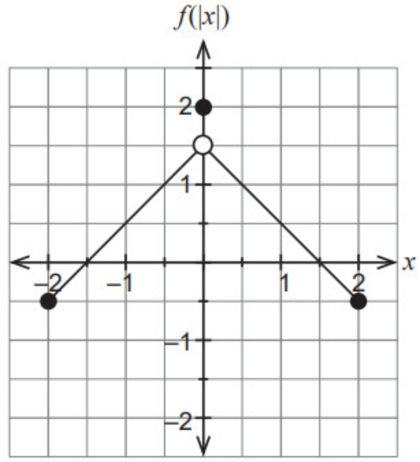
2019 Section 1 Question 4 **Functions and sketching graphs**	Functions f, g and h are defined such that: $f(x) = \frac{1}{x-1}$, $g(x) = x^2$, $h(x) = \sqrt{x}$. (a) Determine the defining rule for $f(h(x))$. (1 mark) (b) Determine the domain for $f(h(x))$. (2 marks) (c) Determine the range for $f(h(x))$. (2 marks) (d) Is it true that $f(h(g(x))) = \frac{1}{x-1} = f(x)$? Justify your answer. (2 marks)

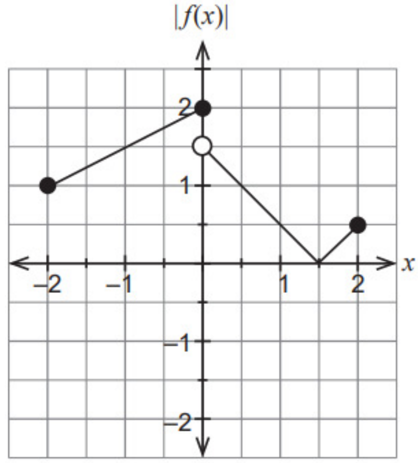
2019 Section 1 Question 5 **Functions and sketching graphs**	The graph of $y = g(x)$ is shown below.  (a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. (3 marks) (b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \leq 4$, determine the defining rule for $y = g^{-1}(x)$. (3 marks)

2019
Section 1
Question 7

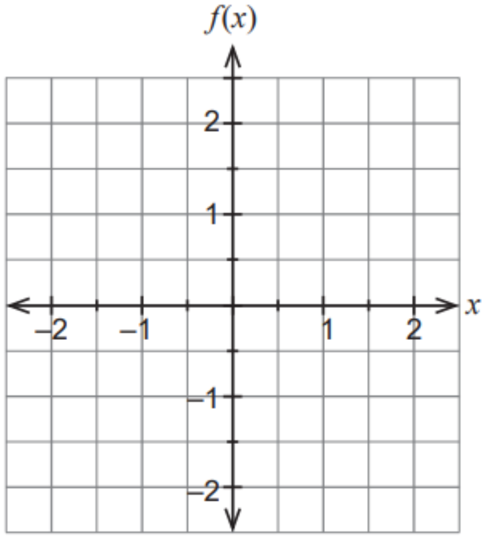
Functions and sketching graphs

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below.

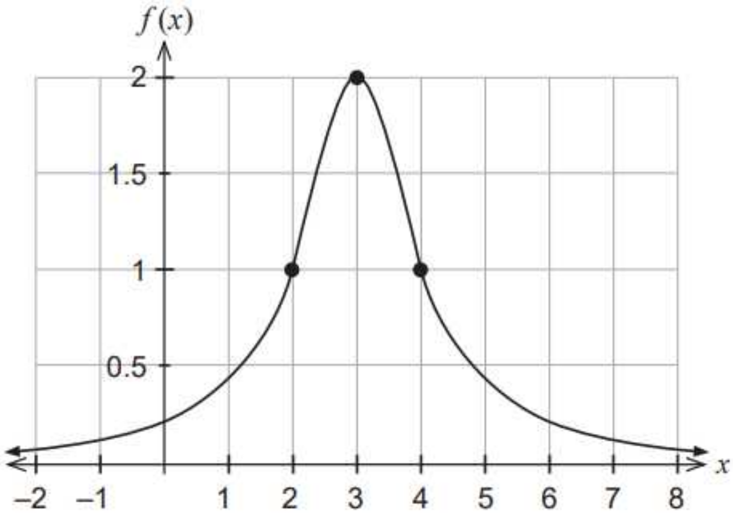




Given that $y = f-1(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below.
Justify your answer considering $y = f-1(x)$.

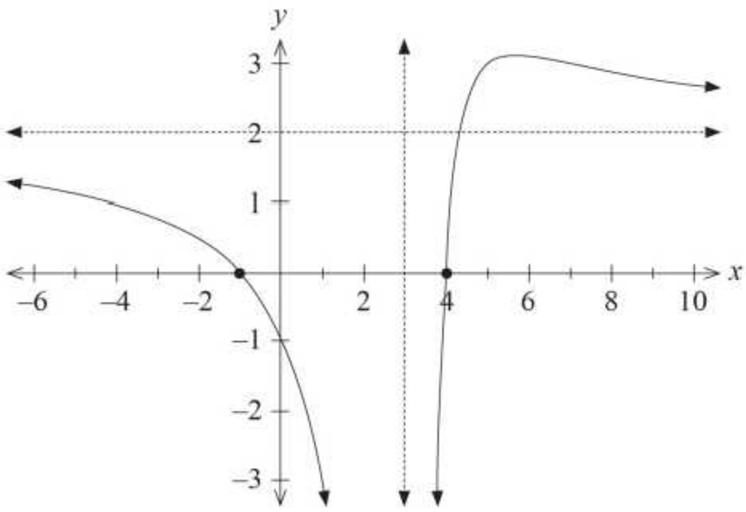


Section 2

2022 Section 2 Question 15 **Functions and sketching graphs**	The graph of a rational function f is shown below. Function f has the form $f(x)=\frac{k}{q(x)}$ with the following properties: • f has no x intercepts or vertical asymptotes • $f(x)\to 0$ for $\lvert x\rvert\to\infty$ • f is symmetric about $x=3$ • function q is quadratic • k is a constant.  Determine the defining rule for f. (4 marks)

2023 Section 1 Question 1 **Functions and sketching graphs**	

The graph of the function $f(x)=\dfrac{k\,(x+a)(x-b)}{(x-c)^2}$ is shown below. The constants $a,\ b,\ c$ and k are positive.



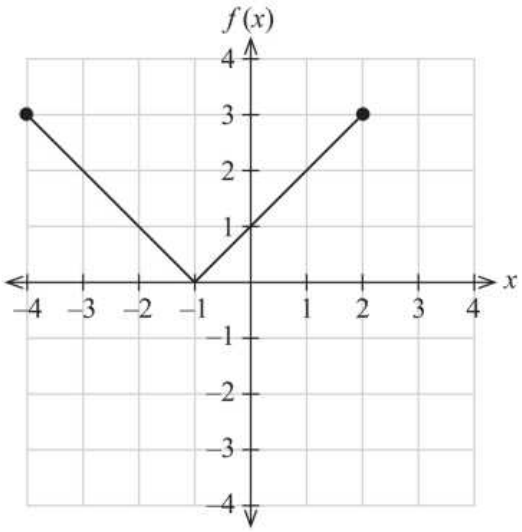
Complete the table below by determining the values for $a,\ b,\ c$ and k.

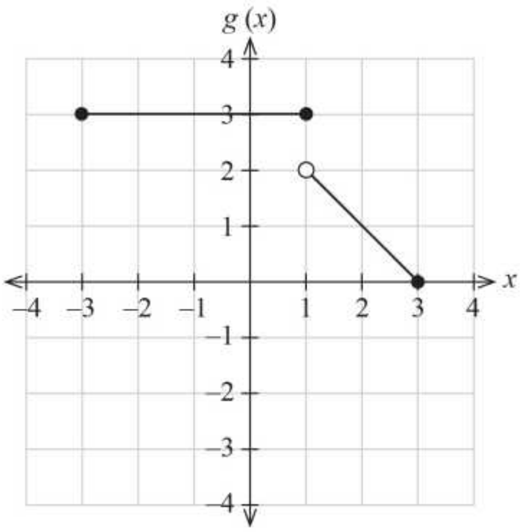
a	b	c	k

a	b	c	k
1	**4**	**3**	**2**

Solution
The x intercepts are $x=-1,\ x=4$ $\quad\therefore\ a=1,\ b=4$ Vertical asymptote is $x=3$ $\quad\therefore\ c=3$ Horizontal asymptote is $y=2$ $\quad\therefore\ k=2$
Specific behaviours
✓ states the value for a and b correctly ✓ states the value for c correctly ✓ states the value for k correctly ✓ provides justification for at least one of the values for $a,\ b,\ c,\ k$

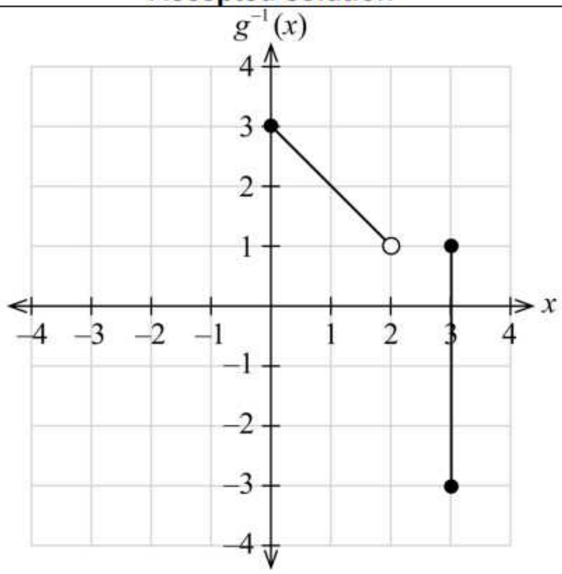
2023 Section 1 Question 4 **Functions and sketching graphs**	The graphs of functions $f(x)$ and $g(x)$ are shown.





(a) Sketch the graph of $y = g^{-1}(x)$ on the axes below. (2 marks)

Solution
Inverse of $g(x)$ does not exist as $g(x)$ is not a one-to-one function. (Graph cannot be sketched).
Specific behaviours
✓✓ identifies inverse of $g(x)$ does not exist or identifies graph cannot be sketched

Accepted solution

Specific behaviours
✓ indicates $x = 3$ for $-3 \le y \le 1$ ✓ indicates $y = 3 - x$ for $0 \le x < 2$

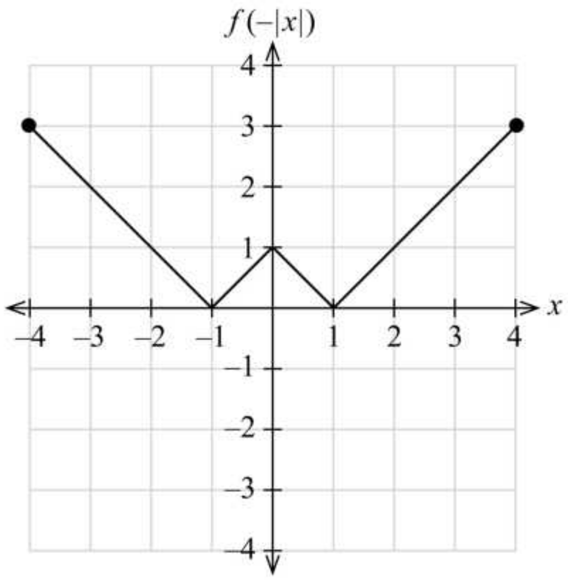
(b) State the value for $g\left(f^{-1}(0)\right)$. (2 marks)

Solution
Let $f^{-1}(0)=x \quad \therefore \; f(x)=0$ From the graph of $y=f(x)$ hence $x=-1$. $g\left(f^{-1}(0)\right) = g(-1) = 3$
Specific behaviours
✓ states that $f^{-1}(0)=-1$ ✓ evaluates $g\left(f^{-1}(0)\right)$ correctly

(c) Determine the set of values of x such that $f(g(x))$ is defined. (2 marks)

Solution
For $f(g(x))$ to be defined then $R_g \subseteq D_f$ i.e. the range of g must be part of the domain of f. This will occur when $g(x) \le 2$ i.e. $1 < x \le 3$.
Specific behaviours
✓ states that $R_g \subseteq D_f$ ✓ states the correct set of values for x

(d) Sketch the graph of $y=f(-\lvert x\rvert)$ on the axes below. (2 marks)

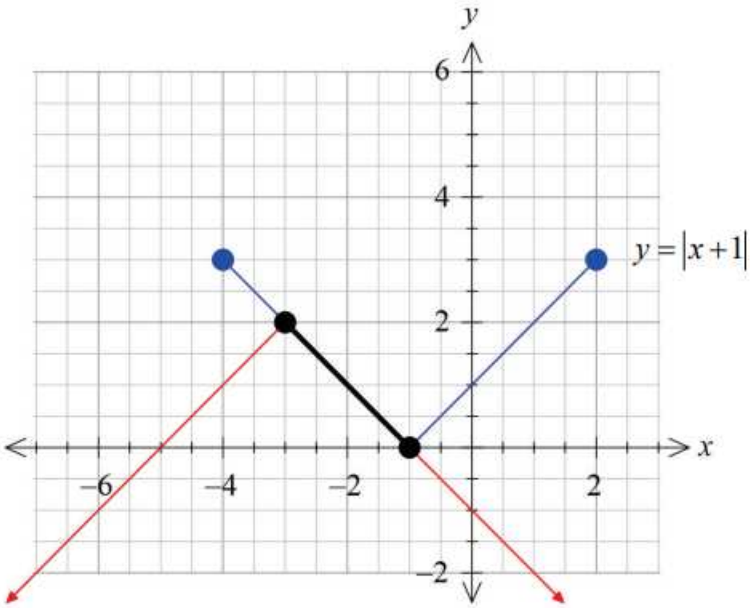


Solution
Shown above.
Specific behaviours
✓ indicates symmetry about $x=0$ ✓ indicates the correct set of points for $-4 \le x \le 4$

(e) The equation $|x+1| = k - |x+a|$ has an infinite number of solutions, with the solution set being $-3 \le x \le -1$. Determine the values of the constants a and k. (3 marks)

Solution

Consider the graph of $y=|x+1|$ and $y=k-|x+a|$ so that they intersect only when $-3 \le x \le -1$.



This intersection will occur when we consider $y=2-|x+3|$.

Hence $k=2$ and $a=3$.

Specific behaviours

- ✓ states the correct value for k
- ✓ states the correct value for a
- ✓ provides appropriate justification (considers the graphs of absolute value functions that yields an intersection only for $-3 \le x \le -1$)

Accept other relevant answers.

2022 Section 1 Question 1

Functions and sketching graphs

Consider functions $f(x)=\sqrt{4-x}$ and $g(x)=\frac{1}{x^2}$.

(a) Determine the exact value of $g(f(-5))$. (2 marks)

Solution

$$g\left(f(-5)\right) = g\left(\sqrt{4-(-5)}\right) = g(3) = \frac{1}{9}$$

Specific behaviours

- ✓ determines $f(-5)$ correctly
- ✓ obtains the correct value for $g\left(f(-5)\right)$

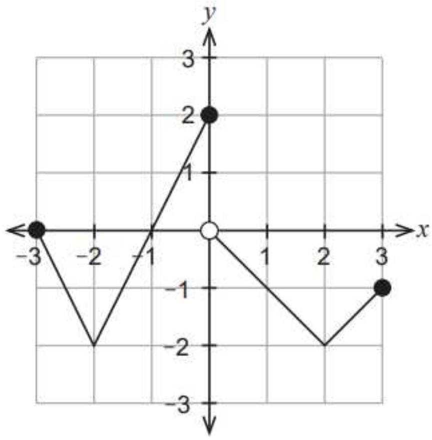
(b) Determine the domain for $f(g(x))$. (3 marks)

Solution
$f(g(x)) = \sqrt{4-\frac{1}{x^2}}$ This will be defined when $4-\frac{1}{x^2} \geq 0$, $x \neq 0$ since $g(x)$ must exist.
i.e. $\frac{1}{x^2} \leq 4$ i.e. $x^2 \geq \frac{1}{4}$ $\therefore D_{fog} = \{x \mid x \geq \frac{1}{2} \cup x \leq -\frac{1}{2}\}$
Specific behaviours
✓ identifies that $f(g(x))$ is defined when $4-\frac{1}{x^2} \geq 0$
✓ states $x \geq \frac{1}{2}$
✓ states $x \leq -\frac{1}{2}$

(c) Explain why function g is not a one-to-one function. (1 mark)

Solution
$g(-2) = g(2) = \frac{1}{4}$ This shows that g maps two values of x to a single value.
Hence g is NOT a one-to-one function BUT is a MANY-to-one function.
Specific behaviours
✓ justifies why g is not a one-to-one function

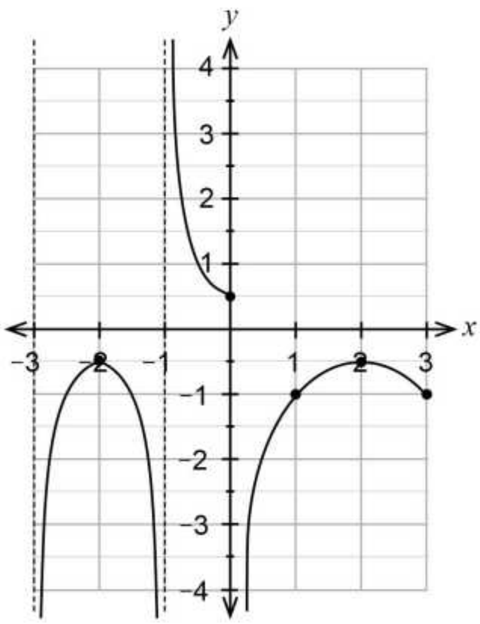
2022 Section 1 Question 2 **Functions and sketching graphs**	The graph of $y = f(x)$ is shown below.



(a) Solve the equation $|f(x)| = x$. (2 marks)

Solution
Equation requires the intersection between $y = \lvert f(x) \rvert$ and $y = x$. This occurs when $0 < x \leq 2$.
Specific behaviours
✓ excludes $x = 0$ and includes $x = 2$ in the solution ✓ states the correct interval of real values for x

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$ on the axes below. (5 marks)

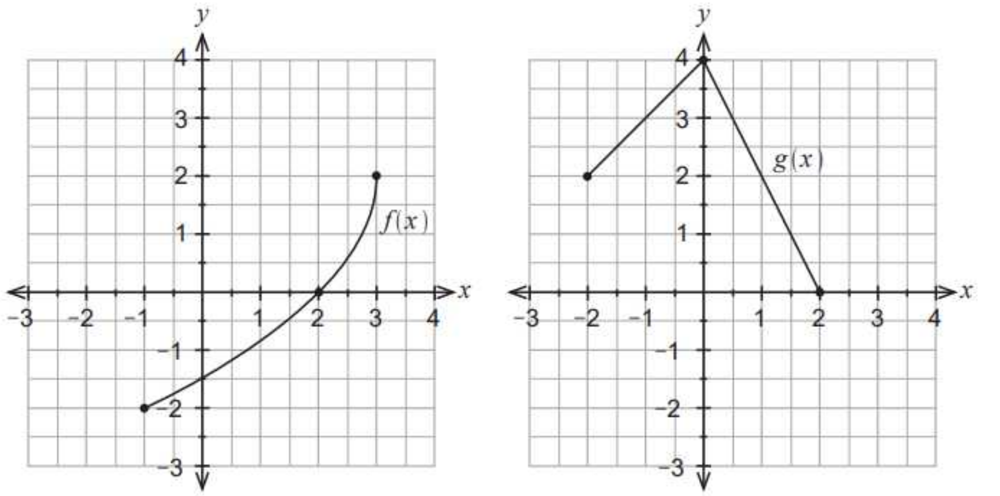


Solution
See graph axes.
Specific behaviours
✓ indicates vertical asymptotes at $x = -3, -1, 0$ ✓ indicates correct function behaviour as $x \to -3$ and $x \to 0^+$ ✓ indicates correct function behaviour as $x \to -1$ ✓ indicates the correct curvature ✓ indicates at least one of the 5 highlighted points

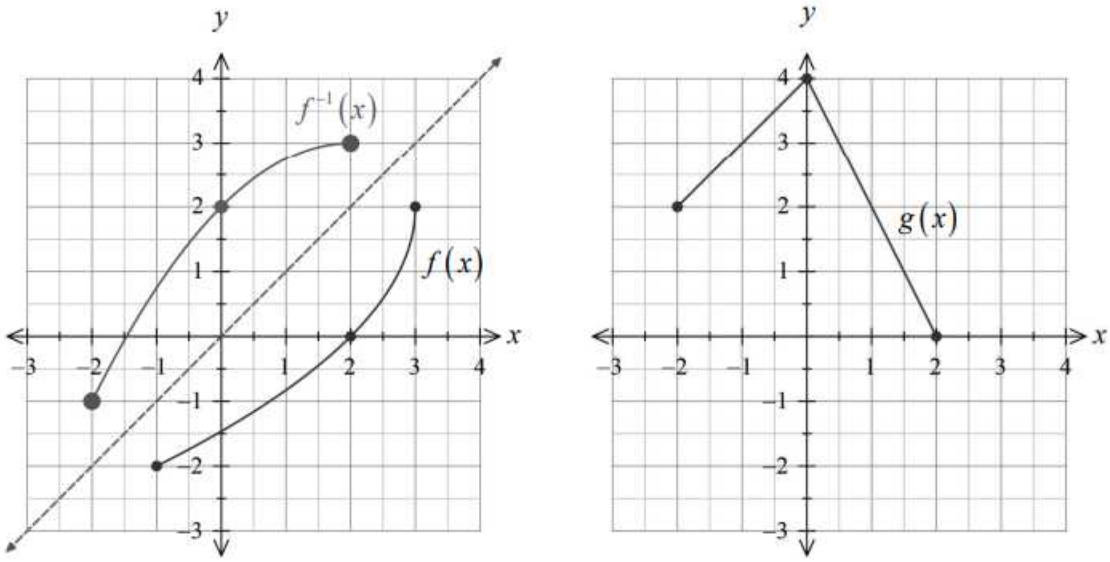
2021 Section 1 Question 2

Functions and sketching graphs

The graphs of functions f and g are shown below.



(a) Sketch the graph of function f^{-1} on the same axes used for function f. (2 marks)



Solution
See above graph axes.
Specific behaviours
✓ indicates a concave down curve that is a reflection of $y = f(x)$ about $y = x$
✓ indicates all the points $(-2,-1)$, $(0,2)$ and $(2,3)$

(b) Explain why the inverse of g is not a function. (1 mark)

Solution
Function g is not a one-to-one function over its domain OR does not pass the horizontal line test.
Specific behaviours
✓ refers to function g not being a one-to-one function

The defining rule for function f is $f(x) = 2 - 2\sqrt{3-x}$ where $-1 \le x \le 3$.

(c) Determine the rule for $y = f^{-1}(x)$. (3 marks)

Solution
$f:\ y = 2 - 2\sqrt{3-x}$ Hence $f^{-1}:\ x = 2 - 2\sqrt{3-y}$ $\sqrt{3-y} = \dfrac{2-x}{2}$ $\therefore\ 3 - y = \left(\dfrac{2-x}{2}\right)^2$ $\therefore\ f^{-1}(x) = 3 - \left(\dfrac{2-x}{2}\right)^2$
Specific behaviours
✓ interchanges x, y to obtain the rule for the inverse   ✓ obtains the correct expression for $\sqrt{3-y}$ ✓ obtains the correct defining rule for $y = f^{-1}(x)$

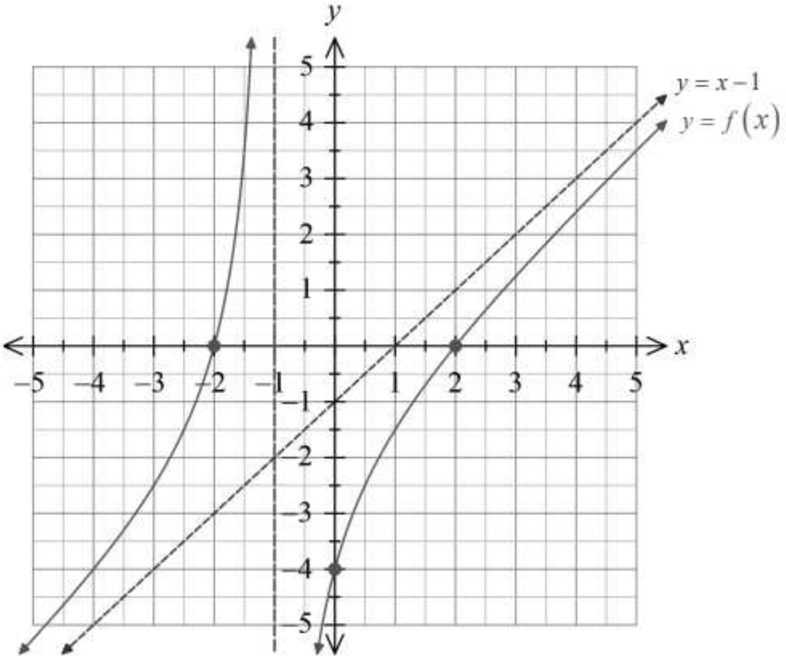
(d) Determine the exact value for $g(f(0))$. (2 marks)

Solution
$g(f(0)) = g\left(2 - 2\sqrt{3}\right)$ $= \left(2 - 2\sqrt{3}\right) + 4$ since $-2 \le 2 - 2\sqrt{3} \le 0$ $= 6 - 2\sqrt{3}$
Specific behaviours
✓ evaluates $f(0)$ correctly   ✓ determines the exact value $6 - 2\sqrt{3}$ correctly

(e) Determine the domain for the function $y = f(g(x))$. Justify your answer. (3 marks)

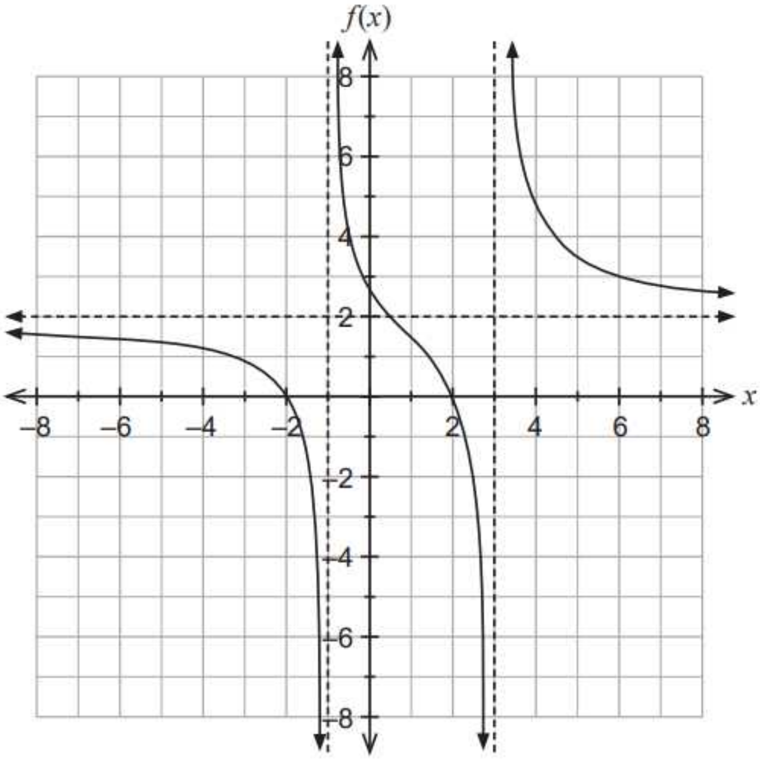
Solution
The range of g must be a SUBSET of the domain of f. $\therefore\ D_{fog} = \{x \mid -2 \le x \le -1,\ 0.5 \le x \le 2\}$ Note that for $-1 < x < 0.5$ $g(x) > 3$ which is not in the domain for function f.
Specific behaviours
✓ states that $-2 \le x \le -1$ ✓ states that $0.5 \le x \le 2$ ✓ justifies the chosen domain correctly

2021 Section 1 Question 4 **Functions and sketching graphs**	Consider the function $f(x) = \frac{x^2-4}{x+1} = x-1 - \frac{3}{x+1}$. Sketch the graph of the function $y = f(x)$ on the axes below. Indicate clearly the x and y intercepts and any asymptotes.



Solution
$f(x) = \frac{(x+2)(x-2)}{(x+1)} = x-1 - \frac{3}{x+1}$ x intercepts occur when $x^2-4=0$ *i.e.* at $x=\pm 2$ y intercept $f(0)=-4$ Vertical asymptote is $x=-1$. As $\lvert x\rvert \to \infty$, $f(x)\to x-1$ (inclined asymptote) Sketch shown above.
Specific behaviours
✓ indicates x intercepts at $x=\pm 2$ ✓ indicates a vertical asymptote at $x=-1$ ✓ indicates $f(0)=-4$ ✓ indicates inclined asymptote $y=x-1$ i.e. $f(x)\to x-1$ for $\lvert x\rvert\to\infty$ ✓ indicates correct curvature in the graph

2020 Section 1 Question 3 **Functions and sketching graphs**	The graph of $y = f(x)$ is shown on the axes below. The defining rule is given by $f(x) = \frac{a(x^2 - b)}{(x+c)(x-d)}$ where a, b, c and d are positive constants.

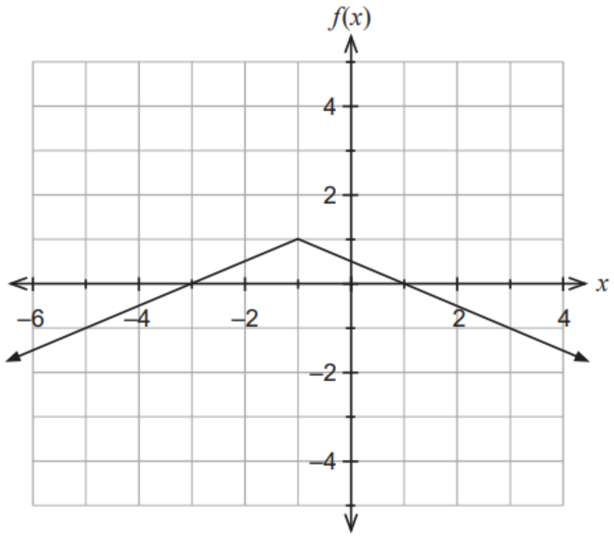


Determine the value of the constants a, b, c and d. Justify your answers. (6 marks)

a	b	c	d
2	4	1	3

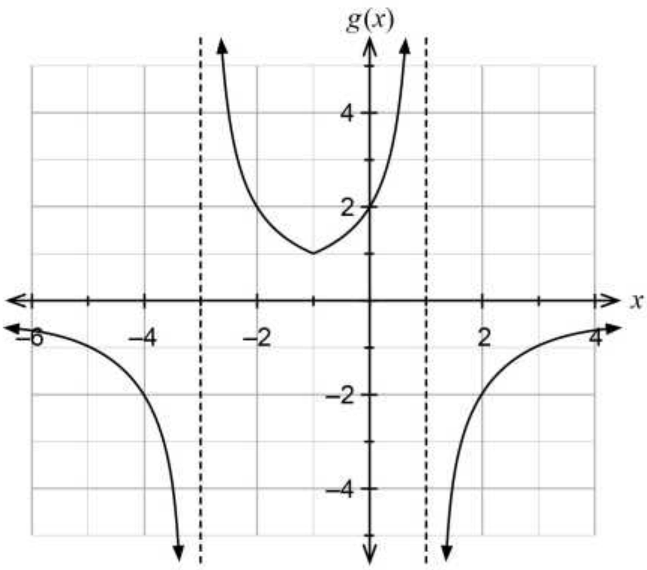
Solution
Horizontal intercepts are $x = -2,\ x = 2 \quad \therefore \quad x^2 - b = (x+2)(x-2)$ i.e. $b = 4$
Vertical asymptotes are $x = -1,\ x = 3 \quad \therefore \quad c = 1,\ d = 3$
Horizontal asymptote is $y = 2 \quad \therefore \quad a = 2$
Specific behaviours
✓ states that $a = 2$
✓ justifies why $a = 2$ (refers to the horizontal asymptote $y = 2$)
✓ states that $b = 4$
✓ justifies why $b = 4$ (refers to two horizontal intercepts)
✓ states that $c = 1,\ d = 3$
✓ justifies why $c = 1,\ d = 3$ (refers to the two vertical asymptotes)

2020 Section 1 Question 5 **Functions and sketching graphs**	The graph of $f(x) = 1 - \frac{\lvert x+1 \rvert}{2}$ is shown below.



(a) Sketch the graph of $g(x) = \frac{1}{f(x)}$ on the axes below.

(4 marks)



Solution
Shown above.
Specific behaviours
✓ indicates vertical asymptotes at $x = -3$ and $x = 1$ ✓ indicates $g(x) < 0$ as $\lvert x \rvert \to \infty$ ✓ indicates $g(x) \geq 1$ for $-3 < x < 1$ ✓ indicates correct graph curvature (cusp at $x = -1$ is not required)

(b) Hence give the domain and range for $h(x) = \dfrac{4}{2 - |x+1|}$.

(3 marks)

Solution
$h(x) = \dfrac{4}{2 - \lvert x+1 \rvert} = \dfrac{4}{2\left(1 - \dfrac{\lvert x+1 \rvert}{2}\right)}$ $= \dfrac{2}{f(x)} = 2g(x)$ Domain $D_h = D_g = \{ x \mid x \neq -3,\ x \neq 1 \}$ Range $R_h = \{ y \mid y < 0,\ y \geq 2 \}$
Specific behaviours
✓ states the correct domain (from function g)   ✓ states the correct range component $y < 0$ ✓ states the correct range component $y \geq 2$

2020 Section 1 Question 6

Functions and sketching graphs

Consider $f(x) = 2\tan(x)$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Let $g(x) = f^{-1}(x)$ be the inverse of function f.

(a) Determine the defining rule for $y = g(x)$. (2 marks)

Solution

Function f: $y = 2\tan(x)$

Hence f^{-1}: $x = 2\tan(y)$ i.e. $\frac{x}{2} = \tan(y)$

$\therefore\ y = \tan^{-1}\left(\frac{x}{2}\right)$ i.e. $g(x) = f^{-1}(x) = \tan^{-1}\left(\frac{x}{2}\right)$ *or* $\arctan\left(\frac{x}{2}\right)$

Specific behaviours

✓ interchanges x, y to form the rule for the inverse function

✓ expresses $f^{-1}(x)$ correctly in terms of x

(b) By using implicit differentiation show that $g'(x)$ can be written in the form $\frac{a}{x^2 + b}$.

(4 marks)

Solution

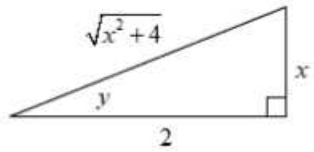
$$\frac{d}{dx}(x) = \frac{d}{dx}(2\tan(y))$$

$$1 = 2(\sec^2 y)\left(\frac{dy}{dx}\right) \quad \dots \ (1)$$

$$\therefore \frac{dy}{dx} = \frac{\cos^2 y}{2} = \frac{1}{2}\left(\frac{2}{\sqrt{x^2+4}}\right)^2 = \frac{2}{x^2+4}$$

Given $\tan(y) = \frac{x}{2}$

$\cos(y) = \frac{2}{\sqrt{x^2+4}}$



OR $$\frac{dy}{dx} = \frac{1}{2\sec^2 y} = \frac{1}{2(1+\tan^2 y)} = \frac{1}{2+2\left(\frac{x}{2}\right)^2} = \frac{2}{x^2+4}$$

Specific behaviours

✓✓ differentiates implicitly correctly to obtain statement (1)

✓ obtains an expression for $\cos(y)$ or $\tan(y)$ correctly in terms of x

✓ obtains a correct simplified expression for $\frac{dy}{dx}$ correctly in the form $\frac{a}{x^2+b}$

(c) Show that $\frac{3x^2+2x+6}{(x^2+4)(x-3)}$ can be expressed as $\frac{q}{x^2+4}+\frac{r}{x-3}$ and hence determine the values for q and r. (3 marks)

Solution
It is required that $q(x-3) + r(x^2+4) = 3x^2+2x+6$
Hence $rx^2 + qx + (4r-3q) = 3x^2+2x+6$
Equating co-efficients we obtain: $r = 3 \quad ... (1)$
$q = 2 \quad ... (2)$
$4r-3q = 6 \quad ... (3)$
Testing $q=2,\ r=3$ in equation (3) : $4(3)-3(2)=6$ is true.
Solving gives $q=2,\ r=3$.
Specific behaviours
✓ forms the equivalence of numerators correctly ✓ solves for q,r correctly ✓ tests the consistency of q,r to obtain the constant 6

(d) Hence determine $\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx$. (4 marks)

Solution
$\int \frac{3x^2+2x+6}{(x^2+4)(x-3)}\,dx = \int \frac{2}{x^2+4} + \frac{3}{x-3}\,dx$
$= \tan^{-1}\left(\frac{x}{2}\right) + 3\ln\lvert x-3\rvert + c$
Specific behaviours
✓ re-writes the integrand in terms of the partial fractions correctly ✓ anti-differentiates correctly using the logarithm of an absolute value ✓ uses the result of part (b) to correctly anti-differentiate ✓ uses a constant of integration

2019 Section 1 Question 4

Functions and sketching graphs

Functions f, g and h are defined such that:

$$f(x)=\frac{1}{x-1},\ g(x)=x^2,\ h(x)=\sqrt{x}.$$

(a) Determine the defining rule for $f(h(x))$. (1 mark)

Solution
$f(h(x)) = \frac{1}{\sqrt{x}-1}$
Specific behaviours
✓ states the correct defining rule

(b) Determine the domain for $f(h(x))$. (2 marks)

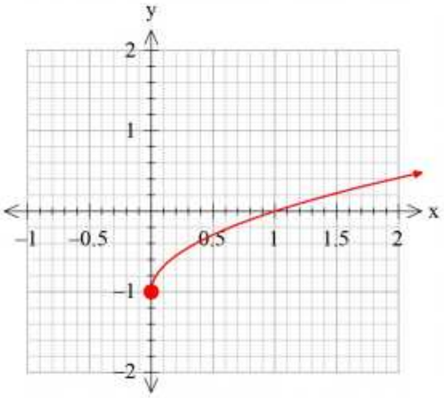
Solution
$D_{foh} = \{ x \mid x \geq 0,\ x \neq 1 \}$
Specific behaviours
✓ states $x \geq 0$ ✓ states $x \neq 1$

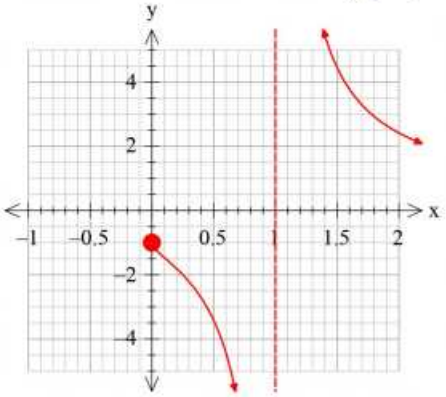
(c) Determine the range for $f(h(x))$. (2 marks)

Solution
When $x > 1$ $\quad f\left(h(x)\right) > 0$ When $0 \leq x < 1$ $\quad -1 \leq \sqrt{x} - 1 < 0 \quad \therefore \quad -1 \geq \dfrac{1}{\sqrt{x}-1}$ Hence $R_{foh} = \{ y \mid y > 0 \ \cup \ y \leq -1 \}$
Specific behaviours
✓ states $y > 0$ ✓ states $y \leq -1$

Alternative Solution

Graph $y = \sqrt{x} - 1$ and then graph its reciprocal function $y = f\left(h(x)\right) = \dfrac{1}{\sqrt{x}-1}$





Hence $R_{foh} = \{ y \mid y > 0 \ \cup \ y \leq -1 \}$

Specific behaviours

✓ states $y > 0$

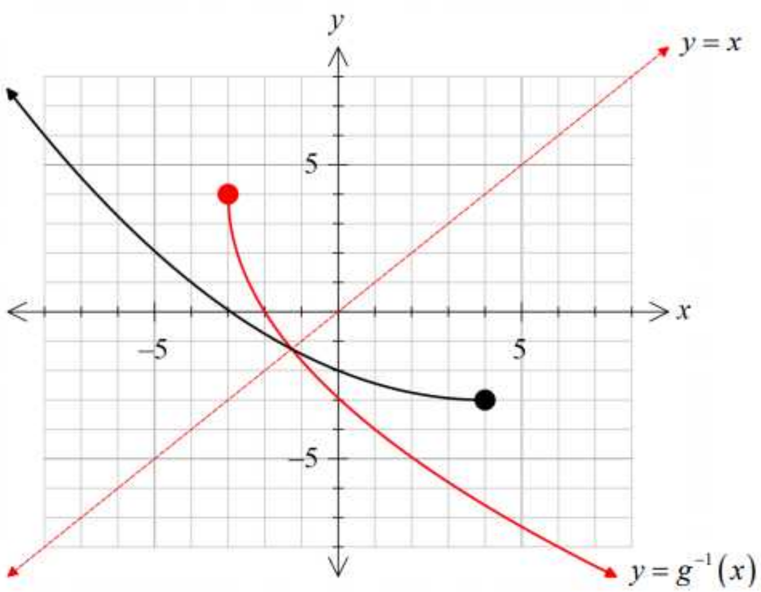
✓ states $y \leq -1$

(d) Is it true that $f(h(g(x))) = \frac{1}{x-1} = f(x)$? Justify your answer. (2 marks)

Solution
The statement is FALSE. $h(g(x)) = \sqrt{x^2} = \lvert x \rvert \geq 0$ Hence $f(h(g(x))) = \frac{1}{\sqrt{x^2}-1} = \frac{1}{\lvert x \rvert - 1}$ $D_{fohog} = \{x \mid x \in \mathbb{R},\ x \neq \pm 1\}$ $R_{fohog} = \{y \mid y > 0 \cup y \leq -1\}$ But $f(x) = \frac{1}{x-1}$ $D_{fohog} = \{x \mid x \in \mathbb{R},\ x \neq 1\}$ $R_{fohog} = \{y \mid y > 0\}$ $\therefore$ $f(h(g(x))) \neq f(x)$ as they have different DOMAIN and RANGE values.
Specific behaviours
✓ states that the statement is false ✓ justifies the statement is false

Alternative Solution
The statement is FALSE. This would be true if $h(g(x)) = x$ i.e. true if $\sqrt{x^2} = x$. But actually $\sqrt{x^2} = \lvert x \rvert \neq x$.
Specific behaviours
✓ states that the statement is false ✓ justifies the statement is false

2019 Section 1 Question 5 **Functions and sketching graphs**	The graph of $y = g(x)$ is shown below.



(a) Sketch the graph of $y = g^{-1}(x)$ on the axes above. (3 marks)

Solution
See above graph axes.
Specific behaviours
✓ indicates the points $(-3,4)$, $(-2,0)$ and $(2,-5)$ ✓ indicates the range $y \le 4$ (arrow on graph not required) ✓ indicates symmetry of $y = g^{-1}(x)$ with $y = g(x)$ about the line $y = x$

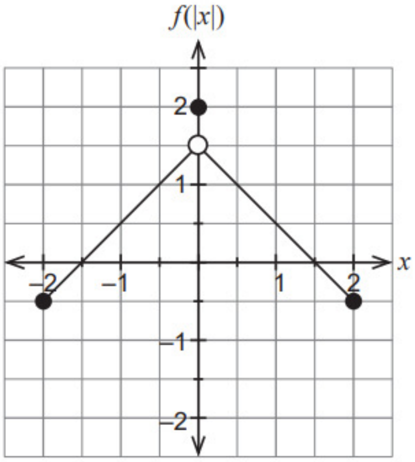
(b) Given that $g(x) = \frac{1}{16}(x-4)^2 - 3$ where $x \le 4$, determine the defining rule for $y = g^{-1}(x)$. (3 marks)

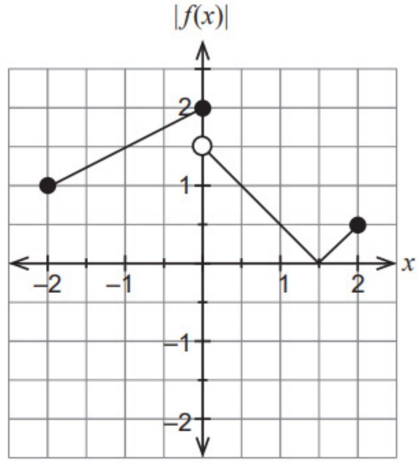
Solution
$g: \quad y = \frac{1}{16}(x-4)^2 - 3 \qquad g^{-1}: \quad x = \frac{1}{16}(y-4)^2 - 3$ $\therefore \quad 16(x+3) = (y-4)^2 \quad \ldots (1)$ $y - 4 = \pm 4\sqrt{x+3}$ Since $R_{g^{-1}} = D_g \; (x \le 4)$ then $g^{-1}(x) = 4 - 4\sqrt{x+3}$
Specific behaviours
✓ interchanges the x, y coordinates to obtain the inverse ✓ manipulates the equation correctly to obtain statement 1 ✓ writes the correct defining rule

2019 Section 1 Question 7

Functions and sketching graphs

The graphs of $y = f(|x|)$ and $y = |f(x)|$ are shown below.

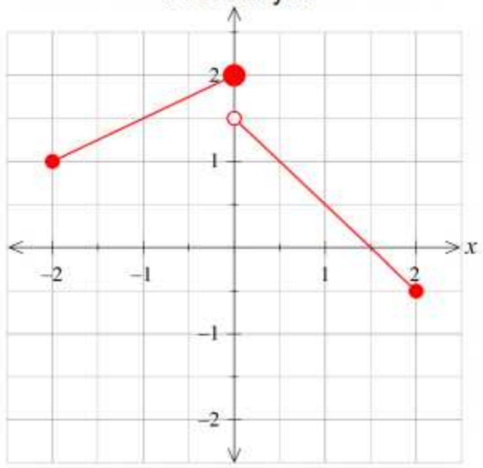




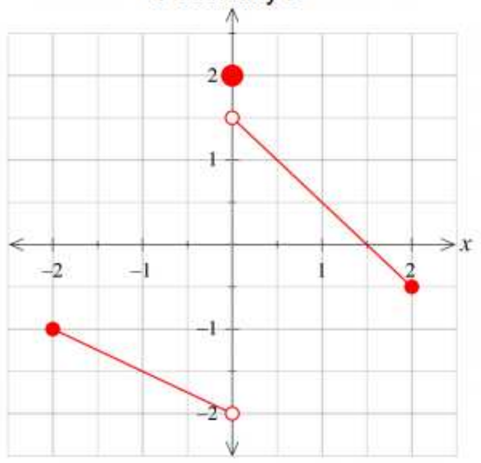
Given that $y = f-1(x)$ is also a function, sketch a possible graph for $y = f(x)$ on the axes below. Justify your answer considering $y = f-1(x)$.

Solution
There are many possibilities for $y = f(x)$. Two of these are:

Possibility A

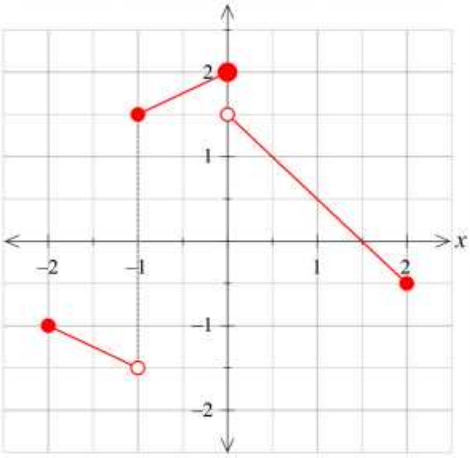
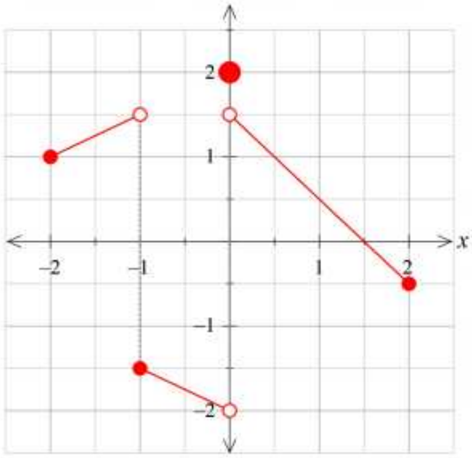


Possibility B



Since $y = f^{-1}(x)$ is a function then $y = f(x)$ over its domain $-2 \le x \le 2$ must be a ONE-TO-ONE function (which does not occur with possibility A or D). Hence $y = f(x)$ could be possibility B or C. Alternatively, function $f(x)$ must satisfy the 'horizontal' line test.

Specific behaviours
✓ indicates the points $(0,2)$, $(1.5,0)$, $(2,-0.5)$
✓ indicates $y = 1.5 - x$ for $0 < x \le 2$
✓ indicates $y = 0.5x + 2$ OR $y = -0.5x - 2$ for $-2 \le x < 0$ or equivalent to obtain $y = f(|x|)$ and $y = |f(x)|$ correctly
✓ justifies that $y = f(x)$ must be a one-to-one function so that $y = f^{-1}(x)$ is a function

	Alternative Solution
	Other possibilities for $y = f(x)$: Possibility C  Possibility D 

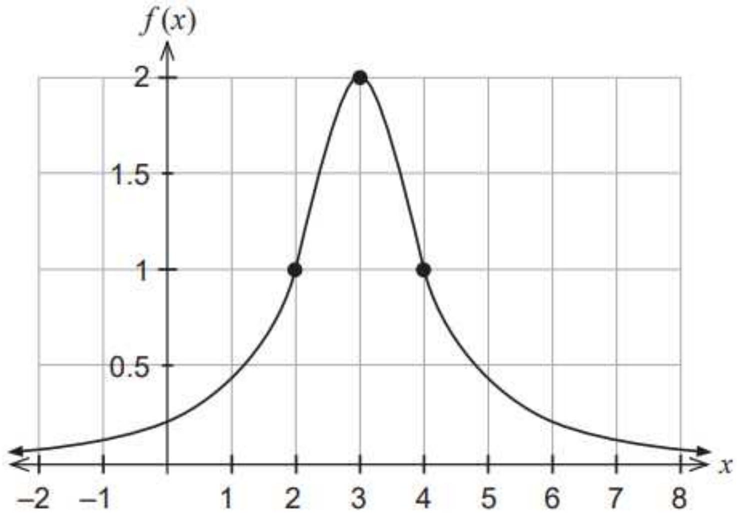
Marking Guide – Section 2

<table>
<tr><td>2022
Section 2
Question
15

Functions
and
sketching
graphs</td><td>

The graph of a rational function f is shown below. Function f has the form $f(x) = \frac{k}{q(x)}$ with the following properties:

- f has no x intercepts or vertical asymptotes
- $f(x) \to 0$ for $|x| \to \infty$
- f is symmetric about $x = 3$
- function q is quadratic
- k is a constant.



Determine the defining rule for f. (4 marks)

Solution

Since q is quadratic with symmetry about $x = 3$ $q(x) = a(x-3)^2 + b$

As there are no vertical asymptotes then $q(x) \neq 0$ $\therefore$ $a, b > 0$ using $k > 0$.

The maximum of f will occur when q is a minimum.

Using $f(3) = 2$ $\quad 2 = \frac{k}{a(3-3)^2 + b}$ $\quad$ *i.e.* $k = 2b$ ... (1)

Using $f(2) = 1$ $\quad 1 = \frac{k}{a(2-3)^2 + b}$ $\quad$ *i.e.* $a + b = k$ ... (2)

Solving $(1),(2)$ simultaneously we obtain $a = b,\ k = 2b$

i.e. $f(x) = \frac{2b}{b(x-3)^2 + b} = \frac{2}{(x-3)^2 + 1} = \frac{2}{x^2 - 6x + 10}$

Specific behaviours

✓ forms a quadratic rule for $q(x)$ that has symmetry about $x = 3$

✓ forms a quadratic rule for $q(x)$ that does NOT have any x intercepts $(a, b > 0)$

✓ forms correct relationships between k and other constants

✓ determines the correct defining rule for $f(x)$

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</table>

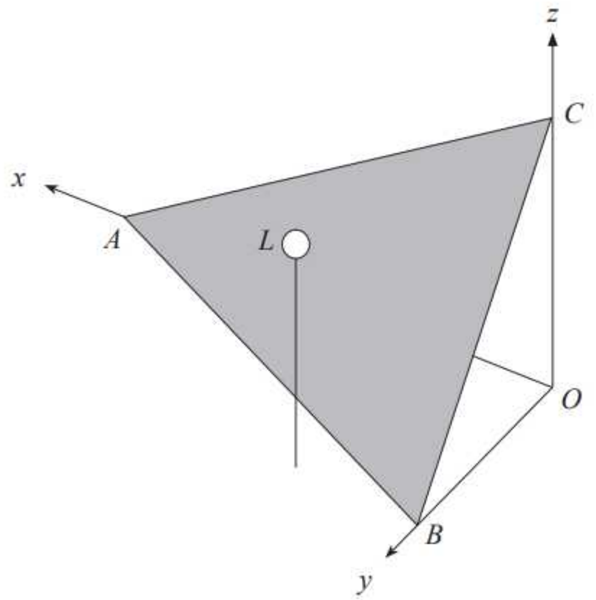
	Alternative Solution Since q is quadratic then let $q(x) = ax^2 + bx + c$ Using $f(3)=2 \quad 2 = \dfrac{k}{9a+3b+c} \quad \ldots (1)$ $f(2)=1 \quad 1 = \dfrac{k}{4a+2b+c} \quad \ldots (2)$ $f(4)=1 \quad 1 = \dfrac{k}{16a+4b+c} \quad \ldots (3)$ Solving $(1),(2),(3)$, simultaneously we obtain $a=a, \quad b=-6a, \quad c=10a, \quad k=2a$ i.e. $f(x) = \dfrac{2a}{ax^2-6ax+10a} = \dfrac{2a}{a(x^2-6x+10)} = \dfrac{2}{x^2-6x+10}$ **Specific behaviours** ✓ uses the rule $q(x)=ax^2+bx+c$ and $f(3)=2, f(2)=f(4)=1$ to form three equations relating k,a,b,c ✓ solves simultaneously to obtain relationships between variables ✓ factors out the common factor between variables ✓ determines the correct defining rule for $f(x)$

Unit 3.3 – Vectors in three dimensions

Section 1

2023 Section 1 Question 5 Vectors in three dimensions	Consider two planes given by their Cartesian equations: $x - 3y + 3z = 9$ $2x + y - z = 4$ (a) Explain why these planes are not parallel. (1 mark) (b) State the geometric interpretation of the solution in the above simultaneous equations. (1 mark) (c) Determine the vector equation for the intersection of these two planes. (3 marks)

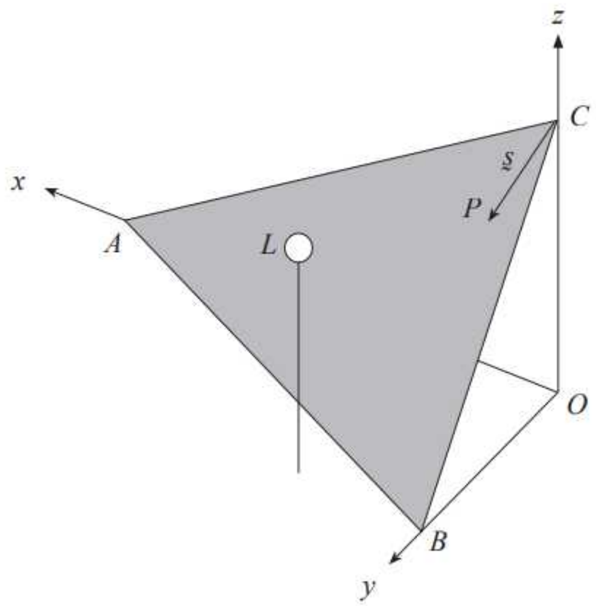
2022 Section 1 Question 5 **Vectors in three dimensions**	Consider the Cartesian equations for three planes: $$\begin{aligned} 2x + 2y + z &= 9 \\ -2x + 2y - 5z &= -13 \\ y - z &= -1 \end{aligned}$$ (a) Show that none of these planes is parallel to another. (2 marks) b) Solve the above equations simultaneously. (3 marks) (c) State the geometric interpretation of the solution obtained in part (b). (1 mark)

2022 Section 2 Question 19 **Vectors in three dimensions**	A downward-sloping ramp is positioned according to the coordinate system shown. $A\,(6, 0, 0)$, $B\,(0, 2, 0)$ and $C\,(0, 0, 3)$ are points on the ramp. A lamp L is positioned on top of a post at $\left(2, 2, \frac{5}{2}\right)$. All dimensions are measured in metres.  (a) Determine the Cartesian equation for the ramp. (2 marks)

At night, the lamp L emits a bright light and illuminates the ramp. The position that is closest to the lamp will be the most brightly illuminated.

(b) Determine the coordinates for the point on the ramp that is the most brightly illuminated. (4 marks)

If a ball is released from point C and is allowed to roll down the ramp, gravity will cause it to follow the path of steepest descent. Suppose the ball is allowed to roll exactly 1 metre from point C to point P, where $\underset{\sim}{s} = \overrightarrow{CP}$ is the direction of the steepest descent down the ramp.

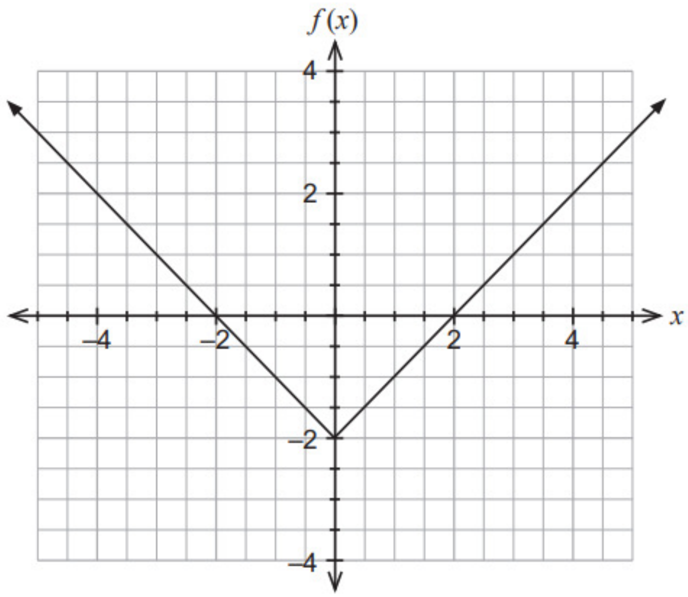
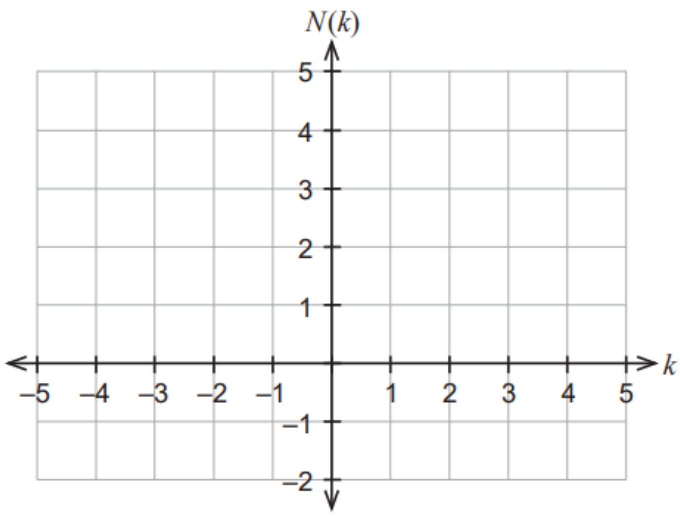


(c) Determine vector $\underset{\sim}{s}$, giving components correct to 0.001. (3 marks)

2020 Section 1 Question 2 **Vectors in three dimensions**	Plane Π has vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. (a) Determine the normal vector $\underset{\sim}{n}$ for plane Π. (3 marks) (b) Determine the Cartesian equation for plane Π. (2 marks)

2020 Section 1 Question 4 **Vectors in three dimensions**	Consider the equations for three planes, each written in Cartesian form: $\Pi_1 \quad x + y + z = 4$ $\Pi_2 \quad x - y - z = 7$ $\Pi_3 \quad y + z = 1$ (a) Explain whether or not any of these planes are parallel. (2 marks)

	(b) Solve the given system of simultaneous equations. (3 marks) (c) Give the geometric interpretation of the solution for this system of equations. (2 marks)

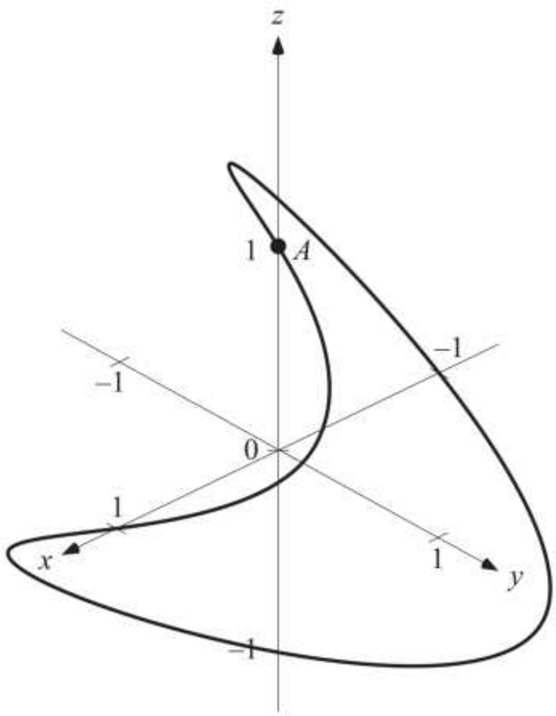
2020 Section 2 Question 21 **Vectors in three dimensions**	The sketch of the graph of $y = f(x)$ is shown below.  Consider the equation $\lvert f(x) \rvert = k$ where k is any real constant. Define function $N(k)$ = the number of real solutions to the equation $\lvert f(x) \rvert = k$. Sketch the graph of function $N(k)$ on the axes below. 

Section 2

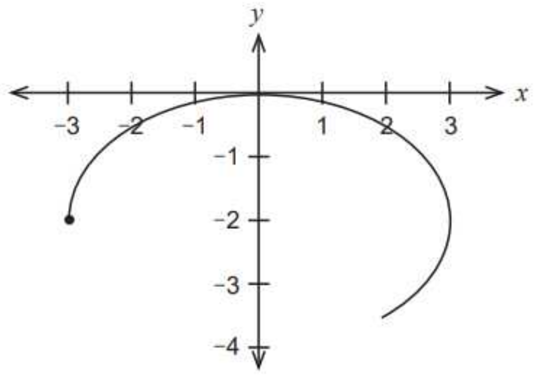
2023 Section 2 Question 9 **Vectors in three dimensions**	The Cartesian equation of a sphere is given as $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$. (a) Write the equation of the sphere in vector form. (3 marks) A line has vector equation $\underset{\sim}{r} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$. (b) Determine the point(s) of intersection between the line and the sphere. (3 marks)

2023 Section 2 Question 14 Vectors in three dimensions	Plane P_1 has Cartesian equation: $z = 2x + y + 4$. Line L has equation given by: $\underset{\sim}{r} = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 2\lambda \end{pmatrix}$. (a) Determine a vector that is perpendicular to plane P_1. (2 marks) (b) Write the equation for plane P_1 in vector form. (2 marks) (c) Determine the acute angle, correct to the nearest degree, between plane P_1 and line L. (3 marks)

	(d) Obtain the Cartesian equation of the plane P_2 that contains the line L and is perpendicular to plane P_1. (4 marks)

2023 Section 2 Question 16 **Vectors in three dimensions**	A fly moves around a path given by a three dimensional curve. The fly's path begins at point A and is shown below. Its position vector is specified by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \sin 2t \\ \cos t \end{pmatrix}$ metres where $0 \le t \le 2\pi$ seconds.  (a) Determine the initial acceleration vector and indicate this clearly on the diagram above. (3 marks)

	(b) Calculate the length of the path taken by the fly, correct to 0.001 metres. (2 marks)

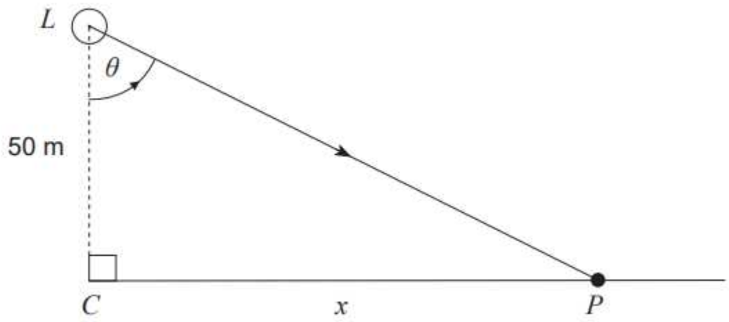
2022 Section 2 Question 10 **Vectors in three dimensions**	The velocity of a particle is given by $\underset{\sim}{v}(t) = \begin{pmatrix} 3\sin t \\ 2\cos t \end{pmatrix}$ where $t \geq 0$. The particle's initial position vector $\underset{\sim}{r}(0) = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$. The path of the particle is shown for the first 4 seconds.  (a) State what the following definite integrals measure about the motion of the particle: (i) $\int_0^1 \underset{\sim}{v}(t)\, dt$ (2 marks)

(ii) $\int_0^{2\pi} |\underset{\sim}{v}(t)|\, dt$ (2 marks)

(b) Determine $\underset{\sim}{r}(t)$. (3 marks)

(c) Determine the Cartesian equation for the path of the particle. (2 marks)

2022 Section 2 Question 11 **Vectors in three dimensions**	A Formula One (F1) racing car has an initial displacement of 192 metres with an initial velocity of 24 metres per second. It accelerates for a period of 11 seconds in a straight line so that its velocity v metres per second and displacement x metres are related by the equation: $v(x) = \frac{x}{8}$ (a) Determine the acceleration a as a function of displacement x i.e. determine $a(x)$. (2 marks) (b) Determine the displacement x as a function of time t . (2 marks) (c) Calculate the top speed reached and the distance travelled during the 11 second period of acceleration. (4 marks)

2021 Section 2 Question 9 Vectors in three dimensions	A beam of light completes three revolutions each minute from a lighthouse L that is 50 metres from a coastline. Determine the speed of the beam of light moving along the coast when it is at point P , 100 metres up the coast, correct to the nearest 0.01 metres per second. (5 marks) 

2021 Section 2 Question 14 **Vectors in three dimensions**	On a Saturday afternoon, three separate family groups visit their local cinema to watch a feature movie. The cinema names this as DollarDay where the ticket prices for adults, children and pensioners are charged in whole dollar amounts.

The table below indicates the number of people in each category and the total paid for each family group.

Group	Adults	Children	Pensioners	Total cost
1	2	4	–	$108
2	3	6	–	$162
3	2	5	2	$152

Let a = the price for each adult ($)
c = the price for each child ($)
p = the price for each pensioner ($)

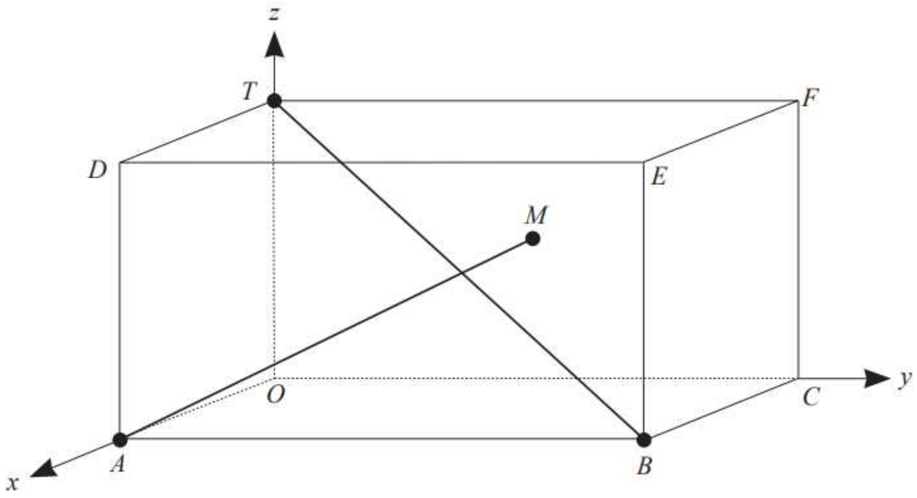
(a) Formulate the equations that can be used to determine the ticket prices. (1 mark)

(b) Using the equations formed, determine the total cost for a group consisting of 1 child accompanied by 2 pensioners. (2 marks)

(c) Solve simultaneously the equations formulated in part (a). (2 marks)

(d) Explain the geometric interpretation of the equations and the simultaneous solution. (2 marks)

	Now assume that the price for an adult is greater than the price for a child and that the price for a pensioner is the lowest priced ticket. (e) Determine the ticket prices for adults, children and pensioners on DollarDay. (3 marks)

2021 Section 2 Question 16 Vectors in three dimensions	A rectangular prism is defined using the coordinate system shown with A (2, 0, 0), C (0, 4, 0) and T (0, 0, 3). Point M is the centre of the planar face $OCFT$ with coordinates (0, 2, 1.5).  (a) Determine the vector equation for the prism's main diagonal $\overleftrightarrow{BT}$. (2 marks) (b) Determine the Cartesian equation of the sphere that contains all vertices of the rectangular prism. (3 marks) (c) Prove, using a vector method, that line $\overleftrightarrow{AM}$ does **not** intersect $\overleftrightarrow{BT}$. (3 marks)

2021 Section 2 Question 19

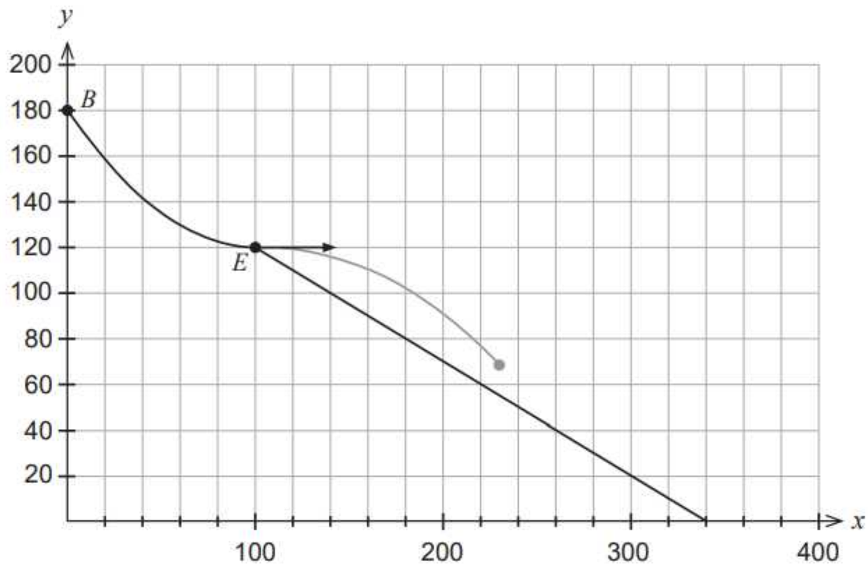
Vectors in three dimensions

Using the correct technique, Olympic ski jumpers can slow down their descent, by creating lift to counteract gravity. These jumpers must land successfully to have their distance recorded and land on sloped ground to prevent serious injury.

A skier begins his descent at point B accelerating down the ramp. At the end of the ramp the skier is travelling horizontally at point E at 32 metres per second (115.2 kilometres per hour).

Let t = the number of seconds in flight after point E $(100, 120)$.
$h(t)$ = the height of the skier above the horizontal ground $y = 0$ (metres)
$x(t)$ = the horizontal position of the skier (metres)

The sloped ground for landing is given by $y = 170 - 0.5x$ where $100 \le x \le 340$.



The ski jumper's suit and skis decrease the horizontal velocity $x'(t)$ so that $x'(t) = 32e^{-0.05t}$.

(a) Show that $x(t) = 740 - 640e^{-0.05t}$. (2 marks)

It is found that the expression for the position vector for the skier during the flight is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 740 - 640e^{-0.05t} \\ 120 - 2.5t^2 \end{pmatrix}$$

(b) Calculate the height of the skier above the sloped ground after 3 seconds of flight, correct to the nearest 0.01 metre. (3 marks)

(c) Determine the vertical lift s (m/s^2) provided by the skier's suit and equipment in the descent if $\frac{d^2h}{dt^2} = s - 9.8$, where s is a constant. (3 marks)

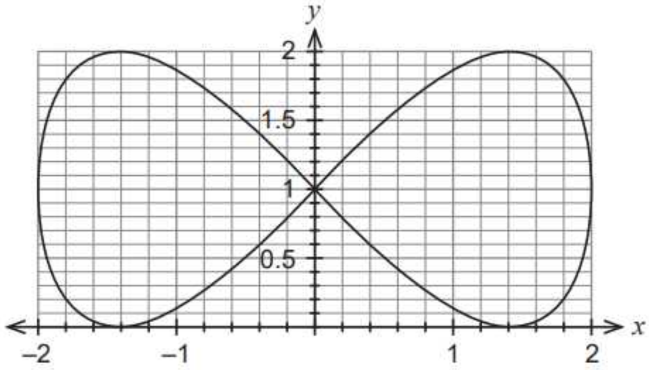
It can be shown that the Cartesian equation for the skier's flight is given by:

$$y = 120 - 1000\left(\ln\left(\frac{740 - x}{640}\right)\right)^2$$

(d) Calculate the time taken for the skier to land on the sloped ground, correct to the nearest 0.01 second. (3 marks)

(e) Calculate the angle at which the skier impacts the sloped ground, correct to the nearest 0.1 degree. (3 marks)

2020 Section 2 Question 14 **Vectors in three dimensions**	A particle travels in a straight line so that its velocity v cm per second and displacement x cm are related by the equation: $v = -0.2x$ (a) Determine the acceleration a in terms of its displacement x. (2 marks) (b) Does the particle's motion constitute simple harmonic motion? Justify your answer. (1 mark) It is known that the initial displacement of the particle is $x = 4$ cm. (c) Determine, correct to the nearest 0.01 second, when the particle has a displacement of 2 cm. (2 marks)

2019 Section 2 Question 13 **Vectors in three dimensions**	The path of a particle is shown below. This particle moves so that its position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} -2\cos\left(\frac{t}{2}\right) \\ 1-\sin(t) \end{pmatrix}$ metres, where t is the number of seconds the particle has been in motion.  (a) Determine the starting position of the particle and mark this as point A on the diagram above. (1 mark) (b) Determine the initial velocity of the particle and illustrate this on the diagram above. (3 marks) (c) Write the expression, in terms of trigonometric functions, for the distance the particle would travel in completing one circuit of the given path. Do **not** evaluate this expression. (3 marks)

	(d) Determine the Cartesian equation for the path of the particle. (3 marks)

2019 Section 2 Question 16 **Vectors in three dimensions**	Plane Π_1 has Cartesian equation $z = 2x + y + 4$. (a) Determine a vector that is normal to plane Π_1 . (2 marks) Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. (b) Determine the point of intersection between line L and plane Π_1 . (3 marks)

	Plane Π_2 contains line L and is perpendicular to plane Π_1 . (c) Determine the vector equation for plane Π_2 . (4 marks) Sphere S has vector equation $\lvert \underset{\sim}{r} - (3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k}) \rvert = \sqrt{35}$. (d) Determine whether line L is a tangent to sphere S. Justify your answer. (3 marks)

2019 Section 2 Question 19 **Vectors in three dimensions**	Two parallel planes Π_1 and Π_2 have their equations given by: $\Pi_1 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = 11$ $\Pi_2 \quad \underset{\sim}{r} \bullet \underset{\sim}{n} = -4 \quad$ where $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. It is known that (2,3,–7) is a point on plane Π_1. Prove the distance d between the point (2,3,–7) and plane Π_2 is given by $d = \dfrac{15}{\sqrt{a^2 + b^2 + c^2}}$.

2023 Section 1 Question 5

Vectors in three dimensions

Consider two planes given by their Cartesian equations:

$$x - 3y + 3z = 9$$
$$2x + y - z = 4$$

(a) Explain why these planes are not parallel. (1 mark)

Solution
The normal vectors for each plane $\begin{pmatrix}1\\-3\\3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ are not scalar multiples of each other. Hence the planes cannot be parallel to each other.
Specific behaviours
✓ explains that the normal vectors are not multiples of each other

(b) State the geometric interpretation of the solution in the above simultaneous equations. (1 mark)

Solution
the two planes will intersect in a line in space
Specific behaviours
✓ states that the planes intersect in a line

(c) Determine the vector equation for the intersection of these two planes. (3 marks)

Solution

$x - 3y + 3z = 9 \quad \ldots (1)$

$2x + y - z = 4 \quad \ldots (2)$

Consider $(1) + 3\times(2)$: $7x + 0y + 0z = 9 + 12$

i.e. $7x = 21$

$\therefore \; x = 3$

Substituting $x = 3$ into (1): $3 - 3y + 3z = 9$

i.e. $z = y + 2$ where $y \in \mathbb{R}$

i.e. there are infinitely many ordered triples for x, y, z.
Hence the intersection of the two planes is a line in space.

Vector equation for this line: $\underset{\sim}{r} = \begin{pmatrix}3\\\lambda\\\lambda+2\end{pmatrix} = \begin{pmatrix}3\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\1\end{pmatrix}$

Specific behaviours

✓ eliminates a variable correctly from the pair of equations
✓ obtains the relationship $z = y + 2$
✓ forms the vector equation of the line using a parameter correctly

2022 Section 1 Question 5 **Vectors in three dimensions**	Consider the Cartesian equations for three planes: $2x+2y+z=9$, $-2x+2y-5z=-13$, $y-z=-1$

Consider the Cartesian equations for three planes:

$$\begin{aligned} 2x+2y+z&=9 \\ -2x+2y-5z&=-13 \\ y-z&=-1 \end{aligned}$$

(a) Show that none of these planes is parallel to another. (2 marks)

Solution
Plane normals are $\begin{pmatrix}2\\2\\1\end{pmatrix}$, $\begin{pmatrix}-2\\2\\-5\end{pmatrix}$ and $\begin{pmatrix}0\\1\\-1\end{pmatrix}$. Since none of these normal vectors are scalar multiples of each other then the planes cannot be parallel to each other.
Specific behaviours
✓ states the normal vectors for each plane ✓ states that none of the normal vectors are multiples of each other

b) Solve the above equations simultaneously. (3 marks)

Solution

$$\begin{aligned} 2x+2y+z &= 9 \quad \ldots (1) \\ -2x+2y-5z &= -13 \quad \ldots (2) \\ y-z &= -1 \quad \ldots (3) \end{aligned}$$

Consider $(1)+(2)$: $4y-4z=-4$

$i.e.\ \ y-z=-1 \quad \ldots (4)$

$y-z=-1 \quad \ldots (3)$

Consider $(4)-(3)$: $0=0$!!

Hence there are an infinite number of solutions to these equations.

Let $z=k$ where $k \in \mathbb{R}$

$$\therefore\ y=k-1$$

$$\therefore\ 2x+2(k-1)+k=9$$

$$i.e.\ 2x=11-3k$$

$$\therefore\ x=\frac{11-3k}{2}$$

$$\underset{\sim}{r} = \begin{pmatrix}\frac{11-3k}{2}\\ k-1\\ k\end{pmatrix} \quad \text{where } k\in\mathbb{R}$$

Specific behaviours

- ✓ eliminates a variable correctly from a pair of equations
- ✓ states that there are an infinite number of solutions
- ✓ expresses correct relationships between variables

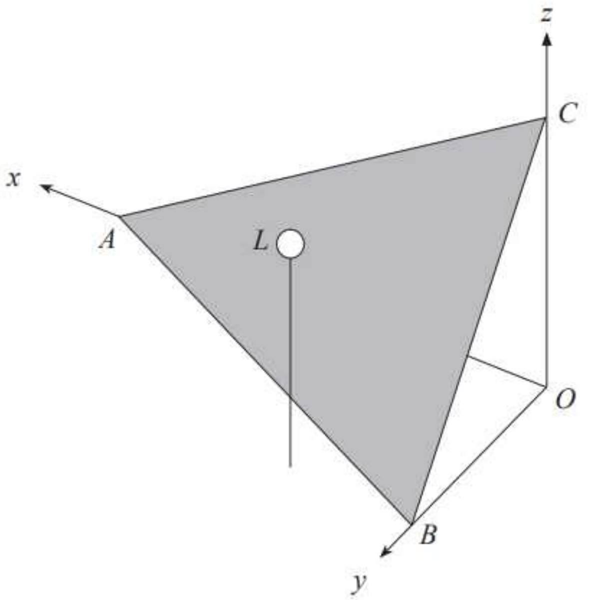
(c) State the geometric interpretation of the solution obtained in part (b). (1 mark)

Solution
The given non-parallel planes intersect in a LINE in space.
Specific behaviours
✓ states the intersection is a line in space

2022 Section 2 Question 19

Vectors in three dimensions

A downward-sloping ramp is positioned according to the coordinate system shown. $A\ (6,0,0)$, $B\ (0,2,0)$ and $C\ (0,0,3)$ are points on the ramp. A lamp L is positioned on top of a post at $\left(2, 2, \frac{5}{2}\right)$. All dimensions are measured in metres.



(a) Determine the Cartesian equation for the ramp. (2 marks)

Solution
Cartesian equation of a plane is $ax+by+cz=d$ Using the ordered pairs we obtain equations: $6a=d$, $2b=d$ and $3c=d$. By choosing $d=6$ we obtain $a=1,\ b=3,\ c=2$ $\therefore$ Equation for the ramp is given by $x+3y+2z=6$ (where $x,y,z\geq 0$)
Specific behaviours
✓ substitutes correctly into the equation $ax+by+cz=d$ to form 3 equations ✓ selects a suitable value for d and solves for the coefficients $a,\ b,\ c$

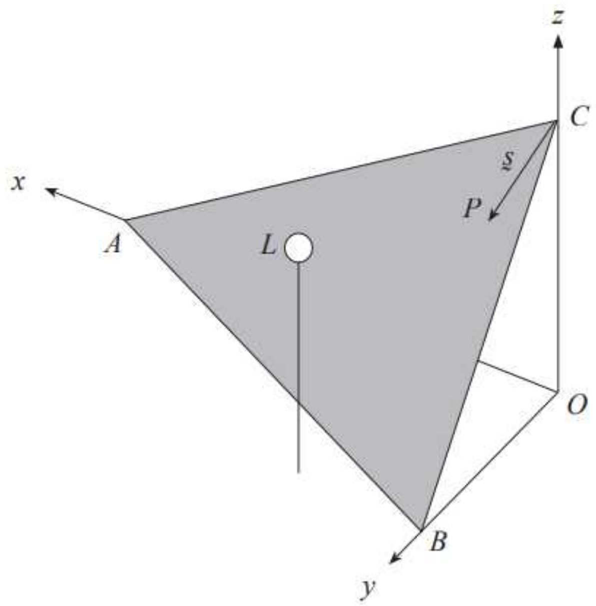
Alternative Solution
Normal vector can be given by the cross product of any 2 vectors that lie in the plane e.g. $\underset{\sim}{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -6 \\ -18 \\ -12 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ Vector equation for plane: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = 6$ i.e. $x+3y+2z=6$
Specific behaviours
✓ determines the normal vector for the plane correctly ✓ forms the Cartesian equation for the plane correctly

At night, the lamp L emits a bright light and illuminates the ramp. The position that is closest to the lamp will be the most brightly illuminated.

(b) Determine the coordinates for the point on the ramp that is the most brightly illuminated. (4 marks)

Solution
The point that is illuminates the most is the point on the plane that is CLOSEST to the point L. This will be on the PERPENDICULAR to the plane from L. Perpendicular line to plane: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ 2.5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 2+3\lambda \\ 2.5+2\lambda \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ Intersection with plane: $(2+\lambda) + 3(2+3\lambda) + 2(2.5+2\lambda) = 6$ i.e. $13 + 14\lambda = 6$ $\therefore \lambda = -0.5$ Hence point that is illuminated the most will be $(1.5,\ 0.5,\ 1.5)$.
Specific behaviours
✓ determines the equation for the line normal to the plane containing L ✓ substitutes correctly into the equation of the plane to solve simultaneously ✓ solves for the value of the parameter λ correctly ✓ states the coordinates for the most illuminated point correctly

If a ball is released from point C and is allowed to roll down the ramp, gravity will cause it to follow the path of steepest descent. Suppose the ball is allowed to roll exactly 1 metre from point C to point P, where $\underset{\sim}{s} = \overrightarrow{CP}$ is the direction of the steepest descent down the ramp.



(c) Determine vector $\underset{\sim}{s}$, giving components correct to 0.001. (3 marks)

Solution
Steepest descent will be determined by the path that goes from C to a point G where G is a point of $\overline{AB}$ where $\overrightarrow{CG} \perp \overline{AB}$. Note that $\underset{\sim}{s} = k\,\overrightarrow{CG}$. Let $\overrightarrow{CG}$ have direction vector $\underset{\sim}{d}$ where $\underset{\sim}{d} \perp \overline{AB}$ and $\underset{\sim}{d} \perp \underset{\sim}{n}$ i.e. $\underset{\sim}{d} = \overrightarrow{AB} \times \underset{\sim}{n}$ $\therefore\ \underset{\sim}{d} = \begin{pmatrix}-6\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\3\\2\end{pmatrix} = \begin{pmatrix}4\\12\\-20\end{pmatrix} = 4\begin{pmatrix}1\\3\\-5\end{pmatrix}$ We require $\lvert\underset{\sim}{s}\rvert = 1$ hence $k\left\lvert\begin{pmatrix}1\\3\\-5\end{pmatrix}\right\rvert = 1$ i.e. $k\left(\sqrt{35}\right) = 1$ $\therefore\ k = \frac{1}{\sqrt{35}}$ Hence $\underset{\sim}{s} = \left(\frac{1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, -\frac{5}{\sqrt{35}}\right)$ i.e. $\underset{\sim}{s} = (0.169,\ 0.507,\ -0.845)$ (3 d.p.) Note: Greatest descent is 0.845 metres.
Specific behaviours
✓ states that the direction of $\underset{\sim}{s}$ is in the direction of the perpendicular to $\overline{AB}$ ✓ determines the correct direction for $\underset{\sim}{d}$ (not the unit vector) ✓ determines the components for $\underset{\sim}{s}$

2020 Section 1 Question 2 **Vectors in three dimensions**	Plane Π has vector equation $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

(a) Determine the normal vector $\underset{\sim}{n}$ for plane Π. (3 marks)

Solution
Normal $\underset{\sim}{n} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0(2)-1(-1) \\ 1(1)-3(2) \\ 3(-1)-0(1) \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$
Specific behaviours
✓ states that the normal vector can be found using a cross product of the plane direction vectors ✓ obtains the correct form for each component of the cross product ✓ determines the cross product correctly (or a multiple of this vector)

(b) Determine the Cartesian equation for plane Π. (2 marks)

Solution
Equation given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix} = -12$ i.e. $x-5y-3z=-12$
Specific behaviours
✓ forms the equation using the normal vector $\underset{\sim}{n}$ correctly ✓ determines the Cartesian equation correctly

2020 Section 1 Question 4 **Vectors in three dimensions**	Consider the equations for three planes, each written in Cartesian form:

$\Pi_1 \quad x+y+z=4$
$\Pi_2 \quad x-y-z=7$
$\Pi_3 \quad y+z=1$

(a) Explain whether or not any of these planes are parallel. (2 marks)

Solution
$\underset{\sim}{n}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ $\underset{\sim}{n}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are the normal vectors for each plane. Since none of the normal vectors are parallel (normal vectors are NOT multiples of each other) then NONE of these planes are parallel.
Specific behaviours
✓ determines the normal vector for each plane ✓ justifies that NONE of the planes are parallel

(b) Solve the given system of simultaneous equations. (3 marks)

Solution
$\Pi_1 \quad x+y+z=4$ $\Pi_2 \quad x-y-z=7$ $\Pi_3 \quad y+z=1$ $(1)+(2): \quad 2x=11 \quad \therefore \quad x=5.5$ $(1)-(3): \quad x=3$ Hence there is NO solution for the system of equations.
Specific behaviours
✓ uses appropriate algebra correctly with TWO pairs of equations   ✓ solves correctly to find the first value for x ✓ deduces that there is no solution

Alternative Solution

$$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_1 + R_2 \\ R_3 \to R_1 - R_3 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 0 & 0 & 11 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

From R_2: $2x=11$ i.e. $x=5.5$

From R_3: $x=3$

Hence there is NO solution for the system of equations.

Note: $$\begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix} \quad R_3 \to R_2 + R_3 \rightarrow \begin{bmatrix} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & 7 \\ 1 & 0 & 0 & 8 \end{bmatrix}$$

From R_3: $x=8$

Specific behaviours

✓ applies at least TWO correct row operations
✓ solves correctly to find the first value for x
✓ deduces that there is no solution

(c) Give the geometric interpretation of the solution for this system of equations. (2 marks)

Solution

There is NO intersection between the three planes. We already know that none of the planes are parallel from part (a).

Hence each PAIR of planes intersect in LINES, but that these lines are PARALLEL to each other.

Note:

$\Pi_1 \quad x+y+z=4$
$\Pi_2 \quad x-y-z=7$
intersect in line $\underset{\sim}{r} = \begin{pmatrix} 5.5 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 5.5,\ \ y+z=1$

$\Pi_1 \quad x+y+z=4$
$\Pi_3 \quad y+z=1$
intersect in line $\underset{\sim}{r} = \begin{pmatrix} 3 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 3,\ \ y+z=1$

$\Pi_2 \quad x-y-z=7$
$\Pi_3 \quad y+z=1$
intersect in line $\underset{\sim}{r} = \begin{pmatrix} 6 \\ \lambda \\ 1-\lambda \end{pmatrix}$ i.e. $x = 6,\ \ y+z=1$

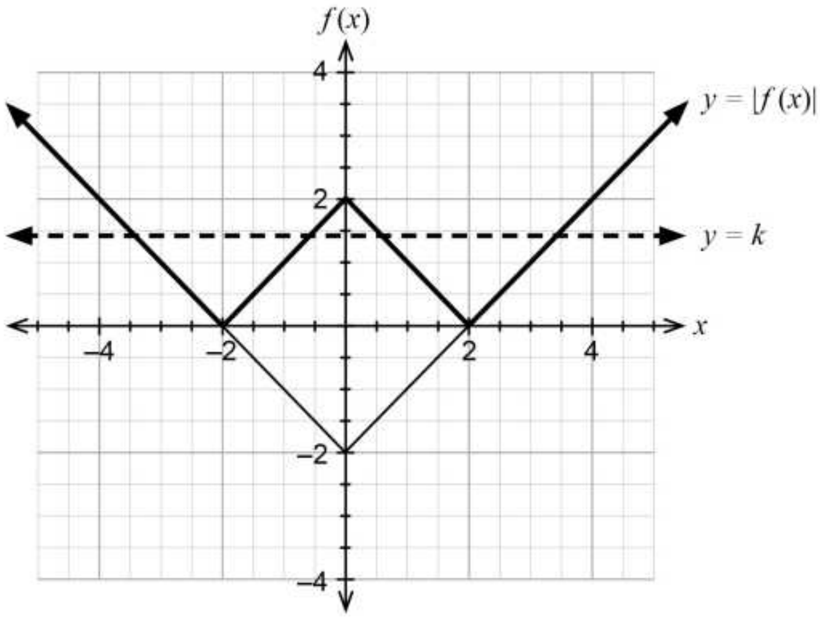
Clearly these lines are PARALLEL since their direction vectors are the same, where $\underset{\sim}{d} = 0\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$. Note that the normal vectors are coplanar since $2\underset{\sim}{n}_3 = \underset{\sim}{n}_1 - \underset{\sim}{n}_2$ i.e. one normal vector is a linear combination of the other two.

✓ states that the three planes have NO intersection
✓ refers to PAIRS of planes intersecting in LINES that are PARALLEL

2020
Section 2
Question 21

Vectors in three dimensions

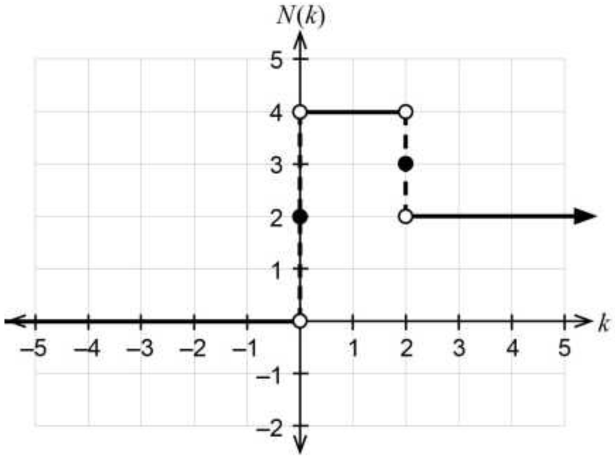
The sketch of the graph of $y = f\,(x)$ is shown below.



Consider the equation $|f\,(x)| = k$ where k is any real constant.

Define function $N(k)$ = the number of real solutions to the equation $|f\,(x)| = k$.

Sketch the graph of function $N(k)$ on the axes below.



Solution
Shown above as a piecewise defined function.
Specific behaviours
✓ indicates the graph of $y=\lvert f(x)\rvert$ is intersected with the horizontal line $y=k$ ✓ indicates $(0,2)$ and $(2,3)$ ✓ indicates $N(k)=0$ for $k<0$ ✓ indicates $N(k)=4$ for $0<k<2$ ✓ indicates $N(k)=2$ for $k>2$

2023 Section 2 Question 9

Vectors in three dimensions

The Cartesian equation of a sphere is given as $x^2+y^2+z^2-4x+2y-6z+5=0$.

(a) Write the equation of the sphere in vector form. (3 marks)

Solution

Completing the square: $(x-2)^2+(y+1)^2+(z-3)^3-2^2-1^2-3^2+5=0$

i.e. $(x-2)^2+(y+1)^2+(z-3)^3 = 9 \quad ..... \quad (1)$

$\therefore$ Centre is $(2,-1,3)$ Radius $=\sqrt{9}=3$

Vector equation of sphere: $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right| = 3$

Specific behaviours

- ✓ completes the square correctly to obtain statement (1)
- ✓ determines the centre and radius correctly
- ✓ forms the vector equation of the sphere correctly

A line has vector equation $\underset{\sim}{r} = \begin{pmatrix} 7 \\ -1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

(b) Determine the point(s) of intersection between the line and the sphere. (3 marks)

Solution

Substitute $\underset{\sim}{r} = \begin{pmatrix} 7+3\lambda \\ -1-\lambda \\ 9+4\lambda \end{pmatrix}$ into $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right| = 3$.

i.e. $\left| \begin{pmatrix} 7+3\lambda \\ -1-\lambda \\ 9+4\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \right| = 3 \quad \therefore \quad \left| \begin{pmatrix} 5+3\lambda \\ -\lambda \\ 6+4\lambda \end{pmatrix} \right| = 3$

i.e. $(5+3\lambda)^2+(-\lambda)^2+(6+4\lambda)^2=9 \quad ... \quad (2)$

Solving using CAS: $\lambda=-1, \quad \lambda=-2$.

Hence the 2 points of intersection are $(4, 0, 5)$ and $(1, 1, 1)$.

Specific behaviours

- ✓ substitutes correctly to obtain statement (2)
- ✓ solves for the values of λ correctly
- ✓ determines the points of intersection correctly (either Cartesian or vector form)

2023 Section 2 Question 14 Vectors in three dimensions	

2023 Section 2 Question 14

Vectors in three dimensions

Plane P_1 has Cartesian equation: $z = 2x + y + 4$.

Line L has equation given by: $\underset{\sim}{r} = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 2\lambda \end{pmatrix}$.

(a) Determine a vector that is perpendicular to plane P_1. (2 marks)

Solution
The normal vector $\underset{\sim}{n}_1$ for plane P_1 will be perpendicular to the plane. The Cartesian equation (in standard form) is $2x + y - z = -4$. Hence $\underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.
Specific behaviours
✓ uses the standard form for the plane $2x + y - z = -4$ ✓ states the normal vector correctly

(b) Write the equation for plane P_1 in vector form. (2 marks)

Solution
Since the standard form for P_1 is $2x + y - z = -4$ then we can write that: $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -4$
Specific behaviours
✓ uses the correct normal vector $\underset{\sim}{n}_1$ ✓ forms the vector equation for plane P_1 correctly

Alternative Solution
Points in plane P_1:

x	y	z
-2	0	0
0	-4	0
0	0	4

Vector equation P_1: $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ i.e. $\underset{\sim}{r} \bullet \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = -4$

Specific behaviours
✓ determines a point in plane P_1 correctly ✓ forms the vector equation for plane P_1 correctly

Alternative Solution
Direction vectors that lie in plane P_1: $\underset{\sim}{v}_1 = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$ $\underset{\sim}{v}_2 = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$ $\underset{\sim}{v}_3 = \begin{pmatrix} 0 \\ -4 \\ -4 \end{pmatrix}$ Vector equation P_1: $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \lambda(\underset{\sim}{v}_1) + \mu(\underset{\sim}{v}_2)$ (or equivalent)
Specific behaviours
✓ determines two vectors that lie in plane P_1 ✓ forms the vector equation for plane P_1 correctly using two parameters

(c) Determine the acute angle, correct to the nearest degree, between plane P_1 and line L. (3 marks)

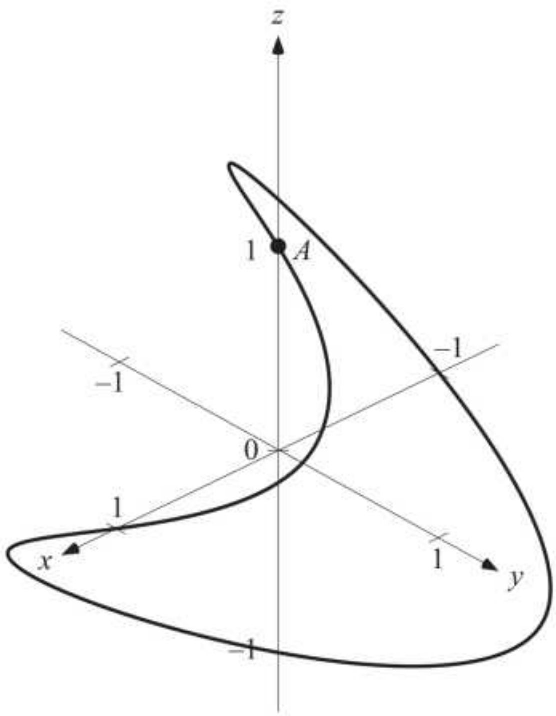
Solution
This angle is determined by finding the angle between the normal to P_1 and line L.   Consider forming the dot product: $\underset{\sim}{n}_1 \bullet \underset{\sim}{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -2+1-2 = -3$ $\therefore \ \lvert\underset{\sim}{n}_1\rvert\lvert\underset{\sim}{d}_1\rvert\cos\theta = -3$ $\therefore \ (\sqrt{6})(\sqrt{6})\cos\theta = -3$ $\therefore \ \cos\theta = -0.5$ i.e. $\lvert\theta\rvert = 60^o$ (solution of the smallest magnitude)   Hence the angle between the line and the plane is $90^o - \lvert\theta\rvert = 30^o$.
Specific behaviours
✓ considers the angle between the plane normal and line direction vector   ✓ determines the angle between the normal and the line correctly   ✓ deduces the acute angle between the plane and the line correctly

	(d) Obtain the Cartesian equation of the plane P_2 that contains the line L and is perpendicular to plane P_1. (4 marks)

Solution
Let the normal vector for plane P_2 be $\underset{\sim}{n}_2$. Since plane $P_2 \perp P_1$ then $\underset{\sim}{n}_2 \perp \underset{\sim}{n}_1$. Also as plane P_2 that contains the line L_1 then $\underset{\sim}{n}_2 \perp \underset{\sim}{d}_1$. Hence let $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$ or use $\underset{\sim}{n}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Any point of line L_1 can be used as the point that will be in plane P_2 i.e. $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Vector equation for plane P_2: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = 1$ i.e. Cartesian equation for plane P_2: $x - y + z = 1$
Specific behaviours
✓ states that the normal vector $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d}_1$ ✓ determines the normal vector $\underset{\sim}{n}_2$ correctly ✓ uses a known point on line L correctly as the point on plane P_2 ✓ determines the Cartesian equation for plane P_2 correctly

	Alternative Solution
	Vectors $\underset{\sim}{n}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\underset{\sim}{d} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ are vectors in plane P_2. Hence plane P_2 is given by: $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\therefore \quad x = 2 - \lambda + 2\mu \quad ...(1)$ $y = 1 + \lambda + \mu \quad ...(2)$ $z = 2\lambda - \mu \quad ...(3)$ $(2)+(3): \quad y + z = 1 + 3\lambda \quad ...(4)$ $(1)-2(2): \quad x - 2y = -3\lambda \quad ...(5)$ Hence $(4)+(5): \quad x - y + z = 1$ i.e. Cartesian equation for plane $P_2: \quad x - y + z = 1$
	Specific behaviours
	✓ writes the equation for plane P_2 using directions $\underset{\sim}{n}_1$ and $\underset{\sim}{d}$ and a point on line L ✓ forms the three parametric equations in terms of parameters λ, μ ✓ eliminates a parameter from a pair of equations ✓ determines the Cartesian equation for plane P_2 correctly

2023 Section 2 Question 16 **Vectors in three dimensions**	A fly moves around a path given by a three dimensional curve. The fly's path begins at point A and is shown below. Its position vector is specified by $\underset{\sim}{r}(t) = \begin{pmatrix} \sin t \\ \sin 2t \\ \cos t \end{pmatrix}$ metres where $0 \le t \le 2\pi$ seconds.



(a) Determine the initial acceleration vector and indicate this clearly on the diagram above. (3 marks)

Solution
$\underset{\sim}{v}(t) = \underset{\sim}{r}'(t) = \begin{pmatrix} \cos t \\ 2\cos 2t \\ -\sin t \end{pmatrix}$ $\quad \underset{\sim}{a}(t) = \underset{\sim}{r}''(t) = \begin{pmatrix} -\sin t \\ -4\sin 2t \\ -\cos t \end{pmatrix}$
$\therefore \ \underset{\sim}{a}(0) = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ i.e. the acceleration vector is acting directly downwards with a magnitude 1 m/sec^2 (shown on diagram).
Specific behaviours
✓ determines the acceleration vector function correctly ✓ calculates the acceleration vector correctly when $t = 0$ ✓ indicates the $\underset{\sim}{a}(0)$ vector correctly on the diagram (an arrow must be evident)

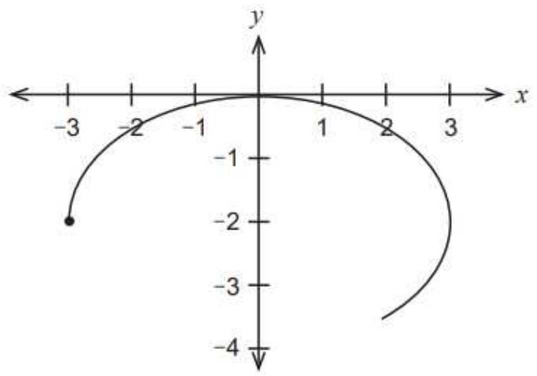
	(b) Calculate the length of the path taken by the fly, correct to 0.001 metres. (2 marks)

Solution
Length of path is the distance travelled in period $0 \le t \le 2\pi$.   Length of path $= \int_0^{2\pi} \lvert \underset{\sim}{v}(t) \rvert \, dt$ $= \int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t + \sin^2 t} \, dt$ $= 10.540734...$ metres   Hence the length of the fly's path is 10.541 metres (correct to 0.001 m)
Specific behaviours
✓ forms the correct expression for the path length (including the speed function)   ✓ calculates the definite integral expression correctly

2022 Section 2 Question 10

Vectors in three dimensions

The velocity of a particle is given by $\underset{\sim}{v}(t) = \begin{pmatrix} 3\sin t \\ 2\cos t \end{pmatrix}$ where $t \ge 0$. The particle's initial position vector $\underset{\sim}{r}(0) = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$. The path of the particle is shown for the first 4 seconds.



(a) State what the following definite integrals measure about the motion of the particle:

(i) $\int_0^1 \underset{\sim}{v}(t) \, dt$ (2 marks)

Solution	
$\int_0^1 \underset{\sim}{v}(t) \, dt = \int_0^1 \frac{d}{dt}(\underset{\sim}{r}(t)) dt = \underset{\sim}{r}(1) - \underset{\sim}{r}(0)$ $= \Delta \underset{\sim}{r}$	This integral gives the change in displacement vector during the first second of motion.
Specific behaviours	
✓ states the integral measures the change in displacement (vector) ✓ states this occurs during the first second of motion	

(ii) $\int_0^{2\pi} |\underset{\sim}{v}(t)|\, dt$ (2 marks)

Solution	
$\int_0^{2\pi} \lvert\underset{\sim}{v}(t)\rvert\, dt = \int_0^{2\pi} Speed(t)\, dt$	This integral gives the distance travelled for the time it takes to complete one circuit of the path of the particle i.e. the perimeter of the ellipse.
Specific behaviours	
✓ refers to the time it takes for the particle to complete one circuit of its path ✓ states the integral measures the distance travelled	

(b) Determine $\underset{\sim}{r}(t)$. (3 marks)

Solution
$\underset{\sim}{r}(t) = \int \begin{pmatrix} 3\sin t \\ 2\cos t \end{pmatrix} dt = \begin{pmatrix} -3\cos t \\ 2\sin t \end{pmatrix} + \underset{\sim}{c}$
Using $\underset{\sim}{r}(0) = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ $\therefore$ $\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3\cos 0 \\ 2\sin 0 \end{pmatrix} + \underset{\sim}{c}$
Hence $\underset{\sim}{c} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ i.e. $\underset{\sim}{r}(t) = \begin{pmatrix} -3\cos t \\ 2\sin t - 2 \end{pmatrix}$
Specific behaviours
✓ anti-differentiates the velocity vector function correctly using a constant vector ✓ determines the vector constant of integration correctly ✓ uses correct vector and mathematics notation

(c) Determine the Cartesian equation for the path of the particle. (2 marks)

Solution
$\underset{\sim}{r}(t) = \begin{pmatrix} -3\cos t \\ 2\sin t - 2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\therefore$ $x = -3\cos t$, $y = 2\sin t - 2$
$\therefore$ $\cos t = -\frac{x}{3}$ and $\sin t = \frac{y+2}{2}$ Using $\cos^2 t + \sin^2 t = 1$ then
$\left(\frac{x}{3}\right)^2 + \left(\frac{y+2}{2}\right)^2 = 1$ i.e. $\frac{x^2}{9} + \frac{(y+2)^2}{4} = 1$ (Equation of an ellipse)
Specific behaviours
✓ forms separate expressions for $\cos t$ and $\sin t$ correctly ✓ uses the Pythagorean identity to eliminate t to form a Cartesian equation

2022 Section 2 Question 11 **Vectors in three dimensions**	A Formula One (F1) racing car has an initial displacement of 192 metres with an initial velocity of 24 metres per second. It accelerates for a period of 11 seconds in a straight line so that its velocity v metres per second and displacement x metres are related by the equation: $v(x) = \frac{x}{8}$ (a) Determine the acceleration a as a function of displacement x i.e. determine $a(x)$. (2 marks)

Solution
Acceleration $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{x}{8}\right)^2\right) = \frac{d}{dx}\left(\frac{x^2}{128}\right) = \frac{x}{64}$ OR $a = v.\frac{dv}{dx} = \left(\frac{x}{8}\right).\frac{1}{8} = \frac{x}{64}$ i.e. $a(x) = \frac{x}{64}$
Specific behaviours
✓ forms a correct expression for $a(x)$ ✓ uses the expression correctly to obtain a in terms of x

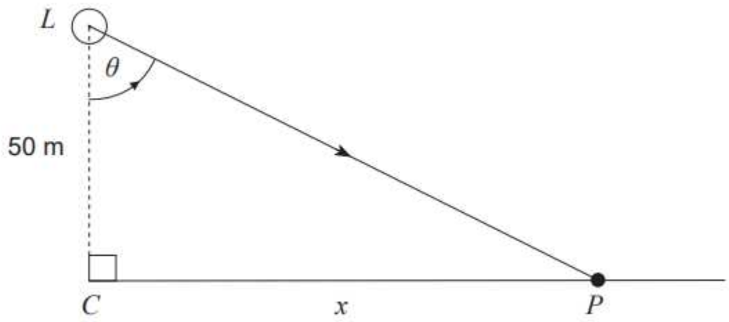
(b) Determine the displacement x as a function of time t . (2 marks)

Solution
We have $v = \frac{dx}{dt} = \frac{x}{8}$ $\therefore \int \frac{1}{x}dx = \int \frac{1}{8}dt$ using separation of variables i.e. $\ln x = \frac{t}{8} + c_1$ Using $x(0) = 192$ $\therefore \ln(192) = c$ $\therefore \ln\left(\frac{x}{192}\right) = \frac{t}{8}$ $\therefore \frac{x}{192} = e^{\frac{t}{8}}$ *i.e.* $x(t) = 192\,e^{\frac{t}{8}}$
Specific behaviours
✓ formulates the differential equation $\frac{dx}{dt} = \frac{x}{8}$ correctly ✓ determines the function $x(t)$ correctly

(c) Calculate the top speed reached and the distance travelled during the 11 second period of acceleration. (4 marks)

Solution
$x(t) = 192\,e^{\frac{t}{8}} \quad \therefore \quad v(t) = 24e^{\frac{t}{8}}$ $v(11) = 24e^{\frac{11}{8}} = 94.921\ldots$ m/sec   Distance travelled $= \Delta x \quad (\text{since } v > 0)$ $= \int_0^{11} v(t)\,dt = x(11) - x(0)$ $= 567.374\ldots$ m   Hence top speed is 95 m/sec and it travels 567 m during this acceleration.
Specific behaviours
✓ determines the velocity function $v(t)$ correctly   ✓ calculates the top speed correctly   ✓ forms the expression for the distance travelled correctly   ✓ calculates the distance travelled correctly

2021 Section 2 Question 9 **Vectors in three dimensions**	A beam of light completes three revolutions each minute from a lighthouse L that is 50 metres from a coastline. Determine the speed of the beam of light moving along the coast when it is at point P, 100 metres up the coast, correct to the nearest 0.01 metres per second. (5 marks)



Solution

$\dfrac{d\theta}{dt} = \dfrac{3\times 2\pi}{60} = \dfrac{\pi}{10}$ radians per second (0.314159…)

In right ΔLCP: $\tan\theta = \dfrac{x}{50}$ $\quad\therefore\ x = 50\tan\theta$

When $x = 100$ $\tan\theta = 2$ i.e. $\sec\theta = \sqrt{5}$ i.e. $\theta = 1.1071..$

$$\therefore\ \frac{dx}{dt} = 50\sec^2\theta \times \frac{d\theta}{dt}$$
$$= 50\left(\sqrt{5}\right)^2\left(\frac{\pi}{10}\right)$$
$$= 25\pi \ \ m/\text{sec}$$
$$= 78.5398...\ \ m/\text{sec}$$

$\therefore$ The beam is moving at a speed of 78.54 metres per second at point P.

Specific behaviours

- ✓ determines the angular rate of change correctly
- ✓ forms the correct expression for the distance x in terms of θ
- ✓ differentiates correctly to relate the rates of change
- ✓ substitutes correctly the value for $\sec\theta,\ \cos\theta\ or\ \theta$
- ✓ calculates the speed correctly to 0.01 (and states the correct units)

2021 Section 2 Question 14

Vectors in three dimensions

On a Saturday afternoon, three separate family groups visit their local cinema to watch a feature movie. The cinema names this as DollarDay where the ticket prices for adults, children and pensioners are charged in whole dollar amounts.

The table below indicates the number of people in each category and the total paid for each family group.

Group	Adults	Children	Pensioners	Total cost
1	2	4	–	\$108
2	3	6	–	\$162
3	2	5	2	\$152

Let a = the price for each adult (\$)
c = the price for each child (\$)
p = the price for each pensioner (\$)

(a) Formulate the equations that can be used to determine the ticket prices. (1 mark)

Solution

$2a+4c = 108 \quad \ldots (1)$

$3a+6c = 162 \quad \ldots (2)$

$2a+5c+2p = 152 \quad \ldots (3)$

Specific behaviours

✓ formulates the equations correctly using variables a, c, p

(b) Using the equations formed, determine the total cost for a group consisting of 1 child accompanied by 2 pensioners. (2 marks)

Solution

Consider equation $(3)-(1)$: $(2a+5c+2p)-(2a+4c) = 152-108$

i.e. $c+2p = 44$

Hence one child and 2 pensioners would cost \$44.

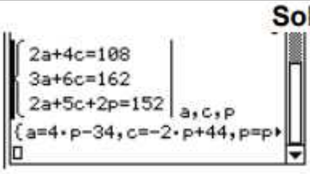
Specific behaviours

✓ uses equations (1) and (3) to determine $c+2p$
✓ states that the total cost is \$44

(c) Solve simultaneously the equations formulated in part (a). (2 marks)

Solution

From CAS:



Given the pensioner price is p: $a=4p-34$ and
$c=44-2p$

Hence there is NO unique solution (many solutions are possible for the ticket prices).

Specific behaviours

✓ obtains the correct relationships between variables
✓ states that there are MANY (non-unique) solutions

(d) Explain the geometric interpretation of the equations and the simultaneous solution. (2 marks)

Solution

Equations (1) and (2) represent that SAME plane in space.

The intersection between the planes (1) and (3) gives a LINE in space, which consists of many points, which is why there is NOT a unique solution.

Specific behaviours

✓ states that the intersection gives a LINE in space
✓ states that two of the planes were identical

Now assume that the price for an adult is greater than the price for a child and that the price for a pensioner is the lowest priced ticket.

(e) Determine the ticket prices for adults, children and pensioners on DollarDay. (3 marks)

Solution

To give sensible values we require $a > 0$ and $c > 0$
i.e. $2p - 34 > 0$ and $44 - 2p > 0$
i.e. $p > 8.25$ and $p < 22$
Given that p is an integer (whole dollars) then $9 \le p \le 21$

Tabulating the 13 possibilities (ordered triples):

a	c	p
2	26	9
6	24	10
10	22	11
14	20	12
18	18	13
22	16	14
26	14	15
30	12	16
34	10	17
38	8	18
42	6	19
46	4	20
50	2	21

Given that $a > c > p$ there is only ONE possibility:

$a = 22$
$c = 16$
$p = 14$

Hence the prices are $22 for adults, $16 for children and $14 for pensioners.

Specific behaviours

✓ determines that $9 \le p \le 21$ where p is an integer (justifies a limitation on one of the variables)
✓ tabulates at least 2 possible ordered triples for a, c, p correctly
✓ states the correct triple of ticket prices

Alternative Solution

From $a+2c=54$ we realise that a must be EVEN.
From $c+2p=44$ we see that c must be EVEN.

Since $a>c$ then $3c<54$ i.e. $c<18$ and is EVEN
i.e. $c\le 16$

Tabulating some possibilities (ordered triples):

a	c	p
22	16	14
26	14	15
30	12	16

Given that $a>c>p$ there is only ONE possibility:

$$a=22$$
$$c=16$$
$$p=14$$

Hence the prices are $22 for adults, $16 for children and $14 for pensioners.

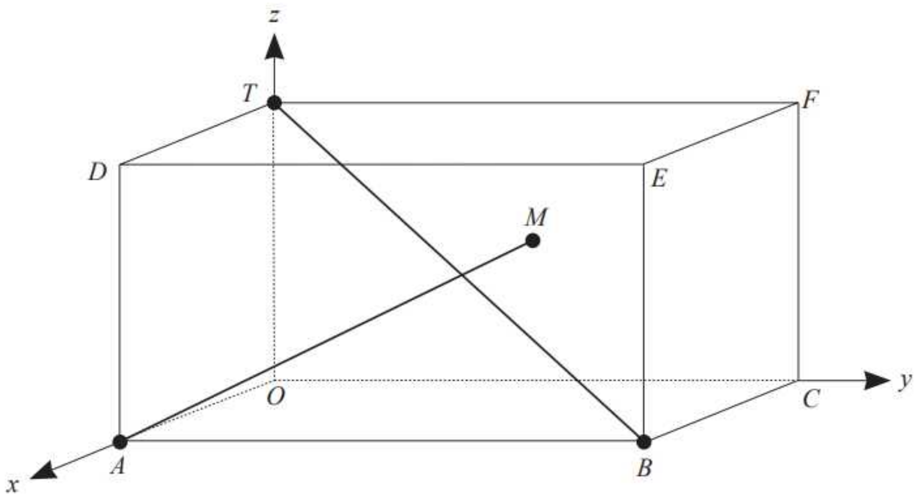
Specific behaviours

✓ determines that $c\le 16$ where c is an even integer (justifies a limitation on one of the variables)
✓ tabulates at least 2 possible ordered triples for a, c, p correctly
✓ states the correct triple of ticket prices

2021 Section 2 Question 16

Vectors in three dimensions

A rectangular prism is defined using the coordinate system shown with A (2, 0, 0), C (0, 4, 0) and T (0, 0, 3). Point M is the centre of the planar face $OCFT$ with coordinates (0, 2, 1.5).



(a) Determine the vector equation for the prism's main diagonal $\overleftrightarrow{BT}$. (2 marks)

Solution
Direction vector for $\overrightarrow{BT}$ $\quad \underset{\sim}{d} = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$
Equation $\overleftrightarrow{BT}$ $\quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 2-2\lambda \\ 4-4\lambda \\ 3\lambda \end{pmatrix}$
Specific behaviours
✓ determines the direction vector for $\overleftrightarrow{BT}$ correctly ✓ forms the vector equation correctly using a parameter

(b) Determine the Cartesian equation of the sphere that contains all vertices of the rectangular prism. (3 marks)

Solution
Sphere centre S will be the midpoint of the main diagonal $\overline{BT}$ i.e. S is $\begin{pmatrix} 1 \\ 2 \\ 1.5 \end{pmatrix}$
$r^2 = CS^2 = (2-1)^2 + (4-2)^2 + (0-1.5)^2 = 7.25 \quad \therefore \; r = \frac{\sqrt{29}}{2} = 2.6925...$
Cartesian equation sphere $(x-1)^2 + (y-2)^2 + (z-1.5)^2 = 7.25$
Specific behaviours
✓ determines the centre of the sphere correctly ✓ determines the radius of the sphere correctly ✓ forms the Cartesian equation for the sphere correctly

(c) Prove, using a vector method, that line $\overleftrightarrow{AM}$ does **not** intersect $\overleftrightarrow{BT}$. (3 marks)

Solution

Equation $\overleftrightarrow{AM}$ $\quad \underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 2 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 2-2\mu \\ 2\mu \\ 1.5\mu \end{pmatrix}$

Lines will intersect if there exists real values for λ, μ where :

$$\begin{pmatrix} 2-2\lambda \\ 4-4\lambda \\ 3\lambda \end{pmatrix} = \begin{pmatrix} 2-2\mu \\ 2\mu \\ 1.5\mu \end{pmatrix}$$

i.e. $\quad 2-2\lambda = 2-2\mu \quad ... (1)$

$\quad 4-4\lambda = 2\mu \quad ... (2)$

$\quad 3\lambda = 1.5\mu \quad ... (3)$

From (1): $\quad \lambda = \mu$

(2): $\quad 4-4\lambda = 2\lambda \quad \therefore \quad \lambda = \frac{2}{3} = \mu$

(3): $\quad 2\lambda = \mu \qquad BUT \quad \mu \neq 2\lambda$

$\therefore$ There are no values λ, μ that give an intersection.

Hence line $\overleftrightarrow{AM}$ does NOT intersect $\overleftrightarrow{BT}$.

Specific behaviours

- ✓ states the condition for the intersection of lines in terms of the parameters
- ✓ forms the 3 equations comparing components for an intersection
- ✓ shows that no such parameter values for λ, μ exist

Alternative Solution

Lines will NOT intersect if the distance of closest approach is greater than ZERO.

Separation vector $\underset{\sim}{v} = \begin{pmatrix} -2\lambda+2\mu \\ -4\lambda-2\mu+4 \\ 3\lambda-1.5\mu \end{pmatrix}$

Closest approach is when the separation vector is PERPENDICULAR to the direction vector of each line.

$\therefore \quad \underset{\sim}{v} \bullet \underset{\sim}{d_1} = 0 \quad 29\lambda - 0.5\mu - 16 = 0 \quad ... (1)$

$\therefore \quad \underset{\sim}{v} \bullet \underset{\sim}{d_2} = 0 \quad 0.5\lambda - 10.25\mu + 8 = 0 \quad ... (2)$

Solving for λ, μ: $\quad \lambda = \frac{56}{99} = 0.5656.... \qquad \mu = \frac{80}{99} = 0.8080...$

$$|\underset{\sim}{v}| = \sqrt{\left(\frac{16}{33}\right)^2 + \left(\frac{4}{33}\right)^2 + \left(\frac{16}{33}\right)^2} = \frac{4}{\sqrt{33}} = 0.6963 ...$$

Hence as $|\underset{\sim}{v}| > 0$ then line $\overleftrightarrow{AM}$ does NOT intersect $\overleftrightarrow{BT}$.

Specific behaviours

- ✓ states the condition for the closest approach
- ✓ forms the 2 equations for the closest approach
- ✓ calculates correctly the closest approach for the 2 lines

2021 Section 2 Question 19

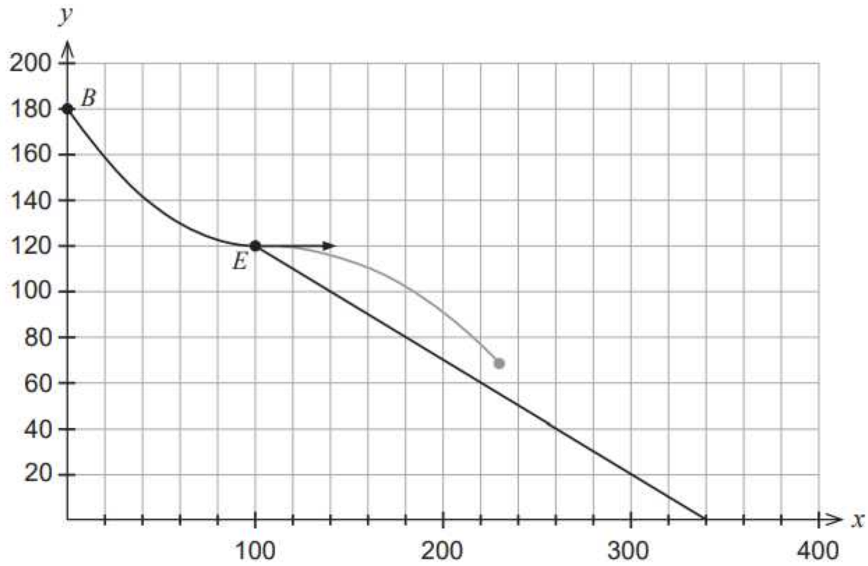
Vectors in three dimensions

Using the correct technique, Olympic ski jumpers can slow down their descent, by creating lift to counteract gravity. These jumpers must land successfully to have their distance recorded and land on sloped ground to prevent serious injury.

A skier begins his descent at point B accelerating down the ramp. At the end of the ramp the skier is travelling horizontally at point E at 32 metres per second (115.2 kilometres per hour).

Let t = the number of seconds in flight after point E $(100, 120)$.
$h(t)$ = the height of the skier above the horizontal ground $y = 0$ (metres)
$x(t)$ = the horizontal position of the skier (metres)

The sloped ground for landing is given by $y = 170 - 0.5x$ where $100 \le x \le 340$.



The ski jumper's suit and skis decrease the horizontal velocity $x'(t)$ so that $x'(t) = 32e^{-0.05t}$.

(a) Show that $x(t) = 740 - 640e^{-0.05t}$. (2 marks)

Solution
$x(t) = \int 32e^{-0.05t}dt = \dfrac{32}{-0.05}e^{-0.05t} + c$ i.e. $x(t) = -640e^{-0.05t} + c$ Using $x(0) = 100$ then $100 = -640(1) + c$ $\therefore$ $c = 740$
Specific behaviours
✓ anti-differentiates $x'(t)$ correctly using a constant   ✓ uses $x(0) = 100$ correctly to determine the constant

It is found that the expression for the position vector for the skier during the flight is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 740 - 640e^{-0.05t} \\ 120 - 2.5t^2 \end{pmatrix}$$

(b) Calculate the height of the skier above the sloped ground after 3 seconds of flight, correct to the nearest 0.01 metre. (3 marks)

Solution
At $t=3$ $\quad r(3) = \begin{pmatrix} 189.1468... \\ 97.5 \end{pmatrix}$
For $x=189.1468..$ Sloped ground $y=170-0.5(189.1468..) = 75.42655...$
Hence skier height ABOVE the ground $= 97.5-75.42655... = 22.073...$ m i.e. The skier is 22.07 metres above the sloped ground after 3 seconds.
Specific behaviours
✓ calculates both components for $\underset{\sim}{r}(3)$ correctly ✓ calculates the sloped ground height correctly ✓ determines the height of the skier above the ground correct to 0.01 metres

(c) Determine the vertical lift s (m/s^2) provided by the skier's suit and equipment in the descent if $\frac{d^2h}{dt^2} = s - 9.8$, where s is a constant. (3 marks)

Solution
Using $\frac{d^2h}{dt^2} = s-9.8$ then $h'(t) = (s-9.8)t + c$
Since $h'(0)=0$ then $c=0$ i.e. $h'(t) = (s-9.8)t$
$\therefore\ h(t) = \frac{(s-9.8)}{2}t^2 + k$ $\quad$ Since $h(0)=120$ then $k=120$
i.e. $h(t) = \frac{(s-9.8)}{2}t^2 + 120$
Hence as $h(t)=120-2.5t^2$ then $\frac{s-9.8}{2} = -2.5$
Solving gives $s=4.8$ ms^{-2}.
Specific behaviours
✓ performs appropriate calculus correctly ✓ forms an equation to determine the value of s ✓ determines the value of s correctly

It can be shown that the Cartesian equation for the skier's flight is given by:

$$y = 120 - 1000\left(\ln\left(\frac{740-x}{640}\right)\right)^2$$

(d) Calculate the time taken for the skier to land on the sloped ground, correct to the nearest 0.01 second. (3 marks)

Solution

We need to determine the landing point on the sloped ground.

Solving simultaneously: $y = 170 - 0.5x$

$$y = 120 - 1000\left(\ln\left(\frac{740-x}{640}\right)\right)^2$$

From CAS: $(100, 120)$ and $(255.915887,\ 42.04205652)$

$\therefore\ 42.04205652 = 120 - 2.5t^2$ OR $255.915887 = 740 - 640e^{-0.05t}$

From CAS: $t = 5.584189949...$ sec

Hence the skier will take 5.58 seconds to land on the sloped ground.

Specific behaviours

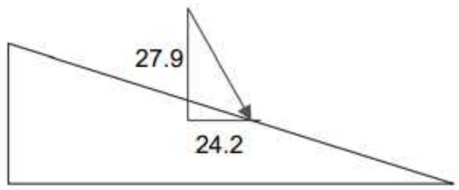
✓ solves simultaneously to correctly determine the landing position
✓ forms the equation to solve for t correctly
✓ solves for t correct to 0.01 seconds

(e) Calculate the angle at which the skier impacts the sloped ground, correct to the nearest 0.1 degree. (3 marks)

Solution

We need to determine the velocity vector $r'(5.5841)$.

$$r'(t) = \begin{pmatrix} 32e^{-0.05t} \\ -5t \end{pmatrix} \qquad \therefore\ \underset{\sim}{r}'(5.58418...) = \begin{pmatrix} 24.2042... \\ -27.9209.. \end{pmatrix}$$



$$\text{Angle of landing} = \tan^{-1}\left(\frac{27.9209}{24.2042}\right) - \tan^{-1}(0.5)$$

$$= 49.0784° - 26.5650°$$

$$= 22.513°$$

Hence the skier lands at an angle of 22.5° to the sloped ground.

Specific behaviours

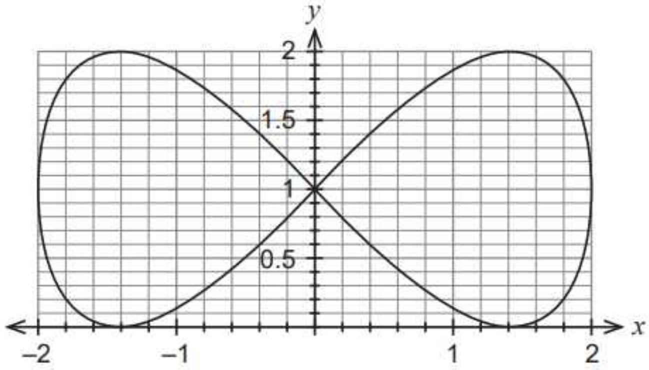
✓ determines the velocity vector $\underset{\sim}{r}'(5.5841)$ for the landing correctly
✓ forms the correct expression to determine the angle of landing
✓ determines the angle of landing correctly

2020 Section 2 Question 14 **Vectors in three dimensions**	A particle travels in a straight line so that its velocity v cm per second and displacement x cm are related by the equation: $v = -0.2x$ (a) Determine the acceleration a in terms of its displacement x. (2 marks) **Solution** Acceleration $a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = (-0.2) \times (-0.2x)$ OR $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(0.02x^2\right) = 0.04x$ i.e. $a = 0.04x$ **Specific behaviours** ✓ uses the chain rule correctly to relate $\frac{dv}{dt}$ in terms of v ✓ obtains a correctly in terms of x (b) Does the particle's motion constitute simple harmonic motion? Justify your answer. (1 mark) **Solution** Since $a = 0.04x$ then we could write $a = (0.2)^2 x$ But the condition for S.H.M. is that $a = -n^2 x$. Hence the motion does NOT follow simple harmonic motion. **Specific behaviours** ✓ justifies why the motion is NOT simple harmonic It is known that the initial displacement of the particle is $x = 4$ cm. (c) Determine, correct to the nearest 0.01 second, when the particle has a displacement of 2 cm. (2 marks) **Solution** We have $v = \frac{dx}{dt} = -0.2x$ $\therefore \int \frac{1}{x}\,dx = \int -0.2\,dt$ using separation of variables i.e. $\ln\lvert x\rvert = -0.2t + c_1$ i.e. $x = e^{-0.2t+c} = k \times e^{-0.2t}$ Using $x(0) = 4$, then $4 = k\left(e^0\right)$ Hence $k = 4$ i.e. $x = e^{-0.2t+c} = 4e^{-0.2t}$ Solving $x(t) = 2$ yields $2 = 4e^{-0.2t}$ i.e. $t = 5\ln 2 = 3.47$ seconds (correct to 0.01 sec) **Specific behaviours** ✓ determines the function $x(t)$ correctly ✓ solves for t correct to 0.01 seconds

2019 Section 2 Question 13

Vectors in three dimensions

The path of a particle is shown below. This particle moves so that its position vector $\underset{\sim}{r}(t)$ is given by $\underset{\sim}{r}(t) = \begin{pmatrix} -2\cos\left(\frac{t}{2}\right) \\ 1 - \sin(t) \end{pmatrix}$ metres, where t is the number of seconds the particle has been in motion.



(a) Determine the starting position of the particle and mark this as point A on the diagram above. (1 mark)

Solution
Substituting $t=0$ $\underset{\sim}{r}(0) = \begin{pmatrix} -2\cos(0) \\ 1-\sin(0) \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
Specific behaviours
✓ indicates the point $(-2,1)$ on the diagram correctly

(b) Determine the initial velocity of the particle and illustrate this on the diagram above. (3 marks)

Solution
$\underset{\sim}{v}(t) = \frac{d\underset{\sim}{r}}{dt} = \begin{pmatrix} \sin\left(\frac{t}{2}\right) \\ -\cos(t) \end{pmatrix}$ Substituting $t=0$ $\underset{\sim}{v}(0) = \begin{pmatrix} \sin\left(\frac{0}{2}\right) \\ -\cos(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
Specific behaviours
✓ determines $\underset{\sim}{v}(t)$ by differentiating BOTH components correctly ✓ evaluates $\underset{\sim}{v}(0)$ correctly ✓ indicates a vector representing $\underset{\sim}{v}(0)$ on the diagram correctly

(c) Write the expression, in terms of trigonometric functions, for the distance the particle would travel in completing one circuit of the given path. Do **not** evaluate this expression. (3 marks)

Solution
One circuit is completed when $-2\cos\frac{t}{2}=-2$ i.e. $t=0,\ 4\pi,\ \ldots$
Distance travelled for one circuit $=\int_0^{4\pi} Speed(t)\,dt = \int_0^{4\pi} \lvert \underset{\sim}{v}(t)\rvert\,dt$ $= \int_0^{4\pi} \sqrt{\sin^2\left(\frac{t}{2}\right)+\cos^2(t)}\,dt$
Specific behaviours
✓ forms a definite integral using the correct limits $t=0,\ 4\pi$ ✓ writes the correct expression for the speed function in terms of t using trigonometric functions ✓ uses correct mathematics notation for vectors and the definite integral

(d) Determine the Cartesian equation for the path of the particle. (3 marks)

Solution
We have $x=-2\cos\left(\frac{t}{2}\right)$ i.e. $\cos\left(\frac{t}{2}\right) = -\frac{x}{2}$ and $y=1-\sin(t)$ i.e. $\sin(t) = 1-y$ $\cos\left(2\left(\frac{t}{2}\right)\right) = 2\cos^2\left(\frac{t}{2}\right)-1 \quad \therefore \quad \cos(t) = 2\left(-\frac{x}{2}\right)^2-1 = \frac{x^2}{2}-1$ Since $\sin^2\theta+\cos^2\theta=1$ then the Cartesian equation becomes $(1-y)^2+\left(\frac{x^2}{2}-1\right)^2 = 1$ Alternative forms: $1-2y+y^2+\frac{x^4}{4}-x^2=0$ $(y-1)^2 = 1-\left(\frac{x^2}{2}-1\right)^2$ or $y = 1 \pm \sqrt{1-\left(\frac{x^2}{2}-1\right)^2}$
Specific behaviours
✓ forms correct expressions for $\cos\left(\frac{t}{2}\right)$ and $\sin(t)$ correctly in terms of x, y ✓ uses the double angle identity to obtain trigonometric equations with the same variable (either both in terms of $\frac{t}{2}$ or t) ✓ uses the identity $\sin^2\theta+\cos^2\theta=1$ correctly to eliminate t to obtain the Cartesian equation

2019 Section 2 Question 16 Vectors in three dimensions	Plane Π_1 has Cartesian equation $z = 2x + y + 4$. (a) Determine a vector that is normal to plane Π_1 . (2 marks)

Solution
Cartesian equation can be written as $2x + y - z = -4$. $\therefore \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -4$ is the vector normal form for plane Π_1. Hence the vector $2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ is perpendicular (the normal) to plane Π_1.
Specific behaviours
✓ re-writes the Cartesian form into vector form OR into standard Cartesian form. ✓ states the vector perpendicular to plane Π_1

Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(b) Determine the point of intersection between line L and plane Π_1 . (3 marks)

Solution
Line L has equation $\underset{\sim}{r} = \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3-\lambda \end{pmatrix}$ Solving simultaneously: $\therefore \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3-\lambda \end{pmatrix} = -4$ i.e. $2(2+\lambda)+1(2\lambda)-1(3-\lambda) = -4$ i.e. $5\lambda + 1 = -4$ i.e. $\lambda = -1$ Hence intersection point is $(1, -2, 4)$.
Specific behaviours
✓ substitutes the expression for line vector correctly into equation for plane ✓ solves correctly for the value of parameter λ ✓ states the point of intersection (can be either in vector or Cartesian form)

Plane Π_2 contains line L and is perpendicular to plane Π_1 .

(c) Determine the vector equation for plane Π_2 . (4 marks)

Solution

Let $\underset{\sim}{n}_2$ be the normal vector for plane Π_2.

Hence $\underset{\sim}{n}_2$ is perpendicular to $\underset{\sim}{d}$ direction vector of the line L AND to the normal $\underset{\sim}{n}_1$ of plane Π_1. Hence $\underset{\sim}{n}_2 = \underset{\sim}{n}_1 \times \underset{\sim}{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

A point in plane Π_2 can be any point on Line L i.e. $(2,0,3)$

Vector equation for Π_2: $\underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 11$

Note: Cartesian equation for Π_2: $x + y + 3z = 11$

Specific behaviours

✓ states that the normal $\underset{\sim}{n}_2$ is perpendicular to both $\underset{\sim}{d}$ and $\underset{\sim}{n}_1$

✓ determines the vector $\underset{\sim}{n}_2$ (by cross product or otherwise)

✓ uses correctly a point of L as a point for plane Π_2

✓ forms the vector equation for plane Π_2 correctly

or

Alternative Solution

The vector equation for plane Π_2 is given by :

$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ i.e. $\underset{\sim}{r} = \begin{pmatrix} 2 + \lambda + 2\mu \\ 2\lambda + \mu \\ 3 - \lambda - \mu \end{pmatrix}$

This is because Π_2 contains line L so it contains the point $(2,0,3)$ and contains the direction vector $(1,2,-1)$ of line L. Also plane Π_2 has to be in a direction that is normal to Π_1 hence we can move in the direction $(2,1,-1)$.

Specific behaviours

✓ uses the known point $(2,0,3)$ in writing the vector equation using two directions

✓ uses two different parameters with the two directions

✓ uses the direction vector $(1,2,-1)$ for line L correctly

✓ uses the normal vector $(2,1,-1)$ for plane Π_1 correctly

Sphere S has vector equation $|\underset{\sim}{r} - (3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})| = \sqrt{35}$.

(d) Determine whether line L is a tangent to sphere S. Justify your answer. (3 marks)

Solution

Examine how many points of intersection there are between the line and the sphere.
To find the intersection solve simultaneously :

$$\left|\begin{pmatrix}2+\lambda\\2\lambda\\3-\lambda\end{pmatrix}-\begin{pmatrix}3\\1\\4\end{pmatrix}\right| = \sqrt{35} \quad i.e. \quad (\lambda-1)^2+(2\lambda-1)^2+(-1-\lambda)^2=35$$

$\therefore \quad 6\lambda^2-4\lambda-32=0$

$\therefore \quad \lambda=-2$ or $\lambda=\frac{8}{3}$ $\quad\therefore$ There are 2 points of intersection.

$\therefore$ The line is NOT a tangent.

Specific behaviours

✓ obtains the equation that determines the intersection
✓ solves correctly to determine the parameter λ
✓ justifies that the 2 points of intersection means the line is not a tangent

or

Alternative Solution

Determine the shortest distance from the centre of the sphere to the line.

Using point A $(2,0,3)$ on line L and centre of sphere C $(3,1,4)$.
Hence $\overline{AC}=(1,1,1)$.

Shortest distance from point C to line L

$$= \frac{|\overline{AC}\times\underset{\sim}{d}|}{|\underset{\sim}{d}|} = \frac{|(1,1,1)\times(1,2,-1)|}{|(1,-2,1)|} = \frac{|(-3,2,1)|}{\sqrt{6}} = \frac{\sqrt{14}}{\sqrt{6}} = \sqrt{\frac{7}{3}} = 1.527...$$

Since the radius of sphere $\sqrt{35} > \sqrt{\frac{7}{3}}$ then there will be two points of intersection.

Specific behaviours

✓ forms the correct expression for the shortest distance
✓ calculates the shortest distance correctly
✓ justifies that the radius > shortest distance means the line is not a tangent

2019 Section 2 Question 19

Vectors in three dimensions

Two parallel planes Π_1 and Π_2 have their equations given by:

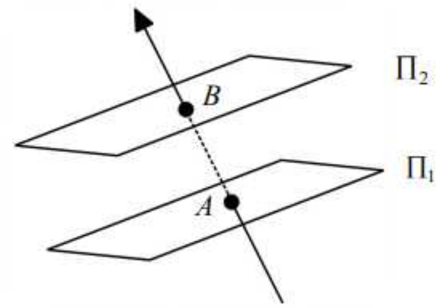
$\Pi_1 \qquad \underset{\sim}{r} \bullet \underset{\sim}{n} = 11$

$\Pi_2 \qquad \underset{\sim}{r} \bullet \underset{\sim}{n} = -4 \qquad$ where $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

It is known that (2,3,–7) is a point on plane Π_1.

Prove the distance d between the point (2,3,–7) and plane Π_2 is given by $d = \dfrac{15}{\sqrt{a^2+b^2+c^2}}$.

Solution



Distance $d = AB$

Equation for line $\overrightarrow{AB}$ is given by:

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{(Equation 1)}$$

Since B is a point in plane Π_2 and line $\overrightarrow{AB}$ then $\left(\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -4$

i.e. $\begin{pmatrix} 2 \\ 3 \\ -7 \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -4 \quad$ (Equation 2)

i.e. $11 + \lambda\left(a^2+b^2+c^2\right) = -4 \qquad \therefore \ \lambda = \dfrac{-15}{a^2+b^2+c^2}$ for position B.

$\lambda\left(a^2+b^2+c^2\right) = -4-11 = -15$

Hence distance $d = \left|\lambda \underset{\sim}{n}\right| = \left|\dfrac{-15}{a^2+b^2+c^2}\right| \left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \dfrac{15}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}$

$= \dfrac{15}{\sqrt{a^2+b^2+c^2}}$

Specific behaviours

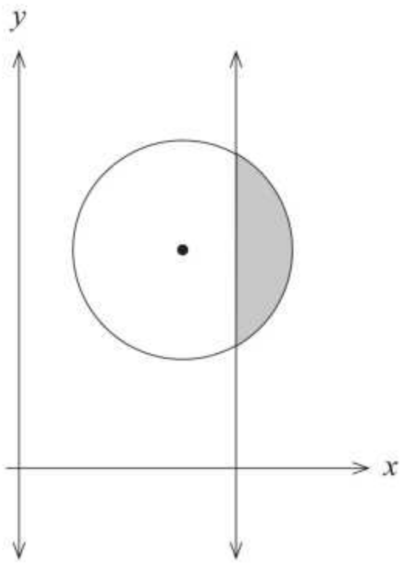
✓ forms the correct equation for the perpendicular to each plane
✓ substitutes correctly into the equation for plane Π_2 to form equation 2
✓ develops the correct expression for the parameter λ
✓ uses the idea that $d = \left|\lambda \underset{\sim}{n}\right|$ to determine the distance expression

Unit 4

Unit 4.1 – Integration and applications of integration

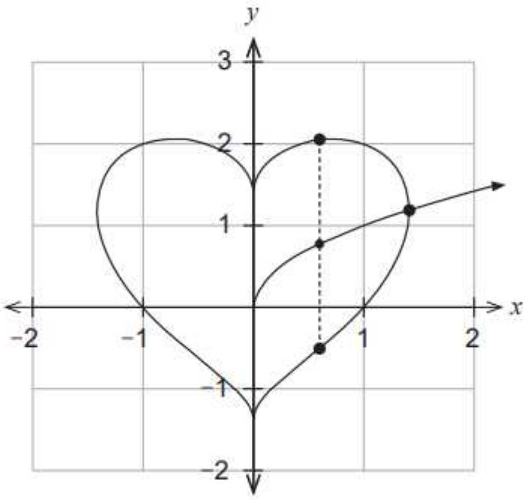
Section 1

2023 Section 1 Question 3 Integration and applications of integration	Using the substitution $x = 119u + 1$, evaluate exactly $\int_1^{120} \left(2 + 4\left(\frac{x+118}{119}\right)^3\right) dx$. (5 marks)

2023 Section 1 Question 7 Integration and applications of integration	The shaded region is bounded by the curve $(x-3)^2+(y-4)^2=4$ and the line $x=4$.  (a) Show that the area of this region is given by the definite integral $\int_4^a 2\sqrt{4-(x-3)^2}\,dx$. State the value for a. (3 marks)

	(b) By using the substitution $x-3=2\sin\theta$, determine the exact value for the area of the shaded region. (6 marks)

2022 Section 1 Question 3 **Integration and applications of integration**	By using one or more of the following identities: $\cos^2 x + \sin^2 x = 1$ $\cos 2x = \cos^2 x - \sin^2 x$ $\sin 2x = 2\sin x \cos x$ evaluate exactly $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$. (5 marks)

2021 Section 1 Question 8 Integration and applications of integration	The heart-shaped figure shown is given by the equation $x^2 + \left(y - \sqrt{\lvert x \rvert}\right)^2 = 2$. For $x \geq 0$, this equation becomes $x^2 + \left(y - \sqrt{x}\right)^2 = 2$. The curve $y = \sqrt{x}$ is also drawn. This heart-shaped curve has the special property that for each x coordinate in its domain its two y coordinates are an equal vertical distance from the curve $y = \sqrt{x}$.  (a) Explain why the domain for the curve given by $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ is $0 \leq x \leq \sqrt{2}$. (2 marks) (b) Show that the total area enclosed by the heart-shaped figure is given by: $Area = 4\int_0^{\sqrt{2}} \sqrt{2 - x^2}\,dx$. (2 marks)

	(c) By using the substitution $x = \sqrt{2}\sin\theta$, evaluate the total area enclosed by the heart-shaped figure, and hence see why it can be said that 'π is at the heart of mathematics'. (5 marks)

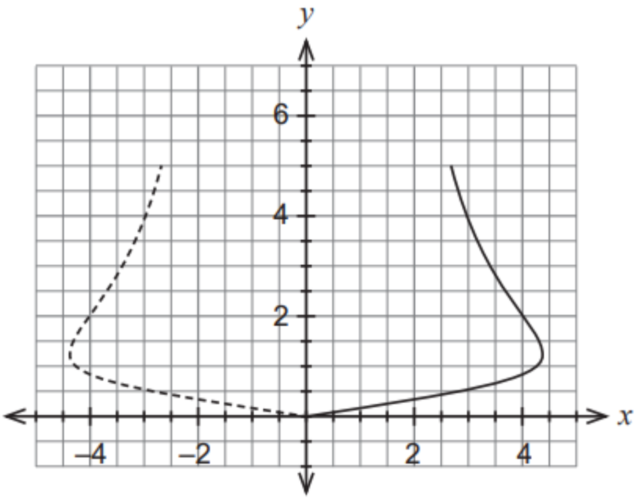
2020 Section 1 Question 7 **Integration and applications of integration**	Evaluate $\int_{-1}^{7} \frac{3x}{\sqrt{x+2}}\,dx$ exactly using the substitution $u = \sqrt{x+2}$. (5 marks)

2019 Section 1 Question 1 **Integration and applications of integration**	Using the identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$, evaluate exactly the definite integral $\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx$. (4 marks)

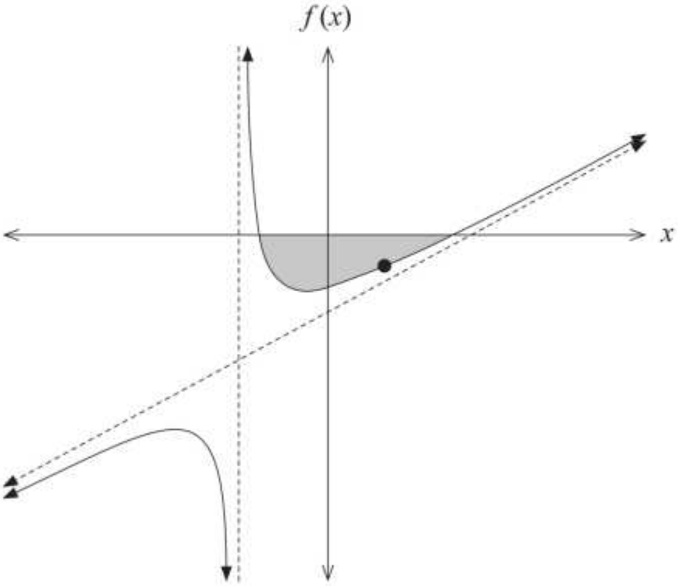
2019 Section 1 Question 3 **Integration and applications of integration**	(a) Given that $\frac{2x^2+5x+6}{x^2(x+3)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+3}$, determine the values of a, b and c. (2 marks)

	(b) Hence determine $\int \frac{2x^2 + 5x + 6}{x^2(x+3)}\,dx$. (3 marks)

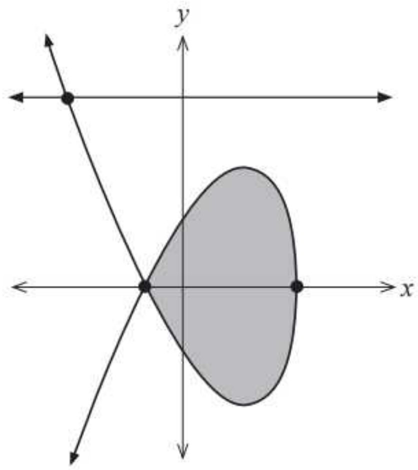
2019 Section 1 Question 6 **Integration and applications of integration**	Using the substitution $x = 2\sin\theta$, evaluate exactly $\int_0^{\sqrt{3}} \sqrt{1 - \frac{x^2}{4}}\,dx$. (6 marks)

2019 Section 1 Question 8 Integration and applications of integration	The top part of a wine glass is modelled by rotating the graph of $x^2 = y^2(36 - x^2 y)$ from $y = 0$ to $y = 5$ about the y axis as shown below. Dimensions are measured in centimetres.  (a) Show that the volume, $V \text{ cm}^3$, when the glass is full is given by $V = \pi \int_0^5 \frac{36y^2}{1+y^3}\,dy.$ (1 mark)

(b) Determine the exact volume $V\,\text{cm}^3$. (4 marks)

2023 Section 2 Question 18 Integration and applications of integration	Function $f(x)$ is a rational function of the form $\dfrac{x^2+bx+c}{x+d}$ with the following properties: • $f(2)=-2$ • $f(x)$ has a vertical asymptote at $x=-3$ and another asymptote with equation $y=x-5$.  (a) Show that $b=-2$, $c=-10$ and $d=3$. (3 marks)

	(b) Calculate the exact area of the shaded region. (3 marks)

2022 Section 2 Question 13 Integration and applications of integration	The equation $x^3 - x^2 - 5x = 3 - y^2$ implicitly defines the curve shown below. The line $y = \sqrt{24}$ intersects this curve as shown below.  It can be shown that the equation $x^3 - x^2 - 5x + 21 = 0$ will determine the intersection between the line $y = \sqrt{24}$ and the implicitly defined curve. (a) Explain, with reference to the graph above, why we know that there is one real and two complex solutions (a conjugate pair) to this cubic equation. (2 marks) (b) Determine the two exact complex solutions to the equation $x^3 - x^2 - 5x + 21 = 0$. (2 marks)

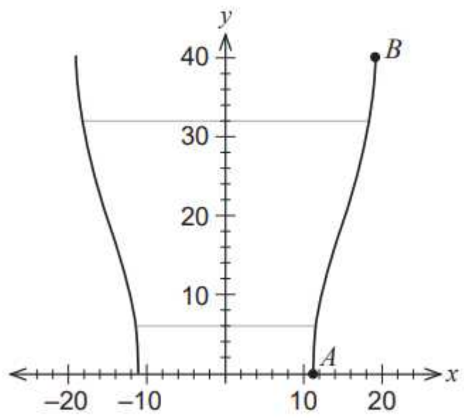
(c) Calculate the area of the shaded region, correct to 0.001 square units. (4 marks)

2022 Section 2 Question 17

Integration and applications of integration

A vase has water with a depth of 6 cm and needs to be filled to a depth of 32 cm. The cross-section of the vase is modelled by the curve AB where $x = 15 - 4\cos\left(\frac{\pi y}{40}\right),\ 0 \le y \le 40,$ and this curve is revolved about the y axis.
All dimensions are in centimetres.

Give all answers in this question to the nearest appropriate unit of measurement.



(a) Calculate the volume of water that needs to be added to increase the depth of water from 6 cm to 32 cm. (3 marks)

Josie, an interior designer, uses a hose to add water to the vase. This hose has a water-saving device that regulates the rate at which water flows into the vase. This rate is given by:

$$\frac{dV}{dt} = 300e^{-\frac{V}{12\,000}}$$

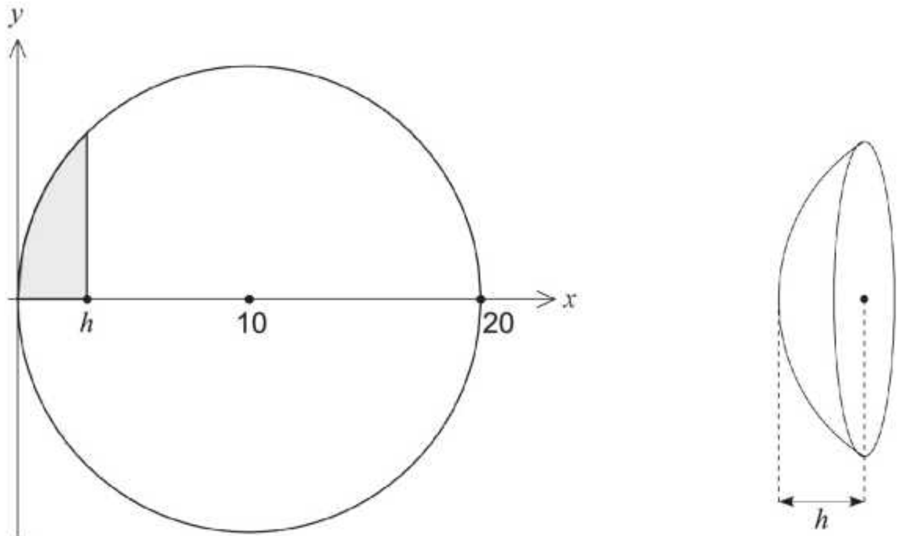
where $V(t)$ = the volume of water (cm^3) poured into the vase after t seconds of flow.

(b) If Josie has already poured 6000 cm^3 , use the increments formula to calculate an approximation for the volume of water she will pour in the next 0.5 seconds. (2 marks)

To prevent an overflow of water, the device can be calibrated to switch off the flow after a set length of time using an in-built timer.

(c) Calculate the rate of flow into the vase at the instant when the depth becomes 32 cm. (2 marks)

	(d) Using separation of variables, obtain the defining rule for $V(t)$. (3 marks)

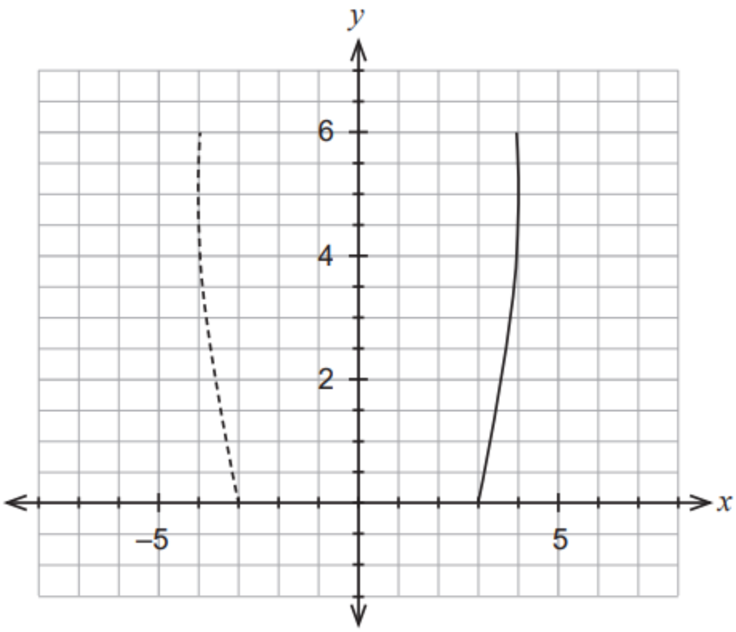
2021 Section 2 Question 13 **Integration and applications of integration**	A solid spherical cap with depth h is part of a solid sphere with radius 10 cm. This cap can be generated by revolving the shaded region about the x axis.  (a) Show that the equation for the circle shown above is $x^2 + y^2 = 20x$. (1 mark)

	(b) Develop an expression for the volume of the spherical cap in terms of h . (4 marks)

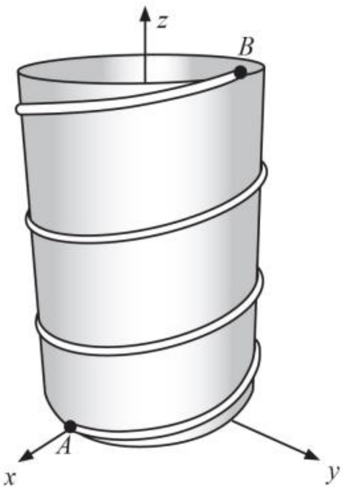
2020
Section 2
Question 9

Integration and applications of integration

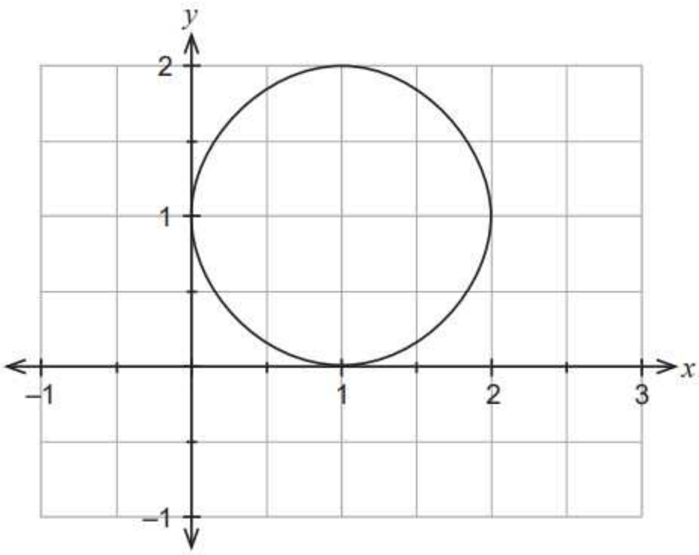
The shape of a small wine glass is modelled by revolving the curve $\sin\left(\frac{y}{\pi}\right) = x - 3$ about the y axis, where $0 \le y \le 6$. All dimensions are in centimetres.



Calculate, correct to the nearest 0.01 cm, the depth of wine in the glass if it is to contain 80% of its maximum volume. (4 marks)

2020 Section 2 Question 12 **Integration and applications of integration**	A cylindrical shaped tower has a path that spirals upwards from the ground to an observation deck at point B as shown in the diagram below. The path begins at point A on the ground and finishes at point B at the top. Let t = time in seconds that a tourist has been walking along the spiral path. The tourist takes 65π seconds to reach point B. The tourist's position on this path at any time t is given by: $\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(0.1t) \\ 10\sin(0.1t) \\ 0.2t \end{pmatrix}$ metres.  (a) Determine the height of the observation deck above the ground, correct to the nearest 0.01 metres. (1 mark) (b) Determine the tourist's velocity $\underset{\sim}{v}(t)$. (2 marks)

	(c) Show that the tourist walks at a constant speed and determine this speed, correct to 0.01 metres per second. (3 marks)

2020 Section 2 Question 16 Integration and applications of integration	The curve shown below is given by the equation $(y-1)^2 = \sin\left(\frac{\pi x}{2}\right)$.  (a) Calculate the area, correct to 0.0001 square units, of the region that forms the interior of this curve. (3 marks) (b) By using the answer to part (a), determine whether this curve is a circle. Explain your answer. (2 marks)

Marking Guide – Section 1

2023 Section 1 Question 3 **Integration and applications of integration**	Using the substitution $x = 119u + 1$, evaluate exactly $\int_1^{120} \left(2 + 4\left(\frac{x+118}{119}\right)^3\right) dx$. (5 marks)

Solution

Using $x = 119u + 1 \quad \therefore \quad dx = 119\,du$

$x + 118 = 119u + 119$

$\therefore \quad \frac{x+118}{119} = \frac{119u+119}{119} = u + 1$

x	1	120
u	0	1

$$\int_1^{120} \left(2 + 4\left(\frac{x+118}{119}\right)^3\right) dx = \int_0^1 \left(2 + 4(u+1)^3\right).119\,du$$

$$= 119\left[2u + (u+1)^4\right]_0^1$$

$$= 119\left[\left(2+2^4\right) - \left(0+1^4\right)\right]$$

$$= 119\,[18 - 1]$$

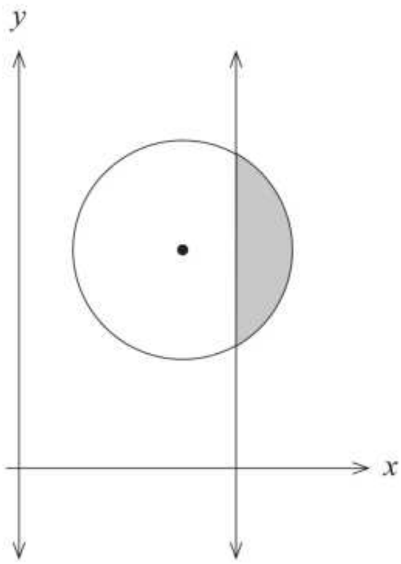
$$= 119 \times 17$$

$$= 2023$$

Specific behaviours

- ✓ writes dx correctly in terms of du
- ✓ changes the limits correctly
- ✓ simplifies the integrand correctly in terms of the variable u
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly

2023 Section 1 Question 7 **Integration and applications of integration**	The shaded region is bounded by the curve $(x-3)^2 + (y-4)^2 = 4$ and the line $x = 4$.



(a) Show that the area of this region is given by the definite integral $\int_4^a 2\sqrt{4-(x-3)^2}\,dx$.
State the value for a. (3 marks)

Solution
Circle radius is 2 units so $a = 3+2 = 5$ From equation of circle $(y-4)^2 = 4-(x-3)^2 \quad \therefore \quad y-4 = \pm\sqrt{4-(x-3)^2}$ $y = 4 \pm \sqrt{4-(x-3)^2}$ Area $= \int_4^5 (y_2 - y_1)\,dx = \int_4^5 \left(4+\sqrt{4-(x-3)^2}\right)-\left(4-\sqrt{4-(x-3)^2}\right)dx$ $= \int_4^5 2\sqrt{4-(x-3)^2}\,dx$
Specific behaviours
✓ states the value for a correctly ✓ obtains $y = 4 \pm \sqrt{4-(x-3)^2}$ correctly from the equation of the circle ✓ forms a difference of y values to obtain the integrand $2\sqrt{4-(x-3)^2}$

(b) By using the substitution $x - 3 = 2\sin\theta$, determine the exact value for the area of the shaded region. (6 marks)

Solution

x	4	5
θ	$\frac{\pi}{6}$	$\frac{\pi}{2}$

$x = 3 + 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta \quad \therefore \quad dx = 2\cos\theta \, d\theta$

$$\int_4^5 2\sqrt{4-(x-3)^2}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sqrt{4-4\sin^2\theta}\,.2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\sqrt{4\cos^2\theta}\,.2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4\cos 2\theta + 4)\,d\theta = \left[2\sin 2\theta + 4\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \left[2(0) + 2\pi\right] - \left[2\left(\frac{\sqrt{3}}{2}\right) + \frac{2\pi}{3}\right]$$

$$= \frac{4\pi}{3} - \sqrt{3} \quad \text{square units}$$

Specific behaviours

✓ changes the limits correctly
✓ obtains dx correctly in terms of $d\theta$
✓ simplifies the integrand in terms of θ correctly (Pythagorean identity)
✓ uses the cosine double angle identity correctly
✓ anti-differentiates the trigonometric function correctly
✓ evaluates the definite integral correctly

2022 Section 1 Question 3 **Integration and applications of integration**	By using one or more of the following identities: $\cos^2 x + \sin^2 x = 1$ $\cos 2x = \cos^2 x - \sin^2 x$ $\sin 2x = 2\sin x \cos x$ evaluate exactly $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx$. (5 marks)

Solution

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2 \, dx &= \int_0^{\frac{\pi}{2}} \left(\sin^2 x + 2\sin x \cos x + \cos^2 x\right) dx \\
&= \int_0^{\frac{\pi}{2}} \left(\left(\sin^2 x + \cos^2 x\right) + 2\sin x \cos x\right) dx \\
&= \int_0^{\frac{\pi}{2}} (1 + \sin 2x) \, dx \\
&= \left[x - \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}} \\
&= \left(\frac{\pi}{2} - \frac{\cos \pi}{2}\right) - \left(0 - \frac{\cos 0}{2}\right) \\
&= \frac{\pi}{2} + \frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= \frac{\pi}{2} + 1
\end{aligned}$$

Specific behaviours

- ✓ expands the integrand correctly
- ✓ uses the Pythagorean identity $\sin^2 x + \cos^2 = 1$
- ✓ uses double angle identity for $\sin 2x$
- ✓ anti-differentiates the trigonometric function correctly
- ✓ evaluates correctly using exact trigonometric values

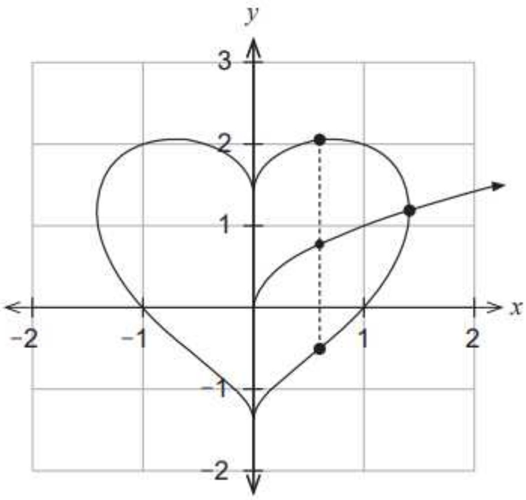
2021 Section 1 Question 8 Integration and applications of integration	

2021 Section 1 Question 8

Integration and applications of integration

The heart-shaped figure shown is given by the equation $x^2 + \left(y - \sqrt{|x|}\right)^2 = 2$.

For $x \geq 0$, this equation becomes $x^2 + \left(y - \sqrt{x}\right)^2 = 2$. The curve $y = \sqrt{x}$ is also drawn.
This heart-shaped curve has the special property that for each x coordinate in its domain its two y coordinates are an equal vertical distance from the curve $y = \sqrt{x}$.



(a) Explain why the domain for the curve given by $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ is $0 \leq x \leq \sqrt{2}$.
(2 marks)

Solution
From $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ $\quad \left(y - \sqrt{x}\right)^2 = 2 - x^2$ $\therefore \; y - \sqrt{x} = \pm\sqrt{2 - x^2}$ *i.e.* $y = \sqrt{x} \pm \sqrt{2 - x^2}$ Hence for $\sqrt{2-x^2}$ and $\sqrt{x}$ to exist we require $2 - x^2 \geq 0$ and $x \geq 0$ i.e. $x^2 \leq 2 \quad \therefore \quad 0 \leq x \leq \sqrt{2}$
Specific behaviours
✓ states that $\sqrt{x}$ must exist ✓ states that $\sqrt{2-x^2}$ must exist or states that $2 - x^2 \geq 0$

Alternative Solution
Intersection of $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ and $y = \sqrt{x}$ is given by : $x^2 + (0)^2 = 2 \quad$ *i.e.* $x^2 = 2 \quad \therefore \; x = \sqrt{2}$ Hence from graph $0 \leq x \leq \sqrt{2}$
Specific behaviours
✓ uses the idea of the intersection of $x^2 + \left(y - \sqrt{x}\right)^2 = 2$ and $y = \sqrt{x}$ ✓ obtains solution $x = \sqrt{2}$

(b) Show that the total area enclosed by the heart-shaped figure is given by:

$$Area = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx.$$

(2 marks)

Solution
From $x^2 + \left(y-\sqrt{x}\right)^2 = 2$ $\qquad \left(y-\sqrt{x}\right)^2 = 2-x^2$ *i.e.* $y_2 - y_s = \sqrt{2-x^2}$ $y_2 - y_1 = 2(y_2 - y_s) = 2\sqrt{2-x^2}$ Area $= \int_{-\sqrt{2}}^{\sqrt{2}} (y_2 - y_1)\,dx = 2\int_0^{\sqrt{2}} (y_2 - y_1)\,dx$ (symmetry about $x=0$) $= 2\int_0^{\sqrt{2}} \left(2(y_2 - y_s)\right)dx$ $= 2\int_0^{\sqrt{2}} 2\sqrt{2-x^2}\,dx = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$
Specific behaviours
✓ indicates symmetry about $x=0$ to obtain one factor of 2   ✓ obtains the integrand $2\sqrt{2-x^2}$ from the two curves

(c) By using the substitution $x = \sqrt{2}\sin\theta$, evaluate the total area enclosed by the heart-shaped figure, and hence see why it can be said that 'π is at the heart of mathematics'. (5 marks)

Solution

Using $x = \sqrt{2}\sin\theta$

x	0	$\sqrt{2}$
u	0	$\frac{\pi}{2}$

$$\frac{dx}{d\theta} = \sqrt{2}\cos\theta \quad \therefore \quad dx = \sqrt{2}\cos\theta\, d\theta$$

$$Area = 4\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \sqrt{2-2\sin^2\theta}\,.\sqrt{2}\cos\theta.\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{2\cos^2\theta}\,.\,\sqrt{2}\cos\theta.\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} 2\cos^2\theta\,d\theta = 4\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta = 4\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 4\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$$

$\therefore \quad Area = 2\pi$ square units

Specific behaviours

- ✓ changes the limits correctly
- ✓ obtains dx in terms of $d\theta$ correctly
- ✓ uses the Pythagorean identity and the cosine double angle identity to simplify the integrand in terms of θ
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly to obtain the correct area

2020 Section 1 Question 7

Integration and applications of integration

Evaluate $\int_{-1}^{7} \frac{3x}{\sqrt{x+2}}\, dx$ exactly using the substitution $u = \sqrt{x+2}$.

(5 marks)

Solution

When $x = -1,\ u = 1$ $\quad \frac{du}{dx} = \frac{1}{2\sqrt{x+2}} \quad \therefore\ dx = 2\sqrt{x+2}\ du$

and $x = 7,\ u = 3$

$$\begin{aligned}
\therefore \int_{-1}^{7} \frac{3x}{\sqrt{x+2}}\,dx &= \int_{1}^{3} \frac{3x}{\sqrt{x+2}}\, 2\sqrt{x+2}\ du \\
&= \int_{1}^{3} 6x\ du \\
&= \int_{1}^{3} 6\left(u^2-2\right) du \\
&= \int_{1}^{3} 6u^2 - 12\ du = \left[2u^3 - 12u\right]_1^3 \\
&= (54-36)-(2-12) \\
&= 28
\end{aligned}$$

Specific behaviours

✓ changes the limits correctly
✓ obtains dx in terms of du correctly
✓ simplifies the integrand correctly
✓ anti-differentiates the integrand correctly
✓ evaluates the definite integral correctly

2019 Section 1 Question 1 Integration and applications of integration	Using the identity $2\sin A\cos B = \sin(A+B)+\sin(A-B)$, evaluate exactly the definite integral $\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx$. (4 marks)

Solution

Using the given identity $2\sin A\cos B = \sin(A+B)+\sin(A-B)$

$$6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right) = 6\times\frac{1}{2}\left(\sin\left(\frac{5x}{2}+\frac{x}{2}\right)+\sin\left(\frac{5x}{2}-\frac{x}{2}\right)\right)$$

$$= 3(\sin 3x+\sin 2x)$$

$$\int_0^{\frac{\pi}{2}} 6\sin\left(\frac{5x}{2}\right)\cos\left(\frac{x}{2}\right)dx = \int_0^{\frac{\pi}{2}}(3\sin 3x+3\sin 2x)dx$$

$$= \left[-\cos 3x-\frac{3\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \left[-\cos\frac{3\pi}{2}-\frac{3\cos\pi}{2}\right]-\left[-\cos 0-\frac{3\cos 0}{2}\right]$$

$$= \left[-(0)-\frac{3(-1)}{2}\right]-\left[-1-\frac{3(1)}{2}\right]$$

$$= \left(\frac{3}{2}\right)-\left(-\frac{5}{2}\right) = 4$$

Specific behaviours

✓ determines the factor 3 in relating the expressions
✓ obtains the integrand terms $\sin 3x+\sin 2x$
✓ anti-differentiates term by term correctly
✓ evaluates the definite integral correctly

2019 Section 1 Question 3 Integration and applications of integration	(a) Given that $\frac{2x^2+5x+6}{x^2(x+3)} = \frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+3}$, determine the values of a, b and c. (2 marks)

Solution

$$\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+3} = \frac{ax(x+3)+b(x+3)+cx^2}{x^2(x+3)}$$

$$= \frac{(a+c)x^2+(3a+b)x+3b}{x^2(x+3)}$$

Hence equating co-efficients we obtain $a+c=2$

$3a+b=5$

$3b=6$

Solving obtains $a=1,\ \ b=2,\ c=1$

Specific behaviours

✓ obtains the numerator correctly in terms of a, b, c in simplifying the fractions
✓ determines the values for a, b, c correctly

	(b) Hence determine $\int \frac{2x^2+5x+6}{x^2(x+3)}\,dx$. (3 marks)

Solution

$$\int \frac{2x^2+5x+6}{x^2(x+3)}dx = \int \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x+3}dx$$

$$= \ln|x| - \frac{2}{x} + \ln|x+3| + c$$

Specific behaviours

✓ expresses the integrand in terms of the partial fractions correctly
✓ anti-differentiates correctly (using absolute value of the natural logarithm)
✓ uses an integration constant

2019 Section 1 Question 6

Integration and applications of integration

Using the substitution $x = 2\sin\theta$, evaluate exactly $\int_0^{\sqrt{3}} \sqrt{1-\frac{x^2}{4}}\,dx$. (6 marks)

Solution

When $x=0,\ \theta=0$ $\quad \frac{dx}{d\theta}=2\cos\theta \quad \therefore\ dx=2\cos\theta\, d\theta$

and $x=\sqrt{3},\ \theta=\frac{\pi}{3}$

$$\therefore \int_0^{\sqrt{3}}\sqrt{1-\frac{x^2}{4}}\,dx = \int_0^{\frac{\pi}{3}}\sqrt{1-\frac{4\sin^2\theta}{4}}\,(2\cos\theta\, d\theta)$$

$$= \int_0^{\frac{\pi}{3}}\sqrt{1-\sin^2\theta}\,(2\cos\theta\, d\theta)$$

$$= \int_0^{\frac{\pi}{3}}\sqrt{\cos^2\theta}\,(2\cos\theta\, d\theta)$$

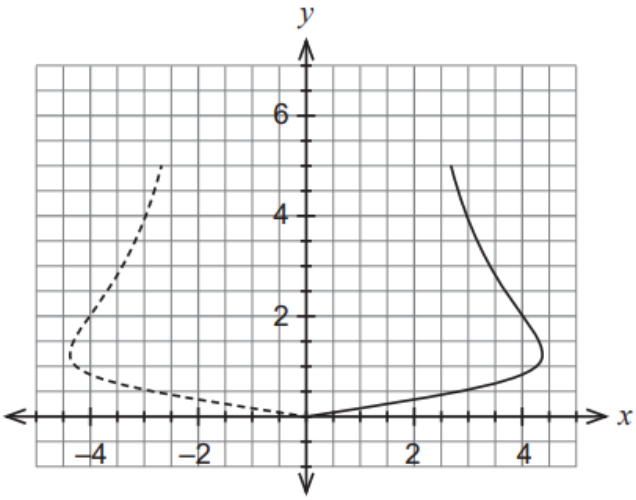
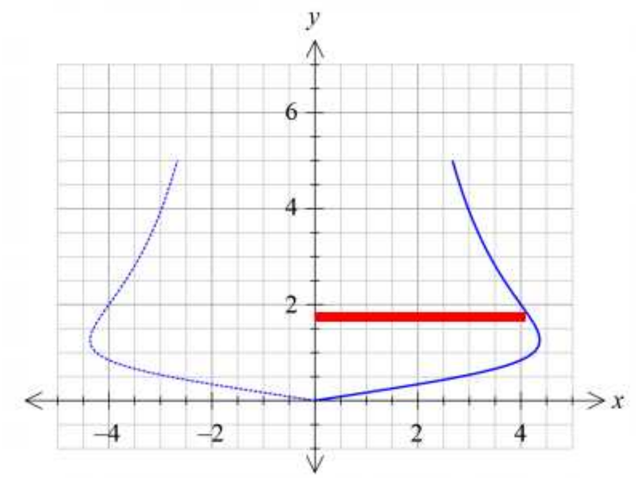
$$= \int_0^{\frac{\pi}{3}}2\cos^2\theta\, d\theta = \int_0^{\frac{\pi}{3}}(1+\cos 2\theta)\, d\theta$$

$$= \left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{3}+\frac{\sqrt{3}}{4}$$

Specific behaviours

✓ changes the limits correctly
✓ obtains dx in terms of $d\theta$ correctly
✓ simplifies the integrand correctly using the identity $\sin^2\theta+\cos^2\theta=1$
✓ uses the $\cos 2\theta$ identity to correctly re-write the integrand
✓ anti-differentiates the integrand correctly
✓ evaluates the definite integral correctly

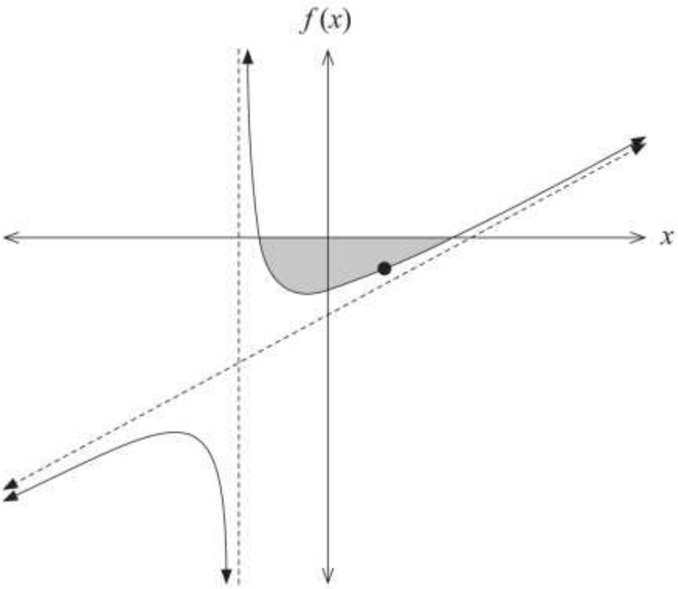
2019 Section 1 Question 8 **Integration and applications of integration**	The top part of a wine glass is modelled by rotating the graph of $x^2 = y^2(36 - x^2 y)$ from $y = 0$ to $y = 5$ about the y axis as shown below. Dimensions are measured in centimetres.  (a) Show that the volume, $V\text{ cm}^3$, when the glass is full is given by $V = \pi\int_0^5 \frac{36y^2}{1+y^3}\,dy.$ (1 mark) 

Solution
From $x^2 = y^2\left(36 - x^2 y\right)$ $\therefore \quad x^2 = 36y^2 - x^2 y^3$ $\therefore \quad x^2 + x^2 y^3 = 36y^2$ i.e. $x^2\left(1+y^3\right) = 36y^2$ gives $x^2 = \dfrac{36y^2}{1+y^3} = \left(\dfrac{6y}{\sqrt{1+y^3}}\right)^2$ Hence $dV = \pi r^2 dy = \pi\left(\dfrac{6y}{\sqrt{1+y^3}}\right)^2 dy$. To obtain the TOTAL of all the possible thin cylindrical disks we add (integrate) over the interval of the possible y values i.e. integrate from $y = 0$ to $y = 5$. Hence volume $V = \int_0^5 dV = \int_0^5 \pi\left(\dfrac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \dfrac{36y^2}{1+y^3}\, dy$
Specific behaviours
✓ obtains the x coordinate correctly from the given curve equation (or x^2)

(b) Determine the exact volume $V\,\text{cm}^3$. (4 marks)

Solution
$V = \int_0^5 \pi\left(\dfrac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \dfrac{36y^2}{1+y^3} dy$ Using $u = 1+y^3$ $\quad \dfrac{du}{dy} = 3y^2 \quad \therefore \; dy = \dfrac{du}{3y^2}$ When $y=0,\ u=1$ $\quad \therefore \; V = \pi\int_1^{126} 36y^2 . \dfrac{1}{u} . \dfrac{du}{3y^2}$ $y=5,\ u=126$ $= \pi\int_1^{126} \dfrac{12}{u} du = \pi\left[12\ln\lvert u\rvert\right]_1^{126}$ $= 12\pi\left(\ln 126\right)$
Specific behaviours
✓ changes the limits correctly ✓ obtains the integrand correctly in terms of u ✓ anti-differentiates correctly (absolute value of natural logarithm not required) ✓ obtains the exact value for the volume in terms of π and a natural logarithm correctly

Alternative Solution
$V = \int_0^5 \pi\left(\frac{6y}{\sqrt{1+y^3}}\right)^2 dy = \pi\int_0^5 \frac{36y^2}{1+y^3}dy$ $\therefore V = \pi\int_0^5 \frac{12(3y^2)}{1+y^3}dy$ $= 12\pi\int_0^5 \frac{d}{dy}(1+y^3).\frac{1}{1+y^3}dy$ $= 12\pi\left[\ln(1+y^3)\right]_0^5$ $= 12\pi(\ln 126)$
Specific behaviours
✓ recognises $36y^2$ as a multiple of the derivative of $1+y^3$ ✓ obtains the numerator as 12 times the derivative of $1+y^3$ ✓ anti-differentiates correctly (absolute value of natural logarithm not required)   ✓ obtains the exact value for the volume in terms of π and a natural logarithm correctly

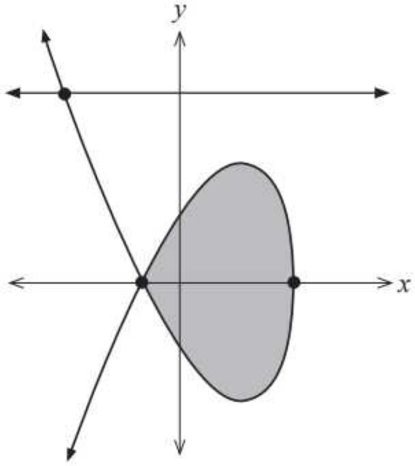
2023 Section 2 Question 18 Integration and applications of integration	Function $f(x)$ is a rational function of the form $\frac{x^2+bx+c}{x+d}$ with the following properties: • $f(2)=-2$ • $f(x)$ has a vertical asymptote at $x=-3$ and another asymptote with equation $y=x-5$.  (a) Show that $b=-2,\ c=-10$ and $d=3$. (3 marks)

Solution
Vertical asymptote for f is $x=-3 \quad \therefore \quad d=3$ Inclined asymptote is $y=x-5$ i.e. For $\lvert x \rvert$ large $f(x) = (x-5) + \frac{e}{x+3}$ for some constant e $\therefore \ f(x) = \frac{(x+3)(x-5)}{x+3} + \frac{e}{x+3} = \frac{x^2-2x-15+e}{x+3}$ Using $f(2)=-2$ then $-2 = \frac{2^2-2(2)-15+e}{2+3}$ Solving gives $e=5$ i.e. $f(x)=\frac{x^2-2x-10}{x+3}$ $\therefore \ b=-2, \ c=-10$.
Specific behaviours
✓ refers to the vertical asymptote to deduce $d=3$ ✓ expresses function $f(x)$ in the form $(x-5) + \frac{e}{x+3}$ ✓ uses $f(2)=-2$ to deduce e and then $b, \ c$

	(b) Calculate the exact area of the shaded region. (3 marks)

Solution
x intercepts given by $x^2 - 2x - 10 = 0$ i.e. $x = 1 - \sqrt{11},\ \ x = 1 + \sqrt{11}$ Area shaded $= -\int_{1-\sqrt{11}}^{1+\sqrt{11}} f(x)\,dx = -\int_{1-\sqrt{11}}^{1+\sqrt{11}} \frac{x^2 - 2x - 10}{x+3}\,dx$ $= -\left(5\ln\left(\frac{4+\sqrt{11}}{4-\sqrt{11}}\right) - 8\sqrt{11}\right)$ ... from CAS $= 8\sqrt{11} - 5\ln\left(\frac{4+\sqrt{11}}{4-\sqrt{11}}\right)$ square units or $8\sqrt{11} - 5\ln\left(4+\sqrt{11}\right) + 5\ln\left(4-\sqrt{11}\right)$
Specific behaviours
✓ determines the exact x intercepts for function $f(x)$ ✓ forms the correct expression for the area in terms of a definite integral ✓ determines the expression for the exact area correctly (or its equivalent)

2022 Section 2 Question 13 **Integration and applications of integration**	The equation $x^3 - x^2 - 5x = 3 - y^2$ implicitly defines the curve shown below. The line $y = \sqrt{24}$ intersects this curve as shown below.



It can be shown that the equation $x^3 - x^2 - 5x + 21 = 0$ will determine the intersection between the line $y = \sqrt{24}$ and the implicitly defined curve.

(a) Explain, with reference to the graph above, why we know that there is one real and two complex solutions (a conjugate pair) to this cubic equation. (2 marks)

Solution
There is only ONE point of intersection between the line $y = \sqrt{24}$ and the curve $(x = -3)$. Since the equation is a cubic with real coefficients, then the other two solutions must be complex (a conjugate pair).
Specific behaviours
✓ states there is just ONE point of intersection between $y = \sqrt{24}$ and the curve ✓ states this equation is cubic with real coefficients

(b) Determine the two exact complex solutions to the equation $x^3 - x^2 - 5x + 21 = 0$. (2 marks)

Solution
$x = -3$ is the real solution. Hence $(x+3)(x^2 - 4x + 7) = 0$ $\therefore\ x = -3$ or $(x-2)^2 + 3 = 0$ $\therefore\ x = 2 \pm \sqrt{3}\,i$ are the two complex solutions
Specific behaviours
✓ states that $x = -3$ is the real solution OR states that $(x+3)$ is a factor ✓ obtains the two complex solutions correctly

(c) Calculate the area of the shaded region, correct to 0.001 square units. (4 marks)

Solution
Intersection with the x axis: Solve $x^3 - x^2 - 5x = 3 - (0)^2$
Hence x intercepts are $x = -1,\ x = 3$
Equation: $y^2 = 3 - (x^3 - x^2 - 5x)$ $\therefore\ y = \pm\sqrt{3 - (x^3 - x^2 - 5x)}$
Area $= 2\int_{-1}^{3} y\,dx = 2\int_{-1}^{3}\sqrt{3-(x^3-x^2-5x)}\,dx = 2(8.5333...) = 17.067$
Specific behaviours
✓ forms a definite integral with respect to x using the correct limits
✓ writes a factor of 2 in the integrand due to the symmetry about $y = 0$
✓ writes the correct integrand for the y value in terms of x
✓ calculates the area correctly

2022 Section 2 Question 17

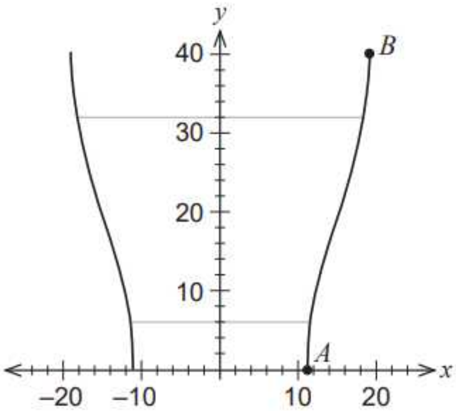
Integration and applications of integration

A vase has water with a depth of 6 cm and needs to be filled to a depth of 32 cm. The cross-section of the vase is modelled by the curve AB where

$x = 15 - 4\cos\left(\frac{\pi y}{40}\right),\ 0 \le y \le 40,$ and this curve is revolved about the y axis.

All dimensions are in centimetres.

Give all answers in this question to the nearest appropriate unit of measurement.



(a) Calculate the volume of water that needs to be added to increase the depth of water from 6 cm to 32 cm. (3 marks)

Solution
Volume water required $= \int_{6}^{32}\pi\left(15 - 4\cos\left(\frac{\pi y}{40}\right)\right)^2 dy = \pi(5763.9368)$
$= 18\,107.9417...$ cm^3
Hence the volume of water required is 18 108 cm^3 (nearest cm^3). Accept 18 litres (nearest litre) provided the calculated value 18 107.94 … is shown.
Specific behaviours
✓ forms a definite integral with the correct limits and using correct notation
✓ uses the correct integrand in terms of y including the factor of π
✓ calculates the correct volume (to nearest unit) using the correct units

Josie, an interior designer, uses a hose to add water to the vase. This hose has a water-saving device that regulates the rate at which water flows into the vase. This rate is given by:

$$\frac{dV}{dt} = 300e^{-\frac{V}{12\,000}}$$

where $V(t)$ = the volume of water (cm^3) poured into the vase after t seconds of flow.

(b) If Josie has already poured 6000 cm^3 , use the increments formula to calculate an approximation for the volume of water she will pour in the next 0.5 seconds. (2 marks)

Solution
$\Delta V \approx \left(\frac{dV}{dt}\right) \times \Delta t = \left(300e^{-\frac{6000}{12000}}\right) \times 0.5$ $= (181.959..) \times 0.5 = 90.979...$ cm^3 Hence she will pour approx. 91 cm^3 (nearest cm^3) in the next 0.5 seconds.
Specific behaviours
✓ substitutes $V = 6000$ and $\Delta t = 0.5$ correctly into the increments formula   ✓ calculates the correct volume

To prevent an overflow of water, the device can be calibrated to switch off the flow after a set length of time using an in-built timer.

(c) Calculate the rate of flow into the vase at the instant when the depth becomes 32 cm. (2 marks)

Solution
Require $\frac{dV}{dt}$ when $V = 18\,107.9417$ cm^3 i.e. $\frac{dV}{dt} = 300e^{-\frac{18\,107.9417}{12\,000}} = 66.3396$ cm^3/sec   The rate of flow when the depth is 32 cm is approx. 66 cm^3/sec.
Specific behaviours
✓ substitutes the total volume from part (a) into the differential equation   ✓ calculates the rate correctly stating the correct units

Alternative Solution
Solve for when $V(t) = 18107.9417...$ This gives $t = 140.8873...$ sec $V'(t) = 12000 \times \frac{1}{\left(\frac{t}{40}+1\right)} \times \frac{1}{40} = \frac{300}{\left(\frac{t}{40}+1\right)}$ Evaluating $V'(140.8873...) = 66.3396...$ cm^3/sec   The rate of flow when the depth is 32 cm is approx. 66 cm^3/sec.
Specific behaviours
✓ solves correctly the value of t that gives the volume obtained in part (a)   ✓ calculates the rate correctly stating the correct units

	(d) Using separation of variables, obtain the defining rule for $V(t)$. (3 marks)

Solution

$$\int e^{\frac{V}{12\,000}}\, dV = \int 300\, dt$$

$$12\,000\, e^{\frac{V}{12\,000}} = 300t + c \qquad \text{Using } V(0)=0$$

$$12000e^{0} = 300(0) + c \qquad \therefore\ c = 12\,000$$

$$12\,000\, e^{\frac{V}{12\,000}} = 300t + 12\,000$$

$$\therefore\ e^{\frac{V}{12\,000}} = \frac{t}{40} + 1 \qquad \therefore\ \frac{V}{12000} = \ln\left(\frac{t}{40} + 1\right)$$

$$i.e.\ V(t) = 12\,000\ \ln\left(\frac{t}{40} + 1\right)$$

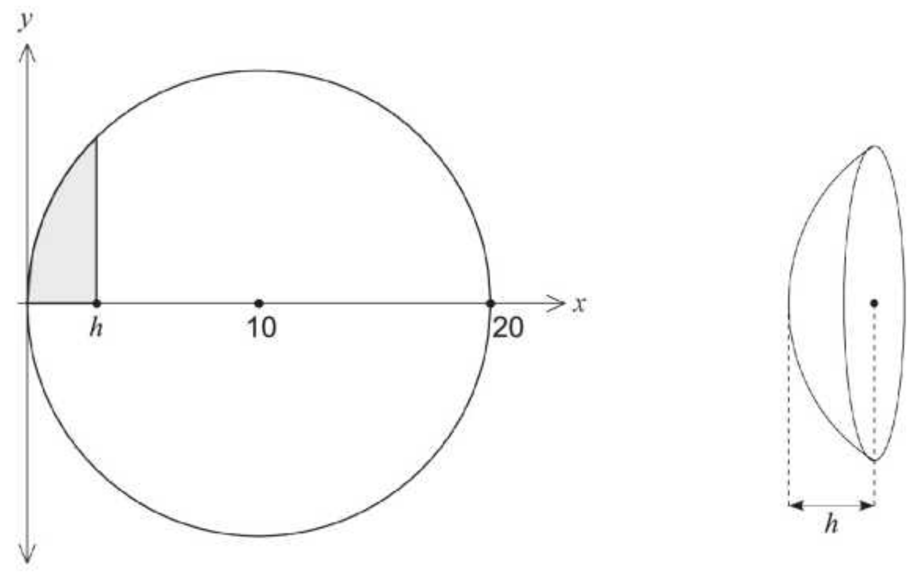
Specific behaviours

✓ separates the variables as an integration statement correctly
✓ anti-differentiates correctly using a constant
✓ determines V as the subject correctly from the anti-derivative statement

2021 Section 2 Question 13

Integration and applications of integration

A solid spherical cap with depth h is part of a solid sphere with radius 10 cm. This cap can be generated by revolving the shaded region about the x axis.



(a) Show that the equation for the circle shown above is $x^2 + y^2 = 20x$. (1 mark)

Solution

Centre is $(10,0)$ with radius 10 $\quad \therefore\ (x-10)^2 + (y-0)^2 = 10^2$

i.e. $x^2 - 20x + 10^2 + y^2 = 10^2$

i.e. $x^2 + y^2 = 20x$

Specific behaviours

✓ forms the equation of the circle in centre-radius form

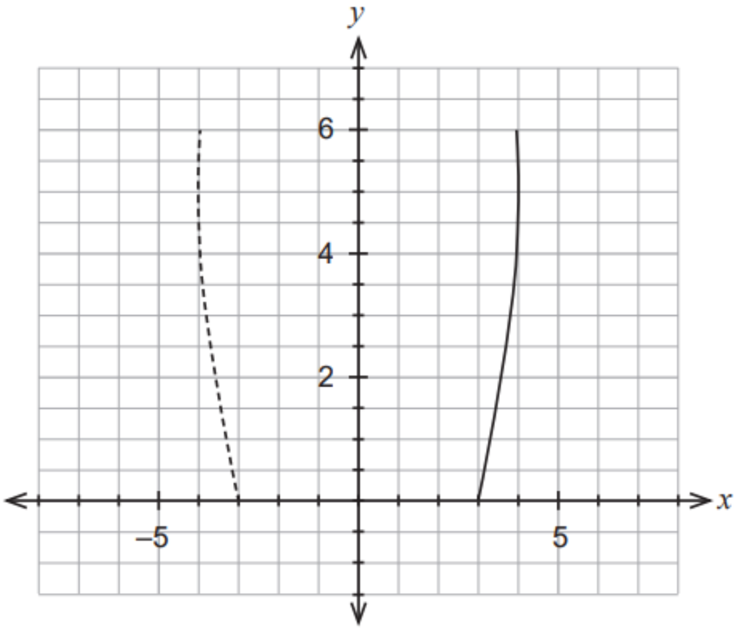
(b) Develop an expression for the volume of the spherical cap in terms of h . (4 marks)

Solution
The circle is given by $x^2 + y^2 = 20x$. Hence $y^2 = 20x - x^2$
Volume $V = \int_0^h \pi y^2 dx = \int_0^h \pi\left(20x - x^2\right)dx = \pi\left[10x^2 - \frac{x^3}{3}\right]_0^h$
$= \pi\left[10h^2 - \frac{h^3}{3}\right] \quad \ldots (1)$
$= \pi h^2\left(10 - \frac{h}{3}\right) \quad \ldots (2)$
Specific behaviours
✓ writes a definite integral with the correct limits and uses correct notation ✓ writes the integrand correctly ✓ anti-differentiates correctly ✓ simplifies the volume expression in terms of h, (equivalent to 1 or 2)

2020 Section 2 Question 9

Integration and applications of integration

The shape of a small wine glass is modelled by revolving the curve $\sin\left(\frac{y}{\pi}\right) = x - 3$ about the y axis, where $0 \le y \le 6$. All dimensions are in centimetres.



Calculate, correct to the nearest 0.01 cm, the depth of wine in the glass if it is to contain 80% of its maximum volume. (4 marks)

Solution

Volume when full $= \int_0^6 \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 259.53228\ldots \text{ cm}^3$

Let h = the depth for 80% volume

Require $\int_0^h \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 0.8 \times 259.53228\ldots$

$\int_0^h \pi\left(\sin\left(\frac{y}{\pi}\right)+3\right)^2 dy = 207.625824\ldots$

Solving using CAS $h = 4.9572\ldots$ cm

i.e. Depth needs to be 4.96 cm for the glass to be 80% full.

Specific behaviours

✓ forms the correct expression to determine the volume when full
✓ determines the volume when full (or 80% full)
✓ forms the correct equation to solve for the 80% volume condition
✓ solves for the depth correct to 0.01 cm

2020 Section 2 Question 12

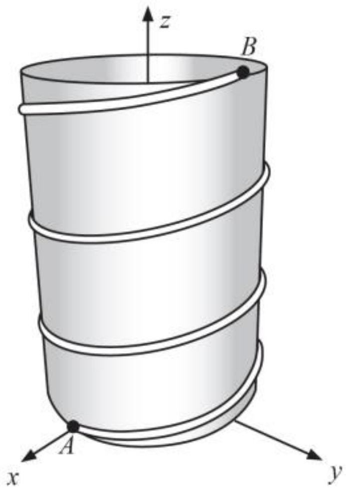
Integration and applications of integration

A cylindrical shaped tower has a path that spirals upwards from the ground to an observation deck at point B as shown in the diagram below. The path begins at point A on the ground and finishes at point B at the top.

Let t = time in seconds that a tourist has been walking along the spiral path. The tourist takes 65π seconds to reach point B.

The tourist's position on this path at any time t is given by:

$$\underset{\sim}{r}(t) = \begin{pmatrix} 10\cos(0.1t) \\ 10\sin(0.1t) \\ 0.2t \end{pmatrix} \text{ metres.}$$



(a) Determine the height of the observation deck above the ground, correct to the nearest 0.01 metres. (1 mark)

Solution
Point B occurs when $t = 65\pi$. Height = $z(B) = 0.2(65\pi) = 13\pi = 40.840...$ metres Hence the height is 40.84 metres (to 0.01 metres).
Specific behaviours
✓ calculates the height correctly (no penalty for incorrect rounding)

(b) Determine the tourist's velocity $\underset{\sim}{v}(t)$. (2 marks)

Solution
$\underset{\sim}{v}(t) = \frac{d}{dt}(\underset{\sim}{r}(t)) = \begin{pmatrix} -\sin(0.1t) \\ \cos(0.1t) \\ 0.2 \end{pmatrix}$
Specific behaviours
✓ considers the derivative of the displacement vector ✓ determines each component correctly

(c) Show that the tourist walks at a constant speed and determine this speed, correct to 0.01 metres per second. (3 marks)

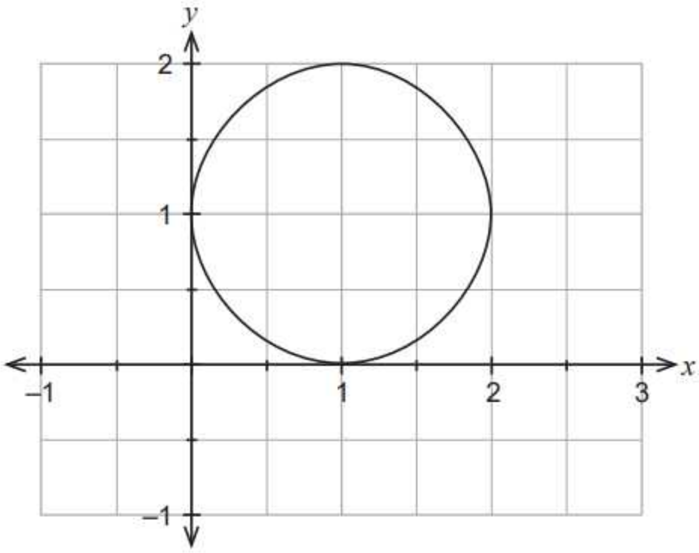
Solution
$Speed(t) = \lvert \underset{\sim}{v}(t) \rvert = \sqrt{(-\sin(0.1t))^2 + (\cos(0.1t))^2 + (0.2)^2}$ $= \sqrt{\sin^2(0.1t) + \cos^2(0.1t) + 0.04}$ $= \sqrt{1 + 0.04}$ $= \sqrt{1.04} = 1.0198\ldots$ Hence the speed is constant and is 1.02 metres per second.
Specific behaviours
✓ forms the correct expression for the speed ✓ uses the trigonometric identity $\sin^2(x) + \cos^2(x) = 1$ ✓ evaluates the speed correctly to 0.01 metres per second

2020 Section 2 Question 16 **Integration and applications of integration**	

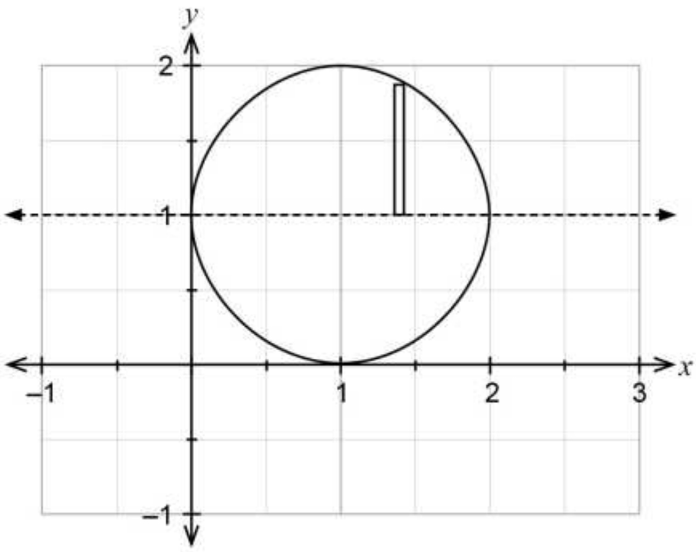
2020
Section 2
Question 16

Integration and applications of integration

The curve shown below is given by the equation $(y-1)^2 = \sin\left(\frac{\pi x}{2}\right)$.



(a) Calculate the area, correct to 0.0001 square units, of the region that forms the interior of this curve. (3 marks)



Solution
The curve is symmetric about the line $y = 1$. Hence the height of the curve above $y = 1$ is given by $y - 1 = \sqrt{\sin\left(\frac{\pi x}{2}\right)}$ Hence using symmetry $Area = 2 \times \int_0^2 \sqrt{\sin\left(\frac{\pi x}{2}\right)}\, dx = 3.0510$ (4 d.p.)
Specific behaviours
✓ forms a definite integral using the correct limits ✓ forms the integrand correctly ✓ evaluates the integral correctly to 0.0001

	(b) By using the answer to part (a), determine whether this curve is a circle. Explain your answer. (2 marks) **Solution** If the curve was a circle, with radius 1 unit then the $Area = \pi(1)^2 = 3.1416$ Hence since the Area from part (a) $\neq \pi$ then the curve is NOT a circle. **Specific behaviours** ✓ compares answer to part (a) with the area of the circle radius 1 unit ✓ concludes that the curve cannot be a circle

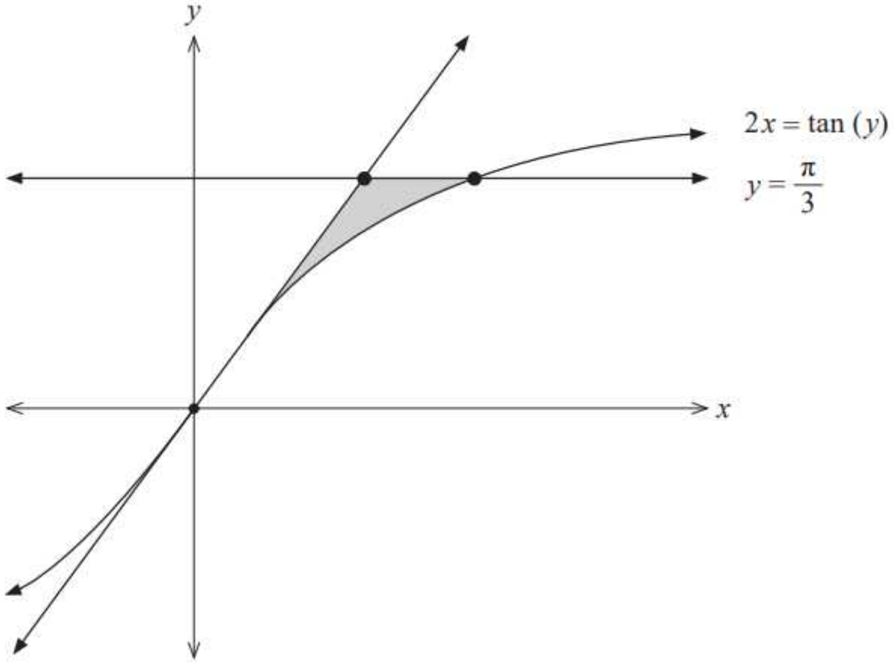
Unit 4.2 – Rates of change and differential equations

Section 1

2022 Section 1 Question 4 Rates of change and differential equations	(a) Function $f(x) = \dfrac{5(x+1)}{(x-1)(x^2+3x+1)}$ can be expressed in the form $\dfrac{a}{x-1} + \dfrac{bx+c}{x^2+3x+1}$. Determine the value of the constants a, b and c. (3 marks)

(b) Hence determine $\int \frac{10x + 10}{(x - 1)(x^2 + 3x + 1)} \, dx$.

(5 marks)

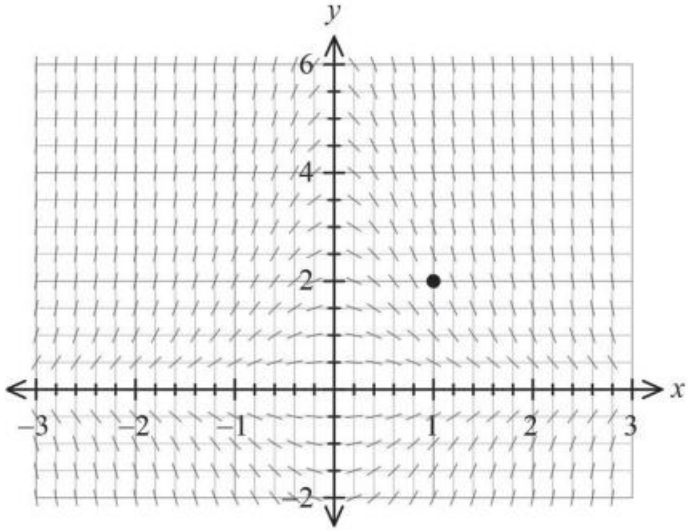
2022 Section 1 Question 7 Rates of change and differential equations	The graph of $2x = \tan(y)$ is shown along with the tangent at $x = 0$. The horizontal line $y = \frac{\pi}{3}$ is also shown.  (a) Using implicit differentiation, determine the equation of the tangent drawn at $x = 0$. (3 marks)

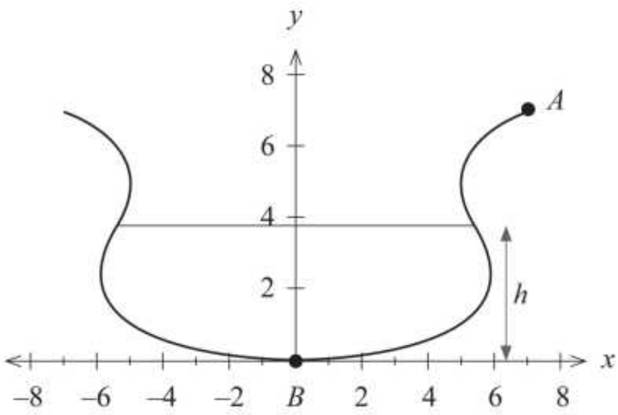
	The shaded region is bounded by the curve $2x = \tan(y)$, the tangent drawn and $y = \frac{\pi}{3}$. (b) Write the expression for the area of the shaded region. (2 marks) (c) Evaluate this area exactly. (3 marks)

2021 Section 1 Question 3 **Rates of change and differential equations**	Using an appropriate substitution, determine the exact value for $\int_2^3 15x\sqrt{x-2}\,dx$. (5 marks)

2021 Section 1 Question 5 **Rates of change and differential equations**	(a) Given that $\dfrac{7x^2 - 12x + 2}{(x-2)(x^2+2)} = \dfrac{a}{x-2} + \dfrac{bx}{x^2+2}$ determine the values of a and b. (2 marks) (b) Hence determine $\int \dfrac{7x^2 - 12x + 2}{(x-2)(x^2+2)}\,dx$. (3 marks)

2020 Section 1 Question 1 Rates of change and differential equations	Evaluate exactly $\int_0^{\pi} \left(4\cos^2 x - \sin x\right) dx$. (3 marks)

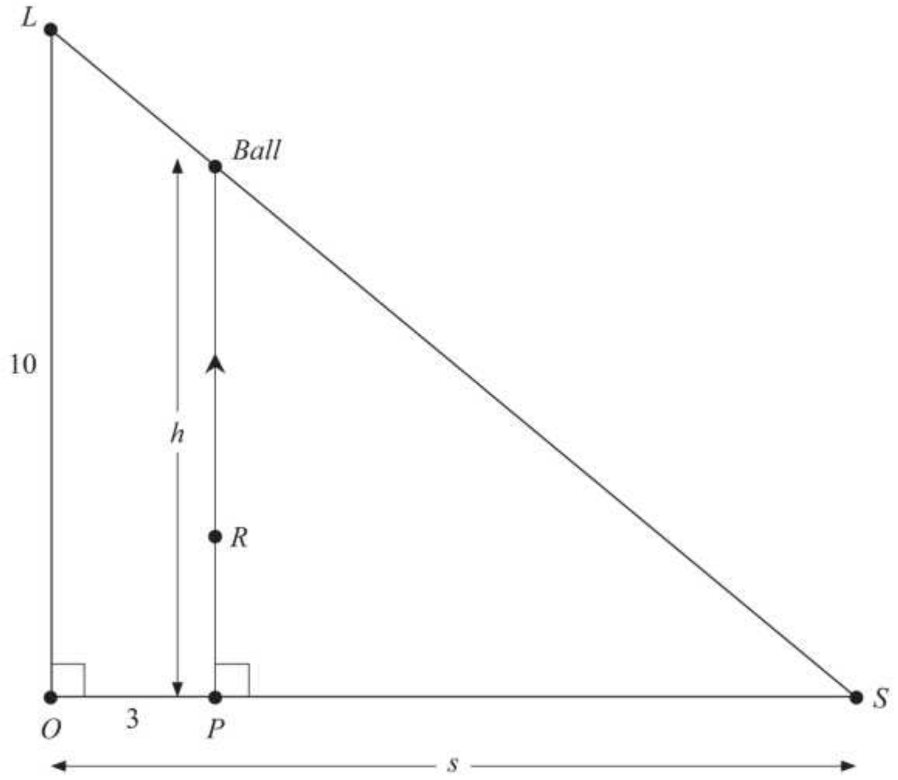
2023 Section 2 Question 11 Rates of change and differential equations	A slope field is given by the equation $\frac{dy}{dx} = k(xy)$ where k is a constant.  (a) The value of the slope field at the point $(1, 2)$ is equal to -4. Determine the value of the constant k. (2 marks) (b) Determine the equation for the solution curve that contains the point (1, 2) and draw this curve on the diagram above. (3 marks)

2023 Section 2 Question 17 **Rates of change and differential equations**	The shape of a decorative vase is modelled by revolving the curve AB about the y axis where $x=\sqrt{y\left(y^2-11y+35\right)}$ with $0 \le y \le 7$. All dimensions are in centimetres.  (a) Determine an integral expression, in terms of h, for the volume of water in the vase if it is filled to a depth of h cm. (2 marks) Water is poured into the initially empty vase at a constant rate of 50 cm^3/s. (b) Determine the time taken to fill the vase to a depth of 6 cm. (2 marks)

With the depth at 6 cm, another 30 cm^3 of water is added to the vase.

(c) Using the increments formula, calculate the approximate change in depth of water in the vase. (3 marks)

2023 Section 2 Question 19 **Rates of change and differential equations**	A ball is projected vertically into the air from point R, so that it will eventually hit the ground at point P, 3 metres from the base of a 10 metre high light at L. At any time t seconds, when the ball is h metres above the ground, it casts a shadow on the ground at point S at a distance s metres from the base of the light.



At any time t it can be shown that $s(10-h)=30$.

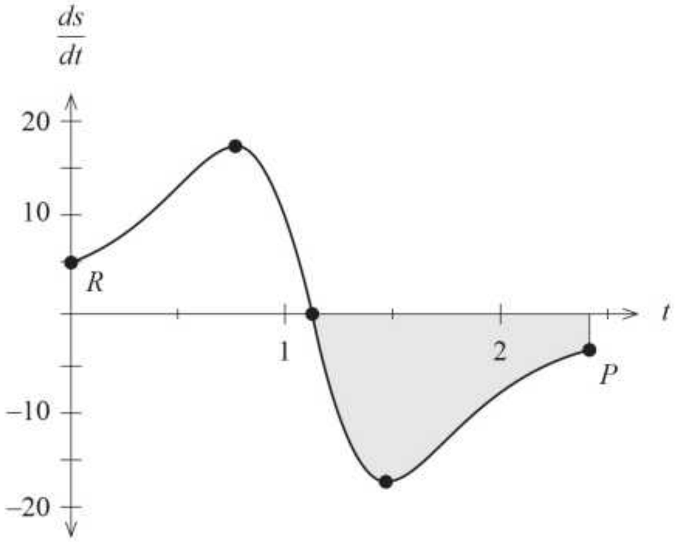
(a) Using implicit differentiation, show that $\frac{ds}{dt}=\frac{30}{(10-h)^2}\times\frac{dh}{dt}$. (3 marks)

At $t = 0.5$ seconds, it is found that the ball is 6.275 metres above the ground and moving upwards at 6.1 metres per second.

(b) By assuming $h''(t) = -9.8$ ms^{-2}, show that $h(t) = 2 + 11t - 4.9t^2$. (3 marks)

(c) Determine the initial speed of the ball's shadow, correct to the nearest 0.01 metres per second.
(3 marks)

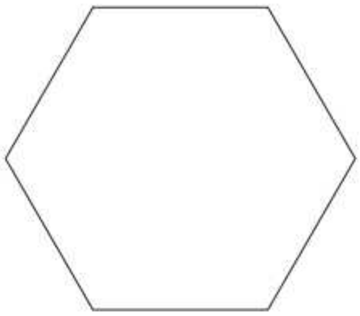
The graph of the function $\frac{ds}{dt}$ against time t is shown below. Point R of this graph corresponds to the ball being thrown into the air, while point P corresponds to the ball hitting the ground.

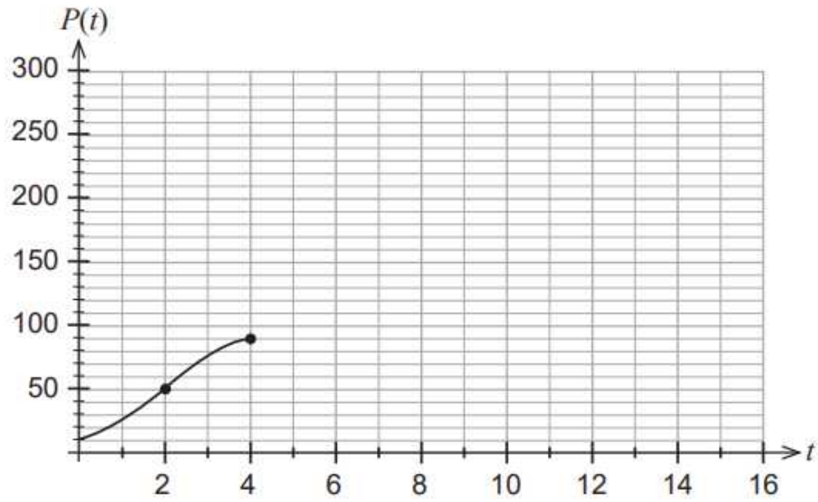


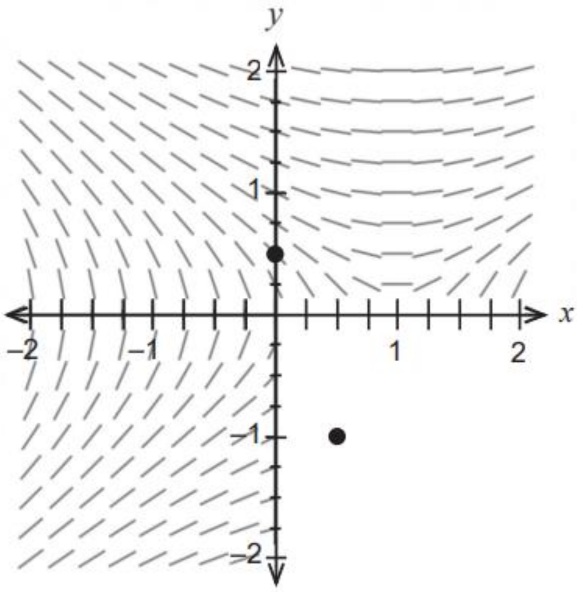
The definite integral $\int_a^b \left(\frac{ds}{dt}\right) dt$ was evaluated so that the area for the shaded region could be determined. This area is 13.4258 square units.

(d) Determine the values for a and b (correct to 0.01 seconds) and describe what this definite integral represents in terms of the motion of the shadow. (4 marks)

	(e) Determine the fastest rate at which the shadow moves (correct to the nearest 0.01 metres per second) and the time when this occurs (correct to the nearest 0.01 seconds). (3 marks)

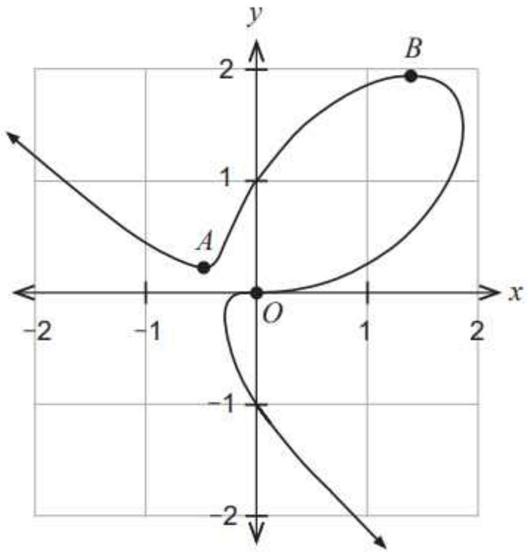
2022 Section 2 Question 8 **Rates of change and differential equations**	A regular hexagon expands so that the length of each side increases at a rate of 0.5 cm per second. Assuming that the polygon maintains its shape, determine the rate at which the area is increasing when the side length is 4 cm. (4 marks) 

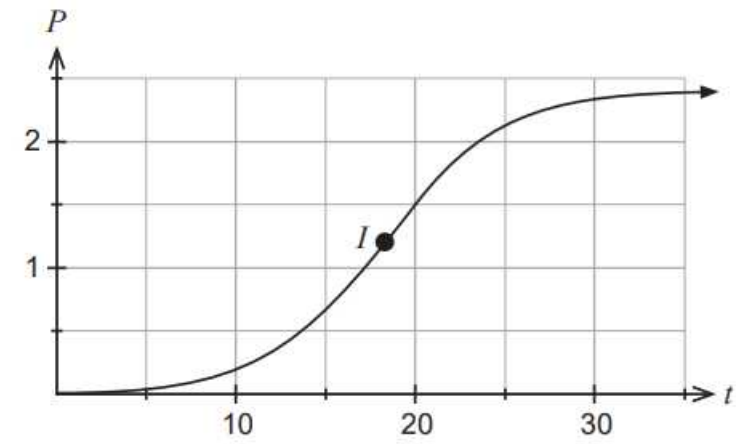
2022 Section 2 Question 16 **Rates of change and differential equations**	An ant colony population P at time t days grows at a rate given by the equation $\frac{dP}{dt} = 0.01P(100-P)$, where $0 \le t \le 4$. The graph of this population is shown below.  (a) For $0 \le t \le 4$, using the growth rate equation explain the variation of the population. (2 marks) At the end of the fourth day, the environment for the ant colony improves dramatically so that its limiting population is increased to 300. (b) Sketch, on the axes above, the expected variation of the population for $t > 4$ days, using the increased limiting population value. (2 marks)

2021 Section 2 Question 10 Rates of change and differential equations	Part of the slope field given by $\frac{dy}{dx} = \frac{x-1}{2y}$ is shown below.  (a) Calculate and draw the slope field at the point (0.5, – 1) . (3 marks)

	(b) Determine the equation of the solution curve that contains the point (0, 0.5) . (3 marks) (c) Draw the solution curve that contains the point (0, 0.5) . (2 marks)

2021 Section 2 Question 12 **Rates of change and differential equations**	The horizontal displacement of a Ferris wheel cabin exhibits simple harmonic motion. The maximum horizontal speed is $\frac{\pi}{2}$ metres per second and its period of motion is exactly 60 seconds. Let $x(t) = A\cos(nt)$ be the horizontal displacement after t seconds. (a) Determine the values of A and n . (3 marks) (b) Determine the horizontal acceleration, correct to the nearest 0.001 m/s^2 , when the horizontal displacement is 10 metres. (3 marks)

2021 Section 2 Question 18 **Rates of change and differential equations**	The equation $x^3 + y^3 = 3xy + y$ implicitly defines the curve shown below.  (a) Using implicit differentiation obtain the expression for $\frac{dy}{dx}$. (3 marks) The slope of the curve at the origin O and points A and B is equal to zero. (b) Show that the equation that determines the x coordinates for points A and B is given by $x^4 - 2x - 1 = 0$ and hence determine the coordinates for point A correct to 0.001. (3 marks)

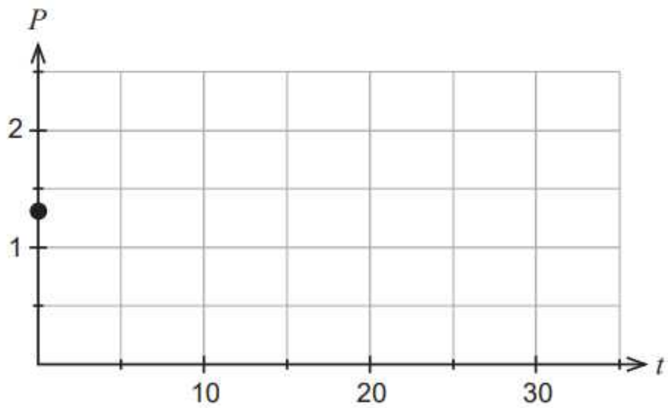
2020 Section 2 Question 19 **Rates of change and differential equations**	The population $P(t)$ of sardines in an ocean, measured in million tonnes after t years, was modelled by the logistic equation: $P(t) = \dfrac{2.4}{1 + 239e^{-0.3t}}$ The graph of this model is shown below. This graph contains a point of inflection at point I.  (a) Calculate the size of the sardine ocean population at $t = 0$. (2 marks) (b) Rewrite the logistic equation in the form $\dfrac{dP}{dt} = rP(k - P)$, stating clearly the values for r and k. (2 marks)

(c) When the sardine population is 500 000 tonnes, use the technique of increments to calculate the approximate change in population in the next month. (3 marks)

(d) Determine the maximum rate of growth of the sardine population. (2 marks)

Suppose that the initial population of sardines was 1.3 million tonnes.

(e) Assuming that the rate of growth is still given by $\frac{dP}{dt} = rP(k-P)$ sketch the graph of the population growth on the axes below. Explain your graph. (2 marks)

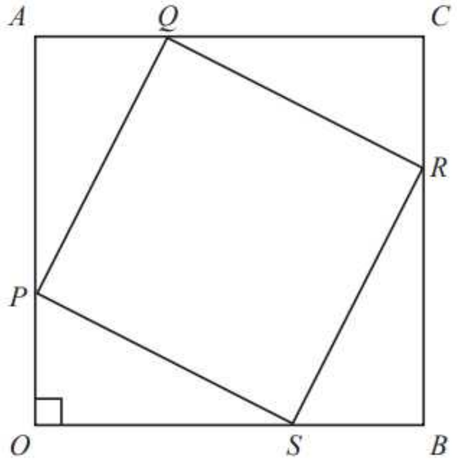


2020 Section 2 Question 20

Rates of change and differential equations

Consider square $OACB$ where point O is the origin. Let the position vectors for points A, B be defined as $\underset{\sim}{a}$, $\underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\underset{\sim}{a}$, $\overrightarrow{AQ} = k\underset{\sim}{b}$, $\overrightarrow{RC} = k\underset{\sim}{a}$ and $\overrightarrow{SB} = k\underset{\sim}{b}$ where $0 \le k \le 1$. This means that points P, Q, R and S are positioned along their respective sides in equal proportion.



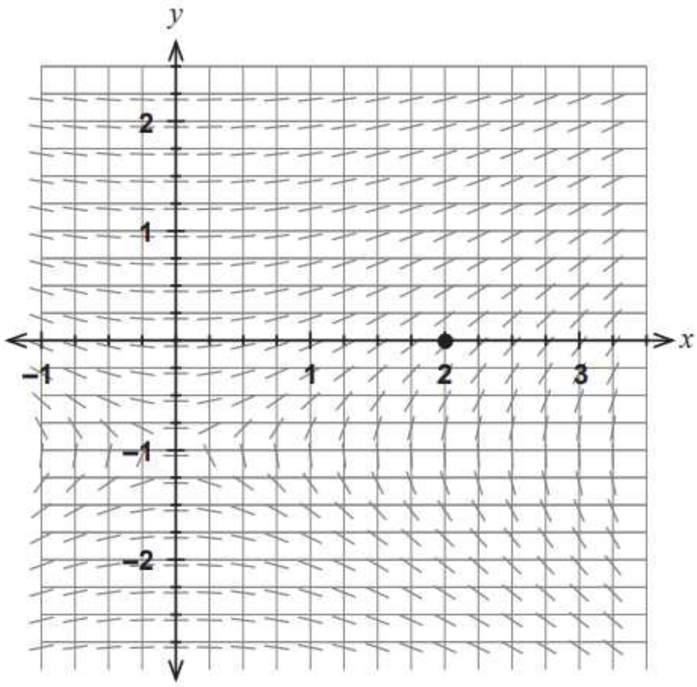
(a) Using vector methods, prove that the size of $\angle PQR = 90°$. (5 marks)

Now suppose that in square $OACB$, it is known that $OA = 10$ cm and that point P is moving away from the origin at a speed of 0.2 cm per second. This means that points Q, R and S are moving at the same speeds along their respective sides.

Let $x =$ the distance OP.

(b) Determine the rate at which the area of square $PQRS$ is changing when $x = 3$ cm.

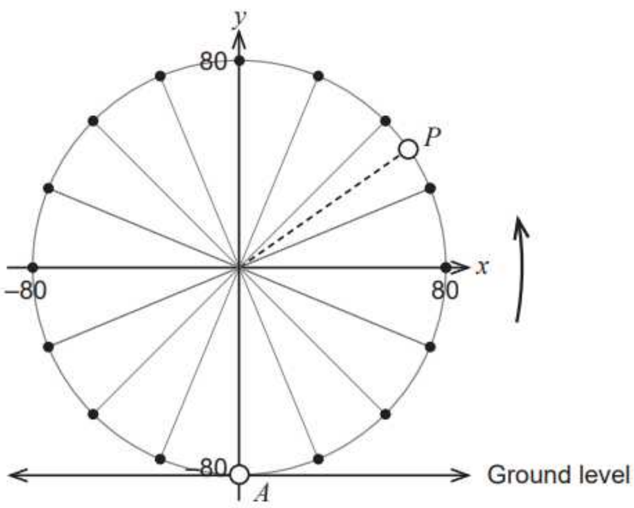
(4 marks)

2019 Section 2 Question 11 **Rates of change and differential equations**	The slope field given by $\frac{dy}{dx} = \frac{x}{2y+2}$ is shown in the diagram below.  (a) Calculate the value of the slope field at the point $(2,0)$. (1 mark)

	(b) On the diagram above, draw the solution curve that contains the point $(2,0)$. (2 marks) (c) Determine the equation for the solution curve that contains the point $(2,0)$. (3 marks)

2019 Section 2 Question 17 **Rates of change and differential equations**	In Australia, the killing of humpback whales was banned in 1963. At the end of 2018, 45 years later, the population P of migrating humpback whales off the coast of Western Australia was estimated at 30 000, i.e. $P(45) = 30\,000$. (a) Assuming that the population of humpback whales had been increasing at an instantaneous rate equal to 10% of the population, estimate the number of humpback whales at the end of 1963. (3 marks) To model the growth in the population from the end of 2018, a marine biologist suggests that the rate of growth be modelled by the equation below. $\frac{dP}{dt} = 0.1P - \frac{P^2}{700\,000}$ The biologist re-defines $P(0) = 30\,000$, i.e. t = number of years from the end of 2018. (b) If $P(t)$ is written in the form $P(t) = \frac{a}{1 + be^{-ct}}$, determine the values of the constants a, b and c. (2 marks) (c) Hence determine the year during which the population of humpback whales off the coast of Western Australia will reach double that estimated at the end of 2018. (2 marks)

	(d) State the major difference in the variation in the population P(t) using the model in part (b) compared with that in part (a). (1 mark)

2019 Section 2 Question 18 **Rates of change and differential equations**	A ferris wheel has a radius of 80 metres and rotates in an anticlockwise direction at a rate of one revolution every 72 seconds. The ferris wheel has 16 cars that are equally spaced around the wheel as shown in the diagram. A coordinate system is set up so that the centre of the ferris wheel is at the origin and the ground level has equation $y = -80$. Passengers begin their ride when a car is at position A $(0,-80)$. Consider a passenger in a car at position P. Let t = the number of seconds the ride has been in progress from position A. θ = the angle in radians that the car has rotated from position A. y = the height of a car above the centre of the ferris wheel (metres).  (a) Show that $\frac{d\theta}{dt} = \frac{\pi}{36}$ radians per second. (1 mark)

(b) Given that $y(\theta) = 80\sin(\theta + \alpha)$, explain why $\alpha = -\dfrac{\pi}{2}$. (1 mark)

(c) Determine how quickly a passenger is moving upward when they are 100 metres above the ground, correct to the nearest 0.01 metres per second. (4 marks)

(d) Show that function $y\ (t)$ satisfies the condition for simple harmonic motion. (2 marks)

A different passenger happens to be in a car that is two cars ahead of a particular car on the ferris wheel.

(e) At what speed, correct to the nearest 0.01 metres per second, is the trailing passenger moving upward when the other passenger is moving downward at exactly the same speed? (3 marks)

2022 Section 1 Question 4

Rates of change and differential equations

(a) Function $f(x)=\dfrac{5(x+1)}{(x-1)(x^2+3x+1)}$ can be expressed in the form $\dfrac{a}{x-1}+\dfrac{bx+c}{x^2+3x+1}$.

Determine the value of the constants a, b and c. (3 marks)

Solution

$$\frac{5x+5}{(x-1)(x^2+3x+1)} = \frac{a(x^2+3x+1)+(x-1)(bx+c)}{(x-1)(x^2+3x+1)}$$

$$= \frac{(a+b)x^2+(3a-b+c)x+(a-c)}{(x-1)(x^2+3x+1)}$$

Equating coefficients:

$a+b=0$

$3a-b+c=5$

$a-c=5$

Solving gives $a=2,\ b=-2,\ c=-3$

i.e. $$\frac{5(x+1)}{(x-1)(x^2+3x+1)} = \frac{2}{x-1} - \frac{(2x+3)}{x^2+3x+1}$$

Specific behaviours

- ✓ forms the correct expression for the equivalent numerator
- ✓ equates coefficients correctly to form 3 linear equations
- ✓ solves correctly to determine a, b and c

(b) Hence determine $\int \frac{10x+10}{(x-1)(x^2+3x+1)}\,dx$.

(5 marks)

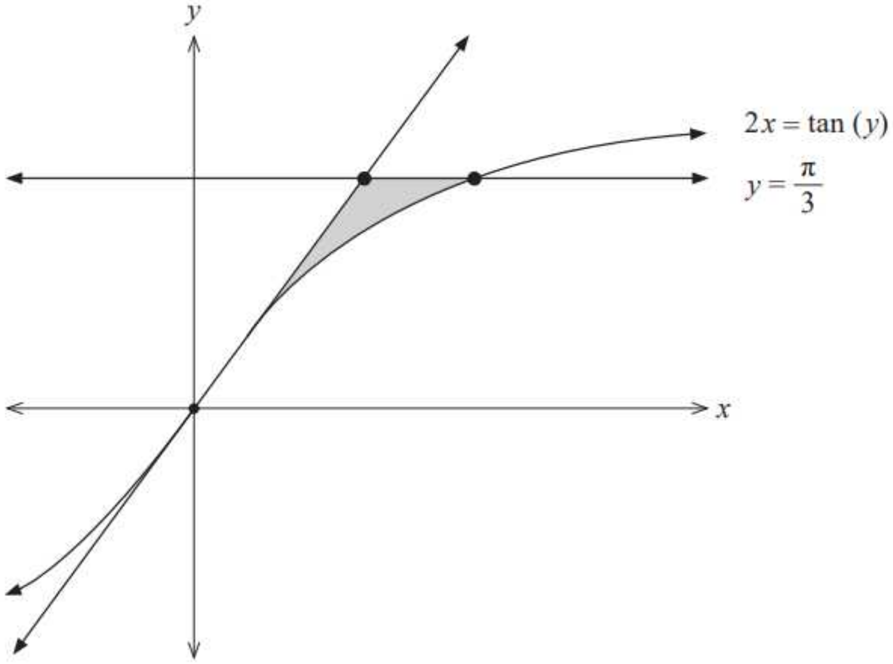
Solution
$\int \frac{10x+10}{(x-1)(x^2+3x+1)}\,dx = 2\int \frac{5x+5}{(x-1)(x^2+3x+1)}\,dx$
$= \int \frac{4}{x-1} - \frac{2(2x+3)}{x^2+3x+1}\,dx$
$= 4\ln\lvert x-1\rvert - 2\ln\lvert x^2+3x+1\rvert + k$
$= \ln\left(\frac{(x-1)^4}{(x^2+3x+1)^2}\right) + k$

Specific behaviours
✓ expresses the given integrand as double $f(x)$
✓ writes the integrand correctly in terms of the partial fractions
✓ anti-differentiates $\frac{a}{x-1}$ correctly using the absolute value of a natural logarithm
✓ anti-differentiates $\frac{bx+c}{x^2+3x+1}$ correctly
✓ uses a constant of integration

2022 Section 1 Question 7

Rates of change and differential equations

The graph of $2x = \tan(y)$ is shown along with the tangent at $x = 0$. The horizontal line $y = \frac{\pi}{3}$ is also shown.



(a) Using implicit differentiation, determine the equation of the tangent drawn at $x = 0$. (3 marks)

Solution
$\frac{d}{dx}(2x) = \frac{d}{dx}(\tan(y)) \quad \therefore \; 2 = \sec^2(y).\frac{dy}{dx}$ $\therefore \; \frac{dy}{dx} = 2\cos^2(y)$ At $y=0$, $\frac{dy}{dx} = 2\cos^2(0) = 2(1) = 2$ Hence equation of the tangent is $y = 2x$ or $x = \frac{y}{2}$.
Specific behaviours
✓ differentiates $2x = \tan(y)$ correctly using implicit differentiation ✓ obtains the correct expression for the derivative ✓ determines the equation of the tangent correctly

The shaded region is bounded by the curve $2x = \tan(y)$, the tangent drawn and $y = \frac{\pi}{3}$.

(b) Write the expression for the area of the shaded region. (2 marks)

Solution
Area $= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\tan(y) - \frac{y}{2}\right) dy$
Specific behaviours
✓ forms a definite integral using correct limits for y with correct notation ✓ forms the integrand correctly

Alternative Solution
Area $= \int_0^{\frac{\pi}{6}} \left(2x - \tan^{-1}(2x)\right) dx \; + \; \int_{\frac{\pi}{6}}^{\frac{\sqrt{3}}{2}} \left(\frac{\pi}{3} - \tan^{-1}(2x)\right) dx$
Specific behaviours
✓ forms two definite integrals using correct limits for x values with correct notation ✓ forms the two integrands correctly

(c) Evaluate this area exactly. (3 marks)

Solution
$Area = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\tan(y) - \frac{y}{2}\right) dy = \int_0^{\frac{\pi}{3}} \left(\frac{1}{2}\frac{\sin y}{\cos y} - \frac{y}{2}\right) dy$ $= \left[-\frac{1}{2}\ln\lvert\cos y\rvert - \frac{y^2}{4}\right]_0^{\frac{\pi}{3}}$ $= \left[-\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{\pi^2}{36}\right] - \left[-\frac{1}{2}\ln(1) - 0\right]$ $= \frac{1}{2}\ln(2) - \frac{\pi^2}{36}$
Specific behaviours
✓ re-writes the tangent function in terms of sine and cosine correctly ✓ anti-differentiates correctly using the logarithm of an absolute value ✓ evaluates correctly in terms of an exact value

2021 Section 1 Question 3 Rates of change and differential equations	Using an appropriate substitution, determine the exact value for $\int_2^3 15x\sqrt{x-2}\,dx$. (5 marks)

Solution

Using $u = x-2$

x	2	3
u	0	1

$$\frac{du}{dx} = 1 \quad \therefore \quad dx = du$$

$$\int_2^3 15x\sqrt{x-2}\,dx = \int_0^1 15(u+2)\left(u^{\frac{1}{2}}\right).du$$

$$= 15\int_0^1 \left(u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right) du$$

$$= 15\left[\frac{2u^{\frac{5}{2}}}{5} + \frac{4u^{\frac{3}{2}}}{3}\right]_0^1 = 15\left[\left(\frac{2}{5}+\frac{4}{3}\right)-(0+0)\right] = 26$$

Specific behaviours

- ✓ changes the limits correctly for the chosen substitution
- ✓ obtains dx in terms of du correctly
- ✓ simplifies the integrand correctly using the chosen substitution
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly

Alternative Solution

Using $u = \sqrt{x-2}$

x	2	3
u	0	1

$$\frac{du}{dx} = \frac{1}{2\sqrt{x-2}} \qquad \therefore \quad dx = 2\sqrt{x-2}\,du = 2u\,du$$

$$\int_2^3 15x\sqrt{x-2}\,dx = \int_0^1 15\left(u^2+2\right)(u).2u\,du$$

$$= 15\int_0^1 \left(2u^4 + 4u^2\right) du$$

$$= 15\left[\frac{2u^5}{5} + \frac{4u^3}{3}\right]_0^1 = 15\left[\left(\frac{2}{5}+\frac{4}{3}\right)-(0+0)\right] = 26$$

Specific behaviours

- ✓ changes the limits correctly for the chosen substitution
- ✓ obtains dx in terms of du correctly
- ✓ simplifies the integrand correctly using the chosen substitution
- ✓ anti-differentiates correctly
- ✓ evaluates the definite integral correctly

2021 Section 1 Question 5 Rates of change and differential equations	(a) Given that $\dfrac{7x^2-12x+2}{(x-2)(x^2+2)} = \dfrac{a}{x-2} + \dfrac{bx}{x^2+2}$ determine the values of a and b. (2 marks)

Solution

$$\frac{a}{x-2} + \frac{bx}{x^2+2} = \frac{a(x^2+2)+bx(x-2)}{(x-2)(x^2+2)} = \frac{(a+b)x^2-2bx+2a}{(x-2)(x^2+2)}$$

Equating coefficients: $a+b=7$

$-2b=-12$

$2a=2$

Solving gives $a=1,\ \ b=6$

Specific behaviours

✓ forms the equivalence of numerators correctly
✓ solves for a,b correctly

(b) Hence determine $\displaystyle\int \frac{7x^2-12x+2}{(x-2)(x^2+2)}\,dx$.

(3 marks)

Solution

$$\int \frac{7x^2-12x+2}{(x-2)(x^2+2)}\,dx = \int\left(\frac{1}{x-2}+\frac{6x}{x^2+2}\right)dx$$

$$= \int\frac{1}{x-2}\,dx + 3\int\frac{2x}{x^2+2}\,dx$$

$$= \ln|x-2| + 3\ln(x^2+2) + k$$

Specific behaviours

✓ re-writes the integrand correctly in terms of the partial fractions
✓ anti-differentiates the $(x-2)^{-1}$ term correctly using the absolute value of a natural logarithm AND uses an integration constant
✓ anti-differentiates the $6x(x^2+2)^{-1}$ term correctly (absolute value not required)

2020 Section 1 Question 1 Rates of change and differential equations	Evaluate exactly $\int_0^{\pi}\left(4\cos^2 x - \sin x\right)dx$. (3 marks)

Solution

$$\int_0^{\pi}\left(4\cos^2 x-\sin x\right)dx = \int_0^{\pi}\left(2\left(1+\cos 2x\right)-\sin x\right)dx$$

$$= \int_0^{\pi}\left(2+2\cos 2x-\sin x\right)dx \quad \ldots (1)$$

$$= \left[2x+\sin 2x+\cos x\right]_0^{\pi}$$

$$= \left(2\pi+0-1\right)-\left(0+0+1\right)$$

$$= 2\pi-2$$

Specific behaviours

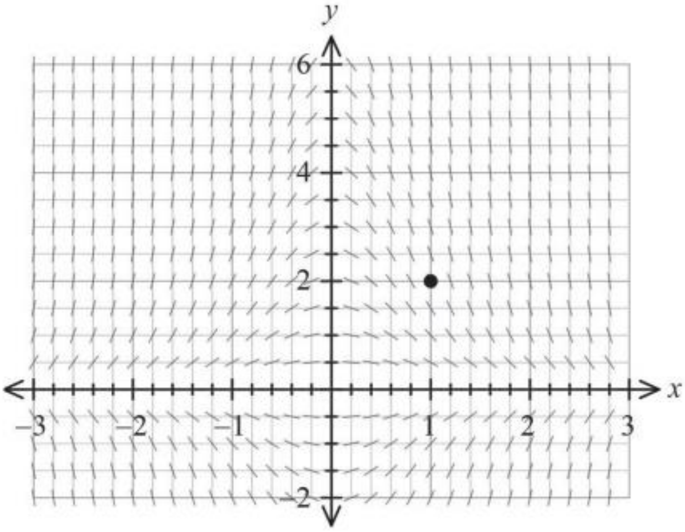
✓ uses the cosine double angle identity to obtain integrand (1)
✓ anti-differentiates the integrand correctly
✓ evaluates the definite integral correctly

Marking Guide – Section 2

2023
Section 2
Question 11

Rates of change and differential equations

A slope field is given by the equation $\frac{dy}{dx} = k(xy)$ where k is a constant.



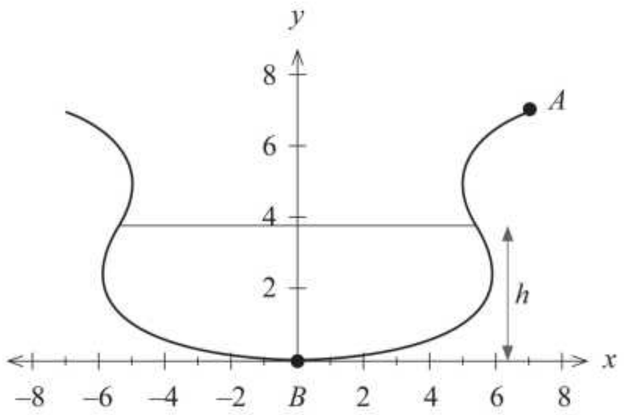
(a) The value of the slope field at the point $(1, 2)$ is equal to -4. Determine the value of the constant k. (2 marks)

Solution
At $(1,2)$ $\frac{dy}{dx} = -4 = k(1)(2)$ $\quad \therefore\ k = -2$ $\quad$ i.e. $\frac{dy}{dx} = -2(xy)$
Specific behaviours
✓ states that $\frac{dy}{dx} = -4$ ✓ solves correctly to determine k

(b) Determine the equation for the solution curve that contains the point (1, 2) and draw this curve on the diagram above. (3 marks)

Solution
From $\frac{dy}{dx} = -2xy$ $\quad \therefore\ \int \frac{dy}{y} = \int -2x\,dx$ ... (1) $\therefore\ \ln y = -x^2 + c$ $\therefore\ y = e^{-x^2+c}$ Using $(1, 2)$ $2 = e^{-1+c}$ $\quad \therefore\ c = \ln 2 + 1 = 1.6931...$ $\therefore\ y = e^{-x^2+\ln 2+1} = 2e\left(e^{-x^2}\right)$ or $2e^{-x^2+1}$
Specific behaviours
✓ separates the variables correctly to form statement (1) ✓ determines the equation for the solution curve through $(1, 2)$ ✓ draws the solution curve with symmetry about $x = 0$ to follow the slope field

2023 Section 2 Question 17 **Rates of change and differential equations**	The shape of a decorative vase is modelled by revolving the curve AB about the y axis where $x=\sqrt{y\left(y^2-11y+35\right)}$ with $0\le y\le 7$. All dimensions are in centimetres.



(a) Determine an integral expression, in terms of h, for the volume of water in the vase if it is filled to a depth of h cm. (2 marks)

Solution
$V(h) = \int_0^h \pi x^2 dy = \int_0^h \pi \,.\, y\left(y^2-11y+35\right) dy$
Specific behaviours
✓ writes a definite integral with correct limits with respect to y ✓ uses the correct integrand (including the factor of π)

Water is poured into the initially empty vase at a constant rate of 50 cm^3/s.

(b) Determine the time taken to fill the vase to a depth of 6 cm. (2 marks)

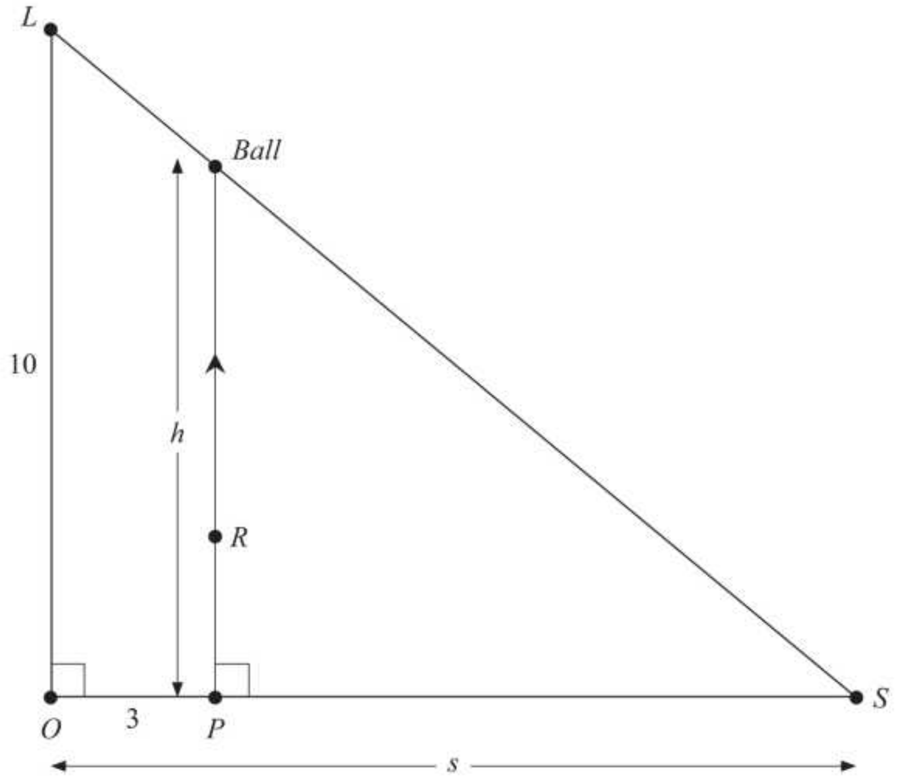
Solution
$V(6) = \int_0^6 \pi \,.\, y\left(y^2-11y+35\right) dy = 162\pi = 508.93800...$ cm^3 $\therefore$ Time $= \dfrac{508.93800...}{50} = 10.1787...$ sec
Specific behaviours
✓ evaluates $V(6)$ correctly ✓ calculates the time taken correctly

With the depth at 6 cm, another 30 cm^3 of water is added to the vase.

(c) Using the increments formula, calculate the approximate change in depth of water in the vase. (3 marks)

Solution
As $V(h) = \int_0^h \pi \cdot y(y^2 - 11y + 35)\, dy$ $\therefore \quad \frac{dV}{dh} = \frac{d}{dh}\left(\int_0^h \pi \cdot y(y^2 - 11y + 35)\, dy\right) = \pi\left(h(h^2 - 11h + 35)\right)$ $= \pi(h^3 - 11h^2 + 35h)$ (using the Fundamental Theorem) $\therefore \quad V'(6) = \pi(6)(6^2 - 11(6) + 35) = 94.24777...$ cm^2 $\Delta V \approx \frac{dV}{dh} \times \Delta h$ $30 = 94.24777... \times \Delta h$ $\Delta h = 0.3183...\ cm$ Hence the depth will increase by 0.32 cm when 30 cm^3 is poured into the vase.
Specific behaviours
✓ differentiates correctly to determine $V'(h)$ ✓ substitutes correctly for ΔV and $V'(6)$ into the increments formula ✓ calculates the change in depth Δh

2023 Section 2 Question 19 **Rates of change and differential equations**	A ball is projected vertically into the air from point R, so that it will eventually hit the ground at point P, 3 metres from the base of a 10 metre high light at L. At any time t seconds, when the ball is h metres above the ground, it casts a shadow on the ground at point S at a distance s metres from the base of the light.



At any time t it can be shown that $s(10-h)=30$.

(a) Using implicit differentiation, show that $\frac{ds}{dt}=\frac{30}{(10-h)^2}\times\frac{dh}{dt}$. (3 marks)

Solution
$\frac{d}{dt}\left(s(10-h)\right) = \frac{d}{dt}(30)$
$\frac{ds}{dt}\times(10-h) + s(-1)\times\frac{dh}{dt} = 0$
$\therefore \frac{ds}{dt} = \frac{s}{(10-h)}\times\frac{dh}{dt}$
Substituting $s=\frac{30}{(10-h)}$, $\frac{ds}{dt} = \frac{30}{(10-h)}\times\frac{1}{(10-h)}\times\frac{dh}{dt} = \frac{30}{(10-h)^2}\times\frac{dh}{dt}$
Specific behaviours
✓ indicates a sum of terms equal to zero ✓ differentiates the left-hand side correctly ✓ substitutes for s correctly to obtain the desired expression for $\frac{ds}{dt}$

At $t = 0.5$ seconds, it is found that the ball is 6.275 metres above the ground and moving upwards at 6.1 metres per second.

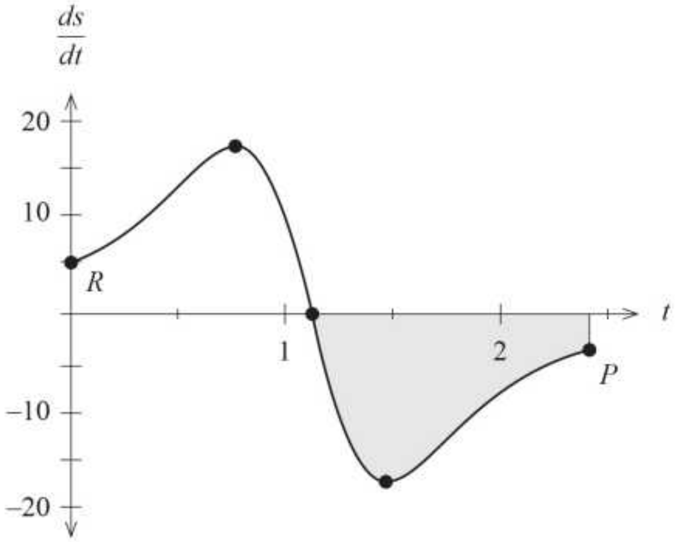
(b) By assuming $h''(t) = -9.8$ ms^{-2}, show that $h(t) = 2 + 11t - 4.9t^2$. (3 marks)

Solution
$h'(t) = \int -9.8\,dt = -9.8t + c$
Using $h'(0.5) = 6.1 \quad \therefore\ 6.1 = -9.8(0.5) + c$
$\therefore\ c = 11$
$h(t) = \int(-9.8t + 11)\,dt = -4.9t^2 + 11t + k$
Using $h(0.5) = 6.275 \quad \therefore\ 6.275 = -4.9(0.5)^2 + 11(0.5) + k$
$\therefore\ k = 2$
Hence $h(t) = 2 + 11t - 4.9t^2$ as required.
Specific behaviours
✓ anti-differentiates correctly to obtain both $h'(t)$ and $h(t)$
✓ forms the equation correctly using $h'(0.5) = 6.1$ to solve for c
✓ forms the equation correctly using $h(0.5) = 6.275$ to solve for k

(c) Determine the initial speed of the ball's shadow, correct to the nearest 0.01 metres per second.
(3 marks)

Solution
When $t = 0$, $h = 2$ and $\frac{dh}{dt} = 11 \quad (s = 3.75)$
$\therefore\ \frac{ds}{dt} = \frac{30}{(10-2)^2} \times (11) = 5.15625$
Hence the ball's shadow initial speed is 5.16 metres per second.
Specific behaviours
✓ determines the correct values for h, $\frac{dh}{dt}$ when $t = 0$
✓ forms the correct expression for $\frac{ds}{dt}$ using the answer from part (a)
✓ calculates the speed correctly to 0.01 metres per second

The graph of the function $\frac{ds}{dt}$ against time t is shown below. Point R of this graph corresponds to the ball being thrown into the air, while point P corresponds to the ball hitting the ground.



The definite integral $\int_a^b \left(\frac{ds}{dt}\right) dt$ was evaluated so that the area for the shaded region could be determined. This area is 13.4258 square units.

(d) Determine the values for a and b (correct to 0.01 seconds) and describe what this definite integral represents in terms of the motion of the shadow. (4 marks)

Solution
At point C: $\frac{ds}{dt} = 0$ when $\frac{dh}{dt} = 0$ i.e. $11 - 9.8t = 0$ i.e. $t = \frac{11}{9.8} = 1.12244898...$ sec $\therefore$ $a = 1.12$ sec At point P: $h = 0$ i.e. $2 + 11t - 4.9t^2 = 0$ Solving gives $t = 2.41398...$ (reject $t < 0$) $\therefore$ $b = 2.41$ sec $\int_{1.12}^{2.41} \left(\frac{ds}{dt}\right) dt = \Delta s = -13.4258 \text{ m}$ This means that the shadow will move 13.43 metres towards the light while the ball is falling from its highest point to hitting the ground.
Specific behaviours
✓ determines the value of a correctly ✓ determines the value of b correctly ✓ states the shadow moves 13.43 metres back towards the light or refers to the negative change in displacement ✓ interprets the ball is falling from its highest point to hitting the ground

(e) Determine the fastest rate at which the shadow moves (correct to the nearest 0.01 metres per second) and the time when this occurs (correct to the nearest 0.01 seconds). (3 marks)

Solution

For any time t sec:

$$\therefore \quad \frac{ds}{dt} = \frac{30}{(10-h)^2} \times \left(\frac{dh}{dt}\right) = \frac{30}{\left(10-\left(2+11t-4.9t^2\right)\right)^2} \times (-9.8t+11)$$

Plotting the graph of the rate $r(t) = \frac{ds}{dt}$ versus t :

There is a maximum at $(0.7699... , 17.4731...)$.

The ball's shadow is moving at the fastest rate at $t = 0.77$ seconds.

The fastest speed for the shadow is 17.47 metres per second.

Specific behaviours

✓ defines the ball's speed function $\frac{ds}{dt}$ correctly as a function of t

✓ states the time for the greatest speed

✓ states the greatest speed correct to 0.01 metres per second

Alternative Solution

For a maximum $\frac{ds}{dt}$ we require $\frac{d^2s}{dt^2} = 0$.

$$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{30}{(10-h)^2} \times \frac{dh}{dt}\right) = \frac{60}{(10-h)^3} \times \left(\frac{dh}{dt}\right)^2 + \frac{30}{(10-h)^2} \times \frac{d^2h}{dt^2}$$

$$= \frac{60}{(10-h)^3} \times (11-9.8t)^2 + \frac{30}{(10-h)^2} \times (-9.8)$$

$$0 = \frac{30}{(10-h)^3}\left[2(11-9.8t)^2 - 9.8(10-h)\right]$$

i.e. Solve $2(11-9.8t)^2 = 9.8\left(10-2-11t+4.9t^2\right)$

Solving gives $t = 0.76995...$ or $t = 1.47494...$ But $t < \frac{11}{9.8} = 1.122$ s

The ball's shadow is moving at the fastest rate at $t = 0.77$ seconds.

Hence the maximum value $s'(0.76995..) = 17.4731...$ m/s.

The fastest speed for the shadow is 17.47 metres per second.

Specific behaviours

✓ differentiates the rate function $\frac{ds}{dt}$ correctly to consider $\frac{d^2s}{dt^2} = 0$

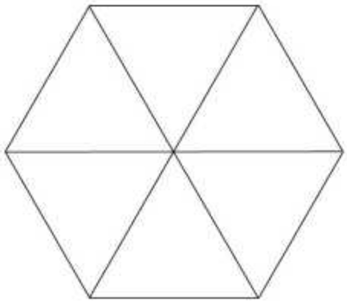
✓ states the time for the greatest speed

✓ states the greatest speed correct to 0.01 metres per second

2022 Section 2 Question 8

Rates of change and differential equations

A regular hexagon expands so that the length of each side increases at a rate of 0.5 cm per second. Assuming that the polygon maintains its shape, determine the rate at which the area is increasing when the side length is 4 cm. (4 marks)



Solution

Let $x =$ side length of the regular hexagon

$$A = 6 \times Area(Equilateral\ \Delta)$$

$$= 6\times\left(\frac{1}{2}\right)\times(x)(x)(\sin 60^o) = 3x^2.\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}x^2}{2}$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

$$= \left(3\sqrt{3}x\right) \times \frac{dx}{dt}$$

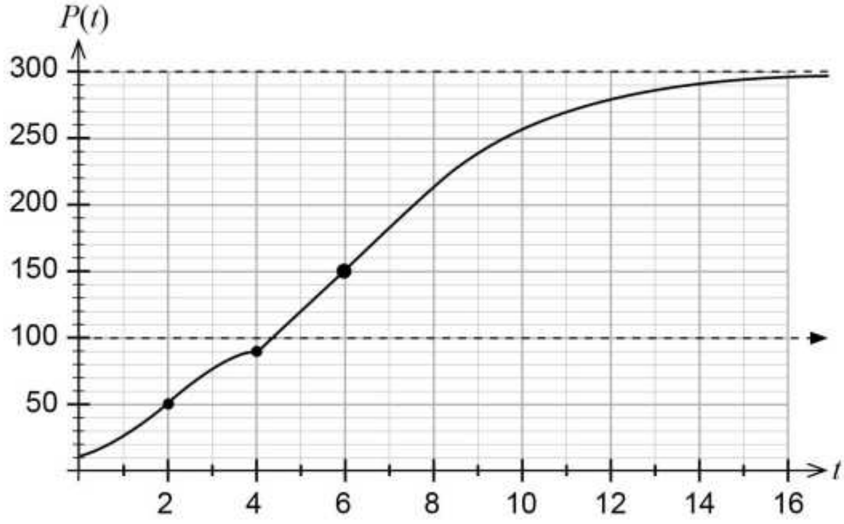
$$= \left(3\sqrt{3}(4)\right)\times(0.5) = 10.3923...\ cm^2/\sec$$

i.e. the area is increasing at a rate of 10.39 cm^2 per second

Specific behaviours

✓ correctly formulates the area in terms of the side length
✓ differentiates correctly to relate the rates of change
✓ substitutes $x = 4$ and $\frac{dx}{dt} = 0.5$ correctly
✓ calculates correctly giving the correct units

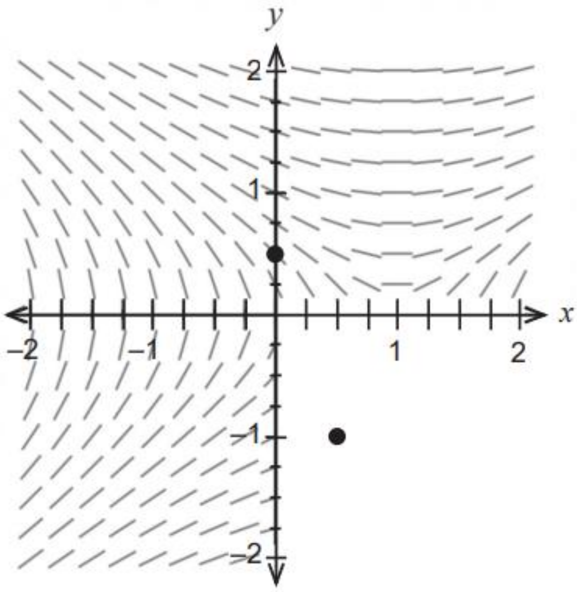
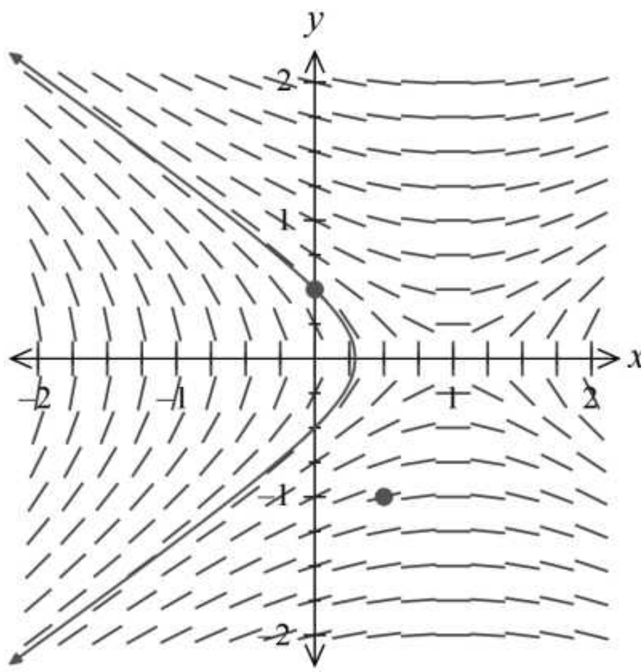
2022 Section 2 Question 16 **Rates of change and differential equations**	An ant colony population P at time t days grows at a rate given by the equation $\frac{dP}{dt} = 0.01P(100 - P)$, where $0 \le t \le 4$. The graph of this population is shown below.



(a) For $0 \le t \le 4$, using the growth rate equation explain the variation of the population. (2 marks)

Solution
Initially when $P < 50$ the growth rate increases i.e. P curve is concave up. When $P > 50$ the growth rate decreases i.e. P curve is concave down. Hence there is a point of inflection when $P = 50$ and then the curve will approach the horizontal asymptote $P = L$.
Specific behaviours
✓ states that the slope (growth rate) increases when $P < 50$ or $t < 2$ ✓ states that the slope (growth rate) decreases when $P > 50$ or $t > 2$

	At the end of the fourth day, the environment for the ant colony improves dramatically so that its limiting population is increased to 300. (b) Sketch, on the axes above, the expected variation of the population for t > 4 days, using the increased limiting population value. (2 marks) **Solution** Shown above **Specific behaviours** ✓ indicates a point of inflection in the graph at $P = 150$ (value of t not important) ✓ indicates the curve flattens out as $P \to 300$

2021 Section 2 Question 10 **Rates of change and differential equations**	Part of the slope field given by $\frac{dy}{dx} = \frac{x-1}{2y}$ is shown below.  (a) Calculate and draw the slope field at the point (0.5, – 1) . (3 marks) 

Solution
At $(0.5, -1)$, $\frac{dy}{dx} = \frac{(0.5)-1}{2(-1)} = 0.25$
Specific behaviours
✓ substitutes correctly into the expression for $\frac{dy}{dx}$ ✓ calculates the slope field correctly ✓ indicates the correct slope orientation at $(0.5, -1)$

(b) Determine the equation of the solution curve that contains the point (0, 0.5) . (3 marks)

Solution
From $\frac{dy}{dx} = \frac{x-1}{2y}$ we obtain $\int 2y\,dy = \int (x-1)\,dx$ i.e. $y^2 = \frac{x^2}{2} - x + c$ Using $(0, 0.5)$, $(0.5)^2 = \frac{0^2}{2} - 0 + c$ $\therefore\ c = 0.25$ $\therefore\ y^2 = \frac{x^2}{2} - x + \frac{1}{4}$ OR $y^2 = \frac{(x-1)^2}{2} + c$ Using $(0, 0.5)$, $(0.5)^2 = \frac{(0-1)^2}{2} + c$ $\therefore\ c = -0.25$ $\therefore\ y^2 = \frac{(x-1)^2}{2} - \frac{1}{4}$
Specific behaviours
✓ separates the variables as an integration statement correctly ✓ anti-differentiates correctly using a constant ✓ determines the anti-derivative constant correctly

(c) Draw the solution curve that contains the point (0, 0.5) . (2 marks)

Solution
Shown on graph.
Specific behaviours
✓ contains the point $(0, 0.5)$ and the curve follows the slope field for $y > 0.5$ ✓ indicates symmetry about $y = 0$ AND indicates the curve is vertical at $y = 0$

2021 Section 2 Question 12 **Rates of change and differential equations**	The horizontal displacement of a Ferris wheel cabin exhibits simple harmonic motion. The maximum horizontal speed is $\frac{\pi}{2}$ metres per second and its period of motion is exactly 60 seconds. Let $x(t) = A\cos(nt)$ be the horizontal displacement after t seconds. (a) Determine the values of A and n . (3 marks)

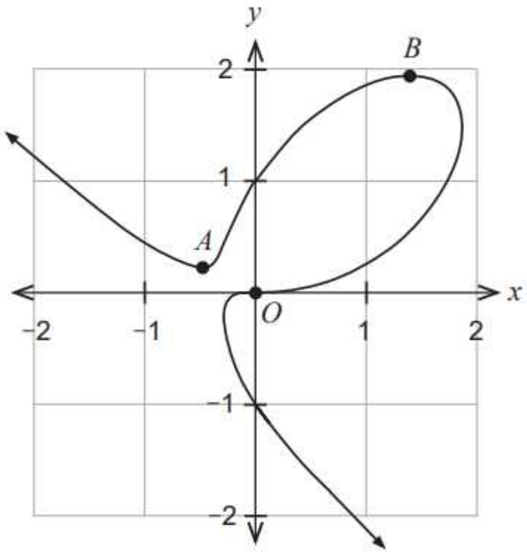
Solution
Period $T = 60 = \frac{2\pi}{n}$ $\therefore$ $n = \frac{\pi}{30}$ $(0.1047....)$
$v(t) = -A\left(\frac{\pi}{30}\right)\sin\left(\frac{\pi t}{30}\right)$ $\therefore$ $Max\ v = A\left(\frac{\pi}{30}\right) = \frac{\pi}{2}$
$\therefore$ $A = 15$
Hence $x(t) = 15\cos\left(\frac{\pi t}{30}\right)$
Specific behaviours
✓ determines n correctly ✓ differentiates and forms the correct expression for the maximum speed ✓ determines A correctly

(b) Determine the horizontal acceleration, correct to the nearest 0.001 m/s^2 , when the horizontal displacement is 10 metres. (3 marks)

Solution
Condition for S.H.M. is $a = -\left(\frac{\pi}{30}\right)^2 x$
$\therefore$ When $x = 10$ $a = -\left(\frac{\pi^2}{900}\right)(10) = -0.10966$ ... metres/sec^2
i.e. acceleration is −0.110 m/sec^2 (3 d.p.)
Specific behaviours
✓ applies the condition for S.H.M. correctly ✓ substitutes $x = 10$ correctly ✓ determines the acceleration correct to 0.001 m/sec^2

Alternative Solution
$\therefore$ When $x = 10$ i.e. $10 = 15\cos\left(\frac{\pi t}{30}\right)$ $\therefore$ $t = 8.0316...$ sec
$\therefore a(t) = -\left(\frac{\pi^2}{900}\right)(15)\cos\left(\frac{\pi t}{30}\right)$ $a(8.0316...) = $ −0.110 m/sec^2 (3 d.p.)
Specific behaviours
✓ solves for t when $x = 10$ correctly ✓ determines the expression for the acceleration correctly ✓ determines the acceleration correct to 0.001 m/sec^2

2021 Section 2 Question 18 **Rates of change and differential equations**	The equation $x^3 + y^3 = 3xy + y$ implicitly defines the curve shown below.



(a) Using implicit differentiation obtain the expression for $\frac{dy}{dx}$. (3 marks)

Solution
$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}(3xy+y)$ $\therefore\ 3x^2 + 3y^2\left(\frac{dy}{dx}\right) = 3x\left(\frac{dy}{dx}\right) + 3y + \left(\frac{dy}{dx}\right)$ ..... (1) i.e. $\frac{dy}{dx} = \frac{3\left(x^2-y\right)}{1+3x-3y^2} = \frac{3\left(y-x^2\right)}{3y^2-3x-1}$
Specific behaviours
✓ obtains the left hand expression of statement (1) ✓ obtains the right hand expression of statement (1) ✓ obtains the correct expression for $\frac{dy}{dx}$ (or its equivalent)

The slope of the curve at the origin O and points A and B is equal to zero.

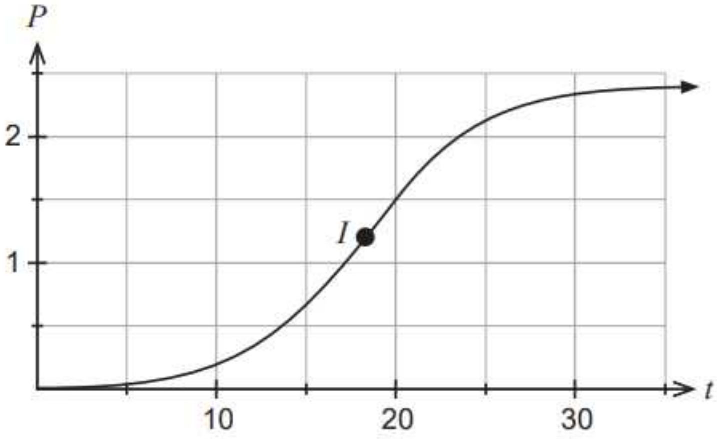
(b) Show that the equation that determines the x coordinates for points A and B is given by $x^4 - 2x - 1 = 0$ and hence determine the coordinates for point A correct to 0.001. (3 marks)

Solution
For a slope of zero we require
$\dfrac{3(x^2 - y)}{1 + 3x - 3y^2} = 0$ *i.e.* $y = x^2$ (numerator zero)
Hence substituting $y = x^2$ into the equation $x^3 + y^3 = 3xy + y$ obtains
$x^3 + (x^2)^3 = 3x(x^2) + x^2$
i.e. $x^3 + x^6 = 3x^3 + x^2$
i.e. $x^6 - 2x^3 - x^2 = 0$
$\therefore\ x^2(x^4 - 2x - 1) = 0$ ... (2)
Since for points A, B $x \neq 0$ then it must be that $x^4 - 2x - 1 = 0$.
Using CAS we obtain point A as $(-0.475,\ 0.225)$ correct to 0.001
Note: point B is $(1.395,\ 1.947)$
Specific behaviours
✓ deduces that $y = x^2$ is required for a zero slope
✓ obtains equation (2) that determines A, B
✓ obtains the coordinates for point A correct to 0.001

2020 Section 2 Question 19 **Rates of change and differential equations**	The population $P(t)$ of sardines in an ocean, measured in million tonnes after t years, was modelled by the logistic equation:

$$P(t) = \frac{2.4}{1+239e^{-0.3t}}$$

The graph of this model is shown below. This graph contains a point of inflection at point I.



(a) Calculate the size of the sardine ocean population at $t = 0$. (2 marks)

Solution
$P(0) = \frac{2.4}{1+239} = 0.01$ million tonnes3 i.e. The population number is 10 000 tonnes at $t = 0$.
Specific behaviours
✓ substitutes $t = 0$ into the logistic equation and calculates correctly ✓ states the answer in tonnes OR states the units of the answer (million tonnes)

(b) Rewrite the logistic equation in the form $\frac{dP}{dt} = rP(k-P)$, stating clearly the values for r and k. (2 marks)

Solution
$k =$ limiting population value $\lim_{t\to\infty} P(t) = 2.4$ Hence $k = 2.4$ From defining rule for $P(t)$, $rk = 0.3$ Hence $r = \frac{0.3}{2.4} = 0.125$ i.e. $\frac{dP}{dt} = 0.125P(2.4-P)$
Specific behaviours
✓ determines value of k correctly ✓ determines value of r correctly

(c) When the sardine population is 500 000 tonnes, use the technique of increments to calculate the approximate change in population in the next month. (3 marks)

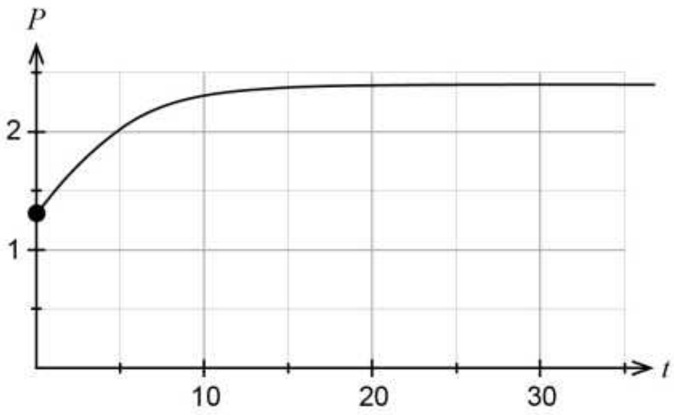
Solution
When $P = 0.5$ million tonnes and $\Delta t = \frac{1}{12}$ years We have $\Delta P \approx \left(\frac{dP}{dt}\right) \times \Delta t = 0.125(0.5)(2.4 - 0.5) \times \frac{1}{12}$ $= (0.11875) \times \frac{1}{12} = 0.0098958...$ million tonnes i.e. Sardine population grows by approximately 9896 tonnes (9900 tonnes)
Specific behaviours
✓ states the correct values for P and Δt ✓ forms the correct expression for ΔP ✓ calculates the correct value for ΔP (units are not necessary)

(d) Determine the maximum rate of growth of the sardine population. (2 marks)

Solution
Maximum occurs at the value of $P = \frac{k}{2} = 1.2$ million tonnes (point of inflection) Note: This occurs when $t = 18.254...$ years. $\therefore \frac{dP}{dt} = 0.125(1.2)(2.4 - 1.2) = 0.18$ million tonnes/year Hence the maximum rate of growth is 180 000 tonnes per year.
Specific behaviours
✓ determines the value of P that yields the maximum rate of growth ✓ calculates the rate of growth correctly stating correct units

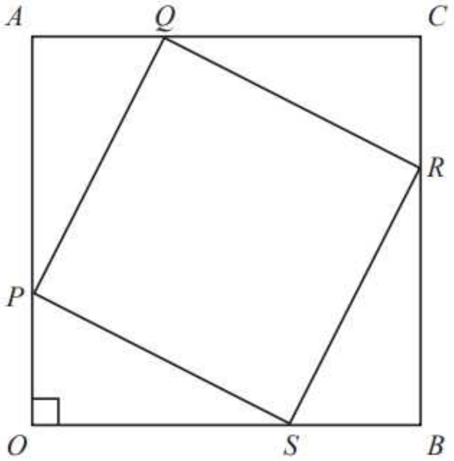
Suppose that the initial population of sardines was 1.3 million tonnes.

(e) Assuming that the rate of growth is still given by $\frac{dP}{dt} = rP(k-P)$ sketch the graph of the population growth on the axes below. Explain your graph. (2 marks)



Solution
Sketch shown above. Since $P(0) > \frac{k}{2} = 1.2$ then $\frac{dP}{dt}$ will be decreasing. $\therefore$ Graph will NOT exhibit any point of inflection and will be concave DOWN as it approaches the horizontal asymptote $P = 2.4$.
Specific behaviours
✓ indicates a curve that is always concave down approaching $P = 2.4$ ✓ justifies using the initial population value $P(0) > 1.2$

2020 Section 2 Question 20 **Rates of change and differential equations**	Consider square $OACB$ where point O is the origin. Let the position vectors for points A, B be defined as $\underset{\sim}{a}$, $\underset{\sim}{b}$ respectively i.e. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Let points P, Q, R and S be defined so that $\overrightarrow{OP} = k\underset{\sim}{a}$, $\overrightarrow{AQ} = k\underset{\sim}{b}$, $\overrightarrow{RC} = k\underset{\sim}{a}$ and $\overrightarrow{SB} = k\underset{\sim}{b}$ where $0 \le k \le 1$. This means that points P, Q, R and S are positioned along their respective sides in equal proportion.



(a) Using vector methods, prove that the size of $\angle PQR = 90°$. (5 marks)

Solution
Vectors $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$ $\quad$ $\overrightarrow{QR} = \overrightarrow{QC} + \overrightarrow{CR}$
$= (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$ $\quad$ $= (1-k)\underset{\sim}{b} + (-k\underset{\sim}{a})$
$= (1-k)\underset{\sim}{b} - k\underset{\sim}{a}$
Consider $\overrightarrow{PQ} \bullet \overrightarrow{QR} = \left((1-k)\underset{\sim}{a} + k\underset{\sim}{b}\right) \bullet \left((1-k)\underset{\sim}{b} - k\underset{\sim}{a}\right)$
$= (1-k)^2 \underset{\sim}{a} \bullet \underset{\sim}{b} - k(1-k)\underset{\sim}{a} \bullet \underset{\sim}{a} + k(1-k)\underset{\sim}{b} \bullet \underset{\sim}{b} - k^2 \underset{\sim}{b} \bullet \underset{\sim}{a}$
Since $s\angle AOB = 90°$ then $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$
Then $\overrightarrow{PQ} \bullet \overrightarrow{QR} = -k(1-k)\underset{\sim}{a} \bullet \underset{\sim}{a} + k(1-k)\underset{\sim}{b} \bullet \underset{\sim}{b}$
$= -k(1-k)\lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{b}\rvert^2$
But since side lengths are equal then $\lvert\underset{\sim}{a}\rvert = \lvert\underset{\sim}{b}\rvert$
Then $\overrightarrow{PQ} \bullet \overrightarrow{QR} = -k(1-k)\lvert\underset{\sim}{a}\rvert^2 + k(1-k)\lvert\underset{\sim}{a}\rvert^2 = 0$
Hence the size of $\angle PQR = 90°$ as required.
Specific behaviours
✓ forms correct expressions for vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$
✓ forms the dot product of vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ using correct notation
✓ expands the dot product expression correctly to obtain 4 terms
✓ uses the dot product $\underset{\sim}{a} \bullet \underset{\sim}{b} = 0$ since $OACB$ is given as a square
✓ uses the information $\lvert\underset{\sim}{a}\rvert = \lvert\underset{\sim}{b}\rvert$ to obtain the dot product ZERO

Now suppose that in square $OACB$, it is known that $OA = 10$ cm and that point P is moving away from the origin at a speed of 0.2 cm per second. This means that points Q, R and S are moving at the same speeds along their respective sides.

Let $x =$ the distance OP.

(b) Determine the rate at which the area of square $PQRS$ is changing when $x = 3$ cm.

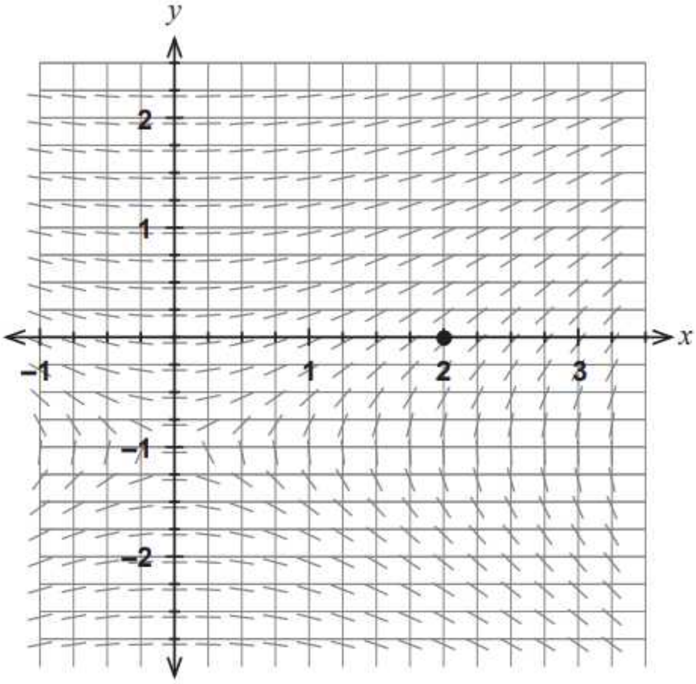
(4 marks)

Solution
Area $\Delta OPS = \frac{1}{2}(x)(10-x)$ Hence area $PQRS$ $A = (10)(10) - 4\times\frac{1}{2}(x)(10-x)$ $= 100 - 20x + 2x^2$ $\frac{dA}{dt} = \frac{dA}{dx}\times\frac{dx}{dt} = (-20+4x)\frac{dx}{dt} = (-20+4(3))(0.2)$ $= -1.6$ cm^2/sec Hence the area is decreasing at a rate of 1.6 cm^2/sec.
Specific behaviours
✓ forms the correct expression for the area of $PQRS$ in terms of x ✓ applies the chain rule to obtain $\frac{dA}{dt}$ in terms of x and $\frac{dx}{dt}$ correctly ✓ substitutes $x = 3$ and $\frac{dx}{dt} = 0.2$ correctly ✓ calculates correctly and states the correct units for the rate of change of area

2019 Section 2 Question 11

Rates of change and differential equations

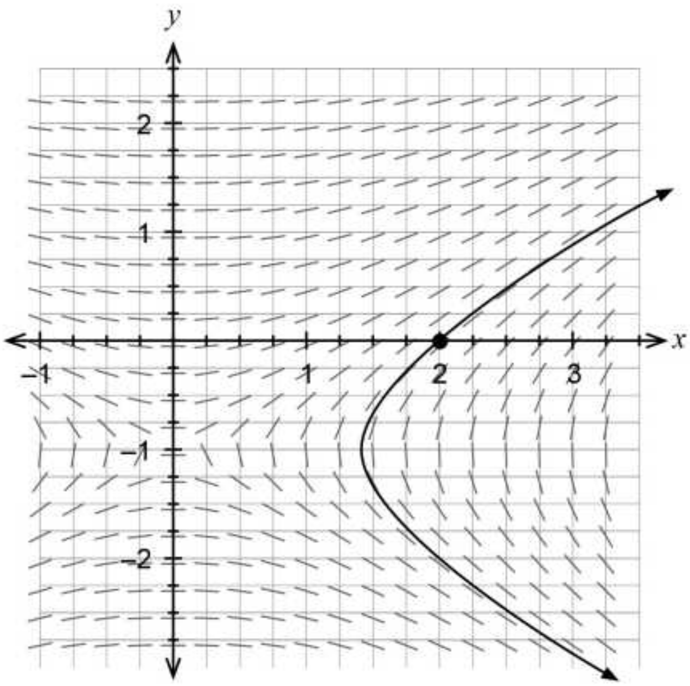
The slope field given by $\frac{dy}{dx} = \frac{x}{2y+2}$ is shown in the diagram below.



(a) Calculate the value of the slope field at the point $(2,0)$. (1 mark)

Solution
Evaluating $\frac{dy}{dx}$ when $x=2,\ y=0$: $\frac{dy}{dx} = \frac{2}{2(0)+2} = 1$ i.e. the slope field at $(2,0)$ has a value of 1.
Specific behaviours
✓ evaluates $\frac{dy}{dx}$ correctly

(b) On the diagram above, draw the solution curve that contains the point $(2,0)$. (2 marks)



Solution
Shown above on the diagram.
Specific behaviours
✓ draws a curve that follows the slope field ✓ draws a curve that is vertical at $y=-1$ or symmetric about $y=-1$

(c) Determine the equation for the solution curve that contains the point $(2,0)$. (3 marks)

Solution
Separating variables obtains $\int(2y+2)\,dy = \int x\,dx$ $\therefore\ y^2+2y = \frac{x^2}{2} + c$ Using $(2,0)$: $0^2+2(0) = \frac{2^2}{2} + c \quad \therefore\ c=-2$ Equation of the solution through $(2,0)$ is $y^2+2y = \frac{x^2}{2} - 2$ Alternatively : $(y+1)^2 = \frac{x^2}{2} - 1$ OR $y = \pm\sqrt{\frac{x^2}{2}-1} - 1$
Specific behaviours
✓ separates the variables correctly ✓ anti-differentiates correctly using a constant ✓ determines the anti-derivative constant correctly

2019 Section 2 Question 17 **Rates of change and differential equations**	In Australia, the killing of humpback whales was banned in 1963. At the end of 2018, 45 years later, the population P of migrating humpback whales off the coast of Western Australia was estimated at 30 000, i.e. $P(45) = 30\ 000$. (a) Assuming that the population of humpback whales had been increasing at an instantaneous rate equal to 10% of the population, estimate the number of humpback whales at the end of 1963. (3 marks)

Solution using $P(45) = 30\ 000$
Rate of change is given by $\frac{dP}{dt} = 0.1P$. This has solution $P(t) = P_0 e^{0.1t}$ Using $P(45) = 30\ 000$ $30\ 000 = P_0 e^{0.1(45)}$ Solving gives $P_0 = 333.26...$ i.e. there were 333 humpback whales at the end of 1963. (answers 300 or 330 are acceptable given the initial information)
Specific behaviours
✓ forms the differential equation correctly to represent the rate of change ✓ writes the specific exponential solution for this differential equation ✓ uses $P(45) = 30\ 000$ to correctly deduce $P(0)$ as an integer value

or

Solution using $P(55) = 30\ 000$
Rate of change is given by $\frac{dP}{dt} = 0.1P$. This has solution $P(t) = P_0 e^{0.1t}$ Using $P(55) = 30\ 000$ $30\ 000 = P_0 e^{0.1(55)}$ Solving gives $P_0 = 122.60...$ i.e. there were 123 humpback whales at the end of 1963. (answers 100 or 120 are acceptable given the initial information)
Specific behaviours
✓ forms the differential equation correctly to represent the rate of change ✓ writes the specific exponential solution for this differential equation ✓ uses $P(55) = 30\ 000$ to correctly deduce $P(0)$ as an integer value

To model the growth in the population from the end of 2018, a marine biologist suggests that the rate of growth be modelled by the equation below.

$$\frac{dP}{dt} = 0.1P - \frac{P^2}{700\ 000}$$

The biologist re-defines $P(0) = 30\ 000$, i.e. t = number of years from the end of 2018.

(b) If $P(t)$ is written in the form $P(t) = \dfrac{a}{1 + be^{-ct}}$, determine the values of the constants a, b and c. (2 marks)

Solution
$\dfrac{dP}{dt} = 0.1P - \dfrac{P^2}{700000} = \dfrac{P}{10} - \dfrac{P^2}{700000} = \dfrac{1}{700000}P(70000-P)$ From formula sheet $\dfrac{dP}{dt} = rP(k-P)$ gives $P = \dfrac{kP_0}{P_0 + (k-P_0)e^{-rkt}}$ As $P(0) = 30\ 000$, then $P(t) = \dfrac{(70000)(30000)}{30000 + (70000-30000)e^{-0.1t}}$ $\therefore\ P(t) = \dfrac{70000}{1 + \frac{4}{3}e^{-0.1t}}$ i.e. $a = 70000,\ b = \dfrac{4}{3},\ c = 0.1$
Specific behaviours
✓ determines the correct value of constant a ✓ determines the correct value of the constants b, c

(c) Hence determine the year during which the population of humpback whales off the coast of Western Australia will reach double that estimated at the end of 2018. (2 marks)

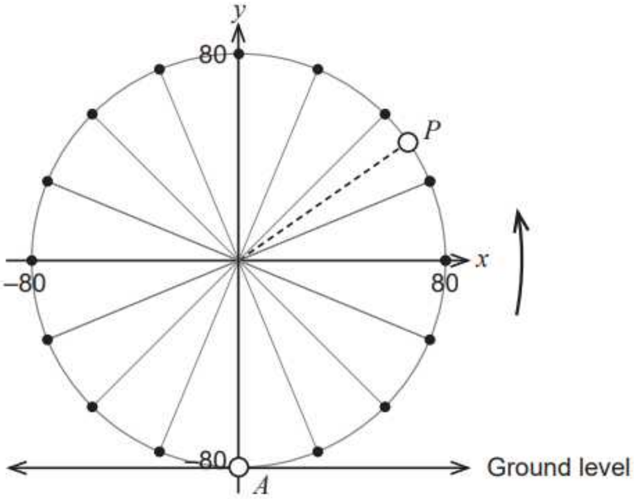
Solution
Require when $P(t) = 60000$.   i.e. Solve $\therefore\ 60000 = \dfrac{70000}{1 + \frac{4}{3}e^{-0.1t}}$ $\therefore\ e^{-0.1t} = \dfrac{1}{8}$ *i.e.* $t = 10(\ln 8)$ Using CAS $t = 20.794....$ years   Hence the population is estimated to double during year 2039.
Specific behaviours
✓ forms the correct equation to solve for t ✓ solves and concludes the calendar year for the population to double

(d) State the major difference in the variation in the population P(t) using the model in part (b) compared with that in part (a). (1 mark)

Solution
The rate of growth in part (b) will increase from $P = 30\ 000$ to $P = 35\ 000$ and then the rate of growth decreases from $P = 35\ 000$ to approach an equilibrium or steady state population of $P = 70\ 000$ i.e. part (b) model represents limited growth.   In part (a), the model will predict that growth will continue without any limit.
Specific behaviours
✓ states a valid property that distinguishes the growth in part (b) from part (a)

2019 Section 2 Question 18 **Rates of change and differential equations**	A ferris wheel has a radius of 80 metres and rotates in an anticlockwise direction at a rate of one revolution every 72 seconds. The ferris wheel has 16 cars that are equally spaced around the wheel as shown in the diagram. A coordinate system is set up so that the centre of the ferris wheel is at the origin and the ground level has equation $y = -80$. Passengers begin their ride when a car is at position A $(0,-80)$. Consider a passenger in a car at position P.

Let t = the number of seconds the ride has been in progress from position A.
θ = the angle in radians that the car has rotated from position A.
y = the height of a car above the centre of the ferris wheel (metres).



(a) Show that $\dfrac{d\theta}{dt} = \dfrac{\pi}{36}$ radians per second.

(1 mark)

Solution
One revolution $(2\pi$ radians$)$ every 72 seconds means $\dfrac{d\theta}{dt} = \dfrac{2\pi}{72} = \dfrac{\pi}{36}$
Specific behaviours
✓ states that the angle changes 2π radians every 72 seconds

(b) Given that $y(\theta) = 80\sin(\theta + \alpha)$, explain why $\alpha = -\dfrac{\pi}{2}$.

(1 mark)

Solution
Using $y(0) = -80$ then $-80 = 80\sin(\alpha)$ i.e. $\sin(\alpha) = -1$ $\therefore$ $\alpha = -\dfrac{\pi}{2}$ OR at position P the angle above the x axis is equal to $\theta - \dfrac{\pi}{2}$ $\therefore$ $\alpha = -\dfrac{\pi}{2}$
Specific behaviours
✓ justifies correctly the value for α

(c) Determine how quickly a passenger is moving upward when they are 100 metres above the ground, correct to the nearest 0.01 metres per second. (4 marks)

Solution
When 100 metres above ground, $y = 100 - 80 = 20$ $\therefore \quad 20 = 80\sin\left(\theta - \frac{\pi}{2}\right) \quad$ *i.e.* $\sin\left(\theta - \frac{\pi}{2}\right) = \frac{1}{4}$ i.e. $\theta = 1.823...$ Hence $\cos\left(\theta - \frac{\pi}{2}\right) = \sqrt{1 - 0.25^2} = \frac{\sqrt{15}}{4} = 0.9682...$ OR $\therefore \quad 20 = 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right) \quad$ *i.e.* $t = 20.8955...$ sec   As $y = 80\sin\left(\theta - \frac{\pi}{2}\right)$ then differentiating both sides with respect to t $\frac{dy}{dt} = \frac{dy}{d\theta} \times \frac{d\theta}{dt} = 80\cos\left(\theta - \frac{\pi}{2}\right) \times \frac{\pi}{36}$ $= 80\left(\frac{\sqrt{15}}{4}\right)\left(\frac{\pi}{36}\right) = 6.7596...$ m/sec   Hence the passenger is moving up at a rate of 6.76 m/sec (nearest 0.01)
Specific behaviours
✓ deduces $y = 20$ ✓ differentiates correctly using the the chain rule to obtain an expression for $\frac{dy}{dt}$ ✓ substitutes either $\cos\left(\theta - \frac{\pi}{2}\right) = \frac{\sqrt{15}}{4}$ or $t = 20.8955$ or $\theta = 1.823...$ ✓ calculates the rate correct to 0.01 m/sec

(d) Show that function $y(t)$ satisfies the condition for simple harmonic motion. (2 marks)

Solution
Need to show that $y''(t) = -n^2\, y(t)$ $y(t) = 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)$ Hence $y'(t) = 80\cos\left(\frac{\pi t}{36} - \frac{\pi}{2}\right) \times \frac{\pi}{36}$ $\therefore \quad y''(t) = 80 \times \frac{\pi}{36} \times \left(-\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right)\right) \times \frac{\pi}{36}$ $= -\left(\frac{\pi}{36}\right)^2 \times 80\sin\left(\frac{\pi t}{36} - \frac{\pi}{2}\right) = -\left(\frac{\pi}{36}\right)^2 y(t)$ Hence function $y(t)$ satisfies the definition for simple harmonic motion.
Specific behaviours
✓ differentiates twice correctly to find $y''(t)$ correctly   ✓ shows that $y''(t) = -n^2\, y(t)$ where it is evident that $n = \frac{\pi}{36}$

A different passenger happens to be in a car that is two cars ahead of a particular car on the ferris wheel.

(e) At what speed, correct to the nearest 0.01 metres per second, is the trailing passenger moving upward when the other passenger is moving downward at exactly the same speed? (3 marks)

Solution
Equal speeds will occur when the 2 cars are equidistant from the y axis. Adjacent cars are separated by $\frac{2\pi}{16} = \frac{\pi}{8}$ radians $\therefore$ For the trailing car $\theta = \frac{7\pi}{8}$ or $t = \frac{63}{2} = 31.5$ sec. $\therefore \; Speed \; = \; y'\left(\frac{7\pi}{8}\right) \; = \; 80\cos\left(\frac{3\pi}{8}\right)\times\frac{\pi}{36}$ $= \; 2.6716....$ m/sec Hence the equal speeds is 2.67 m/sec (nearest 0.01)
Specific behaviours
✓ states the position for each car when equal speeds are achieved ✓ determines the correct value for θ or t for equal speeds ✓ calculates the speed (no penalty for incorrect rounding)

Unit 4.3 – Statistical inference

Section 1

There have been no questions on this topic for this section in the exams of recent years.

Section 2

2023 Section 2 Question 13 **Statistical inference**	A factory produces boxes of breakfast cereal with a labelled weight of 1.00 kg. Let μ denote the population mean and σ denote the population standard deviation of the weights of the boxes. The factory sets the packaging process to a mean weight $\mu = 1.01$ kg with a standard deviation $\sigma = 0.05$ kg. To maintain quality, a random sample of 400 boxes is taken each day and weighed. Let $\overline{X}$ denote the sample mean weight. (a) State the distribution for $\overline{X}$ and its parameters. (3 marks) (b) Determine the probability that the sample mean is more than 5 g above the labelled weight. (2 marks) The sample mean on a particular day is $\bar{x} = 1.05$ kg, while the sample standard deviation is $s = 50$ g. (c) Determine a 95% confidence interval, correct to 0.001 kg, for the population mean weight based on this sample. (2 marks)

Anja, a quality control officer, wants a 95% confidence interval based on a sample size of 100 with a width of no more than 0.1 kg.

(d) What is the maximum standard deviation for this confidence interval? (2 marks)

Over the next 50 days, Ben, who is a data collection agent, takes random samples of size 100 each day and a 95% confidence interval is calculated for each sample. Ten of these 50 intervals (20% of the intervals) have a lower bound that is less than 1.00 kg. Ben claims that this indicates that the mean weight of the packaging is set too low.

(e) Is Ben correct? Justify your response. (2 marks)

2023 Section 2 Question 15 **Statistical inference**	The WeLuvYas Bank extends personal loans to approved customers. A random sample of n personal loans is taken. A 99% confidence interval for the population mean loan μ (in thousands of dollars) based on this sample is $10.2 < \mu < 25.4$. (a) What is the mean personal loan $\bar{x}$ for this sample? (2 marks) (b) Calculate the standard deviation of the sample mean. (2 marks) Ali exclaims excitedly 'everyone here at WeLuvYas is 99% certain that the true population mean is within the interval $10.2 < \mu < 25.4$'. (c) State **two** reasons why Ali is not correct. (2 marks)

	A data analyst discovers that the sample size was actually $2n$. In addition to this, the sample mean was actually \$2000 more than that originally determined. (d) Re-calculate the 99% confidence interval for the population mean on the basis of the updated information. (3 marks)

2022 Section 2 Question 12 **Statistical inference**	The inner diameter of a cylinder in a motor car engine is critical to its performance. Let μ mm denote the population mean cylinder diameter produced by a manufacturing process. A random sample, R_1, of 100 cylinder diameters is taken and the standard deviation for this sample was found to be 1 mm. Let $\bar{X}$ = the sample mean cylinder diameter for sample R_1. (a) State the distribution for $\bar{X}$ and its parameters. (3 marks) (b) What is the probability that $\bar{X}$ differs from μ by more than 0.2 mm. Give your answer correct to 0.001. (2 marks)

	From random sample R_1 , a 95% confidence interval for μ is formed. (c) Calculate the width of this confidence interval, correct to 0.001. (2 marks) Lilian, the production manager, wishes to decrease the width of the confidence interval. She suggests: "We can form sample R_2 by using the data from sample R_1 and then combining this data with itself to form a sample with 200 observations. Using $n = 200$ will decrease the width of the confidence interval." (d) State **two** major problems with using this idea. (2 marks)

2022 Section 2 Question 14 **Statistical inference**	The annual incomes (in thousands of dollars) of a random sample of n Australians is taken. The sample standard deviation is 10.98. A 99% confidence interval I_1 based on this sample is $90 \le \mu \le 94$. (a) Calculate the value of the sample size n . (2 marks)

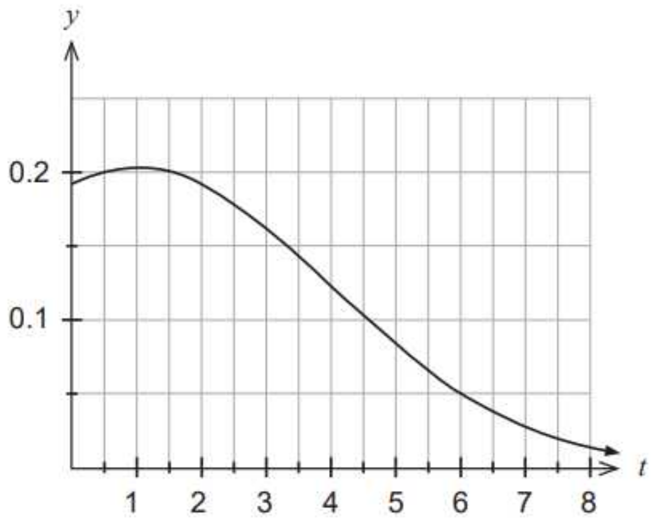
Another random sample of size n is taken and a 99% confidence interval I_2 is calculated.

(b) State **two** aspects in which the intervals I_1 and I_2 may be different. (2 marks)

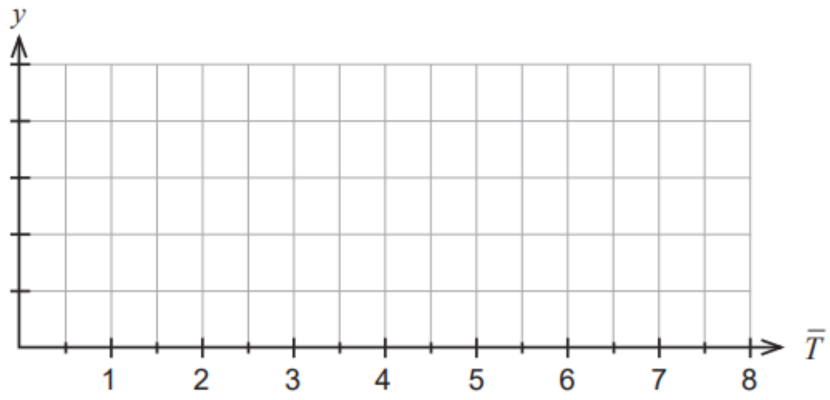
A third random sample of size 50 is taken and a 99% confidence interval I_3 is calculated. James suggests that since interval I_3 is the widest, it is more likely to contain the population mean Australian income μ .

(c) Is James correct? Justify your answer. (2 marks)

2020 Section 2 Question 18 Statistical inference	The mass of chocolate that is placed into each biscuit produced by the BikkiesAreUs company has been observed to be normally distributed with mean $\mu = 7.5$ grams and standard deviation $\sigma = 1.5$ grams. (a) Determine the probability, correct to 0.01, that the total amount of chocolate used for 50 biscuits is less than 365 grams. (4 marks) (b) If the probability that the mean amount of chocolate used per biscuit differs from μ by less than 0.2 grams is 98%, determine n, the number of biscuits that need to be sampled. (3 marks) A competitor company called YouBeautChokkies produces similar biscuits. A sample of 144 biscuits was taken and it was found that the standard deviation of the mass of chocolate used in each biscuit was 1.8 grams and the total amount of chocolate used in the sample of 144 biscuits was 1.09 kg. Charlie Chokka, a representative from the YouBeautChokkies company, stated that “we are using significantly more chocolate for each biscuit than BikkiesAreUs. If you want that real chocolate taste, then buy from us!” (c) Perform the necessary calculations to comment on Charlie’s claim. (4 marks)

2021 Section 2 Question 15 **Statistical inference**	An experiment was conducted to measure how quickly adults respond to the request: 'send me a text message'. Let T = the number of hours taken for an adult to respond and send a text message. It was found that the distribution of the population of response times for adults was given by the probability density function shown below, with mean μ = 3 hours and standard deviation σ = 2.4 hours.  Random samples of size 64 are drawn repeatedly from the population of response times and the sample mean response time $\bar{T}$ is determined for each sample. (a) Calculate, correct to 0.001, the probability that a sample mean response time will be between 150 minutes and 210 minutes. (3 marks)

(b) Sketch the likely distribution of the sample mean $\bar{T}$ (for samples of size 64) on the axes below. (2 marks)



Anika, a teacher at the TekNoCrat School, theorises that as teenagers tend to check their text messages more frequently than adults, then the population mean response time for teenagers will be much lower than the population mean adult response time $\mu = 3$.

Anika is then presented with the sample mean response time for a sample gathered from an unknown source.

Sample size	Sample mean (hours)	Sample standard deviation (hours)
100	2.1	2.7

Calculations are performed and Anika concludes by stating: ‘this sample was clearly not taken from the population of adult response times. It is highly likely that this sample was taken from a sample of 100 teenagers’.

(c) Perform the necessary calculations and comment on Anika’s claim. (4 marks)

2021 Section 2 Question 17 Statistical inference	A researcher is interested in estimating the population mean μ (dollars) that Perth residents had spent via online shopping in December 2020. A random sample of size n gave a sample mean of \$400, a sample standard deviation s and a 95% confidence interval of width \$200. (a) State the 95% confidence interval obtained. (1 mark) (b) Calculate the standard deviation of the sample mean, correct to \$0.01. (2 marks) (c) In terms of n, what sample size would yield a 95% confidence interval of width \$50? Show your reasoning. (2 marks) (d) What is the probability that another sample of size $2n$ would produce a sample mean that differs from μ by more than \$50? (3 marks) Four different confidence intervals (A, B, C and D) are obtained for the mean amount spent via online shopping by Perth residents in December 2020.

Confidence interval	Sample size	Sample standard deviation	Confidence level
A	n	s	95%
B	n	s	99%
C	$2n$	s	95%
D	n	$0.8s$	95%

	(e) Which of the confidence intervals (A, B, C or D) contains μ , the population mean expenditure for online shopping in December 2020? Justify your answer. (2 marks) (f) For each of the following, state the confidence interval that has the smaller width. Justify your answers. (i) A and B. (1 mark) (ii) C and D. (1 mark)

2020 Section 2 Question 17 **Statistical inference**	Members of a random sample of n shoppers at the El Cheepo shopping centre were asked by a consumer researcher how much they had spent in the shopping centre that day. Let μ denote the mean and σ the standard deviation of the amount spent. The standard deviation σ is known from previous research. A 95% confidence interval for μ based on the sample is $150 \le \mu \le 200$ dollars. (a) Determine the sample mean for this sample. (1 mark) (b) Based on this confidence interval, calculate the standard deviation of the sample mean, correct to 0.01. (3 marks) The following week, the researcher again took a random sample of shoppers from the El Cheepo shopping centre, but this time the sample size was doubled. (c) What is the probability that the difference between μ and the sample mean from this sample will be less than \$10? (4 marks)

2019 Section 2 Question 14 **Statistical inference**	Trucks carrying iron ore for the Croc Rock mining company arrive at a weighing station. The service time T per truck is defined to be the time elapsed from the moment a truck enters the station zone, including the time to be positioned and then weighed, up to the time it leaves the zone. It is known that the population mean $\mu(T) = 80$ seconds and the population standard deviation $\sigma(T) = 20$ seconds. At the Croc Rock weighing station, 100 trucks are weighed. (a) State the (approximate) distribution of the sample mean service time per truck for the 100 trucks. (3 marks) (b) What is the probability that the sample mean service time will be more than 83 seconds? (2 marks) Suppose that more than 100 trucks were weighed at the Croc Rock weighing station. (c) How would this affect your answer to part (b)? Explain without recalculation. (2 marks)

	It is desired that the probability that the sample mean service time will be between 80 seconds and 82 seconds is greater than 40%. (d) Determine the minimum number of trucks that will need to be weighed. (3 marks)

2019 Section 2 Question 15 **Statistical inference**	A random sample of n commuters in Melbourne in August 2018 found that the average time to commute to work was 40 minutes. Repeated sampling of the mean indicated that the standard deviation of the sample mean was 3 minutes. (a) Determine a 90% confidence interval for the population mean commuting time μ to work, correct to 0.01 minutes. (3 marks)

Another random sample of $2n$ commuters in November 2018 found that the average time to commute to work was 45 minutes. Assume that both the August and November samples were drawn from the same population.

(b) What is the standard deviation of the sample mean for the November sample, correct to 0.01 minutes? (2 marks)

Suppose that the August and November samples are combined to form a sample with $3n$ commuters. Consider 90% confidence intervals for the following samples for the purpose of determining the population mean commuting time μ.

90% confidence interval	Sample	Size
A	August	n
N	November	$2n$
C	Combined	$3n$

(c) Which of the three confidence intervals, A, N or C, will provide the greatest precision in determining the population mean μ? Justify your answer. (2 marks)

	(d) Which of the three confidence intervals, A, N or C, contains the true value of the population mean μ? Justify your answer. (2 marks)

Marking Guide – Section 1

There have been no questions on this topic for this section in the exams of recent years.

Marking Guide – Section 2

2023 Section 2 Question 13

Statistical inference

A factory produces boxes of breakfast cereal with a labelled weight of 1.00 kg.

Let μ denote the population mean and σ denote the population standard deviation of the weights of the boxes. The factory sets the packaging process to a mean weight $\mu = 1.01$ kg with a standard deviation $\sigma = 0.05$ kg.

To maintain quality, a random sample of 400 boxes is taken each day and weighed. Let $\overline{X}$ denote the sample mean weight.

(a) State the distribution for $\overline{X}$ and its parameters. (3 marks)

Solution
Since $n = 400 > 30$, then $\overline{X} \sim N\left(\mu,\ \sigma_{\overline{X}}^{2}\right)$ i.e. normally distributed and centred with a mean 1.01 kg and standard deviation of the sample mean $\sigma\left(\overline{X}\right) = \dfrac{0.05}{\sqrt{400}} = 0.0025$ kg i.e. $\overline{X} \sim N\left(1.01,\ 0.0025^2\right)$.
Specific behaviours
✓ states that the sample mean will be normally distributed ✓ states the mean of the distribution is 1.01 kg ✓ states the standard deviation is 0.0025 kg

(b) Determine the probability that the sample mean is more than 5 g above the labelled weight. (2 marks)

Solution
$P\left(\overline{X} > 1.005\right) = P\left(Z > \dfrac{1.005 - 1.01}{0.0025}\right) = P(Z > -2)$ $= 0.9772$
Specific behaviours
✓ states the critical weight 1.005 kg ✓ determines the correct probability

The sample mean on a particular day is $\bar{x} = 1.05$ kg, while the sample standard deviation is $s = 50$ g.

(c) Determine a 95% confidence interval, correct to 0.001 kg, for the population mean weight based on this sample. (2 marks)

Solution
95% CI: $1.05 - 1.959964 \times 0.0025 < \mu < 1.05 + 1.959964 \times 0.0025$ $\therefore\ 1.045 < \mu < 1.055$ kg
Specific behaviours
✓ uses the correct critical z score for 95% confidence ✓ determines the bounds correctly to 0.001 kg

Anja, a quality control officer, wants a 95% confidence interval based on a sample size of 100 with a width of no more than 0.1 kg.

(d) What is the maximum standard deviation for this confidence interval? (2 marks)

Solution
Half-width $w = 0.05 = 1.959964 \times \frac{s}{\sqrt{100}}$ Solving gives $s = 0.2551$ kg.
Specific behaviours
✓ uses the correct expression for the half-width ✓ determines the value of the standard deviation correctly

Over the next 50 days, Ben, who is a data collection agent, takes random samples of size 100 each day and a 95% confidence interval is calculated for each sample. Ten of these 50 intervals (20% of the intervals) have a lower bound that is less than 1.00 kg. Ben claims that this indicates that the mean weight of the packaging is set too low.

(e) Is Ben correct? Justify your response. (2 marks)

Solution
Ben is NOT correct. This is a result of random sampling, and the confidence intervals will vary according to the sample data. There is no guarantee that 95% of the confidence intervals will lie above a certain value. All that is guaranteed is that in the long run 95% of the confidence intervals are expected to contain the true population mean μ.
Specific behaviours
✓ states that Ben is not correct ✓ justifies appropriately, based on random sampling

2023 Section 2 Question 15

Statistical inference

The WeLuvYas Bank extends personal loans to approved customers. A random sample of n personal loans is taken. A 99% confidence interval for the population mean loan μ (in thousands of dollars) based on this sample is $10.2 < \mu < 25.4$.

(a) What is the mean personal loan $\bar{x}$ for this sample? (2 marks)

Solution
$\bar{x} = \frac{10.2 + 25.4}{2} = 17.8$ Hence the mean personal loan was $17 800.
Specific behaviours
✓ calculates the midpoint of the confidence interval correctly ✓ states the personal loan amount in dollars

(b) Calculate the standard deviation of the sample mean. (2 marks)

Solution
Half-width $w = 17.8 - 10.2 = 7.6$ $\therefore$ $7.6 = 2.5758 \times \sigma(\overline{X})$ i.e. $\sigma(\overline{X}) = 2.95$ (2 d.p.) i.e. standard deviation is $2950
Specific behaviours
✓ forms the expression for half-width of interval in terms of the standard deviation ✓ calculates the standard deviation correctly

Ali exclaims excitedly 'everyone here at WeLuvYas is 99% certain that the true population mean is within the interval $10.2 < \mu < 25.4$'.

(c) State **two** reasons why Ali is not correct. (2 marks)

Solution
Ali is not correct. It can be said that: 1. A single confidence interval either contains μ or it doesn't. 2. The value of μ is unknown so we do not know if any given CI contains μ. 3. If we repeatedly take samples of size n then we will find that approximately 99% of these intervals will contain the true value of μ. 4. The value of n may be less than 30, meaning that the distribution of the sample mean may not be distributed, hence the 99% confidence interval may not be valid. Note: it is insufficient to only refer to the 'inherent nature of random sampling'.
Specific behaviours
✓ states one reason ✓ states a second reason

A data analyst discovers that the sample size was actually 2*n*. In addition to this, the sample mean was actually $2000 more than that originally determined.

(d) Re-calculate the 99% confidence interval for the population mean on the basis of the updated information. (3 marks)

Solution
The confidence interval changes is two ways. 1. The whole interval is translated upwards by 2. 2. The standard error $\sigma\left(\overline{X}\right)$ and consequently the width of the interval is scaled by a factor of $\frac{1}{\sqrt{2}}$: $w = 2.5758\times\frac{s}{\sqrt{2n}} = \left(2.5758\times\frac{s}{\sqrt{n}}\right)\times\frac{1}{\sqrt{2}}$ $= (7.6)\times\frac{1}{\sqrt{2}} = 5.37$ New confidence interval : $17.8+2-5.37 < \mu < 17.8+2+5.37$ i.e. $14.43 < \mu < 25.17$
Specific behaviours
✓ indicates the midpoint of the confidence interval is increased by 2 ✓ calculates the new standard deviation correctly ✓ calculates the new confidence interval correctly

2022 Section 2 Question 12 **Statistical inference**	The inner diameter of a cylinder in a motor car engine is critical to its performance. Let μ mm denote the population mean cylinder diameter produced by a manufacturing process. A random sample, R_1, of 100 cylinder diameters is taken and the standard deviation for this sample was found to be 1 mm.

Let $\bar{X}$ = the sample mean cylinder diameter for sample R_1.

(a) State the distribution for $\bar{X}$ and its parameters. (3 marks)

Solution
Since $n = 100 > 30$, then $\bar{X} \sim N\left(\mu,\ \sigma_{\bar{X}}^{2}\right)$ i.e. normally distributed and centred with a mean μ and estimated standard deviation of the sample mean $\sigma\left(\bar{X}\right) = \frac{1}{\sqrt{100}} = 0.1$ mm.
Specific behaviours
✓ states that the sample mean will be normally distributed ✓ states the mean of the distribution is μ ✓ states the expected standard deviation is 0.1 mm

(b) What is the probability that $\bar{X}$ differs from μ by more than 0.2 mm. Give your answer correct to 0.001. (2 marks)

Solution
$P\left(\left\lvert\bar{X} - \mu\right\rvert > 0.2\right) = P\left(\lvert z\rvert > 2\right) = 2(0.023) = 0.046$
Specific behaviours
✓ forms the correct probability statement in terms of z ✓ calculates the probability correct to 0.001

From random sample R_1, a 95% confidence interval for μ is formed.

(c) Calculate the width of this confidence interval, correct to 0.001. (2 marks)

Solution
Width $w = 2(k)\left(\sigma\left(\bar{X}\right)\right) = 2(1.96)(0.1) = 0.392$
Specific behaviours
✓ forms the correct expression for the width ✓ calculates the width correctly

	Lilian, the production manager, wishes to decrease the width of the confidence interval. She suggests: "We can form sample R_2 by using the data from sample R_1 and then combining this data with itself to form a sample with 200 observations. Using $n = 200$ will decrease the width of the confidence interval." (d) State **two** major problems with using this idea. (2 marks)

Solution
1. The idea simply replicates (repeats) the data in sample R_1. As such the sample R_2 is no longer random. Therefore the assumptions for using the normal distribution for the sample mean does not hold anymore. 2. Repeating the data values in sample R_1 will not reflect the true random variation in the data (manufacturing process). The confidence interval will therefore not be a true representation of the variation in the sample mean. 3. The sample mean and standard deviation will change if a new larger sample is taken. However, with Lillian's idea these will not change. This will affect both the width and location of the confidence interval. 4. Replicating the sample does NOT decrease the width of the confidence interval. Let the sample observations for R_1 be $x_1, x_2, \ldots, x_{100}$. $\overline{X_2} = \dfrac{\sum_{i=1}^{100} 2x_i}{200} = \overline{X_1}$ similarly $s_1^2 = s_2^2$ so $\overline{X_2} \sim N(\mu,\ 0.1^2)$.
Specific behaviours
✓ states that the R_2 data will no longer be a random sample ✓ states that the assumptions for using the normal distribution for the sample mean does not hold anymore i.e. states any two of the four points outlined in the solution.

2022 Section 2 Question 14 **Statistical inference**	The annual incomes (in thousands of dollars) of a random sample of n Australians is taken. The sample standard deviation is 10.98. A 99% confidence interval I_1 based on this sample is $90 \le \mu \le 94$. (a) Calculate the value of the sample size n. (2 marks)

Solution
For 99% confidence using $k = 2.576$ $2 = (2.576)\left(\dfrac{10.98}{\sqrt{n}}\right)$ Solving gives $n = 200.0029....$ i.e. $n = 200$ OR $n = \left(\dfrac{k \times s}{w}\right)^2 = \left(\dfrac{2.576 \times 10.98}{2}\right)^2 = 200.0029...$ Note: Using $k = 2.58$ gives $n = 200.6245....$ i.e. $n = 201$
Specific behaviours
✓ uses the correct half-width of the interval to correctly form the equation for n ✓ determines the sample size as an integer

Another random sample of size n is taken and a 99% confidence interval I_2 is calculated.

(b) State **two** aspects in which the intervals I_1 and I_2 may be different. (2 marks)

Solution
1. The intervals may have different midpoints, as the sample means from the two sample may be different. 2. The intervals may have different widths, as the sample standard deviations may be different.
Specific behaviours
✓ states the interval midpoints may be different due to the different sample means ✓ states the interval widths may be different due to the different sample standard deviations

A third random sample of size 50 is taken and a 99% confidence interval I_3 is calculated. James suggests that since interval I_3 is the widest, it is more likely to contain the population mean Australian income μ .

(c) Is James correct? Justify your answer. (2 marks)

Solution
No James is NOT correct. The interval is wider because it is based on data with more variation in it. Consequently, the sample mean has more variation. That is, the location of the interval is more variable. We cannot be sure that any confidence interval contains the true mean.
Specific behaviours
✓ states that James is NOT correct ✓ provides correct justification (we cannot be certain that any confidence interval contains the mean)

2020 Section 2 Question 18

Statistical inference

The mass of chocolate that is placed into each biscuit produced by the BikkiesAreUs company has been observed to be normally distributed with mean $\mu = 7.5$ grams and standard deviation $\sigma = 1.5$ grams.

(a) Determine the probability, correct to 0.01, that the total amount of chocolate used for 50 biscuits is less than 365 grams. (4 marks)

Solution
Let $\overline{M}$ = the sample mean for the mass of chocolate per biscuit for 50 biscuits (g) $= N\left(7.5,\ \sigma^2_{\overline{M}}\right)$ where $\sigma_{\overline{W}} = \dfrac{1.5}{\sqrt{50}} = 0.21213\ldots$ For a total of 365 g, the sample mean $\overline{M} = \dfrac{365}{50} = 7.3$ grams per biscuit Require $P\left(\overline{M} < 7.3\right) = 0.1729\ldots = 0.17$
Specific behaviours
✓ states that the sample mean is a normal random variable ✓ states the correct parameters for the normal random variable ✓ calculates the sample mean correctly for the total 365 grams ✓ determines the correct probability (to 0.01)

(b) If the probability that the mean amount of chocolate used per biscuit differs from μ by less than 0.2 grams is 98%, determine n, the number of biscuits that need to be sampled. (3 marks)

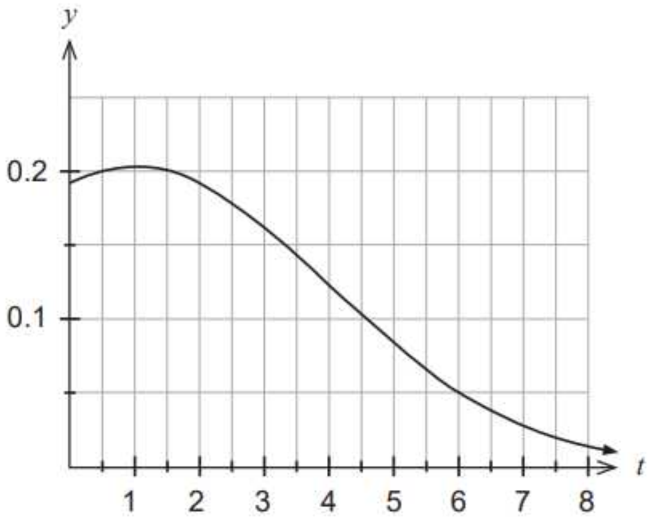
Solution
$\sigma_{\overline{M}} = \frac{1.5}{\sqrt{n}}$ We require $P(-k < z < k) = 0.98$ this gives $k = 2.326$
Hence $2.326\left(\frac{1.5}{\sqrt{n}}\right) < 0.2$ Solving gives $n > 304.32$
i.e. we require at least 305 biscuits to have the sample mean differ by less than 0.2 grams
Specific behaviours
✓ uses the standard z score that represents 98% confidence ✓ forms the correct inequality/equation to solve for n ✓ states the correct minimum integer value for n

A competitor company called YouBeautChokkies produces similar biscuits. A sample of 144 biscuits was taken and it was found that the standard deviation of the mass of chocolate used in each biscuit was 1.8 grams and the total amount of chocolate used in the sample of 144 biscuits was 1.09 kg.

Charlie Chokka, a representative from the YouBeautChokkies company, stated that “we are using significantly more chocolate for each biscuit than BikkiesAreUs. If you want that real chocolate taste, then buy from us!”

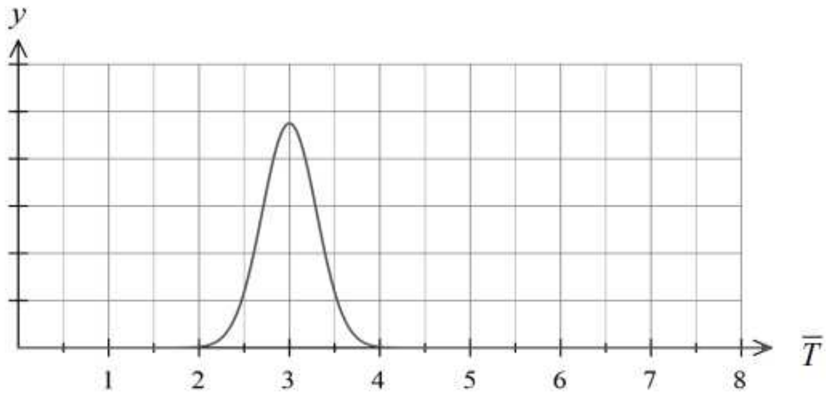
(c) Perform the necessary calculations to comment on Charlie’s claim. (4 marks)

Solution
Let μ_Y = the population mean for the mass of chocolate per biscuit for the YouBeautChokkies company (grams) For the YouBeautChokkies total of 1090 grams, this gives $\overline{M} = 7.56944\ldots$ grams
The distribution for $\overline{M} \sim N\left(7.56944,\ \sigma^2_{\overline{M}}\right)$ where $\sigma_{\overline{M}} = \frac{1.8}{\sqrt{144}} = 0.15$
Confidence Interval for μ_Y 95% level : $7.56944 - 1.96\left(\sigma_{\overline{M}}\right) < \mu_Y < 7.56944 + 1.96\left(\sigma_{\overline{M}}\right)$ i.e. $7.2754 < \mu_Y < 7.8634$
Confidence Interval for μ_Y 99% level : $7.56944 - 2.576\left(\sigma_{\overline{M}}\right) < \mu_Y < 7.56944 + 2.576\left(\sigma_{\overline{M}}\right)$ $7.1830 < \mu_Y < 7.9558$
The BikkiesAreUs population mean $\mu = 7.5$ is WITHIN the confidence interval using $\overline{M} = 7.56944$ and $\sigma = 1.8$. i.e. the claim is NOT vindicated.
i.e. the YouBeautChokkies company are NOT using significantly more chocolate per biscuit than compared to BikkiesAreUs.
Specific behaviours
✓ determines the expected variation using $n = 144$ ✓ determines an appropriate confidence interval for the YouBeautChokkies population mean ✓ states that the BikkiesAreUs population mean 7.5 is within the confidence interval ✓ concludes correctly by writing a comment about the claim

2021 Section 2 Question 15 Statistical inference	An experiment was conducted to measure how quickly adults respond to the request: 'send me a text message'. Let T = the number of hours taken for an adult to respond and send a text message. It was found that the distribution of the population of response times for adults was given by the probability density function shown below, with mean μ = 3 hours and standard deviation σ = 2.4 hours.  Random samples of size 64 are drawn repeatedly from the population of response times and the sample mean response time $\bar{T}$ is determined for each sample. (a) Calculate, correct to 0.001, the probability that a sample mean response time will be between 150 minutes and 210 minutes. (3 marks)

Solution
Since $n = 64$ (significantly large), then $\bar{T} \sim N\left(3,\ \sigma_{\bar{T}}^{\,2}\right)$ where $\sigma_{\bar{T}} = \dfrac{2.4}{\sqrt{64}} = 0.3$ Hence $P\left(2.5 < \bar{T} < 3.5\right) = 0.904$ (3 d.p.)
Specific behaviours
✓ indicates or states that the sample mean is normally distributed ✓ states the parameters of the sample mean distribution ✓ determines the probability correctly to 0.001

(b) Sketch the likely distribution of the sample mean $\overline{T}$ (for samples of size 64) on the axes below. (2 marks)



Solution
Shown above.
Specific behaviours
✓ indicates symmetry about $\overline{T} = 3$ hrs ✓ indicates a standard deviation approx 0.3 hrs (very low density values for $\overline{T} > 4$ and $\overline{T} < 2$)

Anika, a teacher at the TekNoCrat School, theorises that as teenagers tend to check their text messages more frequently than adults, then the population mean response time for teenagers will be much lower than the population mean adult response time $\mu = 3$.

Anika is then presented with the sample mean response time for a sample gathered from an unknown source.

Sample size	Sample mean (hours)	Sample standard deviation (hours)
100	2.1	2.7

Calculations are performed and Anika concludes by stating: ‘this sample was clearly not taken from the population of adult response times. It is highly likely that this sample was taken from a sample of 100 teenagers’.

(c) Perform the necessary calculations and comment on Anika’s claim. (4 marks)

Solution
Let μ_T = the population mean for the response times of the unknown source (hrs) For the sample taken $\overline{T} = 2.1$ hrs and $s = 2.7$ hrs. The distribution for $\overline{T} \sim N\left(2.1,\ \sigma_{\overline{T}}^2\right)$ where $\sigma_{\overline{T}} = \dfrac{2.7}{\sqrt{100}} = 0.27$ 95% Confidence Interval for μ_T : $2.1 - 1.96(0.27) < \mu_T < 2.1 + 1.96(0.27)$ i.e. $1.5708 < \mu_T < 2.6292$ 99% Confidence Interval for μ_T :

$$2.1 - 2.576(0.27) < \mu_T < 2.1 + 2.576(0.27)$$

$$1.40448 < \mu_T < 2.79552$$

The ADULT population mean $\mu = 3$ is NOT WITHIN either confidence interval using $\overline{T} = 2.1$ and $s = 2.7$.

Hence the population from which the sample was drawn has a mean LOWER than the population mean for adult response times. Hence the unknown source MAY have been drawn from a different population BUT we do NOT know it was drawn from a group of teenagers.

Hence Anika's claim CANNOT be accepted.

Specific behaviours
✓ determines the expected variation using $n = 100$
✓ determines an appropriate confidence interval for μ of the unknown source
✓ states that the confidence interval does NOT include the value $\mu = 3$
✓ justifies correctly that Anika's claim cannot be accepted (we do not know the unknown source was a group of teenagers)

2021 Section 2 Question 17

Statistical inference

A researcher is interested in estimating the population mean μ (dollars) that Perth residents had spent via online shopping in December 2020. A random sample of size n gave a sample mean of \$400, a sample standard deviation s and a 95% confidence interval of width \$200.

(a) State the 95% confidence interval obtained. (1 mark)

Solution
95% CI: $400 - \frac{200}{2} \le \mu \le 400 + \frac{200}{2}$ i.e. $300 \le \mu \le 500$
Specific behaviours
✓ states the upper and lower limits of the interval correctly

(b) Calculate the standard deviation of the sample mean, correct to \$0.01. (2 marks)

Solution
Margin of error $= 100$ i.e. $100 = 1.96 \times \sigma(\overline{X})$ $\therefore$ $\sigma(\overline{X}) = 51.02$
Specific behaviours
✓ forms the equation relating the margin of error and the standard deviation correctly ✓ determines the standard deviation correctly to 0.01

(c) In terms of n, what sample size would yield a 95% confidence interval of width \$50? Show your reasoning. (2 marks)

Solution
The interval width is reduced by a factor of 4, so the sample size needs to increase by a factor of $4^2 = 16$ i.e. a sample size of $16n$ is required.
OR As $100 = 1.96 \times \frac{s}{\sqrt{n}}$ i.e. $s = 51.02\sqrt{n}$
Then $\frac{100}{4} = 1.96 \times \frac{s}{4\sqrt{n}}$ i.e. $25 = 1.96 \times \frac{s}{\sqrt{16n}}$
Hence NEW sample size is $16n$.
Specific behaviours
✓ uses an interval width equal to one-quarter the original
✓ states the new sample size in terms of n

(d) What is the probability that another sample of size $2n$ would produce a sample mean that differs from μ by more than \$50? (3 marks)

Solution
$\overline{X} \sim N\left(\mu, \frac{s^2}{2n}\right)$ i.e. $\overline{X} \sim N\left(\mu, \frac{51.02^2}{2}\right)$ $\therefore$ $\sigma(\overline{X}) = 36.0768...$
Require $P(\lvert\overline{X} - \mu\rvert > 50) = P\left(\lvert z\rvert > \frac{50}{36.0768}\right) = P(\lvert z\rvert > 1.3859)$
$= 2(0.083)$
$= 0.166$
Specific behaviours
✓ determines the standard deviation for the sample size $2n$ correctly
✓ forms the correct probability statement
✓ calculates the correct probability

Four different confidence intervals (A, B, C and D) are obtained for the mean amount spent via online shopping by Perth residents in December 2020.

Confidence interval	Sample size	Sample standard deviation	Confidence level
A	n	s	95%
B	n	s	99%
C	$2n$	s	95%
D	n	$0.8s$	95%

(e) Which of the confidence intervals (A, B, C or D) contains μ, the population mean expenditure for online shopping in December 2020? Justify your answer. (2 marks)

Solution
Since the true value of μ is unknown, we CANNOT determine which interval contains the true mean. This is due to the inherent nature of random sampling.
Specific behaviours
✓ states we cannot determine which interval contains μ
✓ states that either μ is unknown OR refers to the nature of random sampling

	(f) For each of the following, state the confidence interval that has the smaller width. Justify your answers. (i) A and B. (1 mark)

Solution
Confidence interval A will have the smaller width since the level of confidence 95% is less than that of B 99%.
Specific behaviours
✓ justifies why A will have the smaller width

(ii) C and D. (1 mark)

Solution
Need to compare the standard deviation of the sample means: $C: \quad \sigma(\overline{X}) = \frac{s}{\sqrt{2n}} = 0.707\left(\frac{s}{\sqrt{n}}\right)$ $D: \quad \sigma(\overline{X}) = \frac{0.8s}{\sqrt{n}} = 0.8\left(\frac{s}{\sqrt{n}}\right)$ Hence confidence interval C will have the smaller width.
Specific behaviours
✓ justifies why C will have the smaller width by correctly comparing the respective standard deviations of the sample mean

2020
Section 2
Question 17

Statistical inference

Members of a random sample of n shoppers at the El Cheepo shopping centre were asked by a consumer researcher how much they had spent in the shopping centre that day. Let μ denote the mean and σ the standard deviation of the amount spent. The standard deviation σ is known from previous research.

A 95% confidence interval for μ based on the sample is $150 \le \mu \le 200$ dollars.

(a) Determine the sample mean for this sample. (1 mark)

Solution
Sample mean $\overline{X} = \frac{150+200}{2} = 175$ Sample mean was $175.
Specific behaviours
✓ calculates the sample mean correctly

(b) Based on this confidence interval, calculate the standard deviation of the sample mean, correct to 0.01. (3 marks)

Solution
Sample mean $\overline{X} = 175$ $\quad \therefore \quad 25 = 1.96 \times \sigma(\overline{X})$ $\therefore \quad \sigma(\overline{X}) = 12.7551...$ i.e. Standard deviation of the sample mean was $12.76 (2 d.p.)
Specific behaviours
✓ determines the critical z score for 95% confidence ✓ forms the equation relating the half-width of the interval and the standard deviation ✓ calculates the standard deviation of the sample mean correct to 0.01

	The following week, the researcher again took a random sample of shoppers from the El Cheepo shopping centre, but this time the sample size was doubled. (c) What is the probability that the difference between μ and the sample mean from this sample will be less than \$10? (4 marks)

Solution

We have that $\frac{\sigma}{\sqrt{n}} = 12.7551$ i.e. $\sigma = 12.7551\sqrt{n}$

For the larger sample $\sigma\left(\overline{X}\right) = \frac{12.7551\sqrt{n}}{\sqrt{2n}} = 9.0192\ldots$

Require $P\left(\left|\overline{X}-\mu\right|<10\right) = 2 \times P\left(0<z<\frac{10}{9.0192}\right) = 2(0.3665\ldots)$

$= 0.73$

Specific behaviours

- ✓ relates the standard deviations for the sample mean between the 2 samples
- ✓ determines the standard deviation of the sample mean for the sample size $2n$
- ✓ forms the correct probability expression
- ✓ calculates the correct probability

2019 Section 2 Question 14 **Statistical inference**	Trucks carrying iron ore for the Croc Rock mining company arrive at a weighing station. The service time T per truck is defined to be the time elapsed from the moment a truck enters the station zone, including the time to be positioned and then weighed, up to the time it leaves the zone. It is known that the population mean $\mu(T) = 80$ seconds and the population standard deviation $\sigma(T) = 20$ seconds. At the Croc Rock weighing station, 100 trucks are weighed. (a) State the (approximate) distribution of the sample mean service time per truck for the 100 trucks. (3 marks)

Solution

$\overline{T}$ is approximately normally distributed as the sample size $n = 100 > 30$

$\overline{T} \sim N\left(80, \frac{20^2}{100}\right) = N(80,\ 4)$ *i.e.* $\sigma\left(\overline{T}\right) = \sqrt{4} = 2$

Specific behaviours

- ✓ states the sample mean is normally distributed (or refers to the Central Limit Theorem)
- ✓ states the correct mean
- ✓ states the correct standard deviation (or variance)

(b) What is the probability that the sample mean service time will be more than 83 seconds? (2 marks)

Solution
$P(\overline{T} > 83) = P\left(z > \frac{83-80}{2}\right) = P(z > 1.5)$ $= 0.067$
Specific behaviours
✓ calculates/uses the correct z scores   ✓ determines the correct probability

Suppose that more than 100 trucks were weighed at the Croc Rock weighing station.

(c) How would this affect your answer to part (b)? Explain without recalculation. (2 marks)

Solution
Since for $n > 100$ would result in $\sigma(\overline{T}) < 2$, then the mean time of 83 minutes would be a more of an extreme sample mean in the normal distribution.   Hence there would be a lower probability that $P(\overline{T} > 83)$.   i.e. the answer to part (b) would be LOWER.
Specific behaviours
✓ States that the answer would decrease   ✓ Justifies by reference to the lower standard deviation of the sample mean OR that 83 minutes becomes a relatively more extreme score

It is desired that the probability that the sample mean service time will be between 80 seconds and 82 seconds is greater than 40%.

(d) Determine the minimum number of trucks that will need to be weighed. (3 marks)

Solution
Require $P(80 < \overline{T} < 82) > 0.4$ i.e. $P(0 < z < k) > 0.4 \quad \therefore \quad k > 1.282$ Hence $\frac{82-80}{\frac{20}{\sqrt{n}}} > 1.282$ Solving gives $n > 164.35$ $\therefore$ At least 165 trucks are required to be weighed.
Specific behaviours
✓ calculates the critical z score for 82 minutes   ✓ forms the inequality or equation to solve for the number of trucks   ✓ states the minimum number of trucks to be weighed as an integer

2019 Section 2 Question 15 Statistical inference	A random sample of n commuters in Melbourne in August 2018 found that the average time to commute to work was 40 minutes. Repeated sampling of the mean indicated that the standard deviation of the sample mean was 3 minutes. (a) Determine a 90% confidence interval for the population mean commuting time μ to work, correct to 0.01 minutes. (3 marks)

Solution
90% confidence interval for μ : $P(-k < z < k) = 0.9$ yields $k = 1.645$ $40 - 1.645(3) < \mu < 40 + 1.645(3)$ i.e. $35.06 < \mu < 44.94$ minutes
Specific behaviours
✓ uses the critical z score for 90% confidence ✓ forms the correct expression for the confidence interval limits (using $\sigma(\overline{X}) = 3$) ✓ determines the upper and lower limit correctly (no penalty for incorrect rounding)

Another random sample of $2n$ commuters in November 2018 found that the average time to commute to work was 45 minutes. Assume that both the August and November samples were drawn from the same population.

(b) What is the standard deviation of the sample mean for the November sample, correct to 0.01 minutes? (2 marks)

Solution
Let $\sigma =$ the population standard deviation for the commuting time (minutes) From August sample we have $\frac{\sigma}{\sqrt{n}} = 3$ i.e. $\sigma = 3\sqrt{n}$ For November sample we require : $\sigma(\overline{X}) = \frac{\sigma}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\left(\frac{\sigma}{\sqrt{n}}\right)$ $= \frac{1}{\sqrt{2}}(3) = 2.12$ min (2 d.p.)
Specific behaviours
✓ forms the correct relationship between the standard deviations OR states that $\sigma = 3\sqrt{n}$ (or its equivalent) ✓ determines the standard deviation of sample mean correct to 0.01 minutes

Suppose that the August and November samples are combined to form a sample with $3n$ commuters. Consider 90% confidence intervals for the following samples for the purpose of determining the population mean commuting time μ.

90% confidence interval	Sample	Size
A	August	n
N	November	$2n$
C	Combined	$3n$

	(c) Which of the three confidence intervals, A, N or C, will provide the greatest precision in determining the population mean μ? Justify your answer. (2 marks) **Solution** C is the most precise, as it is based on the smallest standard error (standard deviation of the sample mean) **Specific behaviours** ✓ states that C will provide the greatest precision ✓ provides a valid reason (smallest standard deviation in the sample mean or largest sample size) (d) Which of the three confidence intervals, A, N or C, contains the true value of the population mean μ? Justify your answer. (2 marks) **Solution** We do not know which interval contains the population mean μ. This is because we do not know the true value of μ OR that by random sampling we cannot be certain which interval contains μ. **Specific behaviours** ✓ states that we do not know which interval contains the population mean ✓ provides a valid reason (either μ is not known OR that by random sampling we cannot be certain which interval contains μ)