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Chapter 1 – Differentiation

Introduction

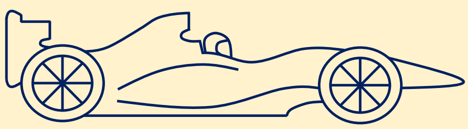
Stepping into the world of Year 12 Mathematics Methods can feel daunting. Exploring **both calculus** and **probability**, Year 12 Methods has two challenging areas of mathematics. However, these are some of the **most important concepts** to understand for real world applications and further study, which is why we dedicate this year to understanding their many intricacies.

This year we want to take you on a **different kind of journey**. Instead of stating the rules of mathematics and expecting you to apply them to problems, we want to show you **how** the concepts and rules were **derived**, **why** they are useful and the **steps** for **applying** them. We hope this gives you a **greater appreciation** for the important role mathematics plays in everyday life.

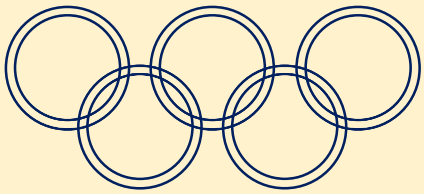
Our goal is to have you **feeling** like you could have **discovered** these concepts **yourself**; that you could have stood in the shoes of the great mathematicians that have come before you and used your understanding to develop some of the most important theorems and concepts to date.

With a strong understanding of **calculus** and **probability**, you will find yourself able to **apply** these concepts to many situations around you.

From establishing the relationship between the **displacement, velocity** and **acceleration** of a **Formula One car** to determining the **probabilities** of **competitors winning** at the **Olympic games**, the applications of the knowledge you will gain are truly endless!



Differentiation of Formula One Car Motion



Binomial Distributions at the Olympics

We begin this great journey with our first major topic: **calculus**.

Calculus explores how the **change** in functions can be measured, and is divided into two key areas: **differentiation** and **integration**. This first chapter on **differentiation** will cover **4 main concepts**:

1. **First Principles and the Power Rule**
2. **Product Rule, Chain Rule and Quotient Rule**
3. **Second Derivative and the Second Derivative Test**
4. **Applications of Differentiation (e.g. Curve Sketching, Small Change and Optimisation)**

Let’s Begin!

1.1 Functions

In mathematics, our goal is to **define quantifiable relationships** between different observations or measurements. Each observation or measurement is called a **variable**, which is a **symbol** (e.g. x or y) used to **express** a **measurement/parameter/property** that may **change**.

For instance, x can be used to represent the **number of apples** in a basket, the **speed** at which **galaxies** are **moving apart**, or **anything else** where the value may change or stay constant.

To mathematically **represent** the **relationship** that **one variable** shares with **another,** we create **functions**.

> A **function** is a mathematical representation of the **relationship** between **two or more variables** such as y and x in: $y = 2x + 5$ or $y = x^3 + 3x^2 + 2x + 5$.

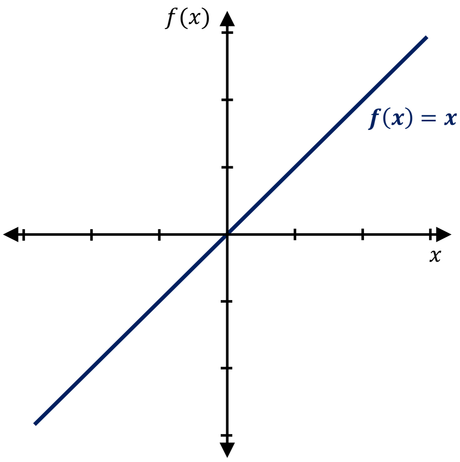
In Year 12 Maths Methods, the **four types** of functions we will explore are **polynomial**, **trigonometric**, **exponential** and **logarithmic functions**. Each function is used to create **relationships** to **describe** the **behaviour of variables** in different situations.

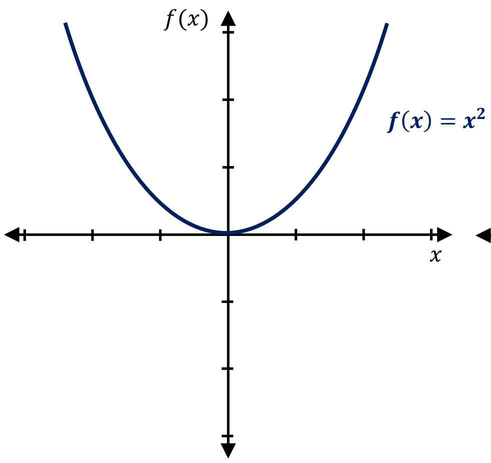
As we learn the **fundamentals** of **differentiation** and **integration** in **chapters one** and **two**, we will **focus solely** on **polynomial functions** for the moment and come back to trigonometric, exponential and logarithmic functions later.

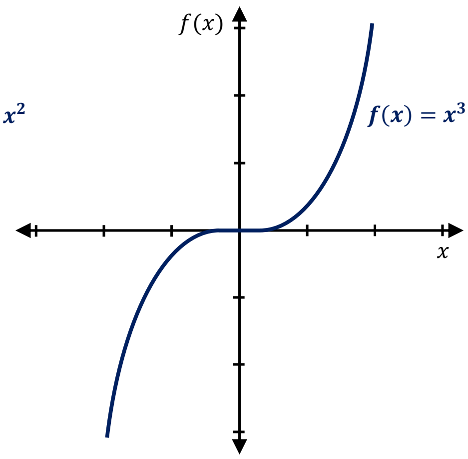
Polynomials

> **Polynomial functions** represent relationships where x is raised to some **positive integer n**. They have the general formula ax^n.

As shown below, **linear (x)**, **quadratic (x^2)** and **cubic (x^3)** functions are all examples of polynomials.

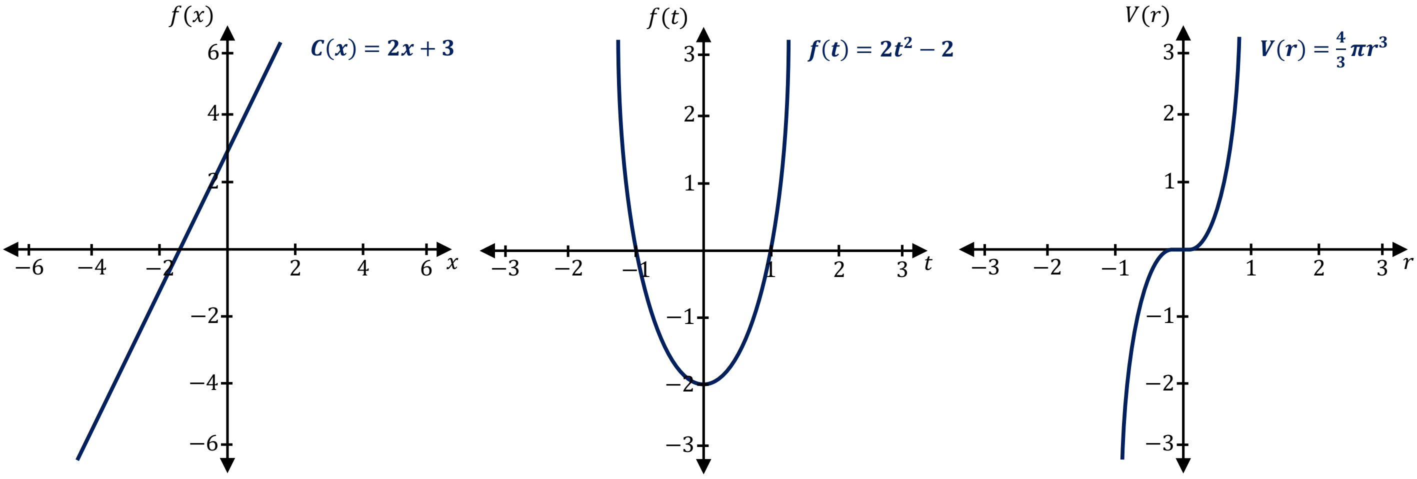






Each of these polynomial functions can be **transformed** to have **gradients** that are **steeper** or **flatter**, or to **shift** them **horizontally** or **vertically**. This is done by **changing** their **coefficients** (i.e. the **numbers in-front** of the variables) or by **adding/removing constants** to the form of the function without destroying the **general formula**. By making these changes, we are able to **create** the polynomials required to **describe** the behaviour of variables in different situations.

For instance, the linear function $\boldsymbol{C(x) = 2x + 3}$ could be used to represent the **cost** $\boldsymbol{(C(x))}$ to transport a **specific number of apples** $\boldsymbol{(x)}$. The **quadratic** $\boldsymbol{f(t) = 2t^2 - 2}$ could be used to represent the **velocity** of a **toy car**. Finally, the cubic function $\boldsymbol{V(r) = \frac{4}{3}\pi r^3}$ could be used to represent the **volume** of a **gym ball** $\boldsymbol{(V(r))}$ as its **radius increases** $\boldsymbol{(r)}$ when air is pumped into the ball. Graphically, these functions would be represented as:



These are just a few examples of the **polynomials** that we will be looking at throughout this year. You will learn how to **apply** these equations to **solve problems** such as **optimising volume**, **tracking displacement**, **velocity** and **acceleration** of objects, and many other **unique situations**!

1.2 Differentiation

Differentiation is the first of the two broad concepts that make up calculus.

> **Differentiation** is the process of determining the **derivative** of a function, which is a **measure** of the **instantaneous rate of change** – also known as the **gradient** of a function.

Like many other mathematical concepts, **differentiation** follows **set rules**. The **rules** for differentiation can be **derived** using **first principles**.

> **First principles** is the process of simplifying a process down to its **fundamental parts**, and **generalising** from there.

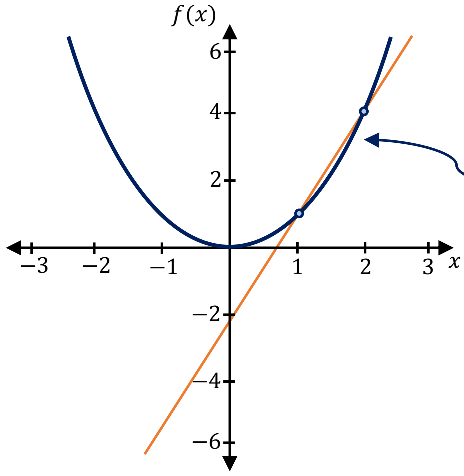
First principles is applied by starting with the **key component(s)** of our definition for 'derivative'. In our case, the **key component** of the derivative definition is **'instantaneous rate of change'**.

Now, let's break this down further. The term **'instantaneous'** means **at a given point**, and the **'rate of change'** of a function is also known as its **gradient**. You might remember the formula for **gradient** is:

$$\boldsymbol{Gradient = \frac{rise}{run} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

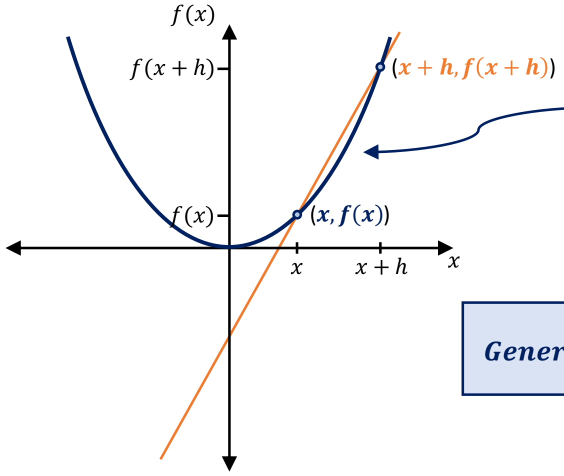
Determining the gradient is a process we are all familiar with from Year 11. Let's walk through it again with the function $f(x) = x^2$.

Suppose we wanted to determine the **gradient** between $x_1 = 1$ and $x_2 = 2$. **Substituting** in the values $x_1 = 1$ and $x_2 = 2$ into $f(x) = x^2$ we determine that $f(x_1) = 1$ and $f(x_2) = 4$. Using **rise over run**, we can calculate the **gradient** between these points to be 3.



$$\begin{aligned} Gradient &= \frac{rise}{run} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{4-1}{2-1} \\ &= 3 \end{aligned}$$

For all functions, a generalisation can be created for the gradient function by using **two general points**: $x_1 = x$ and $x_2 = x + h$. This means that $f(x_1)$ and $f(x_2)$ will **equal** $f(x_1) = f(x)$ and $f(x_2) = f(x+h)$, so **substituting** this into our gradient formula will give us the **generalisation** for the **gradient** of **any two points**.



$$\begin{aligned} Gradient &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{f(x+h) - f(x)}{(x+h) - x} \\ &= \frac{f(x+h) - f(x)}{h} \end{aligned}$$

$$\boxed{General\ Gradient\ Formula = \frac{f(x+h) - f(x)}{h}}$$

While we have calculated the **gradient** between $x = 1$ and $x = 2$ or $x_1 = x$ and $x_2 = x + h$, these both represent the **average rate of change** between these **two points** – not the **instantaneous rate of change** at a **single point** that we are looking for.

To find the **gradient** of a **single point**, we need to reduce the **gap** between x_1 and x_2. This gap is equal to h. We want to make this gap **as close** to **zero as possible**, but **not equal to zero**. The **gap cannot be equal to zero** because then the **denominator** of the gradient function $(x_2 - x_1)$ would **become zero**, which would make the function **undefined**.

$$\therefore \boldsymbol{Gradient} = \frac{\boldsymbol{f(x+h) - f(x)}}{\boldsymbol{h}}$$

Get as close to zero as possible

As we can see above, the **gap** between $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ is determined by the **size** of $\boldsymbol{h}$. Therefore, we need to make $\boldsymbol{h}$ as close to **zero** as possible. Mathematically, we achieve this using **limits**.

> A **limit** is the **value** that a **function** approaches as the **input** approaches some value (e.g. $\boldsymbol{\lim_{h \to 0}}$).

Applying a **limit** of $\boldsymbol{h}$ **approaching zero** to our **generic gradient formula** will give the **instantaneous rate of change** of the function:

> **Instantaneous Rate of Change:** $\dfrac{\boldsymbol{dy}}{\boldsymbol{dx}} = \lim\limits_{h \to 0} \dfrac{\boldsymbol{f(x+h) - f(x)}}{\boldsymbol{h}}$

So, for our function $\boldsymbol{f(x) = x^2}$, if we **substitute** this into the **generic gradient formula** and **apply** the **limit** $\boldsymbol{\lim_{h \to 0}}$, we will get its **derivative**:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

1 **Substitute** $\boldsymbol{f(x) = x^2}$ into the **equation**

2 $$\frac{dy}{dx} = \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

2 **Expand and simplify** the **equation**

3 $$\frac{dy}{dx} = \lim_{h \to 0} 2x + \cancel{h}^{\,0}$$

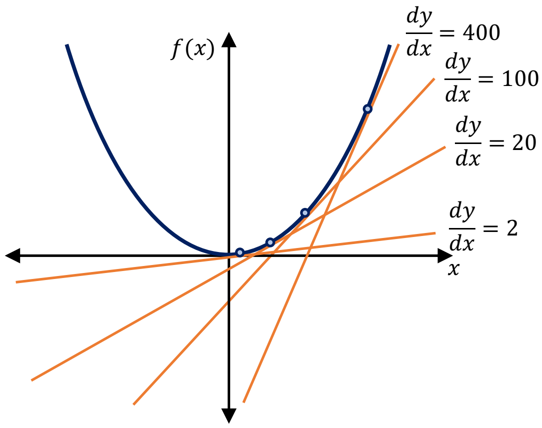
3 **Apply** the $\boldsymbol{\lim_{h \to 0}}$ to make **all** $\boldsymbol{h}$ **values equal to zero**

4 $$\boldsymbol{\frac{dy}{dx} = 2x}$$

4 **Simplify** to the **final derivative**

This derivative equation $\boldsymbol{\frac{dy}{dx} = 2x}$, represents the **instantaneous gradient** at **any point** $\boldsymbol{(x)}$ along the **function** $\boldsymbol{y = x^2}$, and can be used to determine gradient values by **substituting** in values of $\boldsymbol{x}$.

For instance, at $\boldsymbol{x = 1}$ the gradient is $\boldsymbol{2}$, at $\boldsymbol{x = 2.5}$ the gradient is $\boldsymbol{5}$ and at $\boldsymbol{x = 200}$ the gradient is $\boldsymbol{400}$. To illustrate this, we can draw a **tangent line** at each point with a gradient based on $\boldsymbol{\frac{dy}{dx} = 2x}$.



Power Rule

Determining derivatives by first principles is a **slow process**. Instead, we often **differentiate** using **shortcut rules**.

For **polynomials** of the form $\boldsymbol{x^n}$, the shortcut we use is the **power rule**.

Power Rule: If $y = ax^n$ then $\frac{dy}{dx} = \boldsymbol{anx^{n-1}}$

The power rule says that for a **polynomial** $\boldsymbol{ax^n}$, the **derivative** can be determined by **multiplying** the **function** by the **exponent**, and then **subtracting one** from the **exponent**.

The best way to understand the **power rule** is to **apply** it to some polynomials.

Let's start with the function $\boldsymbol{y = x^4}$. Instead of using first principles to determine the derivative, we can apply the **power rule** by **bringing down** the **exponent** $\boldsymbol{4}$ and **multiplying** it with the function, and then **subtracting one** from the exponent. This gives us the **derivative** $\boldsymbol{4x^3}$.

$$y = x^4$$
$$\frac{dy}{dx} = 4x^{4-1}$$
$$\frac{dy}{dx} = 4x^3$$

For a slightly harder example, let's try $\boldsymbol{f(x) = \frac{7x^6}{3}}$. In this case we would **bring down** the **exponent 6** and **multiply** it with the $\boldsymbol{7}$, and then **subtract one** from the exponent to arrive at:

$$f(x) = \frac{7x^6}{3}$$
$$f'(x) = \frac{6 \cdot 7x^{6-1}}{3}$$
$$f'(x) = 14x^5$$

The **power rule** also applies to polynomials with **exponents** that are **fractions** or **negative numbers**. For instance, the differentiation of $\boldsymbol{y = x^{\frac{3}{2}}}$ and $\boldsymbol{y = x^{-3}}$ both follow the same process of **bringing down** the **exponent** $\boldsymbol{\frac{3}{2}}$ **and** $\boldsymbol{-3}$ **respectively** and **multiplying** it by the function, and then **subtracting one** from each exponent:

$$y = x^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{\frac{3}{2}-1}$$
$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{3}{2}-\frac{2}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$$

$$y = x^{-3}$$
$$\frac{dy}{dx} = (-3) \cdot x^{-3-1}$$
$$\frac{dy}{dx} = -3x^{-4}$$
$$\frac{dy}{dx} = -\frac{3}{x^4}$$

A final question we may ask is – what do we do if our polynomial has **constants**? Remember that the value of a constant **never changes**. As the derivative measures the **rate of change**, the **derivative** of a **constant** will therefore **always** be **zero**.

For example, for the function $\boldsymbol{y = x^3 + 4}$, the derivative of $\boldsymbol{x^3}$ will be $\boldsymbol{3x^2}$ and the **derivative** of $\boldsymbol{4}$ will be $\boldsymbol{0}$.

$$y = x^3 + 4$$
$$\frac{dy}{dx} = \boldsymbol{3}x^{3-1} + 0$$
$$\boldsymbol{\frac{dy}{dx} = 3x^2}$$

The **power rule** is just **one** of the **many rules** we are going to explore during our journey through calculus. In the section that follows, we are going to see how the **power rule** can be **applied** along with **other rules** that deal with **more complicated polynomials**.

Worked Example 1

Being forgetful as always, student Tom has forgotten to do his derivatives homework for Teacher Andrew's class! **Help** Tom to **determine** the **derivatives** of the following **functions** and avoid getting a detention!

(a) $y = 2$

$$\boldsymbol{\frac{dy}{dx} = 0}$$

(b) $y = 3x^2 + 9x$

$$\frac{dy}{dx} = \boldsymbol{2}(3)x^{2-1} + 9x^{1-1}$$
$$\boldsymbol{\frac{dy}{dx} = 6x + 9}$$

(c) $f(x) = 3x^{\frac{3}{2}}$

$$\boldsymbol{f'(x) = \frac{3}{2} \times 3}x^{\frac{3}{2}-1}$$
$$\boldsymbol{f'(x) = \frac{9}{2}\sqrt{x}}$$

(d) $y = 3x^{-2} + 2x^2 + 3$

$$\frac{dy}{dx} = \boldsymbol{-2(3)x^{-2-1} + 4x}$$
$$\boldsymbol{\frac{dy}{dx} = -\frac{6}{x^3} + 4x}$$

1.3 Product Rule, Quotient Rule and Chain Rule

The power rule serves as a fantastic starting point for differentiating polynomials in the form of $\boldsymbol{ax^n}$.

However, for functions involving the **multiplication** or **division** of polynomials, or even for a **function within a function**, we need to utilise some **additional rules**.

This is where the **product rule**, **quotient rule** and **chain rule** become important. They provide systematic rules to differentiate **polynomials** that are **multiplied**, **divided** or **within each other respectively**. To understand their uses, we are going to look at **each rule individually**.

Product Rule: If $y = \boldsymbol{u} \times \boldsymbol{v}$ then $\frac{dy}{dx} = \boldsymbol{v\frac{du}{dx} + u\frac{dv}{dx}}$

Quotient Rule: If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain Rule: If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Product Rule

Suppose we have the function $y = (x + 5)(x + 3)$. Using only our existing knowledge, we might choose to **expand** the function and then use the **power rule** to **differentiate** it.

$$y = (x + 5)(x + 3)$$
$$y = x^2 + 8x + 15$$
$$\frac{dy}{dx} = 2x + 8$$

This expansion approach is suitable for simple functions, but it becomes **difficult** for **more complicated functions** such as: $y = (3 + x^2)(2x^2 + 7)$. In these situations it's best to apply the **product rule**.

The **product rule** states that if two functions, u and v, are **multiplied together** ($y = u \times v$), the **derivative** will be v **multiplied** by the **derivative** of u, **added** to u **multiplied** by the **derivative** of v.

Product Rule: If $y = u \times v$ then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

Applying this to our initial example: $y = (x + 5)(x + 3)$, we'll let $u = (x + 5)$ and $v = (x + 3)$. Using the **product rule**, we get:

① $u = x + 5$ $\quad v = x + 3$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 1$

① Determine **derivatives** of u and v

② $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

② **Apply** the **product rule**

③ $\frac{dy}{dx} = (x + 3)(1) + (x + 5)(1)$

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

$\frac{dy}{dx} = x + 3 + x + 5$

④ $\frac{dy}{dx} = 2x + 8$

④ **Expand** and **simplify** to the **final derivative**

Applying this to the second example: $y = (3 + x^2)(2x^2 + 7)$, let's create two functions, u and v, to give $u = (3 + x^2)$ and $v = (2x^2 + 7)$. Using the **product rule**, we get:

① $u = (3 + x^2)$ $\quad v = (2x^2 + 7)$

$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 4x$

① Determine **derivatives** of u and v

② $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

② **Apply** the **product rule**

③ $\frac{dy}{dx} = (2x^2 + 7)2x + (3 + x^2)4x$

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

$\frac{dy}{dx} = 4x^3 + 14x + 12x + 4x^3$

④ $\frac{dy}{dx} = 8x^3 + 26x$

④ **Expand** and **simplify** to the **final derivative**

Quotient Rule

Suppose we have the function, $y = \frac{x+2}{x^2+4}$. Unfortunately, we can't differentiate this using the power rule. Instead we need to use the **quotient rule**.

The **quotient rule** states that if function $\boldsymbol{u}$ is **divided** by function $\boldsymbol{v}$ (i.e. $y = \frac{u}{v}$), then the **derivative** will be $\boldsymbol{v}$ **multiplied** by the **derivative** of $\boldsymbol{u}$, **subtracted** from $\boldsymbol{u}$ **multiplied** by the **derivative** of $\boldsymbol{v}$, all **divided** by the **square** of the **denominator function** ($\boldsymbol{v^2}$).

Quotient Rule: If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Applying the **quotient rule** to the example above, $y = \frac{x+2}{x^2+4}$, we can allocate $\boldsymbol{u}$ and $\boldsymbol{v}$ to be $\boldsymbol{u = x + 2}$ and $\boldsymbol{v = x^2 + 4}$. Using the **quotient rule**, we get:

① $u = x + 2$ $\quad v = x^2 + 4$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

② $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

③ $$\frac{dy}{dx} = \frac{(x^2+4)(1) - (x+2)(2x)}{(x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 4 - 2x^2 - 4x}{(x^2+4)^2}$$

④ $$\frac{dy}{dx} = \frac{-x^2 - 4x + 4}{(x^2+4)^2}$$

① Determine **derivatives** of $\boldsymbol{u}$ and $\boldsymbol{v}$ using the **power rule**

② **Apply** the **quotient rule**

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

④ **Expand** and **simplify** to the **final derivative**

As a second example, consider the **more complicated function**, $y = \frac{4x+6}{x^3-3}$. For this case, we can have $\boldsymbol{u = 4x + 6}$ and $\boldsymbol{v = x^3 - 3}$, and then applying the **quotient rule**, we get:

`

① $u = 4x + 6$ $\quad v = x^3 - 3$

$$\frac{du}{dx} = 4 \qquad \frac{dv}{dx} = 3x^2$$

② $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

③ $$\frac{dy}{dx} = \frac{(x^3-3)(4) - (4x+6)(3x^2)}{(x^3-3)^2}$$

$$\frac{dy}{dx} = \frac{4x^3 - 12 - 12x^3 - 18x^2}{(x^3-3)^2}$$

④ $$\frac{dy}{dx} = \frac{-8x^3 - 18x^2 - 12}{(x^3-3)^2}$$

① Determine **derivatives** of $\boldsymbol{u}$ and $\boldsymbol{v}$ using the **power rule**

② **Apply** the **quotient rule**

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

④ **Expand** and **simplify** to the **final derivative**

Chain Rule

The final rule we will be exploring is the **chain rule**, which applies to **composite functions**.

Composite functions are **functions within functions** ($y(u(x))$). This means the **output** of the **first function** ($u(x)$) becomes the **input** of the **second function** ($y(u)$).

For instance, suppose we have a function $y = u^2 + 2u$, where we know $u(x) = 3x + 2$.

If we wanted to determine the **derivative** of y in terms of x (i.e. $\frac{dy}{dx}$), we **cannot** do this **directly** because y is currently in terms of u. This is where the **chain rule** can be applied.

The **chain rule** states that if function $u(x)$ is contained within y (i.e. $y = f(u)$), then the derivative will be the **derivative** of y in terms of u, **multiplied** by the **derivative** of u in terms of x.

Chain Rule: If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

The chain rule essentially creates $\frac{dy}{dx}$ because **multiplying** $\frac{dy}{du}$ by $\frac{du}{dx}$ cancels out the du, leaving $\frac{dy}{dx}$. Once they are multiplied together, the final step is to **substitute** in u in terms of x.

This probably sounds confusing now, but it will make more sense once you see it in action.

Let's apply the **chain rule** to the example above, where we want to find $\frac{dy}{dx}$ for $y = u^2 + 2u$ given we know $u(x) = 3x + 2$. To do so, we start by determining $\frac{dy}{du}$ and $\frac{du}{dx}$:

① $y = u^2 + 2u$ $u = 3x + 2$

$$\frac{dy}{du} = 2u + 2 \qquad \frac{du}{dx} = 3$$

① Determine the **derivatives** $\frac{dy}{du}$ and $\frac{du}{dx}$

With the values for $\frac{dy}{du}$ and $\frac{du}{dx}$, we can **apply** the **chain rule** by **multiplying** them together.

② $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

③ $$\frac{dy}{dx} = (2u + 2)(3)$$

$$\frac{dy}{dx} = 6u + 6$$

② **Apply** the **chain rule**

③ **Substitute** in **values** of $\frac{dy}{du}$ and $\frac{du}{dx}$

Finally, to have $\frac{dy}{dx}$ in terms of x, we **substitute** $u = 3x + 2$ into the equation:

$$\frac{dy}{dx} = 6u + 6$$

④ $$\frac{dy}{dx} = 6(3x + 2) + 6$$

$$\frac{dy}{dx} = 18x + 18$$

④ **Substitute** in **value** of u in terms of x

Sometimes, we may encounter **polynomials** in which a whole function is **raised** to **an exponent** (e.g., $y = (x+5)^4$). In these situations, the **chain rule** is one **differentiation method** we can use to save us having to expand the polynomial.

To do this, we define the **function** inside the **brackets** as u and **differentiate** it using the **chain rule**. For instance, for $y = (x+5)^4$, we can let $u = x + 5$ and **apply the chain rule** as follows.

① $y = u^4$ $\quad u = x + 5$

$$\frac{dy}{du} = 4u^3 \qquad \frac{du}{dx} = 1$$

① Determine the **derivatives** $\frac{dy}{du}$ and $\frac{du}{dx}$

② $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

② **Apply** the **chain rule**

③ $\dfrac{dy}{dx} = (4u^3)(1)$

$$\frac{dy}{dx} = 4u^3$$

③ **Substitute** in **values** of $\frac{dy}{du}$ and $\frac{du}{dx}$

④ $\dfrac{dy}{dx} = 4(x+5)^3$

④ **Substitute** in **value** of u in terms of x

Whilst applying the chain rule in this manner is valid, like using first principles, it is a **slower process** than using a **specific rule.**

Functions of the form $y = [f(x)]^n$ they can be differentiated using the following formula derived from the **chain rule**:

> **Derived from Chain Rule**: If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

What this tells us is the derivative can be determined by **bringing down** the **exponent** n and **multiplying** it with the function, **subtracting one** from the **exponent**, and finally **multiplying** with the **derivative** of the function **inside** the **brackets** ($f(x)$).

Applying this rule to a new example, let's consider the function $y = (x^2 + 3x)^{\frac{3}{2}}$. To determine the derivative, we start by **bringing down** the **exponent** $\frac{3}{2}$ and **multiplying** it with the function, then **subtracting one** from the **exponent**, and finally **multiplying** the function by the **derivative** of $x^2 + 3x$.

$$y = (x^2 + 3x)^{\frac{3}{2}}$$

① $\dfrac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

① **Apply** the **shortcut rule derived** from **chain rule**

$$\frac{dy}{dx} = \frac{3}{2}(x^2 + 3x)^{\frac{3}{2}-1} \cdot (2x+3)$$

② $\dfrac{dy}{dx} = \dfrac{3}{2}(2x+3)(x^2+3x)^{\frac{1}{2}}$

② **Simplify** to get the **final answer**

And success! You have now learnt all the fundamental differentiation rules that can be applied to polynomial functions. In chapters 3 and 4, we are going to learn how to **apply** these **differentiation techniques** to **trigonometric** and **exponential functions**.

Worked Example 2

Still not convinced that Tom has any idea as to how to do differentiation, Teacher Andrew sets him one final test. If he cannot determine **all** of the following **derivatives** successfully, he will find himself in **detention** for the **rest of the year**! Frightened by the idea of spending an entire year in detention, Tom can barely think!

(a) First Teacher Andrew wants Tom to use the **product rule** to determine the **derivative** of $y = (x^2 + 4)(4x - 6)$. Help show Tom how to **determine** the **derivative**.

$$u = x^2 + 4 \qquad v = 4x - 6$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = 4$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (4x - 6)(2x) + (x^2 + 4)(4)$$

$$\frac{dy}{dx} = 8x^2 - 12x + 4x^2 + 16$$

$$\frac{dy}{dx} = 12x^2 - 12x + 16$$

(b) Impressed by Tom, Andrew challenges Tom to use the **chain rule** to determine the **derivatives** of the following functions. **Help** Tom determine the **derivatives** of the **functions** below.

(i) Determine $\frac{dy}{dx}$ if $y = u^3 + 2$ and $u = 6x^2 + 6$

$$y = u^3 + 2 \qquad u = 6x^2 + 6$$

$$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 12x$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = (3u^2)(12x)$$

$$\frac{dy}{dx} = 3\left(6x^2 + 6\right)^2(12x)$$

$$\frac{dy}{dx} = 36x\left(6x^2 + 6\right)^2$$

(ii) Determine $\frac{dy}{dx}$ if $y = \left(2x^3 - 3\right)^4$

`

$$y = \left(2x^3 - 3\right)^4$$

$$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$\frac{dy}{dx} = 4(2x^3 - 3)^{4-1} \cdot (6x^2)$$

$$\frac{dy}{dx} = 24x^2(2x^3 - 3)^3$$

Problem Set 1 – Differentiation
Progressive Questions

Concept 1

First Principles and Power Rule – Progressive Questions

(5 questions)

Repetitive questions: 1.11 → 1.51 (4 questions)

The Daunting Beginning…

Starring: Rabea, Rupert, Tyler, Tom, Fraser, Alexa Jevon and Kerry

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"What is this place?" remarks Rabea, as he stepped off the Chemistry bus. "Why, it's Mathematics College, a place of prestige and the finest mathematicians in the world!" shouts Rupert.

The newest transfers from Chemistry College: ace-student Tyler and rival Rupert, rogue student Rabea, lucky student Tom and cheerful student Alexa absorb the uniqueness of their new school.

Opposite from them stands Mathematics College's finest: super-student Jevon and school captain Kerry. "Welcome to Mathematics' College, I hope you are as excited as we are to have you all here" says Kerry. "Our first class is with teacher Andrew! He is crazy strict, so don't do anything silly."

First Principles: Q1

Repetitive: 1.11 (1 question)

"Welcome students. My name is teacher Andrew. I operate this classroom on one rule and that rule is that if you misbehave you will find yourself in afterschool detention, and if you do anything wrong you will also find yourself in detention."

"Isn't that two rules?" asks Rabea. The classroom breaks out into laughter. "DETENTION!" screams teacher Andrew. In absolute shock, Rabea walks out of the classroom before he says anything else.

[24 marks]

1. Walking around the college in despair, Rabea walks straight into the janitor's closet without realising! Inside the closet he discovers the room quickly opens into a giant auditorium. A man down the front of the theatre stands at the board doing mathematics in a janitor uniform.

Janitor Peter turns around and notices Rabea. "My boy, you look a bit down. Come sit and do some **first principle derivatives** with me! I'm sure that will cheer you up!" Confused at the sight of the janitor doing complex math, Rabea agrees. Help the pair to determine the **derivatives** of the following functions using **first principles**.

(a) $\lim_{h\to 0} x$ **(4)**

(b) $\lim_{h\to 0} 2x+3$ **(4)**

(c) $\lim_{h\to 0} x^2$ **(4)**

(d) $\lim_{h\to 0} 5x^2$ **(4)**

(e) $\lim_{h\to 0} 4x^2+2x$ **(4)**

(f) $\lim_{h\to 0} 7x^2+11x$ **(4)**

Power Rule: Q2, Q3, Q4, Q5

Repetitive: 1.21 → 1.51 (3 questions)

[11 marks]

2. Clearly enthused by the concept of limits, Janitor Peter decides to capitalise on Rabea's excitement by teaching him about the **power rule**! "Rabea, using the **power rule**, I'd like you to determine the **derivatives** of the following polynomials." Staring at the board, Rabea is confident. Help Rabea determine the **derivatives** of each of the **functions**.

(a) $y = 2$ **(1)**

(b) $f(x) = 3x^2 + 7x$ **(1)**

(c) $y = 9x^6 - 4x^4 + 1$ **(1)**

(d) $y = -\frac{1}{x^2}$ **(2)**

(e) $f(x) = \sqrt{x} - 3$ **(3)**

(f) $f(x) = \frac{2x-3x^4}{x^3}$ **(3)**

[17 marks]

3. Amazed by Rabea's confidence, Janitor Peter asks Rabea to determine the **derivatives** of the following **more complicated polynomials**.

(a) $f(t) = 3t^5 + 2t^2 - 8t$ **(2)**

(b) $y = (x+3)^2 + x^{\frac{5}{2}}$ **(3)**

(c) $f(x) = \frac{(x-3)(x+5)}{x-3}$ **(2)**

(d) $y = \frac{2}{3x^2} - \sqrt{x^3}$ **(3)**

(e) $y = \frac{8}{x^4} + 8x^{\frac{1}{2}}$ **(3)**

(f) $f(x) = \frac{\sqrt{x}-5x^2}{4x^3}$ **(4)**

Captivated by his first maths lesson, Rabea turns to ask Janitor Peter a question. "Janitor Peter, why are you not a mathematics teacher like Teacher Andrew?"

"In time my boy, you will come to see that there is more than meets the eye about this college" hints Janitor Peter.

Exiting the auditorium, Rabea looks down at his timetable for his next class: BIG WORLD MATHEMATICS – Teacher Simon. 'Well, this should be interesting' Rabea thinks.

Walking dismally into their next math class, Teacher Simon is there to greet them with a big smile! Standing at 2.00m, with a curled moustache stretching from ear to ear, and luscious, long hair, Teacher Simon is notorious for being an enthusiastic teacher.

"Wow! Teacher Andrew seems to have given you boys and girls a rough start to the year! I'll tell you what, let's do some big world maths to brighten up your day!" says Simon. Clapping his hands, the floor around them erupts and computers line up against a wall.

[15 marks]

4. Simon quickly splits the students up into pairs, co-mingling the new Chemistry students with the Mathematics' College students. "Students, the **rules** are simple. Get a question **wrong** and **you're out**, last pair standing wins!" Split up into pairs, the students must determine the **derivatives** of the **functions** that appear on their computer screen.

(a) The first pair are lazy student **Fraser** and ace-student **Tyler**. Being casual and laid back, Fraser decides to take a nap, leaving Tyler to do the problems by himself! **Help** Tyler to determine the **derivatives** of the **following functions**.

(i) $f(x) = x^4 - 2x^8$ **(1)**

(ii) $y = \frac{5x}{2x^3}$ **(2)**

(iii) $y = \frac{1}{\sqrt{x^5}}$ **(2)**

(iv) $f(x) = 2x^{-\frac{1}{2}}$ **(3)**

(v) $y = \frac{7x^5+4x-x^2}{x^3}$ **(3)**

(vi) $f(x) = (x^2 + 3x)^2$ **(2)**

(b) After successfully completing the first set of derivatives, the computer screen pops up with the derivative below. Determine the **actual derivative** of this function and whether Tyler and Fraser will **progress** to the next stage. **[2]**

Function	Tyler's Guess	Actual derivative	Progresses? (✓ X)
$y = \frac{x^3 + x^2 - 4x}{x}$	$\frac{dy}{dx} = \frac{3x^2 + 2x - 4}{1}$		

[14 marks]

5. Next up are super student Jevon and cheerful student Alexa. Starting at a new school, Alexa is excited to make some new maths friends. "Hey Jevon, just wanted to say I am super excited to work with you!" says Alexa cheerfully. Always focused, Jevon looks weirdly at her and then returns to the problem in-front of them.

(a) **Help** Jevon and Alexa to determine the **derivatives** of the **following functions**. **[9]**

(i) $y = \frac{1}{4\sqrt{x}} + 5x^{-3}$ **(3)**

(ii) $f(x) = \frac{\sqrt{x^3}}{2x^2} - 4$ **(3)**

(iii) $y = \frac{12x - 10x^3 - \sqrt{2}x^5}{x^5}$ **(3)**

(iv) $f(x) = \left(x^2 + \sqrt{x}\right)^2$ **(3)**

(b) Confident on his first six answers, Jevon is now required to determine the derivative of the following function, **actual derivative** of this function and whether Jevon and Alexa will **progress** to the next stage. **[2]**

Function	Jevon and Wallace's Guess	Actual derivative	Progresses? (✓ X)
$y = \frac{x^2 + 2x - 8}{x + 4}$	$y' = 2 + \frac{2}{x}$		

Absolutely furious that he got a question wrong, Jevon turns to Alexa. "This is all your fault! I am never pairing with you again." Confused, Alexa simply walks off in search of some more friendly students.

Concept 2

Product Rule, Quotient Rule and Chain Rule – Progressive Questions (8 questions)

Repetitive questions: 2.11 → 2.81 (10 questions)

The Start of a Differentiation War

Starring: Teacher Simon, Alexa, Jevon Wallace, Shauna, Tom, Rabea, Kerry and Sparrow

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Struggling to take down the students, Teacher Simon decides he needs to take it up a notch! Adjusting the computer code, Simon has now programmed the computers to make the questions harder and to now include the product rule, quotient rule and chain rule. "Good luck with these questions!" says Teacher Simon.

Product Rule: Q1, Q2

Repetitive: 2.11 → 2.21 (2 questions)

[21 marks]

1. "Hey Wallace, I appreciate you being paired with me and all, but if you can just sit this one out that would be great. I don't want to get any wrong." scoffs Rupert. Looking at the problems in front of him, Rupert powers through them in record time. **Help** to show Wallace the **derivatives** to the **following functions** using the **product rule**.

(a) $y = (x+2)(x-4)$ **(3)**

(b) $y = (3x-4)(2x+1)$ **(3)**

(c) $f(x) = (-2x+2)(x+9)$ **(3)**

(d) $y = (2x-3x^2)(5x-4)$ **(4)**

(e) $y = \sqrt{x}(x^2+2)$ **(4)**

(f) $f(x) = (8x^2+3x)(x^2-4x)$ **(4)**

[24 marks]

2. Nervous as always, lucky student Tom and selfless student Shauna are worried about the problems in-front of them. "Tom, can't you just use your luck to guess each of the derivatives correctly?" asks Shauna. "Shauna, Shauna, Shauna," chuckles Tom, "My luck is limited and I cannot go wasting it on something as small as this competition. We will just have to give up this time." Surrendering to the might of Teacher Simon's coding, they decide to forfeit the competition.

Using the **product rule**, **determine** the **derivatives** Shauna and Tom should have put for the functions below.

(a) $y = (6x+7)(8x+1)$ **(4)**

(b) $y = (x^2-5)(3x^2-8)$ **(4)**

(c) $y = (-2x^2+3x+2)(x+5)$ **(4)**

(d) $y = \left(x^{\frac{5}{2}}+2\right)(2x+1)$ **(4)**

(e) $y = (3x+1)(x^2-7x)$ **(4)**

(f) $f(x) = (6x+2)\left(x^2-\sqrt{x}\right)$ **(4)**

Quotient Rule: Q3, Q4

Repetitive: 2.31 → 2.41 (2 questions)

With many of the pairs already eliminated, only the most daring of the students remain: Rupert and Wallace, and Kerry and Rabea! With two of the top students, Rupert and Kerry, being pinned against each other, Teacher Simon sees this as a fantastic opportunity to continue pushing them! "Alright students, in the next set of questions I'd like you to now use the quotient rule. Good luck!"

[21 marks]

3. Rupert and Wallace are up again, and Rupert yet again laughs at how easy the problems are. Wallace is in awe and asks if he can solve the problems for some practice and Rupert agrees. **Help** Wallace to determine the **derivatives** for the functions below using the **quotient rule**.

(a) $y = \frac{2x+5}{3x+8}$ **(3)**

(b) $y = \frac{6x-9}{4x-2}$ **(3)**

(c) $y = \frac{-x}{2x+6}$ **(3)**

(d) $y = \frac{x^2+6}{x+9}$ **(4)**

(e) $f(x) = \frac{-x+7x^2}{2x+1}$ **(4)**

(f) $y = \frac{-x^2+3x}{3x+5}$ **(4)**

[21 marks]

4. After seeing an amazing performance by the first team, Kerry and Rabea are looking to top it. Kerry smashes through the problems, but rogue student Rabea completely falls behind. **Help** to show Rabea how the **derivatives** for the functions below can be determined using the **quotient rule**.

(a) $y = \frac{2-x}{4+8x}$ **(3)**

(b) $y = \frac{x+2}{x^2-1}$ **(3)**

(c) $f(x) = \frac{x^2-x}{2x+5}$ **(3)**

(d) $f(x) = \frac{2x^2}{x^4+4}$ **(4)**

(e) $y = \frac{4x+6}{x^5+6x}$ **(4)**

(f) $y = \frac{3x^3-2x^2+1}{x^2+5}$ **(4)**

Chain Rule: Q5, Q6, Q7, Q8

Repetitive: 2.51 → 2.81 (4 questions)

Unable to eliminate either of the pairs, teacher Simon decides to take the students outside. He walks them over to a new building and stands in-front of a control panel. "Students, the building you see in-front of you is the shape-shifting building. Designed by Janitor Peter, this building can be manipulated into any shape that you'd like."

"The control panel can only be decrypted by answering some very difficult questions. Rupert, Wallace, Kerry and Rabea, if you can manage to solve your questions without failure then I'll let you manipulate the building and disrupt whatever classes may be going on inside!"

The students grin. Their friends in the Apps class are currently in that building!

[9 marks]

5. Excited by the prospect of being able to disrupt the Apps students' class, Rabea and Kerry get to work. Looking at the panel in-front of them, **help** them use the **chain rule** to determine the **derivatives** of the following functions.

(a) If $y = 2 - u$ and $u = 6x - 6$, determine $\frac{dy}{dx}$ **[3]**

(b) If $y = 2u^2 - 5$ and $u = 2x^2 - 9x$, determine $\frac{dy}{dx}$ **[3]**

(c) If $y = -3u^3 + 5$ and $u = \frac{1}{x}$, determine $\frac{dy}{dx}$ **[3]**

Getting the first questions right, Rabea and Kerry decide to turn the building into the shape of a cube! Inputting the function $y = x^3$, the building suddenly shifts shape. Shouting from inside the building is heard as the Apps students realise what is going on!

[15 marks]

6. Laughing their hearts out, Rupert and Wallace have lost sight of the competition and instead are just excited to change the building shape again. Help Rupert and Wallace to use the rule derived from **chain rule**: if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$ to determine the **derivatives** of the following functions.

(a) $y = (x-3)^3$ **(2)**

(b) $y = (2x^2 + 9)^4$ **(2)**

(c) $f(x) = (3x^2 + 2x)^5$ **(2)**

(d) $f(x) = \sqrt{1 - x^2}$ **(3)**

(e) $y = \frac{1}{(x^2-4x)^3}$ **(3)**

(f) $y = \left(4x^5 + \frac{3}{2}\right)^{\frac{3}{2}}$ **(3)**

Thrilled that they have got the questions right, Rupert and Wallace decide to change the shape of the building into a quadratic to make all of the classrooms' curve sidewards! Inputting the function $y = x^2$, the building curves upwards and more shouting is heard from within the building.

[24 marks]

7. With a big finale in mind, Rabea and Kerry get to work to crack the encryption code of the panel once again! Help Rabea and Kerry to use the **chain rule** as well as **other rules** to determine the **derivatives** of the following functions.

(a) $y = (x+3)^3(x-2)$ **(4)**

(b) $y = \frac{(x-5)^5}{(2x+7)}$ **(4)**

(c) $f(x) = (x+2)^5(3x+4)$ **(4)**

(d) $y = \frac{(x+7)^3}{(4x+5)}$ **(4)**

(e) $y = \frac{(x^2+8)^{\frac{3}{2}}}{(9x+8)}$ **(4)**

(f) $y = (x+6)(x^2-4)^{\frac{1}{2}}$ **(4)**

Successful once again, Rabea goes to input his final function. However, before he manages to input the function, he hears a faint shout and looks around but can't seem to see anyone. "Ouch!" Rabea screams as he reaches down to grab his sore shins.

Standing at 4 foot, 2 inches is student Sparrow. "Who do you think you are?" says Rabea. "Who do I think I am?" Sparrow squeaks. "I am the greatest mathematics applications student on this planet. Who do you think you are!?"

"You have disrupted my learning for the last time Methods student!" screams Sparrow as he scurries back off into the building.

[12 marks]

8. Determined to continue annoying the Apps students, Rupert and Wallace work to crack the encryption code of the panel but are struggling because the questions keep getting harder! **Help Rupert and Wallace** to determine the **derivatives** of the following functions.

(a) $y = \frac{(x^2-9)^4}{(5x-5)}$ **(4)**

(b) $y = (x+3)^{\frac{3}{2}}(x^2-2)$ **(4)**

(c) $y = \frac{x-9}{(2x+2)^6}$ **(4)**

(d) $y = (x^2-4)^6(x+3)$ **(4)**

(e) $y = \frac{(2x^2-5)^{\frac{5}{2}}}{(x+3)}$ **(4)**

(f) $y = (x+4)^{\frac{5}{2}}(2x-9)$ **(4)**

Excited to continue messing with the Apps students, all of the students gather round the panel to input the cubic $y = 3x^4 + x^3 + 2x$ to cause absolute chaos in the building. As they go to enter the function, suddenly from on top of the building eight maths Apps students pop out and bombard the methods students with apple pies!

Running away scared, the methods students can hear the distant laughs of the Apps students' victory! Not willing to accept this, school captain Kerry turns around and screams "We declare war! Be prepared for the mathematics that is to come Apps students."

Problem Set 1 – Differentiation

Repetitive Questions

Concept 1

First Principles and Power Rule– Repetitive Questions

(4 questions)

First Principle: Qs 1.11

[24 marks]

1.11 Determine the **derivatives** of the following functions, using **first principles**.

(a) $\lim_{h \to 0} x$ **(4)**

(b) $\lim_{h \to 0} 3x + 1$ **(4)**

(c) $\lim_{h \to 0} 2x^2 + 1$ **(4)**

(d) $\lim_{h \to 0} x^2 + 3$ **(4)**

(e) $\lim_{h \to 0} 5x^2 + 5$ **(4)**

(f) $\lim_{h \to 0} 4x^2 + 2x$ **(4)**

Power Rule: Qs 1.21, 1.31, 1.41

[11 marks]

1.21 Determine the **derivatives** of the following functions, using the **power rule**.

(a) $y = x^2$ **(1)**

(b) $y = 4x^2 + x$ **(1)**

(c) $f(x) = 7x^4 + 2x^3 + 9$ **(1)**

(d) $y = \frac{2}{x^2} - \sqrt{x}$ **(2)**

(e) $y = \sqrt{x} + 2x^{\frac{3}{2}} - 4x^2$ **(3)**

(f) $f(x) = \frac{2x^3 - 10x}{5x^2}$ **(3)**

[16 marks]

1.31 Determine the **derivatives** of the following functions, using the **power rule**.

(a) $y = 3x^3 - 4x^6 + 6x^7$ **(2)**

(b) $y = (x^2 + 2x)^2$ **(2)**

(c) $f(x) = \frac{(x+6)(x^2+3)}{2x}$ **(3)**

(d) $y = \frac{2}{x^4} - \frac{8}{\sqrt{x}}$ **(3)**

(e) $y = \frac{4}{x^5} - x^{\frac{5}{2}}$ **(3)**

(f) $f(x) = \frac{2\sqrt{x} - 8x^3}{x^{\frac{3}{2}}}$ **(3)**

[12 marks]

1.41 Determine the **derivatives** of the following functions, using the **power rule**.

(a) $y = \frac{x^4}{7} - 2x^{-3}$ **(1)**

(b) $y = -\frac{12}{x^3}$ **(2)**

(c) $f(x) = 5\sqrt{x^3}$ **(2)**

(d) $f(x) = 4x^{-\frac{5}{2}}$ **(3)**

(e) $y = \frac{8x^6 - 5x^3 + 2}{x^4}$ **(2)**

(f) $f(x) = (3x^3 - 2x)^3$ **(2)**

Concept 2

Product Rule, Quotient Rule and Chain Rule – Repetitive Questions (10 questions)

Product Rule: Qs 2.11, 2.21

[21 marks]

2.11 Determine the **derivatives** of the following functions, using the **product rule**.

(a) $y = (2x - 4)(x - 2)$ **(3)**

(b) $y = (2x + 4)(3x + 1)$ **(3)**

(c) $f(x) = (8x + 3)(-7x + 2)$ **(3)**

(d) $y = (3x^2 + 2)(x - 9)$ **(4)**

(e) $f(x) = \sqrt{x}(3x^2 - 5)$ **(4)**

(f) $y = (6x^2 - 4x)(x^2 + 2x)$ **(4)**

[24 marks]

2.21 Determine the **derivatives** of the following functions, using the **product rule**.

(a) $y = (7x - 3)(x^2 - 4)$ **(4)**

(b) $y = (-6x^2 + 2x)(-3x^2 + 9)$ **(4)**

(c) $f(x) = (-x^2 + 3x - 6)(x - 7)$ **(4)**

(d) $y = (x^{\frac{2}{3}} - 8)(5x + 6x^3)$ **(4)**

(e) $f(x) = (x^{\frac{4}{5}} - 2)(2x - 7x^2)$ **(4)**

(f) $y = (x^2 + 2)(x^3 - 4\sqrt{x})$ **(4)**

Quotient Rule: Qs 2.31, 2.41

[21 marks]

2.31 Determine the **derivatives** of the following functions, using the **quotient rule**.

(a) $y = \frac{3x}{x+5}$ **(3)**

(b) $y = \frac{6x-7}{5x-6}$ **(3)**

(c) $f(x) = \frac{-2x}{8x-4x}$ **(3)**

(d) $y = \frac{x^2+4}{8x-4}$ **(4)**

(e) $f(x) = \frac{-6x^2+3x}{2x+2}$ **(4)**

(f) $y = \frac{-x^2-5x}{3x-4}$ **(4)**

[21 marks]

2.41 Determine the **derivatives** of the following functions, using the **quotient rule**.

(a) $y = \frac{x-4}{2x+8}$ **(3)**

(b) $y = \frac{3x+7}{5x^2+3}$ **(3)**

(c) $f(x) = \frac{2x^2-x}{3x-6}$ **(3)**

(d) $y = \frac{2x-x^2}{4x^3}$ **(4)**

(e) $f(x) = \frac{2x^2-4x}{2x^3}$ **(4)**

(f) $y = \frac{6x^3-4x+8}{4x^3-2x}$ **(4)**

Chain Rule: Qs 2.51, 2.61, 2.71, 2.81

[16 marks]

2.51 Determine the **derivative** of the following functions, using the **chain rule**.

(a) If $y = 3 - u$ and $u = 2x + 1$ **(2)**

(b) If $y = 2u + 3$ and $u = 6x - 2$ **(2)**

(c) If $y = 6u^2 - 4$ and $u = 5x^2 - 6$ **(3)**

(d) If $y = 2u^2$ and $u = 2x^2 + 5x - 2$ **(3)**

(e) If $y = u^3$ and $u = x^2 - 6x - 8$ **(3)**

(f) If $y = 4u^3$ and $u = \frac{1}{x} + 4x^3$ **(3)**

[15 marks]

2.61 Determine the **derivative** of the following functions, using the rule **derived** from the **chain rule**:
if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$.

(a) $y = (2x - 4)^3$ **(2)**

(b) $y = (4x^2 - 8)^4$ **(2)**

(c) $f(x) = (2x^2 - x)^5$ **(2)**

(d) $y = (4x + 7x^2)^{\frac{3}{2}}$ **(3)**

(e) $f(x) = \frac{3}{(x^2-7x)^4}$ **(3)**

(f) $y = (6x + 3x^2)^{\frac{3}{2}}$ **(3)**

[24 marks]

2.71 Determine the **derivative** of the following functions, using a combination of **chain rule** and **other rules**.

(a) $y = x^2(x - 4)^3$ **(4)**

(b) $y = (6x - 3)^2\sqrt{x + 5}$ **(4)**

(c) $f(x) = \frac{(2x+4)^3}{5x-4}$ **(4)**

(d) $y = \frac{4x^2-3}{(x+5)^3}$ **(4)**

(e) $f(x) = (x^3 - 5)^{\frac{1}{2}}(2x + 4)$ **(4)**

(f) $y = \frac{(2x-1)^9}{x^3+4}$ **(4)**

[24 marks]

2.81 Determine the **derivative** of the following functions, using a combination of **chain rule** and **other rules**.

(a) $y = (4x - 4)(2 + 9x^2)^6$ **(4)**

(b) $y = \frac{(5x-2)^{\frac{1}{2}}}{7x+x^2}$ **(4)**

(c) $f(x) = \frac{1}{\sqrt{x}}(2x^3 - 9x)$ **(4)**

(d) $y = \frac{-x^3+x^2}{(x+4)^{\frac{3}{2}}}$ **(4)**

(e) $f(x) = 4x^3(7 - 2x)^5$ **(4)**

(f) $y = \frac{(x^4+5x)}{(x^3-4)^{-\frac{1}{2}}}$ **(4)**

1.4 The Second Derivative

The **second derivative** is the **derivative** of the **first derivative** of a function.

The **second derivative** can be determined by taking the **derivative** of the **first derivative function**.

It has the function notation $f''(x)$ or $\frac{d^2y}{dx^2}$.

The process of determining the **second derivative** is the **same** as determining the first derivative. This means we can also utilise the **power rule**, **product rule**, **quotient rule** and **chain rule**.

At a basic level, let's consider we have the function $y = x^3$. To get its **first derivative** we would apply the **power rule** to get $\frac{dy}{dx} = 3x^2$. To determine the **second derivative**, we simply **differentiate** the **first derivative** using the **power rule**.

$$\frac{dy}{dx} = 3x^2$$

$$\frac{d^2y}{dx^2} = (2)3x^{2-1}$$

$$\frac{d^2y}{dx^2} = 6x$$

The same can apply for the product rule and quotient rule. For instance, if we have the functions: $\frac{dy}{dx} = (x+4)(x^2+10)$ and $\frac{dy}{dx} = \frac{7x+3}{x-5}$ we could **apply** the **product rule** and **quotient rule** respectively to get their **second derivatives**:

Product Rule for Second Derivative

$$\frac{dy}{dx} = (x+4)(x^2+10)$$

$$u = (x+4) \qquad v = (x^2+10)$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

$$\frac{d^2y}{dx^2} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d^2y}{dx^2} = (x^2+10)(1) + (x+4)(2x)$$

$$\frac{d^2y}{dx^2} = x^2 + 10 + 2x^2 + 8x$$

$$\frac{d^2y}{dx^2} = 3x^2 + 8x + 10$$

Quotient Rule for Second Derivative

$$\frac{dy}{dx} = \frac{7x+3}{x-5}$$

$$u = 7x+3 \qquad v = x-5$$

$$\frac{du}{dx} = 7 \qquad \frac{dv}{dx} = 1$$

$$\frac{d^2y}{dx^2} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x-5)(7) - (7x+3)(1)}{(x-5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{7x - 35 - 7x - 3}{(x-5)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-38}{(x-5)^2}$$

The **second derivative** is calculated in exactly the **same process** as the **first derivative**, the only difference is the **information each derivative provides**. The **second derivative** will be very useful in the next few sections because it tells us the **instantaneous rate of change** of the **first derivative**.

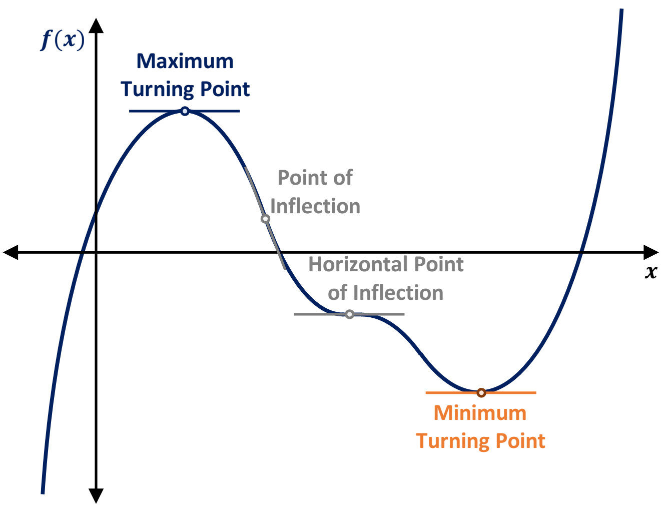
1.5 Curve Sketching

The first application of differentiation we are going to be exploring is **curve sketching** and **derivative sketching**. This section can prove to be a difficult sticking point for students, so read it multiple times and practice through the problem sets.

To learn how to **sketch** a **function curve** or the **derivative** of a function, we must understand what **key points** are.

In Year 12, there are **four points** we are interested in. These are **maximum turning points**, **minimum turning points**, **points of inflection** and **horizontal points of inflection**.

Each point can be seen on the **graph below** and **defined** in the boxes below.



A **maximum turning point** is a **turning point** where the curve is **concave downward** (i.e. gradient going from **positive** to **negative**). At this point, the **value** of the function reaches a **local maximum value** and has a **gradient equal** to **zero** (i.e. $f'(x) = 0$).

A **minimum turning point** is a **turning point** where the curve is **concave upward** (i.e. gradient going from **negative** to **positive**). At this point, the **value** of the function reaches a **local minimum value** and has a **gradient equal** to **zero** (i.e. $f'(x) = 0$).

A **point of inflection (POI)** is a point where the **gradient changes** from **increasing** to **decreasing** or vice-versa.

A **horizontal point of inflection (HPOI)** is a **special type** of a **point of inflection** where the **gradient changes** from **increasing** to **decreasing** or vice-versa, but which also has a **gradient equal** to **zero** at the **point**.

To understand the **nature** of each of these points and why they are important, let's see how they **change** when we **sketch** the **derivative** of a function.

The following table helps us to understand how each of the key points are **transformed** from the **graph of the function** (i.e. $f(x)$) to the graph of the **function's derivative** (i.e. $f'(x)$).

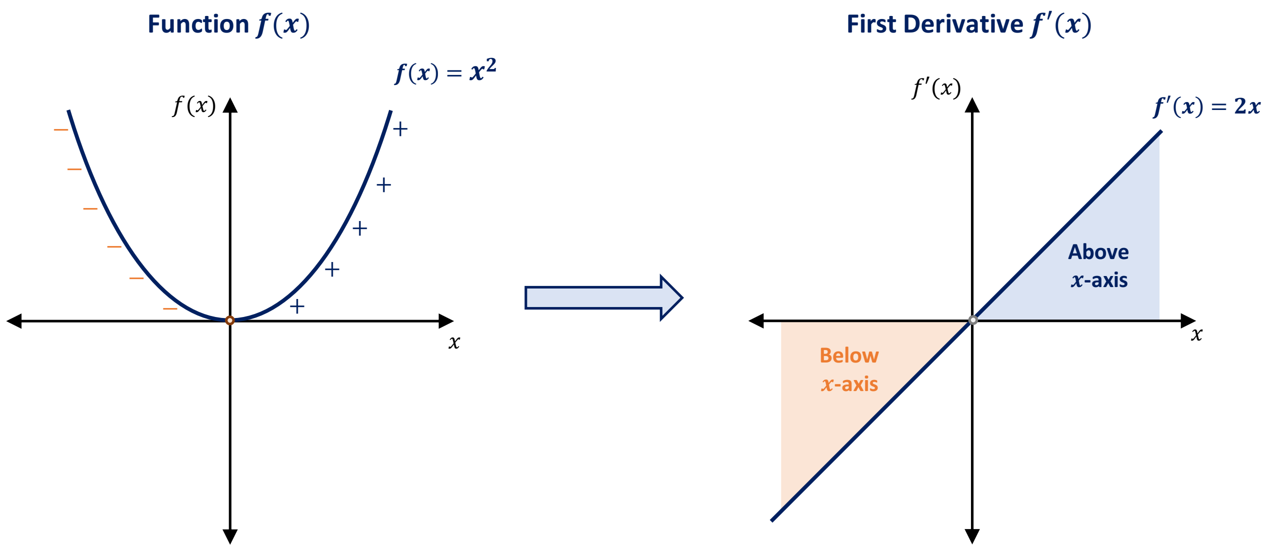
$f(x)$ Key Point	First Derivative $f'(x)$ Features
Maximum/minimum turning point	**x-intercept (root)**
Point of Inflection	**Maximum/minimum turning point**
Horizontal Point of Inflection	**Maximum/minimum turning point** at the **x-axis (root)**

We know that at a **maximum turning point** and **minimum turning point**, the **gradient** is **equal to zero**. Since the **first derivative tracks** the **gradient** of a function, the **turning points** on the **function** $f(x)$ become **x-intercepts** in the function $f'(x)$.

For instance, the function $f(x) = x^2$ has a **minimum turning point** at $x = 0$. Therefore, when sketching the **first derivative** function it will have an **x-intercept** at $x = 0$.

Additionally, the function $f(x) = x^2$ has a **negative gradient prior** to the **minimum turning point** and a **positive gradient after** the **minimum turning point**.

This means the derivative function will be **below** the **x-axis prior** to the **x-intercept** and **above** the **x-axis after** the **x-intercept**.



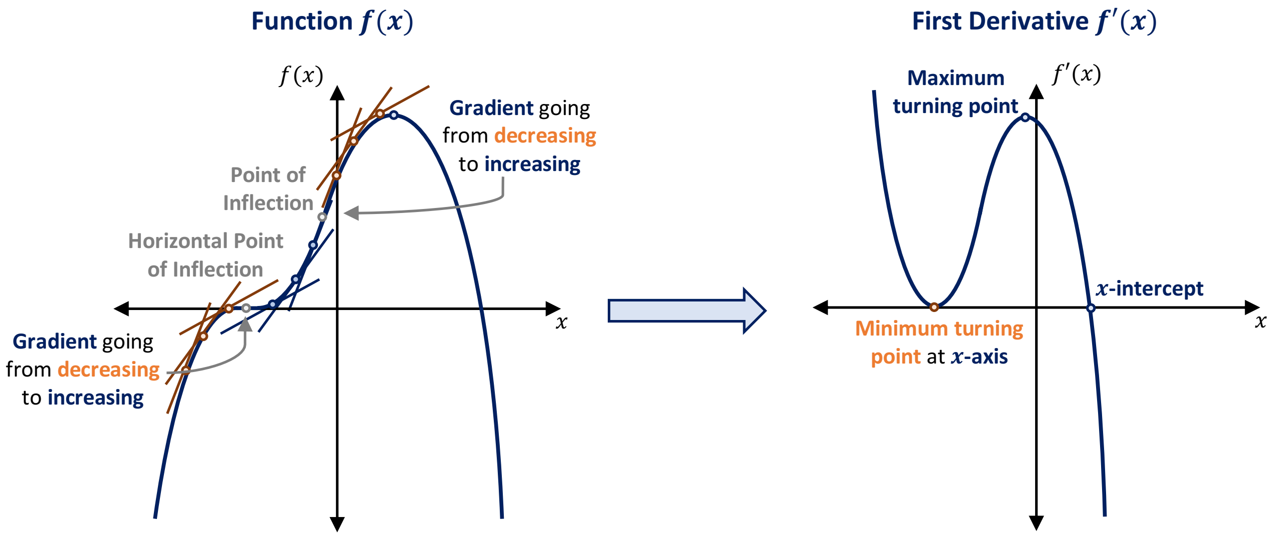
The other two points to consider are the **points of inflection** and **horizontal points of inflection**.

Firstly, for points of inflection on a function $f(x)$, they become a **maximum turning point** or **minimum turning point** for the derivative $f'(x)$.

For points of inflection on $f(x)$, if the **gradient** goes from **increasing** to **decreasing,** it becomes a **maximum turning point** and if the **gradient** goes from **decreasing** to **increasing,** it becomes a **minimum turning point**.

The **horizontal point of inflection** follows an **identical trend** to the **point of inflection**, however, the **maximum** or **minimum turning point** will also be located at the **x-axis**. This is because the **gradient** of a **horizontal point of inflection** is **equal to zero**.

For example, when sketching the derivative of the function $f(x) = 4x^3 - x^4$ below, we can see the **HPOI** has a gradient from **decreasing** to **increasing**, becoming a **minimum turning point** at the **x-axis**. For the **point of inflection**, the **gradient** goes from **increasing** to **decreasing**, so it becomes a **maximum turning point**.

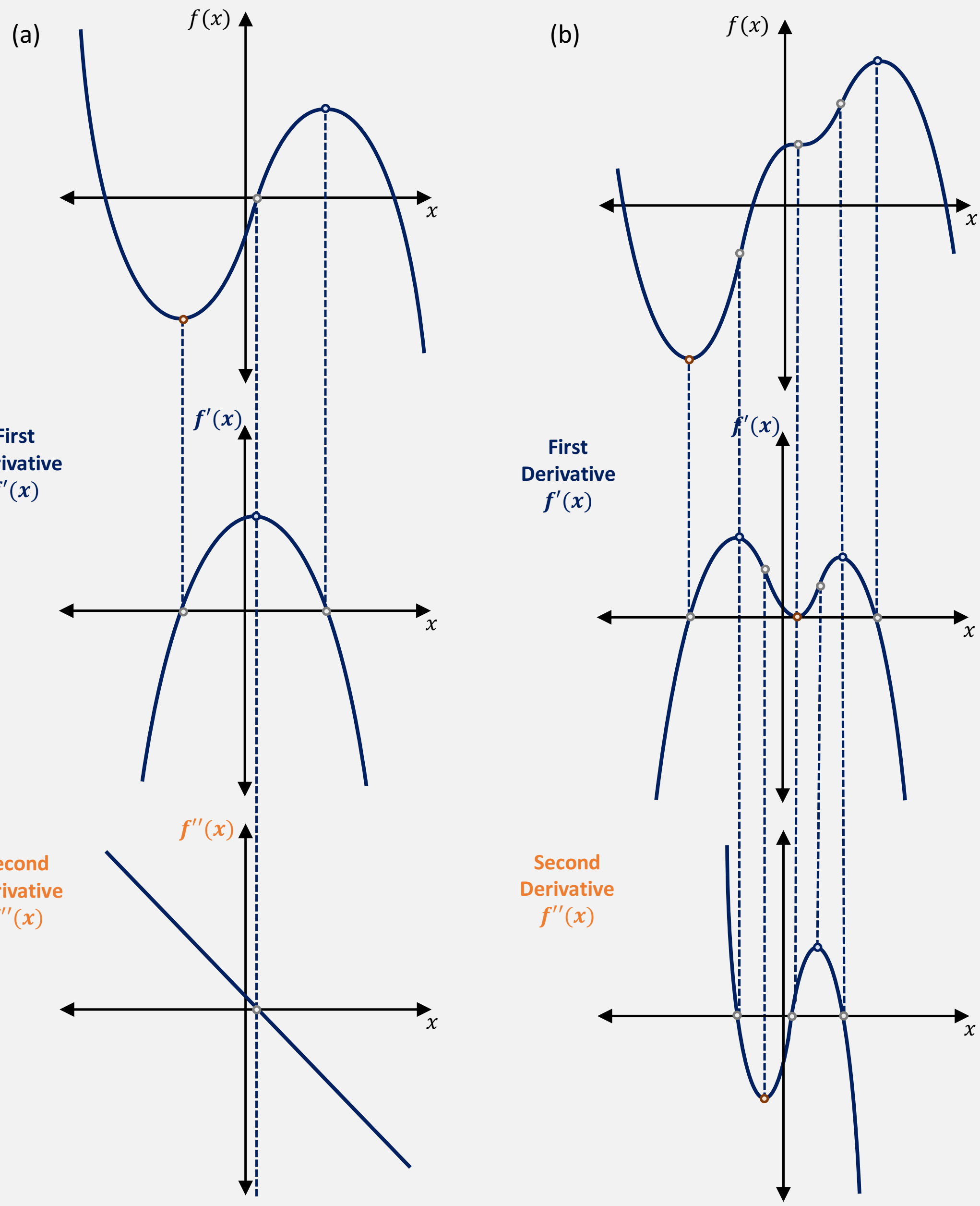


In summary, sketching derivatives of functions can be an overwhelming process that takes practice to solidify. When trying the problems, you can use following table to understand the **transformations each key point undergoes**.

$f(x)$ Key Point	First Derivative $f'(x)$ Features
Maximum/minimum turning point	**x-intercept (root)**
Point of Inflection	**Maximum/minimum turning point**
Horizontal Point of Inflection	**Maximum/minimum turning point** at the **x-axis (root)**
When $f(x)$ has a **positive gradient**	**$f'(x)$ above x-axis**
When $f(x)$ has a **negative gradient**	**$f'(x)$ below x-axis**

Worked Example 1

Whilst Rupert has always been a star maths student, something that he has found himself struggling with is **derivative sketching**. Based on the two function graphs below, help Rupert by sketching the **first derivatives** and **second derivatives** of the functions.



Curve Sketching

In addition to derivative sketching, we can **sketch** our **function** $f(x)$ if we can gather sufficient information about its **key points**. **Stationary points**, x and y **intercepts**, and points of inflection are all important to accurately sketch the graph of the function. Determining **x-intercepts** and **y-intercept** is a process we should be familiar with from Year 11.

To determine the x **-intercepts,** we solve for $f(x) = 0$, and for the y**-intercept,** we solve for $f(x)$ when $x = 0$ (i.e. $f(0)$). Suppose we had to sketch the graph for $f(x) = 9x^3 - 3x^4$. To do this, we would try to find as many key points as we can for the graph. To find the x**-intercepts** we would let $f(x) = 0$ and to find the y**-intercept** we would let $x = 0$.

x-intercepts

$$f(x) = 0$$
$$0 = 9x^3 - 3x^4$$
$$0 = 3x^3(3 - x)$$
$$x = 0, x = 3$$
$\therefore$ x**-intercepts** at $(0, 0)$ and $(3, 0)$

y-intercept

$$f(0) = 9(0)^3 - 3(0)^4$$
$$f(0) = 0$$
$\therefore$ y**-intercept** at $(0, 0)$

To determine the **stationary points** and **points of inflection**, we need to apply **differentiation**.

Maximum turning points, **minimum turning points**, and **horizontal points of inflection** all have a **gradient** of **zero**. Since the **first derivative** is a measure of the **gradient**, if we determine the **first derivative** and let it be **equal to zero** (i.e. $f'(x) = 0$), we can find the x**-coordinate locations** of these **stationary points**.

For instance, given our function, $f(x) = 9x^3 - 3x^4$, we can **differentiate** it and **solve** for $f'(x) = 0$.

$$f(x) = 9x^3 - 3x^4$$
① $f'(x) = 27x^2 - 12x^3$
② **Let** $f'(x) = 0$
$$0 = 27x^2 - 12x^3$$
$$0 = 3x^2(9 - 4x)$$
③ $x = 0, x = \frac{9}{4}$

① Determine the **derivative** $f'(x)$

② Let the **first derivative equal zero**

③ **Solve** for the **stationary points**

Whilst we now know that there are stationary points at $x = 0$ and $x = \frac{9}{4}$, we don't know the **nature** of these **stationary points** – that is, we don't know whether they are **maximum turning points**, **minimum turning points**, or **horizontal points of inflection**.

To identify the **concavity** of the graph, we can use the **second derivative** to determine whether the point is a **maximum turning point**, **minimum turning point** or **horizontal point of inflection**.

> For a point where $f'(x) = 0$, the **second derivative test** states: if $f''(x) > 0$ the point is a **minimum turning point**, if $f''(x) < 0$ it is a **maximum turning point**, and if $f''(x) = 0$ the test is **inconclusive**.

This test is valuable for classifying stationary points that are **maximum** or **minimum** turning points, but **not useful** for **horizontal points of inflection** (which requires an **additional test**).

The **second derivative test** for turning points can be summarised as follows.

Second Derivative Test		
Maximum Turning Point	**Minimum Turning Point**	**Inconclusive**
$f''(x) < 0$	$f''(x) > 0$	$f''(x) = 0$

Applying this **second derivative test** to our function $f(x) = 9x^3 - 3x^4$, we start by determining the **second derivative** and then **substituting** in our **key points** at $x = 0$ and $x = \frac{9}{4}$.

$$f'(x) = 27x^2 - 12x^3$$

① $$f''(x) = 54x - 36x^2$$

① Determine **second derivative**

② $$f''\left(\frac{9}{4}\right) = 54\left(\frac{9}{4}\right) - 36\left(\frac{9}{4}\right)^2$$

$$f''\left(\frac{9}{4}\right) = -\frac{243}{4}$$

② **Substitute** in the **point** $x = \frac{9}{4}$

③ $$f''\left(\frac{9}{4}\right) < 0, \therefore \text{maximum } TP$$

③ Conclude the **nature** of the **key point**

$$f''(0) = 54(0) - 36(0)^2$$

$$f''(0) = 0$$

④ $$\therefore \text{Inconclusive}$$

④ **Substitute** in the **point** $x = 0$ and **conclude** its **nature**

From this test, we can see our **point** at $x = \frac{9}{4}$ is a **maximum turning point**, and our **point** at $x = 0$ is **inconclusive** since $f''(x) = 0$. This means for the **inconclusive point** at $x = 0$, we need to take it **one step further** and **apply another test** – the **first derivative sign test**.

The **first derivative sign test** looks at the gradient of the function on **either side** of a **stationary point** to determine its **nature**. The **points** you choose should be simple ones, like ± 1 either side of the key point. As shown below, based on the **positive/negative nature** on either side, you will be able to deduce whether it is a **maximum turning point**, **minimum turning point** or **horizontal point of inflection**.

Minimum Turning Point

$x = a^-$	$x = a$	$x = a^+$
$-ve$	0	$+ve$

Horizontal Point of Inflection

$x = a^-$	$x = a$	$x = a^+$
$-ve$	0	$-ve$
$+ve$	0	$+ve$

Maximum Turning Point

$x = a^-$	$x = a$	$x = a^+$
$+ve$	0	$-ve$

Applying it to our **inconclusive point** at $x = 0$, we can determine the gradient at $x = -1$ and $x = 1$ to see the **shape** of the **curve** at this point. Substituting $x = -1$ and $x = 1$ into the derivative gives:

$$f'(-1) = 27(-1)^2 - 12(-1)^3$$
$$f'(-1) = 39$$

$$f'(1) = 27(1)^2 - 12(1)^3$$
$$f'(1) = 15$$

From these values of $f'(-1) = 39$ and $f'(1) = 15$ we can see that they are both **positive numbers**, so the curve goes from a **positive gradient** to **zero gradient** to a **positive gradient**, which is the **shape** of a **horizontal point of inflection**.

Horizontal Point of Inflection

$x = -1$	$x = 0$	$x = 1$
$+ve$	0	$+ve$

The **final point** we may need to locate are regular points of inflection, which occur when $f'(x) \neq 0$ and $f''(x) = 0$ and must be **justified** with the **second derivative sign test**. Applying this to our example, if we **start by solving** for $f''(x) = 0$ we can see that there is an inflection point at $x = \frac{3}{2}$, since it wasn't one of our previous points when solving for $f'(x) = 0$.

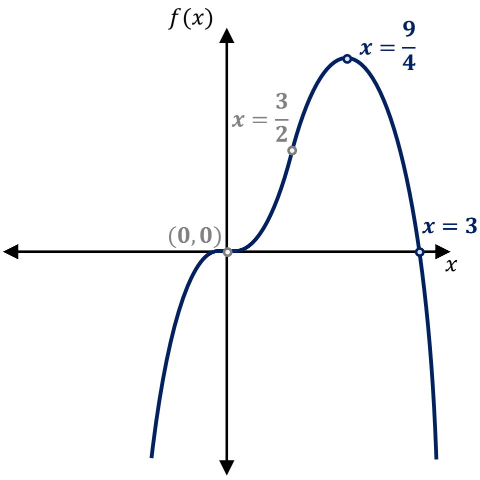
$$f''(x) = 0$$
$$0 = 54x - 36x^2$$
$$0 = x(54 - 36x)$$
$$x = 0, x = \frac{3}{2}$$
$$\therefore \text{potential inflection point at } x = \frac{3}{2}$$

To **justify** that this is a point of inflection we must show that the **concavity changes before** and **after** the **point** by using the **second derivative sign test**. This is the **same process** as the **first derivative sign test** but this time we are showing that the **concavity changes either side of the point** (i.e. positive to negative or vice versa).

As shown below, by **selecting points** either side of $x = 3/2$ we can **justify** that the **concavity changes either side** of the **point** and we can therefore **classify** $x = 3/2$ as a point of inflection.

Point of Inflection		
$x = 1$	$x = 3/2$	$x = 2$
$f''(x) > 0$	$f''(x) = 0$	$f''(x) < 0$

Combining all of the **key points** we have gathered, we can **sketch** the **general shape** of this curve.



To summarise, we can **classify** and **solve** for all **key points** using the following:

Type of Point	$f'(x)$	$f''(x)$	Additional Test
Minimum turning point	$f'(x) = 0$	$f''(x) > 0$	N/A
Maximum Turning Point	$f'(x) = 0$	$f''(x) < 0$	N/A
Point of Inflection	$f'(x) \neq 0$	$f''(x) = 0$	2nd Derivative Sign Test
Horizontal Point of Inflection	$f'(x) = 0$	$f''(x) = 0$	1st Derivative Sign Test

Worked Example 2

Surprised that he was successful in sketching first and second derivatives, Rupert wants to take his sketching skills to a new level by trying to **sketch graphs** from **functions**. To start, he wants to practice **locating** and **classifying** the **stationary points** of the function: $f(x) = \frac{x^4}{16} - \frac{x^3}{2} + 8x - 5$.

(a) **Help** Rupert to determine the **location** of the **stationary points** by solving for $f'(x) = 0$.

$$f(x) = \frac{x^4}{16} - \frac{x^3}{2} + 8x - 5$$

$$f'(x) = \frac{x^3}{4} - \frac{3x^2}{2} + 8$$

Let $f'(x) = 0$

$$0 = \frac{x^3}{4} - \frac{3x^2}{2} + 8$$

$x = -2$ and $x = 4$

(b) **Help** Rupert determine the **nature** of these points by applying the **second derivative test**.

$$f''(x) = \frac{3x^2}{4} - 3x$$

$$f''(-2) = \frac{3(-2)^2}{4} - 3(-2)$$

$$f''(-2) = 9$$

$\therefore x = -2$ is a **minimum turning point**

$$f''(4) = \frac{3(4)^2}{4} - 3(4)$$

$$f''(4) = 0$$

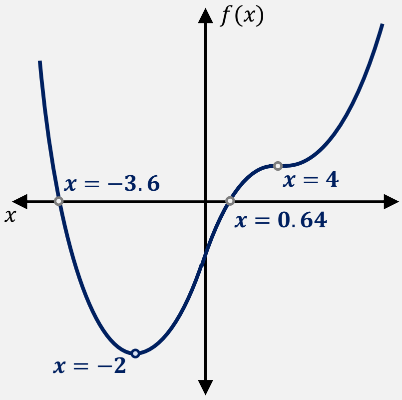
$\therefore x = 4$ is inconclusive

(c) **Help** Rupert to **determine** the **nature** of the inconclusive point at $x = 4$, using the **first derivative test**.

First Derivative Test		
$x = 3$	$x = 4$	$x = 5$
$f'(3) = 1.25$	$f'(4) = 0$	$f'(5) = 1.75$
$+ve$ /	0 —	$+ve$ /

$\therefore x = 4$ is a **horizontal point of inflection**

(d) Based on the **key points** determined and assuming there are **x-intercepts** at $x = -3.6$ and $x = 0.64$, **sketch** the **general shape** of the **curve**.



Worked Example 3

As a final test, Rupert sets himself the task of determining the equation of a function from some simple information. He knows is that the equation has the format $y = ax^3 + bx^2 + cx + d$, with a **y-intercept** at $(0, 6)$, a **maximum turning point** at $x = -2$, and a **point of inflection** at $(1, 2)$. Help Rupert to **determine** its **equation** by determining the **values** of a, b, c and d.

We know $d = 6$ since the **y-intercept** is at $(0, 6)$:

$$y = ax^3 + bx^2 + cx + 6$$

$$y' = 3ax^2 + 2bx + c$$

Substitute in $y'(-2) = 0$ from the **maximum TP**:

$$0 = 3a(-2)^2 + 2b(-2) + c$$

$$0 = 12a - 4b + c$$

$$① \quad \therefore c = 4b - 12a$$

$$y'' = 6ax + 2b$$

Substitute in $y''(1) = 0$ since there is a **POI**:

$$y''(1) = 0$$

$$0 = 6a(1) + 2b$$

$$② \quad b = -3a$$

Substitute ② into ①

$$c = 4(-3a) - 12a$$

$$c = -24a$$

Substitute $y(1) = 2$ to **solve** for a:

$$2 = a(1)^3 + b(1)^2 + c(1) + 6$$

$$2 = a + b + c + 6$$

$$2 = a - 3a - 24a + 6$$

$$a = \frac{2}{13}$$

Use the **value** of a to **solve** for b and c:

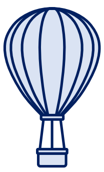
$$b = -\frac{6}{13} \text{ and } c = -\frac{48}{13}$$

$$\therefore y = \frac{2}{13}x^3 - \frac{6}{13}x^2 - \frac{48}{13}x + 6$$

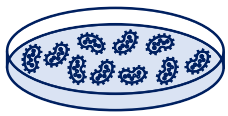
1.6 Rate of Change

As we know, **derivatives** are a **measure** of the **rate of change** of a function.

When a function is used to track a **practical scenario**, the **derivative** of that function will determine the **rate of change** for a certain variable of interest. There is an endless range of applications, from the **rate of change** in the **height** of a **hot air balloon** through to the **rate of change** in the **population** of a **bacteria colony**. By determining the **derivative**, we can determine the **rate of change** at a **given instant**.



Rate of change in the **height** of a **hot air balloon**



Rate of change in the **population** of a **bacteria colony**

When discussing **'rate of change'** there are two concepts we need to distinguish: **instantaneous rate of change** and **average rate of change**.

Instantaneous rate of change is the **rate of change** of a function at a **given instant**. It is determined by **substituting** the **x-coordinate or point of interest** into the **derivative** of the function (i.e. $f'(x)$).

Average rate of change is the **rate of change** of a function over a **period** of **time**. It is determined by dividing the **difference** in the **value** of the function at the **two points** in time, by the **change** in **time** (i.e. $\frac{f(t_2)-f(t_1)}{\Delta t}$).

The best way to understand the difference between **instantaneous rate of change** and **average rate of change** is to apply them.

Consider a **hot air balloon** with a **height** (in **meters**) that follows the equation $h(t) = 0.25t^3 - t^2 - t + 7$, where t is in **seconds**.

Suppose we wanted to know its **instantaneous rate of change** in **height** at certain times, say $t = 1$ and $t = 4$. To do so, we would determine the **derivative** of the function and then **substitute** in the **values** of t.

$$h(t) = 0.25t^3 - t^2 - t + 7$$

① $h'(t) = 0.75t^2 - 2t - 1$

For $t = 1$:

② $h'(1) = 0.75(1)^2 - 2(1) - 1$

$$h'(1) = -2.25m/s$$

For $t = 4$:

③ $h'(4) = 0.75(4)^2 - 2(4) - 1$

$$h'(4) = 3m/s$$

① **Determine** the **derivative** $h'(t)$

② **Substitute** $t = 1$ into $h'(t)$

③ **Substitute** $t = 4$ into $h'(t)$

As we can see, the balloon is initially **decreasing in height** at a **rate** of $2.25\ m/s$ at **1 second** and then reverses to be **increasing in height** at a **rate** of $3m/s$ at **4 seconds**.

If instead, we wanted to know the **average rate of change** in **height** from $t = 1$ and $t = 4$ we start by **substituting** our **values of** t into our **height function h(t)** to determine the **heights** at these times.

For $t = 1$:

$$h(1) = 0.25(1)^3 - (1)^2 - (1) + 7$$
$$h(1) = 5.25m$$

For $t = 4$:

$$h(4) = 0.25(4)^3 - (4)^2 - (4) + 7$$
$$h(4) = 3m$$

Next, to determine the **average rate of change**, we divide the **difference** in these two heights by the **change in time**.

$$Average\ Rate = \frac{h(t_2) - h(t_1)}{\Delta t}$$
$$Average\ Rate = \frac{h(4) - h(1)}{4 - 1}$$
$$Average\ Rate = \frac{3 - 5.25}{4 - 1}$$
$$Average\ Rate = -\ 0.75m/s$$

As we can see, the **average rate of change** gives a completely different result to the **instantaneous rate of change**, since they are **measures** of **two different things**.

Remember, **instantaneous rate of change** measures the **rate of change** at a **given instant**, whereas **average rate of change** measures the **rate of change** over a **period of time**, so they **produce different results**.

Worked Example 1

Bored in detention, Tom is eager to put some of his knowledge of derivatives to practical applications. Tom wants to track the **rate of change** in the **population (P)** of a **bacteria colony** (in **millions**) that follows the equation: $P(t) = \frac{t^3}{750} + 2\sqrt{t}$, where t is **time** in **seconds**. **[CA]**

(a) **Help** Tom determine the **instantaneous rate of change** in the **bacteria population** at $t = 9$ and $t = 10$.

$$P(t) = \frac{t^3}{750} + 2\sqrt{t}$$

$$P'(t) = \frac{t^2}{250} + \frac{1}{\sqrt{t}}$$

$$P'(9) = \frac{(9)^2}{250} + \frac{1}{\sqrt{9}}$$

$$P'(9) = 0.657\ million\ bacteria/s$$

$$P'(10) = \frac{(10)^2}{250} + \frac{1}{\sqrt{10}}$$

$$P'(10) = 0.716\ million\ bacteria/s$$

(b) **Help** Tom determine the **average rate of change** in the **bacteria population** from $t = 9$ to $t = 10$.

$$P(9) = \frac{(9)^3}{750} + 2\sqrt{9}$$

$$P(9) = 6.972\ million\ bacteria/s$$

$$P(10) = \frac{(10)^3}{750} + 2\sqrt{10}$$

$$P(10) = 7.658\ million\ bacteria/s$$

$$Average\ Rate = \frac{P(10) - P(9)}{\Delta t}$$

$$Average\ Rate = \frac{7.658 - 6.972}{10 - 9}$$

$$Average\ Rate = 0.686\ million\ bacteria/s$$

1.7 Rectilinear Motion

One of the most useful applications of differentiation is to the **motion** of objects in **one dimension**.

The **motion** of an object can be described by its **displacement (s)**, **velocity (v)** and **acceleration (a)** in relation to **time (t)**. These are **vector quantities**, so they are **not the same** as distance and speed.

Displacement (s) is the **difference** in an object's **position** relative to a **reference point** (known as the origin) at a **given point in time.** In the metric system, it is typically measured in **meters (m)**.

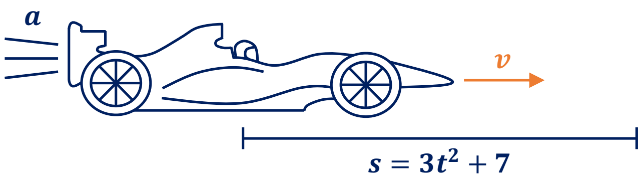
Velocity (v) is **instantaneous rate of change** in the **displacement** of an object and is therefore the **derivative** of **displacement** (i.e $v = \frac{ds}{dt}$). In the metric system, it is typically measured in **meters per second (m/s)**.

Acceleration (a) is **instantaneous rate of change** in the **velocity** of an object and is therefore the **derivative** of **velocity** (i.e $a = \frac{dv}{dt}$). In the metric system, it's typically measured in **meters per second squared (m/s^2)**.

As we can see from these definitions, **displacement**, **velocity** and **acceleration** are **all linked** to one another through **differentiation**. If we wanted to know the **velocity function** of an object, we would **differentiate** the **displacement function**. If we wanted to know the **acceleration function** of an object, we would **differentiate** the **velocity function**.

$$\text{Displacement } (s) \xrightarrow{\frac{ds}{dt}} \text{Velocity } (v) \xrightarrow{\frac{dv}{dt}} \text{Acceleration } (a)$$

For instance, suppose we had a **race car** with the **displacement function**: $s = 3t^2 + 7$ where s is **displacement** in **metres** and t is **time** in **seconds**, and we wanted to know its **velocity function** and **acceleration function**.



To determine the **velocity function**, we would **differentiate** the **displacement function**, and to determine the **acceleration function,** we would **differentiate** the **velocity function**.

Velocity Function

$$s = 3t^2 + 7t$$
$$v = \frac{ds}{dt}$$
$$v = \frac{ds}{dt} = 6t + 7 \ [m/s]$$

Acceleration Function

$$v = 6t + 7$$
$$a = \frac{dv}{dt}$$
$$a = \frac{dv}{dt} = 6 \ [m/s^2]$$

If we wanted to know the **displacement**, **velocity** or **acceleration** at a given point in time, we would just **substitute** in the **time (t)** that we want to know the value for. For instance, after **three seconds** the **motion** of the race car can be described by **substituting** $t = 3$ into each function:

Displacement

$$s = 3t^2 + 7t$$
$$s(3) = 3(3)^2 + 7(3)$$
$$s(3) = 48 \ m$$

Velocity

$$v = 6t + 7$$
$$v(3) = 6(3) + 7$$
$$v(3) = 25 \ m/s$$

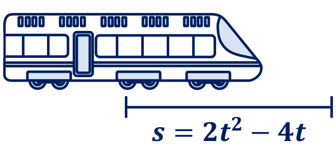
Acceleration

$$a = 6$$
$$a(3) = 6 \ m/s^2$$

Using these equations, we can also determine **other important information** such as when an **object stops**, **changes direction** or when it **returns** to its **starting point**.

Suppose we had a **train**, initially at station A, travelling to station B, with the **displacement function** $s = 2t^2 - 4t$. If we wanted to know the **time(s)** that the train **returned to station** A, we can set the **displacement function equal to zero** (i.e. $s = 0$) and **solve** for t to find the train returns at **2 seconds**:

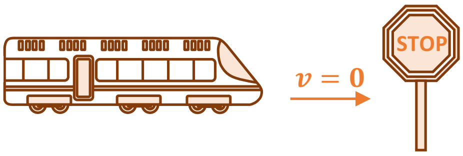
Station A ($s = 0$)



$$s = 2t^2 - 4t$$
$$s(t) = 0$$
$$0 = 2t^2 - 4t$$
$$0 = 2t(t - 2)$$
$$t = 0, t = 2$$

Train returns to station A at 0 and 2 seconds

Additionally, if the **train stopped** along its way to station B we could determine the **time** it **stopped** by first **differentiating** $s(t)$ to get the **velocity function** $v(t)$ and then solving for $v = 0$:



$$s = 2t^2 - 4t$$
$$v = \frac{ds}{dt}$$
$$v = 4t - 4$$
$$v(t) = 0$$
$$0 = 4t - 4$$
$$4 = 4t$$
$$t = 1$$

This **differentiation** and **analysis** of **motion equations** can be done for **any object** that is moving in one dimensional motion. As we will soon see, we can also apply **integration** in rectilinear motion.

Practice Example 1

Tom is bored during detention and decides to duck out to do some practical experiments. In another room, Tom finds a particle accelerator with the **displacement function:** $s = 2t^3 - 3t + 1$. Before Tom fires off the particle, he wants to make some predictions using differentiation.

(a) **Help** Tom to determine the particle's **displacement** after **2 seconds**, the **velocity** after **3 seconds** and the **acceleration** after **4 seconds**.

Displacement

$$s = 2t^3 - 3t + 1$$
$$s(2) = 2(2)^3 - 3(2) + 1$$
$$s(2) = 11\ m$$

Velocity

$$v(t) = \frac{ds}{dt}$$
$$v(t) = 6t^2 - 3$$
$$v(3) = 6(3)^2 - 3$$
$$v(3) = 51\ m/s$$

Acceleration

$$a(t) = \frac{dv}{dt}$$
$$a(t) = 12t$$
$$a(4) = 12(4)$$
$$a(4) = 48\ m/s^2$$

(b) **Help** Tom determine the **times** that the particle is **not accelerating**.

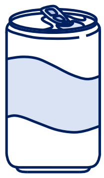
Let $a(t) = 0$

$$0 = 12t$$
$$t = 0s$$

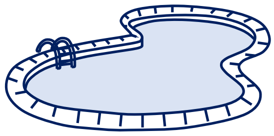
$\therefore$ **Only has zero acceleration at 0 seconds**

1.8 Optimisation

Have you ever looked at the shape of a **swimming pool** or **fizzy drink can** and wondered why it is that **specific shape**? In contrast, have you ever thought about **stock traders** or **big companies** and wondered how they **maximise their investments**? The answer: **optimisation**.



Maximising the Volume of a Can



Maximising the Volume of a Swimming Pool

Maximising the Profit of an Investment

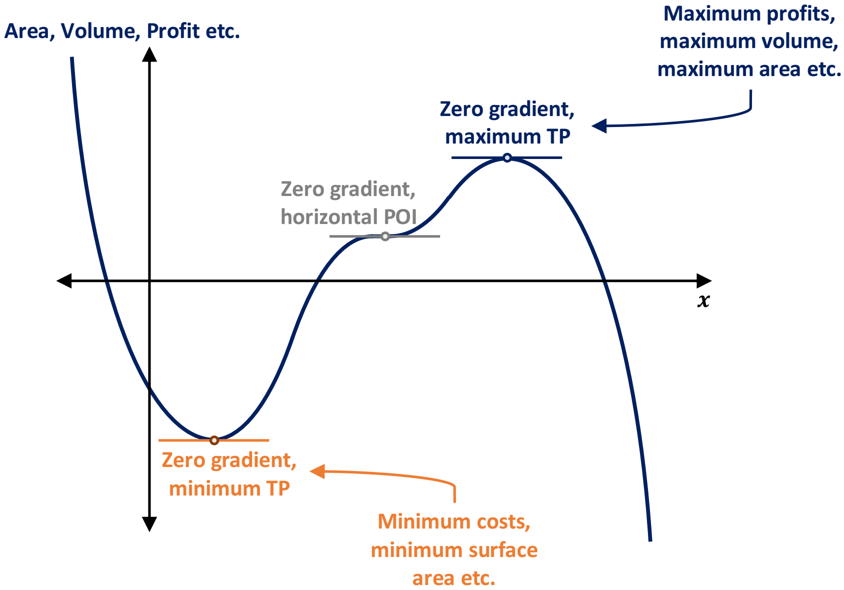
Usually, it is the **optimum dimensions** of a shape or **optimum time** to sell shares, so that it **optimises** the area, volume or profit of an object or situation.

> **Optimisation** is the process of making the **most effective use** of a resource by determining the **maximum** or **minimum** value of a function.

Differentiation can be used to determine the **optimum dimensions** of the objects and parameter around us such as the **optimum timings** to **maximise profits**, provided we know a function which relates the required variables.

For instance, an object will have an **optimum dimension** when its **area** or **volume** equation reaches a **maximum turning point** or **minimum turning point**.

From graph sketching, we know one feature of a **maximum** or **minimum turning point** is that it has a **zero gradient**. Therefore, to find the **maximum** or **minimum turning point we need to find when the gradient is equal zero** (i.e. $f'(x) = 0$).

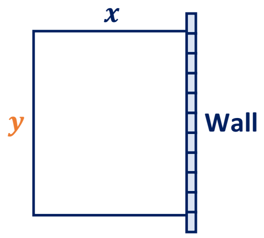


Therefore, if we **differentiate** the function and **solve** for **zero** (i.e. $f'(x) = 0$), we can determine the **location** of **stationary points**. Then, to determine the **nature** of the **stationary point**, we apply our **second derivative test**.

The best way to understand how to **apply the process** is by looking at some **practical examples**.

To start, suppose we had a farmer who had $100\ m$ of **fencing** that he wanted to use to create an enclosure **against a wall** for his sheep with a **maximum area**.

To find this **maximum area**, let's start by defining the **side lengths** of the fencing as x and y.



From the diagram, we know the **area function** will be $A = xy$ and the **fencing perimeter function** will be $P = 2x + y$. Given that $P = 100\ m$, based on the available fencing we can use these two equations to determine the following **single variable equation** for **area**.

$$P = 2x + y$$
$$① \quad 100 = 2x + y$$
$$② \quad y = 100 - 2x$$

$$A = xy$$
$$③ \quad A(x) = x(100 - 2x)$$
$$A(x) = 100x - 2x^2$$

① **Substitute** $P = 100$ into the **perimeter equation**

② **Re-arrange** the **perimeter equation** to be in **terms of** y

③ **Substitute** $y = 100 - 2x$ into the **area equation** to have **area in terms of** x

To determine the **stationary points** of this equation, we can then **differentiate** the **area equation** and solve for $A'(x) = 0$.

$$A'(x) = 100 - 4x$$
$$\text{Let } A'(x) = 0$$
$$0 = 100 - 4x$$
$$x = 25\ m$$

To determine the **nature** of our **stationary point** at $x = 25$, we can **differentiate** $A'(x)$ to get the **second derivative**. As seen below, in this case we **don't** need to **substitute** the value of $x = 25$, since the **second derivative** is a **constant**.

$$A'(x) = 100 - 4x$$
$$A''(x) = -4$$
$$A''(x) < 0, \therefore maximum$$

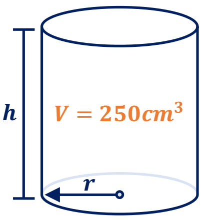
From the second derivative test, we know the dimension of $x = 25$ will give us a **maximum area**, because this **x-coordinate** gives a **local maximum turning point**.

To conclude, we can use our equation $y = 100 - 2x$ to determine the value of y, and then **substitute** in these values to determine the **final maximum area** of $1250m^2$.

Value of y	Maximum Area
$y = 100 - 2(25)$	$A = xy$
$y = 50$	$A = 25 \times 50$
	$A = 1250\ m^2$

$\therefore$ the **maximum area** is $1250\ m^2$ with dimensions of $x = 25\ m$ and $y = 50\ m$

As a more difficult example, suppose we wanted to determine the **minimum surface area** of a **cylindrical** drink can with a **volume** of $V = 250mL = 250\ cm^3$. If we know the **surface area equation** of a **cylinder** to be $S = 2\pi rh + 2\pi r^2$ and the **volume equation** to be $V = \pi r^2 h$, we can use the **volume equation** to generate a **single variable equation** for **surface area**:



$$V = 250cm^3$$
$$① \quad 250 = \pi r^2 h$$
$$② \quad h = \frac{250}{\pi r^2}$$

$$S = 2\pi rh + 2\pi r^2$$
$$③ \quad S = 2\pi r\left(\frac{250}{\pi r^2}\right) + 2\pi r^2$$
$$④ \quad S = \frac{500}{r} + 2\pi r^2$$

① **Substitute** in $V = 250cm^3$ into the **volume equation**

② **Re-arrange** for h

③ **Substitute** in $h = \frac{250}{\pi r^2}$

④ **Simplify** to the final **surface area equation**

With a **single variable equation** for **surface area**, we can now determine the **stationary points** and find the one that **minimises** the **surface area** by finding the **minimum turning point**.

$$S = \frac{500}{r} + 2\pi r^2$$
$$⑤ \quad S' = -\frac{500}{r^2} + 4\pi r$$
$$\text{Let } S' = 0$$
$$0 = -\frac{500}{r^2} + 4\pi r$$
$$⑥ \quad r = 3.414$$

$$S'' = \frac{1000}{r^3} + 4\pi$$
$$S''(3.414) = \frac{1000}{3.41^3} + 4\pi$$
$$⑦ \quad S''(3.414) = 37.70$$
$$S''(3.414) > 0, \therefore \textit{minimum}$$
$$S(3.414) = \frac{500}{3.414} + 2\pi(3.414)^2$$
$$S(3.414) = 219.7\ cm^2$$

⑤ **Differentiate** surface area equation

⑥ Find **values** of r by **solving** for $S' = 0$

⑦ Determine **nature** of **the stationary point** through **second derivative**

⑧ **Finish** with **concluding statement**

⑧ $\therefore$ the **minimum surface area** is $219.7\ cm^2$ for the can to have a **volume** of $250\ mL$

As you can see, once you have your single variable equation, the **steps** for **optimising** in any situation are the **same**, and they can be summarised as shown below. The **difficult part** is being able to **determine** the **single variable equation** each time, which you will discover in the problem sets.

Optimisation Steps

① Determine a **single variable equation** (e.g. $f(x)$) using the information provided.

② **Differentiate** $f(x)$ and **solve** for $f'(x) = 0$ to find the **stationary point locations**.

③ **Differentiate** $f'(x)$ to find the **second derivative** and **apply** the **second derivative test** to determine the **nature** of each stationary point.

④ If required, **substitute** the **x-value** of the **maximum** or **minimum** point into $f(x)$ to find the final answer.

This **same process** applies to the world of **finance** and **business** as well. The principles and steps are the same, as shown in the worked example below.

Worked Example 1

Janitor Peter is looking to fund some of his 'side projects' and needs some more money. Knowing that Kerry has a good knowledge of the finance world, he turns to Kerry and asks her to invest his $\$1,000,000$ in savings!

Analysing the stock market, Kerry has the equation $V(t) = 30t^2 - 3t^3 + 4t + 2$ to track the **value** $\$V$ of the shares of a new technology company. Very excited by these equations, Janitor Peter decides to invest all his money in this technology company. If t is **time** measured in **days**, determine the **number of days** Janitor Peter should wait before selling to **maximise** his **profit** and the **value** of the **share** at that **time**. **[CA]**

$$V(t) = 30t^2 - 3t^3 + 4t + 2$$
$$V'(t) = 60t - 9t^2 + 4$$
$$V'(t) = 0$$
$$0 = 60t - 9t^2 + 4$$
$$t = -0.066,\ t = 6.73$$

$t = -0.066$ is **not valid** since t **cannot** be **less** than **zero**

$$V''(t) = 60 - 18t$$
$$V''(6.73) = 60 - 18(6.73)$$
$$V''(6.73) = -61.14$$
$$V''(6.73) < 0,\ \therefore\ maximum$$
$$V(6.73) = 30(6.73)^2 - 3(6.73)^3 + 4(6.73) + 2$$
$$V(6.73) = \$473$$

∴ **Janitor Peter** should sell his shares after **6.73 days** and their **value** will be $V = \$473$ **per share**

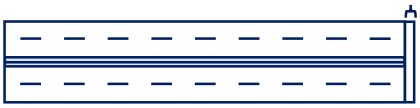
1.9 Small Change

Small change or **incremental change** is exactly what it sounds like – it is **applying calculus** when the **change** in a variable is **small/incremental**. We learnt in Chapter 1.1. that the **definition** of a **derivative** is the **gradient** of a function at a **single point**, but if we take the **gradient** between **two points** that are **very close together**, it **approximates** $\frac{dy}{dx}$ (i.e. $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$).

When there is a **small change** in **one variable** (e.g. δx), the **small change** of the **function** (e.g. δy) can be **approximated** to change based on the formula: $\delta y \approx \frac{dy}{dx}\delta x$

Small change can be relevant to many **practical situations** we may encounter. For instance, if a **large road** is made $2cm$ **longer** the **change in area** is a **small change**, or if **one extra teddy bear** is **produced** in a **mass production line** of teddy bears, the **change of cost** in production is a **small change**.

$\delta x = 2cm$



$\delta A \approx \delta x \frac{dA}{dx}$

$\delta x = 1$

$\delta C \approx \delta x \frac{dC}{dx}$

Change in **area** from **increasing** the **length** of a **road** by **2cm** (i.e. $\delta x = 2$, $\delta A \approx \delta x \frac{dA}{dx}$)

Change in **cost** from **increasing** the production of **teddy bears** by **one** (i.e. $\delta x = 1$, $\delta C \approx \delta x \frac{dC}{dx}$)

When dealing with **small changes**, there is a **formula** we can apply which allows us to **approximate the effect** of the **small change**.

Small Change Formula: $\delta y \approx \frac{dy}{dx}\delta x$

What this formula tells us is that we can **approximate** the change in a variable y, by **multiplying** the **small change in x** (i.e. δx) with the **derivative** (i.e. $\frac{dy}{dx}$) that has the **original value** of x **substituted in**.

For instance, suppose we had the function $y = 9x^3 - 2x^2 + 2$, and wanted to know the **change** in y when x increases from 4 to 4.01. To determine this, we can first determine the **derivative** of this **function** and then **apply** our **small change equation**: $\delta y \approx \frac{dy}{dx}\delta x$.

$$y = 9x^3 - 2x^2 + 2$$
$$① \quad \frac{dy}{dx} = 27x^2 - 4x$$
$$② \quad \frac{dy}{dx}(4) = 27(4)^2 - 4(4)$$
$$\frac{dy}{dx}(4) = 416$$
$$③ \quad \delta y \approx \frac{dy}{dx}\delta x$$
$$\delta y \approx 416 \times (4.01 - 4)$$
$$\delta y \approx 4.16$$

① **Determine** the **derivative** of y

② **Substitute** the **original value** of x into $\frac{dy}{dx}$

③ **Apply the small change formula**

If we compared this to the **actual change** in y by **subtracting** $y(4)$ from $y(4.01)$, we can see that the result is **almost identical**:

$$y(4.01) = 9(4.01)^3 - 2(4.01)^2 + 2$$
$$y(4.01) = 550.17$$

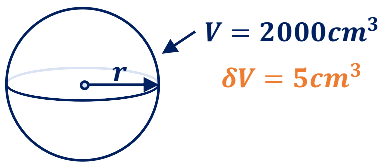
$$y(4) = 9(4)^3 - 2(4)^2 + 2$$
$$y(4) = 546$$

$$\Delta y = y(4.01) - y(4)$$
$$\Delta y = 550.17 - 546$$
$$\Delta y = 4.17$$

The **small change formula** can also be used to **approximate** the **change** in x from a **small change** in y, by **re-arranging** the equation to be in terms of δx.

Re-arranged Small Change Formula: $\delta x \approx \dfrac{\delta y}{dy/dx}$

For instance, let's say we had a **sphere** with a **volume** of $2000cm^3$ and we wanted to know the **change** in **radius** if the **volume** was **increased** to $2005cm^3$.



To start, we need to determine our **original radius**, r_o, which can be determined by **re-arranging** the **volume equation** to **solve** for r.

$$V = \frac{4}{3}\pi r^3$$

① $$r = \sqrt[3]{\frac{V}{\frac{4}{3}\pi}}$$

② $$r_o = \sqrt[3]{\frac{2000}{\frac{4}{3}\pi}}$$

$$r_o = 7.82cm$$

① **Re-arrange** the **volume equation** to **solve** for r

② **Substitute** in $V = 2000cm^3$ to determine r_o

Next, we can **differentiate** the **volume equation** to determine $\frac{dV}{dr}$, and then **re-arrange** our small change equation $\delta V \approx \frac{dV}{dr}\delta r$ into $\delta r \approx \frac{\delta V}{\frac{dV}{dr}}$ to determine the **approximate change** in r.

$$V = \frac{4}{3}\pi r^3$$

① $$\frac{dV}{dr} = 4\pi r^2$$

② $$\frac{dV}{dr}(7.82) = 4\pi(7.82)^2$$

$$\frac{dV}{dr}(7.82) = 768.5$$

③ $$\delta r \approx \frac{\delta V}{\frac{dV}{dr}}$$

④ $$\delta r \approx \frac{5}{768.5}$$

$$\delta r \approx 0.0065cm$$

① **Determine** the **derivative** of V

② **Substitute** the **original value** of r into $\frac{dV}{dr}$

③ **Re-arrange** the **small change formula** to **solve** for δr

④ **Substitute** in $\delta V = 5cm^3$ and $\frac{dV}{dr} = 768.5$ determine δr

Percentage Change

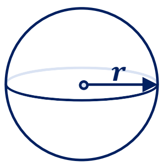
Small change can **also** applied in the context of **percentages**, where the **small percentage change** of a **variable** can be used to **approximate** the **small percentage change** of the **other variable** in a function.

A **percentage change** represents the **small change** in a variable (e.g. δx) **divided** by the **original value** of that **variable** (e.g. x).

$$\boxed{\textbf{Small Percentage Change: } \%\delta x = \frac{\delta x}{x}}$$

Utilising this fact, we can apply our **original small change equation** with the additional step of **dividing** the equation by the **original variable**, so that we are dealing with **percentages**.

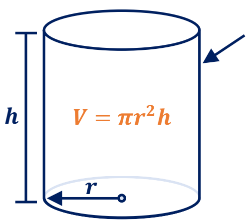
For instance, the **small percentage change** in the **volume** of a **sphere** and **cost** of a **teddy bear** would follow the equations:



$$\%\delta V = \frac{\delta V}{V} \approx \frac{\frac{dV}{dr}\delta r}{V}$$

$$\%\delta C = \frac{\delta C}{C} \approx \frac{\frac{dC}{dx}\delta x}{C}$$

Applying this new equation to a **percentage change** example, suppose we had a **cylinder** with **volume** $V = \pi r^2 h$, and we wanted to know the **change** in **volume** if the **radius** is **increased** by 2%. To start we can determine the **derivative** of the **volume equation** (i.e. $\frac{dV}{dr}$).



$$\frac{\delta r}{r} = 0.02$$

$$V = \pi r^2 h$$
$$\frac{dV}{dr} = 2\pi r h$$

Next, we need to take our **small change equation** and **divide** it by **volume**, V, to get **the change in volume** as a **percentage**. We can then simplify the resulting equation and substitute $\frac{\delta r}{r} = 0.02$ to determine $\frac{\delta V}{V}$.

$$\delta V \approx \frac{dV}{dr}\delta r$$

$$① \quad \frac{\delta V}{V} \approx \frac{\frac{dV}{dr}\delta r}{V}$$

$$② \quad \frac{\delta V}{V} \approx \frac{2\pi r h \times \delta r}{\pi r^2 h}$$

$$③ \quad \frac{\delta V}{V} \approx 2 \times \frac{\delta r}{r}$$

$$④ \quad \frac{\delta V}{V} \approx 2 \times 0.02$$

$$\frac{\delta V}{V} \approx 4\%$$

① **Divide** the small change equation by V to get $\frac{\delta V}{V}$

② **Substitute** in the equations for V and $\frac{dV}{dr}$

③ **Simplify** the **equation** to have $\frac{\delta r}{r}$

④ **Substitute** in $\frac{\delta r}{r} = 0.02$

Worked Example 1

Having finally grasped this new skill of small change, Wallace is eager to put his knowledge to the test and Janitor Peter is eager to guide him.

(a) To get him started, Janitor Peter asks Wallace to determine the **change** in y when x increases from 3 to 3.01 in the function: $y = x^3 + 3x - 2$. **Help** show Wallace how to **solve** this.

$$y = x^3 + 3x - 2$$
$$\frac{dy}{dx} = 3x^2 + 3$$
$$\frac{dy}{dx}(3) = 3(3)^2 + 3$$
$$\frac{dy}{dx}(3) = 30$$
$$\delta y \approx \frac{dy}{dx}\delta x$$
$$\delta y \approx 30 \times (3.01 - 3)$$
$$\delta y \approx 0.3$$

(b) Wallace has a spherical soccer ball with a **volume** of $1500cm^3$, which he pumped up to have a **volume** of $1505cm^3$. **Help** Wallace to **calculate** the **change in radius** (δr) in the soccer ball from this volume change, provided $V = \frac{4}{3}\pi r^3$.

$$V = \frac{4}{3}\pi r^3$$
$$r = \sqrt[3]{\frac{V}{\frac{4}{3}\pi}}$$
$$r_o = \sqrt[3]{\frac{1500}{\frac{4}{3}\pi}}$$
$$r_o = 7.101cm$$
$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$
$$\frac{dV}{dr}(7.101) = 4\pi(7.101)^2$$
$$\frac{dV}{dr}(7.101) = 633.6$$
$$\delta r \approx \frac{\delta V}{\frac{dV}{dr}}$$
$$\delta r \approx \frac{5}{633.6}$$
$$\delta r \approx 0.0079cm$$

(c) Finally, Wallace has a differently shaped ball with a **volume** based on the equation $V = 4r^3$. If Wallace **increases** its **radius** by 1.5%, what is the **approximate percentage change** in **volume**?

$$V = 4r^3$$
$$\frac{dV}{dr} = 12r^2$$
$$\delta V \approx \frac{dV}{dr}\delta r$$
$$\frac{\delta V}{V} \approx \frac{\frac{dV}{dr}\delta r}{V}$$

$$\frac{\delta V}{V} \approx \frac{12r^2 \times \delta r}{4r^3}$$
$$\frac{\delta V}{V} \approx 3 \times \frac{\delta r}{r}$$
$$\frac{\delta V}{V} \approx 3 \times 0.015$$
$$\frac{\delta V}{V} \approx 4.5\%$$

Differentiation Rules

The **differentiation rules** we have looked at for **polynomials** are summarised below.

Rule	Formula
Power Rule	If $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$
Product Rule	If $y = u \times v$ then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
Quotient Rule	If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
Chain Rule	If $y = f(u)$ and $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Derived from Chain Rule	If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

The **second derivative** is the function measuring the **instantaneous rate of change** of the **first derivative**. It has the function notion: $f''(x)$ or $\frac{d^2y}{dx^2}$.

The process of determining the **second derivative** is the **same** as determining the **first derivative**. This means we can also use the **product rule**, **quotient rule** and **chain rule** for the **second derivative**.

Derivative and Graph Sketching

The **transformations** when **sketching** the derivative of a function are summarised as follows.

$f(x)$ Key Point	First Derivative $f'(x)$ Features
Maximum/minimum turning point	**x-intercept (root)**
Point of Inflection	**Maximum/minimum turning point**
Horizontal Point of Inflection	**Maximum/minimum turning point** at the **x-axis (root)**
When $f(x)$ has a **positive gradient**	**$f'(x)$ above x-axis**
When $f(x)$ has a **negative gradient**	**$f'(x)$ below x-axis**

Our goal in **graph sketching** is to locate the **x-intercepts** and **y-intercept**, along with the locations of any **critical points** that include **maximum** and **minimum turning points**, **points of inflection**, and **horizontal points of inflection**.

We start by determining the location of the stationary points by solving for $f'(x) = 0$. Then to **determine** the **nature** of these points we use the **second derivative test**, which is summarised below.

Second Derivative Test		
Maximum Turning Point	**Minimum Turning Point**	**Inconclusive**
$f''(x) < 0$	$f''(x) > 0$	$f''(x) = 0$

When the **second derivative test** is **inconclusive**, we must use the **first derivative test**.

The **first derivative test** determines the function's gradient on either side of a **stationary point** (e.g. $x \pm 1$) to determine its nature. Based on the **positive** or **negative** **nature** on either side, you can deduce whether it's a **maximum turning point**, **minimum turning point** or **horizontal point of inflection**. The type of point can be summarised as follows.

Minimum Turning Point

$x = a^-$	$x = a$	$x = a^+$
$-ve$	0	$+ve$

Horizontal Point of Inflection

$x = a^-$	$x = a$	$x = a^+$
$-ve$	0	$-ve$
$+ve$	0	$+ve$

Maximum Turning Point

$x = a^-$	$x = a$	$x = a^+$
$+ve$	0	$-ve$

Finally, to locate **points of inflection** we look for the points where $f'(x) \neq 0$ and $f''(x) = 0$ and the **second derivative sign test** to **justify** that it is a **point of inflection**.

Rate of Change and Rectilinear Motion

The **derivative** of a function can be used in practical situations to determine the **instantaneous rate of change** and **average rate of change.**

Instantaneous rate of change is the **rate of change** of a function at a **given instant**. It is determined by **substituting** the **x-coordinate** into the **derivative** of the function (i.e. $f'(x)$).

Average rate of change is the **rate of change** of a function over a **period** of **time**. It is determined by dividing the **difference** in the **value** of the function at the **two points** in time, by the **change** in **time** (i.e. $\frac{f(t_2)-f(t_1)}{\Delta t}$)

The **key difference** is that **instantaneous rate of change** measures the **rate of change** at a **given instant**, whereas **average rate of change** measures the **rate of change** over a **period of time**.

Differentiation can also track the **motion** of an object through its **displacement (s)**, **velocity (v)** and **acceleration (a)** in relation to **time (t)**.

$$\text{Displacement } (s) \xrightarrow{\frac{ds}{dt}} \text{Velocity } (v) \xrightarrow{\frac{dv}{dt}} \text{Acceleration } (a)$$

Optimisation

An object will have an **optimum dimension** (cost, weight or size etc.) when its function reaches a **maximum** or **minimum turning point**. Therefore, if we **differentiate** the function and **solve** for **zero** (i.e. $f'(x) = 0$), we can determine the **location** of **critical points**. To determine the **nature** of the **critical point**, we apply our **second derivative test**.

The **steps** for **optimisation** are summarised as follows:

Optimisation Steps

① Determine a **single variable equation** (e.g. $f(x)$) using the information provided.

② **Differentiate** $f(x)$ and **solve** for $f'(x) = 0$ to find the **stationary point locations**.

③ **Differentiate** $f'(x)$ to find the **second derivative** and **apply** the **second derivative test** to determine the **nature** of each stationary point.

④ If required, **substitute** the **x-value** of the **maximum** or **minimum** point into $f(x)$ to find the final answer.

Small Change

Small change tells us that we can **approximate** the change in a variable y, by **multiplying** the **product** of the **small change** in x (i.e. δx) by the **derivative** (i.e. $\frac{dy}{dx}$) with the **original value** of x **substituted in**.

Small Change Formula: $\delta y \approx \frac{dy}{dx}\delta x$

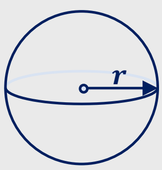
The **small change formula** can also be used to **approximate** the **change** in x from a **small change** in y, by **re-arranging** the equation to be in terms of δx.

Re-arranged Small Change Formula: $\delta x \approx \dfrac{\delta y}{dy/dx}$

Small change can also be expressed in **percentages**. For instance, a **small percentage change** in x can be represented as:

Small Percentage Change: $\%\delta x = \frac{\delta x}{x}$

When dealing with **percentages**, we can apply our **original small change equation** with the **additional** step of **dividing** the equation by the **variable** with the **percentage change**. For instance, the **small percentage change** in the **volume** of a **sphere** and **cost** of a **teddy bear** would follow the equations:



$$\%\delta V = \frac{\delta V}{V} \approx \frac{\frac{dV}{dr}\delta r}{V}$$

$$\%\delta C = \frac{\delta C}{C} \approx \frac{\frac{dC}{dx}\delta x}{C}$$

Problem Set 2 – The Second Derivative and Differentiation Applications Progressive Questions

Concept 1

The Second Derivative and Graph Sketching – Progressive Questions (7 questions)

Repetitive questions: 1.11 → 1.71 (9 questions)

The Differentiation Battle!

Starring: Jevon, Rabea, Tom, Kerry, Sparrow and Jayden

With the war in full effect, a constitution has been set out by Janitor Peter. In this constitution it reads 'Only pranks that involve mathematics may be used. Failure to do so will result in expulsion of the group from the war.'

With their understanding of differentiation locked in, they now want to start applying their knowledge to all of its different applications. In addition to their new knowledge of the second derivative and derivative sketching, Tom and Rabea have their sights set on the 'shape shifting' building.

Before the apps students arrive tomorrow, they decide to practice their second derivatives and derivative sketching.

Second Derivatives: Q1, Q2

Repetitive: 1.11 → 1.22 (3 questions)

[15 marks]

1. Rabea and Tom are already competent in determining the **first derivatives** of functions. **Help** to show Rabea and Tom the **second derivatives** of the following functions.

(a) $y = x^2 - 4x$ **(2)**

(b) $y = 4x^3 + 2x + 4$ **(2)**

(c) $y = 7x^4 - 3x^3 + 10x^2 + 9$ **(2)**

(d) $y = \sqrt{x}$ **(3)**

(e) $y = \frac{4}{x^2}$ **(3)**

(f) $y = \frac{4x^3 - 2x^4}{x}$ **(3)**

[16 marks]

2. Eager to take it up a notch, they reach out to school captain Kerry, who writes up some **harder functions** on the board. Help the boys to determine the **second derivatives** of these functions.

(a) $y = 2x^5 - 6x^3$ **(2)**

(b) $y = 3x - \frac{1}{x}$ **(2)**

(c) $y = \frac{1}{x^{\frac{3}{2}}}$ **(3)**

(d) $y = 6x^3 - \frac{3}{x}$ **(3)**

(e) $y = \frac{7x^3 - 6x^5}{x^4}$ **(3)**

(f) $y = \frac{4}{x^4} - 4\sqrt{x}$ **(3)**

Derivative Sketching: Q3, Q4

Repetitive: 1.31 → 1.41 (2 questions)

With their understandings of second derivatives set, the boys decide to head over to the shape shifting building. The apps students are in class, and the boys pull up the control panel ready to cause some mayhem.

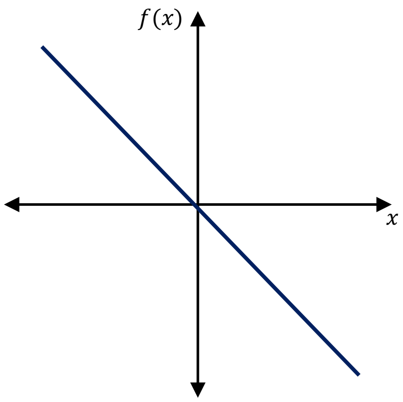
As they are only allowed to prank the apps students if it involves differentiation, the boys decide to turn the building into functions that are the derivatives of other functions.

[11 marks]

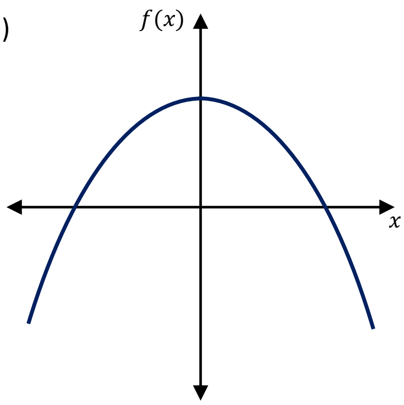
3. To start out, the boys try to **sketch** the **derivative of the functions** below. Help the boys to do this so that they can turn the building into the **shape** of these functions.

(a) On each of the graphs, label the points where the gradient is **positive**, **negative** and **equal to zero**. **[4]**

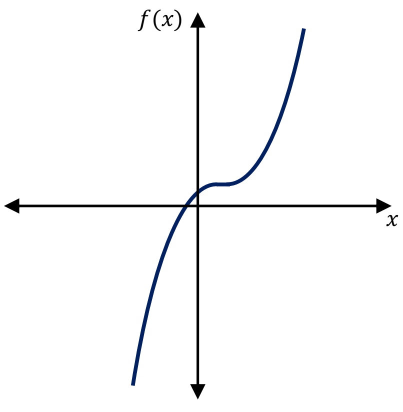
(i)



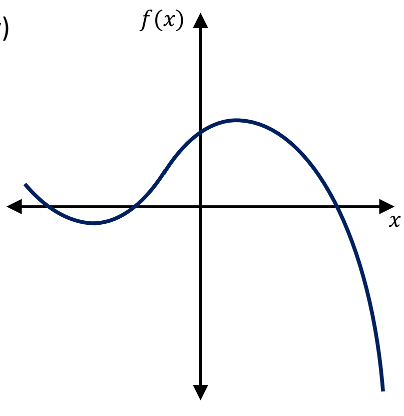
(ii)



(iii)



(iv)



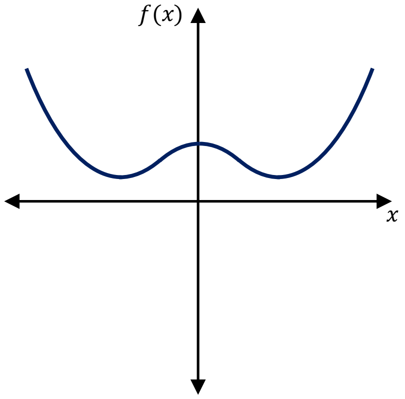
(b) Where possible, **circle** any **stationary points** or **points of inflection**. **[3]**

(c) Sketch the **first derivatives** of each of these functions. **[4]**

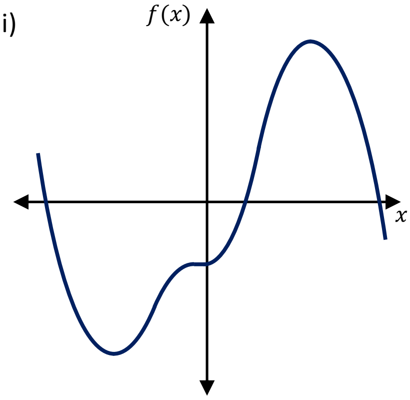
[16 marks]

4. The boys are struggling to contain themselves as they watch the building turn into all different shapes and functions. To take it one step further, they decide to use some even more complicated functions. **Help** the boys to sketch the **first** and **second derivatives** of the **following functions**.

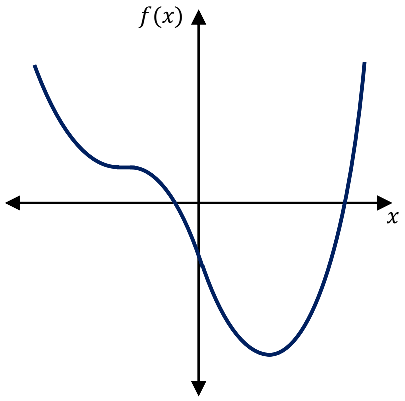
(i)



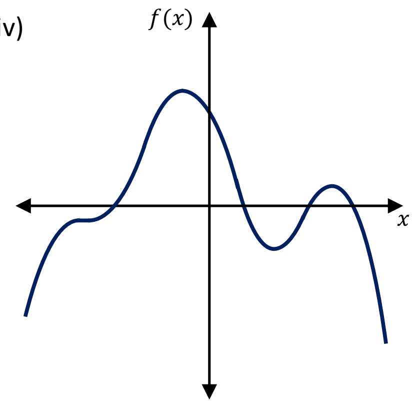
(ii)



(iii)



(iv)



Graph Sketching: Q5, Q6, Q7

Repetitive: 1.51 → 1.71 (4 questions)

Seeing the mayhem that Tom and Rabea are causing, fellow methods student Jevon walks over and is keen to get amongst the chaos. Jevon tells the boys that the boys can expand their math knowledge if they try and determine the equations to random functions Jevon inputs.

Rabea and Tom turn to each other and smile, excited to practice some math and cause mayhem.

[CA] [29 marks]

5. To start out, Jevon wants to check the boys can determine the **stationary points** and **inflection points** of different equations. Based on the equations below, help to find the **x-coordinate** of the **stationary points** and **points of inflection** for the following functions, and **classify** the **nature** of each of these points. Finally, help the boys to create a **rough sketch** of the function.

(a) $y = 3x^2 + 4$	**(4)**	(d) $y = \sqrt{x} + 4x$	**(5)**
(b) $y = x^2 + 2x + 4$	**(5)**	(e) $y = x^4 + 7x^3 + 4x^2 + 2$	**(5)**
(c) $y = \frac{x^3}{3} + \frac{3x^2}{2} + 2x + 8$	**(5)**	(f) $y = x^5 + x^3 + 2x^4$	**(5)**

[12 marks]

6. As a final test, Jevon then asks Rabea and Tom to **sketch** the functions based on the information in the tables below. For each of the **tables below**, **help** the boys to **sketch** a possible **graph** by taking the steps Jevon has outlined (you can assume that POI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$ and HPOI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$)

(a)

Point	A	B	C	D	E	F
$\frac{dy}{dx}$	$-$	0	$+$	$+$	0	$-$
$\frac{d^2y}{dx^2}$	$+$	$+$	$-$	0	$-$	$+$

(i) At what points are their **turning points**? **[1]**

(ii) At what point is the **inflection point**? **[1]**

(iii) **Sketch** a **possible graph** for this function. **[1]**

(b) **Sketch** a **possible graph** for this function. **[3]**

Point	A	B	C	D	E	F	G
$\frac{dy}{dx}$	$-$	0	$-$	$-$	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$+$	0	$+$	0	$+$	$+$	$-$

(c) **Sketch** a **possible graph** for this function. **[3]**

Point	A	B	C	D	E	F	G
$\frac{dy}{dx}$	0	$-$	0	$+$	0	$+$	0
$\frac{d^2y}{dx^2}$	$-$	0	$+$	0	0	0	$-$

(d) **Sketch** a **possible graph** for this function. **[3]**

Point	A	B	C	D	E	F	G
$\frac{dy}{dx}$	$+$	0	$-$	0	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$-$	$-$	0	0	0	$+$	$-$

[28 marks]

7. Satisfied that the boys know their stuff, Jevon begins shape shifting the building and disturbing the apps students! To make sure they are practicing differentiation, Jevon doesn't tell Rabea and Tom the equation he has inputted, and the boys must **figure out** the **equation** based on the **information** they can gather. Each time he changes the shape of the building, Rabea and Tom collect important information. **Based on this information**, help the boys **determine** each **equation**.

(a) The equation has the format $y = ax^2 + bx + c$ and the boys observe that the equation has a y-**intercept** at $(0, 2)$ and a **minimum turning point** at $(1, 4)$. **Help** the boys to **determine** its **equation** by determining the **values** of a, b and c. **[4]**

(b) The equation has the format $y = ax^3 + bx^2 + cx + d$ and the boys observe the equation has a y-**intercept** at $(0, 3)$, a **maximum turning point** at $x = -1$, and a **point of inflection** at $(1, -8)$. Help them to **determine** its **equation** by determining the **values** of a, b, c and d. **[6]**

(c) The equation has the format $y = ax^4 + bx^3 + cx^2 + d$ and the boys observe that the equation has a y-**intercept** at $(0, 6)$, a **minimum turning point** at $x = -2$, an **inflection point** at $x = 1$, and a tangent to the equation $y = 3x + 8$ at $x = -1$. Help the boys to **determine** its **equation** by determining the **values** of a, b, c and d. (Hint: the gradient of the tangent equation is the important part). **[CA] [8]**

(d) The equation has the format $y = ax^5 + bx^4 + cx^3 + dx + e$ and the boys observe that the equation has a y-**intercept** at $(0, 2)$, a **maximum turning point** at $x = -0.5$, a tangent to the equation $y = 3x + 8$ at $x = 1$, and another tangent line $y = 5x - 3$ at $x = 0$. The boys can also identify that the equation has an inflection point at $x = -1$. Help the boys to **determine** its **equation** by determining the **values** of a, b, c, d and e. **[CA] [8]**

Absolutely fed up with all of the disruptions to their lessons, apps leader Sparrow and his infamous side-kicks Jayden and Patrick storm out into the courtyard to face Rabea, Tom and Jevon. "You methods students better watch you back! I'm going to get you back!" shouts Sparrow.

Rabea and Tom turn to each other, and then break out into laughter. "You guys can't, you don't understand differentiation!" replies Rabea.

Sparrow stares at the boys, grabs Jayden and Patrick, and storms off saying "Come on boys, let's go learn some differentiation and get them back!"

Concept 2

Rates of Change and Rectilinear Motion – Progressive Questions (7 questions)

Repetitive questions: 2.11 → 2.71 (7 questions)

The Apps Students Return Fire!

Starring: Sparrow, Jayden, Patrick, Teacher Andrew, Teacher Simon, Janitor Peter

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Enraged to their core, Sparrow, Jayden and Patrick storm into teacher Andrew's office.

"Teacher Andrew, we need to learn some differentiation to get back at the methods students," demands Sparrow.

"Well, I'm not sure if I can help you in the war, but I'll be happy to teach you some rate of change and rectilinear motion which might be of use?" Teacher Andrew replies.

The apps students smile at each other, eager to learn some differentiation and get back at the methods students.

Rate of Change: Q1, Q2, Q3

Repetitive: 2.11 → 2.31 (3 questions)

[CA] [9 marks]

1. To start, teacher Andrew is keen to teach them about **instantaneous rate of change** and **average rate of change**, so that they can **acquire** some projectiles to launch at the methods students. He asks Sparrow and the other apps students to consider the equation which maps the **cost** (in dollars) of a **very rare tomato** over **time** (in years): $\boldsymbol{C(x) = x^3 - 8x^2 + 3}$.

(a) What was the **initial cost** of the tomato and the cost after **ten years**? **[2]**
(b) What is the **average rate of change** in the **cost** of the tomato over the **last ten years**? **[3]**
(c) What is the **instantaneous rate of change** in its **cost** after $\mathbf{12}$ **years**? **[2]**
(d) What is the **average rate of change** of the cost in the $\mathbf{13^{th}}$ **year**? **[2]**

[CA] [11 marks]

2. A second projectile the apps students then consider are **watermelons**, where the n**umber of watermelons available** in the storeroom is tracked by: $\boldsymbol{A(x) = 10 - 0.2x^2}$, where $\boldsymbol{t}$ is the **number of days**.

(a) How many **watermelons** will be available on **day** $\mathbf{5}$? **[1]**
(b) What is the **instantaneous rate of change** in the number of watermelons after the **first week**? **[2]**
(c) What is the **average rate of change** across the **first week**? **[3]**
(d) Briefly discuss, **why** the result of **(b)** is **different** to the result of **(c)**? **[2]**
(e) After how many **days** will **no watermelons** be available? **[3]**

[CA] [9 marks]

3. A third projectile teacher Andrew has suggested are **historical pirate cannonballs**. These are highly sought after, but teacher Andrew believes their fluctuating cost (in dollars) can be tracked by the equation: $C(x) = \frac{2x^3-2}{3}$, where x is **time** in **minutes**.

(a) What is the **cost** of the cannonballs after 2 **minutes**? **[1]**
(b) What is the **average rate of change** during the **tenth minute**? **[3]**
(c) At what **time** is the **instantaneous rate of change** equal to $\$2.25$ **per minute**? **[2]**
(d) At what **time** is the **cost** of a single cannonball equal to $\$1000$? **[2]**
(e) At what time does the **average rate of change** from 2 **minutes** equal to 4.5? **[1]**

Realising that the price of the pirate cannonballs is increasing rapidly every minute, Sparrow, Jayden and Patrick quickly order 5 cannonballs. Instead of buying special tomatoes and watermelons, they decide to go down to their local store and buy ordinary tomatoes and watermelons.

Locked and loaded, the apps students Sparrow, Jayden and Patrick are ready to rain down havoc on the methods building!

Rectilinear Motion: Q4, Q5, Q6, Q7

Repetitive: 2.41 → 2.71 (4 questions)

In order to launch their projectiles, the apps students reach out to teacher Simon who has his very own life size catapult! Teacher Simon happily provides it to them but says, "Remember, under the rules of this war, you must use differentiation when you are firing the projectiles."

"Roger that," replies Jayden. The apps students walk off to the methods building ready to engage.

[CA] [10 marks]

4. To launch their tomatoes at the methods building, they input the equation $s = 3t^2 - 3t + 2$, where s is **displacement** in **meters** and t is **time** in **seconds**. They then begin launching the tomatoes at the building!

(a) How far has the tomato **travelled** after **3 seconds**? **[1]**
(b) What is the **velocity** and **acceleration equations** for the tomato? **[2]**
(c) At what **time** is the tomato travelling at a **velocity** of $v = 6m/s$? **[2]**
(d) What is the **acceleration** of the tomato after 2 **seconds**? **[1]**
(e) If the building is located $100m$ away, at **what time** will the tomatoes reach the building? **[4]**

[CA] [14 marks]

5. With the tomatoes having little impact, the methods students are laughing on. The apps students **adjust** their equation to $s = t^3 - 4t^2 + t + 5$, where t is in **seconds**.

(a) If they need the tomatoes to **travel** $100\ m$, determine the **time** at which they will reach the building. **[2]**
(b) At what **velocity** will the tomatoes be travelling when they **reach the building**? **[2]**
(c) What is the **acceleration** of the tomato after **four seconds**? **[2]**
(d) Do the tomatoes ever **change direction**? If so at what **time(s)**? **[5]**
(e) It appears that this equation causes the tomatoes to **fly backwards**, at what **time(s)** is the tomato back at its **original location**? **[3]**

[CA] [11 marks]

6. Tired of the tomatoes, the apps students decide to upgrade to the **pumpkins**. Still watching on from inside the classroom, the methods students are still confident that the pumpkins won't reach them with their new equation. The apps students input the equation: $s = 12\sqrt{t} - t^2$.

    (a) How far will the pumpkin have travelled after **3 seconds**? **[1]**
    (b) What is the **velocity** after **5 seconds**? **[2]**
    (c) What is the **instantaneous rate of change** in **distance** at 5 seconds? **[2]**
    (d) What is the **average rate of change** in **acceleration** between 4 and 7 seconds? **[3]**
    (e) Will the pumpkins **reach** the building if the building is $100m$ **away**? **[3]**

Watching the whole fiasco from the sidelines, Janitor Peter is disappointed as he has seen that the apps students have failed to apply differentiation correctly.

Being the genius he is, Janitor Peter walks over to the apps students and decides to start helping the apps students, but this time by launching the cannonballs!

The methods students see the chaos that is about to come, and all of them run outside, evacuating the building.

[CA] [11 marks]

7. With everyone evacuated, all of the methods students watch on as Janitor Peter gives the apps students the equation: $s = t^5 + 3t^2 + 6t + 1$ and the apps students load up the catapult with the five cannonballs.

    (a) What will be the **velocity** and **acceleration** equations tracking these cannonballs? **[2]**
    (b) After how many **seconds** has the **acceleration** of the cannonballs reached $12m/s^2$? **[2]**
    (c) If the building is $100m$ **away**, at what **velocity** will the cannonballs be **travelling** when they **hit** the **building**? **[3]**
    (d) What is the **average rate of change** in the **velocity** of the cannonballs between the **time** they are **released** to the **time** they **reach** the **building**? **[4]**

Walking past on his morning walk, teacher Andrew is shocked as he watches the method building collapse to the ground! Furious, he storms over to Janitor Peter.

"What have you done Janitor Peter? You are going to have to pay for a new methods building!" Shouts teacher Andrew.

"No problem," replies Janitor Peter. "I was kind of hoping for a new methods building, and I think it will be a great way for the methods students to practice their optimisation skills."

All of the methods students cheer and raise Janitor Peter up in the air. Teacher Andrew storms out angrily, losing to janitor Peter once again.

Concept 3

Optimisation – Progressive Questions (9 questions)

Repetitive questions: 3.11 → 3.51 (7 questions)

The Optimised Methods Building

Starring: Kerry, Jevon, Rabea, Rupert Teacher Simon and Teacher Andrew

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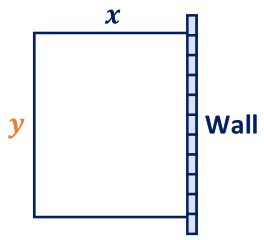
After the destruction of their beloved Methods building, Janitor Peter has mysteriously given the students an unlimited amount of funding to rebuild the Methods building. The only rule is that the building must be designed using optimisation.

Optimisation: Q1, Q2, Q3, Q4, Q5, Q6

Repetitive: 3.11 → 3.51 (7 questions)

[CA] [5 marks]

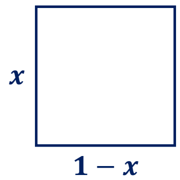
1. Unlike the Janitor, Teachers Andrew and Simon are less convinced the students are ready to handle difficult real-world matters, so they provide a theoretical example on the whiteboard: "If a farmer has $900m$ of **fencing** for his sheep that he can place against a wall, what should the **width** x and **length** y be to provide the **maximum area** for the sheep to roam." Help the students to determine these **optimum dimensions**.



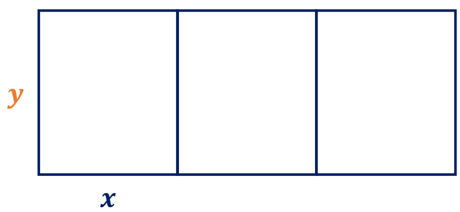
[CA] [8 marks]

2. The teachers are thinking of having a **rectangular courtyard** in the middle of the building, with **fencing** around it.

(a) Show that to **maximise** the **area** of the courtyard the fencing should be a **square** shape by letting the **length** be x and the **width** be $1 - x$. **[4]**

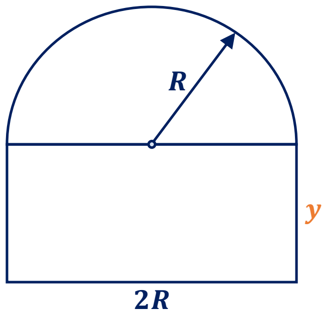


(b) The teachers have designed the **rectangular field** below, which is fenced off into **3 sections** and has **fencing** all around the **outside** of the field as well. If there is $600m$ of fencing available, what is the **maximum possible total area** of the field? **[4]**



[CA] [6 marks]

3. Teacher Simon is coming up with window ideas for his classroom and decides to go with a **semicircle** on top of a **rectangle**, separated by a wooden frame through the middle and surrounded by wooden frame. If there is $6m$ of wooden frame available, use **calculus** to determine the **maximum window area** for sunlight, and the **dimensions** of the window at this **maximum area**. Note: the **perimeter** of a **circle** is $2\pi r$ and the **area** of a **circle** is πr^2.

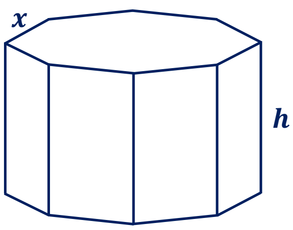


[CA] [5 marks]

4. After watching the teachers and students working on building designs, Rabea and Jevon have opened their own building supply company to try and pinch Peter's funding for themselves. The **daily revenue** of their business is modelled by $R(x) = -2x^3 + 15x^2 + 36x - 100$, for an x **number of sales** and R **dollars**. Use **calculus** to determine the **number of sales** they need to **maximise their profit** and what this **maximum profit** is.

[CA] [11 marks]

5. Teacher Andrew is getting carried away with his freedom of choice and designs an **octagonal prism** as the shape of his classroom. Given that the **area** of an **octagon** is equal to $2x^2(1+\sqrt{2})$, where x is the **length** of each side of the octagon, help Andrew design his building.



(a) Use the diagram to determine an **expression** for the **volume** and **total surface area** of the room, including ceiling and floor (in terms of x, and height h). **[3]**

(b) If **volume** of the room should be $1000m^3$, show that the **surface area** of the room is given by $SA = 4x^2(1+\sqrt{2}) + \frac{4000}{x(1+\sqrt{2})}$. **[3]**

(c) Using **calculus**, determine the **minimum surface area** of the room. **[5]**

[CA] [10 marks]

6. Rupert notices the ridiculous ideas and decides to plan a **rectangular prism** shaped classroom with a **height h**. The **base** of the **rectangular prism** will be a **square** with **length** and **width** x. The **4 sidewalls** will be made from **glass**, which **costs $\$100$ per square metre**, while the square **floor** and the **ceiling** will be made from **wood**, costing **$\$50$ per square metre**. He has been told to achieve an **internal working space of $500m^3$**.

(a) Starting with a diagram, show that the **classroom cost** can be expressed as: $C(x) = 100x^2 + \frac{200000}{x}$. **[5]**

(b) Determine the **minimum cost of materials** for Rupert's room. **[4]**

(c) What will be the **height** of Rupert's room? **[1]**

Difficult Optimisation: Q7, Q8, Q9

Repetitive: 3.11 → 3.51 (7 questions)

[CA] [8 marks]

7. Kerry has run all over Perth to find Rupert's materials, but in search of perfection, Kerry wakes up in a haze and realises she is on Rottnest Island – and needs to get back ASAP! The **total distance** from Rottnest via ferry is $20km$. The ferry has a **maximum set speed** of $40km/h$, but due to choppy conditions, the faster the ferry's speed is set at, the **greater** the **deceleration** from the waves. The **actual speed V** of follows the equation: $V = v - \frac{v^2}{60}$, where v is the **set speed** of the ferry.

(a) Using: $V = \frac{D}{t}$, **re-arrange** this equation to show that the **time** (in minutes) taken by the ferry to complete the trip is given by $t = \frac{20}{v - \frac{v^2}{60}}$. **[2]**

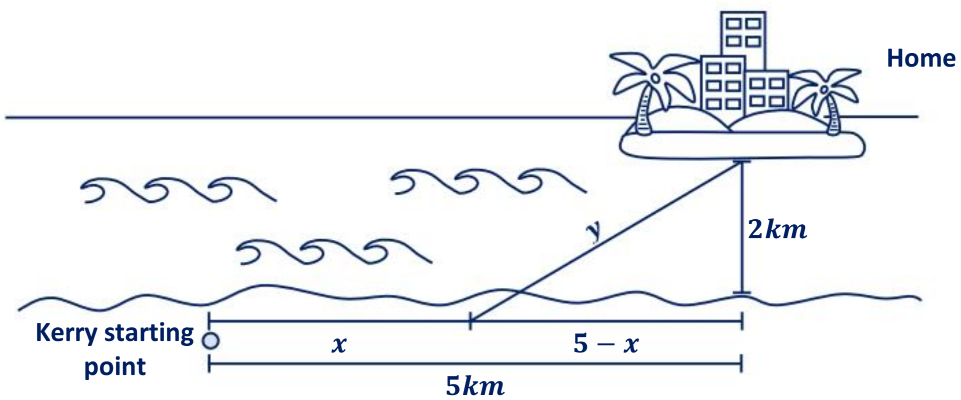
(b) Using calculus, determine the **optimal set speed** of the ferry and therefore the **minimum time taken** to get to shore. **[6]**

[CA] [9 marks]

8. Kerry realises she might be able to **run and swim back home** and needs to decide quickly. The class is **exactly $2km$** across the ocean from the closest point to the Rottnest shore, but Kerry is $5km$ further up the shore from this point. Kerry can **run** at a **speed of $15km/h$** and **swims** at a **speed of $5km/h$**. Again, the equation: $t = \frac{D}{V}$ will be helpful in solving the problem.

(a) Using calculus, **how far** should **Kerry run** before she swims across the river to **minimise** the **time taken**? **[6]**

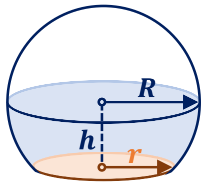
(b) Assuming the **ferry travels** for $20km$ at $15km/hr$, should Kerry take the **ferry** or **run and swim** instead? **[3]**



[CA] [12 marks]

9. Janitor Peter has always dreamed about having a **swimming pool** at Mathematics College, so decides to focus on making the most perfect swimming pool ever constructed – he believes that a **spherical segment pool shape** will do the trick! He also wants the **curved, side walls** of the pools to be made from **glass**.

If we consider the **sphere radius** as $\boldsymbol{R}$, the **radius of the pool floor** as $\boldsymbol{r}$ and the **pool depth** as $\boldsymbol{h}$, then the **volume** of the pool is given by $V = \frac{\pi}{6}h(3R^2 + 3r^2 + h^2)$ and the **surface area** of the walls (excluding bottom surface of pool) is given by $A = 2\pi Rh$. Janitor Peter wants the **area of the pool floor** to be $314.2\ m^2$ and only has a budget for $300m^2$ of **glass** for the walls.



(a) What are the **values or equations** for the **radius of the pool floor** (r) and the **radius of the sphere** (R)? **[4]**

(b) Show that the **volume** of the pool is given by $V = \frac{\pi}{6}h^3 + 50\pi h + \frac{11250}{\pi h}$. **[3]**

(c) Janitor Peter wants this pool to be as **compact as possible**. Use **calculus** to help Janitor Peter to determine the **minimum volume** possible for this pool. **[5]**

Concept 4

Small Change – Progressive Questions (5 questions)

Repetitive questions: 1.11 → 1.33 (3 questions)

The Differentiation Revenge!

Starring: Teacher Andrew, Janitor Peter, Teacher Simon, Rupert, Kerry

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After rebuilding their Methods building, the Methods students decide to give the Apps students a taste of their own medicine. Within their building, they have built their very own catapult that they can use to launch objects at the Apps building.

However, to fall within the 'maths' rules of the war they need to use differentiation, so they decide to use small change to make the perfect projectiles.

Meanwhile, on more serious matters, Teachers Simon and Andrew aim to make a beautiful centrepiece for the nearly completed Methods building!

Small Change: Q1, Q2, Q3, Q4, Q5

Repetitive: 4.11 – 4.51 (6 questions)

Due to the long distance from which they will be firing their catapult, Kerry and Jevon know they must be very exact in the construction of the projectiles to ensure it hits their desired target, the Apps classroom!

[CA] [8 marks]

1. Before finding the perfect projectiles, Kerry and Jevon want to check that their understanding of small change is strong. **Help** Kerry and Jevon determine the **small change** for the following situations:

(a) For the function $y = x^3 - 4x^2 + 2$, determine the **change in y** if the **value of x increases** from 3 to 3.01. **[4]**

(b) For the function $y = 4x^4 - 2x^3 + x^2 - 9$, determine the **change in y** if the **value of x decreases** from 6 to 5.99. **[4]**

[CA] [6 marks]

2. The students also decide they should make a **circular platform** for the catapult. Currently, the **area** of the **circular platform** is $A = 60m^2$ and is based on the equation $A = \pi r^2$.

(a) What was the **original radius r_o** of the platform. **[2]**

(b) What will be the **approximate change** in the **platform area** if the **platform radius** is **increased** by $0.01m$. **[4]**

[CA] [4 marks]

3. As a cheap option for projectiles, Kerry decides to experiment with using **cylindrical cans of corn** with a **volume** based on the equation: $V = \pi r^2 h$. Purchasing them from a local market, the **radius** of the cylinder is $r = 10cm$ and the **height** is $h = 20cm$. If Kerry's research has shown that they should **decrease** the **radius** to $r = 9.98cm$, what will be the **change in the volume** of the cans.

[CA] [12 marks]

4. Using his contacts at Chemistry College, Rabea produces the perfect projectile for their catapult, **perfectly spherical tomatoes**. The initial harvest produced a batch of spherical tomatoes with a **volume** of $350cm^3$. After extensive research and consultation, the Methods students decided on a **slightly larger volume** of $352cm^3$.

(a) Using the **sphere volume formula** of $V = \frac{4}{3}\pi r^3$, determine approximately the **change in radius** of the tomato if the **volume** changes from $350cm^3$ to $352cm^3$. **[6]**

(b) When creating the initial batch of tomatoes, some were left to grow for slightly too long, and had a **radius** 2% **greater** than the other tomatoes. Determine the **approximate percentage change** in the **volume** of the tomatoes. **[6]**

[CA] [12 marks]

5. With a keen eye for mathematics, Jevon has developed his very own **balloon shaped 'Perfect Projectiles'** with a **volume** that follows the equation: $V = \frac{4}{5}r^4$.

(a) If the **original volume** of the perfect projectile is $1000cm^3$ and is reduced to $997cm^3$, determine the **change in the radius**. **[6]**

(b) If the **radius** is **decreased** by 3%, **approximate** the **change in volume**. **[6]**

Problem Set 2 – The Second Derivative and Differentiation Applications Repetitive Questions

Concept 1

The Second Derivative and Derivative Sketching – Repetitive Questions

(9 questions)

Derivative Sketching: Qs 1.11, 1.21, 1.22

[15 marks]

1.11 Determine the second derivatives of the functions and derivatives below.

(a) $y = 3x + 2$ **(1)**

(b) $y = 6x^2 - 2x^3 + 4$ **(1)**

(c) $f(x) = (4x + 2x)(6x^3)$ **(2)**

(d) $f(x) = \sqrt{4x} + \frac{3}{x^6}$ **(2)**

(e) $y = 7x^3 + \frac{4}{x^5} + 9x^{-2}$ **(2)**

(f) $f(x) = \frac{3}{2x+5}$ **(2)**

(g) $y = (2x^2 + 2)(3x + 6)$ **(3)**

(h) $f(x) = (6x + 2)^4$ **(2)**

[16 marks]

1.21 Determine the second derivatives of the functions below.

(a) $y = -3x^3 - x^2$ **(2)**

(b) $y = x + \frac{2}{x^2}$ **(2)**

(c) $y = -\frac{1}{x^{\frac{1}{2}}}$ **(3)**

(d) $y = -\frac{1}{2}x^2 - \frac{1}{x}$ **(3)**

(e) $y = \frac{3x^2+2x^3}{x^4} - 2x$ **(3)**

(f) $y = \frac{4x}{x^3} - \sqrt{x}$ **(3)**

[24 marks]

1.22 Determine the second derivatives of the functions below.

(a) $y = 3x(4x^3 + 6x^2) + 5$ **(3)**

(b) $y = (7x + 2)(x^3 + 4x)$ **(3)**

(c) $f(x) = (6 + 7x)^3 + 4x^{-3}$ **(3)**

(d) $f(x) = \frac{1}{(x+3)^4}$ **(3)**

(e) $y = 8x(2x^2 + 5x)$ **(3)**

(f) $f(x) = \frac{6x+2}{8x}$ **(3)**

(g) $y = \sqrt[3]{2x + 4}$ **(3)**

(h) $f(x) = (3x^2 + 4x)(2x + 5)$ **(3)**

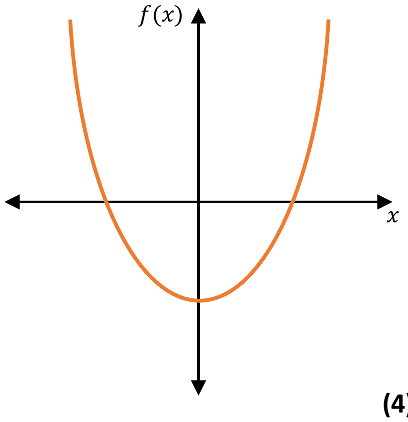
Derivative Sketching: Qs 1.31, 1.41

[16 marks]

1.31 Sketch the **derivatives** and **second derivatives** of the following functions.

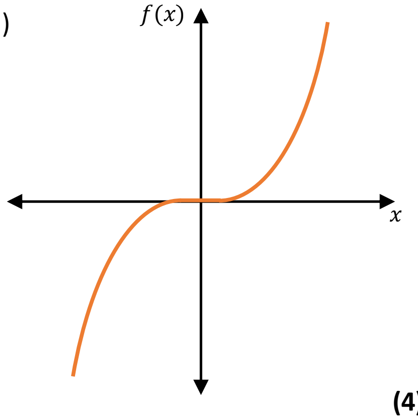
(4)

(i)



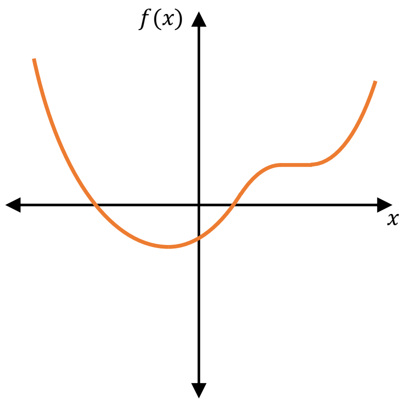
(4)

(ii)



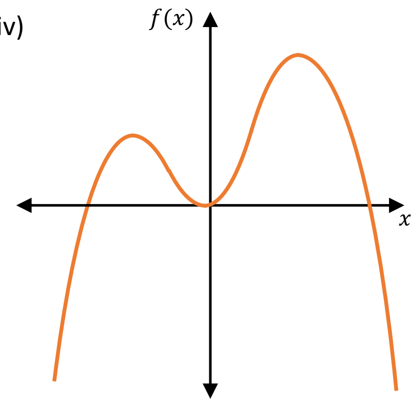
(4)

(iii)



(4)

(iv)

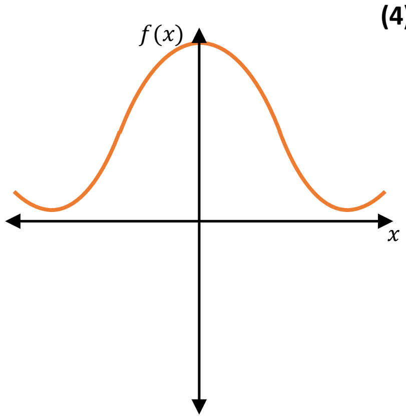


[16 marks]

1.41 Sketch the **derivatives** and **second derivatives** of the following functions.

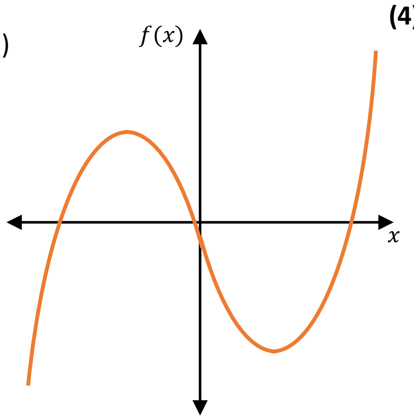
(4)

(i)

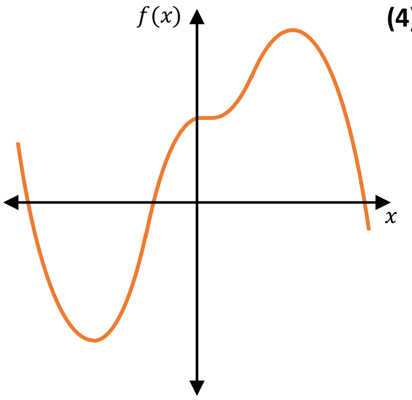


(4)

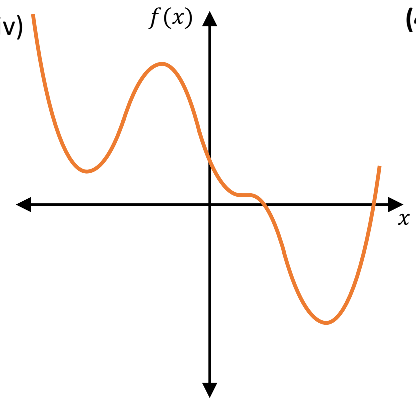
(ii)



(iii) **(4)**



(iv) **(4)**



Graph Sketching: Qs 1.51, 1.61, 1.62, 1.71

[24 marks]

1.51 For each of the functions below, determine the **x-coordinate location** of the **stationary points** and **points of inflection**, and classify the **nature** of each of these points.

(a) $y = x + 2$ **(2)**

(b) $y = 3x^2 + 2x + 4$ **(2)**

(c) $f(x) = \frac{x^2+9}{3x}$ **(3)**

(d) $f(x) = \frac{1}{x^3} - 3x^2 + 2$ **(3)**

(e) $y = (3x^2 + 2)^4$ **(3)**

(f) $f(x) = (2x + 5x)(x^2 + 3x)$ **(3)**

(g) $f(x) = x^4 - \frac{1}{2}x^3 + 3x^2$ **(4)**

(h) $y = x^4 - 2x^3 + x^2 + 6$ **(4)**

[6 marks]

1.61 Based on the **information** provided in the tables below, **sketch** a **possible graph** of the function (you can assume that POI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$ and HPOI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$).

(a)

Point	A	B	C	D	E	F
$\frac{dy}{dx}$	$+$	0	$-$	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$-$	$-$	0	$+$	$+$	$+$

(b)

Point	A	B	C	D	E	F
$\frac{dy}{dx}$	$-$	0	$-$	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$+$	0	$+$	0	$+$	$-$

[6 marks]

1.62 Based on the information provided in the tables below, sketch a possible graph of the function (you can assume that POI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$ and HPOI when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$).

(a)

Point	A	B	C	D	E	F	G
$\frac{dy}{dx}$	$+$	0	$-$	0	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$-$	$-$	0	0	0	$+$	$+$

(b)

Point	A	B	C	D	E	F	G
$\frac{dy}{dx}$	$-$	0	$+$	0	$-$	0	$+$
$\frac{d^2y}{dx^2}$	$+$	$+$	0	$-$	0	$+$	$+$

[CA] [24 marks]

1.71 Based on the information provided below, determine the values of the variables a, b, c and d, and state the final equation.

(a) The equation has the format $y = ax^2 + bx + c$, with a y-intercept at $(0, 2)$ and a maximum turning point at $(1.5, 4.25)$. Determine its equation by finding the values of a, b and c. **[4]**

(b) The equation has the format $y = ax^3 + bx^2 + cx + d$, with a y-intercept at $(0, 6)$, a maximum turning point at $(-1,\ 3)$, and a point of inflection at $x = \frac{2}{3}$. Determine its equation by determining the values of a, b, c and d. **[6]**

(c) The equation has the format $y = ax^3 + bx^2 + cx + d$, with a y-intercept at $(0, 4)$, passes through the point $(2,\ 2)$, a point of inflection at $x = \frac{1}{3}$, and is tangent to the equation $y = 5x$ at $x = 1$. Determine its equation by determining the values of a, b, c and d. **[6]**

(d) The equation has the format $y = ax^4 + bx^3 + cx^2 + d$, with a y-intercept at $(0, 2)$, a minimum turning point at $(1, -\frac{2}{3})$, and a point of inflection at $x = -0.5$, and a tangent to the equation $y = 7x - \frac{34}{3}$ at $x = 2$. Determine its equation by determining the values of a, b, c and d. **[8]**

Concept 2

Rate of Change and Rectilinear Motion – Repetitive Questions

(6 questions)

Rate of Change: Qs 2.11, 2.21, 2.31

[CA] [12 marks]

2.11 Student Wallace has always been very interested in investing and making money. After discovering the importance of rates of change in investing, Wallace decides to create some **equations** to track the values of his stocks. For one of his stocks, its value (in dollars) is tracked by the equation: $V(x) = 0.25t^3 - 2t^2 + 20$, where t is **time** in **days**.

(a) What is the **initial value** of the stock? **[3]**
(b) What is the **average rate of change** of the stock value on the 4^{th} **day**? **[3]**
(c) What is an **equation** describing the **instantaneous rate of change** in the stock value? **[1]**
(d) At what **time(s)** does the stock value **remain unchanged**? **[3]**
(e) At what time does the **value** of the stock drop down to a **minimum value**? **[2]**

[CA] [12 marks]

2.21 A rapidly evolving stock Wallace is looking to invest in has the cost equation: $C(x) = -0.1x^3 + 8x^2 + x^{\frac{1}{2}}$, where x is **time** in **minutes**.

(a) Briefly explain the **difference** between **instantaneous rate of change** and **average rate of change**. **[2]**
(b) What is the **average rate of change** between 5 and 10 **minutes**? **[2]**
(c) Wallace wants to sell the stock once it reaches its peak. At what **time** will the stock reach its **maximum value**? **[3]**
(d) At what **time** is the **instantaneous rate of change** of the stock equal to -5.2? **[2]**
(e) Knowing the stock is going to crash, what will be the **instantaneous rate of change** when the stock cost crashes to **zero dollars**? **[3]**

[CA] [8 marks]

2.31 Wallace has a computer program for one of his more stable stocks, which purchases small amounts of the stock based on the equation: $A = \sqrt{3t+4}$ where A is **amount** in **dollars** and t is **time** in **years**.

(a) What is the **average rate of change** of the amount of stock in the **first year**? **[2]**
(b) After how many **years** will Wallace have $\$15$ in stock? **[2]**
(c) What is the **instantaneous rate of change** when the stock is worth $\$7$? **[2]**
(d) At what time does the **average rate of change** across 2 **years** equal to 1.2? **[2]**

Projectile Motion: Qs 2.41, 2.51, 2.61, 2.71

[CA] [8 marks]

2.41 The displacement of a particle is tracked using the equation: $s = 2t^2 + 3$, where s is **displacement** in **meters** and t is **time** in **seconds**.

(a) What is the **position** of the particle after **three seconds**? **[1]**
(b) State an **equation** that can be used to describe the **velocity** of this particle. **[1]**
(c) What is the **instantaneous rate of change** in the **velocity** of the particle at **2 seconds**? **[2]**
(d) What is the **average change** in the particles' **acceleration** from 3 to 4 seconds? **[2]**
(e) At what **velocity** will the particle be travelling when it is located $50m$ **away**? **[2]**

[CA] [12 marks]

2.51 The displacement of a second particle is tracked using the equation: $s = 3t^3 - 2t^2 + 4t + 7$, where s is **displacement** in **meters** and t is **time** in **seconds**.

(a) What is the **initial displacement** of the particle? **[2]**
(b) What is the **instantaneous rate of change** in **position** at **6 seconds**? **[2]**
(c) At what time(s) is the **acceleration** of the particle at $a = 6m/s^2$? **[2]**
(d) What is the maximum distance the particle will reach in the **first 5 seconds**? **[3]**
(e) What is the **maximum velocity** the particle will reach in the **first 5 seconds**? **[3]**

[CA] [11 marks]

2.61 The displacement of a third particle is tracked using the equation: $s = \frac{x+3}{2x+2}$, where s is **displacement** in **meters** and t is **time** in **seconds**.

(a) What is the **initial position** of the particle? **[3]**
(b) What is its **average change** in **velocity** in the 3^{rd} **second**? **[2]**
(c) At what **time** will the particle reach a distance of $0.7m$? **[1]**
(d) What is the **average change** in **acceleration** from the **time** the particle **starts** to the **time** it reaches $0.7m$? **[2]**
(e) What is the **instantaneous rate of change** in the **acceleration** at the time the particle reaches $65m$? **[3]**

[CA] [13 marks]

2.71 The displacement of a fourth particle is tracked using the equation: $s = \frac{t^3}{2} - 4t^2 + 7t + 9$, where s is **displacement** in **meters** and t is **time** in **seconds**.

(a) What is the **initial velocity** of the particle? **[2]**
(b) At what **time(s)** does the **direction** of the **particle reverse**? **[4]**
(c) What is the **maximum acceleration** of the particle in the first **5 seconds**? **[2]**
(d) What is the **average change** in **position** during the first **4 seconds**? **[2]**
(e) What will be the **acceleration** of the particle when it **reaches a distance** of $10m$? **[3]**

Concept 3
Optimisation – Repetitive Questions
(7 questions)

Optimisation: Qs 3.11, 3.12, 3.21, 3.31, 3.41, 3.42, 3.51

[CA] [4 marks]

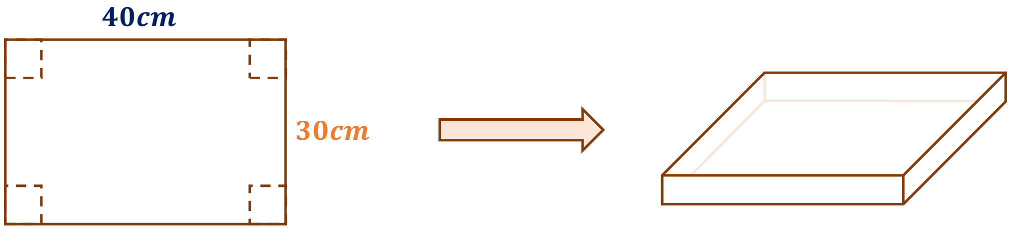
3.11 Use **calculus** to determine the maximum sum of two positive numbers whose **product** is 450.

[CA] [4 marks]

3.12 Use **calculus** to determine **the maximum sum** of **two numbers** whose product is 100, if at least one of the numbers is **negative**.

[CA] [5 marks]

3.21 By cutting out the corners of the $40cm \times 30cm$ cardboard box below, a box can be produced. Find the **height** of the box which gives it **maximum volume**.



[CA] [5 marks]

3.31 The number of goals Alexa scores in a game $G(x)$ depends on the **number of hours** she practices per day, t, and follows the relationship: $G(x) = \frac{50t}{t^3 - 2t + 10}$. What is the **maximum number of goals** Alexa can score this game (to the nearest goal), and how long in hours and minutes should she practice **per day**?

[CA] [5 marks]

3.41 The number of oranges picked per day at Ol' Owen's Orange Orchard is given by $O = 384n - 2n^3$, where n is the **number of staff** at the orchard. The orange vendor, Ol' Owen, wants to know the **maximum** number of oranges they could pick and the **number of staff** he needs to hire to pick these oranges.

[5 marks]

3.42 Suppose that for a new up-and-coming textbook company to **print x thousand textbooks** at a time, the printing cost per textbook (in **dollars**) are $C = \frac{1}{20}x^3 - x^2 - x + 100$. Determine the **number of textbooks** required to have a **minimum printing cost per textbook**.

[6 marks]

3.51 The **volume** of a right scoop of ice cream is given $V = \pi r^2 h$ and the **total surface area** (including the cone) is given by $2\pi rh + 6\pi r^2$. Suppose ice cream scoops must have a **volume** of $200cm^3$, determine the **maximum surface area** of the entire ice cream cone.

Concept 4
Small Change – Repetitive Questions
(6 questions)

Small Change: Qs 4.11 4.12, 4.13, 4.31, 4.32, 4.51

[5 marks]

4.11 A circle's radius changes from $3.2cm$ to $3.19cm$:

(a) What is the **approximate change** in the **area** of the circle? **[3]**
(b) What is the **approximate change** in the **circumference** of the circle? **[2]**

[6 marks]

4.12 The volume of a cube is $A = 6l^2$, where l is the length of the sides. If the **volume of a cube** incrementally increases from $0.41m^3$ to $0.42cm^3$:
Knowing that Use the **incremental change formula** to calculate the **approximate change in side length** of the cube **[3]**

(a) Calculate the **exact change** in **side length** of the cube **[2]**
(b) Determine the **incremental increase** in **surface area** **[1]**

[12 marks]

4.13 Solve the following questions **without a calculator**:

(a) What is the **percentage decrease** in the side length of a cube if the percentage **drop** in volume is 0.6%? **[2]**
(b) If the surface area of a cube is $12m^2$, what is the **approximate final volume** of the cube if its side length decreases by $1cm$? **[3]**
(c) What is the **percentage increase** in the **volume** of a cone if its **radius** increases by 1% and the **height** remains **unchanged**? **[3]**
(d) What is the **approximate value range** of y if $y = 2\sqrt[4]{x^3}$ and $x = 81 \pm 1$? **[4]**

[5 marks]

4.31 If the **height** of a **cylinder** is $5\ units$, what is the **percentage change** in the total surface area if radius increases by 3% from $1000\ units$?

[8 marks]

4.51 The **revenue** generated by a business is $R(x) = \frac{20(x^3 - x^2}{x}$ and the **total cost** of the business is modelled by the function is $C(x) = \frac{2}{3}x^2$ for x items sold (in dollars):

(a) What is the **approximate change** in both the total revenue and total cost by selling 499 items instead of 500? **[5]**
(b) Show whether the business is better off or not in terms of net profit, by selling an extra item after it has **already sold 1000 items** **[3]**

Chapter 2

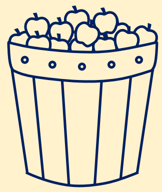
Integration

Chapter 2 – Integration

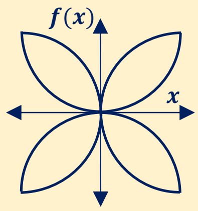
Introduction

Our second big step into the world of calculus is **integration**!

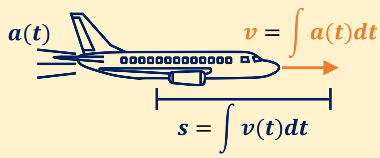
Integration can best be thought of as the **reverse** of **differentiation**. Instead of determining the **derivative** of a function, we are now determining the **antiderivative** of the function. Like differentiation, we can use **integration** in an array of **real-world applications**, from the **net change** in **profit** of **apples**, to the **area** of **complex designs** such as a **flower**, to the **rectilinear motion** of a **plane**.



Net Change in **Profit** from **Changing Number** of **Apples**



Area of **Flower** using **Definite Integrals**



Rectilinear Motion of a **Plane**

To explore **integration**, we are going to look at **five main topics**. These are:

1. **Integration Techniques**
2. **Estimating Area Under the Curve**
3. **Definite Integrals and the Area Under the Curve**
4. **Fundamental Theorem of Calculus**
5. **Applications of Integration – Area Between Curves, Net Change and Rectilinear Motion**

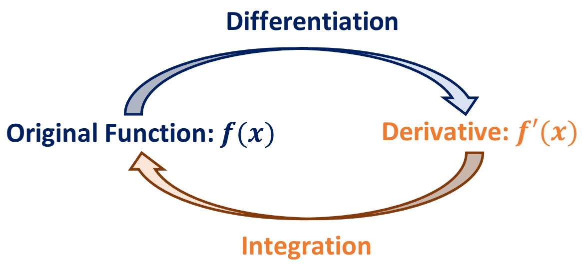
Let's Begin!

2.1 Integration Techniques

Integration is the process of determining the **antiderivative** of a function, also known as the **integral**.

> An **integral** is the **inverse** of the **derivative** of a function and represents the **signed area under a curve**. It is represented using the **symbol** $\int f(x)dx$, such as $\int x^2dx$ and $\int 3x^4dx$.

Understanding **why** an **integral represents** the **area under a curve** is something we will **discuss** in **section** 2.2. However, **for now** our focus is that the **integral** is the **inverse** of a **derivative**, and **integration** is the **process** of determining the **integral**, $f(x)$, from the derivative, $f'(x)$.



We call the **integral** of a function the **antiderivative** because it is the **opposite** of a **derivative**. So for basic polynomials, we can find the integral using the **reverse** of the **power rule**.

$$\boxed{\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad \text{given } n \neq -1}$$

What this **integration rule** tells us is to get the **integral**, we **add one** to the **power** and then **divide the function** by this **new power** ($n+1$). For instance, if we had the derivative $f'(x) = 2x$, to get the antiderivative, we **add one** to the **power** to be x^2 and then **divide the function** by this **new power**.

1 $f(x) = \int 2x dx$ — 1 **State the integral** that will determined

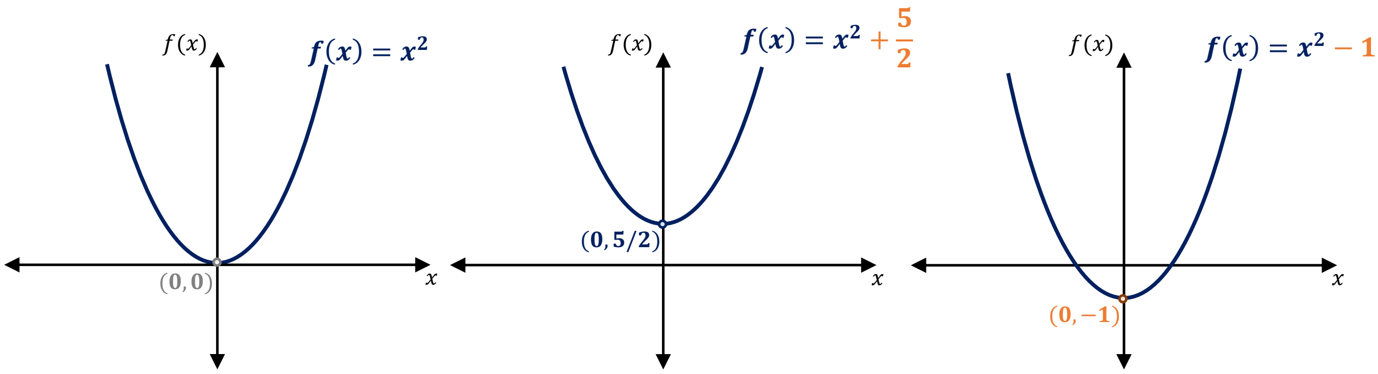
2 $f(x) = \frac{2x^{1+1}}{2} + C$ — 2 **Add one** to the **power** and then **divide** by the **new power**

3 $f(x) = x^2 + C$ — 3 **Simplify** to the **final integral**

For this **integral**, you may have noticed that a **constant** 'C' was **added** to the end of the function.

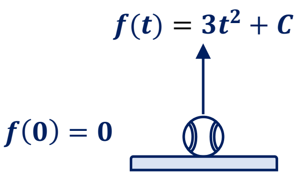
'C' is the **constant of integration** and it represents **any possible constant** that could go at the end of the **integral**.

We have C to represent **an arbitrary constant**, because we know that when we take the **derivative** of a **constant** it becomes **zero**. For instance, if $f(x) = 2$ then $\frac{dy}{dx} = 0$. As a result, in **all** the **integration rules** we explore, we must always **add** C to represent the **unknown constant** of the **original function**. For instance, for $f(x) = x^2 + C$, the C could be an any value such as $C = 0$, $C = \frac{5}{2}$ or $C = -1$.



In many practical situations, the constant C **can be determined** from knowing important information such as the **starting position** or a **position** at a **specific point in time**.

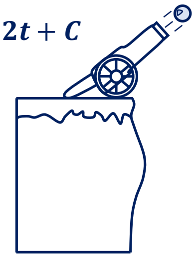
For instance, if you **throw** a **ball into the air from the ground** with the **displacement equation**, $f(t) = 3t^2 + C$, we can determine the **value** of C to be $C = 0$. This is because the **ball starts from the ground** (i.e. $f(0) = 0$), so when we **substitute** in $f(t) = 0$ at $t = 0$ **seconds**, we calculate $C = 0$.



$$f(t) = 3t^2 + C$$
$$f(0) = 0$$
$$0 = 3(0)^2 + C$$
$$C = 0$$
$$\therefore f(t) = 3t^2$$

As an example where $\boldsymbol{C \neq 0}$, imagine you **fire a cannonball off a cliff**, and it has a **height function** $\boldsymbol{h(t)}$ that follows the equation: $\boldsymbol{h(t) = -5t^2 + 2t + C}$. If it has a **height** of **12 meters** after **2 seconds**, you can substitute in $\boldsymbol{f(2) = 12}$ to determine the **value** of $\boldsymbol{C}$ to be $\boldsymbol{C = 28}$:

$$\boldsymbol{h(t) = -5t^2 + 2t + C}$$



$$\begin{gathered} h(t) = -5t^2 + 2t + \boldsymbol{C} \\ \boldsymbol{h(2) = 12} \\ 12 = -5(2)^2 + 2(2) + \boldsymbol{C} \\ 12 = -20 + 4 + \boldsymbol{C} \\ \boldsymbol{C = 28} \\ \therefore h(t) = -5t^2 + 2t + \boldsymbol{28} \end{gathered}$$

The **second integration rule** is for functions of the form: $\int \boldsymbol{(ax+b)^n dx}$. In this case, we use the **rule**:

$$\boxed{\int \boldsymbol{(ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C} \quad \text{given} \quad \boldsymbol{n \neq -1}}$$

This rule shows that to get the **integral**, we **increase** the **power** $\boldsymbol{n}$ by **one** $(\boldsymbol{n+1})$, and then **divide** the function by the **product** of the **co-efficient 'a'** and the **new power** $(\boldsymbol{n+1})$.

For instance, suppose we wanted to **integrate** the function $\boldsymbol{f'(x) = (2x-3)^5}$. We would **apply our rule** above as follows:

$$\begin{aligned} f'(x) &= (2x-3)^5 \\ \boldsymbol{f(x)} &= \int \boldsymbol{(2x-3)^5 dx} \\ \boldsymbol{f(x)} &= \frac{(2x-3)^{5+1}}{\boldsymbol{2 \times 6}} + \boldsymbol{C} \\ \boldsymbol{f(x)} &= \boldsymbol{\frac{(2x-3)^6}{12} + C} \end{aligned}$$

The final rule is for integrals of the form: $\int \boldsymbol{f'(x)[f(x)]^n dx}$. In this case, we use the **reverse** of the **differentiation rule** we derived in the **chain rule section** for differentiation.

$$\boxed{\int \boldsymbol{f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C} \quad \text{given} \quad \boldsymbol{n \neq -1}}$$

Similar to the previous rule, this rule shows that to get the **integral**, we **increase** the **power** $\boldsymbol{n}$ by **one** $(\boldsymbol{n+1})$, and then **divide** the function by the **product** of $\boldsymbol{f'(x)}$ and the **new power** $(\boldsymbol{n+1})$.

For instance, if we integrate the equation: $\boldsymbol{\frac{dy}{dx} = 2x(x^2-4)^3}$, we can use the **reverse chain rule** from above because $\boldsymbol{2x}$ is the **derivative** of $\boldsymbol{f(x) = x^2 - 4}$.

$$\begin{aligned} \frac{dy}{dx} &= 2x(x^2-4)^3 \\ \boldsymbol{y} &= \int \boldsymbol{2x(x^2-4)^3 dx} \\ \boldsymbol{y} &= \frac{2x(x^2-4)^{3+1}}{\boldsymbol{2x \times 4}} + \boldsymbol{C} \end{aligned}$$

$$\begin{aligned} \boldsymbol{y} &= \frac{\cancel{2x}(x^2-4)^4}{\cancel{2x} \times 4} + \boldsymbol{C} \\ \boldsymbol{y} &= \boldsymbol{\frac{(x^2-4)^4}{4} + C} \end{aligned}$$

Linearity of Integrals

When we are **determining difficult integrals**, there are **two important properties** that we can utilise.

The **first property** is that an **integral** can be **separated** into its **individual terms**.

Integration works on a **"per term" basis** which means that an **integral** of one expression **containing multiple terms** can be split into **multiple integrals** each **containing just one term**. This is represented by the rule:

Linearity of Integrals Property: $\int f_1(x) + f_2(x)dx = \int f_1(x)dx + \int f_2(x)dx$

For instance, in the integral below we can see that the **three terms** in **one integral** can be expanded into **three integrals** that can be **evaluated separately**:

$$f(x) = \int 3x^3 + x^{\frac{1}{2}} - 4\, dx$$
$$f(x) = \int 3x^3\, dx + \int x^{\frac{1}{2}}\, dx - \int 4\, dx$$
$$f(x) = \frac{3x^4}{4} + \frac{2}{3}x^{\frac{3}{2}} - 4x + C$$

This property is useful when we want to **integrate parts** of an equation **separately**, and we will use it for various problems in the problem set (particularly in chapters 3 and 4 for trigonometric and exponential functions).

The **second integral property** we can utilise is the fact that the **coefficient** of a term can be **factored outside** of the integral and then **multiplied back in** after the integration.

Coefficients aren't affected by **integration** so they can be **factored out** to help make the **integration** of the **actual term easier**. This can be summarised by the equation:

Integral Coefficient Property: $\int kf(x)dx = k\int f(x)dx$

For instance, in the integral below, we can **factor out** the **coefficient** $\frac{1}{3}$. This makes **integrating** $x^{\frac{3}{2}}$ an easier process. Once integrated we can then **multiply** the **coefficient back in**:

$$f(x) = \int \frac{1}{3}x^{\frac{3}{2}}\, dx$$
$$f(x) = \frac{1}{3}\int x^{\frac{3}{2}}\, dx$$
$$f(x) = \frac{1}{3} \times \frac{2}{5}x^{\frac{5}{2}} + C$$
$$f(x) = \frac{2}{15}x^{\frac{5}{2}} + C$$

This property will **prove valuable** in the problem sets for **more difficult integrals**.

Worked Example 1

Wallace is a student who loves every aspect of mathematics! After struggling with differentiation, Wallace is looking to get an early start on **integration**. **Help** Wallace to **determine** the **following integrals**.

(a) $y = \int 2x\,dx$

$$y = \frac{2x^2}{2} + C$$

$$y = x^2 + C$$

(b) $y = \int 4x^2 + 2x - 2\,dx$

$$y = \frac{4x^3}{3} + \frac{2x^2}{2} - 2x + C$$

$$y = \frac{4x^3}{3} + x^2 - 2x + C$$

(c) $y = \int 4x(2x^2 - 3)^3\,dx$

$$y = \frac{(2x^2 - 3)^4}{4} + C$$

(d) $y = \int 3x^{\frac{1}{2}}\,dx$

$$y = 3x^{\frac{3}{2}} \times \frac{2}{3} + C$$

$$y = 2x^{\frac{3}{2}} + C$$

Worked Example 2

Wallace is annoyed by the fact that all his answers contained the **constant 'C'**. This time, help Wallace to **determine** the **integrals** along with the **value** of C based on the information provided.

(a) Determine $f(x)$ if $f'(x) = x^2 + 4$ and $f(x)$ passes through the point $(1, 4)$

$$f(x) = \int x^2 + 4\,dx$$

$$f(x) = \frac{x^3}{3} + 4x + C$$

Let $f(1) = 4$

$$4 = \frac{(1)^3}{3} + 4(1) + C$$

$$C = -\frac{1}{3}$$

$$\therefore f(x) = \frac{x^3}{3} + 4x - \frac{1}{3}$$

(b) Determine $f(x)$ if $f'(x) = 8(2x - 2)^3$ and $f(x)$ passes through the point $(1, -1)$

$$f(x) = \int 8(2x - 2)^3\,dx$$

$$f(x) = (2x - 2)^4 + C$$

Let $f(1) = -1$

$$-1 = (2(1) - 2)^4 + C$$

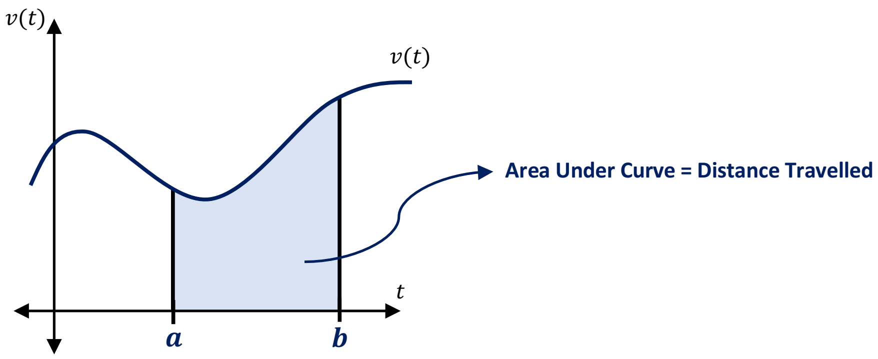
$$-1 = 0 + C$$

$$C = -1$$

$$\therefore f(x) = (2x - 2)^4 - 1$$

2.2 Estimating Area Under the Curve

By definition, an **integral** of a function also represents the **signed area under the curve**. This means that by calculating the integral **between two points** we can calculate the **area under the curve**, which is useful in applications such as calculating the **distance travelled** from a **velocity-time graph**.



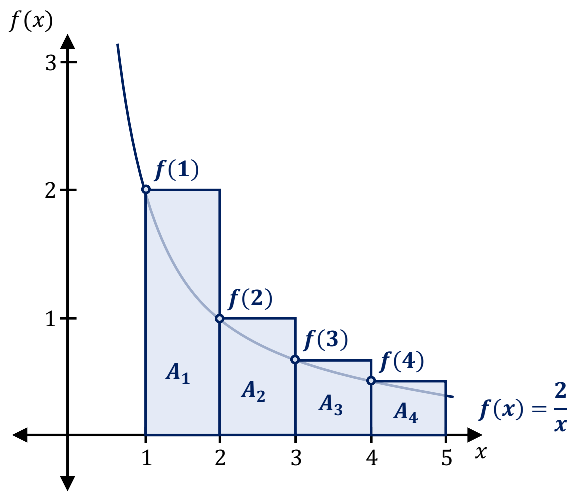
Getting the **exact area under the curve** can be done through using **definite integrals (i.e. $\int_a^b f(x)$)**, however, a great place to start is to learn to **estimate** the **area under a curve first**.

The **Reimann Sum** is an **equation** for the **approximation** of an integral which **sums the rectangles** under the curve with a **width** represented by δx_i, and a **height determined by $f(x_i)$**. The **formula** we will be using is:

Riemann Sum: $Approximate\ Area = \sum_i f(x_i)\delta x_i$

What the Reimann Sum tells us is that we can split the area under a curve into **rectangles** of a **uniform width** and we can **sum them together** to get an **estimate** of the **area**.

For instance, let's consider the equation $f(x) = \frac{2}{x}$. In order to **approximate** the **area under the curve** between $x = 1$ and $x = 5$, let's consider having **four rectangles** with a **width** of $\delta x = 1$ and **varying heights**.



To determine the area approximation, we **sum** the **four rectangle areas** together. That is, we **sum the products** of the **width** of $\delta x = 1$ with the **height** of **each rectangle** determined from **substituting the x-value** into the function $f(x) = \frac{2}{x}$ (e.g. $f(1)$, $f(2)$).

$$Area \approx A_1 + A_2 + A_3 + A_4$$

① $$Area \approx f(1)\delta x + f(2)\delta x + f(3)\delta x + f(4)\delta x$$

② $$Area \approx \frac{2}{1}(1) + \frac{2}{2}(1) + \frac{2}{3}(1) + \frac{2}{4}(1)$$

$$Area \approx 2 + 1 + \frac{2}{3} + \frac{1}{2}$$

③ $$Area \approx \frac{25}{6} = 4.167\ units^2$$

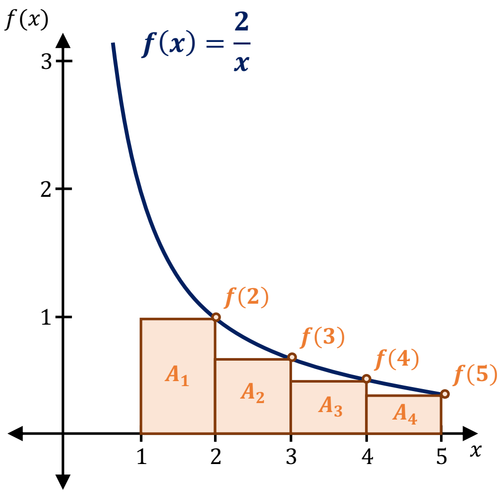
① **Apply** the **Reimann sum formula** to the **four areas**

② **Substitute** in the **x-values** to get the **heights** and **multiply** it by the width of $\delta x = 1$

③ **Simplify** to get the final answer

Unfortunately, this is clearly an **overestimate** of the area. The **rectangles drawn** are known as **'left-rectangles'** because their **height** is based on the **left-hand corner touching the function**.

To **balance out** this **approximation** we can **underestimate the area** using '**right-rectangles**', that is, using rectangles where the **right-corner touches the curve**.



The area of these **four right-rectangles** can be **calculated** in the **same manner** as the **left-rectangles**:

$$Area \approx A_1 + A_2 + A_3 + A_4$$

① $$Area \approx f(2)\delta x + f(3)\delta x + f(4)\delta x + f(5)\delta x$$

② $$Area \approx \frac{2}{2}(1) + \frac{2}{3}(1) + \frac{2}{4}(1) + \frac{2}{5}(1)$$

$$Area \approx 1 + \frac{2}{3} + \frac{1}{2} + \frac{2}{5}$$

③ $$Area \approx \frac{77}{30} = 2.567\ units^2$$

① **Apply** the **Reimann sum formula** to the **four areas**

② **Substitute** in the **x-values** to get the **heights** and **multiply** it by the width of $\delta x = 1$

③ **Simplify** to get the final answer

We now have the **overestimated area** of $4.167\ units^2$ and the **underestimated area** of $2.567\ units^2$. To get a **more accurate area approximation**, we take the **average** of these two results:

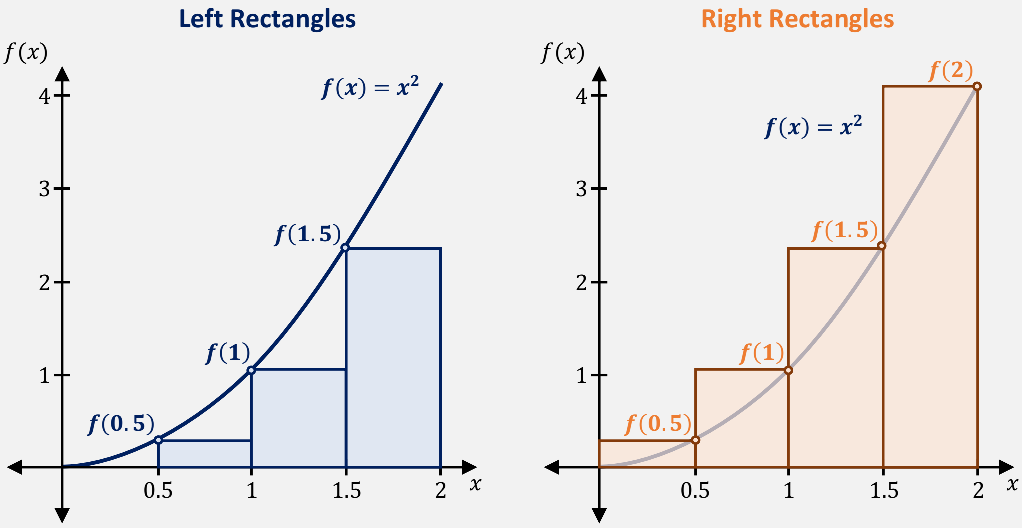
$$Area \approx \frac{4.167 + 2.567}{2}$$

$$Area \approx 3.367\ units^2$$

Estimating the **area under a curve** serves as a **fantastic foundation** for understanding **definite integrals**, which are what we use to **determine the exact area under a curve**.

Worked Example 1

Absolutely falling in love with integration, Wallace wants to **approximate** the **area under the curve** for $f(x) = x^2$ between $x = 0$ and $x = 2$. To do so he draws out the following **left-rectangle graph** and **right-rectangle graph** with **rectangle widths** of $\delta x = 0.5$.



Help Wallace to **approximate** the **area under the curve** for $f(x) = x^2$ between $x = 0$ and $x = 2$.

Left Rectangle Area:

$$Area \approx \delta x\big(f(0) + f(0.5) + f(1) + f(1.5)\big)$$

$$Area \approx (0.5)\big((0)^2 + (0.5)^2 + (1)^2 + (1.5)^2\big)$$

$$Area \approx 0 + 0.125 + 0.5 + 1.125$$

$$Area \approx 1.75\ units^2$$

Right Rectangle Area:

$$Area \approx \delta x\big(f(0.5) + f(1) + f(1.5) + f(2)\big)$$

$$Area \approx (0.5)\big((0.5)^2 + (1)^2 + (1.5)^2 + (2)^2\big)$$

$$Area \approx 0.125 + 0.5 + 1.125 + 2$$

$$Area \approx 3.75\ units^2$$

Area Approximation:

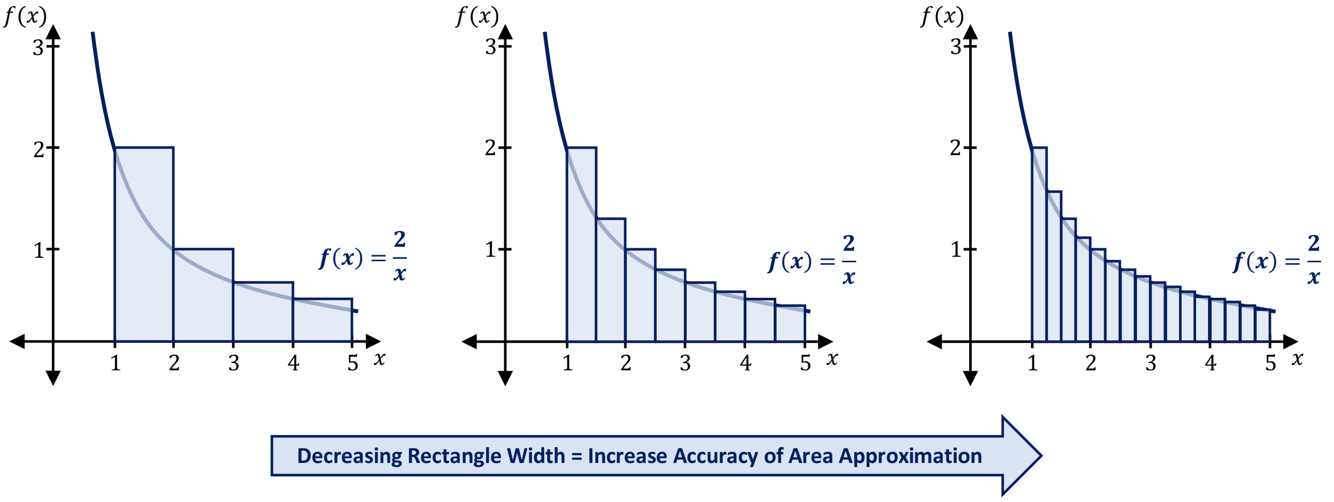
$$Area \approx \frac{1.75 + 3.75}{2}$$

$$Area \approx 2.75$$

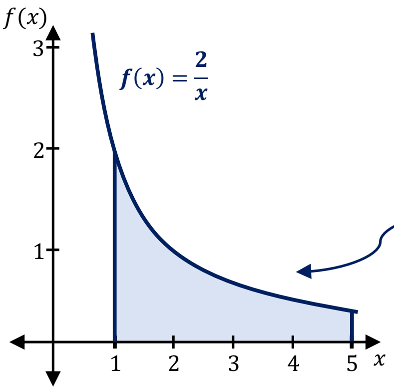
2.3 Definite Integrals and Area Under the Curve

When it comes to approximating area, if we use **rectangles of a smaller width**, the **area approximation** becomes **progressively more accurate**.

For instance, for $f(x) = \frac{2}{x}$ we can see below that **rectangles** with a **width** of $\delta x = 1$ is the **least accurate** and when the width is $\delta x = 0.5$ and $\delta x = 0.25$ the approximations become **progressively more accurate**. If the width was $\delta x = 0$ we would get the **exact area**.



To get an **exact area**, we must make the **width** of the **rectangles approach zero**. As we know from first principles for differentiation, to make a variable **approach zero we apply a limit**. In this case it will be: $\lim_{\delta x \to 0}$ applied to the Reimann Sum formula:



$$Exact\ Area = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\delta x = \int_a^b f(x)dx$$

As can be seen, when **limiting** the **rectangle width** to **zero**, our **exact area** can be expressed as a **definite integral**, $\int_a^b f(x)dx$, which is an **easier way** of **solving** the **area under a curve.**

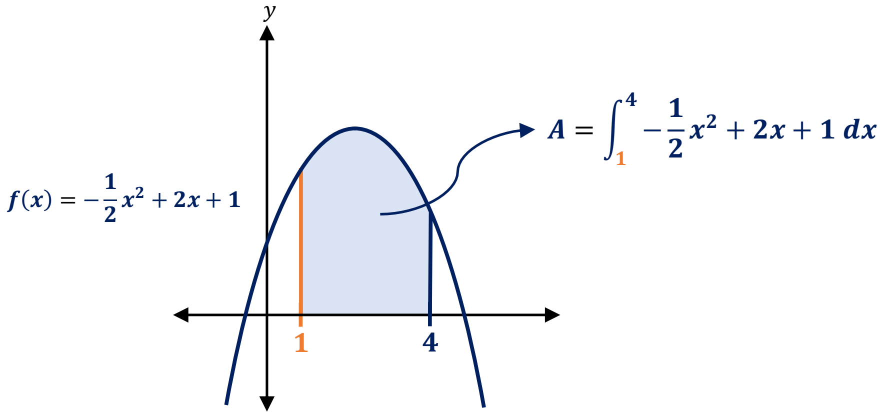
A **definite integral**, $\int_a^b f(x)dx$, calculates the **exact signed area bounded by the function** and the x**-axis** between a and b, where the function, $f(x)$, is continuous for $[a, b]$ and $b > a$.

The **definite integral** is **determined** in two steps.

The **first step** involves **determining** the **integral** of the function, which we are already familiar with. The **second step** involves **substituting** in the **bounds** a and b into the **integrated function** to **determine** the **area under the curve**.

To explore this, let's consider the polynomial $f(x) = -\frac{1}{2}x^2 + 2x + 1$ which we want to know the **area under its curve** between $x = 1$ and $x = 4$.

To determine the **area under the curve**, we would create a **definite integral** and set the **lower bound** as $a = 1$ and the **upper bound** as $b = 4$.



The **first step** is to **integrate** $f(x)$, which we know how to do. However, since we have the **bounds** of $a = 1$ and $b = 4$ we need to place the **integral** in **square brackets** to acknowledge that these **bounds** will be **substituted in**.

① $A = \int_1^4 -\frac{1}{2}x^2 + 2x + 1\,dx$ — ① **Establish** the **definite integral**

② $A = \left[-\frac{1}{6}x^3 + x^2 + x + C\right]_1^4$ — ② **Integrate** the definite integral with the bounds $a = 1$ and $b = 4$

The **second step** is to **substitute** our **bounds** into the **function**, where it is the **upper bound subtract** the **lower bound**. This will be calculated by **substituting** in $x = 4$ into the integral and then **subtracting** the integral after **substituting** in $x = 1$, as follows:

$$A = \left[-\frac{1}{6}x^3 + x^2 + x + C\right]_1^4$$

③ $$A = \left[-\frac{1}{6}(4)^3 + (4)^2 + (4) + C\right] - \left[-\frac{1}{6}(1)^3 + (1)^2 + (1) + C\right]$$

$$A = \left[\frac{28}{3} + C\right] - \left[\frac{11}{6} + C\right]$$

④ $$A = 7.5\ units^2$$

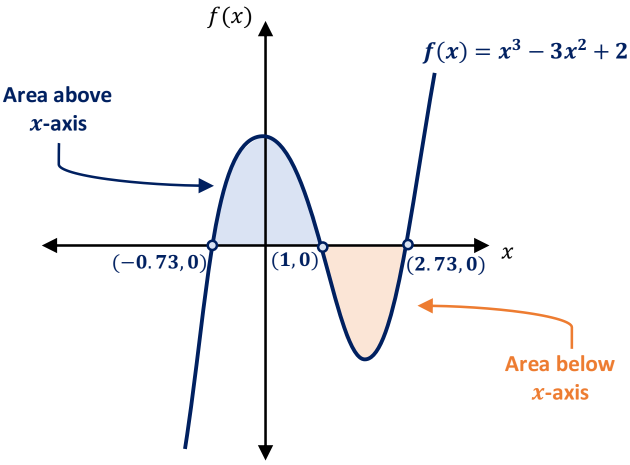
③ **Substitute** in the bounds $a = 1$ and $b = 4$

④ **Simplify** to the final answer

Note that the **constant C** is **always cancelled out** in definite integrals. This means when **calculating definite integrals** there is **no need** to **include 'C'** in the **working out**.

However, what do we do for **calculating areas** for functions that have a **positive area above the x-axis** and a **negative area below the x-axis**?

For instance, suppose we had the function $\boldsymbol{f(x) = x^3 - 3x^2 + 2}$ and we wanted to determine the area between $\boldsymbol{x = -0.73}$ and $\boldsymbol{x = 2.73}$.



We can clearly see that between $\boldsymbol{x = -0.73}$ and $\boldsymbol{x = 1}$, the **area** is **above the x-axis**, and between $\boldsymbol{x = 1}$ and $\boldsymbol{x = 2.73}$, the curve goes **below the x-axis**.

When we **solve** the **definite integral**: $A = \int_{-0.73}^{2.73} x^3 - 3x^2 + 2\, dx$, we get an **area** of $\boldsymbol{0\ units^2}$. This is because the **positive area** of $\boldsymbol{A = 2.25\ units^2}$ **above the x-axis** is **cancelled out** by the **negative area** of $\boldsymbol{A = 2.25\ units^2}$ **below the x-axis**.

To get the **true area** of $\boldsymbol{4.5\ units^2}$, we must apply our **linearity of integrals property** to **split up the definite integral** into its **areas above the x-axis** and **areas below the x-axis**.

Linearity of Definite Integrals: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $\boldsymbol{a < c < b}$

Applying this property, we can **split** our **definite integral**: $A = \int_{-0.73}^{2.73} x^3 - 3x^2 + 2\, dx$ into **two definite integrals**.

The **first definite integral** will be for the **positive area** between $\boldsymbol{x = -0.73}$ and $\boldsymbol{x = 1}$, and then the **second** will be for the **area below** the **x-axis** between $\boldsymbol{x = 1}$ and $\boldsymbol{x = 2.73}$. However, to make the **second area** a **positive area**, we need to place a **negative sign** in-front of it. This **split integral** will look as follows:

1 $$A = \underbrace{\int_{-0.73}^{1} x^3 - 3x^2 + 2\, dx}_{\text{Area above } x\text{-axis}} - \underbrace{\int_{1}^{2.73} x^3 - 3x^2 + 2\, dx}_{\text{Area below } x\text{-axis}}$$

1 **Split** the **integral** into the **area above the x-axis** and the **area below the x-axis**

We can then **evaluate each integral separately** to get the **true area under the curve** between $x = -0.73$ and $x = 2.73$ of $4.5\ units^2$.

$$A = \int_{-0.73}^{1} x^3 - 3x^2 + 2\,dx - \int_{1}^{2.73} x^3 - 3x^2 + 2\,dx$$

② $$A = \left[\frac{x^4}{4} - x^3 + 2x\right]_{-0.73}^{1} + \left[\frac{x^4}{4} - x^3 + 2x\right]_{1}^{2.73}$$

② **Integrate** each definite integral **separately**

③ $$A = \left[\frac{(1)^4}{4} - (1)^3 + 2(1)\right] - \left[\frac{(-0.73)^4}{4} - (-0.73)^3 + 2(-0.73)\right]$$
$$- \left[\frac{(2.73)^4}{4} - (2.73)^3 + 2(2.73)\right] + \left[\frac{(1)^4}{4} - (1)^3 + 2(1)\right]$$

③ **Substitute** in the **bounds** for both definite integrals

④ **Simplify** to the final answer

$$A = 1.25 + 1 + 1 + 1.25$$

④ $$A = 4.5\ units^2$$

If you have access to a calculator, an easy way to determine the area is to **apply** an **absolute value** to the **integral**, which gives the **exact area** irrespective of **whether the area** is **above** or **below** the x**-axis**.

> **Absolute Area Under the Curve:** $A = \int_a^b |f(x)|dx$

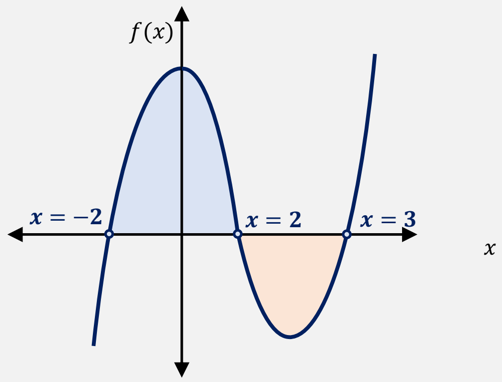
Applying this to our example above, we could get the **same area** as follows:

$$A = \int_{-0.73}^{2.73} |x^3 - 3x^2 + 2|dx = 4.5\ units^2$$

When calculating the area under the curve, **both methods** are **valid**. However, you should **practice both** in-case you don't have a calculator

Worked Example 1

Continuing to explore his interest in integration, Wallace wants to determine the area enclosed between $f(x) = x^3 - 3x^2 - 4x + 12$ and the x**-intercepts**: $x = -2, x = 2$ **and** $x = 3$.

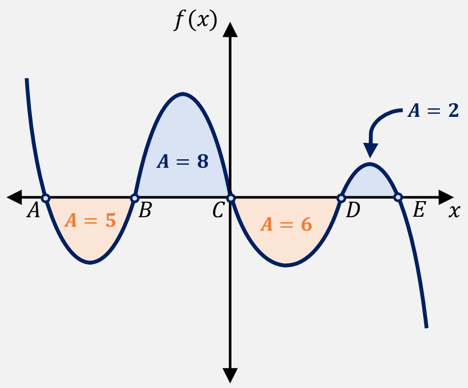


Help Wallace to determine the **area** that is **enclosed** under $f(x)$ between $x = -2$ **and** $x = 3$. **[CA]**

$$A = \int_{-2}^{3} |x^3 - 3x^2 - 4x + 12|dx = 32.75\ units^2$$

Worked Example 2

Teacher Simon is walking past Wallace and notices that he has been studying a lot of integration. To test Wallace's understanding of **definite integrals** and **area under the curve**, he provides the **following curve** with the **areas** between certain points **already calculated**.



Help Wallace to **determine** the **values** that will be calculated for the following scenarios.

(a) Determine the **definite integral**: $\int_A^C f(x)dx$

$$\int_A^C f(x)dx = -5 + 8$$
$$\int_A^C f(x)dx = 3$$

(b) Determine the **area enclosed** between $f(x)$ and the **x-axis** from $x = A$ to $x = C$

$$Area = 5 + 8$$
$$Area = 13\ units^2$$

(c) Determine the **definite integral**: $\int_B^E f(x)dx$

$$\int_B^E f(x)dx = 8 - 6 + 2$$
$$\int_B^E f(x)dx = 4$$

(d) Determine the **area enclosed** between $f(x)$ and the **x-axis** from $x = A$ to $x = E$

$$Area = 5 + 8 + 6 + 2$$
$$Area = 21\ units^2$$

2.4 Fundamental Theorem of Calculus

The **fundamental theorem of calculus** is divided into **two parts**. The first part **we have already covered** through our **definite integrals** earlier which follows the formula:

Fundamental Theorem of Calculus Formula 1: $\int_a^b f(x)dx = F(b) - F(a)$

The first part of the **fundamental theorem of calculus** states that if want to find the **area under a curve $f(x)$** from point a to point b, we determine the **antiderivative $F(x)$** and calculate $F(b) - F(a)$ by substituting in the bounds a and b (which we have **already looked at**).

The **second part** of the **fundamental theorem of calculus** is for determining the **derivatives** of **definite integrals** that are **bounded by a function** such as x or $g(x)$.

$$\underbrace{\frac{d}{dx}\int_0^x 2t^3 - 4t + 2\,dt \quad \text{or} \quad \frac{d}{dx}\int_x^1 8t(t-1)^3\,dt \quad \text{or} \quad \frac{d}{dx}\int_0^{g(x)} 3t^3 - 2tdt}_{\text{Derivatives of definite integrals bounded by } x \text{ or } g(x)}$$

The **second part** of the **fundamental theorem of calculus** has **two rules**. The **first** is if the **definite integral** is **bounded** by x, and the **second** is if the **definite integral** is **bound** by a **function** $g(x)$.

Firstly, if a **definite integral** is **bounded** by x then the **derivative of the integral** will be **equal** to $f(x)$.

Fundamental Theorem of Calculus Formula 2: $f(x) = \frac{d}{dx}\int_a^x f(t)dt$

Essentially, what this means is that to **determine $f(x)$** you **substitute** in the **bound x** for the **variable t**. The best way to understand this formula is to **apply** it to some examples.

Let's suppose we want to find the **derivative** of the **definite integral** of the function: $y = t^2 + 2t$ between the bounds $a = 2$ and $b = x$. This would be written as: $f(x) = \frac{d}{dx}\int_2^x t^2 + 2t\,dt$.

To determine $f(x)$, all we need to do is **apply** the **fundamental theorem of calculus formula** by **substituting** in the **variable x** for the **variable t**.

$$f(x) = \frac{d}{dx}\int_2^x t^2 + 2t\,dt$$

① **Apply:** $f(x) = \frac{d}{dx}\int_a^x f(t)dt$

② $f(x) = x^2 + 2x$

① **Apply** the **fundamental theorem of calculus formula**

② **Substitute** in **variable x** for **variable t**

As you can see, all we did was **substitute** in **variable x** where there is **variable t**. The **lower bound** is **irrelevant** because it is a **constant** and therefore **differentiated** from the $\frac{d}{dx}$ **part** of the formula to **always equal zero**.

As a second, slightly more difficult example, what if the **function x** is the **lower bound** rather than the **upper bound** such as with: $f(x) = \frac{d}{dx}\int_x^2 8t(t-1)^3 dt$

In order to be able to **apply** our **fundamental theorem of calculus formula**: $f(x) = \frac{d}{dx}\int_a^x f(t)dt$, we must **apply** the **definite integral property below** which allows us to **reverse the bounds**.

Definite Integral Property: $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx$

What this property tells us is that if we **reverse the bounds a** and **b**, then the **integral becomes negative**. Applying this our example: $f(x) = \frac{d}{dx}\int_x^2 8t(t-1)^3 dt$ all we need to do is **reverse the bounds** and **place a negative sign** in-front of the **definite integral** and then **follow the usual steps**.

$$f(x) = \frac{d}{dx}\int_x^2 8t(t-1)^3\, dt$$

① $$f(x) = -\frac{d}{dx}\int_2^x 8t(t-1)^3\, dt$$

② Apply: $f(x) = \frac{d}{dx}\int_a^x f(t)dt$

③ $$f(x) = -8x(x-1)^3$$

① **Reverse the bounds** by applying our **integral property**

② **Apply** the **fundamental theorem of calculus formula**

③ **Substitute** in **variable x** for **variable t**

The final consideration is when the integral is **bound** by the function $g(x)$ such as $g(x) = 2x$ or $g(x) = x^2$. In this case, the **derivative** of a **definite integral bounded** by $g(x)$ follows the formula:

Fundamental Theorem of Calculus Formula 3: $\displaystyle\frac{d}{dx}\int_a^{g(x)} f(t)dt = f[g(x)]\cdot g'(x)$

Whilst this formula may appear to be complicated, essentially, what this means is that $g(x)$ will be **substituted** in for **variable t** and the **whole equation** will be **multiplied** by $g'(x)$.

Again, the best way to understand this formula is to **apply** it to an **example**.

Let's suppose we wanted to determine: $f(x) = \frac{d}{dx}\int_0^{2x} 2t - 4t^2 dt$, which is **bound** by $g(x) = 2x$. To determine this, we would **substitute** in $g(x) = 2x$ for t and **multiply it all** by $g'(x) = 2$ as follows:

$$f(x) = \frac{d}{dx}\int_0^{2x} 2t - 4t^2 dt$$

Apply: $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f[g(x)]\cdot g'(x)$

$$f(x) = \left(2(2x) - 4(2x)^2\right)\cdot 2$$

$$\therefore f(x) = 8x - 32x^2$$

Whilst we have covered a vast array of integration concepts, our exploration of integration **does not stop here**. In the next section we are going to explore how **integration** can be **applied to real world applications** such as **area between curves**, **net change** and **rectilinear motion**.

Worked Example 1

Wallace is eager to apply his new knowledge on the **fundamental theorem of calculus**. Using the formula: $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)$ and $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f[g(x)]\cdot g'(x)$, help Wallace to **determine** the following.

(a) **Determine:** $f(x) = \frac{d}{dx}\int_2^x 4t^2 + 4t\,dt$

$$f(x) = \frac{d}{dx}\int_2^x 4t^2 + 4t\,dt$$

Apply: $f(x) = \frac{d}{dx}\int_a^x f(t)dt$

$$f(x) = 4x^2 + 4x$$

(b) **Determine:** $f(x) = \frac{d}{dx}\int_x^5 \frac{5t}{3+t}dt$

$$f(x) = \frac{d}{dx}\int_x^5 \frac{5t}{3+t}dt$$

$$f(x) = -\frac{d}{dx}\int_5^x \frac{5t}{3+t}dt$$

Apply: $f(x) = \frac{d}{dx}\int_a^x f(t)dt$

$$f(x) = -\frac{5x}{3+x}$$

(c) **Determine:** $f(x) = \frac{d}{dx}\int_0^{x^2} 4t^2 + 8t\,dt$

$$f(x) = \frac{d}{dx}\int_0^{x^2} 4t^2 + 8t\,dt$$

Apply: $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f[g(x)]\cdot g'(x)$

$$f(x) = \left(4\left(x^2\right)^2 + 8\left(x^2\right)\right)\cdot 2x$$

$$\therefore f(x) = (4x^4 + 8x^2)\cdot 2x$$

$$\therefore f(x) = 8x^5 + 16x^3$$

Problem Set 3 – Integration
Progressive Questions

Concept 1
Integration Techniques – Progressive Questions

(5 questions)

Repetitive questions: 1.11 → 1.31 (3 questions)

Janitor Peter's Space Station Competition!

Starring: Janitor Peter, Rabea, Kerry, Rupert, Shauna, Wallace, Tyler, Tom and Jevon

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After the Mathematics Methods building was destroyed by the Apps students, the Methods students have had to relocate to Janitor Peter's auditorium. Walking into class one day, the students look at the board and it is covered in complex equations

Rabea inquires "Sir, what are all of those complex equations on the board?"

Janitor Peter promptly replies "Why those are some of my projection calculations for one of my rockets at my space station!"

Jumping with excitement Rabea shouts "Sir can I go to your space station?"

Intrigued by this thought, Janitor Peter makes the following proposition "I'll tell you what… there are eight of you in my class today and I have five spots on my spaceship. Whichever five students prove to be the best at integration can come with me!"

Top students Rabea, Wallace, Kerry, Rupert, Tyler, Tom, Shauna and Jevon all look at one another, excited to compete for the five spots on the spaceship!

Integration Techniques: Q1, Q2, Q3, Q4, Q5
Repetitive: 1.11 → 1.31 (3 questions)

[20 marks]

1. Before establishing the competition, Janitor Peter decides to set the following questions as homework to do over the weekend, **help** the students to determine each of the **integrals**.

(a) $\int x^2\, dx$ **(2)**

(b) $\int 4x^2 - 3x\, dx$ **(2)**

(c) $\int 4x^2 - 3x\, dx$ **(2)**

(d) $\int 7x^4 - 3x^3 + 9\, dx$ **(2)**

(e) $\int \frac{x^3}{3} + 4x^2\, dx$ **(2)**

(f) $\int 2x^{-\frac{1}{2}}\, dx$ **(2)**

(g) $\int \frac{4}{x^2} + \frac{x^3}{3}\, dx$ **(2)**

(h) $\int x^{\frac{1}{3}} - 2x + 4x^3\, dx$ **(2)**

(i) $\int \frac{4}{\sqrt{x}} + \sqrt{x}\, dx$ **(2)**

(j) $\int \frac{3}{2x^2} - \frac{1}{2x^3}\, dx$ **(2)**

[28 marks]

2. After acing their homework over the weekend, Janitor Peter decides that it's time to step up the level of difficulty. **Help** the students to **determine** the following **integrals**.

(a) $\int (2x-2)(4x+4)\,dx$ **(2)**

(b) $\int \frac{2x^5-4x^3}{2x^2}\,dx$ **(3)**

(c) $\int 3x^3(2x-2)\,dx$ **(2)**

(d) $\int \frac{6x^3-2x}{x}\,dx$ **(3)**

(e) $\int (x^2+3)(7x-3)\,dx$ **(3)**

(f) $\int \frac{2-\sqrt{x}}{\sqrt{x}}\,dx$ **(3)**

(g) $\int 4x^4(x^2+2x)\,dx$ **(3)**

(h) $\int \sqrt{x}(x^{\frac{3}{2}}-3x)\,dx$ **(3)**

(i) $\int (3x-2)^2+4\,dx$ **(3)**

(j) $\int \frac{1-3x}{x^3}\,dx$ **(3)**

[30 marks]

3. Continuing to ace the problem sets, Janitor Peter decides to continue pushing the students. For their homework for that evening, Janitor Peter assigns some more difficult integrals. **Help** the students to **determine** the following **integrals**.

(a) $\int (x-3)^4\,dx$ **(3)**

(b) $\int (2+x)^3 dx$ **(3)**

(c) $\int (2x+7)^5\,dx$ **(3)**

(d) $\int 2(4x-5)^7\,dx$ **(3)**

(e) $\int \frac{1}{(x-4)^3}\,dx$ **(3)**

(f) $\int 2x(x^2+7)^5\,dx$ **(3)**

(g) $\int \frac{2}{(7x-8)^4}\,dx$ **(3)**

(h) $\int 16x(4x^2-6)^3\,dx$ **(3)**

(i) $\int \frac{4}{(2x-4)^6}\,dx$ **(3)**

(j) $\int 10x(4-5x^2)^7 dx$ **(3)**

[30 marks]

4. To get ahead of the content for tomorrow's lesson, sneaky student Jevon swipes a page of the question sheet from Janitor Peter's desk! Quick! Before Janitor Peter notices, help Jevon to **solve** the **following integrals**.

(a) $\int 4\sqrt{x}-\frac{5}{x^3}+3\,dx$ **(3)**

(b) $\int \frac{7x^4-2}{3x^2}\,dx$ **(3)**

(c) $\int 2x(x^2-10)^5\,dx$ **(3)**

(d) $\int \frac{2}{(4x-3)^4}\,dx$ **(3)**

(e) $\int (-2x+4)(3x^2-1)\,dx$ **(3)**

(f) $\int \frac{2x^3-x^2}{\sqrt{x}}\,dx$ **(3)**

(g) $\int \frac{5}{3\sqrt{x}}-2x^{\frac{3}{2}}\,dx$ **(3)**

(h) $\int \frac{2}{x^5}+x^{-\frac{3}{2}}-10x\,dx$ **(3)**

(i) $\int (4x^2-4x)^2-7\,dx$ **(3)**

(j) $\int \frac{4}{(x-6)^3}\,dx$ **(3)**

[CA] [24 marks]

5. Overwhelmed by the talent amongst the students, Janitor Peter decides to provide one final problem set before beginning the space competition tomorrow. Janitor Peter asks the students to find **integrals** for the **derivatives** below. **Help** the students by **finding** $\boldsymbol{f(x)}$.

(a) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = 3x^2 - 4x}$ and $\boldsymbol{f(x) = 2}$ at $\boldsymbol{x = 2}$ **(3)**

(b) Determine $f(x)$ if $\boldsymbol{f'(x) = 9x^3 - x + 1}$ and $\boldsymbol{f(x)}$ intersects the point $\boldsymbol{(0, 6)}$ **(3)**

(c) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = (x-2)^2}$ and intersects **y-axis** at $\boldsymbol{f(x) = 5}$ **(3)**

(d) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = (3x-6)^3}$ and intersects **x-axis** at $\boldsymbol{x = 2}$ **(3)**

(e) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = x^3 - 2x + 4}$ and intersects **y-axis** at $\boldsymbol{f(x) = \frac{7}{2}}$ **(3)**

(f) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = \sqrt{x}}$ and intersects **y-axis** at $\boldsymbol{f(x) = \frac{7}{2}}$ **(3)**

(g) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = 10(x-2)^4}$ and $\boldsymbol{f(x) = 2}$ at $\boldsymbol{x = 3}$ **(3)**

(h) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = \frac{4}{(x+7)^5}}$ and $\boldsymbol{f(x)}$ intersects the point $\boldsymbol{(-6, 5)}$ **(3)**

Concept 2

Estimating Area Under Curve – Progressive Questions

(4 questions)

Repetitive questions: 2.11 → 2.31 (3 questions)

Janitor Peter's Space Station Competition!

Starring: Janitor Peter, Rabea, Kerry, Rupert, Shauna, Wallace, Tyler, Tom and Jevon

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The first day of the competition has arrived and Janitor Peter splits up the eight students into four teams. In the first team is Rabea and Wallace, the second team is Tyler and Tom, the third team is Jevon and Rupert, and the fourth team is Kerry and Shauna.

Janitor Peter announces "At the end of this day, one team will be going home, and the remaining three teams will progress to battle it out for the five spots on my ship!"

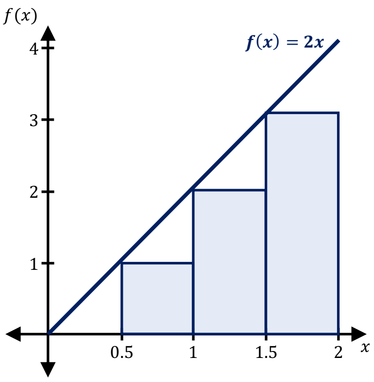
Estimating Area Under Curve: Q1, Q2, Q3, Q4

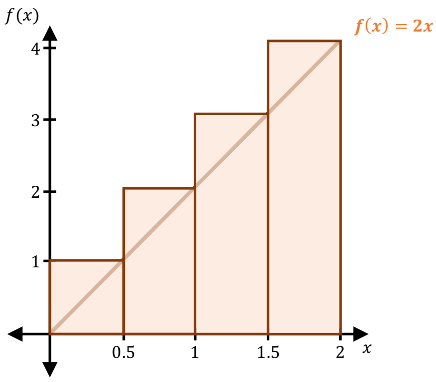
Repetitive: 2.11 → 2.31 (3 questions)

[CA] [8 marks]

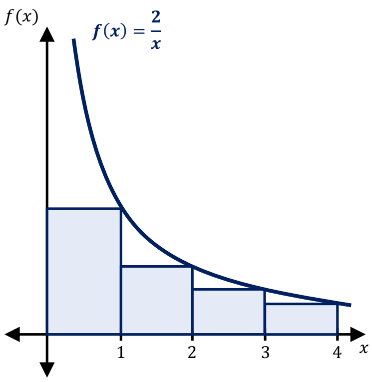
1. For day one of the competition, Janitor Peter wants to test the student's understanding of **estimating** the **area under curves**. To start, he gives Rabea and Wallace two functions: $f(x) = 2x$ and $f(x) = 2/x$. **Help** Rabea and Wallace to **estimate** the **areas under these curves** by using the **left-rectangle** and **right-rectangle** diagrams shown below.

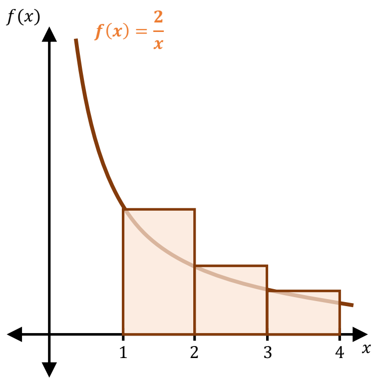
(a) **Estimate the area** under the curve of $f(x) = 2x$ between $x = 0$ and $x = 2$. **[4]**





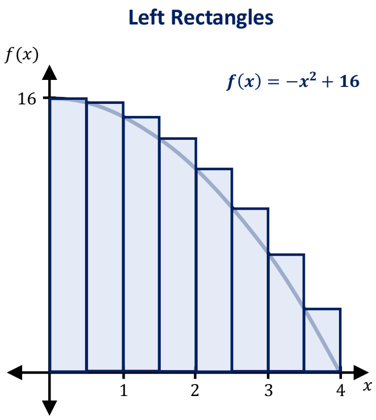
(b) **Estimate the area** under the curve of $f(x) = \frac{2}{x}$ between $x = 0$ and $x = 4$. **[4]**

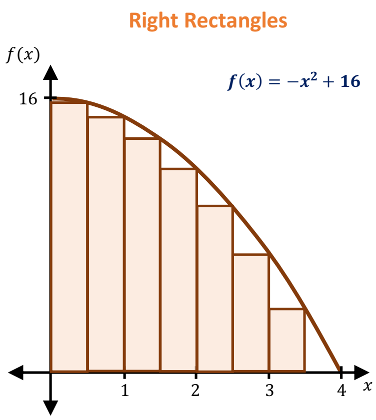




[CA] [5 marks]

2. Moving onto the next team of students, Janitor Peter provides a graph which shows the **distance covered** by a **paper airplane** based on the curve: $f(x) = -x^2 + 16$ from $x = 0$ to $x = 4$. He then asks Tyler and Tom to **estimate the area under the curve** using **left rectangles** and **right rectangles**.





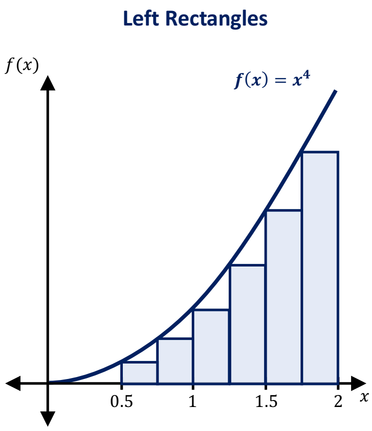
(a) **Help** Tyler and Tom to **estimate the area** of the **left-rectangles** **[2]**

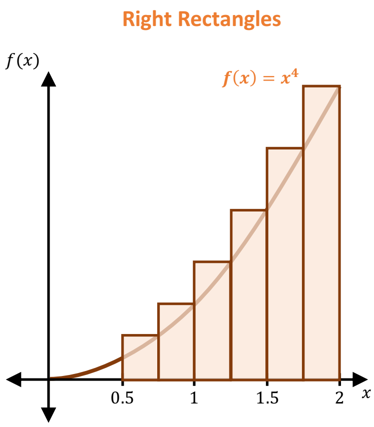
(b) **Help** Tyler and Tom to **estimate the area** of the **right-rectangles** **[2]**

(c) **Help** Tyler and Tom to submit an **estimate** of the **area below** $f(x) = -x^2 + 16$ **[1]**

[CA] [5 marks]

3. For the third pair, Jevon and Rupert, they are given a curve which shows the **velocity of a rocket** in the **first two seconds of flight**. Janitor Peter wants the pair to **estimate** the **distance covered** by the rocket by **estimating the area under the curve** from $x = 0.5$ to $x = 2$.





(a) **Help** Jevon and Rupert to **estimate the area** of the **left-rectangles** **[2]**

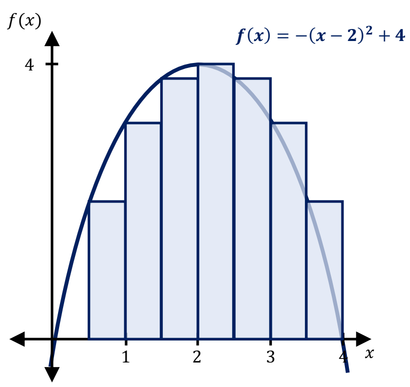
(b) **Help** Jevon and Rupert to **estimate the area** of the **right-rectangles** **[2]**

(c) **Help** Jevon and Rupert to submit an **estimate** of the **area below** $f(x) = x^4$ **[1]**

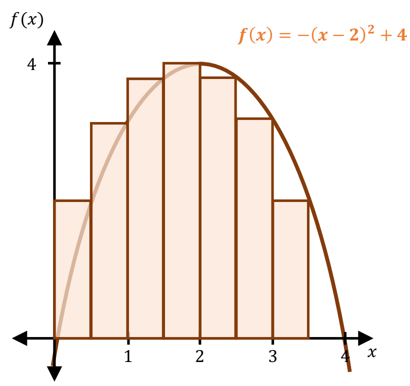
[CA] [5 marks]

4. As the last question for the first day of the competition, Janitor Peter provides a graph which shows the **height of a rocket** over the **first four minutes of flight**. He wants Kerry and Shauna to determine the **area covered** by the rocket by **estimating the area under the curve:** $f(x) = -(x-2)^2 + 4$ from $x = 0$ to $x = 4$.

Left Rectangles



Right Rectangles



(a) **Help** Jevon and Rupert to **estimate the area** of the **left-rectangles** **[2]**

(b) **Help** Jevon and Rupert to **estimate the area** of the **right-rectangles** **[2]**

(c) **Help** Jevon and Rupert to submit an **estimate** of the **area below** $f(x) = -(x-2)^2 + 4$ **[1]**

The scores were in for the first round! "It was very close, everyone!", said Janitor Peter.

"And of course, I can only take three teams to the next round. Unfortunately, Shauna and Kerry, you two came fourth. It was so close!"

Kerry could not believe it. Not once had she lost in a competition! The shame was too much for her as she stormed out of the classroom and slammed the door, leaving Shauna to congratulate everyone else.

Concept 3

Definite Integrals and Area Under Curve – Progressive Questions (9 questions)

Repetitive questions: 3.11 → 3.51 (3 questions)

Janitor Peter's Space Station Competition

Starring: Janitor Peter, Rabea, Kerry, Rupert, Shauna, Wallace, Tyler, Tom and Jevon

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Definite Integrals: Q1, Q2, Q3, Q4

Repetitive: 3.11 → 3.21 (2 questions)

[30 marks]

1. Eyes sparkling, Janitor Peter looked at his excited students who had made it to the final rounds. They were going to be learning outside today! "This might get a little trickier now, students. Let's see who has been listening well!". On his virtual self-erasing whiteboard, Janitor Peter writes out **10 definite integrals**. The teams who get all correct will progress to the next round! Help the teams to **solve** these **definite integrals**.

(a) $\int_0^1 x^2 + 4x \ dx$ **(3)**

(b) $\int_3^6 2x - 4 \ dx$ **(3)**

(c) $\int_2^3 3x^3 - 2x^2 \ dx$ **(3)**

(d) $\int_{-1}^2 \frac{1}{4}x^2 \ dx$ **(3)**

(e) $\int_2^4 (3x-4)^2 \ dx$ **(3)**

(f) $\int_{-2}^0 x^3 - 4x + 2 \ dx$ **(3)**

(g) $\int_1^6 2x - \frac{3}{x^2} \ dx$ **(3)**

(h) $\int_1^3 \frac{x^3 - 4x^4}{x} \ dx$ **(3)**

(i) $\int_1^{-2} 2x + 4 \ dx$ **(3)**

(j) $\int_3^{-1} (3+5x)^4 \ dx$ **(3)**

[30 marks]

2. Although the teams got all the questions correct, Wallace didn't understand how to do them and had to rely on Rabea. **Help** Wallace complete the next set of **definite integrals** so that Rabea doesn't feel like he's the only one carrying the team!

(a) $\int_0^3 x^{\frac{1}{2}} \ dx$ **(3)**

(b) $\int_1^2 2x(x^2-4)^3 \ dx$ **(3)**

(c) $\int_{-1}^1 \frac{1}{(2x-3)^3} \ dx$ **(3)**

(d) $\int_0^2 (4-x^2)^2 \ dx$ **(3)**

(e) $\int_2^4 \frac{1}{x^3} - 9 \ dx$ **(3)**

(f) $\int_0^{-3} x - 4 \ dx$ **(3)**

(g) $\int_1^2 (2+7x)^3 \ dx$ **(3)**

(h) $\int_4^1 \frac{2x^2 - 5x}{\sqrt{x}} \ dx$ **(3)**

(i) $\int_3^2 \frac{1}{(5x-9)^3} \ dx$ **(3)**

(j) $\int_2^{-1} 2x(x^2-5)^4 \ dx$ **(3)**

Before Janitor Peter could move on to the next question, Teacher Andrew strides into the outdoor classroom. The students exchange worried glances, afraid he'll give them another detention! But Teacher Andrew has heard about the competition and wants to go to space too!

[12 marks]

3. Janitor Peter agrees that Teacher Andrew can go to space if he can answer the following **definite integrals** quicker than Janitor Peter. Make sure Teacher Andrew doesn't go to space by helping Janitor Peter answer the **difficult problem** first! Janitor Peter draws a function on the board that is **continuous** between $\mathbf{4 \leq x \leq 10}$. Given that $\int_4^7 \boldsymbol{f(x)dx = 6}$ and $\int_7^{10} \boldsymbol{f(x)dx = \frac{5}{2}}$, help Janitor Peter to solve the following:

(a) $\int_4^{10} -f(x)dx$ **(2)**

(b) $\frac{1}{4}\int_4^{10} f(x)dx$ **(2)**

(c) $\int_4^{10} 2f(x)dx$ **(2)**

(d) $\int_7^4 f(x)dx$ **(2)**

(e) $\int_{10}^7 -3f(x)dx$ **(2)**

(f) $\int_{10}^4 f(x)dx$ **(2)**

It's a close call, but Janitor Peter just finishes before teacher Andrew and he is outraged. He demands another question!

[12 marks]

4. Confident he can win again, Janitor Peter decides to test the same concept! Janitor Peter draws a function on the board that is **continuous** between $\mathbf{-6 \leq x \leq 12}$. Given $\int_{-6}^{-2} \boldsymbol{g(x)dx = 9}$, $\int_{-2}^{4} \boldsymbol{g(x)dx = 4}$ and $\int_4^{12} \boldsymbol{g(x-4) = 6}$, **help** Janitor Peter to **solve** the following:

(a) $\int_{-6}^{-2} -g(x)$ **(2)**

(b) $\int_{-6}^{4} g(x)dx$ **(2)**

(c) $\int_{-2}^{4} \frac{g(x)}{10} dx$ **(2)**

(d) $\int_{12}^{-6} g(x)\, dx$ **(2)**

(e) $\int_4^{-2} -\frac{1}{3}g(x)\, dx$ **(2)**

(f) $\int_4^{-6} -\frac{5}{2}g(x)dx$ **(2)**

Although it was close, Janitor Peter's love and understanding of maths meant he got the questions correct first! Reaching out to shake his competitors hand, Teacher Andrew looks at him with rage and storms off into the distance.

Teacher Andrew is furious that he won't have a seat on the spaceship, but he is confident that he can use other means to get to space…

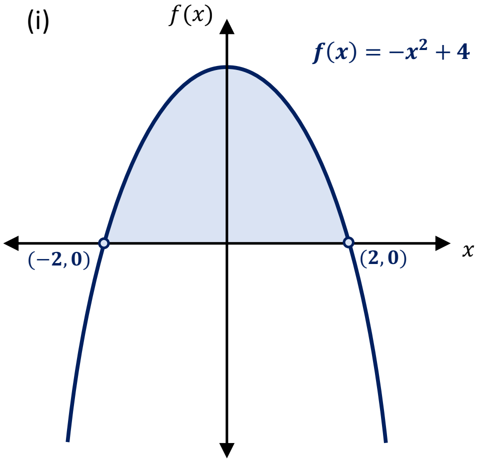
Area Under Curve: Q5, Q6, Q7, Q8, Q9

Repetitive: 3.51 (1 question)

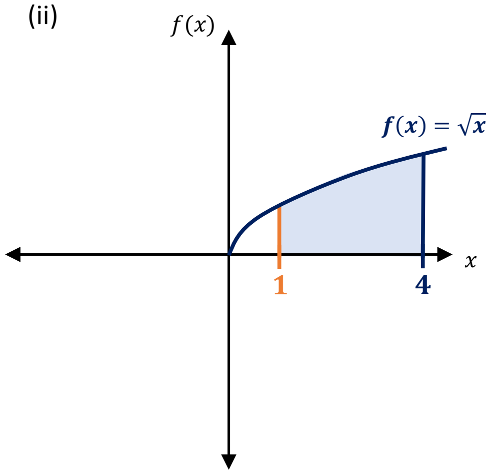
[CA] [24 marks]

5. Exhausted by his mathematics battle with teacher Andrew, Janitor Peter decides to leave the six remaining students to explore the concept of **area under the curve** casually. To start, he asks the six students to work together to **determine** the **shaded area** between the **function** $f(x)$ and the x**-axis**. Help the students use **definite integrals** to **calculate the shaded area** for each function.

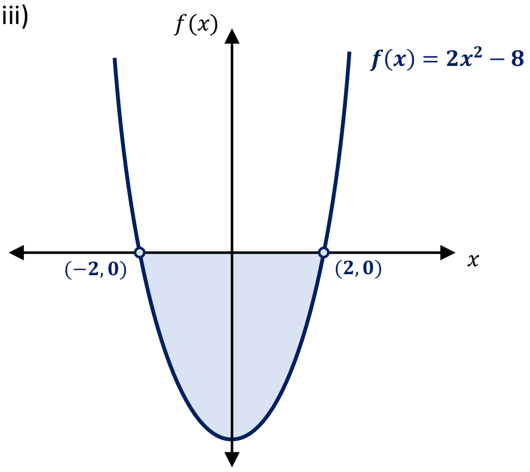
(i)



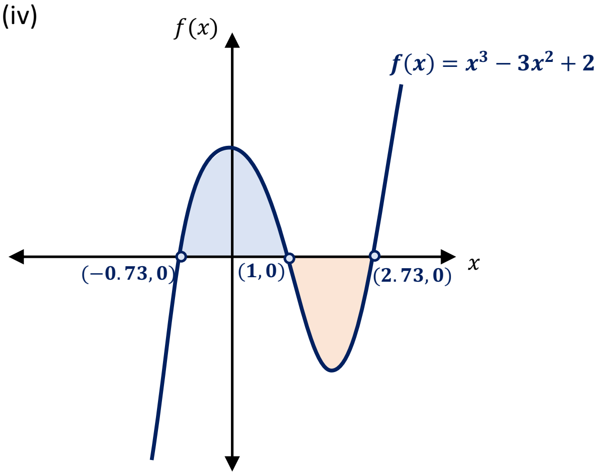
(ii)



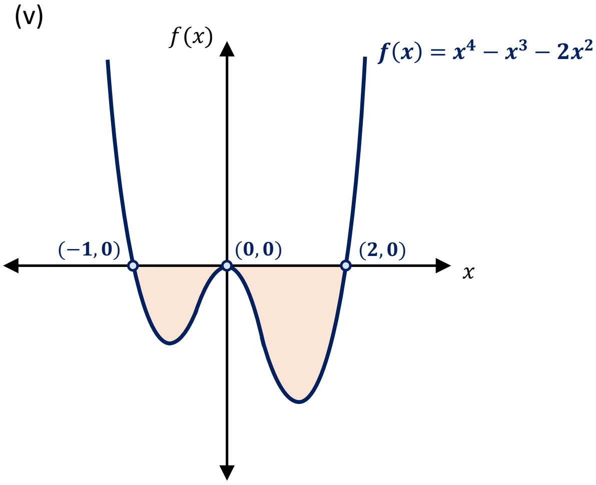
(iii)



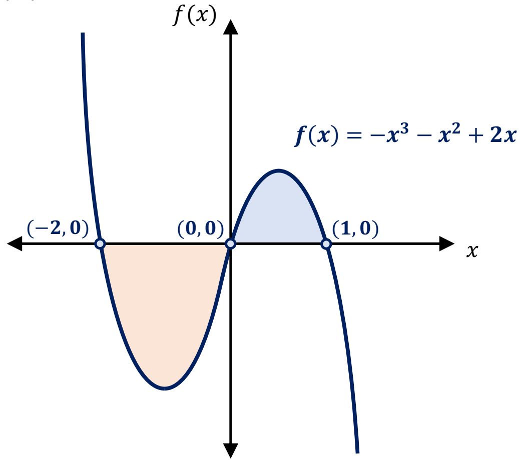
(iv)



(v)



(vi)



[CA] [7 marks]

6. Amazed by the talent of the six methods students, Janitor Peter wants them to apply their knowledge to **more practical situations**. Taking the students outside to a building that $\mathbf{8}$ **meters long** and has a **curved roof** following the **equation** $\boldsymbol{y = -0.5x^2 + 4x}$, help the students to do the following:

(a) **Sketch** a graph the curved roof and determine the **location** of the $\boldsymbol{x}$**-intercepts**. **[3]**

(b) Find the **cross-sectional area bounded by the floor** and **roof**, where the $\boldsymbol{x}$**-axis** represents the floor. **[4]**

[CA] [9 marks]

7. The Maths Applications building has curved roof that follows the **equation** $\boldsymbol{y = -\frac{5}{13}x^2 + 12}$, and has a length of just over $\mathbf{11}$ **metres**.

(a) Illustrate the building's roof in a **graph**, where **the** $\boldsymbol{x}$**-axis represents the floor**. **[3]**

(b) What is the **area** of the **cross section** of this building? **[3]**

(c) What is the **difference in area** if the equation of the roof changed to $\boldsymbol{y = -6x^2 + 6}$ with $\boldsymbol{x}$**-intercepts** at $\boldsymbol{x = -1}$ and $\boldsymbol{x = 1}$ **[3]**

[CA] [10 marks]

8. The Maths Specialist building has a curved roof that follows the **equation** $\boldsymbol{y = -x^2 + 9x}$.

(a) Illustrate the roof in a **graph**, assuming **the** $\boldsymbol{x}$**-axis represents the floor**. **[3]**

(b) Assuming the $\boldsymbol{x}$**-axis** represents the **floor**, what are the **values** of $\boldsymbol{x}$ that the **roof touches the floor**? **[1]**

(c) Find the **area bounded by the** $\boldsymbol{x}$**-axis** and the roof. **[4]**

(d) If the **width** of this roof was $\mathbf{12}$ **metres**, calculate the **volume** of this roof. **[3]**

[CA] [9 marks]

9. Whilst calmly working Jevon gets one of the areas wrong and **throws his book** across the room. The book follows the perfect path of $\boldsymbol{f(x) = -(x-2)^2 + 2}$.

(a) **Sketch** a **graph** the book's journey **[3]**

(b) What **height** did the book reach? **[1]**

(c) At what **values** of $\boldsymbol{x}$ did the **book start on floor** and subsequently **land on the floor**? **[2]**

(d) What is the **area bounded** by the **equation** and the $\boldsymbol{x}$**-axis?** **[3]**

Concept 4

Fundamental Theorem of Calculus – Progressive Questions

(4 questions)

Repetitive questions: 4.11 → 4.31 (2 questions)

Janitor Peter's Space Station Competition!

Starring: Janitor Peter, Rupert, Alexa, Rabea, Jevon, Harriet and Fraser

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Janitor Peter still needs to whittle the six students down to five students. To do so, he decides to setup a mathematics treasure hunt around the school!

Each student has to find the questions, answer them and they will get a clue as to where the next set of questions are. Janitor Peter knows the Fundamental Theorem of Calculus can be tricky, so decides to test the students on that. The student who is slowest to return will be eliminated!

Fundamental Theorem of Calculus: Q1, Q2, Q3, Q4

Repetitive: 4.11 → 4.31 (2 questions)

[16 marks]

1. The first set of questions are hidden behind the statue of Teacher Andrew that he installed himself. Tyler is the first to arrive. **Help** him by completing the problems using the **Fundamental Theorem of Calculus**.

(a) $\frac{d}{dx}\left(\int_0^x t^3\,dt\right)$ **(2)**

(b) $\frac{d}{dx}\left(\int_{-2}^x u^2 + 2u\,du\right)$ **(2)**

(c) $\frac{d}{dx}\left(\int_5^x t^2 + 2t\,dt\right)$ **(2)**

(d) $\frac{d}{dx}\left(\int_2^x \frac{1}{2t} + \frac{5}{\sqrt{t}}\,dt\right)$ **(2)**

(e) $\frac{d}{dx}\left(\int_x^0 t^5 - 6\,dt\right)$ **(2)**

(f) $\frac{d}{dx}\left(\int_{-3}^x t^2 - 3t^3 - 2\,dt\right)$ **(2)**

(g) $\frac{d}{dx}\left(\int_x^0 2u^2 - 5u - 6\,du\right)$ **(2)**

(h) $\frac{d}{dx}\left(\int_1^{2x} \frac{7}{2t^4}\,dt\right)$ **(2)**

[16 marks]

2. Running at record pace Rabea and Rupert are racing to the **second hiding spot**. At the second hiding spot Rabea and Rupert find another set of problems engraved into the wall. **Help** Rabea and Rupert to answer the following set using the **Fundamental Theorem of Calculus**.

(a) $\frac{d}{dx}\left(\int_0^{2x} (1-t)^3\,dt\right)$ **(2)**

(b) $\frac{d}{dx}\left(\int_0^{x^2} 4u^4\,du\right)$ **(2)**

(c) $\frac{d}{dx}\left(\int_{-3}^x (2t+2)^3 - 6\,dt\right)$ **(2)**

(d) $\frac{d}{dx}\left(\int_2^{4x} \frac{1}{\sqrt{2u^2+3}}\,du\right)$ **(2)**

(e) $\frac{d}{dx}\left(\int_x^0 \frac{2}{(t^2-3)^3}\,dt\right)$ **(2)**

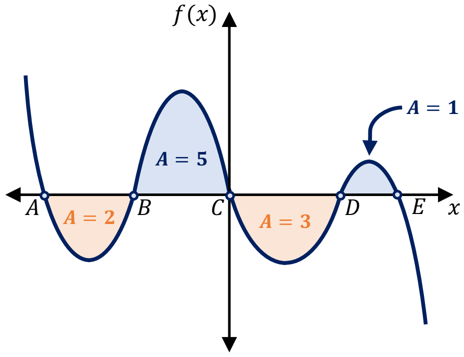
(f) $\frac{d}{dx}\left(\int_{-3}^x (4-3u^3)^7\,dt\right)$ **(2)**

(g) $\frac{d}{dx}\left(\int_9^{x^2} \sqrt{1-t^2}\,dt\right)$ **(2)**

(h) $\frac{d}{dx}\left(\int_{x^2}^0 3u^9 - (2-u)^4\,du\right)$ **(2)**

[4 marks]

3. Jevon and Rupert are some of the top students but when it comes to this challenge, they're struggling a little. Based on the function $\boldsymbol{f(x)}$ below, help them to **answer** the following questions.



(a) Determine $\int_C^E f(x)dx$ **[1]**

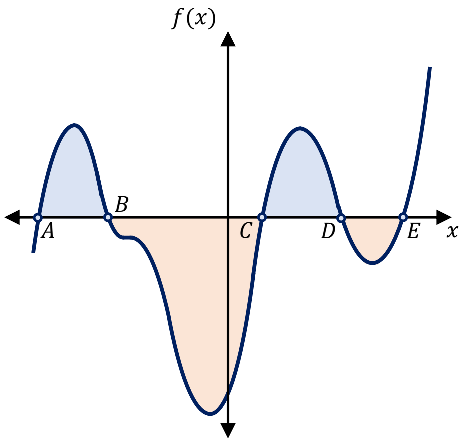
(b) Determine the **area enclosed** between $f(x)$ and the x**-axis** from $x = B$ to $x = D$ **[1]**

(c) Determine $\int_A^E f(x)dx$ **[1]**

(d) Determine the **area enclosed** between $f(x)$ and the x**-axis** from $x = A$ to $x = E$ **[1]**

[4 marks]

4. The students run to the final hiding place. On top of the tables they eat lunch at is a graph, which states that: $\int_A^B f(x)dx = 8$, $\int_B^C f(x)dx = -20$, $\int_C^D f(x)dx = 8$ and $\int_D^E f(x)dx = -4$.



(a) Determine $\int_C^E f(x)dx$ **[1]**

(b) Determine $\int_B^E |f(x)|dx$ **[1]**

(c) Determine the **area enclosed** between $f(x)$ and the x**-axis** from $x = A$ to $x = D$ **[1]**

(d) Determine $\int_A^E |f(x)|dx$ **[1]**

After being slow on the final problem, Jevon comes running to the finishing line in last place. Jevon slams his fist on the table and glares at Rupert. "I should never have trusted you in my team!" The rest of the five students break into a glorious cheer, celebrating the exciting trip they have ahead at the Space Station.

Problem Set 3 – Integration
Repetitive Questions

Concept 1
Integration Techniques – Repetitive Questions
(3 questions)

Integration Techniques: Qs 1.11, 1.21, 1.31

[20 marks]

1.11 **Determine** the following **integrals** without the use of a calculator.

(a) $\int 2xdx$ **(2)**

(b) $\int x - x^2\, dx$ **(2)**

(c) $\int 5x^2\, dx$ **(2)**

(d) $\int \frac{1}{3}x^2 - \frac{3}{2}x\, dx$ **(2)**

(e) $\int \frac{3x^3}{2} - 6x^2\, dx$ **(2)**

(f) $\int \sqrt{x} + x^4\, dx$ **(2)**

(g) $\int \frac{1}{x^3} - \frac{5}{x^2}\, dx$ **(2)**

(h) $\int \frac{4}{3\sqrt{x}} + 3\sqrt{x}\, dx$ **(2)**

(i) $\int 5x^{\frac{3}{2}}\, dx$ **(2)**

(j) $\int \frac{x^2 - x}{x}\, dx$ **(2)**

[27 marks]

1.21 **Determine** the following **integrals** without the use of a calculator.

(a) $\int (4x-3)(x+2)\, dx$ **(2)**

(b) $\int \frac{4x^2 - 5x^3}{2x^2} dx$ **(2)**

(c) $\int 3x^2(x^2 - 2x)\, dx$ **(2)**

(d) $\int \frac{6x^4 - 3x}{3x} dx$ **(3)**

(e) $\int (5x-2)(x^2-4x)\, dx$ **(3)**

(f) $\int \frac{x^2+x}{\sqrt{x}} dx$ **(3)**

(g) $\int 2x^2(9-4x)\, dx$ **(3)**

(h) $\int \sqrt{x}(x^{\frac{5}{2}} - 3\sqrt{x})\, dx$ **(3)**

(i) $\int (x^2-2)^2 + 4x^3\, dx$ **(3)**

(j) $\int \frac{2x - x^2}{x^5} + 2\, dx$ **(3)**

[30 marks]

1.31 **Determine** the following **integrals** without the use of a calculator.

(a) $\int (x+2)^3\, dx$ **(3)**

(b) $\int (9-x)^4 dx$ **(3)**

(c) $\int (5x+4)^3\, dx$ **(3)**

(d) $\int 4(8x-5)^5\, dx$ **(3)**

(e) $\int \frac{1}{(x-5)^4}\, dx$ **(3)**

(f) $\int 4x(2x^2-4)^3\, dx$ **(3)**

(g) $\int \frac{2}{(2x+4)^3}\, dx$ **(3)**

(h) $\int 4x(x^2-6)^4\, dx$ **(3)**

(i) $\int \frac{4}{(3x-4)^6} dx$ **(3)**

(j) $\int 8x(3+4x^2)^4 dx$ **(3)**

Concept 2

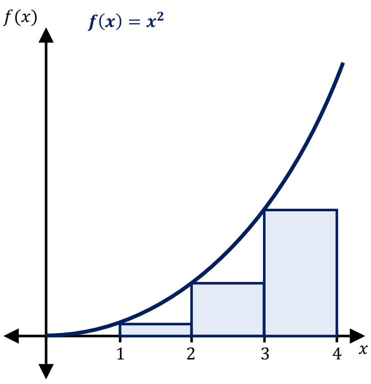
Estimating Area Under the Curve – Repetitive Questions

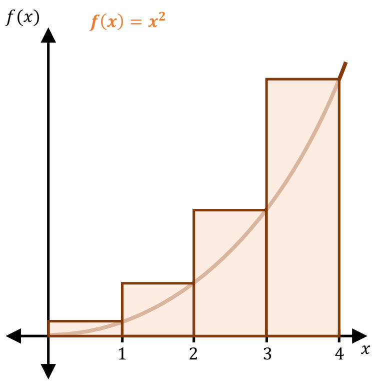
(3 questions)

Estimating Area Under the Curve: Qs 2.11, 2.21, 2.31

[CA] [5 marks]

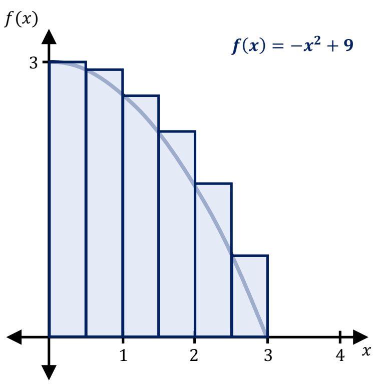
2.11 Using **left-rectangles** and **right-rectangles**, provide an **estimate** for the **area below** $f(x) = x^2$ between $x = 0$ and $x = 4$.

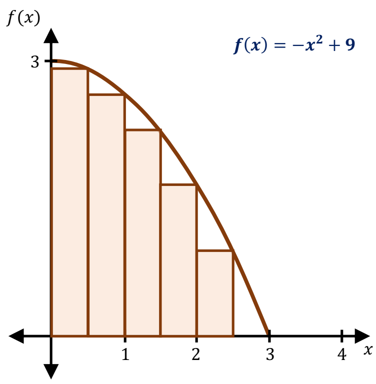




[CA] [5 marks]

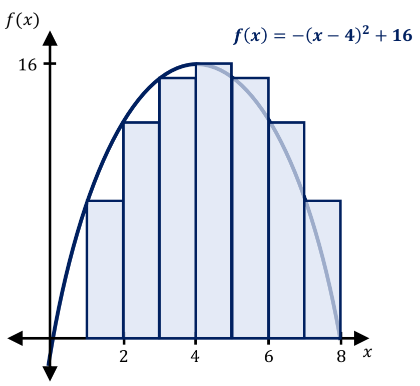
2.21 Using **left-rectangles** and **right-rectangles**, provide an **estimate** for the **area below** $f(x) = -x^2 + 9$ between $x = 0$ and $x = 3$.

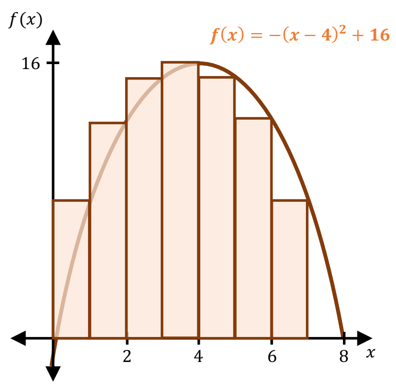




[CA] [5 marks]

2.31 Using **left-rectangles** and **right-rectangles**, provide an **estimate** for the **area below** $f(x) = -(x-4)^2 + 16$ between $x = 0$ and $x = 8$.





Concept 3

Definite Integrals and Area Under Curve – Repetitive Questions

(3 questions)

Definite Integrals: Qs 3.11, 3.21

[30 marks]

3.11 Evaluate the following **definite integrals** 'by hand' and then use a calculator to check your answer.

(a) $\int_0^1 3x^2 + 6\ dx$ **(3)**

(b) $\int_2^5 8x^3 + 4x + 1\ dx$ **(3)**

(c) $\int_2^4 x(2x^2 - 5)\ dx$ **(3)**

(d) $\int_0^3 5x^2 - \sqrt{x}\ dx$ **(3)**

(e) $\int_1^4 (5x - 2)^2\ dx$ **(3)**

(f) $2\int_{-1}^1 x^3 + 3x - 5\, dx$ **(3)**

(g) $\int_{-3}^0 (4 - 3x^2)^2\, dx$ **(3)**

(h) $\int_{-5}^{-2} 3(1 - x)^2\ dx$ **(3)**

(i) $\int_{-1}^1 4x^3 - 2x + 1\ dx$ **(3)**

(j) $\int_{-3}^3 \frac{1}{\sqrt{x}}\, dx$ **(3)**

[30 marks]

3.21 Determine each of the following **definite integrals** and ensure your answers are in **exact values**.

(a) $\int_0^2 \frac{1}{2}x^2\ dx$ **(3)**

(b) $\int_3^5 3x^2 - 2x\ dx$ **(3)**

(c) $\int_{-1}^1 \frac{x^3 - 3x}{2x}\ dx$ **(3)**

(d) $\int_0^2 (8 + 2x)^2\ dx$ **(3)**

(e) $\int_2^4 (x + 3)^3\ dx$ **(3)**

(f) $\int_1^4 7x - \sqrt{x}\ dx$ **(3)**

(g) $\int_1^2 (x^2 - 2)^2\ dx$ **(3)**

(h) $\int_1^3 \frac{2x - 4x^2}{\sqrt{x}}\ dx$ **(3)**

(i) $\int_3^2 \frac{1}{x^3} - x^2\ dx$ **(3)**

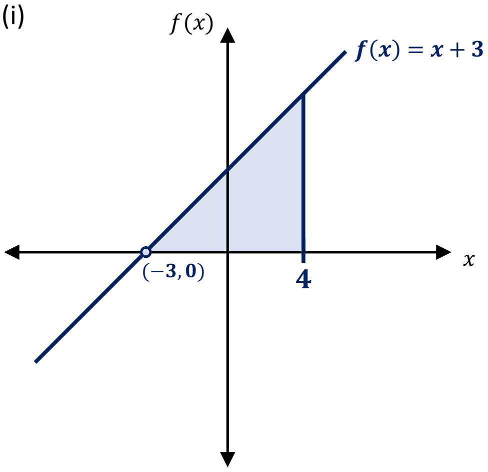
(j) $\int_2^{-1} \frac{5x^4 - 2x^2}{x^2}\ dx$ **(3)**

Area Under the Curve: Qs 3.51

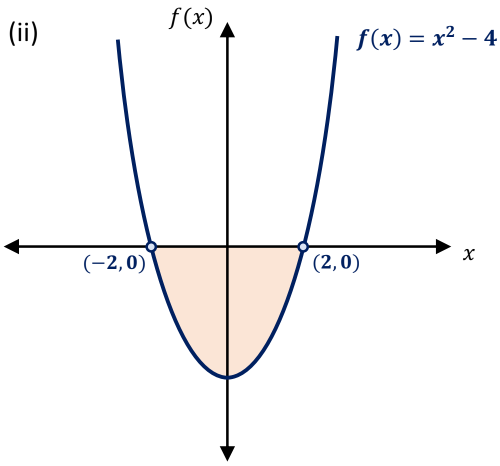
[24 marks]

3.51 Using **definite integrals**, determine the **shaded areas trapped** between the **function** $f(x)$ and the x**-axis** below.

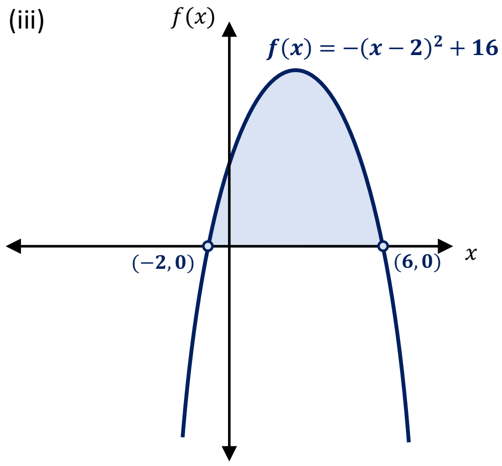
(i)



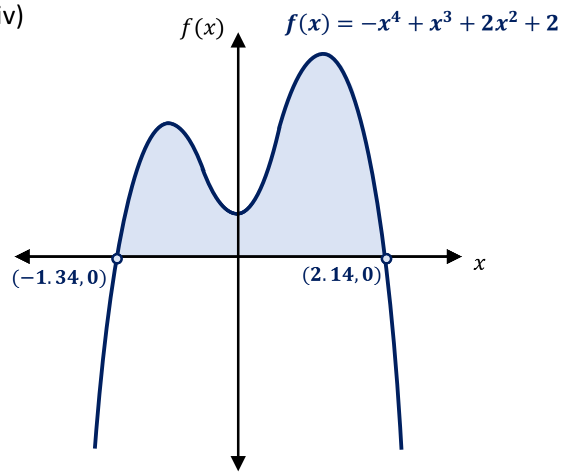
(ii)



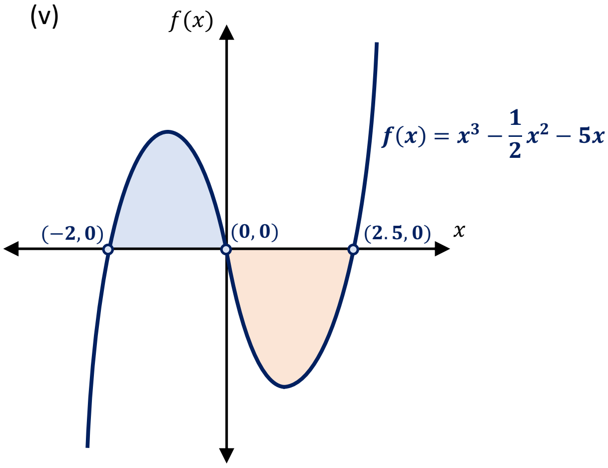
(iii)



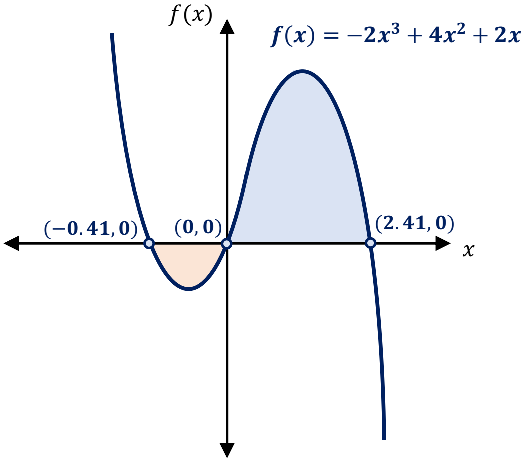
(iv)



(v)



(vi)



Concept 4

Fundamental Theorem of Calculus – Repetitive Questions

(2 questions)

Fundamental Theorem of Calculus: Qs 4.11, 4.31

[16 marks]

4.11 Completing the following problems using the **Fundamental Theorem of Calculus**.

(a) $\frac{d}{dx}\left(\int_0^x t^8\, dt\right)$ **(2)**

(b) $\frac{d}{dx}\left(\int_0^x 2u - u^2\, du\right)$ **(2)**

(c) $\frac{d}{dx}\left(\int_x^0 (3t-t)(2+t)\, dt\right)$ **(2)**

(d) $\frac{d}{dx}\left(\int_1^{2x} \frac{6}{t-2} + 2\, dt\right)$ **(2)**

(e) $\frac{d}{dx}\left(\int_4^{x^2} (1-u^2)^{\frac{1}{2}}\, du\right)$ **(2)**

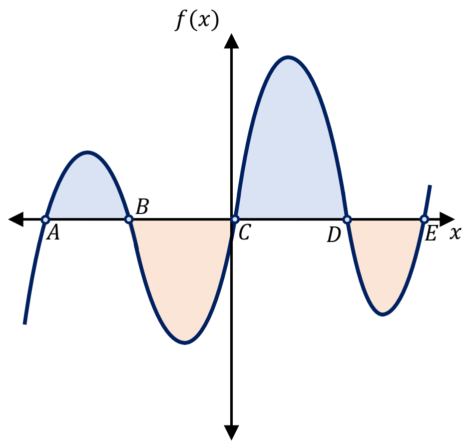
(f) $\frac{d}{dx}\left(\int_2^{3x} 5t - 6t^3\, dt\right)$ **(2)**

(g) $\frac{d}{dx}\left(\int_{x^2}^0 4u^3 - 3u^2 - 6\, du\right)$ **(2)**

(h) $\frac{d}{dx}\left(\int_x^0 \frac{4}{3u^2} + \sqrt[3]{u}\, du\right)$ **(2)**

[4 marks]

4.31 The graph below of $f(x)$ has **specific shaded areas** which are **bound** by the **definite integrals**: $\int_A^B f(x)dx = 3$, $\int_B^C f(x)dx = -4$, $\int_C^D f(x)dx = 7$ and $\int_D^E f(x)dx = -3$.



(a) Determine $\int_A^E f(x)dx$ **[1]**

(b) Determine $\int_B^D |f(x)|dx$ **[1]**

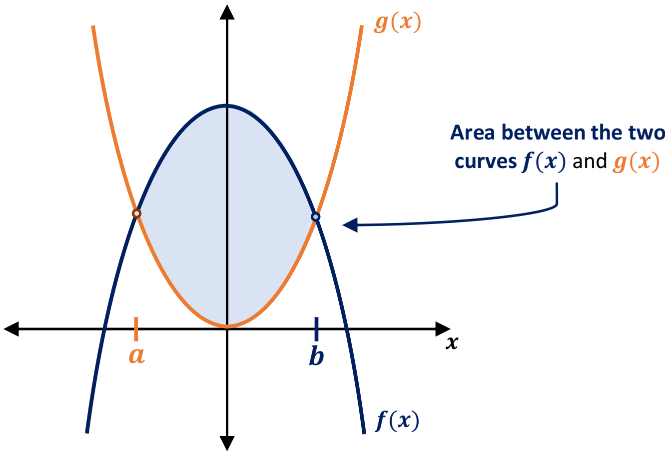
(c) Determine the **area enclosed** between $f(x)$ and the **x-axis** from $x = A$ to $x = D$ **[1]**

(d) Determine $\int_A^E |f(x)|dx$ **[1]**

2.5 Area Between Curves

Now that we have understood the concept of **definite integrals** and **area under the curve**, we can expand on these ideas to **calculate** the **area between two curves**.

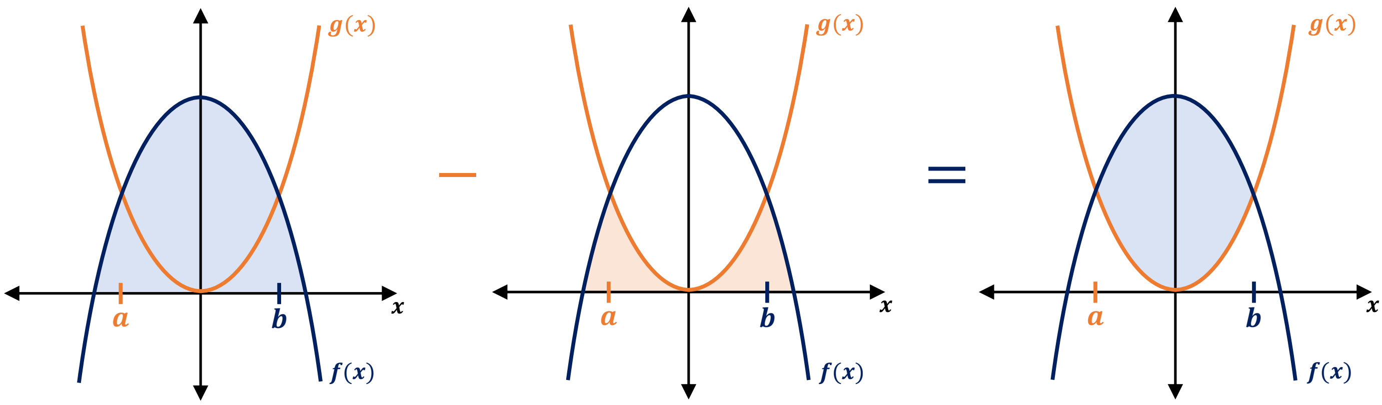
As shown below, the **area between curves** is the area **enclosed between two curves** between **bounds** $\boldsymbol{a}$ and $\boldsymbol{b}$.



When determining the **area between two curves**, $f(x)$ and $g(x)$, where $f(x)$ is the **upper curve**, the **area** can be calculated as the **area** of the **upper curve** ($f(x)$) **minus** the **area** of the **lower curve** ($g(x)$), **irrespective** of whether the curves are **above** or **below** the **x-axis**. This can be represented as the **general formula** below:

$$\textbf{Area Between Curves: } A = \int_a^b [f(x) - g(x)]dx \text{ where } b > a$$

From our example above, what this **definite integral formula** is telling us is that the **area between the curves** $f(x)$ and $g(x)$, is **equal** to the **definite integral** of the **upper curve** $f(x)$ **less** the **definite integral** of the **lower curve** $g(x)$, and it **does not matter** if the curves are **above** or **below** the **x-axis**.



$$\textit{Area under } f(x) - \textit{Area under } g(x) = \textit{Area Between Curves}$$

To complete our example above, we can **calculate** the **area between the functions** if we know the functions are: $f(x) = -x^2 + 8$ and $g(x) = x^2$, and they intersect at $x = -2$ and $x = 2$.

By **subtracting** the **definite integral** of $g(x)$ from the **definite integral** of $f(x)$, this will give us an **area** of $A = \frac{64}{3} units^2$.

$$A = \int_a^b [f(x) - g(x)]dx$$

$$A = \int_{-2}^{2} -x^2 + 8 - x^2 dx$$

① $$A = \int_{-2}^{2} -2x^2 + 8dx$$

② $$A = \left[-\frac{2x^3}{3} + 8x\right]_{-2}^{2}$$

③ $$A = \left(-\frac{2(2)^3}{3} + 8(2)\right) - \left(-\frac{2(-2)^3}{3} + 8(-2)\right)$$

$$A = \frac{32}{3} + \frac{32}{3}$$

④ $$A = \frac{64}{3} units^2$$

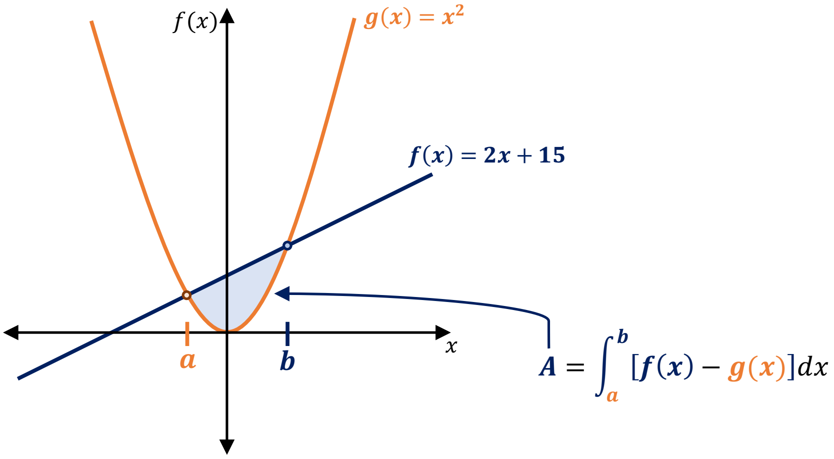
① **Establish** the **definite integral** of $f(x) - g(x)$

② **Integrate** the definite integral with the bounds $a = 2$ and $b = 3$

③ **Substitute** in the bounds $a = -2$ and $b = 2$

④ **Simplify** to the final answer

As a second example, let's consider the **area** trapped between $f(x) = 2x + 15$ and $g(x) = x^2$, however this time **we don't know** the **values** of the bounds a and b.



To determine the **values** of a and b, we need to **determine** the **x-values** at which $f(x)$ and $g(x)$ **intersect each other**. We do this by letting the two functions **equal each other** (i.e. $f(x) = g(x)$) and then **solving** for x. For our example, we can determine that $a = -3$ and $b = 5$:

① $$f(x) = g(x)$$

$$2x + 15 = x^2$$

$$x^2 - 2x - 15 = 0$$

② $$(x - 5)(x + 3) = 0$$

$$x = -3, x = 5$$

③ $$\therefore a = -3 \text{ and } b = 5$$

① **Let** $f(x) = g(x)$

② **Factorise** and **solve for** x

③ **State** the **final bounds** of $a = -3$ and $b = 5$

With our values of $a = -3$ and $b = 5$, we can **evaluate** the **definite integral** by **subtracting** the **lower curve** $g(x) = x^2$ from the **upper curve** $f(x) = 2x + 15$:

$$A = \int_a^b [f(x) - g(x)]dx$$

① $$A = \int_{-3}^{5} 2x + 15 - x^2 dx$$

② $$A = \left[x^2 + 15x - \frac{x^3}{3}\right]_{-3}^{5}$$

③ $$A = \left((5)^2 + 15(5) - \frac{(5)^3}{3}\right) - \left((-3)^2 + 15(-3) - \frac{(-3)^3}{3}\right)$$

$$A = \frac{175}{3} + 27$$

④ $$A = \frac{256}{3}\ units^2$$

① **Establish** the **definite integral** of $f(x) - g(x)$

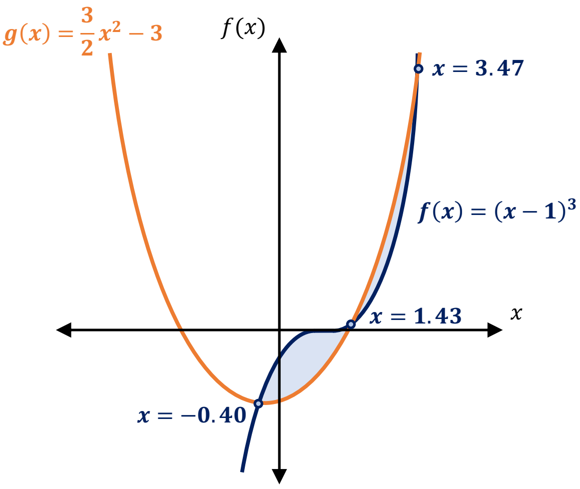
② **Integrate** the definite integral with the bounds $a = -3$ and $b = 5$

③ **Substitute** in the bounds $a = -3$ and $b = 5$

④ **Simplify** to the final answer

As you can see, this process is **very similar** to the process of **determining area under curves**, it just involves us taking the **extra step** of **subtracting** the **lower function** from the **upper function**.

As a third, more difficult example, let's look at how we determine the **area between two functions**: $f(x) = (x-1)^3$ and $g(x) = \frac{3}{2}x^2 - 3$.



As we can see above, $f(x)$ and $g(x)$ create **two key areas**, and the '**upper function**' **alternates** for the two areas. The **first area** is between $x = -0.40$ and $x = 1.43$ and $f(x)$ is the **upper function**, and the **second area** is between $x = 1.43$ and $x = 3.47$ and $g(x)$ is the **upper function**. Therefore, to determine the area between these curves we need to **split the integral up** into **two integrals** and then **calculate each area** as follows:

$$A = \int_{-0.40}^{1.43} f(x) - g(x)dx + \int_{1.43}^{3.47} g(x) - f(x)dx$$

$$A = \int_{-0.40}^{1.43} (x-1)^3 - \left(\frac{3}{2}x^2 - 3\right)dx + \int_{1.43}^{3.47} \left(\frac{3}{2}x^2 - 3\right) - (x-1)^3 dx$$

$$A = 3.04 + 4.01$$

$$A = 7.05\ units^2$$

As a final key point, similar to area under the curve, if we have **access** to a **calculator** then we can simply apply the **absolute value** to the **integral** and then we **don't need to worry** about **splitting up the integral**.

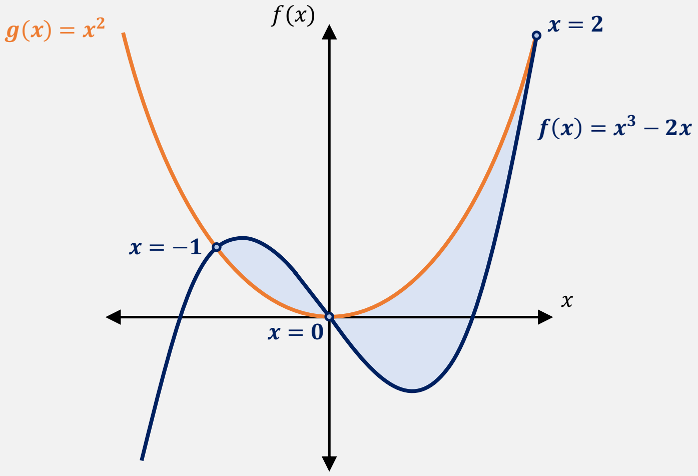
Absolute Area between curves: $A = \int_a^b |f(x) - g(x)|dx$ where $b > a$

For our example above of $f(x) = (x-1)^3$ and $g(x) = \frac{3}{2}x^2 - 3$, this would be:

$$A = \int_{-0.40}^{3.47} \left|(x-1)^3 - \left(\frac{3}{2}x^2 - 3\right)\right| dx = 7.05\ units^2$$

Worked Example 1

Continuing to explore his interest for integration, Wallace, wants to determine the **area trapped** between the two functions: $f(x) = x^3 - 2x$ and $g(x) = x^2$.



Help Wallace to **determine** the **area** trapped between $f(x)$ and $g(x)$.

$$A = \int_{-1}^{0} f(x) - g(x)dx + \int_0^2 g(x) - f(x)dx$$

$$A = \int_{-1}^{0} x^3 - 2x - x^2\ dx + \int_0^2 x^2 - x^3 + 2x\ dx$$

$$A = \left[\frac{x^4}{4} - x^2 - \frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^3}{3} - \frac{x^4}{4} + x^2\right]_0^2$$

$$A = (0) - \left(\frac{(-1)^4}{4} + (-1)^2 - \frac{(-1)^3}{3}\right) - \left(\frac{(2)^3}{3} - \frac{(2)^4}{4} + (2)^2\right) - (0)$$

$$A = \frac{5}{12} + \frac{8}{3}$$

$$A = \frac{37}{12}\ units^2$$

Note, **with a calculator** you could instead do:

$$A = \int_{-1}^{2} \left|x^3 - 2x - x^2\right| dx = \frac{37}{12}\ units^2$$

2.6 Net Change

In certain applications we interested in determining the **net change** of the function **between two points,** such as the **net change in height** in a **hot air balloon** between the **first and the tenth minute** or the **net change** in a **bacteria population** during the **first hour**.

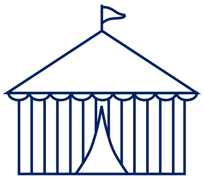
> **Net change** is the difference between **two points** of a function. It uses the **definite integral** to find this **net change** between the **two points**, a and b.

Simply put, the **net change** of a function is the **definite integral** of its **derivative**. So, if we had the **derivative** of a function and we wanted to know its **net change** between points a and b, we would use the **definite integral formula**:

$$\text{Net Change:} \quad f(b) - f(a) = \int_a^b f'(x)\,dx$$

Net change can be applied to **anything** where the **net change** between **two points** needs to be **calculated**, including **stock prices**, **measurements** or **motion**.

For instance, suppose the **rate** at which **people enter** a **large theatre** is modelled by the **rate equation**: $\frac{dP}{dt} = 82t^3 + 30t^2 + 6t$, where P is the **number of people** and t is in **hours**.



$$\text{Population rate of change:} \quad \frac{dP}{dt} = 82t^3 + 30t^2 + 6t$$

If we wanted to know the **net change** in **population** between 0.5 **hours** and 1.5 **hours**, we could use our **net change equation**: $f(b) - f(a) = \int_a^b f'(x)\,dx$.

We would do this by determining the **definite integral** of the **rate equation**, $\frac{dP}{dt}$, with the **lower bound** $t = 0.5$ and the **upper bound** $t = 1.5$. This would be done as follows:

$$P(1.5) - P(0.5) = \int_{0.5}^{1.5} 82t^3 + 30t^2 + 6t\,dt$$

$$P(1.5) - P(0.5) = \left[\frac{41}{2}t^4 + 10t^3 + 3t^2\right]_{0.5}^{1.5}$$

$$P(1.5) - P(0.5) = \left(\frac{41}{2}(1.5)^4 + 10(1.5)^3 + 3(1.5)^2\right) - \left(\frac{41}{2}(0.5)^4 + 10(0.5)^3 + 3(0.5)^2\right)$$

$$P(1.5) - P(0.5) = 141\ people$$

Hopefully the concept of net change shouldn't feel too confusing. It is the **same process** we have been **applying** for **all** of our **definite integrals**.

Worked Example 1

Rabea wants to aim for the stars, so he decides to **hire** a **hot air balloon** which has a **height** described by the **rate equation** of $\frac{dH}{dt} = 4t + 8$ where H is the **height** in **meters** and t is **time** in **minutes**.

(a) Determine the **net change** in **height** between 15 and 30 **minutes**.

$$H(30) - H(15) = \int_{15}^{30} 4t + 8\, dt$$
$$H(30) - H(15) = \left[2t^2 + 8t\right]_{15}^{30}$$
$$H(30) - H(15) = \left(2(30)^2 + 8(30)\right) - \left(2(15)^2 + 8(15)\right)$$
$$H(30) - H(15) = 1470\, m$$

(b) Determine the **average rate of change of height** between 15 and 30 **minutes**.

$$Average\ Rate = \frac{H(30) - H(15)}{15}$$
$$Average\ Rate = \frac{1470}{15}$$
$$Average\ Rate = 98\, m/min$$

(c) Determine the **time** when the balloon will reach $10,000\ m$ if it **started at ground level**.

Let $H(t) - H(0) = 10000$

$$10000 = \int_{0}^{x} 4t + 8\, dt$$

Can solve with calculator if available

$$10000 = \left[2t^2 + 8t\right]_{0}^{x}$$
$$10000 = \left(2(x)^2 + 8(x)\right) - \left(2(0)^2 + 8(0)\right)$$
$$10000 = 2x^2 + 8x$$
$$0 = 2x^2 + 8x - 10000$$

From **solving** on calculator: $x = 68.7$, $x = -72.7$ **(disregard second answer)**

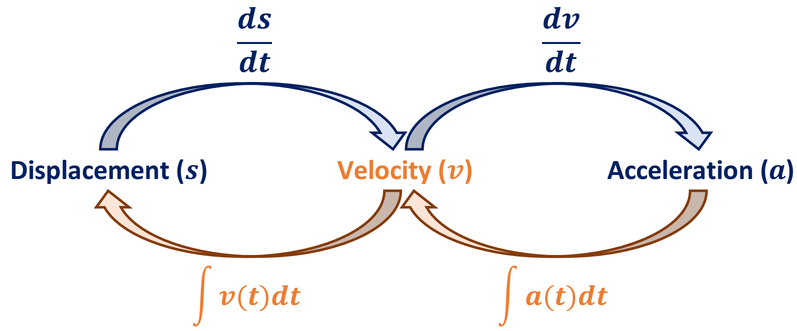
$\therefore$ Hot air balloon will reach $10,000m$ after 68.7 **minutes**.

2.5 Rectilinear Motion

Rectilinear motion tracks the **motion** of objects in **one dimension** through its **displacement (s)**, **velocity (v)** and **acceleration (a)** in relation to **time (t)**.

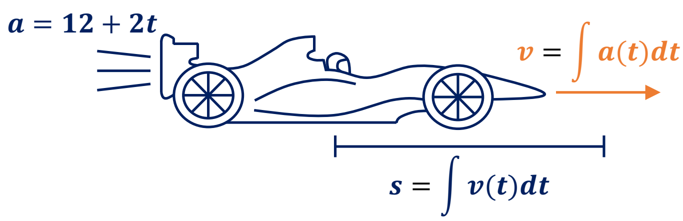
As we explored in differentiation, we can **differentiate** the **displacement function** to get the **velocity function** (i.e. $v(t) = \frac{ds}{dt}$), and we can **differentiate velocity function** to get the **acceleration function** (i.e. $a(t) = \frac{dv}{dt}$).

Alternatively, to reverse this process we can use **integration**. To get the **velocity function** we **integrate** the **acceleration function** (i.e. $v(t) = \int a(t)dt$), and to get the **displacement function** we **integrate** the **velocity function** (i.e. $s(t) = \int v(t)dt$).



Integrating these functions will include C as an **unknown constant**, however, we can often use **information** from the scenario given to determine the **value** of C.

Imagine we have a **race car**, with an **acceleration** tracked by the equation $a(t) = 12 + 2t$. We also know the car **starts at $0m$**, and has a **velocity** of $120m/s$ after **3 seconds**.



To determine the **velocity function** $v(t)$ we can **integrate** the **acceleration function** $a(t)$.

$$v(t) = \int a(t)\,dt$$
$$v(t) = \int 12 + 2t\,dt$$
$$v(t) = 12t + t^2 + C$$

To determine the value of C, we can use the information provided that the car a **velocity** of $120m/s$ after **3 seconds** (i.e. $v(3) = 120$).

$$v(3) = 120$$
$$12(3) + (3)^2 + C = 120$$
$$36 + 9 + C = 120$$
$$C = 75$$
$$\therefore v(t) = 12t + t^2 + 75$$

This **same process** can be applied for the **displacement function** $s(t)$, where we **integrate** the **velocity function** and we **calculate** the **value** of C from **substituting** in $s = 0$ at $t = 0$.

$$s = \int v(t)dt$$
$$s = \int 12t + t^2 + 75dt$$
$$s = 6t^2 + \frac{t^3}{3} + 75t + C$$
$$\text{At } s(0) = 0, C = 0$$
$$\therefore s = 6t^2 + \frac{t^3}{3} + 75t$$

Finally, we can make use of the **net change equation:** $f(b) - f(a) = \int_a^b f'(x)\,dx$, when we want to know the **net change** in **displacement**, **velocity** or **acceleration** between **two times**.

For instance, if the **velocity equation** for a **train** was $v(t) = 4t^2 - 3t + 2$ in m/s, and we wanted to know the **net change** in **displacement** over the **first 10 seconds**, we can apply **net change formula**:

$$s(b) - s(a) = \int_a^b \frac{dv}{dt} dt$$
$$s(10) - s(0) = \int_0^{10} 4t^2 - 3t + 2\, dt$$
$$s(10) - s(0) = \left[\frac{4}{3}t^3 - \frac{3t^2}{2} + 2t\right]_0^{10}$$
$$s(10) - s(0) = \left(\frac{4}{3}(10)^3 - \frac{3(10)^2}{2} + 2(10)\right) - (0)$$
$$s(10) - s(0) = 1203\, m$$

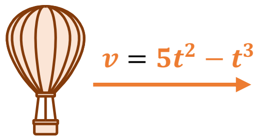
∴ **Net change** in **train distance** in first **10 seconds** is $1203\, m$

The final critical concept to touch on are the concepts of **distance** and **speed**, and **how they differ** from **change in displacement** and **velocity**.

Distance is a measure of **how much ground** an **object has covered**, whereas **change in displacement** is a measure of **how far** an object is from **another position in time**.

Velocity is the **rate of change** of **displacement** with time of an object moving along a path, whereas **speed** is the **absolute value** of velocity and **doesn't account for direction** of travel.

To explore the difference let's consider an example where we have a **hot air balloon**, that travels with a **velocity** that follows the equation: $v = 5t^2 - t^3$, where v is in m/s and t is **time** in **seconds**.



Firstly, if we were to calculate the velocity and speed at 5.5 **seconds** we would get a **velocity** of $-15.125\, m/s$ and a **speed** of $15.125\, m/s$. This is because **speed doesn't consider direction**, so it will **always** be an **absolute value** as shown below:

Velocity

$$v(5.5) = 5(5.5)^2 - (5.5)^3$$
$$\therefore v(5.5) = -15.125\, m/s$$

Speed

$$Speed = \left|5(5.5)^2 - (5.5)^3\right|$$
$$\therefore Speed = 15.125\, m/s$$

Secondly, if we were to **calculate** the **distance travelled** during the **first six seconds** it is must be the **absolute definite integral**: $\int_a^b \left|\frac{dv}{dt}\right| dt$. As we can see, this gives a **different result** to the **displacement** which isn't an absolute definite integral:

Displacement

$$s(6) - s(0) = \int_0^6 5t^2 - t^3 dt$$
$$s(6) - s(0) = 36\, m$$

Distance

$$Distance = \int_0^6 \left|5t^2 - t^3\right| dt$$
$$Distance = 68.2\, m$$

Worked Example 1

Rabea has built himself a race car that has the **acceleration function**: $a(t) = t^2 - \frac{1}{4}t^3$, where a is in m/s^2 and t is **time** in **seconds**. Before driving this race car, Rabea wants to make sure he knows the **velocity function** $v(t)$. **[CA]**

(a) If Rabea knows the race car has an **initial velocity** of $2m/s$, determine $v(t)$.

$$v(t) = \int a(t)dt$$

$$v(t) = \int t^2 - \frac{1}{4}t^3 dt$$

$$v(t) = \frac{t^3}{3} - \frac{t^4}{16} + C$$

Let $v(0) = 2$

$$2 = \frac{(0)^3}{3} - \frac{(0)^4}{16} + C$$

$$C = 2$$

$$\therefore v(t) = \frac{t^3}{3} - \frac{t^4}{16} + 2$$

(b) Help Rabea to determine the **speed** and **velocity** after **6 seconds**?

Velocity

$$v(6) = \frac{t^3}{3} - \frac{t^4}{16} + 2$$

$$v(6) = \frac{(6)^3}{3} - \frac{(6)^4}{16} + 2$$

$$\therefore v(6) = -7m/s$$

Speed

$$Speed = \left|\frac{t^3}{3} - \frac{t^4}{16} + 2\right|$$

$$Speed = \left|\frac{(6)^3}{3} - \frac{(6)^4}{16} + 2\right|$$

$$\therefore Speed = 7m/s$$

(c) Determine the **net change** in **displacement** of the **car** and the **distance travelled** by the car in the first **6 seconds**, and **briefly discuss why the results** are **different**.

Displacement

$$s(b) - s(a) = \int_a^b v(t)dt$$

$$s(6) - s(0) = \int_0^6 \frac{t^3}{3} - \frac{t^4}{16} + 2dt$$

$$s(6) - s(0) = 22.8\ m$$

Distance

$$Distance = \int_a^b |v(t)|dt$$

$$Distance = \int_0^6 \left|\frac{t^3}{3} - \frac{t^4}{16} + 2\right| dt$$

$$Distance = 25.9\ m$$

Distance is a **measure** of the **ground covered by the car**, whereas **displacement** is a **measure** of **how far** the car is **from its starting position**, so they produce **two different results**.

INTEGRATION TOPIC NOTES

Integration

Integration (or **anti-differentiation**) is the **opposite** of **differentiation**. It represents the **signed area under the curve**. The integral of **polynomials** can be found using the reverse power rule:

Reverse Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ given $n \neq -1$

When we take the derivative of a constant, it becomes **zero**. So, when you're determining an **indefinite integral**, always remember to **add C** to represent **the constant of integration**.

The other two rules we can use to integrate polynomials are as follows:

$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ given $n \neq -1$

$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + C$ given $n \neq -1$

Useful Properties for Integrals

The **three key properties** you can utilise when **determining integrals** and **definite integrals** are the **linearity of integrals property**, the **integral coefficient property** and the **reversing of bounds property** for **definite integrals**. These can be summarised by the formulas below:

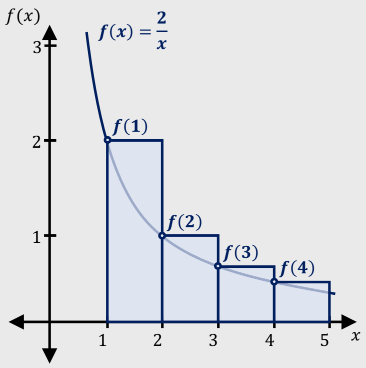
Linearity of Integrals Property: $\int f_1(x) + f_2(x) dx = \int f_1(x) dx + \int f_2(x) dx$

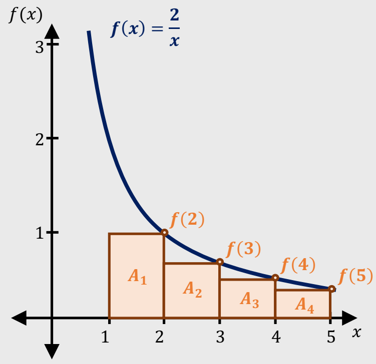
Integral Coefficient Property: $\int kf(x) dx = k \int f(x) dx$

Definite Integral Property: $\int_a^b f(x) dx = -\int_b^a f(x) dx$

Definite Integrals and Area Under the Curve

Definite integrals can be **estimated** by **separating** the area under the graph into **rectangles** and **summing** their areas together. **Left rectangles** involve drawing rectangles where the **top-left corner** of each is touching the function, and right rectangles is where the **top-right corner** is touching the function.





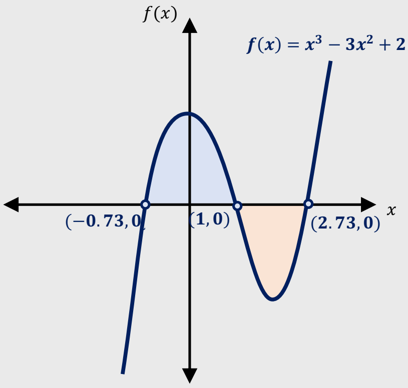
These **areas** are calculated by **multiplying** the **heights** of the rectangles $\boldsymbol{f(x)}$ by their **width** $\boldsymbol{\delta x}$, and then the **area** of the **left-rectangles** can be **added** to the **area** of the **right-rectangles** and **divided** by **two** to get an **area approximation**:

$$\boldsymbol{Area \approx \frac{Left\ Rectangle\ Area + Right\ Rectangle\ Area}{2}}$$

The **exact area under the curve** can be found using **definite integrals**.

A **definite integral**, $\int_a^b f(x)dx$, calculates the **exact singed area bounded by the function** and the $\boldsymbol{x}$**-axis** between $\boldsymbol{a}$ and $\boldsymbol{b}$, where the function, $\boldsymbol{f(x)}$, is continuous for $[a, b]$ and $b > a$.

We can use **definite integrals** to determine the **area under the curve**. If there are multiple areas that go **above** and **below** the $\boldsymbol{x}$**-axis**, we must **split** the **integral** into the **area above the $\boldsymbol{x}$-axis** and the **area below the $\boldsymbol{x}$-axis**, such as for $\boldsymbol{f(x) = x^3 - 3x^2 = 2}$ below:



$$A = \underbrace{\int_{-0.73}^{1} x^3 - 3x^2 + 2\,dx}_{\text{Area above } x\text{-axis}} - \underbrace{\int_{1}^{2.73} x^3 - 3x^2 + 2\,dx}_{\text{Area below } x\text{-axis}}$$

If we have access to a calculator, an easy way to determine the area is to **apply** an **absolute value** to the **integral**, which gives the **exact area** irrespective of **whether the area** is **above** or **below** the $\boldsymbol{x}$**-axis**.

Absolute Area Under the Curve: $A = \int_a^b |f(x)|dx$

Fundamental Theorem of Calculus

The **two key formulas** in relation to the fundamental theorem of calculus are for determining the **derivatives** of **definite integrals** that are **bounded by a function** such as $\boldsymbol{x}$ or $\boldsymbol{g(x)}$. Their formulas are:

Fundamental Theorem of Calculus Formula 2: $f(x) = \dfrac{d}{dx}\displaystyle\int_a^x f(t)dt$

Fundamental Theorem of Calculus Formula 3: $\dfrac{d}{dx}\displaystyle\int_a^{g(x)} f(t)dt = f[g(x)] \cdot g'(x)$

Area Between Curves

The **area between two curves**, $f(x)$ and $g(x)$, where $f(x)$ is the **upper curve**, the **area** can be calculated as the **area** of the **upper curve** ($f(x)$) **minus** the **area** of the **lower curve** ($g(x)$),

$$\text{Area Between Curves: } A = \int_a^b [f(x) - g(x)]dx \quad \text{where } b > a$$

If there are **multiple areas** and the **'upper curve' changes**, then you will need to **split up** the **integral**. However, if you have **access** to a **calculator**, you can apply an **absolute value** to the **integral** and then we **don't need to worry** about **splitting up the integral**.

$$\text{Absolute Area between curves: } A = \int_a^b |f(x) - g(x)|dx \quad \text{where } b > a$$

Net Change and Rectilinear Motion

Net change is the difference between **two points** of a function. It uses the **definite integral** to find this **net change** between the **two points**, a and b.

$$\text{Net Change: } f(b) - f(a) = \int_a^b f'(x)\, dx$$

For **rectilinear motion**, To get the **velocity function** we **integrate** the **acceleration function** (i.e. $v(t) = \int a(t)dt$), and to get the **displacement function** we **integrate** the **velocity function** (i.e. $s(t) = \int v(t)dt$).

Distance is a measure of **how much ground** an **object has covered**, whereas **displacement** is a measure of **how far** an object is **from its origin**.

Speed is the **time rate** at which an object is **moving along a path**, whereas **velocity** is the **rate** and **direction** of an object moving along a path.

When **calculating** the **distance**, we use the **net change** in **displacement formula** but **apply** an **absolute value** to the **definite integral**:

$$\text{Displacement: } s(b) - s(a) = \int_a^b \frac{dv}{dt}dt \qquad \text{Distance: } Distance = \int_a^b \left|\frac{dv}{dt}\right| dt$$

When **calculating** the **speed**, we apply an **absolute value** to the **velocity** determined:

$$\text{Speed: } Speed = |v|$$

Problem Set 4 – Applications of Integration
Progressive Questions

Concept 1

Area Between Curves – Progressive Questions

(6 questions)

Repetitive questions: 1.11 → 1.41 (4 questions)

The Space Station Adventure!

Starring: Janitor Peter, Rabea, Tyler, Tom, Rupert, Wallace and Teacher Andrew

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Entering the space station, Rabea, Tyler, Tom, Rupert and Wallace are blown away by how cool it is. "Come, have a look at our rocket!" says Janitor Peter.

As a mathematics fanatic, Janitor Peter has rigged the launch speed of the rocket to a computer simulated game on the area between curves. That is, the more area between curves the students can determine, the faster the rocket will go.

The students look at one another, excited by the prospect of attempting to make the rocket travel as fast as possible!

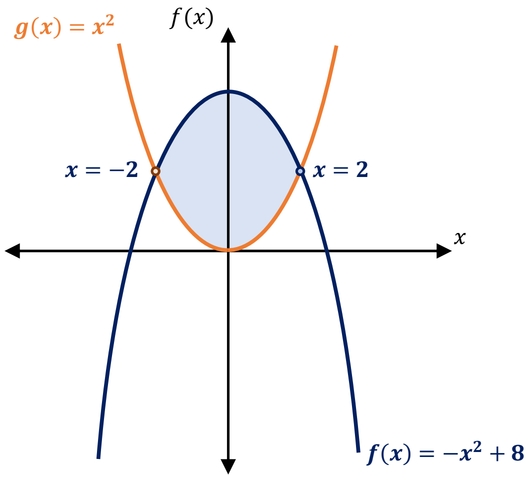
Area Between Curves: Q1, Q2, Q3, Q4, Q5, Q6

Repetitive: 1.11 → 1.41 (4 questions)

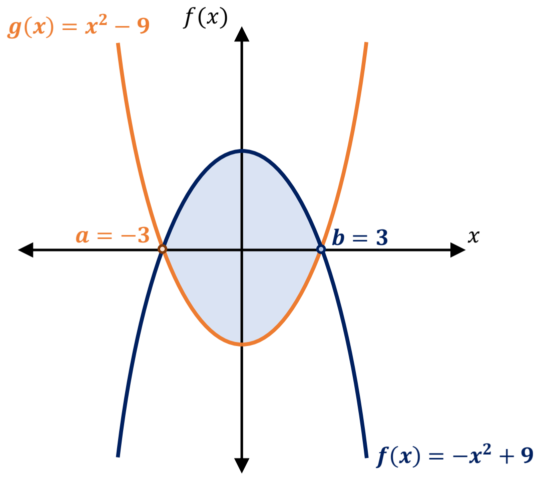
[CA] [24 marks]

1. Eager to make the rocket travel as fast as humanly possible, the five students start working together to determine the **area between the curves** below. Using **definite integrals**, **help** the students to **determine** the **shaded areas below**.

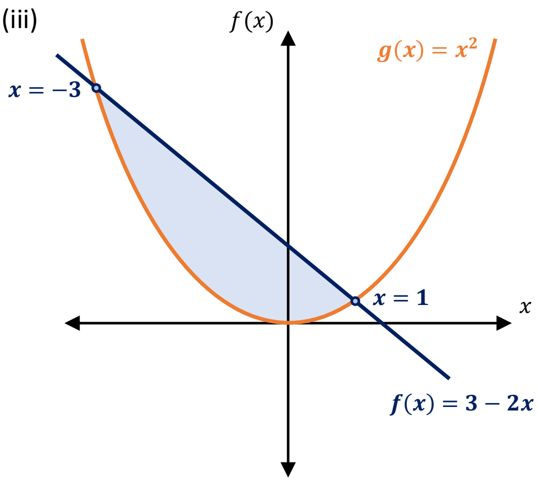
(i)



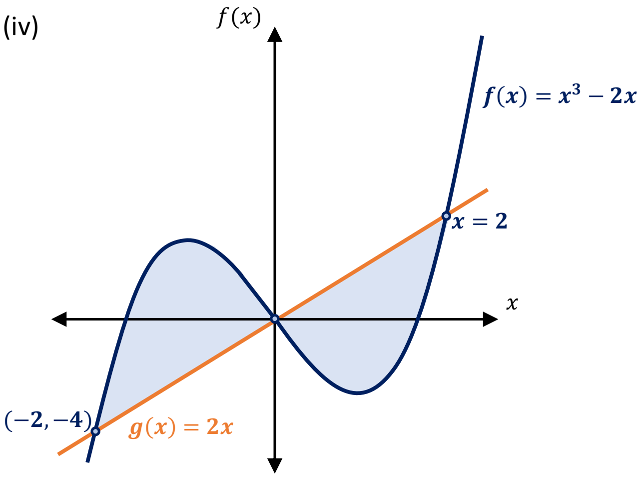
(ii)



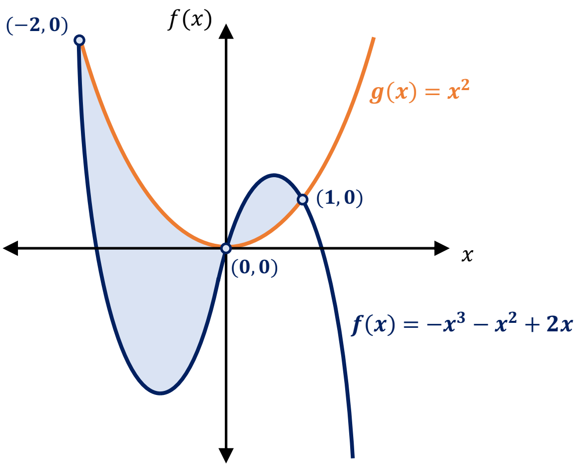
(iii)



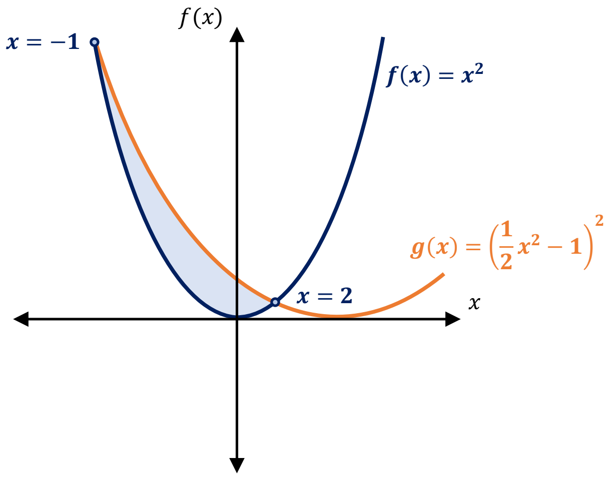
(iv)



(v)



(vi)



[CA] [6 marks]

2. Janitor Peter is amazed that the students have nailed all of the first set of questions. He has never managed to get the rocket to its maximum speed, so is eager to help the students! The next area is comprised of **two functions**: $f(x) = 2x + 5$ and $g(x) = 3x^2$. **Help** Janitor Peter to determine the following:

(a) Find the **points of intersection** between the functions $f(x) = 2x + 5$ and $g(x) = 3x^2$. **[2]**
(b) Write an **definite integral expression** describing the **area trapped between the functions** **[2]**
(c) **Calculate** the **area trapped between these functions**. **[2]**

[CA] [5 marks]

3. Further pushing the rocket velocity, Rabea is eager to give the next area a go. Given that the functions $f(x) = 3x^2 + 5$ and $g(x) = -6x^2 + 14$ make up the entrance, help Rabea **find the x-values** at which the **functions intersect** and therefore the **area trapped between** the **two functions**.

[CA] [6 marks]

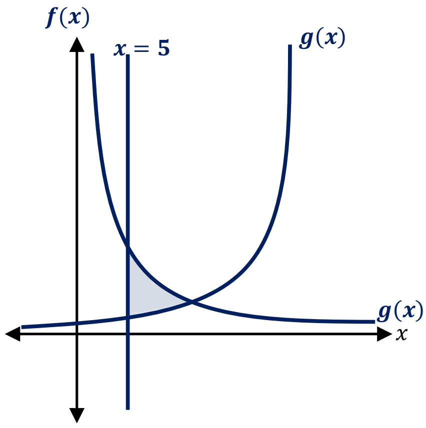
4. The rocket is now set to be at $\mathbf{80\%}$ of its maximum launch velocity and the next area on the screen is bound between the functions: $\boldsymbol{f(x) = 2(x-2)^3 + 6}$ and $\boldsymbol{g(x) = -\left(5x^2 - 2\right) + 15}$. **Help** them answer the following questions!

(a) Determine the **$\boldsymbol{x}$-coordinates** at which the **two functions intersect** **[2]**

(b) Find the **area** trapped **between the functions**, and the **$\boldsymbol{x}$-axis**. **[4]**

[CA] [6 marks]

5. With the velocity set at 90% of its capacity, the students are on the edge of the seats! For their final area it is bounded by the equations, $\boldsymbol{f(x) = \frac{-1}{0.05x-1}}$, $\boldsymbol{g(x) = \frac{1}{0.05x}}$ and $\boldsymbol{x = 5}$. Find the **area bounded by** the **expressions**.



[CA] [7 marks]

6. With the maximum launch speed set, Janitor Peter says "Before I can let you go, students, you will need to design and **create the glass screen** for your **astronaut helmets** so you can breathe in space!" Janitor Peter handed the students his design for them to calculate **how much glass** is **required for each helmet**. The helmet glass design is constructed from the **area** between the functions: $\boldsymbol{f(x) = \frac{x^2}{3} + 3}$ and $\boldsymbol{g(x) = \frac{-x^2}{3} + 3.5}$.

(a) Calculate the **area bound between** the **two expressions**. **[5]**

(b) If all **five students** need their own astronaut helmet and ensuring a $\mathbf{10\%}$ **leniency**, **how much** glass is **required**? **[2]**

With newly constructed space helmets and the rocket set to its maximum speed, Janitor Peter and the five students go to space! However, they are unaware of Teacher Andrew, who has also managed to sneak onto the ship…

Concept 2

Net Change and Rectilinear Motion – Progressive Questions

(10 questions)

Repetitive questions: 2.11→ 2.91 (7 questions)

Lift Off!

Starring: Janitor Peter, Rabea, Tyler, Tom, Rupert, Wallace and Teacher Andrew

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Whilst enjoying their flight orbiting the earth, they notice a large bin containing rotten food scraps. They see this as an opportunity to try and drop the food scraps on the Apps students from space!

Net Change: Q1, Q2, Q3, Q4, Q5

Repetitive: 2.11 → 2.31 (3 questions)

[CA] [5 marks]

1. Determined to make full use of the food scraps on the ship, Wallace suggests that they drop it onto the maths applications building. Excited, Rupert calculates the **instantaneous rate of change** of the **volume of rubbish remaining** to be given by $\frac{dV}{dt} = \frac{1}{(2+t)^2}$ where V **is the mass of rubbish released** in **tons**, and t **is the time** in hours.

(a) What is the **instantaneous rate of change** of **rubbish** held in the spaceship **initially**? **[1]**
(b) How many **kilograms** of **rubbish** will be thrown out during the first **3 hours**. **[2]**
(c) How much rubbish will be **ejected during** the **first hour**. **[2]**

[CA] [8 marks]

2. While the students are busy calculating the remaining mass of rubbish, they don't notice Teacher Andrew, who has somehow snuck on board! Teacher Andrew starts to pour a foreign substance onto the food scraps, trying to dissolve it before it reaches the Earth! The new **instantaneous rate of change** of the **mass** of rubbish **released** is now given by $\frac{dV}{dt} = -t^2 + 5t + 6$ where V **is the mass of rubbish released** in **kilograms** and t is **time** in **hours**.

(a) What is the **net change** over the first **half an hour** in the **mass** of the rubbish? **[2]**
(b) What is the **net change** in the **mass** of the rubbish during the **second hour**? **[2]**
(c) At what **time** did the **release of the rubbish** reach a **maximum rate**? **[4]**

[CA] [6 marks]

3. Teacher Andrew is very angry at the students for going to space and his goal is to bring the rocket back down to Earth so he can immediately give them detention. He busies himself in the engine, trying to change its direction, but suddenly **drops his screwdriver**, and it flies off due to the lack of gravity. Teacher Andrew is furious and starts to think about how expensive it was to produce that screwdriver. He assumes that the **marginal cost** for producing n **screwdrivers** is given by $\frac{dC}{dn} = 0.05n + 3$, where C is the **cost of producing** n **screwdrivers** in **dollars**.

(a) If he knows that the **initial fixed cost is** $\$30$, find the **cost** of producing **70 screwdrivers**. **[2]**
(b) Find the **net change in cost** if n is changed from 70 to 90 screwdrivers. **[2]**
(c) Find the **net change in cost** if n is changed from 90 to 1000 screwdrivers. **[2]**

[CA] [6 marks]

4. Teacher Andrew is jealous of Janitor Peter's ability to build a spaceship and starts to think about building spaceship engines for **profit**. He knows that the **marginal profit of a sale** of $\boldsymbol{n}$ **rocket engines** is given by $\frac{dP}{dn} = n^2 + 0.02n - 2.8$ where $\boldsymbol{P}$ is the **profit** of $\boldsymbol{n}$ **engines** in **dollars**.

(a) Given that if no engines are sold, there is a **loss of $\$60$**, find the **profit when n = 200**. **[2]**
(b) Find **net change** in **profit** if **number of engines sold** is changed from **200 to 400**. **[2]**
(c) Find **net change** in **profit** if **number of engines sold** is changed from **200 to 100**. **[2]**

[CA] [7 marks]

5. Whilst Teacher Andrew is planning his new spaceship business, he forgets that he was a fugitive on board. Janitor Peter finds him and alerts the rest of the students! As the students mull over the possible punishments for Teacher Andrew, the **rate of his heartbeat** begins to **change very rapidly**! The **rate of change** of Teacher Andrew's heartbeat follows the equation, $\frac{dP}{dt} = \frac{1000}{(2+t)^2}$, where $\boldsymbol{P}$ is in **beats per minute** and $\boldsymbol{t}$ is in **minutes**.

(a) Find **instantaneous rate of change** when $t = 4$ **minutes**. **[2]**
(b) Find **net change** of his heartbeat during the **second minute**. **[2]**
(c) Find **average rate of change** in interval $0 < t < 8$ minutes. **[3]**

Seeing Teacher Andrew's distress, the students decide to let this slide, seeing as he only had good intentions for their well-being. They convince him to help them use rectilinear motion to drop biodegradable objects onto the Maths Applications building as they fly by!

Rectilinear Motion: Q6, Q7, Q8, Q9, Q10

Repetitive: 2.61 → 2.91 (4 questions)

[CA] [11 marks]

6. "Students, we must ensure we are not littering, so we need to calculate the rectilinear motion of the rubbish to ensure it lands at the applications building exactly!" says Janitor Peter. The first step for them to be able to **drop rubbish** on the apps building is to fully understand the **motion** of the **rocket**. The rocket moves in a **straight line**, with its **velocity** being tracked by the equation $v = -3t^2 + 24t - 36$ where v is in m/s.

(a) What is the **initial velocity** of the rocket? **[2]**
(b) Find the equation **acceleration** of the rocket. **[2]**
(c) If it is known that the **initial displacement** of the rocket is $-36m$, what is the **displacement equation** for the rocket? **[2]**
(d) Find the **distance travelled** over the **first 4 seconds**. **[2]**
(e) Find the **time** the rocket achieves a **maximum speed**. **[3]**

[CA] [9 marks]

7. After they finally understand the patterns of the rocket, Rabea insists that he gets the honours of throwing the first piece of food scraps onto the Apps students building. He throws the **rotten banana upwards** in a **straight line**. Its **velocity** is given by $v = 23.7 - 9.8t$, where v is in m/s.

(a) Find the **acceleration equation** of the banana. **[2]**
(b) If the displacement after **3 seconds** is **$50m$**, what is the **displacement equation**. **[2]**
(c) What is the **displacement** of the banana after the **first second**? **[2]**
(d) Find the **average velocity** of the banana between $t = 0$ and $t = 5$. **[3]**

[CA] [9 marks]

8. Rabea is so excited by the throwing of the first piece of rubbish, that he grabs the second piece and throws it with all his might. The **displacement** of the rubbish moving in a straight line with an **acceleration** given by $a = 2t + 5$, where a is in m/s^2.

(a) If the **velocity** after 2 **seconds** is $9m/s$, determine the **velocity equation** of the rubbish. **[2]**
(b) What is the **velocity** of the rubbish after 3 **seconds**? **[2]**
(c) If the **initial displacement** of the banana is $4m$, determine the **displacement equation**. **[2]**
(d) Find **net change in displacement** over the **first** 3 **seconds**. **[3]**

[CA] [9 marks]

9. The students finally convince Teacher Andrew to get involved, and hand him the largest piece of rubbish. Reluctantly, he throws and it travels in a **straight line**, with its **velocity** given by $v = t^2 - 3t + 4$, with t being in **seconds**.

(a) What is the **velocity** of the rubbish after 2 **seconds**? **[2]**
(b) If the **initial displacement** is $5m$, find the **displacement** of P after 3 **seconds**. **[2]**
(c) Find the **acceleration** of the rubbish at $t = 3$ **seconds**. **[2]**
(d) Find **net change** in **displacement during the fourth second**. **[3]**

[CA] [13 marks]

10. Teacher Andrew then realises that since they have discarded so much old food, the motion of the rocket had changed as the **mass has decreased**. He tries to work out the new patterns of motion of the rocket. The rocket starts from the **initial point** O, with an **initial velocity** of $35m/s$, and its **acceleration** is given by $a = 4t - 3.$

(a) Determine the **velocity** of the rocket after 3 **seconds**. **[2]**
(b) Find the **displacement** of the rocket after 3 **seconds**. **[2]**
(c) Determine the **change** in **displacement** over the **first** 3 **seconds**. **[3]**
(d) The **total distance travelled** in the **first** 3 **seconds**. **[3]**
(e) Find the **maximum velocity** for when t **is between** 0 **and** 3. **[3]**

Teacher Andrew, Janitor Peter and the students are successfully able to use rectilinear motion and net change to track their flight patterns to make sure their descent back to earth is nice and smooth.

Their trip was truly a bonding experience between teacher and student, and would be a journey that they would remember forever.

However, when the students arrive back to campus, Sparrow and the maths applications are standing outside their building furious about all of the rubbish that has been dumped on it!

"We will get you back for this!" yells Sparrow.

The methods students break out into laughter as they walk off into the methods building and into their next class on exponential functions.

Problem Set 4 –Applications of Integration
Repetitive Questions

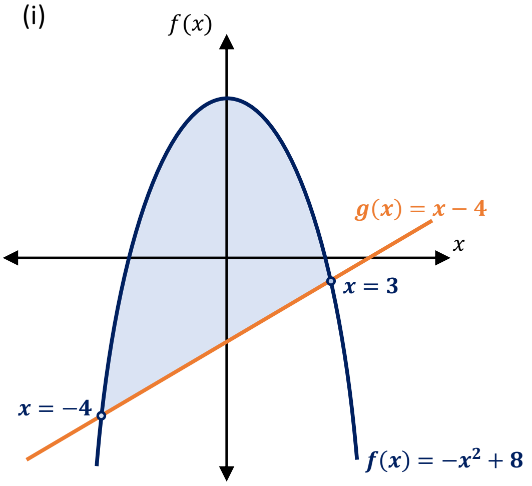
Concept 1
Area Under and Between Curves – Repetitive Questions
(4 questions)

Area Under and Between Curves: Qs 1.11, 1.21, 1.31, 1.41

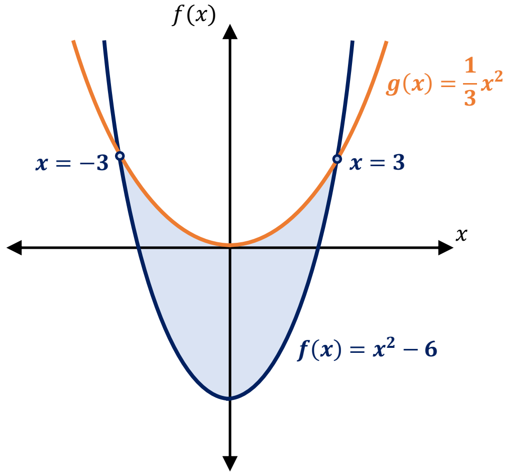
[CA] [24 marks]

1.11 Using the appropriate **definite integrals**, **determine** the **shaded areas trapped between** the **curves** $f(x)$ and $g(x)$ below.

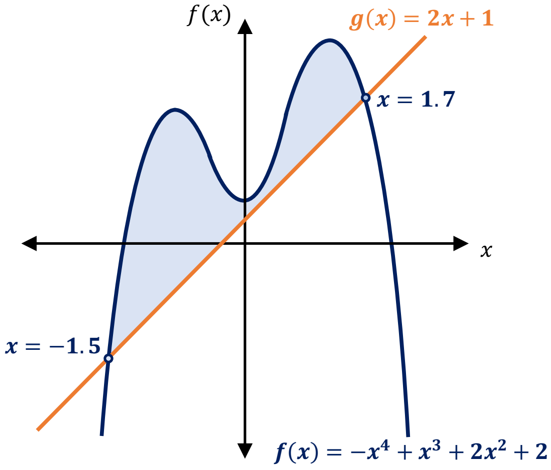
(i)



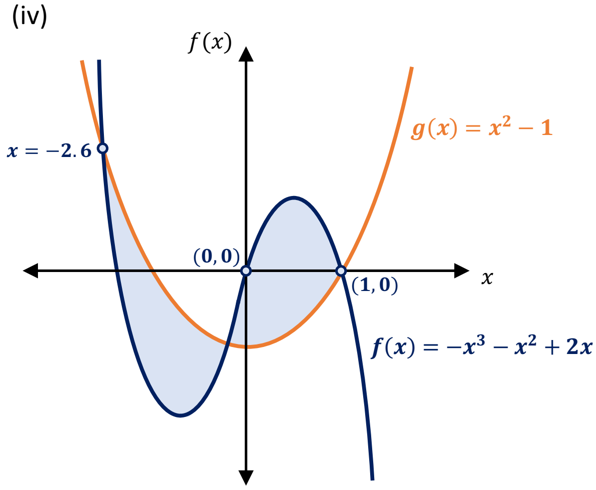
(ii)



(iii)

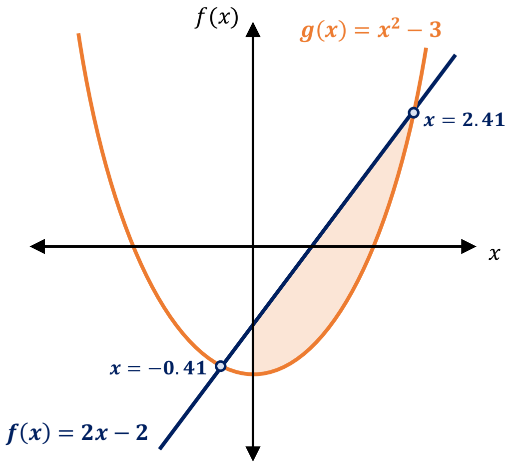


(iv)



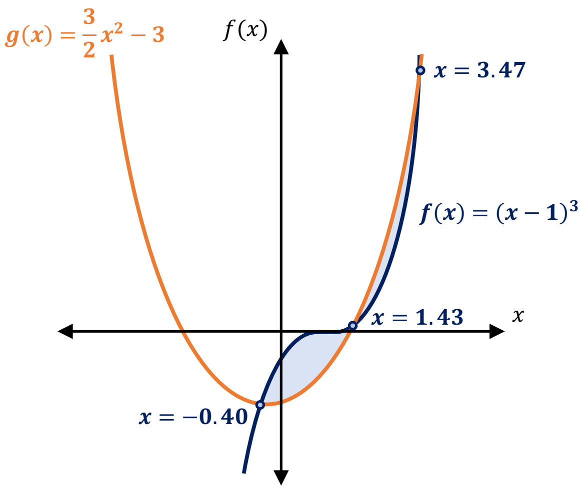
[CA] [4 marks]

1.21 The graph for is shown below. Determine the **area between trapped between $f(x) = 2x - 2$**, $g(x) = x^2 - 3$ and the **y-axis** using **definite integrals**.



[CA] [5 marks]

1.31 The graph for is shown below. Determine the **area between trapped between $f(x) = (x-1)^3$** and $g(x) = \frac{3}{2}x^2 - 3$ using **definite integrals**.



[CA] [8 marks]

1.41 The following graph shows the equations $f(x) = 3x^3 - 2x + 1$ and $g(x) = -6x^2 + 5$.

(a) Find the **area trapped** between the **two equations**. **[4]**

(b) A third line, $y = -x + 2$ is added to the graph. Find the **area bounded** by the **three equations**. **[4]**

Concept 2

Net Change and Rectilinear Motion – Repetitive Questions

(7 questions)

Net Change: Qs 2.11, 2.21, 2.31

[CA] [7 marks]

2.11 The **number of insects** contained within a box in a laboratory follows the function: $f'(t) = 2t^3 - 9t$, where $f(t)$ represents the **number of insects** in the laboratory and t is the **number of hours**:

(a) If the initial population was 30, determine the **number of insects** in the laboratory after 10 **hours**. **[1]**
(b) Find the **instantaneous rate of change** at $t = 3$ and $t = 6$. **[2]**
(c) What is the **net change** in the **number of insects** during the **third hour**? **[2]**
(d) What is the **net change** in the **number of insects** during the **first ten hours**? **[2]**

[CA] [7 marks]

2.21 The engine on a car **consumes fuel** at a **rate** of $\frac{dV}{dt} = 2x$, where V is **volume** in **litres** and t is **time** in **hours**.

(a) **How much fuel** does it **consume** in the first 5 **hours**? **[2]**
(b) What is the **instantaneous rate of change** in the **amount of fuel** after 2 **hours**? **[2]**
(c) If there is 100 **litres of fuel** in the car, after **how many hours** will **all** of the fuel have been consumed? **[3]**

[CA] [8 marks]

2.31 The **fox population** in a forest is **increasing** at a **rate** of $\frac{dA}{dt} = 500t + \frac{1}{3}t^3$ where A is in **per one hundred of foxes** and t is in **years**.

(a) What is the **instantaneous rate of change** in the fox population during the **third year**? **[1]**
(b) How much does the **fox population increase** during the **third year**? **[2]**
(c) Determine how much the fox population will **increase** across the **first** 10 **years**. **[2]**
(d) If the **initial fox population** is 300, determine when the **population** will reach $10,000$. **[3]**

Rectilinear Motion: Qs 2.61, 2.71, 2.81, 2.91

[CA] [8 marks]

2.61 A physics class is measuring the **velocity** of an apple they drop from the top floor of a school. The **apple's velocity** changes according to the **equation:** $v(t) = 3t^2$, where v is in **metres per second** and t is **time** in **seconds**.

(a) What was the **velocity** of the apple **after three seconds**? **[1]**
(b) Assuming the apple has **no initial displacement**, **determine** an **equation** for the **displacement** of the apple. **[3]**
(c) What is the **displacement** of the apple **after four seconds**? **[1]**
(d) What is an **equation** that tracks the **acceleration** of the apple? **[2]**
(e) What is the **acceleration** of the apple **after two seconds**? **[1]**

[CA] [12 marks]

2.71 A particle moves along a **straight line** and its **acceleration** is given by the equation: $s(t) = 6t - 21$, where a is **acceleration** in **meters per second squared** and t is **time** in **seconds**. The **initial displacement** of the particle is 20 **meters** and the **initial velocity** of the particle is $5m/s$.

(a) What is the **initial acceleration** of the particle? **[1]**
(b) At what **time(s)** is the particle at **rest**? **[4]**
(c) What is the **velocity** of the particle **after five seconds**? **[2]**
(d) What is the **net change** in the **displacement** of the particle during the **fourth second**? **[3]**
(e) What is the **displacement** of the particle after **four seconds**? **[2]**

[CA] [13 marks]

2.81 A particle moves along the x**-axis** with the **velocity** given by $v(t) = 3t^2 + 6t - 5$, where v is in **metres per second** and t is **time** in **seconds**.

(a) If the particle is at the **position** $x = 2$ at time $t = 0$, what is the **position** the particle would be at time $t = 1$? **[4]**
(b) What is the **acceleration** of the particle **after eight seconds**? **[2]**
(c) At what **time(s)** is this particle at **rest**? **[3]**
(d) What is the **distance travelled** by the particle across the **first three seconds**? **[2]**
(e) What is the **displacement** of the particle across the **first three seconds**? **[2]**

[CA] [12 marks]

2.91 A firework is thrown and travels in a **straight line**, with its **acceleration** given by $a(t) = 2t^{\frac{1}{2}}$, where a is **acceleration** in **meters per second squared** and t is **time** in **seconds**.

(a) If the **initial displacement** of the firework is **3 meters off the ground** and the **initial velocity** is $0m/s$, determine the **displacement** of the firework after **two seconds**. **[4]**
(b) Find the **instantaneous acceleration** of the firework at **two seconds**. **[2]**
(c) Determine the **average velocity** of the particle between $t = 2$ and $t = 4$. **[3]**
(d) During the **first ten seconds**, determine the **time** at which **greatest rate of change** in **displacement occurs**? **[3]**

Chapter 3

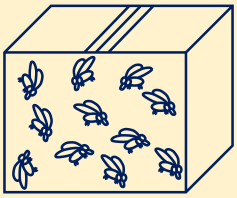
Exponential Functions

Chapter 3 – Exponential Functions

Introduction

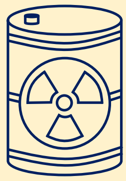
The next type of **function** we are going to explore through the lens of calculus is **exponential functions**.

Exponential functions are really useful for modelling **growth** and **decay**. They are a number raised to a variable exponent (i.e. $f(x) = a^x$) and have a range of **applications**, such as tracking the **growth** of a **fruit fly population**, the **growing interest** owed on a **bank loan**, or the **decay** of **radioactive waste**.



Growth of a Fruit Fly Population

Growth of Interest Owed on a Bank Loan



Decay of Radioactive waste

Just like polynomials, **differentiation** and **integration** can be applied to **exponential functions**. To understand **exponential functions**, we are going to explore **two key topics**:

1. **Differentiation and Integration of Exponential Functions**
2. **Applications of Exponential Functions**

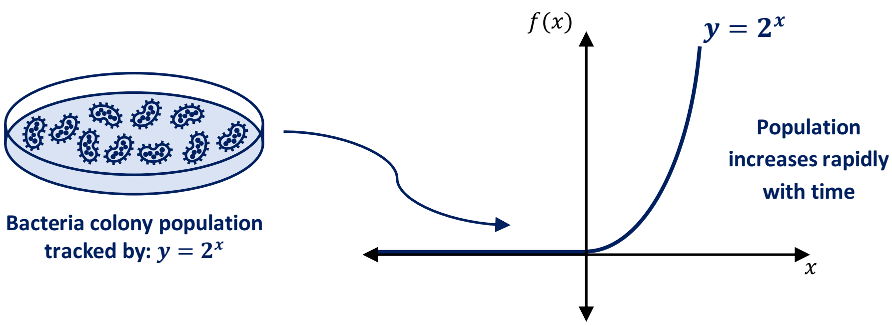
Let's begin!

3.1 Differentiation and Integration of Exponential Functions

Exponential functions are functions with a **number (a) raised** to an **exponent** (x), which is the variable we are **changing**. Therefore, they have the generic function: $f(x) = a^x$.

Exponential functions are fantastic for tracking **compounding growth** and **decay** patterns because the **rate** at which exponentials **increase** or **decrease compounds over time**.

For instance, a **bacteria colony** that **doubles** in **population every minute** can be tracked by the **exponential:** $y = 2^x$, where y is the **bacteria population** and x is **time** in **minutes**. This produces an exponential graph which shows that whilst the **rate of growth starts off slow**, it **compounds** and **increases rapidly as more time passes**.

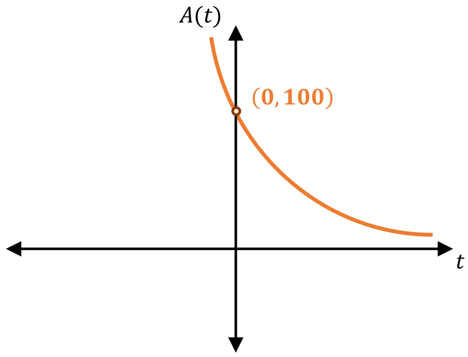


In Year 12, there is a **particular exponential function** we are interested in e^x. This exponential has **Euler's number** (e) as its **base value**.

Euler's number (e) is an **irrational number** (non-repeating, and non-terminating), and is **approximately equal** to $\mathbf{2.71828}$. It is a '**perfect ratio**' for exponentials and is used to track the patterns of many natural phenomena.

The **exponential function** e^x has a similar shape to other exponentials, and it can be used to track many phenomena such as **population growth** or the **decay of a radioactive element**. For instance, $A = 100e^{-t}$ can be used to track the **mass of uranium remaining** as it **decays over t years**.

Decay of Uranium: $A(t) = 100e^{-t}$



Before we understand how to **apply e^x** to **real world applications**, we first need to learn how to **differentiate** and **integrate e^x**.

The **differentiation** of e^x is simple, it states that the **derivative** of e^x is e^x:

If $y = e^x$ then $\frac{dy}{dx} = e^x$

To understand how this rule is brought about, we need to return to **first principles**.

Let's consider, a **general exponential function** of the form: $y = a^x$. Applying the **general instantaneous rate of change formula**: $\frac{d}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, and **substituting a^x** gives the following:

① $\frac{d}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ — ① **State** the **general derivative equation**

② $\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$ — ② **Substitute a^x** into the **equation**

③ $\frac{d}{dx}(a^x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$ — ③ **Simplify** to a **final limit**

Through trial and error, it has been determined that e is a **very unique number** because when you **substitute e** into the limit $\lim_{h \to 0} \frac{a^h - 1}{h}$ you get 1.

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

When you **substitute in values** of $\boldsymbol{a}$ that are close to the **value** of $\boldsymbol{e}$ (e.g. $\boldsymbol{a = 2, a = 2.5}$ or $\boldsymbol{a = 3}$), you don't **quite get a limit** that **equals $\boldsymbol{1}$**:

$\boldsymbol{h}$	$\boldsymbol{a = 2}$ in $\frac{a^h-1}{h}$	$\boldsymbol{a = 2.5}$ in $\frac{a^h-1}{h}$	$\boldsymbol{a = 3}$ in $\frac{a^h-1}{h}$
$\boldsymbol{1}$	$\boldsymbol{1.000000}$	$\boldsymbol{1.500000}$	$\boldsymbol{2.000000}$
$\boldsymbol{0.1}$	$\boldsymbol{0.717735}$	$\boldsymbol{0.959582}$	$\boldsymbol{1.161232}$
$\boldsymbol{0.01}$	$\boldsymbol{0.695555}$	$\boldsymbol{0.920502}$	$\boldsymbol{1.104669}$
$\boldsymbol{0.001}$	$\boldsymbol{0.693387}$	$\boldsymbol{0.916711}$	$\boldsymbol{1.099216}$
$\boldsymbol{0.0001}$	$\boldsymbol{0.693171}$	$\boldsymbol{0.916333}$	$\boldsymbol{1.098673}$
$\boldsymbol{0.00001}$	$\boldsymbol{0.693150}$	$\boldsymbol{0.916295}$	$\boldsymbol{1.098618}$
$\boldsymbol{0.000001}$	$\boldsymbol{0.6931474}$	$\boldsymbol{0.9162912}$	$\boldsymbol{1.098613}$
$\boldsymbol{0.0000001}$	$\boldsymbol{0.6931472}$	$\boldsymbol{0.9162908}$	$\boldsymbol{1.098612}$

However, when you **substitute** in $\boldsymbol{a = e}$, you can see below that as the **value of $\boldsymbol{h}$ approaches zero**, the **value of the limit approaches one**.

Value of $\boldsymbol{h}$	**Result:**
For $\boldsymbol{h = 1}$	$\frac{e^1-1}{1} = \boldsymbol{1.71828}$
For $\boldsymbol{h = 0.01}$	$\frac{e^{0.01}-1}{0.01} = \boldsymbol{1.005016}$
For $\boldsymbol{h = 0.0001}$	$\frac{e^{0.0001}-1}{0.0001} = \boldsymbol{1.00005}$
For $\boldsymbol{h = 0.000001}$	$\frac{e^{0.000001}-1}{0.000001} \approx \boldsymbol{1.0000005}$

Therefore, we can **substitute** in $\lim\limits_{h\to 0}\frac{e^h-1}{h} = \boldsymbol{1}$ to **conclude** that: $\frac{d}{dx}(e^x) = e^x$:

$$\frac{d}{dx}(e^x) = e^x \lim_{h\to 0}\frac{e^h - 1}{h}$$

$$\frac{d}{dx}(e^x) = e^x(1)$$

$$\therefore \frac{d}{dx}(e^x) = e^x$$

Applying $\frac{d}{dx}(e^x) = e^x$ to some examples, this means that the **exponential $\boldsymbol{e^x}$** and its **derivative** will be the **same**. This can be shown through **differentiating $\boldsymbol{y = e^x}$ and $\boldsymbol{y = 3e^x}$** as follows:

$$y = e^x \qquad\qquad y = 3e^x$$

$$\frac{dy}{dx} = e^x \qquad\qquad \frac{dy}{dx} = 3e^x$$

The **second rule** for the **differentiation of exponentials** is **derived** from the **chain rule**. It states that if the **exponent** is a **function** (i.e. $e^{f(x)}$), then the **derivative** is $e^{f(x)}$ **multiplied** by the **derivative of the exponent** $f'(x)$.

Derived from Chain Rule: If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

Applying this to the examples $y = e^{5x+3}$ and $y = e^{x^2-2x+1}$, we can see that this follows a **similar pattern** to rules we derived from the **chain rule** for **polynomials**.

$$y = e^{5x+3}$$

① $\frac{dy}{dx} = f'(x)e^{f(x)}$

② $\frac{dy}{dx} = 5e^{5x+3}$

$$y = e^{x^2-2x+1}$$

① $\frac{dy}{dx} = f'(x)e^{f(x)}$

② $\frac{dy}{dx} = (2x-2)e^{x^2-2x+1}$

① **Apply** the shortcut **rule derived** from **chain rule**

② **State** the **final answer**

We can also apply the **product rule** and **quotient rule** to exponential functions. Let's look at the two functions $y = (3x-1)e^x$ and $y = \frac{e^{2x-2}}{x-6}$ to demonstrate this.

Product Rule

$u = (3x-1)$ $\quad v = e^x$

$\frac{du}{dx} = 3$ $\quad \frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = (e^x)(3) + (3x-1)(e^x)$$

$$\frac{dy}{dx} = 3e^x + 3xe^x - e^x$$

$$\frac{dy}{dx} = 2e^x + 3xe^x$$

Quotient Rule

$u = e^{2x-2}$ $\quad v = x-6$

$\frac{du}{dx} = 2e^{2x-2}$ $\quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-6)(2e^{2x-2}) - (e^{2x-2})(1)}{(x-6)^2}$$

$$\frac{dy}{dx} = \frac{2xe^{2x-2} - 13e^{2x-2}}{(x-6)^2}$$

Integration of Exponential Functions

The **integration exponentials**, we simply do the **reverse** of the **differentiation formulas**.

Firstly, the **integral** of e^x is simply $e^x + C$.

$$\boxed{\int e^x\,dx = e^x + C}$$

Applying this to the examples such as $f'(x) = 5e^x$ and $f'(x) = 8e^x$, the **integral** will be the **same** as the **derivative**, just with the **added C constant**.

$$f'(x) = 5e^x$$

① $f(x) = \int 5e^x\,dx$

② $f(x) = 5e^x + C$

$$f'(x) = 8e^x$$

① $f(x) = \int 8e^x\,dx$

② $f(x) = 8e^x + C$

① **Apply** $\int e^x\,dx = e^x + C$

② **State** the **final answer**

The **second integration rule** is for exponentials of the form: $\int e^{ax+b}dx$. In this case, we use the **rule**:

$$\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + C$$

This rule shows that to get the **integral**, all we need to do is **divide** the exponential function by the **co-efficient 'a'**.

For instance, if we had $f'(x) = e^{2x+1}$ and $f'(x) = e^{-4x-2}$, we could integrate them as follows:

$$f'(x) = e^{2x+1}$$

① $f(x) = \int e^{2x+1}\,dx$

② $f(x) = \dfrac{e^{2x+1}}{2} + C$

$$f'(x) = e^{-4x-2}$$

① $f(x) = \int e^{-4x-2}\,dx$

② $f(x) = -\dfrac{e^{-4x-2}}{4} + C$

① **Apply** $\int e^{ax+b}dx = \dfrac{e^{ax+b}}{a} + C$

② **State** the **final answer**

The **final integration formula** is for **exponentials** that have a **function as their exponent** (i.e. $e^{f(x)}$) and is of the form: $f'(x)e^{f(x)}$. In these cases, the **integral** will simply be the function $e^{f(x)}$ (since we divide by $f'(x)$ using the reverse chain rule).

$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

Applying this to some examples such as $f'(x) = 2xe^{x^2}$ and $\frac{dy}{dx} = 6xe^{3x^2}$, we can see that the **integration** of **exponentials** is essentially just **reversing** the process we had for differentiation.

$$f'(x) = 2xe^{x^2}$$

① $f(x) = \int 2xe^{x^2}\,dx$

② $f(x) = e^{x^2} + C$

$$\frac{dy}{dx} = 12xe^{3x^2}$$

① $y = 2\int 6xe^{3x^2}\,dx$

② $y = 2e^{3x^2} + C$

① **Apply** the **integration rule:** $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$

② **State** the **final answer**

Now that we understand how to **differentiate** and **integrate exponentials**, we can move onto understanding how to **apply exponential functions** to **real world scenarios**.

Worked Example 1

Maths applications student, Sparrow, is looking to understand **exponential functions** so he can use them against the methods students.

(a) **Help** Sparrow to **determine** the **derivatives** of the following exponential functions.

(i) $y = e^{3x}$

$$\frac{dy}{dx} = 3e^{3x}$$

(ii) $y = 2e^{2x}$

$$\frac{dy}{dx} = 2(2)e^{2x}$$

$$\frac{dy}{dx} = 4e^{2x}$$

(iii) $y = e^{5x^2+3}$

$$\frac{dy}{dx} = 10xe^{5x^2+3}$$

(b) Show Sparrow how to use to **product rule**, **quotient rule** and/or **chain rule** to **differentiate** the following exponentials.

(i) $y = 2xe^{2x}$

$$u = 2x \qquad v = e^{2x}$$

$$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{dy}{dx} = e^{2x}(2) + 2x(2e^{2x})$$

$$\frac{dy}{dx} = 2e^{2x} + 4xe^{2x}$$

(ii) $y = \frac{e^x}{x^2}$

$$u = e^x \qquad v = x^2$$

$$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x^2)(e^x) - (e^x)(2x)}{(x^2)^2}$$

$$\frac{dy}{dx} = \frac{x^2e^x - 2xe^x}{x^4}$$

(c) Now **help** Sparrow to use integration to **determine** $f(x)$ for the following functions.

(i) $f'(x) = 3e^x$

$$f(x) = \int 3e^x\,dx$$

$$f(x) = 3e^x + C$$

(ii) $f'(x) = e^{7x}$

$$f(x) = \int e^{7x}\,dx$$

$$f(x) = \frac{e^{7x}}{7} + C$$

(iii) $f'(x) = xe^{2x^2}$

$$f(x) = \frac{1}{4}\int 4xe^{2x^2}\,dx$$

$$f(x) = \frac{1}{4}e^{2x^2} + C$$

3.2 Applications of Exponential Functions

Like polynomials, exponential functions can be applied to **differentiation and integration applications** such as **optimisation**, **area under the curve** and **rectilinear motion**.

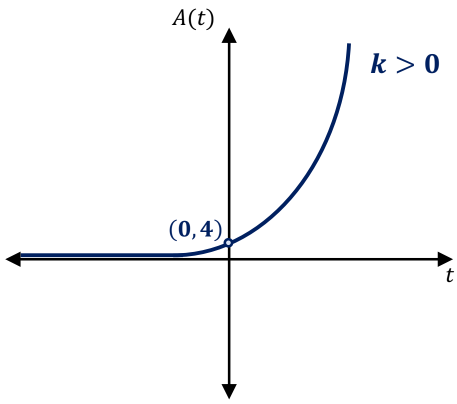
You can explore these applications in the **problem sets**, but an **important application** of exponential functions is **applying** them to **growth** and **decay** scenarios.

> **Exponential functions** of the form $A = A_0 e^{kt}$ can be used to track **growth** and **decay** scenarios that **compound continuously**, where A_0 is the **initial value** at $t = 0$ and k is the **growth/decay rate**.

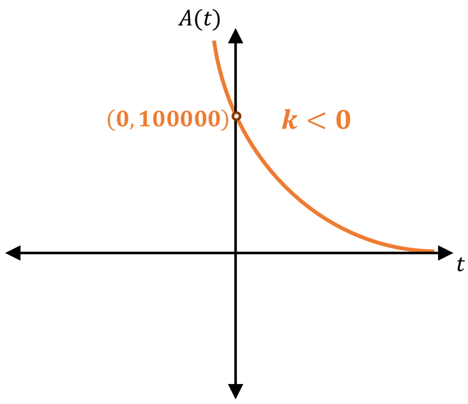
For an exponential function, if k is **greater than zero** (i.e. $k > 0$), then it will be a **growth function**, and if $k < 0$ it will be a **decay function**.

For instance, the growth in the **number of people** a **virus infects** that **compounds continuously over time** can be tracked with a **positive k value** such as 0.05 in $A = 4e^{0.05t}$, whereas the **decline of profits in a company** over **t years** can tracked with a **negative k** value such as -0.4 in the function $A = 100000e^{-0.4t}$.

Spread of Virus: $A = 4e^{0.05t}$



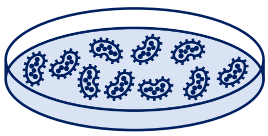
Decline of Company Profits: $A = 100000e^{-0.4t}$



Exponential functions are useful because they can be used to **predict future values** of **populations**, **wealth**, **radioactive decay** and so on.

For instance, suppose we had a **petri dish** filled with **bacteria** that had a **population** that compounded continuously based on the **exponential function**: $P = 10e^{0.04t}$, where t is the **time** in **seconds**.

From our equation, we can see that $A_0 = 10$, so our **initial population** is a **10 bacteria**, and that $k = 0.04$, so our function is growing.



$P = 10e^{0.04t}$
Initial population $P_0 = 10$
Growth rate: $k = 0.04$

If we wanted to know the **population** after $\mathbf{30}$ **seconds**, we can **substitute** in $t = 30$ to get the answer:

$$P = 10e^{0.04t}$$

① Let $t = 30$

$$P = 10e^{0.04(30)}$$

② $\therefore P = 33.2$ bacteria

① **Substitute** in $t = 30$

② **Calculate** the **final answer**

As we can see, the bacteria population **hasn't changed significantly** in the **first thirty seconds**, but if we consider the **population after** $\mathbf{10}$ **minutes** we can see the **exponential growth** take effect.

To determine the **population** after $\mathbf{10}$ **minutes**, let $t = 600\ seconds$:

$$P = 10e^{0.04t}$$

① Let $t = 600$

$$P = 10e^{0.04(600)}$$

② $\therefore P = 2.65 \times 10^{11}$ bacteria

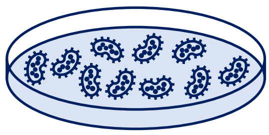
① **Substitute** in $t = 600$

② **Calculate** the **final answer**

Knowing the **exponential function** for a **specific scenario** is fantastic for **calculating future outcomes**, but often we **don't know** the **exponential equation** and just have the **generic equation**: $A = A_0e^{kt}$.

However, if we know the **starting point** at $t = 0$ and a **second point** after **time** t, we can determine the **values** of A_0 and k and therefore, the **exponential function** for that scenario.

Suppose we have a **new dish** of **bacteria** with the generic population equation: $P = P_0e^{kt}$. If we know that at $t = 0$ the **population** was $\mathbf{2}$ **bacteria** and after $\mathbf{120}$ **seconds** the **population** was $\mathbf{1200}$ **bacteria**, we can **substitute** them in to find the **values** of P_0 and k.



$$P = P_0e^{kt}$$
$$P(0) = 2$$
$$P(120) = 1200$$

To start, we can **substitute** in $t = 0$ and $P = 2$ to determine the **value** of P_0:

$$P = P_0e^{kt}$$

① Let $P(0) = 2$

$$2 = P_0e^{k(0)}$$

② $\therefore P_0 = 2$

① **Substitute** in $P = 2$ at $t = 0$

② **Calculate** the **final answer**

Secondly, we can **substitute** in $P = 1200$ at $t = 120$, and use a **calculator** to **solve** for k:

$$P = 2e^{kt}$$

① Let $P(120) = 1200$

$$1200 = 2e^{k(120)}$$

② $k = 0.053308$

$$\therefore P = 2e^{0.053308t}$$

① **Substitute** in $P = 1200$ at $t = 120$

② Use **calculator** to **solve** for k

With our **exponential equation**, $P = 2e^{0.053308t}$, we can now **apply** our understandings of **differentiation** to determine the **instantaneous rate of growth** at different times and populations.

The **instantaneous rate of change** of an exponential function is determined by **calculating** the **derivative** of the **exponential function**. As shown below, the **instantaneous rate of change** of the **exponential function P** is **directly proportional** to **itself** by the **growth rate k**:

$$\boxed{\textbf{Instantaneous Rate of Change: } \frac{dP}{dt} = kP_0e^{kt} = kP}$$

Using our equation, $P = 2e^{0.053308t}$, we can **differentiate** it to get the **instantaneous rate of change equation**:

$$P = 2e^{0.053308t}$$
$$① \quad \frac{dP}{dt} = 0.053308 \times 2e^{0.053308t}$$
$$② \quad \frac{dP}{dt} = 0.053308P$$

① **Differentiate** the **function P**

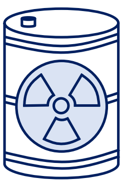
② **Simplify** to final **derivative**

If we then wanted to know the **rate of growth** at a specific **population**, such as when the population is at **100 bacteria**, we could **substitute** in $P = 100$:

$$\left.\frac{dP}{dt}\right|_{P=10} = 0.053308(100)$$
$$\left.\frac{dP}{dt}\right|_{P=10} \approx 5.33\ bacteria/sec$$

All these processes we have just explored can also be applied to **decay equations** (i.e. when $k < 0$).

For instance, suppose we had some **highly radioactive uranium** which **decayed** at a **rate** of $\frac{dA}{dt} = -0.0075A$, and we know the **uranium** had a **mass** of $30g$ after **1000 years** of **decay**.



Radioactive Uranium Decay Rate:

$$\frac{dA}{dt} = -0.0075A$$
$$A(1000) = 30$$

To **determine** the **exponential function** in the form $A = A_0e^{kt}$ from $\frac{dA}{dt} = -0.0075A$, we need to determine the **values** of A_0 and k.

Firstly, we can deduce that $k = -0.0075$ from $\frac{dA}{dt} = -0.0075A$, since we know the **derivative form** of **exponentials** in **base e** is in the form $\frac{dA}{dt} = kA \Leftrightarrow A = A_0e^{kt}$:

$$\therefore A = A_0e^{-0.0075t}$$

To determine A_o, we can use **substitute** in $A = 30g$ at $t = 1000$ to **solve** for A_0:

$$\text{Let } A(1000) = 30$$
$$① \quad 30 = A_0e^{-0.0075(1000)}$$
$$A_0 = \frac{30}{e^{-0.0075(1000)}}$$
$$② \quad A_0 = 54241g$$
$$\therefore A = 54241e^{-0.0075t}$$

① **Substitute** in $A = 30$ at $t = 1000$

② Use **calculator** to **solve** for A_0

With this equation, $A = 54241e^{-0.0075t}$, suppose we wanted to know **after how many years** of **decay** will it have a **mass** of $20g$. To do so, we can **substitute** in $A = 20$ and then **use our calculator** to **solve** for t:

$$\text{Let } A = 20$$
$$20 = 54241e^{-0.0075t}$$
$$t = 1054\ years$$

Solve for t using calculator

Exponential functions are fantastic in their **many applications**, and in the **problem sets** you will see many of the ways they can be used to **predict** future outcomes in different situations.

Worked Example 1

Student Sparrow is ready to put his knowledge of exponentials to the test and wreak havoc in the methods class! To do so, Sparrow has a box of **600 fruit flies** that **grow** in **population (P)** based on the **equation**: $P = 600e^{0.04t}$, where t is **time** in **minutes**. **[CA]**

(a) How **long** should Sparrow leave the flies in the box to reach a **population** of 8000?

$$P = 600e^{0.04t}$$
$$\text{Let } P = 8000$$
$$8000 = 600e^{0.04t}$$
$$t = 64.8\ mins$$

Solve for t using calculator

(b) What is the **instantaneous rate of growth** when the **fruit fly population** reaches 8000?

$$\frac{dP}{dt} = kP$$
$$\frac{dP}{dt} = 0.04P$$
$$\left.\frac{dP}{dt}\right|_{P=8000} = 0.04(8000)$$
$$\left.\frac{dP}{dt}\right|_{P=8000} = 320\ flies/min$$

(c) When Sparrow released the flies into the methods classroom, he found that the population grew at a **faster rate** than he expected! If the population grew from an **initial population** of 8000 to **$200,000$ flies** in **one hour**, determine the **new exponential equation**.

$$P = 8000e^{kt}$$
$$\text{Let } P = 200,000 \text{ and } t = 60$$
$$200000 = 8000e^{k(60)}$$
$$k = 0.05365$$
$$\therefore P = 8000e^{0.05365t}$$

Solve for k using calculator

EXPONENTIAL FUNCTIONS TOPIC NOTES

Exponentials

Exponential functions are functions with an **integer (a) raised** to the **exponent** (x), which is the variable we are **changing**. Therefore, they have the generic function: $f(x) = a^x$.

In Year 12, there is a **particular exponential function** we are interested in: e^x. This exponential has **Euler's number** (e) as its **base value**.

Euler's number (e) is an **irrational number** (non-repeating, and non-terminating) and is **approximately equal** to 2.71828. It is a '**perfect ratio**' for exponentials and is used to track the patterns of many natural phenomena.

Differentiation and Integration of Exponential Functions

The **derivative** of $y = e^x$ is e^x:

Exponential Rule: If $y = e^x$ then $\frac{dy}{dx} = e^x$

In the case where we have exponentials of the form $e^{f(x)}$, our rule to find the derivative of this expression is derived from the **chain rule**:

Derived from Chain Rule: If $y = e^{f(x)}$ then $\frac{dy}{dx} = f'(x)e^{f(x)}$

We can also apply this to the **product rule** and **quotient rule** to exponential functions.

There are three key rules for the integration of exponentials. Firstly, the **integral** of e^x is $e^x + C$.

$$\int e^x\,dx = e^x + C$$

Secondly, for exponentials of the form: $\int e^{ax+b}dx$. The **integral** will be the exponential function **divided** by the **co-efficient 'a'**:

$$\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + C$$

Thirdly, for exponentials of the form $f'(x)e^{f(x)}$, the **integral** will be the function $e^{f(x)}$ **divided** by the **derivative** of the exponent $f'(x)$:

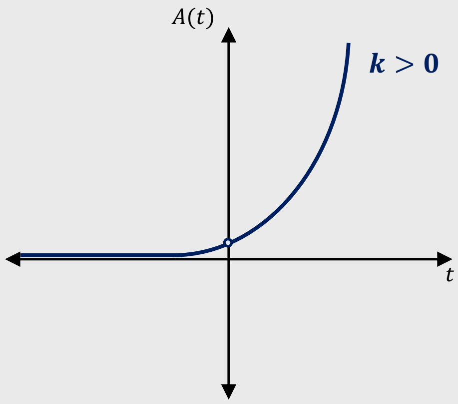
$$\int f'(x)e^{f(x)}dx = e^{f(x)} + C$$

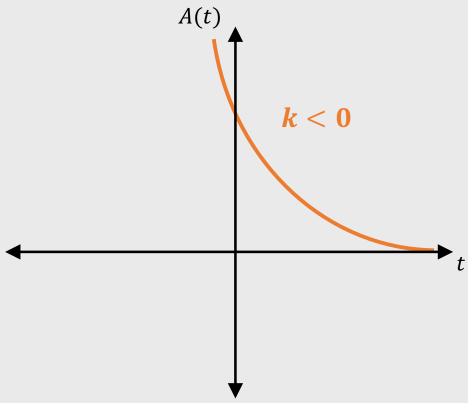
Applications of Exponential Functions

Exponential functions can be used to model **growth** and **decay** scenarios.

Exponential functions in the form $A = A_0 e^{kt}$ can be used to track **growth** and **decay** scenarios that **compound continuously**, where A_0 is the **initial value** at $t = 0$ and k is the **growth/decay rate**.

If $k > 0$ then it will be a **growth function**, and if $k < 0$ it will be a **decay function**.





When faced with a practical problem, if we know the **starting point** and a **second point** at some **time** t, we can determine the values of A and k. This means we can determine the **exponential function** of the form: $P = P_0 e^{kt}$ for that scenario.

The **instantaneous rate of change** of an exponential function is determined by **calculating** the **derivative** of the **exponential function**.

Instantaneous Rate of Change: $\frac{dP}{dt} = kP_0 e^{kt} = kP$

The value of k should always be **solved to multiple decimal places** because the **output** can be **significantly affected** by the size of k.

Problem Set 5 – Exponential Functions
Progressive Questions

Concept 1

Differentiation and Integration of Exponential Functions – Progressive Questions (8 questions)

Repetitive questions: 1.11 → 1.81 (7 questions)

The Space Aftermath

Starting: Teacher Andrew, Rabea, Tyler, Tom, Rupert and Wallace

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Following their incredible journey through space and beyond, Rabea, Tyler, Tom, Rupert and Wallace are back in class studying exponential functions, but all they can think about is their awe-inspiring adventure to outer space. Whilst they excitedly chatter amongst themselves about their space odyssey, their classmates begin to feel incredibly envious and excluded.

Differentiation of Exponential Functions: Q1, Q2, Q3, Q4

Repetitive: 1.11 → 1.31 (3 questions)

[20 marks]

1. "Welcome back to class, astronauts," Teacher Andrew says, as he ushers Rabea, Alexa, Rupert, Fraser and Harriet to their seats. "You may have been to space, but you still have to study exponential functions." The students groan as they are hit by reality once more. **Help** the students **differentiate** the following **exponential functions**!

(a) $f(x) = e^x$ **(2)**

(b) $y = 3e^x$ **(2)**

(c) $y = 4e^x + 3x$ **(2)**

(d) $f(x) = 6e^{2x}$ **(2)**

(e) $f(x) = e^{-3x}$ **(2)**

(f) $f(x) = 3e^{0.5x} + 2x^2$ **(2)**

(g) $y = e^{2x^2} + 4$ **(2)**

(h) $f(x) = 4e^{3x-2}$ **(2)**

(i) $y = 5e^{3x^2+7}$ **(2)**

(j) $f(x) = \frac{1}{e^{4x}}$ **(2)**

[20 marks]

2. Rabea is finding it extremely difficult to concentrate. "Hey Tyler," he whispers. "Remember that time in space when…" Suddenly, Teacher Andrew places down another page of **exponential functions** on his desk. **Help** Rabea to **differentiate** the following **exponentials**.

(a) $f(x) = \frac{1}{e^{2x}}$ **(2)**

(b) $y = 4x^3 - e^{-0.5x^2}$ **(2)**

(c) $f(x) = e^{x^2-2x+1}$ **(2)**

(d) $f(x) = 4e^{x^3-2x^2}$ **(2)**

(e) $f(x) = e^{-3x}$ **(2)**

(f) $f(x) = \frac{2}{3e^{3x}} + x^5$ **(2)**

(g) $y = e^{-2x+5x^4+4}$ **(2)**

(h) $y = e^{\sqrt{x}}$ **(2)**

(i) $f(x) = e^{4x^2} - e^{3x}$ **(2)**

(j) $y = \frac{1}{e^{4x}} + 4e^{7x^3-3x^2}$ **(2)**

[35 marks]

3. Whilst Teacher Andrew is busy dealing with Rabea, he fails to realise that the four other moonwalkers were chattering away excitedly in another corner. "I think I want my wedding in space," Alexa gushes. Teacher Andrew sighs and produces another page of exponentials. **Help** them to find the **derivatives** of the following functions using the **product rule** and **quotient rule**.

(a) $f(x) = xe^{2x}$ **(3)**

(b) $y = \frac{e^x}{x^2}$ **(3)**

(c) $f(x) = xe^{6x^3}$ **(3)**

(d) $f(x) = \frac{2x+2}{e^x}$ **(3)**

(e) $f(x) = -x^3e^{2x}$ **(3)**

(f) $f(x) = e^x(2+x)^5$ **(4)**

(g) $y = \frac{5x^2}{e^{3x}}$ **(4)**

(h) $y = \sqrt{x}e^{2x}$ **(4)**

(i) $f(x) = \frac{e^{5x^2}}{5x}$ **(4)**

(j) $y = (1+x)^3e^{2x}$ **(4)**

[CA] [15 marks]

4. Having been in space for so long, Rupert is really struggling with these problem sets. **Help** Rupert to determine the **gradient** of various exponential functions at the **following values**.

(a) Determine the **gradient** of $\boldsymbol{f(x) = e^{2x}}$ at the **point** $\boldsymbol{(0,1)}$ **[3]**

(b) Determine the **gradient** of $\boldsymbol{f(x) = 4e^{4x}}$ at the **point** $\boldsymbol{(0,4)}$ **[3]**

(c) Determine the **gradient** of $\boldsymbol{f(x) = 3e^{x^2}}$ at the **point** $\boldsymbol{(1,3e)}$ **[3]**

(d) Determine the **gradient** of $\boldsymbol{y = xe^x}$ at the **point** $\boldsymbol{(2,2e^2)}$ **[3]**

(e) Determine the **gradient** of $\boldsymbol{y = \frac{1}{e^{3x}}}$ at the **point** $\boldsymbol{(\frac{1}{3},\frac{1}{e})}$ **[3]**

Integration of Exponential Functions:

Repetitive: 1.51 → 1.81 (4 questions)

[25 marks]

5. As the five space adventurers continue to chatter amongst themselves, the rest of the students begin to feel rather envious. "Just because you went to space doesn't make you guys superior from the rest of us..." Kerry mutters. Teacher Andrew agrees and gives the five students a **lunchtime detention of integration**. **Help** the students through this detention by **integrating** the **following functions**.

(a) $\int e^x\,dx$ **(2)**

(b) $\int 9e^x dx$ **(2)**

(c) $\int e^{2x}\,dx$ **(2)**

(d) $\int 4e^{4x} dx$ **(2)**

(e) $\int e^{6x}\,dx$ **(2)**

(f) $\int 2xe^{x^2} dx$ **(3)**

(g) $\int 8xe^{x^2+3}\,dx$ **(3)**

(h) $\int 5e^{4x-3}\,dx$ **(3)**

(i) $\int 7e^{-3x}\,dx$ **(3)**

(j) $\int xe^{\frac{1}{2}x^2} dx$ **(3)**

[25 marks]

6. Falling back in-love with mathematics, Rupert is eager to continue pushing himself even after the other four students have left the detention. **Help** Rupert to **integrate** the following functions.

(a) $\int \frac{1}{e^{2x}}\, dx$ **(2)**

(b) $\int 8xe^{2x^2+5} dx$ **(2)**

(c) $\int \frac{e^{4x}}{4} + e^{-2x}\, dx$ **(2)**

(d) $\int \frac{1}{4} e^{4x} dx$ **(2)**

(e) $\int \frac{1}{6} e^{6x}\, dx$ **(2)**

(f) $\int e^{4x} - x^2 dx$ **(3)**

(g) $\int 2xe^{4x^2}\, dx$ **(3)**

(h) $\int \frac{4}{e^{5x}}\, dx$ **(3)**

(i) $\int \frac{1}{2} e^{2x}\, dx$ **(3)**

(j) $\int 4xe^{2x^2} - \frac{1}{x^3} dx$ **(3)**

[30 marks]

7. “I think we may have gone a little too far,” Rabea whispers to his four space friends. The group decide to try to make it up to the class by doing everyone’s homework. **Help** the group to **solve** the following **definite integrals** that were set for homework.

(a) $\int_0^1 e^x\, dx$ **(3)**

(b) $\int_2^3 3e^x\, dx$ **(3)**

(c) $\int_0^4 e^{2x}\, dx$ **(3)**

(d) $\int_{-1}^2 2xe^{x^2}\, dx$ **(3)**

(e) $\int_2^4 6xe^{3x^2}\, dx$ **(3)**

(f) $\int_{-2}^0 e^{-2x}\, dx$ **(3)**

(g) $\int_1^3 e^{2x+4}\, dx$ **(3)**

(h) $\int_0^{-1} \frac{1}{4} e^{3x}\, dx$ **(3)**

(i) $\int_1^{-2} 2x^2 + 4e^{2x}\, dx$ **(3)**

(j) $\int_3^{-1} 4xe^{x^2+2} dx$ **(3)**

[CA] [15 marks]

8. The second part of the homework involves determining the **using integration** to **determine $f(x)$**. **Help** the students to do **determine $f(x)$** for each of the following scenarios.

(a) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = e^x}$ and $\boldsymbol{f(x) = 2 + e}$ at $\boldsymbol{x = 1}$ **[3]**

(b) Determine $f(x)$ if $\boldsymbol{f'(x) = 6e^x}$ and $\boldsymbol{f(x)}$ intersects the point $\boldsymbol{(0, 8)}$ **[3]**

(c) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = 8xe^{x^2}}$ and $\boldsymbol{f(x) = 4e^4 - 3}$ at $\boldsymbol{x = 2}$ **[3]**

(d) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = \frac{1}{3}e^{3x}}$ and intersects y**-axis** at $\boldsymbol{f(x) = \frac{19}{9}}$ **[3]**

(e) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = 2x^2 + 4e^{2x}}$ and intersects y**-axis** at $\boldsymbol{f(x) = 8}$ **[3]**

“Hey everyone, we just wanted to apologise for our arrogant behaviour today. We would like to make it up to you all by inviting you to return to space with us for another adventure!” Rabea declares. To their surprise, their idea is welcomed by a round of applause. It is evident that their class is excited to return to space for a second space adventure.

Concept 2

Applications of Exponential Functions – Progressive Questions (12 questions)

Repetitive questions: 2.11 → 2.131 (14 questions)

Reach for the Stars

Starring: Janitor Peter, Methods Students

"How are we going to go to space if we don't even have a spaceship?" Wallace eagerly asks.

"Well, we might have to build our own…" Rabea says. "Luckily, we have the guidance of Janitor Peter, who is rather experienced in this field!"

The students approach Janitor Peter, who is busying himself in his office, avoiding his janitor duties at all costs. "Hey Peter," Wallace asks. "How might one go about building a spaceship?"

Janitor Peter pauses, before answering. "The first step is to find the funding. I'm willing to help you, but unfortunately my last spaceship almost bankrupted me, so we'll have to apply for another bank loan."

Applications of Exponential Functions: Q1, Q2, Q3, Q4, Q5, Q6

Repetitive: 2.11 → 2.62 (7 questions)

[CA] [6 marks]

1. Janitor Peter places several bank documents on the desk. One particular loan that strikes the students is a bank loan with an $\mathbf{80\%}$ **return per annum** on the investment.

(a) If the students have $\mathbf{\$1,000}$ as an **initial investment**, what is an **exponential function** of the form: $\boldsymbol{A = A_0 e^{kt}}$ to describe how their money will **increase** over $\boldsymbol{t}$ **years**? **[2]**

(b) What will be the **value** of their investment after $\mathbf{2}$ **years**? **[2]**

(c) After **how many years** will their investment be worth $\mathbf{\$1,000,000}$? **[2]**

[CA] [8 marks]

2. Realising that they need to make money faster than a risky bank loan, the students decide that they need to find other ways to make money. Wallace decides to kickstart their spaceship funding by setting up a **cake stand** to **baked goods**. The **profits** made each day the bake stand remains open **increases continually** at $\mathbf{4.5\%}$ **per day**, with the **profits** $\$\boldsymbol{P}$ such that: $\boldsymbol{\frac{dP}{dt} = 0.045}$.

(a) If **initial profit** they had was $\mathbf{\$100}$, **how much** will Wallace have made after **6 days**? **[2]**

(b) How much Wallace have made after **two months**? **[2]**

(c) How many **days** will Wallace have to work at the bake stand to have made $\mathbf{\$1000}$? **[2]**

(d) If the growth rate was found to be $\mathbf{8\%}$ **per day** rather than $\mathbf{4.5\%}$, **how many days** would it take to make $\mathbf{\$1,000}$? **[2]**

[CA] [7 marks]

3. Whilst Wallace sells cakes, Alexa decides to **sell pet ladybirds**. This species of ladybirds has a **population growth factor** such that $\frac{dP}{dt} = 0.87P$, where P is the **population** at **time** t in **days** since Alexa brought them to the Mathematics College.

(a) How long will it take the **original population** to **double**? **[1]**
(b) If the **initial population** was 8, how many ladybirds will Alexa have after **one week**? **[2]**
(c) What will the ladybird **population** be when the population grows at a rate of **100 ladybirds per day**? **[2]**
(d) How many **days** will it take for the ladybird population to **grow at a rate** of **100 ladybirds per day**? **[2]**

[CA] [7 marks]

4. Noticing the success of Alexa's ladybird store, Fraser decides to start selling a unique bacteria to the science students. The **number of bacteria** in Fraser's possession **increases** at a **rate** of $\frac{dN}{dt} = 1.1N$, where N is the **number of bacteria** and t is **time** in **minutes**.

(a) If **initial bacteria population** is **8 bacteria**, how many **minutes** will it take for the bacteria population to reach **one million**? **[2]**
(b) If the **bacteria population** reached **one million** after **20 minutes**, what was the **actual value** of the **growth rate k**? **[2]**
(c) Fraser accidentally spills a bottle of disinfectant into his bacteria storage container and the bacteria population starts **decreasing** at a **rate** of $\frac{dN}{dt} = -0.25N$. If the population was at **4 million** to begin with, **how many minutes** will it take for the population to **decrease** to $500,000$ bacteria? **[3]**

[CA] [5 marks]

5. The collective funds raised by Wallace, Alexa and Fraser have allowed the students to afford the fuel for their spaceship. The **fuel** required is a **highly radioactive substance** called Newranium, which is said to have a **decay rate** of **11% per day**.

(a) **How many days** will pass before the Newranium remaining is 75% of its **original amount**? **[2]**
(b) If Wallace puts a $2g$ piece of Newranium in the spaceship, after **how many days** will only $600mg$ of Newranium be remaining? **[2]**
(c) How much Newranium will be left **after 6 weeks**? **[1]**

[CA] [8 marks]

6. Having some Newranium left-over, the students notice that the **value** of Newranium is **rapidly increasing** at a **rate** of $\frac{dP}{dt} = 0.12P$, where P is **profit in dollars** and t is **time** in **days**.

(a) If the **initial value** of the Newranium is $\$2,000$, how much will it be worth in **three days**? **[2]**
(b) If after **six days** the value of the Newranium is $\$6,000$, what is the **actual value** of k? **[2]**
(c) With this new growth rate, **how long** will it take for its **value** to **double**? **[2]**
(d) After **how many days** will the **value** of the Newranium be **$\$1$ million**? **[2]**

Selling the Newranium at its peak value, the students have managed to gather sufficient funding to build their rocket that they will fly off into space in!

General Applications of Exponential Functions: Q7, Q8, Q9, Q10, Q11, Q12

Repetitive: 2.71 → 2.121 (7 questions)

Now that the students have accumulated enough funding to begin building their spaceship, they start to take measurements for the various components needed for their spaceship.

[CA] [10 marks]

7. With their spaceship design in mind, the students work out that the spaceship will have a launch **velocity** that follows the equation: $v = \frac{500}{7}e^{0.1t}$, where t is **time** in **seconds**. The **initial position** of the spaceship is at 7 **metres** above the ground.

(a) What is the **velocity** of the spaceship after **six seconds**? **[1]**
(b) What will be the **acceleration** of the spaceship **one second** after take-off? **[2]**
(c) What will be the **displacement** of the spaceship from the ground after 4 **seconds**. **[3]**
(d) Determine the spaceship's **displacement** after 3 **seconds**. **[1]**
(e) Determine **net change** in **displacement** of the spaceship during the **fifth second**. **[3]**

[CA] [10 marks]

8. As a safety precaution, the students want to **install parachutes** into the spaceship for a safe exit in the case of an emergency. The students purchase parachutes that follow the **acceleration equation**: $a = 0.1e^{0.1t}$, where t is **time** in **minutes**. The **initial displacement** is 200 **metres** and their **velocity** after 10 **minutes** is $12.72m/min$.

(a) What is the **initial acceleration** of the students in parachutes? **[1]**
(b) What will be the **velocity** of the students after 6 **minutes**? **[2]**
(c) What will be the **net change** in **displacement** of the students during the 15^{th} **minute**? **[2]**
(d) What will be the **displacement** of the students in parachutes after 10 **minutes**. **[3]**
(e) Assuming the parachutist lands after 8.5 **minutes**, what will be their **speed** on landing? **[2]**

[CA] [10 marks]

9. When purchasing the raw materials to construct the spaceship, they want to construct it using a special metal called 'Nitanium', which has a **value** with an **instantaneous rate of change** given by the equation: $\frac{dV}{dt} = t^2 - e^{0.2t}$, where V is the **value** in **dollars per kilogram** and t is **time** in **hours**.

(a) What will be the **instantaneous rate of change** in the **value** of Nitanium after 6 **hours**? **[2]**
(b) What is the **net change** over the first **half an hour** in the value of the Nitanium? **[2]**
(c) What is the **net change** in the **value** of the Nitanium during the **tenth hour**? **[2]**
(d) If the **initial value** is $\$20$, what is the **maximum value** it reaches during the first 10 **hours**? **[4]**

[CA] [7 marks]

10. To make a few extra dollars before purchasing the Nitanium metal, the students decide to setup a **premium lemonade stand** with a **profit in dollars** following the equation: $P(x) = 2\left(e^{\frac{x}{50}} + 1\right)$, where x is the **number of cups of expensive lemonade sold**.

(a) Using the **incremental formula**, what is the **approximate change** in price between the $100th$ and $102nd$ cup they sell? Hence, estimate the price of the $102nd$ cup? **[4]**
(b) Is this the **right time** to use the **incremental formula**? Calculate the **actual** price of the $102nd$ **cup** they sell to support your answer. **[3]**

[CA] [16 marks]

11. The students decide that the body piece of their spaceship will be constructed using key parts in the areas created by **exponential functions** and the **x-axis**. To determine **how much material they need** for each of the parts, help the students to **calculate** the **following areas**:

(a) Calculate the **area under the curve $y = e^{2x}$** between $x = 0$ and $x = 1.5$ **[4]**

(b) Calculate the **area under the curve $y = e^{-2x}$** between $x = 0$ and $x = 2$ **[4]**

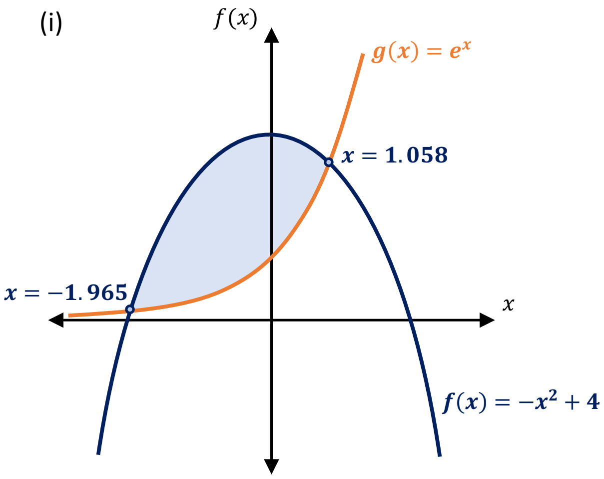
(c) Calculate the **area under the curve $y = e^{-\frac{1}{3}x}$** from $x = 0$ to $x = \infty$ **[4]**

(d) Calculate the **trapped between the x-axis** and $y = -2e^{x}$ from $x = -2$ to $x = 0$ **[4]**

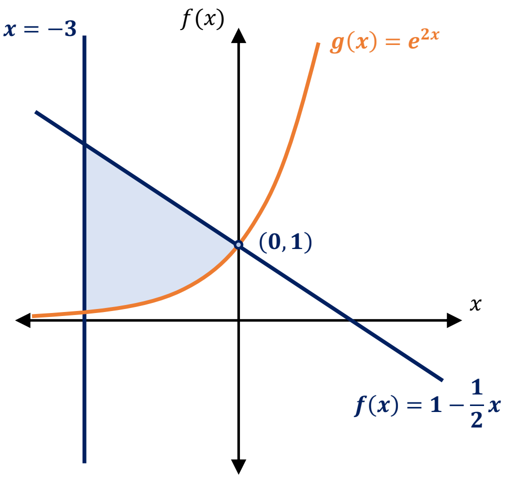
[CA] [16 marks]

12. For the exterior of the spaceship, they are eager to use the shapes created between exponential functions and other functions. To determine **how much material they need**, help them to **determine** the **shaded areas below** using **definite integrals**.

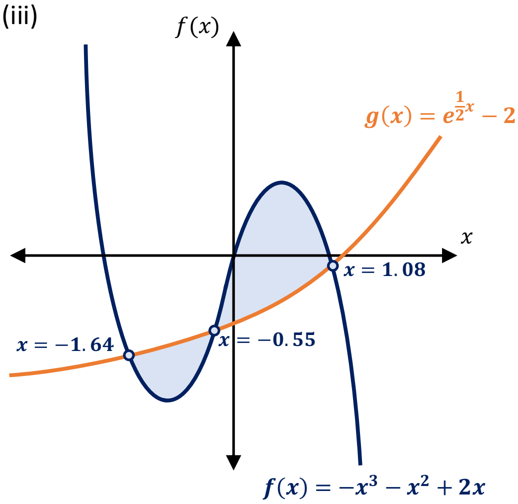
(i)



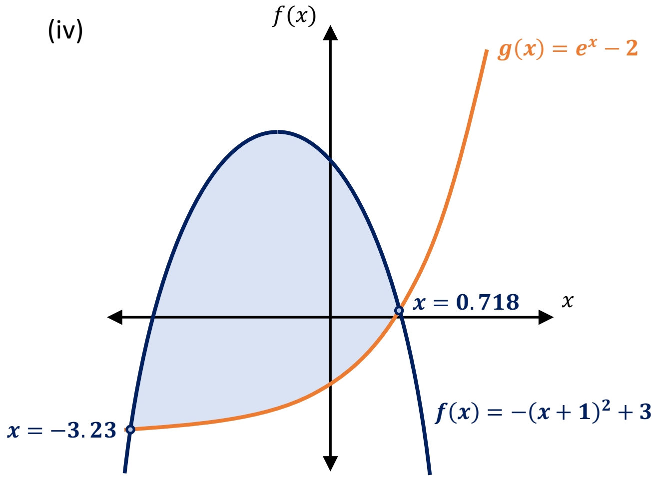
(ii)



(iii)



(iv)



"We're finally going to be real astronauts!" Harriet cries. With the spaceship constructed, the students blast off the ground and into space with the hopes of reaching a distant planet with a theme park known as Trigonometric Mountain. With all of the rides, shows and games operating on trigonometric functions, the students can't wait!

Problem Set 5 – Exponential Functions
Repetitive Questions

Concept 1
Differentiation and Integration of Exponential Functions – Repetitive Questions (7 questions)

Differentiation of Exponential Functions: Qs 1.11, 1.21, 1.31

[20 marks]

1.11 Shauna is looking to catch-up to the rest of the class on exponential functions. **Help** Shauna to **differentiate** the following **exponential functions**.

(a) $f(x) = -3e^{x}$ **(2)**

(b) $y = 10e^{2x}$ **(2)**

(c) $y = -3x^2 - 4e^{6x}$ **(2)**

(d) $f(x) = -\frac{1}{2}e^{2x}$ **(2)**

(e) $f(x) = e^{-5x}$ **(2)**

(f) $f(x) = 7e^{0.5x}$ **(2)**

(g) $y = e^{-0.75x} - 3x^5$ **(2)**

(h) $f(x) = 5e^{3x^2-3}$ **(2)**

(i) $y = -8e^{2-4x^2}$ **(2)**

(j) $f(x) = \frac{1}{e^{7x}}$ **(2)**

[20 marks]

1.21 Continuing to push herself, **help** Shauna to **differentiate** the following more difficult **exponential functions**.

(a) $f(x) = e^{-2x+4x^3}$ **(2)**

(b) $y = \frac{1}{2\sqrt{x}} - 3e^{6x}$ **(2)**

(c) $f(x) = 5e^{2x-x^2}$ **(2)**

(d) $f(x) = -9e^{\frac{1}{4}x-2}$ **(2)**

(e) $f(x) = -4e^{-\frac{2}{3}x} + \frac{1}{2}x^{\frac{3}{2}}$ **(2)**

(f) $f(x) = \frac{e^{-7x}}{2}$ **(2)**

(g) $y = \frac{2}{5e^{3x}} - 2x^{-4}$ **(2)**

(h) $y = e^{-2x^2+4} - e^{2x}$ **(2)**

(i) $f(x) = -e^{-5x^2} + e^{\sqrt{x}}$ **(2)**

(j) $y = 6e^{7x-2x^2} + \frac{1}{e^{4x}}$ **(2)**

[35 marks]

1.31 Help Shauna to find the **derivatives** of the following questions using the **product rule** and **quotient rule**.

(a) $f(x) = -2xe^{x}$ **(3)**

(b) $y = x^3e^{3x}$ **(3)**

(c) $f(x) = \frac{x}{e^{x}}$ **(3)**

(d) $f(x) = \frac{5x-2}{e^{2x}}$ **(3)**

(e) $f(x) = -2xe^{5x}$ **(3)**

(f) $f(x) = e^{x}(2-x)^4$ **(4)**

(g) $y = (x-3)^3e^{2x}$ **(4)**

(h) $y = \frac{\sqrt{x}}{e^{3x}}$ **(4)**

(i) $f(x) = \frac{e^{x}}{6x-3}$ **(4)**

(j) $y = e^{2x}x^{\frac{3}{2}x}$ **(4)**

Integration of Exponential Functions: Qs 1.51, 1.61, 1.71, 1.81

[25 marks]

1.51 Rupert also realises that he has fallen behind so he decides to work with Shauna on **integration**. Help Rupert and Shauna to **integrate** the following **exponential functions**.

(a) $\int 3e^{x}\,dx$ **(2)**

(b) $\int -5e^{2x}dx$ **(2)**

(c) $\int 6e^{2x}\,dx$ **(2)**

(d) $\int 8e^{\frac{1}{2}x}dx$ **(2)**

(e) $\int -3e^{-3x}\,dx$ **(2)**

(f) $\int 8e^{2x}dx$ **(3)**

(g) $\int -2x^{x^2-2}\,dx$ **(3)**

(h) $\int 4e^{-\frac{1}{4}x}\,dx$ **(3)**

(i) $\int 8e^{-4x+2}\,dx$ **(3)**

(j) $\int 4xe^{2x^2}dx$ **(3)**

[25 marks]

1.61 Rupert and Shauna are eager to keep going, **help** them to **integrate** the following functions.

(a) $\int e^{-2x}\,dx$ **(2)**

(b) $\int 2xe^{x^2}dx$ **(2)**

(c) $\int e^{-2x}+e^{4x}\,dx$ **(2)**

(d) $\int -\frac{1}{2}e^{\frac{1}{2}x}dx$ **(2)**

(e) $\int 5xe^{x^2-2}\,dx$ **(2)**

(f) $\int 4xe^{4x^2}-3x^3dx$ **(3)**

(g) $\int 4xe^{x^2-7}\,dx$ **(3)**

(h) $\int \frac{8}{e^{2x}}\,dx$ **(3)**

(i) $\int \frac{1}{2}e^{-6x}\,dx$ **(3)**

(j) $\int 9xe^{x^2}-\frac{1}{3x^2}dx$ **(3)**

[30 marks]

1.71 Help Rupert and Shauna to **solve** the following **definite integrals** that were set for homework.

(a) $\int_2^4 e^{x}\,dx$ **(3)**

(b) $\int_{-1}^{0} 2e^{x}\,dx$ **(3)**

(c) $\int_0^5 3e^{3x}\,dx$ **(3)**

(d) $\int_4^0 e^{-2x}\,dx$ **(3)**

(e) $\int_2^4 2xe^{x^2}\,dx$ **(3)**

(f) $\int_{-2}^{0} e^{-\frac{1}{3}x}\,dx$ **(3)**

(g) $\int_1^3 8xe^{2x^2-4}\,dx$ **(3)**

(h) $\int_0^{-1} \frac{1}{2}e^{3x}\,dx$ **(3)**

(i) $\int_2^{-2} 2x^2+4e^{2x}\,dx$ **(3)**

(j) $\int_3^{-1} -4xe^{2x^2-3}\,dx$ **(3)**

[CA] [12 marks]

1.81 The second part of the homework involves determining the **using integration** to **determine** $\boldsymbol{f(x)}$. **Help** Rupert and Shauna to do **determine** $\boldsymbol{f(x)}$ for each of the following scenarios.

(a) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = e^{2x}}$ and $\boldsymbol{f(x) = 1.5}$ at $\boldsymbol{x = 0}$ **[3]**
(b) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = -2e^{x}}$ and $\boldsymbol{f(x)}$ intersects the point $\boldsymbol{(1, -2e-2)}$ **[3]**
(c) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = 4xe^{x^2}}$ and $\boldsymbol{f(x) = 2e^4 - 3}$ at $\boldsymbol{x = 2}$ **[3]**
(d) Determine $\boldsymbol{f(x)}$ if $\boldsymbol{f'(x) = e^{-\frac{1}{2}x}}$ and intersects the point $\boldsymbol{(3, -2e^{-\frac{3}{2}} - 1)}$ **[3]**

Concept 2

Applications of Exponential Functions – Repetitive Questions (14 questions)

Applications of Exponential Functions: 2.11, 2.21, 2.31, 2.41, 2.51, 2.61, 2.62

[CA] [9 marks]

2.11 The **stock price** of a popular cryptocurrency is **rising** at a **rate** of 25% **per month** and will **continue** to **increase** at this rate.

(a) If the **initial stock price** is $\$1$, what is an **exponential function** of the form: $A = A_0 e^{kt}$ to describe how the **value of the stock** will **increase** over t **months**. **[2]**
(b) How much will the stock be **valued** at after **half a year**? **[2]**
(c) After **how many months** will the **value** of the **stock** be **worth** $\$10,000$? **[2]**
(d) If the **growth rate** of the stock was instead 50% **per month**, **how many months** would it take for the **value** of the stock to be **worth** $\$1$ **million**? **[3]**

[CA] [9 marks]

2.21 The value of a new computer **depreciates** after it is purchased. Suppose that the value of the computer **depreciates** according to the **exponential decay model**. If the **value** of the **computer** is $\$1,500$ at the **end of one year** and that its value has been **decreasing** at the **rate of** 5% **per month**.

(a) Determine the **value** of the of the computer when it was **new**. **[3]**
(b) What is the **value** of the computer after **two years**? **[2]**
(c) After **how many months** would the **value of the computer** be worth $\$500$? **[2]**
(d) If the computer was instead **worth** $\$750$ **after six months**, determine the **true value** of k. **[2]**

[CA] [8 marks]

2.31 A certain type of **bacteria**, given a favourable growth medium, has a rapid **growth rate** of $\frac{dN}{dt} = 1.22N$, where N is the **number of bacteria** and t is **time** in **hours**.

(a) If the **initial bacteria population** is 20 **bacteria**, **how many hours** will it take for the **population** to reach a **population of** 100? **[2]**
(b) What will be the **population** of the bacteria **after one day**? **[1]**
(c) After **how many hours** will the **bacteria colony** reach a **population of** 10 **million**? **[2]**
(d) If a chemical is applied to **decay** the bacteria population at a rate of $\frac{dN}{dt} = -0.33N$, determine **how long it will take** for the bacteria population to **decrease** from 3 **million to** $100,000$. **[3]**

[CA] [4 marks]

2.41 A national park in Japan has a population of 1000 **foxes** in the year 2015. Conservationists are concerned because the fox population is **decreasing** at the **rate of** 7.5% **per year**. If the population **continues** to **decrease at this rate**, how long would it take until the **population** is only 250 **foxes**?

[CA] [5 marks]

2.51 An eSports team creates a website to analyse match statistics. Their business plan states that their goal is to accumulate **one million users** by the **end of 5 years**. If they can achieve this goal, their website will be endorsed by the official game creators. The **initial number of users** who signed up before the website was launched was **one thousand**. **Find** the **monthly exponential growth rate** needed if the user base must accumulate to **one million users** by the **end of 60 months**.

[CA] [8 marks]

2.61 The total number views to date of the most viewed video online, "Baby Shark" by PinkFong, has been **increasing exponentially** according to the exponential growth function $\boldsymbol{y = 100e^{0.009669t}}$, where $\boldsymbol{y}$ represents the **number of views** and $\boldsymbol{t}$ represents the **time** measured in **days** since the video was posted. If the video was posted on the **18th of June 2016**:

(a) Calculate **how many views** the video had **after 400 days**? **[2]**
(b) Calculate **how many days** it took for **one million** people to view the video? **[3]**
(c) Calculate **how many days** it took for **one billion** people to view this video? **[3]**

[CA] [11 marks]

2.62 Pufferfish and jellyfish are both animals that live underwater. A body of water contains a certain **number of pufferfish**. At a certain point in time, **10 jellyfish** are introduced into the **body of water** and start eating the pufferfish. The **number of pufferfish**, P, decreases according to $\boldsymbol{P = 375 - e^{\frac{1}{25}t}}$, where $\boldsymbol{t}$ represents the **time in months** after the jellyfish are introduced. The population of jellyfish, J, **increases** according to $\boldsymbol{\frac{dJ}{dt} = \frac{1}{50}J}$.

(a) **How many pufferfish** were in the **body of water** when the jellyfish were introduced? **[4]**
(b) **When** will the **population of pufferfish** be **equal to ten**? **[2]**
(c) Determine the **time** when the **rate of increase** of **jellyfish equal** to the **rate of decrease** of **pufferfish**? **[3]**
(d) Determine **when** the **number of jellyfish equal** to the **number of pufferfish**? **[2]**

General Applications of Exponential Functions: 2.71, 2.81, 2.82, 2.91, 2.111, 2.121

[CA] [8 marks]

2.71 **Aspirin** is a medication used to reduce pain, fever, or inflammation. After an **Aspirin tablet** is ingested, the amount entering the bloodstream is modelled by: $\boldsymbol{M(t) = 100te^{-\frac{1}{2}t}}$, where $\boldsymbol{M}$ is in $\boldsymbol{mg}$ and $\boldsymbol{t}$ is in **number of hours** after its absorption into the blood stream.

(a) **How much aspirin** is in the bloodstream at $\boldsymbol{t = 0}$ and $\boldsymbol{t = 5}$? **[2]**
(b) When is **amount** of aspirin in the bloodstream a **maximum**, and what is the **maximum**? **[4]**
(c) After **how many hours** will there be $\boldsymbol{9mg}$ of **aspirin remaining**? **[2]**

[CA] [10 marks]

2.81 An **invasive species** is introduced into the wild. The size of the **population**, $\boldsymbol{P}$, of the invasive species can be modelled by the function $P = \frac{200}{\left(1+19e^{-\frac{1}{2}t}\right)^2}$, where t is in **weeks**.

(a) What is the **population** after **7 days**? **[2]**
(b) Find the **rate of growth** of the **size of population**. **[3]**
(c) Determine the **range** of the **function P**, **justifying** your answer. **[2]**
(d) Hence, **find** the **size** of the population when it is growing at its **fastest** and **slowest** rate. **[3]**

[CA] [14 marks]

2.82 A falling object cannot keep accelerating forever, and due to **resistance** caused by the air, reaches a **terminal speed**. If the **velocity** of such an object, t **seconds** after the fall commences, is represented by $v\ m/s$ where $v = 100(1 - e^{-0.1t})$.

(a) What is the **velocity** of the object after **5 seconds**. **[3]**
(b) What will be the **equation** for the **acceleration** of the object? **[2]**
(c) If the **initial displacement** of the object was **zero meters**, what is the **equation** for the **displacement** of the object? **[2]**
(d) What will the **change in velocity** of the object be during the **third second**? **[2]**
(e) What is the **net change** in the **displacement** of the object during the **first 60 seconds**? **[2]**
(f) Determine the **terminal velocity**. **[3]**

[CA] [14 marks]

2.91 An tennis ball is launched off a building using a tennis ball machine, such that the **acceleration** for the tennis ball follows the **acceleration equation**: $a = 0.2e^{0.05t}$, where t is **time** in **seconds**. The **initial position** of the tennis ball is **2 meters** and the **initial velocity** is $2m/s$.

(a) What is the **initial acceleration** of the tennis ball? **[1]**
(b) What is the **velocity** of the tennis ball **3 seconds**. **[3]**
(c) What will be the **net change** in the **velocity** of the tennis ball during the **first 5 seconds**? **[2]**
(d) What will be the **net change** in the **displacement** of the tennis ball in the **first 5 seconds**? **[3]**
(e) What will be the tennis ball's **displacement** and **velocity after 5 seconds**? **[3]**
(f) What is the **net change** in the **acceleration** during the **fourth second**? **[2]**

[CA] [8 marks]

2.101 A new cryptocurrency has a **price** that has an **instantaneous rate of change** given by the equation: $\frac{dP}{dt} = 2t + e^{0.04t}$, where P is the **price** in **dollars** and t is **time** in **hours**.

(a) What will be the **instantaneous rate of change** in the **price** of the cryptocurrency **after three hours**? **[2]**
(b) What is the **net change** in the **price** of the cryptocurrency during the **first three hours**? **[2]**
(c) Does the **value** of the cryptocurrency **ever start decreasing**? **[2]**
(d) What is the **net change** in the **price** of the cryptocurrency during the **ninth hour**? **[2]**

[CA] [16 marks]

2.111 Over the school holidays, Jevon is working on his passion project of fixing up the family car. For some of the key components in the car, Jevon is using the shapes created by **exponential functions** and the **x-axis**. To determine **how much material he will need** for each of the parts, help Jevon to **calculate** the **following areas**:

(a) Calculate the **area under the curve** $y = e^x$ between $x = 0$ and $x = 1$ **[4]**

(b) Calculate the **area under the curve** $y = e^{2x} - 1$ between $x = -2$ and $x = 0$ **[4]**

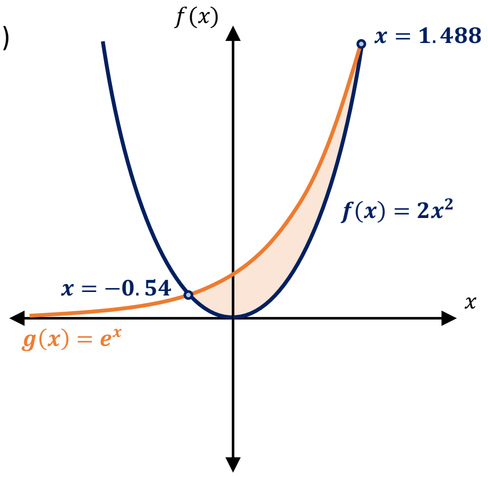
(c) Calculate the **area under the curve** $y = e^{-\frac{1}{4}x} - 4$ from $x = 2$ to $x = 6$ **[4]**

(d) Calculate the **trapped between the x-axis** and $y = -2e^x + 8$ from $x = -4$ to $x = -2$ **[4]**

[CA] [16 marks]

2.121 Jevon is also eager to create some more unique components to upgrade his family car. To determine **how much material** he will need for each component, help him to **determine** the **shaded areas below** using **definite integrals**

(i)



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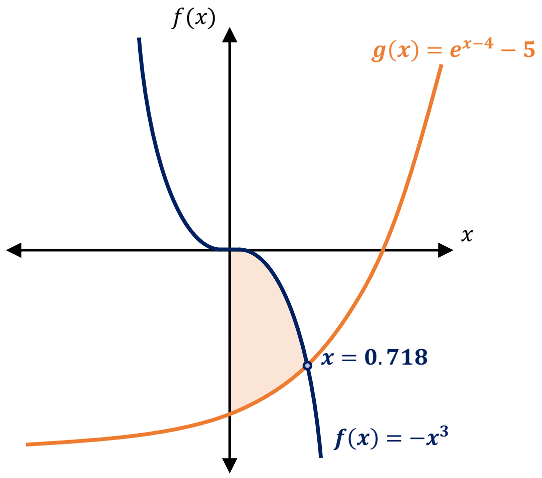

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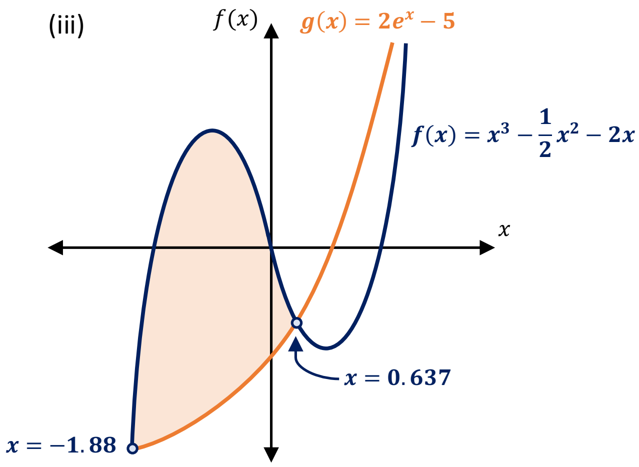

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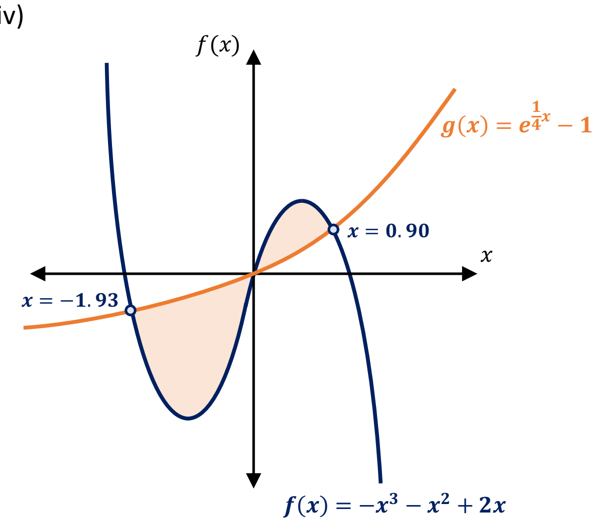
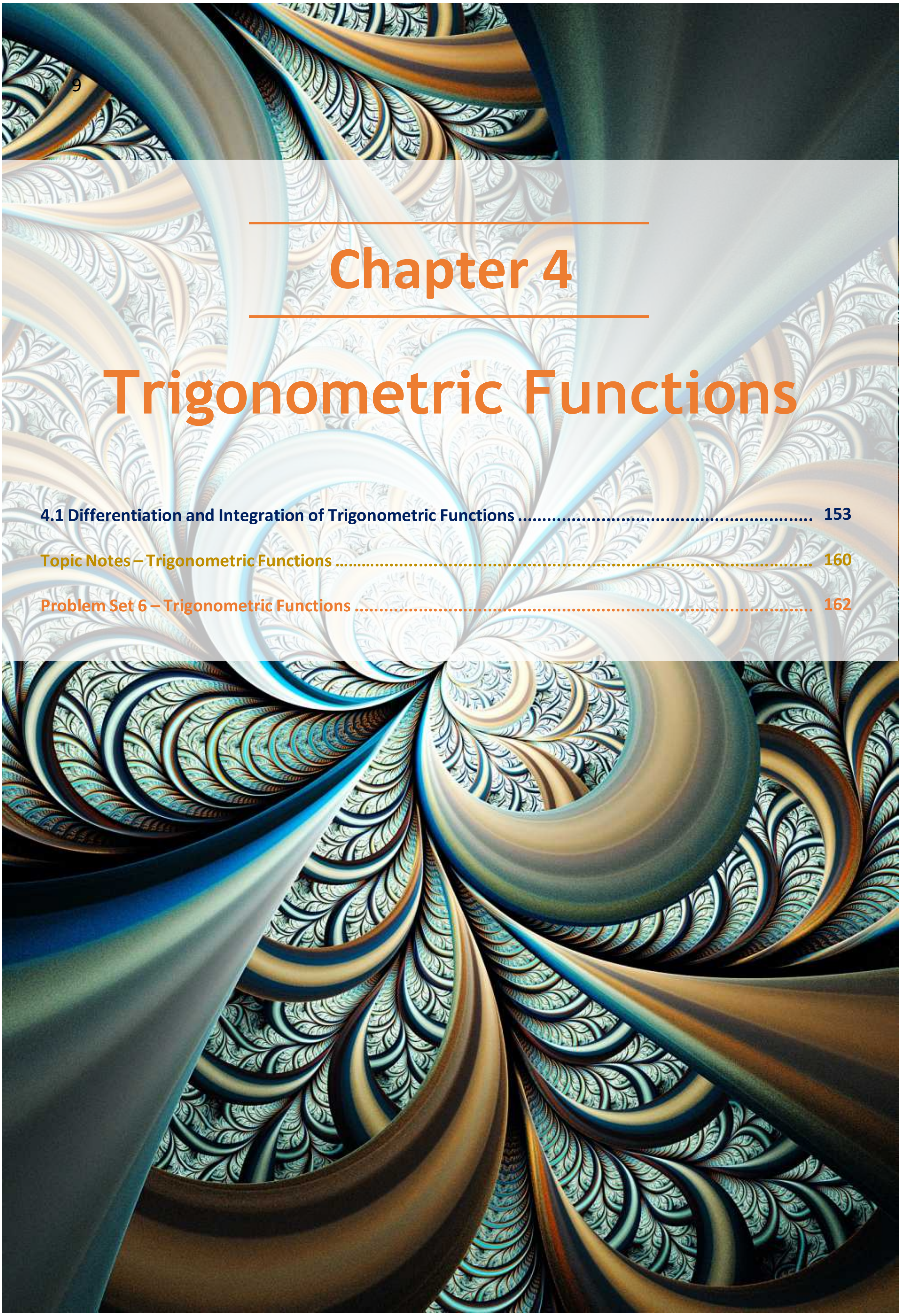




Chapter 4

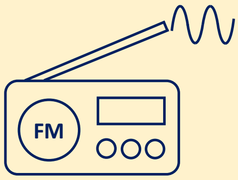
Trigonometric Functions

Chapter 4 – Trigonometric Functions

Introduction

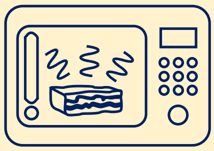
So far, we have explored the world of calculus through polynomials and exponentials. However, we can also **apply calculus** to **trigonometric functions.**

Trigonometric functions are **wave functions**. They are **repeating patterns** that can be used to track the **patterns of waves** in the **ocean**, the **sound** from our **headphones** or **microwaves** for **cooking food**.



Radio Waves transmitting to radio

Sound waves passing through headphones



Microwaves cooking your food

Just like polynomials and exponentials, **differentiation** and **integration** can be **applied to trigonometric functions**. To understand **trigonometric functions**, we are going to explore **three main topics**. These are:

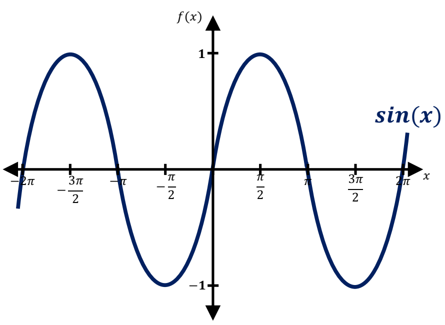
1. **Trigonometric Functions**
2. **Differentiation and Integration of Trigonometric Functions**
3. **Applications of Trigonometric Functions**

Let's begin!

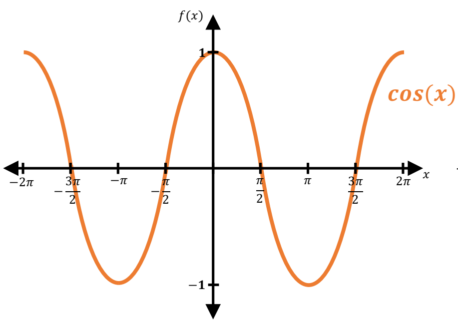
4.1 Differentiation and Integration of Trigonometric Functions

Trigonometric functions are functions that we are familiar with from Year 11. $\boldsymbol{sin(x)}$, $\boldsymbol{cos(x)}$ and $\boldsymbol{tan(x)}$ are all functions that should ring a bell. You might **remember their shape**, shown below:

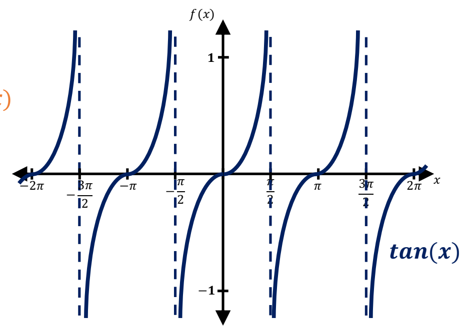
$\boldsymbol{f(x) = sin(x)}$



$\boldsymbol{f(x) = cos(x)}$



$\boldsymbol{f(x) = tan(x)}$



In Year 11, we were focussed on their **shape** and the **transformations** that can be applied. In **Year 12**, we are now **focussed** on the **differentiation** and **integration** of these functions.

Differentiation of Trigonometric Functions

In Year 12, we only have specific rules for the **differentiation** of $\boldsymbol{sin(x)}$ and $\boldsymbol{cos(x)}$, so we will set aside $\boldsymbol{tan(x)}$ for now and **return to it briefly** when we explore **quotient rule**.

The rules for **differentiating** $\boldsymbol{sin(x)}$ and $\boldsymbol{cos(x)}$ are relatively straightforward and are summarised as follows:

If $y = sin(x)$ then $\frac{dy}{dx} = cos(x)$	If $y = cos(x)$ then $\frac{dy}{dx} = -sin(x)$

From these rules, we can see that the **derivative** for $\boldsymbol{sin(x)}$ is $\boldsymbol{cos(x)}$, and the **derivative** for $\boldsymbol{cos(x)}$ is $-\boldsymbol{sin(x)}$.

Similar to the exponentials section, to understand how these rules are brought about, we need to use **first principles**.

Applying the **general instantaneous rate of change formula**: $\frac{d}{dx} = \lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$, when we **substitute** in $\boldsymbol{sin(x)}$ and $\boldsymbol{cos(x)}$, we produce **two key limits**: $\lim\limits_{h\to 0}\frac{\sin h}{h}$ and $\lim\limits_{h\to 0}\frac{1-\cos h}{h}$, as shown:

For $\frac{d}{dx}(sin(x))$:

$$\begin{aligned}\frac{d}{dx}(sin(x)) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{sin(x+h)-sin(x)}{h}\\ &= \lim_{h\to 0}\frac{sin(x)\,cos(h)+cos(x)\,sin(h)-sin(x)}{h}\\ &= \lim_{h\to 0}\frac{cos(x)sin(h)}{h}-\lim_{h\to 0}\frac{sin(x)-sin(x)cos(h)}{h}\\ &= cos(x)\lim_{h\to 0}\frac{sin(h)}{h}-sin(x)\lim_{h\to 0}\frac{1-cos(h)}{h}\end{aligned}$$

For $\frac{d}{dx}(cos(x))$:

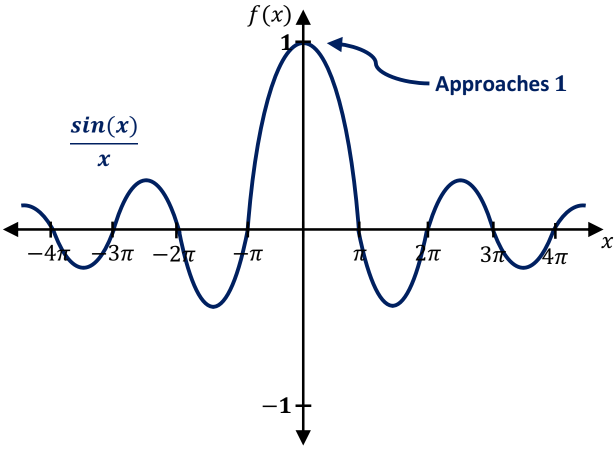
$$\begin{aligned}\frac{d}{dx}(cos(x)) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{cos(x+h)-cos(x)}{h}\\ &= \lim_{h\to 0}\frac{cos(x)\,cos(h)-sin(x)\,sin(h)-cos(x)}{h}\\ &= \lim_{h\to 0}\frac{cos(x)\,cos(h)-cos(x)}{h}-\lim_{h\to 0}\frac{sin(x)sin(h)}{h}\\ &= -cos(x)\lim_{h\to 0}\frac{1-cos(h)}{h}-sin(x)\lim_{h\to 0}\frac{sin(h)}{h}\end{aligned}$$

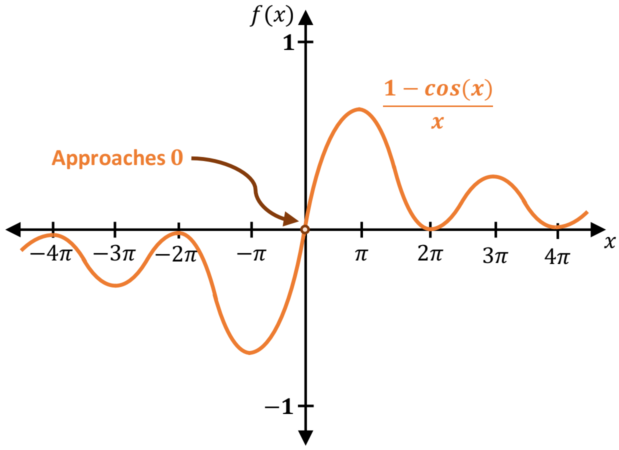
Like exponentials, we can calculate the **values** these **two key limits**: $\lim\limits_{h\to 0}\frac{\sin h}{h}$ and $\lim\limits_{h\to 0}\frac{1-\cos h}{h}$, as $\boldsymbol{h}$ **approaches zero**.

h	$\lim\limits_{h\to 0}\frac{\sin h}{h}$	$\lim\limits_{h\to 0}\frac{1-\cos h}{h}$
0.1	0.998334167	0.049958347
0.01	0.999983333	0.004999958
0.001	0.999999833	0.000500000
0.0001	0.999999998	0.00005
0.00001	1.000000000	0.000005
0.000001	1.000000000	0
0.0000001	1.000000000	0

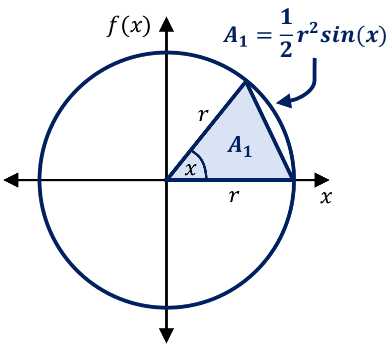
If we **approach zero** from the **left-side** (i.e. $-0.1, -0.01$ ...), the **same result** is yielded of $\lim_{h \to 0} \frac{\sin h}{h}$ **approaching one** and $\lim_{h \to 0} \frac{1-\cos h}{h}$ **approaching zero**.

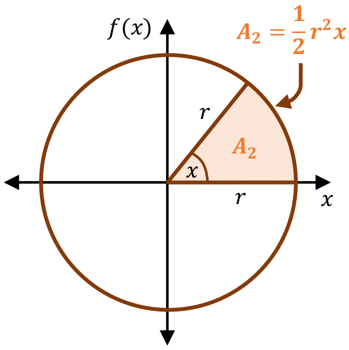
This can also be **justified graphically**, where we can see that as both of the graphs **approach** $x = 0$, the function $\frac{\sin x}{x}$ **approaches 1** and the function $\frac{1-\cos x}{x}$ **approaches 0**:

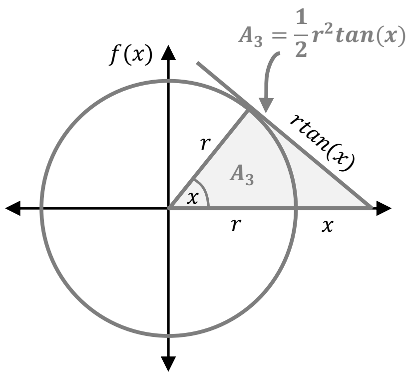




Finally, $\frac{\sin x}{x}$ **approaching 1** can be **proven geometrically** with the **unit circle**. Below we can create **three key areas** where $r = 1$ and x is the **angle** in **radians**, the **area of a triangle** is: $A = \frac{1}{2}bh$ and the **area** of a **sector** is: $A = \frac{1}{2}r^2x$. By using these **areas** and **limits** we can conclude that $\frac{\sin x}{x} = 1$.







$$A_1 < A_2$$
$$\frac{1}{2}r^2\sin(x) < \frac{1}{2}r^2x$$
$$\sin(x) < x$$
$$\frac{\sin(x)}{x} < 1$$

$$A_2 < A_3$$
$$\frac{1}{2}r^2x < \frac{1}{2}r^2\tan(x)$$
$$x < \frac{\sin(x)}{\cos(x)}$$
$$\cos(x) < \frac{\sin(x)}{x}$$

$$\lim_{x \to 0} \cos(x) < \lim_{x \to 0} \frac{\sin(x)}{x} < 1$$
$$1 < \lim_{x \to 0} \frac{\sin(x)}{x} < 1$$
$$\therefore \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Whilst some of those proofs may feel confusing, essentially what you need to know is that we can **conclude** that: $\lim_{h\to 0}\frac{\sin h}{h} = 1$ and $\lim_{h\to 0}\frac{1-\cos h}{h} = 0$. **Substituting** these into our equations determined earlier from **first principles**, we can conclude what the **derivatives** of $sin(x)$ and $cos(x)$ are:

For $\frac{d}{dx}(sin(x))$:

$$\frac{d}{dx}(sin(x)) = cos(x)\lim_{h\to 0}\frac{sin(h)}{h} - sin(x)\lim_{h\to 0}\frac{1-cos(h)}{h}$$

$$\frac{d}{dx}(sin(x)) = cos(x)\,(1) - sin(x)(0)$$

$$\therefore \frac{d}{dx}(sin(x)) = cos(x)$$

For $\frac{d}{dx}(cos(x))$:

$$\frac{d}{dx}(cos(x)) = -cos(x)\lim_{h\to 0}\frac{1-cos(h)}{h} - sin(x)\lim_{h\to 0}\frac{sin(h)}{h}$$

$$\frac{d}{dx}(sin(x)) = -cos(x)\,(0) - sin(x)(1)$$

$$\therefore \frac{d}{dx}(cos(x)) = -sin(x)$$

Thus, through first principles it can be concluded that the derivative of $sin(x)$ is $cos(x)$, and the **derivative** of $cos(x)$ is $-sin(x)$.

Applying this to some examples, this means that the **derivatives** for $y = 3sin(x)$, $y = 5cos(x)$ and $y = -cos(x)$ will be:

$$y = 3sin(x) \qquad\qquad y = 5cos(x) \qquad\qquad y = -cos(x)$$

$$\frac{dy}{dx} = 3cos(x) \qquad\qquad \frac{dy}{dx} = -5sin(x) \qquad\qquad \frac{dy}{dx} = sin(x)$$

From Year 11, you may recall that **sine** and **cosine** functions can be written in the **general form** $sin(ax+b)$ and $cos(ax+b)$.

To **differentiate** these functions, we **apply** the **same formulas** we have just used, with the **additional step** of **multiplying** the **derivative** by the **coefficient** a. That is:

If $y = sin(ax+b)$ then $\frac{dy}{dx} = a\,cos(ax+b)$	If $y = cos(ax+b)$ then $\frac{dy}{dx} = -a\,sin(ax+b)$

Applying this to some examples, we can see it is an **almost identical process**:

$$y = -sin(4x) \qquad\qquad y = 2cos(3x) \qquad\qquad y = -cos(2x)$$

$$\frac{dy}{dx} = -4cos(4x) \qquad\qquad \frac{dy}{dx} = -6sin(3x) \qquad\qquad \frac{dy}{dx} = 2sin(2x)$$

Similar to exponentials, we can also apply the **product rule**, **quotient rule** and **chain rule** to **trigonometric functions**.

With regard to **product rule**, let's consider the example of $y = sin(x)(cos(x) + 4)$. Let's have $u = sin(x)$ and $v = cos(x) + 4$. Using the **product rule**, we get:

① $u = sin(x)$ $\quad$ $v = cos(x) + 4$

$$\frac{du}{dx} = cos(x) \qquad \frac{dv}{dx} = -sin(x)$$

② $$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

③ $$\frac{dy}{dx} = (cos(x) + 4)cos(x) + sin(x)(-sin(x))$$

④ $$\frac{dy}{dx} = cos^2(x) + 4cos(x) - sin^2(x)$$

① Determine **derivatives** of u and v

② **Apply** the **product rule**

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

④ **Expand** and **simplify** to the **final derivative**

Using the **quotient rule**, we can find the derivative of $y = \frac{sin(x)}{x^2}$, letting $u = sin(x)$ and $v = x^2$:

① $u = sin(x)$ $\quad$ $v = x^2$

$$\frac{du}{dx} = cos(x) \qquad \frac{dv}{dx} = 2x$$

② $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

③ $$\frac{dy}{dx} = \frac{x^2(cos(x)) - (sin(x))(2x)}{(x^2)^2}$$

④ $$\frac{dy}{dx} = \frac{x^2cos(x) + 2xsin(x)}{x^4}$$

① Determine the **derivatives** of u and v

② **Apply** the **quotient rule**

③ **Substitute** in **values** of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

④ **Expand** and **simplify** to the **final derivative**

Finally, suppose we had the function, $y = sin^3(x)$. To **differentiate** this function we would **apply our understanding** of **chain rule**.

The function $y = sin^3(x)$ can also be written as: $y = \left(sin(x)\right)^3$, which means we can **use** our **shortcut rule** from **chain rule below**:

> **Derived from Chain Rule**: If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

What this tells us is the derivative can be determined by **bringing down** the **power** n and **multiplying** it with the **function**, **subtracting one** from the **power**, and finally **multiplying** the function by the **derivative** of the function **inside** the **brackets** (i.e. $f'(x)$).

Applying this to $y = sin^3(x)$ we get:

$$y = sin^3(x)$$

① $$\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$$

$$\frac{dy}{dx} = 3sin^{3-1}(x) \cdot cos(x)$$

② $$\frac{dy}{dx} = 3sin^2(x)cos(x)$$

① **Apply** the **shortcut rule derived** from **chain rule**

② **Simplify** to get the **final answer**

To instil this concept, let's also apply this formula to the **differentiation** of $y = cos^4(x)$.

$$y = cos^4(x)$$
$$① \quad \frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$$
$$\frac{dy}{dx} = (4)\left(-sin^{4-1}(x)\right)$$
$$② \quad \frac{dy}{dx} = -4sin^3(x)$$

① Apply the shortcut rule derived from chain rule

② Simplify to get the final answer

Finally, we will consider is a **specific example** of the quotient rule – the differentiation of $tan(x)$. By definition, the **tangent function** is simply **sine divided** by cosine. In other words:

$$\boxed{tan(x) = \frac{sin\,(x)}{cos\,(x)}}$$

From this definition, $tan(x)$ can be seen as a function fitting the form of $y = \frac{u}{v}$, and hence we can apply the quotient rule to **differentiate** $tan(x)$ as follows:

$$y = tan(x)$$
$$① \quad y = \frac{sin\,(x)}{cos\,(x)}$$
$$② \quad u = sin(x) \qquad v = cos(x)$$
$$\frac{du}{dx} = cos(x) \qquad \frac{dv}{dx} = -sin(x)$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$③ \quad \frac{dy}{dx} = \frac{cos(x)(cos(x)) - (sin(x))(-sin(x))}{(cos(x))^2}$$
$$\frac{dy}{dx} = \frac{cos^2(x) + sin^2(x)}{cos^2(x)}$$
$$④ \quad \frac{dy}{dx} = \frac{1}{cos^2(x)}$$

① Apply the identity of $tan(x)$

② Determine the derivatives of u and v

③ Substitute in values of u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$

④ Apply the identity: $cos^2(x) + sin^2(x) = 1$ to get the final derivative

Integration of Trigonometric Functions

Integrating sine and cosine functions is simply the reverse of **differentiation**. That is, $sin(x)$ **integrates** to become $-cos(x)$, and $cos(x)$ integrates to become $sin(x)$. These formulas can be summarised as:

$$\boxed{\int sin(x)\,dx = -cos(x) + C} \qquad \boxed{\int cos(x)\,dx = sin(x) + C}$$

When the **trigonometric functions** are in the form $sin(ax + b)$ or $cos(ax + b)$ the only **additional step** is to **divide the integral** by a. These integrals can be summarised as follows:

$$\boxed{\int sin(ax + b)dx = -\frac{1}{a}cos(ax + b) + C} \qquad \boxed{\int cos(ax + b)dx = \frac{1}{a}sin(ax + b) + C}$$

Let's **apply** these formulas to an **array of examples**. To begin, let's **integrate** $\frac{dy}{dx} = 4sin(x)$ and $f'(x) = 5cos(x)$.

$$\frac{dy}{dx} = 4sin(x)$$
$$y = \int 4sin(x)\,dx$$
$$y = -4cos(x) + C$$

$$f'(x) = 5cos(x)$$
$$f(x) = \int 5cos(x)\,dx$$
$$f(x) = 5sin(x) + C$$

Taking it up a notch, let's **integrate** $\frac{dy}{dx} = -5sin(5x)$ and $f'(x) = cos(4x+6)$.

$$\frac{dy}{dx} = -5sin(5x)$$
① $$y = \int -5sin(5x)\,dx$$
$$y = \frac{1}{5}\left(5cos(5x)\right) + C$$
② $$y = cos(5x) + C$$

$$f'(x) = cos(4x+6)$$
① $$f(x) = \int cos(4x+6)\,dx$$
② $$f(x) = \frac{1}{4}sin(4x+6) + C$$

① **Apply** the **integration formula** from **above**

② **State** the **final answer**

As we close out this short chapter, we don't need to explore in detail the **applications of trigonometric calculus** because they are **identical** to the **polynomial applications** we explored in **chapters 1 and 2**.

From **small change** to **rectilinear motion** and **area between curves**, these are concepts we have already learnt about. As a result, we don't need to cover them in detail here – you'll get the chance to **practice applying** these concepts to **trigonometric functions** in the **problem set**.

So, congratulations! We have officially **come to the end of calculus** for **Unit 3**. We will return with **logarithms** in **Unit 4**. Pat yourself on the back and celebrate by doing the **trigonometric problem set**!

Practice Example 1

Fraser fell asleep during class so he is a little lost on the **trigonometric content**. He remembers the calculus content but needs some help **applying** it to **trigonometric functions**.

(a) **Help** Fraser to **determine** the **derivatives** of the following trigonometric functions.

(i) $y = sin(2x)$

$$\frac{dy}{dx} = 2cos(2x)$$

(ii) $y = -cos\left(\frac{1}{6}x\right)$

$$\frac{dy}{dx} = \frac{1}{6}sin(\frac{1}{6}x)$$

(iii) $y = sin^4(x)$

$$\frac{dy}{dx} = 4sin^{4-1}(x) \cdot cos(x)$$
$$\frac{dy}{dx} = 4sin^3(x)cos(x)$$

(b) Now **help** Sparrow to **integrate** of the following functions.

(i) $f'(x) = 4cos(2x)$

$$f(x) = \int 4cos(2x)\,dx$$
$$f(x) = 2sin(2x) + C$$

(ii) $f'(x) = sin(2x-7)$

$$f(x) = \int sin(2x-7)\,dx$$
$$f(x) = -\frac{1}{2}cos(2x-7) + C$$

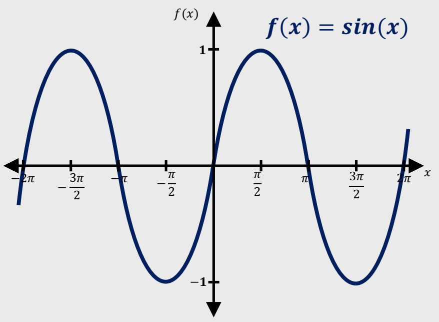
(iii) $f'(x) = -cos(8x)$

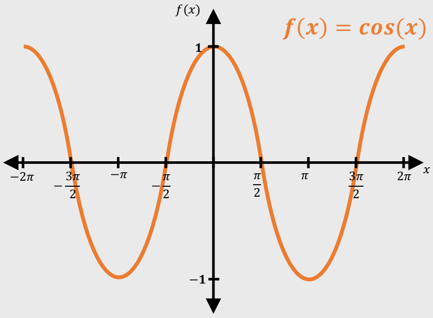
$$f(x) = \int -cos(8x)\,dx$$
$$f(x) = -\frac{1}{8}sin(8x) + C$$

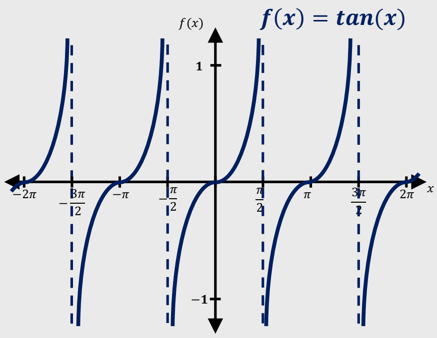
TRIGONOMETRIC FUNCTIONS TOPIC NOTES

Trigonometric Functions

In Year 11 we looked at trigonometric functions in some detail. However, for Year 12 you will need to know how to **differentiate** and **integrate** trigonometric functions! The following graphs show each trigonometric function.







Differentiating Trigonometric Functions

For **differentiating trigonometric functions**, the **derivative** for $sin(x)$ is $cos(x)$, and the **derivative** for $cos(x)$ is $-sin(x)$:

If $y = sin(x)$ then $\frac{dy}{dx} = cos(x)$

If $y = cos(x)$ then $\frac{dy}{dx} = -sin(x)$

To **differentiate trigonometric functions** of the form: $sin(ax + b)$ and $cos(ax + b)$, we **apply** the **same formulas** we have just used, but with the **additional step** of **multiplying** the **derivative** by the **coefficient** a.

If $y = sin(ax + b)$ then $\frac{dy}{dx} = a\,cos(ax + b)$

If $y = cos(ax + b)$ then $\frac{dy}{dx} = -a\,sin(ax + b)$

We can use the **product rule**, **quotient rule** and **chain rule** to differentiate trigonometric functions. The following rule is derived from the **chain rule** for functions such as $y = sin^3(x)$ and $y = cos^4(x)$.

Derived from Chain Rule: If $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

Also remember that $tan(x)$ is equal to $\frac{sin(x)}{cos\,(x)}$ and you apply the **quotient rule** to **differentiate** it.

$$tan(x) = \frac{sin\,(x)}{cos\,(x)}$$

Integrating Trigonometric Functions

Integrating sine and **cosine** functions is simply going to be the **reverse** of **differentiation**. Remember to always include the **constant** C.

That is $\boldsymbol{sin(x)}$ **integrates** to become $\boldsymbol{-cos(x)}$, and $\boldsymbol{cos(x)}$ integrates to become $\boldsymbol{sin(x)}$. These formulas can be summarised as:

$$\boxed{\int \boldsymbol{sin(x)\, dx = -cos(x) + C}}$$

$$\boxed{\int \boldsymbol{cos(x)\, dx = sin(x) + C}}$$

For **trigonometric functions** of the form $\boldsymbol{sin(ax+b)}$ or $\boldsymbol{cos(ax+b)}$ the only **additional step** is to **divide the integral** by $\boldsymbol{a}$.

$$\boxed{\int \boldsymbol{sin(ax+b)dx = -\frac{1}{a}cos(ax+b) + C}}$$

$$\boxed{\int \boldsymbol{cos(ax+b)dx = \frac{1}{a}sin(ax+b) + C}}$$

Problem Set 6 – Trigonometric Functions Progressive Questions

Concept 1

Differentiation and Integration of Trigonometric Functions – Progressive Questions (8 questions)

Repetitive questions: 1.11 → 1.81 (7 questions)

A Trip to Trigonometric Mountain!

Starring: Janitor Peter, Rabea, Harriet, Tyler, Fraser, Alexa and Rupert

---

"There it is!" exclaims Rabea.

"Students this is Trigonometric Mountain! The biggest theme park in our universe with rides that defy gravity and follow trigonometric functions, and a wave park with huge trigonometric waves." says Janitor Peter.

The students are over the moon with excitement! They grab their astronaut helmets and run off their spaceship ecstatic about all the fun they will have on Trigonometric Rides.

Differentiation of Exponential Functions: Q1, Q2, Q3, Q4

Repetitive: 1.11 → 1.31 (3 questions)

[20 marks]

1. Before the students can enter the theme park, the entry gate requires the students to answer a series of questions. Help the students **differentiate** the following **trigonometric functions**!

(a) $f(x) = sin(x)$ **(2)**

(b) $y = cos(x)$ **(2)**

(c) $y = 5sin(x)$ **(2)**

(d) $f(x) = 6cos(2x)$ **(2)**

(e) $f(x) = -\cos(3x)$ **(2)**

(f) $f(x) = -\frac{1}{2}sin(2x)$ **(2)**

(g) $y = -4\,cos(4x - 9)$ **(2)**

(h) $f(x) = 2sin\,(-x)$ **(2)**

(i) $y = cos\,(7x - 4)$ **(2)**

(j) $f(x) = -12sin(3x)$ **(2)**

[20 marks]

2. Looking at the high prices of the entry tickets, Rabea decides to bet the gatekeeper that he can **differentiate trigonometric functions** faster than him for a free ticket. The gatekeeper agrees! Help Rabea to **differentiate** the following **trigonometric functions** faster than the gatekeeper

(a) $f(x) = 2sin\,(-2x)$ **(2)**

(b) $y = \frac{1}{2}cos(4 - 8x)$ **(2)**

(c) $f(x) = 3sin\,(5 - 2x)$ **(2)**

(d) $f(x) = sin(2x) - cos(x)$ **(2)**

(e) $f(x) = 2\,sin^3(x)$ **(2)**

(f) $f(x) = 4\,cos^2(x)$ **(2)**

(g) $y = -5cos\,(-5x - 8)$ **(2)**

(h) $y = \sqrt{sin(x)} + x^2$ **(2)**

(i) $f(x) = -\,sin^2(x)$ **(2)**

(j) $y = -\cos\,(9x) + \frac{3}{x^3}$ **(2)**

[35 marks]

3. Inside the park, Harriet and Tyler are both keen surfers so they run over to the **wave park**. Before stepping into the water, they have the option to **adjust** the **shape of the waves** based on a series of **trigonometric functions**. To make sure they select a wave with a **slower rate of change** they need to **differentiate** the functions provided. Help Harriet and Tyler to use the **product rule** and **quotient rule** to find the **derivatives** of the following **trigonometric functions**.

(a) $f(x) = xsin(x)$ **(3)**

(b) $y = \frac{cos(x)}{x^2}$ **(3)**

(c) $f(x) = cos(2x)x$ **(3)**

(d) $f(x) = \frac{sin\,(x)}{cos\,(x)}$ **(3)**

(e) $f(x) = cos(x)(sin(2x) - 9)$ **(3)**

(f) $f(x) = \frac{1-cos(x)}{2x}$ **(4)**

(g) $y = tan(x)$ **(4)**

(h) $y = sin(2x)\sqrt{x}$ **(4)**

(i) $f(x) = x^3\cos(2x - 1)$ **(4)**

(j) $y = \frac{sin\,(6x-2)}{x-1}$ **(4)**

[CA] [15 marks]

4. Fraser and Alexa want to pick the roller coaster with the **fastest instantaneous rate of change**. They are given a **list of trigonometric functions** that govern roller coaster rides. Help them to determine the **gradient** of the equation at the **following values** to help them find the fastest ride!

(a) Determine the **gradient** of $\boldsymbol{f(x) = sin(x)}$ at the **point** $\boldsymbol{(\pi, 0)}$ **(3)**

(b) Determine the **gradient** of $\boldsymbol{f(x) = cos(x)}$ at the **point** $\boldsymbol{(2\pi, 1)}$ **(3)**

(c) Determine the **gradient** of $\boldsymbol{f(x) = sin(4x) + 2}$ at the **point** $\boldsymbol{(\frac{3\pi}{4}, 2)}$ **(3)**

(d) Determine the **gradient** of $\boldsymbol{y = -cos(4x)}$ at the **point** $\boldsymbol{(0, 1)}$ **(3)**

(e) Determine the **gradient** of $\boldsymbol{y = tan(x)}$ at the **point** $\boldsymbol{(\pi, 0)}$ **(3)**

Integration of Exponential Functions: Q5, Q6, Q7, Q8

Repetitive: 1.51 → 1.81 (4 questions)

[25 marks]

5. Rupert is eager to ride the biggest ride at the park known as the 'Vomit Comet Flyer'. To enter the ride they are required to **prove they are worthy**. Rupert is given a **series of integrals** of **trigonometric functions**. Help him to **solve them** so that he can go on the ride.

(a) $\int sin(x)\,dx$ **(2)**

(b) $\int cos(2x)dx$ **(2)**

(c) $\int 4sin(4x)dx$ **(2)**

(d) $\int -3cos(2x)dx$ **(2)**

(e) $\int 2cos(2x + 5)dx$ **(2)**

(f) $\int -2cos(2x) - sin(x)dx$ **(3)**

(g) $\int \frac{1}{4}sin\left(\frac{1}{2}x\right) - x^2\,dx$ **(3)**

(h) $\int -sin(2x - 4)dx$ **(3)**

(i) $\int 7cos\,(-x)dx$ **(3)**

(j) $\int -\frac{1}{8}sin(x)dx$ **(3)**

[26 marks]

6. Hanging out at the wave park, the machine that Harriet and Tyler used to set the shape of the waves they would surf has malfunctioned. It starts spitting out waves of random sizes. So they can know the **size of the waves** as they approach, help Harriet and Tyler to solve to **integrate** the following **gradient functions** of the waves approaching.

(a) $\int -2sin(4x)\,dx$ **(2)**

(b) $\int -3cos(7x-2)dx$ **(2)**

(c) $\int cos(3x) - \frac{1}{2}sin(2x)dx$ **(3)**

(d) $\int -4sin(\frac{4}{3}x)dx$ **(2)**

(e) $\int \frac{1}{2}cos(\frac{1}{2}x)\,dx$ **(2)**

(f) $\int sin(x) + \frac{1}{2}cos(2x)\,dx$ **(3)**

(g) $\int \frac{1}{2}cos(4x) - \frac{7}{2}sin(x)dx$ **(3)**

(h) $\int -6cos(8x) - \frac{1}{\sqrt{x}}dx$ **(3)**

(i) $\int cos(-x) - \frac{2}{3x^3} - 2dx$ **(3)**

(j) $\int -\frac{1}{8}sin\left(\frac{1}{4}x\right) - 2\sqrt{x}dx$ **(3)**

[30 marks]

7. Walking around the theme park, Janitor Peter notices an incredible rollercoaster that shape shifts into **any trigonometric function** you can think of. Before riding this rollercoaster, Janitor Peter wants to make sure his **integration skills** are still up to scratch. **Help** Janitor Peter to **determine** the following **definite integrals**.

(a) $\int_0^{\pi} sin(x)\,dx$ **(3)**

(b) $\int_{-\pi/2}^{\pi/2} cos(x)\,dx$ **(3)**

(c) $\int_0^{\pi/6} cos(6x)\,dx$ **(3)**

(d) $\int_{-2\pi}^{2\pi} -sin(\frac{1}{2}x)\,dx$ **(3)**

(e) $\int_0^{\pi} sin(x-\pi)\,dx$ **(3)**

(f) $\int_{-2\pi}^{\pi} -2cos(\frac{1}{2}x)\ dx$ **(3)**

(g) $\int_{\pi}^{0} sin(3x)\,dx$ **(3)**

(h) $\int_{\pi/4}^{-\pi/4} 2sin(4x)\,dx$ **(3)**

(i) $\int_{\pi}^{2\pi} -cos(x-\frac{\pi}{2})\ dx$ **(3)**

(j) $\int_{\pi/2}^{\pi} -sin(2x-\pi)\,dx$ **(3)**

[CA] [15 marks]

8. Still continuing to release randomly shaped waves, Tyler sees this as a perfect opportunity to practice his **integration skills** by **determining the equations** governing the approaching waves. **Help** Tyler to do this by **finding $f(x)$** for each of the following **gradient functions**.

(a) Determine $f(x)$ if $f'(x) = sin(x)$ and $f(x) = -2$ at $x = 0$ **(3)**

(b) Determine $f(x)$ if $f'(x) = -2cos(x)$ and $f(x)$ intersects the point $(-\frac{\pi}{2}, 4)$ **(3)**

(c) Determine $f(x)$ if $f'(x) = -sin(2x)$ and $f(x) = -1$ at $x = \frac{\pi}{4}$ **(3)**

(d) Determine $f(x)$ if $f'(x) = cos(x-\frac{\pi}{2})$ and intersects the point $(-\pi, 0)$ **(3)**

(e) Determine $f(x)$ if $f'(x) = sin(x+\pi)$ and intersects y-axis at $f(x) = 3$ **(3)**

Concept 2

Applications of Trigonometric Functions – Progressive Questions (6 questions)

Repetitive questions: 2.11 → 2.61 (6 questions)

Escape from Trigonometric Mountain!

Starring: Janitor Peter, Rabea, Harriet, Tyler, Fraser, Alexa and Rupert

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After an incredible day at Trigonometric Mountain the students are exhausted and eager to return home to Earth. However, when they get back to the ship they realise that key parts of the ship were damaged on landing.

In order to get back home safely, they will first need to repair the damaged components on their ship…

Applications of Trigonometric Functions: Q1, Q2, Q3, Q4, Q5, Q6

Repetitive: 2.11 → 2.61 (6 questions)

[CA] [16 marks]

1. Instead of using exponential functions to construct replacement parts for the spaceship, Rupert decides to construct pieces in the shapes of **areas under trigonometric functions**. To determine **how much material** Rupert will need to construct the parts, **help** Rupert to calculate the following **areas**:

(a) Calculate the **area under the curve** $y = sin(x)$ between $x = 0$ and $x = \pi$ **[4]**

(b) Calculate the **area under the curve** $y = cos(2x)$ between $x = 0$ and $x = \frac{\pi}{2}$ **[4]**

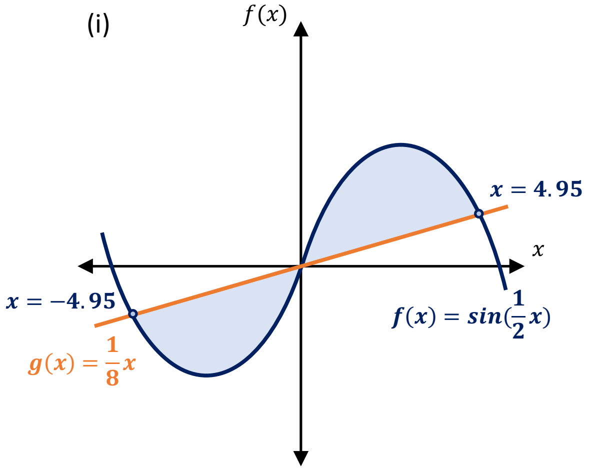
(c) Calculate the **area under the curve** $y = sin\left(\frac{1}{2}x\right) + 2$ from $x = -\pi$ to $x = \pi$ **[4]**

(d) Calculate the **area under the curve** $y = cos\left(\frac{1}{2}x\right)$ from $x = -2\pi$ to $x = 2\pi$ **[4]**

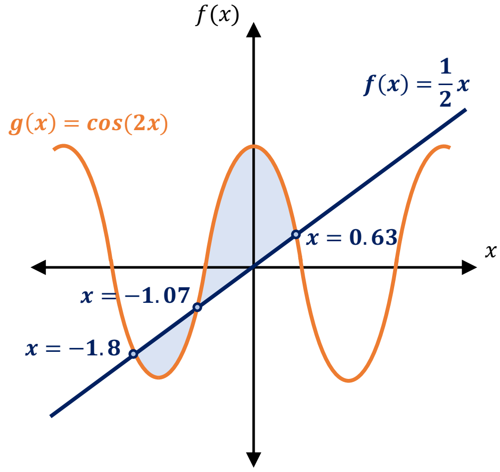
[CA] [16 marks]

2. For some additional parts that will go on the exterior of the spaceship, help Rupert to **determine** the **shaded areas below** using **definite integrals**.

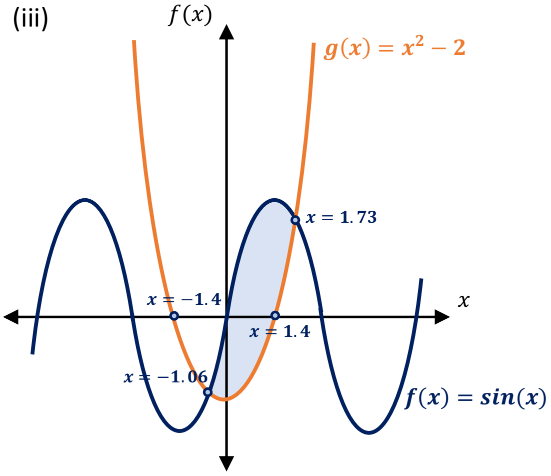
(i)



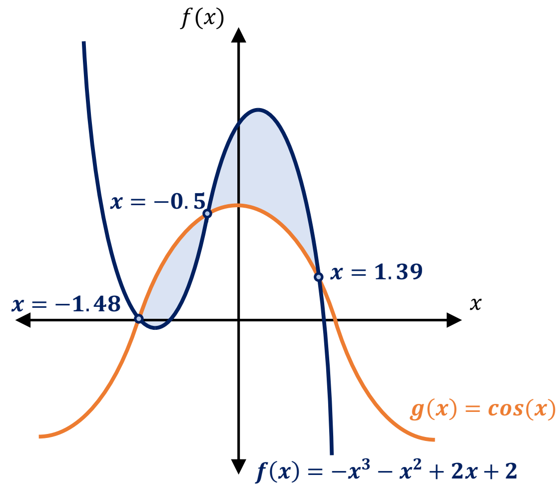
(ii)



(iii)



(iv)



[CA] [9 marks]

3. With a modified spaceship and a different gravity field from being on a different planet, the students determine that the spaceship will have a **launch acceleration** that follows the **equation:** $a = -2\cos(2t)$, where t is **time** in **seconds**. The **initial position** of the spaceship is **15 metres** above the ground and the **initial launch velocity** is $20m/s$.

(a) What is the **velocity** of the spaceship after **six seconds**? **[1]**
(b) What will be the **acceleration** of the spaceship **one second** after take-off? **[2]**
(c) What will be the **displacement** of the spaceship after **9 seconds**. **[3]**
(d) What will be the **net change** in **displacement** of the spaceship during the **ninth second**. **[3]**

[CA] [11 marks]

4. To make the spaceship hit record speeds, the students decide to **install an additional engine** into the spaceship! The students purchase an engine that allows the spaceship to follow the **velocity equation**: $v = sin(2t) + 3\,cos(2t) + 30$, where t is **time** in **seconds**. The **initial displacement** of the rocket when this **engine turns on** is **200 metres**.

(a) What will be the **velocity** of the students after **12 seconds**? **[2]**
(b) What is the **initial acceleration** of the rocket when this engine turns on? **[2]**
(c) What will be the **net change** in **displacement** of the spaceship during the $12th$ **second**? **[2]**
(d) What will be the **displacement** of the students in spaceship after **30 seconds**. **[3]**
(e) If the ship reaches **deep space** after **2 minutes**, what **speed** will they travel to Earth at? **[2]**

[CA] [11 marks]

5. As they are approaching Earth the students have realised that their **fuel reserves** are **depleting** at a **rate** of: $\frac{dA}{dt} = 100cos\left(\frac{1}{16}t\right)$, where A is the **amount of fuel lost** in **litres** and t is **time** in **hours**.

(a) What will be the **instantaneous rate of change** in the **amount of fuel** after **two hours**? **[2]**
(b) What is the **instantaneous rate of change** in the **amount of fuel** at the **twentieth hour**? **[2]**
(c) What is the **net change** over the first **hour** in the amount of fuel? **[2]**
(d) What is the **net change** in the **amount of fuel** during the **tenth hour**? **[2]**
(e) If there are **2000 litres of fuel** in the tank, **how many hours** does the **amount of fuel** reach **zero**? **[3]**

[CA] [9 marks]

6. The students also realise that they are having a **water shortage** on their trip back to Earth. The **volume of water** remaining is depleting at a rate of $\frac{dA}{dt} = cos(x) - x + 100$, where V is **volume** in **litres** and t is **time in hours**.

(a) At what **instantaneous rate** is the **water storage depleting initially**? **[1]**
(b) What is the **instantaneous rate of change** in the **amount of water** after **ten hours**? **[2]**
(c) What is the **net change** in the **volume of water** over the during the **fifth hour**? **[2]**
(d) What will be the **instantaneous rate of change** in the **volume of water** after **two hours**? **[2]**
(e) If there is 5000 **litres of water**, after **how many hours** will they have **run out of water**? **[2]**

Making it back to Earth with just enough fuel and water, the Maths Methods students are relieved to be safely on the bus back to Mathematics College.

When the students arrive back at the college, Rabea notices that there are flyers are hanging everywhere. He grabs one of the flyer's and calls over all of the other students.

"Hey team, come take a look at this! The Maths Applications students are putting on a carnival this weekend to make some money! " shouts Rabea

"Looks like we are going to have to use probability to turn the odds in our favour to make sure that we win all of the prizes!" grievously says Rupert.

All of the methods students break out into laughter excited to use probability to take down the Maths Applications students once again…

Problem Set 6 – Trigonometric Functions
Repetitive Questions

Concept 1
Differentiation and Integration of Trigonometric Functions – Repetitive Questions (7 questions)

Differentiation of Trigonometric Functions: Qs 1.11, 1.21, 1.31

[20 marks]

1.11 Having somehow fallen behind in class, **help** star student Tyler to **differentiate** the following **trigonometric functions**.

(a) $f(x) = 2sin(x)$ **(2)**

(b) $y = \frac{1}{2}cos(2x)$ **(2)**

(c) $y = -cos(2x)$ **(2)**

(d) $f(x) = \frac{1}{2}sin(5x-2)$ **(2)**

(e) $f(x) = -\frac{1}{2}sin\left(\frac{1}{2}x\right)$ **(2)**

(f) $f(x) = \frac{1}{4}sin(3x-9)$ **(2)**

(g) $y = -9\,sin\left(\frac{1}{2}x - 9\right)$ **(2)**

(h) $f(x) = 5cos(-3x)$ **(2)**

(i) $y = sin(-3x+9)$ **(2)**

(j) $f(x) = -12sin(-\frac{1}{4}x)$ **(2)**

[25 marks]

1.21 Continuing to push himself, **help** Tyler to **differentiate** the following more difficult **functions**.

(a) $f(x) = sin^3(x)$ **(2)**

(b) $y = \frac{1}{2\sqrt{x}} - cos(x)$ **(2)**

(c) $f(x) = 4sin(3-3x)$ **(2)**

(d) $f(x) = 4cos^2(x)$ **(2)**

(e) $f(x) = -5cos\,(-3x)$ **(2)**

(f) $f(x) = 3\,sin^4(x) - 2x$ **(3)**

(g) $y = \sqrt{cos(x)} - \frac{2}{x^4}$ **(3)**

(h) $y = cos^3(x) - e^{2x}$ **(3)**

(i) $f(x) = -cos(-8x) + \frac{2}{3}x^3$ **(3)**

(j) $y = cos^4(x) - \frac{1}{x^{2/3}}$ **(3)**

[35 marks]

1.31 Determine the **derivatives** of the following functions using the **product rule** and **quotient rule**.

(a) $f(x) = 2xcos(x)$ **(3)**

(b) $y = \frac{sin(2x)}{e^x}$ **(3)**

(c) $f(x) = \frac{2cos(x)}{x^3}$ **(3)**

(d) $f(x) = \frac{5x-2}{sin(x)}$ **(3)**

(e) $f(x) = cos(x-9)e^{5x}$ **(3)**

(f) $f(x) = 2x^3cos(3x-1)$ **(4)**

(g) $y = e^x sin(-2x)$ **(4)**

(h) $y = \frac{\sqrt{x}}{cos(x)}$ **(4)**

(i) $f(x) = \frac{sin(4x)}{6x-3}$ **(4)**

(j) $y = sin(2x)x^{\frac{3}{2}x}$ **(4)**

Integration of Trigonometric Functions: Qs 1.51, 1.61, 1.71, 1.81

[25 marks]

1.51 Rabea is eager to join in with Tyler's study session. Help them to **integrate** the following **trigonometric functions**.

(a) $\int -cos(x)\, dx$ **(2)**

(b) $\int sin(3x)dx$ **(2)**

(c) $\int 2cos(5x)dx$ **(2)**

(d) $\int -5sin(-3x)dx$ **(2)**

(e) $\int sin(-8x-9)dx$ **(2)**

(f) $\int sin(2x) - cos(\frac{1}{2}x)dx$ **(3)**

(g) $\int 2sin(4x) - x^3 dx$ **(3)**

(h) $\int -cos(3x-2)dx$ **(3)**

(i) $\int sin(-x) + x^2 dx$ **(3)**

(j) $\int -\frac{1}{8}sin(\frac{1}{4}x)dx$ **(3)**

[27 marks]

1.61 Eager to keep going, **help** them to **integrate** the following functions.

(a) $\int -\frac{1}{2}sin(-4x)\, dx$ **(2)**

(b) $\int -2cos(4x-6)dx$ **(2)**

(c) $\int sin(2x) - \frac{1}{6}cos(3x)dx$ **(3)**

(d) $\int -4sin(2x) + \sqrt{x}\, dx$ **(3)**

(e) $\int -5sin(\frac{1}{5}x)\, dx$ **(2)**

(f) $\int cos(2x) + \frac{1}{2x^3}dx$ **(3)**

(g) $\int \frac{1}{2}sin(-\frac{1}{3}x)dx$ **(3)**

(h) $\int -3sin(5x) - \frac{1}{\sqrt{x}}dx$ **(3)**

(i) $\int -2cos(-2x) + 2dx$ **(3)**

(j) $\int -\frac{1}{8}cos\,(\frac{1}{2}x)dx$ **(3)**

[30 marks]

1.71 Help Tyler and Rabea to **solve** the following **definite integrals** that were set for homework

(a) $\int_0^{\pi/2} cos(x)\, dx$ **(3)**

(b) $\int_0^{\pi} sin(x)\, dx$ **(3)**

(c) $\int_0^{\pi/6} cos(3x)\, dx$ **(3)**

(d) $\int_{-2\pi}^{\pi} -sin(\frac{1}{4}x)\, dx$ **(3)**

(e) $\int_{-3\pi/2}^{-\pi/2} cos(x+\pi)\, dx$ **(3)**

(f) $\int_{-\pi/4}^{\pi/4} 3cos(2x)\ dx$ **(3)**

(g) $\int_0^{3\pi/2} \frac{1}{3}cos(x)\, dx$ **(3)**

(h) $\int_{\pi/3}^{0} -sin(3x)\, dx$ **(3)**

(i) $\int_{-\pi/3}^{-\pi/6} sin(3x-\pi)\, dx$ **(3)**

(j) $\int_{\pi}^{\pi/4} cos(2x - \frac{\pi}{2})\, dx$ **(3)**

[CA] [12 marks]

1.81 The second part of the homework involves determining the **using integration** to **determine** $f(x)$. **Help** Rabea and Tyler to do **determine** $f(x)$ for each of the following scenarios.

(a) Determine $f(x)$ if $f'(x) = sin(x)$ and $f(x) = \pi$ at $x = 2$ **[3]**

(b) Determine $f(x)$ if $f'(x) = cos(\frac{1}{2}x)$ and $f(x)$ intersects *y*-axis at $f(x) = -2$ **[3]**

(c) Determine $f(x)$ if $f'(x) = 2sin(4x)$ and $f(x) = \pi/4$ at $x = 0.5$ **[3]**

(d) Determine $f(x)$ if $f'(x) = \frac{1}{2}sin(x-\pi)$ and intersects the point $(\frac{\pi}{2}, -1.5)$ **[3]**

Concept 2

Applications of Trigonometric Functions – Repetitive Questions (6 questions)

General Applications of Exponential Functions: 2.11, 2.21, 2.31, 2.41, 2.51, 2.61

[CA] [16 marks]

2.11 Star student Kerry is looking to build her very own motorbike for when she graduates from Mathematics College. To create key external parts, Kerry decides to is using the shapes created by **exponential functions** and the **x-axis**. To determine **how much material she will need** for each of the parts, **help** Kerry to **calculate** the **following areas**:

(a) Calculate the **area under the curve** $y = cos(x)$ between $x = 0$ and $x = \frac{3\pi}{2}$ **[4]**

(b) Calculate the **area under the curve** $y = -sin(\frac{1}{2}x)$ between $x = -2\pi$ and $x = 0$ **[4]**

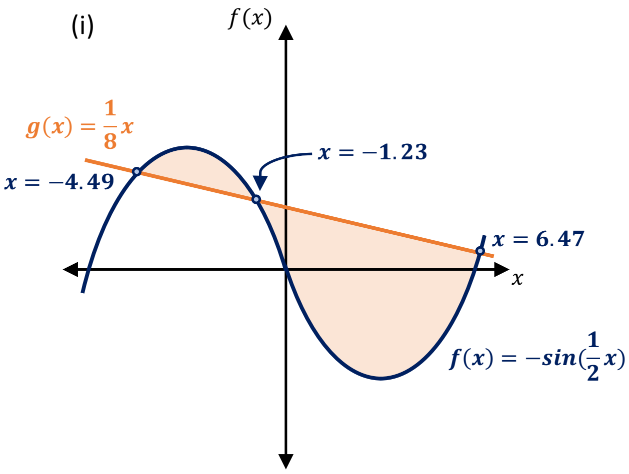
(c) Calculate the **area under the curve** $y = \frac{1}{4}cos(2x)$ from $x = 0$ to $x = \pi$ **[4]**

(d) Calculate the **area under the curve** $y = 2 + sin(x)$ from $x = 0$ to $x = \frac{3\pi}{2}$ **[4]**

[CA] [16 marks]

2.21 Kerry also needs to create some more complicated internal components for her motorbike's engine. To determine **how much material** she will need for each component, help Kerry to **determine** the **shaded areas below** using **definite integrals**.

(i)



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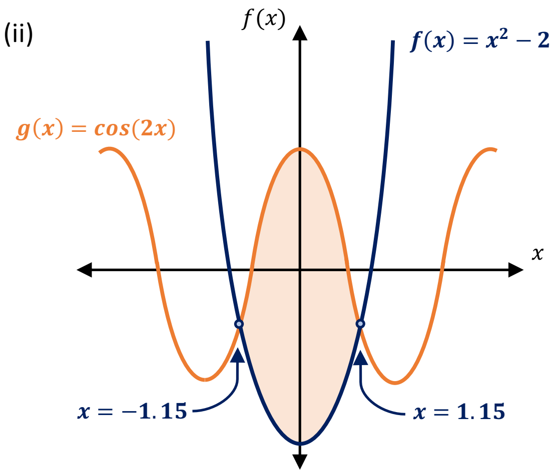

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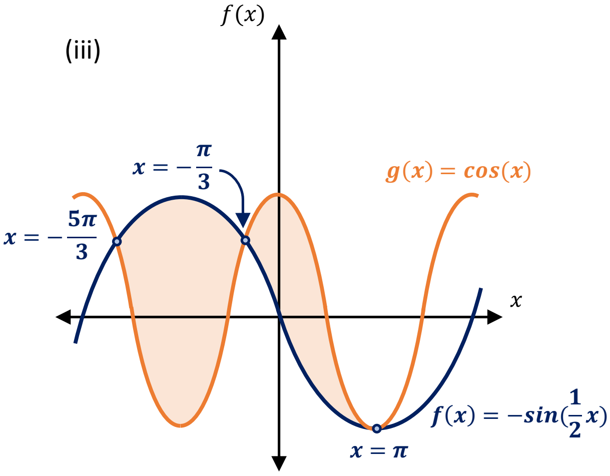
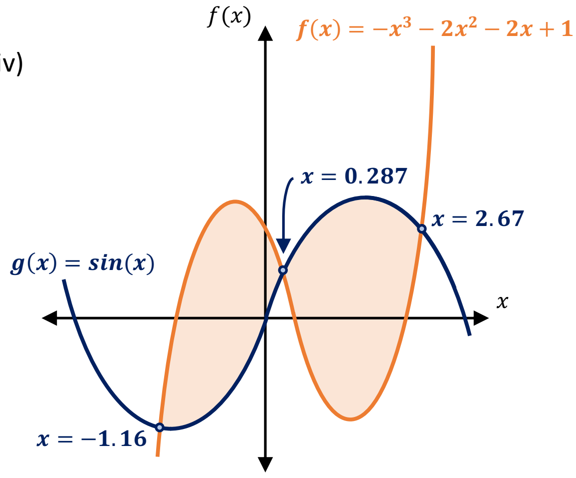


(iv)



[CA] [9 marks]

2.31 Kerry begins experimenting with trigonometric functions in relation to **rectilinear motion**. Along a lake she **grabs a stone** and throws it such that it has a **velocity** that follows the **velocity equation**: $v = cos(4t) + 3sin(2t) + 3$, where v is in m/s and t is **time** in **seconds**. The **initial displacement** of the stone is **3 meters**.

(a) What is the **initial velocity** of the stone? **[1]**
(b) What will the **velocity** of the stone be **after four seconds**? **[1]**
(c) What will the **acceleration** of the stone be **after three seconds**? **[2]**
(d) After **four seconds**, what is the **displacement** of the stone? **[2]**
(e) What is the **net change** in the **displacement** of the stone in the **fifth second**? **[3]**

[CA] [13 marks]

2.41 Having fun throwing stones with trigonometric patterns, Kerry picks up another stone and throws it such that it follows the **acceleration equation**: $a = -10sin(4t)$, where t is **time** in **seconds**. The **initial position** of the stone is **1 meter** above the ground and the **initial velocity** is $4m/s$.

(a) What will be the **acceleration** of the stone after **one second**? **[1]**
(b) What is the equation tracking the **velocity** of the stone? **[2]**
(c) What will be the **net change** in the **displacement** of the stone **during the third second**? **[3]**
(d) What will be the **net change** in the **distance** of the stone **during the first three seconds**? **[3]**
(e) What is the **acceleration**, **velocity** and **displacement** of the stone **after six seconds**? **[4]**

[CA] [9 marks]

2.51 After having fun with rectilinear motion, Kerry decides to head home. The **volume of fuel** left in the **family car tank** is tracked by the equation: $\frac{dA}{dt} = 50sin\left(\frac{1}{8}t\right)$, where A is the **amount of fuel** in **litres** and t is **time** in **hours** between

(a) What will be the **instantaneous rate of change** in the **amount of fuel** after **30 minutes**? **[2]**
(b) What is the **instantaneous rate of change** in the **amount of fuel** after **one hour**? **[2]**
(c) What is the **net change** during the **second hour** in the amount of fuel? **[2]**
(d) If there was **100 litres**, after **how many hours** does the **amount of fuel** reach **zero litres**? **[3]**

[CA] [10 marks]

2.61 In a swimming pool on a hot day, the **height** of the pool changes as at a **rate** based on the equation: $\frac{dH}{dt} = cos\left(\frac{1}{8}x\right) + 2$, where H is **height** in **centimetres** and t is **time in hours**.

(a) What is the **instantaneous rate of change** in the **height** of the pool after **one hour**? **[1]**

(b) What is the **net change in the height of the pool during the third hour**? **[2]**

(c) What will be the **instantaneous rate of change** in the **volume of water** after **36 hours**? **[2]**

(d) After **how many hours** will the pool reach a **minimum height**? **[3]**

(e) What is the **net change** in the **height** of the pool during the **first 10 hours**? **[2]**

Chapter 5

Discrete Random Variables

Chapter 5 – Discrete Random Variables

Introduction

As our journey through calculus temporarily comes to a close for Unit 3, we will now **dive** into the **world of probability**.

Probability is a big topic with many areas, but the main aim of this topic is to take everyday **events** and establish the **probability distribution** of the outcomes that can occur. It is a topic often feared by students because it uses lots of uncommon terminology, definitions, and notations.

Similar to calculus, we hope to walk you through probability in a way that helps you to **understand** where all of the concepts and equations come from. By exploring probability in this way, we hope you will come to the end of your journey feeling like you too could have **invented** the various areas of probability.

There are **five key areas** to our Year 12 journey of probability: **discrete random variables**, **Bernoulli and binomial distributions**, **continuous random variables**, **normal distributions** and **sample proportions**.

In this first chapter on **discrete random variables**, we are going to be exploring the probability of events that occur in **discrete amounts**. From **rolling dice** and **dealing cards**, to **calculating profits** and **carnival games**, the journey into discrete random variables is going to be an exciting one!

Probabilities on Dealing Cards and Rolling Dice

Profits on Carnival Games

Marbles in a Bag

For discrete random variables, we have divided the chapter into **four main concepts**:

1. **Random Variables**
2. **Discrete Random Variables**
3. **Mean, Variance and Standard Deviation of Discrete Random Variables**
4. **Applications of Discrete Random Variables**

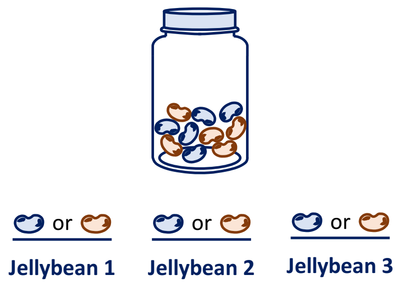
Let's Begin!

5.1 Random Variables

A **random variable** is a **variable X** that takes on the **possible outcomes** of an **event x**, such as the **random variable X** representing the **value rolled** on a **six-sided die** (i.e. $x = 1, 2, 3, 4, 5$ or 6).

For any event that takes place, a **probability distribution** of a **random variable** tells us what the **possible values** are for **the outcome x,** and the **probabilities** of those possible outcomes occurring.

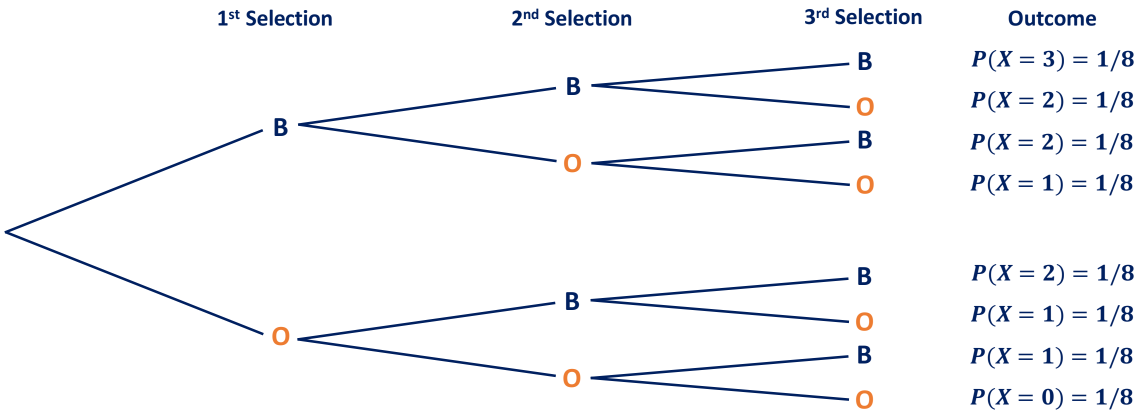
For instance, suppose we have **10 jellybeans** in a jar, **five blue** and **five orange**, and we were to **draw three jellybeans** with **replacement**, meaning we put the jellybeans back after checking to see what colour we took out. The **possible outcomes** of this experiment are:



Possible Outcomes

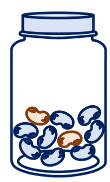
1. **Blue, Blue, Blue**
2. **Blue, Blue, Orange**
3. **Blue, Orange, Blue**
4. **Orange, Blue, Blue**
5. **Blue, Orange, Orange**
6. **Orange, Blue, Orange**
7. **Orange, Orange, Blue**
8. **Orange, Orange, Orange**

If the **random variable X** represents the **number** of **blue jellybeans** selected then X can take on the **values** of $x = 0, 1, 2$ or 3. Since there is a **0.5 probability** of **selecting** a **blue** or **orange** jellybean, from the **tree diagram** below, we can see there are **eight possible outcomes** with each having a $\frac{1}{8}$ **probability of occurring**.



We can see that the probability of **selecting zero blue jellybeans** is $\frac{1}{8}$, the **probability** of **selecting one or two blue jellybeans** is $\frac{3}{8}$ and the **probability** of **selecting three blue jellybeans** is $\frac{1}{8}$. From this information, we can create a **probability distribution** for the **number of blue jellybeans selected**:

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



$X =$ number of blue jellybeans selected

With this distribution, we can use it to **calculate** the **probabilities** of **different events occurring**.

For instance, the **probability** of **selecting less than two blue jellybeans** (i.e. $P(X < 2)$) would be the **sum** of $P(X = 0)$ and $P(X = 1)$:

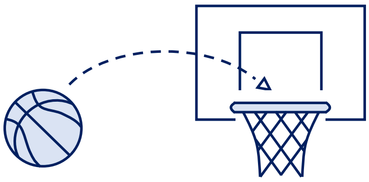
$$P(X < 2) = P(X = 0) + P(X = 1)$$
$$P(X < 2) = \frac{1}{8} + \frac{3}{8}$$
$$\therefore P(X < 2) = \frac{1}{2}$$

Beyond the table format, a probability distribution can also be represented using a **probability distribution function**.

> A **probability distribution function** (e.g. $p(x)$ or $f(x)$) is a **function** that gives the **probability** of each **outcome occurring**, such as $p(x) = \frac{x}{15}$ for $x = 1, 2, 3, 4, 5$.

For instance, suppose the **number of successful basketball shots** (X) out of **three** was represented by the **probability function** below:

$$P(X = x) = \frac{4-x}{6} \quad \text{for} \quad x = 1, 2, 3$$



X = number of successful basketball shots

By **substituting** in a **value** of x, we can **calculate** the **probability** of an event occurring. For instance, the **probability** of getting **three successful basketball shots** (i.e. $P(X = 3)$) would be:

$$P(X = 3) = \frac{4 - 3}{6}$$
$$P(X = 3) = \frac{1}{6}$$

Similarly, if we wanted the probability of **shooting at least two shots successfully** ($P(X \geq 2)$), we can **substitute** in the **values** of $x = 2$ and $x = 3$ to determine their **probabilities** and then **add** them together:

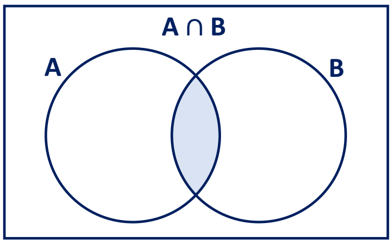
$$P(X = 2) = \frac{4 - 2}{6} \qquad\qquad P(X = 3) = \frac{4 - 3}{6}$$
$$P(X = 2) = \frac{1}{3} \qquad\qquad P(X = 3) = \frac{1}{6}$$

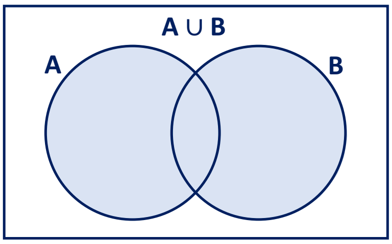
$$P(X \geq 2) = P(X = 2) + P(X = 3)$$
$$P(X \geq 2) = \frac{1}{3} + \frac{1}{6}$$
$$\therefore P(X \geq 2) = \frac{1}{2}$$

You may also recall from Year 11 that we explored the '**AND**' and '**OR**' rules of probability.

The term '**AND**' means the **probability** of events A **and** B **both occurring**. It is represented by $P(A \cap B)$.

The term '**OR**' means the **probability** of **either event** A **or event** B **occurring**. It is represented by $P(A \cup B)$.



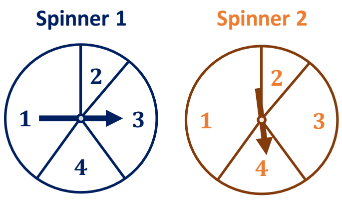


For each of these events, we can **calculate** the **probability** of them occurring for different scenarios that can either be **independent events** or **mutually exclusive events**. **Mutually exclusive events** are two events that **cannot occur at the same time** (e.g. you can't get both a head and tail from a single coin flip) whereas **independent events** mean that **one event** is **unaffected** by the occurrence of the other event (e.g. rolling two dice).

When **calculating** $P(A \cap B)$ and $P(A \cup B)$ for two **independent events** or **mutually exclusive events**, the **formulas** used are:

Distribution	$P(A \cap B)$	$P(A \cup B)$
Independent Events	$P(A \cap B) = P(A) \times P(B)$	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mutually Exclusive Events	$P(A \cap B) = 0$	$P(A \cup B) = P(A) + P(B)$

Applying this to an example of **independent events**, let's suppose we have **two identical spinners** that are **spun** at the **same time**. The variable X represents the **value spinner 1 lands on** (i.e. $x = 1, 2, 3, 4$) and the **variable** Y represents the **value spinner 2 lands on**. Below we can see the **probability distribution** for X and Y are the **same**:



X = value landed on by spinner 1
Y = value landed on by spinner 2

x	1	2	3	4
$P(X = x)$	0.4	0.1	0.3	0.2

y	1	2	3	4
$P(Y = y)$	0.4	0.1	0.3	0.2

If both spinners are spun at the same time, the **outcomes of each spinner** is **independent** of each other. Hence, the **probability** of **spinner 1 landing** on a **one** and **spinner 2 landing** on a **three**, $P(X = 1 \cap Y = 3)$, will be **calculated** by using the formula as follows:

$$P(A \cap B) = P(A) \times P(B)$$
$$P(X = 1 \cap Y = 3) = P(X = 1) \times P(Y = 3)$$
$$P(X = 1 \cap Y = 3) = 0.4 \times 0.3$$
$$P(X = 1 \cap Y = 3) = 0.12$$

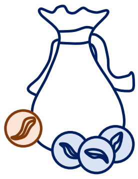
One final important area you may recall from Year 11 is **conditional probability**.

Conditional probability is the **probability** that an **event A** occurs **given** that another **event B** occurs (i.e. $P(A|B)$).

For **two independent events**, the **probability** of **event A** occurring **given event B** occurs is calculated using the **formula**:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For instance, let's consider a scenario where we have a **bag** with an **assortment** of **blue marbles** and **orange marbles**. **Three marbles** are then selected **with replacement** and X represents the **number of orange marbles selected**. The following probability **distribution** has been created for the **variable** X:



$X =$ number of orange marbles selected

x	0	1	2	3
$P(X = x)$	0.064	0.288	0.432	0.216

With this table, we can apply **conditional probability**. For instance, calculating the **probability** of selecting **less than three orange marbles given at least one orange marble** has been selected: $P(X < 3|X \geq 1)$. Applying our conditional probability formula will give:

$$P(X < 3|X \geq 1) = \frac{P(1 \leq X < 3)}{P(X \geq 1)}$$

$$P(X < 3|X \geq 1) = \frac{0.288 + 0.432}{0.288 + 0.432 + 0.216}$$

$$P(X < 3|X \geq 1) = 0.769$$

As a second example, we could calculate the probability of **selecting three orange marbles given** that **at least two orange marbles** have been selected: $P(X = 3|X \geq 2)$. This would be calculated as:

$$P(X = 3|X \geq 2) = \frac{P(X = 3)}{P(X \geq 2)}$$

$$P(X = 3|X \geq 2) = \frac{0.216}{0.432 + 0.216}$$

$$P(X = 3|X \geq 2) = 0.333$$

Hopefully most of this content will feel familiar from Year 11. From here on out we are going to be diving into the **Year 12 probability content**, starting with **discrete random variables**.

5.2 Discrete Random Variables

Random variables can be classified into two main types: **discrete random variables** and **continuous random variables**.

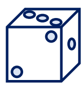
> **Discrete random variables** are variables with a **distinct number of values** for the outcomes of the experiment, such as the **number rolled on a dice** or the **number of customers** in a **store**.

> **Continuous random variables** are variables with an **infinite number of values** that **fall between an interval**, such as the **mass of apples** at a grocery store or the **time taken** to **run a marathon**.

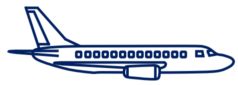
For the remainder of Unit 3 we will be focussing only on **discrete random variables**. We will explore **continuous random variables** in Unit 4.

Like our previous examples on **random variables**, we **assign a variable** such as X, which takes on **values of x** that are the **possible outcomes** for the **experiment**:

$X =$ Number of Heads from Three Coin Flips
$x = 1, 2$ or 3



$Y =$ Value on a Six-Sided Die
$y = 1, 2, 3, 4, 5$ or 6



$Z =$ Number of Passengers on a Plane
$z = 1, 2 \ldots 150$

These **discrete random variables** are a **type of random variable** and as a result we can **apply all** of the **probability principles** we have learnt earlier such as **and/or probability** and **conditional probability**.

Additionally, in some situations we might have a **probability distribution** where **certain probabilities** in our distribution are **unknown**.

For instance, suppose we were playing a **basketball toss game** at a carnival, where the objective was to **score** as many **baskets** as possible **using three basketballs**. If X represents the **number of shots** that were **successful** we could have the following **probability distribution**, where k is **unknown**.

x	0	1	2	3
$P(X = x)$	0.20	0.40	k	0.10

To determine the **value** of k, random variables have **two key rules** we can utilise.

The first is that **all the probabilities must be greater than zero** and the second is that **all the probabilities must sum to a total of one**. These **two properties** can be **summarised** as:

> **Discrete random variable properties:** ① $P(X = x) \geq 0$ and ② $\sum P(X = x) = 1$

For our scenario, the **particularly useful rule** for **determining** the **value** of k is the fact that all of the **probabilities sum to one** ($\sum P(X=x)=1$). **Applying** this to our scenario gives $k = 0.30$:

$$\text{①}\ \sum P(X=x)=1$$
$$0.20+0.40+k+0.1=1$$
$$k=1-0.70$$
$$\text{②}\ k=0.30$$

① **Apply** the **property** that **all** the **probabilities must sum to one**

② **Determine** the **value** of k

As a more difficult example, what if we didn't know the **probabilities of successfully shooting $x=2$** and **$x=3$ shots**. These **unknown probabilities** are represented as q and p respectively.

x	0	1	2	3
$P(X=x)$	0.20	0.40	q	p

In order to **determine** the **values** of q and p, it is **not enough** to apply the property: $\sum P(X=x)=1$. We need **two pieces of information** to **solve** for **two variables**.

If it was **known** that the $P(X<3|X\geq 1)=0.80$, we can determine the values of q and p by creating **two simultaneous equations**:

Equation 1

$$\sum P(X=x)=1$$
$$0.2+0.4+q+p=1$$
$$\text{①}\ p=0.4-q$$

Equation 2

$$P(X<3|X\geq 1)=0.80$$
$$\frac{P(1\leq X<3)}{P(X\geq 1)}=0.80$$
$$\text{②}\ \frac{0.4+q}{0.4+q+p}=0.80$$

Applying our knowledge of **simultaneous equations**, we can **substitute** equation **1** into equation **2** to determine the **value** of q, and then use this to determine the value of p.

Substitute ① into ②

$$\frac{0.4+q}{0.4+q+(0.4-q)}=0.8$$
$$\frac{0.4+q}{0.8}=0.8$$
$$q=0.8\times 0.8-0.4$$
$$q=0.24$$

Substitute $q=0.24$ into ①

$$p=0.4-0.24$$
$$p=0.16$$
$$\therefore q=0.24,\ p=0.16$$

Beyond **determining** the **values** of **unknown probabilities**, we can **apply all** of the **general probability concepts** we learnt in the random variables section, such as in the worked example below.

Worked Example 1

Student Kerry is well known as a great athlete but has just recently started learning the sport of **cricket**. She wants to look at **how many runs** she can **hit off each ball bowled** to her.

She knows that the **number of runs** she scores is **independent** from her score on the last ball, and comes up with the following table for the **chance she scores** a certain **number of runs** on each ball:

x	1	2	3	4	6
$P(X=x)$	0.43	0.12	0.05	0.13	p

(a) Help Kerry to complete her probability distribution by **determining** the **value** of p.

$$\sum P(X=x)=1$$
$$0.43+0.12+0.05+0.13+p=1$$
$$p=1-0.73$$
$$p=0.27$$

(b) If Kerry knows she must score **at least 3 runs** to **secure the win** for her team, determine the **probability** her team wins.

$$P(X\geq 3)=P(X=3)+P(X=4)+P(X=6)$$
$$P(X\geq 3)=0.05+0.13+0.27$$
$$P(X\geq 3)=0.45$$

(c) Kerry is playing another game not long after and she is eager to show off her skills and finish the game with a **six**. **Given she** will **at least 2 runs**, determine the **probability** she **hits a six**.

$$P(X=6|X\geq 2)=\frac{P(X=6)}{P(X\geq 2)}$$
$$P(X=6|X\geq 2)=\frac{0.27}{0.12+0.05+0.13+0.27}$$
$$P(X=6|X\geq 2)=0.474$$

5.3 Mean, Variance and Standard Deviation of DRV's

One of the best ways to describe **probability distribution** is by determining its **mean**, **variance**, and **standard deviation**. These are defined as follows:

The **mean**, $\boldsymbol{\mu}$, also known as the **expected value**, $\boldsymbol{E(X)}$, is the **average outcome** of a random variable. It is the **average value** you would **expect** if you continually repeated an event.

Standard deviation (σ) is a measure of the **spread** of **results from the mean**. A **small** $\boldsymbol{\sigma}$ indicates the data points are **spread close** to the **mean**, whereas a **large** $\boldsymbol{\sigma}$ indicates the data is **widely spread** from the mean.

Variance, $\boldsymbol{Var(X)}$, is also a measure of the **spread** of the **results from the mean**. It is equal to the **standard deviation squared** (i.e. $Var(X) = \sigma^2$).

In Year 12, we are **particularly interested** in knowing how to **calculate** the **mean**, **variance**, and **standard deviation** for the different distributions we will explore.

For **discrete random variables** they can be calculated with the **following formulas**:

Mean: $\boldsymbol{\mu = E(X) = \sum xp(x)}$

Variance: $Var(X) = \sum (x - E(X))^2 p(x)$ or $Var(X) = E(X^2) - [E(X)]^2$

Standard Deviation: $\boldsymbol{\sigma = \sqrt{Var(X)}}$

The best way to understand these formulas is to **apply** them to some **examples**.

Let's consider that we are **conducting a raffle**, where if you land on the **winning number** you **win a prize**.

If there are **four possible numbers** in the raffle (i.e. $x = 1, 2, 3$ **and** 4) and each number has a **different probability** of coming up, we can create a **discrete distribution** where $\boldsymbol{X}$ **represents** the **number drawn**:

$\boldsymbol{x = 1, 2, 3, 4}$
(number drawn)

x	1	2	3	4
$P(X = x)$	0.10	0.30	0.20	0.40

By using the **probabilities** and **values of** $\boldsymbol{x}$ above, we can **apply our formulas** above to **calculate** the **mean**, **variance**, and **standard deviation**.

To do so, let's start by **calculating** the **mean**, $E(X)$ with the formula: $E(X) = \sum xp(x)$. What this formula tells us is that the **mean** is the **sum** of each **value** of x **multiplied** by its **probability**. So for our raffle example, the **average number** that will be drawn will be:

$$E(X) = \sum xp(x)$$

① $E(X) = 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.4$

② $E(X) = 2.9$

① **Sum** each **value of x multiplied** with its **probability**

② **Calculate** the **final answer**

For the variance, $Var(X)$, we can apply one of the following formulas: $Var(X) = \sum(x - E(X))^2 p(x)$ or $Var(X) = E(X^2) - [E(X)]^2$.

Both formulas are **equivalent** to one another so it doesn't matter which we use. However, **applying** the **second formula**: $Var(X) = E(X^2) - [E(X)]^2$ is **typically easier**. For now, we will look at both.

The **first formula**: $Var(X) = \sum(x - E(X))^2 p(x)$, calculates variance by **squaring the difference** between x and the **mean** (i.e. $(x - E(X))^2$) and then **multiplying** by its **probability**, $p(x)$. We apply this to **each value of x** and then **sum** the results together.

The **second formula**: $Var(X) = E(X^2) - [E(X)]^2$, calculates variance by **summing** each value of x **squared** (x^2) **multiplied** by its **probability** (i.e. $E(X^2)$) and then **subtracting** the **mean squared** (i.e. $[E(X)]^2$). As you can see the second formula is more straightforward, but let's **apply both** to our example with a mean of $\mu = E(X) = 2.9$.

Formula 1

$$Var(X) = \sum(x - E(X))^2 p(x)$$

$$Var(X) = (1 - 2.9)^2 \times 0.1 + (2 - 2.9)^2 \times 0.3 + (3 - 2.9)^2 \times 0.2 + (4 - 2.9)^2 \times 0.4$$

$$Var(X) = 0.361 + 0.243 + 0.002 + 0.484$$

$$Var(X) = 1.09$$

Formula 2

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = 1^2 \times 0.1 + 2^2 \times 0.3 + 3^2 \times 0.2 + 4^2 \times 0.4 - 2.9^2$$

$$Var(X) = 0.1 + 1.2 + 1.8 + 6.4 - 2.9^2$$

$$Var(X) = 1.09$$

Finally, now that we know the **variance**, we can calculate **standard deviation** using the formula: $\sigma = \sqrt{Var(X)}$. For this formula all we need to do is **square-root** the **variance**:

$$\sigma = \sqrt{Var(X)}$$

① $\sigma = \sqrt{1.09}$

② $\sigma = 1.04$

① **Substitute** in the **value** $Var(X) = 1.09$

② **Calculate** the **final answer**

Applying the **formulas** for $E(X)$, $Var(X)$ and σ to different scenarios just takes practice, so try working through the example below.

Worked Example 1

Janitor Peter was cleaning his auditorium when he noticed two discrete distributions left on the whiteboard. To summarise these distributions, **help** Janitor Pete find the **mean** and **standard deviation** for the following **two** distributions. **[CA]**

(a)

x	0	1	2	3
$P(X = x)$	0.15	0.40	0.30	0.15

Mean μ or $E(X)$:

$$\mu = E(X) = \sum xp(x)$$

$$E(X) = 0 \times 0.15 + 1 \times 0.40 + 2 \times 0.30 + 3 \times 0.15$$

$$E(X) = 1.45$$

To **calculate σ** start by **calculating $Var(X)$**:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = 0^2 \times 0.15 + 1^2 \times 0.40 + 2^2 \times 0.30 + 3^2 \times 0.15 - 1.45^2$$

$$Var(X) = 2.95 - 1.45^2$$

$$Var(X) = 0.8475$$

Substitute in $Var(X) = 0.8475$ to determine σ:

$$\sigma = \sqrt{Var(X)}$$

$$\sigma = \sqrt{0.8475}$$

$$\sigma = 0.921$$

(b) $p(x) = \frac{2x^2}{60}\ for\ x = 1, 2, 3, 4$

Mean $E(X)$:

$$E(X) = \sum xp(x)$$

$$E(X) = 1 \times \frac{2(1)^2}{60} + 2 \times \frac{2(2)^2}{60} + 3 \times \frac{2(3)^2}{60} + 4 \times \frac{2(4)^2}{60}$$

$$E(X) = \frac{10}{3}$$

To **calculate σ** start by **calculating $Var(X)$**:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$Var(X) = 1^2 \times \frac{2}{60} + 2^2 \times \frac{8}{60} + 3^2 \times \frac{18}{60} + 4^2 \times \frac{32}{60} - \left(\frac{10}{3}\right)^2$$

$$Var(X) = \frac{59}{5} - \left(\frac{10}{3}\right)^2$$

$$Var(X) = \frac{31}{45}$$

Substitute in $Var(X) = 31/45$ to determine σ:

$$\sigma = \sqrt{Var(X)}$$

$$\sigma = \sqrt{31/45}$$

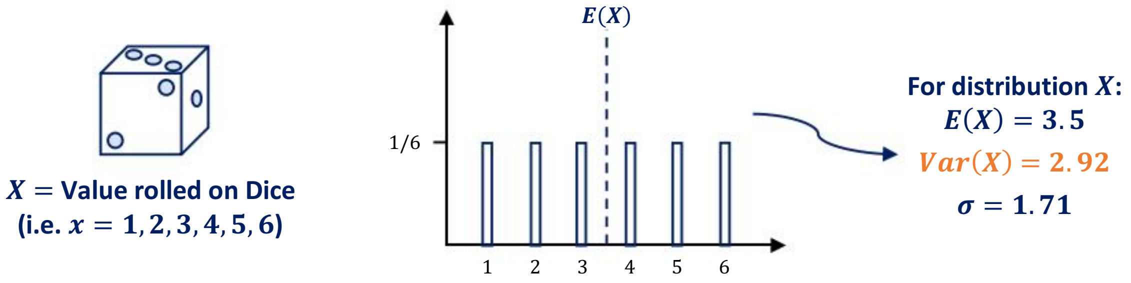
$$\sigma = 0.830$$

Linearity of Discrete Random Variables

When it comes to **mean**, **variance**, and **standard deviation**, one concept that is fundamental to understand is that **their values** can **undergo transformations** when the **distribution is transformed**.

To illustrate this point, suppose we have a **fair six-sided die**, where $\boldsymbol{X}$ is the **value rolled** on the dice (i.e. $\boldsymbol{x = 1, 2, 3, 4, 5, 6}$).

Applying our formulas from the previous section, we would calculate that the **mean** is $\boldsymbol{E(X) = 3.5}$, the **variance** is $\boldsymbol{Var(X) = 2.92}$ and the **standard deviation** is $\boldsymbol{\sigma = 1.71}$. On the graph below, the **blue dotted line** shows $\boldsymbol{3.5}$, the **mean** of the dice roll, and the spread of results.



Now, imagine we **increased the values** on the **six-sided die** by **four** or **doubled the values** on the dice. To **calculate** the **mean**, **variance**, and **standard deviation** of this new distribution, instead of having to apply our original formulas we could instead **apply the concept** of **linear transformations**.

There are two types of linear transformations: a **change of scale** or a **change of origin**.

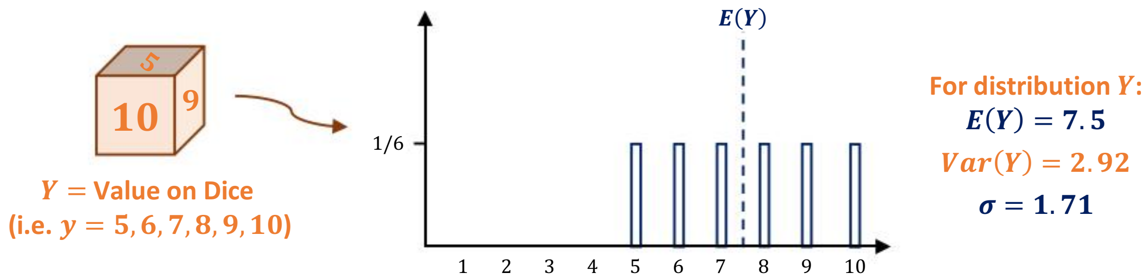
A **change of scale** is **stretching/contracting** a distribution by **multiplying** it by a **constant** ($\boldsymbol{a}$).

A **change of origin** is **shifting** each value of a distribution by **adding or subtracting** a **constant** ($\boldsymbol{b}$).

Based on these **two characteristic changes** to **distribution** $\boldsymbol{X}$, a **new distribution** $\boldsymbol{Y}$ is created based on the **linear transformation formula**:

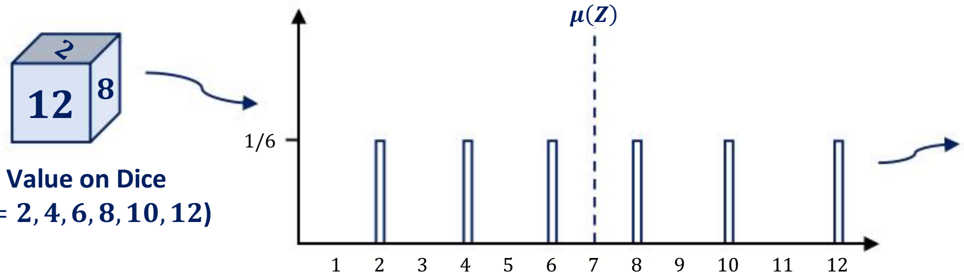
$$\boldsymbol{Y = aX + b}$$

Let's bring it back to our example to understand how these **two linear transformations** affect the values of $\boldsymbol{E(X)}$, $\boldsymbol{Var(X)}$ and $\boldsymbol{\sigma}$. For the **change in origin** of **increasing all values** on the **dice by four**, our **transformed distribution** is: $\boldsymbol{Y = X + 4}$. The **effect** this has can be displayed as follows:



As we can see, the change in origin affects the **mean** by **increasing** it by 4, from 3.5 to 7.5. However, it has no effect on the **spread of the distribution**, so the variance and **standard deviation** remain the **same**.

Let's now consider the effect of a **change in scale** on the original die, so now each die value is **worth double**. This means the **transformed distribution** can now be defined as $Z = 2X$.



Z = Value on Dice
(i.e. $z = 2, 4, 6, 8, 10, 12$)

For distribution Z:
$E(Z) = 7$
$Var(Z) = 11.68$
$\sigma = 3.42$

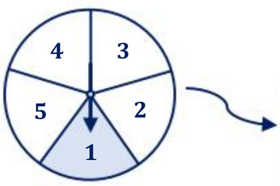
As seen above, **doubling the values** on the dice **doubles the mean** from 3.5 to 7. It also **doubles** the **standard deviation** from 1.71 to 3.42, and it quadruples the variance from 2.92 to 11.68.

The effect a **change in scale** (a) and change in origin (b) have on the **mean**, variance, and **standard deviation** are **always the same**. These **effects** can be **summarised** with the **following formulas**:

Distribution X	Mean ($E(X)$)	Variance ($Var(X)$)	Standard Deviation (σ)
Linear Transformation ($Y = aX + b$)	$E(Y) = aE(X) + b$	$Var(Y) = a^2 Var(X)$	$\sigma(Y) = \lvert a \rvert \sigma(X)$

Applying these **formulas** to an **example,** let's imagine that we have a **carnival game** where players have to **spin a spinner** with **five equal segments** and X is the **value landed on** by the spinner.

For $x = 1, 2, 3, 4, 5$, the **probability** of landing on a **specific number** is $1/5$ and can be represented in the table below:



X = Value on spinner
(i.e. $x = 1, 2, 3, 4, 5$)

x	1	2	3	4	5
$P(X = x)$	0.2	0.2	0.2	0.2	0.2

Applying our original formulas we would **calculate** the **mean**, variance, and **standard deviation** of **distribution X** to be:

For distribution X:
$E(X) = 3$
$Var(X) = 2$
$\sigma = 1.41$

If the **values on the spinner** are then **doubled** and **increased by three** (i.e. $Y = 2X + 3$), we can **apply** our **linear transformation equations** to calculate the **new mean**, **variance**, and **standard deviation** to be:

Mean $E(Y)$:

$$E(Y) = aE(X) + b$$
Substitute $E(X) = 3, a = 2, b = 3$
$$E(Y) = 2(3) + 3$$
$$E(Y) = 9$$

Variance $Var(Y)$:

$$Var(Y) = a^2Var(X)$$
Substitute $Var(X) = 2$ and $a = 2$
$$Var(Y) = 2^2(2)$$
$$Var(Y) = 8$$

Standard Deviation $\sigma(Y)$:

$$\sigma(Y) = |a|\sigma(X)$$
Substitute $\sigma(X) = 1.41$ and $|a| = 2$
$$\sigma(Y) = 2(1.41)$$
$$\sigma(Y) = 2.82$$

By **applying** the **linear transformations formulas**, we provide ourselves with a **shortcut** to determining the **mean**, **variance**, and **standard deviation** for a **transformed distribution**.

Practice applying these **formulas** in the worked example below and in the problem sets.

Worked Example 2

Janitor Peter is looking to understand linear transformations. To experiment, Janitor Peter decides to make **linear transformations** to his **distribution X** to make a **new distribution Y**. If $E(X) = 1.5$ and $\sigma(X) = 0.8$, based on the linear transformations below, determine the **new mean** and **standard deviation**.

(a) $Y = 3X$

Mean $E(Y)$:

$$E(Y) = aE(X) + b$$
Substitute $E(X) = 1.5$ and $a = 3$
$$E(Y) = 3(1.5)$$
$$E(Y) = 4.5$$

Standard Deviation $\sigma(Y)$:

$$\sigma(Y) = |a|\sigma(X)$$
Substitute $\sigma(X) = 0.8$ and $|a| = 3$
$$\sigma(Y) = 3(0.8)$$
$$\sigma(Y) = 2.4$$

(b) $Z = 4X + 2$

Mean $E(Z)$:

$$E(Z) = aE(X) + b$$
Substitute $E(X) = 1.5$, $a = 4$, $b = 2$
$$E(Z) = 4(1.5) + 2$$
$$E(Z) = 8$$

Standard Deviation $\sigma(Z)$:

$$\sigma(Z) = |a|\sigma(X)$$
Substitute $\sigma(X) = 0.8$ and $|a| = 4$
$$\sigma(Z) = 4(0.8)$$
$$\sigma(Z) = 3.2$$

5.4 Applications of Discrete Random Variables

As we have already seen, **discrete random variables** are applicable to many **real-world situations** and as we have progressed through this chapter, we have learnt how to **apply all** of **the probability concepts**.

In relation to real word applications, **one final concept** we will explore is **designing the probability distribution** to be **relevant** to the **situation** and then **applying all the concepts** we have learnt.

For instance, let's consider a classic carnival game-scenario, the **basketball toss**! In this game, a player **pays** $\$2$ to **shoot five basketballs**, and if they successfully shoot **four basketballs,** they win a **small prize** worth $\$3$ and if they successfully shoot **all five basketballs**, they win a **large prize** worth $\$10$.

After recording the results of $\mathbf{1000}$ **games**, a **probability distribution** for the **number of successful shots made** (X) was **produced**:

x	1	2	3	4	5
$P(X=x)$	0.35	0.30	0.20	0.10	0.05

Small Prize Worth $\$3$

Big Prize Worth $\$10$

Whilst this probability distribution is helpful for the participants, as the **owner** of the **carnival station**, if you wanted to know the **expected profit per game** (i.e. $E(X)$), then we would need to **adjust the distribution** to be **based on profit**, not number of shots.

To **adjust the distribution** to be **profit based**, we can do this by considering the **owner earns** $\$2$ upfront for each game, but then **loses** $\$3$ from the **small prize** when **four shots** are successfully made and **loses** $\$10$ from the **big prize** when **five shots** are successfully made.

Therefore, for $x = 1$, $x = 2$ and $x = 3$ the owner **earns** $\$2$, for $x = 4$ the owner **loses** $\$1$, and for $x = 5$ the owner **loses** $\$8$.

If Y represents the **amount** the owner **makes per game**, the new probability distribution will be:

y	2	-1	-8
$P(Y=y)$	0.85	0.10	0.05

To calculate the owner's **expected profit per game**, we apply our **expected value/mean formula**:

$$\boxed{\text{Mean: } \mu = E(X) = \sum xp(x)}$$

Substituting in the values above, we get the following **expected profit**:

$$\boldsymbol{E(Y)} = \sum xp(y)$$
$$\boldsymbol{E(Y)} = 2 \times 0.85 + (-1) \times 0.10 + (-8) \times 0.05$$
$$\boldsymbol{E(Y)} = \$\mathbf{1.20}$$

This means the owner will make an **average** of $\$\mathbf{1.20}$ **per game**.

If we wanted to know the owner's **expected profit** over $\mathbf{1000}$ **games**, we would **multiply** the **expected profit per game** by the **number of games**:

$$\boldsymbol{Profit} = \boldsymbol{E(Y)} \times \mathbf{1000}$$
$$\boldsymbol{Profit} = \mathbf{1.20} \times \mathbf{1000}$$
$$\boldsymbol{Profit} = \$\mathbf{1200}$$

∴ expect a $\$\mathbf{1200}$ **profit** from $\mathbf{1000}$ **games**

The owner may also want to know the **minimum price** they can **charge per game** to **break even**. In other words, what **entry price** should they charge for the game to make $\boldsymbol{E(Y) = 0}$.

We can see how this can been solved in the **worked example** below.

Worked Example 1

Plotting as always, Sparrow is looking for a way to make some extra cash using carnival games. Sparrow sets up a game of **wheel of fortune**, which has **five values** of **different segments**:

$$\boldsymbol{p(x)} = \frac{\boldsymbol{x+3}}{\mathbf{30}} \text{ for } \boldsymbol{x} = \mathbf{1, 2, 3, 4, 5}$$

(a) Determine the **probability distribution** of $\boldsymbol{X}$ by filling in the table below.

$\boldsymbol{x}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$
$\boldsymbol{P(X = x)}$	$\frac{\mathbf{4}}{\mathbf{30}}$	$\frac{\mathbf{5}}{\mathbf{30}}$	$\frac{\mathbf{6}}{\mathbf{30}}$	$\frac{\mathbf{7}}{\mathbf{30}}$	$\frac{\mathbf{8}}{\mathbf{30}}$

(b) If Sparrow gives the winner a $\$\mathbf{5}$ prize if they land on the $\boldsymbol{x} = \mathbf{4}$, and a $\$\mathbf{8}$ **prize** if they land on $\boldsymbol{x} = \mathbf{5}$, determine how much Sparrow should **charge per game** to **break even. [CA]**

Let y be the **amount** Sparrow charges per game

$$\boldsymbol{E(Y) = 0}$$
$$0 = y \times \frac{4+5+6}{30} + (y-5) \times \frac{7}{30} + (y-8) \times \frac{8}{30}$$
$$0 = \frac{\mathbf{15}}{\mathbf{30}}y + \frac{\mathbf{7}}{\mathbf{30}}y - \frac{35}{30} + \frac{\mathbf{8}}{\mathbf{30}}y - \frac{64}{30}$$
$$0 = y - \frac{99}{30}$$
$$y = \$\mathbf{3.30}$$

∴ Sparrow should **charge** $\$\mathbf{3.30}$ per game to break even

DISCRETE RANDOM VARIALBES – TOPIC NOTES

Random Variables

A **random variable** is a **variable** that takes on the **possible outcomes** of an event, such as the **random variable x** representing the **value rolled** on a **six-sided die** (i.e. $x = \mathbf{1, 2, 3, 4, 5}$ or $\mathbf{6}$).

For any event that takes place, a **probability distribution** of a **random variable** tells us what the **possible values** are for **variable X** are and the **associated probabilities** of those values occurring. These can be represented using **tree diagrams** and **probability distributions**.

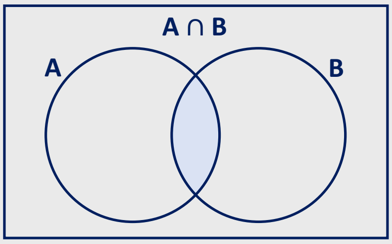
x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This probability distribution can be used to calculate the **probability** of a selection of events occurring, for example, $P(X < 2)$, where the **probability** of **all values less than 2** are added together.

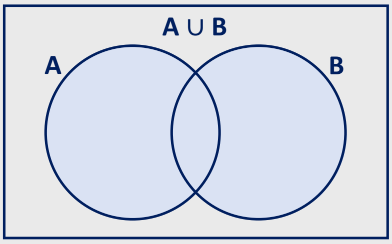
Probability distribution functions are also useful ways of representing the probability of certain outcomes.

A **probability distribution function** (e.g. $p(x)$ or $f(x)$) is a **function** that gives the **probability** of each **outcome occurring**, such as $p(x) = \frac{x}{15}$ for $x = \mathbf{1, 2, 3, 4, 5}$.

It is also important to remember the terminology regarding '**AND**' and '**OR**' for independent and mutually exclusive events:



'**AND**' is represented by $P(A \cap B)$.



'**OR**' is represented by $P(A \cup B)$.

Events can either be **independent events** or **mutually exclusive events**. **Mutually exclusive events** are two events that **cannot occur at the same time** whereas **independent events** mean that **one event** is **unaffected** by the occurrence of the other event.

When **calculating** $P(A \cap B)$ and $P(A \cup B)$ for two **independent events** or **mutually exclusive events**, the **formulas** are as follows:

Distribution	$P(A \cap B)$	$P(A \cup B)$
Independent Events	$P(A \cap B) = P(A) \times P(B)$	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Mutually Exclusive Events	$P(A \cap B) = 0$	$P(A \cup B) = P(A) + P(B)$

We also need to understand the concept of **conditional probability**.

Conditional probability is the **probability** that an **event A** occurs **given** that another **event B** occurs (i.e. $P(A|B)$).

For **two independent events**, the **probability** of **event A** occurring **given event B** occurs is calculated using the **formula**:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Discrete Random Variables

Discrete random variables are variables with a **distinct number of values** for the outcomes of the experiment, such as the **number rolled from a dice** or the **number of customers** in a **store**.

Sometimes we might have a **probability distribution** where **certain probabilities** in our distribution are **unknown**. Let's call this unknown probability k.

x	0	1	2	3
$P(X = x)$	0.20	0.40	k	0.10

To determine the **value** of k, random variables have **two key rules** we can use which can be **summarised** as:

Discrete random variable properties: ① $P(X = x) \geq 0$ and ② $\sum P(X = x) = 1$

Mean, Variance and Standard Deviation of DRV's

Mean, **variance**, and **standard deviation** for discrete random variables can be calculated using the formulas below.

Mean: $\mu = E(X) = \sum xp(x)$

Variance: $Var(X) = \sum (x - E(X))^2 p(x)$ or $Var(X) = E(X^2) - [E(X)]^2$

Standard Deviation: $\sigma = \sqrt{Var(X)}$

The values of **mean**, **variance**, and **standard deviation** can undergo **linear transformations** to form a **transformed distribution** (e.g. **distribution Y**).

The **two linear transformations** are a **change in scale**, which is **multiplying** the distribution by a **constant (a)**, and a **change in origin** is shifting the distribution by **adding or subtracting** a **constant** (b).

$$Y = aX + b$$

A **change of scale** is **stretching/contracting** a distribution by **multiplying** it by a **constant** (a).

A **change of origin** is **shifting** each value of a distribution by **adding or subtracting** a **constant** (b).

The impact **Linear transformations** have on the values of **mean**, **variance**, and **standard deviation** can be summarised as:

Distribution X	**Mean ($E(X)$)**	**Variance ($Var(X)$)**	**Standard Deviation (σ)**
Linear Transformation ($Y = aX + b$)	$E(Y) = aE(X) + b$	$Var(Y) = a^2Var(X)$	$\sigma(Y) = \lvert a\rvert\sigma(X)$

Problem Set 7 – Discrete Random Variables
Progressive Questions

Concept 1
Discrete Random Variables – Progressive Questions

(8 questions)

Repetitive questions: 1.11 → 1.92 (11 questions)

The Mathematics College Carnival!

Starring: Teacher Neil, Rupert, Sarah, Rabea

Over the last few years, the mathematics applications students have gathered money through their annual Mathematics College Carnival!

With carnival games galore, thousands of members of the local community all come down to have some fun at the carnival. Despite the community looking to have fun, the methods students are looking to see how they can use probability to win all the prizes…

Discrete Probability Distributions: Q1, Q2, Q3, Q4, Q5, Q6

Repetitive: 1.11 → 1.92 (11 questions)

Leading the methods students is Rupert. "Students, I would like each of you to go to a carnival game and play it as many times as you can. It will be costly, but we will be sure to get back at the apps students soon.", says Rupert. The methods students look at each other's concerned expressions but decide to trust in their leader and head out to gather the data they need.

[10 marks]

1. Before heading out to their games, Rupert wants to check the students will only play games that have a **discrete random variable distribution**. Using chalk, he writes up the following **five distributions** and some **true** and **false statements**. For each distribution, **explain** to the students **why** the statement is **correct**/**incorrect** based on whether it **is**/**isn't** distributions of **discrete random variables**.

(a) "This is **not** a discrete random variable distribution because all of the **probabilities** are the same." **[2]**

x	1	2	3	4	5
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

(b) "This **is** a **discrete random variable distribution** because the **probabilities sum** to **one** and the **two negatives cancel each other out**." **[2]**

x	1	2	3	4	5
$P(X = x)$	-0.2	0.3	-0.05	0.5	0.35

(c) "This **is** a discrete random variable distribution as all of the probabilities are **positive** and **sum to one**." **[2]**

$$P(X = x) = \frac{x}{12} \text{ for } x = 3, 4, 5$$

(d) "This is **not** a discrete random variable distribution because the **values** of x are **negative**." **[2]**

x	-8	-7	-6	-5	-4	-3
$P(X = x)$	0.03	0.7	0.15	0.02	0.12	0.08

(e) "This is **not** a discrete random variable distribution because a probability distribution can only have **one function**." **[2]**

$$P(X = x) = \begin{cases} 0.2 & x = 0, 1, 2, 3 \\ \dfrac{x}{20} & x = 4, 5 \end{cases}$$

[CA] [8 marks]

2. Heading out to her first game, **Kerry** has always dreamed of becoming a professional basketball player and decides to show off her skills at the **basketball toss game**. Spending $100 at the game, she collects the following probabilities for scoring x **number of basketballs** out of **five shots**. Unfortunately, she forgot the data for successfully shooting **two balls**!

x	0	1	2	3	4	5
$P(X = x)$	0.32	0.4	k	0.1	0.06	0.02

(a) Determine the **value** of k for Kerry. **[2]**
(b) Curious to test her probability knowledge, **help** Kerry to determine $P(X > 1 | X \leq 4)$. **[2]**
(c) To help report her findings back to Rupert, determine the **mean** and **variance** of this distribution. **[4]**

[CA] [8 marks]

3. Ace-student **Tyler** is eager to use his brilliant brain to take on the **rubber duckies**! There are hundreds of **rubber duckies** and he thinks he can select the ones that will give him the big prizes! Spending over $400 on the game, Tyler has **very low success rates** but manages to create the following **discrete distribution function**. He ranks the prizes from **one (big)** through to **five (small)**.

$$p(x) = \frac{2x^2+3}{k} \text{ for } x = 1, 2, 3, 4, 5$$

(a) Determine the **value** of k for Tyler. **[2]**
(b) Determine the **probability** Tyler gets the **biggest** or **second biggest prize** on his next go. **[2]**
(c) Calculate the **mean** and **variance** for this distribution. **[4]**

[CA] [14 marks]

4. Looking to engage his brain, lazy Fraser decides to try his luck at a simple game of **bingo**. In this simplified version, there are only **five numbers** and all players have the **numbers** $x = 3, 4, 5$. If **any** of his number comes up, he wins a **prize** with **five** giving the **biggest prize**! Sitting there for two hours, his numbers barely come up and he creates the following distribution.

x	1	2	3	4	5
$P(X = x)$	p	q	0.05	0.02	0.01

(a) If Fraser knows that $P(X < 4 | X \geq 2) = 0.05$, determine the **values** of p and q. **[4]**
(b) Determine the **expected value** and **standard deviation** for Fraser's game. **[5]**
(c) If the **expected value** of the bingo was instead to be $E(X) = 1.52$, determine the **new value** of p (assuming the value of q remains **constant**). **[3]**
(d) If the **expected value** was changed to **part (c)**, would the distribution still be **discrete**? **[2]**

[CA] [8 marks]

5. Staring down the infamous **ladder climb**, instead of spending his own money, **Jevon** sets out a chair and decides to watch people attempt the ladder climb for two hours! Recording the **number of rungs** each person makes it up (out of a total of 6), Jevon records the following **distribution**.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{14}{25}$	$\frac{4}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{1}{25}$	$\frac{1}{25}$

(a) What is the **probability** the next player makes it up **at most two rungs**? **[2]**
(b) Determine $P(X \leq 5 | X > 3)$. **[2]**
(c) What is the **mean** and **variance** of this ladder climb? **[4]**

[CA] [6 marks]

6. Looking to conduct some research against the leader of the apps class, Rupert approaches the stall that Sparrow is running! "Hello Sparrow, I would like to play **300 games** of your **bottle knock over game**" demands Rupert. Gladly accepting Rupert's money, he gives Rupert **300 balls** to play the bottle knock over. In this game Rupert must knock over **six bottles** stacked up in a pyramid to **win prizes** based on the number of bottles he knocks over. With x being the **number of bottles** he **knocks over**, he records the following **distribution function**:

$$P(X \geq x) = \frac{2x^2+x+1}{k} \text{ for } x = 1, 2, 3, 4, 5, 6$$

(a) Determine the **value** of k for this distribution. **[2]**

(b) Create a **distribution** for the bottle toss in a **table** format. **[2]**

(c) Determine the probability that Rupert knocks **over 6 bottles**, given he knocks over **at least 3 bottles**. **[2]**

Linearity of Discrete Random Variables: Q7, Q8

Repetitive: 1.1 → 1.2 (3 questions)

[CA] [12 marks]

7. Furthering his understanding of discrete random variables, the ringleader Sparrow has realised that he can **linearly** change the **mean** and **standard deviation** of the **amount of money** they make.

If Sparrow knows his basketball game to have a **mean** of $\mu = 0.6$, a **variance** of $Var(X) = 0.004$ and a **standard deviation** of $\sigma = 0.02$, determine the new **mean**, **variance** and **standard deviation** based on the following changes.

(a) $Y = 2X$ **[3]**

(b) $Y = 5X + 5$ **[3]**

(c) $Y = -2X + 4$ **[3]**

(d) $Y = -\frac{1}{4}X + 1$ **[3]**

[CA] [18 marks]

8. After realising that this linear change worked for his basketball game, Sparrow brings together all the applications students and tells them to **shift** their **means** and **standard deviations linearly**. To put his brain to the test, Sparrow decides to go round and determine their **linearity equations** for fun! Based on the information below, **help** Sparrow to **determine** the **linearity equations**.

(a) For the **ladder climb**, the original **mean** and **variance** was $\mu = 12$ and $\sigma^2 = 4$ respectively. After Jayden made the change, it became $\mu = 20$ and $\sigma^2 = 8$. Determine the **linearity equation**, in the form $W = AX + B$. **[6]**

(b) For the **duck pond game**, the original **mean** and **standard deviation** was $\mu = 3$ and $\sigma = 1$ respectively. After Liam made the change, it became $\mu = 6$ and $\sigma = 0.5$. Determine the **linearity equation**, in the form $Y = AX + B$. **[6]**

(c) For the **ring toss onto bottle tops**, the original **mean** and **variance** was $\mu = 27$ and $\sigma^2 = 9$ respectively. After Sparrow made the change, it became $\mu = 3$ and $\sigma^2 = 1$. Determine the **linearity equation**, in the form $Z = AX + B$. **[6]**

Concept 2

Applications of Discrete Random Variables –Progressive Questions (4 questions)

Repetitive questions: 2.11 → 2.41 (4 questions)

The Mathematics College Carnival Comeback!

Starring: Teacher Neil, Rupert, Sarah, Rabea

Returning as a group, the methods students are all doubting their plan to takedown the apps students. Despite being cleaned out, Rupert struts back confidently as he is now ready to implement his plan.

"Team, the apps students are currently laughing at us, but they won't be for long! Now is our time to strike back! We have the data, now all we need to do is shift the games in our favour and we will have our revenge!"

Excited by the prospect of revenge, the methods students head back out to their stations, ready to clean house.

Applications of Discrete Random Variables: Q1, Q2, Q3, Q4

Repetitive: 2.11 → 2.41 (4 questions)

[CA] [8 marks]

1. Returning to her basketball shooting game, Kerry is excited to start winning her money back. Shooting much more successfully this time, she produces the following data for the **number of basketballs out of five** that she successfully shot.

x	1	2	3	4	5
$P(X = x)$	0.2	0.2	0.1	0.3	0.2

(a) Kerry wins the **big prize** if she scores **more** than **three** of the **five shots**. Calculate the **probability** that she wins the **big prize**. **[1]**

(b) If the game **costs** $\$2$ to play and the **big prize costs** $\$5$, calculate the **expected profit** the apps students will make from this game. **[2]**

(c) If Kerry plays 200 **games**, how much **money** should she **expect to win** based on the **value of the prizes**? **[2]**

(d) Although it's too late, how much should the apps students have **charged per game** if they wanted to **break even**. **[3]**

[CA] [8 marks]

2. Having lost over $\$400$ at the **rubber duckies** initially, ace-student Tyler is eager to clean the stall out of their prizes. To rig the game in his favour, he puts on his **thermogradient glasses** which measure the **amount of heat** on each of the **duckies**. He notices the ones that have been touched the **least** have the **biggest prizes**, and this helps him to produce a new probability distribution based on the function:

$$p(x) = \frac{x+1}{20} \text{ for } x = 1, 2, 3, 4, 5$$

(a) If x represents the **amount** each **prize** is worth, determine Tyler's **expected profit** if each game **costs** $\$2$ to play. **[2]**

(b) If Tyler managed to convince the apps students to **lower** the cost of the game, and he made $\$250$ from **200 games**, approximate how much each game **cost**. **[3]**

(c) How much should the apps students **charge** for the game to **break-even**? **[3]**

[CA] [10 marks]

3. Heading over to the simplified **bingo game**, lazy Fraser is excited to start winning some of his money back. Instead of having the numbers 3, 4 and 5, Fraser decides just to have the **number 1**. If his number shows up, he wins the **big teddy bear** which **costs** the **stall** $\$10$. Having more success this time, he produces the following **distribution**.

x	1	2	3	4	5
$P(X = x)$	$\frac{8}{20}$	$\frac{5}{20}$	$\frac{3}{20}$	$\frac{2}{20}$	$\frac{2}{20}$

(a) If Fraser **pays $\$3$ per game** and **wins a prize valued at $\$10$** every time the **number one** pops up, determine the amount he can **expect** to **win per game**. **[3]**

(b) How much should the apps students **charge per game** to ensure the game breaks **even**? **[3]**

(c) Realising how much money Fraser is winning, the apps students adjust the game such that **two numbers appear,** and for Fraser this must be **numbers 1 and 2**. Determine the apps students new expected **profit**/**loss** per game. **[4]**

[CA] [6 marks]

4. Annoyed at how much money he has lost, Sparrow decides to offer up all of the remaining carnival toys and money, in exchange for ownership of the new methods building if he wins against Rupert! Afraid of what is to come, Rupert hesitant but decides to accept. The game involves a **random number generator** that **spits out numbers** between 0 and 5, which follows the following probability distribution:

$$P(X = x) = \frac{x^2+4}{75} \text{ for } x = 1, 2, 3, 4, 5$$

(a) If **Rupert wins** if the **number is greater** than three, determine the **probability** Rupert **wins**. **[3]**

(b) If they play **100 games**, and the **winner** of the wager is the person who **wins** the **most games**, determine whether Rupert is **expected** to **win**? **[3]**

Losing the final game, Sparrow and the apps students are forced to give all of their toys to the methods students! Sparrow is furious!

Meanwhile, the methods students are walking away laughing with truckloads of toys, satisfied with their work.

Problem Set 7 – Discrete Random Variables
Repetitive Questions

Concept 1
Discrete Random Variables – Repetitive Questions
(11 questions)

Discrete Probability Distributions: Qs 1.11, 1.12, 1.13, 1.21, 1.31, 1.41, 1.42, 1.51, 1.71, 1.81, 1.82, 1.91, 1.92

[10 marks]

1.11 For each distribution and statement below, **explain why** the statement is **correct**/**incorrect** based on whether it **is**/**isn't** a **discrete random variables**.

(a) "This is **not** a discrete random variable distribution because the **values of x** are in the **wrong order** in the table." **[2]**

x	6	5	4	3	2
$P(X=x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{4}$

(b) "This is **not** a discrete random variable distribution because it the probabilities **do not sum to one** and one of the probabilities is **negative**." **[2]**

x	-1	0	1	2	3
$P(X=x)$	-0.4	0.7	0.04	0.2	0.46

(c) "This **is** a discrete random variable distribution as all of the probabilities are **positive** and **sum to one** and **none** of the probabilities are negative." **[2]**

$$p(x) = \frac{x}{7} \text{ for } x = 1.1, 2, 3.9$$

(d) "This is **not** a discrete random variable distribution because the **values** of x are **negative**." **[2]**

x	-3	-2	-1	0	1	2
$P(X=x)$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{8}$

(e) "This is **not** a discrete random variable distribution because a probability distribution can only have **one function**." **[2]**

$$P(X=x) = \begin{cases} 0.1 & x = 3, 4, 5, 6 \\ \frac{x}{5} & x = 1, 2 \end{cases}$$

[CA] [10 marks]

1.21 The table below represents the probability distribution of an **unfair five sided dice**, with X representing the **value landed on the dice**. However, there is **no value** for the **value of six**.

x	3	4	5	6	7
$P(X = x)$	0.41	0.19	0.02	m	0.22

(a) For this unfair dice, determine the **value** of m. **[2]**

(b) What is the **probability** of the dice landing on a **value greater than four**. **[2]**

(c) Determine **probability** of the dice landing on a **value greater than four**, **given** the value is **less than seven**. **[2]**

(d) Determine the **mean** and **variance** of this distribution. **[4]**

[CA] [11 marks]

1.31 Kerry has created a **bias spinner** with numbers **one through to six**. With x representing the **value landed** on by the **spinner**, it has the probability distribution function:

$$p(x) = \frac{5x - 3}{k} \text{ for } x = 1, 2, 3, 4, 5, 6$$

(a) Determine the **value** of k. **[2]**

(b) Calculate the **mean** and **variance** of this distribution. **[4]**

(c) What is the **probability** that the spinner lands on a **value greater than two**? **[2]**

(d) What is the **probability** that the spinner lands on a **value of five**, **given** that the value it lands on is **greater than one**? **[3]**

[CA] [10 marks]

1.41 Wallace is working at a motorbike shop over the summer holidays, and the following table describes the probability distribution of the **discrete random variable** X, which is the **number of cars** he **sells during a week**.

x	1	2	3	4	5
$P(X = x)$	p	0.20	0.15	0.05	q

(a) If Wallace knows that $P(X = 5 | X \geq 4) = 0.1$, determine the values p and q. **[4]**

(b) What is the **probability** that Wallace **sells five motorbikes** during the week? **[1]**

(c) To avoid getting frowned upon by his boss, what is the **probability** that Wallace **sells more than one car** during the week? **[2]**

(d) **Given** Wallace has already **sold three cars** this week, what is the **probability** that **he sells another two**? **[3]**

[CA] [12 marks]

1.42 The following table describes the probability distribution of the **discrete random variable X**, which is the **number of successful basketball shots** Fraser makes in his high school basketball game.

x	0	1	2	3	4	5
$P(X=x)$	0.20	0.1	0.1	0.05	m	n

(a) If $P(X<2|X\leq 4)=0.8$, evaluate m and n. **[4]**

(b) Determine the **expected value** and **standard deviation** of this distribution. **[4]**

(c) What is the **probability** that Fraser makes **four or five successful basketball shots**? **[2]**

(d) What is the **probability** that Fraser makes **three shots at most**, **given** he has already **made one shot**? **[2]**

[CA] [9 marks]

1.51 The probability of **selecting a specific number** from a **bag of numbers** can be described by the **discrete random variable X**, following the distribution below.

x	1	2	3	4	5	6
$P(X=x)$	$\frac{13}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{3}{30}$	$\frac{1}{30}$	$\frac{2}{30}$

(a) What is the **probability** that the **number three** is **pulled out of the bag**? **[1]**

(b) What is the **probability** that the number pulled out of the bag is **greater than two**? **[2]**

(c) Determine $P(X\leq 4|X>1)$. **[2]**

(d) What is the **mean** and **variance** of this distribution? **[4]**

[CA] [10 marks]

1.71 The following is a probability function for the **random variable X**.

$$P(X=x)=\begin{cases}\dfrac{8-x}{k} & x=0,1,2\\ \dfrac{3x-2}{k} & x=3,4,5\end{cases}$$

(a) Determine the **value** of k for this distribution. **[2]**

(b) Create a **distribution** for the bottle toss in a **table** format. **[2]**

(c) Determine the **mean** and the **standard deviation** of this distribution. **[4]**

(d) Determine $P(X\leq 3|X>0)$. **[2]**

[CA] [8 marks]

1.81 It is known that the **mean** of the **discrete random variable X** is $\mu=6$ with **standard deviation** $\sigma=0.9$. Determine the **new mean** and **standard deviation** for **distribution Y**.

(a) $Y=3X$ **[2]**

(b) $Y=6X+4$ **[2]**

(c) $Y=-\frac{1}{8}X+\frac{1}{4}$ **[2]**

(d) $Y=\sqrt{2}X+2$ **[2]**

[CA] [2 marks]

1.82 The **mean** and **standard deviation** of a biased dice roll game (variable **X**) is $\mu = 4$ and $\sigma = 1.5$. The total winnings are found by **doubling** the dice roll and **adding four**, otherwise known as $Y = 2X + 4$. What is the **mean** and **standard deviation** of the total winnings?

[CA] [12 marks]

1.91 Using the information below, determine the **linearity equations** in the form $Y = AX + B$

(a) The original **mean** and **variance** of a card game was $\mu = 6$ and $\sigma^2 = 2$. Changes were made so that the **mean** became $\mu = 12$ and **variance** $\sigma^2 = 1.5$. Determine the **linearity equation**. **[6]**

(b) The **mean** of a darts game was $\mu = 9$ and **variance** was $\sigma^2 = 3$. Changes were made, causing the **mean** to become $\mu = 12$ and **variance** $\sigma^2 = 2$. Determine the **linearity equations**. **[6]**

[CA] [12 marks]

1.92 using the information below, determine the **linearity equations** in the form $Y = AX + B$

(a) The **mean** and **variance** of an investment scheme was $\mu = 120$ and $\sigma^2 = 25$. Changes were made so that the **mean** became $\mu = 150$ and **variance** became $\sigma^2 = 20$. Determine the **linearity equation. [6]**

(b) Initially, the **mean** of a different investment was $\mu = 150$ and **variance** was $\sigma^2 = 30$. Changes were made, causing the **mean** to become $\mu = 160$ and **variance** became $\sigma^2 = 35$. Determine the **linearity equations. [6]**

Concept 2

Applications of Discrete Random Variables – Repetitive Questions

(4 questions)

Applications of Discrete Random Variables: Qs 2.11, 2.21, 2.31, 2.41

[CA] [8 marks]

2.11 Rabea is back at the carnival for some more fun. One simple game that he stumbles upon is a **raffle**, where the **number on the ticket** represents **how valuable the prize** is that you win. With x representing the **value of the prize in dollars**, the following probability distribution was created:

x	1	2	3	4	5
$P(X = x)$	0.15	0.30	0.10	0.20	0.35

(a) What is the **probability** that Rabea **wins a prize** worth **more than two dollars**? **[1]**

(b) If each raffle **ticket costs \$3**, calculate the **profit** Rabea will **expect** to make per game. **[2]**

(c) If Rabea plays 300 **games**, how much money should he **expect to win** based on the **value of the prizes**? **[2]**

(d) If the raffle ticket owners wanted to make an **overall profit** from the carnival game, what is the **lowest price** they should have charged for **each raffle ticket**? **[3]**

[CA] [11 marks]

2.21 The next game that Rabea decides to play is the **rubber duckies**, where Rabea must select one of the **hundreds of rubber duckies** in a pond and the number written underneath it will be the **value in dollars** of the **prize** that he receives. The following probability distribution function is created:

$$p(x) = \frac{3x+4}{87} \text{ for } x = 1, 2, 3, 4, 5, 6$$

(a) If x represents the **value** of each **prize** in a carnival game, find Rabea's **expected profit** per game if each game costs $\$4$ to play. **[2]**

(b) What is the **variance** and **standard deviation** of each game? **[3]**

(c) How much should the carnival owners charge to at least **break-even**? **[3]**

(d) The carnival game had a deal day where they **lowered** the **entry cost**. This meant a player **won** $\$100$ after 100 **games**. Approximately how much did a game **cost**? **[3]**

[CA] [6 marks]

2.31 In another game that Rabea plays is the **ladder climb**, where x represents the **number of rungs** that Rabea climbs up. In this game, if the player makes it up **four rungs** he will **win a prize worth** $\$5$ and if he **makes it up five rungs** he will **win** a **prize worth** $\$10$.

x	1	2	3	4	5
$P(X = x)$	$\frac{4}{12}$	$\frac{3}{12}$	$\frac{3}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

(a) What is the **probability** that Rabea **doesn't win a prize**? **[1]**

(b) If Rabea pays $\$3$ to **play this game**, how much can Rabea **expect** to **make**/**lose** from this game? **[2]**

(c) If Rabea does the **ladder climb** 50 **times**, how much should he **expect** to **make**/**lose**? **[2]**

(d) How much **money** should the owners of the game **charge** to ensure they **at least break-even** from each game? **[1]**

[CA] [8 marks]

2.41 As a final game, Rabea plays a game that has **two fair 6-sided dice**. If the two dice are rolled, and the **sum** of the two numbers is **less than 6**, the player **wins**.

(a) Given that x is the **sum** of the **two dice**, what is the **probability** that Rabea **wins**? **[2]**

(b) If Rabea is **charged** $\$3$ to **play**, but **profits** by $\$5$ if he **wins**, determine Rabea's **expected profit**/**loss**. **[3]**

(c) Rabea is dissatisfied by the **expected value** of his game, so he asks for a **change** in the rules so that the he **wins** $\$10$ if the **sum** of the two numbers is **less than 4**. How much can he **expect** to **gain**/**lose** over the next 100 **games?** **[3]**

Chapter 6

Bernoulli and Binomial Distributions

Chapter 6 – Bernoulli and Binomial Distributions

Introduction

The next type of distributions we will explore for Unit 3 are **Bernoulli** and **Binomial Distributions**. These distributions are used for situations where there are only **two possible outcomes** for an event: **success** or failure.

We have broken this chapter up into **three main concepts**:

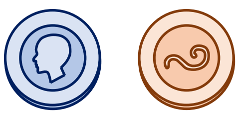
1. **Bernoulli Distributions**
2. **Binomial Distributions**
3. **Mean, Variance and Standard Deviation of Binomial Distributions**

Let's Begin!

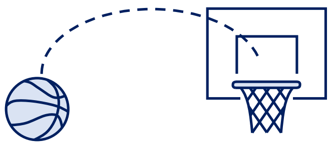
6.1 Bernoulli Distributions

A **Bernoulli trial** is an event that only has **two possible outcomes**, a **success** or failure, and is **only performed once**.

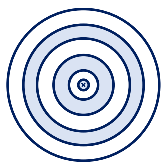
Most events that occur can be viewed as having a **success** or failure outcome, and that allows us to apply a **Bernoulli trial**. From **flipping** a **coin**, to **shooting** a **basketball** into a hoop, to **hitting a bullseye** in archery, all these situations are **Bernoulli trials** because they have a **success** and failure outcome and are **performed once**.



Success $=$ **Head**, Failure $=$ Tail



Success $=$ **Shot In**, Failure $=$ Miss

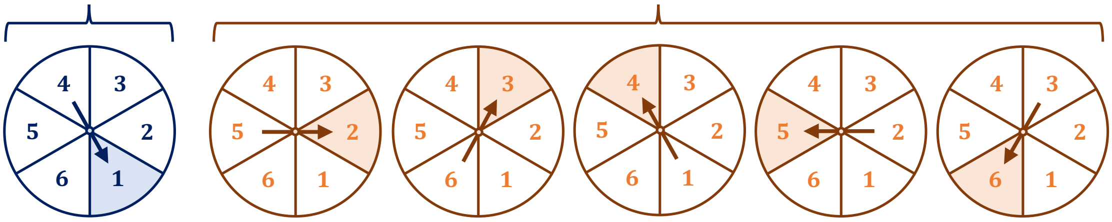


Success $=$ **Bullseye**, Failure $=$ Miss

To explore Bernoulli trials, consider we have a **spinner** with **six equal segments**, where the **success outcome** is **landing on** a $\mathbf{1}$ and the failure outcome is not landing on a 1. Therefore, the probability of **success** (i.e. **landing on** $\mathbf{1}$) is $\frac{1}{6}$, and the probability of failure (i.e. landing on $2, 3, 4, 5$ or 6) is $\frac{5}{6}$.

Success
$$P = \frac{1}{6}$$

Failure
$$P = \frac{5}{6}$$



As a Bernoulli trial, X, we can allocate the **failure outcome** a value of $x = 0$ and the **success outcome** a value of $x = 1$. This creates the following distribution:

x	0	1
$P(X = x)$	$\frac{5}{6}$	$\frac{1}{6}$

In this table, what do you **notice** about the **relationship** between **probabilities** of the **success** and **failure**?

Since the probabilities **must sum** to 1, if the **probability** of **success** is p, then the **probability of failure** is $1 - p$. This idea is **consistent** for **all Bernoulli trials**, no matter the situation because there are only two outcomes. We can summarise this into a probability distribution:

x	0	1
$P(X = x)$	$1 - p$	p

Using this **probability distribution** and our knowledge of **mean**, **variance**, and **standard deviation**, we can determine their **general formulas** for Bernoulli trials.

Recall from the previous chapter, the **mean ($E(X)$)** is the **sum** of the **possible values for X**, **multiplied** by their **associated probabilities**.

$$\text{Mean: } \mu = E(X) = \sum xp(x)$$

For Bernoulli trials, let's see what happens if we **multiply** $(1 - p)$ by 0 (i.e. **failure**), and p by 1 (i.e. **success**), and then **add** these together.

$$E(X) = (1 - p)(0) + p(1)$$
$$E(X) = 0 + p$$
$$\therefore E(X) = p$$

This calculation has shown us that the **mean** of a **Bernoulli trial** is simply p, the **probability** of **success**.

Let's now look at the **variance** and **standard deviation**. You will recall, **variance** is calculated by **summing** $(x - \mu)^2$ **multiplied** by its **probability**, $p(x)$, for **each value** of x. The **standard deviation** (σ) is simply calculated as the **square root** of the **variance**.

$$\text{Variance: } Var(X) = \sum (x - E(X))^2 p(x)$$

$$\text{Standard Deviation: } \sigma = \sqrt{Var(X)}$$

Using our values, let's **substitute** them into the **variance formula**.

$$Var(X) = \sum (x - E(X))^2 p(x)$$
$$Var(X) = (0 - p)^2(1 - p) + p(1 - p)^2 p$$
$$Var(X) = p(1 - p)(p + 1 - p)$$
$$Var(X) = p(1 - p)$$

Note that sometimes for simplicity, $(1-p)$ is denoted as q. Thus, if we let $q = 1 - p$, we get:

$$Var(X) = p(1-p)$$
$$\therefore Var(X) = pq$$

The **standard deviation** is the **square root of the variance**, so the standard deviation is:

$$\sigma = \sqrt{pq}$$

Therefore, in summary, our **formulas** for **Bernoulli distributions** are:

Mean: $E(X) = p$	Variance: $Var(X) = pq$	Standard Deviation: $\sigma = \sqrt{pq}$

Like Discrete Random Variables, we can apply these formulas for any Bernoulli trial. For instance, in our earlier spinner example, **landing on** a 1 was a **success** and **not landing on** a 1 was a **failure**. Knowing, $p = \frac{1}{6}$ and $q = \frac{5}{6}$, our **mean**, **variance** and **standard deviation** will be:

Mean ($\mu(X)$)

$$E(X) = p$$
$$E(X) = \frac{1}{6}$$

Variance ($Var(X)$)

$$Var(X) = pq$$
$$Var(X) = \frac{1}{6} \times \frac{5}{6}$$
$$Var(X) = \frac{5}{36}$$

Standard Deviation (σ)

$$\sigma = \sqrt{pq}$$
$$\sigma = \sqrt{\frac{1}{6} \times \frac{5}{6}}$$
$$\sigma = \frac{\sqrt{5}}{6}$$

The concepts of **Bernoulli Distributions** set the foundations for the closely related and more practical concept of **binomial distributions**.

Worked Example 1

Ace student Rupert is an expert at **archery** and holds himself to an extremely high standard. He considers **hitting a bullseye** to be a **success** and **everything else** to be a **failure**. After recording the results of his last **300 attempts**, Rupert records that he has a **fixed probability** of $p = 0.2$ of **hitting a bullseye** and a **probability** of $q = 0.8$ of **hitting elsewhere**.

Realising this is a **Bernoulli trial**, help Rupert to calculate the **mean**, **variance**, and **standard deviation** of his attempt at hitting a bullseye. **[CA]**

Mean ($E(X)$)

$$E(X) = p$$
$$E(X) = 0.2$$

Variance ($Var(X)$)

$$Var(X) = pq$$
$$Var(X) = 0.2 \times 0.8$$
$$Var(X) = 0.16$$

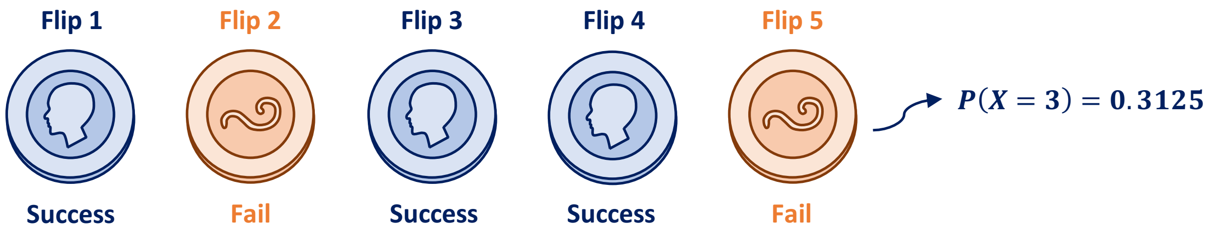
Standard Deviation (σ)

$$\sigma = \sqrt{pq}$$
$$\sigma = \sqrt{0.2 \times 0.8}$$
$$\sigma = 0.40$$

6.2 Binomial Distributions

A **binomial random variable** is the **number** of **successful outcomes** (x) for a given number of **independent Bernoulli trials** (n). They are represented with the **notation**, $X \sim B(n, p)$ or $X \sim Bin(n, p)$.

In other words, a **binomial distribution** is just a **Bernoulli trial** but repeated n **number of times**, such as **flipping a coin** where a **heads** is the **successful outcome** and **tails** is a **failure**, but **flipping it 5 times**.



For binomial distributions the next logical question to ask is, how do we create a **binomial distribution** that models the **probabilities** of different outcomes for **variable** X over n **trials**?

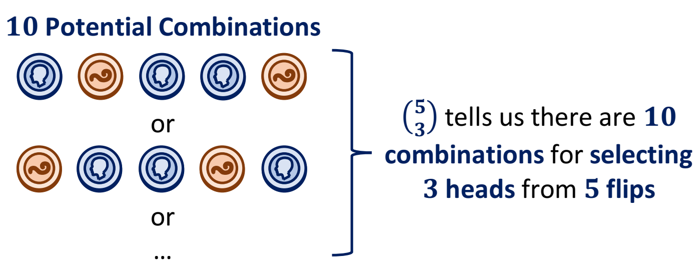
The answer – the **binomial distribution function**.

Binomial Distribution Function: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

The **variables** in this function should be mostly familiar, where n is the **number of trials**, x is the **number of successful trials**, and p is the **probability of success** of a single trial. To understand this function, let's break it down into its **two components**: $\binom{n}{x}$ and $p^x(1-p)^{n-x}$

The **first component**, $\binom{n}{x}$ (also denoted as nCx) comes from **combinatorics**, which is based on the idea that if there is n **trials** and x **successful trials**, there is a **set number** of **combinations** in which this can occur. We often say "n **choose** x" to describe $\binom{n}{x}$.

For instance, for our coin example above, where **three** out of the **five flips successfully land on heads**, there are **10 different ways** of **choosing three heads** from **five flips**:



The **combinatorics formula**, $\binom{n}{x}$, is **calculated** using the formula:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

In this formula, the **exclamation mark** $(!)$ is a **factorial symbol**. **Factorial symbols** (e.g. $n!$) mean that the **true value** of $n!$ is the **product** of the **integer multiplied** by all **positive integers less than itself**. This is shown for $3!$ and $5!$ below:

For $3!$	For $5!$
$3! = 3 \times 2 \times 1$	$5! = 5 \times 4 \times 3 \times 2 \times 1$

So applying the formula, $\binom{n}{x} = \frac{n!}{x!(n-x)!}$, to our **coin flipping example** of 3 **successful trials** out of 5 **total trials** (i.e. $\binom{5}{3}$) we would determine that there are 10 **possible combinations**:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

$$\binom{5}{3} = \frac{5!}{3!\,(5-3)!}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!}$$

$$\binom{5}{3} = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1}$$

$$\binom{5}{3} = \frac{5 \times 4 \times \cancel{3 \times 2 \times 1}}{\cancel{3 \times 2 \times 1} \times 2 \times 1}$$

$$\binom{5}{3} = \frac{20}{2}$$

$$\binom{5}{3} = 10$$

> Note, the nCr formula can be easily performed by your **calculator**, so use it when permitted.

The **second half**: $p^x(1-p)^{n-x}$, is derived from the fact that in a **single trial**, we can either have a **success** with **probability**, p, or a **failure** with **probability**, $1-p$.

Therefore, in a **binomial distribution** with n **trials** and x **successful trials**, the **probability** of x **successful trials** occurring is p^x, and the **probability** of the rest of the trials being failures is $(1-p)^{n-x}$.

Thus, **combining these two components** we get our **binomial distribution formula**:

> **Binomial Distribution Function**: $P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

For our **coin flip example**, the probability of getting **3 heads out of 5 flips** is determined as follows:

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$P(X = 3) = \binom{5}{3} 0.5^3 (0.5)^{5-3}$$

$$P(X = 3) = 10 \times 0.5^3 \times 0.5^2$$

$$P(X = 3) = 0.3125$$

Beyond **calculating** the **probability** of landing on **heads three times**, we can **apply** this formula for **all** of the **number of heads** for $x = 0$, $x = 1$, $x = 2$, $x = 4$ and $x = 5$ and then create a **probability distribution**:

For 0 Heads:

$$P(X = 0) = \binom{5}{0} \times 0.5^0 \times 0.5^5$$
$$P(X = 0) = 0.03125$$

For 1 Head:

$$P(X = 1) = \binom{5}{1} \times 0.5^1 \times 0.5^4$$
$$P(X = 1) = 0.15625$$

For 2 Heads:

$$P(X = 2) = \binom{5}{2} \times 0.5^2 \times 0.5^3$$
$$P(X = 2) = 0.3125$$

For 3 Heads:

$$P(X = 3) = \binom{5}{3} \times 0.5^3 \times 0.5^2$$
$$P(X = 3) = 0.3125$$

For 4 Heads:

$$P(X = 4) = \binom{5}{4} \times 0.5^4 \times 0.5^1$$
$$P(X = 4) = 0.15625$$

For 5 Heads:

$$P(X = 5) = \binom{5}{5} \times 0.5^5 \times 0.5^0$$
$$P(X = 5) = 0.03125$$

x	0	1	2	3	4	5
$P(X = x)$	0.03125	0.15625	0.3125	0.3125	0.15625	0.03125

Additionally, we can also use the formula: $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$, to determine **certain probability inequalities**, such as the **probability** of getting **more than 3 heads** (i.e. $P(X > 3)$). We can do this by **summing the probabilities** for $x = 4$ and $x = 5$ as follows:

① $P(X > 3) = P(X = 4) + P(X = 5)$

② $P(X > 3) = \binom{5}{4} \times 0.5^4 \times 0.5^1 + \binom{5}{5} \times 0.5^5 \times 0.5^0$

$P(X > 3) = 0.15625 + 0.03125$

③ $P(X > 3) = 0.1875$

① **Sum** the **probabilities** for $P(X = 4)$ and $P(X = 5)$

② **Substitute** in the **binomial distribution formula**

③ **Simply** to the **final answer**

We can also apply **conditional probability formula**: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, such as **calculating** the **probability** of having **less than five heads given at least three flips** are a **head**:

$$P(X < 5 | X \geq 3) = \frac{P(3 \leq X < 5)}{P(X \geq 3)}$$

$$P(X < 5 | X \geq 3) = \frac{P(X = 3) + P(X = 4)}{P(X = 3) + P(X = 4) + P(X = 5)}$$

$$P(X < 5 | X \geq 3) = \frac{0.3125 + 0.15625}{0.3125 + 0.15625 + 0.03125}$$

$$P(X < 5 | X \geq 3) = 0.9375$$

Hopefully, you can see that **binomial distributions** can **apply** the **same probability principles** that we applied in discrete random variables. The worked example below should help to **further illustrate** this.

Worked Example 1

Fraser has developed a fascination for botany and flowers. In particular, his garden at home is filled with **white roses** and **red roses**. After counting up all of the **white and red roses**, Fraser determines that **60% of the roses** are **white**. He also knows that he has so many roses that even if he picks several roses, the chance of getting a certain colour will **stay the same**. If Fraser **sells** these **roses 12 at a time** and X represents the **number of white roses selected** such that $X \sim Bin(12, 0.6)$, **help** Fraser to **determine** the following. **[CA]**

(a) The **probability** that Fraser selects **9 white roses**.

$$P(X = 9) = \binom{12}{9} 0.60^{9} \times 0.40^{3}$$
$$P(X = 9) = 0.142$$

(b) The **probability** that Fraser **selects 8 red roses**.

$$P(X = 4) = \binom{12}{4} 0.60^{4} \times 0.40^{8}$$
$$P(X = 4) = 0.042$$

(c) The **probability** that Fraser selects **at least 2 white roses**.

$$P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$$
$$P(X \geq 2) = 1 - \binom{12}{0} 0.60^{0} \times 0.40^{12} - \binom{12}{1} 0.60^{1} \times 0.40^{11}$$
$$P(X \geq 2) = 1 - 1.678 \times 10^{-5} - 3.020 \times 10^{-4}$$
$$P(X \geq 2) \approx 0.9997$$

(d) Determine the **probability** that Fraser **doesn't** select a **white rose** until the **fifth rose**.

Since the **order is known**:

$$0.40^{4} \times 0.60^{1} = 0.01536$$

(e) The **probability** that Fraser selects **12 white roses**, **given** he selects **at least 10 white roses**.

$$P(X = 12 | X \geq 10) = \frac{P(X = 12)}{P(X = 10) + P(X = 11) + P(X = 12)}$$

$$P(X = 12 | X \geq 10) = \frac{\binom{12}{12} 0.60^{12} \times 0.40^{0}}{\binom{12}{10} 0.60^{10} \times 0.40^{2} + \binom{12}{11} 0.60^{11} \times 0.40^{1} + \binom{12}{12} 0.60^{12} \times 0.40^{0}}$$

$$P(X = 12 | X \geq 10) = \frac{2.177 \times 10^{-3}}{0.0639 + 0.0174 + 2.177 \times 10^{-3}}$$

$$P(X = 12 | X \geq 10) = 0.0261$$

Mean, Variance and Standard Deviation of Binomial Distributions

Like **discrete random variables** and **Bernoulli trials**, every **binomial distribution** has a **mean**, **variance** and **standard deviation** that helps us to better understand the nature of the distribution.

Since a **binomial distribution** is just a **Bernoulli trial repeated n times**, the **mean**, **variance** and **standard deviation** will be the **formulas** for the **Bernoulli trial multiplied** by n.

This gives us the following **formulas** for a **binomial distribution**:

	Mean ($E(X)$)	Variance ($Var(X)$)	Standard Deviation (σ)
Binomial Distribution	$E(X) = np$	$Var(X) = npq$	$\sigma = \sqrt{npq}$

To understand how to **apply** these formulas, let's continue with our **coin flipping example**.

In this example, a **coin** is **flipped five times** and **landing on a head** is considered a **success** and the **probability of success** is 0.5. Therefore, for $X \sim B(5, 0.5)$, we can use our **mean**, **variance** and **standard deviation formulas** by **substituting** $n = 5$, $p = 0.5$ and $q = 0.5$ as follows:

Mean ($E(X)$)

$$E(X) = np$$
$$E(X) = 5 \times 0.5$$
$$E(X) = 2.5$$

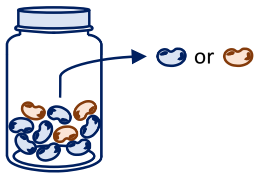
Variance ($Var(X)$)

$$Var(X) = npq$$
$$Var(X) = 5 \times 0.5 \times 0.5$$
$$Var(X) = 1.25$$

Standard Deviation (σ)

$$\sigma = \sqrt{npq}$$
$$\sigma = \sqrt{5 \times 0.5 \times 0.5}$$
$$\sigma = 1.12$$

As another example, let's consider we have a **jar** of **10 jellybeans** with **7 blue jellybeans** and **3 orange jellybeans**, and a **successful outcome** is **selecting a blue jellybean** from the jar. If there are **10 trials with replacement** and the **probability of picking a blue jellybean (success)** is 0.70 then $X \sim B(10, 0.70)$.



If X represents the selection of a **blue jellybean**
$X \sim B(10, 0.70)$

Using our formulas and **substituting** in $n = 10$, $p = 0.70$ and $q = 0.30$, we can determine the **mean**, **variance** and **standard deviation** to be:

Mean ($E(X)$)

$$E(X) = np$$
$$E(X) = 10 \times 0.70$$
$$E(X) = 7$$

Variance ($Var(X)$)

$$Var(X) = npq$$
$$Var(X) = 10 \times 0.70 \times 0.30$$
$$Var(X) = 2.1$$

Standard Deviation (σ)

$$\sigma = \sqrt{npq}$$
$$\sigma = \sqrt{10 \times 0.70 \times 0.30}$$
$$\sigma = 1.45$$

These formulas can be used for **any binomial distributions**, and in our final problem set of Unit 3 we are going to see a **wide range of scenarios** that we can **apply binomial distributions** to.

Worked Example 2

Fraser is in a hurry at a florist and spots a bucket of roses of which **two fifths** are **red**, whilst the rest are **white**. As he is selecting $\mathbf{12}$ **roses**, the florist continues to **replace** them **straight away** such that $\mathbf{60}\%$ of the **roses** are **white**. **[CA]**

(a) Letting $\boldsymbol{X}$ be the **number** of **white roses** selected such that $\boldsymbol{X \sim Bin(12, 0.6)}$, determine the **number** of **white roses** Fraser should **expect** to select.

Mean $(\boldsymbol{E(X)})$

$$\boldsymbol{E(X) = np}$$

Substitute $\boldsymbol{n} = \mathbf{12}$ and $\boldsymbol{p} = \mathbf{0.6}$

$$\boldsymbol{E(X) = 12 \times 0.60}$$

$$\boldsymbol{E(X) = 7.2}$$

(b) **Help** Fraser to determine the **variance** and **standard deviation** of this binomial distribution.

Variance $(\boldsymbol{Var(X)})$

$$\boldsymbol{Var(X) = npq}$$

$$\boldsymbol{Var(X) = 12 \times 0.6 \times 0.40}$$

$$\boldsymbol{Var(X) = 2.88}$$

Standard Deviation $(\boldsymbol{\sigma})$

$$\boldsymbol{\sigma = \sqrt{npq}}$$

$$\boldsymbol{\sigma = \sqrt{12 \times 0.6 \times 0.4}}$$

$$\boldsymbol{\sigma = 1.70}$$

BERNOULLI AND BINOMIAL DISTRIBUTIONS – TOPIC NOTES

Bernoulli Distributions

A **Bernoulli trial** is an event that only has **two possible outcomes**, a **success** or **failure**, and is **only performed once**.

We can create a table to illustrate a Bernoulli trial. As a Bernoulli trial, X, we can allocate the **failure outcome** a value of $x = 0$ and the **success outcome** a value of $x = 1$. This creates the following general **Bernoulli distribution**.

x	0	1
$P(X = x)$	$1 - p$	p

From the equations for **mean**, **variance,** and **standard deviation** and the table above, the following equations can be derived, specific to the **Bernoulli Distribution**.

Mean: $E(X) = p$

Variance: $Var(X) = pq$

Standard Deviation: $\sigma = \sqrt{pq}$

Binomial Distributions

A **binomial random variable** is the **number** of **successful outcomes** (x) for a given number of **independent Bernoulli trials** (n). They are represented with the **notation**, $X \sim B(n, p)$ or $X \sim Bin(n, p)$.

The **Binomial Distribution** models the **probabilities** of different outcomes for **variable** X over n **trials** and is illustrated by the following equation.

Binomial Distribution Function: $P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$

The **combinatorics part of the formula**, $\binom{n}{x}$, that if there is n **trials** and x **successful trials**, there is a **set number** of **combinations** in which this can occur. It is **calculated** using the formula:

$$\binom{n}{x} = \frac{n!}{x!\,(n - x)!}$$

There are also **criteria** that must be met for a **distribution** to be referred to as **binomial**. These are:

Criteria of a Binomial Distribution
1. There is a **success** and **failure** outcome
2. The **number of trials** (n) is **fixed**
3. Each trial is **independent**
4. The **probability of success** is **constant** for each trial

Like the **Bernoulli Distribution**, the **Binomial Distribution** also has equations for determining the **mean**, **variance,** and **standard deviation**, as illustrated below.

	Mean ($E(X)$)	Variance ($Var(X)$)	Standard Deviation (σ)
Binomial Distribution	$E(X) = np$	$Var(X) = npq$	$\sigma = \sqrt{npq}$

The following table illustrates the rules for both **Bernoulli Trials** and the **Binomial Distribution**.

Distribution	Mean ($E(X)$)	Variance ($Var(X)$)	Standard Deviation (σ)
Bernoulli Trial	$E(X) = p$	$Var(X) = pq$	$\sigma = \sqrt{pq}$
Binomial Distribution	$E(X) = np$	$Var(X) = npq$	$\sigma = \sqrt{npq}$

Problem Set 8 – Bernoulli and Binomial Distributions Progressive Questions

Concept 1

Bernoulli and Binomial Distributions – Progressive Questions (4 questions)

Repetitive questions: 1.11 → 1.41 (4 questions)

Going for Gold at the Olympics!

Starring: Teacher Andrew, Teacher Simon, Tyler, Fraser, Tom, Kerry, Harriet, Shauna

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The Mathematics College has been well-known for producing some of the finest mathematics students in the world. Recently, Teacher Simon entered his students into the Olympics to participate as the official mathematicians for the Olympics!

The students are supporting the Australian athletes through using Bernoulli and Binomial Distributions to turn the odds of the sports in their favour and help Australia win their first ever Olympics.

Bernoulli and Binomial Distributions: Q1, Q2, Q3, Q4

Repetitive: 1.11 → 1.41 (4 questions)

[CA] [4 marks]

1. As part of the application process, the students are given a few scenarios to analyse. They must correctly **identify** whether the following scenarios are **Bernoulli trials** or **not** to guarantee their interview to become the official mathematicians of the Olympics. Can you help them **tick** the **correct boxes**?

Scenario	Yes	No
Flipping a coin with **heads** being a **success** and **tails** being a **fail**.		
Drawing coloured batons from a bag containing **4 orange batons** and **3 blue batons** and x represents the **number of blue batons selected**.		
A **spinner** is spun with **numbers one through to four** and x represents the **value** of the **number landed** on (i.e. $x = 1, 2, 3, 4$)		
A **spinner** with one hundred sections is spun where a **success** is the spinner **lands on one** and a **failure** is it **lands on another number**.		

[CA] [5 marks]

2. Following their successful application, the students are invited to a formal interview with the Australian Olympics panel. For their first interview question, the students are given a **seven-sided dice** where a **success** occurs if the value of the dice (x) is $x = 1, 2, 3$ and a **fail** is $x = 4, 5, 6, 7$.

(a) **Help** the students to **identify** the **type** of **probability distribution** this is and its **parameters**. **[2]**

(b) What is the **mean**, **variance** and **standard deviation** of this distribution? **[3]**

[CA] [9 marks]

3. As a second part of the interview, the students are handed a **multiple-choice quiz** based on the history of the Olympic games with **five possible answers per question** (i.e. **a through to e**). The students have no clue about history, so they know **every answer** will be a guess.

(a) If there is only one question and $x = 1$ if they get the **answer correct** and $x = 0$ if they get the **answer wrong**, what type of **distribution** is this and what are its **parameters**? **[2]**

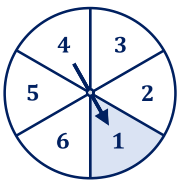
(b) **What** is the **mean**, **variance** and **standard deviation** of this distribution? **[3]**

(c) If they are handed a new test with **twenty questions** and Y represents the **number of questions** they get **right**, what **type** of distribution is this and what are its **parameters**? **[2]**

(d) Determine the **mean** and **variance** for **distribution Y**. **[2]**

[CA] [6 marks]

4. The Australian Olympic Panel are tasked with selecting their mathematicians for the event. Unable to choose from the wide array of candidates, they decide to leave it up to chance. Each mathematics group is **allocated a number on a spinner** from 1 to 6 with the **maths students** from Mathematics College **allocated the number** 1.



Success if spinner lands on 1

(a) If the students at Mathematics College have a $\frac{1}{6}$ **probability** of being chosen, what is the **mean** and **variance** of this distribution if the spinner is **spun once**? **[2]**

(b) The Olympic committee decide to spin the spinner **100 times** and see which team pops up the most. If Y represents the **number of times the students** at **Mathematics College** are **landed on**, what **type** of distribution is this and what are its **parameters**? **[2]**

(c) Determine the **mean** and **variance** for **distribution Y**. **[2]**

Success! Teacher Janet rushes to the students in excitement as they receive the news that they have been granted the position of the official mathematicians of the Olympics.

Excited to be the official mathematicians for the Olympics, the methods students are eager to apply their knowledge of Bernoulli and Binomial distributions to help Australia win their first ever Summer Olympic Games!

Concept 2

Applications of Binomial Distributions – Progressive Questions (7 questions)

Repetitive questions: 2.11 → 2.71 (9 questions)

Going for Gold at the Olympics!

Starring: Teacher Andrew, Teacher Simon, Tyler, Fraser, Tom, Kerry, Harriet, Shauna

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The Olympics have officially begun and the students are determined to start to shift the odds in their favour so that Australia can start to win in their events. Their plan is to achieve this through manipulating the expected value, variance, and by finding the probabilities of certain outcomes using Binomial distributions.

Applications of Binomial Distributions: Q1, Q2, Q3, Q4, Q5, Q6, Q7

Repetitive: 2.11 → 2.71 (9 questions)

[CA] [7 marks]

1. It is the first event of the day, the **female javelin event**. Tyler is analysing data, which shows that the chosen athlete, Sarah, has a **success rate** of $\mathbf{0.78}$. Each athlete has **6 throws** in the event and the **random variable** $\boldsymbol{Y}$ denotes the number of **successful throws** in the event. Assist Tyler to predict Sarah's performance.

(a) What is the **expected value** and **variance** of Sarah's performance? **[2]**
(b) Find the **probability** that Sarah performs **exactly 5 successful** throws. **[1]**
(c) What is the **probability** Sarah has **more than** $\mathbf{1}$ **successful throw**? **[2]**
(d) Another athlete in the same event achieved **4 successful throws** in this event, what is the **probability** that Sarah throws **more than 4 successful throws** and **wins** this event? **[2]**

[CA] [6 marks]

2. Tyler and his friends leap with excitement when they find out that Sarah won the javelin event! For the next event, Teacher Simon asks Fraser to observe the volleyball tournament to record the **success rate of points** of the Australian team. Fraser concludes that the volleyball team **wins points** at a **success rate** of $\mathbf{0.6}$. Fraser uses the **random variable** $\boldsymbol{X}$ to denote the **number of points** the **Australian volleyball team win per game**.

(a) If there is an **average of** $\mathbf{100}$ **points** played per game, what **expected value** and **variance** should Fraser report for the Australian team? **[2]**
(b) If in their first Olympic match against Germany they need exactly $\mathbf{65}$ **points** to **win**. What is the **probability** that **Australia win**? **[1]**
(c) In their semi-finals against Columbia, what is the **probability** they **win** $\mathbf{71}$ **or** $\mathbf{72}$ **points**? **[2]**
(d) In the grand final they are up against the feared United Kingdom team. What is the **probability** they **win** $\mathbf{80}$ **points**? **[1]**

[CA] [9 marks]

3. The athlete returned with yet another gold medal! Teacher Simon turns to Kerry to confirm the next win. Kerry is working with Adam, an **excellent archery athlete**, whose previous data shows his **probability of scoring** in the **inner circle** is $\mathbf{0.55}$. Kerry uses a **random variable** $\boldsymbol{X}$ to define the **number of times** Adam **successfully scores** in the **inner circle**.

(a) In a game where Adam shoots **$\mathbf{10}$ arrows**, **how many** would he **expect** to land in the inner circle? **[1]**

(b) What is the **probability** that Adam shoots **$\mathbf{7}$ or $\mathbf{8}$ inner circles**? **[2]**

(c) In a game where Adam scores **no less** than **$\mathbf{7}$ inner circles**, what is the **probability** that he scores **$\mathbf{9}$ inner circles**? **[3]**

(d) Across the day, Adam competes in **$\mathbf{12}$ games** against 12 different countries. Find the **probability** that Adam scores **at least $\mathbf{7}$ inner circles in $\mathbf{6}$ games**. **[3]**

[CA] [9 marks]

4. Shauna is more interested in athlete injuries than the actual sports. She decides to start exploring athlete injuries and discovers that $\mathbf{32\%}$ of athletes that have had **prior injuries**, **experience a minor injury** during the Olympics. Given that there are **$\mathbf{40}$ Australian athletes** who have had prior injuries, let $\boldsymbol{Y}$ be the **number of those athletes** who will experience a **minor injury**.

(a) What is the **probability** that **ten** of those **athletes** have an **injury**? **[2]**

(b) What is the **probability** that **more than two** of the athletes experience a **minor injury**? **[2]**

(c) What is the probability that **twenty or twenty one** athletes experience a **minor injury**? **[2]**

(d) A sample of athletes are randomly selected from **another country**, which has a much higher **injury rate** of $\mathbf{75\%}$ amongst $\mathbf{25}$ previously injured athletes. **Given** that there **more than $\mathbf{22}$ players** were injured, what is the **probability** that all **$\mathbf{25}$ players** will be injured? **[3]**

[CA] [11 marks]

5. Lazy Fraser notices that $\mathbf{98\%}$ of athletes **arrive on time** to their events. Despite often being late to class, Fraser is determined to ensure that all of the Australian athletes attend their events at the correct time. Additionally, the Olympic committee has a policy that if athletes are **late to their event** the athlete will be **disqualified**.

(a) Of the **$\mathbf{238}$ athletes**, what is the **probability at least $\mathbf{237}$** of them **arrive on time** to their event? **[2]**

(b) What is the **probability** that **at least one Australian athlete** will be **disqualified**? **[1]**

(c) Of the **$\mathbf{238}$ athletes**, what is the **probability** that **less than three** of the athletes are disqualified, **given** that **one athlete** has **already been disqualified**. **[3]**

(d) Fraser notices that one specific athlete in fencing is particularly lazy with a $\mathbf{62\%}$ **chance** of **being late** to their event. If they have **twelve different opponents to face**, what is the **probability** this athlete is **disqualified**. **[2]**

(e) For another country, Fraser identifies that they have a **record breaking lateness rate** of $\mathbf{0.14}$. If they have **$\mathbf{212}$ athletes**, what is the probability that **$\mathbf{14}$ athletes** are **disqualified**? **[3]**

[CA] [12 marks]

6. Lucky student Tom has always dreamed of becoming an Olympic athlete, particularly in the sport of diving. To gain some inspiration, Tom decides to analyse the Australian diving scores. Diving is scored on a **scale of 1 to 10**, and Tom decides that a **successful dive** is a score of **8 or greater**. He determines that Australian divers have a **0.43 probability** of having a **successful dive**.

(a) Of the **three attempts** a diver gets, what's the probability they have **at least one successful dive**? **[3]**

(b) Tom determines that a diver needs **at least two successful dives** to qualify to the next round of the competition. What is the **probability** an Australian diver **moves to the next round**? **[2]**

(c) Australia's best diver has a much **higher success rate** of **0.85**. What is the **probability** that diver has **at least two successful dives**? **[2]**

(d) If Australia's best diver needs to **win five rounds** to make it to the grand finale, what is the **probability** that Australia's best diver **wins all five rounds**? **[2]**

(e) Australia's best diver makes it to the grand finale. If they need **all three dives** to be **successful** what is the **probability** this occurs, **given** the diver has already had **one successful dive**? **[3]**

[CA] [10 marks]

7. Coming into the final event of the Olympic games, Australia is tied for first place in the medal count with America with a total of 112 medals each. The final event is the **women's grand final** for **soccer** and its Australia vs the United States. After the first 90 minutes and overtime, the game is all-tied up at 3-3 and it will go to a **penalty shootout**.

After running some analytics for the preliminary and knockout games, Harriet has calculated that Australia has a **0.63 success rate** when taking a **penalty shot**.

(a) If there are **five penalty shots** per team, **how many shots** can Australia **expect** to score? **[2]**

(b) Of the five penalty shots taken, what is the **probability** Australia **scores all five**? **[1]**

(c) What is the probability Australia scores the **first three goals** but **misses** the **fourth** and the **fifth shot**? **[2]**

(d) The typical range to **win** is between **3 and 5 successful shots**. What is the **probability** Australia is within this range? **[2]**

(e) The United States closes out the penalty shootout with **3 successful shots**. What is the **probability** that **Australia score more than 4 goals** given they have already scored **two successful shots**? **[3]**

And success! Australia successfully beat the United States in the soccer grand final and are crowned the winners of the Olympic Games! The methods students and teachers are over the moon for their amazing effort! The only person not celebrating is Teacher Andrew, who is jealous that he was rejected from the Mathematics team. In celebration, Tyler shouts gloriously "Could mathematics get any better?"

Spiteful Teacher Andrew replies "It won't. I will make sure of that." The students are shocked by this response. Teacher Andrew walks off with an evil laugh.

One things for sure, Teacher Andrew is going to be up to some trouble in the second half of the semester…

Problem Set 8 – Bernoulli and Binomial Distributions Repetitive Questions

Concept 1

Bernoulli and Binomial Distributions – Repetitive Questions

(4 questions)

Bernoulli and Binomial Distributions: Qs 1.11, 1.21, 1.31, 1.41

[4 marks]

1.11 From the table below, **determine** whether or not the following scenarios are **Bernoulli distributions** by **ticking** the **correct boxes**.

Scenario	Yes	No
Rolling a six-sided dice with **one** as a **success** and other values as a fail.		
Selecting coloured sweets from a bag of 4 strawberry flavoured and **9 banana** flavoured which are replaced after every choice.		
Randomly selecting pens from a pencil case and considering their colour.		
Flipping a coin to for netball with **heads** being a **success** and tails being a fail.		

[CA] [10 marks]

1.21 In a simple game between Fraser and Harriet, they have an **even, ten-sided dice** where a **success** occurs if the **value** on the **dice** is **even** and a fail if the value on the dice is an odd number.

(a) If the dice is **rolled once**, **help** the students to **identify** the **type** of **probability distribution** this is and its **parameters**. **[2]**
(b) What is the **mean**, variance and **standard deviation** of this distribution? **[3]**
(c) If the dice is **rolled ten times**, **help** the students to **identify** this new **type** of **probability distribution** and its **parameters**. **[2]**
(d) What is the **mean**, variance and **standard deviation** of this new distribution? **[3]**

[CA] [9 marks]

1.31 Continuing to explore their interest in Bernoulli and Binomial distributions, Fraser sets up a multiple-choice quiz for Harriet with **four possible answers per question** (i.e. **a through to d**) about university level mathematics. With no prior understanding about university level maths, Harriet is forced to **guess for each question**, with a **success** being getting the **correct answer** and a fail being getting the incorrect answer.

(a) If there is only one question, **help** Harriet to identify type of **distribution** this is and its **parameters**. **[2]**
(b) **What** is the **mean**, variance and **standard deviation** of this distribution? **[3]**

(c) If Fraser gives Harriet a longer test with **ten questions**, what **type** of distribution is this and what are its **parameters**? **[2]**
(d) Determine the **mean** and **variance** for this new **distribution**. **[2]**

[CA] [6 marks]

1.41 Harriet and Fraser decide to take their fun outside and onto the soccer field with a penalty shootout. Being on the Mathematics College soccer team, Harriet has a 55% **chance** of **scoring a goal** when Fraser is goal-keeper. If X represents the **number of goals scored** and a **success** is **scoring a goal** and a **fail** is **missing**, **help Harriet** to determine the following:

(a) If Harriet takes **one penalty shot**, what is the **mean** and **variance** of this distribution? **[2]**
(b) If Harriet takes **ten penalty shots**, what **type** of distribution is this and what are its **parameters**? **[2]**
(c) What is the **mean** and **variance** for this distribution? **[2]**

Concept 2

Applications of Binomial Distributions – Repetitive Questions (9 questions)

Applications of Binomial Distributions: Qs 2.11, 2.12, 2.21, 2.31, 2.32, 2.32, 2.41, 2.51, 2.61, 2.71

[CA] [10 marks]

2.11 As a top level soccer player, Shauna has a **success rate** of 0.84 for scoring a goal in a penalty shootout. If she takes **5 shots** in a penalty shootout and **X represents** the **number of goals she scores**:

(a) What is the **expected value** and **variance** of Shauna's performance? **[2]**
(b) What's the **probability** that Shauna will **score exactly 3 goals**? **[1]**
(c) What is the **probability** that Shauna will **score less than two goals**? **[3]**
(d) What is the **probability** that Shauna will **more than two goals**? **[2]**
(e) What is the **probability** that Shauna will have incredible penalty shootout and **score all five goals**? **[2]**

[CA] [9 marks]

2.12 After conducting a survey at Mathematics College, it was determined that 87% of students in a year group **have siblings**. **Twenty students** are randomly selected to be in sample group. The random **variable X** is the **number of those students** who **have a sibling**.

(a) If Kerry predicts that there will be **fifteen of the students with siblings**, what is the **probability** that her prediction will be **correct**? **[1]**
(b) What is the **expected number** of students with siblings in the sample group? **[1]**
(c) What is the **probability** that **at least 18 students** in the sample group have a sibling? **[3]**
(d) Given there are **at least 18 students with siblings** in the sample group, what is the **probability** that **all 20 students have siblings**? **[3]**
(e) What will be the **variance** of this distribution? **[1]**

[CA] [11 marks]

2.21 The Australian rugby team is being analysed to determine how effective their tackles are. After observing games for a season, it was determined that their tackles had a **success rate** of 0.90**.** The **random variable** X was used to denote the **number of successful tackles** the team performs.

(a) A total of 80 **important tackles** were performed within a game. Define the **appropriate distribution** for the **number of successful tackles** made by the Australian rugby team. **[1]**
(b) What is the **probability** that **at least** 77 of the tackles were **successful**? **[2]**
(c) What is the probability that no more than 77 of the tackles were successful, **given at least** 75 of the tackles were **successful**. **[3]**
(d) What was the **expected value**, variance and **standard deviation** for the **number of successful tackles** made during the game? **[3]**
(e) In a different game, it was observed that the **expected value** for the same team was 30. **How many tackles** were made by the Australian team assuming the **same success rate**? **[2]**

[CA] [8 marks]

2.31 Janitor Peter, plants **tomato crops** every year and previous data shows the probability of having a crop dying early is 0.15.

(a) In a season where 20 crops are grown, how many would you **expect** to survive to harvest? **[1]**
(b) What is the probability that Janitor Peter grows 10 **crops successfully**? **[1]**
(c) In this season there has already been **one crop die early**. What is the **probability** that less than four crops die early? **[3]**
(d) During the whole summer season, there were 8 **individual planting sessions** of 20 **crops per session**. What is the **probability** that **at most one crop dies** in 6 of the planting sessions. **[3]**

[CA] [7 marks]

2.32 Over the winter break, Kerry decides to become a chef, however, she isn't a very good cook. The **probability** of one of her **meals being edible** is 0.30**.** Her boss decides to order some meals to see how good her cooking is, using the **random variable** X to define the **number of times** her cooking has been **edible**.

(a) If her boss **orders seven meals**, how many would you **expect** to be **edible**? **[1]**
(b) What is the **probability** that Kerry prepares **more than one edible meal**? **[2]**
(c) What is the **probability** that out of **seven meals** that Kerry prepares for her boss, **at least five** of them are **edible**? **[2]**
(d) The boss asks Kerry to prepare **seven meals** on Tuesday and **seven meals** on Wednesday. What is the **probability** that there are **at least two edible meals** on each day? **[2]**

[CA] [11 marks]

2.41 In a singing TV show, 15% of the applicants make it past the **first round of auditions**. If in a specific group there are 12 **applicants** and X is the **number of applicants** that make it through to the next round of auditions, **help determine** the following:

(a) What is the **mean**, variance and **standard deviation** of **distribution** X? **[3]**
(b) What's the **probability** that **all** 12 **of the applicants** make it through the **first round** of auditions? **[1]**

(c) What is the **probability** that **at least nine applicants** are **unsuccessful** in making it to the next round of auditions? **[2]**

(d) **Given** that they must let **at least two** of the applicants through to the next round, what is the probability that **at least four** of the contestants make it through? **[3]**

(e) In a different group of applicants, the **expected number of applicants** to make it through was 15. What is the **size of this group** assuming the **same success rate**? **[2]**

[CA] [9 marks]

2.51 It is determined that with **two of hours study per day**, 68% of those students will obtain an ATAR of **more than** 95. If X represents the **number of students** in a **sample of 10 students** who achieve an ATAR of **more than** 95, help to determine the following:

(a) What is the **number of students expected** to achieve an ATAR of **more than** 95? **[1]**

(b) What is the **probability** that **seven** of the **students** achieve an ATAR of **more than** 95? **[1]**

(c) What is the **probability** that **at least three** of the students in the sample group, achieve an ATAR of **more than** 95? **[2]**

(d) **Given** that **no less than two students** achieve an ATAR of more than 95, what is the **probability** that **at least eight** do? **[3]**

(e) If in a different group, the **expected number of students** to get an ATAR of more than 95 was 15, what is the **size** of the group assuming a **lower success rate** of 0.52? **[2]**

[CA] [9 marks]

2.61 A phone company has a track record of producing fully functioning phones 99% **of the time**. Let X define the **number of phones produced** that are **fully functioning**.

(a) In a sample of 100 **phones**, what is the **probability** that **at least** 98 of them are **fully functioning**. **[2]**

(b) What is the **number of phones** that are **expected** to be working and what is the **standard deviation**? **[2]**

(c) **Given** that 95 of the **phones work**, what is the **probability** that **all** 100 **phones** will be working? **[3]**

(d) In a different batch it is **only expected** that 49 of the phones will work. **How many phones** are in this batch if the **functioning rate** is 0.98? **[2]**

[CA] [9 marks]

2.71 A robot produces **green**, **blue**, **orange**, **white** and **red coloured toys** for a company known as 'Larger Toys' and their highest toy in demand is the **green coloured toys**. Let X be the **number green coloured toys** that are **produced by the robot**.

(a) If the robot **produces green toys** 87% of the time, what is the **probability** that **8 green toys** are produced in a batch of 9 toys? **[2]**

(b) What is the probability that **at least 7 green toys** are produced in a batch of 9 toys? **[2]**

(c) If there are **8 batches of toys**, with **9 toys in each batch**, what is the **probability** that **6 of these batches** have **at least 7 green toys**? **[3]**

(d) Out of the **8 batches**, what is the **expected number of batches** to have **at least 7 green toys** in them? **[2]**