nelson maths.

12

Mathematical Methods

Units **3** & **4**

Lachlan Grierson
Sam Hodgson
Anna Wethereld

Series editor
Stephen Swift



QLD



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Contributing authors: Ross Brodie, Regina Edwards, Stephen Swift

LEARNING DISCOVERY

TRIGONOMETRIC FUNCTIONS



This dappled light effect can be caused by light passing through transparent media such as air and soap bubbles. When the light bends at the edges of these mediums, it causes a rainbow-like pattern of colours. Models of sine and cosine functions often represent light waves.

QLD

Nelson

Nelson Maths 12 Mathematical Methods
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ISBN 9780170484985

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Permissions/Photo researcher: Lumina Datamatics
Cover: iStock.com/Eerik
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National Library of Australia Cataloguing-in-Publication Data
A catalogue record for this book is available from the National Library of Australia

Cengage Learning Australia
Level 5, 80 Dorcas Street
South Melbourne, Victoria Australia 3205

For learning solutions, visit **cengage.com.au**

Printed in Australia by Ligare Pty Limited.
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Preface

Nelson Maths 11–12 QLD is a new QCE mathematics series that is backed by research into the science of learning. The design and structure of the series has been informed by teacher advice and evidence-based pedagogy, with the focus on preparing QCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past QCE exam questions and ensuring that all theory and examples are precisely mapped to the QCAA syllabus and exams.
- To reduce the **cognitive load** for 21st century learners, explanations are clear and concise, using the technique of **chunking** text with accompanying diagrams and infographics.
- The exercise structure of **Recap**, **Mastery** and **Exam practice** leads students from procedural fluency to **higher-order thinking** using the learning techniques of **interleaving**.
- **Exam practice** includes exam-style questions and past QCE exam questions with **examiner advice and feedback**.
- The cumulative structure of Exercise **Recaps** and **Cumulative examinations** in every chapter is built on the learning and memory techniques of **spacing** and **retrieval**, as recommended by the new syllabus.

About the authors

Lachlan Grierson has taught Mathematics and Information Technology in schools and at TAFE for over 20 years, most recently at Emerald and Warwick Christian Colleges. He was a QCAA Panellist for Information Processing Technology on the Gold Coast, and recently wrote topic tests for *Nelson Maths 7–10*.

Sam Hodgson teaches at Brisbane State High School. He has taught Mathematical Methods and marked its external exam extensively since it was introduced in 2019. Sam works at a large academic school where he also teaches Specialist Mathematics, is involved in Fast-Track Mathematics programs, and coordinates the Year 10 Mathematical Methods Preparation course.

Anna Wethereld has been a leader in schools, curriculum, assessment and teacher training for over 25 years, including at Brisbane State High School, QCAA and Queensland College of Teachers. She was a Panellist for Mathematics C (Specialist Maths) for Brisbane North, a Head of Mathematics and Deputy Principal. As the state's Mathematics Curriculum Manager at NESA, Anna led Mathematics K–12 in NSW and coordinated the development of its Years 11–12 syllabuses.

Series editor **Stephen Swift** taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School. He was the lead author of *Nelson QMaths 7–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

Contributing authors

Amy Hawke and **Tara Kuhn** wrote many of the worksheets.
Brandon Pettis wrote the worked solutions.
Scott Smith, **Katie Jackson** and **Cherylanne Saywell** created the video tutorials.
Joel Speranza created the *Exam question analysis* videos.
Tom Sprenger, **Tania Eastcott** and **Elizabeth Nabhan** wrote the topic tests.

Syllabus reference grid

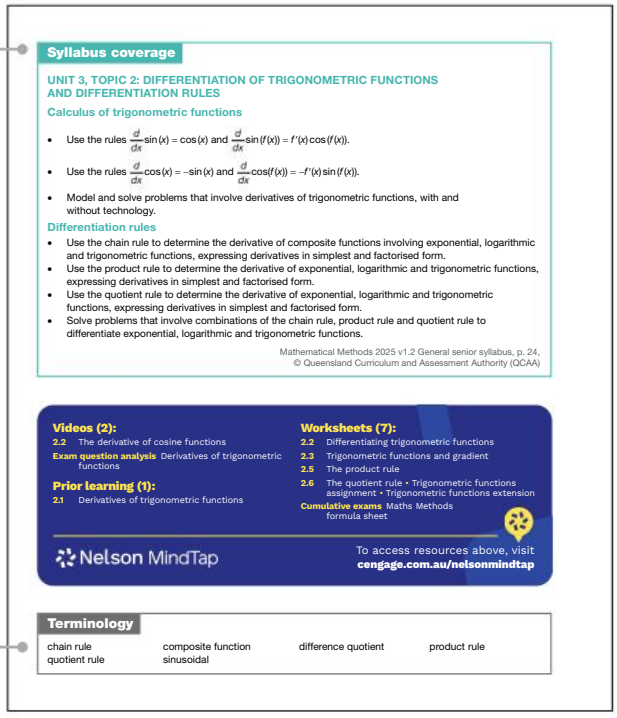
Topics and subtopics	*Nelson Maths 12* *Mathematical Methods QLD* chapter	
UNIT 3: FURTHER CALCULUS AND INTRODUCTION TO STATISTICS		
1 Differentiation of exponential and logarithmic functions		
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Calculus of logarithmic functions	1	Derivatives of exponential and logarithmic functions
2 Differentiation of trigonometric functions and differentiation rules		
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1 Further integration		
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Applications of integration	6	Integration and areas
2 Trigonometry		
Cosine and sine rules	7	The sine and cosine rules
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Confidence intervals for proportions	10	Confidence intervals for proportions

Source: Mathematical Methods 2025 v1.2 General senior syllabus pp. 24-30 © Queensland Curriculum and Assessment Authority (QCAA)

About this book

At the beginning of each chapter

Syllabus coverage and extracts are shown at the front of the chapter, along with a listing of **Nelson MindTap** chapter resources.



Terminology previews the keywords and phrases from within the chapter.

In each chapter

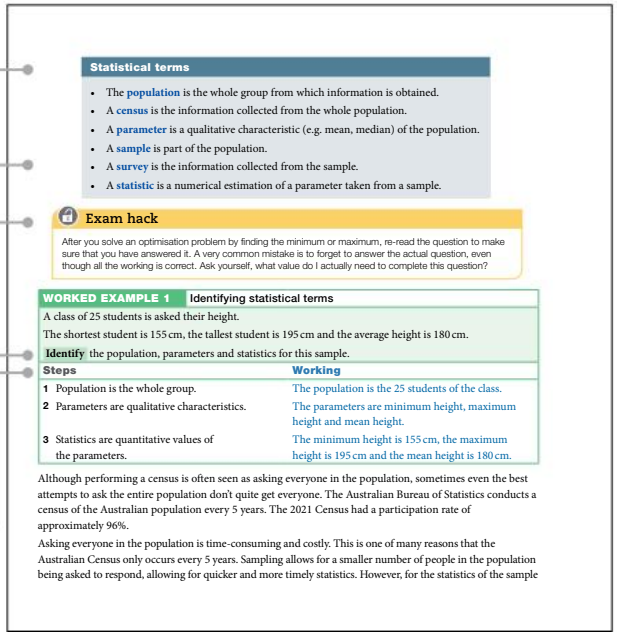
Important facts and formulas are highlighted in a shaded box.

Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Exam hacks highlight valuable exam hints and common student errors.

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more verbs and details.

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.



In each chapter exercise, **Recap** questions revise skills from the previous exercise (useful for lesson starters), **Mastery** questions provide skill practice linked to worked examples, and **Exam practice** applies learned skills to exam-style problems and past QCE questions, including multiple choice, short response, problem-solving, simple and complex questions.

Past QCE exam questions are clearly tagged.
For example, © QCAA 2023 1Q18 means 2023 Exam Paper 1 Question 18.
For multiple choice questions, the percentage of students who chose the correct answer is shown (except for the 2020 exam), for example, 68%.

KEY

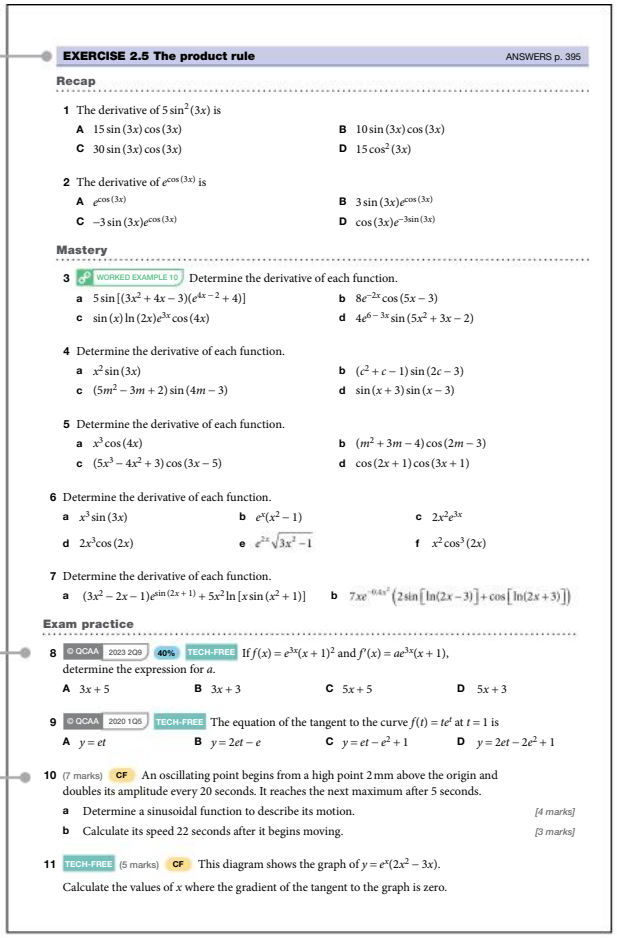
2021	2021 exam year
2020S	QCAA sample question
1	Paper 1
2	Paper 2

Complex familiar CF and complex unfamiliar CU questions are clearly tagged and colour-coded.

Investigations explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.

Technology promotes ICT in the classroom, using spreadsheets and the internet.

Graphics calculator instructions are provided for the TI-84 Plus CE, Casio fx-CG50 AU and TI-Nspire CX calculators.



EXERCISE 2.5 The product rule ANSWERS p. 395

Recap

1 The derivative of $5\sin^2(3x)$ is
A $15\sin(3x)\cos(3x)$ **B** $10\sin(3x)\cos(3x)$
C $30\sin(3x)\cos(3x)$ **D** $15\cos^2(3x)$

2 The derivative of $e^{\cos(3x)}$ is
A $e^{\cos(3x)}$ **B** $3\sin(3x)e^{\cos(3x)}$
C $-3\sin(3x)e^{\cos(3x)}$ **D** $\cos(3x)e^{-3\sin(3x)}$

Mastery

3 WORKED EXAMPLE 10 Determine the derivative of each function.
a $5\sin[(3x^2+4x-3)(e^{4x-2}+4)]$ **b** $8e^{-2x}\cos(5x-3)$
c $\sin(x)\ln(2x)e^{3x}\cos(4x)$ **d** $4e^{6-3x}\sin(5x^2+3x-2)$

4 Determine the derivative of each function.
a $x^2\sin(3x)$ **b** $(c^2+c-1)\sin(2c-3)$
c $(5m^2-3m+2)\sin(4m-3)$ **d** $\sin(x+3)\sin(x-3)$

5 Determine the derivative of each function.
a $x^3\cos(4x)$ **b** $(m^2+3m-4)\cos(2m-3)$
c $(5x^3-4x^2+3)\cos(3x-5)$ **d** $\cos(2x+1)\cos(3x+1)$

6 Determine the derivative of each function.
a $x^3\sin(3x)$ **b** $e^x(x^2-1)$ **c** $2x^2e^{3x}$
d $2x^3\cos(2x)$ **e** $e^{2x}\sqrt{3x^2-1}$ **f** $x^2\cos^3(2x)$

7 Determine the derivative of each function.
a $(3x^2-2x-1)e^{\sin(2x+1)}+5x^2\ln[x\sin(x^2+1)]$ **b** $7xe^{-0.4x^2}\left(2\sin\left[\ln(2x-3)\right]+\cos\left[\ln(2x+3)\right]\right)$

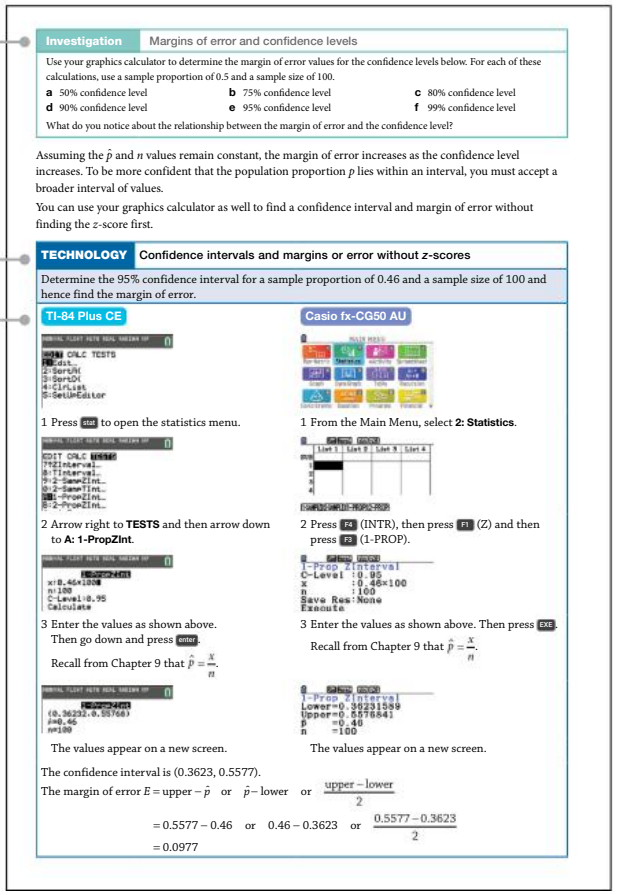
Exam practice

8 © QCAA 2023 2Q9 40% TECH-FREE If $f(x)=e^{3x}(x+1)^2$ and $f'(x)=ae^{3x}(x+1)$, determine the expression for a.
A $3x+5$ **B** $3x+3$ **C** $5x+5$ **D** $5x+3$

9 © QCAA 2020 1Q5 TECH-FREE The equation of the tangent to the curve $f(t)=te^t$ at $t=1$ is
A $y=et$ **B** $y=2et-e$ **C** $y=et-e^2+1$ **D** $y=2et-2e^2+1$

10 (7 marks) CF An oscillating point begins from a high point 2 mm above the origin and doubles its amplitude every 20 seconds. It reaches the next maximum after 5 seconds.
a Determine a sinusoidal function to describe its motion. *[4 marks]*
b Calculate its speed 22 seconds after it begins moving. *[3 marks]*

11 TECH-FREE (5 marks) CF This diagram shows the graph of $y=e^x(2x^2-3x)$.
Calculate the values of x where the gradient of the tangent to the graph is zero.



Investigation Margins of error and confidence levels

Use your graphics calculator to determine the margin of error values for the confidence levels below. For each of these calculations, use a sample proportion of 0.5 and a sample size of 100.
a 50% confidence level **b** 75% confidence level **c** 80% confidence level
d 90% confidence level **e** 95% confidence level **f** 99% confidence level
What do you notice about the relationship between the margin of error and the confidence level?

Assuming the $\hat{p}$ and n values remain constant, the margin of error increases as the confidence level increases. To be more confident that the population proportion p lies within an interval, you must accept a broader interval of values.
You can use your graphics calculator as well to find a confidence interval and margin of error without finding the z-score first.

TECHNOLOGY Confidence intervals and margins or error without z-scores

Determine the 95% confidence interval for a sample proportion of 0.46 and a sample size of 100 and hence find the margin of error.

TI-84 Plus CE

1 Press stat to open the statistics menu.

2 Arrow right to **TESTS** and then arrow down to **A: 1-PropZInt**.

3 Enter the values as shown above. Then go down and press enter.
Recall from Chapter 9 that $\hat{p}=\frac{x}{n}$.

The values appear on a new screen.

The confidence interval is (0.3623, 0.5577).
The margin of error E = upper $-\hat{p}$ or $\hat{p}-$ lower or $\frac{\text{upper}-\text{lower}}{2}$
$=0.5577-0.46$ or $0.46-0.3623$ or $\frac{0.5577-0.3623}{2}$
$=0.0977$

Casio fx-CG50 AU

1 From the Main Menu, select **2: Statistics**.

2 Press F4 (INTR), then press F1 (Z) and then press F2 (1-PROP).

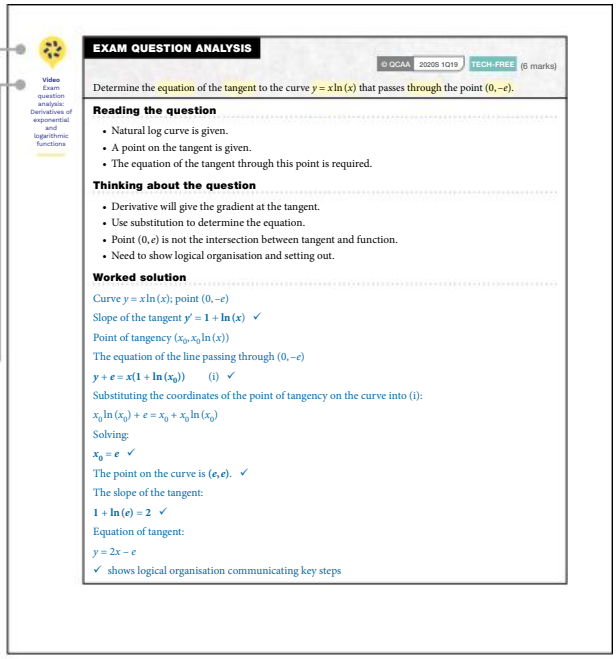
3 Enter the values as shown above. Then press EXE.
Recall from Chapter 9 that $\hat{p}=\frac{x}{n}$.

The values appear on a new screen.

At the end of each chapter

Exam question analysis leads students through a past QCE or exam-style question that exemplifies the chapter, discusses how to approach the question, provides advice on interpreting the question, highlights keywords, describes common student errors and presents a fully worked solution and marking key.

Each exam question analysis is accompanied by an video walkthrough explained by **Joel Speranza**.



Video
Exam question analysis: Derivatives of exponential and logarithmic functions

EXAM QUESTION ANALYSIS

© QCAA 2020S 1Q19 TECH-FREE (6 marks)

Determine the equation of the tangent to the curve $y = x\ln(x)$ that passes through the point $(0, -e)$.

Reading the question

- Natural log curve is given.
- A point on the tangent is given.
- The equation of the tangent through this point is required.

Thinking about the question

- Derivative will give the gradient at the tangent.
- Use substitution to determine the equation.
- Point $(0, e)$ is not the intersection between tangent and function.
- Need to show logical organisation and setting out.

Worked solution

Curve $y = x\ln(x)$; point $(0, -e)$

Slope of the tangent $y' = \mathbf{1 + \ln(x)}$ ✓

Point of tangency $(x_0, x_0\ln(x))$

The equation of the line passing through $(0, -e)$

$\boldsymbol{y + e = x(1 + \ln(x_0))}$ (i) ✓

Substituting the coordinates of the point of tangency on the curve into (i):

$x_0\ln(x_0) + e = x_0 + x_0\ln(x_0)$

Solving:

$\boldsymbol{x_0 = e}$ ✓

The point on the curve is $\boldsymbol{(e, e)}$. ✓

The slope of the tangent:

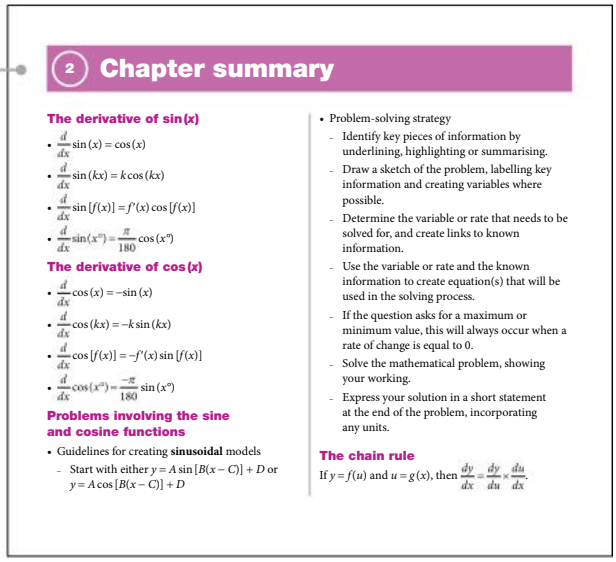
$\boldsymbol{1 + \ln(e) = 2}$ ✓

Equation of tangent:

$y = 2x - e$

✓ shows logical organisation communicating key steps

Chapter summary for easy reference.



2 Chapter summary

The derivative of sin(x)

- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\sin(kx) = k\cos(kx)$
- $\frac{d}{dx}\sin[f(x)] = f'(x)\cos[f(x)]$
- $\frac{d}{dx}\sin(x^o) = \frac{\pi}{180}\cos(x^o)$

The derivative of cos(x)

- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\cos(kx) = -k\sin(kx)$
- $\frac{d}{dx}\cos[f(x)] = -f'(x)\sin[f(x)]$
- $\frac{d}{dx}\cos(x^o) = \frac{-\pi}{180}\sin(x^o)$

Problems involving the sine and cosine functions

- Guidelines for creating **sinusoidal** models
  - Start with either $y = A\sin[B(x - C)] + D$ or $y = A\cos[B(x - C)] + D$

- Problem-solving strategy
  - Identify key pieces of information by underlining, highlighting or summarising.
  - Draw a sketch of the problem, labelling key information and creating variables where possible.
  - Determine the variable or rate that needs to be solved for, and create links to known information.
  - Use the variable or rate and the known information to create equation(s) that will be used in the solving process.
  - If the question asks for a maximum or minimum value, this will always occur when a rate of change is equal to 0.
  - Solve the mathematical problem, showing your working.
  - Express your solution in a short statement at the end of the problem, incorporating any units.

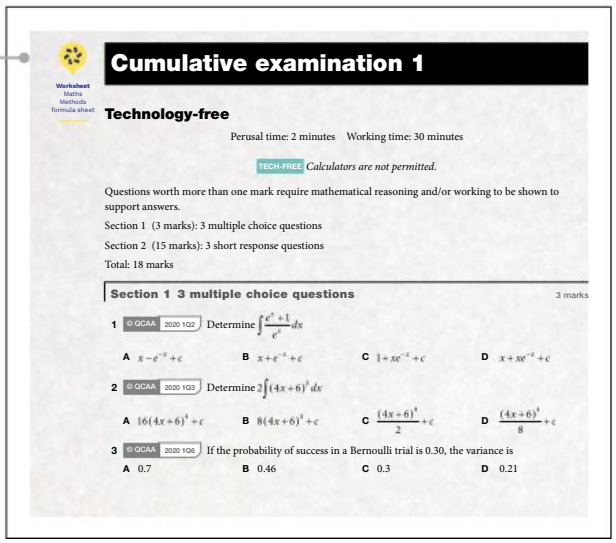
The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Cumulative examinations 1 and **2** are mini exams based on the format of the QCE examination papers 1 and 2, with around 50% of questions focusing on the chapter in which they appear and 50% on previous chapters, including complex familiar and unfamiliar questions.

Cumulative examination 1 contains technology-free questions in multiple choice and short response format: 30 minutes, 18 marks.

Cumulative examination 2 contains technology-active questions in multiple choice and short response format: 30 minutes, 18 marks.



Worksheet
Maths Methods formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

Total: 18 marks

Section 1 3 multiple choice questions 3 marks

1 © QCAA 2020 1Q2 Determine $\int \frac{e^x + 1}{e^x} dx$

A $x - e^{-x} + c$ B $x + e^{-x} + c$ C $1 + xe^{-x} + c$ D $x + xe^{-x} + c$

2 © QCAA 2020 1Q3 Determine $2\int (4x + 6)^3 dx$

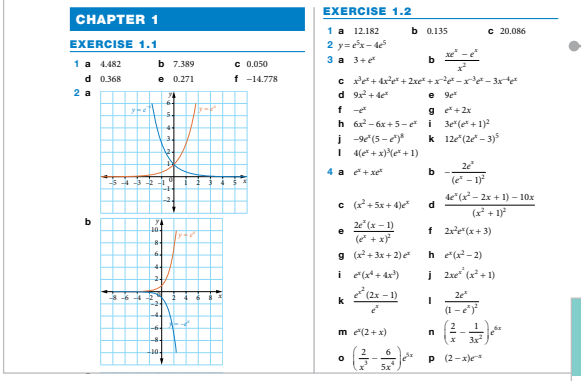
A $16(4x + 6)^4 + c$ B $8(4x + 6)^4 + c$ C $\frac{(4x + 6)^4}{2} + c$ D $\frac{(4x + 6)^4}{8} + c$

3 © QCAA 2020 1Q6 If the probability of success in a Bernoulli trial is 0.30, the variance is

A 0.7 B 0.46 C 0.3 D 0.21

At the end of the book

Answers


CHAPTER 1

EXERCISE 1.1

1 a 4.482 b 7.389 c 0.050
d 0.368 e 0.271 f −14.778

2 a

b

EXERCISE 1.2

1 a 12.182 b 0.135 c 20.086

2 $y = e^5x - 4e^5$

3 a $3 + e^x$ b $\frac{xe^x - e^x}{x^2}$

c $x^3e^x + 4x^2e^x + 2xe^x + x^{-2}e^x - x^{-3}e^x - 3x^{-4}e^x$

d $9x^2 + 4e^x$ e $9e^x$

f $-e^x$ g $e^x + 2x$

h $6x^2 - 6x + 5 - e^x$ i $3e^x(e^x + 1)^2$

j $-9e^x(5 - e^x)^8$ k $12e^x(2e^x - 3)^2$

l $4(e^x + x)^3(e^x + 1)$

4 a $e^x + xe^x$ b $-\frac{2e^x}{(e^x - 1)^2}$

c $(x^2 + 5x + 4)e^x$ d $\frac{4e^x(x^2 - 2x + 1) - 10x}{(x^2 + 1)^2}$

e $\frac{2e^x(x - 1)}{(e^x + x)^2}$ f $2x^2e^x(x + 3)$

g $(x^2 + 3x + 2)e^x$ h $e^x(x^2 - 2)$

i $e^x(x^4 + 4x^3)$ j $2xe^{x^2}(x^2 + 1)$

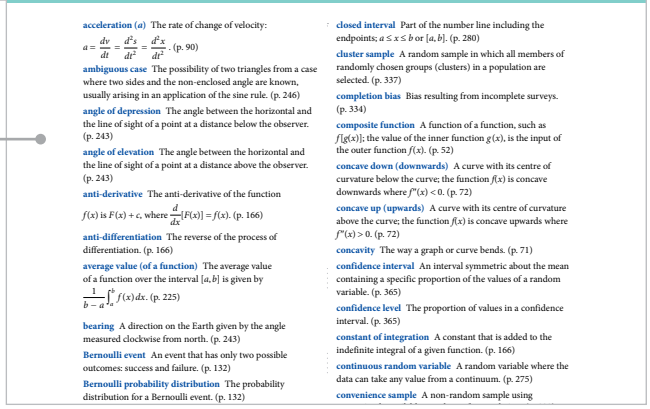
k $\frac{e^{x^2}(2x - 1)}{e^x}$ l $\frac{2e^x}{(1 - e^x)^2}$

m $e^x(2 + x)$ n $\left(\frac{2}{x} - \frac{1}{3x^2}\right)e^{6x}$

o $\left(\frac{2}{x^3} - \frac{6}{5x^4}\right)e^{5x}$ p $(2 - x)e^{-x}$

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students)

Glossary and index


acceleration (a) The rate of change of velocity: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}$. (p. 90)

ambiguous case The possibility of two triangles from a case where two sides and the non-enclosed angle are known, usually arising in an application of the sine rule. (p. 246)

angle of depression The angle between the horizontal and the line of sight of a point at a distance below the observer. (p. 243)

angle of elevation The angle between the horizontal and the line of sight of a point at a distance above the observer. (p. 243)

anti-derivative The anti-derivative of the function $f(x)$ is $F(x) + c$, where $\frac{d}{dx}[F(x)] = f(x)$. (p. 166)

anti-differentiation The reverse of the process of differentiation. (p. 166)

average value (of a function) The average value of a function over the interval $[a, b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. (p. 225)

bearing A direction on the Earth given by the angle measured clockwise from north. (p. 243)

Bernoulli event An event that has only two possible outcomes: success and failure. (p. 132)

Bernoulli probability distribution The probability distribution for a Bernoulli event. (p. 132)

closed interval Part of the number line including the endpoints; $a \le x \le b$ or $[a, b]$. (p. 280)

cluster sample A random sample in which all members of randomly chosen groups (clusters) in a population are selected. (p. 337)

completion bias Bias resulting from incomplete surveys. (p. 334)

composite function A function of a function, such as $f[g(x)]$; the value of the inner function $g(x)$, is the input of the outer function $f(x)$. (p. 52)

concave down (downwards) A curve with its centre of curvature below the curve; the function $f(x)$ is concave downwards where $f''(x) < 0$. (p. 72)

concave up (upwards) A curve with its centre of curvature above the curve; the function $f(x)$ is concave upwards where $f''(x) > 0$. (p. 72)

concavity The way a graph or curve bends. (p. 71)

confidence interval An interval symmetric about the mean containing a specific proportion of the values of a random variable. (p. 365)

confidence level The proportion of values in a confidence interval. (p. 365)

constant of integration A constant that is added to the indefinite integral of a given function. (p. 166)

continuous random variable A random variable where the data can take any value from a continuum. (p. 275)

convenience sample A non-random sample using

A combined **Glossary and index**

Nelson MindTap

Nelson MindTap is an online learning space that provides students with interactive learning and assessment experiences. Margin links in the student book signpost multimedia student resources found on Nelson MindTap*.

Videos

Skillsheets

Worksheets

For students:

- **Engage** with the **online eBook** by adding notes, highlights, bookmarks, and using the **Search** and **Read Aloud** (in Australian voice) functions.
- **Watch videos** featuring expert teacher advice to unpack worked examples and deepen your understanding.
- **Revise** using pre-chapter **prior learning sheets, skillsheets** and **worksheets** to practise your skills and build your confidence.
- **Navigate** your own learning path, accessing the content and support as you need it.

For teachers:

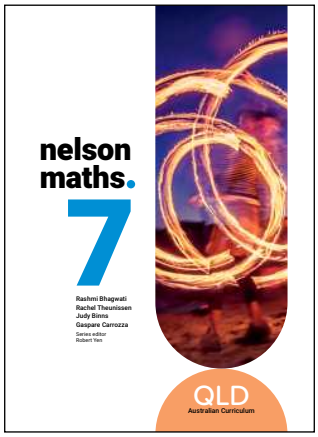
- **Access topic tests, teaching plans** and **worked solutions** to each exercise set.
- **Use course customisation** to tailor content and assign tasks to groups of students or the whole class.
- **Integrate** content directly within your school's LMS for ease of access.
- Help build your students' exam readiness with **Cognero Assess** – a test bank containing hundreds of questions and answers to create, assign or export formative and summative tests.

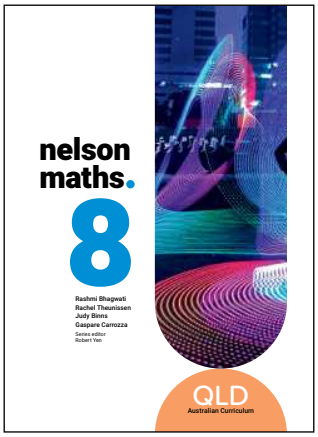
* Complimentary access to these resources is only available to schools that use this book as part of a class set, book hire or booklist. Not available for single purchases. Contact your Cengage Learning Consultant for information about access and conditions.

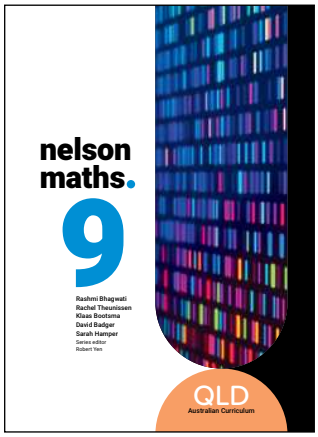
Security & privacy:

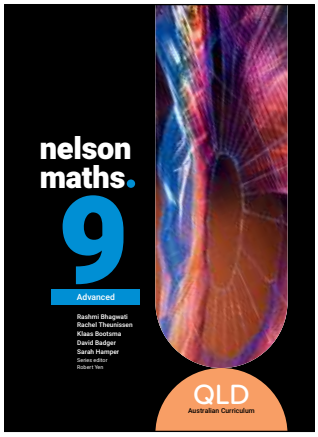
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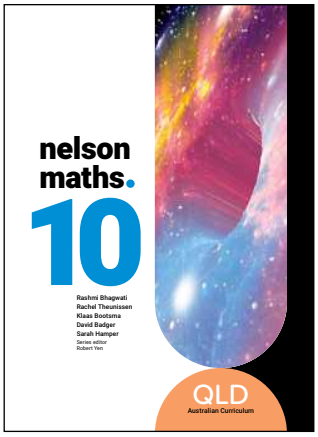
Nelson Maths 7–12 QLD series

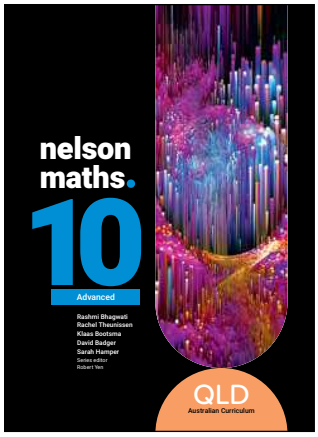


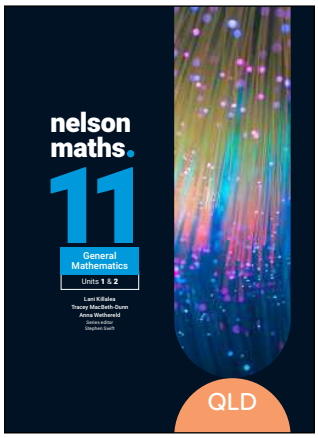


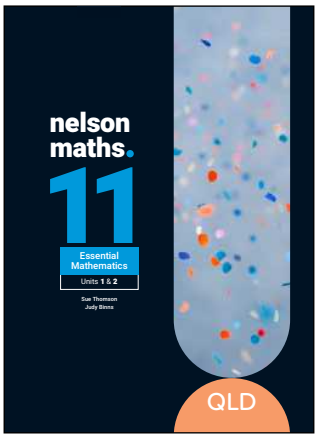


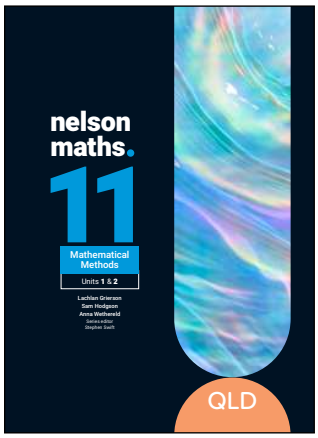


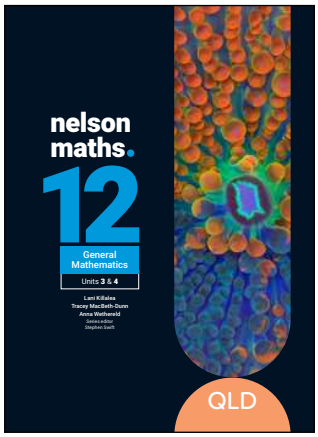


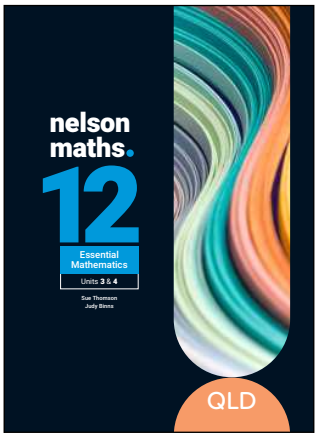


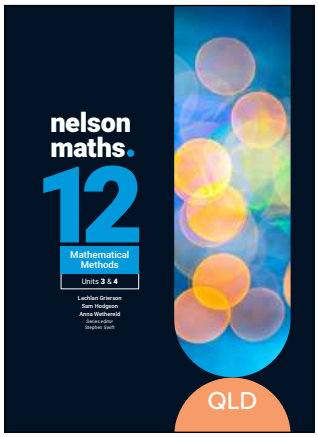












List of symbols and abbreviations

$=$	is equal to
$\equiv$	is identical to
$\neq$	is not equal to
$\approx$	is approximately equal to
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\in$	is an element of
$\cup$	union
$\cap$	intersection
$\pm$	plus or minus
π	pi ≈ 3.14
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\notin$	is not an element of
$\sum$	the sum of
$\ldots$	and so on
$\{\}, \varnothing$	the null set
U	the universal set
R	the real numbers
R^+	the positive real numbers
N	the natural numbers
$^\circ$	degree
∞	infinity
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\propto$	is proportional to
Δ	the discriminant
$\sqrt{\ }$	square root, radical sign
$\bar{A}, A'$	not A, the complement of A
:	such that
S	sample space
$P(A)$	the probability of A
$P(A\|B)$	the probability of A given B
$n(A), \|A\|$	the cardinality (number of elements) of set A
LHS	left-hand side
RHS	right-hand side
m	gradient of a straight line
c	y-intercept of a straight line
(x, y)	Cartesian coordinates, ordered pair
$^nC_r, \binom{n}{r}$	the number of combinations of r objects from n objects
$n!$	factorial n
$\lim_{h \to 0}$	the limit as $h \to 0$
$\to$	gets closer to
$\frac{dy}{dx}, f'(x)$	the derivative of $y, f(x)$
wrt	with respect to
x, s	displacement
v	velocity
a	acceleration
$\log_{10}$	common logarithm
ln, $\log_e$	natural logarithm
$\log_a(b)$	the logarithm of b to the base a
cos	cosine
sin	sine
tan	tangent
$\mu = E(X)$	expected value of the random variable X
$\text{Var}(X)$	variance of the random variable X
$\sigma_X = \text{SD}(X)$	standard deviation of the random variable X
$\hat{p}$	sample proportion
$X \sim N(\mu, \sigma^2)$	X is a random normal variable with mean μ and variance σ^2

List of mathematical verbs

A glossary of 'doing words' commonly found in mathematics problems

Cognitive verbs are highlighted in light green. See QCAA's glossary of cognitive verbs for more details.

analyse: study and state in detail the relationship of parts of a situation

apply: use knowledge or a procedure in a given situation

calculate: find a numerical value

classify: identify and sort into categories

comment: express an opinion or judgement about a statement or calculation

communicate: transmit information to others

compare: state similarities and differences and their significance

consider: take into account; think carefully

construct: draw an accurate diagram or logically arrange items or ideas

convert: change from one form to another

deduce: use logic and reason to prove things

define: give the meaning of or identify in exact terms

demonstrate: show to be correct

describe: state the features of a situation, object, pattern, event, etc.

determine: find the answer or make a decision using evidence

discuss: give reasons based on evidence for and against a position or proposal

establish: introduce and develop a result

evaluate: find the value of or state the application, strengths and limitations of a solution

examine: investigate the details and assumptions of a situation

expand: remove brackets or change a product to an extended sum of terms

explain: state the meaning in logical detail

explore: examine or state the details and assumptions of a situation

factorise: convert to factors or change a sum of terms to a product of factors

formulate: give mathematical expression to an idea or situation

hence find/prove: find an answer or prove a result using previous answers or information supplied

identify: state the type, name or distinguishing feature of an item or situation

interpret: state a conclusion or trend from given information

investigate: establish facts, trends or conclusions from collected information

isolate a variable: express a formula or equation with one variable only, usually on the left-hand side (LHS)

justify: give reasons or evidence for an answer or conclusion

modify: change to accommodate different information

obtain: find an answer or conclusion

predict: use given information or results to deduce what will happen under different conditions

prove: use logical steps to establish the truth of a result, statement or assumption

rationalise the denominator: To simplify a fraction involving a surd by making its denominator rational (that is, not a surd)

recall: remember (and state)

recognise: use knowledge to identify features of a situation

show that: in questions where the answer is given, use mathematical reasoning to prove that the answer is true

simplify: reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

sketch: draw a diagram that shows the general shape and includes relevant features

solve: find the answer or explanation for a problem, particularly the values of variables in equations or inequalities

substitute: replace a variable by a number to calculate an answer

test: check that a statement or result is correct

translate (to mathematical form): express a situation as mathematical relationships

verify: check a solution or result, usually referring to the given situation

write/state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

CHAPTER

1 DERIVATIVES OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Syllabus coverage

Nelson MindTap chapter resources

Terminology

1.1 The natural exponential function $y = e^x$

1.2 Derivatives involving e^x

1.3 The natural logarithmic function $y = \ln(x)$

1.4 Exponential and logarithmic equations

1.5 The derivative of $\ln(x)$

1.6 Problems involving exponential and logarithmic functions

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Prior learning
Derivatives of exponential and logarithmic functions

Syllabus coverage

UNIT 3, TOPIC 1: DIFFERENTIATION OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Calculus of exponential functions

- Estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$, using technology, for various values of $a > 0$.
- Recognise that e is the unique number a for which the above limit is 1.
- Recognise and determine the qualitative features of the graph of $y = e^x$, including asymptote and intercept.
- Use the rules $\frac{d}{dx} e^x = e^x$ and $\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$.

Calculus of logarithmic functions

- Recognise and determine the qualitative features of the graph of $y = \ln(x) = \log_e(x)$, including asymptote and intercept.
- Recognise and use the inverse relationship of the functions $y = e^x$ and $y = \ln(x)$.
- Solve equations involving exponential and logarithmic functions with base e, with and without technology.
- Use the rules $\frac{d}{dx} \ln(x) = \frac{1}{x}$ and $\frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)}$.
- Model and solve problems that involve derivatives of exponential and logarithmic functions, with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 24,
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Videos (3):

1.6 Exponential decay • Exponential growth

Exam question analysis Derivatives of exponential and logarithmic functions

Prior learning (1):

1.1 Derivatives of exponential and logarithmic functions

Worksheets (14):

1.1 Exponential functions • Derivatives of exponential functions

1.4 Solving exponential equations • Exponential equations • Logarithm laws • Logarithms review

1.5 Exponential and logarithmic functions • Derivatives of logarithmic functions • Differentiating exponential and logarithmic functions

1.6 Exponential functions assignment • Exponential functions extension • Derivatives of logarithmic functions assignment • Derivatives of logarithmic functions extension

Cumulative exams Maths Methods formula sheet

Puzzles (2):

1.4 Logarithms - Solving equations 1 • Logarithms - Solving equations 2

TI-Nspire CX II instructions (1):

1.1, 1.3, 1,4, 1.6 Derivatives of exponential and logarithmic functions

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

Euler's number	exponential	exponential decay	exponential function
exponential growth	inverse function	irrational number	ln
log–log graph	natural logarithm	transcendental number	

1.1 The natural exponential function $y = e^x$

You can find the **derivative** of $f(x) = a^x$ if $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$ exists.

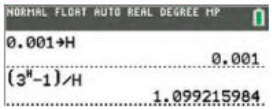
If it does exist, the derivative of a^x is just a^x multiplied by a constant.

Investigation — Derivatives of a^x

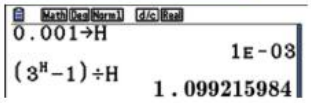
You can investigate the derivative of a^x using your graphics calculator.

1 Find the values of $\frac{a^h - 1}{h}$ for $a = 0.5, 1.5, 2, 2.5, 2.7, 2.72, 3$ and 3.5 using $h = 0.001$ to estimate the values of $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$. An example for $a = 3$ is shown below.

TI-84 Plus CE

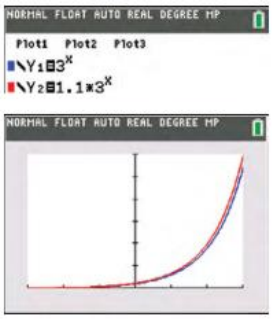


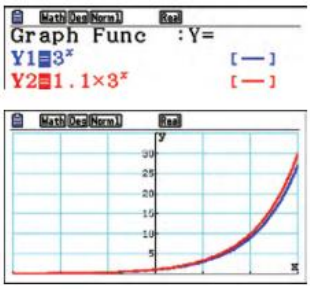
Casio fx-CG50 AU



2 Now use these estimates to find an approximation to the derivative of a^x for different values of a.

3 Using your results, graph each exponential function and its derivative on your calculator. Here is an example for $a = 3$.





4 What happens to the value of the limit as a increases up to 3?
What happens to the value of the limit as a increases beyond 3?

For values of a close to 3, the $\lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$ is close to 1. In fact, there exists a number for which the limit equals 1. This special constant, symbolised by the letter 'e', is often called **Euler's number** and is a very important number in mathematics. The Swiss mathematician Leonhard Euler (1707–1783) later proved that this constant is an **irrational number**.

In algebra, it is common to use base 10 when dealing with **exponential functions**.
However, in calculus, it is more convenient to use base e.

e is a **transcendental number**, which means that it is *not* the root of any algebraic equation with rational coefficients.

e is the number such that $\lim_{h \to 0}\frac{e^h - 1}{h} = 1$.

Worksheets
Exponential functions
Derivatives of exponential functions

TI-Nspire
Derivatives of exponential and logarithmic functions

Consider the derivative of e^x, given by $\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$

$$= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \quad [\text{since } e^{x+h} = e^x e^h]$$

$$= \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h} \right)$$

$$= e^x \qquad \left[\text{since } \lim_{h \to 0} \frac{e^h - 1}{h} = 1\right]$$

Derivative of e^x

The derivative of $f(x) = e^x$ is e^x.

Investigation Euler's number

1 Enter the function $Y_1 = \left(1 + \frac{1}{X}\right)$ in the graphing section of your graphics calculator.
2 Use WINDOW (or V-WINDOW) to set $X_{min} = -1$, $X_{max} = 1$, $X_{scl} = 0.1$, $Y_{min} = 0$, $Y_{max} = 4$, $Y_{scl} = 1$ and graph the function.
3 Trace the graph from left to right and observe how the value of the function changes.
4 Adjust the values for X to $X_{min} = -1$, $X_{max} = 10$, $X_{scl} = 1$ and trace the value of the function for large values of X. Note how the value of the function behaves.
5 Repeat using $X_{min} = -1$, $X_{max} = 100$, $X_{scl} = 10$ and observe the value of the function.
6 Comment on your findings.

The number e may be defined as the following **limit**.

Definition of e

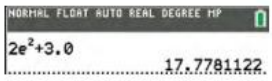
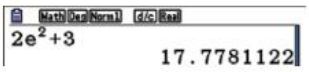
$$e = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = 2.718\,281\ldots$$

This property makes $f(x) = e^x$ so important in mathematics, physics, engineering, economics and biology that it is called *the* exponential function.

TI-Nspire
Derivatives of exponential and logarithmic functions

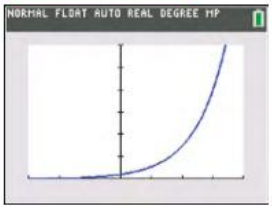
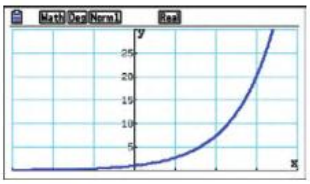
WORKED EXAMPLE 1 Evaluating e in equations

Use your calculator to evaluate $2e^2 + 3$, correct to 2 decimal places.

Steps	Working	
1 Enter the equation in your calculator.	TI-84 Plus CE	Casio fx-CG50 AU
		
	On a calculator page, type the equation and press ENTER. Write one of the numbers (the 2 or the 3) as a decimal to force an approximate answer.	Use the Run-Matrix menu. Type the equation and press EXIT.
2 Write the answer.	$2e^2 + 3 \approx 17.78$	

WORKED EXAMPLE 2 Graphing $y = e^x$

Use your calculator to draw the graph of $y = e^x$.

Steps	Working
1 Enter the equation in your calculator. 2 Display the graph.	**TI-84 Plus CE**  1 Use a graph page. 2 Press y= and type the function into Y_1. 3 Set the WINDOW to $-3 \le x \le 4$ and $0 \le y \le 30$ and press Graph.
	Casio fx-CG50 AU  1 Use the Graph menu. 2 Set the V-Window to $-3 \le x \le 4$ and $0 \le y \le 30$ and press EXIT. 3 Type the function into Y_1 and select DRAW.

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WORKED EXAMPLE 3 Tangent to the curve $y = e^x$

Determine the exact equation of the tangent to the curve $y = e^x$ at the point $(3, e^3)$.

Steps	Working
1 Find the slope at $(3, e^3)$.	At $x = 3$, $\frac{dy}{dx} = e^x = e^3$, so $m = e^3$.
2 Write the equation of the line.	$y - y_1 = m(x - x_1)$
3 Substitute values.	$y - e^3 = e^3(x - 3)$
4 Write in standard form, $ax^3 + bx^2 + cx + d = 0$.	$e^3x - y - 3e^3 + e^3 = 0$
5 Simplify and write as a statement.	The equation of the tangent to the curve $y = e^x$ at the point $(3, e^3)$ is $y = e^3x - 2e^3$.

From Worked examples 1–3, you will be aware of the following properties:

- The value of $y = e^x$ at any point on the graph is equal to the gradient of the tangent at that point.
- The value of $y = e^x$ and the instantaneous rate of change are the same for any given value of x.
- The value of $y = e^x$ at any point is equal to the area under the curve up to that point.

EXERCISE 1.1 The natural exponential function $y = e^x$

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Mastery

1 WORKED EXAMPLE 1 Evaluate each expression correct to 3 decimal places.

a $e^{1.5}$ **b** e^2 **c** e^{-3}

d $\frac{1}{e}$ **e** $2e^{-2}$ **f** $-2e^2$

2 WORKED EXAMPLE 2 On the same set of axes, sketch each pair of exponential functions.

a $y = e^x$ and $y = e^{-x}$ **b** $y = e^x$ and $y = -e^x$

c $y = e^x$ and $y = 2e^x$ **d** $y = e^x$ and $y = e^{2x}$

3 WORKED EXAMPLE 3 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(2, e^2)$.

4 For $y = e^x$, find the gradient of the tangent to the curve at the point where $x = 0.65$, correct to 2 decimal places.

5 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(4, e^4)$.

Exam practice

6 © QCAA 2023 1Q1 70% TECH-FREE $e^{\ln(x)}$ is equivalent to

A 0 **B** 1 **C** x **D** $\frac{1}{x}$

7 Evaluate $3e^2$, correct to 3 decimal places.

A 8.154 **B** 14.154 **C** 22.167 **D** 24.462

8 Evaluate $e^{2.5}$, correct to 3 decimal places.

A 10.156 **B** 12.182 **C** 18.546 **D** 24.364

9 (1 mark) Evaluate $4e^{-3}$, correct to 3 decimal places.

10 (4 marks) Sketch $y = 2e^x$ and $y = -e^x$ on the same set of axes.

1.2 Derivatives involving e^x

The derivative of e^x

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$$

WORKED EXAMPLE 4 Derivative of e^x

Calculate the derivative of each function.

a $3x^2 + 6e^x$ **b** $y = (2 + e^x)^5$

Steps	Working
a 1 Use the linear sum.	$\frac{d}{dx}(3x^2 + 6e^x) = 3\frac{d}{dx}(x^2) + 6\frac{d}{dx}(e^x)$
2 Simplify.	$= 6x + 6e^x$
b 1 Write the function.	$y = (2 + e^x)^5$
2 Let $u = 2 + e^x$ for the chain rule.	Let $u = 2 + e^x$ and $y = u^5$.
3 Calculate $\frac{dy}{du}$.	$\frac{dy}{du} = 5u^4$
4 Calculate $\frac{du}{dx}$.	$\frac{du}{dx} = e^x$
5 Use the chain rule.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
6 Substitute for u and simplify.	$= 5u^4 \times e^x$
7 Write the result.	$\frac{dy}{dx} = 5e^x(2 + e^x)^4$

WORKED EXAMPLE 5 Product and quotient rules

Calculate the derivative of each function.

a $f(x) = x^2e^x$ **b** $y = \dfrac{e^x + 1}{2e^x - 1}$

Steps	Working
a 1 Write the function.	$f(x) = x^2e^x$
2 Let $u = x^2$ and $v = e^x$.	$f(x) = uv$
3 Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.	$u' = 2x$ and $v' = e^x$
4 Use the product rule.	$f'(x) = uv' + vu'$
5 Substitute for u, u', v' and v.	$= x^2e^x + e^x \times 2x$
6 Write the result	$f'(x) = xe^x(x + 2)$
b 1 Write the function.	$y = \dfrac{e^x + 1}{2e^x - 1}$
2 Let $u = e^x + 1$ and $v = 2e^x - 1$.	$= \dfrac{u}{v}$
3 Find $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$.	$u' = e^x$ and $v' = 2e^x$
4 Use the quotient rule.	$y' = \dfrac{vu' - uv'}{v^2}$
5 Substitute for u, u', v' and v.	$= \dfrac{\left(2e^x - 1\right)e^x - \left(e^x + 1\right) \times 2e^x}{\left(2e^x - 1\right)^2}$
6 Expand brackets in the numerator.	$= \dfrac{2e^{2x} - e^x - 2e^{2x} - 2e^x}{\left(2e^x - 1\right)^2}$
7 Simplify and write the result.	$y' = \dfrac{-3e^x}{\left(2e^x - 1\right)^2}$

WORKED EXAMPLE 6 The chain rule

Calculate the derivative of $y = e^{x^2 + 2x + 5}$.

Steps	Working
1 Write the equation.	$y = e^{x^2 + 2x + 5}$
2 Let $u = x^2 + 2x + 5$.	$y = e^u$
3 Find $\dfrac{dy}{du}$.	$\dfrac{dy}{du} = e^u$
4 Find $\dfrac{du}{dx}$.	$\dfrac{du}{dx} = 2x + 2$
5 Use the chain rule.	$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$
6 Substitute for known values.	$= e^u \times (2x + 2)$
7 Substitute for u.	$= (2x + 2)e^{x^2 + 2x + 5}$
8 Simplify to get your answer.	$\dfrac{dy}{dx} = 2(x + 1)e^{x^2 + 2x + 5}$

WORKED EXAMPLE 7 The product rule

Calculate the derivative of $y = x^2e^{2x}$.

Steps	Working
1 Write the equation.	$y = x^2e^{2x}$
2 Let $u = x^2$ and $v = e^{2x}$.	$y = uv$
3 Find $\frac{du}{dx}$ and $\frac{dv}{dx}$.	$u' = 2x$ and $v' = 2e^{2x}$
4 Use the product rule.	$y' = uv' + vu'$
5 Substitute for u, u', v' and v.	$= x^2 2e^{2x} + e^{2x} \times 2x$
6 Simplify.	$= 2x^2e^{2x} + 2xe^{2x}$
7 Write your answer.	$y' = 2xe^{2x}(x + 1)$

EXERCISE 1.2 Derivatives involving e^x

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Recap

1 Evaluate each expression correct to 3 decimal places.

a $e^{2.5}$ **b** e^{-2} **c** e^3

2 Find the exact equation of the tangent to the curve $y = e^x$ at the point $(5, e^5)$.

Mastery

3 WORKED EXAMPLE 4 Find the derivative of each function.

a $3x + e^x$ **b** $\frac{e^x}{x}$ **c** $x^3e^x + x^2e^x + x^{-2}e^x + x^{-3}e^x$

d $3x^3 + 4e^x + 5$ **e** $9e^x$ **f** $-e^x$

g $e^x + x^2$ **h** $2x^3 - 3x^2 + 5x - e^x$ **i** $(e^x + 1)^3$

j $(5 - e^x)^9$ **k** $(2e^x - 3)^6$ **l** $(e^x + x)^4$

4 WORKED EXAMPLE 5 Find the derivative of each function.

a xe^x **b** $\frac{e^x + 1}{e^x - 1}$ **c** $(x^2 + 3x + 1)e^x$ **d** $\frac{4e^x + 5}{x^2 + 1}$

e $\frac{e^x - x}{e^x + x}$ **f** $2x^3e^x$ **g** $(x^2 + x + 1)e^x$ **h** $x^2e^x - 2xe^x$

i x^4e^x **j** $x^2e^{x^2}$ **k** $\frac{e^{x^2}}{e^x}$ **l** $\frac{1 + e^x}{1 - e^x}$

m $e^x(1 + x)$ **n** $\frac{e^{6x}}{3x}$ **o** $\frac{2e^{5x}}{5x^3}$ **p** $\frac{x - 1}{e^x}$

5 WORKED EXAMPLE 6 Find the derivative of each function.

a e^{x^2} **b** e^{-x^2+4} **c** $e^x(e^x + e)$ **d** $5e^{4x-1}$

e e^{6x-2} **f** e^{7-5x} **g** $2e^{4x+1}$ **h** $4e^{6-2x}$

i $-3e^{9x+4}$ **j** $y = -2e^{6-4x}$ **k** $10e^{6-8x}$ **l** $-5e^{2x+4}$

m $6e^{-7x+1}$ **n** $2e^{\frac{x}{3}+1}$ **o** $3e^{2-\frac{x}{5}}$ **p** $-4e^{\frac{x}{5}+4}$

q $5e^x - 1$ **r** $4e^{\frac{1}{x}}$ **s** $e^{2x-1} - e^{4x^2}$ **t** $\frac{4e^{-3x}}{e^{5x+1}}$

u $(e^{2x-1})^5$ **v** $e^{(x-1)^3}$

6 WORKED EXAMPLE 7 Find the derivative of each function.

a $y = e^{x^2+2x}$ **b** $y = e^{x^2-3x-2}$ **c** $y = e^{x^2-4x}$

d $f(x) = e^{2-6x}$ **e** $f(x) = e^{6-3x+5x^2}$ **f** $g(x) = e^{x^3+4x}$

g $h(x) = 3e^{4x^2-5x}$ **h** $y = -5e^{1-2x-5x^2}$ **i** $y = e^{(2x+1)^2}$

j $y = e^{(4-x)^5}$ **k** $y = e^{(x+5)^{-2}}$ **l** $y = e^{\sqrt{x-4}}$

m $f(x) = e^{(x+4)^{\frac{1}{3}}}$ **n** $g(x) = e^{(x^2+4x)^2}$ **o** $y = 3xe^{x^2+4}$

p $y = 2x^2e^{x-4}$ **q** $f(x) = (e^x - e^{x+1})^2$ **r** $y = 4x^3e^x$

7 Find the derivative of each function.

a $4x + e^x$ **b** $\frac{e^x}{2x}$ **c** $(e^x + 2)^2$

d $(e^x + 3x)^3$ **e** $2xe^x$ **f** $\frac{e^x+2}{e^x-2}$

g $2x^2e^x - 4xe^x$ **h** $2x^2e^{x^2}$ **i** $\frac{2x-2}{2e^x}$

j $(e^x + 2)^2$

8 Find the derivative of each function.

a $2e^{x^2}$ **b** e^{8-3x} **c** $4e^{2x+1}$

d $y = -e^{4-2x}$ **e** $-2e^{\frac{x}{4}+5}$ **f** $\frac{2e^{-2x}}{e^{4x+1}}$

g $y = 2e^{x^2-2x}$ **h** $y = -3e^{1-3x-2x^2}$ **i** $y = e^{(2x+2)^{-2}}$

j $y = 2e^{\sqrt{x-4}}$ **k** $y = 2e^{\frac{1}{x}}$

Exam practice

9 © QCAA 2023 1Q2 85% TECH-FREE If $f(x) = e^{6-2x}$, determine the value of $f'(2)$.

A e^2 **B** $2e^2$ **C** $-e^2$ **D** $-2e^2$

10 © QCAA 2020S 2Q2 A population of bacteria after t hours is given by $P(t) = 5000e^{0.18t}$.
The rate of increase of the population (to the nearest unit) at 15 minutes is

A 74 399 bacteria/hour. **B** 13 392 bacteria/hour.

C 5230 bacteria/hour. **D** 941 bacteria/hour.

11 TECH-FREE For the curve with the equation $y = (3x^2 - 6x)e^x$, find the x-coordinate of the x-intercepts.

A 0, –2 **B** 0, 0 **C** 0, 1 **D** 0, 2

12 (6 marks) TECH-FREE CF Sketch the graph of $y = (3x^2 - 6x)e^x$, including any turning points.

1.3 The natural logarithmic function $y = \ln(x)$

In Year 11, Chapter 9, *Logarithms*, you learnt about **common logarithms**. These have base 10 and are shown as $\log_{10}(x)$ or just $\log(x)$.

Natural logarithms are base e and are shown as **ln** (x).

Natural logarithms

- The **natural logarithm** of a number is the logarithm to the base $e = 2.718\ldots$
- It is shown as $\ln(x) = \log_e(x)$.
- $\ln(x) = a$ if and only if $x = e^a$.

WORKED EXAMPLE 8 Using your calculator with ln (x)

Express $\ln(7.2) \approx 1.97$ in index form.

Steps	Working
1 Write the equation.	$\ln(7.2) \approx 1.97$
2 Write it as a power.	$e^{1.97} \approx 7.2$

Investigation Values of natural logarithms

Use your calculator for this investigation.

Find the value of these natural logarithms.

- $\ln(e)$
- $\ln(1)$
- $\ln\left(\frac{1}{e}\right)$
- $\ln(5)$, $\log(5)$ and $\frac{\ln(5)}{\log(5)}$
- $\ln(87)$, $\log(87)$ and $\frac{\ln(87)}{\log(87)}$
- $\ln(0.063)$, $\log(0.063)$ and $\frac{\ln(0.063)}{\log(0.063)}$

What can you say about the ratio $\frac{\ln(x)}{\log(x)}$?

Find the value of $y = e^{\frac{\ln(5)}{\log(5)}}$.

What does this mean about the ratio $\frac{\ln(5)}{\log(5)}$?

WORKED EXAMPLE 9 Evaluating ln (x)

Calculate each expression correct to 4 decimal places.

a $\ln(247.8)$ **b** $\ln(0.749)$ **c** $\ln(0.000\,456)$ **d** $\ln(-5.37)$

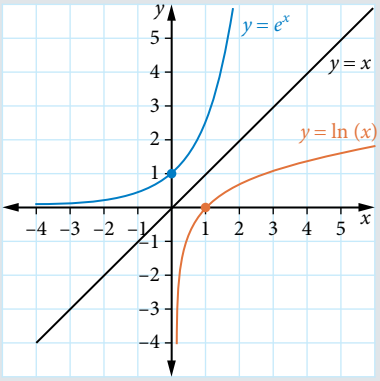
Steps	Working
a Use your calculator.	$\ln(247.8) \approx 5.5126$
b Use your calculator.	$\ln(0.749) \approx -0.2890$
c Use your calculator.	$\ln(0.000\,456) \approx -7.6930$
d Only positive numbers have logarithms.	$\ln(-5.37)$ is not defined.

You can use the exponential function to reverse logarithms.

Exponential and logarithmic functions

$\ln(x) = y$ is equivalent to $x = e^y$. This means that the coordinates of the natural logarithmic and exponential functions are interchanged.

- The natural logarithmic function and the exponential function are **inverse functions**.
- The point (a, b) on the graph of $y = \ln(x)$ corresponds to the point (b, a) on the graph of $y = e^x$. The coordinates of corresponding points are exchanged.
- Their graphs are the reflections of each other in the line $y = x$.



WORKED EXAMPLE 10 Logarithmic equations

Solve each logarithmic equation, correct to 3 significant figures.

a $\ln(x) = 2.47$ **b** $\ln(y) = -5.2$

Steps	Working
a Use your calculator.	$x = e^{2.47} \approx 11.8$
b Use your calculator.	$y = e^{-5.2} \approx 0.005\,52$

Investigation Logarithmic and exponential graphs

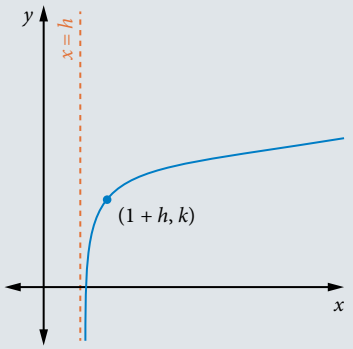
Use your graphics calculator for this investigation.

- Draw the graphs of $y = e^x$ and $y = \ln(x)$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^{x+2}$ and $y = \ln(x) - 2$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^x + 3$ and $y = \ln(x - 3)$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?
- Draw the graphs of $y = e^{x-1} + 2$ and $y = \ln(x - 2) + 1$ on the same axes.
  - What are the x- and y-intercepts of the graphs?
  - What are the asymptotes of the graphs?

What can you conclude about the x- and y-intercepts and the asymptotes of natural logarithm and exponential graphs?

The graph of $y = \ln(x - h) + k$

- The graph of $y = \ln(x - h) + k$ has a vertical asymptote at $x = h$.
- The point $(1, 0)$ on $y = \ln(x)$ is transformed to $(1 + h, k)$.
- $y = \ln(x - h) + k$ is the graph of $y = \ln(x)$ translated h units horizontally and k units vertically.
  The translations are $h > 0$ right, $h < 0$ left, $k > 0$ up, $k < 0$ down.

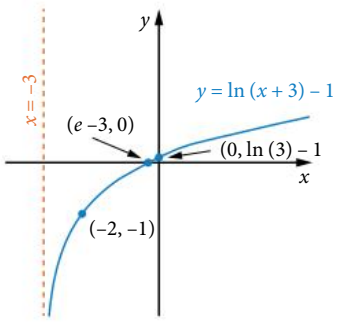
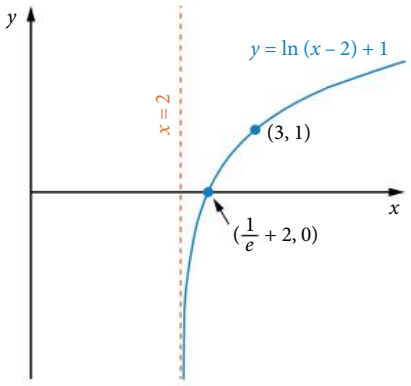


When you sketch the graphs of natural logarithmic or exponential functions, show the appropriate intercepts and asymptotes, and points corresponding to $(0, 1)$ or $(1, 0)$ on the graphs.

WORKED EXAMPLE 11 Graphing a natural logarithmic function

a **Sketch** the graph of $y = \ln(x + 3) - 1$ and state the domain and range.

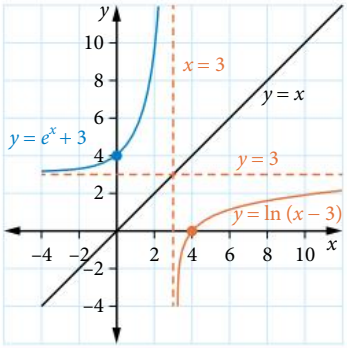
b **Sketch** the graph of $y = \ln(x - 2) + 1$ and state the domain and range.

Steps	Working
a **1** Write the equation in logarithmic form: $y = \ln(x - h) + k$	$y = \ln(x + 3) - 1$ $= \ln(x - (-3)) + (-1)$
2 State the important features of the logarithmic function, using h and k.	$y = \ln(x - (-3)) + (-1)$ passes through $(-2, -1)$ and has a vertical asymptote at $x = -3$.
3 Find the y-intercept if possible.	At $x = 0$, $y = \ln(3) - 1 \approx 0.099$
4 Find the x-intercept if possible.	$0 = \ln(x - 3) - 1$ $\ln(x + 3) = 1$ $x + 3 = e^1 = e$ $x = e - 3 \approx -0.282$
5 Use the information to sketch the graph. Label the important features.	
6 State the domain and range.	The domain is $(-3, \infty)$ and the range is R.
b **1** It is in standard form already.	$y = \ln(x - 2) + 1$
2 State the important features of the logarithmic function, using h and k.	$y = \ln(x - 2) + 1$ passes through $(3, 1)$ and has a vertical asymptote at $x = 2$.
3 Find the y-intercept if possible.	$\ln(0 - 2)$ does not exist, so there is no y-intercept.
4 Find the x-intercept if possible.	$0 = \ln(x - 2) + 1$ $\ln(x - 2) = -1$ $x = e^{-1} + 2 = \frac{1}{e} + 2$ The x-intercept is $\left(\frac{1}{e} + 2, 0\right)$.
5 Use the information to sketch the graph. Label the important features.	
6 State the domain and range.	The domain is $x > 2$ and range is R.

WORKED EXAMPLE 12 Inverse of a logarithmic function

a **Determine** the inverse of $y = \ln(x - 3)$.

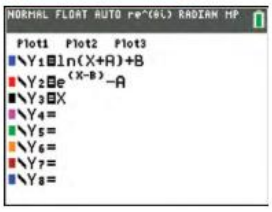
b **Sketch** the graph of $y = \ln(x - 3)$ and its inverse on the same axes.

Steps	Working
a **1** Write in exponential form.	$e^y = x - 3$
2 Make x the subject.	$x = e^y + 3$
3 The coordinates are reversed in the inverse.	The inverse is $y = e^x + 3$.
b **1** Describe the logarithmic function.	$y = \ln(x - 3)$ has a vertical asymptote at $x = 3$, and $(1, 0)$ is transformed to $(4, 0)$.
2 Describe the exponential function.	$y = e^x + 3$ has a horizontal asymptote at $y = 3$, and $(0, 1)$ is transformed to $(0, 4)$.
3 Find the intercepts.	$y = \ln(x - 3)$ has an intercept at $(4, 0)$ and $y = e^x + 3$ has an intercept at $(0, 4)$.
4 Sketch the functions on the same axes. The exponential function crosses $x = 3$ at $(3, e^3 + 3) \approx (3, 23)$. The logarithmic function crosses $y = 3$ at about $(23, 3)$.	

Investigation Intersections of logarithmic functions and their inverses

Some logarithmic functions and their inverses intersect each other. You can use your graphics calculator to find the point(s) of intersection.

TI-84 Plus CE



1 Press **3** STO→ ALPHA **A** to make $A = 3$.

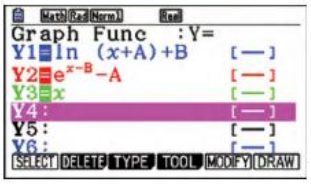
2 Similarly, store $B = 2$ in the memory.

3 Press **y =** and put $y = \ln(x + A) + B$ into Y_1, $y = e^{(x-B)} - A$ into Y_2 and $y = x$ into Y_3.

4 Press **Graph** to draw the graphs.

5 Use **trace**, **zoom** or **CALC** to find the intersection(s).

Casio fx-CG50 AU



1 In the Run-Matrix menu, press **3** → ALPHA **A** to make $A = 3$.

2 Similarly, store $B = 2$ in the memory.

3 Press **MENU** **5** Graph and put $y = \ln(x + A) + B$ into Y1, $y = e^{(x-B)} - A$ into Y2 and $y = x$ into Y3.

4 Press **F6** (DRAW) to see the graphs.

5 Use Trace, Zoom or G-Solv to find the intersection(s).

All calculators

Try entering in different values of A and B to investigate the intersections of other logarithmic functions and their inverses. Hint: Try setting $A = 0$ and changing B, then setting $B = 0$ and changing A before changing both.

Are the intersections always on $y = x$?

Why/Why not?

Are there values of A and B for which there is only one intersection?

What values have 2 intersections?

What values have no intersections?

TI-Nspire
Derivatives of exponential and logarithmic functions

EXERCISE 1.3 The natural logarithmic function $y = \ln(x)$

ANSWERS p. 390

Recap

1 On the same set of axes, sketch both $y = e^x$ and $y = e^{\frac{1}{2}x}$.

2 Find the derivative of $g(x) = \sqrt{e^{4x-1}}$.

A $4e^{8x-2}$ **B** $2e^{2x-\frac{1}{2}}$ **C** $2e^{4x-1}$ **D** $4e^{x-\frac{1}{4}}$

Mastery

3 WORKED EXAMPLE 8 Write each expression in index form.

a $\ln(47.5) \approx 3.86$ **b** $\ln(0.63) \approx -0.46$

c $7.648 \approx \ln(2096)$ **d** $9.22 \approx \ln(10\,100)$

e $\ln(0.000\,489) \approx -7.62$

4 Write each expression in logarithmic form.

a $e^{-5.89} \approx 0.002\,77$ **b** $109 \approx e^{4.69}$

c $1.4 \approx e^{0.37}$ **d** $e^{9.47} \approx 13\,000$

e $e^{-2.3} \approx 0.1$

5 WORKED EXAMPLE 9 Evaluate each expression, correct to 3 significant figures.

a $\ln(36.8)$ **b** $\ln(0.075)$

c $\ln(498)$ **d** $\ln(2.12)$

e $\ln(0.000\,082)$

6 WORKED EXAMPLE 10 Find, correct to 3 significant figures, the value whose natural logarithm is

a 3.85 **b** −6.78 **c** 1.68

d −0.95 **e** 0.53

7 Simplify each expression.

a $\ln(3xy) + \ln(4y)$ **b** $\ln(6pq^2) - 2\ln(q)$

c $\ln(p+q) - \ln(pq)$ **d** $2\ln(xy) - \ln(y) + 1$

e $2 - \ln(e)$ **f** $\ln(m^2 - n^2) - \ln(2mn)$

8 WORKED EXAMPLE 11 Sketch the graph of $y = \ln(x+1) - 3$ and state its domain and range.

9 Sketch the graph of $y = \ln(x-4) + 1$ and state its domain and range.

10 Sketch the graph of $y = \ln(x+2) + 3$ for $-1 \leq x < 2$ and state its range over this domain.

11 Sketch the graph of $y = 3 - \ln(x-2)$ and state its domain and range.

12 State the asymptotes of

a $\ln(x+2)$ **b** $\ln(x-4)$ **c** $2\ln(x+1) - 4$ **d** $\ln(2x-1)$

13 WORKED EXAMPLE 12

a Determine the inverse of $y = e^{x+2}$.

b Sketch the graph of $y = e^{x+2}$ and its inverse on the same axes.

14 **a** Determine the inverse of $y = e^{x-3}$.

b Sketch the graph of $y = e^{x-3}$ and its inverse on the same axes.

15 **a** Determine the inverse of $y = e^x + 2$.

b Sketch the graph of $y = e^x + 2$ and its inverse on the same axes.

16 **a** Determine the inverse of $y = e^x - 4$.

b Sketch the graph of $y = e^x - 4$ and its inverse on the same axes.

Exam practice

17 Write $\ln(28.4) = 3.35$ in index form.

A $28.4 = e^{3.35}$ **B** $28.4 = e^{25.05}$ **C** $3.35 = e^{28.4}$ **D** $25.05 = e^{3.35}$

18 Evaluate $\ln(13.8)$ correct to 4 decimal places.

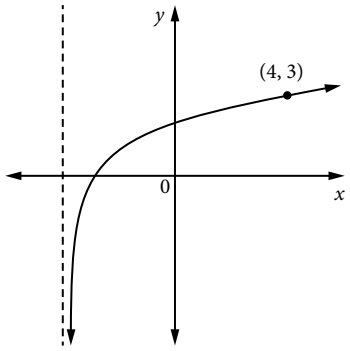
A 3.3500 **B** 2.6247 **C** 13.8000 **D** not defined

19 Evaluate $\ln(0)$ correct to 4 decimal places.

A 0.0000 **B** 1.0000 **C** 2.7183 **D** not defined

20 (4 marks) CF Sketch the graph of $y = \ln(1 - x) + 2$ and state its domain and range.

21 © QCAA 2024 1Q16 TECH-FREE (4 marks) CF The graph is of the form $y = \log_a(x + b)$.
A point on the graph (4, 3) is labelled. The line $x = -4$ is an asymptote.



There is a point $P(x_P, y_P)$ on the graph where y_P is twice the value of the y-intercept of the curve.
Determine the value of x_P.

Worksheets
Solving exponential equations

Exponential equations

Logarithm laws

Logarithms review

Puzzles
Logarithms - Solving equations 1

Logarithms - Solving equations 2

1.4 Exponential and logarithmic equations

From the definition of $\ln(x)$, if $y = \ln(x) = \log_e(x)$, then $x = e^y$.

You can state this in words as "$\ln(x)$ is the index to which e must be raised to obtain x".

Just as with a general base, the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are **inverse functions**.

Properties of natural logarithms

$$e^{\ln(a)} = a, \text{ for } a > 0$$
$$\ln(e^a) = a$$

Natural logarithms follow the same laws as general logarithms.

$$\ln(xy) = \ln(x) + \ln(y)$$
$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$
$$\ln(x^n) = n\ln(x)$$
$$\ln(1) = 0$$
$$\ln(e) = 1$$

WORKED EXAMPLE 13 Logarithmic equation

Solve $\ln(2x + 1) = 3$.

Steps	Working
1 Exponentiate both sides.	$\ln(2x + 1) = 3$ $2x + 1 = e^3$
2 Solve for x.	$2x = e^3 - 1$ $x = \frac{e^3 - 1}{2}$

Exam hack

If the question does not ask for an approximate answer, you must give an exact answer.

TI-Nspire
Derivatives of exponential and logarithmic functions

WORKED EXAMPLE 14 Combined equation

Solve $e^x = x^2$, correct to 3 decimal places.

Steps	Working

You must solve this using approximations.

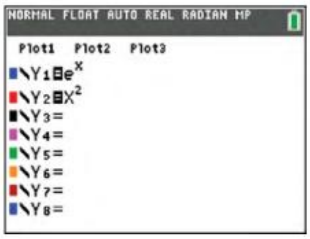
1 Select the **Graph** mode or press the y= button.

2 Input the functions:

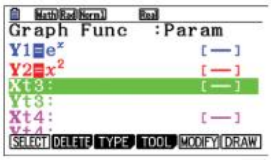
$Y1 = e^x$

$Y2 = x^2$

TI-84 Plus CE



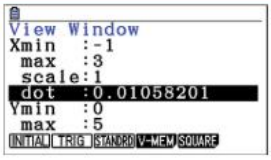
Casio fx-CG50 AU



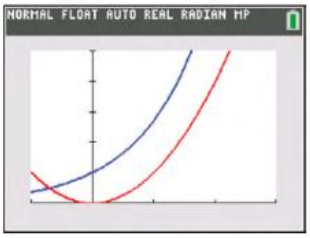
9780170484985

3 Set the viewing window Xmin = −1, Xmax = 3, Ymin = 0, Ymax = 5.

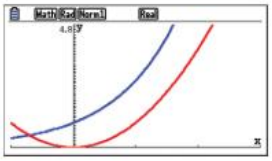




4 Graph the equations.

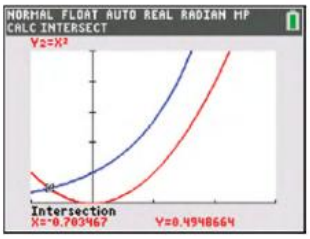


Press **draw** to graph both equations.

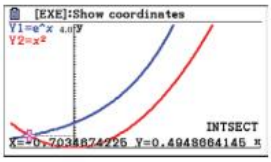


Press **GRAPH** to display the graphs.

5 Find the points of intersection.



Use the **G-Solve** option and then choose **Intersect** to find the intersection point.



Use **2nd** + **Calc** and select **5: intersect** to find the intersection point, following the on-screen prompts.

6 State the result.

$x \approx -0.703$

EXERCISE 1.4 Exponential and logarithmic equations

ANSWERS p. 391

Recap

1 Solve each logarithmic equation, correct to 3 significant figures.

a $\ln(a) = 2.98$

b $\ln(b) = -3.02$

2 **a** Find the inverse of $y = \ln(x + 2)$.

b Sketch the graph of $y = \ln(x + 2)$ and its inverse on the same axes.

Mastery

3 WORKED EXAMPLE 13 Solve each exponential equation for x.

a $e^x = e^5$

b $e^{2x} = 1$

c $e^{x+2} = e^4$

d $e^{-x} = e^{-2}$

e $e^{3x} = e^9$

4 WORKED EXAMPLE 14 Solve each equation for x.

a $\ln(x) = 3$

b $\ln(e^3) = x$

c $\ln(x^2) = 0$

d $\ln(x) + \ln(2) = \ln(6)$

e $\ln(5x) = \ln(5) + \ln(3)$

5 Solve for x, correct to 3 decimal places if necessary.

a $e^{x^2} = 20$

b $2e^x = 15$

c $e^{x-3} + e^x = 10$

d $e^{2x} - 5e^x + 6 = 0$

e $e^x = x^2 + 2$

f $e^{x^2-4x} = 10$

6 Solve each logarithmic equation for x, correct to 3 decimal places if necessary.

a $\ln(x + 5) = 1.7$

b $\ln(4x) - \ln(2) = 3.5$

c $\ln(x) + \ln(x + 1) = 0.5$

d $\ln(e^x + 1) = 2.3$

Exam practice

7 TECH-FREE Solve for x: $e^{2x} = e^{x+3}$.

A $x = 3$ **B** $x = \frac{3}{2}$ **C** $x = 1$ **D** $x = -3$

8 TECH-FREE Given $\ln(x + 1) - \ln(x - 1) = \ln(3)$, find x.

A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 4$

9 TECH-FREE Determine x given that $e^{x^2} = e^x$.

A $x = 0$ or $x = 1$ **B** $x = 0$ or $x = 2$ **C** $x = 1$ or $x = 2$ **D** $x = 1$

10 (8 marks) Solve for x, correct to 3 decimal places.

a $3\ln(2x) = 4$ *[2 marks]*

b $e^{\frac{x}{2}} - 7x = 0$ *[2 marks]*

c $\ln(3x + 4) = 2$ *[2 marks]*

d $e^{3x} = 7x^2$ *[2 marks]*

11 (8 marks) CF Solve for x.

a $e^x = \frac{1}{e^3}$ *[2 marks]*

b $\ln(e^{x+1}) = 7$ *[2 marks]*

c $e^{x^2} = e$ *[2 marks]*

d $\ln(x) = -\ln(4)$ *[2 marks]*

12 © QCAA 2024 1Q4 68% TECH-FREE Simplify $y = 2\ln(e^x)$.

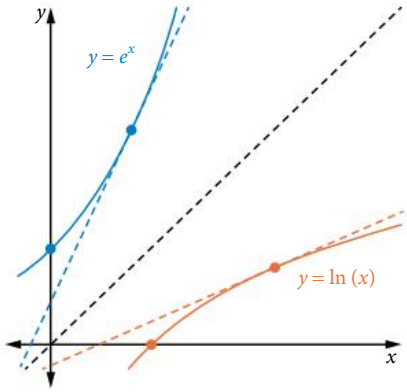
A $y = 2x$ **B** $y = 2^x$ **C** $y = \frac{2}{x}$ **D** $y = x^2$

1.5 The derivative of ln(x)

Consider the graphs of $y = \ln(x)$ and $y = e^x$.

$\ln(2.23) = 0.8$, so (2.23, 0.8) is on the graph of $y = \ln(x)$.

Also, $e^{0.8} = 2.23$, so (0.8, 2.23) is the corresponding point on the graph of $y = e^x$.



Draw a tangent to e^x at (0.8, 2.23).

$\frac{d}{dx}(e^x) = e^x$, so the slope of the tangent is also 2.23.

The equation of the tangent is

$y - 2.23 = 2.23(x - 0.8)$

$y = 2.23x - 2.23(0.8) + 2.23$

$y = 2.23x + 0.45$

The y-intercept of the tangent is $(0, 0.45)$.

Note that $0.45 = 2.23 \times (1 - 0.8)$, which is used to calculate the slope below.

Now draw the tangent to $y = \ln(x)$ at $(2.23, 0.8)$.

By symmetry, the x-intercept of this tangent must be $(0.45, 0)$.

The slope of the tangent to $y = \ln(x)$ at $(2.23, 0.8)$ is given by

$$m = \frac{0.8 - 0}{2.23 - 0.45} = \frac{0.8}{2.23 - 2.23(1 - 0.8)} = \frac{0.8}{2.23(0.8)} = \frac{1}{2.23}.$$

Hence the derivative of $y = \ln(x)$ at (2.23, 0.8) is $\frac{1}{2.23}$.

If you choose any other point of $y = \ln(x)$, you will get the same pattern.

The derivative of $y = \ln(x)$ at any point $(a, \ln(a))$ is $\frac{1}{a}$.

The derivative of ln(x)

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

WORKED EXAMPLE 15 The derivative of ln(x)

Calculate the derivative of $y = \ln(x)$ at $x = 5$.

Steps	Working
1 Write the general rule.	$\frac{dy}{dx} = \frac{1}{x}$
2 Find the value for $x = 5$.	When $x = 5$, $\frac{dy}{dx} = \frac{1}{5} = 0.2$.
3 Write the answer.	The derivative of $y = \ln(x)$ at $x = 5$ is 0.2.

Worksheets
Exponential and logarithmic functions
Derivatives of logarithmic functions
Differentiating exponential and logarithmic functions

You can use all the differentiation rules for derivatives involving $\ln(x)$.

WORKED EXAMPLE 16 Differentiating functions involving $\ln(x)$

Calculate the derivative of $4\ln(x) + 9x + 2$.

Steps	Working
1 Use the general derivative rules.	$\frac{d}{dx}[4\ln(x) + 9x + 2] = 4\frac{d}{dx}[\ln(x)] + 9 + 0$
2 Use the derivative rule for $\ln(x)$.	$= \frac{4}{x} + 9$
3 Write the answer.	$= \frac{4+9x}{x}$

WORKED EXAMPLE 17 The chain, product and quotient rules

Calculate the derivative of each function.

a $\ln(5x)$ **b** $\ln(3x^4)$ **c** $x^3\ln(x)$ **d** $\frac{\ln(x)}{x^3}$

Steps	Working
a 1 Use the chain rule.	Let $u = 5x$. Then $\ln(5x) = \ln(u)$. $\frac{d}{dx}[\ln(5x)] = \frac{d}{du}[\ln(u)] \times \frac{du}{dx}$ $= \frac{1}{u} \times 5$
2 Substitute $u = 5x$.	$= \frac{1}{5x} \times 5$
3 Simplify.	$= \frac{1}{x}$
b 1 Use the chain rule.	Let $u = 3x^4$. Then $\ln(3x^4) = \ln(u)$. $\frac{d}{dx}[\ln(3x^4)] = \frac{d}{du}[\ln(u)] \times \frac{du}{dx}$ $= \frac{1}{u} \times 12x^3$
2 Substitute $u = 3x^4$ and simplify.	$= \frac{1}{3x^4} \times 12x^3$ $= \frac{4}{x}$
c 1 Use the product rule.	Let $u(x) = x^3$ and $v(x) = \ln(x)$. $u' = 3x^2 \quad v' = \frac{1}{x}$ $\frac{d}{dx}[x^3\ln(x)] = uv' + vu'$
2 Substitute for u, u', v' and v.	$= x^3 \times \frac{1}{x} + \ln(x) \times 3x^2$ $= x^2 + 3x^2\ln(x)$

d **1** Use the quotient rule.

Let $u(x) = \ln(x)$ and $v(x) = x^3$.

Then $u' = \frac{1}{x}$ and $v' = 3x^2$.

2 Write the formula for the derivative.

$$\frac{d}{dx}\left[\frac{\ln(x)}{x^3}\right] = \frac{vu' - uv'}{v^2}$$

3 Substitute for u, u', v' and v.

$$= \frac{x^3 \times \frac{1}{x} - \ln(x) \times 3x^2}{(x^3)^2}$$

4 Simplify.

$$= \frac{x^2 - 3x^2 \ln(x)}{x^6}$$

$$= \frac{1 - 3\ln(x)}{x^4}$$

You might find the answers for parts **a** and **b** above surprising.

Always look for shortcuts using the logarithm laws when you **differentiate** logarithmic functions; for example:

- $\ln(5x) = \ln(5) + \ln(x)$, whose derivative is $0 + \frac{1}{x}$ because $\ln(5)$ is a constant.
- $\ln(3x^4) = \ln(3) + \ln(x^4) = \ln(3) + 4\ln(x)$, so its derivative is $0 + \frac{4}{x}$.

The derivative of ln (*f*(*x*))

$$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$$

WORKED EXAMPLE 18 Derivative of ln (*f*(*x*)) 1

Given $y = \ln(e^x + 2x + 1)$, **determine** $\frac{dy}{dx}$.

Steps	Working
1 Set up the function and its derivative.	Let $f(x) = e^x + 2x + 1$. Then $f'(x) = e^x + 2$.
2 State the rule.	$\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$
3 Use the rule.	$= \frac{e^x + 2}{e^x + 2x + 1}$

WORKED EXAMPLE 19 Derivative of ln (*f*(*x*)) 2

Differentiate $\ln(3x^4 + 2x^2 - 3)$.

Steps	Working
1 Set up the function and its derivative.	Let $f(x) = 3x^4 + 2x^2 - 3$. Then $f'(x) = 12x^3 + 4x$.
2 State and use the rule.	$\frac{d}{dx}[\ln(3x^4 + 2x^2 - 3)] = \frac{f'(x)}{f(x)}$ $= \frac{12x^3 + 4x}{3x^4 + 2x^2 - 3}$

EXERCISE 1.5 The derivative of ln(x)

ANSWERS p. 391

Recap

1 Solve $e^{x-1} = e^2$.

2 Solve $\ln(x^2 + 3x) = 1.8$, correct to 3 decimal places.

Mastery

3 WORKED EXAMPLE 16 Find the derivative of $y = \ln(x)$ at each value.

a $x = 0.5$ **b** $x = 2$ **c** $x = 4$

d $x = 20$ **e** $x = 0.01$

4 WORKED EXAMPLE 17 Find the derivative of each function.

a $7\ln(x)$ **b** $5\ln(x)$ **c** $\ln(x) + x + x^2$

d $6x^3 - 7\ln(x)$ **e** $x - x^2 + 9\ln(x)$

5 WORKED EXAMPLE 18 Find the derivative of each function.

a $\ln(6x)$ **b** $\ln(-4x)$ **c** $\ln\left(-\frac{3x}{4}\right)$

d $4\ln(3x)$ **e** $5\ln(3x) - 2\ln\left(\frac{x}{3}\right)$

6 Find the derivative of each function.

a $4\ln(x^2)$ **b** $3\ln(x^2 + 3x + 2)$ **c** $2\ln\left(\sqrt{3x-1}\right)$

d $3\ln(x^2) - 2\ln\left(\sqrt{x}\right)$ **e** $2\ln((3x-4)^3)$

7 Differentiate each function.

a $x\ln(x)$ **b** $x^2\ln(x^2)$ **c** $(2x^2 + 3x + 1)\ln(2x + 1)$

d $x^2\ln(x) + x\ln(x^2)$ **e** $\sqrt{x^2+5}\ln(x^2 + 5)$

8 Differentiate each function.

a $\dfrac{\ln(x^2+3)}{x+3}$ **b** $\dfrac{\ln(x+3)}{\ln(x-3)}$ **c** $\dfrac{x^2}{\ln(x)}$

d $\dfrac{\ln(x^2+2x+5)}{(x+1)^2}$ **e** $\dfrac{\ln\left(\sqrt{3x-1}\right)}{\sqrt{3x+1}}$

9 Find the derivative of each function.

a $\ln(e^x)$ **b** $x\ln(xe^x)$ **c** $e^x\ln(x)$

d $e^{2x+1}\ln(2x+1)$ **e** $3e^{4x+3} - e^{3x}\ln(4x+3)$

10 WORKED EXAMPLE 19 Find the derivative of each function.

a $\ln(3x^2 + 2x + 4)$ **b** $\ln(5e^x + 7x^2e^x)$ **c** $\ln(\ln(x))$

d $\ln(x^5)$ **e** $\ln(4x^3) + 3\ln(x^4)$

11 Find $\dfrac{dy}{dx}$ for each function.

a $y = \ln((x-5)^3)$ **b** $y = \ln((4x+3)^5)$ **c** $y = \ln((3x^2 + 4x + 2)^3)$

d $y = \ln((x^2 + 2x + 1)^5)$ **e** $y = \ln((2x^2 + 3x + 1)^4)$

12 Find the derivative of each function.

a $\ln(4x^3 - 5x^2 + 2x - 3)$ **b** $\ln(x^5 + 3x^4 - 2x)$ **c** $\ln(7x^4 + 3x^3 - 2x^2 + 5x - 6)$

13 Find the derivative of each function.

a $\ln((x+3)^2(x-5)^3)$

b $[\ln(x+3)^2][\ln(x-5)^3]$

c $\dfrac{\ln(x+3)^2}{\ln(x-5)^3}$

d $\ln\left[\dfrac{(x+3)^2}{(x-5)^3}\right]$

14 Find $\dfrac{dy}{dx}$ for each function.

a $y=\ln\left(\sqrt{x^2-3x+4}\right)$

b $y=\ln(4x-5)^{\frac{3}{2}}$

c $y=[\ln(4x-5)]^{\frac{3}{2}}$

d $y=\sqrt[3]{\ln(x^3)}$

15 Find the derivative of each function.

a $\ln(x^{-1})$

b $[\ln(x)]^{-1}$

c $\ln(x^2-3x+5)^{-4}$

d $\ln(x^3-3x^2+5x+4)^{-0.5}$

16 Find the derivative of $(3x^2+2x+1)\ln(x^3+x^2+x-5)$.

17 Find the derivative of $(x^3+x^2+x-5)\ln(3x^2+2x+1)$.

18 (2 marks) Find the derivative of each function.

a $y=\ln(x^4)$

b $f(x)=\dfrac{\ln(x)}{x^3}$

Exam practice

All questions TECH-FREE

19 © QCAA 2020S 1Q5 The slope of the tangent to the graph of $y=\ln(x^2)$ at $x=e^2$ is

A $\dfrac{1}{e^2}$ **B** $\dfrac{2}{e^2}$ **C** $\dfrac{4}{e^4}$ **D** $\dfrac{1}{e^4}$

20 Find the derivative of $\ln(3x+2)$.

A $\dfrac{1}{3}$ **B** $\dfrac{1}{3x}$ **C** $\dfrac{3}{3x+2}$ **D** $\dfrac{3}{2}$

21 © QCAA 2021 1Q14 MODIFIED (4 marks)

Consider the function $f(x)=\ln(2x+5)$, for $x>\dfrac{-5}{2}$.

a Determine $f'(x)$. *[1 mark]*

b Determine the x-intercept of the graph of $f(x)$. *[2 marks]*

c Determine the gradient of the tangent to the graph of $f(x)$ at the x-intercept. *[1 mark]*

22 (2 marks) Find the derivative of $\dfrac{\ln(2x+5)^4}{\ln(x^2+3)^3}$.

23 (2 marks) Find the derivative of $y=\ln\left(\sqrt{6x^3-2x+5}\right)$.

24 © QCAA 2023 2Q15 MODIFIED (4 marks)

Determine the derivative of $f(x)=\ln(x^3)+\ln(x+4)^2$.

Express the derivative as a single fraction in its simplest and factorised form.

25 (3 marks) CF Given $g(x)=\ln[f(x)]$, $f(5)=8$ and $f'(5)=2$, find $g'(5)$.

26 (3 marks) CF Find a general expression for the derivative of $\ln[(ax+b)^n]$.

Worksheets
Exponential functions assignment

Exponential functions extension

Derivatives of logarithmic functions assignment

Derivatives of logarithmic functions extension

1.6 Problems involving exponential and logarithmic functions

Many natural phenomena are modelled using exponential functions of the form

$$f(t) = Ae^{kt}$$

where A and k are constants.

Exponential growth and decay

$f(t) = A \times e^{kt}$ is used to model **exponential growth** over time t, where A is the initial quantity of $f(t)$ and $k > 0$. It can also be used to model **exponential decay** for $k < 0$. Sometimes a base other than e is used.

Video
Exponential decay

WORKED EXAMPLE 20 Derivative as the rate of change

The concentration of a drug measured in mg/mL in a patient's bloodstream t hours after it is administered is given by the exponential function $C(t) = 3.87e^{-0.85t}$.

Calculate

a the initial concentration.

b the concentration after 1 hour.

c the rate of change of concentration.

d the rate of change of concentration after 3 hours.

Steps	Working
a 1 Substitute $t = 0$.	$C(t) = 3.87e^{-0.85 \times 0}$
	$= 3.87$
2 Write the result.	The initial concentration is 3.87 mg/mL.
b 1 Substitute $t = 1$.	$C(t) = 3.87e^{-0.85 \times 1}$
	$= 1.654\ldots$
2 Write the result.	The concentration after 1 hour is about 1.65 mg/mL.
c 1 Find the derivative.	$C(t) = 3.87e^{-0.85t}$
2 Let $u = -0.85t$.	$C(u) = 3.87e^{u}$
3 Calculate $\frac{dC}{du}$.	$\frac{dC}{du} = 3.87e^{u}$
4 Calculate $\frac{du}{dt}$.	$\frac{du}{dt} = -0.85$
5 Use the chain rule.	$\frac{dC}{dt} = \frac{dC}{du} \times \frac{du}{dt}$
6 Substitute the known values.	$= 3.87e^{u} \times (-0.85)$
7 Substitute for u and simplify.	$= -3.2895e^{-0.85t}$
8 Write the answer as a statement.	The rate of change is $C'(t) = -3.2895e^{-0.85t}$ mg/mL per hour.
d 1 Substitute $t = 3$.	$C'(t) = -3.2895e^{-0.85t}$
	$= -3.2895e^{-0.85 \times 3}$
	$= -0.2568\ldots$
2 Write the answer.	The rate of change of concentration after 3 hours is about −0.257 mg/mL.

WORKED EXAMPLE 21 Exponential growth

The number of gum trees with red flowers in a tropical region of Australia was studied. The equation $F(t) = 1000 \times 2^{0.06t}$ was used in the model, where F is the number of trees and t is the time in years.

Determine

a the initial number of trees.

b the number of trees after 5 years.

c the rate of increase in the number of trees after 5 years.

d how long it took for the number of trees to double.

Steps		Working
a	**1** Substitute $t = 0$.	$F(t) = 1000 \times 2^{0.06 \times 0}$ $= 1000$
	2 Write the answer.	There were 1000 trees at the beginning.
b	**1** Substitute $t = 5$.	$F(t) = 1000 \times 2^{0.06 \times 5}$ $= 1231.1444\ldots$
	2 Write the answer.	There were about 1231 trees after 5 years.
c	**1** Use the derivative of an exponential function for $t = 5$.	$F(t) = 1000 \times 2^{0.06 \times t}$
	2 Let $u = 0.06t$.	$F(u) = 1000 \times 2^u$
	3 Find $\frac{dF}{du}$.	$\frac{dF}{du} = 1000 \times \log_e(2) \times 2^u$
	4 Find $\frac{du}{dt}$.	$\frac{du}{dt} = 0.06$
	5 Use the chain rule.	$\frac{dF}{dt} = \frac{dF}{du} \times \frac{du}{dt}$
	6 Substitute known values.	$= 1000 \times \log_e(2) \times 2^u \times 0.06$
	7 Substitute $t = 5$.	$= 60 \times \log_e(2) \times 2^{0.06t}$ $F'(5) = 51.201\ldots$
	8 Write the result.	The rate of increase after 5 years is approximately 51 trees per year.
d	**1** 2000 is double the initial number of trees. Let $F(t) = 2000$.	$2000 = 1000 \times 2^{0.06t}$
	2 Divide both sides by 1000.	$2 = 2^{0.06t}$
	3 Equate indices.	$0.06t = 1$
	4 Solve for t.	$t = 16.667$
	5 Write the answer.	It took approximately 16 years and 8 months for the number of trees to double.



Video
Exponential growth

WORKED EXAMPLE 22 Growth of bacteria

Scientists know that, given sufficient space and nutrients, the number of bacteria in a culture grows according to the function $N(t) = ae^{bt}$, where N is the number of bacteria and t is time in minutes. A particular culture had 2500 bacteria present initially. After 20 minutes, the culture had grown to 8400 bacteria.

a **Determine** the function for the number of bacteria present in the culture.

b **Determine** how many bacteria will be present 1 hour after the initial count was made.

c **Calculate** the rate of change of the number of bacteria in the culture.

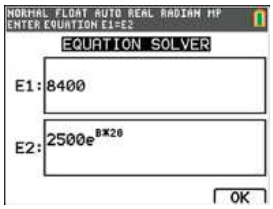
d **Assess** how long will it take for the number of bacteria in the culture to double.

Steps	Working
a 1 Substitute $(0, 2500)$ and solve for a.	$2500 = ae^{b \times 0}$
	$2500 = a$
2 Substitute $(20, 8400)$ and solve for b.	$8400 = 2500e^{20b}$
	$e^{20b} = \frac{8400}{2500} = 3.36$
	$20b = \log_e(3.36)$
	$b = \frac{\log_e(3.36)}{20} = 0.0605\ldots$
3 Write the function.	$N(t) = 2500e^{0.0605\ldots t}$
b 1 Evaluate N when $t = 60$.	$N(60) = 2500e^{0.0605\ldots \times 60}$
	$= 94\,832.64$
2 Write the result.	There are about 95 000 bacteria after an hour.
c Find the derivative.	$N'(t) = abe^{bt}$
	$= 2500 \times 0.0605\ldots e^{0.0605\ldots t}$
	$\approx 151e^{0.0605t}$
d 1 The doubled number is 5000. Solve for t.	$5000 = 2500e^{0.0605\ldots t}$
	$e^{0.0605\ldots t} = 2$
	$0.0605\ldots t = \ln(2)$
	$t = \frac{\ln(2)}{0.0605\ldots}$
	$= 11.438\ldots$
2 Write the result.	It takes about 11 minutes for the number of bacteria to double.

Use a calculator to evaluate the unknowns.

TI-84 Plus CE

1 Press math and choose **Numeric Solver**.
Enter the equation and press **OK**.



2 Press **solve**.



Casio fx-CG50 AU

1 Select **EQUATION** and select **SOLVER**.
Enter the equation and the values.



2 Press **solve**.

A power relationship between variables is easily determined by plotting logarithms of the variables. This is called a **log–log graph**.

Suppose $\ln(y) = 2\ln(x) + 1.6$.

Then $\ln(y) = \ln(x^2) + \ln(e^{1.6}) = \ln(e^{1.6}x^2)$, so $y = e^{1.6}x^2 \approx 5x^2$.

Log graphs

If $\ln(y) = a\ln(x) + b$, then y is a power function of x given by $y = e^b x^a$.

WORKED EXAMPLE 23 Logarithmic scales

Andrea left the concentrations and quantity of sulfuric acid, hydrogen peroxide, sodium thiosulfate and starch in the 'clock reaction' constant. She recorded the times taken for the solution to turn blue for different concentrations of 50 mL of potassium iodide (KI) solution.

Use a log–log graph to **investigate** the relationship between

a concentration and reaction time.

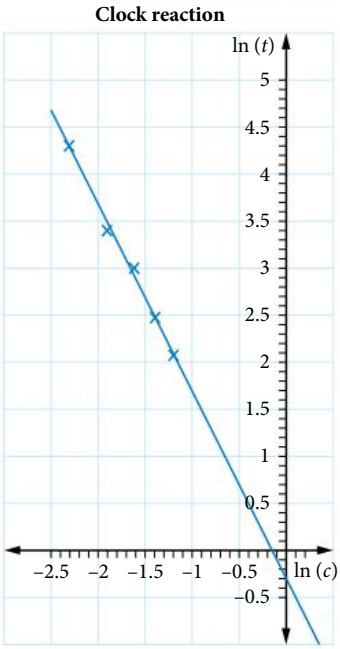
b concentration and rate of reaction.

KI concentration (molarity), c	0.3	0.25	0.2	0.15	0.1
Reaction time (seconds), t	8	12	20	30	75

Steps	Working
a 1 Find the logarithms for each variable.	

$\ln(c)$	−1.20	−1.39	−1.61	−1.90	−2.30
$\ln(t)$	2.08	2.48	3.00	3.40	4.32

2 Plot the graph and draw a line of best fit.



3 Choose some points on the line.

The line passes through $(-2, 3.7)$ and $(0, -0.3)$.

4 Find the slope and the $\ln(t)$-intercept.

$$m = \frac{-0.3 - 3.7}{0 - (-2)} = -2, \ln(t)\text{-intercept} = -0.3$$

5 Write the equation.

$$\ln(t) = -2\ln(c) - 0.3$$

6 Rearrange to make t the subject.

$$t = e^{-0.3}c^{-2} \approx 0.74c^{-2} = \frac{0.74}{c^2}$$

7 Write the relationship.

The reaction time is inversely proportional to the square of the concentration of potassium iodide.

b 1 Write an expression for the rate of reaction.

$$\text{rate of reaction} = \frac{\text{quantity}}{\text{time}}$$

2 Substitute the time.

$$= \frac{\text{quantity}}{0.74c^{-2}}$$

$$= \frac{\text{quantity}}{0.74} \times c^2$$

3 Write the relationship.

The rate of reaction is proportional to the square of the concentration of potassium iodide.

TECHNOLOGY Exponential relationships

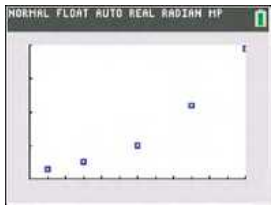
The table shows the radius of a colony of bacteria on an agar plate over 12 hours.

Time t (h)	1	3	6	9	12
Radius r (cm)	0.3	0.5	1.0	2.2	3.9

Use your graphics calculator to find the relationship between radius and time.

The relationship seems like it might be exponential.

TI-84 Plus CE



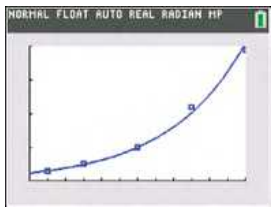
1 Enter the data in L1 and L2.

2 Use stat plot, and choose Plot 1. Set to use L1 and L2 and make the window $0 \le X \le 12$ and $0 \le Y \le 4$. Press graph.



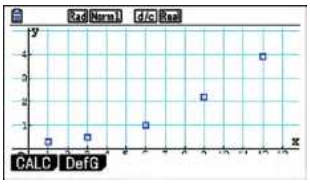
3 To find the exponential line of best fit, press stat and choose CALC, 0: ExpReg. Make XList L1, YList L2 and Store RegEQ Y1 and Calculate.

4 To see the exponential line of best fit, press graph again.



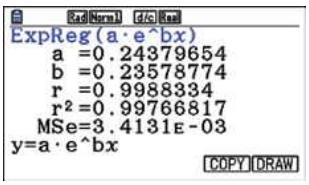
5 The TI finds the exponential $y = ab^x$, so to change to the form $y = ae^{kx}$, you need to write $k = \ln(b)$.

Casio fx-CG50 AU

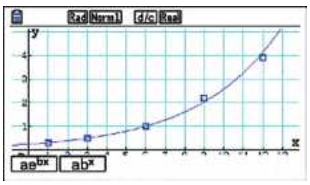


1 In the Statistics menu, enter the data from the table above in L1 and L2.

2 SET GRAPH1 to use List 1 and List 2 and make the V-WINDOW $0 \le X \le 12$ and $0 \le Y \le 4$. EXIT and press F1 (GRAPH1).



3 To find the exponential line of best fit, press F1 (CALC) and choose EXP and ae^{bx}.



4 To see the exponential line of best fit, press F6 (DRAW).

TI-Nspire
Derivatives of exponential and logarithmic functions

EXERCISE 1.6 Problems involving exponential and logarithmic functions ANSWERS p. 392

Recap

1 Find the derivative of $y = \ln(x)$ at $x = 15$.

2 $f(x) = \ln(x^3 + 3x^2 - 8)$. Find $f'(x)$.

Mastery

3 WORKED EXAMPLE 21 The decay of a uranium sample is given by $M(t) = 200e^{-0.012t}$, where M is its mass in grams and t is the time in years.

a What was the initial mass of the sample?

b What will the mass of the sample be after

i 1 year? **ii** 10 years? **iii** 100 years?

c What is the rate of decay after

i 1 year? **ii** 10 years? **iii** 100 years?

4 WORKED EXAMPLE 22 An advertising campaign of a product helped increase sales of the product according to the formula $S(t) = 200e^{0.15t}$, where S is the number of sales and t is the time in weeks.

a How many sales were made at the beginning of the campaign?

b How many sales were made after 6 weeks of the campaign?

c What was the rate at which the sales were made after 6 weeks?

d The advertising campaign finished after 12 weeks. What was the rate of sales at this time?

5 WORKED EXAMPLE 23 A breeding program is implemented to increase the number of fish in the Tamar River. The model for the population of fish over time t months is $F(t) = 500 + 2e^{0.1t}$.

a How many fish are there at the start of the program?

b What is the population after

i 6 months? **ii** 5 years?

c Find a formula for the rate at which the population increases over time.

d At what rate is the fish population increasing after

i 6 months? **ii** 5 years?

6 The temperature T (°C) of a chemical that cools down over t minutes is given by $T(t) = 95e^{-0.25t}$.

a Find the temperature after

i 1 minute. **ii** 5 minutes.

b Find the rate at which the temperature changes after

i 1 minute. **ii** 5 minutes.

7 WORKED EXAMPLE 24 Daniel timed a chemical reaction for different concentrations of sodium nitrate and calculated the rates of reaction shown below.

Concentration, c (M)	0.01	0.02	0.05	0.1	0.2
Rate of reaction, r (g/s)	0.11	0.31	1.11	4	10

Use a log graph to find the power relationship.

8 The period and radius of the planets' orbits are shown below. Use a graph to find the relationship between period and distance from the Sun.

Planet	Period (days)	Distance (million km)
Mercury	88	58
Venus	225	108
Earth	365	150
Mars	687	228
Jupiter	4 333	778
Saturn	10 759	1 427
Uranus	30 685	2 870
Neptune	60 191	4 497
Pluto*	90 465	5 900

*Pluto was reclassified as a dwarf planet in 2006.

9 Marc and Helen Bornstein claimed that people in big cities walk faster than those in small towns. The table below shows the average walking speeds, w, of people in a number of cities and the population, n, at the time. Use a graph to find a possible relationship.

City	Population	Speed (m/s)
Psychro, Crete	370	0.84
Itea, Greece	2500	0.69
Corte, Corsica	5500	1.01
Bastia, France	49 000	1.49
Netanya, Israel	71 000	1.31
Iraklion, Greece	78 000	1.17
New Haven, USA	138 000	1.34
Jerusalem, Israel	305 000	1.35
Brno, Czech Republic	342 000	1.47
Athens, Greece	867 000	1.59
Munich, Germany	1 340 000	1.71
Brooklyn, USA	2 602 000	1.54

10 The area of rainforest in a tropical location is defined by the function $A(t) = 80\,000e^{-0.02t}$, where A is the area in hectares (10 000 m^2) and t is the time in years.

a What is the initial area of rainforest?

b How many hectares will there be after

i 1 year? **ii** 10 years? **iii** 100 years?

c At what rate is the rainforest decreasing after

i 1 year? **ii** 10 years? **iii** 100 years?

11 An influenza virus spreads through a major city according to the function $N(t) = 85e^{0.08t}$, where N is the number of infected people and t is time in days.

a How many people are initially affected?

b How many people are infected after

i 5 days? **ii** 2 weeks? **iii** 12 weeks?

c What is the rate of infection after

i 5 days? **ii** 2 weeks? **iii** 12 weeks?

d Use your calculator to determine how long it would take for there to be 100 000 cases.

12 Use a log–log graph to find the relationship between the variables in the table below.

t	3.8	5.8	8.1	12.4	15.4	18.5	18.9
m	17.6	42.9	77.1	118	160	270	212

13 Find the relationship between the variables in the table below.

m	−1.3	0.3	1.2	2.6	5.5	6.9
n	2.5	5.5	20	66	210	670

14 A survey of birds in a wetlands area determined that their numbers were increasing in accordance with the function $B(m) = 1000e^{0.03m}$, where B is the number of birds and m is the number of months. After how long will there be approximately 3000 birds?

15 A city's population is modelled by the function $P(t) = 2000e^{0.03t}$, where t is time in years.

a By what percentage has the population increased after 5 years?

b How many years will it take for the population to

i double? **ii** triple? **iii** quadruple?

16 A car tyre with a small puncture is inflated to a pressure of 25 units. 12 hours later, the pressure has deflated to 20 units. The pressure, P, after t hours is given by $P = P_0e^{-kt}$.

a After how many hours would the pressure drop to 15 units?

b Determine the rate of pressure loss at these times.

i $t = 0$ **ii** $t = 12$ **iii** $t = 24$

17 Galileo noticed that the time a chandelier in Pisa Cathedral took to make one swing seemed to be constant. He timed the swings of pendulums of different lengths and divided by the number of swings to measure the period (the time for one complete swing). His observations were similar to those below. Use a graph to find the relationship.

Length, L (m)	0.3	0.5	1	1.5	2
Period, T (s)	1.1	1.4	2.0	2.5	2.8

Exam practice

18 TECH-FREE Find the derivative of $y = 7^x$.

A $7^x\log_{10}(7)$ **B** $7^x\log_e(7)$ **C** $x^7\log_e(7)$ **D** $7^x\log_e(x)$

19 TECH-FREE Find the derivative of $y = 2^{-3x}$.

A $-3\times2^{-3x}\log_e(2)$ **B** $-3x\times-3^2\log_e(x)$

C $-3\times2^{-3x}\log_{10}(2)$ **D** $-3\times2^{-3x}\log_e(0.008)$

20 (3 marks) A substance is heated to 80 °C and then left to cool in a laboratory with a room temperature of 20 °C. The temperature C at time t minutes is given by $C = 20 + 60e^{-0.25t}$.

What is the rate of cooling at

a $t = 0$? **b** $t = 5$? **c** $t = 10$?

21 (3 marks) The formula for the decay of a radioactive substance over t years is given by $R(t) = R_0e^{-0.05t}$.

The half-life is the time taken for a substance to decay to half of the original quantity.
What is the approximate half-life of this substance?

22 © QCAA 2024 2Q13 (8 marks) The number of termites in a particular nest can be modelled by $N(t) = \dfrac{A}{2+e^{-t}}$, where A is a constant and t represents time (months) since the nest first became a visible mound above ground level.

It is estimated that when the mound first became visible, the population was 3×10^5 termites.

a Determine the value of A. *[1 mark]*

b Determine the number of termites in the nest half a year after the mound became visible. *[2 marks]*

c Determine the time in months after the mound became visible for the initial population to increase by 130 000 termites. Express the time as a decimal. *[2 marks]*

d Develop a formula for the rate of change in the number of termites at any time after the mound became visible. Express your formula as a fraction. *[2 marks]*

e Determine the rate of change in the number of termites five months after the mound became visible. *[1 mark]*

23 (3 marks) The number of apps sold by a software company after t months is given by $A(t) = 2500e^{0.36t}$. If each app costs \$2.50, how long will it take for the company to make \$500 000?

24 (4 marks) CF To be effective, a particular antibiotic must have a concentration of at least 5 ppm (parts per million) in the blood. The antibiotic is removed from the blood by the kidneys, and the rate of removal depends on the concentration. The concentration in a patient's blood was measured and the results are shown. The initial amount was enough to make the concentration 20 ppm. Levels above 30 ppm are dangerous.

Use a suitable graph to find the relationship and determine how often patients should be given the antibiotic.

Time, t (h)	0.5	1	1.5	2	2.5	3
Concentration, c (ppm)	18	16	14	13	11	10

25 © QCAA 2023 1Q19 TECH-FREE (7 marks) CU Jaxon and Shari each own a shop and have recorded the number of customers entering their shop on two consecutive days.

	Day 1	Day 2
Jaxon's customers	40	30
Shari's customers	10	20
Total	50	50

The number of daily customers for each shop can be modelled by the equation $y = A\ln(Bx)$, where x is the day and y is the number of customers. The constants A and B are different for each shop.

Determine algebraically whether the total number of customers for Jaxon and Shari's shops will be the same every day in the future.

26 (4 marks) CU When she started, Cordelia could sew 20 shirts in a day, but after a fortnight she increased her production to 30 shirts a day. Use a model such as $s = a\ln(t + 1)$ or $s = a\ln(t + 1) + b$, where t is time in weeks, to find how much longer it will take Cordelia to reach her target of 40 shirts a day.

27 (4 marks) CU Daniella set up two magnets to repel each other and measured the force by the compression of a spring. Her results are shown in the table below.

Use a suitable graph to find the relationship between separation, x, and the force, F.

Separation, x (cm)	10	8	6	4	3
Force, F (mm of compression)	1	1.5	3	6	11

28 (4 marks) CU A sniffer dog is being trained to identify drugs in bags at an airport. In each session, the trainer puts some drugs in 10 out of 200 bags. At the start, the dog only finds 2 bags containing drugs, but after 5 days, it manages to find 6 of them.

Use a logarithmic model to find how long it will take the dog to find all the bags containing drugs.

Video
Exam question analysis: Derivatives of exponential and logarithmic functions

EXAM QUESTION ANALYSIS

© QCAA | 2020S 1Q19 | TECH-FREE (6 marks)

Determine the equation of the tangent to the curve $y = x\ln(x)$ that passes through the point $(0, -e)$.

Reading the question

- Natural log curve is given.
- A point on the tangent is given.
- The equation of the tangent through this point is required.

Thinking about the question

- Derivative will give the gradient at the tangent.
- Use substitution to determine the equation.
- Point $(0, e)$ is not the intersection between tangent and function.
- Need to show logical organisation and setting out.

Worked solution

Curve $y = x\ln(x)$; point $(0, -e)$

Slope of the tangent $\boldsymbol{y' = 1 + \ln(x)}$ ✓

Point of tangency $(x_0, x_0\ln(x))$

The equation of the line passing through $(0, -e)$

$\boldsymbol{y + e = x(1 + \ln(x_0))}$ (i) ✓

Substituting the coordinates of the point of tangency on the curve into (i):

$x_0\ln(x_0) + e = x_0 + x_0\ln(x_0)$

Solving:

$\boldsymbol{x_0 = e}$ ✓

The point on the curve is $\boldsymbol{(e, e)}$. ✓

The slope of the tangent:

$\boldsymbol{1 + \ln(e) = 2}$ ✓

Equation of tangent:

$y = 2x - e$

✓ shows logical organisation communicating key steps

1 Chapter summary

Derivatives of exponential functions

- $f'(x) = a^x \lim_{h \to 0}\left(\frac{a^h - 1}{h}\right)$
- $e = \lim_{n \to +\infty}\left(1 + \frac{1}{n}\right)^n = 2.718281\ldots$
- $f(x) = e^x$ is known as the **exponential function**.
- The derivative of e^x is itself: $\frac{d}{dx}(e^x) = e^x$
- The derivative of a^x: $\frac{d}{dx}(a^x) = a^x \ln(a)$
- Exponential growth and decay are modelled by the function $f(x) = Ae^{kx}$, where A and k are constants. For $k > 0$, it is **exponential growth**; for $k < 0$, it is **exponential decay**.

Derivatives of logarithmic functions

- The **natural logarithm** of a number is the logarithm to the base $e = 2.718\ldots$
- It is shown as $\ln(x) = \log_e(x)$.
- $\ln(x) = a$ if and only if $x = e^a$.

Natural logarithm laws

If $y = \ln(x) = \log_e(x)$, then $x = e^y$.

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- $\ln(x^n) = n\ln(x)$
- $\ln(1) = 0$
- $\ln(e) = 1$

Graphing natural logarithms

- The natural logarithmic function and the exponential function are **inverse functions**. The point (a, b) on the graph of $y = \ln(x)$ corresponds to the point (b, a) on the graph of $y = e^x$. Their graphs are reflections of each other in the line $y = x$.

  For the graph of $y = \ln(x - h) + k$
  - there is a vertical asymptote at $x = h$
  - the point $(1, 0)$ on $y = \ln(x)$ is transformed to $(1 + h, k)$.

Derivatives

- The derivative of the natural logarithm is $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$.
- The derivative of $\ln(f(x))$ is $\frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$.

Log–log graphs

- If $\ln(y) = a\ln(x) + b$, then y is a power function of x given by $y = e^b x^a$.

Worksheet
Maths Methods formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (16 marks): 4 short response questions

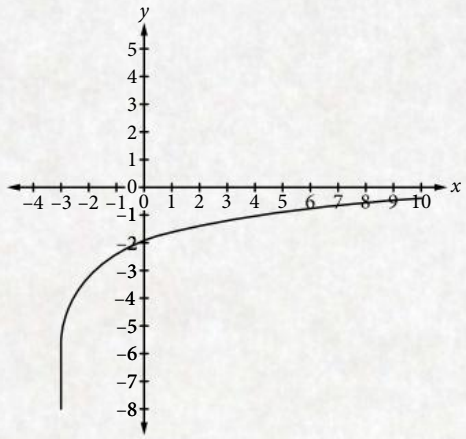
Total: 19 marks

Section 1 3 multiple choice questions

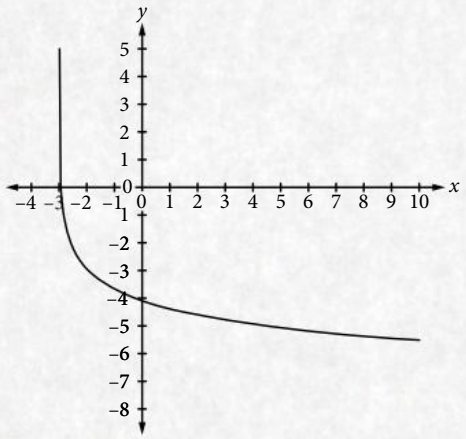
3 marks

1 © QCAA 2022 1Q6 Which graph represents the function $f(x) = -3 - \ln(x + 3)$?

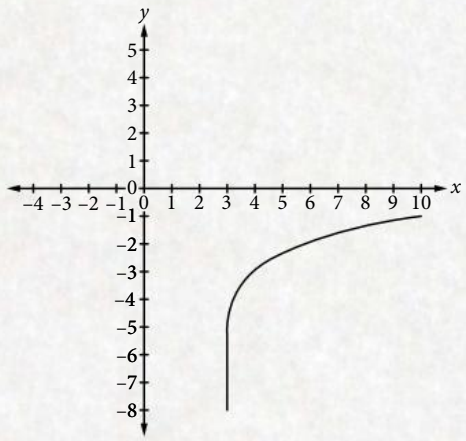
a



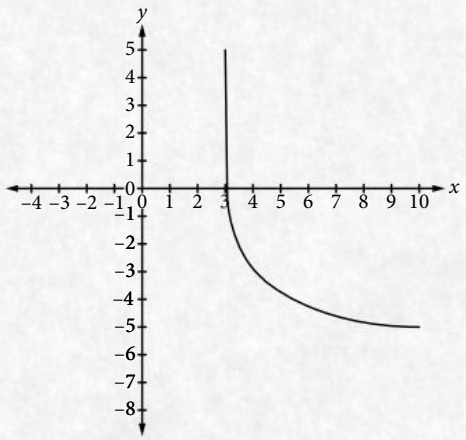
b



c



d



2 Find the derivative of the function $y = -3(2x - 6)^3$ and evaluate it at $x = 2$.

A 72 **B** 27 **C** 0 **D** −72

3 © QCAA 2023 1Q6 Substitutions for h are used to estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$.

Which sequence is the most appropriate?

A −4, −2, −1, −0.5, −0.25, −0.125 …

B −0.05, −0.1, −0.2, −0.4, −0.8 …

C 2, 1, 0, −1, −2, −3 …

D 1, 2, 3, 4, 5, 6 …

Section 2 4 short response questions

16 marks

4 (3 marks) Differentiate $y = \dfrac{x-1}{x+1}$. Write your answer in the simplest form.

5 (4 marks) Solve $2x^2 + 12x + 10 = 0$.

6 © QCAA 2021 1Q12b (3 marks) Solve for x in $\log_{10}(x + 3) + \log_{10}(x - 3) = \log_{10}(9x - 29)$.

7 © QCAA 2020 1Q17 (6 marks) The volume of water in a tank is represented by a function of the form $V(t) = Ae^{kt}$, where V is in litres and t is in minutes.

Initially, the volume is 100 litres and it is decreasing by 50 litres per minute.

Determine the time at which the volume is decreasing at the rate of $\dfrac{50}{7}$ litres per minute.

Express your answer in the form $\ln(a)$.

Worksheet
Maths Methods formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are not permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2021 2Q2 A substance is being heated such that its temperature T in °C after t minutes is given by the function $T = 2e^{0.5t}$.

The first integer value of t for which the instantaneous rate of change of temperature is greater than 100°C per minute is

A $t = 10$ **B** $t = 9$ **C** $t = 8$ **D** $t = 7$

2 © QCAA 2020S 2Q1 The limit of $\frac{12^h - 1}{h}$ as h approaches 0 is closest to

A 0.0 **B** 1.0 **C** 2.5 **D** 3.0

3 $(3.56 \times 10^3) \times (4.25 \times 10^4)$ simplifies to

A 7.81×10^{12} **B** 7.81×10^7 **C** 1.5085×10^8 **D** 1.5085×10^{12}

Section 2 4 short response questions

15 marks

4 (3 marks) Determine the gradient of the tangent to $y = (2x + 5)^2(3x - 1)$ at $x = 8$.

5 (4 marks) A multiple choice quiz has 6 questions on it, each with 4 possible options. Calculate the percentage chance that by simply guessing the correct answer for each question, you will get all 6 questions correct.

6 © QCAA 2020S 2Q18 (5 marks) The amount of a certain drug in the bloodstream, M (mg), at any time t (hours) is modelled closely by

$$M(t) = Ate^{-bt} \text{ where } A \text{ and } b \text{ are parameters.}$$

a Determine the exact values of A and b if the maximum amount of the drug in the bloodstream was 120 mg at $t = 2$ hours. *[3 marks]*

b Evaluate the reasonableness of your solution. *[2 marks]*

7 © QCAA 2021 2Q18 (3 marks)

The number of animals in a population (in thousands) is modelled by the function P such that $P(t) = \frac{100}{1 + 4e^{-t}}$, where t is in years.

Determine the number of animals in the population when the population is growing the fastest.

CHAPTER

2 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS



Syllabus coverage
Nelson MindTap chapter resources
Terminology
2.1 The derivative of $\sin(x)$
2.2 The derivative of $\cos(x)$
2.3 Problems involving the sine and cosine functions
Sinusoidal models
2.4 The chain rule
2.5 The product rule
2.6 The quotient rule
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Derivatives of trigonometric functions

Syllabus coverage

UNIT 3, TOPIC 2: DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS AND DIFFERENTIATION RULES

Calculus of trigonometric functions

- Use the rules $\frac{d}{dx}\sin(x) = \cos(x)$ and $\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$.
- Use the rules $\frac{d}{dx}\cos(x) = -\sin(x)$ and $\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$.
- Model and solve problems that involve derivatives of trigonometric functions, with and without technology.

Differentiation rules

- Use the chain rule to determine the derivative of composite functions involving exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form.
- Use the product rule to determine the derivative of exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form.
- Use the quotient rule to determine the derivative of exponential, logarithmic and trigonometric functions, expressing derivatives in simplest and factorised form.
- Solve problems that involve combinations of the chain rule, product rule and quotient rule to differentiate exponential, logarithmic and trigonometric functions.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 24,
© Queensland Curriculum and Assessment Authority (QCAA)

Videos (2):

2.2 The derivative of cosine functions

Exam question analysis Derivatives of trigonometric functions

Prior learning (1):

2.1 Derivatives of trigonometric functions

Worksheets (7):

2.2 Differentiating trigonometric functions

2.3 Trigonometric functions and gradient

2.5 The product rule

2.6 The quotient rule • Trigonometric functions assignment • Trigonometric functions extension

Cumulative exams Maths Methods formula sheet

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

chain rule	composite function	difference quotient	product rule
quotient rule	sinusoidal		

2.1 The derivative of sin(x)

Investigation — Difference quotients of sin(x)

The **difference quotient** is $\frac{f(x+h)-f(x)}{h}$.

The derivative of a function is the limit of the difference quotient as $h \to 0$.

Use your calculator to calculate the following values, ensuring that your calculator's angle setting is radians.

Complete the table of values below for $h = 0.01$.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\frac{\sin(x+h)-\sin(x)}{h}$									
$\cos(x)$									

Look at the values of the difference quotient and the value of $\cos(x)$.

Repeat the table for $h = 0.001$. What happens as h gets closer to 0?

The derivative of sin(x)

$$\frac{d}{dx}\sin(x) = \cos(x)$$

WORKED EXAMPLE 1 — The derivative of sin(x)

Determine the derivative of

a $5\sin(x)$ **b** $\sin(2x)$ **c** $-4\sin(3x)$ **d** $\sin(x°)$

Steps	Working
a 1 Recall that $\frac{d}{dx}\left[kf(x)\right] = k\frac{d}{dx}f(x)$.	$\frac{d}{dx}5\sin(x) = 5\frac{d}{dx}\sin(x)$
2 Apply the derivative of sine rule.	$\frac{d}{dx}5\sin(x) = 5\cos(x)$
b 1 Use the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$y = \sin(u)$, where $u = 2x$
2 Apply the chain rule and the derivative of sine rule.	$\frac{d}{dx}\sin(2x) = \cos(u) \times 2 = 2\cos(2x)$
c 1 Use the chain rule and $\frac{d}{dx}\left[kf(x)\right] = k\frac{d}{dx}f(x)$.	$\frac{d}{dx}\left[-4\sin(3x)\right] = -4\frac{d}{dx}\sin(u)$, where $u = 3x$
2 Apply the chain rule and the derivative of sine rule.	$\frac{d}{dx}\left[-4\sin(3x)\right] = -4\cos(u) \times 3 = -12\cos(3x)$

d **1** First convert the degree angle to a radian angle.

$$\sin(x^\circ) = \sin\left(\frac{\pi x}{180}\right)$$

2 Use the chain rule.

$$y = \sin(u), \text{ where } u = \frac{\pi x}{180}$$

3 Apply the chain rule and the derivative of sine rule.

$$\frac{d}{dx}\sin(u) = \frac{\pi}{180}\cos(u)$$
$$= \frac{\pi}{180}\cos(x^\circ)$$

You always differentiate trigonometric functions with the angle in **radians**. To differentiate a trigonometric function with the angle in **degrees**, you must first convert the angle to radians before differentiating.

Derivatives of sine functions

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\sin(kx) = k\cos(kx)$$

$$\frac{d}{dx}\sin(f(x)) = f'(x)\cos(f(x))$$

$$\frac{d}{dx}\sin(x^\circ) = \frac{\pi}{180}\cos(x^\circ)$$

WORKED EXAMPLE 2 Derivatives of sin [$f(x)$]

Determine the derivatives of these functions.

a $f(x) = \sin(3x^2 + x - 5)$ **b** $g(x) = \sin^4(4 - x)$

Steps	Working
a **1** Use the chain rule.	$f(u) = \sin(u)$, where $u = 3x^2 + x - 5$
2 Apply the chain rule and the derivative of sine rule.	$f'(x) = \cos(u) \times (6x + 1)$ $= (6x + 1)\cos(3x^2 + x - 5)$
b **1** Use the chain rule twice.	$g(u) = \sin^4(u)$, where $u = 4 - x$ $g(v) = v^4$, where $v = \sin(u)$
2 Apply the chain rule and the derivative of sine rule.	$g'(u) = 4v^3 \times \cos(u)$ $= 4\sin^3(u) \times \cos(u)$ $g'(x) = 4\cos(u) \times \sin^3(u) \times -1$ $= -4\cos(4 - x)\sin^3(4 - x)$

More complex examples of the chain, product and quotient rules will be met later in this chapter.

EXERCISE 2.1 The derivative of sin(x)

ANSWERS p. 394

Mastery

1 WORKED EXAMPLE 1 Determine the derivative of each function.

a $6\sin(x)$ **b** $-3\sin(x)$ **c** $5 - \sin(x)$

d $\sin(4\theta)$ **e** $\sin(7\theta)$ **f** $\frac{1}{4}\sin(8x)$

g $\frac{5}{8}\sin(10t)$ **h** $\sin(6x^\circ)$

2 WORKED EXAMPLE 2 Determine the derivative of each function.

a $\sin(\alpha^3)$ **b** $f(\alpha) = \sin(4\alpha^2)$ **c** $y = \sin(2x^3 - 7)$

d $3\sin(y^2 - 5y)$ **e** $2\sin(t^3 - 3t^2 + t - 1)$ **f** $3\sin(\sqrt{x})$

g $\sin\left(\sqrt{x - x^3}\right)$ **h** $f(x) = \sqrt{4\sin(3x)}$

3 Determine the derivative of each function.

a $y = 3\sin(4t)$ **b** $f(p) = -5\sin(3p)$ **c** $r(\theta) = 7\sin(2\theta°)$

d $y = \sin(4 - x)$ **e** $z = \sin(3v - 5)$ **f** $v(t) = \sin(3 - 5t)$

g $4\sin(2t + 3)$ **h** $6\sin(m - 4)$

4 Determine the derivative of each function.

a $y = \sin[(3x - 2)^3]$ **b** $\sin[(5t + 4)^6]$ **c** $\sin[(2m - 7)^{-1}]$

d $g(v) = \sin[(5 - v)^{-3}]$ **e** $-5\sin^4(3n + 1)$ **f** $a(t) = 4\sin^3(6 - 5t)$

g $f(x) = 5\sin[(2x + 7)^{-4}]$ **h** $y = -3\sin^2(4x + 5)$

5 Determine the derivative of each function.

a $[\sin(t)]^3$ **b** $2[\sin(4t)]^5$

c $g(x) = \sin^2(x) + 2\sin(x) + 3$ **d** $y = [5 - 2\sin(x)]^3$

e $y = 4\sin^2(5x - 3)$ **f** $h(t) = [1 - 3\sin(t) + \sin(3t)]^2$

g $\sin^4(2g^3)$ **h** $-4\sin^5[(1 - 3f)^{-2}]$

Exam practice All questions TECH-FREE

6 The derivative of $y = 4\sin(3x^3 - 2x)$ is

A $4(9x^2 - 2) \times \cos(3x^3 - 2x)$ **B** $36 \times \cos(3x^3 - 2x)$

C $4(9x^2 - 2) \times \sin(3x^3 - 2x)$ **D** $4 \times \cos(9x - 2)$

7 The derivative of $y = 4\sin^3(2x - 1)$ is

A $12\sin^2(2x - 1) \times \cos(2x - 1)$ **B** $12\cos^2(2x - 1)$

C $12\sin^2(2x - 1)$ **D** $24\sin^2(2x - 1) \times \cos(2x - 1)$

8 © QCAA 2020 1Q11a MODIFIED (1 mark) Determine the derivative of $y = \dfrac{1}{\sin(x)}$.

9 (1 mark) P is a point on the curve $y = 4 + 3\sin(2x)$, where $0 \le x \le \dfrac{2\pi}{3}$, and the tangent to P is parallel to the line $y = 2 - 6x$. Determine the coordinates of P.

10 (2 marks) Given the function $f(x) = kx + \sin(k^2x)$, where k is a non-zero constant, determine the values of x in terms of k for which $f'(x) = 0$ in the interval $[0, \pi]$.

11 © QCAA 2024 1Q9 59% At a certain location, the temperature (°C) can be modelled by the function $T = 5\sin\left(\dfrac{\pi}{12}x\right) + 23$, where x is the number of hours after sunrise.

Determine the rate of change of temperature (°C/hour) when $x = 4$.

A $\dfrac{5\pi}{48}$ B $\dfrac{5\pi}{24}$ C $\dfrac{5\pi\sqrt{3}}{24}$ D $\dfrac{5\pi\sqrt{3}}{6}$

Worksheet
Differentiating trigonometric functions

2.2 The derivative of cos(x)

Investigation — Transforming sin(x) into cos(x)

- Sketch $y = \sin(x)$ for $0 \le x \le 4\pi$ on your graphics calculator. Identify the coordinates of the first and second maximum points.
- Sketch $y = \cos(x)$ for $0 \le x \le 4\pi$ on your graphics calculator. Identify the coordinates of the first and second maximum points.
- Sketch $y = \sin\left(\frac{\pi}{2} - x\right)$ for $0 \le x \le 4\pi$ on your graphics calculator.

What do you notice about the graphs of $y = \sin(x)$ and $y = \cos(x)$?

What do you notice about the graphs of $y = \cos(x)$ and $y = \sin\left(\frac{\pi}{2} - x\right)$?

The graphs of $\sin(x)$ and $\cos(x)$ have a similar shape. In fact, $\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$.

Using the chain rule, $\frac{d}{dx}\cos(x) = \frac{d}{dx}\sin\left(\frac{\pi}{2} - x\right)$

$$= \frac{d}{d\left(\frac{\pi}{2} - x\right)}\sin\left(\frac{\pi}{2} - x\right) \times \frac{d}{dx}\left(\frac{\pi}{2} - x\right)$$

$$= \cos\left(\frac{\pi}{2} - x\right) \times -1$$

$$= -\sin(x)$$

Derivative of cos(x)

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Video
The derivative of cosine functions

WORKED EXAMPLE 3 — The derivative of cos(x)

Determine the derivative of

a $2\cos(x)$ **b** $\cos(3x)$ **c** $-3\cos(4x)$ **d** $\cos(x°)$

Steps	Working
a 1 Use $\frac{d}{dx}[k \times f(x)] = k \times \frac{d}{dx}f(x)$.	$\frac{d}{dx}2\cos(x) = 2 \times \frac{d}{dx}\cos(x)$
2 Apply the derivative of sine rule.	$\frac{d}{dx}2\cos(x) = -2\sin(x)$
b 1 Use the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$y = \cos(u)$, where $u = 3x$
2 Apply the chain rule and the derivative of sine rule.	$\frac{d}{dx}\cos(3x) = -\sin(u) \times 3$ $= -3\sin(3x)$
c 1 Use the chain rule and $\frac{d}{dx}[k \times f(x)] = k \times \frac{d}{dx}f(x)$.	$\frac{d}{dx}[-3\cos(4x)] = -3 \times \frac{d}{dx}\cos(u)$, where $u = 4x$
2 Apply the chain rule and the derivative of sine rule.	$\frac{d}{dx}[-3\cos(4x)] = -3 \times -\sin(u) \times 4$ $= 12\sin(4x)$

d 1	First convert the degree angle to a radian angle.	$\cos(x°) = \cos\left(\frac{\pi x}{180}\right)$
2	Use the chain rule.	$y = \cos(u)$, where $u = \frac{\pi x}{180}$
3	Apply the chain rule and the derivative of sine rule.	$\frac{d}{dx}\cos(u) = -\sin(u) \times \frac{\pi}{180}$ $= \frac{-\pi}{180} \times \sin(x°)$

Derivatives of cosine functions

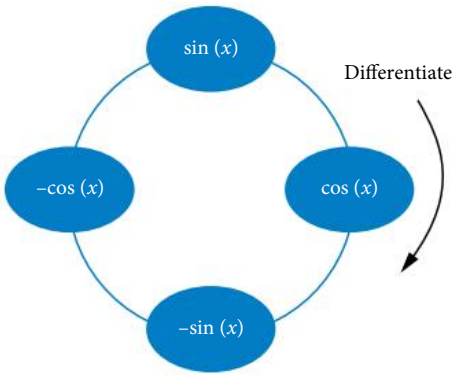
$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\cos(kx) = -k\sin(kx)$$

$$\frac{d}{dx}\cos(f(x)) = -f'(x)\sin(f(x))$$

$$\frac{d}{dx}\cos(x°) = \frac{-\pi}{180} \times \sin(x°)$$

If you differentiate $\sin(x)$ repeatedly, you end up returning to $\sin(x)$. There is a cyclical relationship between the derivatives of sine and cosine functions, as shown in the diagram.



WORKED EXAMPLE 4 Derivatives of cos (*f*(*x*))

Determine the derivative of each function.

a $f(x) = \cos(x^2 - 2x + 6)$ **b** $g(x) = \cos^3(5x - 1)$

Steps	Working
a 1 Use the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$f(u) = \cos(u)$, where $u = x^2 - 2x + 6$
2 Apply the chain rule and the derivative of sine rule.	$f'(x) = -\sin(u) \times (2x - 2)$ $= -(2x - 2)\sin(x^2 - 2x + 6)$
b 1 Use the chain rule.	$g(u) = \cos^3(u)$, where $u = 5x - 1$ $g(v) = v^3$, where $v = \cos(u)$
2 Apply the chain rule and the derivative of sine rule.	$g'(u) = 3v^2 \times -\sin(u)$ $= -3\cos^2(u)\sin(u)$ $g'(x) = -3\cos^2(u)\sin(u) \times 5$ $= -15\cos^2(5x - 1)\sin(5x - 1)$

EXERCISE 2.2 The derivative of cos(x)

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Recap

1 The derivative of $\sin(5x-3)$ is

A $5\sin(5x-3)$ **B** $5\cos(5x-3)$

C $5\cos(x)$ **D** $5\sin(x)$

2 The derivative of $\sin(4x^2-x+1)$ is

A $\cos(8x-1)$ **B** $(8x-1)\cos(4x^2-x+1)$

C $(4x^2-x+1)\cos(4x^2-x+1)$ **D** $(8x-1)\sin(x^2-x+1)$

Mastery

3 WORKED EXAMPLE 3 Determine the derivative of each function.

a $4\cos(x)$ **b** $-5\cos(x)$ **c** $7-2\cos(x)$ **d** $\cos(3a)$

e $\cos(-6\theta)$ **f** $\frac{1}{3}\cos(12x)$ **g** $\frac{5}{6}\cos(10t)$ **h** $\cos(5m)$

4 WORKED EXAMPLE 4 Determine the derivative of each function.

a $\cos(x^3)$ **b** $k(n)=\cos(5n^4)$

c $y=\cos(5x^2-9)$ **d** $6\cos(2y^2-5y-5)$

e $4\cos(3t^3+4t^2+3t+2)$ **f** $5\cos\left(\sqrt{x}\right)$

g $\cos\left(\sqrt{x^5-x^2}\right)$ **h** $f(x)=\sqrt{3\cos(3x)}$

5 Determine the derivative of each function.

a $v=8\cos(3t)$ **b** $g(x)=-2\cos(4x)$ **c** $r(\theta)=8\cos(3\theta°)$

d $3\cos(m-5)$ **e** $b=\cos(5v-3)$ **f** $v(t)=\cos(5-3t)$

g $6\cos(4t+7)$ **h** $y=\cos(7-2x)$

6 Determine the derivative of each function.

a $y=\cos[(4x-5)^2]$ **b** $\cos[(3t+2)^5]$ **c** $\cos[(m-3)^{-1}]$

d $g(v)=\cos[(4-v)^{-4}]$ **e** $-3\cos[(2n+5)^6]$ **f** $a(t)=3\cos[(5-4t)^4]$

g $f(x)=8\cos[(3x+1)^{-3}]$ **h** $y=-2\cos[(7x+3)^{-5}]$

7 Determine the derivative of each function.

a $[\cos(\alpha)]^4$ **b** $3[\cos(2t)]^4$ **c** $f(x)=\cos^2(x)+3\cos(x)-2$

d $y=[6-5\cos(x)]^4$ **e** $y=4\cos^2(3x-5)$ **f** $h(t)=[3-\cos(2t)+\cos(t)]^2$

g $-5\cos^6[(1-f)^{-3}]$ **h** $\cos^3(4g^5)$

Exam practice

All questions TECH-FREE

8 The derivative of $3\cos(-3x)$ is

A $-3\sin(-3x)$ **B** $-3\cos(-3)$ **C** $9\cos(-3x)$ **D** $9\sin(-3x)$

9 Given that $f(x)=-2\cos\left(\frac{3x}{2}\right)$, the value of $f'\left(\frac{\pi}{2}\right)$ is

A $-\sqrt{2}$ **B** $\frac{3\sqrt{2}}{2}$ **C** $\sqrt{2}$ **D** $-\frac{3\sqrt{2}}{2}$

10 The derivative of $3\cos^4(2x+5)$ is

A $-24\sin^3(2x+5)$

B $-12\cos^3(2x+5)\times\sin(2x+5)$

C $-24(2x+5)\sin^3(2x+5)$

D $-24\cos^3(2x+5)\times\sin(2x+5)$

11 (1 mark) P is a point on the curve $y=4+3\cos(2x)$, where $0\le x\le\frac{3\pi}{4}$, and the tangent to P is parallel to the line $y=-2+3\sqrt{3}x$.

Determine the coordinates of P.

12 (7 marks) A particle moves so that its displacement is $x(t)=4\cos\left(\frac{t}{3}\right)$, $0\le t\le 2\pi$, where x is in metres and t is in seconds.

a Determine any times within the domain when the particle is at a displacement of 0. *[1 mark]*

b Given that velocity is the derivative of displacement, determine the equation for the velocity of the particle. *[1 mark]*

c Determine any times within the domain when the particle is stationary. *[1 mark]*

d Given that acceleration is the derivative of velocity, determine the equation for the acceleration of the particle. *[1 mark]*

e Describe what is happening to the particle at $t=\frac{3\pi}{2}$. *[3 marks]*

13 (3 marks) CF Given that the displacement of a particle is defined as $\cos(kt+b)$ for constants k and b, that velocity v is the derivative of displacement, and acceleration a is the derivative of velocity, create a function for a written in terms of t and x.

2.3 Problems involving the sine and cosine functions

Worksheet
Trigonometric functions and gradient

WORKED EXAMPLE 5 Derivative of $\tan(x)$

Determine the derivative of $\tan(x)$.

Steps	Working
1 Recall the relationship between sine, cosine and **tangent**.	$\tan(x)=\frac{\sin(x)}{\cos(x)}$
2 Recall the quotient rule.	$\frac{d}{dx}\tan(x)=\frac{vu'-uv'}{v^2}$ where $u=\sin(x)$ and $v=\cos(x)$
3 Substitute values into the quotient rule.	$\frac{d}{dx}\tan(x)=\frac{\cos(x)\times\cos(x)+\sin(x)\times\sin(x)}{\cos^2(x)}$
4 Simplify.	$\frac{d}{dx}\tan(x)=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}$ $=\frac{1}{\cos^2(x)}$ Alternatively, $\frac{d}{dx}\tan(x)=\frac{\cos^2(x)}{\cos^2(x)}+\frac{\sin^2(x)}{\cos^2(x)}=1+\tan^2(x)$

You use the derivatives of trigonometric functions to find the **equation of a tangent** to the graph of a trigonometric function.

WORKED EXAMPLE 6 Equation of a tangent

Determine the equation of the tangent to $f(x) = 5\sin(2x)$ at $x = \frac{\pi}{6}$.

Steps	Working
1 First, calculate $f\left(\frac{\pi}{6}\right)$ and state the coordinates of the point.	$f\left(\frac{\pi}{6}\right) = 5\sin\left(\frac{\pi}{3}\right) = \frac{5\sqrt{3}}{2}$ Therefore, point $\left(\frac{\pi}{6}, \frac{5\sqrt{3}}{2}\right)$ lies on the function and on the tangent.
2 Determine the derivative of $f(x)$.	$f'(x) = 10\cos(2x)$
3 Substitute the x value into $f'(x)$ to determine the gradient of the tangent.	$f'\left(\frac{\pi}{6}\right) = 10\cos\left(\frac{\pi}{3}\right) = 5$
4 Substitute known information into the straight-line equation, $y = mx + c$.	$y - \frac{5\sqrt{3}}{2} = 5\left(x - \frac{\pi}{6}\right)$

Sinusoidal models

Sinusoidal models involve the **periodic function sine** or **cosine**.

Guidelines for creating sinusoidal models

- Start with either $y = A\sin[B(x - C)] + D$ or $y = A\cos[B(x - C)] + D$
  where A is the **amplitude**, the **period** $= \frac{2\pi}{B}$, the **phase shift** is C and the **central (mean) value** is D.
- If your model starts from the central value, choose the sine function.
- If your model starts from either a maximum or minimum value, choose the cosine function.
- If your model starts anywhere else, select the trigonometric function that ensures the C value is as close to 0 as possible. You should avoid using the C value as much as possible as it adds extra complexity when solving.
- Remember that A is the dilation in the y direction, B is the dilation in the x direction, C is the translation in the x direction and D is the translation in the y direction.

The suggested problem-solving strategy below is also generally helpful with modelling problems.

Problem-solving strategy

- Identify key pieces of information by underlining, highlighting or summarising.
- Draw a sketch of the problem, labelling key information and creating variables where possible.
- Determine the variable or rate that needs to be solved for and create links to known information.
- Use the variable or rate and the known information to create equation(s) that will be used in the solving process.
- If the question asks for a maximum or minimum value, this will always occur when a rate of change is equal to 0.
- Solve the mathematical problem, showing your working.
- Express your solution in a short statement at the end of the problem, incorporating any units.

WORKED EXAMPLE 7 Creating and solving sinusoidal models

A ship is waiting to enter a port at 7 am on a Monday morning. The ship can't enter the port safely until the tide rises to meet the minimum depth requirement. At 7 am on Monday morning, it is low tide in the port with a depth of 12 m. The previous high tide of 23 m at the port was at 6:30 pm on Sunday night.

a **Determine** a sinusoidal model for the depth, d, of the port, where t is the number of hours after 7 am Monday morning.

b **Calculate** all times on Monday that the tide is falling at a rate of 1 m h^{-1}.

Steps	Working
a 1 Determine values for A, B and D.	Time between high tide and low tide is 12.5 hours. $\therefore \frac{1}{2}$ period $= 12.5$ period $= 25$ $25 = \frac{2\pi}{B}$ $B = \frac{2\pi}{25}$ Median value D can be found by averaging the high tide and low tide values. $D = \frac{12+23}{2}$ $= \frac{35}{2}$ $= 17.5$ Difference between high tide and low tide represents 2 times the amplitude. $2A = 23 - 12$ $2A = 11$ $A = 5.5$
2 Select the appropriate sinusoidal model.	Let t be number of hours after 7 am Monday. Given that $t = 0$ is at the minimum value, select the cosine function with a negative A value. $d = A\cos[B(t - C)] + D$
3 Substitute in known values.	Given that $t = 0$ is the minimum value, no phase shift is required, therefore $C = 0$. $\therefore d = -5.5\cos\left(\frac{2\pi t}{25}\right) + 17.5$
b 1 Determine the derivative of the depth equation.	$d'(t) = -5.5 \times -\sin\left(\frac{2\pi t}{25}\right) \times \frac{2\pi}{25}$ $= \frac{11\pi}{25}\sin\left(\frac{2\pi t}{25}\right)$
2 Substitute in the rate and state the domain $(-7 \leq t \leq 17)$ for Monday.	Falling by 1 m h^{-1} means $d'(t) = -1$. $\therefore -1 = \frac{11\pi}{25}\sin\left(\frac{2\pi t}{25}\right)$, when $-7 \leq t \leq 17$

3 Solve for values of t.

Test whether values are within domain.

$$\sin\left(\frac{2\pi t}{25}\right) = -\frac{25}{11\pi}$$

$$\sin(\theta) = -\frac{25}{11\pi}, \text{ when } -\frac{14\pi}{25} \le t \le \frac{34\pi}{25}$$

$$\theta = -\pi + 0.8088,\ 0 - 0.8088,\ \pi + 0.8088,\ 2\pi - 0.8088$$

$$\frac{2\pi t}{25} = -2.3328,\ -0.8088,\ 3.9504,\ 5.4744$$

$$t = -9.2820,\ -3.2180,\ 15.7180,\ 21.7820$$

Ignore the first and last values, as they are out of the domain.

4 Interpret solutions.

Therefore, there are two times on Monday when the tide is falling by 1 m h^{-1}. Those times are 3:47 am and 10:43 pm.

EXERCISE 2.3 Problems involving the sine and cosine functions

ANSWERS p. 395

Recap

1 TECH-FREE The derivative of $\cos(2x + 1)$ is

A $-2\sin(2x + 1)$ **B** $2\sin(2x + 1)$

C $-2\sin(x)$ **D** $-2\cos(x)$

2 TECH-FREE The derivative of $\cos(x^2 - 3x)$ is

A $-\sin(2x - 3)$ **B** $-(x^2 - 3x)\sin(2x - 3)$

C $(2x - 3)\sin(x^2 - 3x)$ **D** $-(2x - 3)\sin(x^2 - 3x)$

Mastery

3 WORKED EXAMPLE 5 TECH-FREE Determine the derivative of each function.

a $9\tan(x)$ **b** $-2\tan(x)$ **c** $5 - \tan(x)$

d $\tan(5m)$ **e** $\tan(x^4)$ **f** $k(n) = \tan(3n^2)$

g $y = \tan(4x^2 - 25)$ **h** $7\tan(3x^2 - 6x + 3)$

4 TECH-FREE Determine the derivative of each function.

a $\tan(-4\theta)$ **b** $\frac{2}{5}\tan(10x)$ **c** $\frac{4}{9}\tan(15t)$

d $\tan(3m°)$ **e** $8\tan(4t^3 - 3t^2 + 5t - 1)$ **f** $6\tan\left(\sqrt{x}\right)$

g $\tan\left(\sqrt{x^6 - x^4}\right)$ **h** $f(x) = \sqrt{8\tan(2x)}$

5 WORKED EXAMPLE 6 TECH-FREE Determine the equation of the tangent to the curve $y = 3\cos(x)$ at the point where $x = \frac{2\pi}{3}$.

6 WORKED EXAMPLE 7 A surveyor with a theodolite at the top of a cliff 120 m above the water observes that a boat moving directly away from him is at an angle of depression of 35° and that the angle of depression is decreasing at 2° per minute. Find the speed of the boat in km h^{-1}.

Exam practice

7 TECH-FREE The derivative of $2\tan(2x)$ is

A $\frac{1}{\cos^2(2x)}$ **B** $\frac{2}{\cos^2(x)}$ **C** $\frac{4}{\cos^2(x)}$ **D** $\frac{4}{\cos^2(2x)}$

8 The equation of the tangent to $5\sin(3x+2)$ at $x=\frac{3\pi}{4}$ is

A $y=-14.05x+1.74$ **B** $y=2.36x+1.74$

C $y=1.74x+34.87$ **D** $y=-14.05x+34.87$

9 The equation of the normal to $5\sin(3x+2)$ at $x=\frac{3\pi}{4}$ is

A $y=0.07x+1.58$ **B** $y=2.36x+1.74$

C $y=0.07x+1.74$ **D** $y=1.74x+1.58$

10 TECH-FREE Sam arrives at the beach at exactly 9 am. It happens to be high tide, with the height of high tide being 5.2 m. The next high tide is in 12 hours time. Low tide is expected to be 1.8 m. A sinusoidal model for the height h of the tide at the beach t hours after 9 am is

A $h=-1.7\cos\left(\frac{\pi t}{6}\right)+3.5$ **B** $h=-1.7\sin\left(\frac{\pi t}{6}\right)+3.5$

C $h=1.7\cos\left(\frac{\pi t}{6}\right)+3.5$ **D** $h=1.7\cos(12t)+3.5$

11 (7 marks) An oscillating point begins 8 mm below the origin and rises to a maximum of 6 mm above the origin 5 seconds later. It continues to oscillate between the two extremes.

a Write a sinusoidal function to describe its movement. *[4 marks]*

b Find its speed 12 seconds after the motion begins. *[3 marks]*

12 TECH-FREE (6 marks) When a stone is dropped into a still pond, ripples move outwards from the point where the stone was dropped. At a point X, a distance of 1.8 m from the point where the stone was dropped, the water apparently moves up and down so that the difference in height between its highest and lowest points is 0.6 cm. Three ripples pass point X each second.

a Write a sinusoidal (sine or cosine) expression for the up-and-down movement of the surface at point X. *[2 marks]*

b Find an expression for the velocity with which the surface moves up and down at point X. *[4 marks]*

13 TECH-FREE (5 marks) CF Determine the equation of the normal to the curve $f(x)=\sin(2x)$ at the point $\left(\frac{\pi}{2},0\right)$.

14 (8 marks) CF The curves $y=\sin(2x)$ and $y=\cos(2x)$ intersect at $x=\frac{\pi}{8}$. Determine the angle between the curves at this point.

15 TECH-FREE (7 marks) CU The centre of the light cast by a light suspended by a chain 3 m from a cathedral roof moves backwards and forwards across the floor with a speed of $2\sin(1.5t)$ m s^{-1}. The light is 2 m above the floor. Find an expression for the movement of the light, which is almost horizontal. Hence find the greatest angle of the chain from the vertical.

2.4 The chain rule

Use the **chain rule** when a function is within another function. These are also called **composite functions**.

Derivative rules

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln[f(x)] = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\sin[f(x)] = f'(x)\cos[f(x)]$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\cos[f(x)] = -f'(x)\sin[f(x)]$$

The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

WORKED EXAMPLE 8 The chain rule

Determine the derivative of each function.

a $y = \ln[\sin(x)]$ **b** $y = e^{\cos(5x)}$ **c** $y = \sqrt{3x^2 + x - 4}$ **d** $y = \sin^2(e^x)$

Steps	Working
a 1 Recall the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$y = \ln(u)$, where $u = \sin(x)$
2 Apply the chain rule, the derivative of a logarithm rule and the derivative of sine rule.	$\frac{dy}{du} = \frac{1}{u}$ $\frac{du}{dx} = \cos(x)$ $\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)}$
b 1 Recall the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$	$y = e^u$, where $u = \cos(5x)$
2 Apply the chain rule, the derivative of an exponential rule and the derivative of sine rule.	$\frac{dy}{du} = e^u$ $\frac{du}{dx} = -5\sin(5x)$ $\frac{dy}{dx} = -5\sin(5x)e^{\cos(5x)}$
c 1 Use index laws to write the function with fractional power.	$y = (3x^2 + x - 4)^{0.5}$ $y = u^{0.5}$, where $u = 3x^2 + x - 4$

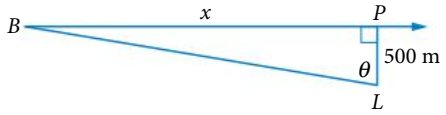
2 Apply the chain rule and the derivative of a power rule.	$\frac{dy}{du} = 0.5u^{-0.5}$ $\frac{du}{dx} = 6x+1$ $\frac{dy}{dx} = \frac{1}{2u^{0.5}} \times (6x+1)$ $\frac{dy}{dx} = \frac{6x+1}{2\sqrt{3x^2+x-4}}$
d Apply the chain rule, the derivative of an exponential rule, the derivative of sine rule and the derivative of a power rule.	$y = u^2$, where $u = \sin(e^x)$ $u = \sin(v)$, where $v = e^x$ $\frac{dv}{dx} = e^x$ $\frac{du}{dv} = \cos(v)$ $\frac{du}{dx} = e^x \cos(e^x)$ $\frac{dy}{du} = 2u$ $\frac{dy}{dx} = 2u \times e^x \cos(e^x)$ $\frac{dy}{dx} = 2e^x \cos(e^x) \sin(e^x)$

WORKED EXAMPLE 9 Using the chain rule to solve problems

The *MV Sweetpea* is travelling in a straight line at a constant speed so that it will pass a lighthouse (point L) at point P, which is 500 m away from the lighthouse.

When the angle at the lighthouse between the boat's current position and P is 83°, the angle is decreasing at a rate of 0.8° per minute.

Given that a nautical mile (n.m.) is approximately 1852 m, **determine** the boat's speed in knots (nautical miles per hour).

Steps	Working
1 Draw a diagram and assign appropriate variables.	 Let x be the distance between the boat and point P. Let θ be the angle at the lighthouse between the boat's current position and point P.
2 Identify key information and manipulate into appropriate forms for use.	When $\theta = 83°$, $\frac{d\theta}{dt} = -0.8°$ per min. $\frac{d\theta}{dt} = -\frac{0.8\pi \times 60}{180}$ $= -\frac{0.8\pi}{3}$ h^{-1} In $\triangle BPL$, $\tan(\theta) = \frac{x}{500}$ $x = 500\tan(\theta)$ m $= \frac{500}{1852}\tan(\theta)$ n.m. $\frac{dx}{d\theta} = \frac{500}{1852}(1 + \tan^2(\theta))$ n.m.

3 Create the equation to be solved.

$$v = \frac{dx}{dt}$$
$$= \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$
$$= \frac{500}{1852}(1 + \tan^2(\theta)) \times -\frac{0.8\pi}{3} \text{ n.m. h}^{-1}$$

4 Substitute $\theta = \frac{83\pi}{180}$ into the speed equation.

$$v = \frac{500}{1852}\left(1 + \tan^2\left(\frac{83\pi}{180}\right)\right) \times -\frac{0.8\pi}{3} \text{ n.m. h}^{-1}$$
$$= -15.23 \text{ knots}$$

5 Interpret solution.

Since speed does not involve direction, the boat is travelling at approx. 15.23 knots.

EXERCISE 2.4 The chain rule

ANSWERS p. 395

Recap

1 The derivative of $\tan(5x - 2)$ is

A $5 + 5\tan^2(5x - 2)$

B $\frac{5(\sin^2(5x-2) + \cos^2(5x-2))}{\sin^2(5x-2)}$

C $1 + \tan^2(5x - 2)$

D $5(1 + \tan^2(x))$

2 The equation of the tangent of $f(x) = 5\sin(x)$ at $x = \frac{\pi}{3}$ is

A $y = -\frac{5}{2}x + \frac{5\sqrt{3}}{2} + \frac{5\pi}{6}$

B $y = \frac{5}{2}x + \frac{\pi}{3} - \frac{25\sqrt{3}}{4}$

C $y = \frac{1}{2}x + \frac{5\sqrt{3}}{2} - \frac{\pi}{6}$

D $y = \frac{5}{2}x + \frac{5\sqrt{3}}{2} - \frac{5\pi}{6}$

Mastery

3 WORKED EXAMPLE 8 Determine the derivative of $\sin(\ln(x)) - \ln(\sin(x)) + \sin(e^x) - e^{\sin(x)}$.

4 Differentiate each function.

a $e^{(5x-6)^3}$

b $\ln[(5x + 4)^2]$

c $\sin[(y - 2)^7]$

d $-3[\sin(5q + 8)]^4$

e $\cos[7(7 - 5a)^3]$

f $\cos[\sin^6(5g^3 + 3)]$

5 WORKED EXAMPLE 9 The bob of a clock pendulum swings 8 cm from the centre to each side. It takes 1 second to go from the centre and return to the centre. Write a sinusoidal function to describe the sideways motion of the bob and find its speed when the pendulum is vertical.

Exam practice

6 TECH-FREE The derivative of $\cos\left(\sqrt[3]{x^2 - 3}\right)$ is

A $\frac{-2x\sin\left(\sqrt[3]{x^2-3}\right)}{3\sqrt[3]{\left(x^2-3\right)^2}}$

B $-\sin\left(\frac{2x}{3\times\sqrt[3]{(x^2-3)^2}}\right)$

C $\frac{-2\sin\left(\sqrt[3]{x^2-3}\right)}{3\times\sqrt[3]{(x^2-3)^2}}$

D $\frac{-\sin\left(\sqrt{x^2-3}\right)}{3\left(\sqrt{x^2-3}\right)}$

7 TECH-FREE The derivative of $\ln[(3x - 1)^4]$ is

A $\frac{12}{3x-1}$

B $12\ln[(3x - 1)^3]$

C $\frac{4}{(3x-1)^3}$

D $\frac{12}{(3x-1)^5}$

8 © QCAA 2022 1Q13a TECH-FREE (1 mark) Determine the derivative of $f(x) = 3e^{2x+1}$.

9 TECH-FREE (1 mark) Determine the derivative of $y = \sqrt{\sin(x)}$.

10 © QCAA 2021 1Q11a MODIFIED TECH-FREE (2 marks) Determine the derivative with respect to x of the function $y = (e^x + 1)^3$.

11 (5 marks) CF From a point on the Storey Bridge 80 m above the roadway, a night-time observer can see the lights of a car moving towards them from Kangaroo Point. If the car is moving at 54 km h^{-1}, determine the rate of increase of the angle of depression of the lights when the car is 300 m away from a point directly below the observer.

12 (5 marks) CU The centre of the string of a musical instrument is plucked so that it is 4.4 mm away from the rest position and then let go. It vibrates at a frequency of 210 Hz. Write a sinusoidal function for the position of the centre of the string in the first 3 seconds of movement, before the sound begins to fade. Calculate how fast the string is moving at the extremes of its motion after 2 seconds.

13 © QCAA 2024 1Q2 92% TECH-FREE Determine $\frac{dy}{dx}$ for the function $y = e^{\sin(x)}$

A $\cos(x)\, e^{\sin(x)}$

B $\sin(x)\, e^{\cos(x)}$

C $e^{\sin(x)}$

D $e^{\cos(x)}$

2.5 The product rule

Worksheet
The product rule

The product rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

WORKED EXAMPLE 10 The product rule

Determine the derivative of each function.

a $y = (x^2 - 5x + 1)\sin(2x)$

b $f(x) = 5e^{4x-2}\cos(3x^2 + 4x - 3)$

c $y = \ln(3x)\cos(3x)$

d $f(x) = 2x\cos\left(\sqrt[3]{x}\right)$

e $y = 3e^{4x+1}\cos^2(5x - 3)$

Steps	Working
a 1 Use the product rule: $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$	$y = uv$, where $u = x^2 - 5x + 1$ and $v = \sin(2x)$
2 Apply the product rule, the derivative of a power rule and the derivative of sine rule.	$\frac{du}{dx} = 2x - 5$ and $\frac{dv}{dx} = 2\cos(2x)$ $\frac{dy}{dx} = (x^2 - 5x + 1) \times 2\cos(2x) + \sin(2x) \times (2x - 5)$
b 1 Use the product rule.	$y = uv$, where $u = 5e^{4x-2}$ and $v = \cos(3x^2 + 4x - 3)$
2 Apply the product rule, the derivative of an exponential rule and the derivative of cosine rule.	$\frac{du}{dx} = 20e^{4x-2}$ and $\frac{dv}{dx} = -(6x + 4)\sin(3x^2 + 4x - 3)$ $\frac{dy}{dx} = 5e^{4x-2} \times -(6x + 4)\sin(3x^2 + 4x - 3)$ $+ \cos(3x^2 + 4x - 3) \times 20e^{4x-2}$

c Apply the product rule, the derivative of a logarithm rule and the derivative of sine rule.	$y = uv$, where $u = \ln(3x)$ and $v = \cos(3x)$ $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = -3\sin(3x)$ $\frac{dy}{dx} = \ln(3x) \times -3\sin(3x) + \cos(3x) \times \frac{1}{x}$
d Apply the product rule, the derivative of a power rule and the derivative of cosine rule.	$y = uv$, where $u = 2x$ and $v = \cos\left(x^{\frac{1}{3}}\right)$ $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = -\sin\left(x^{\frac{1}{3}}\right) \times \frac{1}{3x^{\frac{2}{3}}}$ $\frac{dy}{dx} = 2x \times \left(\frac{-\sin\left(\sqrt[3]{x}\right)}{3\left(\sqrt[3]{x^2}\right)}\right) + \cos\left(\sqrt[3]{x}\right) \times 2$ $\frac{dy}{dx} = \frac{-2x\sin\left(\sqrt[3]{x}\right)}{3\left(\sqrt[3]{x^2}\right)} + 2\cos\left(\sqrt[3]{x}\right)$
e **1** Write as a product of 2 functions. **2** Find the derivatives. **3** Use the product rule. **4** Factorise.	$y = uv$, where $u = 3e^{4x+1}$, $v = \cos^2(5x-3)$ $u' = 12e^{4x+1}$, $v' = 2\cos(5x-3) \times -5\sin(5x-3)$ $\frac{dy}{dx} = u'v + uv'$ $= 12e^{4x+1}\cos^2(5x-3) + 3e^{4x+1} \times -10\cos(5x-3)\sin(5x-3)$ $= 6e^{4x+1}\cos(5x-3)[2\cos(5x-3) - 5\sin(5x-3)]$

EXERCISE 2.5 The product rule

ANSWERS p. 395

Recap

1 The derivative of $5\sin^2(3x)$ is

A $15\sin(3x)\cos(3x)$ **B** $10\sin(3x)\cos(3x)$

C $30\sin(3x)\cos(3x)$ **D** $15\cos^2(3x)$

2 The derivative of $e^{\cos(3x)}$ is

A $e^{\cos(3x)}$ **B** $3\sin(3x)e^{\cos(3x)}$

C $-3\sin(3x)e^{\cos(3x)}$ **D** $\cos(3x)e^{-3\sin(3x)}$

Mastery

3 WORKED EXAMPLE 10 Determine the derivative of each function.

a $5\sin[(3x^2 + 4x - 3)(e^{4x-2} + 4)]$ **b** $8e^{-2x}\cos(5x-3)$

c $\sin(x)\ln(2x)e^{3x}\cos(4x)$ **d** $4e^{6-3x}\sin(5x^2 + 3x - 2)$

4 Determine the derivative of each function.

a $x^2\sin(3x)$ **b** $(c^2 + c - 1)\sin(2c - 3)$

c $(5m^2 - 3m + 2)\sin(4m - 3)$ **d** $\sin(x + 3)\sin(x - 3)$

5 Determine the derivative of each function.

a $x^3\cos(4x)$ **b** $(m^2 + 3m - 4)\cos(2m - 3)$

c $(5x^3 - 4x^2 + 3)\cos(3x - 5)$ **d** $\cos(2x + 1)\cos(3x + 1)$

6 Determine the derivative of each function.

a $x^3 \sin(3x)$ **b** $e^x(x^2 - 1)$ **c** $2x^2e^{3x}$

d $2x^3\cos(2x)$ **e** $e^{2x}\sqrt{3x^2 - 1}$ **f** $x^2\cos^3(2x)$

7 Determine the derivative of each function.

a $(3x^2 - 2x - 1)e^{\sin(2x+1)} + 5x^2\ln[x\sin(x^2+1)]$ **b** $7xe^{-0.4x^2}\left(2\sin\left[\ln(2x-3)\right] + \cos\left[\ln(2x+3)\right]\right)$

Exam practice

8 © QCAA 2023 2Q9 40% TECH-FREE If $f(x) = e^{3x}(x+1)^2$ and $f'(x) = ae^{3x}(x+1)$, determine the expression for a.

A $3x + 5$ **B** $3x + 3$ **C** $5x + 5$ **D** $5x + 3$

9 © QCAA 2020 1Q5 TECH-FREE The equation of the tangent to the curve $f(t) = te^t$ at $t = 1$ is

A $y = et$ **B** $y = 2et - e$ **C** $y = et - e^2 + 1$ **D** $y = 2et - 2e^2 + 1$

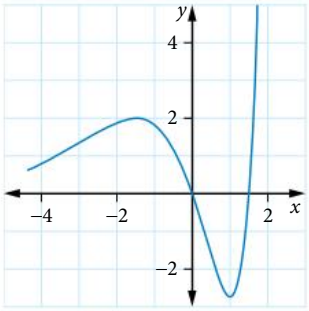
10 (7 marks) CF An oscillating point begins from a high point 2 mm above the origin and doubles its amplitude every 20 seconds. It reaches the next maximum after 5 seconds.

a Determine a sinusoidal function to describe its motion. *[4 marks]*

b Calculate its speed 22 seconds after it begins moving. *[3 marks]*

11 TECH-FREE (5 marks) CF This diagram shows the graph of $y = e^x(2x^2 - 3x)$.

Calculate the values of x where the gradient of the tangent to the graph is zero.



12 (7 marks) CU A revolving searchlight is 80 m from a straight prison wall. At a point 120 m from the point on the wall directly opposite the searchlight, the light appears to move along the wall at 13.6 m s^{-1}. Determine how long the searchlight takes to complete one revolution.

13 © QCAA 2024 1Q6 76% TECH-FREE Differentiate $y = \ln(x)\cos(x)$ with respect to x.

A $\dfrac{\cos(x)}{x}$

B $-\dfrac{\sin(x)}{x}$

C $\dfrac{\cos(x)}{x} + \ln(x)\sin(x)$

D $\dfrac{\cos(x)}{x} - \ln(x)\sin(x)$

2.6 The quotient rule

The quotient rule

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

WORKED EXAMPLE 11 The quotient rule

Determine the derivative of each function.

a $y = \dfrac{2t^2 - 5t + 1}{3\cos(2t)}$ **b** $f(q) = 7\sin\left(\ln\left(\dfrac{3q-6}{2q+3}\right)\right)$ **c** $y = \dfrac{\sin(3x)}{\sin(2x)}$ **d** $f(x) = \dfrac{(x+1)e^{2x}}{\cos(x)}$

Steps	Working
a 1 Use the quotient rule.	$y = \frac{u}{v}$, where $u = 2t^2 - 5t + 1$ and $v = 3\cos(2t)$
2 Apply the quotient rule, the derivative of a power rule and the derivative of cosine rule.	$\frac{du}{dt} = 4t - 5$ and $\frac{dv}{dt} = -6\sin(2t)$ $\frac{dy}{dt} = \frac{3\cos(2t)\times(4t-5) + (2t^2 - 5t + 1)\times 6\sin(2t)}{(3\cos(2t))^2}$ $\frac{dy}{dt} = \frac{3(4t-5)\cos(2t) + 6(2t^2 - 5t + 1)\sin(2t)}{9\cos^2(2t)}$ $\frac{dy}{dt} = \frac{(4t-5)\cos(2t) + 2(2t^2 - 5t + 1)\sin(2t)}{3\cos^2(2t)}$
b 1 Use the quotient rule.	Let $p = \frac{3q-6}{2q+3}$ $p = \frac{u}{v}$, where $u = 3q - 6$ and $v = 2q + 3$
2 Apply the quotient rule and the derivative of a power rule.	$u' = 3$ and $v' = 2$ $p'(q) = \frac{3(2q+3) - 2(3q-6)}{(2q+3)^2}$ $= \frac{21}{(2q+3)^2}$
3 Relate the derivative of p back to the original f function using the chain rule.	$f(p) = 7\sin(\ln(p))$ $= f'(p) = 7\cos(\ln(p))\times\frac{1}{p}$ $f'(q) = f'(p)\times p'(q)$ $= 7\cos(\ln(p))\times\frac{1}{p}\times\frac{21}{(2q+3)^2}$ $= 147\cos\left(\ln\left(\frac{3q-6}{2q+3}\right)\right)\times\frac{2q+3}{3q-6}\times\frac{1}{(2q+3)^2}$ $= 49\cos\left(\ln\left(\frac{3q-6}{2q+3}\right)\right)\times\frac{1}{q-2}\times\frac{1}{2q+3}$ $= \frac{49\cos\left(\ln\left(\frac{3q-6}{2q+3}\right)\right)}{(q-2)(2q+3)}$

c **1** Use the quotient rule.

$y = \frac{u}{v}$, where $u = \sin(3x)$ and $v = \sin(2x)$

2 Apply the product rule, the derivative of a power rule and the derivative of cosine rule.

$\frac{du}{dx} = 3\cos(3x)$ and $\frac{dv}{dx} = 2\cos(2x)$

$$\frac{dy}{dx} = \frac{\sin(2x) \times 3\cos(3x) - \sin(3x) \times 2\cos(2x)}{(\sin(2x))^2}$$

$$\frac{dy}{dx} = \frac{3\sin(2x)\cos(3x) - \sin(3x)\cos(2x)}{\sin^2(2x)}$$

d **1** Use the quotient rule.

$y = \frac{u}{v}$, where $u = (x+1)e^{2x}$ and $v = \cos(x)$

2 Apply the product rule, the derivative of a power rule and the derivative of cosine rule.

$\frac{du}{dx} = 2(x+1)e^{2x} + e^{2x}$ and $\frac{dv}{dx} = -\sin(x)$

$$\frac{dy}{dx} = \frac{\cos(x) \times (2(x+1)e^{2x} + e^{2x}) + \sin(x) \times (x+1)e^{2x}}{(\cos(x))^2}$$

$$\frac{dy}{dx} = \frac{e^{2x}\left[(2x+3)\cos(x) + (x+1)\sin(x)\right]}{\cos^2 x}$$

EXERCISE 2.6 The quotient rule

ANSWERS p. 396

Recap

1 The derivative of $2x^2\cos(4x)$ is

A $-16x\sin(4x)$

B $4x\cos(4x) + 8x^2\sin(4x)$

C $4x\cos(4x) - 8x^2\sin(4x)$

D $\frac{4x\cos(4x) + 8x^2\sin(4x)}{\cos^2(4x)}$

2 The derivative of $\ln(3x^2)\sin(3x^2)$ is

A $\frac{\frac{2\sin(3x^2)}{x} - 6\cos(3x^2)\ln(3x^2)}{\sin^2(3x^2)}$

B $\frac{2\sin(3x^2)}{x} + 6x\cos(3x^2)\ln(3x^2)$

C $\frac{12\cos(3x^2)}{x}$

D $\frac{2\sin(3x^2)}{x} - 6\cos(3x^2)\ln(3x^2)$

Mastery

3 WORKED EXAMPLE 11 Determine the derivative of each function.

a $\frac{x^3 - 3x^2 + 4}{\sin(4-x)}$

b $\frac{5\sin(1-2x)}{\sin(3x+2)}$

c $\frac{2x^2 - 3x^3 + 2}{\cos(5-3x)}$

d $\frac{3\cos(2-x)}{\cos(3+x)}$

4 Determine the derivative of each function.

a $\frac{\cos(5x-3)}{3x+2}$

b $\frac{5\cos(3-4x)}{(3x-5)^3}$

c $\frac{\sin(3x-5)}{x+4}$

d $\frac{2\sin(2x+3)}{(4x-1)^2}$

5 Determine the derivative of each function.

a $7\sin\left(\frac{(3q-6)^3}{(2q+3)^2}\right)$

b $\frac{1 - e^x\sin(x)}{1 + e^x\cos(x)}$

c $\frac{3\sin^2\left[(3x-2)^4\right]}{5\cos^3(x^3 - x^2 + 5)}$

d $\frac{\sin(1+e^{-2x})\cos(1-e^x)}{\tan(1+e^{-x})}$

Exam practice

6 TECH-FREE The derivative of $y = \dfrac{x-1}{x+3}$ is

A $\dfrac{4}{(x-1)^2}$ **B** $\dfrac{-4}{(x+3)^2}$ **C** $\dfrac{4}{(x+3)^2}$ **D** $\dfrac{-1}{3}$

7 TECH-FREE The derivative of $y = \dfrac{e^{2x}}{\cos(2x)}$ is

A $\dfrac{2e^{2x}(\cos(2x)+\sin(2x))}{\cos^2(2x)}$ **B** $\dfrac{-2e^{2x}(\cos(2x)+\sin(2x))}{\sin^2(2x)}$

C $\dfrac{e^{2x}}{-\sin(2x)}$ **D** $\dfrac{2e^{2x}(\cos(2x)+\sin(2x))}{\sin^2(2x)}$

8 TECH-FREE The derivative of $y = \dfrac{3x^2+x-1}{\ln(4x)}$ is

A $\dfrac{(6x+1)\ln(4x)-\dfrac{(3x^2+x-1)}{x}}{(3x^2+x-1)^2}$ **B** $\dfrac{(6x+1)\ln(4x)-(3x^2+x-1)}{x\left[\ln(4x)\right]^2}$

C $\dfrac{\dfrac{(3x^2+x-1)}{x}-(6x+1)\ln(4x)}{(3x^2+x-1)^2}$ **D** $\dfrac{(6x+1)\ln(4x)-\dfrac{(3x^2+x-1)}{x}}{\left[\ln(4x)\right]^2}$

9 The value of derivative of $y = \dfrac{3x^2+x-1}{\ln(4x)}$ when $x = 2$ is

A 0.1215 **B** 1.6226 **C** −0.1215 **D** 4.7485

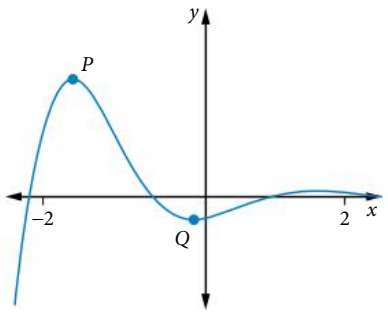
10 © QCAA 2021 1Q11b MODIFIED TECH-FREE (3 marks) Determine the derivative with respect to x of the function $y = \dfrac{\sin(x)}{x^2}$. (Give your answer in simplest form.)

11 © QCAA 2022 1Q13b MODIFIED TECH-FREE (3 marks) Given that $g(x) = \dfrac{\ln(x)}{x}$, determine the simplest value of $g'(e)$.

12 (7 marks) CF This diagram shows the graph of $f(x) = \dfrac{\cos(2x+3)}{2e^x}$.

A local maximum occurs at P and a local minimum occurs at Q.

Calculate the coordinates of P and Q, correct to 3 decimal places.



EXAM QUESTION ANALYSIS

© QCAA 2022 2Q13 (4 marks)

A sandy beach has a fence on one side and ocean on the other. The width of the beach is the distance (in metres) from the fence to the water's edge. The width, $w(t)$, at a certain point is given by

$$w(t) = a + b\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right),\ 0 \le t \le 24$$

where t is time (in hours) since 6 am. The width of the beach is 8 metres at 8 am and 3 metres at 5 pm.

a Determine a and b. *[2 marks]*

b Determine the rate of change of the width of the beach at 8 am and the first time after this when this rate of change is repeated. *[2 marks]*

Reading the question

- It is sinusoidal.
- a is the median value and b the amplitude.
- Rate of change is the derivative.
- It is repeated – look at the graph.

Thinking about the question

- Values at 8 am and 5 pm give two equations to solve.
- Find the derivative and graph to find the repeated time.

Worked solution (✓ = 1 mark)

a 8 am is at $t = 2$, 5 pm is at $t = 11$

$$8 = a + b\sin\left(\frac{\pi}{6}\times 2 - \frac{\pi}{3}\right) = a + b\sin(0) = a$$

Hence $\boldsymbol{a = 8}$ ✓

$$3 = 8 + b\sin\left(\frac{\pi}{6}\times 11 - \frac{\pi}{3}\right) = 8 + b\sin\left(\frac{3\pi}{2}\right) = 8 - b$$

$\boldsymbol{b = 5}$ ✓

b $w(t) = 8 + 5\sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$

$$w'(t) = 5 \times \frac{\pi}{6}\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right) = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$$

At $t = 2$ (8 am), $w'(t) = \frac{5\pi}{6}\cos\left(\frac{\pi}{6}\times 2 - \frac{\pi}{3}\right) = \frac{5\pi}{6}\ \text{m h}^{-1}$

At 8 am, the rate of change of the width is $\boldsymbol{\frac{5\pi}{6}\ \text{m h}^{-1}}$ ✓

From a graphics calculator graph, this is repeated at $t = 14$ or **8 pm** ✓

Video Exam question analysis: Derivatives of trigonometric functions

2 Chapter summary

The derivative of sin(x)

- $\frac{d}{dx}\sin(x) = \cos(x)$
- $\frac{d}{dx}\sin(kx) = k\cos(kx)$
- $\frac{d}{dx}\sin[f(x)] = f'(x)\cos[f(x)]$
- $\frac{d}{dx}\sin(x^\circ) = \frac{\pi}{180}\cos(x^\circ)$

The derivative of cos(x)

- $\frac{d}{dx}\cos(x) = -\sin(x)$
- $\frac{d}{dx}\cos(kx) = -k\sin(kx)$
- $\frac{d}{dx}\cos[f(x)] = -f'(x)\sin[f(x)]$
- $\frac{d}{dx}\cos(x^\circ) = \frac{-\pi}{180}\sin(x^\circ)$

Problems involving the sine and cosine functions

- Guidelines for creating **sinusoidal** models
  - Start with either $y = A\sin[B(x - C)] + D$ or $y = A\cos[B(x - C)] + D$
  - If the model starts from the central value, choose the sine function.
  - If the model starts from either a maximum or minimum value, choose the cosine function.
  - If the model starts anywhere else, select the trigonometric functions that ensures the C value is as close to 0 as possible. Avoid using the C value as much as possible, as it adds extra complexity when solving.
  - A is the dilation in the y direction, B is the dilation in the x direction, C is the translation in the x direction, D is the translation in the y direction.
- Problem-solving strategy
  - Identify key pieces of information by underlining, highlighting or summarising.
  - Draw a sketch of the problem, labelling key information and creating variables where possible.
  - Determine the variable or rate that needs to be solved for, and create links to known information.
  - Use the variable or rate and the known information to create equation(s) that will be used in the solving process.
  - If the question asks for a maximum or minimum value, this will always occur when a rate of change is equal to 0.
  - Solve the mathematical problem, showing your working.
  - Express your solution in a short statement at the end of the problem, incorporating any units.

The chain rule

If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

The product rule

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The quotient rule

$$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (4 marks): 4 multiple choice questions

Section 2 (16 marks): 3 short response questions

Total: 20 marks

Worksheet
Maths Methods formula sheet

Section 1 4 multiple choice questions

4 marks

1 © QCAA 2022S 1Q3 If $y = \cos^2(3x)$, then $\frac{dy}{dx} =$

A $-6\sin(3x)\cos(3x)$

B $-2\sin(3x)\cos(3x)$

C $2\sin(3x)\cos(3x)$

D $6\sin(3x)\cos(3x)$

2 The derivative of $(4x^2 - x + 7)\cos(x)$ is

A $(4x^2 - x + 7)(-\sin(x)) + (8x - 1)\cos(x)$

B $-\sin(x)(8x - 1)$

C $(8x - 1)(4x^2 - x + 7) - \sin(x)\cos(x)$

D $\frac{(8x-1)\cos(x)+(4x^2-x+7)\sin(x)}{\cos^2(x)}$

3 The derivative of $e^{\sin(6x)}$ is

A $\cos(6x)\,e^{\sin(6x)}$

B $\sin(6x)\,e^{6\cos(6x)}$

C $6\cos(6x)\,e^{x}$

D $6\cos(6x)\,e^{\sin(6x)}$

4 The derivative of $\ln\left(\frac{\sin(2x)}{x}\right)$ is

A $\frac{2x\cos(2x)-\sin(2x)}{x^2}$

B $\frac{x}{\sin(2x)}$

C $\frac{2x\cos(2x)-\sin(2x)}{x\sin(2x)}$

D $\frac{\sin(2x)-2x\cos(2x)}{x\sin(2x)}$

Section 2 3 short response questions

16 marks

5 (5 marks) Find the equation of the tangent to the curve $y = \ln(e + 3x)$ at $x = 0$.

6 (6 marks) Determine $f'\left(\frac{\pi}{4}\right)$ for each function.

a $f(x) = \ln[\sin(2x)]$ *[3 marks]*

b $y = \frac{\cos(2x)}{e^{-x}}$ *[3 marks]*

7 © QCAA 2023 1Q16 (5 marks) Solve the equation $4 + 7e^{-2x} = 3e^{2x}$.

Worksheet
Maths Methods formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (16 marks): 4 short response questions

Total: 19 marks

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2023 2Q1 If $f(x) = \sin(3x)$, determine the value of $f'\left(\frac{\pi}{8}\right)$.

A 2.772 **B** 1.148 **C** 0.929 **D** 0.383

2 © QCAA 2023 2Q8 The number of koalas in a conservation park is modelled by $N = 15\ln(7t + 1)$, $t \geq 1$, where t represents the time (years) since the park opened. There were 20 koalas in the park when it opened.

Determine the approximate rate of change in the number of koalas when $t = 3$.

A 46 **B** 26 **C** 25 **D** 5

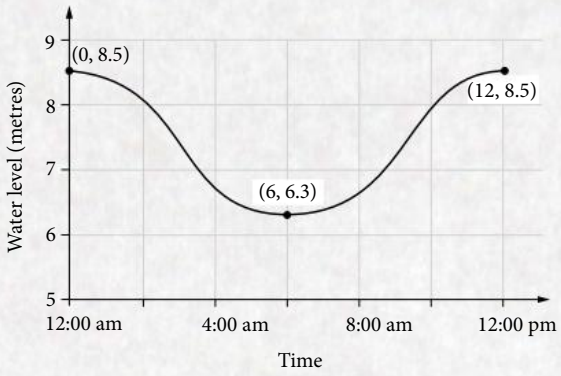
3 If $f(x) = e^{2x}\cos(2x)$, the value of $f'(5)$ is

A −60 929.29 **B** −18 461.76 **C** −12 997.84 **D** 970 330 390.8

Section 2 4 short response questions

16 marks

4 © QCAA 2023 2Q12 (4 marks) The graph shows the water level under a bridge over a 12-hour period.



a Determine the equation of the cosine function that models the water level as a function of time after 12:00 am. *[1 mark]*

b How long in the 12-hour period shown is the rate of change of water level more than 0.55 metres per hour? *[3 marks]*

5 (3 marks) Determine $f'(2)$ for $f(x) = (2x^2 - 4x + 1)\ln(6x - 2)$.

6 (5 marks) The velocity $v(t)$ (in m s^{-1}) of a falling object at time t (in seconds) is modelled by the equation $v(t) = A\ln(bt)$, where A and b are parameters. The velocity reaches a maximum of $20\ \text{m s}^{-1}$ at $t = 4$ s.

Determine the exact values of A and b, and evaluate whether the model is reasonable for describing the velocity of the falling object.

7 (4 marks) The speed, v metres per second, of a drag car from its starting position is given by $v = 5000[\cos(0.002t^2) + \sin(0.002t)] - 5000$, $t > 0$. When the car starts to slow down, it has run out of fuel. At this instant, the parachute is deployed to stop the car.

Determine the speed of the drag car when the parachute is deployed and how many seconds after the parachute is deployed before the drag car comes to a stop.

CHAPTER

3 FURTHER APPLICATIONS OF DIFFERENTIATION



Syllabus coverage

Nelson MindTap chapter resources

Terminology

3.1 The second derivative

3.2 Concavity and points of inflection

Concavity

Points of inflection

3.3 The second derivative test

Stationary points

3.4 Curve sketching

Graph of the derivative function

3.5 Straight-line motion

3.6 Optimisation

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170484985

Syllabus coverage

UNIT 3, TOPIC 3: FURTHER APPLICATIONS OF DIFFERENTIATION

The second derivative and applications of differentiation

- Understand the concept of the second derivative as the rate of change of the first derivative function.
- Recognise acceleration as the second derivative of displacement position with respect to time.
- Understand the concepts of concavity and points of inflection and their relationship with the second derivative.
- Understand and use the second derivative test for finding local maxima and minima.
- Sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection.
- Model and solve optimisation problems from a wide variety of fields using first and second derivatives, where the function to be optimised is either given or to be developed.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 25,
© Queensland Curriculum and Assessment Authority (QCAA)

Prior learning
Further applications of differentiation

Videos (4):

3.2 Concavity • Points of inflection

3.4 Curve sketching

Exam question analysis Further applications of differentiation

Prior learning (1):

3.1 Further applications of differentiation

Worksheets (23):

3.1 The second derivative • First and second derivatives • Higher derivatives

3.2 Concavity

3.4 Gradient functions 1 • Gradient functions 2 • Curve sketching with derivatives • Curve sketching 1 • Curve sketching 2 • Further curve sketching • Further graphs

3.6 Greatest and least values • Optimisation • Optimisation – cost and revenue • Optimisation – area and volume • Optimisation – mixed problems • Applications of optimisation • Optimisation problems • Further optimisation problems • Starting maxima and minima problems • Second derivative assignment • Second derivative problems

Cumulative exams Maths Methods formula sheet

Puzzle (1):

Chapter summary Second derivative find-a-word

TI Nspire CX II instructions (1):

3.1 Further applications of differentiation

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

acceleration	concave down	concave up	concavity
decreasing	displacement	first derivative	global maximum
global minimum	horizontal point of inflection	increasing	local maximum
local minimum	nature	optimisation	point of inflection
second derivative	second derivative test	stationary point	turning point
velocity			

Worksheets
The second derivative

First and second derivatives

Higher derivatives

3.1 The second derivative

For the function $y = f(x)$, the **first derivative** is the rate of change of y with respect to x.
The **second derivative** is the rate of change of the first derivative function with respect to x.
In other words, the second derivative is the derivative of the first derivative.

The first and second derivatives

For the function $y = f(x)$

- the first derivative is written as y' or $f'(x)$ or $\frac{dy}{dx}$
- the second derivative is written as y'' or $f''(x)$ or $\frac{d^2y}{dx^2}$.

WORKED EXAMPLE 1 Finding the second derivative

Determine the second derivative, $\frac{d^2y}{dx^2}$, for $y = (2x + 5)^3$.

Steps	Working
1 Differentiate once for first derivative.	$y = (2x + 5)^3$
This is a simple application of the chain rule.	$y' = 3 \times 2(2x + 5)^2$
	$y' = 6(2x + 5)^2$
2 Differentiate again for second derivative.	$y'' = 2 \times 2 \times 6(2x + 5)$
	$y'' = 24(2x + 5)$

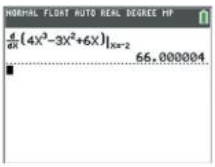
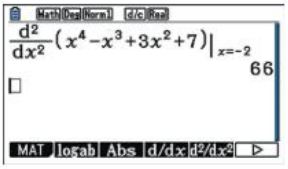
WORKED EXAMPLE 2 Evaluating the second derivative

Determine the value of $f''(x)$ for the function $f(x) = x^4 - x^3 + 3x^2 + 7$ at the point where $x = -2$.

Steps	Working
1 Differentiate once for first derivative.	$f(x) = x^4 - x^3 + 3x^2 + 7$
	$f'(x) = 4x^3 - 3x^2 + 6x$
2 Differentiate again for second derivative.	$f''(x) = 12x^2 - 6x + 6$
3 Let $x = -2$.	$f''(-2) = 12(-2)^2 - 6(-2) + 6$
4 Evaluate.	$f''(-2) = 66$

You can use your calculator to determine the value of second derivative.

TI-Nspire
Further applications of differentiation

	TI-84 Plus CE	Casio fx-CG50 AU
		
1 Enter the function.	1 Press math and select **8:nDeriv(**.	1 Select the F1 mode.
	2 Enter x to complete $\frac{d}{dx}$ and then enter the **first derivative** in the brackets.	2 Press F4 (MATH) and then press F5 (d^2/dx^2).
		3 Enter the **original function** in the brackets.
2 Enter the x value and evaluate.	3 Enter $x = -2$ and press enter. The result is displayed.	4 Enter $x = -2$ and press EXE. The result is displayed.

Exam hack

Remember that the product, chain and quotient rules are in the QCAA Formula book.

WORKED EXAMPLE 3 Using the differentiation rules to find the second derivative

Determine the second derivative for each function.

a $g(x) = x\sin(x)$ **b** $y = e^{1-2x}$ **c** $f(t) = \ln(3+t^2)$

Steps	Working
a 1 Choose u and v (or $f(x)$ and $g(x)$).	Let $u = x$ and $v = \sin x$.
2 Differentiate each part.	$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \cos x$
3 Apply the product rule to find $g'(x)$.	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $g'(x) = x\cos x + \sin x$
There is an alternative version of the product rule: If $y = uv$ then $y' = vu' + uv'$. This is not in the QCAA Formula book, but it is a commonly used form.	
4 Differentiate each term separately to find $g''(x)$ by applying the product rule to the first term.	$g''(x) = \frac{d}{dx}\left[(x\cos x) + \sin x\right]$ $= (-x\sin x + \cos x) + \cos x$
5 Write the answer.	$g''(x) = 2\cos x - x\sin x$
b 1 Let u or $g(x)$ be the 'inside' function.	Let $u = 1 - 2x$
2 Write the original function in terms of u or $g(x)$. Differentiate each.	$y = e^u$ $\frac{dy}{du} = e^u, \ \frac{du}{dx} = -2$
3 Apply the chain rule to find y'.	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $= e^u \times (-2)$ $= -2e^u$
4 Apply the chain rule again to find y''.	$y'' = -2e^u \times (-2)$ $= 4e^u$
5 Substitute for u or $g(x)$ and write the answer.	$y'' = 4e^{1-2x}$
c 1 Let u or $g(x)$ be the 'inside' function.	Let $u = 3 + t^2$
2 Write the original function in terms of u or $g(x)$. Differentiate each.	$f(t) = \ln u$ $\frac{dy}{du} = \frac{1}{u}, \ \frac{du}{dt} = 2t$
3 Apply the chain rule to find $f'(t)$.	$\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt}$ $= \frac{1}{u} \times 2t$ $f'(t) = \frac{2t}{u}$

4 Substitute for u or $g(x)$ and write the answer. $f'(t) = \dfrac{2t}{3+t^2}$

5 Choose u and v (or $f(x)$ and $g(x)$) as $f'(t) = \dfrac{u}{v}$. Differentiate each part. Let $u = 2t$ and $v = 3 + t^2$

$$\frac{du}{dt} = 2, \quad \frac{dv}{dt} = 2t$$

6 Apply the quotient rule to find $f''(t)$.

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dt} - u\dfrac{dv}{dt}}{v^2}$$

$$= \frac{(3+t^2)\times 2 - (2t)\times(2t)}{(3+t^2)^2}$$

7 Expand and simplify the numerator.

$$= \frac{6 + 2t^2 - 4t^2}{(3+t^2)^2}$$

$$f''(t) = \frac{6 - 2t^2}{(3+t^2)^2}$$

Exam hack

If the function is not linear, don't simplify the expression unless you are asked to.

EXERCISE 3.1 The second derivative

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Mastery

1 WORKED EXAMPLE 1 Determine the second derivative of each function.

a $f(x) = x^5 - 3x^3 + 7x + 11$

b $f(x) = \sin(2x)$

c $y = 2e^{5-4x}$

d $y = \ln(7 - 5x)$

e $g(x) = (2 - 3x)^4$

f $y = 4e^{3x+1}$

2 WORKED EXAMPLE 2 Determine the value of $f''(x)$ for each function.

a $f''(2)$ if $f(x) = x^5 - 2x^2 + 11$

b $f''(-2)$ if $f(x) = x^2 - x^{-1} + 12x$

c $h''\left(\dfrac{\pi}{3}\right)$ if $h(x) = \sin(4x)$

d $y''(3)$ if $y = \ln(5x^2 - 1)$

e $y''(0.5)$ if $y = e^{4x^2+2}$

f $f''\left(\dfrac{\pi}{4}\right)$ if $f(x) = 4x - \cos(3x)$

3 WORKED EXAMPLE 3 Determine $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each function.

a $f(x) = (1 - 6x)^5$

b $f(t) = \dfrac{2t+1}{t-1}$

c $g(w) = \sqrt{w^2+1}$

d $f(x) = x^2 \sin(2x)$

e $h(m) = \dfrac{m+2}{m^2+2m}$

f $g(p) = \ln(p^3 e^p)$

4 If $f(x) = \dfrac{x^3 - 1}{x}$, find $f'(2)$ and $f''(-3)$.

Exam practice All questions TECH-FREE

5 (2 marks) If $g(x) = 2x^5 + 9x$, find $g''(x)$.

6 (2 marks) Given $y = 2x\ln(x)$, calculate $y'(e)$ and $y''(1)$.

7 © QCAA 2022 1Q13c MODIFIED (5 marks) Determine the second derivative of $h(x) = x\sin(x)$. (Give your answer in simplest form.)

8 (2 marks) CF Find the value of x for which $\frac{d^2y}{dx^2} = 5$, given $y = 4x^3 - 3x^2 + 5x$.

9 (5 marks) CU Find the second derivative of $f(x) = \tan(4x)$.

10 (4 marks) CU Determine the value of a and b (where a and b are greater than 0) if $f(x) = (x + a)^2(2x - b)$ and $f'(2) = 40$ and $f''(2) = 34$.

11 © QCAA 2024 1Q11 (6 marks)

a Determine the second derivative of $y = x^3 - 3x^2$. *[2 marks]*

b Use your result from part **a** to calculate the value of the second derivative when $x = -1$. *[1 mark]*

c Determine the x- and y-coordinates of the point on the graph of $y = x^3 - 3x^2$ for which the rate of change of the first derivative is zero. *[3 marks]*

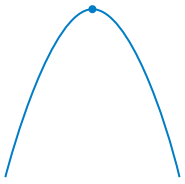
3.2 Concavity and points of inflection

Worksheet
Concavity

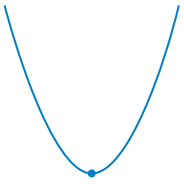
Concavity

In Year 11, Chapter 10, *Applications of derivatives*, you learnt that:

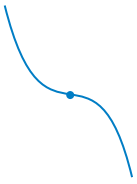
- a function is **increasing** when it has a positive gradient $\left(\frac{dy}{dx} > 0\right)$ and its graph is heading in an upwards direction (↗) from left to right
- a function is **decreasing** when it has a negative gradient $\left(\frac{dy}{dx} < 0\right)$ and its graph is heading in a downwards direction (↘) from left to right
- a **stationary point** on a graph of a function is a point where the gradient is zero $\left(\frac{dy}{dx} = 0\right)$, so it is neither increasing nor decreasing
- there are 3 types of stationary points: **local maximum**, **local minimum** and **horizontal point of inflection** (shown below).



Local maximum



Local minimum

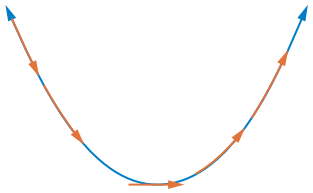


Horizontal point of inflection

When describing the shape of a continuous function or determining the nature of stationary points, it can be useful to consider concavity. **Concavity** relates to how the gradient of a function is changing.

The graph below illustrates a function that is **concave up** (or upwards). The gradient of the graph is increasing from left to right. So, the rate of change of the gradient is positive, $f''(x) > 0$.

Minimum stationary points are concave up in nature.

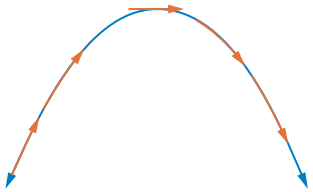


Remember the derivative of the derivative is the second derivative, which is the rate of change of the gradient function.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$$

The graph below shows a function that is **concave down** (or downwards). The gradient of the graph is decreasing, so the rate of change of the gradient is negative, $f''(x) < 0$.

Maximum stationary points are concave down in nature.



Concavity

For the function $y = f(x)$:

- where $f''(x) > 0$, the gradient of the function $f(x)$ is increasing, and the graph is concave up



- where $f''(x) < 0$, the gradient of the function $f(x)$ is decreasing, and the graph is concave down.



Video
Concavity

WORKED EXAMPLE 4 Determining concavity

Determine where the curve of $f(x) = x^3 - 12x^2 - 4x + 3$ is concave up.

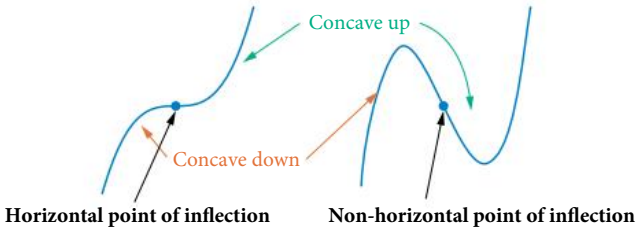
Steps	Working
1 Determine the first derivative.	$f(x) = x^3 - 12x^2 - 4x + 3$ $f'(x) = 3x^2 - 24x - 4$
2 Determine the second derivative.	$f''(x) = 6x - 24$
3 Identify the test for a curve to be concave up.	The curve is concave up when $f''(x) > 0$.
4 Substitute for $f''(x)$ and solve the inequality.	$\therefore 6x - 24 > 0$ $6x > 24$ $x > 4$
5 State the answer.	The curve of $f(x) = x^3 - 12x^2 - 4x + 3$ is concave up when $x > 4$.

Points of inflection

At a **point of inflection**, the graph of a function *bends* and the concavity changes, either from concave up to concave down, or from concave down to concave up. This means that, as you move from the left to the right of the point of inflection, the sign of $f''(x)$ changes.

At the point of inflection, the function is neither concave up nor concave down.

There are 2 types of points of inflection: a horizontal (or stationary) point of inflection, and a non-horizontal (or non-stationary) point of inflection.



In both cases, the tangent to the curve at the point of inflection crosses the curve.

At stationary points of inflection, the gradient is 0.

Points of inflection

- A point of inflection on the graph of $f(x)$ is a point where the concavity changes and $f''(x) = 0$.
- At a point of inflection, the curve is neither concave up nor concave down.
- You must check that the sign of $f''(x)$ is different on either side of the point of inflection.

WORKED EXAMPLE 5 Determining points of inflection 1

Determine any points of inflection on the curve of $y = x^3 - 2x^2 + 9x - 12$.

Steps	Working
1 Determine the first derivative.	$y = x^3 - 2x^2 + 9x - 12$ $y' = 3x^2 - 4x + 9$
2 Determine the second derivative.	$y'' = 6x - 4$
3 Identify the condition for a point of inflection.	At a point of inflection $y'' = 0$.
4 Substitute for y'' and solve for x.	$0 = 6x - 4$ $x = \frac{2}{3}$
5 Check the sign of y'' on either side of the x.	x: 0 | $\frac{2}{3}$ | 1 y'': -4 ($y'' < 0$) | 0 | 2 ($y'' > 0$)
6 Use the definition of a point of inflection.	The concavity changes from concave downwards ($y'' < 0$) to concave upwards ($y'' > 0$) at $x = \frac{2}{3}$.
7 Find the y-coordinate at the point of inflection.	At $x = \frac{2}{3}$: $y = \left(\frac{2}{3}\right)^3 - 2\left(\frac{2}{3}\right)^2 + 9\left(\frac{2}{3}\right) - 12$ $y = -\frac{178}{27}$
8 State the answer.	A point of inflection occurs at $\left(\frac{2}{3}, -\frac{178}{27}\right)$.

Video
Points of inflection

WORKED EXAMPLE 6 Determining points of inflection 2

Consider the function $g(x) = x^3(x - 4)$.

a **Determine** any points of inflection on the curve.

b **Identify** the type of point(s) of inflection and discuss the concavity around each point.

Steps	Working
a 1 Determine the first derivative.	$g(x) = x^3(x-4) = x^4 - 4x^3$ $g'(x) = 4x^3 - 12x^2$
2 Determine the second derivative.	$g''(x) = 12x^2 - 24x$ $= 12x(x-2)$
3 Identify the condition for a point of inflection.	At a point of inflection, $g''(x) = 0$.
4 Substitute for $g''(x)$ and solve for x.	$\therefore 12x(x-2) = 0$ $12x = 0$ or $x - 2 = 0$ $x = 0$ or $x = 2$
5 Check how the concavity (the sign) changes on each side of the points of inflection at $x = 0$ and $x = 2$. Use values such as $x = -1$ and $x = 1$ for $x = 0$ and $x = 1$ and $x = 3$ for $x = 2$. You can also use a table like that in Worked example 5.	$g''(-1) = 36 > 0$ (concave up) $g''(1) = -12 < 0$ (concave down) $\therefore$ at $x = 0$ there is a point of inflection. $g''(3) = 36 > 0$ (concave up) $\therefore$ at $x = 2$ there is a point of inflection.
6 Find the y-coordinate at each point of inflection.	$g(0) = 0$ $g(2) = -16$
7 State the answer.	There are points of inflection at $(0, 0)$ and $(2, -16)$.
b 1 To determine the type of point of inflection, calculate $g'(0)$ and $g'(2)$.	$g'(0) = 0$, $\therefore$ horizontal point of inflection. $g'(2) = -16 \neq 0$, $\therefore$ non-horizontal point of inflection.
2 Discuss the type of point and concavity changes at each point.	At $(0, 0)$ there is a horizontal point of inflection at which the graph changes from concave up to concave down. At $(2, -16)$. there is a non-horizontal point of inflection at which the graph changes from concave down to concave up.

EXERCISE 3.2 Concavity and points of inflection

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Recap

1 If $h(x) = \dfrac{x^2}{x+2}$, $h''(3)$ is

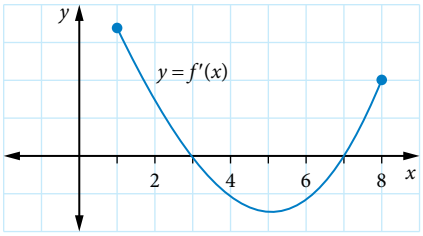
A 0.037 **B** 0.064 **C** 0.125 **D** 0.296

2 Find $f'(x)$ and $f''(x)$ if $f(x) = 3x^3 - \dfrac{1}{2}x$.

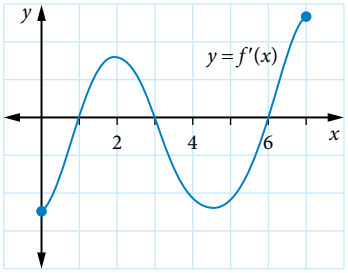
Mastery

3 Each diagram below shows the graph of the first derivative, $f'(x)$, of the function $f(x)$.
In each case, state the intervals for which $f(x)$ is increasing and for which $f(x)$ is decreasing.

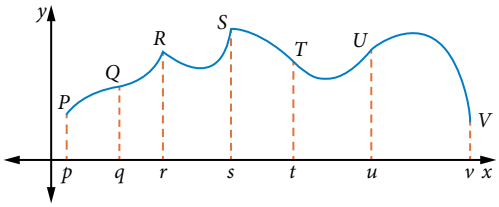
a



b



4 The graph of $f(x)$ is shown.



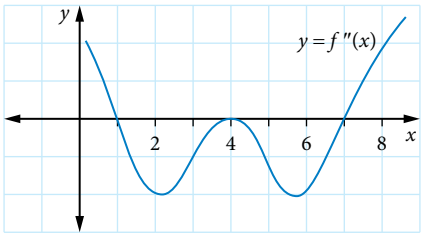
State the intervals for which the curve is

a concave upwards **b** concave downwards.

5 WORKED EXAMPLE 4 Identify the interval for which the curve $f(x) = 2x^3 - 9x^2 - 24x + 30$ is

a concave upwards **b** concave downwards.

6 The graph of the second derivative, $f''(x)$, of the function $f(x)$ is shown.
State the x-coordinate of any point of inflection, giving reasons for your answers.



7 WORKED EXAMPLE 5 Determine any points of inflection on the curve $y = (x - 2)^3$.

8 WORKED EXAMPLE 6 Determine any points of inflection and their type on the curve $y = x^3 - 6x^2 + 3$.

9 Identify the points of inflection and the intervals of concavity for the curve $f(t) = t + \sin(t)$ for $-2\pi \leq t \leq 2\pi$.

Exam practice

10 TECH-FREE The graph of $y = 2x^3 - x^4$ has a horizontal point of inflection at

A $x = 0$ **B** $x = 1$ **C** $x = 1.5$ **D** $x = 2$

11 TECH-FREE If $f'(x) > 0$ and $f''(x) < 0$ for all x in a given interval in the domain of $f(x)$, then the graph of $f(x)$ over this interval would have a curve that

A decreases, having increasing gradient with increasing x.

B increases, having increasing gradient with increasing x.

C decreases, having decreasing gradient with increasing x.

D increases, having decreasing gradient with increasing x.

12 TECH-FREE At the point at $x = -\frac{16}{3}$, the graph of $y = 2(x+2)(x+7)^2$

A changes from concave up to concave down.
B changes from concave down to concave up.
C has a stationary point of inflection.
D has a local maximum.

13 © QCAA 2020S 2Q7 The graph of $y = x^3 - 6x^2$ is concave downwards when

A $x < -2$ **B** $x < 0$ **C** $x < 2$ **D** $x < 4$

14 © QCAA 2020 2Q10 The approximate value of x at which the graph of the function $y = x^3 + 6x^2 + 7x - 2\cos(x)$ changes concavity is

A −3.26 **B** −2.85 **C** −2.20 **D** −1.89

15 © QCAA 2022 1Q15 MODIFIED (4 marks) TECH-FREE The derivative of a function is given by $f'(x) = e^x(x-5)$. Determine the interval on which the graph of $f(x)$ is both decreasing and concave up.

16 (4 marks) CF TECH-FREE Explain why the graph of $y = x^4 + 4$ has no points of inflection even though $y''(x) = 0$ has a solution.

17 (4 marks) CF TECH-FREE If $y = x^3 - ax^2 + 5x + 9$ has a point of inflection at $x = 2$, find the value of a.

3.3 The second derivative test

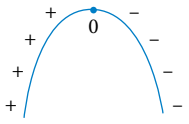
Stationary points

Turning points are found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point.

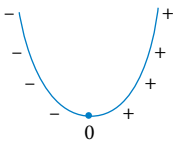
At a maximum turning point, the gradient changes from positive to negative. A **local maximum turning point** represents the highest value of the function in the local vicinity.

At a minimum turning point, the gradient changes from negative to positive. A **local minimum turning point** represents the lowest value of the function in the local vicinity.

A local maximum or minimum turning point may not actually be the absolute maximum or minimum value of the function. Such points are called the **global maximum** or **global minimum** points.



Local maximum turning point



Local minimum turning point

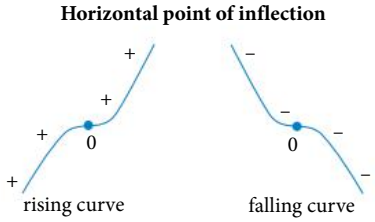
A **horizontal point of inflection** can also be found where $f'(x) = 0$; however, the sign of the gradient does not change on either side of the point.

A horizontal point of inflection *on a rising curve* has $\frac{dy}{dx} > 0$ on both sides of the stationary point.

At the point, the curve changes from concave down ($f''(x) < 0$) to concave up ($f''(x) > 0$).

A horizontal point of inflection *on a falling curve* has $\frac{dy}{dx} < 0$ on both sides of the stationary point.

At the point, the curve changes from concave up ($f''(x) > 0$) to concave down ($f''(x) < 0$).



The sign of the second derivative can also be used to identify the *nature* of stationary points as either a local maximum or minimum turning point. This is called the **second derivative test**.

The second derivative test for maximum and minimum points

For the function $y = f(x)$, if $f'(x) = 0$ and

- $f''(x) > 0$, the turning point is a minimum.
- $f''(x) < 0$, the turning point is a maximum.

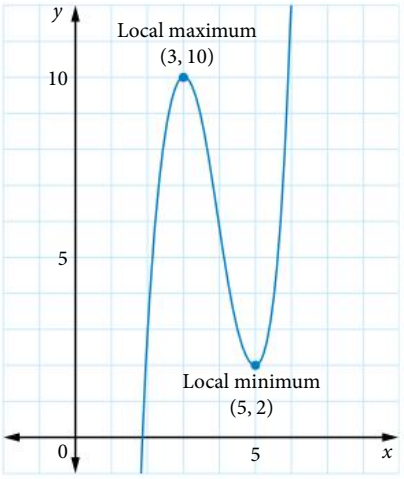
Consider the function $f(x) = (x - 2)(x - 5)^2 + 2$.

$f'(x) = 3(x - 3)(x - 5)$ and $f''(x) = 6(x - 4)$.

The graph of $f(x)$ has stationary points at $(3, 10)$ and $(5, 2)$.

At $x = 3$, $f'(x) = 0$ and $f''(x) < 0$, hence it is a local maximum turning point. The graph is also concave down.

At $x = 5$, $f'(x) = 0$ and $f''(x) > 0$, hence it is a local minimum turning point. The graph is also concave up.



WORKED EXAMPLE 7 Using the second derivative test

Determine the coordinates and nature of any stationary points on the graph of $f(x) = 2 + 3x - x^3$.

Steps	Working
1 Determine the first derivative.	$f(x) = 2 + 3x - x^3$ $f'(x) = 3 - 3x^2$
2 Solve $f'(x) = 0$ to find any stationary points.	$0 = 3 - 3x^2$ $0 = 3(1 - x)^2$ $0 = (1 - x)(1 + x)$ $x = 1$ or $x = -1$
3 Find the corresponding y coordinates.	$f(1) = 4$ and $f(-1) = 0$
4 Determine the second derivative.	$f''(x) = -6x$
5 Calculate $f''(-1)$ and $f''(1)$ to determine the nature of the stationary points.	$f''(1) = -6$, $f''(x) < 0$ $f''(-1) = 6$, $f''(x) > 0$
6 Apply the second derivative test.	When $x = -1$, $f'(-1) = 0$ and $f''(-1) > 0$, so there is a local minimum at $(-1, 0)$. When $x = 1$, $f'(1) = 0$ and $f''(1) < 0$, so there is a local maximum at $(1, 4)$.
7 State the result.	The graph of $f(x) = 2 + 3x - x^3$ has a local minimum at $(-1, 0)$ and a local maximum at $(1, 4)$.

If $f''(x) = 0$, the second derivative test is inconclusive and there are 2 different ways you can determine the nature of stationary points.

- Investigate the concavity on both sides of the stationary point using the second derivative. A maximum stationary point will be concave down on both sides ($f''(x) < 0$). A minimum will be concave up on both sides ($f''(x) > 0$).
- Use the **sign test** (also called the **first derivative test**). Check the sign of the gradient on both sides of the stationary point.

WORKED EXAMPLE 8 Identifying the nature of stationary points

Determine the coordinates and nature of any stationary points on the graph of $y = (x - 2)(x + 1)^3$.

Steps	Working
1 Determine the first derivative.	$y = (x - 2)(x + 1)^3$ Let $u = x - 2$ and $v = (x+1)^3$ $\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 3(x+1)^2$ $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ $y' = (x-2) \times 3(x+1)^2 + 1(x+1)^3$ $y' = (x+1)^2(4x-5)$
2 Solve $y' = 0$ to find any stationary points.	$0 = (x+1)^2(4x-5)$ $\therefore x = -1$ or $x = 1.25$
3 Find the corresponding y coordinates.	$y(-1) = 0$ and $y(1.25) = -8.54$
4 Determine the second derivative.	$y'' = 12x^2 + 6x - 6$
5 Calculate $f''(-1)$ and $f''(1.25)$ to determine the nature of the stationary points.	$f''(-1) = 0$ $f''(1.25) = 6, f''(x) > 0$
6 Apply the second derivative test to $f''(1.25)$.	When $x = 1.25, f'(1.25) = 0$ and $f''(1.25) > 0$, so there is a local minimum at $(1.25, -8.54)$.
7 Identify the nature of the point at $x = -1$ using either the sign test or checking the concavity. Write a conclusion.	**Using the sign test** x: -2 | -1 | 0 y': -13 | 0 | -5 The gradient remains negative. **Using concavity** x: -2 | -1 | 0 y'': 30 | 0 | -6 The curve changes from concave up to concave down.
8 State the result.	The graph of $y = (x - 2)(x + 1)^3$ has a local minimum at $(1.25, -8.54)$ and a horizontal point of inflection at $(-1, 0)$ on a falling curve.

Stationary points

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	$f''(x)$ at the stationary point	Type of stationary point
+	0	−	$f''(x) < 0$	Maximum 
−	0	+	$f''(x) > 0$	Minimum 
+	0	+	$f''(x) = 0$	Horizontal point of inflection 
−	0	−	$f''(x) = 0$	Horizontal point of inflection 

EXERCISE 3.3 The second derivative test

ANSWERS p. 397

Recap

1 The graph of $f(x) = x^3 - 3x^2 - 9x + 1$ is concave upwards when

A $x < -1$ **B** $x < 1$ **C** $x > 1$ **D** $x > 4$

2 Identify the points of inflection and the intervals of concavity for the curve

$$y = 3x^5 - 5x^3 + 4x + 3 \text{ for } -2 < x < 2.$$

Mastery

3 WORKED EXAMPLE 7 Identify any stationary points on each curve and determine their nature.

a $y = 3x^2 - 9x + 8$ **b** $y = 3x^4 - 16x^3 + 18x^2$

c $y = e^{-x} + x$ **d** $y = \dfrac{x+1}{x-1}$

4 WORKED EXAMPLE 8 Identify any stationary points on each curve and determine their nature.

a $y = x^3 - 3x^2 + 3x$ **b** $y = 2x + \dfrac{2}{x-1}$

c $f(x) = 12x^2 - x^3 + 69 - 48x$ **d** $y = x^2e^{-x}$

5 Identify all stationary points on the curve $y = x^4 - 2x^3 + 6$.

6 Identify any stationary points on the curve $y = x + \cos(x)$ for $0 \leq x \leq 2\pi$.

Exam practice

All questions TECH-FREE

7 For the graph of $f(x) = 9 + 8x - 2x^2$, the point at $x = 2$

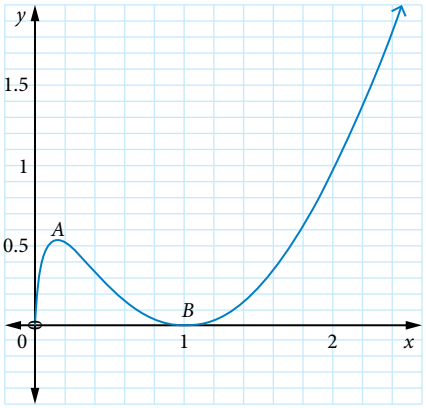
A is a local minimum. **B** is a local maximum.

C is a horizontal point of inflection. **D** is a non-stationary point.

8 © QCAA 2020 1Q13 (7 marks) A function is defined as $f(x) = x(\ln(x))^2$, $x > 0$.

The graph of the function is shown and has a local maximum at point A and a global minimum at point B.

The derivative of the function is given by $f'(x) = 2\ln(x) + (\ln(x))^2$, $x > 0$.



a Verify that there is a stationary point at $x = 1$. *[2 marks]*

b Determine the coordinates of point A. *[3 marks]*

The graph of the function has a point of inflection at $x = e^p$.

c Determine p. *[2 marks]*

9 (4 marks) CF The curve $y = x^2 + ax - 12$ has a stationary point at $x = 2$.

a Find the value of a. *[2 marks]*

b What type of stationary point occurs at $x = 2$? *[2 marks]*

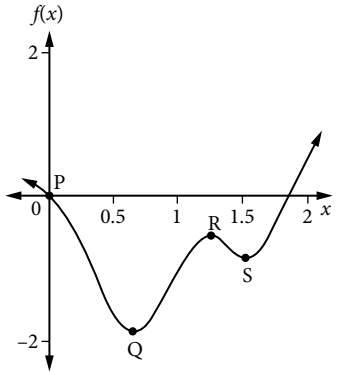
10 (4 marks) CF The curve $y = ax^3 + bx^2 + 6x + 5$ has a horizontal point of inflection at $(1, 7)$. Find the value of a and b.

11 (2 marks) CF Explain why the function $h(x) = x^2 + \sqrt{x} + 2$ has no stationary points. Justify your answer.

12 © QCAA 2021 1Q20 (7 marks) CU The population of rabbits (P) on an island, in hundreds, is given by $P(t) = t^2\ln(3t) + 6$, $t > 0$, where t is time in years.

Determine the intervals of time when the population is increasing and the intervals when it is decreasing.

13 © QCAA 2024 2Q9 65% It is known that $f'(x) = 0$ and $f''(x) < 0$ for one of the labelled points on the graph of $f(x)$.



Which point matches this description?

A P **B** Q **C** R **D** S

3.4 Curve sketching

Graphs of the derivative function

The relationship between a function and its derivative can provide valuable and accurate information about the function and its shape. Consider the graph of $y = f(x)$ and $y = f'(x)$.

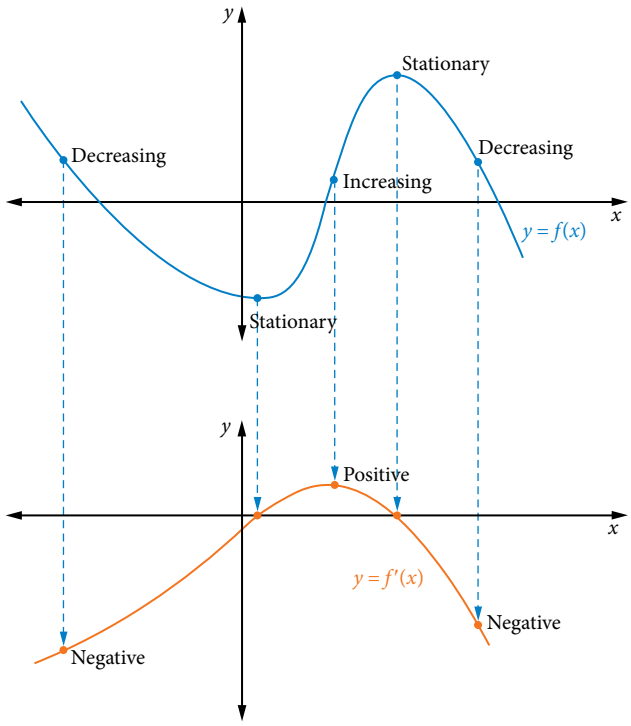
The **first derivative** can be used to

- calculate the gradient at any point
- determine where a function is increasing or decreasing
- identify where stationary points occur.

The graph of $y = f'(x)$ shows these elements visually.

For instance, on the graph of $y = f(x)$ there is a local maximum stationary point. The gradient to the left of this point is increasing and to the right decreasing.

Translating the maximum stationary point from the graph of $y = f(x)$ to the graph of $y = f'(x)$, you can see the stationary point is now the x-intercept. To the left of this x-intercept, the graph is positive (above the x-axis) and to the right it is negative (below the x-axis).



Video Curve sketching

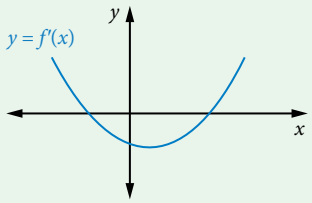
Worksheets
Gradient functions 1
Gradient functions 2
Curve sketching with derivatives
Curve sketching 1
Curve sketching 2
Further curve sketching
Further graphs

Graph of $f(x)$ and $f'(x)$

	increasing		decreasing		Stationary point		
Graph of $f(x)$					Minimum	Maximum	Point of inflection or
Graph of $f'(x)$	Positive ($f'(x) > 0$), above the x-axis		Negative ($f'(x) < 0$)), below the x-axis		x-intercept $\left(\frac{dy}{dx} = 0\right)$		
$f''(x)$ value	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) = 0$ (concavity changes)

WORKED EXAMPLE 9 Graphing $f(x)$ using the graph of $f'(x)$

The graph of $f'(x)$ is shown. **Sketch** a possible graph of $f(x)$.



Steps

1 The derivative is positive (above the x-axis) at first, so the function is increasing.

2 The first x-intercept $\left(\frac{dy}{dx}=0\right)$ on the derivative graph is just before the y-axis, so the function has a stationary point.

3 Between the 2 x-intercepts the derivative graph is negative (below the x-axis), so the function is decreasing.

4 As the derivative function goes from positive to negative around the first x-intercept, there is a maximum stationary point on the graph of $f(x)$ at that point.

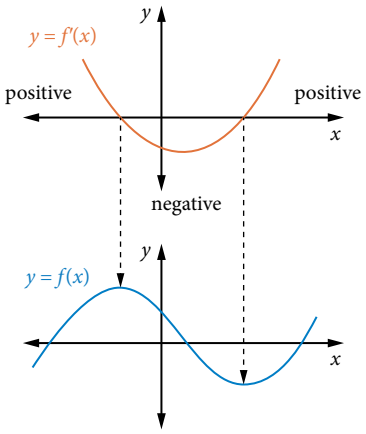
5 The second x-intercept $\left(\frac{dy}{dx}=0\right)$ on the derivative graph is just after the y-axis, so the function again has a stationary point.

6 After the second x-intercept, the derivative graph is positive, so the function is increasing again.

7 As the derivative function goes from negative to positive around the second x-intercept, there is a minimum stationary point on the graph of $f(x)$ at that point.

Graph the original function $f(x)$ so that the points match the relevant points from the derivative function $f'(x)$.

Working



This is just one possible graph. It could be higher or lower (and have 1, 2 or 3 zeros).

When sketching graphs, it is useful to determine your own preferred order to identify key features before drawing the function.

Curve sketching

1 **Stationary points:** Use the first derivative to locate stationary points. The first derivative can also be used to determine the intervals in which the function is increasing or decreasing.

2 **Nature of stationary points:** Use the second derivative to identify the nature of the stationary points. The second derivative can also be used to determine the intervals in which the function is concave upwards or downwards.

3 **Points of inflection:** Use the second derivative to identify any points of inflection ($f''(x) = 0$).

4 **Intercepts:** Determine the x- and y-intercepts. Find the x-intercept(s) by solving $f(x) = 0$, if possible. Find the y-intercept by evaluating $f(0)$.

5 **Discontinuities** and other special features: Determine the nature of the graph where $f(x)$ is undefined, particularly to find whether there is a vertical asymptote.

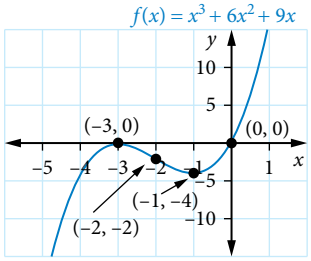
You do not need to check for discontinuities for polynomial functions as they are defined for all values of x.

6 **Endpoints:** Determine the behaviour of the function as $x \to \pm\infty$. Consider if the function has a limit as $x \to \pm\infty$. This is a horizontal asymptote.

7 **Sketch:** Draw the function, labelling all important features.

WORKED EXAMPLE 10 Curve sketching 1

Sketch the graph of $f(x) = x(x+3)^2$, labelling all important features.

Steps	Working
1 Expand the function and consider the general shape. Consider expanding or factorising functions if it makes them easier to deal with. In this case it is quicker to expand and differentiate rather than use the product rule.	$f(x) = x^3 + 6x^2 + 9x$ When expanded, the leading coefficient is $1 > 0$, so the function is cubic. 
2 Determine the first derivative.	$f'(x) = 3x^2 + 12x + 9$
3 Solve $f'(x) = 0$ for stationary points.	$0 = 3x^2 + 12x + 9$ $0 = 3(x^2 + 4x + 3)$ $0 = 3(x+3)(x+1)$ $x = -3$ or $x = -1$
4 Calculate the y values of the stationary points and state the coordinates.	$f(-3) = 0$ and $f(-1) = -4$ Stationary points are at $(-3, 0)$ and $(-1, -4)$.
5 Determine the second derivative.	$f''(x) = 6x + 12$
6 Find $f''(x)$ at the stationary points and use the second derivative test to determine the nature of the stationary points.	$f''(-3) = -6$ As $f''(-3) < 0$, it is a local maximum. $f''(-1) = 6$ As $f''(-1) > 0$, it is a local minimum.
7 Identify any possible points of inflection.	Let $f''(x) = 0$ $0 = 6x + 12$ $x = -2$
8 Calculate the y values of the point of inflection and state the coordinates.	$f(-2) = -2$ There is a possible point of inflection at $(-2, -2)$.
9 Check the concavity around the point of inflection and write a conclusion.	<table><tr><td>x</td><td>−2.5</td><td>−2</td><td>−1.5</td></tr><tr><td>$f''(x)$</td><td>−3</td><td>0</td><td>3</td></tr></table> The curve changes from concave downwards to concave upwards, so $(-2, -2)$ is a point of inflection.
10 Find the x-intercept(s) by solving $f(x) = 0$.	$0 = x(x+3)^2$ $x = 0$ or $x = -3$ The intercepts are at $(0, 0)$ and $(-3, 0)$.
11 Check the behaviour at the endpoints of $f(x)$ as $x \to \pm\infty$.	As $x \to -\infty$, $f(x) \to -\infty$ As $x \to \infty$, $f(x) \to \infty$
12 Sketch the curve, showing all important features.	

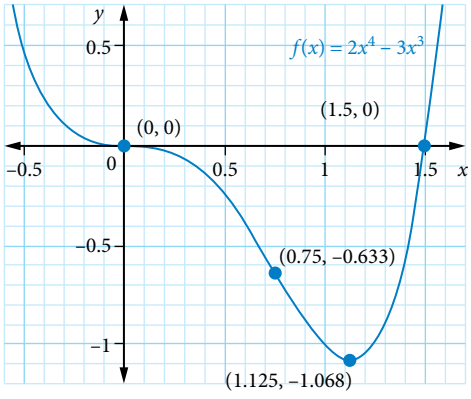
Exam hack

If an x-intercept occurs at 0, you have also found the y-intercept and do not need to prove it separately.

WORKED EXAMPLE 11 Curve sketching 2

Sketch the graph of $f(x) = 2x^4 - 3x^3$, labelling all key features.

Where necessary state answers correct to 3 decimal places.

Steps	Working
1 Determine the first derivative.	$f(x) = 2x^4 - 3x^3$ $f'(x) = 8x^3 - 9x^2$
2 Solve $f'(x) = 0$ for stationary points.	$f'(x) = x^2(8x - 9)$ $0 = x^2(8x - 9)$ $x = 0$ or $x = 1.125$
3 Calculate the y values of the stationary points and state the coordinates.	$f(0) = 0$ and $f(1.125) = -1.068$ Stationary points are at $(0, 0)$ and $(1.125, -1.068)$.
4 Determine the second derivative.	$f''(x) = 24x^2 - 18x$
5 Find $f''(x)$ at the stationary points and use the second derivative test to determine the nature of the stationary points. Remember if $f''(x) = 0$ you need to check the concavity on either side to determine the nature of the point.	$f''(0) = 0$ Possible point of inflection. x: –0.5, 0, 0.5 $f''(x)$: 15, 0, –3 There is a horizontal point of inflection at $(0, 0)$ as the curve changes from concave up to concave down. $f''(1.125) = 10.125$. As $f''(1.125) > 0$ it is a local minimum at $(1.125, -1.068)$.
6 Identify any points of inflection and verify the concavity changes around the point. You do not need to verify the point at $x = 0$.	Let $f''(x) = 0$. $0 = 24x^2 - 18x$ $0 = 6x(4x - 3)$ $x = 0$ or $x = 0.75$ x: 0.5, 0.75, 1 $f''(x)$: –3, 0, 6
7 Calculate the y values of the point of inflection and write a conclusion.	$f(0.75) = -0.633$ A point of inflection occurs at $(0.75, -0.633)$ as curve changes from concave down to concave up.
8 Find the x- and y-intercepts.	$0 = 2x^4 - 3x^3$ $0 = x^3(2x - 3)$ $x = 0$ or $x = 1.5$ The intercepts are at $(0, 0)$ and $(1.5, 0)$.
9 Check the behaviour at the endpoints of $f(x)$ as $x \to \pm\infty$.	As $x \to -\infty$, $f(x) \to \infty$ As $x \to \infty$, $f(x) \to \infty$
10 Sketch the curve, showing all important features.	

Exam hack

Do not rely on your graphing calculator when sketching graphs! Use it to identify the shape of the graph and confirm the values of significant features.

WORKED EXAMPLE 12 Curve sketching 3

Sketch the graph of $f:(-\infty,0.6]\to R$, $f(x)=(2x-1)e^{3x}$, labelling all key features. State the range of f. Where necessary state answers correct to 3 decimal places.

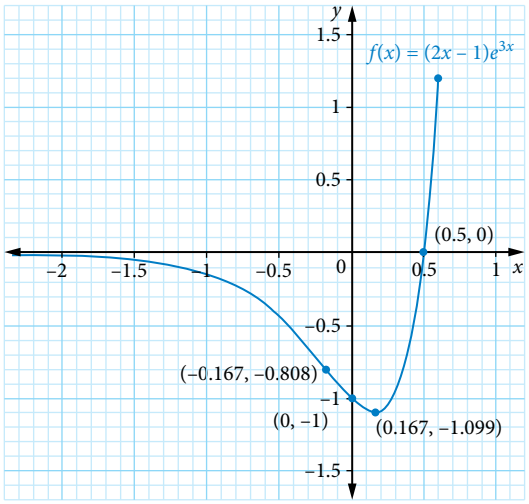
Steps	Working
1 Determine the first derivative using the product rule.	Let $u=2x-1$ and $v=e^{3x}$. $\frac{du}{dx}=2 \qquad \frac{dv}{dx}=3e^{3x}$ $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ $f'(x)=3(2x-1)e^{3x}+2e^{3x}$
2 Solve $f'(x)=0$ for stationary points.	$f'(x)=e^{3x}(6x-1)$ $0=e^{3x}(6x-1)$ $x=\frac{1}{6}\approx 0.167$
3 Calculate the y values of the stationary points and state the coordinates.	There is a stationary point at $(0.167,-1.099)$.
4 Determine the second derivative using the product rule.	$f''(x)=6e^{3x}+3e^{3x}(6x-1)$ $=3e^{3x}(6x+1)$
5 Find $f''(x)$ at the stationary points and use the second derivative test to determine the nature of the stationary points.	$f''(0.167)=9.892$. As $f''(0.167)>0$, it is a local minimum at $(0.167, -1.099)$.
6 Identify any points of inflection and verify the concavity changes around the point.	Let $f''(x)=0$. $0=3e^{3x}(6x+1)$ $x=\frac{-1}{6}\approx -0.167$

x	-0.2	-0.167	-0.1
$f''(x)$	-0.329	0	0.889

7 Calculate the y values of the point of inflection and write a conclusion.	$f(-0.167)=-0.808$ A point of inflection occurs at $(-0.167,-0.808)$ as curve changes from concave down to concave up.
8 Find the x- and y-intercepts.	Let $f(x)=0$. $0=(2x-1)e^{3x}$ $x=0.5$ $f(0)=-1$ The intercepts are at $(0.5,0)$ and $(0,-1)$.
9 Check the behaviour at the endpoints of $f(x)$ as $x\to-\infty$ and $x=0.6$.	As $x\to-\infty$, $f(x)\to 0$. There is a horizontal asymptote at $y=0$. $f(0.6)=1.21$
10 State the range.	The range is approximately $[-1.099, 1.21]$.

11 Sketch the curve, showing all important features.

Remember to put a closed circle at $x = 0.6$ as it is included in the domain of the function.



Exam hack

The range is not always the y values at the endpoints. You must also compare the y values of the local minimum and maximum points.

EXERCISE 3.4 Curve sketching

ANSWERS p. 397

Recap

1 The function $g(x)$ satisfies the following conditions.

- $g'(x) = 0$ at $x = 1$ and $x = 5$
- $g'(x) < 0$ where $x < 1$
- $g'(x) > 0$ where $1 < x < 5$ and $x > 5$

Which of the following is true about the graph of $g(x)$?

A It has a maximum at $x = 1$.

B It has a horizontal point of inflection at $x = 1$.

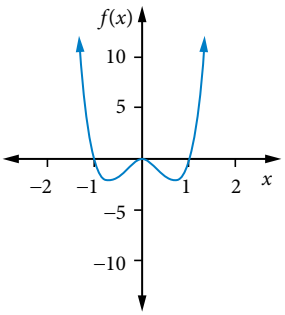
C It has a horizontal point of inflection at $x = 1$ and a minimum at $x = 5$.

D It has a horizontal point of inflection at $x = 5$.

2 Find any stationary points on the curve $y = x^4 - 4x^3 + 10$ and determine their nature.

Mastery

3 WORKED EXAMPLE 9 The graph of $f(x)$ is shown. Sketch the graph of $f'(x)$.



4 WORKED EXAMPLE 10, 11 Sketch the graphs of each function, labelling all key features.

a $g(x) = 10x - x^2 - 21$

b $h(x) = -x^3 - 1$

c $f(x) = x^3 + 3x^2 - 24x$

d $f(x) = 5 + 5x^3 - 3x^5$

e $f(x) = 7 + 2x^2 - x^4$

f $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$

5 WORKED EXAMPLE 12 Sketch the graph of each function, labelling all key features.

a $f(x) = xe^{-3x}$

b $f(x) = \dfrac{1}{x-3}$

c $g(x) = x^{-1}e^{x}$

d $g(x) = 1 - \dfrac{1}{x}$

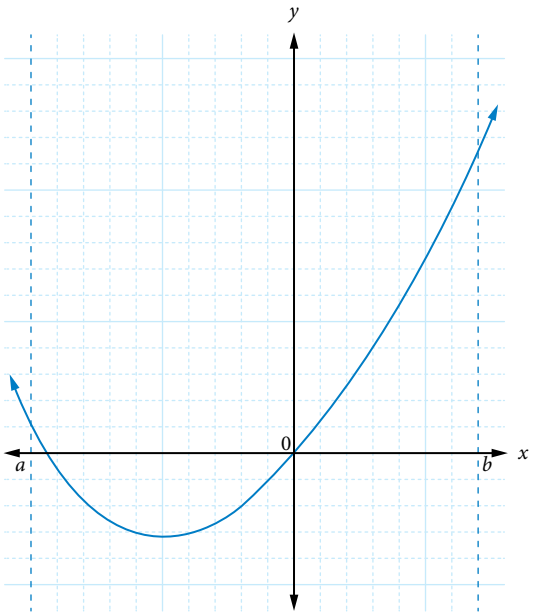
e $f(x) = (x+1)x^{\frac{2}{3}}$

f $f(x) = \cos(x) - x$ for $0 < x < 3\pi$

Exam practice

All questions TECH-FREE

6 © QCAA 2022 1Q1 34% Consider the graph of $f'(x)$ for $a \le x \le b$.



Which statement describes all the local maxima and minima of the graph of $f(x)$ over $a \le x \le b$?

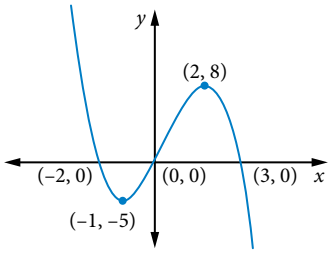
A one local minimum and one local maximum

B one local minimum and two local maxima

C one local minimum only

D one local maximum only

7 The graph of $y = f(x)$ is shown.



$f'(x)$ is positive when

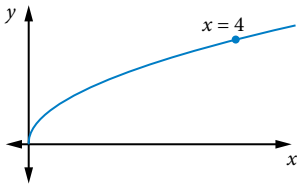
A $-1 < x < 2$

B $-1 \le x \le 2$

C $x < -1$ or $x > 2$

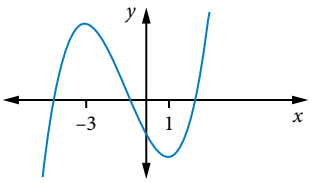
D $x < -2$ or $0 < x < 3$

8 The graph of $y = h(x)$ is shown. The x- and y-axes have the same scale.
Which one of the following inequalities is correct?



A $0 < h''(4) < h'(4) < h(4)$
B $0 < h''(4) < h(4) < h'(4)$
C $h''(4) < 0 < h'(4) < h(4)$
D $h''(4) < 0 < h(4) < h'(4)$

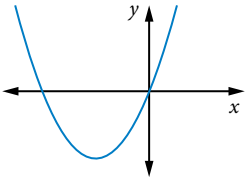
9 The graph of a section of the curve of $y = g(x)$ is shown. There is a local maximum at $x = -3$ and a local minimum at $x = 1$.



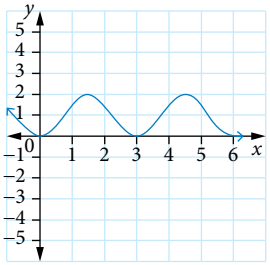
Which of the following correctly represents the relationship?

A $g(0) < g''(1) < g'(-3)$
B $g(0) < g'(-3) < g''(1)$
C $g''(1) < g(0) < g'(-3)$
D $g'(-3) < g(0) < g''(1)$

10 (2 marks) Given the graph of $f(x)$ below, sketch the graph of its derivative function.



11 © QCAA 2020 1Q16 (4 marks) CF Consider the following graph of $f(x)$.



Identify the graph of the second derivative $f''(x)$ from the graphs in Diagram 1, Diagram 2 and Diagram 3.

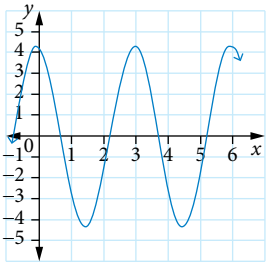


Diagram 1

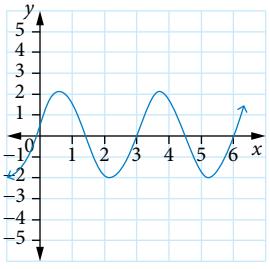


Diagram 2

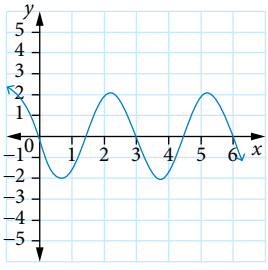


Diagram 3

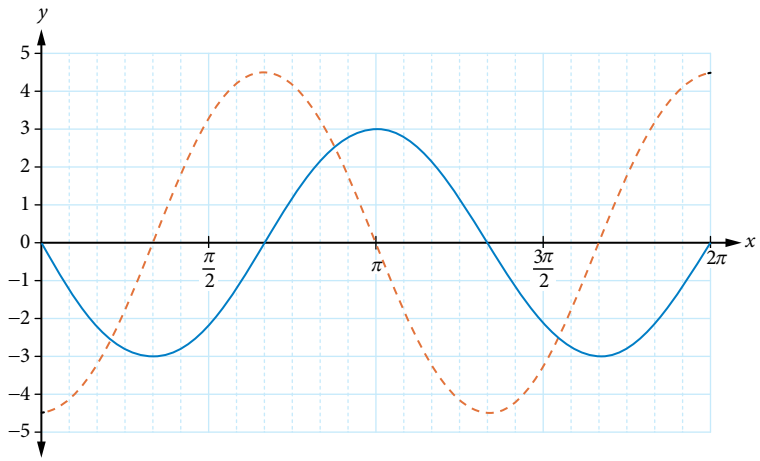
Justify your decisions using mathematical reasoning.

12 (4 marks) Sketch the graph of the function $g(x) = x^3 + 3x^2 + 2$ labelling the stationary points, point of inflection and y-intercept. Do NOT determine the x-intercepts of the curve.

13 © QCAA 2022 1Q16 (3 marks) CF

A section of the graphs of the first and second derivatives of a function are shown.

Sketch a possible graph of the function on the same axes over the domain $0 \leq x \leq 2\pi$.
Explain all reasoning used to produce the sketch.



14 (4 marks) CF Given the function $y = x^3 + 6x^2 - 5$, determine the global maximum and minimum values over the domain $-7 \leq x \leq 1$.

15 (5 marks) CF

a Use calculus to show why the function $f(x) = 1 - \sqrt[3]{x^2}$ has no stationary points. *[2 marks]*

b Sketch the graph of the function. *[3 marks]*

16 (7 marks) CF Sketch the graph of the function $f(x) = xe^{-2x}$ labelling all key points. Show full working to support your sketch.

17 (5 marks) CU A continuous function, g, satisfies the conditions below. The function has exactly two stationary points.

- $g(-1) = g'(-1) = g''(-1) = 0$
- $g\left(\frac{1}{2}\right) = 0, g''\left(\frac{1}{2}\right) > 0$
- $g(x)$ is decreasing when $x < -1$ and $-1 < x < \frac{1}{2}$

Using the information, draw a sketch of the function.

18 (5 marks) CU The graph of the function $y = a + bx + cx^2 + dx^3$ has a turning point at $(0, 0)$.
The curve slopes downwards at an angle of 45° to the x-axis as it passes through the point $(2, 1)$.
Find the value of a, b, c and d and hence sketch the curve labelling all key features.

3.5 Straight-line motion

Kinematics is the study of the motion of a particle in a straight-line using **displacement** (x or s), **velocity** (v) and **acceleration** (a).

Displacement, velocity and acceleration

For straight-line motion

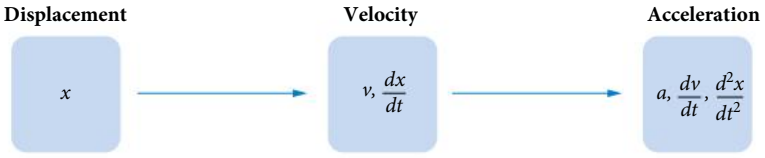
- the **position** of a particle is measured relative to a chosen origin
- **displacement**, $x(t)$ or $s(t)$, is the change in a position at time t from the chosen origin
- **velocity**, $v(t)$, is the rate of change of displacement with respect to time, so $v(t) = \frac{dx}{dt}$
- **acceleration**, $a(t)$, is the rate of change of velocity with respect to time, so $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.

Displacement is different from distance as it relates to the both distance and direction of the particle.

Displacement is a 'signed distance' so it can be positive or negative.

Velocity is a 'signed speed' so it can be positive or negative.

As velocity is the derivative of displacement, and acceleration is the derivative of velocity, then acceleration is the second derivative of displacement with respect to time.



WORKED EXAMPLE 13 Straight-line motion

The displacement position of a particle travelling in a straight line is given by $x(t) = 2t^3 - 3t^2 + 5$, where x is in metres and t is in seconds.

a Find an expression for $a(t)$.

b **Determine** when the acceleration of the particle will reach $30\,\text{m s}^{-2}$.

Steps	Working
a 1 Determine the first derivative, $v(t) = \frac{dx}{dt}$.	$x(t) = 2t^3 - 3t^2 + 5$ $v(t) = 6t^2 - 6t$
2 Determine the second derivative, $a(t) = \frac{dv}{dt}$.	$\therefore a(t) = 12t - 6$
b 1 Let $a = 30$ and solve for t.	$30 = 12t - 6$ $36 = 12t$ $t = 3$
2 State the result.	The acceleration of the particle will reach $30\,\text{m s}^{-2}$ after 3 seconds.

Both the behaviour of the functions for displacement, velocity and acceleration and the shapes of their graphs provide information about the motion of a particle.

Displacement, velocity and acceleration in a straight line

	Positive	Negative	Zero
Displacement	$x(t) > 0$: the particle is located to the right of (or above) the origin.	$x(t) < 0$: the particle is located to the left of (or below) the origin.	$x(t) = 0$: the particle is at the origin.
Velocity	$v(t) > 0$: the particle is moving to the right (or upwards).	$v(t) < 0$: the particle is moving to the left (or downwards).	$v(t) = 0$: the particle is not moving, or changing direction if the sign of $v(t)$ changes around this point.
Acceleration	$a(t) > 0$: the particle is accelerating to the right (or upwards).	$a(t) < 0$: the particle is accelerating to the left (or downwards).	$a(t) = 0$: the particle is not accelerating; its velocity is constant.

- When $v(t)$ and $a(t)$ have the **same sign** (both positive or both negative), then the particle is **increasing in speed**.
- When $v(t)$ and $a(t)$ have **different signs** (one positive, the other negative), then the particle is **decreasing in speed**.

WORKED EXAMPLE 14 Interpreting straight-line motion 1

The position of an object, x metres above the ground after t seconds, is given by $x(t) = t^3 - 6t^2 + 9t$ for $t > 0$.

a **Determine** the initial displacement of the object.

b At what time does the object change direction?

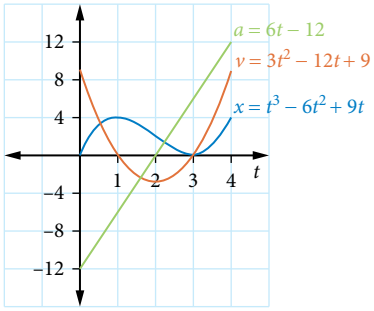
c **Describe** the motion of the object at $t = 2$.

d **Sketch** the graphs for displacement, velocity and acceleration of the object on the same set of axes for $0 \leq t \leq 4$.

e **Identify** when the object is increasing in speed (speeding up).

f **Identify** when the object is decreasing in speed (slowing down).

Steps	Working
a 1 Let $t = 0$ and solve for x.	$x(t) = 0^3 - 6(0)^2 + 9(0) = 0$
2 State the result.	The object is initially located at the origin.
b 1 Determine the first derivative, $v(t) = \frac{dx}{dt}$.	$x(t) = t^3 - 6t^2 + 9t$ $v(t) = 3t^2 - 12t + 9$
2 The object changes direction at the turning points of $x(t)$, which occur when $v(t) = 0$. Let $v(t) = 0$ and solve for t.	$0 = 3t^2 - 12t + 9$ $0 = 3(t^2 - 4t + 3)$ $0 = 3(t - 1)(t - 3)$ $t = 1$ or $t = 3$
3 Determine the second derivative, $a(t) = \frac{dv}{dt}$.	$\therefore a(t) = 6t - 12$
4 Use either the second derivative test or the sign test to determine the nature of the stationary points to verify the velocity changes sign at the point.	$a(1) = -6$. As $a(1) < 0$ it is a local maximum at $(1, 6)$. $a(3) = 6$. As $a(3) > 0$ it is a local minimum at $(3, 0)$. **Using the sign test** t: 0, 1, 2, 3, 4 v: 9, 0, −3, 0, 9 The gradient changes sign around both points.
5 State the result.	The velocity changes sign at $t = 1$ and $t = 3$, so the object changes direction at these points.

c 1 To describe the motion of the object, determine the displacement, velocity and acceleration when $t = 2$.	$x(2) = 2$ $v(2) = -3$ $a(2) = 0$
2 Interpret the results and write a statement.	At $t = 2$ the object is 2 metres above the origin. It is travelling downwards at a velocity of $3\,\text{m s}^{-1}$ and the acceleration of $0\,\text{m s}^{-2}$ is changing from down to up. As $a(2) = 0$, the object is neither slowing down nor speeding up..
d Draw the graphs of $x(t)$, $v(t)$ and $a(t)$ over the domain $0 \leq t \leq 4$.	
e The object is speeding up when $v(t)$ and $a(t)$ have the same sign. Identify when this occurs using the graph.	The particle is speeding up when $1 < t < 2$ and $3 < t < 4$.
f The object is slowing when $v(t)$ and $a(t)$ have different signs. Identify when this occurs using the graph.	The particle is slowing down when $0 < t < 1$ and $2 < t < 3$.

WORKED EXAMPLE 15 Interpreting straight-line motion 2

A rock is dropped into a tank of water, such that its height, h metres, after t seconds is given by $h(t) = 3 - 0.2t^2$.

a **Calculate** the height, velocity and acceleration of the rock at 2 seconds.

b **Interpret** the answer for the velocity and acceleration from part **a** in the context of the question.

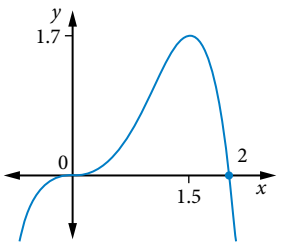
Steps	Working
a 1 Calculate $h(2)$.	$h(2) = 3 - 0.2(2)^2 = 2.2\,\text{m}$
2 Determine the first derivative, $v(t) = \frac{dx}{dt}$ and evaluate at $t = 2$.	$h(t) = 3 - 0.2t^2$ $v(t) = -0.4t$ $v(2) = -0.4(2) = -0.8\,\text{m s}^{-1}$
3 Determine the second derivative, $a(t) = \frac{dv}{dt}$ and evaluate at $t = 2$.	$a(t) = -0.4$ $\therefore a(2) = -0.4\,\text{m s}^{-2}$
b Comment on the results.	The rock is moving downwards at a speed of $0.8\,\text{m s}^{-1}$. It is moving at a constant acceleration of $0.4\,\text{m s}^{-2}$. The rock is increasing in speed.

EXERCISE 3.5 Straight-line motion

ANSWERS p. 399

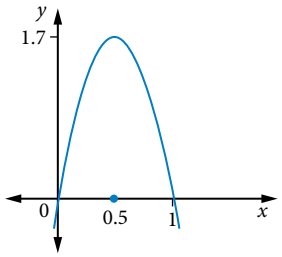
Recap

1 The graph of the function $y = f(x)$ is shown.

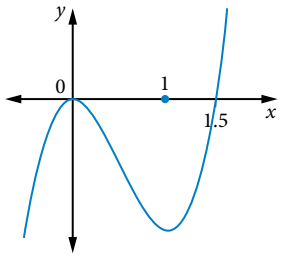


Which graph could be the derivative function $y = f'(x)$?

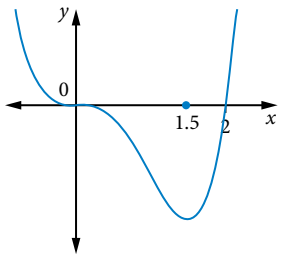
A



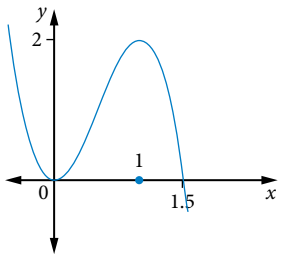
B



C



D



2 Sketch the graph of the function $g(x) = 3x - x^3 - 2$ labelling all key features.

Mastery

3 WORKED EXAMPLE 13 The position of an object is given by $x(t) = t^3 - 4t^2 + 11t - 7$, where t is time in seconds and x is in metres.

- **a** Find an expression for the acceleration of the object.
- **b** What is the initial acceleration of the object?
- **c** What is the acceleration of the object after 3 s?
- **d** When will the acceleration reach 20 m s^{-2}?

4 An object is projected vertically upwards from a platform. The approximate position of the object in metres is given by $s(t) = 17t - 5t^2 + 40$, where t is time in seconds.

- **a** How high above the ground is the platform?
- **b** Find an expression for the velocity and acceleration of the object at time t.
- **c** Explain what the expression for the acceleration means.
- **d** After how long does the object begin falling back towards the ground?

5 The position of an object is given by $x(t) = \dfrac{2t+3}{4t-5}$, where x is in metres and t is in seconds. What is the acceleration after 3 s?

6 The position of a particle is given by $x(t) = \sin(2t) - 2\sin(t)$, where x is in centimetres and t is in seconds. What is the acceleration after 2 s?

7 WORKED EXAMPLE 14 The displacement of a particle is given by $x(t) = \frac{t^3}{3} - \frac{5t^2}{2} + 4t$ for $t > 0$, where t is in seconds and x is in metres.

a Find expressions for the velocity (v) and acceleration (a) of the particle at time t.

b What is the particle's velocity and acceleration when $t = 3$?

c Plot the graphs of x, v and a for $0 \leq t \leq 5$ on the same set of axes.

d Between which times is the particle slowing down?

e Between which times is the particle speeding up?

8 WORKED EXAMPLE 15 A ball is thrown into the air such that its height, h metres, after t seconds is given by $h(t) = 3t - t^3 + 2t^2$ for $0 \leq t \leq 3$.

a Find the height, velocity and acceleration of the ball when $t = 2$ seconds.

b Interpret the answer from part **a** for velocity and acceleration in the context of the question.

Exam practice

9 © QCAA 2020S 2Q8 A particle travels along a straight line with a velocity of $v(t) = 3e^{\left(\frac{-t}{2}\right)} \times \sin(2t)\,\text{m s}^{-1}$. Determine the acceleration of the particle at $t = 2$ seconds.

A $-1.03\,\text{m s}^{-2}$ **B** $-0.01\,\text{m s}^{-2}$ **C** $2.49\,\text{m s}^{-2}$ **D** $3.51\,\text{m s}^{-2}$

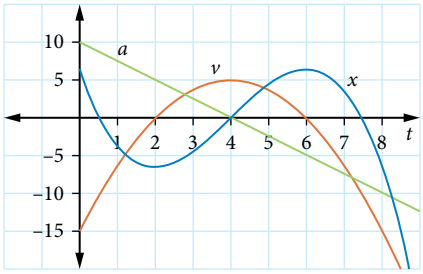
10 © QCAA 2020 2Q9 The displacement of a particle (in metres) at time t (in seconds) is represented by the function

$$s(t) = t\ln(t),\ 0 < t < 4$$

Determine the approximate acceleration of the particle at time $t = 3$.

A $-0.66\,\text{m s}^{-2}$ **B** $-0.33\,\text{m s}^{-2}$ **C** $0.33\,\text{m s}^{-2}$ **D** $0.66\,\text{m s}^{-2}$

11 TECH-FREE (3 marks) The diagram shows graphs of the position (x), velocity (v) and acceleration (a) of an object moving in a straight line.



a At what times does the direction of the motion of the object change? *[1 mark]*

b Between which times is the particle speeding up? *[1 mark]*

c Between which times is the particle slowing down? *[1 mark]*

12 TECH-FREE (5 marks) A particle is moving such that its displacement is given by $s(t) = t^2 - 2t + 1$, where s is in metres and t is in seconds.

a Show that its acceleration is constant and find its value. *[2 marks]*

b Determine when the particle will be at rest. *[2 marks]*

c What will the particle's displacement be at that time? *[1 mark]*

Exam hack

If the object is at rest, then it is not moving, therefore its velocity is zero.

13 (6 marks) The position of a particle is given by $x(t) = \sin(t) - 0.5t$ for $0 \leq t \leq 2\pi$, where t is in seconds and x is in metres.

- **a** What is the particle's velocity and acceleration (correct to 2 decimal places) when $t = 4$? *[2 marks]*
- **b** Plot the graphs of x, v and a for $0 \leq t \leq 2\pi$ on the same set of axes. *[2 marks]*
- **c** Between which times is the particle slowing down? *[1 mark]*
- **d** Between which times is the particle speeding up? *[1 mark]*

14 (2 marks) The displacement of a windup toy truck is given by $x(t) = 12xe^{-x}$. Determine its acceleration after travelling for 3 seconds. Round the answer correct to 3 decimal places.

15 (3 marks) CF The equation of the displacement of a particle moving in a straight line is given by $s(t) = 4t^4 - 6t^3 + 5t$, where s is measured in metres and t in seconds. Determine the displacement when the acceleration is 0.

16 (5 marks) CF The displacement position of an object is given by $s(t) = pt^2 + qt + r$, where p, q and r are constants and displacement is in metres and time in seconds.

Given that $a(4) = -1$, $v(4) = -11$ and $s(4) = -21$, determine the value of p, q and r.

3.6 Optimisation

Many practical problems involve finding either maximum or minimum quantities, such as minimising costs or maximising profits in a business. The goal in these situations is to obtain the best (optimal) solution to the problem. The process of determining the maximum or minimum quantity is called **optimisation**.

Investigation Sideshow alley costs and considerations

Luke is the owner–operator of a carnival ride that he sets up in country and city shows such as the Stanthorpe show and the Brisbane Ekka.

The ride has the capacity to take up to 90 people, but Luke knows from experience that he only gets about a third of the ride's capacity when he charges his normal price of $15 for a ride.

He also knows that each time he runs the ride, it costs the same amount regardless of how many people have bought tickets. When Luke reduces the price of the ride, he attracts more customers. He wants to make the ride more profitable and has an idea that if he gets the price right, he can achieve this. He has learned that charging too little means that he loses money.

Recently, Luke has noticed that rides costing only $10 are usually full, and he is sure that his ride would be full every time if he dropped his price to that figure.

Assuming that the number of customers will increase in direct proportion to the price drop, decide what would be the most profitable price at which to offer his ride.

How much better off would Luke be for each ride compared with the takings at the fare of $15?

Investigate other methods of approaching this problem by considering different assumptions.

A common approach to solving optimisation problems is to use differentiation.

Although there is no hard and fast way to solve situations involving optimisation, following a planned process is important.

Worksheets
Greatest and least values
Optimisation
Optimisation – cost and revenue
Optimisation – area and volume
Optimisation – mixed problems
Applications of optimisation
Optimisation problems
Further optimisation problems
Starting maxima and minima problems
Second derivative assignment
Second derivative problems

Steps for solving optimisation problems

1. Read the question, and take note of words such as maximum/minimum, least/highest, largest/smallest.
2. Draw a sketch or diagram (where appropriate) and label it using as few variables as possible.
3. Identify or set up a function to describe what needs to be maximised or minimised. Make sure you communicate what each variable represents.
4. Make sure the quantity to be maximised or minimised is in terms of one variable only.
5. Find stationary points by differentiating the function and solving $f'(x) = 0$.
6. Determine the nature of the stationary point(s). This can be done either by
   - using the second derivative or
   - testing the gradient on either side of the stationary point.
7. If the domain is restricted, check the endpoints.
8. Identify the optimal solution.
9. Re-read the question to make sure you have answered what is asked and that your solution is reasonable.

WORKED EXAMPLE 16 Optimisation problems 1

A small manufacturing company can sell all the items it produces at a price of \$8 per item. The production cost function for producing n items per week (in dollars) is given by

$$C(n) = 100000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3$$

a What should the weekly production target be to maximise profits?

profit $P(n)$ = revenue (price (p) × number of items sold (n)) – cost

b What is the maximum profit that can be made?

Steps	Working
a 1 Write the profit function and simplify.	$P(n) = 8n - (100\,000 + 8n - 0.0156n^2 + 2 \times 10^{-6}n^3)$ $= -100\,000 + 0.0156n^2 - 2 \times 10^{-6}n^3$
2 Determine the first derivative.	$P'(n) = 0.0312n - 6 \times 10^{-6}n^2$
3 Solve $P'(n) = 0$ for the stationary points.	$0 = 0.0312n - 6 \times 10^{-6}n^2$ $0 = n$ or $0 = 0.0312 - 6 \times 10^{-6}n$ $n = 0$ or $n = 5200$
4 Identify if both points are reasonable.	The stationary point at $n = 0$ is not reasonable, so the stationary point occurs when $n = 5200$.
5 Determine the second derivative.	$P''(n) = 0.0312 - 12 \times 10^{-6}n$
6 Find $P''(5200)$ and use the second derivative test to determine the nature of the stationary point.	$P''(5200) = -0.0312$ As $P''(5200) < 0$ it is a maximum.
7 State the result.	The maximum profit is made when 5200 items are produced per week.
b 1 Calculate the profit, $P(5200)$.	$P(5200) = -100\,000 + 0.0156(5200)^2 - 2 \times 10^{-6}(5200)^3$ $= 40\,608$
2 State the result.	The maximum profit of \$40 608 can be made when 5200 items are produced weekly.

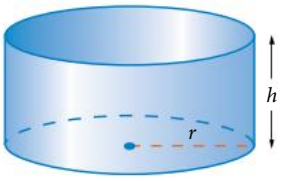
When solving problems involving area and volume, take care when drawing the diagram.

Exam hack

Make sure you are familiar with all perimeter, area, surface area and volume formulas in the QCAA Formula book.

WORKED EXAMPLE 17 Optimisation problems 2

A cylindrical metal can is open at the top. It must be designed to have a capacity of 1078 cm^3. Because the outside of the can must be galvanised, the manufacturer wants the surface area of the can to be minimised to save money. **Determine** the optimal dimensions of the can.

Steps	Working
1 Sketch a diagram and state key information.	$V = 1078\text{ cm}^3$ 
2 Set up an equation for the surface area of the can.	$S = \pi r^2 + 2\pi rh$
3 As the equation has two unknowns (r and h), express one variable in terms of the other.	$V = \pi r^2 h$ $1078 = \pi r^2 h$ $h = \dfrac{1078}{\pi r^2}$
Usually the 'extra' number in the question gives a way to eliminate a variable. In this case $V = 1078$ gives a substitution for h or r.	
It doesn't matter which variable you use, but using r will involve a square root, so use h.	
4 Substitute for h in the equation for surface area and simplify.	$S = \pi r^2 + 2\pi r\dfrac{1078}{\pi r^2}$ $= \pi r^2 + \dfrac{2156}{r}$
5 Determine the first derivative.	$S' = 2\pi r - \dfrac{2156}{r^2}$
6 Solve $S' = 0$ for the stationary points.	$0 = 2\pi r - \dfrac{2156}{r^2}$ $0 = \dfrac{2\pi r^3 - 2156}{r^2}$ $0 = 2\pi r^3 - 2156$ $r^3 = 343.138\ldots$ $r \approx 7\text{ cm}$
7 Determine the second derivative.	$S'' = 2\pi + \dfrac{4312}{r^3}$
8 Find $S''(7)$ and use the second derivative test to determine the nature of the stationary point.	$S''(7) = 18.849\ldots$ As $S''(7) > 0$ it is a minimum.
9 Calculate h.	$h = \dfrac{1078}{\pi(7)^2}$ $\approx 7\text{ cm}$
10 State the result.	A 1078 cm^3 cylinder with minimal surface area will have a radius and height of about 7 cm.

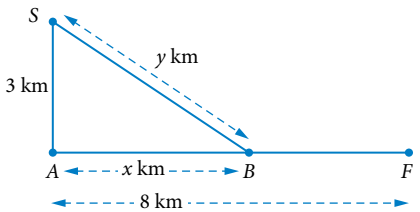
Exam hack

When looking for connections between variables in practical situations, consider using Pythagoras' theorem, trigonometry or similar figures, as they often have a way of linking things.

WORKED EXAMPLE 18 Solving optimisation problems

Sarah is a competitor in a race. She is on a paddle board 3 km from the nearest point A on a straight beach. The finish line (her destination) is 8 km along the beach from A. Sarah knows that in the current conditions she can paddle at $4\,\text{km h}^{-1}$ and jog along the beach at $5\,\text{km h}^{-1}$.

Determine the point on the beach between A and the finish line towards which Sarah should paddle to reach the finish line in the least possible time.

Steps	Working
1 Sketch a diagram.	
2 Identify any conditions, restrictions or assumptions.	Sarah (S) will paddle to point B located between points A and F (the finish line). $\therefore 0 \le x \le 8$
3 Express y in terms of x by using Pythagoras' theorem.	$y^2 = x^2 + 9$ $y = \sqrt{x^2 + 9}$
4 Set up an equation for the time to reach the finish line.	$T(x) = \frac{y}{4} + \frac{BF}{5}$ $T(x) = \frac{\sqrt{x^2+9}}{4} + \frac{8-x}{5}$
Remember: speed $= \frac{\text{distance}}{\text{time}}$, so time $= \frac{\text{distance}}{\text{speed}}$.	
5 Determine the first derivative. Use the chain rule and simplify.	$T'(x) = \frac{\frac{1}{2}(2x)\left(x^2+9\right)^{-\frac{1}{2}}}{4} - \frac{1}{5}$ $T'(x) = \frac{x}{4\sqrt{x^2+9}} - \frac{1}{5}$
6 Solve $T'(x) = 0$ for the stationary points.	$0 = \frac{x}{4\sqrt{x^2+9}} - \frac{1}{5}$ $0 = \frac{5x - 4\sqrt{x^2+9}}{20\sqrt{x^2+9}}$ $0 = 5x - 4\sqrt{x^2+9}$ $\frac{5x}{4} = \sqrt{x^2+9}$ $\frac{25x^2}{16} = x^2 + 9$ $25x^2 = 16x^2 + 144$ $9x^2 = 144$ $x = \pm 4$, ignore the negative distance as it is outside the domain ($0 \le x \le 8$).

7 Determine the nature of the stationary point.

Given the complexity of $T'(x)$ use the sign test rather than the second derivative test to verify the nature.

x	3	4	5
$T'(x)$	−0.023	0	0.014

The sign changes around $x = 4$, so a minimum point exists.

8 Calculate the minimum time taken when $x = 4$.

$$T(4) = \frac{\sqrt{4^2 + 9}}{4} + \frac{8 - 4}{5}$$

$$T(4) = \frac{41}{20} \approx 2.05 \text{ hours}$$

9 Determine the distances Sarah will paddle and jog.

Paddle: $y(4) = \sqrt{x^2 + 9} = 5\,\text{km}$

Jog: $BF(4) = 8 - x = 4\,\text{km}$

10 State the answer.

Sarah can reach the finish line in the least amount of time if she paddles 5 km to point B, which is 4 km along the beach from point A in the direction of the finish line, and then jogs 4 km.

Exam hack

After you solve an optimisation problem by finding the minimum or maximum, re-read the question to make sure that you have answered it. A very common mistake is to forget to answer the actual question, even though all the working is correct. Ask yourself, what value do I actually need to complete this question?

EXERCISE 3.6 Optimisation

ANSWERS p. 400

Recap

1 The displacement of a particle (in metres) at time t (in seconds) is represented by the function

$$s(t) = \frac{t + 3}{2t + 1}$$

Determine the approximate acceleration of the particle at time $t = 2$.

A $-0.375\,\text{m s}^{-2}$ **B** $-0.4\,\text{m s}^{-2}$ **C** $0.375\,\text{m s}^{-2}$ **D** $0.16\,\text{m s}^{-2}$

2 If the displacement of an object is given by $x(t) = 4(t - 1)(t + 2)(t + 3)$ metres, where t is time in seconds, determine the function for the acceleration of the object.

Mastery

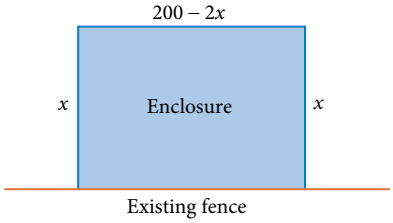
3 WORKED EXAMPLE 16 A company sells all the units it produces at \$4 per unit. The total cost, C, for producing x units is given in dollars by

$$C(x) = 50 + 1.3x + 0.001x^2$$

a Write the profit function.

b Determine the number of units that should be sold so that the profit is a maximum.

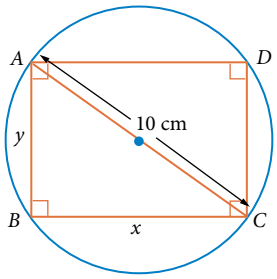
4 WORKED EXAMPLE 17 A rectangular area is enclosed using 200 m of movable fencing and an existing fence as one side of the enclosure, as shown in the diagram.



a Write an expression for the area of the enclosure, A.

b Determine the maximum area that can be enclosed using the movable fencing.

5 WORKED EXAMPLE 18 A rectangle is drawn so that it is enclosed in a circle of diameter 10 cm as shown.



a Write an expression for the area of the rectangle in terms of x only.

b Find the area of the largest rectangle that can be enclosed by the circle.

6 A printer has a contract to print 200 000 copies of a small catalogue. It costs \$8 per hour to run the printing press, which will produce 1200 impressions per hour. Each impression prints x catalogues, x being the number of plates (metal copies of the set type) that are used on the press. Each plate costs \$2.

a Write an equation for h, the number of hours for which each press must work.

b The total cost of production is the sum of the running costs of the press and the cost of making plates. Write a function for the total cost of production.

c How many plates should be made so that the job is done most economically?

7 An electric train uses power at the rate of $250\,000 + x^3$ (J h^{-1}, joules per hour), where x is the speed of the train in km h^{-1} over the speed range of 20–80 km h^{-1}.

a If the distance travelled is d, write an equation for the time taken, t, in hours.

b Write the power consumption, P, as a function of x.

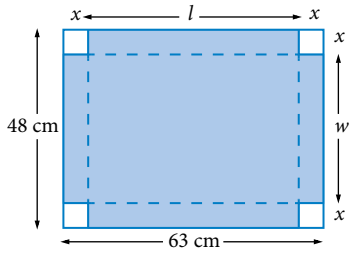
c Find the speed at which the train should travel to minimise the use of electricity.

8 The sum of 2 non-negative numbers is 12. Determine the value of the 2 numbers for which their product is a maximum.

9 The slant edge of a right rectangular cone is 6 cm in length. Determine the height of the cone when the volume is a maximum.

Exam practice

10 A rectangular piece of cardboard 48 cm wide and 63 cm long has squares of length x cm cut out of each corner as shown. The remaining cardboard is folded up to make a rectangular box with an open top.



The volume of the box is a maximum when x is equal to

A 9 **B** 10 **C** 28 **D** 30

11 TECH-FREE (6 marks) The amount of a particular medicine in a person's bloodstream t hours after it has been taken is given by $A = 2t^2 - t^3$, for $0 \leq t \leq 2$. Determine the time when the amount of medicine in the person's bloodstream is at its maximum.

12 (6 marks) The Silverado Company produces x tonnes of silver per month at a total cost of C dollars, given by

$$C(x) = 2000 - 75x - 5x^2 + \frac{x^3}{3}$$

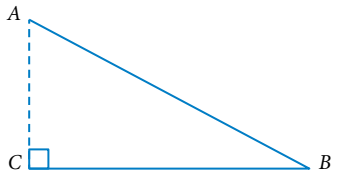
Determine the production level for which costs are minimised. Use calculus techniques to verify that your production level corresponds to minimised costs.

13 © QCAA 2023 1Q17 TECH-FREE (6 marks) CF A chemical is added to the water in a swimming pool at 10:00 am to prevent algae. The amount of chemical absorbed into the water over time t (hours) is represented by

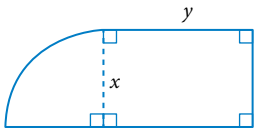
$$A = 10t^2 - 4t^3, 0 \leq t \leq 1\frac{2}{3}$$

Determine the time of day when the rate of absorption of the chemical is at its maximum. Use calculus techniques to verify that your time corresponds to a maximum rate.

14 (7 marks) CF A piece of wire 6 metres long is to be bent to form the base and hypotenuse of a right-angled triangle ABC, as shown. Using calculus techniques, determine the maximum possible area of the triangle.



15 (7 marks) CF A pool is to be built in the shape of a rectangle and a quarter of a circle, as shown. The pool must have an area of 49 m^2 and the smallest possible perimeter (P).

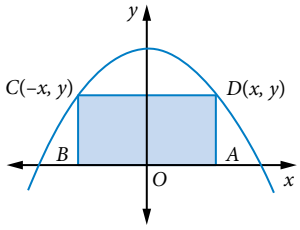


a Show that the perimeter of the pool is given by $P(x) = \frac{98}{x} + 2x$. *[3 marks]*

b Determine the smallest possible perimeter for the pool. Use calculus techniques to verify that your perimeter is the minimum possible. *[4 marks]*

16 (7 marks) CU What are the dimensions of the cone of greatest volume that can just fit inside a sphere of radius 2 m? Justify your answer using calculus techniques.

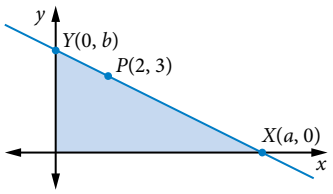
17 TECH-FREE (6 marks) CU Rectangle $ABCD$ has vertices A and B on the x-axis and vertices C and D on the graph of $y = 9 - 3x^2$. The coordinates of D are (x, y), where x and y are positive real numbers; the coordinates of C are $(-x, y)$. Determine the maximum area of rectangle $ABCD$ and the value for x for which this occurs.



18 TECH-FREE (6 marks) CU P is the point on the line $x + y - 4 = 0$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P and this minimum length.

19 (7 marks) CU A straight line passes through the points $P(2, 3)$, $X(a, 0)$ and $Y(0, b)$, where $a > 2$.

By first showing that $b = \dfrac{3a}{a-2}$, determine the minimum area of the triangle formed by the line and the coordinates axes.



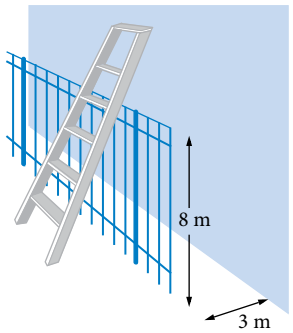
20 (10 marks) CU A remote village, V, is located on a small island. 3 km due west of a point, M, on the coast of the mainland. A data hub, D, is located on the mainland 5 km due north of M. The shoreline of the mainland runs directly north from M to D.

A fibre-optic cable needs to be laid to provide high-speed internet to the village. It costs \$1000 per kilometre to lay the cable along the shore and \$2600 per kilometre to lay the cable underwater. The cable can only be laid in straight sections.

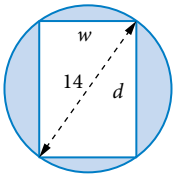
a Determine the minimum cost to lay the cable from the data centre to the village. Justify your answer using calculus techniques. *[7 marks]*

b A different form of cable is discovered that means that the cost of laying the cable underwater can be reduced to \$1100 per kilometre. Determine the path for laying the cable to minimise the cost in this case. *[3 marks]*

21 (7 marks) CU An 8 m high fence stands 3 m from a large vertical wall. Find the length of the shortest ladder that will reach over the fence to the wall behind, as shown in the diagram.



22 (7 marks) CU A steel company constructs rectangular beams. The beams will be cut from a cylindrical steel rod. The diameter of the cross-section of the steel rod (which is also the diagonal of the cross-section of the beam) is 14 cm. The strength, S, of the beam is proportional to the product of the width of the beam (w cm) and the square of its depth (d cm), and is given by $S = kwd^2$, where k is a constant of proportionality.



a By first showing that $S = k(196 - w^2)$, determine the dimensions (width and depth) of the steel beam that will give it the greatest strength.

b A proposal to modify the beam to be square in shape but still with a diagonal length of 14 cm has been rejected. Verify whether this was the correct decision by determining if the rectangular beam is at least 8% stronger than the square beam.

23 (9 marks) CU A city's electricity consumption and renewable energy generation changes throughout the year, influenced by seasonal weather patterns.

The city's electricity consumption can be modelled by the function $C(t) = a\sin\left(\frac{t\pi}{6}\right) + b$,

where t is the number of months and $0 \leq t \leq 12$. The city's electricity consumption reaches a maximum of 10 000 kWh when $t = 3$ and a minimum of 6000 kWh when $t = 9$.

The city's renewable energy generation can be modelled by

$$G(t) = -400\cos\left(\frac{(t-1)\pi}{6}\right) + 5000.$$

a Determine the values of a and b in the electricity consumption function $C(t)$. *[2 marks]*

b Determine when the city's electricity consumption and renewable energy generation are both decreasing. *[2 marks]*

c Determine the rate of change in electricity consumption when the renewable energy generation reaches its peak. *[5 marks]*

Video
Exam question analysis: Further applications of differentiation

EXAM QUESTION ANALYSIS

© QCAA 2023 1Q18 TECH-FREE (4 marks)

A person enters the lowest carriage of a miniature Ferris wheel with a six-metre diameter. The bottom carriage is one metre off the ground. When top speed is reached, it takes three seconds for a carriage to travel from the lowest to the highest point of the ride. It is claimed that:

> *The vertical motion of the Ferris wheel produces a maximum vertical acceleration on each rider that is more than half the acceleration of free fall.*

Free fall occurs when gravity is the only force acting, resulting in an acceleration of $9.8\,\text{m}\,\text{s}^{-2}$.

Evaluate the reasonableness of the claim.

Reading the question

- The motion will be a periodic model, starting at the minimum position.
- The diameter is 6 m and the period is 6 seconds.
- Start 1 metre off the ground.

Thinking about the question

- Find the periodic model: cos as when $t = 0$, the graph will be at a minimum.
- Find the amplitude, period and phase shifts.
- Take into account the centre of motion.
- Find the velocity and acceleration functions.
- Find the maximum acceleration.
- Evaluate the claim.

Worked solution (✓ = 1 mark)

As the motion is circular in nature, it can be modelled by a periodic model. ✓

At $t = 0$, the Ferris wheel is at its lowest point: $\therefore$ use a negative cosine function.

$h(t) = a\cos(b(t+c)) + d$

Diameter = 6 m, so amplitude = 3: $\therefore a = -3$

Period = 6 seconds: $T = \dfrac{2\pi}{b}$

$$b = \frac{2\pi}{6}$$

$$\therefore b = \frac{\pi}{3}$$

There is no phase shift: $\therefore c = 0$

The centre of motion is at the height of the axis of the Ferris wheel, which is 4 m above the ground:
$\therefore d = 4$

$$\therefore \boldsymbol{h(t) = -3\cos\left(\frac{\pi}{3}t\right) + 4} \quad ✓$$

$$v(t) = 3 \times \frac{\pi}{3}\sin\left(\frac{\pi}{3}t\right)$$

$$\therefore v(t) = \pi\sin\left(\frac{\pi}{3}t\right)$$

$$a(t) = \pi \times \frac{\pi}{3}\cos\left(\frac{\pi}{3}t\right)$$

$$\therefore a(t) = \frac{\pi^2}{3}\cos\left(\frac{\pi}{3}t\right)$$

The amplitude of $a(t)$ is $\frac{\pi^2}{3}$, so the maximum vertical acceleration is $\frac{\pi^2}{3}$.

Maximum acceleration $= \frac{\pi^2}{3} \approx \mathbf{3.2898}\ldots \text{ m s}^{-2}$ ✓

Half the acceleration of free fall $= \frac{1}{2} \times 9.8 \text{ m s}^{-2}$

$= 4.9 \text{ m s}^{-2}$

$3.2898\ldots < 4.9$

The **claim is not reasonable** as the maximum acceleration of the Ferris wheel (3.3 m s^{-2}) is less than half the acceleration of free fall. ✓

Puzzle
Second derivative find-a-word

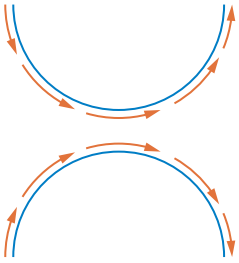
3 Chapter summary

The second derivative

- The second derivative is the rate of change of the first derivative function.
- For the function $y = f(x)$:
  - the **first derivative** $= y' = f'(x) = \dfrac{dy}{dx}$
  - the **second derivative** $= y'' = f''(x) = \dfrac{d^2y}{dx^2}$

Concavity

- Where $f''(x) > 0$, the gradient of the function $f(x)$ is increasing, and the graph is **concave up**.
- Where $f''(x) < 0$, the gradient of the function $f(x)$ is decreasing, and the graph is **concave down**.



Points of inflection

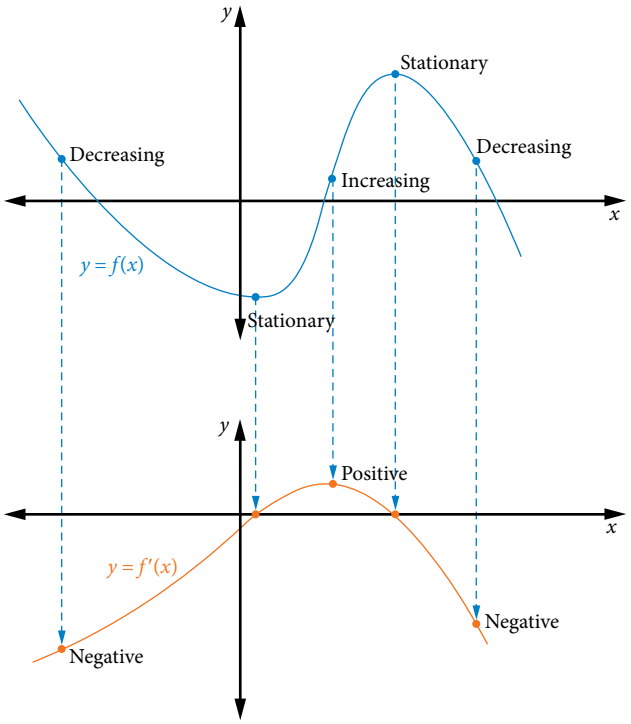
- A **point of inflection** on the graph of $f(x)$ is a point where the concavity changes and $f''(x) = 0$.
- $f''(x)$ has different signs on either side of a point of inflection.
- A **horizontal or** (stationary point) **of inflection** is located where $f'(x) = 0$ and $f''(x) = 0$.
- A **non-horizontal point of inflection** is located where $f'(x) \neq 0$ and $f''(x) = 0$.

Stationary points and the second derivative test

- A **turning point** (**local maximum** or **minimum** point) is found where $f'(x) = 0$ and also where the sign of the gradient changes on either side of the stationary point.
- A **horizontal point of inflection** is found where $f'(x) = 0$ and the concavity changes. The sign of the gradient stays the same on both sides of the stationary point.
- The **second derivative test**: For the function $y = f(x)$, if $f'(x) = 0$ and
  - if $f''(x) > 0$, the turning point is a minimum.
  - if $f''(x) < 0$, the turning point is a maximum.
- The **sign test** (also called the **first derivative test**) can be used to identify the nature of a stationary point.

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	$f''(x)$ at the stationary point	Type of stationary point
+	0	−	$f''(x) < 0$	Maximum
−	0	+	$f''(x) > 0$	Minimum
+	0	+	$f''(x) = 0$	Horizontal point of inflection
−	0	−	$f''(x) = 0$	Horizontal point of inflection

Graph of the derivative function



	increasing		decreasing		Stationary point		
Graph of $f(x)$					Minimum	Maximum	Point of inflection or
Graph of $f'(x)$	Positive ($f'(x) > 0$), above the *x-axis*		Negative ($f'(x) < 0$)), below the *x-axis*		x-intercept $\left(\frac{dy}{dx} = 0\right)$		
$f''(x)$ value	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) = 0$ (concavity changes)

Curve sketching

1 **Stationary points:** Use the first derivative to locate stationary points. The first derivative can also be used to determine the intervals in which the function is increasing or decreasing.

2 **Nature of stationary points:** Use the second derivative to identify the nature of the stationary points. The second derivative can also be used to determine the intervals in which the function is concave upwards or downwards.

3 **Points of inflection:** Use the second derivative to identify any points of inflection ($f''(x) = 0$).

4 **Intercepts:** Determine the x- and y-intercepts. Find the x-intercept(s) by solving $f(x) = 0$, if possible. Find the y-intercept by evaluating $f(0)$.

5 **Discontinuities** and other special features: Determine the behaviour at points where $f(x)$ is undefined, particularly to find whether there is a vertical asymptote.

> You do not need to check for discontinuities for polynomial functions as they are defined for all values of x.

6 **Endpoints:** Determine the behaviour of the function as $x \to \pm\infty$. Consider if the function has a limit as $x \to \pm\infty$. This is a horizontal asymptote.

7 **Sketch:** Draw the function, labelling all important features.

Displacement, velocity and acceleration

- **Displacement**, $x(t)$, is the change in a position at time t from the chosen origin.
- **Velocity**, $v(t)$, is the rate of change of displacement with respect to time, so $v(t) = \frac{dx}{dt}$.
- **Acceleration**, $a(t)$, is the rate of change of velocity with respect to time, so $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$.
- As velocity is the derivative of displacement, and acceleration is the derivative of velocity, then acceleration is the second derivative of displacement with respect to time.

	Positive	Negative	Zero
Displacement	$x(t) > 0$: the particle is located to the right of (or above) the origin.	$x(t) < 0$: the particle is located to the left of (or below) the origin.	$x(t) = 0$: the particle is at the origin.
Velocity	$v(t) > 0$: the particle is moving to the right (or upwards).	$v(t) < 0$: the particle is moving to the left (or downwards).	$v(t) = 0$: the particle is not moving, or changing direction if the sign of $v(t)$ changes around this point.
Acceleration	$a(t) > 0$: the particle is accelerating to the right (or upwards).	$a(t) < 0$: the particle is accelerating to the left (or downwards).	$a(t) = 0$: the particle is not accelerating; its velocity is constant.

- When $v(t)$ and $a(t)$ have the **same sign** (both positive or both negative), then the particle is **increasing in speed**.
- When $v(t)$ and $a(t)$ have **different signs** (one positive, the other negative), then the particle is **decreasing in speed**.

Optimisation problems

1. Read the question, take note of words such as maximum/minimum, least/highest, largest/smallest.
2. Draw a sketch or diagram (where appropriate) and label it using as few variables as possible.
3. Identify or set up a function to describe what needs to be maximised or minimised. Make sure you communicate what each variable represents.
4. Make sure the quantity to be maximised or minimised is in terms of one variable only.
5. Find stationary points by differentiating the function and solving $f'(x) = 0$.
6. Determine the nature of the stationary point(s). This can be done either by
   - using the second derivative, or
   - testing the gradient on either side of the stationary point.
7. If the domain is restricted, check the endpoints.
8. Identify the optimal solution.
9. Re-read the question to make sure you have answered what is asked and that your solution is reasonable.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

[TECH-FREE] *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

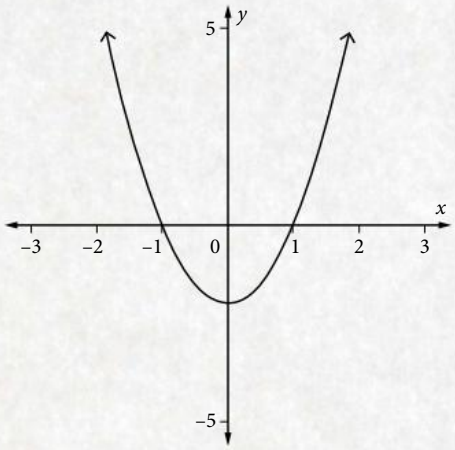
Total: 18 marks

Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

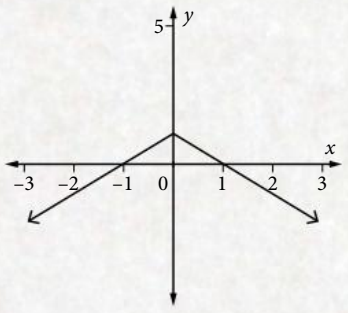
3 marks

1 © QCAA 2021 1Q5 The graph of $f''(x)$ is shown.

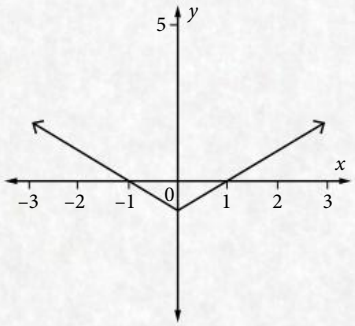


Which of the following could be the graph of $f'(x)$?

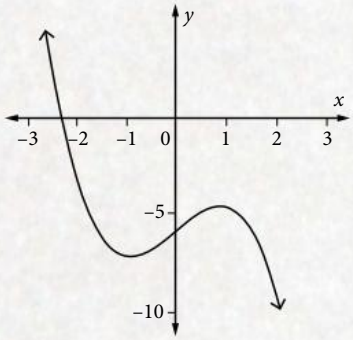
A



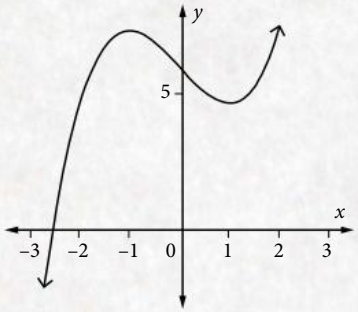
B



C



D



2 © QCAA 2020S 1Q3 MODIFIED If $y = \sin^2(4x)$, then $\frac{dy}{dx} =$

A $8\sin(4x)\cos(4x)$ **B** $4\sin(4x)\cos(4x)$

C $-8\sin(4x)\cos(4x)$ **D** $-4\sin(4x)\cos(4x)$

3 © QCAA 2021 1Q4 The second derivative of the function $f(x)$ is given by $f''(x) = \frac{2x}{1+x^2}$.
The interval on which the graph of $f(x)$ is concave up is

A $x < 0$ **B** $x \leq 0$ **C** $x > 0$ **D** $x \geq 0$

Section 2 3 short response questions

15 marks

4 © QCAA 2022 1Q13ab MODIFIED (4 marks)

a Determine the derivative of $f(x) = 3e^{2x+1}$. *[1 mark]*

b Given that $g(x) = \frac{\ln x}{x}$, determine the simplest value of $g'(e)$. *[3 marks]*

5 © QCAA 2020S 1Q15 MODIFIED (6 marks) Consider the function: $f(x) = x^2e^{(-x+4)}$.

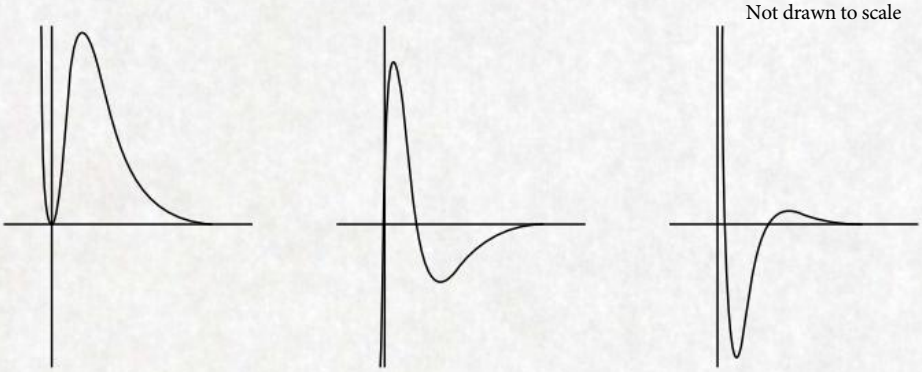
The first and second derivatives are:

$f'(x) = x(2-x)e^{(-x+4)}$

$f''(x) = (x^2 - 4x + 2)e^{(-x+4)}$

The graphs of $f(x)$, $f'(x)$ and $f''(x)$ are shown below:

Not drawn to scale



$f(x)$ $f'(x)$ $f''(x)$

a Determine the coordinates of the local maximum of $f(x)$. *[2 marks]*

b Use the second derivative test to verify the nature of the stationary point from part **a**. *[2 marks]*

c Determine the x-coordinates of any point/s of inflection. *[2 marks]*

6 © QCAA 2020S 1Q16 (5 marks) Determine the equation of the tangent to the function $y = x^3 + ax^2 + bx + 1$ at the point of inflection $(1, 6)$.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2021 2Q8 The displacement (in metres) of a particle is given by $s(t) = -3\cos(t) + 2\sin(t)$, where t is in seconds.

The instantaneous velocity of the particle at time $t = \frac{\pi}{2}$ seconds is

A $-3\,\text{m s}^{-1}$ **B** $-2\,\text{m s}^{-1}$ **C** $2\,\text{m s}^{-1}$ **D** $3\,\text{m s}^{-1}$

2 © QCAA 2022 2Q2 Identify the correct features of the function $f(x) = xe^x$.

A $f'(-1) = 0,\ f''(-1) < 0$ **B** $f'(-1) = 0, f''(-1) > 0$

C $f'(-1) < 0,\ f''(-1) < 0$ **D** $f'(-1) < 0, f''(-1) > 0$

3 © QCAA 2022 2Q3 The derivative of the function $f(x)$ is given by $f'(x) = \sin(x^3)$ for the domain $-1.8 < x < 1.8$.

The number of points of inflection that the graph of $f(x)$ has on this interval is

A 1 **B** 3 **C** 4 **D** 5

Section 2 4 short response questions

15 marks

4 (4 marks) The Jolly Juice Company is designing a new juice bottle that is perfectly cylindrical and has a capacity of 500 mL. (Hint: $1\,\text{mL} = 1\,\text{cm}^3$)

a If the bottle has a base radius of r cm, show that the surface area S of the bottle is given by $S = 2\pi r^2 + \frac{1000}{r}$. *[1 mark]*

b Find the value of r for which S is a minimum. *[3 marks]*

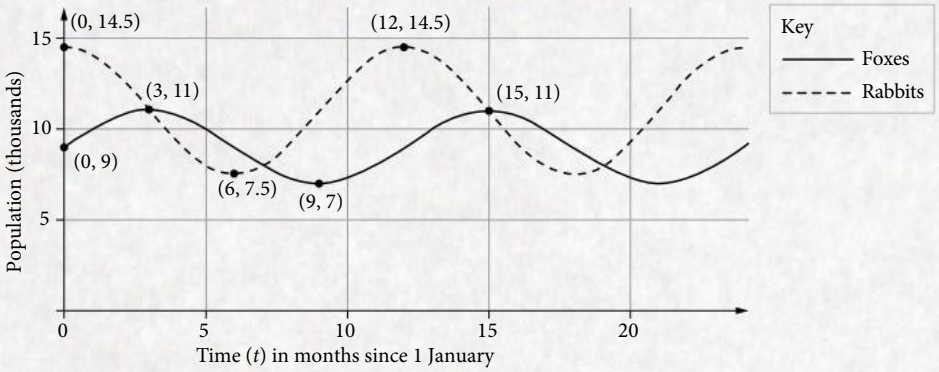
5 © QCAA 2023 2Q15 (4 marks) Determine the derivative of $f(x) = \ln x^2 + \ln(x-5)^3$.
Express the derivative as a single fraction in its simplest and factorised form.

6 © QCAA 2021 2Q17 (4 marks) Rabbits and foxes are among two species of mammals that live on an isolated island.

Rabbits represent a significant food source for the foxes.

The populations of rabbits and foxes were monitored each month for two years.

The graph shows the population of foxes (in thousands) and the population of rabbits (in thousands), at any time t (in months) over the two years. The two populations can be modelled using trigonometric functions.



Jane believes that there were periods of time over the two years when the total population of foxes and rabbits on the island exceeded 25 000.

Evaluate the reasonableness of Jane's claim.

7 (3 marks) The position of an object is given by $x = t\cos(4t)$, where x is the position, in metres, of the object from a fixed point after t seconds. Find the acceleration of the object after $\frac{\pi}{3}$ seconds. Express your answer correct to 3 decimal places.

CHAPTER

4 DISCRETE RANDOM VARIABLES AND THE BINOMIAL DISTRIBUTION



Syllabus coverage
Nelson MindTap chapter resources
Terminology
4.1 Discrete random variables
4.2 Uniform and non-uniform discrete random variables
4.3 The mean and variance of discrete random variables
4.4 Problems involving discrete random variables
4.5 Bernoulli distributions
4.6 Problems involving Bernoulli distributions
4.7 Binomial distributions
4.8 The mean and variance of binomial distributions
4.9 Problems involving binomial distributions
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Discrete random variables and the binomial distribution

Syllabus coverage

UNIT 3, TOPIC 5: DISCRETE RANDOM VARIABLES

General discrete random variables

- Understand the concepts of a discrete random variable and its associated probability function, and its use in modelling data.
- Use relative frequencies obtained from data to determine point estimates of probabilities associated with a discrete random variable.
- Recognise uniform discrete random variables and use them to model random phenomena with equally likely outcomes.
- Recognise non-uniform discrete random variables and use them to model random phenomena.
- Determine and use the mean (expected value) of a discrete random variable as a measurement of centre, $E(X) = \mu = \Sigma p_i x_i$ where p_i is the probability of outcome x_i occurring.
- Determine and use the variance of a discrete random variable as a measure of spread, $Var(X) = \Sigma p_i (x_i - \mu)^2$ where p_i is the probability of outcome x_i occurring, μ is the mean.
- Determine and use the standard deviation of a discrete random variable, $\sqrt{Var(X)}$ as a measure of spread.
- Model and solve problems that involve discrete random variables and associated probabilities, with and without technology.

Bernoulli distributions

- Use a Bernoulli random variable as a model for two-outcome situations.
- Identify contexts suitable for modelling by Bernoulli random variables.
- Recognise and determine the mean p and variance $p(1 - p)$ of the Bernoulli distribution with parameter p.
- Model and solve problems that involve Bernoulli random variables and associated probabilities, with and without technology.

Binomial distributions

- Understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes', r, in n independent Bernoulli trials, with the same probability of success p in each trial.
- Identify contexts suitable for modelling by binomial random variables.
- Determine and use the probabilities $P(X = r) = \binom{n}{r} p^r (1 - p)^{n - r}$ associated with the binomial distribution with parameters n and p.
- Calculate the mean np and variance $np(1 - p)$ of a binomial distribution using technology and algebraic methods.
- Use the language of probability, including *at most*, *at least*, *no more than*, *no less than*, *inclusive* and *between*.
- Model and solve problems that involve binomial distributions and associated probabilities with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 26,
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Videos (2):

4.7 Binomial probability

Exam question analysis Discrete random variables and the binomial distribution

Prior learning (1):

4.1 Discrete random variables and the binomial distribution

Worksheets (14):

4.1 Random variables • Discrete probability distributions 1 • Discrete probability distributions 2

4.3 Expected values • Using expected values • Expected value • Variance and standard deviation

4.5 The Bernoulli distribution

4.6 Geometric probability distributions

4.7 Binomial probability experiments • Using the binomial probability distribution

4.8 Binomial probability – Mean and standard deviation

4.9 Applications of binomial probability

Cumulative exams Maths Method formula sheet

Spreadsheet (1):

4.8 Binomial distribution

Puzzle (1):

Chapter summary Discrete random variables find-a-word

TI-Nspire CX II instructions (1):

4.3, 4.7 Discrete random variables and the binomial distribution

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

Bernoulli event
Bernoulli probability distribution
Bernoulli random variable
Bernoulli trial
binomial distribution
binomial experiment
binomial random variable
discrete probability distribution
discrete variable
expected value
fair game
geometric probability distribution
house percentage
odds
point estimate
probability function
random variable
standard deviation
uniform variable
variance

Worksheets
Random variables
Discrete probability distributions 1
Discrete probability distributions 2

4.1 Discrete random variables

A **random variable** has a numerical value determined by the outcome of a chance experiment. It can be **discrete** or **continuous**. A random variable is usually shown by a capital letter (such as X) and specific values by the lowercase letter (such as x).

A **discrete probability distribution** shows probabilities of a **discrete random variable**. It is also called a **probability function**. You write the values of a probability function as $P(X = x)$ or $p(x)$. The values are all non-negative and the sum of the values $\Sigma p(x) = 1$.

An example of a discrete random variable can be found by tossing a coin twice and counting how many tails are thrown. In this case, X = number of tails.

$P(\text{Tails} = x)$

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

x represents the individual outcomes of event X occurring: 0, 1 or 2 tails being thrown.
$P(X = x)$ represents the probability that outcome x occurs for random variable X.

Put more simply, it is the probability of each of the outcomes occurring.

A probability function cannot have negative values, or have a sum of values not equal to 1.

WORKED EXAMPLE 1 Discrete probability distributions

Determine whether each function could be a discrete probability distribution.

a

x	−2	0	2	4	6	8
$f(x)$	0.3	0.2	0.1	0.1	0.2	0.3

b $f(x) = \frac{x}{15}, x = 1, 2, 3, 4, 5$

c $f(x) = \frac{x^2}{60}, x = -4, -3, -2, -1, 0, 1, 2, 3, 4$

Steps	Working
a 1 Check the values.	All the values are positive, either 0.1, 0.2 or 0.3
2 Find the sum.	$\Sigma f(x) = 0.3 + 0.2 + 0.1 + 0.1 + 0.2 + 0.3$ $= 1.2$
3 Write the answer.	$f(x)$ is not a probability function because the sum is not 1.
b 1 Check the values.	x: 1, 2, 3, 4, 5 $f(x)$: $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}$
2 Find the sum.	$\Sigma f(x) = \frac{1}{15} + \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = \frac{15}{15} = 1$
3 Write the answer.	$f(x)$ could be a probability function because all the values are non-negative and the sum is 1.

c **1** Check the values.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	$\frac{16}{60}$	$\frac{9}{60}$	$\frac{4}{60}$	$\frac{1}{60}$	$\frac{0}{60}$	$\frac{1}{60}$	$\frac{4}{60}$	$\frac{9}{60}$	$\frac{16}{60}$

2 Find the sum.

$$\sum f(x) = \frac{16+9+4+1+0+1+4+9+16}{60} = \frac{60}{60} = 1$$

3 Write the answer.

$f(x)$ could be a probability function because all the values are non-negative, and the sum is 1.

A **point estimate** is the probability of a single possibility. For example, you can estimate the probability of a person having blue eyes by finding the relative frequency of blue eyes.

WORKED EXAMPLE 2 Estimating probabilities

Calculate the probability of a Year 11 student having 4 siblings from the results for a Year 11 class shown below.

Number of siblings	0	1	2	3	4	5
Number of students	3	6	6	3	5	1

Steps	Working
1 Find the total frequency.	$\sum f = 24$
2 Estimate the desired probability.	$P(4) = \frac{5}{24} = 0.2083\ldots$
3 Write the answer.	The probability of a Year 11 student having 4 siblings is about 0.2.

Investigation Year 12 probabilities

Your teacher may want you to estimate probabilities of Year 12 traits such as number of siblings, mode of transport to school, etc.

Consider how the data could be collected.

How accurate would the information collected be?

How typical of Year 12 students across Australia would your estimates be?

EXERCISE 4.1 Discrete random variables

ANSWERS p. 400

Mastery

1 Classify each variable as discrete or continuous.

a Mass of a bus

b Number of passengers in a bus

c Number of bus stops on a route

d Distance between bus stops

e Price of a washing machine

f Capacity of a washing machine

g Energy star rating of a washing machine

h Volume of water used by a washing machine

2 WORKED EXAMPLE 1 State whether each table of values could represent a probability distribution, giving reasons for your answer.

a

x	1	2	3
$f(x)$	0.3	0.4	0.2

b

x	−3	−1	1	3
$f(x)$	0.1	0.2	0.3	0.4

c

x	0	2	6	8
$f(x)$	0.4	−0.1	0.2	0.5

3 a State whether $p(x) = \frac{5x^3}{12} - \frac{x^4}{24} - \frac{3}{4} - \frac{35x^2}{24} + \frac{25x}{12}$ for $x = 1, 2, 3, 4$ could be a probability function, giving reasons.

b Comment on the nature of the function.

4 WORKED EXAMPLE 2 The number of As, Bs, Cs, Ds and Es in a Year 11 Maths Methods class at the end of semester 1 is shown in the table.

Grade	A	B	C	D	E
Number of students	3	6	10	4	2

a Estimate the probability that a student received an A.

b Estimate the probability that a student in the class was awarded a C or better.

5 Of 300 people seen at a fracture clinic, 57 were diagnosed with sprains (not fractures), 102 with broken (fractured) arms, 47 with fractures of leg bones and 48 with other fractures. The rest had no serious problem.

a Estimate the probability that a new patient will have a broken arm.

b Estimate the probability that a new patient will be cleared of fractures.

6 The number of offences committed in Queensland one year is shown.

Type	Against person	Theft	Fraud	Drug	Public order	Other
Number	12 000	17 000	2000	23 000	20 000	15 000

a What is the probability that an offence is against a person?

b What is the probability that an offence involves theft?

c What is the probability that an offence is a public order offence?

Exam practice

Use the following information for Questions 7 and 8.

The ice-cream flavours eaten by a Year 12 Maths Methods class are shown in the table.

Ice-cream flavour	Chocolate	Strawberry	Vanilla	Caramel	Mint
Number of students	10	6	8	1	5

7 TECH-FREE Estimate the probability that a student chooses mint ice-cream.

A 5.0 **B** 1.0 **C** 0.2 **D** 0.167

8 TECH-FREE Estimate the probability that a student chooses a Neapolitan (chocolate, strawberry or vanilla) flavour.

A 0.7 **B** 0.8 **C** 0.9 **D** 1.0

9 TECH-FREE (4 marks) CF Construct a discrete probability distribution on the domain {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} such that the values decrease steadily from 0 to 9 and $p(9) \neq 0$.

10 (3 marks) CF Of the 500 people attending a music festival, 250 bought water, 180 bought t-shirts, 37 bought soft drinks, 30 bought other merchandise and the others did not buy anything.

a Estimate the probability that a new customer will buy a drink. *[1 mark]*

b Estimate the probability that a customer will not buy anything. *[2 marks]*

4.2 Uniform and non-uniform discrete random variables

Worksheets
Expected values
Using expected values
Expected value
Variance and standard deviation

When tossing a normal die, there are six possible outcomes: 1, 2, 3, 4, 5 and 6. Each outcome has the same probability. As there are 6 possible values, all the probabilities are $\frac{1}{6}$. This is an example of a **uniform variable**. As it is also discrete, it is a **uniform discrete random variable**.

Uniform discrete probability distributions

The probability function of a uniform discrete random variable X is a constant.
If X has n equally likely values, then $P(X = x) = \frac{1}{n}$.

Tossing a coin, where the outcomes tails and heads are assigned the values 0 and 1, produces a uniform discrete random variable. The probability function is $p(x) = \frac{1}{2}$ for $x = 0, 1$.

WORKED EXAMPLE 3 Uniform random variables

The variable N is a number chosen at random from the numbers 1, 2, …, 20.

a **Define** the probability function.

b **Comment** on whether it is uniform.

c **Calculate** the probability that n is over 12.

Steps	Working
a Find the probability of each number.	'At random' means equally likely, so $p(x) = \frac{1}{20}$ for $x = 1, 2, \ldots, 20$.
b State the answer.	All the values are the same, so N is a uniform discrete random variable.
c Find the probability.	There are 8 outcomes over 12, so $P(n > 12) = \frac{8}{20} = 0.4$.

Consider the card game Rummy. Aces have a score of 1, number cards score their own number, and kings, jacks and queens score 10 points. The variable R is the score values and is a discrete random variable.

If two cards are chosen **at random**, for example, with scores of 4 and 10, $p(4) = \frac{1}{13}$ but $p(10) = \frac{4}{13}$, so R is not uniform.

Remember from your work on basic probability that choosing outcomes '**at random**' means the outcomes are equally likely.

WORKED EXAMPLE 4 Non-uniform variables 1

One marble each from a cup containing 16 marbles numbered 1 to 16 is given at random to each of 7 people. The players receive m points, given by the remainder when the number is divided by 6.

a **Calculate** the probability function.

b **Comment** on whether it is uniform.

c **Calculate** the probability that m is below 2.

Steps	Working
a Find the probability of each remainder.	$p(0) = \frac{2}{16}$ for 6 and 12 $p(1) = \frac{3}{16}$ for 1, 7 and 13 $p(2) = \frac{3}{16}$ for 2, 8 and 14 $p(3) = \frac{3}{16}$ for 3, 9 and 15 $p(4) = \frac{3}{16}$ for 4, 10 and 16 $p(5) = \frac{2}{16}$ for 5 and 11
b State the answer.	Not all the values are the same, so M is not a uniform discrete random variable.
c Find the probability.	$P(m < 2) = \frac{2}{16} + \frac{3}{16}$ $= \frac{5}{16}$

Sometimes you are interested in the probability of repeating an experiment, say, n times before a particular outcome occurs. For example, you might want to know the probability of rolling a normal die 10 times before you get a 6. This is the same as the probability of failing to get 6 nine times in a row, and then getting a 6. Because the individual events are independent, you can simply multiply the probabilities.

$$P(\text{10 rolls to get a six}) = \left(\frac{5}{6}\right)^9 \times \frac{1}{6} = \frac{5^9}{6^{10}} \approx 0.0323$$

This is a discrete probability situation, but it is not finite. There is always a probability of getting any number of rolls, even though it might be small. This is an example of an infinite discrete random probability.

WORKED EXAMPLE 5 Non-uniform random variables 2

a **Define** the probability function for the random variable N from tossing a normal coin n times in a row before the first head appears.

b Use the distribution to **calculate** the probability that a coin is tossed 9 times before a head appears.

Steps	Working
a **1** State the probabilities of success and failure.	$p = q = \frac{1}{2}$
2 State the probability function.	$P(n) = \left(\frac{1}{2}\right)^n \times \frac{1}{2} = \frac{1}{2^{n+1}}$
b Find the probability.	$P(9) = \frac{1}{2^{n+1}}$ $= \frac{1}{1024}$

EXERCISE 4.2 Uniform and non-uniform discrete random variables

ANSWERS p. 401

Recap

1 Determine whether each function could be a discrete probability distribution, giving reasons.

a

x	1	2	3	4	5
$p(x)$	0.1	0.3	0.1	0.1	0.3

b

x	–2	1	3	7	22
$p(x)$	0.2	0.1	0.3	0.25	0.15

c

x	2	3	4	5	6
$p(x)$	–0.1	0.2	0.3	0.2	0.4

2 Estimate the probability of a student chosen at random from a school swimming squad being aged 15, from the information in the table.

Age	12	13	14	15	16	17	18	19
Number of students	8	7	7	6	9	5	3	5

Mastery

3 WORKED EXAMPLE 3 A number p is chosen at random from the prime numbers from 1 to 20.

a Find the probability function.

b State whether it is uniform.

c Find the probability that p is under 6.

d Find the probability that p is between 14 and 16 inclusive.

4 WORKED EXAMPLE 4 A number is chosen at random from the numbers 1 to 20 and divided by 5. The number n is obtained by adding the remainder to the quotient.

a Find the probability function of N.

b State whether it is uniform.

c Find the probability that n is under 6.

d Find the probability that n is between 3 and 6 inclusive.

5 WORKED EXAMPLE 5 A marble is drawn at random from a bag containing 8 green and 4 red marbles. The marble is replaced before another marble is drawn.

a State the distribution of N, the random variable for the number of marbles drawn before you get the first red one.

b Find $P(5)$.

c What is $P(8)$?

6 m is the remainder when a number from 1 to 24 is divided by 8.

a Find the probability function of M.

b State whether it is uniform.

c Find the probability that m is under 8.

d Find the probability that m is between 4 and 6 inclusive.

7 n is the remainder when a number from 1 to 24 is divided by 9.

a Find the probability function of N.

b State whether it is uniform.

c Find the probability that n is under 8.

d Find the probability that n is between 4 and 6 inclusive.

Exam practice

8 TECH-FREE A card is dealt at random from a normal pack of cards. The card is replaced and the pack shuffled before another card is dealt.

If N is the random variable for the number of cards drawn before you get the first heart, $p(n) =$

A $\frac{1^n}{3^{n+1}}$ **B** $\frac{2^n}{2^{n+1}}$ **C** $\frac{3^n}{4^{n+1}}$ **D** $\frac{4^n}{3^{n+1}}$

9 © QCAA 2021 1Q9 TECH-FREE 51% A basket contains 10 green apples and 30 red apples. Three apples are drawn at random from the basket with replacement. Determine the probability that exactly two of the three apples are green.

A $\frac{3}{64}$ **B** $\frac{9}{64}$ **C** $\frac{10}{64}$ **D** $\frac{27}{64}$

10 © QCAA 2021 2Q6 61% When seeds of a certain variety of flower are planted, the probability of each seed germinating is 0.8.

If eight seeds are planted, what is the probability that at least six seeds will germinate?

A 0.797 **B** 0.503 **C** 0.294 **D** 0.001

11 TECH-FREE (6 marks) A player rolls a fair die. If a prime number occurs, the player wins that number of dollars; but if a non-prime number occurs, the player loses that number of dollars. Let X be the number of dollars the player stands to win.

a Construct the probability distribution for X. *[3 marks]*

b Draw a graph of the distribution. *[3 marks]*

12 TECH-FREE (5 marks) A game consists of rolling 3 dice. If all 3 dice show the same number, you win \$20. If 2 numbers are the same, you win \$5. If all 3 are different, you lose \$5. X is the random variable representing the amount you win.

a Find the probability distribution of X. *[3 marks]*

b What is the probability of winning? *[2 marks]*

13 TECH-FREE (5 marks) CF The individual prices of lollies in a pick-and-pay display were as follows. All prices are in cents.

5	10	5	10	20	5	5	10	20	20	15	5	5	10	10
20	15	50	50	20	25	25	15	5	10	10	5	10	5	20

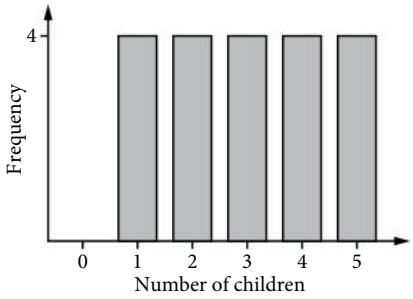
a Construct the probability distribution function for the prices P of lollies chosen at random. *[3 marks]*

b What is the probability of getting a lolly worth less than 20 cents? *[2 marks]*

14 TECH-FREE (5 marks) CU How many times must you roll a die for the probability of getting at least one 6 to exceed 0.9?

15 © QCAA 2024 1Q7 TECH-FREE 63% Twenty families are selected to participate in a lifestyle study related to family size.

The number of children in these families is uniformly distributed as shown.



A random sample of five families is chosen from this group, without replacement. A possible mean number of children in the sample is

A 5.0 **B** 2.0 **C** 1.0 **D** 0.0

4.3 The mean and variance of discrete random variables

Worksheets
Expected values
Using expected values
Expected value
Variance and standard deviation

Consider the discrete random variable X for the age of a student chosen at random from a group of 5 Year 11 students of ages 16, 17, 17, 18 and 18.

The probability distribution of X is shown below.

x	16	17	18
$P(X = x)$	0.2	0.4	0.4

The **mean** of their ages is $\frac{16+17+17+18+18}{5} = 17.2$.

If a student is chosen randomly, n times, you would expect to get the average.

Notice that $17.2 = \frac{16\times1+17\times2+18\times2}{5}$

$$= 16 \times \frac{1}{5} + 17 \times \frac{2}{5} + 18 \times \frac{2}{5}$$

$$= \sum xp(x)$$

Now think of the general case with k different scores x_i of frequencies f_i, where the total frequency is $\sum f_i = n$. If you construct a discrete variable X by choosing at random, you get this probability distribution.

x	x_1	x_2	x_3	…	x_i	…	x_k
$P(X = x)$	$\frac{f_1}{n}$	$\frac{f_2}{n}$	$\frac{f_3}{n}$	…	$\frac{f_i}{n}$	…	$\frac{f_k}{n}$

The mean of the data is given by $\sum \frac{fx}{n} = \sum x\frac{f}{n} = \sum xf(x)$.

This rule is used for any discrete random variable, even if it doesn't come from data.

Expected value of a discrete random variable

The **expected value** of a discrete random variable X is defined by $\mu = E(X) = \sum xp(x)$.

WORKED EXAMPLE 6 Finding expected values 1

Calculate the expected value of a discrete random variable with values 1 to 8 and probabilities $\frac{1}{8}$.

Steps	Working
1 Use the formula.	$\mu = E(x) = \sum xp(x)$
2 Substitute values.	$= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + \ldots + 8 \times \frac{1}{8}$
3 Solve.	$= 4.5$
4 State the answer.	The expected value is 4.5.

Calculating the expected value for a uniform distribution is a little easier than for a non-uniform distribution.

WORKED EXAMPLE 7 Finding expected values 2

A 'Lucky' lottery has first, second and third prizes of \$100 000, \$10 000 and \$5000, two prizes of \$1000, two of \$500, 10 of \$200, 20 of \$100, 100 of \$50, 600 of \$25, 570 of \$15 and 2480 of \$10. There are 270 000 tickets costing \$2.20 each.

Determine the expected value for the prizes and the expected value of buying a ticket.

Steps	Working
1 Use the formula to find the expected value of the prizes.	$E(x) = \sum xp(x)$ $= \$100\,000 \times \frac{1}{270\,000} + \$10\,000 \times \frac{1}{270\,000}$ $+ \ldots + \$10 \times \frac{2480}{270\,000}$
2 Substitute values.	$= \frac{1}{270\,000} \times (1 \times \$100\,000 + 1 \times \$10\,000 + \ldots$ $+ 2480 \times \$10)$
3 Simplify.	$= \frac{\$175\,350}{270\,000}$
4 Calculate.	$= \$0.6494\ldots$
5 Subtract the ticket cost.	Expected value of ticket $\approx \$0.65 - \$2.20$ $= -\$1.55$
6 State the answer.	The expected prize value is about 65 cents and the expected value of buying a ticket is about −\$1.55.

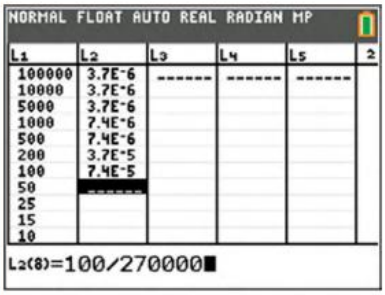
A Lucky lottery is not so lucky after all! Since lotteries are set up to make money for the promoters, this is not surprising. You should always expect to lose.

Notice in the above example that it is easier to add up the total value of the prizes to calculate the expected value than to directly follow the formula.

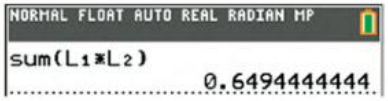
TECHNOLOGY Expected value

You can use your graphics calculator to find the expected value of a random normal variable.
The following instructions show how to do this for the lottery prize in the previous example.

TI-84 Plus CE

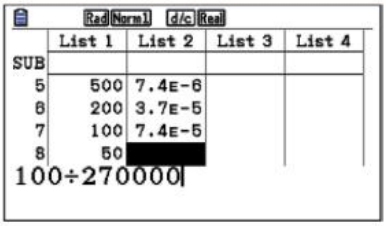


1 Press stat and enter and choose **1: edit**.
2 Move up to L_1 and press clear and, if you need to, remove old data. Do the same for L_2 if needed.
3 Put the values of 100 000, 10 000, ..., 10 in L_1.
4 Put the probabilities as 1 ÷ 270 000, ..., 2480 ÷ 270 000 in L_2.

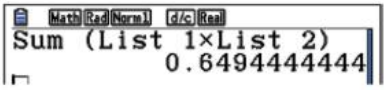


5 Press **quit** (2nd mode) and **list** (2nd stat) and move across to **MATH** and choose **5: sum(**.
6 Put in ($L_1 \times L_2$) to get the expected value.
Use 2nd **1** for L_1.

Casio fx-CG50 AU



1 Choose **2** Statistics and if you need to remove old data, use F6 F4 and F1 (DEL-ALL).
2 Do the same for List 2 if needed.
3 Put the values of 100 000, 10 000, ..., 10 in List 1.
Put the probabilities as 1 ÷ 270 000, ..., 2480 ÷ 270 000 in List 2.



4 Press MENU and choose **1: Run-Matrix**.
5 Press OPTN F6 F6 and F1 (Sum).
6 Put in (List 1 × List 2) to get the expected value.
Use OPTN F1 to get List 1.

TI-Nspire
Discrete random variables and the binomial distribution

Variance and standard deviation of a discrete random variable

The **variance** of a discrete random variable X is defined as

$$Var(X) = \sum(x - \mu)^2 p(x) = \sum x^2 p(x) - \mu^2$$

where $\mu = E(X)$ is the expected value of X.

The **standard deviation** of a discrete random variable X is defined as

$$\sigma_X = \text{SD}(X) = \sqrt{Var(X)}.$$

WORKED EXAMPLE 8 Variance and standard deviation 1

Calculate the variance and standard deviation of a discrete random variable with values 1 to 8 and probabilities $\frac{1}{8}$.

Steps	Working
1 Find the expected value.	$\mu = \sum xp(x) = 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + \ldots + 8 \times \frac{1}{8} = 4.5$
2 Use the formula for variance.	$Var(X) = \sum x^2 p(x) - \mu^2$
3 Substitute in values.	$= 1^2 \times \frac{1}{8} + 2^2 \times \frac{1}{8} + \ldots + 8^2 \times \frac{1}{8} - 4.5^2$
4 Simplify.	$= \frac{1}{8} \times (1^2 + 2^2 + \ldots + 8^2) - 20.25$ $= 25.5 - 20.25$ $= 5.25$

5 Find the standard deviation.	$SD(X) = \sqrt{5.25} = 2.291\ldots$
6 State the answer.	The variance is 5.25 and the standard deviation is about 2.29.

WORKED EXAMPLE 9 Variance and standard deviation 2

Determine the variance and standard deviation of the random variable M with the probability distribution shown.

x	2	3	5	7	11
$p(x)$	0.1	0.2	0.3	0.2	0.2

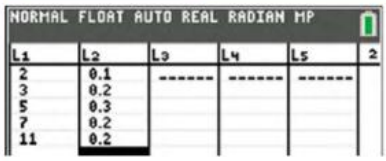
Steps	Working
1 Find the expected value.	$\mu = \Sigma xp(x)$ $= 2 \times 0.1 + 3 \times 0.2 + \ldots + 11 \times 0.2$ $= 5.9$
2 Use the formula for variance.	$Var(X) = \Sigma x^2 p(x) - \mu^2$ $= 4 \times 0.1 + 9 \times 0.2 + \ldots + 121 \times 0.2 - 5.9^2$ $= 43.7 - 34.81$ $= 8.89$
3 Find the standard deviation.	$SD(X) = \sqrt{8.89} = 2.981\ldots$
4 State the answer.	The variance is 5.9 and the standard deviation is about 2.98.

TI-Nspire
Discrete random variables and the binomial distribution

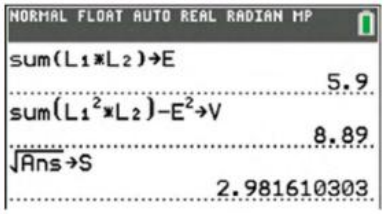
TECHNOLOGY Variance and standard deviation

You can use your graphics calculator to find the variance and standard deviation of a random normal variable. The following instructions show how to do this for the above example.

TI-84 Plus CE



1 Put the values into L_1 and L_2 as shown on page 125.



2 Press **quit**.

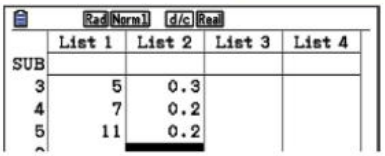
3 Find the expected value as shown on page 125 and store it in **E**.

4 Use sum($L_1^2 \times L_2$) – E^2 to find the variance.

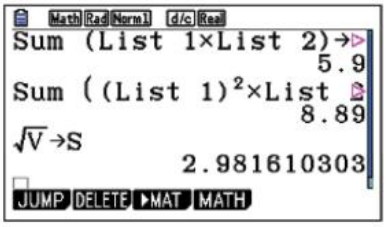
5 Find the square root of the answer to get the standard deviation.

6 Store the values as shown if you need them for other calculations.

Casio fx-CG50 AU



1 Put the values into List 1 and List 2 as shown on page 125.



2 Press MENU and choose **1: Run-Matrix**.

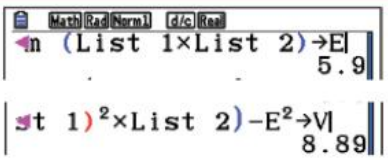
3 Find the expected value as shown on page 125 and store in **E**.

4 Use sum((List 1)^2 × List 2) – E^2 to find the variance.

You need the brackets around List 1.

5 Find the square root of the answer to get the standard deviation.

6 Press F⇔D to get a decimal answer.



7 Store the values as shown if you need them for other calculations.

Note that the rest of the display for the first two calculations is shown at the bottom.

EXERCISE 4.3 The mean and variance of discrete random variables

ANSWERS p. 401

Recap

1 Find the probability function for the random variable X, a number chosen at random from the odd numbers from 7 to 23 inclusive, and comment on the nature of the variable.

2 A number is chosen at random between 25 and 45 inclusive and the digits are added together to give q.

a Find the probability distribution of the random variable Q.

b Find $P(q < 9)$.

Mastery

Use your graphics calculator as directed by your teacher for this exercise.

3 WORKED EXAMPLE 6 Find the expected value for choosing a number at random from 5 to 25 inclusive.

4 Find the expected value for the discrete random variable of this distribution.

X	2	3	5	7	11	13	17	19
$P(X = x)$	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0.125

5 WORKED EXAMPLE 7 Find the expected value for the discrete random variable of each distribution.

a

x	0	5	10	15
$P(X = x)$	0.5	0.1	0.2	0.2

b

x	2	3	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

c

x	−5	−4	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

d

x	1	3	4	5
$p(x)$	0.4	0.1	0.2	0.3

6 The results out of 40 for a test are shown below.

22	36	12	16	35	30	25	28	26	20
18	15	32	27	21	13	8	30	17	27

Find the expected value of the variable S for the test results.

7 Results in a subject, out of 100, are shown below.

32	45	68	90	45	63	55	17	50	60
36	67	70	85	55	65	67	57	43	80

a Find the expected value of the variable F for the test results.

b The test results are rescaled to a mark M out of 30.
Find the expected value of the variable M.

8 WORKED EXAMPLE 8 Find the variance and standard deviation of a discrete random variable with values 1 to 10 and probabilities $\frac{1}{10}$.

9 Find the variance and standard deviation of a discrete random variable with values 6 to 15 and probabilities $\frac{1}{10}$.

10 Find the variance and standard deviation of a discrete random variable with values 3, 6, 9, 11, 13, 15, 17, 19 and probabilities $\frac{1}{8}$.

11 WORKED EXAMPLE 9 Find the variance and standard deviation of the random variable for each probability distribution.

a

x	0	5	10	15
$P(X = x)$	0.5	0.1	0.2	0.2

b

x	2	3	11
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

c

x	−5	−4	1	2
$p(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

Exam practice

Use the following distribution to answer Questions 12 and 13.

x	1	3	4	5
$p(x)$	0.4	0.1	0.2	0.3

12 What is the expected value of x in the distribution?

A 1 **B** 1.25 **C** 1.5 **D** 3

13 What is the variance?

A 0.75 **B** 1.04 **C** 3 **D** 9

14 What is the expected value from rolling a normal 6-sided die?

A 3 **B** 3.5 **C** 4 **D** 4.5

15 (2 marks) Two normal dice are thrown. S is the equal or smaller number of the numbers on the dice. Find the expected value of S.

16 (4 marks) A lottery has first, second and third prizes of $200 000, $20 000 and $5000. There are 7 prizes of $1000, 10 of $500, 25 of $100, 75 of $75, 200 of $50, 6000 of $40, 700 of $20 and 2800 of $12. Tickets cost $5.50 and there are 200 000 tickets.

Find your expected value of buying a ticket.

17 (3 marks) Find the variance and standard deviation from rolling a normal 6-sided die.

18 (4 marks) CF Two normal dice are thrown. T is the equal or smaller number of the numbers on the dice. Find the variance and standard deviation of T.

4.4 Problems involving discrete random variables

You have calculated the expected value of a few lotteries and games with entry fees. Commercial gambling games are set up so the expected value for players is negative. The average amount that players win is less than the entry fee. Commercial gambling games usually have the amount you win set up as **odds**.

Odds and fair games

The odds for a bet or game is the ratio of the amount won to the amount bet (or entry fee).

A **fair game** is one with an expected value of 0.

If you won a bet of $10 at odds of 5 : 1, you would win $50 and also get your bet back.

In the case of games played for money, the expected value includes any entry fee.

WORKED EXAMPLE 10 Is it a fair game?

A game is played with 2 dice. It costs $2 to play.

If the total is 11 or 12 you are paid $12 and if the total is 2, 3 or 4 you are paid $6. You lose on other totals.

Consider whether the game is fair.

Steps	Working
1 Write the probabilities of each total.	x: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 $p(x)$: $\frac{1}{36}$, $\frac{2}{36}$, $\frac{3}{36}$, $\frac{4}{36}$, $\frac{5}{36}$, $\frac{6}{36}$, $\frac{5}{36}$, $\frac{4}{36}$, $\frac{3}{36}$, $\frac{2}{36}$, $\frac{1}{36}$
2 Choose a random variable.	Let X be the amount won on a roll of the dice.
3 Write the probabilities of each result.	$p(\$10) = \frac{2}{36} + \frac{1}{36} = \frac{3}{36}$ $p(\$4) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36}$ $p(-\$2) = \frac{4}{36} + \frac{5}{36} + \ldots + \frac{3}{36} = \frac{27}{36}$
4 Find the expected value.	$E(X) = \$10 \times \frac{3}{36} + \$4 \times \frac{6}{36} + -\$2 \times \frac{27}{36}$ $= \$\frac{30 + 24 - 54}{36}$ $= \$0$
5 Write your conclusion.	Since the expected value is $0, the game is fair.

Casinos, lotteries, bookmakers and other betting agencies set games so they win in the long run.

House percentage

The **house percentage** is the proportion of money bet that casinos and other betting agencies win in the long run, expressed as a percentage.

WORKED EXAMPLE 11 The house percentage

A game consists of a wheel divided into 12 equal sectors numbered 1 to 12. You may place a \$2 bet on any number. If your number comes up, you get \$18 back. That is, you win \$16 and get your \$2 back.

a **Determine** the house percentage.

b How much should you win for the game to be considered fair?

Steps	Working
a 1 Choose the random variable.	Let B be the amount won on a bet.
2 Write the probability distribution.	b (\$): 16, −2 $p(b)$: $\frac{1}{12}$, $\frac{11}{12}$
3 Find the expected value.	$E(B) = 16 \times \frac{1}{12} + (-2) \times \frac{11}{12}$ $= -0.5$
4 Express as a percentage.	House percentage $= \frac{\$0.50}{\$2} \times 100\% = 25\%$
b 1 Choose a variable for the amount won.	Let $\$a$ be the amount won on a bet.
2 Recalculate the expected value.	$E(B) = a \times \frac{1}{12} + (-2) \times \frac{11}{12}$
3 Solve $E(B) = 0$.	$a \times \frac{1}{12} + (-2) \times \frac{11}{12} = 0$ $a = 22$
4 Write the answer.	You would have to win \$22 for the game to be fair.

EXERCISE 4.4 Problems involving discrete random variables

ANSWERS p. 401

Recap

1 Find the expected value of choosing a number at random from the numbers between 5 and 20 inclusive that are divisible by 2 or 3 or both.

2 Find the variance and standard deviation of the probability distribution shown.

x	1	2	3	5	8	13
$p(x)$	0.05	0.2	0.3	0.2	0.15	0.1

Mastery

3 WORKED EXAMPLE 10 A game is played with 2 normal dice. It costs \$5 to enter, and you win \$13 if the bigger number shown is 3 or less. Otherwise, you lose.

a Is the game fair?

b What would you need to win to make the game fair?

4 WORKED EXAMPLE 11 It costs $3 to play a game with 2 dice. You throw the dice and win $6 if you get a total of 2, 3 or 4. You lose for a total of 5, 6, 7, 8 or 9, Otherwise, you toss again until you get total from 2 to 9.

a What is the house percentage?

b What would you need to win to make the game fair?

5 A charity raffle has 500 tickets for $10 each and 3 prizes worth $200, $100 and $50.

Find the expected value of buying a ticket and the house percentage.

6 Life insurance companies effectively make a bet with their clients that they will live beyond a given age. For term insurance, the bet is that they will live for the next year. This means that if the person insured dies within a year of taking out the policy, the beneficiary is paid. You are not allowed to buy insurance unless you have an interest in the event not occurring. This means you can only take out life insurance for yourself or someone else you want to live, such as your spouse or a valued employee.

Term insurance for a 20-year-old non-smoker for $1 000 000 cover costs about $55 a month for women and $75 a month for men. It is actually paid for the whole year in advance.

According to the ABS, the probability of a 20-year-old dying in the next year is 0.000 27 for a woman and 0.000 65 for a man.

a Calculate the expected value and the house percentage for these life insurance policies.

b Explain the differences between the cost for women and men.

c Why are the ratio of the costs not the same as the ratios of the probabilities?

7 In the American version of the game of roulette, a ball travels round a wheel and lands on one of 38 coloured numbers. Half the numbers from 1 to 36 are red and the others are black. There are 2 green numbers, 0 and 00, which are considered neither odd nor even.

The odds for some of the possible bets are shown below.

odd, even, red or black 1 : 1	single numbers 35 : 1
2 numbers 17 : 1	4 numbers 8 : 1

Work out the house percentage for each bet, remembering that 0 and 00 are not odd, even, red or black. Explain your answer.

Exam practice

8 TECH-FREE A dice game costs $2 to enter. If you roll a double (the same number on both dice), you win $10, but otherwise you lose.

What is the house percentage?

A 0% (fair game) **B** 33.33% **C** 16.67% **D** 8.33%

9 TECH-FREE A 'lucky envelope' game has 500 envelopes. 2 envelopes have a $100 note inside, 3 have a $50 note, 5 have a $20 and 10 have a $10 note. It costs $2 to pick an envelope. The house percentage is

A 55% **B** 45% **C** 35% **D** 65%

10 (4 marks) CF A sideshow alley game has a square table marked with a grid of squares, each one 25 cm long. You play by throwing round plastic discs from behind the counter towards the square table. The discs are 20 cm in diameter. You win if a plastic token lands completely within a square, not overlapping any lines. It costs $2 per throw. The prize for a win is $20.

Find the house percentage.

11 (5 marks) CU A chocolate wheel is divided into 52 equal sections. 24 of the sections have heart symbols, 12 have club symbols, 8 have anchor symbols, 4 have smiley face symbols, 2 have pirate flag symbols and 2 have Australian flag symbols. Players may bet on any symbol before the wheel is spun. After the wheel comes to rest, bets are paid at these odds: hearts 1 : 1, clubs 3 : 1, anchors 5 : 1, smiley faces 11 : 1 and flags 23 : 1. Odds of 3 : 1 means if you win, you get your bet back, plus 3 times the bet. What is the house percentage?

Worksheet
The Bernoulli distribution

4.5 Bernoulli distributions

A **Bernoulli event** is an event with only 2 possible outcomes, usually called **success** and **failure**.
A distribution with only 2 possible outcomes is called a **Bernoulli probability distribution**.

A Bernoulli probability distribution is a discrete probability distribution with only 2 possible outcomes – success and failure. The probability of success, p, and probability of failure, q, are fixed.

For a Bernoulli event:

$$p + q = 1$$

Bernoulli variables

The **Bernoulli random variable** X has the value of 1 for success and 0 for failure.

For a discrete probability distribution that has a Bernoulli random variable X:

$$P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

This is for a single trial only.

WORKED EXAMPLE 12 Identifying Bernoulli distributions

Determine whether each situation would result in a Bernoulli distribution.

a Taking a driving test and recording whether you pass the test

b Tossing a coin 5 times and recording the number of heads

c Rolling a 4-sided die and recording the number that occurs

d Drawing a card from a standard deck and recording whether it is a heart

Steps	Working
a 1 State the outcomes.	There are 2 possible outcomes: pass or fail.
2 State the result.	This is a Bernoulli distribution.
b 1 State the outcomes.	There are 6 possible outcomes: 0, 1, 2, 3, 4, 5.
2 State the result.	This is not a Bernoulli distribution.
c 1 State the outcomes.	There are 4 possible outcomes: 1, 2, 3 and 4.
2 State the result.	This is not a Bernoulli distribution.
d 1 State the outcomes.	There are 2 possible outcomes: heart or not a heart.
2 State the result.	This is a Bernoulli distribution.

In section 4.3 of this chapter, you saw how to calculate the expected value, $E(X)$, and variance, $Var(X)$, for a discrete random variable. You also used the results

$$E(X) = \mu \qquad \text{and} \qquad Var(X) = \sigma^2$$

where μ is the mean and σ is the standard deviation, SD(X).

Bernoulli distributions

$$E(X) = \mu = p$$
$$Var(X) = \sigma^2 = p(1-p)$$
$$\sigma = \sqrt{p(1-p)}$$

WORKED EXAMPLE 13 Bernoulli distributions

A drug has been developed for the treatment of skin cancer. The drug is found to be effective in 81% of cases. The effectiveness of the drug = X.

a **Construct** a probability distribution table for X.

b **Calculate** $E(X)$.

c **Calculate** the standard deviation of X, correct to 4 decimal places.

Steps	Working
a 1 There are 2 possible outcomes: the drug is effective (success) and the drug is not effective (failure).	
2 Write the probability of success.	$p = P(\text{effective}) = 0.81$
3 Construct the distribution.	x: 0, 1 $P(X = x)$: 0.19, 0.81
b State the rule for expected value.	$E(X) = p = 0.81$
c 1 Calculate the variance by substituting for known value and evaluate.	$Var(X) = p(1-p)$ $= 0.81 \times 0.19$ $= 0.1539$
2 Calculate the standard deviation. Take the square root.	$\sigma^2 = Var(X) = 0.1539$ $\sigma = \sqrt{0.1539}$ $= 0.392\,300\ldots$
3 Round off and state the result.	$\sigma \approx 0.3923$

EXERCISE 4.5 Bernoulli distributions

ANSWERS p. 401

Recap

1 A game is played with one person throwing two normal dice. If the total is even, the person throwing the dice wins and if the total is odd the other person wins. Is the game fair, and does it make any difference if the players take it in turns?

2 It costs $5 to play a game that uses many packs of cards with all except the cards numbered 2 to 6 removed. Two cards are dealt and if the total is 7, 8 or 9 the house wins, otherwise you win. What is the house percentage?

Mastery

3 WORKED EXAMPLE 12 Determine whether each statement results in a Bernoulli distribution.

a Spinning a spinner on a wheel with equal-sized segments numbered 1 to 12 and recording whether the number is odd

b Asking a randomly selected person if they have had a flu injection this year

c Choosing a marble from a bag containing red marbles, blue marbles and green marbles and recording the colour

d Tossing a coin and recording if the outcome is a head

e Drawing a card from a standard deck and recording if the selection is an ace

f Rolling a die and recording if the number is a multiple of 3

g Drawing a card from a standard pack and noting its suit

h Selecting a person at random and recording their age

4 For each Bernoulli distribution identified in question **3**

i identify a successful outcome

ii determine the probability of success (except for part **b**).

5 WORKED EXAMPLE 13 Calculate the probability of success in each Bernoulli distribution.

a Picking a dark chocolate from a bag containing 5 milk chocolates, 6 dark chocolates and 4 white chocolates

b Drawing a picture card (K, Q or J) from a standard deck

c Selecting a black sock from a drawer containing 6 red socks, 2 brown socks, 8 black socks and 4 green socks

d Getting a 4 or a 2 when a normal die is rolled

e Getting 2 heads when 2 coins are tossed

6 A multiple choice question has 5 possible answers, A, B, C, D and E.

Caleb guesses the answer to the question. Let X be that he guesses the correct answer.

a Construct a probability distribution table for X.

b Find $E(X)$.

c Find the standard deviation of X.

7 Calculate the variance of the random variable in a Bernoulli distribution with each probability value.

a $p = \frac{1}{4}$ **b** $p = 0.4$ **c** $q = \frac{2}{5}$ **d** $p = \frac{11}{14}$ **e** $p = 70\%$

8 Calculate the standard deviation of the random variable in a Bernoulli distribution with each probability value.

a $p = \frac{1}{3}$ **b** $p = 30\%$ **c** $p = \frac{7}{15}$ **d** $q = \frac{12}{25}$ **e** $p = 0.25$

Exam practice

Use the following information to answer Questions 9–11.

A discrete random variable, U, has a Bernoulli distribution as follows.

U	0	1
$P(U = u)$	0.32	0.68

9 Find $E(U)$.

A 0.2176 **B** 0.4624 **C** 0.4665 **D** 0.68

10 Find $Var(U)$.

A 0.2176 **B** 0.4624 **C** 0.4665 **D** 0.68

11 Find $SD(U)$.

A 0.2176 **B** 0.4624 **C** 0.4665 **D** 0.68

12 (4 marks) Daria is a professional tennis player. She knows from experience that there is a 28% chance of serving an ace on any serve. Y is the discrete random variable that represents Daria serving an ace.

a Construct a probability distribution table for Y. *[2 marks]*

b Calculate $E(Y)$ and $Var(Y)$. *[2 marks]*

13 (4 marks) There is a 79% chance that it will rain on any given day in Tully during the wet season in January and February.

a Construct a probability distribution table for X, where X is the probability that it rains on any given day during the wet season. *[2 marks]*

b Find the standard deviation of X. *[2 marks]*

14 (4 marks) Z is a discrete random variable that has a Bernoulli distribution. It is known that the variance for this distribution is 0.24. The probability of success is p.

a Find p if $p > 1 - p$. *[2 marks]*

b Find $E(Z)$. *[2 marks]*

15 (4 marks) CF Y is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation of Y is 0.4854.

a Find the variance, correct to 4 decimal places. *[2 marks]*

b Find the probability of success, correct to 4 decimal places, if $P(\text{success}) > P(\text{failure})$. *[2 marks]*

4.6 Problems involving Bernoulli distributions



Worksheet
Geometric probability distributions

A **Bernoulli trial** is when a Bernoulli event is conducted. Bernoulli trials are independent, so the probability of success (p) or failure (q) remains the same for each trial.

In some situations, you might be interested in how many Bernoulli trials are needed until the first success occurs. For example, you would probably want to know how many games your favourite team is likely to play to have its first win. The pattern for this is a sequence of failures and then one success:

$$F, F, F, \ldots, S$$

WORKED EXAMPLE 14 Bernoulli distributions

Alex is a netball player who knows from experience that she has a 65% chance of shooting a goal from any attempt. What is the probability that it will take Alex exactly 4 attempts to shoot her first goal? **Calculate** the answer correct to 2 significant figures.

Steps	Working
1 Write the sequence of attempts.	F, F, F, S
2 Calculate p.	$p = 65\% = 0.65$
3 Calculate q.	$p + q = 1$ $q = 1 - 0.65$ $= 0.35$
4 Calculate the probability of the stated sequence.	$P(FFFS) = 0.35 \times 0.35 \times 0.35 \times 0.65$ $= 0.027\,86\ldots$
5 Round off and state the result.	The probability that Alex needs 4 attempts to shoot her first goal is about 0.028 or 2.8%.

The distribution shown in the above example is a **geometric probability distribution**.

The probability distribution is given by $P(X = x) = q^{x-1}p$.

It is the number of trials needed *until* the first success is achieved.

EXERCISE 4.6 Problems involving Bernoulli distributions

ANSWERS p. 402

Recap

1 State whether each situation results in a Bernoulli distribution.

- **a** Rolling a die and recording whether the number displayed is a 6
- **b** Recording the length of time a customer spends in a queue waiting to make a purchase
- **c** Drawing a card from a standard deck without replacement and recording whether it is a king
- **d** Recording whether the viewers of a film are male

2 A confectioner tells its customers that 40% of chocolates in a box of their 'Premier' chocolates are truffles. Bart bought a box of 'Premier' chocolates and selected one. Let Y be the discrete random variable that Bart chose a truffle.

- **a** Construct a probability distribution table for Y.
- **b** Find $E(Y)$.
- **c** Find $\text{SD}(Y)$.

Mastery

3 State whether each situation gives a geometric distribution.

- **a** A couple have 3 boys. They want to know the probability of their next child being a girl.
- **b** About 10% of the population has an O negative blood type. A technician at the blood bank wants to know how many samples will need to be tested before type O negative is detected.
- **c** A bag contains 5 red, 7 green and 4 black marbles. A marble is drawn, its colour is noted and then it is set aside before another marble is drawn. You want to find the number of marbles that will be selected before a green marble is drawn.
- **d** A card is selected from a well-shuffled deck, its suit is noted and then the card is returned to the deck before another card is drawn. You want to determine the average number of cards that will need to be drawn before the first heart occurs.
- **e** A coin is tossed and the outcome noted. You are interested in how many times the coin will need to be tossed before 2 tails or 3 heads come up.

4 WORKED EXAMPLE 14 What is the probability that it will take exactly 5 rolls of a normal die to first get a 6?

5 Shauna is sitting for a multiple choice test in which each question has 5 possible answer choices. She randomly selects the answer for each question. Find the probability that Shauna's first correct answer is to question

a 3 **b** 5 **c** 8

6 A fair coin is tossed a number of times until it lands heads. What is the probability of the coin landing heads on the 4th toss?

7 About 70% of the time, access to an online booking website for a popular concert is too busy.

If X denotes the number of times you need to log on until you get through, find

a $P(X = 4)$ **b** $P(3 \leq X \leq 6)$ **c** $P(X \leq 5)$

8 A market researcher at a shopping centre is surveying people to find if they have a subscription to a TV streaming service. It is known that the probability that a person has a subscription to a TV streaming service is 0.2. Let X denote the number of people surveyed until the researcher finds the first person who has a subscription to a TV streaming service.

a What is the probability that the market researcher must survey 4 people to find one who has a subscription to a TV streaming service?

b What is the probability that the market researcher must survey more than 6 people to find one who has a subscription to a TV streaming service?

Exam practice

9 In a particular game with 2 dice, you can only begin if you roll a double (the same number on both dice) The probability that you need more than 6 attempts before you start is

A 0.1667 **B** 0.3349 **C** 0.6651 **D** 1.0

10 A basketball player has a 65% chance of making a free throw. What is the probability that it takes the player 2 or 3 throws to make the first free throw?

A 0.3071 **B** 0.35 **C** 0.65 **D** 0.6929

11 What is the probability of rolling a sum of 7 in fewer than 3 rolls of a pair of dice?

A $\frac{1}{12}$ **B** $\frac{7}{36}$ **C** $\frac{11}{36}$ **D** $\frac{21}{36}$

12 (6 marks) A particular type of insect has a 28% chance of dying on any one day and this probability remains constant for about 10 days. If Y denotes the number of days that an insect of this type survives until it dies, calculate

a $P(Y = 1)$ *[1 mark]*

b $P(3 \leq Y \leq 5)$ *[1 mark]*

c the probability that an insect of this type survives its first 3 days *[2 marks]*

d the probability that an insect of this type dies on its fourth day. *[2 marks]*

13 (4 marks) In a particular game with 2 dice, you can only begin if you roll a double. Find the probability that

a you start on your first go *[1 mark]*

b you need 4 attempts to start *[1 mark]*

c you start within 3 attempts *[1 mark]*

d you need more than 5 attempts before you start. *[1 mark]*

14 (2 marks) In a bottling plant, it is known that 25% of bottles are underfilled.

If X is the number of underfilled bottles before a correctly filled bottle is found, calculate

a $P(X = 4)$ *[1 mark]*

b $P(2 \leq X \leq 7)$ *[1 mark]*

15 (4 marks) An archer knows from experience that she can hit the target zone 80% of the time.

Let Z be the number of shots taken to hit the target. Calculate each probability.

a $P(Z = 3)$ *[1 mark]*

b $P(Z = 10)$ *[1 mark]*

c $P(Z < 5)$ *[1 mark]*

d $P(Z \geq 5)$ *[1 mark]*



Worksheets
Binomial probability experiments

Using the binomial probability distribution

4.7 Binomial distributions

The **binomial distribution** represents the number of successes in n independent Bernoulli trials for some given value of n. The binomial distribution results from the performance of a **binomial experiment**.

- There is a fixed number, n, of repeated trials.
- Each trial has exactly 2 outcomes: success (p) and failure (q).
- Each trial is independent (the probability of success stays the same).

The **binomial random variable** is the number of successes for n trials.

WORKED EXAMPLE 15 Binomial experiments

Identify whether each situation is a binomial experiment.

a Tossing a fair coin many times and noting the number of tails that occur

b Selecting and replacing a pen from a basket containing 5 red and 7 black pens 12 times, and recording the number of red pens drawn

c Selecting 25 people at random from the population and recording their age in years

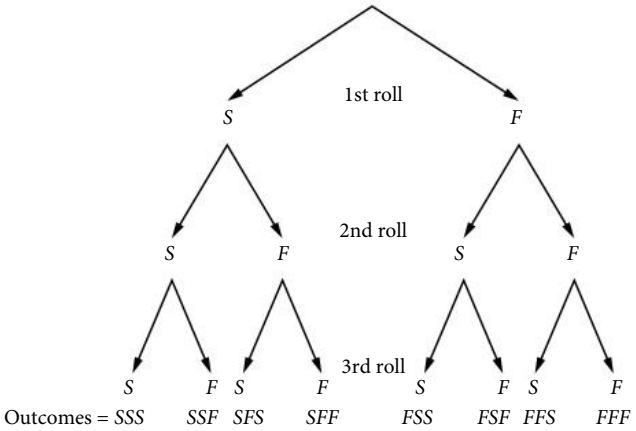
d Selecting 15 people at random from the population and noting if they have more than one sibling

Steps		Working
a	**1** Is the number of trials fixed?	There are 'many' trials, so n is not fixed.
	2 State the result.	This is not a binomial experiment.
b	**1** Is the number of trials fixed?	There are 12 trials.
	2 Are there exactly 2 outcomes?	The outcomes are 'red' (success) and 'not red' (failure).
	3 Is p the same for each trial?	Pens are replaced, so $p = \frac{5}{12}$ for every trial.
	4 Are trials independent?	Replacement means the trials are independent.
	5 State the result.	This is a binomial experiment.
c	**1** Is the number of trials fixed?	There are 25 trials.
	2 Are there exactly 2 outcomes?	There are many different outcomes.
	3 State the result.	This is not a binomial experiment.
d	**1** Is the number of trials fixed?	There are 15 trials.
	2 Are there exactly 2 outcomes?	The outcomes are 'more than 1 sibling' (success) and 'not more than 1 sibling' (failure).
	3 Is p the same for each trial?	For a large population, the probability that a person has more than 1 sibling is the same for every trial.
	4 Are trials independent?	People are selected randomly, so the trials are independent.
	5 State the result.	This is a binomial experiment.

Consider the binomial experiment of rolling a die 3 times. A success (S) is a composite number (that is, 4 or 6) and a failure (F) is a non-composite number. This is a binomial experiment where:

- $P(S) = P(\text{composite}) = p = \frac{2}{6} = \frac{1}{3}$
- $P(F) = P(\text{non-composite}) = q = \frac{2}{3}$

There are many different ways to determine the outcomes of the experiment. For example, a tree diagram gives the following.



The outcomes and probabilities are summarised in the table.

Number of successes	Possible outcomes	Probability
0	*FFF*	$1 \times \left(\frac{2}{3}\right)^3 = \frac{8}{27}$
1	*SFF, FSF, FFS*	$3 \times \left(\frac{2}{3}\right)^2 \left(\frac{1}{3}\right) = \frac{12}{27}$
2	*SSF, SFS, FSS*	$3 \times \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{6}{27}$
3	*SSS*	$1 \times \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

You can see that the values of the binomial distribution are actually the terms of the binomial expansion:

$$(p+q)^3 = {}^3C_0p^3 + {}^3C_1p^2q + {}^3C_2pq^2 + {}^3C_3q^3$$

This means that rather than using a tree diagram or another method, it is possible to calculate the probabilities of the various terms of a binomial distribution using the binomial expansion.

$$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_nb^n$$

To apply the binomial expansion to the binomial distribution, you replace a with p and b with q and let n be the number of trials in the experiment.

Remember, ${}^nC_r$ can be written as $\binom{n}{r}$.

The binomial distribution

For the binomial random variable X, the probability of x successes from n trials is

$$P(X=x) = \binom{n}{x}p^xq^{n-x} \quad \text{or} \quad P(X=x) = \binom{n}{x}p^x(1-p)^{n-x}$$

where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$

WORKED EXAMPLE 16 Binomial distributions

A binomial distribution has the probability function

$$P(X = x) = \binom{6}{x}(0.2)^x(0.8)^{6-x}$$

a **Determine** the number of trials in this experiment.

b **Calculate** the probability of success in any trial.

c **Organise** the distribution as a table.

Steps	Working
a Compare the function with the definition.	$n = 6$, so there are 6 trials.
b Compare the function with the definition.	The probability of success, $p = 0.2$
c 1 Write the probability function.	$P(X = x) = \binom{6}{x}(0.2)^x(0.8)^{6-x}$
2 Let $x = 0$.	$P(X = 0) = \binom{6}{0}(0.2)^0(0.8)^{6-0}$
3 Evaluate.	$= 1 \times 1 \times (0.8)^6$ $= 0.262\,144$
4 Let $x = 1$.	$P(X = 1) = \binom{6}{1}(0.2)^1(0.8)^{6-1}$
5 Evaluate.	$= 6 \times 0.2 \times (0.8)^5$ $= 0.393\,216$

6 Do the remaining calculations and put the results in a table.

x	0	1	2	3	4	5	6
$P(X = x)$	0.2621	0.3932	0.2458	0.0819	0.0154	0.001 536	0.000 064

The binomial probability function provides a convenient way for working out the probabilities of any number of successes in a binomial experiment.

Video Binomial probability

WORKED EXAMPLE 17 Binomial probability

Ramy sits for a test containing 10 multiple choice questions, each with 5 choices.
Calculate the probability that Ramy scores 7 correct answers after guessing every answer.

Steps	Working
1 Calculate the probability of success.	$p = \frac{1}{5}$
2 Calculate the probability of failure.	$q = 1 - \frac{1}{5} = \frac{4}{5}$
3 Identify n.	There are 10 trials, so $n = 10$.
4 Write the probability distribution function.	$P(X = x) = \binom{n}{x}p^x q^{n-x}$
5 Substitute for known values and evaluate.	$P(X = 7) = \binom{10}{7}\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^3$ $= 0.000\,786\ldots$
6 Round off and state the result.	The probability of guessing 7 correct answers is about 0.000 79 or 0.079%.

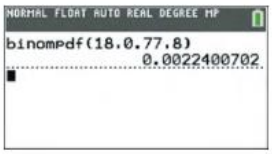
Using your graphics calculator simplifies calculation of binomial probabilities.

WORKED EXAMPLE 18 Binomial probabilities on a calculator

Calculate the probability of obtaining 8 successes in a binomial experiment with 18 trials, where $p = 0.77$.

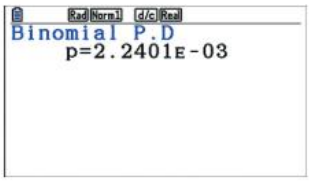
Steps	Working
1 State the parameters.	$n = 18$ $p = 0.77$ $x = 8$

TI-84 Plus CE



1 Press 2nd vars ([distr]), then select **A:binompdf(** (binomial probability distribution function).
2 Enter the parameters.
3 Press ENTER to display the result.

Casio fx-CG50 AU



1 Select the Statistics screen from the main menu.
2 Press F5 (DIST) then F5 (BINOMIAL), then F1 (Bpd) (binomial probability distribution).
3 Press F2 (Var) (if necessary) and then enter the parameters.

2 State the result.	The probability of 8 successes is about 0.0022.

TI-Nspire
Discrete random variables and the binomial distribution

Sometimes you will need to interpret questions to work out what values are required.

WORKED EXAMPLE 19 Probabilities and binomials

In a binomial experiment with 8 trials, the probability of success in any trial is 0.6.

If X is the random variable that represents the number of successes, **calculate** the probability that

a X is at least 5
b X is less than 3
c X is at most 4
d X is greater than 1

Steps	Working
a 1 List the parameters.	$n = 8$ $p = 0.6$
2 'At least 5' means 5 or more.	$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7) + P(X = 8)$
3 Use the formula.	$= \binom{8}{5}(0.6)^5(0.4)^3 + \binom{8}{6}(0.6)^6(0.4)^2 + \binom{8}{7}(0.6)^7(0.4) + \binom{8}{8}(0.6)^8$
4 Evaluate using a calculator.	$= 0.594\,086\ldots$
5 State the result.	The probability of at least 5 successes is about 0.5941.
b 1 'Less than 3' does not include 3.	$P(X < 3) = P(X = 2) + P(X = 1) + P(X = 0)$ $= \binom{8}{2}(0.6)^2(0.4)^6 + \binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8$ $= 0.049\,807\ldots$
2 State the result.	The probability of fewer than 3 successes is about 0.0498.

c	**1** 'At most 4' means 4 or fewer.	$P(X \leq 4) = P(X = 4) + P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0)$ $= \binom{8}{4}(0.6)^4(0.4)^4 + \binom{8}{3}(0.6)^3(0.4)^5 + \binom{8}{2}(0.6)^2(0.4)^6 + \binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8$ $= 0.405\,913\ldots$
	2 State the result.	The probability of at most 3 successes is about 0.4059.
d	**1** State the condition.	$P(X > 1) = P(X = 2) + P(X = 3) + \ldots + P(X = 8)$
	2 Use counting techniques.	$= 1 - P(X \leq 1)$ $= 1 - [P(X = 1) + P(X = 0)]$
	3 Use the formula.	$= 1 - \left[\binom{8}{1}(0.6)(0.4)^7 + \binom{8}{0}(0.4)^8\right]$
	4 Evaluate using a calculator.	$= 0.991\,480\ldots$
	5 State the result.	The probability of more than 1 success is about 0.9915.

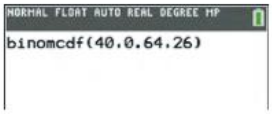
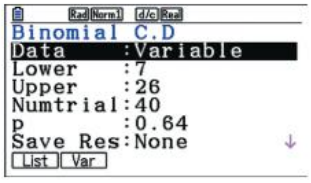
In addition to the binomial **probability density function** (PDF), your calculator also has a binomial probability **cumulative distribution function** (CDF). While the probability distribution function of say $P(X = 4)$ gives the probability of exactly 4 successes, the cumulative distribution function of $P(X = 4)$ gives the sum of the probabilities of exactly 0, 1, 2, 3 and 4 successes.

The table shows the binomial probability distribution and binomial probability cumulative distribution for $n = 6$ and $p = 0.25$.

X	PDF	CDF
0	0.1780	0.1780
1	0.3560	0.5340
2	0.2966	0.8306
3	0.1318	0.9624
4	0.0330	0.9954
5	0.0044	0.9998
6	0.0002	1.0000

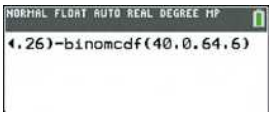
WORKED EXAMPLE 20 More probabilities with technology

A binomial experiment has 40 trials. The probability of success in any trial is 0.64. If the random variable X is the number of successes, use a graphics calculator to **determine** the probability that X is between 7 and 26.

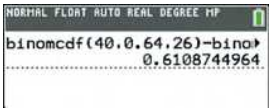
Steps	Working
1 'Between' means that 7 and 26 are included. Write the condition.	$7 \leq x \leq 26$
2 State the parameters.	$n = 40$ $p = 0.64$
3 The TI-84 Plus CE uses only an upper bound for x in the cumulative distribution calculation. You need to subtract the cumulative distribution total up to $(x = 7 - 1 = 6)$ from the cumulative distribution total up to $x = 26$.	**TI-84 Plus CE**  1 Press 2nd vars ([distr]), then select **B:binomcdf**((binomial cumulative distribution function). 2 Enter the parameters with $x = 26$, then press ENTER.
	Casio fx-CG50 AU  1 Select the Statistics screen from the main menu. 2 Press F5 (DIST) then F5 (BINOMIAL), then F2 (Bcd) (binomial cumulative distribution).

TI-Nspire
Discrete random variables and the binomial distribution

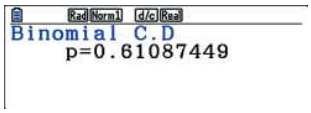
The Casio calculator uses an upper and lower bound for x.



3 Press (−) and then press 2nd vars ([distr]), then select **B:binomcdf**((binomial cumulative distribution function).



4 Enter the parameters with $x = 6$, then press ENTER. Press ENTER to display the result.



3 Press F2 (Var) (if necessary) and then enter the parameters.

4 State the result.

The probability that X is between 7 and 26 is about 0.6109.

EXERCISE 4.7 Binomial distributions

ANSWERS p. 402

Recap

1 It is known that 9% of the components produced on a certain assembly line have at least one defect. Components are randomly sampled from this line one at a time and tested. What is the probability that the first non-defective component is found

a on the third trial?

b before the fourth trial?

2 The probability that a random blood donation is type B is 0.1. What is the probability that the first type B sample is collected after more than 5 samples have been collected?

Mastery

3 WORKED EXAMPLE 15 Which of the following situations are binomial experiments? For those that are not, indicate why not.

a Tossing a fair coin 2000 times and counting the number of times a head appears

b Drawing a card from a standard deck with replacement 50 times and noting the number of times a diamond is drawn

c Tossing a fair coin and counting the number of tosses before a head appears

d Selecting 40 people at random from the population and recording their eye colour

e Checking 5 heavy machinery operators for alcohol consumption from a workforce of 20 operators of whom it is known that 5% consume alcohol before work

f A shipment of 50 appliances contains 2 defective appliances. A retailer tests 15 of the appliances (without replacement) and counts the number that fail to meet specifications.

g Rolling a normal die 100 times and counting the number of times a 3 is rolled

h An experienced archer knows that she has a 20% chance of hitting the inner bullseye. She shoots 10 arrows and records the number of times the inner bullseye is hit.

4 Evaluate each expression, correct to 4 places if necessary.

a $\binom{5}{3}(0.5)^3(0.5)^2$ **b** $\binom{8}{2}(0.6)^2(0.4)^6$ **c** $\binom{6}{3}(0.25)^3(0.75)^3$

d $\binom{9}{5}(0.72)^5(0.28)^4$ **e** $\binom{7}{0}(0.55)^0(0.45)^7$ **f** $\binom{15}{10}(0.88)^{10}(0.12)^5$

5 WORKED EXAMPLE 16 Identify the parameters p, q and n for each binomial probability distribution.

a $\binom{12}{2}(0.5)^2(0.5)^{10}$ **b** $\binom{15}{10}(0.1)^{10}(0.9)^5$ **c** $\binom{7}{1}(0.6)(0.4)^6$

d $\binom{25}{11}(0.23)^{11}(0.77)^{14}$ **e** $\binom{8}{8}(0.47)^8$ **f** $\binom{20}{0}(0.33)^{20}$

6 Calculate the probability distribution for each of the following and graph the result.

a $P(X = x) = \binom{10}{x}(0.2)^x(0.8)^{10-x}$

b $P(X = x) = \binom{10}{x}(0.5)^x(0.5)^{10-x}$

c $P(X = x) = \binom{10}{x}(0.8)^x(0.2)^{10-x}$

7 Comment on the symmetry of each distribution drawn in Question **6**.

8 A binomial variable, X, has the probability function

$$P(X = x) = \binom{5}{x}(0.35)^x(0.65)^{5-x}$$

a How many trials are in this experiment?

b What is the probability of success in any trial?

c Show the distribution as a table.

d Plot the distribution and comment on its symmetry.

9 A binomial variable, X, has the probability function

$$P(X = x) = \binom{8}{x}(0.72)^x(0.28)^{8-x}$$

a How many trials are in this experiment?

b What is the probability of success in any trial?

c Show the distribution as a table.

d Plot the distribution and comment on its symmetry.

10 Consider a binomial experiment with $n = 5$, $p = 0.6$, and $x =$ the number of successes.
Use the formula for $P(X = x)$ to find the probability for

a $x = 0$ **b** $x = 3$ **c** $x = 5$

11 Consider a binomial experiment with $n = 12$, $p = 0.32$, and $x =$ the number of successes.
Use the formula for $P(X = x)$ to find the probability for

a $x = 3$ **b** $x = 9$ **c** $x = 11$

12 WORKED EXAMPLE 17 A die is rolled 10 times. Calculate the probability that a 6 results exactly 4 times.

13 WORKED EXAMPLE 18 Use a graphics calculator to calculate the probability of

a 8 successes in a binomial experiment with 12 trials, where $p = 0.82$

b 3 successes in a binomial experiment with 15 trials, where $p = 0.26$

c 5 successes in a binomial experiment with 11 trials, where $p = 0.54$

d 4 successes in a binomial experiment with 14 trials, where $p = 0.18$.

14 In a binomial experiment, $n = 25$, $p = 0.635$ and $X =$ number of successes. Calculate the probability that

a $X = 15$ **b** $X = 7$ **c** $X = 20$

d $X = 7$ or 15 **e** $X = 20$ or 21

15 A binomial experiment has 20 trials and $p = 0.385$.
Calculate the probability that

a exactly 12 successes occur

b exactly 6 successes occur

c either 9 or 10 successes occur

d 5 or 6 successes occur.

16 WORKED EXAMPLE 19 In a binomial experiment with 6 trials, the probability of success in any trial is 0.4. If X is the random variable that represents the number of successes, calculate the probability that

a X is at least 3 **b** X is greater than 4

c X is at most 5 **d** X is less than 3

17 In a binomial experiment with 4 trials, the probability of success in any trial is 0.8.
If X is the random variable that represents the number of successes, calculate the probability that

a X is 3 **b** X is less than 3

c X is no more than 3 **d** X is greater than 3

18 A binomial experiment has 7 trials and $p = 0.3$. Calculate the probability that

a exactly 2 successes occur

b fewer than 4 successes occur

c at least 5 successes occur

d at most 2 successes occur.

19 WORKED EXAMPLE 20 A binomial experiment has 50 trials, where the probability of success in any trial is 0.47. If the random variable X is the number of successes, use a graphics calculator to find the probability that

a X is from 11 to 30 **b** X is from 5 to 45

c $X > 30$ **d** $X \leq 30$

Exam practice

20 © QCAA 2020 1Q8 TECH-FREE A test includes six multiple choice questions. Each question has four options for the answer. If the answers are guessed, the probability of getting at most two questions correct is represented by

A $\binom{6}{0}0.25^0 \times 0.75^6 + \binom{6}{1}0.25^1 \times 0.75^5$

B $\binom{6}{0}0.25^0 \times 0.75^6 + \binom{6}{1}0.25^1 \times 0.75^5 + \binom{6}{2}0.25^2 \times 0.75^4$

C $1 - \left(\binom{6}{0}0.25^0 \times 0.75^6 + \binom{6}{1}0.25^1 \times 0.75^5\right)$

D $1 - \left(\binom{6}{0}0.25^0 \times 0.75^6 + \binom{6}{1}0.25^1 \times 0.75^5 + \binom{6}{2}0.25^2 \times 0.75^4\right)$

21 © QCAA 2020S 1Q1 TECH-FREE A random variable X is binomially distributed with 10 trials and a probability of success equal to 0.25 at each attempt. The value of $P(X \geq 1)$ is equal to

A $\left(\frac{1}{4}\right)^{10}$ **B** $\left(\frac{3}{4}\right)^{10}$ **C** $1-\left(\frac{1}{4}\right)^{10}$ **D** $1-\left(\frac{3}{4}\right)^{10}$

22 © QCAA 2023 1Q13 TECH-FREE (5 marks) At a certain airport, the departure of one in five international flights is delayed every day. The status of any flight is independent of other flights.

One international flight is selected at random each day for three days. Each selection is recorded as either 'delayed' or 'not delayed'.

a State two conditions that make this context suitable for modelling using a binomial random variable. *[2 marks]*

b Calculate the probability that at least two of the selected flights were delayed. *[3 marks]*

23 (4 marks) In a binomial experiment with 35 trials, the probability of success in any trial is 0.74. Calculate the probability that

a between 20 and 30 successes occur *[1 mark]*

b between 15 and 25 successes occur *[1 mark]*

c no more than 7 successes occur *[1 mark]*

d at least 18 successes occur. *[1 mark]*

24 (4 marks) Consider a binomial experiment with $n = 9$, $p = 0.75$ and $X =$ the number of successes.

Find the probability that

a X is less than 4 *[1 mark]*

b X equals 4 *[1 mark]*

c X is greater than 3 and less than 7 *[1 mark]*

d X is greater than or equal to 3 and less than or equal to 7. *[1 mark]*

25 (9 marks) A binomial distribution has the probability function

$$P(X = x) = \binom{8}{x}(0.38)^x(0.62)^{8-x}$$

a How many trials are in this experiment? *[1 mark]*

b What is the probability of success in any trial? *[1 mark]*

c Show the distribution as a table. *[7 marks]*

26 (3 marks) A bag contains 3 blue and 7 red marbles. A marble is randomly selected and replaced. This process is repeated 9 times. Calculate the probability that a blue marble is selected 5 times.

27 (4 marks) CF In a binomial experiment with 11 trials, the probability of success in any trial is 0.7. If X is the random variable that represents the number of successes, calculate the probability that

a X is at least 4 *[1 mark]*

b X is no more than 5 *[1 mark]*

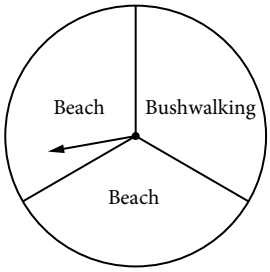
c X is greater than 2 *[1 mark]*

d X is at most 3. *[1 mark]*

28 (4 marks) CU A binomial experiment has 50 trials, for which the probability of success in any trial is 0.32. If the random variable Z is the number of successes, use a graphics calculator to find the probability that Z is between 15 and 25.

29 © QCAA 2024 1Q12 TECH-FREE (6 marks)

Each day over a three-day long weekend, a family spins a pointer on a circular board to decide whether they will spend the day at the beach or bushwalking. The circular board consists of three equal sections.



a Determine the probability that the family will spend all three days bushwalking. *[1 mark]*

b Determine the following binomial probabilities, expressed as fully simplified fractions.

i Exactly two days will be spent at the beach *[2 marks]*

ii Fewer than three days will be spent at the beach *[3 marks]*

Worksheet
Binomial probability – Mean and standard deviation

Spreadsheet
Binomial distribution

4.8 The mean and variance of binomial distributions

Investigation Shapes of binomial distributions

Download the spreadsheet 'Binomial distribution' from Nelson MindTap to investigate the shapes of binomial distributions for different values of n and p.

1 What do you find about the shape of the distribution for larger values of n?

2 What happens to the graph as the value of p gets closer to 0?

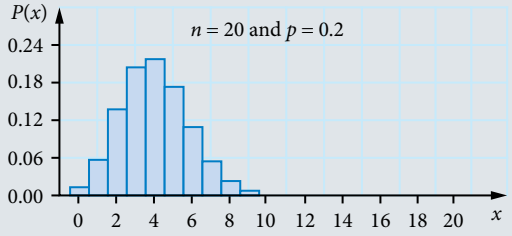
3 What happens to the graph as the value of p gets closer to 1?

4 How does the expected value change as n and p are changed?

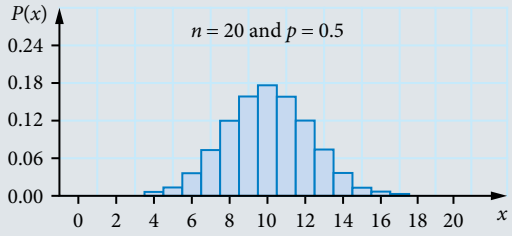
This investigation shows that the value of p for a binomial distribution influences the symmetry of the graph of the distribution.

The shape of the binomial distribution

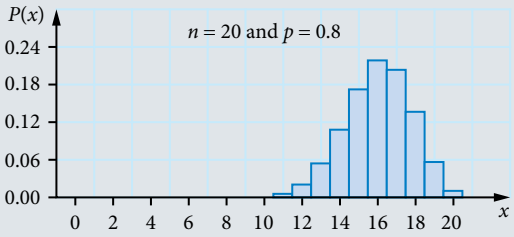
For $p < 0.5$, the distribution is **positively skewed**; that is, it is skewed to the right. The 'tail' is on the positive (right) side.



For $p = 0.5$, the distribution is **symmetrical**.



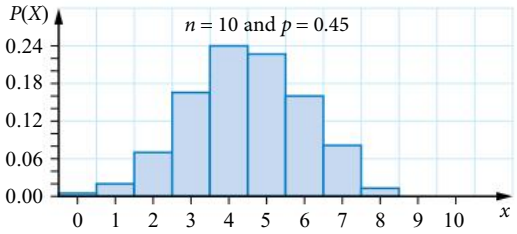
For $p > 0.5$, the distribution is **negatively skewed**; that is, it is skewed to the left. The 'tail' is on the negative (left) side.



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WORKED EXAMPLE 21 Shape of a binomial distribution

Sketch a histogram of the binomial distribution for 10 trials in which $p = 0.45$ and comment on the skewness of the distribution.

Steps	Working
1 State the parameters.	$n = 10$ $p = 0.45$
2 Use a calculator to calculate the probabilities.	$P(X = 0) = 0.002\,532\ldots$ $P(X = 1) = 0.020\,724\ldots$ $P(X = 2) = 0.076\,302\ldots$ $P(X = 3) = 0.166\,478\ldots$ $P(X = 4) = 0.238\,366\ldots$ and so on.
3 Draw the histogram.	
4 Describe the skewness of the graph.	The graph is positively skewed.

A binomial distribution has n Bernoulli trials.

Each trial has an expected value of p, so the expected value of n trials is the sum of all of them, np.

Similarly, the variance of each trial is $p(1 - p)$, as this is the variance of a Bernoulli trial.

The variance of n trials is just the sum of all of them, so the variance of the binomial distribution is $np(1 - p)$.

Expected value, variance and standard deviation of a binomial distribution

For a binomial distribution, if X is the number of successes in n trials, each with probability of success p and probability of failure q, the **expected value**, **variance** and **standard deviation** are

$$E(X) = \mu = np$$

$$Var(X) = np(1 - p) = npq$$

$$\text{SD}(X) = \sigma = \sqrt{np(1-p)} = \sqrt{npq}$$

WORKED EXAMPLE 22 Expected value, variance and standard deviation

In a particular binomial experiment with 90 trials, the probability of success in any trial is 0.16. The random variable X is the number of successes.

a **Determine** the expected value of X.

b **Determine** the probability that X is within one standard deviation of the mean.

Steps	Working
a 1 Identify the parameters.	$n = 90$ $p = 0.16$
2 Write the rule for expected value.	$E(X) = np$
3 Substitute for known values and evaluate.	$= 90 \times 0.16$ $= 14.4$
4 State the result.	The expected value of X is either 14 or 15.

b 1	Write the rule for standard deviation.	$\sigma = \sqrt{np(1-p)}$
2	Substitute for known values and evaluate.	$= \sqrt{90 \times 0.16 \times 0.84}$ $= 3.477\,93\ldots$
3	Round off.	≈ 3.48
4	Write an expression for the values within 1 standard deviation of the mean.	$\mu - \sigma < X < \mu + \sigma$
5	Substitute for known values and evaluate.	$14.4 - 3.48 < X < 14.4 + 3.48$ $10.92 < X < 17.88$
6	Identify the required values of X.	Only the values from 11 to 17 are within 1 standard deviation of the mean.
7	State the required probability.	$P(\mu - \sigma < X < \mu + \sigma) = P(11 \leq X \leq 17)$
8	Use the binomial cumulative distribution function on your calculator.	≈ 0.6870
9	State the result.	There is about a 69% chance that X is within one standard deviation of the mean.

WORKED EXAMPLE 23 The mean and variance

A binomial random variable, Z, has a mean of 16 and a variance of 5.76.

a **Calculate** the probability of success, p.

b **Determine** the number of trials, n.

Steps		Working
a 1	Write the rules for the mean and variance.	$\mu = np$ $Var(Z) = np(1-p)$
2	Substitute the known values.	$16 = np$ [1] $5.76 = np(1-p)$ [2]
3	Substitute for np in [2].	$5.76 = 16(1-p)$
4	Solve for p.	$\frac{5.76}{16} = 1 - p$ $p = 1 - \frac{5.76}{16}$ $p = 0.64$
5	State the result.	The probability of success is 0.64.
b 1	Write the rule for the mean.	$\mu = np$
2	Substitute for known values.	$16 = n \times 0.64$
3	Solve for n.	$n = 25$
4	State the result.	There are 25 trials.

EXERCISE 4.8 The mean and variance of binomial distributions

ANSWERS p. 403

Recap

1 Determine whether each experiment is a binomial experiment.

a Players are dealt a card from a well-shuffled pack. The first player with a spade wins.

b Players are dealt a card from a well-shuffled pack. Each player's card is replaced before the next card is dealt. The first player with a spade wins.

c Each player throws a die 5 times. The player with the most 6s wins.

d The number of male children in families is counted.

e The number of male children in families with 5 children is counted.

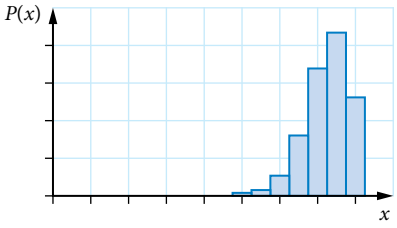
2 Use a graphics calculator to calculate the probability of obtaining 12 successes in a binomial experiment with 24 trials, where $p = 0.68$.

Mastery

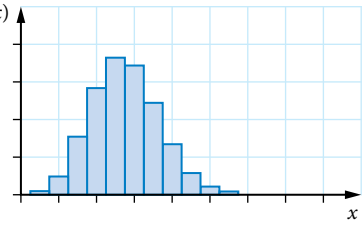
3 The diagrams below show a number of probability distributions for a binomial experiment with 15 trials. Match each diagram to one of these probabilities.

0.1 0.5 0.9 0.3 0.7

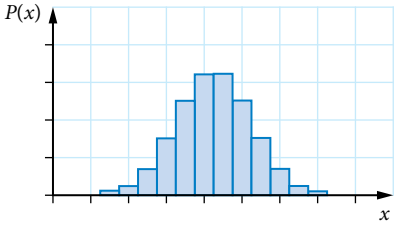
a



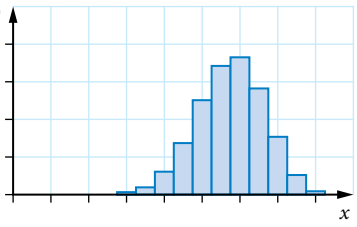
b



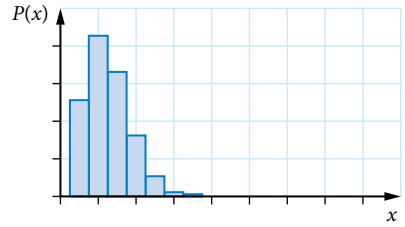
c



d



e



4 WORKED EXAMPLE 21 Draw a histogram of the binomial distribution of 8 trials in which $p = 0.25$ and comment on the skewness of the distribution.

5 Draw a histogram of the binomial distribution of 12 trials in which $p = 0.78$ and comment on the skewness of the distribution.

6 Draw a histogram of the binomial distribution of 10 trials in which $p = 0.44$ and comment on the skewness of the distribution.

7 WORKED EXAMPLE 22 A binomial experiment has 150 trials for which the probability of success in each trial is 0.38. If the random variable X is the number of successes, calculate

a $E(X)$ **b** $Var(X)$

8 Calculate the mean and standard deviation of a binomial distribution with

a $n = 15$ and $p = 0.26$

b $n = 40$ and $p = 0.82$

c $n = 110$ and $p = 0.34$

d $n = 88$ and $p = 0.17$

9 A binomial experiment has 20 trials for which the probability of success is $\frac{3}{14}$.
The random variable Z is the number of successes.

a What is the expected value of Z?

b What is the probability that Z is within 1 standard deviation of the mean?

c What is the probability that Z is within 2 standard deviations of the mean?

10 A binomial experiment has 180 trials for which the probability of success is 0.27.
The random variable X is the number of successes.

a What is the expected value of X?

b What is the probability that X is within 2 standard deviations of the mean?

11 WORKED EXAMPLE 23 A binomial random variable, X, has a mean of 7.6 and a variance of 4.712.

a Calculate the probability of success.

b Find the number of trials.

12 A binomial random variable, Z, has a mean of 23.1 and a variance of 7.854.

a Calculate the probability of success.

b Determine the number of trials.

Exam practice

Use the following information to answer Questions 13 and 14.

A binomial experiment is completed 25 times and has an expected value of 11.

13 Find the probability of success.

A 0.44 **B** 2.482 **C** 6.16 **D** 37.945

14 Find the variance for the total number of successes.

A 0.44 **B** 2.482 **C** 6.16 **D** 37.945

15 © QCAA 2020 1Q6 TECH-FREE If the probability of success in a Bernoulli trial is 0.30, the variance is

A 0.70 **B** 0.46 **C** 0.30 **D** 0.21

16 © QCAA 2020S 2Q12 (5 marks) People with type O blood are said to be 'universal donors'. In Australia, 9% of the population has this blood type.
On a given day, a random group of 45 people volunteer to donate blood.

a Identify why this context is suitable for modelling as a binomial distribution. *[1 mark]*

b Determine the mean and standard deviation of the number of people who are universal donors. *[2 marks]*

c Determine the probability that no more than 3 of the donors are universal donors. *[2 marks]*

17 (4 marks) A random binomial variable Z has a mean of 36 and a standard deviation of 3. What is the probability of success?

18 (4 marks) It is known that 70% of a shipment of car parts will pass a safety check. Car parts are randomly selected and their safety checked. What is the smallest number of car parts that must be selected to ensure that the probability that at least one will pass the safety check is more than 0.9?

19 (4 marks)

a Draw graphs of the binomial distributions for $p = 0.1, 0.3, 0.5, 0.7$ and 0.9 for $n = 12$. *[3 marks]*

b Comment on the shapes of the graphs. *[1 mark]*

20 (4 marks) CU A binomial random variable, Y, has a mean of 5.4 and a variance of 2.97.

a Find the probability of success, p. *[2 marks]*

b Determine the number of trials, n. *[2 marks]*

21 © QCAA 2024 2Q17 (3 marks) CF

At a particular orchard, 3% of fruit is bruised during picking. After picking, the fruit is packed into boxes, each containing four pieces of fruit.

A grocery shop orders 140 boxes of fruit to sell to its customers.

Determine the expected number of boxes that will contain bruised fruit. *[3 marks]*

4.9 Problems involving binomial distributions

There are many different applications of the binomial probability distribution, particularly for decision making. Before you treat a particular situation as a binomial experiment, you should check that it has the required properties.

Worksheet
Applications of binomial probability

WORKED EXAMPLE 24 Applying binomial distributions

40% of the residents in a regional city are in favour of the construction of a traffic roundabout in the main street. Suppose 15 residents are randomly selected. **Calculate** the probability that

a 7 favour the construction of the roundabout

b fewer than 4 favour the construction of the roundabout

c more than 2 favour the construction of the roundabout.

Steps	Working
a **1** Check this situation against the properties of a binomial experiment.	There is a fixed number of trials ($n = 15$). Each trial can be assumed to be independent. There are only 2 outcomes: support and non-support for the roundabout. The probability of success is constant ($p = 0.4$).
2 State the conclusion.	This is a binomial experiment.
3 Write the parameters.	$n = 15$ $p = 0.4$
4 Let the random variable X be the number of successes.	Required probability $= P(X = 7)$
5 Write the probability function.	$P(X = 7) = \binom{15}{7}(0.4)^7(0.6)^8$
6 Evaluate.	$= 0.177\,083\ldots$
7 State the result.	The probability that 7 people favour the construction of the roundabout is about 0.177 (17.7%).

b **1**	'Fewer than 4' means $X \leq 3$.	$P(X \leq 3) = P(X = 3) + P(X = 2) + P(X = 1) + P(X = 0)$
2	Use the probability function.	$= \binom{15}{3}(0.4)^3(0.6)^{12} + \binom{15}{2}(0.4)^2(0.6)^{13} + \binom{15}{1}(0.4)(0.6)^{14} + \binom{15}{0}(0.6)^{15}$
3	Evaluate.	$= 0.090\,50\ldots$
4	State the result.	The probability that fewer than 4 people favour the construction of the roundabout is about 0.091 (9.1%).
c **1**	'More than 2' means $X \geq 3$.	$P(X \geq 3) = P(X = 3) + P(X = 4) + \ldots + P(X = 15)$
		$= 1 - P(X \leq 2)$
2	Use the probability function.	$= 1 - \binom{15}{2}(0.4)^2(0.6)^{13} - \binom{15}{1}(0.4)(0.6)^{14} - \binom{15}{0}(0.6)^{15}$
		$= 1 - 0.027\,114\ldots$
		$= 0.972\,885\ldots$
3	State the result.	The probability that more than 2 people favour the construction of the roundabout is about 0.973 (97.3%).

WORKED EXAMPLE 25 Probability model

An aptitude test is required of all applicants for a position as a courier. The test consists of 15 multiple choice questions, each with 5 choices. Seth decides to randomly guess the answer to each question.

a **Determine** many answers Seth should expect to correctly guess.

b **Calculate** the probability that the number of correct answers guessed is within 1 standard deviation of the mean.

Steps		Working
a **1**	This is a binomial experiment. Identify the parameters.	$n = 15$ $p = \frac{1}{5} = 0.2$
2	Let the random variable X be the number of successes.	
3	Write the rule for expected value.	$E(X) = np$
4	Substitute for known values and evaluate.	$= 15 \times 0.2$ $= 3$
5	State the result.	The applicant should expect to correctly guess 3 answers.
b **1**	Calculate the standard deviation.	$\sigma = \sqrt{npq}$
2	Substitute for known values.	$= \sqrt{15 \times 0.2 \times 0.8}$
3	Evaluate and round off.	≈ 1.55
4	Write an expression for the values within 1 standard deviation of the mean.	$\mu - \sigma < X < \mu + \sigma$
5	Substitute for known values and evaluate.	$3 - 1.55 < X < 3 + 1.55$ $1.45 < X < 4.55$
6	Identify the required values of X.	Only the values from 2 to 4 are within 1 standard deviation of the mean.

7 State the required probability.	$P(\mu - \sigma < X < \mu + \sigma) = (2 \le X \le 4)$
8 Use the binomial CDF on your calculator.	≈ 0.6686
9 State the result.	There is about a 67% chance that the number of correct answers guessed is within one standard deviation of the mean.

WORKED EXAMPLE 26 Problems involving the binomial distribution

The probability of an archer hitting a target with any shot is 0.4.

a If the archer shoots 8 times, **determine** the probability of hitting the target at least twice.

b **Calculate** the number of times the archer must shoot to be sure that the probability of hitting the target at least once is greater than 75%.

Steps	Working
a 1 Identify the parameters.	$n = 8$ $p = 0.4$
2 Write the required probability.	$P(X \ge 2) = P(X = 2) + P(X = 3) + \ldots + P(X = 8)$ $= 1 - P(X < 2)$
3 Use the probability function.	$= 1 - \binom{8}{1} \times (0.4)^1 \times (0.6)^7 - \binom{8}{0} \times (0.6)^8$ $= 1 - 0.106\,375\ldots$ $= 0.893\,624\ldots$
4 State the result.	There is about an 89% chance of hitting the target at least twice.
b 1 Write the required probability.	P(hitting target at least once) $= P(X \ge 1)$ $= 1 - P$(not hitting the target) $= 1 - P(X = 0)$ $= 1 - (0.6)^n$
2 Apply the condition.	$1 - (0.6)^n > 0.75$
3 Interpret this result.	You need to find the smallest value of n for which $(0.6)^n < 0.25$.
4 Try values of n until the condition is satisfied.	When $n = 0$, $(0.6)^0 = 1 > 0.15$ When $n = 1$, $(0.6)^1 > 0.25$ When $n = 2$, $(0.6)^2 > 0.25$ When $n = 3$, $(0.6)^3 = 0.216 < 0.25$
5 State the result.	The archer must fire 3 shots to be sure that the probability of hitting the target at least once is greater than 75%.

EXERCISE 4.9 Problems involving binomial distributions

ANSWERS p. 404

Recap

1 a Draw graphs of the binomial distributions for $n = 4$, 8, 12 and 16 for $p = 0.6$.

b Comment on the shapes of the graphs.

2 In a particular binomial experiment with 120 trials, the probability of success in any trial is 0.35. The random variable X is the number of successes.

a What is the expected value of X?

b What is the probability that X is within one standard deviation of the mean?

Mastery

3 WORKED EXAMPLE 24 A card is drawn and replaced 7 times from an ordinary pack of 52 playing cards. Find the probability that

a 3 diamonds are drawn

b at least one diamond is drawn

c no more than 2 diamonds are drawn

d at most 4 diamonds are drawn.

4 A school computer network consisting of 30 computers was attacked by a computer virus. The virus enters each computer with probability 0.4, independently of other computers. Find the probability that the virus enters

a exactly 5 computers

b at least 10 computers

c no more than 12 computers

d from 5 to 20 computers.

5 WORKED EXAMPLE 25 It is known that 45% of people have blood type O. A random sample of 20 people have their blood type tested.

a Find the probability that more than 7 will have blood type O.

b How many people would you expect to have blood type O?

c What is the probability that the number of people with blood type O is within 1 standard deviation of the mean?

d What is the probability that the number of people with blood type O is within 2 standard deviations of the mean?

6 It is known that approximately 18% of all people have high blood pressure.

The blood pressure of a random sample of 85 people is measured.

a What is the expected number of people with high blood pressure?

b What is the probability that the number of people with high blood pressure is within one standard deviation of the mean?

7 WORKED EXAMPLE 26 A bag contains 5 red, 2 green and 8 blue marbles.

a If a marble is drawn 5 times with replacement, what is the probability of drawing a red marble at least twice?

b How many times must a marble be drawn (with replacement) to be sure that the probability of drawing a red marble at least once is greater than 90%?

8 A large shipment of widgets contains 20% that are defective.

a If 5 widgets are randomly selected from the shipment, what is the probability that none is defective?

b What is the smallest number of widgets that should be selected to ensure that the probability of at least one defective widget being selected is more than 0.95?

9 It is known that 35% of teenagers work in part-time jobs. 20 teenagers are randomly selected and asked if they have a part-time job. Z is the random variable that defines the number of teenagers who have a part-time job.

a Find $P(Z \leq 7)$.

b Find $E(Z)$.

c What is $\text{SD}(Z)$?

d Calculate $P(\mu - \sigma < Z < \mu + \sigma)$.

10 How many aces would you expect to get from 60 selections of a normal pack of cards?

11 The probability that a lettuce seed germinates is about 0.8. About how many seeds should you plant if you want 500 lettuce seedlings?

12 A machine has a probability of 0.05 of manufacturing a defective part.

a What is the expected number of defective parts in a sample of 20 parts manufactured by this machine?

b What is the standard deviation for this situation?

13 A new medication is found to reduce cholesterol levels in 85% of people who use it. The medication is given to 20 people who suffer from high cholesterol. The discrete random variable Y is the number of people who experience reduced cholesterol levels after taking the new medication.

a Find the probability that no more than 10 people experience reduced cholesterol levels.

b Find $E(Y)$ and $\text{SD}(Y)$.

c Calculate $P(\mu - 2\sigma < Y < \mu + 2\sigma)$.

14 It is known that 8% of printed circuit boards made by the AOK company are defective. The company's quality control manager wishes to see whether the percentage of defective boards in a current batch has decreased. The random variable X is the number of defective circuit boards. A sample of size 50 is to be selected. If there is no change from the 8% figure, find

a $E(X)$ **b** $Var(X)$ **c** $\text{SD}(X)$

15 The probability that a car battery fails within 2 years of purchase is about 60%.

a From a sample of 420 batteries, how many would you expect to last longer than 2 years?

b What is the standard deviation of the number of failures in 2 years?

16 A probability distribution for the number of successes, X, is shown in the table.

X	0	1	2	3	4
$P(X = x)$	0.1296	0.3456	0.3456	0.1536	0.0256

Calculate

a $E(X)$ **b** $Var(X)$ **c** $\text{SD}(X)$

17 A telemarketer knows that for any call to a customer there is a 15% chance of making a sale.

a Find the probability that no sales are made in 8 calls.

b Find the probability that more than 4 sales are made in 20 calls.

c A telemarketer is required to achieve a mean of at least 5 sales each day. What is the least number of calls that will need to be made to achieve this target?

d Calculate the least number of calls that the telemarketer will need to make for the probability of making at least 1 sale to be greater than 95%.

18 A law firm has 8 lawyers who all work in the same office. Each lawyer is equally likely to work from home as in the office. How many desks must be in the office so that there are enough desks for the lawyers in the office at least 90% of the time?

19 The probability of a basketball player sinking a basket from the free throw line is 0.45. What is the least number of shots that the player must take to ensure that the probability of sinking a basket at least twice is more than 0.95?

Exam practice

20 A software company has 10 developers who all work in the same office. Each developer is equally likely to work from home or in the office on any given day. How many desks must be in the office so that there are enough desks for the developers in the office at least 85% of the time?

A 6 **B** 7 **C** 8 **D** 9

21 The probability that a tomato seed germinates is about 0.75. About how many seeds should you plant if you want to grow 400 tomato seedlings?

A 480 seeds **B** 520 seeds **C** 560 seeds **D** 600 seeds

22 How many picture cards would you expect to get from 40 selections of a standard pack of playing cards?

A 6 **B** 7 **C** 8 **D** 9

23 (3 marks) Eggs are classified by weight as small, medium or large. From long experience, the poultry farmer knows that 28% of all eggs produced at the farm are classified as being large. 15 eggs are randomly selected from a conveyor belt on which the eggs travel to be classified. Calculate the probability that

a 8 are classified as large *[1 mark]*

b fewer than 5 are classified as large *[1 mark]*

c more than 3 are classified as large. *[1 mark]*

24 (3 marks) For a large population, the month of birth is equally distributed throughout the year. This means that the probability of the date of birth of a randomly chosen person being in any particular month may be taken as $\frac{1}{12}$.

A group of 24 people is randomly selected.

a How many people would you expect to be born in March? *[1 mark]*

b What is the probability that the number of people born in March is within 1 standard deviation of the mean? *[2 marks]*

25 (4 marks) The probability of an experienced chess player winning any game is known to be 0.7.

a If the chess player has 7 games in a tournament, what is the probability of winning at least twice? *[2 marks]*

b How many games must the chess player play to be sure that the probability of winning at least once is greater than 80%? *[2 marks]*

26 (4 marks) X is a discrete random variable that has a Bernoulli distribution. It is known that the standard deviation for this distribution is 0.477.

a Find the variance of X, correct to 4 decimal places. *[2 marks]*

b Find the probability of success, p, if $p > 1 - p$. *[2 marks]*

27 © QCAA 2021 1Q17 TECH-FREE (3 marks) CF

In any five-day working week Leonardo either catches a bus to work or uses another form of transportation. On average, he catches the bus to work on three of the five days. His decision on any given day is independent of his decision on any other day.

Determine the probability that Leonardo catches a bus to work on exactly one day in a given five-day working week.

28 (4 marks) CF The mean of a binomial distribution is 30 and the standard deviation is $\sqrt{12}$. Calculate the probability of between 20 and 30 successes for this distribution.

29 (5 marks) CF A manufacturer decides to reject a large shipment of components if there is more than one unusable item in a sample of 10 randomly selected components. It is known that 10% of the shipment is unusable. What is the probability that the manufacturer will reject the shipment?

EXAM QUESTION ANALYSIS

(5 marks)

A quarter of the large number of fridge magnets in a souvenir shop are predominantly green.
A tourist has picked some out at random. The owner correctly tells her that the probability she has 2 green ones is about 0.2966.
The tourist puts them back and picks out two more than she did previously.
What is the probability that she has 5 predominantly green fridge magnets (correct to 4 decimal places)?

Reading the question

- This is an (approximately) binomial distribution.
- The number of trials is unknown.
- The probability of 2 successes is known.
- 4 decimal places are required in the answer.

Thinking about the question

- Success is getting a green fridge magnet.
- The expression for the number of successes must be solved to find n.
- Then probability for 5 successes from $n + 2$ trials is needed.

Worked solution (✓ = 1 mark)

$P(X = 2) \approx 0.2966$

$\boldsymbol{P(X = 2) = \binom{n}{2} 0.25^2 \times 0.75^{n-2}}$ ✓

For $P(X = 2) < 1$, she must have **picked at least 3 magnets**. ✓

Use trial and error, either with expression or **using ${}^nC_x$ or binomialpdf** on a graphics calculator. ✓

$n = 2$: $P(X = 2) = 0.0625$

$n = 3$: $P(X = 2) = 0.140\ldots$

$n = 4$: $P(X = 2) = 0.210\ldots$

$n = 5$: $P(X = 2) = 0.263\ldots$

$n = 6$: $P(X = 2) = 0.296\,63\ldots$

She initially **picked 6 fridge magnets**. ✓

For 8 fridge magnets $P(X = 5) = \binom{8}{5} 0.25^5 \times 0.75^3 = 0.023\,071\ldots$

The probability that she has 5 predominantly green fridge magnets is **about 0.0231**. ✓

Video
Exam question analysis: Discrete random variables and the binomial distribution

Puzzle
Discrete random variables find-a-word

4 Chapter summary

Discrete random variables

- A **random variable** has a numerical value determined by the outcome of a chance experiment. A random variable is usually shown by a capital letter and specific values by the lowercase letter.
- A discrete **probability distribution** shows probabilities of a discrete random variable. It is also called a **probability function**. The probabilities are shown as $P(X = x)$ or $p(x)$.
- $p(X = x) \geq 0$ and $\Sigma p(x) = 1$.
- A **point estimate** is an estimate of the probability of a single outcome (of a discrete variable) occurring.

Uniform and non-uniform discrete random variables

- The probability function of a **uniform discrete random variable** X with n values is given by $P(X = x) = \frac{1}{n}$.
- Outcomes chosen '**at random**' from a discrete set are equally likely.

The mean and variance of discrete random variables

- The **expected value** of a discrete random variable X is defined by $\mu = E(X) = \Sigma x p(x)$.
- The **variance** of a discrete random variable X is defined as $Var(X) = \Sigma(x - \mu)^2 p(x) = \Sigma x^2 p(x) - \mu^2$, where $\mu = E(X)$ is the expected value of X.
- The **standard deviation** of a discrete random variable X is defined as $\sigma_X = \text{SD}(X) = \sqrt{Var(X)}$.

Problems involving discrete random variables

- A **fair game** is one with an expected value of 0.
- The **odds** for a bet is the ratio of the amount won to the amount bet.
- The **house percentage** is the proportion of money bet that casinos or other betting agencies win in the long run expressed as a percentage.

Bernoulli distributions

- A **Bernoulli event** has only two possible outcomes – **success** and **failure**.
- A **Bernoulli probability distribution** is the distribution for a Bernoulli event.
- $P(\text{success}) = p$ and $P(\text{failure}) = q$ are fixed and $p + q = 1$.
- The **Bernoulli random variable**, X, is given the values of 1 for success and 0 for failure.
- $P(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$
- For a Bernoulli distribution:
  - $E(X) = \mu = p$ and
  - $Var(X) = \sigma^2 = p(1 - p)$.
- A **Bernoulli trial** is when a Bernoulli event is conducted. Bernoulli trials are independent.
- The **geometric probability distribution** of a discrete random variable X is the distribution of x trials to achieve the first success in an experiment of repeated Bernoulli trials: $P(X = x) = q^{x-1}p$.

Binomial variables

- The **binomial distribution** represents the number of successes in n independent Bernoulli trials.
  - There is a fixed number, n, of repeated trials.
  - Each trial is independent and has exactly 2 outcomes – success (p) and failure (q).
  - The probability of success is the same for each trial.
- The **binomial random variable** is the number of successes for n trials, where the probability of x successes from n trials is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x}$$

or

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

- The graph of a binomial distribution is:

  positively skewed (skewed to the right) when $p < 0.5$

  symmetrical when $p = 0.5$

  negatively skewed (skewed to the left) $p > 0.5$.
- For a binomial experiment, if X is the number of successes in n trials, each with probability of success p:

  expected value $= E(X) = \mu = np$

  variance $= Var(X) = np(1 - p) = npq$

  standard deviation $= \text{SD}(X) = \sigma = \sqrt{np(1-p)} = \sqrt{npq}$

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2023 1Q6 Substitutions for h are used to estimate the limit of $\frac{a^h - 1}{h}$ as $h \to 0$.
Which sequence is the most appropriate?

A $-4, -2, -1, -0.5, -0.25, -0.125 \ldots$

B $-0.05, -0.1, -0.2, -0.4, -0.8 \ldots$

C $2, 1, 0, -1, -2, -3 \ldots$

D $1, 2, 3, 4, 5, 6 \ldots$

2 © QCAA 2022 1Q2 A binomial random variable arises from the number of successes in n independent Bernoulli trials. A context **not** suitable for modelling using a binomial variable is recording the number of

A heads when a coin is tossed 12 times.

B left-handed people in a sample of 100 people.

C times a player hits a target from 20 shots where each shot is independent of all other shots.

D red marbles selected when three marbles are drawn without replacement from a bag containing four blue and five red marbles.

3 © QCAA 2021 1Q6 A random variable X is the number of successes in a Bernoulli experiment with n trials, each with a probability of success p and a probability of failure q. The probability distribution table of X is shown.

k	$P(X = k)$
0	$\frac{1}{81}$
1	$\frac{8}{81}$
2	$\frac{24}{81}$
3	$\frac{32}{81}$
4	$\frac{16}{81}$

Which values of n, p and q will generate this probability distribution?

A $n = 4, p = \frac{2}{3}, q = \frac{1}{3}$

B $n = 4, p = \frac{1}{3}, q = \frac{2}{3}$

C $n = 5, p = \frac{2}{3}, q = \frac{1}{3}$

D $n = 5, p = \frac{1}{3}, q = \frac{2}{3}$

Section 2 4 short response questions

15 marks

4 © QCAA 2022 1Q12 (3 marks) The probability that a debating team wins a debate can be modelled as a Bernoulli distribution. Given that the probability of winning a debate is $\frac{4}{5}$

a Determine the mean of this distribution. *[1 mark]*

b Determine the variance of this distribution. *[1 mark]*

c Determine the standard deviation of this distribution. *[1 mark]*

5 © QCAA 2022 1Q15 MODIFIED (4 marks) The derivative of a function is given by $f'(x) = e^x(x - 6)$. Determine the interval on which the graph of $f(x)$ is both decreasing and concave up.

6 © QCAA 2020S 1Q16 MODIFIED (4 marks) Determine the equation of the tangent to the function $y = x^3 + ax^2 + bx + 2$ at the point of inflection $(2, 4)$.

7 © QCAA 2023 1Q15 (4 marks) In a certain game, players throw one water balloon at a target. There is a one in four chance of hitting the target.

a State the probabilities of all the possible outcomes for one throw at the target. *[2 marks]*

b Let H be the discrete random variable for one of the outcomes. Determine the mean and variance of the distribution of random variable H when 20 players throw a water balloon at the target. *[2 marks]*

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2023 2Q2 The probability of hitting a bullseye on a standard dartboard is 1 in 1250. What is the probability of hitting a bullseye exactly once in 10 attempts?

A $\binom{9}{1}\left(\frac{1}{1250}\right)^1 \times \left(\frac{1249}{1250}\right)^9$

B $\binom{9}{1}\left(\frac{1}{1250}\right)^9 \times \left(\frac{1249}{1250}\right)^1$

C $\binom{10}{1}\left(\frac{1}{1250}\right)^1 \times \left(\frac{1249}{1250}\right)^9$

D $\binom{10}{1}\left(\frac{1}{1250}\right)^9 \times \left(\frac{1249}{1250}\right)^1$

2 © QCAA 2023 2Q4 MODIFIED The displacement (m) of a moving particle is given by $d = e^{0.4t} - 1$, where t is time (s). The acceleration ($m\,s^{-2}$) of the particle when $t = 6$ is

A 0.0 **B** 0.64 **C** 1.764 **D** 3.2

3 © QCAA 2020 2Q7 The records of a shoe manufacturer show that 10% of shoes made are defective. Assuming independence, the probability of getting 2 defective shoes in a batch of 20 is

A 0.1937 **B** 0.2852 **C** 0.3917 **D** 0.6083

Section 2 4 short response questions

15 marks

4 © QCAA 2020 2Q11 (4 marks) A sugar company samples the packets of sugar it produces and finds that 5% of packets are underweight. Consider a batch of 20 packets.

a Determine how many packets of sugar the company can expect to be underweight in a batch of 20 packets. *[1 mark]*

b Determine the variance of the batch. *[1 mark]*

c Determine the probability that at most one of the packets of sugar in the batch is underweight. *[2 marks]*

5 © QCAA 2023 2Q15 (4 marks) Determine the derivative of $f(x) = \ln x^2 + \ln (x-5)^3$.

Express the derivative as a single fraction in its simplest and factorised form.

6 © QCAA 2020S 2Q7 MODIFIED (4 marks) Describe what is happening to the graph of $y = x^3 - 6x^2$ at $x < 2$.

7 (3 marks) 3 dice are rolled together. Find the probability that the total of the numbers on the dice is less than 7.

CHAPTER

5 ANTI-DIFFERENTIATION

Syllabus coverage

Nelson MindTap chapter resources

Terminology

5.1 The integrals of x^n and $(ax + b)^n$

The anti-derivative

Properties of integration

The integral of $(ax + b)^n$

5.2 The integrals of e^x and $\frac{1}{x}$

Integrating e^x

Integrating $\frac{1}{x}$ and $\frac{1}{ax+b}$

Integrating $y = \frac{f'(x)}{f(x)}$

Integration by recognition

5.3 The integrals of $\sin(x)$ and $\cos(x)$

5.4 Straight-line motion

5.5 Problems involving indefinite integrals

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170484985

Syllabus coverage

UNIT 3, TOPIC 4: INTRODUCTION TO INTEGRATION

Anti-differentiation

- Recognise anti-differentiation as the reverse of differentiation.
- Use the notation $\int f(x)\,dx$ for anti-derivatives or indefinite integrals.
- Use the formula $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
- Use the formula $\int e^x dx = e^x + c$.
- Use the formula $\int \frac{1}{x}\,dx = \ln(x) + c$, for $x > 0$.
- Use the formulas $\int \sin(x)\,dx = -\cos(x) + c$ and $\int \cos(x)\,dx = \sin(x) + c$.
- Understand and use the formulas $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$ and $\int k f(x)\,dx = k\int f(x)\,dx$.
- Determine indefinite integrals of the form $\int f(ax + b)\,dx$.
- Determine $f(x)$ given $f'(x)$ and an initial condition $f(a) = b$.
- Determine displacement given velocity and the initial value of displacement.
- Determine displacement given acceleration and the initial values of displacement and velocity.
- Model and solve problems that involve indefinite integrals, with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus p. 25,
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Prior learning
Anti-differentiation

Videos (3):
- **5.2** Integrating exponential functions
- **5.4** Displacement and velocity as integrals
- **Exam question analysis** Anti-differentiation

Prior learning (1):
- **5.1** Anti-differentiation

Worksheets (12):
- **5.1** Anti-derivatives 1 • Antiderivatives • Anti-differentiation • The chain rule • Integrating functions • Finding indefinite integrals 1
- **5.2** Integration of reciprocals • Finding indefinite integrals 2
- **5.4** Displacement, velocity and acceleration
- **5.5** Anti-differentiation assignment • Anti-differentiation problems
- **Cumulative exams** Maths Methods formula sheet

TI-Nspire CX II instructions (1):
- **Exam question analysis** Anti-differentiation

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

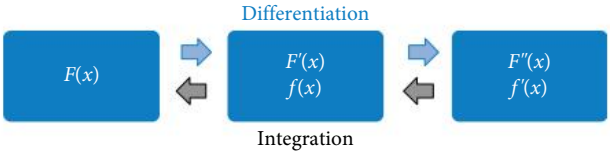
anti-derivative
anti-differentiation
constant of integration
indefinite integral
integral
integrand
integration
integration by recognition
primitive

5.1 The integrals of x^n and $(ax + b)^n$

The anti-derivative

Anti-differentiation (also called **integration**) is the reverse process of finding the **derivative**.

If $f(x)$ is the derivative function of $F(x)$, then $F(x)$ is called the **anti-derivative** or **integral** or **primitive** of $f(x)$.



Investigation — Undoing differentiation

If you know the derivative, $f(x)$, you can undo the process of differentiation or work backwards from it to find a possible primitive function.

1 Copy and complete these 3 tables.

$f(x)$	$f'(x)$
$5x$	
	3
$x2$	
$3x2$	
	$4x$

$f(x)$	$f'(x)$
$\frac{1}{4}x^3$	
	$6x^2$
	$\frac{3}{2}x^2$
$\frac{1}{2}x^4$	
	$16x^3$

$f(x)$	$f'(x)$
	x^3
$5x^2 - 4x$	
$5x^4 - 7x^2$	
	$6x + 11$
	$9x^2 - 8x + 1$

2 Consider your results. Are there any patterns?

3 In your own words describe how the integral function can be determined using the derivative.

Consider the function $F(x) = x^2 + 3$. The derivative of $F(x)$ is $2x$.

Now consider the function $G(x) = x^2 - 1$. The derivative of $G(x)$ is also $2x$.

That's because the derivative of a constant term (a number) is 0. Any anti-derivative could have a constant term, so the anti-derivative of $2x$ has many answers, of the form $x^2 + c$, where c is a constant term. Always write '$+ c$' on the end to show you have included it. The c stands for any number and is called the **constant of integration**.

So, the general anti-derivative of $2x$ is $x^2 + c$.

Algebraically this is written as $\int 2x\,dx = x^2 + c$.

- The symbol $\int$ is read as 'the integral of'.
- 'dx' means 'with respect to x'.

Because c can take on any value, there is an infinite number of anti-derivatives, so the anti-derivative is called an **indefinite integral**.

The integral of ax^n

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c, \text{ where } n \neq -1$$

ax^n is called the **integrand**.

Exam hack

To find the integral of any power function of the form ax^n, 'add 1 to the power of x and divide by the new power'.

Using this rule, you can see that $\int 1\,dx = \int x^0\,dx = \frac{x^{0+1}}{0+1} + c = x + c$.

WORKED EXAMPLE 1 Finding the anti-derivative

Determine the anti-derivative of each function.

a $f(x) = 4x^7$

b $y = 3x^2 - 10x + 4$

Steps	Working
a 1 Write the function as a derivative.	$\frac{dy}{dx} = 4x^7$
2 Write the integral to be found. Integrate using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$	$F(x) = \int 4x^7 dx$ $= \frac{4x^{7+1}}{7+1} + c$ $= \frac{4x^8}{8} + c$
3 Simplify and state the result.	$F(x) = \frac{x^8}{2} + c$
b 1 Write the function as a derivative.	$\frac{dy}{dx} = 3x^2 - 10x + 4$
2 Write the integral to be found. Integrate each term using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$	$y = \int 3x^2 - 10x + 4\,dx$ $= \frac{3x^{2+1}}{2+1} - \frac{10x^{1+1}}{1+1} + 4x + c$ $= \frac{3x^3}{3} - \frac{10x^2}{2} + 4x + c$
3 Simplify and state the result.	$y = x^3 - 5x^2 + 4x + c$

Exam hack

You can check your answer by differentiating it to see if you get the original function.

The integral of 4 is $4x$, by differentiating the RHS or writing 4 as $4x^0$ and using the formula.

Properties of integration

As integration is the reverse of **differentiation**, it has a similar set of properties.

Properties of integrals

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\int k f(x)\,dx = k \int f(x)\,dx$$

Exam hack

Make sure you are familiar with the QCAA Formula book. Be aware it uses shorthand versions of some formulas. For example, it represents $\int (f(x) \pm g(x))\,dx$ as $\int (f(x) + g(x))dx = \int f(x)\,dx + \int g(x)\,dx$.

WORKED EXAMPLE 2 Finding the indefinite integral

a **Determine** $\int \frac{2}{5x^3}dx$.

b Find y if $\frac{dy}{dx} = (3x-5)(2x+1)$.

c If $f(x) = \frac{2x^3 - 3\sqrt{x^7} + 4}{x^3}$, find $F(x)$.

Steps	Working
a 1 Integrate using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$.	Let $F(x) = \int \frac{2}{5x^3}dx = \frac{2}{5}\int x^{-3}dx$
Factorising out a constant can make the expression easier to work with.	$= \frac{2}{5}\left(\frac{x^{-3+1}}{-3+1}\right) + c$
	$= \frac{2}{5}\left(\frac{x^{-2}}{-2}\right) + c$
2 Simplify and state the result.	$F(x) = -\frac{1}{5x^2} + c$
Exam hack Write your answer in the same form as the original function – if it has all positive powers, your answer should too.	
b 1 Expand and simplify the derivative function.	$\frac{dy}{dx} = (3x-5)(2x+1)$
	$= 6x^2 + 3x - 10x - 5$
	$\frac{dy}{dx} = 6x^2 - 7x - 5$
2 Write the integral to be found and integrate each term using the formula $\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$.	$y = \int 6x^2 - 7x + 5\,dx$
	$= \frac{6x^{2+1}}{2+1} - \frac{7x^{1+1}}{1+1} + 5x + c$
	$= \frac{6x^3}{3} - \frac{7x^2}{2} + 5x + c$
3 Simplify and state the result.	$= 2x^3 - 3.5x^2 + 5x + c$

c **1** Simplify the derivative function using index laws and subtracting powers.

Recall $\sqrt[b]{x^a} = x^{\frac{a}{b}}$

$$f(x) = \frac{2x^3 - 3\sqrt{x^7} + 4}{x^3}$$
$$= \frac{2x^3 - 3x^{\frac{7}{2}} + 4}{x^3}$$
$$= 2 - 3x^{\frac{1}{2}} + \frac{4}{x^3}$$

2 Write the integral to be found and integrate each term using the formula.

$$F(x) = \int \left(2 - 3x^{\frac{1}{2}} + \frac{4}{x^3}\right) dx$$
$$= 2x - \frac{3x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{4x^{-3+1}}{-3+1} + c$$
$$= 2x - \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4x^{-2}}{-2} + c$$
$$F(x) = 2x - 2\sqrt{x^3} - \frac{2}{x^2} + c$$

WORKED EXAMPLE 3 Find y when given $\frac{dy}{dx}$

Find y if $\frac{dy}{dx} = 4x - 5$ and the point $(3, 2)$ is a point on the graph of y.

In this question you will be able to determine the value of c.

Steps	Working
1 Write the integral to be found.	$y = \int (4x - 5)\, dx$
2 Integrate each term and simplify.	$= \frac{4x^2}{2} - 5x + c$
	$y = 2x^2 - 5x + c$
3 Find the value of c by substituting in $x = 3$ and $y = 2$.	$2 = 2(3)^2 - 5(3) + c$
	$c = -1$
4 State the answer including the value of c.	$\therefore y = 2x^2 - 5x - 1$

The integral of $(ax + b)^n$

To find the derivative of $y = (4x - 7)^3$ you use the chain rule.

Let $y = u^3, \frac{dy}{du} = 3u^2$

Let $u = 4x - 7, \frac{du}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 3u^2 \times 4$$
$$= 12u^2$$
$$= 12(4x - 7)^2$$

Reverse this process to find the anti-derivative:

$\int 12(4x-7)^2\,dx = (4x-7)^3 + c$

$\therefore \int (4x-7)^2\,dx = \frac{1}{12}(4x-7)^3 + c$

This aligns with the general statement 'Add 1 to the power and divide by the new power'; however, it needs to be extended to include 'divide by the derivative of the bracketed expression'.

Now, consider the function $f(x) = (ax+b)^{n+1}$

Applying the chain rule: $f'(x) = a(n+1)(ax+b)^n$

Hence in reverse: $\int a(n+1)(ax+b)^n\,dx = (ax+b)^{n+1} + c$

So, $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$

This rule will only apply if the expression inside the brackets $(ax + b)$ is linear.

The integral of $(ax + b)^n$

$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$, where $n \neq -1$

This is the reverse chain rule.

This rule is not in the QCAA Formula book.

WORKED EXAMPLE 4 Finding the integral of $(ax + b)^n$

Find $\int (7x-1)^5\,dx$.

Steps	Working
1 Integrate using the formula $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$.	$\int (7x-1)^5\,dx = \frac{1}{7\times(5+1)}(7x-1)^{5+1} + c$
2 Simplify.	$\int (7x-1)^5\,dx = \frac{1}{42}(7x-1)^6 + c$

EXERCISE 5.1 The integral of x^n and $(ax + b)^n$

ANSWERS p. 405

Mastery

1 WORKED EXAMPLE 1 Find the anti-derivative of $f(x) = x^3 - 10x^4 - 6x^2 + 12$.

2 Determine each integral.

a $\int (8x-7)\,dx$

b $\int (4)\,dx$

c $\int (7-u^3+u^2)\,du$

d $\int (z-z^3+8)\,dz$

3 WORKED EXAMPLE 2 Find $F(x)$ if $f(x) = \frac{8x^7 + 3x^4}{x^2}$, $x \neq 0$.

4 Find each indefinite integral.

a $\int (0.7t - 0.4t^2 + 0.6)\,dt$

b $\int (4(x+3)^2 + 8)\,dx$

c $\int (2\sqrt{x^3})\,dx$

d $\int g(g+2)(4g-5)\,dg$

e $\int \left(5x^3 - \frac{5}{x^3}\right)dx$

f $\int \left(\frac{3x^2 - 7x^3}{2x}\right)dx, x \neq 0$

5 Determine

a $F(x)$ if $f(x) = (3-x)^2$

b y if $\frac{dy}{dx} = 5\sqrt[3]{x^4}$

c v if $\frac{dv}{dt} = 4.9(t-5)^2$

d $G(x)$ if $g(x) = (5x-4)(2x+3)$

6 The gradient of the function $f(x)$ is given by $\frac{3}{7x^5}$. Determine $f(x)$.

7 WORKED EXAMPLE 3 Determine the function $x(t)$, if $\frac{dx}{dt} = 0.6t^2 - 2t + 4$ and $x = -2$ when $t = 1$.

8 WORKED EXAMPLE 4 Find each indefinite integral.

a $\int (5x+8)^3 dx$

b $\int \left(\frac{1}{(2x-11)^2} \right) dx$

c $\int (\sqrt[3]{25-2x}) dx$

d $\int \left(\frac{(3x+6)^4}{9x+18} \right) dx$

Exam practice

All questions TECH-FREE

9 © QCAA 2023 1Q4 62% If the gradient of the function $f(x)$ is given by $\frac{20}{x^3}$, then $f(x)$ is equal to

A $-\frac{60}{x^4} + c$ **B** $-\frac{5}{x^4} + c$ **C** $-\frac{10}{x^2} + c$ **D** $-\frac{40}{x^2} + c$

10 Determine $5\int (2x-3)^4 dx$.

A $10(2x-5)^5 + c$ **B** $2(2x-5)^5 + c$ **C** $\frac{(2x-5)^5}{5} + c$ **D** $\frac{(2x-5)^5}{2} + c$

11 Find the function $g(x)$ if $g'(x) = 2x$ and $g(2) = 1$.

A $g(x) = x^2$ **B** $g(x) = 2x^2 + 1$ **C** $g(x) = x^2 - 3$ **D** $g(x) = 2x^2 - 3$

12 (2 marks) Determine an anti-derivative of the expression $6x^2 + 4x + 1$.

13 (2 marks) Find an anti-derivative of $\frac{1}{(1-x)^3}$ with respect to x.

14 (2 marks) Determine $\int \left(x - \frac{1}{x} \right)^2 dx$.

15 (2 marks) Determine $\int (32-5x)^4 dx$.

16 (2 marks) If $\frac{dy}{dx} = 3\sqrt{x+4}$ and $y = 0$ when $x = 5$, find an expression for y.

17 (3 marks) CF If $f(x) = \int (6x - 12x^2) dx$ and $f(-1) = 5$, find $f(1)$.

18 (2 marks) CF If $f'(x) = g'(x) + 4$, $f(0) = 3$ and $g(0) = 2$, show that $f(x) = g(x) + 4x + 1$.

19 © QCAA 2024 1Q1 93% Determine $\int x^4 dx$.

A $4x^3 + c$ **B** $5x^5 + c$ **C** $\frac{1}{3}x^3 + c$ **D** $\frac{1}{5}x^5 + c$

Worksheets
Integration of reciprocals

Finding indefinite integrals 2

5.2 The integrals of e^x and $\frac{1}{x}$

Integrating e^x

You already know that $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{ax}) = ae^{ax}$.

It follows that $\int e^x dx = e^x + c$ and $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$.

Extending this

$$y = e^{ax+b}$$

$$\frac{dy}{dx} = ae^{ax+b}$$

$$\therefore \int ae^{ax+b} dx = \frac{ae^{ax+b}}{a} + c$$

$$= e^{ax+b} + c$$

So $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$.

Integrals of e^x

$$\int e^x dx = e^x + c$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$$

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$$

The integral of $f(x) = e^{ax+b}$ is an example of this general rule:

The integral of functions of the form $f(ax + b)$

$\int f(ax+b)dx = \frac{1}{a}F(ax+b) + c$, where $F(x)$ is the integral of $f(x)$.

Video
Integrating exponential functions

WORKED EXAMPLE 5 Finding the integral of an exponential function

Determine each integral.

a $\int 12e^{2-3x} dx$ **b** $\int \frac{4 + e^{4x} + e^x}{3e^{2x}} dx$ **c** $\int x^2 + 5e^{4x} dx$

Steps	Working
a Integrate using the formula $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$, where $a = -3$ and $b = 2$.	$\int 12e^{2-3x} dx = \frac{12e^{2-3x}}{-3} + c$ $= -4e^{2-3x} + c$

b **1** Write the integral to be found and simplify by subtracting powers.

$$\int \frac{4+e^{4x}+e^{x}}{3e^{2x}}dx = \frac{1}{3}\int \frac{4}{e^{2x}}+e^{2x}+e^{-x}dx$$

2 Integrate using the formula $\int e^{ax}dx = \frac{1}{a}e^{ax}+c$.

$$= \frac{1}{3}\int 4e^{-2x}+e^{2x}+e^{-x}dx$$

$$= \frac{1}{3}\left(\frac{4e^{-2x}}{-2}+\frac{e^{2x}}{2}-e^{-x}\right)+c$$

$$= \frac{1}{3}\left(-2e^{-2x}+\frac{1}{2}e^{2x}-e^{-x}\right)+c$$

$$= -\frac{2}{3}e^{-2x}+\frac{1}{6}e^{2x}-\frac{1}{3e^{x}}+c$$

c Integrate using the rules of integration.

$$\int x^{2}+5e^{4x}dx = \frac{x^{2+1}}{3}+\frac{5e^{4x}}{4}+c$$

$$= \frac{x^{3}}{3}+\frac{5e^{4x}}{4}+c$$

WORKED EXAMPLE 6 Finding $f(x)$ given $f'(x)$ and a point

Find $f(x)$ if its gradient function is $f'(x)=6e^{3x}$ and $f(0)=7$.

Steps	Working
1 Integrate $f'(x)$ to find $f(x)$.	$f(x)=\int 6e^{3x}dx$
	$=\frac{6e^{3x}}{3}+c$
	$=2e^{3x}+c$
2 Find the value of c by substituting in $x=0$ and $y=7$.	$7=2e^{3(0)}+c$
	$=2+c$
	$c=7-2$
	$=5$
3 State the answer including the value of c.	$f(x)=2e^{3x}+5$

Integrating $\frac{1}{x}$ and $\frac{1}{ax+b}$

$$\frac{d}{dx}(\ln(x))=\frac{1}{x}$$

and $\frac{d}{dx}(\ln(ax+b))=\frac{a}{ax+b}$

So $\int \frac{1}{x}dx = \ln(x)+c$.

And $\int \frac{1}{ax+b} = \frac{1}{a}\ln(ax+b)$.

Remember that $\log_e x = \ln x$.

Note that in this course, this integral is defined only if $x > 0$, as this is the domain of $\ln(x)$.

In this course, only cases where the denominator is positive are considered.

Integrals of $\frac{1}{x}$ and $\frac{1}{ax+b}$

$$\int \frac{1}{x}\,dx = \ln(x) + c, \text{ where } x > 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b)$$

The integral of $\frac{1}{ax+b}$ is an example of the general rule:

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c$$

WORKED EXAMPLE 7 Finding the integral of $\frac{1}{x}$ and $\frac{1}{ax+b}$

Determine each integral.

a $\int \frac{5}{2x}\,dx$ **b** $\int \frac{3}{4x+1}\,dx$ **c** $\int \frac{(x+5)^2}{x}\,dx$

Steps	Working
a 1 Simplify by taking out the factor.	$\int \frac{5}{2x}\,dx = \frac{5}{2}\int \frac{1}{x}\,dx$
2 Integrate using $\int \frac{1}{x}\,dx = \ln(x) + c$.	$= \frac{5}{2}\ln(x) + c$
b 1 Simplify by taking out the factor.	$\int \frac{3}{4x+1}\,dx = 3\int \frac{1}{4x+1}\,dx$
2 Integrate using $\int \frac{1}{ax+b} = \frac{1}{a}\ln(ax+b)$.	$= \frac{3\ln(4x+1)}{4} + c$
c 1 Expand and simplify the integrand.	$\int \frac{(x+5)^2}{x}\,dx = \int \frac{x^2+10x+25}{x}\,dx$
	$= \int x + 10 + \frac{25}{x}\,dx$
2 Integrate using the rules of integration.	$= \frac{x^2}{2} + 10x + 25\ln x + c$

Integrating $y = \frac{f'(x)}{f(x)}$

Given $\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$, you can write a corresponding integral statement.

The integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) + c \text{ for } f(x) \neq 0$$

When integrating any function of the form $\frac{g(x)}{f(x)}$, check if the numerator $g(x)$ is a multiple of the derivative of the denominator $f(x)$. If so, the integral will be a natural logarithmic function.

WORKED EXAMPLE 8 Finding the integral of $\frac{f'(x)}{f(x)}$

Determine each integral.

a $\int \frac{8x+3}{4x^2+3x-8}dx$, where $4x^2+3x-8>0$ **b** $\int \frac{30x}{5x^2+6}dx$, where $5x^2+6>0$

Steps	Working
a 1 Test if the numerator is the derivative of the denominator.	$\frac{d}{dx}(4x^2+3x-8)=8x+3$
2 It is, so the integral is a logarithmic function. Integrate using the formula $\int \frac{f'(x)}{f(x)}dx=\ln(f(x))+c$ for $f(x)>0$.	$\int \frac{8x+3}{4x^2+3x-8}dx=\ln(4x^2+3x-8)+c$
The condition $4x^2+3x-8>0$ ensures the resulting natural logarithm is defined.	
b 1 Test if the numerator is the derivative of the denominator.	$\frac{d}{dx}(5x^2+6)=10x$
2 The numerator is a multiple of the derivative of the denominator. Integrate and adjust the integral.	$\int \frac{30x}{5x^2+6}dx=3\int \frac{10x}{5x^2+6}dx$ $=3\ln(5x^2+6)+c$

Integration by recognition

Applying the rules of integration may not always be straightforward or even possible in more complex situations. In these cases, **integration by recognition** is used. You find the derivative of a related function and use this derivative to find the required anti-derivative.

You may need to differentiate a related function and then use the result to find the anti-derivative.

In these cases, you will need to use the fact that if $f(x)=g'(x)$, then $\int g'(x)dx=f(x)+c$.

WORKED EXAMPLE 9 Integration by recognition 1

Given that $y=e^{x^5}$, **calculate** $\frac{dy}{dx}$ and **use** this result to find $\int x^4e^{x^5}$.

Steps	Working
1 Find $\frac{dy}{dx}$.	$y=e^{x^5}$ $\frac{dy}{dx}=5x^4e^{x^5}$
2 State the result.	$\frac{d}{dx}(e^{x^5})=5x^4e^{x^5}$
3 Write as an integral statement.	$\int 5x^4e^{x^5}dx=e^{x^5}+c$
4 Adjust the integral to suit the problem by removing the factor.	$5\int x^4e^{x^5}dx=e^{x^5}+c$

5 Compare this statement to the integral required, $\int x^4 e^{x^5}$, and state the result.

$$\int x^4 e^{x^5}\,dx = \frac{1}{5}e^{x^5} + c$$

Exam hack

Don't forget to add *c*. If you don't include it in your working, make sure you write it at the last step!

WORKED EXAMPLE 10 Integration by recognition 2

Find $f'(x)$ for $f(x) = \ln(x^2 - 9)$, and **use** this result to find $\int \frac{x}{x^2 - 9}\,dx$, where $x > 3$.

Steps	Working
1 Find $f'(x)$ using $\frac{f'(x)}{f(x)}$.	$f'(x) = \frac{2x}{x^2 - 9}$
2 Write the corresponding integral.	$\int \frac{2x}{x^2 - 9}\,dx = \ln(x^2 - 9) + c$
3 Adjust the integral to suit the problem.	$\frac{1}{2}\int \frac{2x}{x^2 - 9}\,dx = \frac{1}{2}\ln(x^2 - 9) + c$
4 Compare this statement to the integral required and state the result.	$\int \frac{x}{x^2 - 9}\,dx = \frac{1}{2}\ln(x^2 - 9) + c$

EXERCISE 5.2 The integrals of e^x and $\frac{1}{x}$

ANSWERS p. 405

Recap

1 An anti-derivative of $x^3 + 2x$ is

A $x^4 + 2x^2$ **B** $4x^4 + x^2$ **C** $\frac{x^4}{4} + \frac{x^2}{2}$ **D** $\frac{x^4}{4} + x^2$

2 Determine the function $h(x)$ if $h'(x) = 4x^5 + 3$ and $h(1) = 4$.

Mastery

3 WORKED EXAMPLE 5 Find each integral.

a $\int (12e^{-4x})dx$ **b** $\int (3e^{3x+1})dx$

c $\int (e^{2x+1} - e^{-x})dx$ **d** $\int (10e^{5x} + 6x)dx$

e $\int \left(5e^{-2x} + e^{\frac{x}{4}}\right)dx$ **f** $\int (2x^2 - 6e^{3x})dx$

4 Determine each integral.

a $\int (1 + e^x)^2\,dx$ **b** $\int (2e^x - 3)^2\,dx$

c $\int \left(\frac{e^x + 1}{e^x}\right)dx$ **d** $\int \left(\frac{e^{2x} + 2e^x + 1}{e^x}\right)dx$

5 WORKED EXAMPLE 6 Find $F(x)$ if $f(x) = F'(x) = 10e^{2x+3}$ and $F(-1.5) = 7$.

6 WORKED EXAMPLE 7 Find each indefinite integral.

a $\int \frac{4}{x}dx$ for $x>0$

b $\int\left(\frac{7}{9x}\right)dx$ for $x>0$

c $\int\left(\frac{5}{2x+3}\right)dx$ for $x>\frac{-3}{2}$

d $\int\left(\frac{2}{5-3x}\right)dx$ for $x>\frac{5}{3}$

e $\int\left(\frac{1}{3x}\right)dx$ for $x>0$

f $\int\left(\frac{6}{9x+2}\right)dx$ for $x>\frac{-2}{9}$

7 Determine

a $\int(x+4)^{-1}dx$

b $\int\left(\frac{x}{3x+4x^2}\right)dx$

c $\int\left(\frac{x^2+2x-3}{x}\right)dx$

d $\int\left(\frac{3x^2-8x+12}{2x}\right)dx$

8 WORKED EXAMPLE 8 Find each integral.

a $\int\left(\frac{2+15x^2}{2x+5x^3}\right)dx$ where $2x+5x^3>0$

b $\int\left(\frac{4(9x^2+12x)}{3x^3+6x^2+1}\right)dx$ where $3x^3+6x^2+1>0$

9 If $\frac{dy}{dx}=\frac{9}{4-3x}$ where $x>\frac{-4}{3}$ and $y=9$ when $x=1$, find an expression for y in terms of x.

10 WORKED EXAMPLE 9 Consider $y=e^{x^4}$.

a Calculate $\frac{dy}{dx}$.

b Use the result from part **a** to find $\int x^3e^{x^4}$.

11 WORKED EXAMPLE 10 Differentiate $\ln(x^2-21)$, where $x^2-21>0$, and hence determine $\int\left(\frac{6x}{x^2-21}\right)dx$.

12 Determine $\int\left(x-\frac{1}{x}\right)^3dx$ for $x>0$.

Exam practice

All questions TECH-FREE

13 © QCAA 2021 1Q3 80% Determine $\int 10e^{4x}dx$.

A $\frac{10e^{4x+1}}{4x+1}+c$ **B** $40e^{4x}+c$ **C** $\frac{5}{2}e^{4x}+c$ **D** $2e^{5x}+c$

14 © QCAA 2020 1Q9 Determine $\int\frac{x+1}{x^2+2x}dx$.

A $\ln\left(\frac{1}{2x+2}\right)+c$ **B** $\ln(2x+2)+c$ **C** $\frac{1}{2}\ln(x^2+2x)+c$ **D** $2\ln(x^2+2x)+c$

15 © QCAA 2020S 1Q11 MODIFIED (1 mark) Determine $\int 3e^{4x}dx$.

16 (2 marks) Determine $\int\frac{(2+e^x)^2}{2e^x}dx$.

17 (2 marks) Determine $\int(2x(5x^2+6x)^{-1})dx$.

18 (2 marks) Determine $\int\left(\frac{3x^2+4x}{2x^3+4x^2-5}\right)dx$, where $2x^3+4x^2-5>0$.

19 (2 marks) Find y if $\frac{dy}{dx}=x^2-\frac{1}{x}$ and the graph of the function passes through $\left(1,\frac{7}{3}\right)$.

20 (3 marks) Determine $\frac{d}{dx}(e^{x^2+3})$ and use this result to find $\int xe^{x^2+3}dx$.

21 (3 marks) CU Differentiate $y=x^2\ln(3x)$ where $x>0$, and hence determine $\int x\ln(3x)\,dx$.

22 (3 marks) CU Determine $\int 2x^4e^{x^5-3}dx$.

23 (4 marks) CU Determine $\int\frac{2e^{2x}+3e^x-2}{e^x+2}dx$.

24 (4 marks) CU The function $y=f(x)$ has the derivative $\frac{dy}{dx}=e^{kx}$, where k is a constant. The tangent drawn at $(1,e^4)$ passes through the origin.

Determine the equation of the function.

5.3 The integrals of sin(x) and cos(x)

The derivatives of trigonometric functions are reversed to find their anti-derivatives.

Trigonometric function	Derivative
$y=\sin(x)$	$\frac{dy}{dx}=\cos(x)$
$y=\cos(x)$	$\frac{dy}{dx}=-\sin(x)$
$y=\sin(ax+b)$	$\frac{dy}{dx}=a\cos(ax+b)$
$y=\cos(ax+b)$	$\frac{dy}{dx}=-a\sin(ax+b)$

Integrals of sin(x) and cos(x)

$$\int\cos(x)\,dx=\sin(x)+c$$

$$\int\sin(x)\,dx=-\cos(x)+c$$

$$\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+c$$

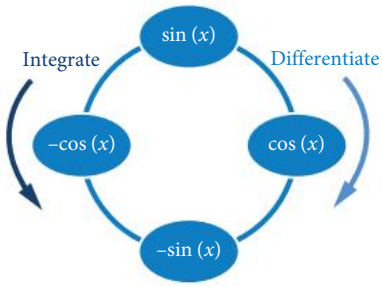
$$\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+c$$

The integrals of $\sin(ax+b)$ and $\cos(ax+b)$ are another example of the general rule

$$\int f(ax+b)\,dx=\frac{1}{a}F(ax+b)+c$$

where $F(x)$ is the integral of $f(x)$.

The relationship between differentiating and integrating $\sin(x)$ and $\cos(x)$ can be shown visually.



WORKED EXAMPLE 11 Finding the integral of sin (x) and cos (x)

Find the anti-derivative of each function.

a $y = \sin(3x) + \cos(5x)$ **b** $g(x) = 3\cos\left(\frac{-x}{2}\right)$ **c** $f'(x) = 3x^2 - \cos(12x - 7) + 6e^{2x}$

Steps	Working
a 1 Integrate each term using the rules of integration.	$Y = \int \sin(3x)\,dx + \int \cos(5x)\,dx$ $= -\frac{1}{3}\cos(3x) + \frac{1}{5}\sin(5x) + c$
2 State the answer.	The anti-derivative is $Y = -\frac{1}{3}\cos(3x) + \frac{1}{5}\sin(5x) + c$.
b 1 Integrate using the rule $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$.	$\int 3\cos\left(\frac{-x}{2}\right)dx = \frac{3}{\frac{-1}{2}}\sin\left(\frac{-x}{2}\right) + c$
2 Simplify and state the answer.	$G(x) = -6\sin\left(\frac{-x}{2}\right) + c$
c 1 Integrate each term using the rules of integration.	$\int (3x^2 - \cos(12x-7) + 6e^{2x})\,dx$ $= x^3 - \frac{1}{12}\sin(12x-7) + \frac{6}{2}e^{2x} + c$
2 Simplify and state the answer.	$f(x) = x^3 - \frac{1}{12}\sin(12x-7) + 3e^{2x} + c$

Exam hack

Remember to check your answer by differentiating it.

WORKED EXAMPLE 12 Integration by recognition 3

Differentiate $x\cos(x)$ and hence **determine** $\int 4x\sin(x)\,dx$.

Steps	Working
1 Write the expression as a function.	Let $y = x\cos(x)$.
2 Find $\frac{dy}{dx}$ using the product rule.	$\frac{d}{dx}(x\cos(x)) = \cos(x) - x\sin(x)$
3 Rearrange to isolate the required subject.	$x\sin(x) = \cos(x) - \frac{d}{dx}(x\cos(x))$
4 Write the integral statement.	$\int x\sin(x)\,dx = \int \cos(x)\,dx - \int \frac{d}{dx}(x\cos(x))\,dx$
5 Integrate each term.	$= \sin(x) - x\cos(x) + c$
6 Adjust the integral to suit the problem by multiplying by 4.	$\int 4x\sin(x)\,dx = 4\sin(x) - 4x\cos(x) + c$

EXERCISE 5.3 The integrals of sin(x) and cos(x)

ANSWERS p. 406

Recap

1 Find $\int (e^{2x} - e^x)^2 dx$.

2 Given that $\frac{dy}{dx} = \frac{7}{2x-4}$ and $y = 5$ when $x = -2$, find an expression for y in terms of x.

Mastery

3 WORKED EXAMPLE 11 Find each integral.

a $\int 10\sin(5x)\,dx$

b $\int \frac{2}{3}\cos\left(\frac{x}{3}\right)dx$

c $\int (-8\sin(2x+3))dx$

d $\int 4\sin\left(\frac{1}{2}(5x-6)\right)dx$

4 Determine

a $\int \sin(7-8x)\,dx$

b $\int \left(\sin\left(\frac{x}{2}\right) - 2\cos\left(\frac{x}{2}\right)\right)dx$

c $\int (1+x+\cos(x-5))\,dx$

d $\int (2e^{4x} - \sin(2x))\,dx$

e $\int \left(2\cos(5x) + 3\sin\left(\frac{x}{3}\right)\right)dx$

f $\int (1+3x+\sin(2x+3))\,dx$

5 WORKED EXAMPLE 12 Differentiate $2x\sin(x)$ and hence determine $\int 8x\cos(x)\,dx$.

6 Given that $f'(x) = \sin(4x) + \cos(2x)$ and $f(0) = 5$, find the function $f(x)$.

7 Determine the anti-derivative of $\frac{2}{3x^2} - \cos\left(\frac{3\pi x}{2}\right)$.

8 The gradient of a tangent to a curve is given by $\frac{dy}{dx} = \cos(3x) + 4$.

Determine a possible equation of the curve y.

Exam practice

9 TECH-FREE Determine $\int 3\sin(9x)\,dx$

A $-3\cos(9x) + c$ **B** $-\frac{1}{3}\cos(9x) + c$ **C** $\frac{1}{3}\cos(9x) + c$ **D** $3\cos(9x) + c$

10 TECH-FREE Determine $\int 2\cos\left(\frac{-\pi x}{5}\right)dx$.

A $\frac{-10}{\pi}\sin\left(\frac{-\pi x}{5}\right) + c$ **B** $\frac{-5}{\pi}\sin\left(\frac{-\pi x}{5}\right) + c$ **C** $\frac{10}{\pi}\sin\left(\frac{-\pi x}{5}\right) + c$ **D** $\frac{5}{\pi}\sin\left(\frac{-\pi x}{5}\right) + c$

11 TECH-FREE (1 mark) Determine $\int \cos(5x-6)\,dx$.

12 TECH-FREE (2 marks) Determine $\int \left(5\sin(4x) - 8\cos\left(\frac{x}{4}\right)\right)dx$.

13 TECH-FREE (2 marks) Determine $\int \left(\frac{1}{2}\cos(4x+5) - 2e^{-x}\right)dx$.

14 TECH-FREE (2 marks) Determine $\int (6e^{3x} - 9\cos(3x))\,dx$.

15 TECH-FREE (3 marks) If $f(x) = x\cos(4x)$ then $f'(x) = \cos(4x) - 4x\sin(4x)$.

Use this fact to determine the indefinite integral of $x\sin(4x)$.

16 (3 marks) CF The gradient of a tangent to a curve $y = F(x)$ is given by $f(x) = a\cos\left(4x + \frac{\pi}{4}\right)$, where $a \in R$.

Given $f\left(\frac{\pi}{3}\right) = 8$ and $F\left(\frac{\pi}{3}\right) = 5$, determine a and hence determine the function of the curve $F(x)$.

Round answers to 3 decimal places where appropriate.

17 © QCAA 2024 2Q18 (5 marks) CU An object experiencing straight-line motion along a path has an acceleration (m s^{-2}) defined by the function $a(t) = 3\sin(2t)$ where t is time (s) since the object begins moving, $t \geq 0$.

When $t = 0$, both displacement and velocity are zero.

On the path is a motion sensor that is able to detect motion up to 2 metres away.

The object passes directly by the motion sensor when $t = 3$.

Determine the average velocity of the object while it moves through the range of the sensor.

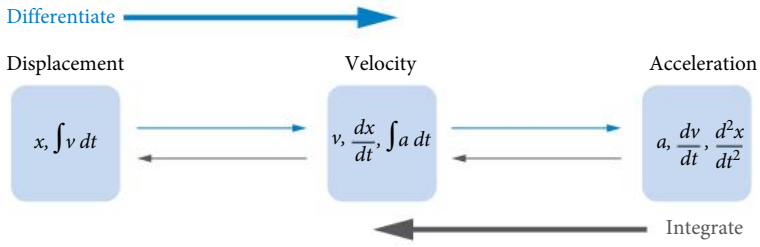
5.4 Straight-line motion

Worksheet
Displacement, velocity and acceleration

In Chapter 3, *Further applications of differentiation*, you learnt that:

- velocity is the derivative of displacement, $v(t) = \frac{dx}{dt}$
- acceleration is the derivative of velocity, $a(t) = \frac{dv}{dt}$
- acceleration is the second derivative of displacement, $a(t) = \frac{d^2x}{dt^2}$.

This means that the reverse process can be used to find displacement, $x(t) = \int v(t)\,dt$, and velocity, $v(t) = \int a(t)\,dt$.



Displacement and velocity as integrals

$$x(t) = \int v(t)\,dt$$
$$v(t) = \int a(t)\,dt$$

If you know an expression for acceleration, you will need to anti-differentiate it once to find an equation for velocity and again to find an equation for displacement. Each time you integrate, there will be a constant of integration you have to calculate using additional information.

Video
Displacement and velocity as integrals

WORKED EXAMPLE 13 Displacement and velocity as integrals 1

The acceleration of a particle travelling in a straight line is given by $a(t) = 4t$.

a Find an expression for the velocity $v(t)$ if $v = 0$ when $t = 2$.

b Find an expression for the displacement $x(t)$ if the particle started at the origin.

Steps	Working
a 1 Integrate the acceleration function to find the velocity function.	$v(t) = \int 4t\,dt$
	$v(t) = 2t^2 + c$
2 Find the value of c by substituting in $t = 2$ and $v = 0$.	$0 = 2(2)^2 + c$
	$c = -8$
3 State the result.	$\therefore v(t) = 2t^2 - 8$
b 1 Integrate the velocity function to find the displacement function.	$x(t) = \int 2t^2 - 8\,dt$
	$x(t) = \frac{2}{3}t^3 - 8t + d$
2 Find d by substituting in $t = 0$ and $x = 0$.	$0 = \frac{2}{3}(0)^3 - 8(0) + d$
	$d = 0$
3 State the result.	$\therefore x(t) = \frac{2}{3}t^3 - 8t$

Use d for the constant as c has already been used.

WORKED EXAMPLE 14 Displacement and velocity as integrals 2

An object moves in a straight line so that its acceleration (m s^{-2}) at time t seconds is given by $a(t) = 6t + 4$ for $t \geq 0$. If $v(0) = 4$ and $x(0) = 7$, **determine** the velocity and position of the particle when $t = 2$.

Steps	Working
1 Integrate the acceleration function to find the velocity function.	$v(t) = \int 6t + 4\,dt$
	$= 3t^2 + 4t + c$
2 Find the value of c by substituting in $t = 0$ and $v = 4$.	$4 = 3(0)^2 + 4(0) + c$
	$c = 4$
3 State $v(t)$ including the value of c.	$\therefore v(t) = 3t^2 + 4t + 4$
4 Calculate $v(2)$.	$v(2) = 3(2)^2 + 4(2) + 4$
	$= 24\,\text{m s}^{-1}$
5 Integrate the velocity function to find the displacement function.	$x(t) = \int (3t^2 + 4t + 4)\,dt$
	$= t^3 + 2t^2 + 4t + d$
6 Find the value of d by substituting in $t = 0$ and $x = 7$.	$7 = (0)^3 + 2(0)^2 + 4(0) + d$
	$7 = d$
7 State $x(t)$ including the value of d.	$x(t) = t^3 + 2t^2 + 4t + 7$
8 Calculate $x(2)$.	$x(2) = (2)^3 + 2(2)^2 + 4(2) + 7$
	$= 31\,\text{m}$
9 State the result.	At $t = 2$, the object has a velocity of $24\,\text{m s}^{-1}$ and a displacement of 31 m; its position is 31 m to the right of the origin.

An object that is subject to the force of gravity has an acceleration of $-9.8\,\text{m s}^{-2}$; that is, $9.8\,\text{m s}^{-2}$ downwards. The positive direction is taken to be upwards and the negative direction is downwards.

WORKED EXAMPLE 15 Straight-line motion involving gravity

An object, 9 m above the ground, is projected vertically upwards with a velocity of $25\,\text{m s}^{-1}$.

a What is the velocity of the particle at 2 s?

b When does the particle come to rest for the first time?

c What is the position of the particle at 3 s?

Steps	Working
a 1 Determine the equation for acceleration using gravity.	$a(t) = -9.8$
2 Integrate the acceleration function to find the velocity function.	$v(t) = \int (-9.8)\,dt$ $= -9.8t + c$
3 Find the value of c by substituting in $t = 0$ and $v = 25$.	$25 = -9.8(0) + c$ $c = 25$
4 State $v(t)$ including the value of c.	$\therefore v(t) = -9.8t + 25$
5 Calculate $v(2)$.	$v(2) = -9.8(2) + 25$ $= 5.4\,\text{m s}^{-1}$
6 State the result.	At 2 s, the velocity of the object is $5.4\,\text{m s}^{-1}$ upwards.
b 1 The object will be at rest when $v = 0$. Use this to find t.	$0 = -9.8t + 25$ $t = \dfrac{-25}{-9.8}$ ≈ 2.551
2 Round the answer and state the result.	The object first comes to rest at about 2.55 s.
c 1 Integrate the velocity function to find the displacement function.	$x(t) = \int (-9.8t + 25)\,dt$ $= -4.9t^2 + 25t + d$
2 Find the value of d by substituting in $t = 0$ and $x = 9$.	$9 = -4.9(0)^2 + 25(0) + d$ $d = 9$
3 State $x(t)$ including the value of c.	$x(t) = -4.9t^2 + 25t + 9$
4 Calculate $x(3)$.	$x(3) = -4.9(3)^2 + 25(3) + 9$ $= 39.9\,\text{m}$
5 State the result.	At $t = 3$, the object has a displacement of 39.9 m.

EXERCISE 5.4 Straight-line motion

ANSWERS p. 406

Recap

1 Determine $\int 3\cos\left(\frac{x}{2}\right)dx$.

A $-6\sin\left(\frac{x}{2}\right)+c$ **B** $-\frac{3}{2}\sin\left(\frac{x}{2}\right)+c$ **C** $\frac{3}{2}\sin\left(\frac{x}{2}\right)+c$ **D** $6\sin\left(\frac{x}{2}\right)+c$

2 Determine $\int (\cos(3x) + \sin(3x))\,dx$.

Mastery

3 WORKED EXAMPLE 13 The acceleration of an object as a function of time, t, is given by $a(t) = 3t + 1$. Find the velocity, $v(t)$, of the object if it started at rest.

4 The velocity, v cm s^{-1}, of a particle moving in a straight line at time t seconds is given by

$$v = 4t^2 - 10t - 6, t \geq 0$$

Initially the particle is 27 cm to the right of a fixed origin.

a Obtain an expression for the particle's displacement at time t seconds.

b How far from its initial position is the particle after the first second?

c Determine the position of the particle when its velocity is zero.

5 The velocity of a particle moving in a horizontal line is given by $v = \frac{dx}{dt} = 6 - 9e^{-3t}$, where t is the time in seconds and x is the displacement in metres.

a What is the initial ($t = 0$) velocity of the particle?

b Find the equation of the displacement (x) in terms of t if it is known that initially the particle was 4 metres to the left of the origin.

6 WORKED EXAMPLE 14 An object moves in a straight line so that its acceleration (m s^{-2}) at time t seconds is given by $a(t) = 5 - 8t$ for $t \geq 0$. If its initial velocity is 7 m s^{-1} and $x = 7.5$ m when $t = 3$, calculate the velocity and position of the particle when $t = 1$.

7 The acceleration (a m s^{-2}) of a particle moving in a straight line after t seconds is given by $a = 6t - 18$, where $t \geq 0$. At $t = 1$, the velocity of the particle is 9 m s^{-1} and its displacement is 20 m. Find an equation for

a v, the velocity of the particle after t seconds

b x, the displacement of the particle from the origin after t seconds.

8 WORKED EXAMPLE 15 An object on top of a cliff 20 m above the ground is projected vertically upwards with a velocity of 10 m s^{-1}.

a What is the velocity of the object after 2 s?

b When does the object come to rest for the first time?

c The greatest height above the ground is reached when the object stops going up and begins to fall. What is the greatest height reached by the object?

d What is the height of the object after 3 s?

Exam practice

9 © QCAA 2020S 1Q14 TECH-FREE (5 marks) Let $v(t) = \frac{1}{\pi} + 3\sin(t), t \geq 0$ represent the velocity of an object moving in a straight line.

At $t = \frac{\pi}{3}$, the position of the object is 4.

a Determine the acceleration function. *[2 marks]*

b Determine the displacement function. *[3 marks]*

10 TECH-FREE (5 marks) An object moves in a straight line and its velocity (m s^{-1}) at time t is given by $v(t) = t^2 - t - 6$. At $t = 2$, the particle is 10 m to the left of the origin.

a Determine the initial position of the object. *[4 marks]*

b Determine the acceleration function for the object. *[1 mark]*

11 TECH-FREE (5 marks) A particle is moving in a straight line. At $t = 0$, its displacement $x = 1$ and its velocity $v = \frac{1}{2}$. The acceleration of the particle is given by $a = \frac{1}{4}\cos(4t)$.

a Find the velocity of the particle at time t. *[2 marks]*

b Find the position of the particle at time t. *[3 marks]*

12 (6 marks) The acceleration of an object is given by $a = 3.5t + 0.6\,\text{m s}^{-2}$.
The object begins from rest and accelerates for 30 seconds.

a Find an expression for the velocity of the object. *[2 marks]*

b Find the velocity reached after 10 seconds. *[1 mark]*

c Find an expression for the displacement. *[2 marks]*

d Find the displacement after 10 seconds. *[1 mark]*

13 (6 marks) CF A particle accelerates horizontally according to the equation $a(t) = -e^{4t}\,(\text{cm s}^{-2})$.
Initially, the particle is 4 cm to the right of the origin, moving with a velocity of $\frac{3}{4}\,\text{cm s}^{-1}$.

a Find the velocity of the particle after 1 s (correct to 2 decimal places). *[3 marks]*

b Find the position of the particle after 2 s (correct to 2 decimal places). *[3 marks]*

14 (4 marks) CF The velocity of an object at time t seconds is given by $v(t) = 2t^2 - \sqrt{t+1} + 1\,(\text{m s}^{-1})$.
If $x(0) = 4$, what is the position of the particle 4 seconds later?

15 (4 marks) CU A body is travelling such that its velocity is given by $v(t) = 6\sin(3t)\,(\text{m s}^{-1})$ for $0 \le t \le \frac{\pi}{2}$. Find the *total distance* travelled from $t = 0$ to $t = \frac{\pi}{2}$ s.

5.5 Problems involving indefinite integrals

Worksheets
Anti-differentiation assignment
Anti-differentiation problems

There are many types of applications involving indefinite integrals.
For instance, you may be provided with a rate of change in the form of the derivative function and some initial conditions. Integrating the function and using the initial set of conditions enables you to determine the original function.

WORKED EXAMPLE 16 Finding y given $\frac{dy}{dx}$ and a point

If $\frac{dy}{dx} = \frac{4}{2x-1}$ and $y = 5$ when $x = 1$, **determine** an expression for y in terms of x.

Steps	Working
1 Integrate $\frac{dy}{dx}$ to find y.	$y = \int \frac{4}{2x-1}\,dx = 4\int \frac{1}{2x-1}\,dx$ $y = 2\ln(2x-1) + c$
2 Find the value of c by substituting in $x = 1$ and $y = 5$.	$5 = 2\ln(2(1)-1) + c$ $5 = 0 + c$ $c = 5$
3 State the answer including the value of c.	$y = 2\ln(2x-1) + 5$

WORKED EXAMPLE 17 Applying integration

A colony of bacteria in a petri dish is growing at a rate given by $\frac{dP}{dt} = 30e^{0.1t}$, where $P(t)$ is the number of bacteria after t hours.

a **Determine** the rule for $P(t)$ if initially there were 200 bacteria.

b How many bacteria are there after 5 hours?

Steps	Working
a 1 Integrate $\frac{dP}{dt}$ to find $P(t)$.	$P(t) = \int 30e^{0.1t}\,dt$
	$P(t) = 300e^{0.1t} + c$
2 Find the value of c by substituting in $t = 0$ and $P = 200$.	$200 = 300e^{0.1(0)} + c$
	$-100 = c$
3 State the answer including the value of c.	$P(t) = 300e^{0.1t} - 100$
b 1 Find $P(5)$.	$P(5) = 300e^{0.1(5)} - 100 \approx 394.6$
2 State the answer.	After 5 hours there are approximately 395 bacteria.

WORKED EXAMPLE 18 Straight-line motion

An object is thrown upwards with a velocity of 18 m s^{-1} from a height of 16 m and is acted on by gravity alone.

a Find expressions for the velocity and displacement at time t.

b Find, correct to 2 decimal places

i when the velocity is 0

ii the height of the object when its velocity is 0

iii when the object hits the ground

iv the velocity of the object when it hits the ground.

Steps	Working
a 1 Find $a(t)$.	$a(t) = -9.8$
2 Integrate $a(t)$ to find $v(t)$.	$v(t) = \int -9.8\,dt$
	$= -9.8t + c$
3 Find the value of c by substituting in $t = 0$ and $v = 18$.	$18 = -9.8(0) + c$
	$18 = c$
4 State the function including the value of c.	$v(t) = -9.8t + 18$
5 Integrate $v(t)$ to find $x(t)$.	$x(t) = \int (-9.8t + 18)\,dt$
	$= -4.9t^2 + 18t + d$
6 Find the value of d by substituting in $t = 0$ and $x = 16$.	$16 = -4.9(0)^2 + 18(0) + d$
	$16 = d$
7 State the function including the value of c.	$x(t) = -4.9t^2 + 18t + 16$

b i 1 Find t when $v = 0$.

$$0 = -9.8t + 18$$

$$t = \frac{18}{9.8}$$

$$\approx 1.836\ldots$$

2 State the answer.

The velocity is 0 at about 1.84 seconds.

ii 1 Find x when $t \approx 1.836\ldots$

$$x(t) = -4.9(1.836\ldots)^2 + 18(1.836\ldots) + 16$$

$$\approx 32.530\ldots$$

2 State the answer.

The height when $v = 0$ is about 32.53 m.

iii 1 Let $x = 0$.

$$0 = -4.9t^2 + 18t + 16$$

2 Use the quadratic formula to solve for t.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-18 \pm \sqrt{18^2 - 4 \times -4.9 \times 16}}{2 \times -4.9}$$

$$t = 4.413\ldots \text{ or } t = -0.739\ldots$$

Disregard $t = -0.739\ldots$ as $t > 0$. So $t = 4.413\ldots$

3 State the answer.

The object hits the ground at about 4.41 seconds.

iv 1 Find $v(4.413\ldots)$

$$v(4.413\ldots) = -9.8(4.413\ldots) + 18$$

$$= -25.247\ldots \text{ m s}^{-1}$$

2 State the answer.

The velocity of the object when it hits the ground is -25.25 m s^{-1}.

EXERCISE 5.5 Problems involving indefinite integrals

ANSWERS p. 406

Recap

1 The velocity of an object travelling in a straight line is given by $v(t) = -t^2 + 2t$.
Determine an expression for the displacement $x(t)$ if $x = 0$ when $t = 2$.

2 An object moves in a straight line so that its acceleration (m s^{-2}) at time t seconds is given by $a = 7 - 4t$ for $t \geq 0$. If the initial velocity is 12 m s^{-1} and $x = 59.5$ m when $t = 3$, calculate the velocity and position of the particle when $t = 2$.

Mastery

3 Given that $y = e^{x^5}$, calculate $\frac{dy}{dx}$ and use this result to find $\int (2x^4 e^{x^5})\,dx$.

4 WORKED EXAMPLE 16 If $\frac{dy}{dx} = \frac{16}{9 - 4x}$ and $y = -5$ when $x = 1$, find an equation for y in terms of x.

5 Calculate $\frac{d}{dx}\left(\sqrt{x^2 - 8}\right)$ and use the result to find $\int \left(\frac{5x}{3\sqrt{x^2 - 8}}\right) dx$.

6 WORKED EXAMPLE 17 The height of a plant is increasing at a rate given by $\frac{dh}{dt} = \frac{30}{t + 1}$, $t \geq 0$, where h is the height of the plant in centimetres after t days.

a Find the rule for $h(t)$ if the plant was 10 cm tall initially.

b What is the height of the plant after 7 days?

7 WORKED EXAMPLE 18 An object on top of a building 35 m tall is projected vertically upwards with a velocity of 32 m s^{-1}.

a What is the velocity of the particle at 3 s?

b When (correct to 2 decimal places) does the particle come to rest for the first time?

c What (correct to 2 decimal places) is the greatest height reached by the object?

d What is the position of the particle at 4 s?

Exam practice

8 (3 marks) TECH-FREE Calculate $\frac{d}{dx}(3x^4+1)^3$ and use the result to determine $\int (x^3(3x^4+1)^2)\,dx$.

9 (2 marks) TECH-FREE Determine $\int \left(\frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}}\right) dx$.

10 (3 marks) CF The population of rabbits on a farm is growing at a rate given by $P'(t) = 5t^2 + 2t + 20$, where P is the population after t months.

a Determine the rule for $P(t)$, if initially there were 30 rabbits on the farm. *[2 marks]*

b When will the population of rabbits exceed 300? *[1 mark]*

11 (4 marks) CF The point (2, 19) lies on the graph of the function $y = f(x)$. If $\frac{dy}{dx} = 8x^2 - 4x + 7$, find the equation of the function and its value at $x = -2$.

12 (3 marks) CF The velocity of the bottom of the pendulum in a grandfather clock is given by

$$v = 56.7\cos(6.3t)$$

where t is time in seconds and v is in cm s^{-1}. Find an expression for the displacement from the centre if at time $t = 0$ the displacement is 0 cm. Justify your reasoning.

13 (5 marks) CF A pebble falls from the top of a cliff and strikes the ground 2.5 s after it begins to fall. Calculate the height of the cliff, correct to 2 decimal places.

14 (5 marks) CF The acceleration (cm s^{-2}) of a particle is given by $a(t) = e^{2t} - 5$. If the particle started from rest, how far is the particle from its starting point and which way is it moving 2 seconds later?

15 (4 marks) CF The air molecules in a flute move backwards and forwards about their initial position as a note is played. The average acceleration is given by

$$a = L\sin(0.016t)$$

where a is in m s^{-2} and t is time in seconds.

L is a quantity determined by the loudness of the note. At time $t = 0$, the molecules at the far end are moving at their fastest. Use anti-derivatives to find equations for the velocity and displacement of these molecules at time t, and comment on the directions of the acceleration and displacement of the molecules.

16 (7 marks) CU An object is projected vertically upwards from the ground with an initial velocity of 210 m s^{-1}.

a What is the greatest height reached by the object? *[3 marks]*

b When does the particle return to the ground? *[4 marks]*

17 (11 marks) CU Two particles A and B are moving to the right along a straight line. Particle A is accelerating according to the equation $a_A(t) = \frac{3}{4}$ and particle B is accelerating according to the equation $a_B(t) = \frac{2}{3}$. The velocity and displacement of the particles is given by $v(t)$ and $x(t)$ respectively.

If $v_A(0) = \frac{1}{2}$, $x_A(2) = 2$, $v_B(1) = \frac{5}{6}$ and $x_B(1) = \frac{1}{2}$, when will the 2 particles have the same position?

18 (4 marks) CU A particle moves in a straight line and its velocity in m s^{-1} is given by $v(t) = 6t^2 - 18t + 12$, for $0 \leq t \leq 3$, where t is time in seconds.

Find the total distance travelled by a particle in the 3 seconds of its motion.

19 © QCAA 2020 1Q20 (6 marks) TECH-FREE CU At the end of the first stage of its growth cycle, a species of tree has a height of 5 metres and a trunk radius of 15 cm.

In the second stage of its growth cycle, the tree stays at this height for the next 10 years. However, the growth rate of the trunk radius (in cm per year) varies over the 10 years and is given by the function

$$r(t) = 1.5 + \sin\left(\frac{\pi t}{5}\right)$$

Assume the density (mass per unit volume) of the tree trunk is approximately $1\ \text{g cm}^{-1}$ and that the tree trunk is in the shape of a cylinder.

Determine the ratio of the mass of the trunk at the end of the second stage to its mass at the end of the first stage.

20 (6 marks) CU The electric tilt-train from Gladstone to Rockhampton has an acceleration of $0.69\ \text{m s}^{-2}$ and a braking deceleration of $1.05\ \text{m s}^{-2}$. The train operates at a top speed of $160\ \text{km h}^{-1}$. There are no regular stops on the 109 km from Gladstone to Rockhampton. How long should it take to travel between Gladstone and Rockhampton?

Video
Exam question analysis: Anti-differentiation

EXAM QUESTION ANALYSIS

© QCAA | 2020 2Q12 (7 marks)

The rates of change in population for two cities are given by

$$\text{City A: } A'(t) = \frac{45}{t+1}$$

$$\text{City B: } B'(t) = 105e^{0.03t}$$

where t is the number of years since 2018 and both $A'(t)$ and $B'(t)$ are measured in people per year. At the beginning of 2018, City A had a population of 5000, and City B had a population of 3500.

a Determine the population models for both cities. *[3 marks]*

b Use the information in part **a** to predict the population of City B at the beginning of 2028. *[1 mark]*

c Use the information in part **a** to predict the year in which the population of the two cities will be the same. *[3 marks]*

Reading the question

- There are two cities and two population rates of change.
- t is the number of years since 2018.
- At $t = 0$, the population of City A was 5000.
- At $t = 0$, the population of City B was 3500.

Thinking about the question

- Integrate the rate of change in population function for each city.
- Find the constant of integration for the population function of each city.
- Find the population of City B at $t = 10$.
- When the populations are the same, their functions will be equal.

Worked Solution (✓ = 1 mark)

a $A(t) = 45\ln(t+1) + c$ ✓

$B(t) = 3500e^{0.03t} + c$ ✓

Given $A(0) = 5000$ and $B(0) = 3500$

The population functions for the two cities are:

$A(t) = 45\ln(t+1) + 5000$ ✓ for the population functions of the two cities

$B(t) = 3500e^{0.03t}$

b $B(10) = 3500e^{0.03(10)}$

$B(10) = 4724$ ✓

c $45\ln(t+1) + 5000 = 3500e^{0.03t}$ ✓

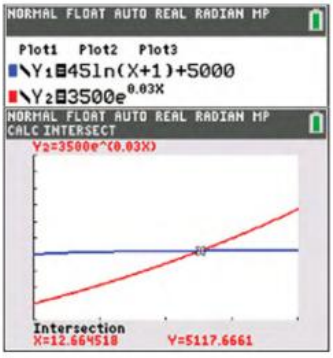
Using my calculator and finding the point of intersection

$t = 12.66$ ✓

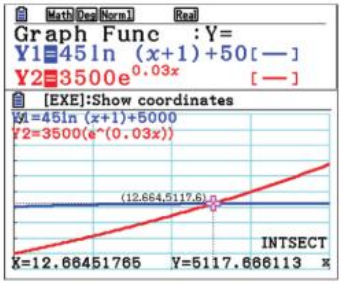
Year $= 2018 + 12.66 \approx 2030$

During 2030 the populations will be the same. ✓

TI-84 Plus CE



Casio fx-CG50 AU



TI-Nspire
Anti-differentiation

5 Chapter summary

The integral of x^n and $(ax + b)^n$

- **Anti-differentiation** (also called **integration**) is the reverse process of finding the derivative.
- The **indefinite integral** of the function $f(x)$ is $F(x)+c$, where c is the **constant of integration**.
- The symbol $\int$ represents the integral, so $F(x)=\int f(x)dx$, where $F'(x)=f(x)$.
- $\int ax^n dx = \frac{ax^{n+1}}{n+1}+c$, where $n \neq -1$
- Properties of integrals:
  - $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
  - $\int k f(x)dx = k\int f(x)dx$
- $\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1}+c$, where $n \neq -1$.
- The general rule for the integral of functions of the form $f(ax+b)$ is $\int f(ax+b)dx = \frac{1}{a}F(ax+b)+c$, where $F(x)$ is the integral of $f(x)$.

The integrals of e^x

- $\int e^x dx = e^x + c$
- $\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
- $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + c$

The integrals of $\frac{1}{x}$ and $\frac{1}{ax+b}$

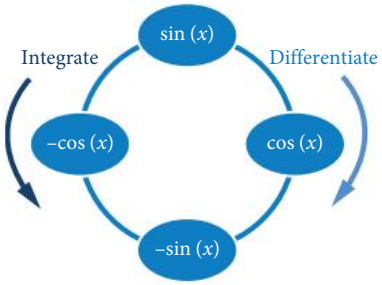
- $\int \frac{1}{x}dx = \ln(x)+c$, where $x>0$
- $\int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b)$

Integral of $\frac{f'(x)}{f(x)}$

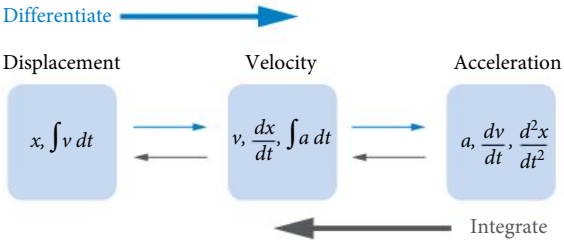
- $\int \frac{f'(x)}{f(x)}dx = \ln(f(x))+c$ for $f(x)\neq 0$

The integrals of $\sin(x)$ and $\cos(x)$

- $\int \cos(x)dx = \sin(x)+c$
- $\int \sin(x)dx = -\cos(x)+c$
- $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$
- $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)+c$



Straight-line motion



- $x(t)=\int v(t)dt$, and $v(t)=\int a(t)dt$

Worksheet
Maths Methods formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2020 1Q2 Determine $\int \frac{e^x+1}{e^x}dx$

A $x-e^{-x}+c$ **B** $x+e^{-x}+c$ **C** $1+xe^{-x}+c$ **D** $x+xe^{-x}+c$

2 © QCAA 2020 1Q3 Determine $2\int(4x+6)^3\,dx$

A $16(4x+6)^4+c$ **B** $8(4x+6)^4+c$ **C** $\frac{(4x+6)^4}{2}+c$ **D** $\frac{(4x+6)^4}{8}+c$

3 © QCAA 2020 1Q6 If the probability of success in a Bernoulli trial is 0.30, the variance is

A 0.7 **B** 0.46 **C** 0.3 **D** 0.21

Section 2 3 short response questions

15 marks

4 © QCAA 2020 1Q12 (5 marks) An object is moving in a straight line from a fixed point. The object is at the origin initially.

The acceleration a (in m s^{-2}) of the object is given by

$$a(t) = \pi\cos(\pi t) \qquad t \geq 0, \text{ where } t \text{ is time in seconds.}$$

The velocity at $t = 1$ is $0.5\ \text{m s}^{-1}$

a Determine the initial acceleration. *[1 mark]*

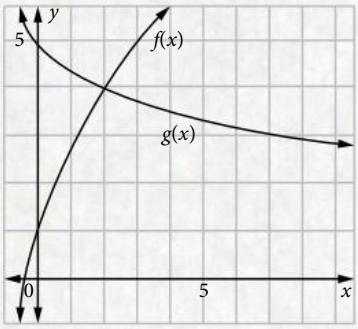
b Determine the initial velocity. *[2 marks]*

c Determine the displacement after one second. *[2 marks]*

5 © QCAA 2020 1Q18 (6 marks) The function $f(x)$ has the form given by $f(x) = 3\log_2(x + a) + b$.

The function $g(x)$ has the form given by $g(x) = -\log_3(x + c) + 5$.

A section of the graphs of the two functions is shown.



Determine the values of a, b and c.

6 (4 marks) $f'(x) = e^{2x}(2\cos(3x) - 3\sin(3x))$ and $f\left(\frac{\pi}{3}\right) = 0$.

Find the equation of the tangent to $f(x)$ at $x = -\frac{\pi}{3}$.

Worksheet
Maths Methods formula sheet

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

Total: 18 marks

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2021 2Q10 An object travels in a straight line so that its velocity at time t seconds is given by

$$v(t) = 2t + \sin(2t)$$

Determine the expression for acceleration as a function of time.

A $a(t) = 2 + 2\cos(2t)$

B $a(t) = 2 - \frac{1}{2}\cos(2t)$

C $a(t) = t^2 + 2\cos(2t)$

D $a(t) = t^2 - \frac{1}{2}\cos(2t)$

2 © QCAA 2021 2Q7 Determine $f(x)$, given $f'(x) = 6x^2 + \frac{1}{x^2} + \frac{1}{x}$ and $f(1) = 5$.

A $f(x) = 2x^3 + \frac{3}{x^3} + \ln(x) - 1$

B $f(x) = 2x^3 - \frac{1}{x} + \ln(x) + 4$

C $f(x) = 2x^3 - \frac{1}{x} + \frac{2}{x^2} + 2$

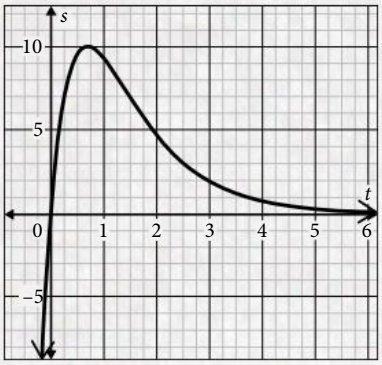
D $f(x) = 2x^3 + \frac{3}{x^3} + \frac{2}{x^2} - 2$

3 © QCAA 2020 2Q5 An object moves in a straight line with a velocity v given by

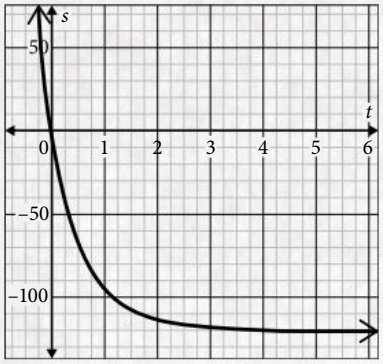
$$v(t) = 40(e^{-t} - e^{-2t})\,\text{m s}^{-1} \text{ where } t \geq 0$$

The object is at the origin initially. The displacement–time graph in the first 6 seconds is

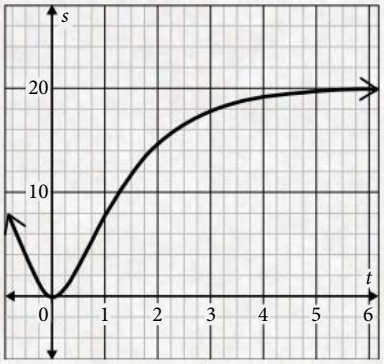
A



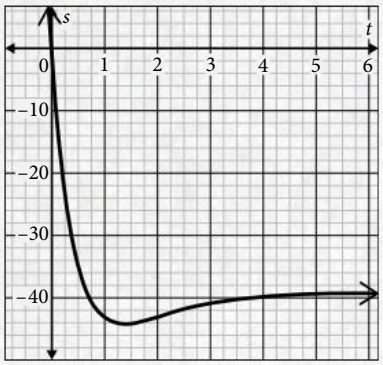
B



C



D



Section 2 3 short response questions

15 marks

4 © QCAA 2022 2Q11 (7 marks) A salesperson has a 20% probability of making a sale to each customer who enters the store. Each sale is independent of all other sales.

a Determine the mean number of sales on a day where 25 customers enter the store. *[2 marks]*

b Determine the standard deviation of the number of sales on a day where 25 customers enter the store. *[2 marks]*

c Determine the minimum number of customers who would have to enter the store to have an 88% chance or more of making at least one sale. *[3 marks]*

5 (4 marks) Solve the equation $2\ln(x) - \ln(2x + 3) - \ln(e^4) = 0$, correct to 3 decimal places.

6 (4 marks) A cylindrical tank full of water had a capacity of 25 000 litres. Two pipes at the bottom of the tank are used to supply different applications. The larger pipe has twice the capacity of the other.

The valves for both pipes are opened at the same time. After 30 minutes, 5000 litres has drained from the tank. The volume of water in the tank decreases exponentially.

The larger valve is then turned off.

How long will it take for another 5000 litres to drain?

CHAPTER

6 INTEGRATION AND AREAS



Syllabus coverage

Nelson MindTap chapter resources

Terminology

6.1 Approximating areas under curves
- The rectangle method
- The trapezoidal method

6.2 Definite integrals
- Fundamental theorem of calculus
- Properties of definite integrals

6.3 Areas under curves
- Signed areas
- Combined areas above and below a curve

6.4 Areas between curves

6.5 Problems involving definite integrals
- Total and marginal change
- Straight-line motion

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170484985

Syllabus coverage

UNIT 4, TOPIC 1: FURTHER INTEGRATION

Fundamental theorem of calculus and definite integrals

- Use sums of the form $\sum_i f(x_i)\delta x_i$ to estimate the area under the curve $y = f(x)$.
- Recognise the definite integral $\int_a^b f(x)\,dx$ as a limit of sums of the form $\sum_i f(x_i)\delta x_i$.
- Understand the fundamental theorem of calculus, $\int_a^b f(x)dx = F(b) - F(a)$, and use it to calculate definite integrals.
- Use the definite integral $\int_a^b f(x)dx$ to determine the area under the curve $y = f(x)$ between $x = a$ and $x = b$ if $f(x) > 0$ over this interval.

Applications of integration

- Calculate the area enclosed by a curve and the x-axis over a given domain, with and without technology.
- Calculate the area between curves, with and without technology.
- Use the trapezoidal rule, $\int_a^b f(x)dx \approx \frac{w}{2}\left[f(x_0) + 2(f(x_1) + f(x_2) + f(x_3) + \ldots f(x_{n-1})) + f(x_n)\right]$, where $w = \frac{b-a}{n}$, to approximate an area and the value of a definite integral, with and without technology.
- Calculate total change by integrating instantaneous or marginal rates of change, with and without technology.
- Model and solve problems that involve definite integrals, including motion problems, with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 28
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Prior learning
Integration and areas

Videos (3):
- **6.2** Definite integrals
- **6.3** Area under a curve
- **Exam question analysis** Integration and areas

Prior learning (1):
- **6.1** Integration and areas

Worksheets (13):
- **6.1** Areas using rectangles • Trapezoidal rule
- **6.2** Definite integrals • Finding definite integrals
- **6.3** Calculating physical areas
- **6.4** Calculating areas between curves • Areas between curves 1 • Areas between curves 2
- **6.5** Total change • Displacement, velocity and acceleration • Integration assignment • Integration problems
- **Cumulative exams** Maths Methods formula sheet

TI-Nspire CX II instructions (1):
- **6.1, 6.2, 6.3, 6.4, Exam question analysis** Integration and areas

Nelson MindTap
To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

average value
centred rectangle
definite integral
difference function
fundamental theorem of calculus
indefinite integral
left endpoint
left rectangle
limit
marginal change
marginal rate of change
physical area
right endpoint
right rectangle
signed area
total change
trapezoidal rule

Worksheets
Areas using rectangles

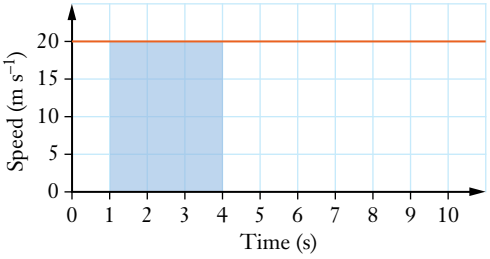
Trapezoidal rule

6.1 Approximating areas under curves

Understanding what the area under a graph of a curve represents is important. It provides a way to analyse changing quantities and to measure total quantities, such as the amount of fuel consumed by a car, the flow of water through a pipe, or the revenue generated from sales over time.

Investigation Speed and distance

The speed–time graph below represents a car travelling at $20\,\text{m s}^{-1}$ ($72\,\text{km h}^{-1}$).

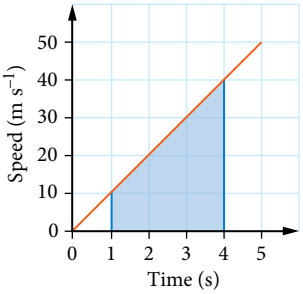


What is the shaded area under the graph between 1 and 4 seconds?

What distance does the car travel between 1 and 4 seconds?

What connection can you see between the area and the distance?

The speed–time graph below represents a racing car whose speed increases steadily after the start.



What is its speed at $t = 1$?

What is its speed at $t = 4$?

What is the average speed between 1 and 4 seconds?

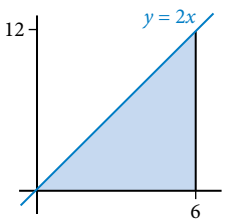
What is the distance it travels between 1 and 4 seconds?

What is the area under the graph between 1 and 4 seconds?

What can you say about the relationship between the area under a speed–time graph and the distance travelled?

The rectangle method

One method for finding the area under the graph of a curve is to use a series of rectangles (also called vertical slices). The more rectangles used, the closer the area approximates the actual area.

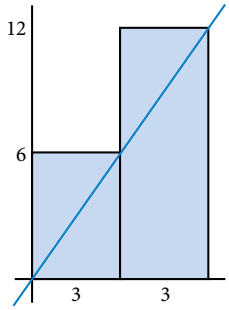
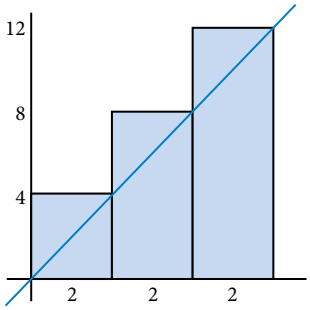
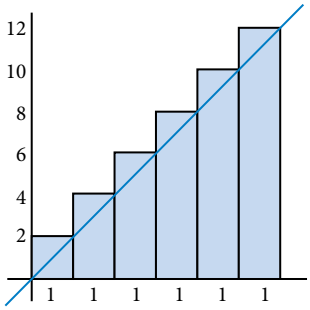
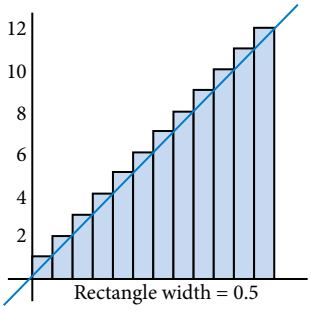


The graph of $y = 2x$ and the area under the line from $x = 0$ to $x = 6$ is shown.

The area is $A = \frac{1}{2}bh = 36$ units^2.

9780170484985

Now consider the following table which illustrates how the area could be approximated using 2, 3, 6 and 12 rectangles.

Approximate area			
			
2 rectangles	3 rectangles	6 rectangles	12 rectangles
$3 \times 6 + 3 \times 12$ $= 54$	$2 \times 4 + 2 \times 8 + 2 \times 12$ $= 48$	$1 \times 2 + 1 \times 4 + 1 \times 6$ $+ 1 \times 8 + 1 \times 10 + 1 \times 12$ $= 42$	$0.5 \times 1 + 0.5 \times 2 + \ldots$ $+ 0.5 \times 11 + 0.5 \times 12$ $= 39$

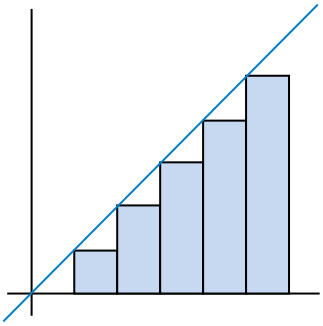
The greater the number of rectangles the closer the approximate area is to the actual area. The approximate area is also greater than the actual area.

In the diagrams in the table above, the top right corners of the rectangles touch the curve. They are called **right rectangles**.

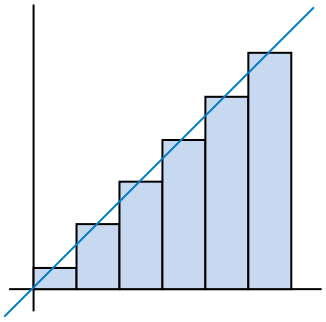
Left rectangles have their top left corner touching the curve.

Centred rectangles have the middle of their top touching the curve.

Depending on the curve and the rectangle position used, the approximate area will be greater or less than the actual area.



Left rectangles



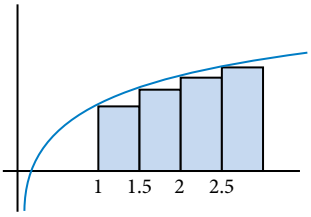
Centred rectangles

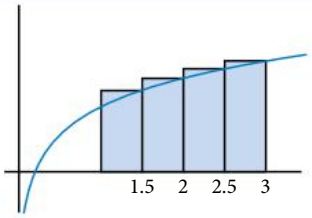
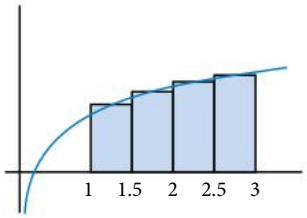
WORKED EXAMPLE 1 Approximating areas using the rectangle method

Find an approximation to the area under the curve $f(x) = \ln(5x)$ between $x = 1$ and $x = 3$.

Use rectangles of width of $\frac{1}{2}$ unit and

a left rectangles **b** right rectangles **c** centred rectangles.

Steps	Working
a 1 Sketch the graph. Draw in rectangles that touch the curve at the top left corner. The point where the rectangle touches the curve is the x and y values used in the area calculations.	
2 State the x values for each rectangle.	The x values are 1, 1.5, 2 and 2.5.
3 Determine the height of each rectangle using $f(x) = \ln(5x)$.	$f(1) \approx 1.609$, $f(1.5) \approx 2.015$ $f(2) \approx 2.303$, $f(2.5) \approx 2.526$

4	Find the area of each rectangle using A = width × height.	$A_1 = \frac{1}{2} \times 1.609 \approx 0.805$ $A_2 = \frac{1}{2} \times 2.015 \approx 1.007$ $A_3 = \frac{1}{2} \times 2.303 \approx 1.151$ $A_4 = \frac{1}{2} \times 2.526 \approx 1.263$
5	Find the total area by adding the areas of the rectangles.	$A_T = 0.805 + 1.007 + 1.151 + 1.262$ $= 4.226$
6	State the result.	The total of the left rectangles is 4.226 units^2.
b 1	Sketch the graph with right rectangles.	
2	State the x values for each rectangle.	The x values are 1.5, 2, 2.5 and 3.
3	Determine the height of each rectangle.	$f(1.5) \approx 2.015$, $f(2) \approx 2.303$, $f(2.5) \approx 2.526$, $f(3) \approx 2.708$
4	Find the area of each rectangle.	$A_1 = \frac{1}{2} \times 2.015 \approx 1.007$ $A_2 = \frac{1}{2} \times 2.303 \approx 1.151$ $A_3 = \frac{1}{2} \times 2.526 \approx 1.263$ $A_4 = \frac{1}{2} \times 2.708 \approx 1.354$
5	Find the total area.	$A_T = 1.007 + 1.151 + 1.263 + 1.354$ $= 4.775$
6	State the result.	The total of the right rectangles is 4.775 units^2.
c 1	Sketch the graph with centre rectangles.	
2	State the x values for the centre of each rectangle.	The x values are 1.25, 1.75, 2.25 and 2.75.
3	Determine the height of each rectangle.	$f(1.25) \approx 1.833$, $f(1.75) \approx 2.169$, $f(2.25) \approx 2.42$, $f(2.75) \approx 2.621$

4 Find the area of each rectangle.

$$A_1 = \frac{1}{2} \times 1.833 \approx 0.916$$

$$A_2 = \frac{1}{2} \times 2.169 \approx 1.085$$

$$A_3 = \frac{1}{2} \times 2.42 \approx 1.21$$

$$A_4 = \frac{1}{2} \times 2.621 \approx 1.311$$

5 Find the total area.

$$A_T = 0.916 + 1.085 + 1.21 + 1.311$$
$$= 4.522$$

6 State the result.

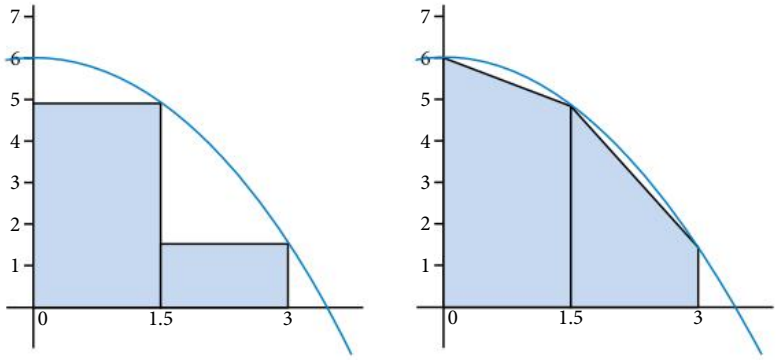
The total of the right rectangles is 4.522 units^2.

The left rectangles underestimated the area, while the right rectangles overestimated the area. The centred rectangles gave the closest approximation to the actual area. However, in more complex situations or when using more rectangles, this method becomes tedious so, in general, only the **left rectangles** or **right rectangles** area methods are used.

The trapezoidal method

The accuracy of an approximation to the area under the curve can be improved by using trapeziums rather than rectangles.

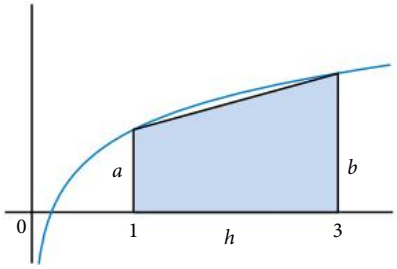
This is because a trapezium reduces the space between the curve and the shape.



The area of a trapezium can be found using $A = \frac{1}{2}(a+b)h$, where a and b are the parallel heights of the trapezium where it touches the curve and h is the width of the trapezium.

The area of a trapezium between 1 and 3 for $\ln(5x)$ is

$$A = \frac{1}{2}(f(1) + f(3)) \times 2$$
$$= \frac{1}{2}(\ln(5) + \ln(15)) \times 2$$
$$= 4.318 \text{ units}^2$$



From Worked example 1, the approximate area under the curve is between 4.226 units^2 and 4.775 units^2. So, even using a single trapezium, the approximate area is more accurate.

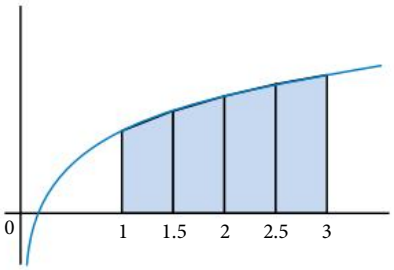
Extending this situation to use 4 trapeziums, the approximate area is:

$$A_T = \left[\frac{1}{2}(f(1)+f(1.5))\times 0.5\right]+\left[\frac{1}{2}(f(1.5)+f(2))\times 0.5\right]+\left[\frac{1}{2}(f(2)+f(2.5))\times 0.5\right]+\left[\frac{1}{2}(f(2.5)+f(3))\times 0.5\right]$$

Notice that each value for a and b is used twice in A_T except for the ends at $x = 1$ and $x = 3$.
So A_T can be simplified to

$$A_T = \frac{0.5}{2}\left[f(1)+f(3)+2\times(f(1.5)+f(2)+f(2.5))\right]$$

$$= 4.501 \text{ units}^2$$



The trapezoidal rule gives the average of the left and right rectangle areas.

This approach can easily be converted into a general formula for the approximation of area using trapeziums (also called **trapezoids**).

The trapezoidal rule

The area under the curve $f(x)$ on the interval $[a, b]$, where $f(x) \geq 0$ with n trapeziums can be found using

$$\int_a^b f(x)dx \approx \frac{w}{2}\left[f(x_0)+2(f(x_1)+f(x_2)+f(x_3)+\ldots f(x_{n-1}))+f(x_n)\right]$$

where $w = \dfrac{b-a}{n}$.

An alternative version of the **trapezoidal rule** is

$$\int_a^b f(x)dx \approx \frac{w}{2}\left[E+2M\right]$$

where E is the sum of the end values of the function ($f(x_0)$ and $f(x_n)$, and M is the sum of the middle values ($f(x_1), f(x_2), f(x_3) + \ldots f(x_{n-1})$).

Exam hack

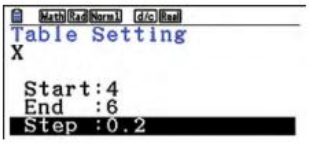
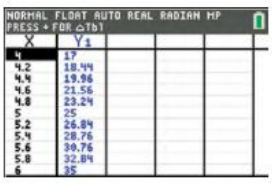
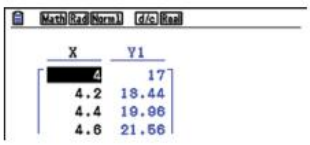
The QCAA Formula book has the full version of the trapezoidal rule, so you do not need to memorise it precisely.

WORKED EXAMPLE 2 Approximating areas using the trapezoidal rule

Determine the approximate area under the curve $f(x) = x^2 - x + 5$ between $x = 4$ and $x = 6$ using 10 trapeziums.

Steps	Working
1 Find the width of each trapezium using $w = \frac{b-a}{n}$.	$w = \frac{6-4}{10}$ $= 0.2$
2 State the x values for each trapezium.	The x values are 4, 4.2, 4.4, 4.6, 4.8, 5, 5.2, 5.4, 5.6, 5.8 and 6.
3 Determine the height of each trapezium parallel using $f(x) = x^2 - x + 5$.	$f(4) = 17$, $f(4.2) = 18.44$, $f(4.4) = 19.96$ $f(4.6) = 21.56$, $f(4.8) = 23.24$, $f(5) = 25$ $f(5.2) = 26.84$, $f(5.4) = 28.76$, $f(5.6) = 30.76$ $f(5.8) = 32.84$, $f(6) = 35$ You can use your calculator to determine the function values (the height of the trapezium parallels).
4 Find the values of E and M.	$E = 17 + 35 = 52$ $M = 18.44 + 19.96 + 21.56 + 23.24 + 25 + 26.84 + 28.76 + 30.76 + 32.84$ $= 227.4$
5 Find the total area using the formula.	$\int_a^b f(x)dx \approx \frac{w}{2}[E + 2M]$ $\approx \frac{0.2}{2}[52 + 2 \times 227.4]$ ≈ 50.68
6 State the result.	The approximate area under the curve $f(x)$ between $x = 4$ and $x = 6$ is 50.68 units^2.

Using a calculator to determine the trapezium parallels:

	TI-84 Plus CE	Casio fx-CG50 AU
1 Enter the function $f(x) = x^2 - x + 5$.	Press Y= and enter the function in Y_1.	Use the table menu to enter the function in Y1.
2 Set up the x values.	 Use **tblset** (2nd WINDOW) to set up the x values.	 Choose **SET** to set up the x values.
3 View the y values.	 Use table (2nd Graph) to view the table. Use the cursor controls to move up and down the table.	 Press EXIT and choose TABLE to see the values. Use the cursor controls to move up and down the table.

TI-Nspire Integration and areas

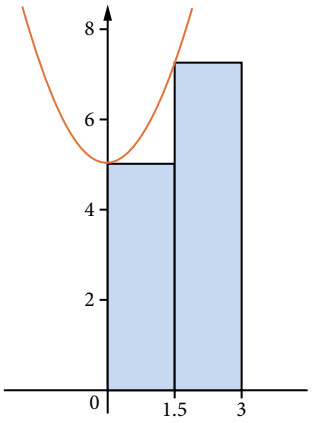
EXERCISE 6.1 Approximating areas under curves

ANSWERS p. 407

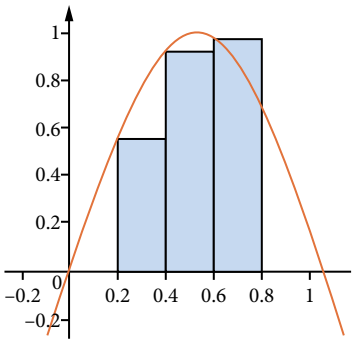
Mastery

1 WORKED EXAMPLE 1 Use left rectangles to find an approximation for the area under the curve. Round answers to 3 decimal places.

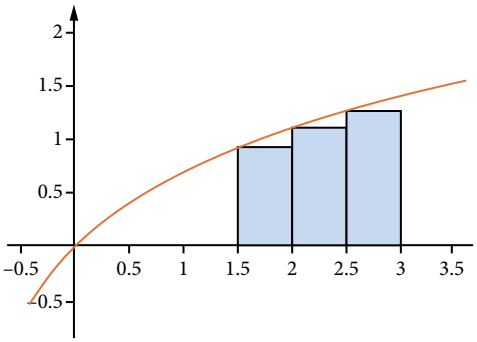
a $y = x^2 + 5$ between $x = 0$ and $x = 3$ using 2 rectangles



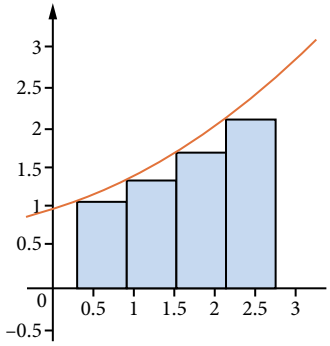
b $y = \sin(3x)$ between $x = 0$ and $x = 0.8$ using 4 rectangles.
(Note: the first rectangle is so low it doesn't show up well on the diagram.)



c $y = \ln(x + 1)$ between $x = 1.5$ and $x = 3$ using 3 rectangles.

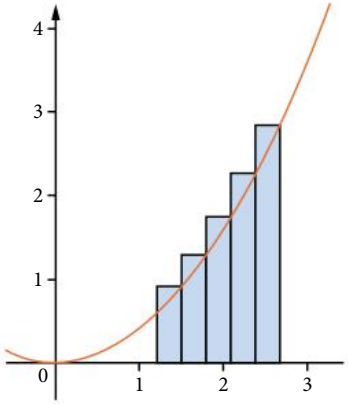


d $y = 2^{\frac{x}{2}}$ between $x = 0.3$ and $x = 2.8$ using 4 rectangles.

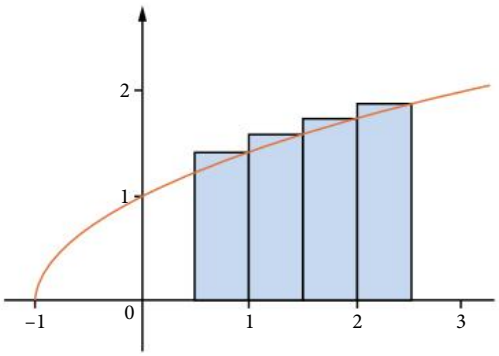


2 Use right rectangles to find an approximation for each area. Round answers to 2 decimal places.

a $y = 3x^2$ between $x = 1.2$ and $x = 2.7$ using 5 rectangles.



b $f(x) = \sqrt{x+1}$ between $x = 0.5$ and $x = 2.5$ using 4 rectangles.



3 Use centred rectangles of width 0.4 to find an approximate area under the curve $f(x) = 3x^2 - 2x + 5$ between $x = 1$ and $x = 3$.

4 WORKED EXAMPLE 2 Use the trapezoidal method to find an approximation for the area under the curve. Round answers to 3 decimal places.

a $y = x^3 + 2x^2 - 8x - 3$ between $x = 3$ and $x = 5$ using 5 strips

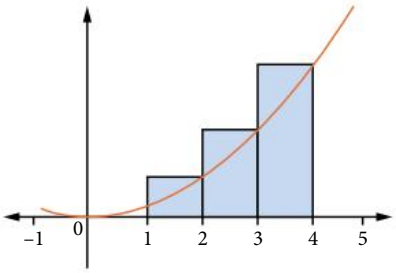
b $g(x) = \dfrac{1}{x+1}$ between $x = 2$ and $x = 4$ using 4 strips

c $f(x) = \cos(x)$ between $x = 0$ and $x = 1.5$ using 3 strips

d $y = e^x$ between $x = 0.6$ and $x = 2.4$ using 4 strips

Exam practice

5 © QCAA 2020S 1Q2 TECH-FREE The approximate area under the curve $y = 2x^2$ between $x = 1$ and $x = 4$ is found using rectangles (of width equal to one unit), as shown in the diagram.



What is the approximate area found using these rectangles?

A 58 units^2 **B** 45 units^2 **C** 42 units^2 **D** 28 units^2

6 © QCAA 2022 1Q9 TECH-FREE 29% The approximate area under the curve $f(x) = \sqrt{2x+1}$ between $x = 0$ and $x = 4$ using the trapezoidal rule with four strips is

A $2+\sqrt{3}+\sqrt{5}+\sqrt{7}$ **B** $2+2\left(\sqrt{3}+\sqrt{5}+\sqrt{7}\right)$

C $4+2\left(\sqrt{3}+\sqrt{5}+\sqrt{7}\right)$ **D** $4+\sqrt{3}+\sqrt{5}+\sqrt{7}$

7 © QCAA 2020 2Q3 Using the trapezoidal rule with four sections, the approximate area under the curve $y = x^2 + 5$ between $x = 0$ and $x = 2$ is

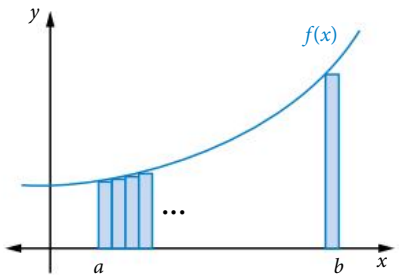
A 12.67 units^2 **B** 12.75 units^2 **C** 13.00 units^2 **D** 13.75 units^2

8 © QCAA 2021 2Q4 46% Using the trapezoidal rule with an interval size of 1, the approximate value of the integral $\int_0^3 0.5^x\,dx$ is

A 1.25 **B** 1.26 **C** 1.31 **D** 1.88

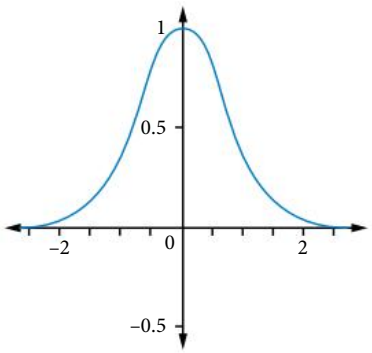
9 (2 marks) An approximation to an area under a curve can be found by adding the area of n rectangles of width h units that lie under the curve $f(x)$ between $x = a$ and $x = b$, as shown.

Under what conditions will the approximation be most accurate in terms of the values of n and h?



10 © QCAA 2020S 1Q18 TECH-FREE (6 marks) Determine the area of the largest rectangle that has one side on the x-axis and two vertices on the curve $y = e^{-x^2}$. Express your answer in simplest form.

The curve is sketched.



11 © QCAA 2024 2Q11 (4 marks) State the trapezoidal rule and use it with six strips to determine an approximate value of the definite integral for the curve of $f(x) = 4(x - 3)^2$ from $x = 0$ to $x = 3$.

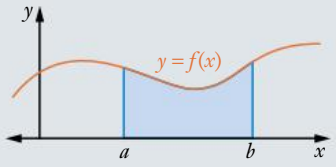
Show all substitutions made into the rule.

6.2 Definite integrals

Definite integrals

Given a continuous function $f(x)$ on the interval $[a, b]$, where $f(x) \geq 0$ for all x in the interval, the area enclosed by the function, the lines $x = a$ and $x = b$ and the x-axis is called the **definite integral** and is written as $\int_a^b f(x)dx$.

$$\therefore A = \int_a^b f(x)dx$$

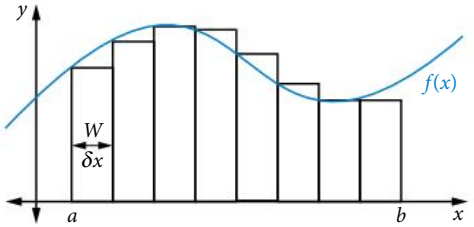


Worksheets
Definite integrals
Finding definite integrals

Fundamental theorem of calculus

The area under a curve can be approximated by the sum of the areas of rectangles (or trapeziums) of equal widths, w. As the number of shapes increases:

- the width of the shape decreases
- the sum of the areas of the shapes will approach a **limit**.



As w gets increasingly smaller it is shown as δx, which means very small change in x.

The area (A) under the curve $y = f(x)$ between $x = a$ and $x = b$ can be approximated by the sum of the areas of rectangles of width δx and height $f(x)$.

This can be written algebraically as

$$A \approx \sum_{x=a}^{b} f(x)\delta x$$

where $\sum_{x=a}^{b}$ means the sum of the expression for the value of x between a and b.

This can be written in a more general form as $\sum_i f(x_i)\delta x_i$, where i represents the number of slices (or subintervals) or points on the x-axis.

δx can be written as Δx.

δ is the lowercase Greek letter *delta*.

Δ is the capital Greek letter *delta*.

Σ is the capital Greek letter *sigma*.

Σ means 'the sum of".

Definite and indefinite integrals

As the number of rectangles increases, the width of the rectangles will approach 0 ($\delta x \to 0$).

This means $A \approx \lim_{\delta x \to 0} \sum_i f(x_i)\delta x_i$ becomes the **definite integral** $A = \int_a^b f(x)dx$.

- The definite integral $\int_a^b f(x)dx$ with limits at a and b is the area between the curve $f(x)$ and the x-axis. It is read as 'the integral of $f(x)$ between a and b with respect to x'.
- The **fundamental theorem of calculus** connects the anti-derivative with the area under a curve, and states that the **definite integral** of $f(x)$ from a to b is given by

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

  where $F(x)$ is the anti-derivative of $f(x)$.
- The **indefinite integral** (anti-derivative) is a **function**, while the definite integral is a **number**.

Video
Definite integrals

WORKED EXAMPLE 3 Calculating definite integrals

Evaluate each definite integral.

a $\int_0^4 (3x^2 + 1)dx$ **b** $\int_4^7 (6x^2 - 10x + 3)dx$ **c** $\int_0^4 \frac{5}{x+2}dx$

Steps	Working
a 1 Integrate $3x^2 + 1$.	$\int_0^4 (3x^2 + 1)dx = \left[\frac{3x^3}{3} + x\right]_0^4$
2 Evaluate the integral by substituting the limits, $x = 0$ and $x = 4$, and subtract. $\left[F(x)\right]_a^b = F(b) - F(a)$	$= ((4)^3 + (4)) - ((0)^3 + (0))$ $= 68 - 0$ $= 68$
The '+ c' is not required when finding the definite integral, as it will cancel itself out in the subtraction step.	
3 State the result.	$\int_0^4 (3x^2 + 1)dx = 68$
b 1 Integrate $6x^2 - 10x + 3$.	$\int_4^7 (6x^2 - 10x + 3)dx = \left[\frac{6x^3}{3} - \frac{10x^2}{2} + 3x\right]_4^7$ $= \left[2x^3 - 5x^2 + 3x\right]_4^7$
2 Substitute the limits $x = 4$ and $x = 7$, and subtract. $\left[F(x)\right]_a^b = F(b) - F(a)$	$= (2(7)^3 - 5(7)^2 + 3(7)) - (2(4)^3 - 5(4)^2 + 3(4))$ $= 462 - 60$ $= 402$
3 State the result.	$\int_4^7 (6x^2 - 10x + 3)dx = 402$

c 1 Integrate $\frac{5}{x+2}$.

$$\int_0^4 \frac{5}{x+2}\,dx = \left[5\ln(x+2)\right]_0^4$$

2 Evaluate at each limit and simplify.

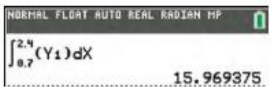
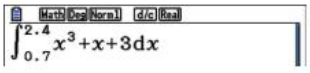
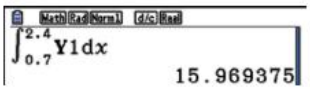
$$= 5\ln(6) - 5\ln(2)$$
$$= 5[\ln(6) - \ln(2)]$$
$$= 5\ln(3)$$

> Leave your answers as exact values unless you are asked for an approximation.

You can use your calculator to evaluate the definite integrals.

WORKED EXAMPLE 4 Calculating definite integrals using technology

Calculate the integral of $x^3 + x + 3$ from $x = 0.7$ to $x = 2.4$.

Steps	Working
	TI-84 Plus CE
1 Enter the integral function $x^3 + x + 3$ including the limits $x = 0.7$ and $x = 2.4$.	 Press **math** and choose **9: fnInt(.** Enter the integral including the variable x at the end.
2 Evaluate the integral.	 Press **enter** to evaluate.
	Casio fx-CG50 AU
	 Press **MENU** and choose **Run-Matrix.** Press **OPTN**, choose **CALC**, **∫dx** and complete the integral.
	 Press **exe** to evaluate.

Properties of definite integrals

Properties of the definite integral

$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Factorise a **constant out**: A constant factor can be removed from an integral.
$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits**: Reversing the order of the limits changes the sign of the definite integral.
$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms**: A sum or difference of terms can be integrated separately.
$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ where $a < b < c$.	**Split limits**: The limits of an integral can be split.

TI-Nspire Integration and areas

WORKED EXAMPLE 5 Using the properties of definite integrals

a Given $\int_2^4 f(x)\,dx = 4$, **evaluate** $\int_4^2 (f(x)+x)\,dx$.

b If $\int_{-1}^3 f(x)\,dx = 8$, and $\int_3^6 f(x)\,dx = 10$, **determine** $\int_{-1}^6 f(x)\,dx$.

c If $\int_1^5 g(x)\,dx = 15$, **evaluate** $\int_1^5 2(g(x)+3)\,dx$.

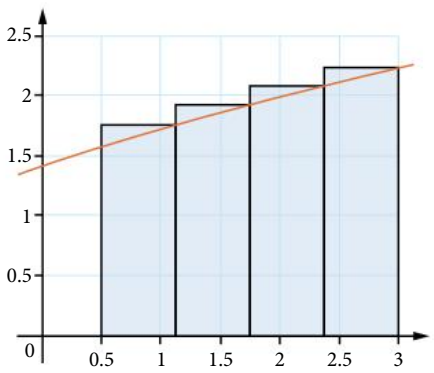
Steps	Working
a 1 Simplify by splitting the terms.	$\int_4^2 (f(x)+x)\,dx = \int_4^2 f(x)\,dx + \int_4^2 x\,dx$
2 Swap the limits.	$= -\int_2^4 f(x)\,dx - \int_2^4 x\,dx$
3 Substitute $\int_2^4 f(x)\,dx = 4$.	$= -4 - \left[\frac{x^2}{2}\right]_2^4$
4 Evaluate and simplify.	$= -4 - (8-2)$ $= -10$
b As the limits are split, use $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ where $a < b < c$.	$\int_{-1}^6 f(x)\,dx = \int_{-1}^3 f(x)\,dx + \int_3^6 f(x)\,dx$ $= 8 + 10$ $= 18$
c 1 Factorise the constant.	$\int_1^5 2(g(x)+3)\,dx = 2\int_1^5 (g(x)+3)\,dx$
2 Simplify by splitting the terms.	$= 2\left[\int_1^5 g(x)\,dx + \int_1^5 3\,dx\right]$
3 Substitute $\int_1^5 g(x)\,dx = 15$.	$= 2\left[15 + \left[3x\right]_1^5\right]$
4 Evaluate and simplify.	$= 2[15 + (15-3)]$ $= 54$

EXERCISE 6.2 Definite integrals

ANSWERS p. 407

Recap

1 Determine the approximate area under the curve $y = \sqrt{x+2}$ between $x = 0.5$ and $x = 3$ using 4 rectangles of equal width, as shown in the diagram. Give the approximation to 2 decimal places.



2 Using the trapezoidal rule with three strips, determine the approximate area under the curve $y = 3x^2 - 5x + 2$ between $x = 2$ and $x = 5$.

Mastery

3 WORKED EXAMPLE 3 Evaluate each definite integral.

a $\int_1^{10} (9x+7)\,dx$ **b** $\int_5^{10} 4x^{-2}\,dx$ **c** $\int_{-1}^{2} 6e^{3t}\,dt$

d $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} \cos(2x)\,dx$ **e** $\int_{-1}^{5} \left(\frac{6}{3g+4}\right) dg$ **f** $\int_1^{10} \left(4x+3-\frac{2}{x}\right) dx$

4 WORKED EXAMPLE 4 Evaluate each definite integral using technology. Give approximations to 3 decimal places.

a $\int_0^5 3e^{x^2+1}\,dx$ **b** $\int_0^5 (x^3+x^2+x+1)\,dx$

c $\int_0^{\pi} \left[\sin^2(x)+\cos^2(x)\right] dx$ **d** $\int_{10}^{100} \log_{10}(x)\,dx$

5 Evaluate each definite integral.

a $\int_0^6 8\,dx$ **b** $\int_0^{\pi} \sin(x)\,dx$ **c** $\int_1^4 e^{3V}\,dV$

d $\int_{-12}^{12} (20-m)\,dm$ **e** $\int_{2.4}^{5.8} \left(\frac{3}{x^4}\right) dx$ **f** $\int_0^5 \left(\frac{1}{r+1}\right) dr$

6 Using technology, find the values of $\int_{-1}^{0} (3+2x-x^2)\,dx$, $\int_0^3 (3+2x-x^2)\,dx$ and $\int_{-1}^{3} (3+2x-x^2)\,dx$. Comment on your findings.

7 WORKED EXAMPLE 5 Given $\int_2^5 f(x)dx = 7$, evaluate $\int_2^5 4f(x)\,dx$.

8 If $\int_0^3 h(x)\,dx = 7$, evaluate $\int_0^3 (5-4h(x))\,dx$.

Exam practice

9 © QCAA 2023 1Q5 TECH-FREE 67% Determine $\int_1^3 \frac{1}{2x}\,dx$.

A $\frac{1}{2}\ln 6$ **B** $\frac{1}{2}\ln 5$ **C** $\frac{1}{2}\ln 4$ **D** $\frac{1}{2}\ln 3$

10 © QCAA 2020S 1Q4 TECH-FREE If $\int_{-2}^{0} f(x)\,dx = 4$ and $\int_0^3 f(x)\,dx = -10$, which of the following is true?

A $\int_{-2}^{3} f(x)\,dx = -14$ **B** $\int_{-2}^{3} f(x)\,dx = -6$ **C** $\int_{-2}^{3} f(x)\,dx = 6$ **D** $\int_{-2}^{3} f(x)\,dx = 14$

11 © QCAA 2020S 1Q6 TECH-FREE Calculate $\int_{-1}^{7} \frac{dx}{\sqrt{(2+x)}}$.

A 1 **B** $\frac{4}{3}$ **C** 2 **D** 4

12 © QCAA 2023 2Q6 52% $\int_a^{5a} \frac{1}{x+a}\,dx$, $a \neq 0$ is

A 1.7918 **B** 1.6094 **C** 1.3863 **D** 1.0986

13 TECH-FREE (2 marks) Evaluate $\int_0^1 (x - x^2)\,dx$.

14 TECH-FREE (2 marks) Evaluate $\int_1^4 (2 + \sqrt{x})\,dx$.

15 TECH-FREE (1 marks) Simplify $\int_{-3}^{2} (4x - x^2)\,dx + \int_2^3 (4x - x^2)\,dx$ to a single integral. Do not evaluate.

16 (2 marks) Evaluate $\int_3^8 e^{5n}\,dn$.

17 (2 marks) Explain why you cannot determine $\int_0^3 \frac{1}{x^2}\,dx$.

18 (2 marks) If $\int_2^{10} g(x)\,dx = 7$ and $\int_{10}^{6} g(x)\,dx = 3$ determine the value of $\int_2^6 g(x)\,dx$.

19 © QCAA 2022 1Q17 TECH-FREE (4 marks) CF

Determine the value of b given $\int_a^b 3x^2\,dx = 117$ and $\int_a^{b-1} 3x^2\,dx = 56$ for $b > 1$.

20 © QCAA 2024 2Q10 83% The velocity (m s^{-1}) at time t (s) of an object is given by $v(t) = 0.4\,t^2 + 3t$ for $t \geq 0$. The change in displacement (m) of the object from four to five seconds is

A 15.43 **B** 21.63 **C** 32.53 **D** 54.17

Video
Area under a curve

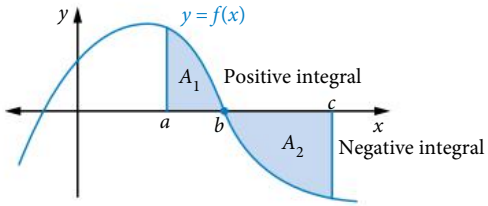
Worksheet
Calculating physical areas

6.3 Areas under curves

Signed areas

The definite integral $\int_a^b f(x)\,dx$ calculates the area enclosed between the graph of $y = f(x)$, the x-axis and the limits $x = a$ and $x = b$.

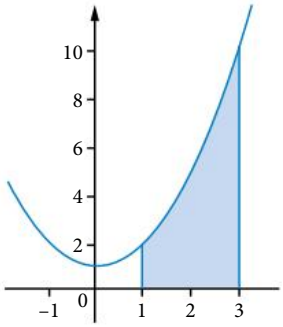
Definite integrals can be either positive or negative; therefore, areas calculated using the definite integral are called **signed areas**.



- If the area is located above the x-axis, then the definite integral will be positive and it will give the area under the curve. For instance, $A_1 = \int_a^b f(x)\,dx$, and $\int_a^b f(x)\,dx > 0$.
- If the area is located below the x-axis, then the definite integral will be negative. It will give the area above the curve. As $f(x) < 0$, $\int_b^c f(x)\,dx < 0$, so $A_2 = -\int_b^c f(x)\,dx$.
- Alternatively, reversing the limits of integration, $A_2 = \int_c^b f(x)\,dx$ or simply $A_2 = \left|\int_b^c f(x)\,dx\right|$.
- If the graph is both partially above and below the x-axis, then the graph and the integral must be split. The area from $x = a$ to $x = c$ is $A_1 + A_2 = \int_a^b f(x)\,dx + \left|\int_b^c f(x)\,dx\right|$.

WORKED EXAMPLE 6 Finding the area under a curve

Find the area enclosed between the curve $f(x) = x^2 + 1$ and the x-axis between $x = 1$ and $x = 3$.

Steps	Working
1 Sketch the graph and check the sign of the area required.	
2 Write the integral required to find the area.	$\int_1^3 (x^2+1)\,dx$
3 Integrate and simplify (if needed).	$\int_1^3 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^3$
4 Substitute in the limits and evaluate.	$= \left(\frac{(3)^3}{3}+(3)\right) - \left(\frac{(1)^3}{3}+(1)\right)$ $= 12 - 1\frac{1}{3}$ $= 10\frac{2}{3}$ units^2

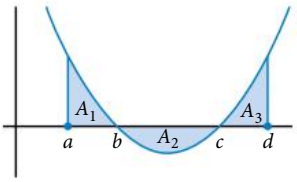
Exam hack

Always sketch a graph first to identify whether you are finding a positive or a negative area or a combination of both.

Combined areas above and below a curve

When finding the area under a curve using integration, always check to see if the graph is above the x-axis. If the graph has areas both above and below the x-axis, break the area into separate regions and create an integral for each region.

Consider the graph of the curve $f(x)$ on the right.



A_1 and A_3 are both above the x-axis so they would have positive values.
A_2 is below the x-axis so it would have a negative value.

The total shaded area $= A_1 + (-A_2) + A_3$

$$= \int_a^b f(x)\,dx + \left(-\int_b^c f(x)\,dx\right) + \int_c^d f(x)\,dx$$

(Changing the sign makes a negative area become positive.)

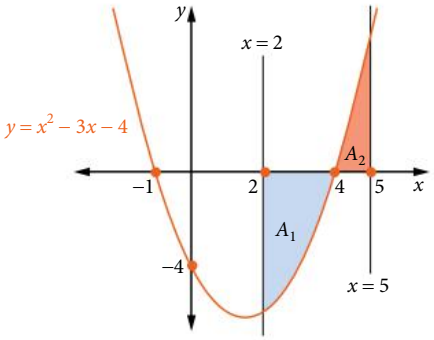
Or simply use the absolute value: total shaded area $= \int_a^b f(x)\,dx + \left|\int_b^c f(x)\,dx\right| + \int_c^d f(x)\,dx$.

An alternative method to deal with areas located below the x-axis is to swap the limits. In this case the limits b and c are swapped.

So the sum of the shaded areas then becomes $\int_a^b f(x)\,dx + \int_c^b f(x)\,dx + \int_c^d f(x)\,dx$.

WORKED EXAMPLE 7 Finding areas above and below curves

Find the area enclosed between the curve $f(x) = x^2 - 3x - 4$ and the x-axis between $x = 2$ and $x = 5$.

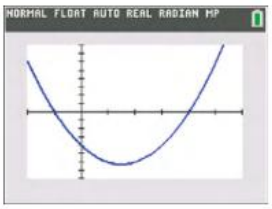
Steps	Working
1 Sketch the graph showing the required area and important points. In the TECH-FREE exam, you must show working for important points such as the x- and y-intercepts.	
2 Find the x-intercept that splits the area into the area above the x-axis and the area below.	$0 = x^2 - 3x - 4$ $= (x + 1)(x - 4)$ $x = -1$ or $x = 4$ $x = 4$ is where the area is split.
3 Write the integrals required to find the area.	$\text{Area} = -\int_2^4 (x^2 - 3x - 4)\,dx + \int_4^5 (x^2 - 3x - 4)\,dx$
4 Evaluate the integrals.	$= -\left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_2^4 + \left[\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right]_4^5$ $= -\left(-18\frac{2}{3} - \left(-11\frac{1}{3}\right)\right) + \left(-15\frac{5}{6} - \left(-18\frac{2}{3}\right)\right)$ $= -\left(-7\frac{1}{3}\right) + 2\frac{5}{6}$ $= 10\frac{1}{6}$ units^2.
5 State the answer.	The area enclosed is $10\frac{1}{6}$ units^2.

TI-Nspire Integration and areas

To determine the area enclosed between two curves on your calculator:

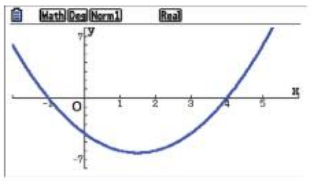
1 Enter the function $x^2 - 3x - 4$ and adjust the view.

TI-84 Plus CE



Press Y= and enter the function into Y_1.
Set the window to $-2 \le x \le 6$ and $-8 \le y \le 8$, and graph the function.

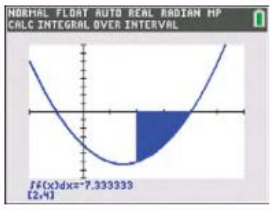
Casio fx-CG50 AU



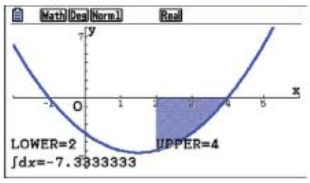
Press MENU and choose **Graph**.
Enter the function into Y1.
Set **V-Window** to $-2 \le x \le 6$ and $-8 \le y \le 8$, and graph the function.

9780170484985

2 Find the area of the first region from limits $x = 2$ and $x = 4$.

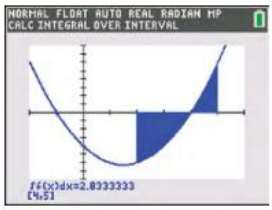


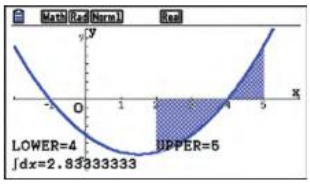
Select **Calc** then **7: ∫f(x)dx.** Set the limits (the upper and lower limits).



Select **G-Solv** then **∫dx.** Set the limits (the upper and lower bound).

3 Find the area of the second region from $x = 4$ and $x = 5$.





4 Add the areas to find the total area. Ignore the signs.

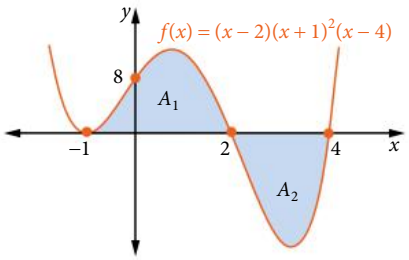
Area $\approx 7.333 + 2.822$
≈ 10.166 units^2

You can set up the Casio to find a physical area instead of finding the 2 parts separately, but it is better in the exam to show the 2 parts.

When using your calculator to find combined areas, find each area separately. In the previous example, if you had assumed Area $= \int_2^5 (x^2 - 3x - 4)\,dx$ without checking for areas above and below the x-axis, you would have calculated a value of $-4\frac{1}{2}$ units^2 rather than the actual physical area of $10\frac{1}{6}$ units^2.

WORKED EXAMPLE 8 Finding areas bounded by the x-axis and a curve

Determine the area bounded by the curve $f(x) = (x - 2)(x + 1)^2(x - 4)$ and the x-axis.

Steps	Working
1 Find the x-intercepts as the area will be between them.	The zeros are at 2, 4 and -1.
2 Find the y-intercept.	The y-intercept is at $(0, 8)$.
3 Find the sign changes around each x-intercept.	$f(x)$ is positive before and after $x = -1$. $f(x)$ is positive before and negative after $x = 2$. $f(x)$ is negative before and positive after $x = 4$.
4 Sketch the graph showing the required area and important points.	
5 Expand $f(x)$ to make the integration possible.	$f(x) = (x - 2)(x + 1)^2(x - 4)$ $= (x - 2)(x - 4)(x^2 + 2x + 1)$ $= (x^2 - 6x + 8)(x^2 + 2x + 1)$ $= x^4 - 4x^3 - 3x^2 + 10x + 8$

6 Write the integrals required to find the area.

$$\text{Area} = \int_{-1}^{2} (x^4 - 4x^3 - 3x^2 + 10x + 8)\,dx - \int_{2}^{4} (x^4 - 4x^3 - 3x^2 + 10x + 8)\,dx$$

7 Evaluate the integrals.

$$= \left[\frac{x^5}{5} - x^4 - x^3 + 5x^2 + 8x\right]_{-1}^{2} - \left[\frac{x^5}{5} - x^4 - x^3 + 5x^2 + 8x\right]_{2}^{4}$$

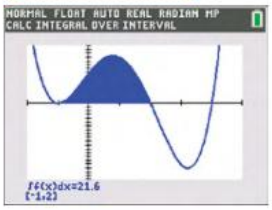
$$= \left(18\frac{2}{5} - \left(-3\frac{1}{5}\right)\right) - \left(-3\frac{1}{5} - 18\frac{2}{5}\right)$$

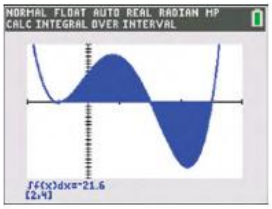
$$= 43\frac{1}{5} \text{ units}^2$$

8 Check using your calculator.

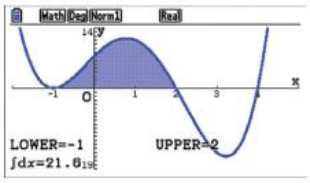
TI-Nspire Integration and areas

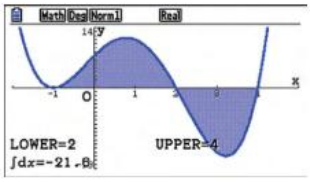
TI-84 Plus CE





Casio fx-CG50 AU





9 State the answer.

The area enclosed is $43\frac{1}{5}$ units^2.

EXERCISE 6.3 Areas under curves

ANSWERS p. 407

Recap

1 Evaluate $\int_{5}^{10} \frac{1}{(x-4)^2}\,dx$.

2 If $\int_{0}^{4} g(x) = 25$ and $\int_{0}^{4} (-3g(x) + bx)\,dx = 45$, determine the value of b.

Mastery

3 WORKED EXAMPLE 6 Find the area enclosed between the curve $f(x) = 7 - 6x^3$ and the x-axis between $x = -2$ and $x = 1$.

4 Find the area between the graph of each function and the x-axis. Give the answers correct to 3 decimal places.

a $y = x^2 - 5x + 8$ from $x = 1$ to $x = 4$

b $g(x) = e^{3x+2} + 1$ from $x = -2$ to $x = 0$

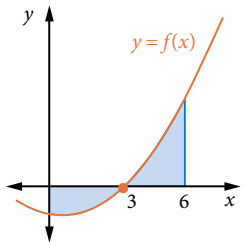
c $y = 4x^3 - 3x^2 + 6x - 2$ between $x = 1$ and $x = 3$

d $h(x) = \frac{1}{4x}$ from $x = -3$ to $x = -1$

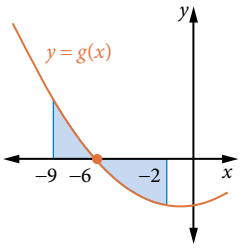
5 WORKED EXAMPLE 7 Find the area enclosed by $y = 7x^2 + 19x - 6$, $x = -5$, $x = -2$ and the x-axis.

6 For each graph, state the integral used to calculate the shaded physical area.

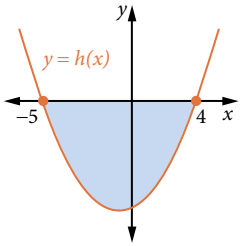
a



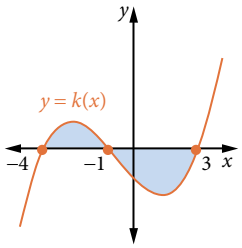
b



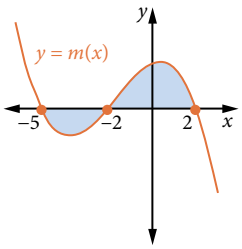
c



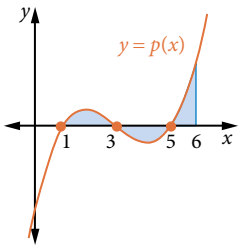
d



e



f



7 Using technology, determine the area bounded by the curve and x-axis.

a $y = 6x - 4 - x^2$ **b** $y = x^3 + 3x^2 - 10x - 3$

8 WORKED EXAMPLE 8 Find the area enclosed by $f(x) = 3x^2 - 12$ and the x-axis.

9 Find the physical area bounded by the x-axis and $y = (5 - x)(x + 1)(x - 3)$.

Exam practice

10 © QCAA 2022 1Q3 TECH-FREE 67% The area between the curve $y = 9 - x^2$ and the x-axis is

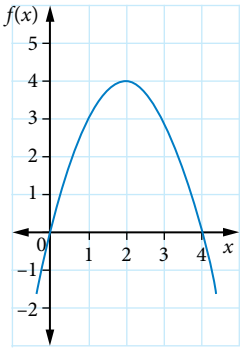
A 12 units^2 **B** 18 units^2 **C** 36 units^2 **D** 54 units^2

11 © QCAA 2020 2Q3 Let R be the region enclosed by the graph of $y = xe^x$, the x-axis, and the lines $x = -1$ and $x = 1$.

The area of R is closest to

A 0.74 **B** 1.26 **C** 2.35 **D** 3.09

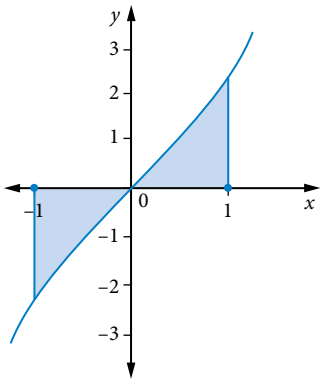
12 © QCAA 2023 1Q12 TECH-FREE (4 marks) The region bounded by the x-axis and the curve of $f(x) = x(4 - x)$ represents the plan of a garden bed. All measurements are in metres.



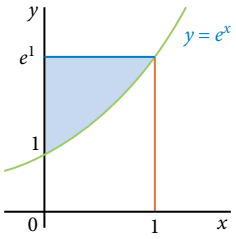
a Estimate the area of the garden bed using sums of the form $\sum_i f(x_i)\delta x_i$ where $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, and $x_4 = 4$. *[1 mark]*

b Use a definite integral to determine the area of the garden bed. *[3 marks]*

13 TECH-FREE (4 marks) Consider the graph of function $y = e^x - e^{-x}$.
Find the area enclosed by the graph and $x = -1$ and $x = 1$ as an exact value.



14 TECH-FREE (2 marks) By first stating an expression that could be used to calculate the shaded region shown, determine the exact shaded area.



15 © QCAA 2021 2Q11 (5 marks) Consider the function $f(x) = e^x \sin(x)$, $0 \le x \le 2\pi$.

a State the exact values of the x-intercepts of the graph of $f(x)$. *[2 marks]*

b Write an expression for the area enclosed between the graph of $f(x)$ and the x-axis. *[2 marks]*

c Determine the area enclosed between the graph of $f(x)$ and the x-axis to the nearest square unit. *[1 mark]*

16 (3 marks) Determine the area under the curve $y = 2(x-1)^3$ from $x = -1$ to $x = 2$.

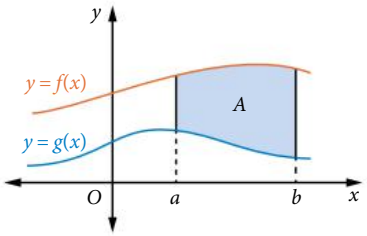
17 (3 marks) Determine the area enclosed by $y = \sin(3x)$ and the x-axis between $x = 0$ and $x = 2\pi$.

18 © QCAA 2024 1Q5 TECH-FREE 58% Determine $\int_a^b 2\cos(x)\,dx$, where $a = \frac{\pi}{3}$ and $b = \frac{\pi}{2}$.

A $1 - \frac{\sqrt{3}}{2}$ **B** $\frac{\sqrt{3}}{2} - 1$ **C** $2 - \sqrt{3}$ **D** $\sqrt{3} - 2$

6.4 Areas between curves

Often an enclosed area involves more than one curve and an axis. The approach taken to solve these types of problems will depend on whether the curves intersect over the required interval.



Consider the continuous functions $y = f(x)$ and $y = g(x)$ over the interval $a \le x \le b$.

Given $f(x) > g(x)$, the shaded area, A, between $f(x)$ and $g(x)$ could be found by finding the area under each curve from $x = a$ to $x = b$ and subtracting:

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx$$

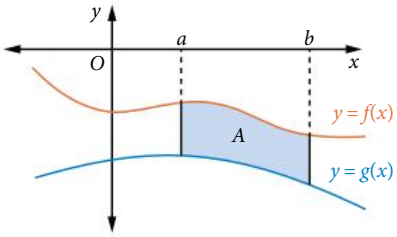
Using the properties of definite integrals, this simplifies to

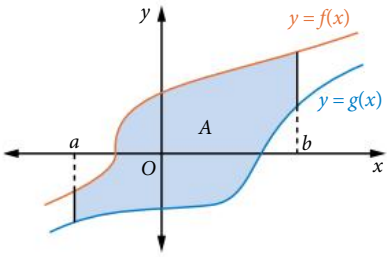
$$A = \int_a^b \left[f(x) - g(x)\right]dx$$

$f(x)$ is the **upper** function and $g(x)$ is the **lower** function over the interval as $f(x) > g(x)$.

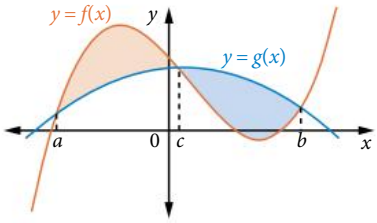
$$\therefore A = \int_a^b \left[\textbf{upper} - \textbf{lower}\right]dx$$

The area between 2 curves that do not intersect can be found using this approach, regardless of the position of the curves, as long you subtract the lower function from the higher function.





If the curves intersect over the interval, you need to find the point(s) of intersection and consider each area separately, noting which function is greater. In this graph:



- $f(x) > g(x)$ over the interval $a \le x \le c$
- $g(x) > f(x)$ over the interval $c \le x \le b$.

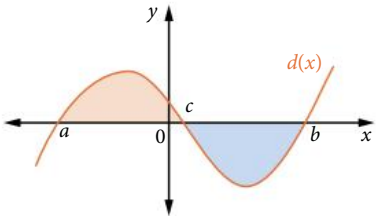
So, the shaded area, A_T, over the interval $a \le x \le b$ is given by

$$A_T = \int_a^c \left[f(x) - g(x)\right]dx + \int_c^b \left[g(x) - f(x)\right]dx$$

When finding the area between curves using $A = \int_a^b \left[f(x) - g(x)\right]dx$,

$f(x) - g(x)$ is called the **difference function** $d(x)$.

$$A = \int_a^b \left[f(x) - g(x)\right]dx = \int_a^b d(x)dx$$



The zeros of the difference function are the x values of the points of intersection of the original functions.

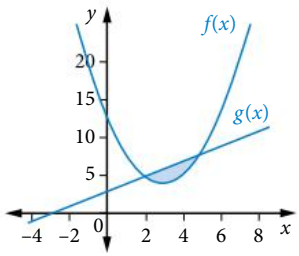
Worksheets
Calculating areas between curves
Areas between curves 1
Areas between curves 2

Area between curves

$$\text{Area} = \int_a^b \left[\text{upper} - \text{lower}\right] dx = \int_a^b \left[f(x) - g(x)\right] dx$$

WORKED EXAMPLE 9 Finding an area enclosed between curves

Find the area enclosed by the curve $f(x) = x^2 - 6x + 13$ and $g(x) = x + 3$.

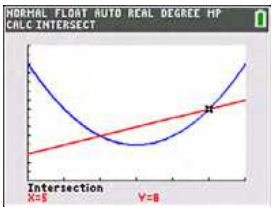
Steps	Working
1 Sketch the graph showing the required area. Always draw a sketch to determine which is the upper and which is the lower function.	
2 Find the points of intersection by solving the equations simultaneously.	$x + 3 = x^2 - 6x + 13$ $0 = x^2 - 7x + 10$ $0 = (x - 2)(x - 5)$ $x = 2$ or $x = 5$
3 Identify the upper and lower functions. Over the interval, $g(x) \geq f(x)$. Set up the integral using $A = \int_a^b \left[f(x) - g(x)\right] dx$ and simplify.	$A = \int_2^5 \left[(x+3) - (x^2 - 6x + 13)\right] dx$ $= \int_2^5 \left[-x^2 + 7x - 10\right] dx$
4 Integrate and evaluate.	$= \left[\frac{-x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$ $= \left(\frac{-(5)^3}{3} + \frac{7(5)^2}{2} - 10(5)\right) - \left(\frac{-(2)^3}{3} + \frac{7(2)^2}{2} - 10(2)\right)$ $= 19\frac{1}{2} - 15$ $= 4.5 \text{ units}^2$
5 State the answer.	The area enclosed is 4.5 units^2.

Verify the answer using technology if available.

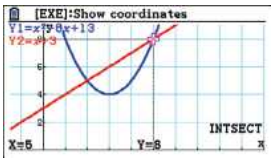
TI-84 Plus CE | **Casio fx-CG50 AU**

Method 1

1 Enter the functions Y_1 and Y_2 and graph them.
Set the window to see the intersections.
Find the intersections.

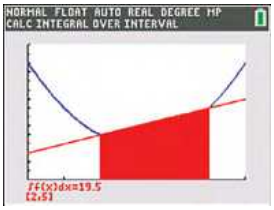


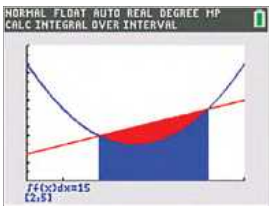
Use [calc] to find the intersections.



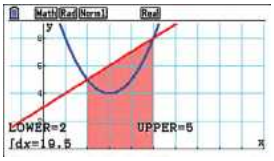
Use the Graph menu.
Use G-Solv to find the intersections.

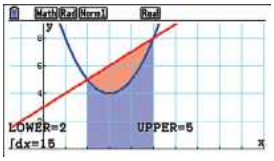
2 Find the integrals of each function between the intersections (from $x = 2$ to $x = 5$).
Use up and down to switch between the functions.
Subtract the areas to find the required area.





Use [calc].

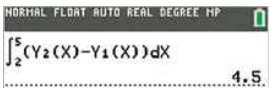




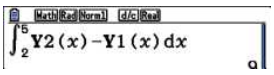
Use G-Solv.

Method 2

Enter the functions and find the intersections as in Step **1** of Method 1. Find the integral of $Y_2 - Y_1$ from 2 to 5 (the intersections).



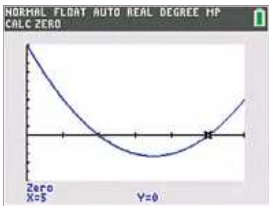
Quit the graph to find the integral. Press vars and use Y-VARS and Function to get Y_1 and Y_2.



Press MENU and choose Run-Matrix. Use MATH for the integral. Press VARS and GRAPH to get Y.

Method 3

1 Enter the difference function $d(x) = f(x) - g(x) = x^2 - 7x + 10$ in Y_1. Graph the function and find the zeros. They are the x values of the intersections.

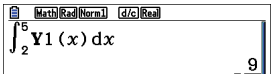


2 Find the integral of Y_1 from 2 to 5 (the zeros).
A negative sign shows the second function is uppermost. Ignore the sign.



Quit the graph to find the integral.





Press MENU and choose Run-Matrix to find the integral.
Press VARS and GRAPH to get Y.

TI-Nspire
Integration and areas

WORKED EXAMPLE 10 Finding areas between curves using technology 1

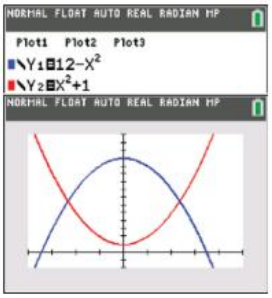
Find the area between $f(x) = 12 - x^2$ and $g(x) = x^2 + 1$ and the lines at $x = -1$ and $x = 2$.

Steps

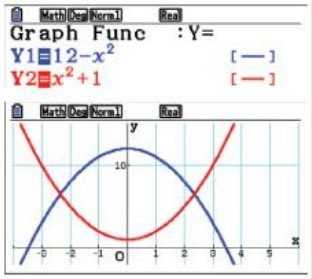
1 Sketch the graph showing the required area.

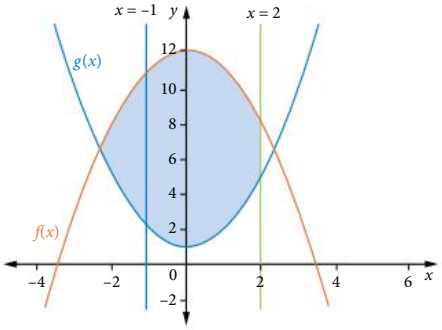
Working

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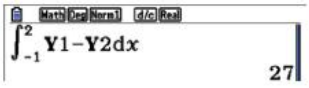
2 Identify the upper and lower functions. Over the interval, $f(x) > g(x)$.

3 Set up the integral using $A = \int_a^b \left[f(x) - g(x) \right] dx$.

$$A = \int_{-1}^{2} \left[(12 - x^2) - (x^2 + 1) \right] dx$$

4 Evaluate the integral.





5 State the answer.

The area enclosed is 27 units^2.

TI-Nspire
Integration and areas

WORKED EXAMPLE 11 Finding areas between curves using technology 2

Determine the area enclosed by the functions $f(x) = (x + 1)(x - 1)(x - 2)$ and $g(x) = 4x^2 - 28$.

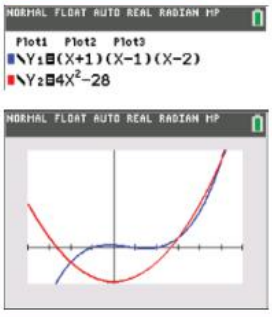
Steps

1 Sketch the graphs.

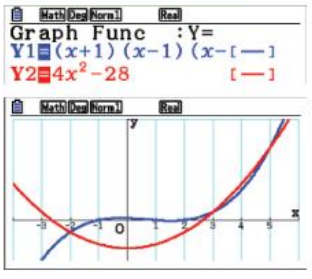
2 Find and state the points of intersection.

Working

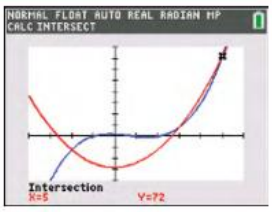
TI-84 Plus CE

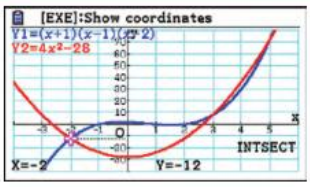


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3 Set up the integral statement over the interval using $A = \int_a^b \left[f(x) - g(x) \right] dx$.

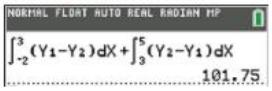


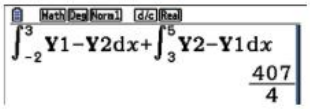


The functions intersect at $(-2, -12)$, $(3, 8)$ and $(5, 72)$.

$$A = \int_{-2}^{3} \left[((x+1)(x-1)(x-2)) - (4x^2 - 28) \right] dx + \int_{3}^{5} \left[(4x^2 - 28) - ((x+1)(x-1)(x-2)) \right] dx$$

4 Evaluate the integral statement.





5 State the answer.

The area enclosed is 101.75 units^2.

EXERCISE 6.4 Areas between curves

ANSWERS p. 408

Recap

1 Find the area enclosed between the curve $f(x) = 15 + 8x - 6x^2$ and the x-axis between $x = -1$ and $x = 2$.

2 Find the area under the curve $y = (2x+1)^{-\frac{1}{3}}$ from $x = 0$ to $x = 13$.

Mastery

3 WORKED EXAMPLE 9 Find the area enclosed by each pair of curves.

a $2x + y = 12$ and $y = 2x^2 - 12x + 20$

b $y = 2x - 1$ and $f(x) = 2x + 8 - x^2$

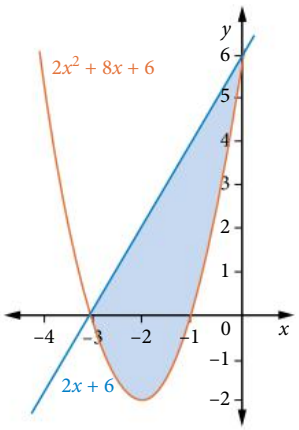
4 WORKED EXAMPLES 10, 11 Find the area between

a $f(x) = 3 - x^3$, $g(x) = x^2 - 9$, $x = -3$ and $x = 1$

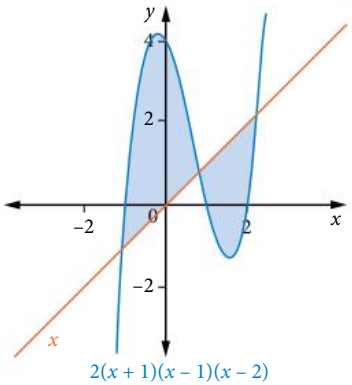
b $f(x) = 2x^2 - 8$, $g(x) = x^2 - 2x + 6$, $x = -3$ and $x = 2$

5 Calculate each shaded area.

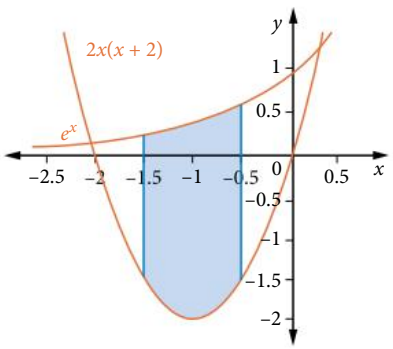
a



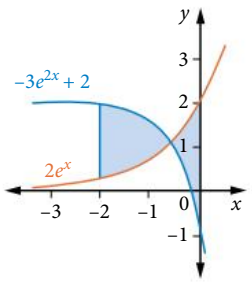
b



c



d



6 Find the area enclosed between the graphs of $y = 3x^2 + 2x - 21$ and $y = (x+1)(x+3)(x-3)$.

Exam practice

7 TECH-FREE The area enclosed by the curves $y = 2x + 1$ and $h(x) = x^2$ is

A $\frac{6\sqrt{2}}{3}$ **B** $\frac{5\sqrt{2}}{3}$ **C** $\frac{10\sqrt{2}}{3}$ **D** $\frac{8\sqrt{2}}{3}$

8 TECH-FREE The area enclosed between the graphs of $f(x) = 4x^2 + 4x + 5$ and $y = 5 - 2x$ is given by

A $\int_{-1.5}^{0} (-6x - 4x^2)\,dx$ **B** $\int_{-1.5}^{0} (4x^2 + 2x)\,dx$

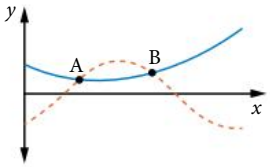
C $\int_{-1.5}^{0} (4x^2 + 6x)\,dx$ **D** $\int_{-1.5}^{0} (4x^2 - 6x)\,dx$

9 TECH-FREE (5 marks) Consider the functions $f(x) = x^2$ and $g(x) = 3x$.

a Determine the x-coordinates of the points of intersection of the graphs of the two functions. *[2 marks]*

b Use the results from part **a** to calculate the area enclosed by the graphs of $f(x)$ and $g(x)$. *[3 marks]*

10 © QCAA 2023 2Q13 (4 marks) The curved lines represent graphs of the equations $y = x^2 - 4x + 8$ and $y = 10\cos(x + 10)$.



a Determine the coordinates of the points of intersection A and B. *[1 mark]*

b State an integral expression representing the area enclosed by the two graphs. *[2 marks]*

c Determine the area enclosed by the two graphs. *[1 mark]*

11 (4 marks) Find the area enclosed between $f(x) = 4x + 13$ and $g(x) = 4x^2 + 12x + 8$.

12 (4 marks) Find the area enclosed between $f(x) = (x + 2)(x - 2)^2$ and $y = x + 2$.

13 (4 marks) Find the area between $f(x) = x^2 - 2x + 2$, $g(x) = 4 + 4x - x^2$, the y-axis and $x = 3$.

Worksheets
Total change
Displacement, velocity and acceleration
Integration assignment
Integration problems

6.5 Problems involving definite integrals

Total and marginal change

When given the function for the instantaneous rate of change of a quantity such as area or velocity, integration allows you to determine the amount by which the quantity has changed over a particular time interval, or to find the formula for the quantity, if given initial conditions.

The amount by which the quantity has changed over time is called the **total change** (or net change).

For example, if the rate at which a population is changing over time is given by $\frac{dP}{dt}$, then the total change in population over a specific period from $t = a$ to $t = b$ would be $\int_a^b \left(\frac{dP}{dt}\right) dt$.

To find the **average value** of the population from $t = a$ to $t = b$, calculate $\frac{1}{b-a}\int_a^b \left(\frac{dP}{dt}\right)dt$.

Average value of a function

The average value of a function $f(x)$, for an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

WORKED EXAMPLE 12 Finding the total change

Storm water flows into a large tank at a rate given by the function $\frac{dV}{dt} = 10t^2 + 20$ litres per minute, where V is the amount of water in the tank t minutes after a storm has started.

a **Determine** the rate of water flow at 4 minutes after a storm has started.

b What is the total volume of water that flows into the tank during the first 5 minutes?

c What is the average rate of flow of water into the tank over the first 5 minutes?

Steps	Working
a 1 Substitute $t = 4$ into $\frac{dV}{dt}$ to find the rate of flow.	$\frac{dV}{dt} = 10(4)^2 + 20$ $= 180$
2 State the answer.	The rate of flow of water at 4 minutes is 180 litres per minute.
b 1 The total volume is the area under the graph of $\frac{dV}{dt}$ from $t = 0$ to $t = 5$.	Total volume $= \int_0^5 (10t^2 + 20)dt$
2 Integrate and evaluate.	$= \left[\frac{10}{3}t^3 + 20t\right]_0^5$ $= \left(\frac{10}{3}(5)^3 + 20(5)\right) - \left(\frac{10}{3}(0)^3 + 20(0)\right)$ ≈ 516.67
3 State the answer.	The total volume of water that flows into the tank in the first 5 minutes is approximately 516.67 L.
c 1 Determine the average.	Average flow $= \frac{1}{5-0}\int_0^5 (10t^2 + 20)\,dx$ ≈ 103.33
2 State the answer.	The average rate of water flow over the first 5 minutes is 103.33 litres per minute.

In economics, the way in which costs, revenue and profits change is important. The instantaneous rate of change with respect to the number of items is referred to as the **marginal rate of change**.

The **marginal change** in a quantity is the cost of producing one more item. For example, if the cost function for producing x products is given by $C(x)$ dollars, then the marginal cost is approximately $\frac{dC}{dx}$ dollars per product.

revenue = price per item × number of items

$R(x) = pn$

profit = revenue − cost

$P(x) = R(x) - C(x)$

$= pn - C(x)$

WORKED EXAMPLE 13 Finding a marginal change

A toy manufacturer's cost of producing x toys is given by $C(x) = 200\ln(x+1)$, where C is the cost in dollars. Each toy is sold for $75.

a **Determine** the marginal cost for producing the 50th toy.

b What is the total cost of producing the first 50 toys?

c What is the average cost of producing the first 50 toys?

d Find the marginal cost and marginal profit for producing the 200th toy and explain what each value represents.

Steps	Working
a 1 Find the marginal cost function $\frac{dC}{dx}$.	$C(x) = 200\ln(x+1)$ $\frac{dC}{dx} = \frac{200}{x+1}$
2 Substitute $x = 50$ into $\frac{dC}{dx}$ to find the marginal cost for the 50th toy.	$\frac{dC}{dx} = \frac{200}{51}$ $= 3.9215\ldots$
3 State the answer.	The marginal cost for producing the 50th item is approximately $3.92.
b 1 The total cost is the area under the graph of $\frac{dC}{dx}$ from $x = 0$ to $x = 50$. Write the integral statement.	Total cost $= \int_0^{50}\left[\frac{200}{x+1}\right]dx$
2 Integrate and evaluate.	$= \left[200\ln(x+1)\right]_0^{50}$ $= (200\ln(50+1)) - (200\ln(0+1))$ $= 786.3651\ldots$
3 State the answer.	The total cost for producing the first 50 items is approximately $786.36.
c 1 Determine the average cost.	Average cost $= \frac{1}{50-0}\int_0^{50}\left[200\ln(x+1)\right]dx$ $= 15.7273$
2 State the answer.	The average cost per item is approximately $15.73.

d 1 Using the marginal cost function, substitute $x = 200$ and evaluate.

$$C'(200) = \frac{200}{201}$$
$$= 0.9950\ldots$$

2 Interpret the result.

The marginal cost of producing the 200th toy is approximately \$0.99. This means that the cost to produce the 200th toy is about \$0.99.

3 State the profit function.

$P(x)$ = revenue − cost
$= 75x - (200 \ln(x+1))$
$P(x) = 75x - 200 \ln(x+1)$

4 Determine the marginal profit function.

$$P'(x) = 75 - \frac{200}{x+1}$$

5 Substitute $x = 200$ and evaluate.

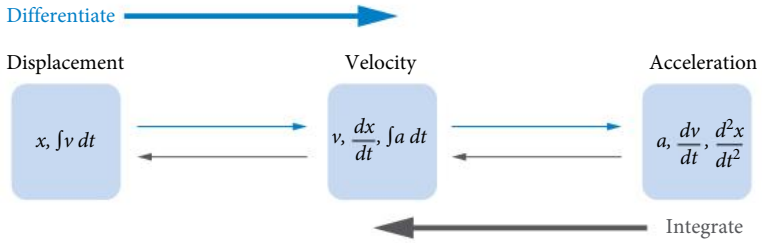
$$P'(200) = 75 - \frac{200}{200+1}$$
$$= 74.0049\ldots$$

6 Interpret the result.

The marginal profit for producing the 200th toy is approximately \$74.01. This means that producing and selling the 200th toy will result in a profit of about \$74.01.

Straight-line motion

The relationships between displacement, velocity and acceleration are:



These relationships can be applied to situations involving motion and total change.

When given either the graph or the function of velocity with respect to time, you may be able to find:

- displacement: the definite integral or signed area, $x(t) = \int v(t)dt$
- total distance travelled: the area under the graph of the curve $v(t)$
- acceleration: $\frac{dv}{dt}$ or the gradient of the velocity function.

Total change in displacement and distance travelled

Consider the velocity, $v(t)$, of an object moving between $t = a$ and $t = b$.

The displacement can be positive or negative.

- total change in displacement $= \int_a^b v(t)dt$

The distance travelled must always be positive.

- distance travelled $= \int_a^b |v(t)|dt$

WORKED EXAMPLE 14 Integration and straight-line motion

An object is moving in a straight line such that its acceleration after t seconds is given by $a(t) = 2t - 6\,\text{m s}^{-2}$. After 2 seconds the velocity of the object is $0\,\text{m s}^{-1}$.

Determine

a an expression for the velocity of the object

b the change in displacement of the object between $t = 1$ and $t = 3$ and interpret your answer

c the total distance travelled in between $t = 1$ and $t = 3$ seconds

Steps	Working
a 1 Integrate the acceleration function to find the velocity function.	$v(t) = \int (2t - 6)dt$ $= t^2 - 6t + c$
2 Find c by substituting in $t = 2$ and $v = 0$.	$0 = (2)^2 - 6(2) + c$ $c = 8$
3 Write the equation for velocity.	$\therefore v(t) = t^2 - 6t + 8$
b 1 Find the change in displacement from $t = 1$ to $t = 3$.	$\int_1^3 (t^2 - 6t + 8)dt$ $= \left[\frac{t^3}{3} - 3t^2 + 8t\right]_1^3$ $= 6 - 5\frac{1}{3}$ $= \frac{2}{3}$
2 Interpret the answer.	Between $t = 1$ and $t = 3$ the object has moved $\frac{2}{3}$ m right of the origin.
c 1 Determine if the object has changed direction between $t = 1$ and $t = 3$. Either draw a sketch of $v(t)$ or determine when $v(t) = 0$.	$0 = t^2 - 6t + 8$ $= (t - 2)(t - 4)$ $\therefore t = 2$ and $t = 4$
2 The object changes direction, so either split the integral statement OR use the formula $\int_a^b \lvert v(t)\rvert dt$.	distance $= \int_1^2 (t^2 - 6t + 8)dx - \int_2^3 (t^2 - 6t + 8)dx$ OR distance $= \int_1^2 \lvert t^2 - 6t + 8\rvert dx$
3 Integrate and evaluate.	$= \left(1\frac{1}{3}\right) - \left(-\frac{2}{3}\right)$ $= 2$
4 State the answer.	The total distance travelled is 2 m from $t = 1$ to $t = 3$.

EXERCISE 6.5 Problems involving definite integrals

ANSWERS p. 408

Recap

1 Find the area enclosed between the curve $f(x) = x^2 - 2x - 2$ and $g(x) = x + 2$.

2 Find the area between the curves $f(x) = 3x^2 - 8x - 3$ and $g(x) = 2x^2 - 5x + 7$.

Mastery

3 WORKED EXAMPLE 12 The temperature of a cup of coffee is cooling according to Newton's law of cooling. The rate of change of the temperature $T(t)$ (in degrees Celsius) at time t (in minutes) is given by $T'(t) = -10e^{-0.1t}$, where t is the time in minutes since the coffee was served.

- **a** What is the net change in the temperature of the coffee in the first 10 minutes?
- **b** What is the average rate of change in temperature between $t = 2$ and $t = 15$?
- **c** If the coffee's initial temperature was 90°C, find the temperature function $T(t)$.

4 WORKED EXAMPLE 13 The marginal cost of producing a particular item is given by $6x^2 - 114x + 780$, where x is in thousands and the cost is in thousands of dollars. The set-up cost before making any items is $15 000. The items are sold for $820 each.

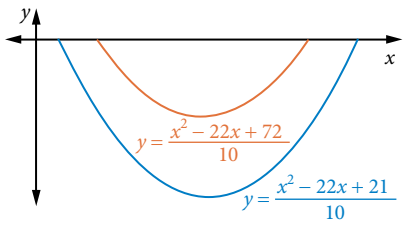
- **a** What is the marginal cost of producing the 100th item?
- **b** What is the cost of producing the first 100 items?
- **c** What is the cost function?
- **d** What is the profit function?
- **e** Find the maximum profit and the number of items that must be produced to achieve it.

5 WORKED EXAMPLE 14 An object is thrown upwards at 15 m s^{-1} from a height of 2 m. Its acceleration is given by the function $a(t) = -9.8$ m s^{-2}. Find the following, correct to 2 decimal places.

- **a** an expression for the velocity of the object
- **b** the time to reach maximum height
- **c** an expression for its height
- **d** the maximum height
- **e** the total distance travelled by the object in the first 2 seconds

6 A sweeping 'circular' driveway has been designed using two parabolas as its edges, to allow room for parking near the house. The x-axis is the edge of the roadway, as shown in the diagram.

Estimate the cost of concreting to a depth of 15 cm with concrete costing $426 per cubic metre.



Exam practice

7 © QCAA 2020 2Q6 Oil is leaking from a tanker at the rate of $r(t) = 9000e^{-0.2t}$ litres per hour, where t is in hours.

Determine how much oil leaks from the tanker (to the nearest litre) from time $t = 0$ to time $t = 10$.

A 38 910 litres **B** 8756 litres **C** 7782 litres **D** 1556 litres

8 TECH-FREE (2 marks) The marginal cost function for making x components per hour is given by $\$(100 - 4x)$ for up to 20 components per hour. It costs \$800 to set up the machinery for a one-hour run.

a What is the total cost function? *[1 mark]*

b What is the cost of making 10 components per hour? *[1 mark]*

9 (4 marks) The rate at which water drains from a tank is given by

$$\frac{dV}{dt} = -500\cos\left(\frac{\pi t}{6}\right)$$

where $V(t)$ is the volume of water (in litres) at time t (in hours).

a What is the rate at which the tank is draining at $t = 10$ minutes? *[1 mark]*

b What is the net change in the water volume during the first 5 hours? *[2 marks]*

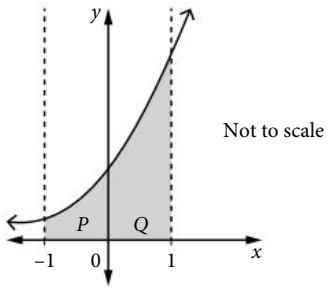
c If the tank initially held 2000 litres, find the function for $V(t)$. *[1 mark]*

10 (3 marks) CF The marginal cost of producing USB drives, in dollars per day, is given by $0.000\,015n^2 - 0.011n + 2.5$ per day for up to 1500 drives per day.
It costs \$2000 per day even if no drives are produced.

Compare the average and marginal costs of producing 800 drives per day and comment on your results.

11 © QCAA 2021 1Q18 TECH-FREE (4 marks) CF The graph of $y = f(x)$, where $f(x)$ is the quadratic function $f(x) = ax^2 + bx + 4$, is shown.

Two regions of the area between the graph of $y = f(x)$ and the x-axis are shaded.



Region P has an area of $\frac{13}{6}$ units^2 and Region Q has an area of $\frac{43}{6}$ units^2.

Determine the values of a and b.

12 (4 marks) CF A fashion designer with outlets in all major cities in Australia has worked out that if she makes more than 50 of any particular item, the price she can charge decreases sharply.

The marginal revenue function for a particular dress is given by

$$R'(x) = 2750 - 75x + 1.5x^2 - 0.01x^3$$

and the total cost function is

$$C(x) = 4200 + 505x$$

Find the number she should make to maximise profit and the maximum profit.

13 © QCAA 2023 2Q16 **CF** (6 marks) A particle is moving in a straight line. The velocity (m s^{-1}) of the particle is given by $v(t) = \dfrac{20\sin(2t)}{6 - 5\cos(2t)}$, $t \geq 0$, where t is time (s) after moving from its initial position. The initial position of the particle is +6.0 m from the origin.

a Use calculus methods to determine an equation for the position of the particle from the origin at any time t. *[3 marks]*

b Determine the position of the particle relative to the origin when it first reaches maximum velocity. *[3 marks]*

14 (6 marks) **CF** It takes an elevator 20 seconds to ascend from the ground floor of a building to the fifth floor. The velocity of the elevator during its ascent is given by $v(t) = \dfrac{2\pi}{5}\sin\left(\dfrac{\pi t}{20}\right)$.

The velocity, v, is measured in metres per second, and time, t, is measured in seconds.

a Determine the acceleration of the elevator during its ascent and provide a sketch of the acceleration function for $0 \leq t \leq 20$.

b With reference to your answer from part **a**, explain what is happening to the velocity of the elevator in the intervals $0 < t < 10$ and $10 < t < 20$.

c Suppose that the ground floor has displacement $x = 0$. Determine the displacement function of the elevator and hence determine the height above the ground floor of the fifth floor.

15 © QCAA 2022 2Q17 **CF** (4 marks) A snail is travelling along a straight path from point A. The snail's velocity (cm min^{-1}) is modelled by $v(t) = 1.4\ln(1 + t^2)$, where t is time (in minutes) for $0 \leq t \leq 15$.

An ant passes point A 12 minutes after the snail and follows the snail's path. The ant moves with a constant acceleration of $2\ \text{cm min}^{-2}$ and passes the snail at $t = 15$ minutes.

Determine the ant's velocity at point A.

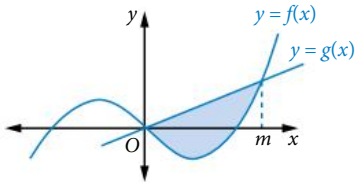
16 (6 marks) **CF** The shape of a shade sail is formed by three intersecting parabolas. The parabolas have equations $y = x^2 - 8x + 16$, $y = x^2 + 8x + 16$ and $y = 2 - 0.125x^2$, where all dimensions are in metres. The cost of the shade cloth is $\$30/\text{m}^2$. Find the cost of the sail and comment on your result.

17 (3 marks) **CF** The learning curve for the production of wheelbarrows in a factory is given by $f(x) = 2.8x^{-0.3}$, where x is the number of wheelbarrows produced and $f(x)$ is the number of work hours needed to produce the xth wheelbarrow. Find the time needed to produce the second 50 wheelbarrows.

18 TECH-FREE (4 marks) **CU** Consider the graph of $y = f(x)$ and $y = g(x)$.

$$f(x) = x^3 - ax,\ a > 0 \text{ and } g(x) = ax,\ a > 0$$

$f(x)$ and $g(x)$ intersect at $x = 0$ and $x = m$. The shaded area shown is 64. Determine the values of a and m.



19 (4 marks) CU A new school drop-off zone is being designed as a sweeping driveway with its entrance and exit joining the road (x-axis). The drop-off zone can be modelled as the area between the positive x-axis, the positive y-axis and the curves $y = \frac{12 - x^2}{4}$ and $y = \frac{36 - x^2}{4}$ (in metres), $y \geq 0$.

Estimate the cost of concreting the drop-off zone to a depth of 10 cm with concrete costing \$394 m–3.

20 (6 marks) CU A vehicle's cam shaft has a cross section that can be approximated by the area between the curves $y = 0.75x^2 - 3.75x + 6$ and $y = 5x - 1 - x^2$, where x is in centimetres.

The shaft is 32 cm long. It is made of high-tensile steel with a density of 7920 kg m^{-3}.

Find the mass of the shaft.

21 (6 marks) CU The acceleration of a fireworks skyrocket increases as it ascends because the mass decreases as the fuel is burned. It is given by the expression $a = \frac{400}{4.5 - t} - 9.8\,\text{m s}^{-2}$ for the first 2 seconds.

The fuel is all spent after 2 seconds, when the rocket explodes in a blaze of colour. Approximately how high does the rocket get before the explosion? (Hint: Use technology to integrate the velocity.)

22 © QCAA 2023 2Q19 (6 marks) CU Over a suitable domain, a hill has a cross-sectional area given by $\int h(x)\,dx = \frac{a}{b}e^{bx} + c$, where:

- a, b and c are constants, $b \neq 0$
- $h(x)$ represents vertical distance (m), x represents horizontal distance (m).

It is known that $h(0) = 1.22$ and $h(40) = 25$.

Where the gradient of the hill is 0.86 there is a tree stump. A second tree stump is located further up the hill. The difference in hill gradient between the two tree stumps is 0.44.

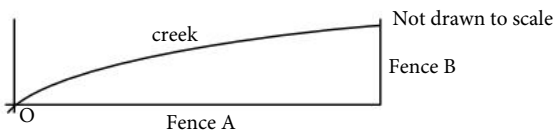
A surveyor predicts that the vertical distance separating the two tree stumps is between 7.5 m and 8.5 m. Evaluate the reasonableness of this prediction.

23 © QCAA 2020S 2Q20 (5 marks) CU A farmer has a paddock with straight fences on two sides (fence A and fence B) perpendicular to each other and bounded by a creek on the other side. Fence B is 4 kilometres long. The creek boundary can be modelled using

$$d = \ln(5x + e) - 1$$

where d is the perpendicular distance in kilometres from fence A to the creek and x is the distance in kilometres along fence A from the point O.

The farmer wants to divide his paddock area in half with a straight fence, parallel to fence B.



Determine where the farmer should locate the fence.

EXAM QUESTION ANALYSIS

© QCAA 2020 2Q19 (7 marks)

Consider the following information when completing this question.

The length of a curve $y = f(x)$ over the interval $[a, b] = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

You are driving along a road with a vertical distance above sea level D (in metres) given by the function

$$D(x) = 300 + \ln(x^2 - 3x + e)$$

where x is the horizontal distance from an initial point of measurement (in kilometres) at sea level.

Assume that if x is positive you are east of the initial point of measurement and if x is negative you are west of the initial point of measurement.

You start your drive along the road at a horizontal distance of 10 kilometres west of the initial point of measurement and drive until you are at a horizontal distance of 10 kilometres east of the initial point.

Determine the time you spend driving downhill, if you drive downhill at an average speed of 40 km h^{-1}.

Reading the question

- The vertical distance of the road above sea level is given by $D(x)$.
- The initial point of measurement is the origin. So east $x > 0$ and west $x < 0$.
- You start 10 km west of the origin.
- Drive until you are 10 km east of origin.
- Average speed downhill is 40 km h^{-1}.
- $\text{speed} = \dfrac{\text{distance}}{\text{time}}$

Thinking about the question

- Need the units for $D(x)$ and $f(x)$ to be the same.
- Determine where you are driving downhill.
- Draw a sketch of the situation.
- Determine the distance driving downhill.
- Determine the time driving downhill.

Video Exam question analysis: Integration and areas

Worked solution (✓ = 1 mark)

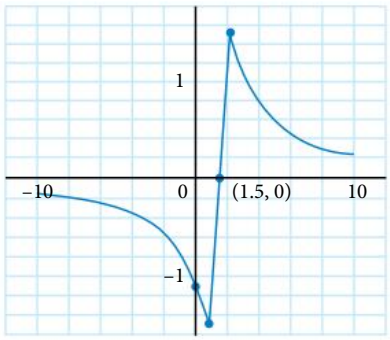
Change $D(x)$ into km:

$$D(x) = \frac{300 + \ln(x^2 - 3x + e)}{1000}$$

When driving downhill, $D(x)$ will be decreasing and so $D'(x)$ will be negative.

You need to determine the x values at which this occurs. ✓

Sketch of $D'(x)$ below.



✓

$D'(x)$ is decreasing when $-10 \leq x \leq 1.5$. ✓

$$\text{Distance} = \int_{-10}^{1.5} \sqrt{1 + (D'(x))^2}\, dx$$ ✓

$$= \mathbf{11.5} \text{ kilometres}$$ ✓

$$\text{Time} = \frac{\text{distance}}{\text{speed}}$$

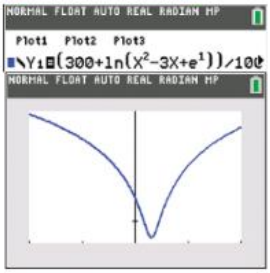
$$= \frac{11.5}{40}$$

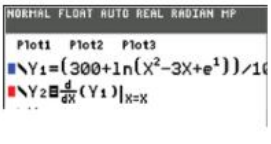
$$= \mathbf{0.29} \text{ hours}$$ ✓

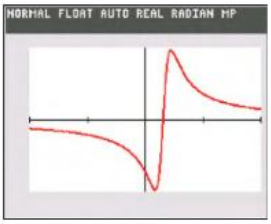
You are travelling downhill for 0.29 hours (17.25 minutes).

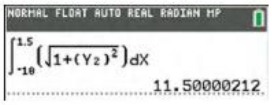
✓ for logical organisation communicating key steps

TI-84 Plus CE

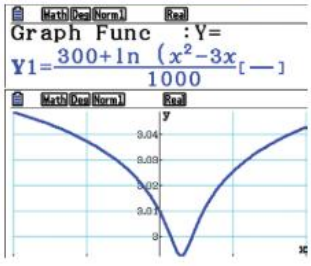


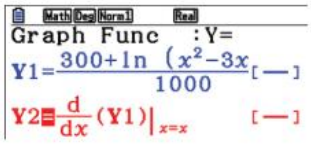


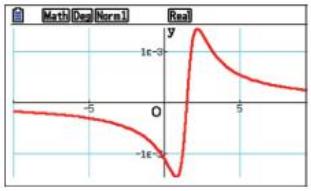


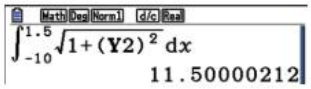


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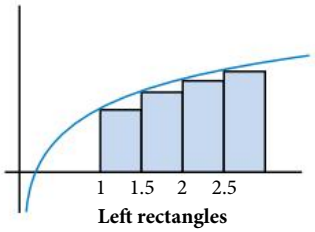
A common error for students answering this QCE exam question was not making $D(x)$ and x the same unit. As a result, the distance was calculated as 13.25 km and the time as 0.33 hours.

TI-Nspire
Integration and areas

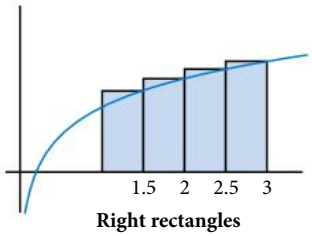
6 Chapter summary

Approximating areas under curves

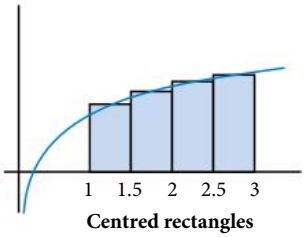
- The area under the graph of a curve can be approximated using a series of rectangles.



Left rectangles



Right rectangles



Centred rectangles

- As the number of rectangles increases, the width of the rectangles will approach 0 ($\delta x \to 0$).

  This means $A \approx \lim\limits_{\delta x \to 0} \sum\limits_{i} f(x_i)\delta x_i$ becomes

  the **definite integral** $A = \int_a^b f(x)\,dx$.

- The trapezoidal rule is

$$\int_a^b f(x)\,dx \approx \frac{w}{2}\Big[f(x_0)+2\Big[f(x_1)+f(x_2)+f(x_3)+\cdots f(x_{n-1})\Big]+f(x_n)\Big]$$

  where $w = \dfrac{b-a}{n}$.

- An alternative version of the trapezoidal rule is

$$\int_a^b f(x)\,dx \approx \frac{w}{2}\big[E+2M\big]$$

  where E is the sum of the end values of the function ($f(x_0)$ and $f(x_n)$) and M is the sum of the middle values $f(x_1), f(x_2), f(x_2), \ldots f(x_{n-1})$.

Definite integrals

- The **definite integral** of $f(x)$ with **limits** at a and b is given by

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b)-F(a)$$

  where $F(x)$ is the anti-derivative of $f(x)$.

- The **indefinite integral** (anti-derivative) is a function, while the definite integral is a number.

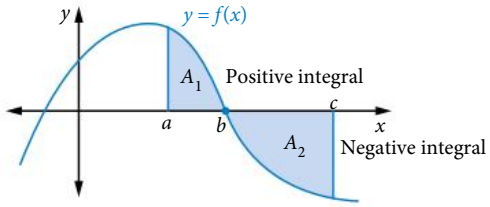
- The **fundamental theorem of calculus** is

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

Properties of the definite integral	
$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Factorise a **constant out**: A constant factor can be removed from an integral.
$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$	**Swap limits:** Reversing the order of the limits changes the sign of the definite integral.
$\int_a^b (f(x)\pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	**Split terms**: A sum or difference of terms can be integrated separately.
$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ where $a < b < c$.	**Split limits**: The limits of an integral can be split.

Areas under curves

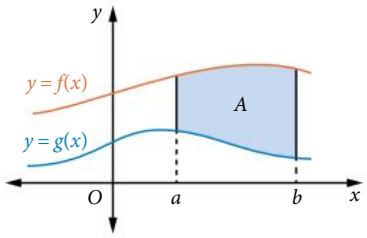
- The definite integral $\int_a^b f(x)\,dx$ gives the **signed area** enclosed between the graph of $y = f(x)$, the x-axis and the limits $x = a$ and $x = b$.



- If the area is located above the x-axis, then the definite integral will be positive and it will give the area under the curve.
- If the area is located below the x-axis, then the definite integral will be negative. It will give the area above the curve.
- If the graph is both partially above and below the x-axis, then the graph and the integral must be split.

Areas between curves

- Area $= \int_a^b [\text{upper} - \text{lower}]\,dx = \int_a^b [f(x) - g(x)]\,dx$



Total and marginal change

- The amount by which a quantity has changed over time is called the **total change** (or net change).
- The **average value** of a function $f(x)$, for an interval $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.
- In economic situations, the instantaneous rate of change with respect to the number of items is referred to as the **marginal rate of change**.
  It relates to costs, revenue and profits, where:
  revenue = price per item × number of items
  $R(x) = pn$
  profit = revenue − cost
  $P(x) = R(x) - C(x)$
  $P(x) = pn - C(x)$
- The **marginal change** in a quantity is the change in the quantity from producing one more item.

Straight-line motion

- velocity $= v(t) = \int a(t)\,dt$
- displacement $= x(t) = \int v(t)\,dt$
- Displacement can be positive or negative.
  - total change in displacement $= \int_a^b v(t)\,dt$
- Distance travelled must always be positive.
  - distance travelled $= \int_a^b |v(t)|\,dt$

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

Total: 18 marks

Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2023 1Q3 A bag contains 10 buttons of the same shape and size in different colours: 5 blue, 3 green and 2 red.

If 3 buttons are randomly drawn from the bag, which probability can be calculated using the binomial distribution?

A P(3 green) with replacement

B P(3 blue) without replacement

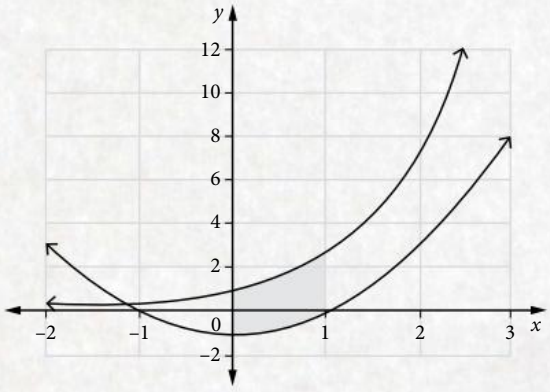
C P(2 green and 1 red) with replacement

D P(2 red and 1 blue) without replacement

2 © QCAA 2021 1Q7 Determine $\int_1^3 (2x+3)\,dx$.

A 2 **B** 4 **C** 14 **D** 16

3 © QCAA 2020 1Q1 The graphs of $f(x) = e^x$ and $g(x) = x^2 - 1$ are shown.



The area of the shaded section bounded by these graphs between the lines $x = 0$ and $x = 1$ is

A $1 - e$ **B** $e - 2$ **C** $e - \frac{5}{3}$ **D** $e - \frac{1}{3}$

Section 2 3 short response questions

15 marks

4 © QCAA 2021 1Q13 (5 marks) Consider the functions $f(x) = x^2$ and $g(x) = 4x$.

a Determine the x-coordinates of the points of intersection of the graphs of the two functions. *[2 marks]*

b Use the results from part **a** to calculate the area enclosed by the graphs of $f(x)$ and $g(x)$. *[3 marks]*

5 © QCAA 2022 1Q14 (6 marks) The rate that water fills an empty vessel is given by

$$\frac{dV}{dt} = 0.25e^{0.25t} \text{ (in litres per hour)}, 0 \leq t \leq 8\ln(6),$$

where t is time (in hours).

a Determine the function that represents the volume of water in the vessel (in litres). *[2 marks]*

The vessel is full when $t = 8\ln(6)$.

b Determine the volume of water, to the nearest litre, the vessel can hold when full. *[2 marks]*

The table shows the approximate rate the water flows into the vessel at certain times.

t	$\frac{dV}{dt}$
0	0.25
1	0.32
2	0.41
3	0.53

c Use information from the table and the trapezoidal rule to determine the approximate volume of water in the vessel after three hours. *[2 marks]*

6 © QCAA 2023 1Q15 (4 marks) In a certain game, players throw one water balloon at a target. There is a one in four chance of hitting the target.

a State the probabilities of all the possible outcomes for one throw at the target. *[2 marks]*

b Let H be the discrete random variable for one of the possible outcomes. Determine the mean and variance of the distribution of random variable H when 20 players throw a water balloon at the target. *[2 marks]*

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

Total: 18 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2020S 2Q2 A population of bacteria after t hours is given by $P(t) = 5000e^{0.18t}$.

The rate of increase of the population (to the nearest unit) at 15 minutes is

A 74 399 bacteria/hour.
B 13 392 bacteria/hour.
C 5230 bacteria/hour.
D 941 bacteria/hour.

2 © QCAA 2020S 2Q4 The area of the region enclosed by the graphs of $y = \dfrac{x}{\sqrt{x+1}}$ and $y = 2x$ is

A 1.267 units^2
B 0.467 units^2
C 2.000 units^2
D 0.029 units^2

3 © QCAA 2022 2Q7 A marble moves in one direction in a straight line with velocity $v = 2\ln(t+1)$ (in metres per second) where t is time (in seconds) since the marble passed through the origin.

Determine the distance from the origin the marble has rolled after 4 seconds.

A 0.40 m
B 1.60 m
C 3.22 m
D 8.09 m

Section 2 3 short response questions

15 marks

4 © QCAA 2021 2Q13 (7 marks) The amount of gravel (in tonnes) sold by a construction company in a given week is a continuous random variable X and has a probability density function defined by:

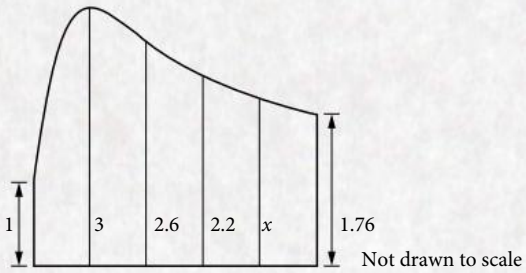
$$f(x) = \begin{cases} c(1-x^2), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a Show that $c = \dfrac{3}{2}$. *[1 mark]*

b Determine $P(X < 0.25)$. *[2 marks]*

c Determine the variance of X. *[4 marks]*

5 © QCAA 2020 2Q15 MODIFIED (4 marks) A field is divided into five sections as shown. The width of each section is 1 metre. The perpendicular height, in metres, of each section is given in the diagram. The area of the field was approximated using the trapezoidal rule and found to be 11.12 m^2.



a Determine the height marked x on the diagram. *[2 marks]*

b Determine the area of the field, given the shape of the field is modelled by the function

$$f(x) = \frac{4x}{x^2 + 1} + 1,\ 0 \le x \le 5$$ *[1 mark]*

c Determine the percentage difference between the trapezoidal approximation and the exact integral value. *[1 mark]*

6 © QCAA 2022 2Q19 (4 marks) Flying foxes enter and leave a fruit-growing region every evening. The rate at which the flying foxes enter the region is modelled by the function

$$A(t) = 42\sin\left(0.03t - \frac{\pi}{3}\right) + 71,\ 0 \le t \le 240$$

The rate at which the flying foxes leave the region is modelled by the function

$$L(t) = 42\sin\left(0.04t - \frac{\pi}{3}\right) + 42,\ 0 \le t \le 240$$

Both $A(t)$ and $L(t)$ are measured in animals per minute and t is measured in minutes after 7 pm.

There are 100 flying foxes in the region at 7 pm.

Determine the maximum number of flying foxes in the region and the time that this occurs.

CHAPTER

7 THE SINE AND COSINE RULES



Syllabus coverage

Nelson MindTap chapter resources

Terminology

7.1 The sine rule
- Right-angled trigonometry review
- The sine rule
- The ambiguous case of the sine rule

7.2 The cosine rule

7.3 The area of a triangle

7.4 Problems involving trigonometry

7.5 3D problems involving trigonometry

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Prior learning
The sine and cosine rules

Syllabus coverage

UNIT 4, TOPIC 2: TRIGONOMETRY

Cosine and sine rules

- Use the sine rule (ambiguous case is required), $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$, where a, b and c are the side lengths of the triangle and A, B and C are the corresponding opposite angles.
- Use the cosine rule, $c^2 = a^2 + b^2 - 2ab\cos(C)$.
- Use the formula $area = \frac{1}{2}bc\sin(A)$ to calculate the area of a triangle.
- Model and solve problems that involve the sine rule, cosine rule and the area formula in two- and three-dimensional contexts (including bearings, directions and angles of elevation and depression), with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 28
© Queensland Curriculum and Assessment Authority (QCAA)

Videos (7):

7.1 True bearings • The sine rule
7.2 The cosine rule • The cosine rule for angles
7.3 The sine area formula
7.4 The sine and cosine rules
Exam question analysis The sine and cosine rules

Prior learning (1):

7.1 The sine and cosine rules

Worksheets (14):

7.1 A page of bearings • Bearings match-up • Elevations and bearings • The sine rule – Finding lengths of sides • The sine rule – Finding angles • Sine rule problems
7.2 Cosine rule problems • Finding an unknown side • Finding an unknown angle • The cosine rule – angles and sides
7.3 Areas of triangles
7.5 Sine and cosine rules assignment • Sine and cosine rule problems
Cumulative exams Maths Methods formula sheet

TI-Nspire CX II instructions (1):

7.1 The sine and cosine rules

Nelson MindTap
To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

ambiguous case
angle of depression
angle of elevation
bearing
cosine rule
hypotenuse
included angle
sine rule
solving a triangle
three-figure bearing

7.1 The sine rule

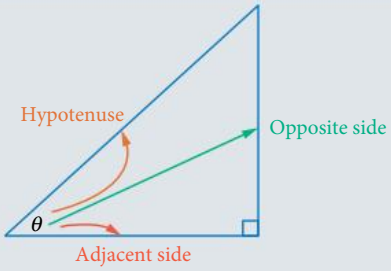
Right-angled trigonometry review

The trigonometric ratios

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Many practical situations involving triangles use bearings and angles of elevation and depression.

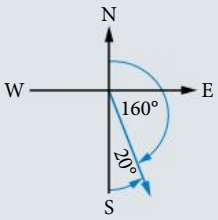
Bearings, angles of elevation and depression

A **bearing** is an angle that shows the direction of one point from another.

A **three-figure bearing** (or **true bearing**) is the angle measured clockwise from north; e.g. 160°.

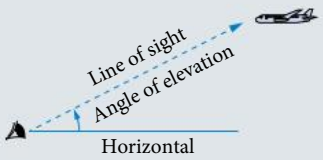
A **compass bearing** is the angle measured from north or south towards the east or west; e.g. S 20° E.

In the diagram 160° = S 20° E.



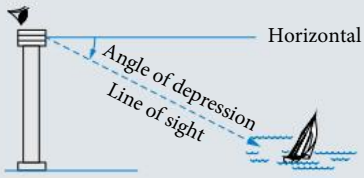
Angle of elevation – Looking up

The angle of elevation of an object is the angle you must look up from the horizontal to see the object.



Angle of depression – Looking down

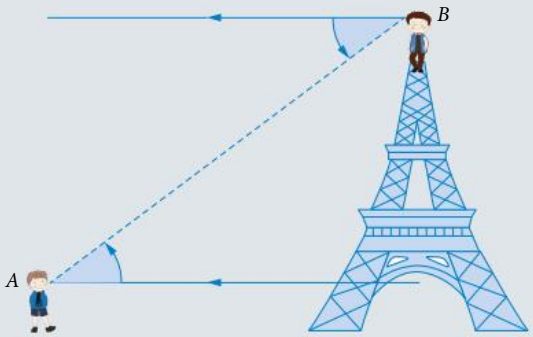
The angle of depression of an object is measured downwards from the horizontal.



Video
True bearings
The sine rule
Worksheets
A page of bearings
Bearings match-up
Elevations and bearings
The sine rule – Finding lengths of sides
The sine rule – Finding angles
Sine rule problems

Angles A and B are alternate angles.

Angle of elevation A to B = Angle of depression B to A



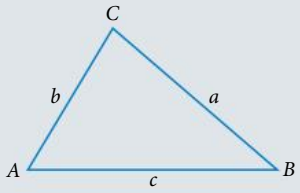
Remember that $1° = 60'$ (minutes) and $1' = 60''$ (seconds).

The sine rule

The **sine rule** can be used to find unknown sides and angles in non-right-angled triangles.

The sine rule

For any triangle ABC



A, B, C are the angles and a, b, c are the sides opposite each angle respectively.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The sine rule relates each side of a triangle to its opposite angle.

The longest side is opposite the largest angle.

The shortest side is opposite the smallest angle.

The sine rule is used when the problem has either:

- 2 sides and one non-included angle, OR
- 2 angles and one side.

When labelling shapes such as triangles, the naming convention is to use capital letters for corners (vertices) and lowercase for the opposite sides. For example, in the triangle $\triangle ABC$ above, $AC = b$, $AB = c$ and $BC = a$.

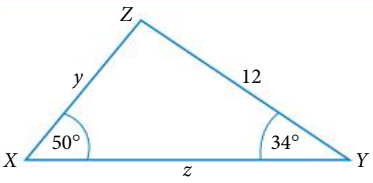
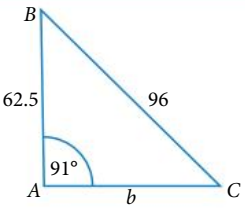
WORKED EXAMPLE 1 The sine rule

Solve each triangle.

a $\triangle XYZ$, where $X = 50°$, $Y = 34°$ and $x = 12$

b $\triangle ABC$, where $A = 91°$, $a = 96$ and $c = 62.5$

Finding all the unknown sides and angles is called **solving a triangle**.

Steps	Working
a 1 Draw a diagram labelling all known and unknown elements.	
2 Identify connected pairs and use this to determine which value to find first. Know X and x, and Y and so use these to find y.	$\frac{y}{\sin(Y)} = \frac{x}{\sin(X)}$
3 Substitute known values.	$\frac{y}{\sin(34°)} = \frac{12}{\sin(50°)}$
4 Rearrange to isolate y and evaluate. Make sure your calculator is set to degrees.	$y = \frac{12\sin(34°)}{\sin(50°)}$ $y = 8.759\ldots$
5 Find Z.	$Z = 180° - (50° - 34°)$ $= 96°$
6 Find z. If possible, use connected pairs given in the question to maximise accuracy.	$\frac{z}{\sin(Z)} = \frac{x}{\sin(X)}$
7 Rearrange to isolate z and evaluate.	$z = \frac{12\sin(96°)}{\sin(50°)}$ $= 15.579\ldots$
8 Round appropriately and state the answers.	$Z = 96°$, $y = 8.8$ and $z = 15.6$
b 1 Draw a diagram labelling all known and unknown elements.	
2 Identify connected pairs. Know A, a and c so find C. You can invert the sine rule to make it easier to find an unknown angle.	$\frac{\sin(A)}{a} = \frac{\sin(C)}{c}$

3 Substitute, rearrange and evaluate.

$$\frac{\sin(91°)}{96} = \frac{\sin(C)}{62.5}$$

$$\sin^{-1}\left(\frac{62.5\sin(91°)}{96}\right) = C$$

$$C = 40.612\ldots°$$

4 Find B.

$$B = 180° - 91° - 40.612\ldots°$$

$$B = 48.387\ldots°$$

5 Find b.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$

$$b = \frac{96\sin(48.387\ldots°)}{\sin(91°)}$$

$$= 71.785\ldots$$

6 Round appropriately and state the answers. $B = 48.4\,°$, $b = 71.8$ and $C = 40.6°$

Exam hack

You cannot use the sine rule in cases where you have 2 sides and the included angle. In such cases you will use the cosine rule, which you will explore in the next section.

Investigation Ambiguous triangles

Draw each triangle described.

$\triangle ABC$ with $B = 45°$, $a = 8$ cm, and $b = 6$ cm

$\triangle EFG$ with $F = 26°$, $e = 8$ cm, and $f = 10$ cm

$\triangle PQR$ with $Q = 76°$, $p = 8$ cm, and $q = 9$ cm

In which cases is more than one triangle possible?

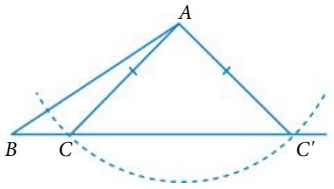
Can you work out when 2 sides and the angle not between the sides gives a unique answer?

The ambiguous case of the sine rule

The **ambiguous case** of the sine rule is when there are 2 possible answers.

It only happens when finding an angle, not a side.

For example, the diagram shows that given the sizes of 2 sides AB, AC and one angle B for a triangle, there are 2 possible triangles: $\triangle ABC$ and $\triangle ABC'$, where B is a common angle, AB is a common side, and $AC = AC'$ are matching equal sides. Angle C is ambiguous because it can have 2 values. It is obtuse in $\triangle ABC$ and acute in $\triangle ABC'$.



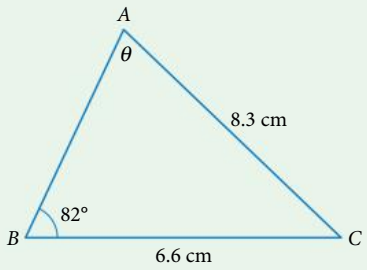
In this case, both the acute and obtuse angles (C' and C) are correct, which means there are 2 possible triangles. Sometimes, only the acute angle is correct because the obtuse angle is incorrect. You need to test which is correct by considering the angle sum of a triangle (180°).

WORKED EXAMPLE 2 The ambiguous case of the sine rule

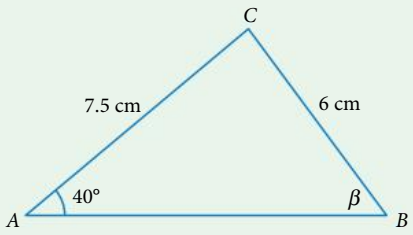
Calculate the unknown angle for each triangle.

If the ambiguous case applies, sketch both triangles.

a



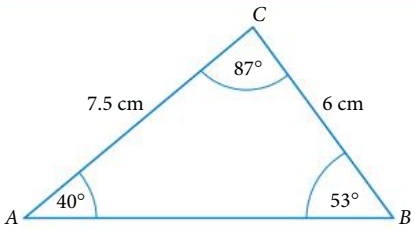
b

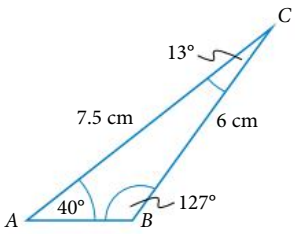


Steps	Working
a 1 List the known or identified values.	$B = 82°$, $b = 8.3$ and $a = 6.6$
2 Know B, b and a, so find A. Put the unknown in the numerator.	$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$
3 Substitute, rearrange and evaluate.	$\frac{\sin(\theta)}{6.6} = \frac{\sin(82°)}{8.3}$ $\theta = \sin^{-1}\left(\frac{6.6\sin(82°)}{8.3}\right)$ $= 51.947…°$
	TI-84 Plus CE NORMAL FLOAT AUTO REAL DEGREE MP sin⁻¹(6.6sin(82)/8.3) 51.94710821
	Casio fx-CG50 AU sin⁻¹ (6.6sin (82)÷8. 51.94710821
5 Find the alternate obtuse angle.	$\theta = 51.947…°$ or $128.052…°$
6 Check if obtuse angle is possible.	$82° + 128.052…° = 210.052 … ° > 180°$, so reject the obtuse answer 128°.
7 As the sum is >180 this is not an ambiguous case.	θ can only be $51.947…°$
8 Round appropriately.	$\theta = 52°$
b 1 List the known or identified values.	$A = 40°$, $a = 6$ cm and $b = 7.5$ cm
2 Know A, a and b, so find B.	$\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$
3 Substitute, rearrange and evaluate.	$\frac{\sin(40°)}{6} = \frac{\sin(\beta)}{7.5}$ $\beta = \sin^{-1}\left(\frac{7.5\sin(40°)}{6}\right)$ $= 53.464…°$
4 Find the alternate obtuse angle.	$\beta = 53.464…°$ or $126.535…°$
5 Check if obtuse angle is possible.	$40° + 126.535…° = 166.535° < 180°$, so the obtuse case is possible.
6 Round and write the answers.	$\beta = 53°$ or $127°$

TI-Nspire
The sine and cosine rules

7 Sketch the triangles.





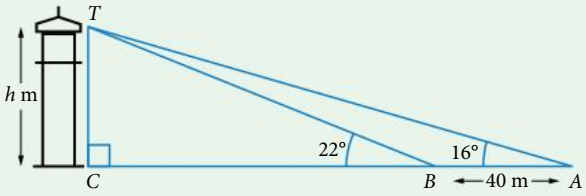
$C = 180° - 40° - 127° = 13°$

or $C = 180° - 40° - 53° = 87°$

Sometimes in more complex sine rule problems, you need to find a side in one triangle so you can then find a side in another triangle. The intermediate side you need is usually located between the known side and the side you need to find. You might have to to combine the sine rule with trigonometry of right-angled triangles.

WORKED EXAMPLE 3 Solving problems using the sine rule

The top of a lighthouse is observed at an angle of elevation of 16°. 40 m closer to the lighthouse the angle of elevation increases to 22°. What is the height of the lighthouse?



Steps	Working
a 1 Identify the length needed (h) and the side located between it and the given side (AB). Identify other known values.	Height of tower (TC) $= h$, $AB = 40$ m, $BT = ?$ $\angle TAB = 16°$, $\angle CBT = 22°$
2 TB is opposite the 16° angle ($\angle TAB$) and the 40 m side (AB) is opposite $\angle ATB$. Find $\angle ATB$.	$\angle ATB = 22°$ (exterior angle in $\triangle ABT$) $- 16°$ $\angle ATB = 6°$
3 Use the sine rule in $\triangle ABT$ to find BT.	$\frac{BT}{\sin(16°)} = \frac{40}{\sin(6°)}$ $BT = \frac{40\sin(16°)}{\sin(6°)}$ $BT = 105.478\ldots$
4 Use the right-angled triangle $\triangle BCT$ to find h.	$\sin(22°) = \frac{h}{BT}$ $h = 105.478\ldots \times \sin(22°)$ $h = 39.512\ldots$
5 Round appropriately and state the answers.	The lighthouse is approximately 39.5 m tall.

Exam hack

The sine rule works for both right-angled triangles and non-right-angled triangles.

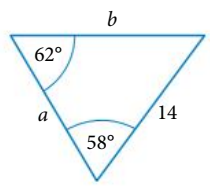
EXERCISE 7.1 The sine rule

ANSWERS p. 408

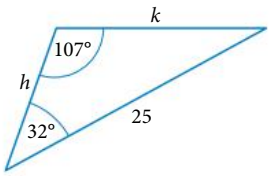
Mastery

1 WORKED EXAMPLE 1 Calculate the value of the variables in each triangle.

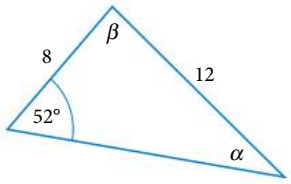
a



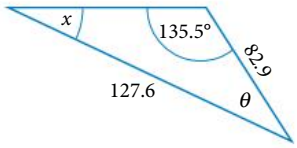
b



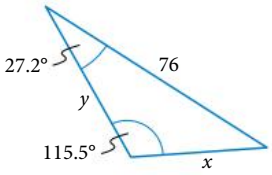
c



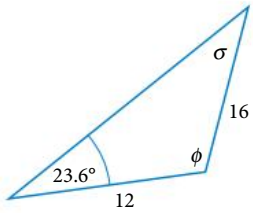
d



e



f



2 Solve each triangle.

a $\triangle XYZ$, where $Z = 65°$, $y = 11$ m and $z = 16.2$ m

b $\triangle PQR$, where $P = 35.3°$, $Q = 52.8°$ and $q = 67.5$ cm

c $\triangle DEF$, where $F = 111°$, $f = 12.5$ km and $d = 8.96$ km

d $\triangle ABC$, where $B = 124.1°$, $C = 18.7°$ and $c = 94.6$ cm

3 WORKED EXAMPLE 2 In $\triangle ABC$, $BC = 18$ cm, $AC = 15$ cm and $B = 38°$. Find both possible values of A and make a sketch of each possible $\triangle ABC$.

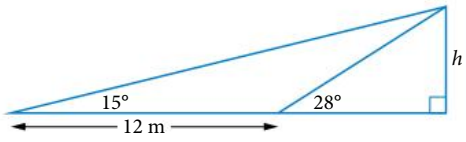
4 In $\triangle DEF$, $f = 30$ cm, $d = 35$ cm and $F = 18°$. Find possible values for e and D.

5 Find the length of the longest side of $\triangle PQR$ if $P = 31°42'$, $Q = 28°50'$ and $PR = 42$ mm.

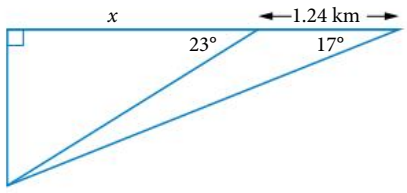
6 In $\triangle XYZ$, $x = 2.3$ cm, $z = 3.7$ cm and $X = 37.7°$. Solve the triangle.

7 WORKED EXAMPLE 3 Find the unknown in each of the following.

a



b



Exam practice

8 TECH-FREE In triangle PQR, angles P and Q and sides q and r are known. Which formula could be used to find the unknown side, p?

A $p = \frac{1}{2}qr\sin P$

B $p = \sin^{-1}\left(\frac{q}{r}\right)$

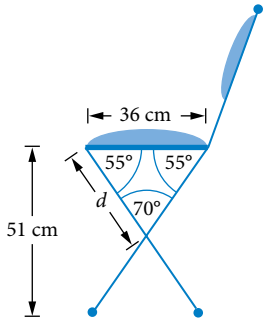
C $p = \frac{q\sin P}{\sin Q}$

D $p^2 = q^2 + r^2 - 2qr\sin P$

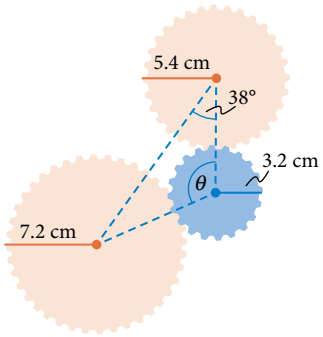
9 In $\triangle PQR$, $p = 8$ cm, $q = 6$ cm, and $P = 45°$. What is the size of Q?

A 30° **B** 32° **C** 45° **D** 60°

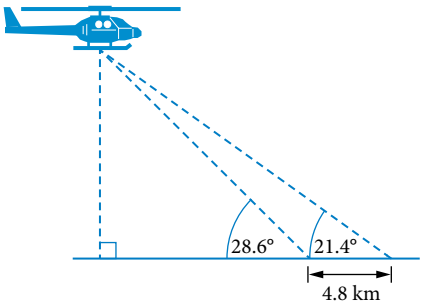
10 (2 marks) The folding chair shown in the diagram has a seat that is 36 cm deep. The legs of the chair are inclined at an angle of 55° to the seat and join at an angle of 70°. The seat is 51 cm above the floor. Find the distance from the seat, d, at which the legs are joined.



11 (3 marks) CF Three gears are arranged as shown. The radii of the gears are 7.2 cm, 5.4 cm and 3.2 cm. Calculate θ.



12 (4 marks) CF A helicopter is sighted at the same time by 2 ground observers who are 4.8 km apart, as shown in the diagram. The angles from the observers to the helicopter (measured to the horizontal) are 21.4° and 28.6°. How high is the helicopter?



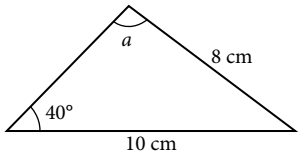
13 (5 marks) CU A plane is flying from Brisbane to Dalby at a speed of 150 km h^{-1}. The pilot thinks the altimeter, which shows a height of 15 000 feet (4600 m), is faulty. Looking ahead, the navigator sees the top of a hill marked as 405 m on the map at an angle of depression of 13°. Two minutes later, the angle of depression has increased to 20.6°. What is the actual altitude of the plane, and what should the navigator tell the pilot about their altimeter?

14 (5 marks) CU A person standing at point Z on the bank of a river observes a tree at point X on the opposite bank at a bearing of 292.17°T. The person then walks 885 m down the bank to point Y and notes that the tree is now at a bearing of 285.33°T. Find the distance between the tree and the initial position, Z, of the observer.

15 © QCAA 2024 2Q7

Identify the possible values for a in the triangle.

Not to scale



A 13.5° or 126.5° **B** 53.5° or 126.5°

C 53.5° or 86.5° **D** 13.5° or 86.5°

7.2 The cosine rule

The **cosine rule** is used when:

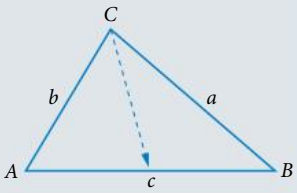
- 2 sides and the included angle are known – to find the unknown side
- all 3 sides of the triangle are known – to find any angle in the triangle.

The cosine rule

In any triangle ABC

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

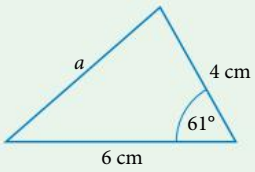


If $C = 90°$, then $\cos(C) = \cos(90°) = 0$, so the cosine rule becomes $c^2 = a^2 + b^2$, which is Pythagoras' theorem.

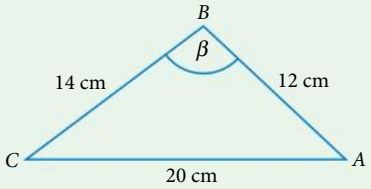
WORKED EXAMPLE 4 The cosine rule

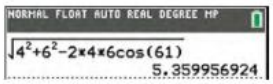
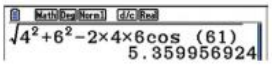
Find the unknown value for each diagram.

a



b



Steps	Working
a 1 Rewrite the cosine rule in the form to find side a.	$a^2 = b^2 + c^2 - 2bc\cos(A)$
2 Solve for a.	$a^2 = 4^2 + 6^2 - 2 \times 4 \times 6\cos(61°)$ $a = \sqrt{4^2 + 6^2 - 2 \times 4 \times 6\cos(61°)}$
3 Use your calculator.	**TI-84 Plus CE**  **Casio fx-CG50 AU** 
4 Round and state the answer.	$a \approx 5.36$ cm.

Videos
The cosine rule
The cosine rule for angles

Worksheets
Cosine rule problems
Finding an unknown side
Finding an unknown angle
The cosine rule - angles and sides

b 1 Use the cosine rule of the form $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ to find angle β.

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac}$$

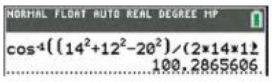
When you calculate the angle given 3 sides, remember it is the side opposite the angle that is subtracted.

2 Solve for β.

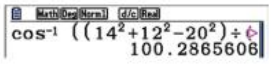
$$\cos\beta = \frac{14^2 + 12^2 - 20^2}{2 \times 14 \times 12}$$

$$\beta = \cos^{-1}\left(\frac{14^2 + 12^2 - 20^2}{2 \times 14 \times 12}\right)$$

TI-84 Plus CE



Casio fx-CG50 AU



3 Round and state the answer.

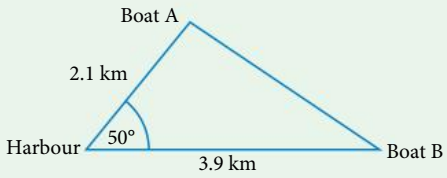
$\beta \approx 100.29°$.

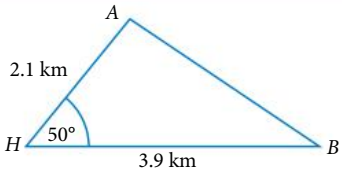
WORKED EXAMPLE 5 Solving problems using the cosine rule

Two fishing boats depart from the harbour on paths that differ by 50°. Boat A travels 2.1 km then puts down its anchor and starts fishing, Boat B travels 3.9 km and does the same.

a How far apart are the 2 boats at this time?

b What is the angle formed at Boat B between the line of sight to the harbour and Boat A?



Steps	Working
a 1 Draw a diagram labelling all known values.	
2 The distance between the boats is side AB. Use the cosine rule of the form $c^2 = a^2 + b^2 - 2ab\cos(C)$ to find side AB.	$AB^2 = HA^2 + HB^2 - 2(HA)(HB)\cos(H)$
3 Substitute known values. Rearrange to isolate AB and evaluate.	$AB^2 = 2.1^2 + 3.9^2 - 2 \times 2.1 \times 3.9\cos(50°)$ $AB = \sqrt{2.1^2 + 3.9^2 - 2 \times 2.1 \times 3.9\cos(50)}$ $AB = 3.015\ldots$
4 Round and state the answer.	The distance between Boats A and B is approximately 3 km.
b 1 Find B using the sine rule.	$\dfrac{\sin(50°)}{3.015\ldots} = \dfrac{\sin(B)}{2.1}$ $B = \sin^{-1}\left(\dfrac{2.1\sin(50°)}{3.015\ldots}\right)$ $B = 32.244\ldots°$
2 Round and state the answer.	The angle formed between the line of sight to the harbour and Boat A from Boat B is approximately 32°.

EXERCISE 7.2 The cosine rule

ANSWERS p. 409

Recap

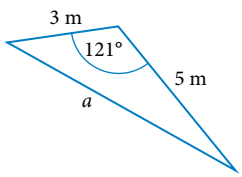
1 In $\triangle ABC$, $A = 42°$, $b = 36$ m, $a = 28$ m, find B.

2 Solve $\triangle KMN$, given $M = 35°$, $m = 29$ m, $n = 19$ m.

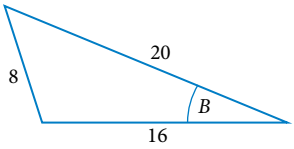
Mastery

3 WORKED EXAMPLE 4 Calculate the value of the unknown in each triangle.

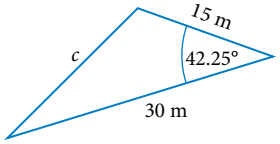
a



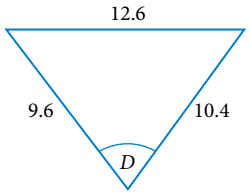
b



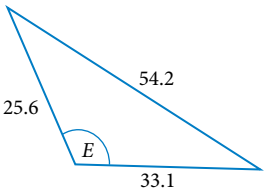
c



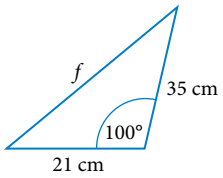
d



e



f



4 Solve each triangle.

a $\triangle DAG$, where $D = 121°$, $a = 3$ m and $g = 5$ m

b $\triangle UVW$, where $W = 55°$, $u = 45$ cm and $v = 50$ cm

c $\triangle XYZ$, where $x = 7.2$ m, $y = 12.5$ m and $z = 8.3$ m

d $\triangle ABC$, where $a = 8.5$ m, $b = 20$ m and $c = 15.2$ m

e $\triangle LMN$, where $L = 42.3°$, $m = 15.4$ km and $n = 12.9$ km

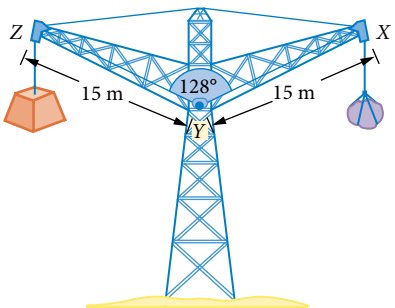
f $\triangle REF$, where $E = 168.2°$, $f = 192$ mm and $r = 151$ mm

5 In $\triangle XYZ$, $x = 8.8$, $y = 11$ and $z = 17.6$. Calculate the smallest angle.

6 In $\triangle PQR$, $p = \sqrt{5}$, $q = 1$ and $r = 2$. Calculate the largest angle.

7 WORKED EXAMPLE 5 A crane with 2 arms, each 15 m long, is shown. The arms are separated by an angle of 128°. A load is attached at point X and the counterweight is attached at point Z.

Find XZ.

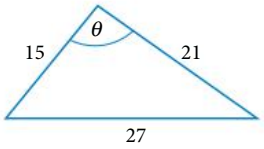


8 A weight is supported by cables attached to a horizontal beam. The cables are of lengths 12 m and 9 m. They are attached to the beam at points 18 m apart. Find the angle that each cable makes with the beam.

9 A hiker encounters a swamp at point X that is 4.2 km from point Y on the opposite side of the swamp. To travel around the swamp, the hiker walks 3.4 km from X on a bearing of 325°T and then walks 1.8 km to point Y. Find the bearing of X from Y.

Exam practice

10 TECH-FREE Which of the following expressions is true for $\cos\theta$?



A $\cos\theta = \dfrac{21^2+27^2-15^2}{2\times21\times27}$

B $\cos\theta = \dfrac{21^2-15^2+27^2}{2\times21\times15}$

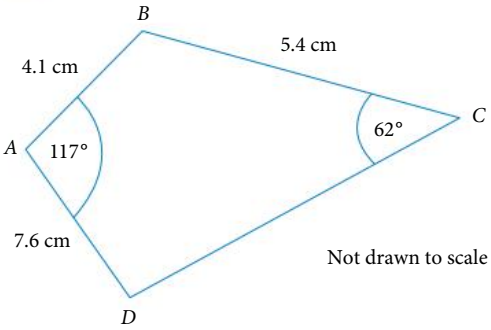
C $\cos\theta = \dfrac{21^2+15^2-27^2}{2\times21\times15}$

D $\cos\theta = \dfrac{15^2+27^2-21^2}{2\times15\times27}$

11 In $\triangle FGH$, $f = 7$ cm, $g = 10$ cm and $\angle H = 120°$. What is the length of side h?

A 12 cm **B** 14 cm **C** 16 cm **D** 18 cm

12 © QCAA 2020 2Q18 (4 marks) CF The diagram shows the quadrilateral $ABCD$.



Determine the perimeter of the quadrilateral.

13 (5 marks) CF A yacht sails on a triangular course. The first leg is 18 km NE and the second leg is 15 km at 168°T. What is the third leg?

7.3 The area of a triangle

Video
The sine area formula

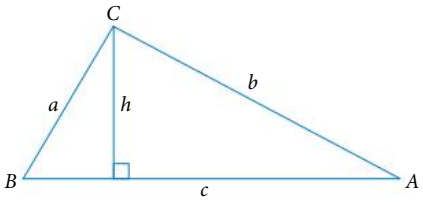
Worksheet
Areas of triangles

The area of a right-angled triangle is given by $\text{Area} = \frac{1}{2}\times\text{base}\times\text{height}$.

The height, h, of $\triangle ABC$ below can be found using $\sin(A) = \dfrac{h}{b}$

$$h = b\sin(A)$$

hence, the area of $\triangle ABC = \frac{1}{2}cb\sin(A)$.



Area of a triangle sine formula

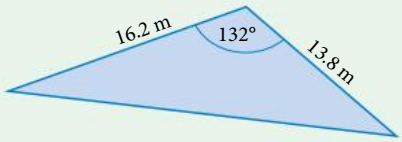
Area of $\triangle ABC = \frac{1}{2}bc\sin A$,

where A is the included angle between sides b and c.

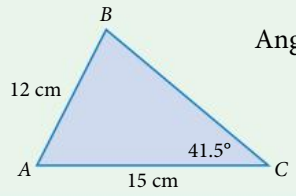
WORKED EXAMPLE 6 Area of a triangle

Determine the area of each triangle, correct to 2 decimal places.

a



b

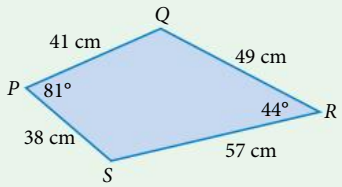


Angle B is acute.

Steps	Working
a 1 Write the area formula.	$\text{Area} = \frac{1}{2}bc\sin(A)$
2 Substitute values and evaluate.	$= \frac{1}{2} \times 16.2 \times 13.8 \sin(132°)$ $= 83.068\ldots$
3 Round appropriately and state the answer.	The area is about $83.07\ \text{m}^2$.
b 1 Find angle B using the sine rule.	$\frac{\sin(B)}{b} = \frac{\sin(C)}{c}$ $\sin(B) = \frac{15\sin(41.5°)}{12}$ $B = 55.921\ldots°$
2 Find angle A.	$A = 180° - 41.5° - 55.921\ldots°$ $A = 82.578\ldots°$
3 Write the area formula.	$\text{Area} = \frac{1}{2}bc\sin(A)$
4 Substitute values and evaluate.	$= \frac{1}{2} \times 15 \times 12\sin(82.578\ldots°)$ $= 89.245\ldots$
5 Round appropriately and state the answer.	The area is about $89.25\ \text{cm}^2$.

WORKED EXAMPLE 7 Area of shapes

Determine the area of this shape, correct to 2 decimal places.



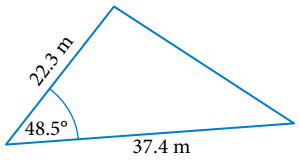
Steps	Working
1 Identify the 2 triangles and write the area as a sum.	Area of $PQRS$ = area of $\triangle PQS$ + area of $\triangle QRS$
2 Write the area formulas.	$= \frac{1}{2}PQ \times PS\sin(P) + \frac{1}{2}RQ \times RS\sin(R)$
3 Substitute values and evaluate.	$= \frac{1}{2} \times 41 \times 38\sin(81°) + \frac{1}{2} \times 49 \times 57\sin(44°)$ $= 769.409\ldots + 970.090\ldots$ $= 1739.499\ldots$
4 Round appropriately and state the answer.	The area is about $1739.50\ \text{cm}^2$.

EXERCISE 7.3 The area of a triangle

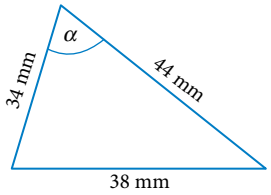
ANSWERS p. 409

Recap

1 Find the value of the unknown side in this triangle.



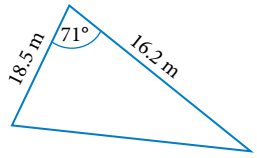
2 Find the value of α in this triangle.



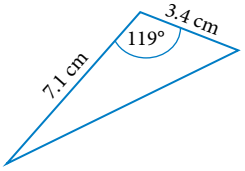
Mastery

3 WORKED EXAMPLE 6 Find the area of each triangle.

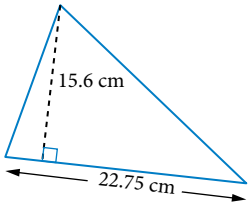
a



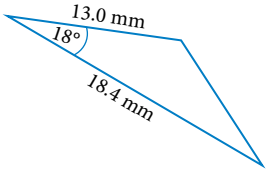
b



c



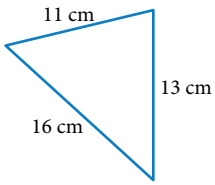
d



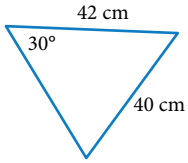
4 A triangle has sides of length 10 cm and 12 cm with an included angle of 65°. Find the area of the triangle.

5 Find the area of each triangle, correct to 2 decimal places.

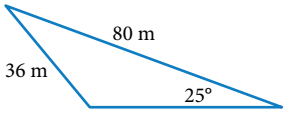
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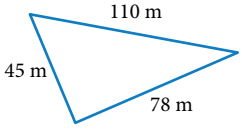
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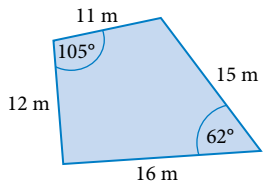


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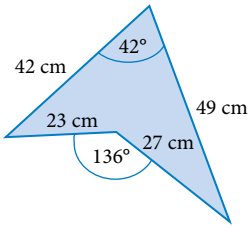


6 WORKED EXAMPLE 7 Find the area of each shape.

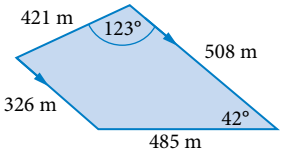
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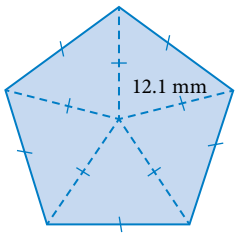
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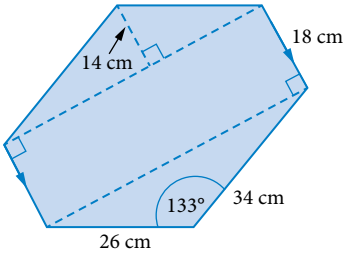
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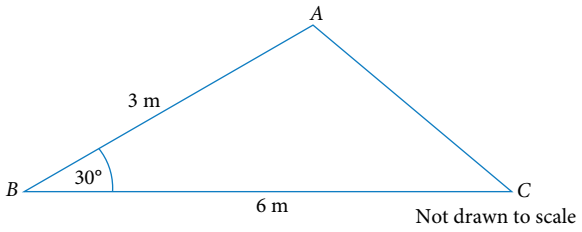
7 A triangle has an angle of 35° and 2 sides of length 9 cm and 14 cm.
Calculate the areas of the 4 possible triangles, correct to 1 decimal place.

Exam practice

8 What is the area of a triangular field that has sides of length 35 m and 47 m with an included angle of 62°?

A 712.46 m^2 **B** 726.22 m^2 **C** 740.58 m^2 **D** 755.31 m^2

9 © QCAA 2020 1Q14 (3 marks) TECH-FREE Determine the area of the triangle shown.



10 (3 marks) CF The area of triangle ABC is 64 cm^2, and 2 side lengths are 15 and 10 cm.
Find the size of the included angle between the 2 known sides.

11 (3 marks) CF A farmer plans to fence off a triangular paddock. The sides of the paddock are 42 m, 30 m and 25 m. What is the area of the paddock?

Video
The sine and cosine rules

7.4 Problems involving trigonometry

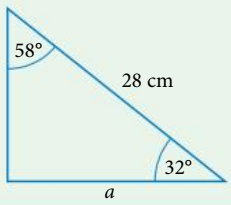
Solving problems with trigonometry

- Look for solutions using right-angled triangles first.
- Use the sine rule for problems involving 2 sides and 2 angles.
  - Be careful of ambiguous cases.
- Use the cosine rule for problems involving 2 sides and the included angle or 3 sides.
- If the known side is in a different triangle from an unknown side, finding the side common to the triangles is often the intermediate step.
- Use a combination of the sine and cosine rule in the one problem.
- Make sure larger/smaller side lengths are opposite the larger/smaller angles.

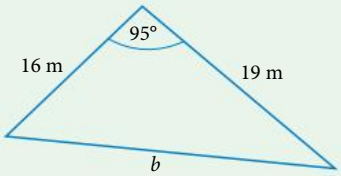
WORKED EXAMPLE 8 Recognising triangle rules

Find the value of the variable in each diagram. Round answers to one decimal place.

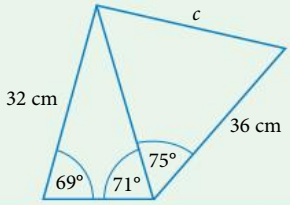
a



b



c



Steps	Working
a 1 Find the unknown angle.	unknown angle $= 180° - (58° + 32°)$ $= 90°$
2 Find the unknown side a using a right-angled triangle method.	$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
3 Substitute, rearrange and evaluate.	$\cos(32°) = \frac{a}{28}$ $a = 28\cos(32°)$ $a = 23.745\ldots$
4 Round and state the answer.	$a \approx 23.7$ cm
b 1 Find the unknown side using the cosine rule.	$b^2 = a^2 + c^2 - 2ac\cos(B)$
2 Substitute, rearrange and evaluate.	$b^2 = 16^2 + 19^2 - 2 \times 16 \times 19 \times \cos(95°)$ $= 669.990\ldots$ $b = 25.884\ldots$
3 Round and state the answer.	$b \approx 25.9$ m
c 1 Find the common side between the 2 triangles using the sine rule.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$
2 Let x be the common side. Substitute, rearrange and evaluate.	$\frac{x}{\sin(69°)} = \frac{32}{\sin(71°)}$ $x = \frac{32\sin(69°)}{\sin(71°)}$ $= 31.595\ldots$

3 Find c using the cosine rule.

$c^2 = x^2 + 36^2 - 2 \times x \times 36 \times \cos(75°)$
$= 1705.515\ldots$
$c = \sqrt{1705.515\ldots}$
$= 41.297\ldots$

> When using the cosine rule in a second step, evaluate the first answer, and keep the 'exact' answer on your calculator.

4 Round and state the answer.

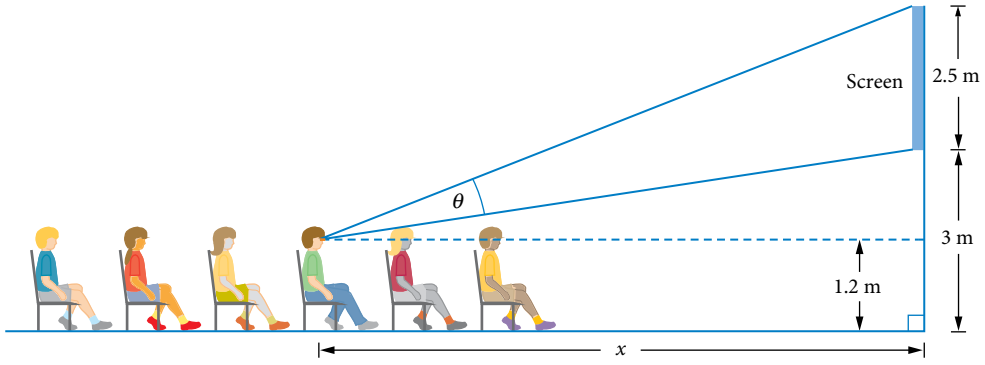
$c \approx 41.3$ cm

Investigation | Viewing distances and angles

The set up for a presentation in a school hall is shown.

The screen is 2.5 m high and mounted so that the bottom of the screen is 3 m from the floor.

Eye level for the audience is approximately 120 cm.



Determine the distance x (to the nearest 10 cm) that will give the viewer the maximum viewing angle θ. Use a table of values like the one below, or a spreadsheet, to find the optimal distance.

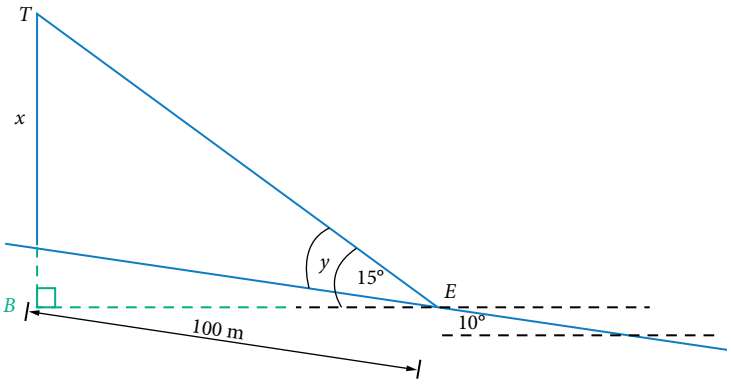
x (m)	1	2	3	4	5	6	7	8
θ								

Cinemas have a sloping floor to make it easier for viewers to see the screen. Using the same dimensions for the screen, adjust the seating so that the floor slopes up towards the back. Investigate the changes of angle with flat and sloping floors of different angles.

WORKED EXAMPLE 9 | Solving 2D triangles

A vertical flagpole stands on the side of a slope with a 10° incline. At a point 100 m down the slope from the base of the flagpole, the angle of elevation to the top of the flagpole is 15°.

Determine the height of the flagpole.

Steps	Working
1 Draw a sketch of the situation, labelling all important points. Identify and label the required side x and the opposite angle y. Label the angle opposite the 100 m, T. Label B.	
2 Find the unknown angle, y.	$y = 15° - 10°$ $= 5°$
3 Find the unknown angle, T.	$T = 180° - (90° + 15°)$ $= 75°$

4 Find x using the sine rule. $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)}$

5 Substitute, rearrange and evaluate.

$$\frac{x}{\sin(5°)} = \frac{100}{\sin(75°)}$$

$$x = \frac{100\sin(5°)}{\sin(75°)}$$

$$= 9.023\ldots$$

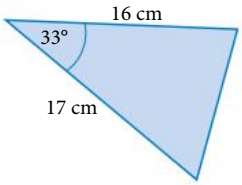
6 Round and state the answer. The flagpole is about 9 m tall.

EXERCISE 7.4 Problems involving trigonometry

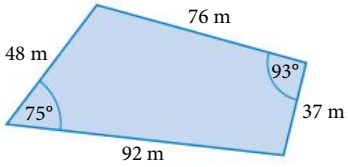
ANSWERS p. 409

Recap

1 What is the area of the triangle?



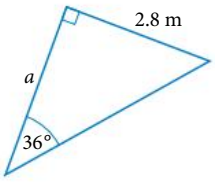
2 What is the area of this quadrilateral?



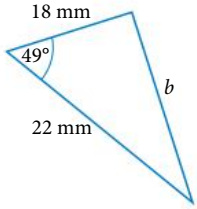
Mastery

3 WORKED EXAMPLE 8 Find the value of the variable in each triangle.

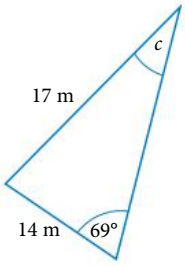
a



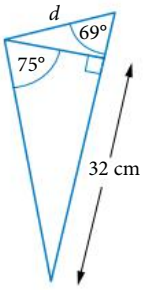
b



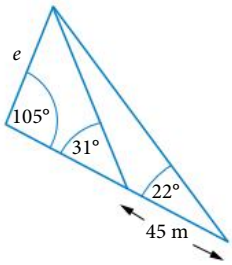
c



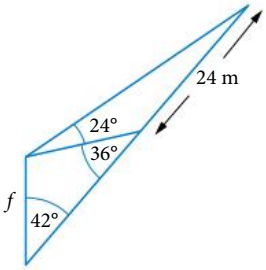
d



e



f



4 WORKED EXAMPLE 9 A building in the city is 85 m tall. The angle of depression from the top of this building to a fire alarm on the face of a building on the opposite side of the street is 58°20′. The street is 15 m wide. Calculate the height of the fire alarm above the ground and the angle of elevation from the base of the first building to the fire alarm.

5 From the cockpit of a plane cruising 1520 m above the ground, the angles of depression to the tops of 2 identical 20 m high structures directly ahead are 50° and 70° respectively. How far apart are the structures?

6 An observer sees that the angle of elevation to the top of a tower, 80 m high and due south of their position, is 37°. The top of another tower, 60 m high and due west of their position, is at an angle of elevation of 28°. Calculate the horizontal distance between the towers.

7 Arnie is jogging along a straight road running east–west from A to B, points that are 10 km apart, when he sees a church in the distance. The bearing from A to the church is N 48° E and from B the church is at a bearing of N 67° W. Find the distance from A to the church.

8 An aircraft flies on a course of 322°T for 250 km and then changes course to 244°T to an airstrip, where it lands. If the airstrip is 480 km from the starting position, calculate the distance travelled from the time the plane changed course until it landed, and the bearing from the airstrip to the starting position.

Exam practice

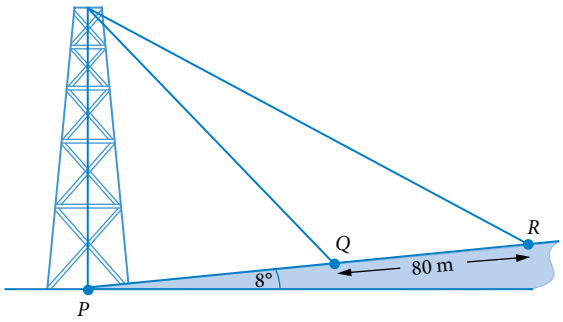
9 (3 marks) Two points, A and B, on the same bank of a river are 120 m apart. Point C is on the opposite bank. $\angle CAB$ and $\angle CBA$ are measured to be 55° and 33° respectively. Find the width of the river.

10 (3 marks) CF A coastal surveillance vessel detects 2 suspect boats on the radar screen. The first is 60 km away from the surveillance vessel on a bearing of 062.2°T. The second is 42 km away on a bearing of 296.58°T. How far apart are the 2 suspect boats?

11 (4 marks) CF The top of a hill is at an angle of elevation of 7°30′. At 100 m closer to the hill on a level surface, the angle of elevation is 8°. What is the height of the hill?

12 (4 marks) CU A tower is erected at the base of a road that is inclined at a constant angle of 8° to the horizontal. Angles of elevation to the top of the tower are measured at 2 points that are 80 m apart, some distance up the slope away from the tower. The angles are found to be 25° and 8° respectively.

Find the height of the tower.



Worksheets
Sine and cosine rules assignment

Sine and cosine rule problems

7.5 3D problems involving trigonometry

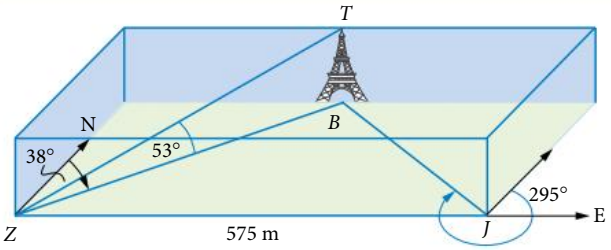
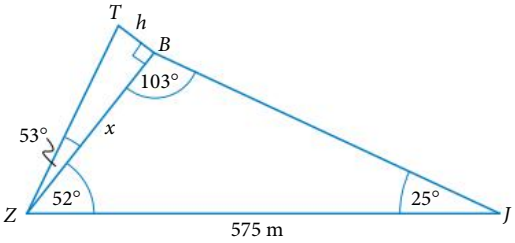
Solving 3D triangle problems

- Sketch the situation, labelling important points. It can be helpful to draw the 3D sketch inside a box.
- Translate (unfold) the sketch into 2D form, labelling all important points.
- Identify and label unknowns and intermediate sides and angles.
- Look for solutions for unknowns using right-angled triangle rules, and the sine and cosine rules.
- Write the solution to the problem in an appropriate form.

WORKED EXAMPLE 10 3D problems 1

Zack observes the top of the Eiffel Tower on a bearing of 038°T and at an angle of elevation of 53°. Jill is 575 m due east of Zack, and she determines the bearing of the Eiffel Tower to be 295°T.

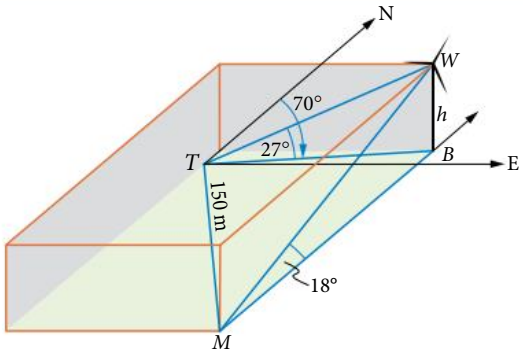
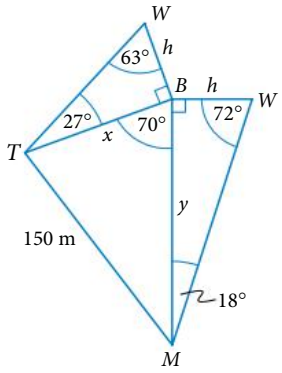
Determine the height of the Eiffel tower.

Steps	Working
1 Sketch the situation – drawing it in a box can help with 3D perspective. Label all important points including the vertices of the triangles.	
2 Find unknown angles in $\triangle ZBJ$.	$\angle JZB = 52°$ (complementary) $\angle BJZ = 295° - 270° = 25°$ $\angle ZBJ = 103°$ (sum of angles in $\triangle ZBJ$)
3 Create a 2D sketch by unfolding the diagram to get 2 flat triangles joined along ZB.	
4 Use the sine rule in $\triangle ZBJ$ to find a statement for x.	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$ $\frac{x}{\sin(25°)} = \frac{575}{\sin(103°)}$ $x = \frac{575\sin(25°)}{\sin(103°)}$
5 Use $\triangle ZBT$ to find h.	$\tan(53°) = \frac{h}{x}$ $h = x\tan(53°)$
6 Substitute x and evaluate.	$= \frac{575\sin(25°)\tan(53°)}{\sin(103°)}$ $= 330.961\ldots$
7 Round and state the answer.	The Eiffel Tower is approximately 331 m tall.

WORKED EXAMPLE 11 3D problems 2

Maria and Tom are 150 m apart on a straight level road. A wind turbine is due north of Maria and on a bearing of 070°T from Tom. The top of the turbine (excluding the blades) appears at an angle of elevation of 27° to Tom and 18° to Maria.

Find the height of the wind turbine above the road.

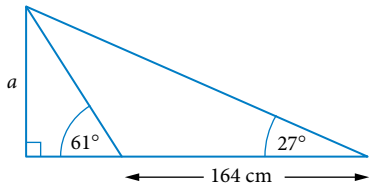
Steps	Working
1 Sketch the situation. Drawing it in a box can help with 3D perspective. Label all important points including the vertices of the triangles.	
2 Find unknown angle in $\triangle TBM$, $\triangle WBM$ and $\triangle TWB$.	$\angle TBM = 70°$ (alternate angle to $\angle NTB$) $\angle BWM = 180° - 90° - 18° = 72°$ $\angle TWB = 180° - 90° - 27° = 63°$
3 Create a 2D sketch by unfolding the diagram, showing WB on the 2 flat triangles joined to $\triangle TBM$.	
4 Find x in $\triangle WBT$.	$\tan(63°) = \frac{x}{h}$ $x = h\tan(63°)$
5 Find y in $\triangle WBM$.	$\tan(72°) = \frac{y}{h}$ $y = h\tan(72°)$
6 Use the cosine rule in $\triangle TBM$. Substitute x and y.	$c^2 = a^2 + b^2 - 2ab\cos(C)$ $150^2 = x^2 + y^2 - 2xy\cos(70°)$ $150^2 = [h\tan(63°)]^2 + [h\tan(72°)]^2 - 2[h\tan(63°)] \times [h\tan(72°)]\cos(70°)$
7 Simplify by factorising.	$= h^2\tan^2(63°) + h^2\tan^2(72°) - 2h^2\tan(63°)\tan(72°)\cos(70°)$ $= h^2[\tan^2(63°) + \tan^2(72°) - 2\tan(63°)\tan(72°)\cos(70°)]$
8 Rearrange and evaluate.	$150^2 = h^2 \times 9.192...$ $h = \frac{150}{\sqrt{9.192}}$ $= 49.474...$
9 Round and state the answer.	The height of the wind turbine is approximately 49 m.

EXERCISE 7.5 3D problems involving trigonometry

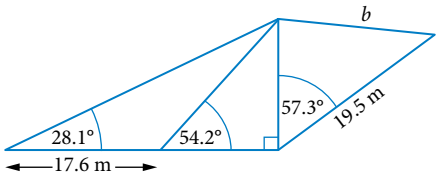
ANSWERS p. 409

Recap

1 Find the value of a.

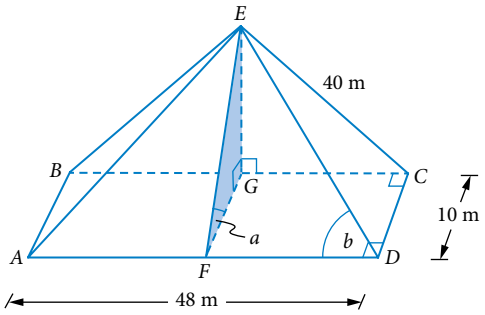


2 What is the length of b in this diagram?



Mastery

3 WORKED EXAMPLES 10, 11 A tent is constructed in the shape of a rectangular-based pyramid $ABCDE$ with the vertex E vertically above G, the midpoint of BC, as shown in the diagram.



a Calculate the angle that the triangular face AED makes with the base of the pyramid.

b Find the angle between the lines DE and AD.

4 From a lighthouse 150 m above sea level, the lighthouse keeper observes a boat due east at an angle of depression of 25° and another boat due south at an angle of depression of 32°.
Find the bearing and distance of the second boat from the first boat.

5 An aircraft flying due east at 385 km h^{-1} at a constant height of 1500 m passes over a village at a given instant. One minute later, a person who is 1.6 km south of the village sees the plane.
In what direction and at what angle of elevation does the person see the plane?

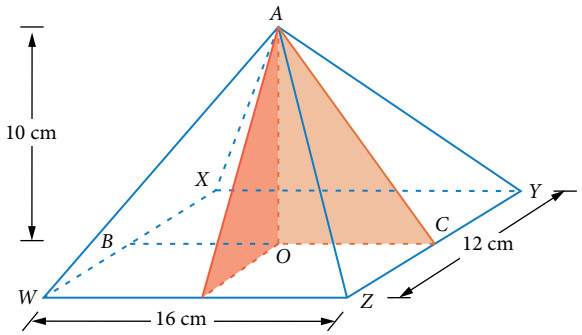
6 A straight stick 2 m long is stuck in the ground and is inclined at 60° to the horizontal, pointing in an easterly direction. Calculate the length and direction of the shadow of the stick when the sun is due north at an elevation of 30°.

7 Miguel is 90 m due west of Consuella on a level road. He sees that the angle of elevation of a kite flown by their son is 30°, while Consuella sees that it is due north at an angle of elevation of 38°.
What is the height of the kite above the ground?

9780170484985

Exam practice

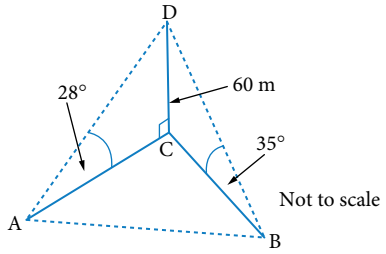
8 (4 marks) CF A rectangular-based pyramid with a base measuring 16 cm by 12 cm and a height of 10 cm is to be constructed using Perspex sheeting.



Calculate the angle that each of the triangular faces *AYZ* and *AZW* makes with the horizontal base.

9 (4 marks) CF Two groups of people set off to spotlight possums at night. One group spots a possum. The tree that has the possum is at a bearing of N 28° W. The angle of elevation of their spotlight beam is 38°. At this instant, the other group, which is 100 m west of the first group, sights the same tree at a bearing of N 29° E. How high up the tree is the possum?

10 © QCAA 2021 2Q16 (4 marks) CF In the diagram DC represents a 60-metre vertical tower. A and B are two points in the same horizontal plane as the foot C of the tower. The angle above the horizontal from A to D is 28° and the angle above the horizontal from B to D is 35°.



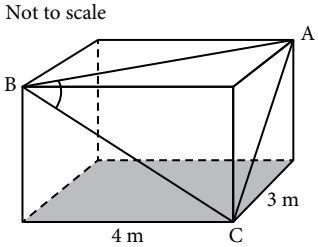
The bearing of C from A is 050°T and the bearing of C from B is 300°T.
Determine the distance between A and B to the nearest metre.

11 (5 marks) CU A hillside is considered to have a uniform slope of 22° with the horizontal. A hiker, starting from point *O*, walks due north up the hillside for 120 m to point *C*, then walks due west to point *B*. The hiker then walks 170 m back to the starting point. Calculate the hiker's angle of descent.

12 (5 marks) CU A surveyor notes that the top of a water tower is at a bearing of 062°T. After walking 200 m directly east, she finds that the tower is at a bearing of 059°10′T. The angle of elevation of the top of the tower is then 4°31′. What is the height of the tower?

13 © QCAA 2024 1Q18 TECH-FREE (5 marks) CU The diagram shows some dimensions of a large storage container that is a rectangular prism. The angle ABC is 60°.

A person requires a container that is at least 4 metres in height.



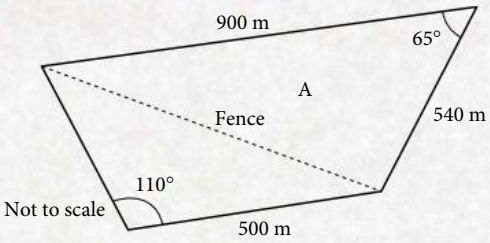
Make a justified decision about whether this storage container meets the person's requirements.

Video
Exam question analysis: The sine and cosine rules

EXAM QUESTION ANALYSIS

© QCAA 2023 2Q14 (7 marks)

A fence divides a paddock into two triangular sections as shown.



a Determine the length of the fence. *[1 mark]*

b Calculate the area of triangular section A. *[1 mark]*

c Determine the total area of the paddock. *[5 marks]*

Reading the question

- There are two triangles with a connected side (fence).
- Triangle with A, has two sides and the included angle.
- Other triangle has one angle and one side.

Thinking about the question

- Find the length of the fence using cosine rule.
- Find area of triangular section *A*.
- Find the length of the unknown side of the paddock.
- Find second area and then the total area.

Worked solution (✓ = 1 mark)

a Use the cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

$$c = \sqrt{900^2 + 540^2 - 2\times 900\times 540\times \cos(65°)}$$

$$= \mathbf{831.1528\,m}\ ✓$$

b $A = \frac{1}{2}ab\sin(C)$

$$A = \frac{1}{2}\times 900\times 540\times \sin(65°)$$

$$= \mathbf{220\,232.7922\,m^2}\ ✓$$

c

Method 1

Use the sine rule to find the angle opposite the 500 m side of the second triangle.

$$\frac{\sin(D)}{500} = \frac{\sin(110)}{831.1528}$$ ✓

$$\sin(D) = 500 \times \frac{\sin(110)}{831.1528}$$

$$D = \sin^{-1}\left(500 \times \frac{\sin(110)}{831.1528}\right)$$

$$= 34.4228°$$

Determine the remaining angle in the triangle.

$180 - (110 + 34.4228)$

$= \mathbf{35.5772°}$ ✓

Determine the area of the 110° triangle.

$$A = \frac{1}{2} \times 831.1528 \times 500 \times \sin(35.5772)$$

$= \mathbf{120\,891.0424\ m^2}$ ✓

Determine the total area.

total area = 220 232.7922 + 120 891.0424

$= \mathbf{341\,123.8346\ m^2}$ ✓

Method 2

Use the cosine rule to find the missing side length of the second triangle.

$c^2 = a^2 + b^2 - 2ab\cos(C)$

$\mathbf{831.1528^2 = a^2 + 500^2 - 2 \times a \times 500 \times \cos(110)}$ ✓

$a = \mathbf{514.599\ m}$ ✓

Determine the area of the 110° triangle:

$$A = \frac{1}{2} \times 500 \times 514.599 \times \sin(110)$$

$= \mathbf{120\,891.2207\ m^2}$ ✓

Determine the total area:

total area = 220 232.7922 + 120 891.2207

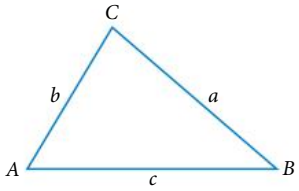
$= \mathbf{341\,124.0129\ m^2}$ ✓

✓ for logical organisation communicating key steps in both methods

7 Chapter summary

The sine rule

- In any triangle ABC

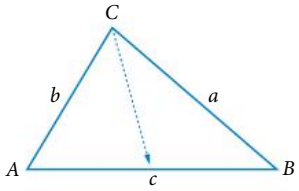


A, B, C are the angles and a, b, c are the sides opposite each angle respectively.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The cosine rule

- In any triangle ABC



$c^2 = a^2 + b^2 - 2ab\cos(C)$ ← Use to find an unknown side

$\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$ ← Use to find an unknown angle

The area of a triangle

- Area of $\triangle ABC = \frac{1}{2}bc\sin(A)$, where A is the included angle between sides b and c.

Solving 2D and 3D triangle problems

- Sketch the situation, labelling important points. It can be helpful to draw the 3D sketch inside a box.
- If the situation is 3D, translate (unfold) the sketch into 2D form, labelling all important points.
- Identify and label unknowns and intermediate sides and angles.
- Look for solutions for unknowns using right-angled triangle rules, and the sine and cosine rules. When using the sine rule, be careful of the ambiguous case.
- If the known side is in a different triangle from an unknown side, finding the side common to the triangles is often the intermediate step.
- Write the solution to the problem in an appropriate form.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 In triangle PQR, the sides p, q and r are known.
Which formula finds the unknown angle, Q?

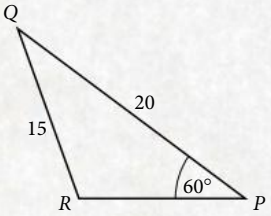
A $q^2 = p^2 - r^2 - 2pr\cos(Q)$

B $\sin(Q) = \dfrac{p^2 + r^2 - q^2}{2pr}$

C $q^2 = p^2 + r^2 - 2pr\sin(Q)$

D $\cos(Q) = \dfrac{r^2 + p^2 - q^2}{2pr}$

2 Which of the following expressions is equal to $\sin(R)$?



A $\dfrac{15}{20\sin(60°)}$

B $\dfrac{15\sin(60°)}{20}$

C $\dfrac{20\sin(60°)}{15}$

D $\dfrac{20}{15\sin(60°)}$

3 © QCAA 2023 1Q9 Determine $\int_0^3 \pi \sin\left(\dfrac{\pi}{3}x\right) dx$.

A 3

B 6

C –3

D –6

Section 2 3 short response questions

15 marks

4 © QCAA 2023 1Q11 (4 marks) Two random samples (A and B) were obtained using two different Bernoulli experiments. Each Bernoulli trial in the random samples was recorded as 1 (for success) or 0 (for failure). The results are shown.

A	1	1	1	1	0	1	1	0	1	1
B	0	0	1	1	1	0	1	1	0	0

In sample A, for each trial the mean is 0.8 and the variance is 0.16.

a Use the sample B results to determine the mean and variance for each trial in sample B. *[2 marks]*

b Compare the variability about the means of samples A and B. *[2 marks]*

5 © QCAA 2021 1Q15 (4 marks) In the isosceles triangle ABC, angle C is 120° and side a is 4 cm.

a Draw the triangle, indicating all given information. *[1 mark]*

b Calculate the area of the triangle in cm^2. (Give your answer in simplest form.) *[3 marks]*

6 © QCAA 2022 1Q19 (7 marks) Two triangles are said to be similar if their corresponding angles are congruent and the corresponding sides are in proportion, e.g. if $\triangle UVW$ is similar to $\triangle XYZ$ then

$$\angle U = \angle X,\ \angle V = \angle Y \text{ and } \angle W = \angle Z \text{ and } \frac{UV}{XY} = \frac{VW}{YZ} = \frac{UW}{XZ}$$

Two parallel walls AB and CD, where the northern ends are A and C respectively, are joined by a fence from B to C. The wall AB is 20 metres long, the angle $ABC = 30°$ and the fence BC is 10 metres long.

A new fence is being built from A to a point P somewhere along CD. The new fence AP will cross the original fence BC at O.

Let $OB = x$ metres, where $0 < x \leq 10$.

Determine the value of x that minimises the total area enclosed by $\triangle OBA$ and $\triangle OCP$.
Verify that this total area is a minimum.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 3 short response questions

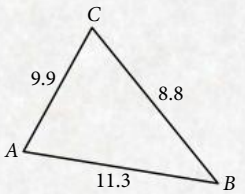
Total: 18 marks

Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

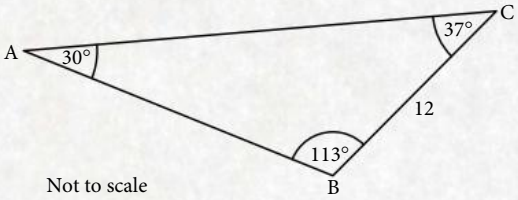
1 © QCAA 2020 2Q8 Determine the size of angle A in the triangle.



Not drawn to scale

A 48.5° **B** 61.4° **C** 118.6° **D** 131.5°

2 © QCAA 2022 2Q9 Determine the length of side AB in triangle ABC.



Not to scale

A 22.13 **B** 14.44 **C** 9.97 **D** 7.82

3 © QCAA 2022 2Q1 The position (in cm) of a particle is given by $x = \cos(4t)$, where t is time in seconds. The velocity of the particle when $t = 5$ is

A $1.6323\,\text{cm s}^{-1}$ **B** $0.4081\,\text{cm s}^{-1}$ **C** $-0.9129\,\text{cm s}^{-1}$ **D** $-3.6518\,\text{cm s}^{-1}$

Section 2 3 short response questions

15 marks

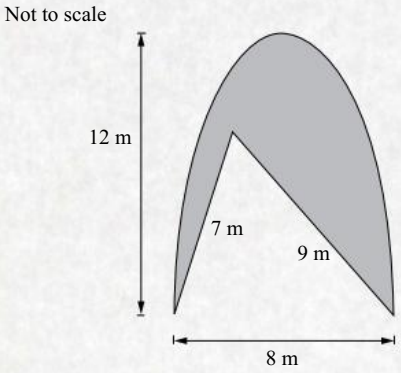
4 © QCAA 2022 2Q15 (7 marks) A hiker begins her journey at a youth hostel (H) and walks for 8 km on a bearing of 052°T to her lunch stop (L). She then walks on a bearing of 210°T for 5.2 km until she reaches a campsite (C).

Determine the direction she would need to walk in a straight line to return directly to the youth hostel.

5 © QCAA 2021 2Q18 MODIFIED (3 marks) The number of animals in a population (in thousands) is modelled by the function P such that $P(t) = \dfrac{300}{1+2e^{-3t}}$, where t is in years.

Determine the number of animals in the population when the population is growing the fastest.

6 © QCAA 2023 2Q18 (5 marks) A company makes windows using glass that has a mass of 5.6 kg per square metre. A customer orders an unusual window in a partial parabolic shape, as shown.



Determine the mass of the window.

CHAPTER

8 CONTINUOUS RANDOM VARIABLES AND THE NORMAL DISTRIBUTION



Syllabus coverage

Nelson MindTap chapter resources

Terminology

8.1 Continuous random variables

8.2 Probability density functions
- Cumulative distribution functions
- Types of continuous random distributions

8.3 The mean and variance of continuous random variables

8.4 Normal distributions

8.5 *z*-scores

8.6 Quantiles and probabilities

8.7 Problems involving normal distributions

Exam question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Prior learning
Continuous random variables and the normal distribution

Syllabus coverage

UNIT 4, TOPIC 3: CONTINUOUS RANDOM VARIABLES AND THE NORMAL DISTRIBUTION

General continuous random variables

- Use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable.
- Understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts.
- Calculate the expected value, $E(X) = \mu = \int_{-\infty}^{\infty} xp(x)\,dx$, of a continuous random variable where $p(x)$ is the probability density function.
- Calculate the variance, $Var(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\,dx$, and standard deviation σ, of a continuous random variable.
- Understand standardised normal variables (z-values, z-scores) and use these to compare samples.

Normal distributions

- Identify contexts, e.g. naturally occurring variations, that are suitable for modelling by normal random variables.
- Recognise features of the graph of the probability density function of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution.
- Recognise and use the link between the normal distribution and the notation $X \sim N(\mu, \sigma^2)$.
- Calculate probabilities and quantiles associated with a given normal distribution, using technology.
- Model and solve problems that involve normal distributions, with and without technology (distribution tables are not required).

Mathematical Methods 2025 v1.2 General senior syllabus, p. 29
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Videos (4):

8.2 Probability density functions
• Cumulative distribution functions

8.5 z-scores

Exam question analysis Continuous random variables and the normal distribution

Prior learning (1):

8.1 Continuous random variables and the normal distribution

Worksheets (13):

8.1 Discrete and continuous random variables

8.2 Probability density function • Uniform and triangular probability distribution functions

8.4 The normal distribution
• The normal curve

8.5 The standard normal curve
• Areas under the normal curve • z-scores
• Applying the normal distribution

8.7 Normal distribution worded problems 1
• Normal distribution worded problems 2
• Continuous random variables assignment
• Continuous random variables extension

Cumulative exams Maths Methods formula sheet

TI-Nspire CX II instructions (1):

8.3, 8.4, 8.5, 8.6 Continuous random variables and the normal distribution

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

closed interval	continuous random variable	cumulative distribution function (cdf)
inverse cumulative distribution function	normal distribution	open interval
probability density function (pdf)	quantile	standard normal distribution
standard score	triangular distribution	uniform distribution
variance	z-score (z-value)	

8.1 Continuous random variables

Chapter 4, *Discrete random variables and the binomial distribution,* established that a discrete random variable is a variable whose value depends on the outcome of a probability experiment.

A discrete random variable can take on discrete values, such as the number of rainy days in a week, while a continuous random variable can take on a continuous range of values.

For example, if X represents the length of time (in seconds) of a video, then X is a continuous random variable.

Continuous random variables

A **continuous random variable** is a random variable with values from an interval, which could be all the real numbers. A **continuous probability distribution (or function)** shows the probabilities of a continuous random variable. The values of the **probability function** are written as $P(X = x)$ or $p(x)$.

A random variable is represented by a capital letter, while the corresponding lowercase letter is used to represent a specific value of the variable.

For the random variable X, the values of the **probability function** are written as $P(X = x)$ or $p(x)$.

Worksheet
Discrete and continuous random variables

A continuous random variable is usually shown in class groups. When you state someone's height as 172 cm, you are implying a grouping of class width 1 cm from 171.5 to 172.5 cm.

Probability histograms of grouped data are like those of discrete probability distributions. The vertical axis is adjusted to reflect the class width, ensuring that the height matches the assumed probability for discrete values within each class. Probability histograms of continuous variables must have a total area of 1, regardless of how the data is grouped.

WORKED EXAMPLE 1 Probability histograms

The heights of a group of 17-year-olds, measured to the nearest centimetre, are shown.

162	161	158	169	161	180	166	159	171	163
166	163	181	170	178	175	164	169	174	160

a **Construct** a probability histogram of the height, H, with a class width of 2 cm.

b **Construct** a probability histogram of H with a class width of 10 cm.

c Find the total area of each histogram.

Steps

a **1** Tabulate the data with a class width of 2 cm. Since each column has a class width of 2 cm, the relative frequency must be divided by 2 to get the correct column height.

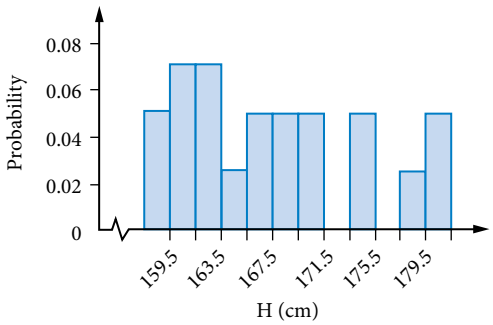
Thus for the class 165.5–167.5, the value 0.05 is the assumed probability of an individual score, so $P(H = 166) = 0.05$.

Working

H (cm)	f	Relative frequency	Column height
157.5–159.5	2	0.1	0.05
159.5–161.5	3	0.15	0.075
161.5–163.5	3	0.15	0.075
163.5–165.5	1	0.05	0.025
165.5–167.5	2	0.1	0.05
167.5–169.5	2	0.1	0.05
169.5–171.5	2	0.1	0.05
171.5–173.5	0	0	0
173.5–175.5	2	0.1	0.05

H (cm)	f	Relative frequency	Column height
175.5–177.5	0	0	0
177.5–179.5	1	0.05	0.025
179.5–181.5	2	0.1	0.05
Total	20	1	

2 Draw the probability histogram.



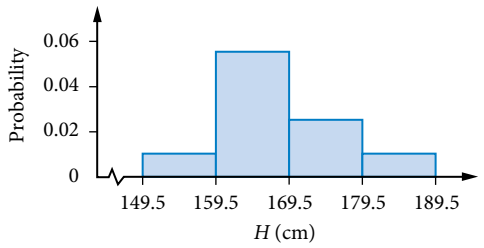
b 1 Retabulate the data with a class width of 10 cm.

Since each column has a class width of 10 cm, the relative frequency must be divided by 10 to get the correct column height.

In this case, the assumed probability of the height of 166 cm is $P(H = 166) = 0.055$.

H (cm)	f	Relative frequency	Column height
149.5–159.5	2	0.1	0.01
159.5–169.5	11	0.55	0.055
169.5–179.5	5	0.25	0.025
179.5–189.5	2	0.1	0.01
Total	20	1	

2 Draw the histogram.



c 1 Calculate the total area for the first histogram.

$\text{Area}_1 = 0.05 \times 2 + 0.075 \times 2 + \ldots + 0.05 \times 2$
$= 1$

2 Calculate the total area for the second histogram.

$\text{Area}_2 = 0.01 \times 10 + 0.055 \times 10 + 0.025 \times 10 + 0.01 \times 10$
$= 1$

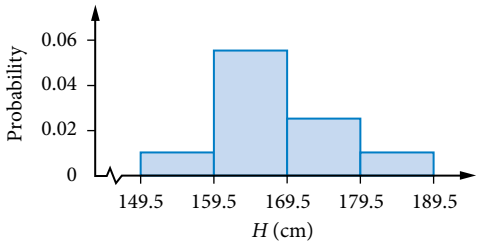
3 State the result.

The area under each histogram is 1 and represents the sum of the probabilities of all outcomes.

The area over an interval can be used to estimate the probability of a score lying in that interval, just as it does for discrete probabilities. For part **a** of the previous example, the midpoints of the classes are 158.5, 160.5, 162.5, … You should notice that the height of the graph at 162.5, for example, is not the probability of getting a score of 162.5.

WORKED EXAMPLE 2 Finding probabilities

Use the histogram in Worked example 1 part **b** to find the probability of choosing a person at random from the group with a height between 161 cm and 164 cm.

Steps	Working
1 Identify where the required heights occur.	Heights between 161 cm and 164 cm occur in the class 159.5–169.5.
2 The required probability is the area of the rectangle with the height of the class 159.5–169.5 and the width between 161 cm and 164 cm.	
3 State the height of the rectangle.	height = 0.055
State the width of the rectangle.	width = 164 − 161 = 3 cm
Calculate the required probability and evaluate.	$P(161 < h < 164) = 0.055 \times 3$ $= 0.165$

EXERCISE 8.1 Continuous random variables

ANSWERS p. 409

Mastery

1 WORKED EXAMPLE 1 The IQs (intelligence quotient test scores) of some Year 12 students are shown.

90	111	89	123	137	97	120	101	101	101	110	108	101	85	103
121	92	130	109	112	121	139	100	83	141	82	94	94	119	104
95	87	108	104	101	115	106	93	90	128	107	110	98	138	

(*Note*: IQ is standardised, so the population has an average IQ of 100.)

a Considering IQ as a continuous variable, construct a probability histogram of intelligence, I, with class widths of 10.

b Find the total area of the histogram.

2 Some students checked their watches against the school clock and found that they were out by the following amounts (in seconds).

26	37	21	45	21	32	23	34	17	46	21	44	27	38	30	36	29	66	22	34
30	42	17	43	29	33	30	41	18	61	26	49	55	68	27	58	18	52	28	32

a Construct probability histograms of the times, T, with class widths of 4 s and 10 s, starting at 15.5 and 9.5 respectively.

b Find the total area of each histogram.

3 WORKED EXAMPLE 2 Use the probability histogram from question **1** to find the probability of choosing a person at random from the group with an IQ from 115 to 125 (inclusive).

4 Use the 10 s class-width histogram constructed for question **2** to find the probability of choosing a person at random whose watch was from 24 to 32 s out.

5 The weights of a group of 17-year-olds, measured to the nearest kilogram, are listed.

63 58 66 82 88 84 70 67 104 66 67 90 70 55 66 98 49 72 76 46

a Construct probability histograms of the weight, W, with class widths of 5 kg and 10 kg.

b Find the total area of each histogram.

c Use the 10 kg class histogram to find the probability of choosing a person at random from the group with a weight between 64 kg and 76 kg.

Exam practice

6 The heights of a group of 16-year-olds, in centimetres, are given.

150 155 160 165 170 175 180 185 190 195 200 205

Using a class width of 10 cm, find the probability of choosing a person at random from the group with a height between 165 cm and 180 cm.

A $\frac{1}{6}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$

7 The ages, measured in years, of participants in a marathon were found to be:

21 26 23 38 32 29 41 35 24 31 28 37

Using a class width of 5 years, find the probability of choosing a participant at random with an age between 25 years and 36 years.

A $\frac{1}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{4}$ **D** $\frac{1}{6}$

8 The scores of students in a math test out of 100 are as follows:

56 72 68 85 90 75 80 62 74 77 69 88

Using a class width of 10 points, find the probability of selecting a student at random with a score between 70 and 84.

A $\frac{1}{4}$ **B** $\frac{1}{3}$ **C** $\frac{5}{12}$ **D** $\frac{1}{2}$

9 (5 marks) The times recorded for a group of students running a 100 m sprint, measured to the nearest second, are shown.

17	14	20	14	15	14	16	18	19	24	12	15	20
20	18	22	20	19	18	20	15	23	18	16	16	13

a Construct probability histograms of the times, T, with class widths of 2 s and 5 s. *[3 marks]*

b Find the total area of each histogram. *[1 mark]*

c Use the 5 s class histogram to find the probability of choosing a person at random from the group with a time between 13 s and 18 s. *[1 mark]*

10 (6 marks) CF The heights (in cm) of some Year 12 students are recorded.

150	162	155	170	175	149	160	158	164	172	150
161	159	148	165	160	157	171	174	178	180	145
152	159	163	151	168	169	153	154	166	176	177
174	172	165	162	158	149	150	175	173	178	

a Considering height as a continuous variable, construct a probability histogram of heights, H, with class widths of 5 cm. *[3 marks]*

b Find the total area of the histogram. *[1 mark]*

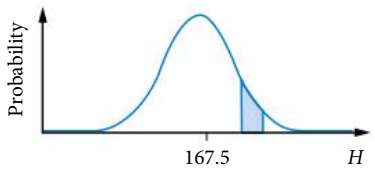
c Find the probability of choosing a person at random from the group with a height between 160 cm and 170 cm (inclusive). *[2 marks]*

8.2 Probability density functions

Worksheets
Probability density function
Uniform and triangular probability distribution functions

In the previous section, you looked at the heights of a group of 17-year-olds, measured to the nearest centimetre. If you made very exact measurements of a very large number of people's heights and used very small class intervals, you would obtain a probability histogram that looked like a smooth curve.

It would look like the graph shown here. The total area under the graph would be 1, and the area of an interval on the graph would still be the probability of getting a score in that interval. The height of the graph at 166.34, for example, is not the probability of getting a score of 166.34.



In fact, the probability of getting any exact score must be zero. For continuous variables, it only makes sense to find probabilities over an interval. The function that we use to obtain the probabilities of continuous variables using the area has a special name.

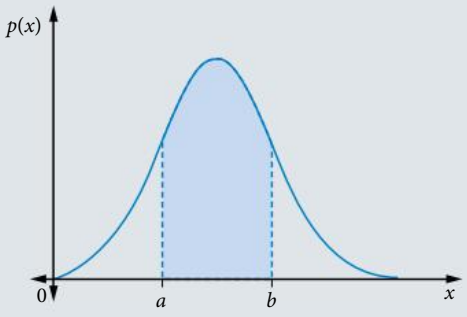
Probability density function

For a continuous random variable X, the **probability density function (pdf)** $p(x)$ is a function whose area over an interval gives the probability of X over that interval; that is,

$$P(a \leq X \leq b) = \int_a^b p(x)\,dx$$

The probability density function must satisfy the following conditions:

- $p(x) \geq 0$ for all $x \in [a, b]$
- $\int_a^b p(x)\,dx = 1$



It is important to define the interval over which a variable is being considered.

Closed and open intervals

The **closed interval** $a \leq x \leq b$ is written $[a, b]$.

The **open interval** $a < x < b$ is written (a, b).

The interval $a \leq x < b$ is written $[a, b)$, and the interval $a < x \leq b$ is written $(a, b]$.



Video
Probability density functions

WORKED EXAMPLE 3 Probability density functions

Determine whether each function could be a probability density function over the given interval.

a $f(x) = \frac{1}{5}$ for $0 \leq x \leq 5$

b $f(x) = 3x^2$ for the interval $[0, 1]$

c $f(x) = 3e^{-x}$ for the interval $[0, 2]$

d $f(x) = \frac{1}{2}\sin(x)$ for $0 \leq x \leq \pi$

Steps	Working
a 1 Check that $f(x) \geq 0$.	$\frac{1}{5} \geq 0$ for $0 \leq x \leq 5$
2 Calculate $\int_0^5 f(x)\,dx$.	$\int_0^5 \frac{1}{5}\,dx = \left[\frac{x}{5}\right]_0^5$ $= \frac{5}{5} - \frac{0}{5}$ $= 1$
3 Write the conclusion.	Since $f(x) \geq 0$ and $\int_0^5 f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq 5$.
b 1 Check that $f(x) \geq 0$.	$3x^2 \geq 0$ for $0 \leq x \leq 1$
2 Calculate $\int_0^1 f(x)\,dx$.	$\int_0^1 3x^2\,dx = [x^3]_0^1$ $= 1^3 - 0^3$ $= 1$
3 Write the conclusion.	Since $f(x) \geq 0$ and $\int_0^1 f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq 1$.
c 1 Check that $f(x) \geq 0$.	Since $e^{-x} \geq 0$ for all $x \in R$, $3e^{-x} \geq 0$ for $0 \leq x \leq 2$.
2 Calculate $\int_0^2 f(x)\,dx$.	$\int_0^2 3e^{-x}\,dx = [-3e^{-x}]_0^2$ $= -3e^{-2} + 3e^{-0}$ $= 2.5939\ldots$
3 Write the conclusion.	Since $\int_0^2 f(x)\,dx \neq 1$, $f(x)$ is not a pdf for $0 \leq x \leq 2$.

d 1	Check that $f(x) \geq 0$.	$\frac{1}{2}\sin(x) \geq 0$ for $0 \leq x \leq \pi$
2	Calculate $\int_0^{\pi} f(x)\,dx$.	$\int_0^{\pi} \frac{1}{2}\sin(x)\,dx = \left[-\frac{1}{2}\cos(x)\right]_0^{\pi}$ $= -\frac{1}{2}\cos(\pi) + \frac{1}{2}\cos(0)$ $= 1$
3	Write the conclusion.	Since $f(x) \geq 0$ and $\int_0^{\pi} f(x)\,dx = 1$, $f(x)$ is a pdf for $0 \leq x \leq \pi$.

You can multiply any positive continuous function by a constant to make a probability density function over an interval.

WORKED EXAMPLE 4 Value at an interval

Find the value of k that makes $f(x)$ a probability density function for the stated interval and state $f(x)$.

a $f(x) = k(x^2 - 1)$ for $0 \leq x \leq 3$

b $f(x) = ke^{-8x}$ for $x > 0$

Steps		Working
a 1	$f(x)$ is a pdf for the interval $0 \leq x \leq 3$, so use the properties of a pdf.	$\int_0^3 f(x)\,dx = 1$
2	Replace $f(x)$.	$\int_0^3 k(x^2 - 1)\,dx = 1$
3	Use the properties of integrals.	$k\int_0^3 (x^2 - 1)\,dx = 1$
4	Calculate the definite integral.	$k\left[\frac{x^3}{3} - x\right]_0^3 = 1$ $k\left[\left(\frac{3^3}{3} - 3\right) - \left(\frac{0^3}{3} - 0\right)\right] = 1$ $6k = 1$
5	Solve for k.	$k = \frac{1}{6}$
6	State the result.	$f(x) = \frac{1}{6}(x^2 - 1)$ for $0 \leq x \leq 3$
b 1	Use the fact that $f(x)$ is a pdf for the interval $x > 0$.	$\int_0^{\infty} f(x)\,dx = 1$
2	Replace $f(x)$.	$\int_0^{\infty} ke^{-8x}\,dx = 1$
3	Use the properties of integrals.	$k\int_0^{\infty} e^{-8x}\,dx = 1$

4 Because the terminal value is ∞, you need to find the limit as $n \to \infty$.

$$k \times \lim_{n\to\infty}\left[-\frac{e^{-8x}}{8}\right]_0^n = 1$$

$$k \times \lim_{n\to\infty}\left(-\frac{e^{-8n}}{8}+\frac{1}{8}\right) = 1$$

$$k \times \lim_{n\to\infty}\left(-\frac{1}{8e^{8n}}+\frac{1}{8}\right) = 1$$

$$k \times \left[\lim_{n\to\infty}\left(-\frac{1}{8e^{8n}}\right)+\frac{1}{8}\right] = 1$$

$$k \times \left(0+\frac{1}{8}\right) = 1$$

$$\frac{k}{8} = 1$$

5 Solve for k.

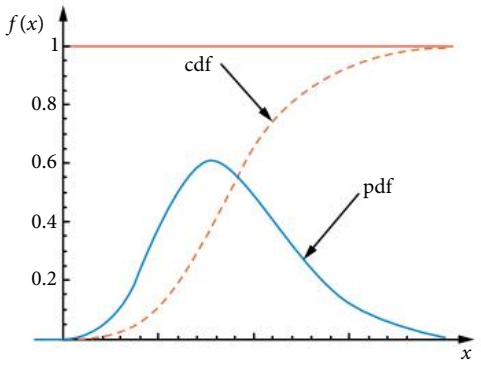
$$k = 8$$

6 State the result.

$$f(x) = 8e^{-8x} \text{ for } x > 0$$

Cumulative distribution functions

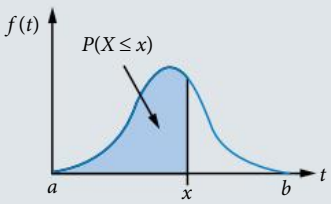
The **cumulative distribution function, cdf,** gives the probability of finding a continuous random variable at a value less than a given value.



Cumulative distribution functions

Given the continuous random variable X, with a probability density function $f(x)$, defined on the interval (a, b), the **cumulative distribution function** (**cdf**) is given by:

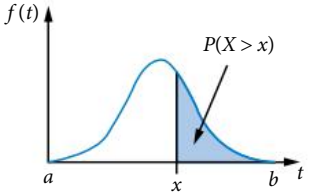
$$F(x) = \int_a^x f(t)dt$$



So, $P(X \le x) = F(x) = \int_a^x f(t)\,dt$, as shown in the diagram.

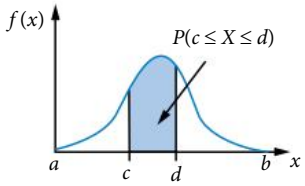
A number of results follow from this definition.

$$P(X > x) = 1 - F(x) = 1 - \int_a^x f(t)\,dt$$



$P(c \le X \le d) = F(d) - F(c)$

$$\int_a^d f(x)\,dx - \int_a^c f(x)\,dx = \int_c^d f(x)\,dx$$



WORKED EXAMPLE 5 Cumulative distribution functions

The probability density function for a continuous random variable X is $f(x) = \frac{1}{x^2}$, for $x \ge 1$.

a Find the cumulative density function for X.

b **Calculate**

i $P(X \le 2)$ **ii** $P(3 \le X \le 6)$ **iii** $P(X > 10)$

Steps	Working
a 1 Write the cdf.	$F(x) = \int_a^x f(t)\,dt$
2 Substitute for $f(x)$.	$F(x) = \int_1^x \frac{1}{t^2}\,dt$
	$= \left[-\frac{1}{t}\right]_1^x$
	$= -\frac{1}{x} + \frac{1}{1}$
3 State the result.	$F(x) = 1 - \frac{1}{x}$ for $x \ge 1$
b i 1 Write the probability using the cdf.	$P(X \le 2) = F(2)$
2 Use the answer from part **a** to find $F(2)$.	$= 1 - \frac{1}{2}$
	$= \frac{1}{2}$
3 State the result.	$P(X \le 2) = \frac{1}{2}$
ii 1 Write the probability using the cdf.	$P(3 \le X \le 6) = F(6) - F(3)$
2 Use the answer from part **a** to find $F(6)$ and $F(3)$.	$= \left(1 - \frac{1}{6}\right) - \left(1 - \frac{1}{3}\right)$
	$= -\frac{6}{6} - \frac{2}{3}$
	$= \frac{1}{6}$
3 State the result.	$P(3 \le X \le 6) = \frac{1}{6}$

Video
Cumulative distribution functions

iii **1** Write the probability using the cdf. $P(X > 10) = 1 - F(10)$

2 Use the answer from part **a** to find $F(10)$.

$$= 1 - \left(1 - \frac{1}{10}\right)$$

$$= 1 - 1 + \frac{1}{10}$$

$$= \frac{1}{10}$$

3 State the result. $P(X > 10) = \frac{1}{10}$

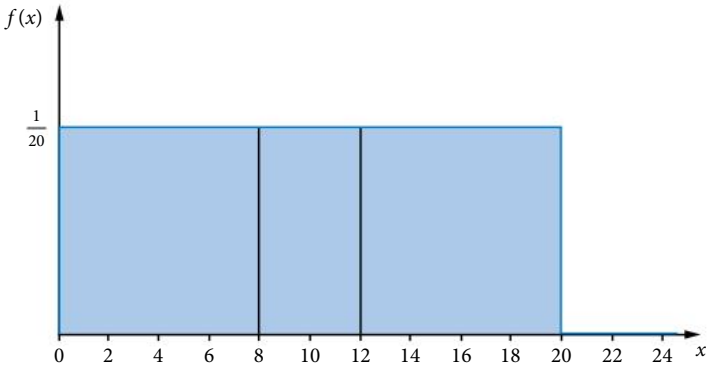
The cumulative distribution function in the previous example, $F(x) = 1 - \frac{1}{x}$ for $x \geq 1$,

can also be written more fully as $F(x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 - \frac{1}{x} & \text{for } x \geq 1 \end{cases}$

Types of continuous random distributions

You arrive at the ground level of a building, and you need to take an elevator to a higher level. It is known that once you press the 'up' button to call the elevator, the time taken for the elevator to arrive is equally likely to be any time between 0 and 20 seconds.

If X is the random variable defined as the number of seconds it takes for the elevator to arrive, then the graph of the probability density function $p(x)$ is shown below.



Because the times are equally likely, the probability density function is $f(x) = \frac{1}{20}$.

The graph of the probability density function is rectangular in shape. As you would expect, the area under the pdf is $\frac{1}{20} \times 20 = 1$.

$P(8 \leq X \leq 12) = \frac{1}{20} \times (12 - 8) = \frac{1}{5}$

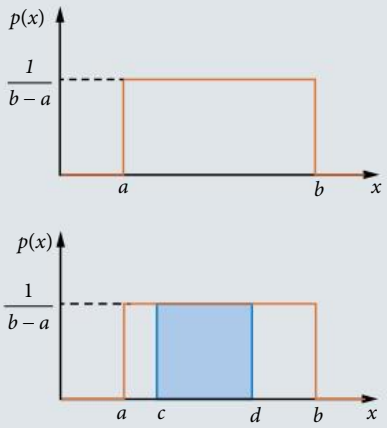
This is an example of a **continuous uniform distribution** and, because of its shape, it is sometimes called a rectangular distribution.

Uniform distributions

A continuous random probability variable X, which is defined for the domain $[a, b]$, has a **uniform distribution** $p(x)$ if $p(x)$ is constant over the domain of X.

The uniform distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$



Also:

$$P(c \le X \le d) = \frac{d-c}{b-a}$$

WORKED EXAMPLE 6 Uniform distribution

Dev dropped a $2 coin on the way to the bus stop, 600 m from home. It is equally likely that Dev dropped it anywhere on the way. Let X be the distance (in metres) from home where the coin was dropped.

a Write the probability density function $p(x)$.

b Find the probability that Dev dropped it at least 200 m from home.

Steps	Working
a 1 Draw a diagram.	Home x Bus-stop 0 m 600 m
2 Write the conditions.	'Equally likely' over the distance means that the distribution is uniform. The domain is $[0, 600]$.
3 Write the distribution.	$p(x) = \dfrac{1}{b-a}$
4 Substitute.	$= \dfrac{1}{600-0}$ $= \dfrac{1}{600}$
5 Write the distribution fully.	$p(x) = \begin{cases} \frac{1}{600} & \text{for } 0 \le x \le 600 \\ 0 & \text{elsewhere} \end{cases}$
b 1 Write the condition.	'At least 200 m from home' means between 200 m and 600 m from home.
2 Find the probability.	$P(200 \le X \le 600) = \dfrac{d-c}{b-a}$ $= \dfrac{600-200}{600}$
3 Evaluate.	$= \dfrac{2}{3}$
4 State the result.	The probability that the coin was dropped at least 200 m from home is $\frac{2}{3}$.

Another distribution with a graph that has a distinctive shape is the **triangular distribution**.

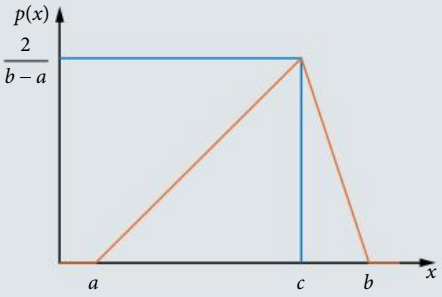
Triangular distributions

A **triangular distribution** for the continuous random probability variable X has the following parameters:

minimum value = a

peak value = c

maximum value = b



The triangular distribution $f(x)$ is defined as

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$

The triangular distribution, like all probability density functions, has the property that the area under the function is 1. You can use this property to show that the maximum value of the probability distribution function is $\dfrac{2}{b-a}$ and this occurs at the peak value of c.

Referring to the diagram in the definition:

$$\text{Area} = \frac{\text{base} \times \text{height}}{2} = 1$$

$$\text{So, } \frac{(b-a) \times \text{height}}{2} = 1$$

$$\text{height} = \frac{2}{b-a}$$

WORKED EXAMPLE 7 Triangular distributions

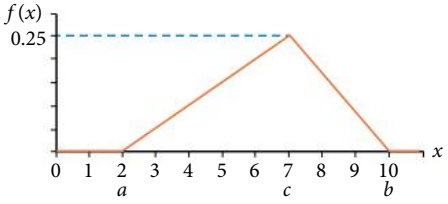
A continuous random variable X has a triangular distribution for $2 \le x \le 10$ and a peak value that occurs at $x = 7$.

a Draw the graph of the probability density function.

b State the probability density function.

c **Calculate**

i $P(X \le 4)$ **ii** $P(X \ge 6)$ **iii** $P(5 \le X \le 8)$

Steps	Working
a 1 State the parameters.	$a = 2$, $b = 10$ and $c = 7$
2 Calculate the maximum value of the function.	$f(c) = \dfrac{2}{b-a}$
3 Substitute.	$= \dfrac{2}{10-2}$
4 Evaluate.	$= 0.25$
5 Draw the graph.	

b 1 State the pdf for a triangular distribution.

$$f(x)=\begin{cases}\dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a<x<c\\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c<x<b\end{cases}$$

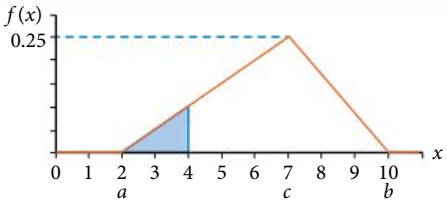
2 Substitute for known values.

$$=\begin{cases}\dfrac{2(x-2)}{(10-2)(7-2)} & \text{for } 2<x<7\\ \dfrac{2(10-x)}{(10-2)(10-7)} & \text{for } 7<x<10\end{cases}$$

3 Evaluate and state the result.

$$f(x)=\begin{cases}\dfrac{1}{20}(x-2) & \text{for } 2<x<7\\ \dfrac{1}{12}(10-x) & \text{for } 7<x<10\end{cases}$$

c i 1 You might find it helpful to sketch a diagram.



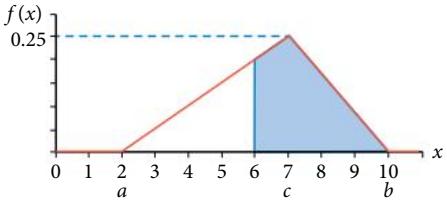
2 Integrate the pdf to find the probability.

$$P(X\le 4)=\int_2^4 \frac{1}{20}(x-2)dx$$

3 Evaluate.

$$=\frac{1}{10}$$

ii 1 Sketch a diagram.



2 Write the required probability.

$$P(X\ge 6)=1-P(X<6)$$

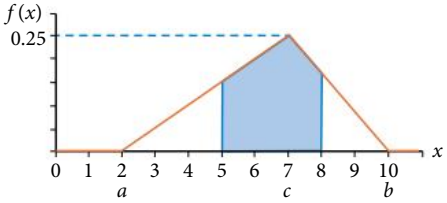
3 Integrate the pdf to find the probability.

$$P(X\ge 6)=1-\int_2^6 \frac{1}{20}(x-2)dx$$

4 Evaluate.

$$=\frac{3}{5}$$

iii 1 Sketch a diagram.



2 State the required probability.

$$P(5\le X\le 8)=P(5\le X\le 7)+P(7\le X\le 8)$$

3 Integrate the pdf to find the probability.

$$P(5\le X\le 8)=\int_5^7 \frac{1}{20}(x-2)\,dx+\int_7^8 \frac{1}{12}(10-x)\,dx$$

$$=\frac{2}{5}+\frac{5}{24}$$

4 State the result.

$$P(5\le X\le 8)=\frac{73}{120}$$

EXERCISE 8.2 Probability density functions

ANSWERS p. 410

Recap

1 The thumb lengths of 60 students, measured to the nearest millimetre, were as follows:

38	50	38	40	35	52	45	50	40	32	40	45
43	40	45	45	55	37	50	45	45	55	50	35
45	50	40	40	50	41	41	40	40	46	45	52
45	45	48	45	45	35	45	45	40	45	40	40
47	70	55	51	45	35	52	32	40	43	45	42

a Construct a probability histogram of the length L, with a class width of 5 mm.

b Find the total area of the histogram.

c Use the histogram to find the probability of choosing a person at random from the group with a thumb length between 38 mm and 47 mm.

Mastery

2 WORKED EXAMPLE 3 Determine whether each function could be a probability density function over the given interval.

a $f(x) = 4e^{-4x}$ for the interval $[0, \infty]$

b $f(x) = 2x^3$ for the interval $[-1, 1]$

c $f(x) = \frac{1}{4}$ for $-2 \le x \le 2$

d $f(x) = 3e^{-8x}$ for the interval $[0, 3]$

e $f(x) = \frac{1}{2}\cos(x)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

f $f(x) = 2(x - 1)$ for the interval $[1, 2]$

g $f(x) = 3x^2$ for the interval $[0, 1]$

h $f(x) = 0.5x$ for the interval $[0, 2]$

3 WORKED EXAMPLE 4 Find the value of k that makes $f(x)$ a probability density function for the stated interval and state $f(x)$.

a $f(x) = k(1 - x^2)$ for $0 \le x \le 1$

b $f(x) = \frac{k}{x^3}$ for $3 \le x \le 6$

c $f(x) = k(x - 5)$ for $5 \le x \le 11$

d $f(x) = k(1 + x^2)$ for $-2 \le x \le 1$

e $f(x) = kx^3$ for $0 \le x \le 1$

f $f(x) = ke^{-\frac{x}{10}}$ for $x \ge 0$

4 WORKED EXAMPLE 5 The probability density function for a continuous random variable X is $f(x) = 4x^3$ for $0 \le x \le 1$.

a Write the cumulative density function for X.

b Calculate

i $P\left(X \le \frac{1}{3}\right)$ **ii** $P\left(\frac{1}{2} \le X \le \frac{2}{3}\right)$ **iii** $P\left(X \le \frac{1}{2}\right)$

5 The probability density function for a continuous random variable X is $f(x) = \frac{2}{x^2}$ for $1 < x < 2$. Find the cumulative distribution function.

6 WORKED EXAMPLE 6 The probability density function for a continuous random variable X is $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \le x \le 1$.

a Write the cumulative density function for X.

b Calculate

i $P(X \le 0.5)$ **ii** $P(-0.5 \le X \le 0.5)$ **iii** $P(X > 0.5)$

7 A probability density function is given as $f(x) = k(x^3 - 1)$ for $1 \leq x \leq 2$.

a Calculate the value of k.

b Write the cumulative density function for X.

c Calculate

i $P(X \leq 1.5)$ **ii** $P(1 \leq X \leq 1.8)$ **iii** $P(X > 1.5)$

8 A cumulative distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \dfrac{x+1}{2} & \text{for } -1 \leq x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

Calculate

a $P\left(X \leq \frac{1}{2}\right)$ **b** $P\left(X > \frac{1}{2}\right)$ **c** $P\left(-\frac{1}{2} \leq X \leq \frac{3}{4}\right)$

9 Consider the probability function for a continuous random variable X, which is defined as $f(x) = \frac{1}{6}e^{-\frac{x}{6}}$ for $x \geq 0$.

a Show that this function is a valid pdf.

b Calculate

i $P(X \leq 2)$ **ii** $P(1 \leq X \leq 5)$

10 Consider the probability function for a continuous random variable X, which is defined as $f(x) = 30x^2(1 - x)^2$ for $0 \leq x \leq 1$.

a Show that this function is a valid pdf.

b Calculate

i $P\left(X \leq \frac{1}{3}\right)$ **ii** $P\left(\frac{1}{3} \leq X \leq \frac{1}{2}\right)$

11 Find the uniform probability density function for the random variable X defined on each domain.

a $[0, 10]$ **b** $[2, 8]$ **c** $[5, 18]$ **d** $[10, 100]$ **e** $[100, 500]$

12 The length of time, in seconds, for which a 2-month-old baby smiles is equally likely to be any value between 0 and 18 seconds. Let X be the smiling time.

a Write the probability density function $p(x)$.

b Calculate the probability that a randomly chosen 2-month-old baby smiles between 3 and 15 seconds.

13 The length of time, in minutes, that a person must wait for a bus is uniformly distributed between 0 and 18 minutes. What is the probability that a person waits fewer than 10.5 minutes for the bus?

14 WORKED EXAMPLE 7 Find the triangular probability density function for the random variable X defined by each set of parameters.

a minimum value = 0, maximum value = 12, and peak value = 6

b minimum value = 3, maximum value = 15, and peak value = 9

c minimum value = 8, maximum value = 21, and peak value = 12

d minimum value = 5, maximum value = 17, and peak value = 13

e minimum value = 7, maximum value = 21, and peak value = 15

15 A continuous random variable X has a triangular distribution defined for $1 \leq x \leq 8$ with a peak value at $x = 3$.

a Draw the graph of the probability density function.

b State the probability density function.

c Calculate

i $P(X \leq 2)$ **ii** $P(X \geq 5)$ **iii** $P(4 \leq X \leq 7)$

16 A continuous random variable X has a triangular distribution with a minimum value of 2, a maximum value of 14 and a peak value at $x = 5$.

a State the probability density function.

b Calculate

i $P(X \leq 4)$ **ii** $P(X \geq 7)$ **iii** $P(3 \leq X \leq 12)$

Exam practice

17 Find the value of k that makes the function $f(x) = k(3x^2 - 2)$ a probability density function for the interval $0 \leq x \leq 2$.

A 4 **B** 1 **C** 0.5 **D** 0.25

18 Find the value of k that makes the function $f(x) = ke^{-5x}$ a probability density function for $x > 0$.

A 1 **B** 5 **C** 10 **D** 25

19 The probability density function for a continuous random variable X is

$$f(x) = \frac{3x^2}{8} \text{ for } 0 \leq x \leq 2$$

The cumulative density function for X, $F(X) =$

A $\frac{x^6}{8}$ **B** $\frac{x^3}{8}$ **C** $\frac{6x}{8}$ **D** $\frac{3x^2}{4}$

20 (2 marks) For the cumulative density function from question **19**, calculate

a $P(X \leq 1)$ *[1 mark]*

b $P(0.5 \leq X \leq 2)$ *[1 mark]*

21 (2 marks) Unreinforced concrete paths may get their first cracks anywhere along the path with equal probability. A concrete path 7.4 m long is constructed from the back door to a clothes line. D is the distance (in metres) from the back door where a crack appears.

a Construct the probability density function for D. *[1 mark]*

b Find the probability that a crack develops more than 3 m from the back door. *[1 mark]*

22 (2 marks) For the cumulative distribution function given in question **7**, what is the value of k such that $P(X \leq k) = 0.6$?

23 (3 marks) A pdf is defined as $f(x) = kx$ for $0 \leq x \leq 2$.

a Calculate

i $P(0 \leq X \leq 1.5)$ **ii** $P\left(\frac{1}{3} \leq X \leq \frac{1}{2}\right)$ *[2 marks]*

b Fully describe the cdf. *[1 mark]*

24 (2 marks) An unreliable clock stops at random at any time during the day.

a If X is the time in hours at which the clock stops, construct a probability density function for X. *[1 mark]*

b Find the probability that the clock will stop between 1:00 pm and 3:45 pm. *[1 mark]*

25 (3 marks) For the situation described in question **12**, find the probability that a randomly selected two-month-old baby smiles for more than 12 seconds if you know that this baby always smiles for more than 8 seconds.

26 (4 marks) CF A fast food franchise is planning a new outlet in a popular street mall. Melanie is trying to decide whether or not to take up the franchise. She decides to use a triangular distribution to model future weekly sales. From experience, it is known that weekly sales will fluctuate between $2000 and $8000, with the most likely weekly sales being around $5000. Melanie also knows that weekly sales of at least $4000 need to be made to cover the rent. What is the probability that weekly sales will be sufficient to cover the rental costs?

27 (4 marks) CF Dani catches a bus to work each day. The longest time it takes to get from the bus stop near home to the one near Dani's workplace is 38 minutes and the shortest time is 25 minutes. The most common time for this commute is 30 minutes. What is the probability that the commute takes at least 28 minutes?

28 (4 marks) CU A machine manufactures stainless steel screws that are meant to be 15 mm long. The manufacturing process is such that although the majority of the screws are 15 mm long, their lengths range from 14.8 mm to 15.2 mm. Calculate the probability that a randomly selected screw is:

a less than 14.94 mm long *[2 marks]*

b more than 15.15 mm long. *[2 marks]*

8.3 The mean and variance of continuous random variables

The mean, **median**, variance, standard deviation and range are used for both discrete and continuous probability functions.

For a discrete random variable

$$\text{expected value (mean)}, E(X) = \mu$$

$$= \sum_{x=1}^{n} x_n \times P(X = x_n)$$

This definition can be extended to a continuous random variable, but the integral is used instead of the sum.

Expected value

For the continuous random variable X defined on the interval $[a, b]$, with probability density function $p(x)$, the **expected value** (mean) is defined as

$$E(X) = \mu = \int_a^b x\, p(x)\, dx$$

WORKED EXAMPLE 8 Expected value

Calculate the expected value for a uniform distribution defined on the interval $[3, 28]$.

Steps	Working
1 Write the pdf.	$p(x) = \frac{1}{b-a}$
2 Substitute for known values.	$= \frac{1}{28-3}$
3 Evaluate.	$p(x) = 0.04$
4 Write the formula for expected value.	$E(X) = \int_a^b x\, p(x)\, dx$
5 Substitute for known values.	$= \int_3^{28} x \times 0.04\, dx$
6 Evaluate.	$= \left[0.02x^2\right]_3^{28}$ $= 0.02 \times 28^2 - 0.02 \times 3^2$ $= 15.5$
7 State the result.	The expected value is 15.5.

You can see that for the uniform distribution in Worked example 8, the expected value is the centre of the distribution. This is true for all uniform distributions.

Expected value for a uniform distribution

For the continuous random variable X with a uniform distribution defined on the interval $[a, b]$, the expected value is given by $E(X) = \mu = \frac{a+b}{2}$.

In the case of a symmetrical triangular distribution, the expected value is also the centre of the distribution. This is not true for asymmetrical triangular distributions.

WORKED EXAMPLE 9 Triangular distribution expected value

Calculate the expected value for a triangular distribution defined on the interval $[2, 12]$ whose peak value is at 8.

Steps	Working
1 State the pdf for a triangular distribution.	$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \end{cases}$
2 Substitute for known values.	$= \begin{cases} \frac{2(x-2)}{(12-2)(8-2)} & \text{for } 2 \le x \le 8 \\ \frac{2(12-x)}{(12-2)(12-8)} & \text{for } 8 \le x \le 12 \end{cases}$
3 Evaluate and state the result.	$f(x) = \begin{cases} \frac{1}{30}(x-2) & \text{for } 2 \le x \le 8 \\ \frac{1}{20}(12-x) & \text{for } 8 \le x \le 12 \end{cases}$

4 Write the rule for expected value.	$E(X) = \int_a^b x\,p(x)\,dx$
5 $p(x)$ has 2 parts.	$= \int_2^8 x \times \frac{(x-2)}{30}\,dx + \int_8^{12} x \times \frac{(12-x)}{20}\,dx$
6 Evaluate.	$= \left[\frac{x^2(x-3)}{90}\right]_2^8 + \left[\frac{-x^2(x-18)}{60}\right]_8^{12}$
	$= 7\frac{1}{3}$
7 State the result.	The expected value is $7\frac{1}{3}$.

The same process can be used for other probability density functions.

WORKED EXAMPLE 10 Expected value for a continuous random variable

Calculate the expected value for the continuous random variable X with the probability density function $p(x) = 3x^2$ for $0 \leq x \leq 1$.

Steps	Working
1 Write the rule for expected value.	$E(X) = \int_a^b x\,p(x)\,dx$
2 Substitute for known values.	$= \int_0^1 x \times 3x^2\,dx$
3 Evaluate.	$= \int_0^1 3x^3\,dx$
	$= \left[\frac{3x^4}{4}\right]_0^1$
	$= \frac{3(1^4)}{4} - 0$
	$= \frac{3}{4}$
4 State the result.	The expected value is $\frac{3}{4}$.

The **variance** of the discrete random variable X is given by

$$Var(X) = \sum p(x)(x-\mu)^2$$

As with the expected value, the variance of a continuous random variable is calculated using the integral over the interval for which the variable is defined.

Variance

For the continuous random variable X defined on the interval $[a, b]$, with probability density function $p(x)$ and $E(X) = \mu$, the **variance** is defined as:

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$$

The alternative formula for variance below may be more convenient for some calculations.

$$Var(X) = E(X^2) - [E(X)]^2$$

Exam hack

The alternative formula is not in the QCAA Formula book, so you can either learn it or use the one in the Formula book.

As was the case with discrete random variables, the standard deviation is the square root of the variance.

Standard deviation

For the continuous random variable X, the **standard deviation** is defined as:

$$\text{SD}(X) = \sigma = \sqrt{Var(X)}$$

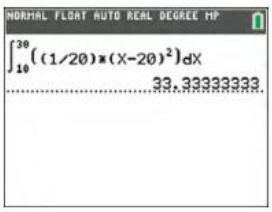
WORKED EXAMPLE 11 Variance and standard deviation

Calculate the variance and standard deviation of a continuous random variable with a uniform distribution over the interval $[10, 30]$.

Steps	Working
1 Calculate the expected value.	$E(X) = \frac{a+b}{2}$ $= \frac{10+30}{2}$ $= 20$
2 Write the pdf.	$p(x) = \frac{1}{b-a}$
3 Substitute for known values.	$= \frac{1}{30-10}$
4 Evaluate.	$p(x) = \frac{1}{20}$
5 Write the rule for variance.	$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$
6 Substitute for known values.	$= \int_{10}^{30} \frac{1}{20}(x-20)^2\,dx$ $= \frac{100}{3}$
7 Find the standard deviation.	$\text{SD}(X) = \sqrt{Var(X)}$
8 Evaluate.	$= \sqrt{\frac{100}{3}}$ $= 5.773\,502\ldots$
9 Round off and state the result.	The variance is about 33.33 and the standard deviation is about 5.77.

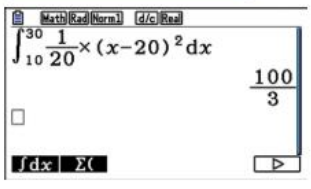
It is convenient to use a calculator to calculate the integral.

TI-84 Plus CE



1 Press MATH and then select **9: fnInt(** (finite integral).
2 Enter the required information and press ENTER to display the result.

Casio fx-CG50 AU



1 From the Run-Matrix screen, press F4 (Math), F6 (▷) and then F1 (∫dx).
2 Press EXE to display the result.

TI-Nspire
Continuous random variables and the normal distribution

WORKED EXAMPLE 12 Variance and standard deviation of a triangular distribution

Calculate the variance and standard deviation of a continuous random variable with a symmetrical triangular distribution over the interval $[10, 40]$.

Steps	Working
1 The triangular distribution is symmetrical.	$E(X) = \text{peak value}$
	$= \frac{a+b}{2}$
	$= \frac{10+40}{2}$
	$= 25$
2 Use the formula for the pdf of a triangular distribution where $a = 10$, $b = 40$ and $c = 25$.	$f(x) = \begin{cases} \frac{2(x-10)}{(40-10)(25-10)} & \text{for } 10 \le x \le 25 \\ \frac{2(40-x)}{(40-10)(40-25)} & \text{for } 25 \le x \le 40 \end{cases}$
3 Evaluate and state the result.	$f(x) = \begin{cases} \frac{1}{225}(x-10) & \text{for } 10 \le x \le 25 \\ \frac{1}{225}(40-x) & \text{for } 25 \le x \le 40 \end{cases}$
4 Use the formula for variance.	$Var(X) = \int_a^b p(x)(x-\mu)^2 dx$
5 The function has 2 parts.	$= \int_{10}^{25} \frac{1}{225}(x-10)(x-25)^2 dx$
	$+ \int_{25}^{40} \frac{1}{225}(40-x)(x-25)^2 dx$
6 Use a calculator to evaluate.	$= 37.5$
7 Find the standard deviation.	$SD(X) = \sqrt{37.5}$
8 Evaluate.	$= 6.123\,724\ldots$
9 Round off and state the result.	The variance is 37.5 and the standard deviation is about 6.12.

WORKED EXAMPLE 13 Variance of pdf

Calculate the variance of a continuous random variable with the probability density function $f(x) = 2(1 - x)$ for $0 \le x \le 1$.

Steps	Working
1 Check that $f(x)$ is a valid pdf.	$2(1-x) = 2 - 2x \ge 0$ for $0 \le x \le 1$
	$\int_0^1 2(1-x)dx = 2\int_0^1 (1-x)dx$
	$= 2\left[x - \frac{x^2}{2}\right]_0^1$
	$= 2\left(1 - \frac{1}{2} - 0\right)$
	$= 1$
2 Write the conclusion.	$f(x) = 2(1-x)$ for $0 \le x \le 1$ is a valid pdf.
3 Calculate the expected value.	$E(X) = \int_a^b xp(x)dx$
4 Substitute for known values.	$= \int_0^1 x\left[2(1-x)\right]dx$

5 Evaluate.

$$E(X) = 2\int_0^1 (x - x^2)\,dx$$
$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$= 2\left(\frac{1^2}{2} - \frac{1^3}{3} - 0\right)$$
$$= \frac{1}{3}$$

6 Use the formula for variance.

$$Var(X) = \int_0^1 2(1-x)\left(x - \frac{1}{3}\right)^2 dx$$
$$= 2\int_0^1 (1-x)\left(x^2 - \frac{2x}{3} + \frac{1}{9}\right)dx$$

7 Simplify.

$$= 2\int_0^1 \left(x^2 - \frac{2x}{3} + \frac{1}{9} - x^3 + \frac{2x^2}{3} - \frac{x}{9}\right)dx$$
$$= 2\int_0^1 \left(-x^3 + \frac{5x^2}{3} - \frac{7x}{9} + \frac{1}{9}\right)dx$$

8 Evaluate.

$$= 2\left[-\frac{x^4}{4} + \frac{5x^3}{9} - \frac{7x^2}{18} + \frac{x}{9}\right]_0^1$$
$$= 2\left(-\frac{1}{4} + \frac{5}{9} - \frac{7}{18} + \frac{1}{9} - 0\right)$$
$$= \frac{1}{18}$$

EXERCISE 8.3 The mean and variance of continuous random variables

ANSWERS p. 411

Recap

1 Which functions could be probability density functions?

a $f(x) = \frac{3x^2}{8}$ for the interval $[0, 2]$

b $f(x) = \frac{1}{8}$ for the interval $[0, 4]$

c $f(x) = \frac{3}{10}(2 - x^2)$ for $-1 \le x \le 1$

d $f(x) = 4e^{-2x}$ for $x > 0$

2 A continuous random variable X has a triangular distribution for $3 \le x \le 12$ and a peak value that occurs at $x = 9$.

a Draw the graph of the probability density function.

b State the probability density function.

c Calculate

i $P(X \le 6)$ **ii** $P(X \ge 11)$ **iii** $P(6 \le X \le 10)$

Mastery

3 WORKED EXAMPLE 8 A continuous random variable X has a uniform distribution on the interval $[5, 22]$.

a Calculate the probability density function for X.

b What is the expected value for the distribution?

4 What is the expected value for a uniform continuous random variable X on the interval $[12, 44]$?

5 Calculate the expected value for a continuous random variable X with a uniform distribution on the interval $[25, 86]$.

6 WORKED EXAMPLE 9 A continuous random variable X has a triangular distribution on the interval $[4, 18]$ with a peak value at $x = 12$.

a Calculate the probability density function for X.

b What is the expected value for the distribution?

7 A continuous random variable X has a triangular distribution on the interval $[2, 5]$ with a peak value at $x = 5$. What is the expected value for the distribution?

8 The parameters for some triangular distributions are listed below, where a = minimum value, b = maximum value and c = peak value. Calculate the expected value of each distribution.

a $a = 5, b = 12, c = 10$ **b** $a = 10, b = 60, c = 40$

c $a = 18, b = 47, c = 32$ **d** $a = 25, b = 56, c = 33$

9 WORKED EXAMPLE 10 Calculate the expected value for the continuous random variable X with the probability density function $p(x) = 4e^{-4x}$ for $x \geq 0$.

10 Calculate the expected value for the continuous random variable X with the probability density function $p(x) = 2(1 - x)$ for $0 \leq x \leq 1$.

11 Calculate the expected value for the continuous random variable X with the probability density function

a $p(x) = 8x^{-3}$ for $x \geq 2$ **b** $p(x) = 3x^2$ for $0 \leq x \leq 1$

c $p(x) = \frac{3}{10}(3x - x^2)$ for $0 \leq x \leq 2$

12 WORKED EXAMPLE 11 A continuous random variable X has a uniform distribution over the interval $[8, 24]$.

a Find the probability density function for X.

b Calculate the expected value of X.

c Calculate the variance and standard deviation for the distribution.

13 The intervals over which a number of uniform continuous variables are defined are described below. In each case, find the expected value, variance and standard deviation of the variable.

a $[0, 20]$ **b** $[3, 12]$ **c** $[15, 35]$ **d** $[20, 70]$ **e** $[0.5, 2.5]$

14 WORKED EXAMPLE 12 A continuous random variable X has a symmetrical triangular distribution over the interval $[6, 22]$.

a Find the probability density function for X.

b Calculate the expected value of X.

c Calculate the variance and standard deviation for the distribution.

15 A continuous random variable X has a symmetrical triangular distribution defined by $p(x)$ over the interval $[0.4, 2.8]$. Calculate

a $p(x)$ **b** $E(X)$ **c** $Var(X)$

16 A continuous random variable X has a triangular distribution defined by $p(x)$ over the interval $[8, 26]$ with a peak value at $x = 15$. Calculate

a $p(x)$ **b** $E(X)$ **c** $Var(X)$

17 WORKED EXAMPLE 13 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = \frac{x}{2} + 2$ for $-4 \leq x \leq -2$.

18 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 2x^{-2}$ for $1 \leq x \leq 2$.

19 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$.

Exam practice

20 © QCAA 2023 1Q7 TECH-FREE 61% Determine the mean of the continuous random variable X with the probability density function

$$f(x) = \begin{cases} \frac{1}{8}x, & 0 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

A $\frac{1}{8}$ **B** $\frac{3}{8}$ **C** $\frac{1}{2}$ **D** $\frac{8}{3}$

21 © QCAA 2023 1Q10 TECH-FREE 62% The continuous random variable Y has the probability density function

$$f(x) = \begin{cases} 1 + y, & 0 \leq y \leq \sqrt{3} - 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine $P\left(0 \leq Y \leq \frac{1}{2}\right)$.

A $\frac{1}{5}$ **B** $\frac{3}{8}$ **C** $\frac{5}{8}$ **D** $\frac{3}{4}$

22 The standard deviation of a continuous random variable with a uniform distribution over the interval $[5, 12]$ is approximately

A 2.021 **B** 3.031 **C** 4.042 **D** 5.052

23 (2 marks) Calculate the variance and standard deviation of a continuous random variable with a symmetrical triangular distribution over the interval $[12, 34]$.

24 (3 marks) CF 25 lettuce seeds are scattered at random in a round seed-raising tray 30 cm in diameter. Construct the probability density function for the distance R of a seedling from the centre of the tray and, hence, find the expected distance from the centre of the tray.

Hint: The probability density function is uniform with respect to area.

25 (4 marks) CF Max works in the inner city and uses a parcel pick-up service to transport important documents. The shortest time he has had to wait for a parcel to be collected is 5 minutes and the average time is 15 minutes. Calculate the standard deviation of the waiting time.

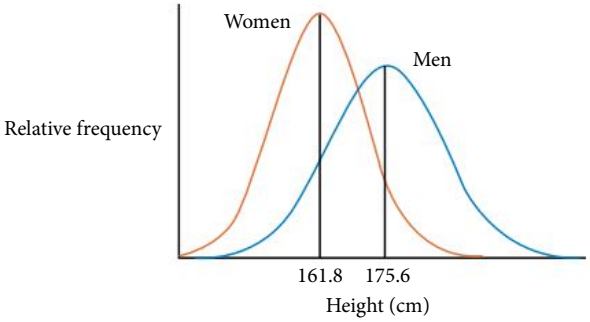
26 (4 marks) CU A continuous random variable X has a uniform distribution that is defined over the interval $[8, b]$. If $Var(X) = 18\frac{3}{4}$, find $E(X)$.

27 (4 marks) CU A continuous random variable X has a symmetrical triangular distribution with a minimum value of 3. If $Var(X) = 13\frac{1}{2}$, find $E(X)$.

8.4 Normal distributions

The **normal distribution** is one of the most widely known and used distributions in statistics. This is because the normal distribution approximates many natural phenomena such as height, weight, length and time.

For example, the average Australian male (18 years and over) is 175.6 cm tall and the average Australian female is 161.8 cm tall. Of course, there are many males and females who are taller than these averages and many who are shorter. The graphs of the relative frequencies of the heights of these groups are shown below.



Each of these graphs show the characteristic *bell-shaped curve* that is typical of a normal distribution. The graph of a normal distribution is symmetrical about the mean.

Normal distributions

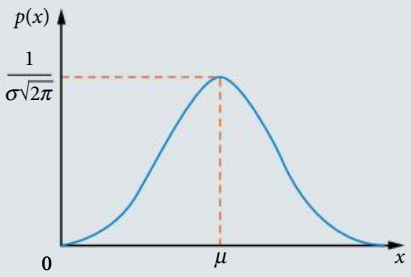
The probability density function for the **normal distribution** is given by:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the mean and σ is the standard deviation of the distribution.

Features of the normal distribution include:

- the distribution is symmetrical about the mean
- the mean, median and mode are equal
- the area under the curve is 1
- the majority of values lie around the mean
- there are only 2 parameters: μ and σ
- the maximum value occurs at $x = \mu$, so $p(x) = \frac{1}{\sigma\sqrt{2\pi}}$.



The normal distribution with mean μ and variance σ^2 is shown as $N(\mu, \sigma^2)$.

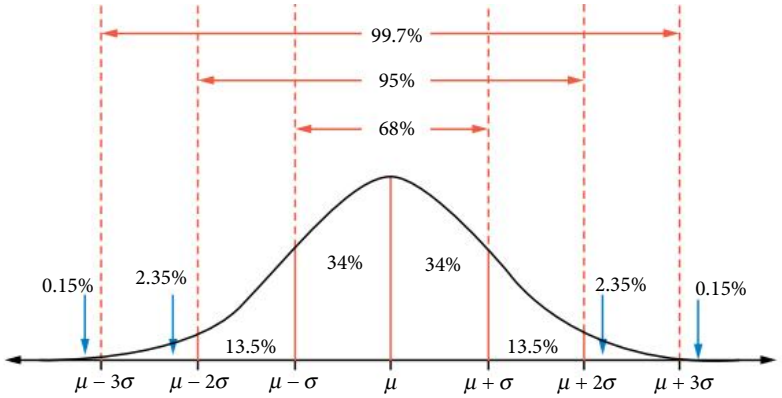
The *standard* normal distribution is $N(0, 1)$.

Worksheets
The normal distribution
The normal curve

$X \sim N(4, 6)$ means that the continuous random variable X has a normal distribution with mean 4 and variance 6. Although the mean and standard deviation for a normal distribution may change, the area under the curve remains constant.

- About 68% of the population falls within 1 standard deviation of the mean.
- About 95% of the population falls within 2 standard deviations of the mean.
- About 99.7% of the population falls within 3 standard deviations of the mean.

The normal curve



WORKED EXAMPLE 14 Normal distribution

a What is the probability density function of the variable $X \sim N(40, 49)$?

b Find the mean and standard deviation of the probability density function for a normal distribution given by $f(x) = \frac{3}{2\sqrt{2\pi}} e^{-\frac{9}{8}(x-2)^2}$.

Steps	Working
a 1 Write the values of μ and σ.	$\mu = 40,\ \sigma^2 = 49$ $\mu = 40,\ \sigma = 7$
2 Write the formula for a normal distribution.	$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
3 Substitute for known values.	$= \frac{1}{7\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-40}{7}\right)^2}$
4 Simplify.	$= 0.056\,991\ldots\ e^{-0.010204\ldots(x-40)^2}$
5 Round off.	$\approx 0.0570\, e^{-0.010204(x-40)^2}$
b 1 Write the pdf.	$f(x) = \frac{3}{2\sqrt{2\pi}} e^{-\frac{9}{8}(x-2)^2}$
2 Compare with $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$.	$\frac{1}{\sigma\sqrt{2\pi}} = \frac{3}{2\sqrt{2\pi}}$, so $\sigma = \frac{2}{3}$ $e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = e^{-\frac{9}{8}(x-2)^2}$, so $\mu = 2$
3 State the result.	$\mu = 2$ and $\sigma = \frac{2}{3}$

The area under the probability density function is the probability of values in the interval. Most graphics calculators have the probability distribution function and cumulative distribution function for normal distributions as built-in functions.

The lower and upper bounds of the normal distribution are $-\infty$ and ∞ respectively.

Exam hack

If your calculator does not allow ∞ as an input, you can use 1E99 = 1×10^{99} as a sufficiently close approximation for ∞.

WORKED EXAMPLE 15 Probabilities and the normal distribution

A continuous random variable $X \sim N(80, 400)$.

a Find the probability that $X < 100$.

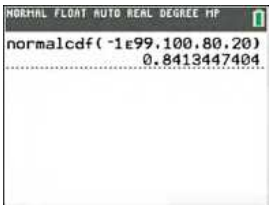
b **Calculate** $P(30 < X < 115)$.

Steps

Working

a 1 Use your calculator.

TI-84 Plus CE

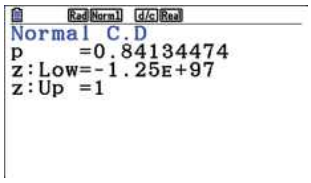


1 Press 2nd vars (dist) and then select **2: normalcdf(**.

2 Enter (-) **1** 2nd EE **99** for the lower bound and then enter the other required information.

Press ENTER to display the result.

Casio fx-CG50 AU



1 From the Statistics screen, press F5 (Dist), F1 (Norm), F2 (Ncd). If necessary, select **Variable** for the data source.

2 For the lower bound, press (-) 1, then select **E** from the Catalogue (SHIFT 4), then **99**. Enter the other required information and press EXE to display the result.

2 State the result.

$P(X < 100) \approx 0.8413$

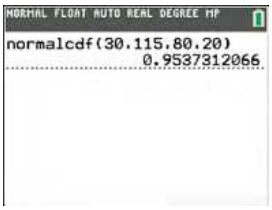
b 1 Repeat the steps previously described with:

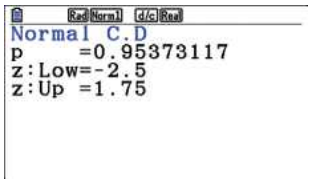
Lower bound = 30

Upper bound = 115

$\mu = 80$

$\sigma = 20$





2 State the result.

$P(30 < X < 115) \approx 0.9537$

TI-Nspire Continuous random variables and the normal distribution

EXERCISE 8.4 Normal distributions

ANSWERS p. 412

Recap

1 Calculate the expected value for

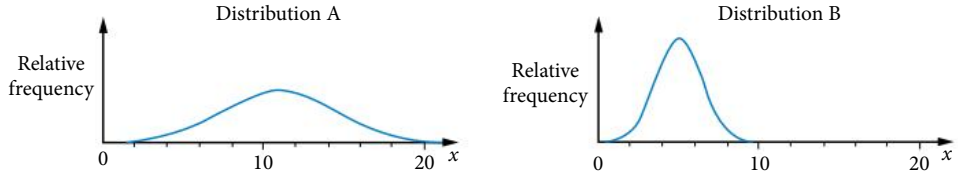
a a uniform distribution defined on the interval $[4, 26]$

b a triangular distribution defined on the interval $[3, 10]$ with a peak value at 5.

2 Calculate the expected value and variance of a continuous random variable with the probability density function $f(x) = 2e^{-2x}$ for $x > 0$.

Mastery

3 The diagram below shows 2 normal distribution curves.



a Which has the larger mean?

b Which has the smaller standard deviation?

4 WORKED EXAMPLE 14 What is the probability density function for each of the following normal distributions?

a $\mu = 3$ and $\sigma = \frac{3}{4}$

b $\mu = 20$ and $\sigma = 5$

c $\mu = 34.5$ and $\sigma = 7.6$

d $\mu = 128$ and $\sigma = 32$

e $\mu = 1.7$ and $\sigma = 0.4$

5 What is the mean and standard deviation of the probability density function for each normal distribution?

a $f(x) = \frac{1}{3\sqrt{2\pi}} e^{\frac{-(x-12)^2}{18}}$

b $p(x) = \frac{2}{\sqrt{2\pi}} e^{-2(x-2)^2}$

c $p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-2)^2}{2}}$

d $0.066\,49\, e^{\frac{-(x-20)^2}{72}}$

e $0.086\,73 e^{-0.02363(x-18.5)^2}$

f $1.1398 e^{-4.08163(x-1.2)^2}$

6 WORKED EXAMPLE 15 A continuous random variable $X \sim N(115, 784)$.

a Calculate the probability that $x \leq 80$.

b Calculate the probability that $x \leq 130$.

c Find $P(100 \leq X \leq 130)$.

d Find $P(X \geq 146)$.

7 A continuous random variable $X \sim N(26.8, 22.09)$. Calculate:

a $P(X < 20)$

b $P(X > 22)$

c $P(X > 40)$

d $P(22 < X < 32)$

e $P(19 < X < 42)$

8 Scores on a particular IQ test are known to be normally distributed with a mean of 115 and a standard deviation of 18.

a Determine

i $\mu \pm \sigma$

ii $\mu \pm 2\sigma$

iii $\mu \pm 3\sigma$

b Find

i $P(\mu - \sigma < X < \mu - \sigma)$

ii $P(\mu - 2\sigma < X < \mu - 2\sigma)$

iii $P(\mu - 3\sigma < X < \mu - 3\sigma)$

c Find

i $P(X < 70)$

ii $P(X > 160)$

9 For each random normal variable X with the specified mean and standard deviation, calculate the probability indicated.

a $\mu = 38$ and $\sigma = 64$, $P(22 < X < 45)$

b $\mu = 1058$ and $\sigma = 289$, $P(876 < X < 1243)$

c $\mu = 8.46$ and $\sigma = 2.07$, $P(5.32 < X < 7.98)$

d $\mu = 0.64$ and $\sigma = 0.14$, $P(0.45 < X < 0.79)$

10 What is the approximate probability that a normally distributed value lies between

a 2σ below the mean and 1σ above the mean?

b the mean and the value 2σ above the mean?

c the mean and the value 1σ below the mean?

Exam practice

11 $X \sim N(25, 16)$. The mean and standard deviation of X are, respectively

A 5 and 4 **B** 25 and 16 **C** 5 and 16 **D** 25 and 4

12 The mean and standard deviation for the random normal variable X are $\mu = 287.65$ and $\sigma = 68.73$. The probability $P(146.9 < X < 188.6)$ is

A 0.0545 **B** 0.109 **C** 0.4455 **D** 0.545

13 What is the approximate probability that a normally distributed value lies between the mean and the value 3σ below the mean?

A 0.0015 **B** 0.4985 **C** 0.5 **D** 0.997

14 © QCAA 2023 2Q3 62% In a certain normal distribution curve, 95% of the area lies between the values 50.32 and 113.68. The mean of this distribution is 82.

Determine the standard deviation.

A 16.16 **B** 21.12 **C** 31.68 **D** 63.36

15 (2 marks) A continuous random variable $X \sim N(18.5, 11.56)$.

a Find the probability that $X < 9.6$. *[1 mark]*

b Calculate $P(12 < X < 21)$. *[1 mark]*

16 © QCAA 2020 2Q14 (6 marks) Let X denote the time in minutes between the arrival of trains at a station. The cumulative distribution function of X is defined by

$$F(x) = \begin{cases} 2 - \dfrac{10}{x}, & 5 \le x \le 10 \\ 0, & \text{otherwise} \end{cases}$$

a Determine the probability density function of X. *[3 marks]*

b Determine the probability that $5 < X < 7$. *[1 mark]*

c Determine the mean time between the arrival of trains at the station. *[2 marks]*

17 (3 marks) CF A continuous random variable, X, is known to be normally distributed with a mean of 85 and a standard deviation of 12. Find the range between which approximately

a 68% of the values lie *[1 mark]*

b 95% of the values lie *[1 mark]*

c 99.7% of the values lie. *[1 mark]*

18 (4 marks) CU The heights of a plantation of seedlings are normally distributed. In one plantation, 84% of the seedlings are less than 148 cm high and 16% are less than 106 cm high.

a Find the mean and standard deviation of the seedlings. *[2 marks]*

b What proportion of the seedlings are between 90 cm and 130 cm high? *[2 marks]*

19 © QCAA 2024 2Q5 78% The mass (g) of adult kookaburras in a certain region is normally distributed with a mean of 300 g and a standard deviation of 13 g. Select the correct statement about the mass of adult kookaburras.

A 34% are between 287 g and 313 g

B 68% are between 274 g and 326 g

C 95% are between 261 g and 326 g

D 99.7% are between 261 g and 339 g

Worksheets
The standard normal curve
Areas under the normal curve
z-scores
Applying the normal distribution

8.5 z-scores

Which is better: a score of 47 on a test that is known to be normally distributed with a mean of 40 and a standard deviation of 5 or a score of 44 on a test with a mean of 38 and a standard deviation of 7?

You can tell which is better by changing the scores to a standardised form. The distribution for each score is changed to the **standard normal distribution**. The variable in a standard normal distribution is always shown by Z. A particular value of Z is denoted by z.

The standard normal distribution has a mean of 0 and a standard deviation of 1. The value of Z (the Z-value) shows how many standard deviations the original value is from the mean.

Find the difference between the corresponding x value and the mean and divide this result by the standard deviation. This is the ***z*-score** (***z*-value** or **standard score**).

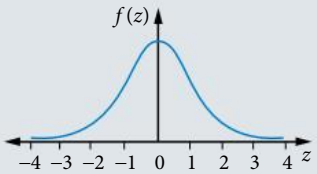
z-score

$$z = \frac{x - \mu}{\sigma}$$

The standard normal distribution

If $z = \frac{x - \mu}{\sigma}$, $\mu = 0$ and $\sigma = 1$, the probability density function for the standard normal distribution is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$



- The distribution is symmetrical about the mean.
- The mean, median and mode all equal 0.
- The area under the curve is 1.
- The majority of values lie around the mean.

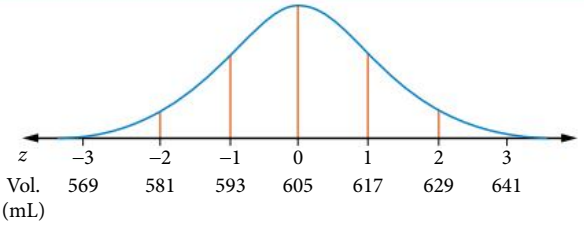
WORKED EXAMPLE 16 Finding *z*-scores

A machine fills bottles with vinegar. The volume of vinegar filled by the machine is normally distributed with a mean of 605 mL and a standard deviation of 12 mL. The intended volume of vinegar in filled bottles is 605 mL. Two bottles are randomly selected and found to have volumes of 593 mL and 619 mL respectively.

a **Sketch** the graph of the distribution, showing volumes and *z*-scores.

b What proportion of filled bottles would contain more vinegar than the first bottle selected?

c Which of the selected bottles is more accurately filled?

Steps	Working
a Draw the standard normal distribution curve. Next, calculate the volumes that correspond to the *z*-scores. $z = 0 \leftrightarrow \mu = 605$ $z = 1 \leftrightarrow \mu + \sigma = 605 + 12 = 617$ $z = -1 \leftrightarrow \mu - \sigma = 605 - 12 = 593$ $z = 2 \leftrightarrow \mu + 2\sigma = 605 + 24 = 629$ etc.	
b 1 Calculate the *z*-score.	$z = \frac{x-\mu}{\sigma}$ $= \frac{593-605}{12}$ $= -1$
2 Refer to the approximate area under the standard normal distribution.	$P(X > 593) = 1 - P(X \leq 593)$ $\approx 1 - (0.50 - 0.34)$ $\approx 1 - 0.16$ ≈ 0.84
c 1 State the *z*-score for the first bottle.	$z_1 = -1$
2 Calculate the *z*-score for the second bottle.	$z_2 = \frac{x-\mu}{\sigma}$ $= \frac{619-605}{12}$ ≈ 1.1667
3 Compare the *z*-scores.	$\lvert z_1 \rvert < \lvert z_2 \rvert$
4 Write the conclusion.	The first bottle is more accurately filled because its volume is closer to the intended volume (605 mL).

Video *z*-scores

WORKED EXAMPLE 17 More *z*-scores

It is known that the scores on a test are normally distributed with $\mu = 58$ and $\sigma = 12$.

a What is the *z*-score for a mark of 40?

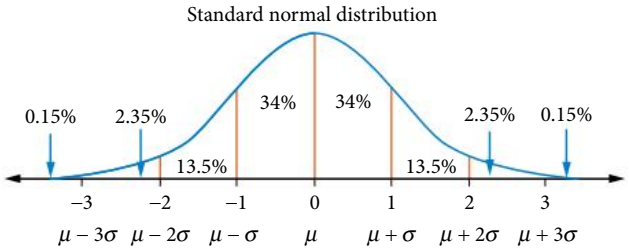
b What is the actual score for a *z*-score of 1.8?

Steps	Working
a 1 Use the rule for *z*-scores.	$z = \frac{x-\mu}{\sigma}$
2 Substitute for known values.	$= \frac{40-58}{12}$
3 Evaluate.	$= -1.5$
4 State the result.	A score of 40 is the same as a *z*-score of −1.5.

b 1	Use the rule for z-scores.	$z = \dfrac{x-\mu}{\sigma}$
2	Substitute for known values.	$1.8 = \dfrac{x-58}{12}$
3	Evaluate.	$1.8 \times 12 = x - 58$
4	Solve for x.	$x = 58 + 21.6$
		$= 79.6$
5	State the result.	A z-score of 1.5 is the same as a score of 79.6.

The distribution of the area under the standard normal distribution is the same as for the normal distribution.

- About 68% of the population falls within 1 standard deviation of the mean.
- About 95% of the population falls within 2 standard deviations of the mean.
- About 99.7% of the population falls within 3 standard deviations of the mean.

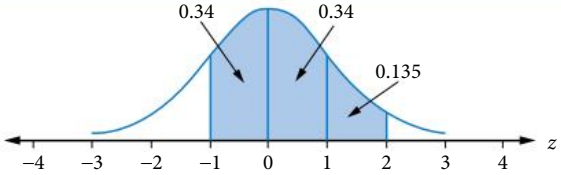
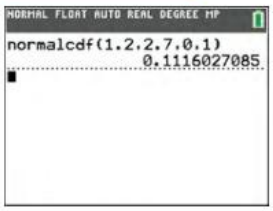
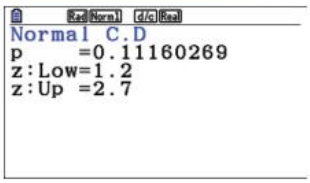


WORKED EXAMPLE 18 Finding probabilities with z-scores

For the standard normal distribution, find

a $P(-1 < Z < 2)$

b $P(1.2 < Z < 2.7)$

Steps	Working
a 1 Draw a diagram showing the required area (probability).	
2 Refer to the areas under a standard normal distribution.	$P(-1 < Z < 2) = 0.34 + 0.34 + 0.135$ $= 0.815$
b 1 Use your calculator to find the probability.	**TI-84 Plus CE**  Press 2nd vars (dist) and then select **2: normalcdf(**. Enter the values: Lower bound = 1.2 Upper bound = 2.7 $\mu = 0$, $\sigma = 1$
	Casio fx-CG50 AU  From the Statistics screen press F5 (Dist), F1 (Norm), F2 (Ncd). If necessary, select **Variable** for the data source. Enter the values: Lower bound = 1.2 Upper bound = 2.7 $\mu = 0$, $\sigma = 1$
2 State the result.	$P(1.2 < Z < 2.7) \approx 0.1116$

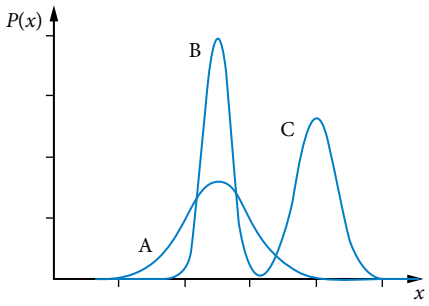
TI-Nspire
Continuous random variables and the normal distribution

EXERCISE 8.5 *z*-scores

ANSWERS p. 412

Recap

1 Consider the normal distribution curves shown.



a Which has the greatest mean?

b Which normal curve has the greatest standard deviation?

Give reasons for your answers.

2 a What is the probability density function for a normal distribution with $\mu = 20$ and $\sigma = 5$?

b What is the mean and standard deviation of the probability density function for a normal distribution given by $f(x) = \frac{4}{3\sqrt{2\pi}} e^{-\frac{8}{9}(x-20)^2}$?

Mastery

3 WORKED EXAMPLE 16 A Little Athletics club keeps records of all its athletics competitions. It was found that the time for girls to run 400 m is normally distributed with a mean of 95 seconds and a standard deviation of 6 seconds. Mandy and her friend Zoe ran against each other in a recent 400 m race and their scores were 89 s and 107 s respectively.

a Sketch the graph of the distribution showing times and *z*-scores.

b Mark Mandy's and Zoe's times on separate graphs.

c Calculate the *z*-scores for Mandy and Zoe and mark these on the graphs.

d Shade the area under the respective graph to represent performances that were better than those of Mandy.

e Shade the area under the respective graph to represent performances that were worse than those of Zoe.

f Of all the Little Athletics runners who participated in this event, approximately what proportion would have performed better than Mandy?

g If 100 Little Athletics runners had participated in this event, about how many would have performed better than Zoe?

4 A worker at a farm packs trays of strawberries. The weights of strawberries in the trays are normally distributed with a mean of 508 g and a standard deviation of 12 g.

The farm manager rejects any trays of strawberries weighing less than 496 g or more than 532 g.

a Sketch a graph of the distribution showing the weights of trays of strawberries and the corresponding *z*-scores.

b Shade the area under the graph that represents the weights of trays that are rejected.

c If 1000 trays of strawberries are packed in the day, about how many will be rejected?

5 WORKED EXAMPLE 17 A set of scores is normally distributed with $\mu = 62$ and $\sigma = 8$. Calculate the z-score for each score.

a 78 **b** 54 **c** 74

d 38 **e** 88 **f** 50

6 Test results are found to be normally distributed with $\mu = 48$ and $\sigma = 10$. Calculate the test result for each z-score.

a 1 **b** −2.4 **c** 1.8

d −1.5 **e** 3.2 **f** −2.8

7 A random normal variable X has a mean of 10.6 and a standard deviation of 3.4. Find the z-score for each value of X.

a 8 **b** 9.6 **c** 11.2 **d** 12.7 **e** 6.3

f 14 **g** 7 **h** 11.1 **i** 10.2 **j** 10.63

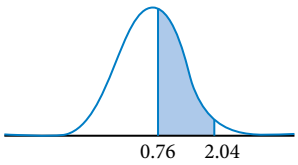
8 A random normal variable X has a mean of 2307 and a standard deviation of 240. Find the z-score for each value of X.

a 2800 **b** 2100 **c** 2120 **d** 2409 **e** 2343

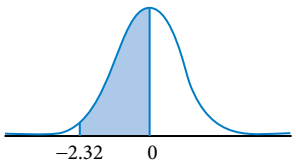
f 1874 **g** 2035 **h** 2714 **i** 3010 **j** 1625

9 WORKED EXAMPLE 18 Find the probability represented by the shading on each graph of the standard normal distribution.

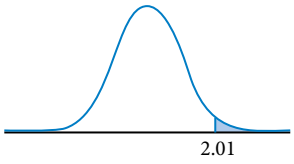
a



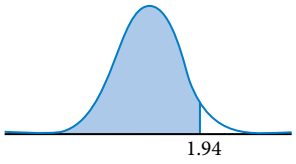
b



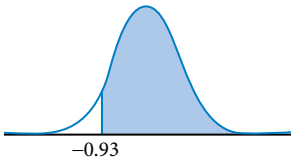
c



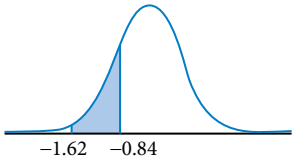
d



e



f



10 For the standard normal variable Z, find the approximate value for

a $P(0 < Z < 1)$ **b** $P(-2 < Z < 0)$ **c** $P(-1 < Z < 1)$

d $P(-2 < Z < 1)$ **e** $P(-2 < Z < 2)$

11 For the standard normal variable Z, find each probability.

a $P(0 < Z < 0.88)$ **b** $P(0 < Z < 1.76)$

c $P(0 < Z < 0.863)$ **d** $P(0 < Z < 2.038)$

12 Z is the standard normal variable. Find

a $P(Z < -1.07)$ **b** $P(Z > 2.15)$

c $P(Z > 1.96)$ **d** $P(-2.07 < Z < 1.63)$

13 Find each probability.

a $P(Z < -1.305)$ **b** $P(Z > 0.623)$

c $P(Z > 0.596)$ **d** $P(-1.307 < Z < 2.6)$

Exam practice

14 © QCAA 2021 1Q2 TECH-FREE 68% The table shows the time a technician has spent servicing photocopiers.

Time (in minutes)	Frequency
$0 \le t \le 5$	10
$5 \le t \le 10$	20
$10 \le t \le 15$	30
$15 \le t \le 20$	40
$20 \le t \le 25$	100

What is the probability that a given service required at least 10 minutes but less than 20 minutes?

A 0.15 **B** 0.35 **C** 0.70 **D** 0.85

15 © QCAA 2021 1Q8 TECH-FREE 56% The continuous random variable X has the probability density function

$$f(x) = \begin{cases} \dfrac{3}{x^2}, & 1 \le x \le \dfrac{3}{2} \\ 0, & \text{otherwise} \end{cases}$$

The mean of X is

A $\ln\left(\frac{3}{2}\right)$ **B** $\ln\left(\frac{27}{8}\right)$ **C** $\ln\left(\frac{9}{2}\right)$ **D** 1

16 For the standard normal variable Z, find $P(Z < -1.646 \text{ or } Z > 0.831)$.

A 0.6324 **B** 0.4498 **C** 0.2529 **D** 0.1265

17 © QCAA 2020 1Q4 TECH-FREE Pulse rates of adult men are approximately normally distributed with a mean of 70 and a standard deviation of 8. Which of the following choices correctly describes how to determine the proportion of men that have a pulse rate greater than 78?

A Determine the area to the left of $z = 1$ under the standard normal curve.

B Determine the area to the right of $z = 1$ under the standard normal curve.

C Determine the area to the right of $z = -1$ under the standard normal curve.

D Determine the area between $z = -1$ and $z = 1$ under the standard normal curve.

18 © QCAA 2020S 1Q7 TECH-FREE The distribution of marks in three subjects is given below. The mark received by a student in each of the subjects is also shown.

Subject	Mean	Standard deviation	Student mark
Science	60	15	70
Mathematics	68	6	72
Music	65	9	72

Standardised z-scores were used to compare the results and showed the student

A performed equally well in mathematics and science.

B performed equally well in mathematics and music.

C performed better in mathematics than in science.

D performed better in mathematics than in music.

19 © QCAA 2021 2Q14 (7 marks) The heights of students at School A are normally distributed with a mean of 165 cm and a standard deviation of 15 cm.

a Determine the probability that a student chosen at random from School A is shorter than 180 cm. *[1 mark]*

b Determine the minimum integer value of the height of a student who is in the top 2% of this distribution. *[3 marks]*

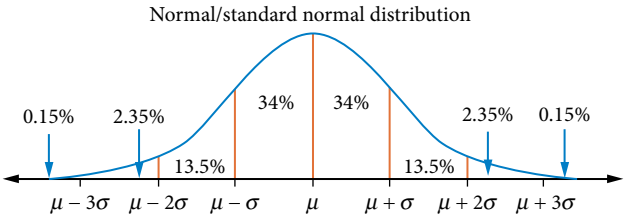
The heights of students at School B are also normally distributed. A student at School B has the same height as the height determined in part **b** but their corresponding z-score is 3.

c Determine which student's height ranks higher in terms of percentile for their school. *[3 marks]*

20 (5 marks) CF Callum scored 18 out of 25 for English and 15 out of 20 for Maths Methods. The means were 15 and 13 and the standard deviations were 8 and 5 for English and Maths respectively. Use z-scores to find the subject he performed better in.

21 (6 marks) CU Dania, who is 17, has an IQ of 110 and a height of 175 cm. The average IQ is 100 and the average height of 17-year-old females is 171 cm with a standard deviation of 12 cm. The standard deviation for IQ is 15. Which is more unusual: Dania's height as a 17-year-old, or her IQ?

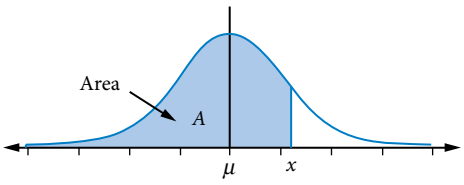
8.6 Quantiles and probabilities



For the standard normal distribution, the area between the mean and $z = 1$ is about 0.34, the area between $z = 1$ and $z = 3$ is about 0.16 and so on.

You can use your calculator to determine the X-value or Z-value that corresponds to a particular area under the distribution curve.

The **inverse cumulative distribution function** is the value that corresponds to a given area under the normal or standard normal distribution curve as shown here.

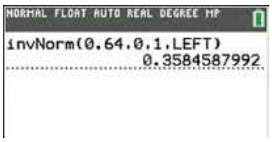


So, $P(X < x) = A$.

TECHNOLOGY Inverse normal distribution

Calculate the z-score that corresponds to an area of 0.64.

TI-84 Plus CE



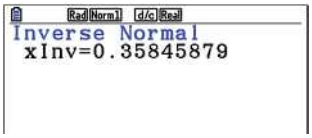
1 Press 2nd vars (dist) and then select **3: invNorm(**.

2 Enter the values:

Area = 0.64, $\mu = 0$, $\sigma = 1$

3 Set the Tail to Left.

Casio fx-CG50 AU



1 From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select Variable for the data source.

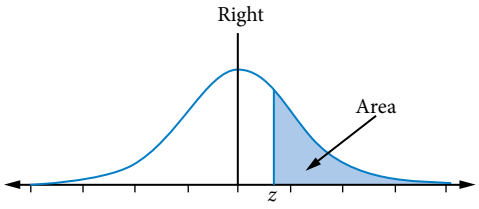
2 Set the Tail to Left.

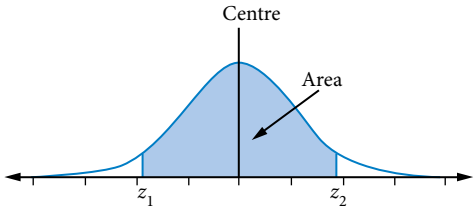
3 Enter the values:

Area = 0.64, $\mu = 0$, $\sigma = 1$

So, $P(Z < 0.3585) \approx 0.64$

1 You can see that the 'Tail' can be set to 'left', 'centre' or 'right'. The area for the 'right' and 'centre' settings are shown below.





Try each of these settings to see how they work.

2 Use your calculator to calculate the value of a in each of the following.

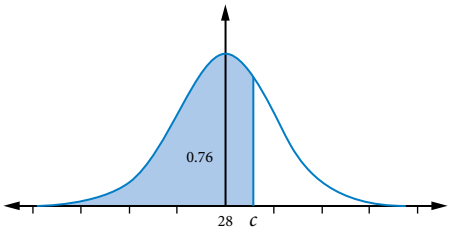
a $P(Z < a) = 0.35$ **b** $P(Z > a) = 0.67$ **c** $P(-a < Z < a) = 0.84$

WORKED EXAMPLE 19 Normal distributions using technology 1

If X is a normally distributed random variable, **use** a calculator to find

a c given that $P(X < c) = 0.76$, where $\mu = 28$ and $\sigma = 6$

b w given that $P(X > w) = 0.47$, where $\mu = 42.7$ and $\sigma = 11.8$

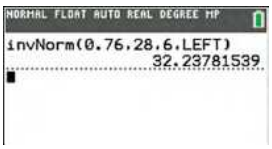
Steps	Working
a 1 Draw a sketch to help visualise the problem. Show the required area and the approximate position of c.	

TI-Nspire Continuous random variables and the normal distribution

TI-Nspire
Continuous random variables and the normal distribution

2 Use your calculator to find the probability.

TI-84 Plus CE

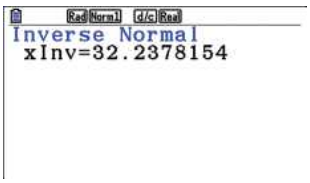


1 Press 2nd vars (dist) and then select **3: invNorm(**.

2 Enter the values:
Area = 0.76
$\mu = 28$
$\sigma = 6$

3 Set the Tail to Left.

Casio fx-CG50 AU



1 From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select **Variable** for the data source.

2 Set the Tail to Left.

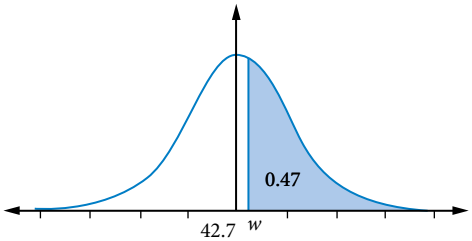
3 Enter the values:
Area = 0.76
$\mu = 28$
$\sigma = 6$

3 State the result.

If $P(X < c) = 0.76$ for $\mu = 28$ and $\sigma = 6$, $c \approx 32.24$.

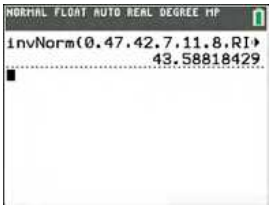
b 1 Sketch a diagram to help visualise the problem.

Show the required area and the approximate position of w.



2 Use your calculator to find the probability.

TI-84 Plus CE

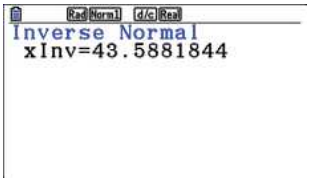


1 Press 2nd vars (dist) and then select **3: invNorm (**.

2 Enter the values:
Area = 0.47
$\mu = 42.7$
$\sigma = 11.8$

3 Set the Tail to Right.

Casio fx-CG50 AU



1 From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select **Variable** for the data source.

2 Set the Tail to Right.

3 Enter the values:
Area = 0.47
$\mu = 42.7$
$\sigma = 11.8$

3 State the result.

If $P(X > w) = 0.47$, where $\mu = 42.7$ and $\sigma = 11.8$, $w \approx 43.59$.

Quantiles (or **fractiles**) divide a probability distribution into equal fractions of the total population. A quantile is the value of a statistical variable below which a particular fraction of the scores lie.

Quartiles divide a set of values into quarters:

- 25% of all the values lie below the first quartile (Q_1 = lower quartile).
- 50% of all the values lie below the second quartile (Q_2 = middle quartile).
- 75% of all the values lie below the third quartile (Q_3 = upper quartile).

The second quartile is also called the **median**.

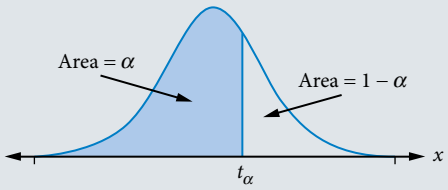
Similarly, **deciles** divide values into tenths and **percentiles** into hundredths.

40% of all values lie below D_4 and 73% lie below P_{73}.

Quantiles

For a continuous random variable X, a quantile t_α is the value of x such that $P(X < t_\alpha) = t_\alpha$, where $0 < \alpha < 1$.

t_α is the value of x that divides the distribution in the ratio $\alpha : 1 - \alpha$.



For example, the median (middle quantile, 50th percentile) is the value of x such that $\alpha = 0.5$, i.e. $P(X < x) = 0.5$.

WORKED EXAMPLE 20 Normal distributions using technology 2

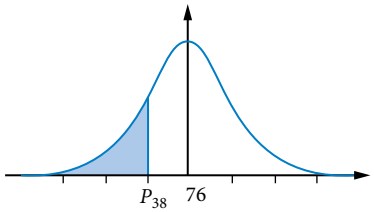
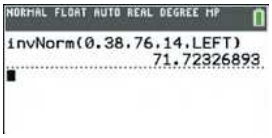
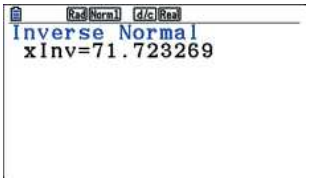
X is a normally distributed random variable with a mean of 76 and a standard deviation of 14.

Calculate

a the 38th percentile (P_{38})

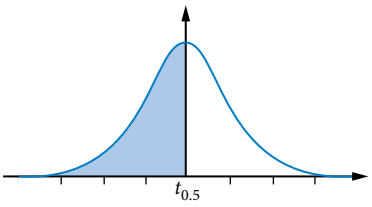
b the median

c a, given that $P(X > a) = 0.235$

Steps	Working
a 1 Sketch a diagram to help visualise the problem. $P_{38} = t_{0.38}$	
2 Use the calculator's Inverse Normal function.	**TI-84 Plus CE**  1 Press 2nd vars (dist) and then select **3: invNorm(**. 2 Enter the values: Area = 0.38 $\mu = 76$ $\sigma = 14$ 3 Set the Tail to Left. **Casio fx-CG50 AU**  1 From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select **Variable** for the data source. 2 Set the Tail to Left. 3 Enter the values: Area = 0.38 $\mu = 76$ $\sigma = 14$
3 State the result.	$P_{38} \approx 71.72$

TI-Nspire
Continuous random variables and the normal distribution

b 1 Sketch a diagram to help visualise the problem.



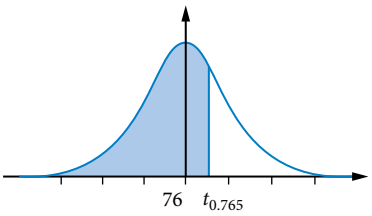
2 For the normal distribution, the median and the mean are the same.

Median = 76

c 1 Sketch a diagram to help visualise the problem.

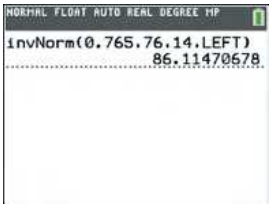
$P(X > a) = 0.235$ is the same as $P(X < a) = 1 - 0.235 = 0.765$.

You want to find $t_{0.765}$.



2 Use the calculator's Inverse Normal function.

TI-84 Plus CE



1 Press 2nd vars (dist) and then select **3: invNorm(**.

2 Enter the values:

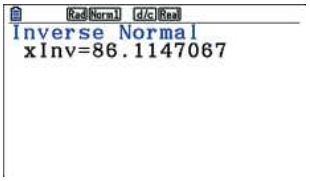
Area = 0.765

$\mu = 76$

$\sigma = 14$

3 Set the Tail to Left.

Casio fx-CG50 AU



1 From the Statistics screen, press F5 (Dist), F1 (Norm), F3 (InvN). If necessary, select **Variable** for the data source.

2 Set the Tail to Left.

3 Enter the values:

Area = 0.765

$\mu = 76$

$\sigma = 14$

3 State the result.

For $P(X > a) = 0.235$, $a \approx 86.11$

TI-Nspire
Continuous random variables and the normal distribution

In the above example, the calculation could have been completed with an area = 0.235 and the tail set to 'right'.

For a normal distribution, it is possible to calculate the values of the mean or standard deviation using a known quantile and the standard normal distribution. This is done by first calculating the z-score and then using the formula $z = \frac{x - \mu}{\sigma}$ to calculate the unknown mean or standard deviation.

WORKED EXAMPLE 21 Using the z score

a The continuous random variable X is normally distributed with a mean of 8.6.
If $t_{0.36} = 7.7$, find the standard deviation of the distribution.

b Variable X is normally distributed so that the 84th percentile is 45.16 and the standard deviation is 6.8.
Find the mean of the distribution.

Steps		Working
a	**1** Write the probability statement.	$P(X < t_\alpha) = \alpha$
	2 Substitute for known values.	$P(X < 7.7) = 0.36$
	3 Find the corresponding z-score.	$P(Z < z) = 0.36$
	4 Use a calculator as previously described.	$z \approx -0.3585$
	5 Write the formula for z.	$z = \frac{x - \mu}{\sigma}$
	6 Substitute for known values.	$-0.3585 = \frac{7.7 - 8.6}{\sigma}$
	7 Evaluate.	$\sigma = \frac{0.9}{0.3585}$ $= 2.510\,460\ldots$
	8 State the result.	The standard deviation is about 2.51.
b	**1** Write the probability statement.	$P(X < 45.16) = 0.84$
	2 Find the corresponding z-score.	$P(Z < z) = 0.84$
	3 Use a calculator as previously described.	$z \approx 0.9945$
	4 Write the formula for z.	$z = \frac{x - \mu}{\sigma}$
	5 Substitute for known values.	$0.9945 = \frac{45.16 - \mu}{6.8}$
	6 Evaluate.	$\mu = 45.16 - (0.9945 \times 6.8)$ $= 38.3974$
	7 State the result.	The mean is about 38.40.

EXERCISE 8.6 Quantiles and probabilities

ANSWERS p. 413

Recap

1 The lengths of tailor fish are normally distributed, with a mean of 20 cm and a standard deviation of 4 cm. Two tailor fish are caught and found to have lengths of 23.5 cm and 17.4 cm respectively.

a Sketch the graph of the distribution, showing lengths and z-scores.

b What proportion of tailor would be longer than the first fish caught?

c Which of the caught fish is closer in length to a typical tailor?

2 It is known that the stem lengths of a crop of flowers are normally distributed with a mean of 46 cm and a standard deviation of 5 cm.

a What is the z-score for a stem length of 37 cm?

b What is the stem length for a z-score of 2.1?

Mastery

3 WORKED EXAMPLE 19 Let X be a normal random variable with $\mu = 10$ and $\sigma = 3$. Find

a c such that $P(X < c) = 0.95$ **b** k such that $P(X < k) = 0.28$

4 Let X be a normal random variable with $\mu = 45$ and $\sigma = 11$. Find

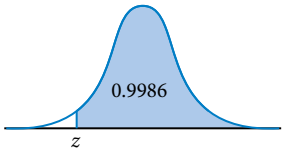
a c such that $P(X < c) = 0.42$ **b** k such that $P(X < k) = 0.876$

5 Let X be a normal random variable with $\mu = 105$ and $\sigma = 14$. Find

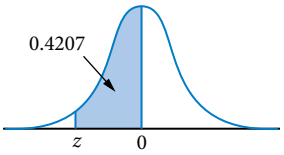
a c such that $P(X > c) = 0.35$ **b** k such that $P(X > k) = 0.86$

6 Find the standard normal score, Z, for each diagram.

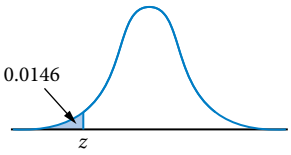
a



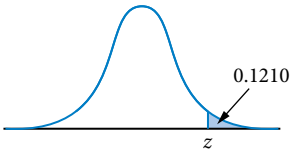
b



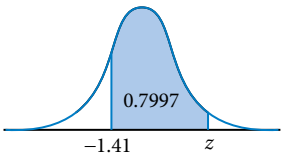
c



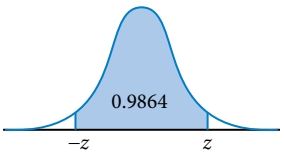
d



e



f



7 Z has a standard normal distribution. Find the value of a if

a $P(0 < Z < a) = 0.3710$ **b** $P(0 < Z < a) = 0.4933$

c $P(Z < a) = 0.7019$ **d** $P(Z < a) = 0.6776$

e $P(Z > a) = 0.821$ **f** $P(Z > a) = 0.4308$

8 Use your graphics calculator to find each value for the standard normal distribution.

a $P(-1.356 < Z < 1.873)$ **b** $P(Z < 0.823)$

c $P(Z > -2.306)$ **d** $P(Z < 1.6)$

e $P(Z < -0.628 \text{ or } Z > 1.636)$

9 For the set of standard normal scores, Z, find the value of a given that

a $P(Z < a) = 0.4093$ **b** $P(Z < a) = 0.9056$

c $P(Z > a) = 0.503$ **d** $P(Z > a) = 0.2657$

e $P(-a < Z < a) = 0.866$ **f** $P(-a < Z < a) = 0.2106$

10 WORKED EXAMPLE 20 For a standard normal distribution, calculate

a the 45th percentile **b** P_{88}

c the 0.53 quantile **d** the third quartile (Q_3)

e the 5th decile

11 X is a normally distributed random variable with a mean of 58 and a standard deviation of 12. Calculate

a P_{37} **b** P_{72} **c** the 0.65 quantile

d Q_1 **e** $t_{0.86}$

12 X is a normally distributed random variable with a mean of 115 and a standard deviation of 18. Calculate

a $t_{0.32}$ **b** $t_{0.96}$ **c** $t_{0.14}$

d $t_{0.77}$ **e** $t_{0.82}$

13 WORKED EXAMPLE 21 The continuous random variable X is normally distributed with a mean of 100. If $t_{0.75} = 110.12$, find the standard deviation of the distribution.

14 The variable X is normally distributed so that the 38th percentile is 30.47 and the standard deviation is 5. Find the mean of the distribution.

15 X is distributed normally with a standard deviation of 4.5, and $t_{0.68} = 38.60$.
Find the mean for X, correct to one decimal place.

Exam practice

16 For a standard normal distribution, calculate D_2.

A −1.213 **B** −0.8416 **C** 0.2 **D** 0.8267

17 © QCAA 2021 1Q10 TECH-FREE 83% Handspans of teenagers are approximately normally distributed, with a mean of 15 cm and a standard deviation of 2 cm.
Which of the following groups is expected to be largest?

A teenagers with handspans that are between 7 cm and 11 cm
B teenagers with handspans that are between 11 cm and 15 cm
C teenagers with handspans that are between 13 cm and 17 cm
D teenagers with handspans that are between 17 cm and 21 cm

18 © QCAA 2021 2Q1 86% The scores obtained on a test can be assumed to be normally distributed with a mean of 102 and a standard deviation of 19.
What proportion of scores are over 113?

A 0.2813 **B** 0.5789 **C** 0.7187 **D** 0.8216

19 (2 marks) X is a normally distributed variable with a mean of 18.7 and the 83rd percentile is 21.37.
Find the value of σ, giving your answer correct to one decimal place.

20 (2 marks) X is distributed normally with $\sigma = 22.8$ and $Q_1 = 111.02$.
Find μ, giving your answer correct to one decimal place.

21 (4 marks) Give your answers correct to one decimal place.

a The continuous random variable X is normally distributed with a mean of 22.6. If $t_{0.58} = 23.69$, find the standard deviation of the distribution. *[2 marks]*

b The variable X is normally distributed so that the 45th percentile is 12.11 and the standard deviation is 2.3. Find the mean of the distribution. *[2 marks]*

22 (3 marks) X is distributed normally such that $t_{0.47} = 37.649$ and $t_{0.72} = 45.086$. Find the mean and the standard deviation of X, giving your answers correct to one decimal place.

23 (3 marks) X is distributed normally such that $P(X < 63.589) = 0.324$ and $P(X > 79.848) = 0.258$.
Find the mean and the standard deviation of X, giving your answers correct to one decimal place.

24 © QCAA 2022 2Q16 (4 marks) CF The time spent waiting in a queue at a certain supermarket is given by $(X + 11)$ minutes, where X is a random variable with a probability density function

$$f(x) = \begin{cases} \dfrac{a(4-x^2)}{32}, & -2 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the probability of waiting between 10 and 12 minutes in a queue at this supermarket.

25 © QCAA 2020 2Q16 (4 marks) CF Bottles of soft drink should contain a volume with a mean of 591 mL, but some variation is expected. Any bottle at or below the 20th percentile of the volume distribution is rejected. A percentile is a measure in statistics that shows the values below which a given percentage of observations occur.

Thirty-five per cent of the bottles contain 593 mL or more of soft drink.

Assuming the volumes are normally distributed, determine the smallest volume (in mL) that will be accepted.

Worksheets
Normal distribution worded problems 1
Normal distribution worded problems 2
Continuous random variables assignment
Continuous random variables extension

8.7 Problems involving normal distributions

The normal distribution can be used to model many situations involving large populations. The heights and weights of people, the length of time a phone battery lasts, cholesterol levels of adults, and the weight of sugar in bags filled by a machine are all approximately normal distributions.

WORKED EXAMPLE 22 Probability problems

A survey indicates that Australians use their mobile phones for an average of 22 months before upgrading to a new one. The distribution is approximately normal, with a standard deviation of 4.5 months.

a Approximately what percentage of mobile phone users in Australia would you expect to keep their mobile phones for less than 2 years before upgrading?

b What is the probability that a randomly selected mobile phone user keeps their phone for between 15 months and $2\frac{1}{2}$ years before upgrading?

Steps	Working
a 1 State the parameters.	$\mu = 22$ $\sigma = 4.5$
2 Write the required proportion as a probability statement.	$P(-\infty < X \le 24)$
3 Use your calculator cdf to calculate the required probability.	$P(-\infty < X \le 24) = 0.671\,639...$
4 Round off and state the result.	About 67.2% of mobile phone users will keep their phones for less than 2 years before upgrading.
b 1 Write the required proportion as a probability statement.	$P(15 < X \le 30)$
2 Use your calculator cdf to calculate the required probability.	$P(15 < X \le 30) = 0.902\,372\ldots$
3 Round off and state the result.	About 90.2% of mobile phone users will keep their phones for between 15 months and $2\frac{1}{2}$ years before upgrading.

If you say that someone is 178 cm tall, it is assumed that the measurement has been given to the nearest centimetre. This means that the person's actual height is somewhere between 177.5 cm and 178.5 cm.

WORKED EXAMPLE 23 Probability models

The chest measurements of 18-year-old male footballers are normally distributed with a mean of 96 cm and a standard deviation of 7 cm. **Determine** the probability that an 18-year-old male footballer has a chest measurement of 99 cm.

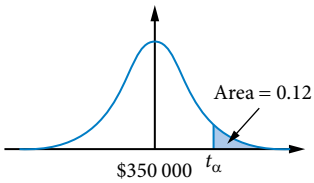
Steps	Working
1 Write the required probability.	$P(98.5 \leq X < 99.5)$
2 Use your calculator cdf to calculate the required probability.	$P(98.5 \leq X < 99.5) = 0.051\,954\ldots$
3 Round off and state the result.	There is about a 5.2% chance that an 18-year-old male footballer has a chest measurement of 99 cm.

WORKED EXAMPLE 24 Business models

A small business owner knows that the annual total sales are normally distributed with a mean of $350 000 and a standard deviation of $58 000.

a What is the probability that the company's sales will fall within $65 000 of the mean?

b **Calculate** the annual sales figure that has only a 12% chance of being exceeded.

Steps	Working
a 1 Calculate the boundary values.	Lower bound = $350 000 − $65 000 = $285 000 Upper bound = $350 000 + $65 000 = $415 000
2 Write the required proportion as a probability statement.	$P(285\,000 < X < 415\,000)$
3 Use your calculator cdf to calculate the required probability.	$P(285\,000 < X < 415\,000) = 0.737\,579\ldots$
4 State the result.	There is about a 73.8% chance that annual sales will be within $65 000 of the mean.
b 1 Sketch a diagram to help visualise the problem.	
2 Find the required quantile.	$P(X > t_\alpha) = 0.12$ So $P(X < t_\alpha) = 1 - 0.12$ $= 0.88$
3 Use the inverse normal function on your calculator.	$t_\alpha = 418\,149.233\ldots$
4 State the result.	There is only a 12% chance of exceeding annual sales of about $418 149.

EXERCISE 8.7 Problems involving normal distributions

ANSWERS p. 413

Recap

1 If X is a normally distributed random variable, use a calculator to find

a b, given that $P(X < b) = 0.25$, where $\mu = 32$ and $\sigma = 8$

b m, given that $P(X > m) = 0.85$, where $\mu = 27.6$ and $\sigma = 6.3$

2 X is a normally distributed random variable with a mean of 54 and a standard deviation of 11. Calculate

a the 63rd percentile (P_{63})

b the third quartile

c a, given that $P(X > a) = 0.44$

Mastery

3 WORKED EXAMPLE 22 The cholesterol levels of Australian adults are assumed to be normally distributed with a mean of 5.6 millimoles per litre (mmol/L) and a standard deviation of 1.2 mmol/L. What is the probability that a randomly selected Australian adult has a cholesterol level

a less than 4.3 mmol/L?

b between 4.9 mmol/L and 5.8 mmol/L?

c greater than 8.5 mmol/L?

4 The average rainfall in a northern Queensland town in January is 268.5 mm, with a standard deviation of 84.5 mm. What is the probability that the rainfall for a particular January is

a greater than 220 mm?

b less than 190 mm?

5 Suppose that X is a random variable representing the reaction time, in seconds, in a simulated driving experiment for adults who have been given a specific amount of alcohol. Assuming that X has a normal distribution with a mean of 0.8 s and a standard deviation of 0.2 s, find each probability.

a $P(X < 0.5)$

b $P(X > 1.2)$

c $P(0.75 < X < 1.25)$

6 WORKED EXAMPLE 23 A survey was conducted to find the number of hours per week that adults spend on home computers. Analysis of the survey results showed that the number of hours was normally distributed, with a mean of 8 hours and a standard deviation of 1 hour. Find the probability that the time spent on the home computer by a randomly selected adult is

a less than 5.5 hours per week

b between 4.5 and 9.5 hours per week

c more than 10 hours per week

d 6 hours per week.

7 The time that gym members use a treadmill is normally distributed, with a mean of 24 minutes and a standard deviation of 7 minutes. A gym member is randomly selected. Find the probability that the gym member uses a treadmill for the following times.

a less than 17 minutes

b between 16 and 27 minutes

c more than 28 minutes

d 30 minutes

8 WORKED EXAMPLE 24 Scores for a public service examination are normally distributed, with a mean of 78 and a standard deviation of 7.5. To be eligible for a public service position, a candidate must score in the top 5%. What is the lowest score a candidate can earn and still be eligible for employment?

9 Mensa is a non-profit organisation open to people who score at the 98th percentile or higher on a standardised IQ test.

a If IQs are normally distributed, with a mean of 100 and a standard deviation of 16, what is the minimum IQ required for admission to Mensa?

b If three individuals are chosen at random from the general population, what is the probability that none satisfy the minimum requirement for Mensa?

10 Antoine had a score of 65% for English, but only 57% for Physics. The class mean and standard deviation for English were 62% and 8% respectively; for Physics they were 52% and 10%. In which subject did Antoine appear to do better? Explain your reasoning.

11 In a study of people's digestion rates, normal meals were found to have a mean digestion time of 9.4 hours with a standard deviation of 2.1 hours. Based on this data, what proportion of people can be expected to digest a normal meal in 8 hours or less?

12 The hourly earnings of plumbers are normally distributed with a mean of \$56 per hour and a standard deviation of \$14.

a From a group of 20 plumbers, how many would you expect to be earning over \$70 per hour?

b From 50 plumbers, how many would you expect to be earning less than \$60 per hour?

c What is the interquartile range ($Q_3 - Q_1$) for the hourly earnings of plumbers?

13 Two basketball players have average game totals of 25 points and 29 points respectively. The first player has a standard deviation of 9 points and the second a standard deviation of 5 points.

Is the first or the second player more likely to score more than 37 points in a particular game?

14 The 'life' of young women's fashions measures how long a store can expect a fashion item to sell in sufficient quantities to be worth keeping in stock. For some types of fashions, it is about 2 months (60 days), but with a wide variation, reflected in the standard deviation of 19 days.

What percentage of these fashions would be expected to

a have a life of over 3 months (90 days)?

b have a life of less than a month?

c be a '9-day-wonder' (less than 10 days)?

15 Ten-year-old girls can do an average of 11.2 push-ups with a standard deviation of 5.4 push-ups. Which is more unusual for 10-year-old girls: being able to do only 2 push-ups or being able to do 17?

16 The daily output of a production line varies according to a normal distribution with a mean of 163 units of production and a standard deviation of 4.5 units. What is the probability that the daily output will

a fall below 160 units?

b exceed 170 units?

c be between 160 and 165 units?

Exam practice

17 The amount of time per week a student uses a computer in the school computer lab is normally distributed, with a mean of 6.2 h and a standard deviation of 0.9 h. Find the probability that a randomly selected student uses a lab computer for fewer than 4 hours per week.

A 0.7531 **B** 0.5056 **C** 0.2591 **D** 0.0073

18 © QCAA 2020 1Q7 TECH-FREE The life expectancy (in years) of an electronic component can be represented by the probability density function

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

The probability that the component lasts between 1 and 10 years is

A 0.010 **B** 0.100 **C** 0.900 **D** 0.990

19 © QCAA 2020S 2Q9 The birth mass of babies is normally distributed with a mean 3500 grams and a standard deviation 500 grams.

The probability that the birth mass of a baby is less than 3200 grams is

A 0.01 **B** 0.06 **C** 0.22 **D** 0.27

20 (4 marks) The weights of adult male Australian cattle dogs are normally distributed, with a mean of 15 kg and a standard deviation of 3 kg. A cattle dog is randomly selected.

a Find the probability that the dog's weight is less than 13 kg. *[1 mark]*

b Find the probability that the dog's weight is between 13 kg and 17 kg. *[1 mark]*

c Find the probability that the dog's weight is more than 17 kg. *[1 mark]*

d Find the weight of a dog that is heavier than 85% of cattle dogs. *[1 mark]*

21 (4 marks) CF A continuous random variable, X, has a probability density function defined by

$$f(x) = k(x-1)(2-x) \text{ for } 0 \leq x \leq 2$$

Find the value of k.

22 (4 marks) CF A continuous random variable, X, has a probability density function defined by

$$f(x) = \frac{2}{11}\left(\frac{1}{3}x + 2\right) \text{ for } -2 \leq x \leq 1$$

Calculate the median.

23 (3 marks) CF The lengths of bolts produced by a factory are approximately normally distributed. Bolts shorter than 18.25 cm and longer than 19.85 cm are rejected as unacceptable. A quality control report notes that, on average, 4% of bolts are rejected as undersized and 6% are rejected as oversized. Find the mean and standard deviation of the distribution.

24 (4 marks) CF Which is more unusual: a mass of 140 g or a diameter of 12 cm for apples that are normally distributed with averages of 150 g and 10 cm, and standard deviations of 8 g and 1.5 cm respectively? Explain your answer.

25 (4 marks) CF A filling machine delivers gelatine powder into packets. The amount of gelatine in the packets is normally distributed with a standard deviation of 6 g. The advertised weight of a packet is 375 g.

a If the mean is 380 grams, find the proportion of packets that will be underweight (less than 375 g). *[2 marks]*

b Find the required mean value to ensure that only 2% of packets are underweight. *[2 marks]*

26 (5 marks) CF A manufacturer produces batteries for which the life span is normally distributed. The average life of a battery is 4000 hours and the standard deviation is 60 hours.

a What percentage of batteries have a life span of more than 4130 hours? *[2 marks]*

b Would it be unusual for a battery to have a life span that is more than 4130 hours? Explain your answer. *[3 marks]*

27 © QCAA 2021 2Q20 (3 marks) CU The random variable B is normally distributed with a mean of 0 and a standard deviation of 1. Determine the probability that the quadratic equation $x^2 + 3x + 2B = 0$ has real roots.

28 (5 marks) CU A retailer sells a brand of car tyre that has a life expectancy that is normally distributed, with a mean life of 40 000 km and a standard deviation of 3500 km. The retailer wants to give a guarantee for free replacement of tyres that don't wear well. How should the guarantee be stated if the retailer is willing to replace approximately 10% of the tyres sold?

29 (5 marks) CU A car muffler is designed so that its lifetime (in months) is approximately normally distributed with a mean of 39.6 months and a standard deviation of 5.7 months. The manufacturer has decided to use a marketing strategy in which any muffler that does not last for at least 27 months is replaced for free. Should the manufacturer be concerned that the marketing strategy might have a significant impact on profitability? Explain your answer.

30 © QCAA 2024 2Q19 (6 marks) CU The normal distribution probability density function is $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, with the parameters mean, μ, and standard deviation, σ.

The speeds of electric scooter (e-scooter) riders on a particular section of a bike path are approximately normally distributed with a mean of 18 km/h. It is known that $p(10) = 0.0135$.

The speed limit for e-scooters on this section of bike path is 23 km/h.

A speed camera is set up and records the speeds of 75 e-scooter riders. Every rider travelling faster than the speed limit is given a $143 fine. Before setting up the speed camera, the following suggestion was made.

The total of the fines expected to be issued will be more than $1500.

Evaluate the reasonableness of this suggestion.

Video
Exam question analysis: Continuous random variables and the normal distribution

EXAM QUESTION ANALYSIS

© QCAA 2020S 2Q19 (5 marks)

Consider the following information when completing this question.

If X is normally distributed with mean μ and standard deviation, σ then

$$X \sim N(\mu, \sigma^2)$$

If X and Y are two independent random normal variables such that

$$X \sim N(\mu_1, \sigma_1^2) \text{ and } Y \sim N(\mu_2, \sigma_2^2)$$

then

$$X - Y \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

and

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Contestant A and Contestant B have trained to navigate an obstacle course.

The times for each contestant to run the course are independent of each other.

On any given day, the time to run the obstacle course for each contestant is normally distributed.

Let A be the run time (in minutes) for contestant A and B be the run time (in minutes) for contestant B with

$A \sim N(80, 10^2)$

$B \sim N(78, 12^2)$

Determine the probability that contestant B runs the obstacle course faster than contestant A.

Reading the question

- Recognise use of normally distributed variables
- Difference of times for two contestants given
- Need to correctly determine the values for the mean and standard deviation

Thinking about the question

- Justify procedure used to determine the solution.
- Use an appropriate mathematical representation to communicate approach.
- Solve the problem to find the probability.

Worked solution (✓ = 1 mark)

For B to run faster than A, B's time must be less than A's time.

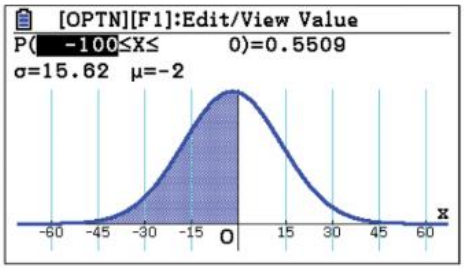
This means that $B - A < 0$.

The difference between their times is a normal distribution, given by:

$B - A \sim N(78 - 80, 12^2 + 10^2)$ ✓ recognises the use of a normally distributed variable

mean $= -2$

standard deviation $= \sqrt{244}$ ✓



Contestant B is faster when

$\textbf{time}_{\text{contestant B}} - \textbf{time}_{\text{contestant A}} < 0$ ✓ justifies procedure

Use normal probability application on a graphics calculator:

lower limit = –100, upper limit = 0, ✓

mean = –2, σ = 15.6205

There is an **approximately 55%** chance that contestant B will run the obstacle course faster than contestant A. ✓ solves the problem

8 Chapter summary

Continuous random variables

- A **continuous random variable** is a random variable with values from an interval or continuum. A **continuous probability distribution (or function)** shows the probabilities of a continuous random variable. The values of the **probability function** (or **probability distribution**) are written as $P(X = x)$ or $p(x)$.

Probability density functions

- For a continuous random variable X, the **probability density function** (**pdf**) $p(x)$ over an interval $[a, b]$ is defined as

  $P(a \le X \le b) = \int_a^b p(x)\,dx$, $p(x) \ge 0$ for all $x \in [a, b]$

  and $\int_a^b p(x)\,dx = 1$.

- The **closed interval** $a \le x \le b$ is written $[a, b]$. The **open interval** $a < x < b$ is written (a, b). The interval $a \le x < b$ is written $[a, b)$, and the interval $a < x \le b$ is written $(a, b]$.
- For X defined on the interval (a, b) with a probability density function $f(x)$, the **cumulative distribution function** (**cdf**) is given by

  $F(x) = \int_a^x f(t)dt$.

- A **uniform distribution**, $p(x)$, on the interval $[a, b]$, is constant over the domain of X. The uniform distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

- A **triangular distribution**, $f(x)$, with minimum value (a), peak value (c), and maximum value (b) is defined as:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$

The mean and variance of continuous random variables

- The **expected value** for the continuous random variable X on the interval $[a, b]$, with pdf $p(x)$, is defined as $E(X) = \mu = \int_a^b x\,p(x)\,dx$. For a uniform distribution, $E(X) = \mu = \dfrac{a+b}{2}$.
- For the continuous random variable X on the interval $[a, b]$, with $E(X) = \mu$, the **variance** is $Var(X) = \int_a^b p(x)(x-\mu)^2\,dx$ and the **standard deviation** is $\text{SD}(X) = \sigma = \sqrt{Var(X)}$.
- For a uniform distribution on the interval $[a, b]$, $Var(X) = \dfrac{(b-a)^2}{12}$ and $\text{SD}(X) = \dfrac{b-a}{2\sqrt{3}}$.
- For the continuous random variables X and Y such that $Y = aX + b$

  $E(aX + b) = aE(X) + b$

  $Var(aX + b) = a^2 Var(X)$.
- The probability density function for the **normal distribution** is $p(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where the mean = μ and the standard deviation = σ. The mean, median and mode are equal.

Normal distributions

- For a normal distribution
  - $P(\mu - \sigma \le X \le \mu + \sigma) \approx 0.68$
  - $P(\mu - 2\sigma \le X \le \mu + 2\sigma) \approx 0.95$
  - $P(\mu - 3\sigma \le X \le \mu + 3\sigma) \approx 0.99$
- $X \sim N(\mu, \sigma^2)$ means the continuous random variable X has a normal distribution with mean μ and variance σ^2.

z-scores

- The *z*-**score** (or *z*-**value**) is $z = \dfrac{x-\mu}{\sigma}$.
- The probability density function for the **standard normal distribution** is $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

Quantiles and probabilities

- The **inverse cumulative normal distribution function** calculates the value that corresponds to a given area under the normal or standard normal distribution curve.
- **Quantiles** (or **fractiles**) divide a probability distribution into equal groups. **Quartiles** divide a set of values into quarters, **deciles** divide values into tenths and **percentiles** into hundredths. The second quartile (Q_2) is the **median**. 40% of all values lie below D_4 and 73% lie below P_{73}.
- For a continuous random variable X, a quantile t_α is the value of x such that $P(X < t_\alpha) = \alpha$, where $0 < \alpha < 1$.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (15 marks): 4 short response questions

Total: 18 marks

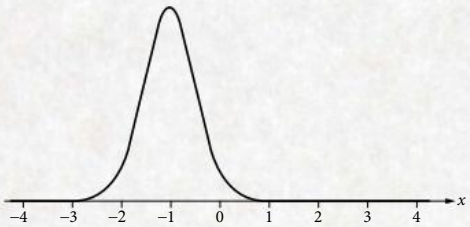
Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

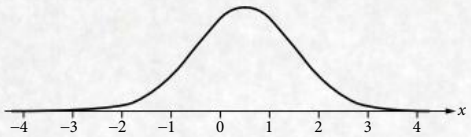
3 marks

1 © QCAA 2022 1Q5 Which normal distribution curve represents a normal distribution with a mean of 1 and a standard deviation of 0.5?

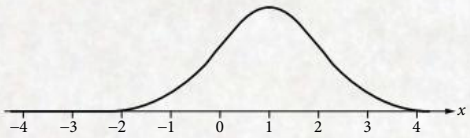
A



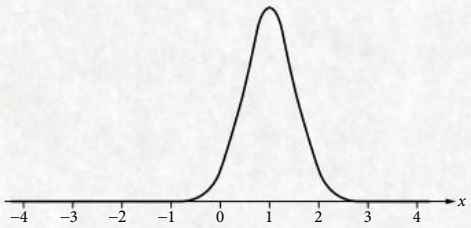
B



C



D

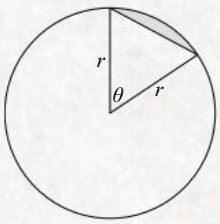


2 © QCAA 2022 1Q4 The weekly amount of money a company spends on repairs is normally distributed, with a mean of \$1200 and a standard deviation of \$100.

Given that $P(Z \leq -2.5) = 0.0062$ and $P(Z > 1) = 0.1587$, where Z is a standard normal random variable, determine the probability that the weekly repair costs will be between \$950 and \$1300.

A 0.6525 **B** 0.6587 **C** 0.8351 **D** 0.8413

3 © QCAA 2022 1Q7 A circle with radius r and internal angle θ has a shaded segment as shown.



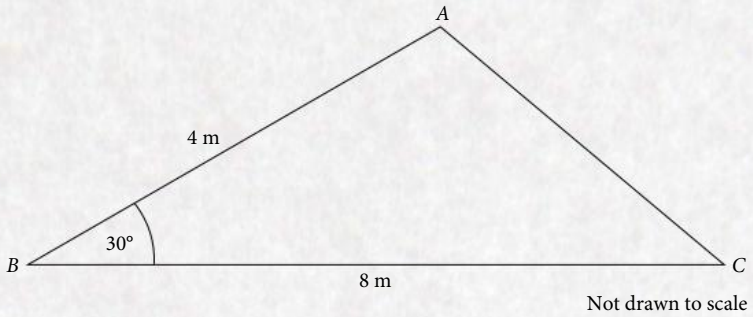
If θ is in radians, the area of the shaded segment is

A $\frac{r^2}{2}\left(\frac{\theta\pi}{180} - \sin(\theta)\right)$ **B** $\frac{r^2}{2}\left(\theta - \sin(\theta)\right)$ **C** $\frac{r^2}{4}\left(\frac{\theta\pi}{90} - 1\right)$ **D** $\frac{r^2}{2}(\theta - 1)$

Section 2 3 short response questions

15 marks

4 © QCAA 2020 1Q14 MODIFIED (4 marks) Determine the area of the triangle shown and the length of AC.



Not drawn to scale

5 © QCAA 2020S 1Q17 (7 marks) A continuous random sample variable X has a probability density function f, defined by

$$f(x)=\begin{cases} a+bx^2 & 0\leq x\leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E(X)=\frac{3}{5}$, determine a and b.

6 © QCAA 2022 1Q18 (4 marks) A percentile is a measure in statistics showing the value below which a given percentage of observations occur.

The continuous random variable X has the probability density function

$$f(x)=\begin{cases} 2x-2, & 1\leq x\leq 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the 36th percentile of X.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (16 marks): 4 short response questions

Total: 19 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2020S 2Q6 The side lengths of a triangle are in the ratio 2 : 3 : 4. The smallest angle in this triangle is

A 0.51° **B** 0.81° **C** 28.96° **D** 46.57°

2 © QCAA 2022 2Q5 A continuous random variable X has the probability density function

$$f(x)=\begin{cases}\dfrac{\cos(x)}{2}, & \dfrac{-\pi}{2}\le x\le\dfrac{\pi}{2}\\ 0, & \text{otherwise}\end{cases}$$

The standard deviation of X is

A 0.467 **B** 0.684 **C** 1.211 **D** 1.467

3 © QCAA 2022 2Q6 A stall at the school fete sells cups of lemonade. Assuming the amount of lemonade in a cup is normally distributed with a mean of 60 mL and a standard deviation of 3 mL, 80% of the cups contain more than

A 52.4 mL **B** 57.5 mL **C** 61.6 mL **D** 62.5 mL

Section 2 4 short response questions

16 marks

4 © QCAA 2022 2Q12 (4 marks) Suppose that the distance travelled by vehicles in a year can be modelled by a normal distribution. In 2021 vehicles travelled a mean of 13 700 km with a standard deviation of 3400 km.

a Determine the probability that a vehicle chosen at random travelled less than 12 000 km in 2021. *[2 marks]*

b Determine the value of x where 60% of vehicles travelled less than x km in 2021. *[2 marks]*

5 © QCAA 2020S 2Q16 (4 marks) Radar station R picks up signals from two ships.

Ship A is stationary and on a bearing 041°T from R and is 65 kilometres away.

Ship B is on a bearing 295°T from R and is 53 kilometres away.

Determine the time (in hours) for ship B to travel to ship A if ship B can travel at 30 kilometres per hour.

6 © QCAA 2020S 2Q17 (3 marks) During one 30-day period, the rate at which pollution passes into a nearby lake is measured every six days and the results are given in the table below.

Day (t)	0	6	12	18	24	30
Rate of pollution in units per day $p(t)$	7	8	10	13	17	22

This information can be modelled as a quadratic function.

Determine the total amount of pollution entering the lake during this 30-day period.

7 © QCAA 2023 2Q17 (5 marks) Model bridges were constructed for a competition. The models that could support the heaviest loads before collapsing were given awards.

The load results of the competition were normally distributed, with a mean of 1.36 kg and a standard deviation of 0.12 kg.

Three awards categories were used: honours for the top 15% of load results; distinction for the next 15%; and a commendation for the next 15%.

The model bridge constructed by Finley only just missed out on a commendation. Kirby's model bridge only just qualified for honours. Determine the difference, to the nearest gram, between the loads supported by Finley and Kirby's models.

CHAPTER

9 SAMPLING AND PROPORTIONS



Syllabus coverage
Nelson MindTap chapter resources
Terminology
9.1 Random samples and bias
9.2 Random sampling
9.3 Sample proportions
9.4 Variation of samples
9.5 Sample size and normality
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Sampling and proportions

Syllabus coverage

UNIT 4, TOPIC 4: SAMPLING AND PROPORTIONS

Random sampling

- Understand the concept of a random sample.
- Understand sources of bias in samples, and procedures to ensure randomness.
- Identify and use procedures to ensure randomness.
- Recognise the use of graphical displays of real and simulated data of random samples from various types of distributions, including uniform, Bernoulli, binomial and normal.

Sample proportions

- Understand the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and the standard deviation $\sqrt{p(1-p)/n}$ of the sample proportion $\hat{p}$, where n is the sample size.
- Recognise and use the approximate normality of the distribution of $\hat{p}$ for large samples.
- Use repeated random sampling data, for a variety of values of p and a range of sample sizes, to examine the distribution of $\hat{p}$ and the approximate standard normality of $\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}}$, where the closeness of the approximation depends on both n and p.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 30,
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Videos (2):
9.2 Stratified sampling
Exam question analysis Sampling and proportions

Prior learning (1):
9.1 Sampling and proportions

Worksheets (4):
9.3 Sample proportions
9.5 Sample proportion calculations
• Sample proportion probabilities
Cumulative exams Maths Methods formula sheet

TI Nspire CX II instructions (1):
9.2, 9.4, 9.5 Sampling and proportions

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Terminology

biased sample
census
cluster sample
completion bias
convenience sample
design bias
fair sample
interviewer bias
judgement (purposive) sample
parameter
population
quota sample
random sample
recall bias
reporting bias
response bias
sample
sample proportion
selection bias
self-selection sample
simple random sample
statistic
stratified random sample
survey
systematic sample

9780170484985

9.1 Random samples and bias

Imagine asking a small group of people what their favourite type of pizza is. This is most likely an easy process. How would this be different from asking every person in Brisbane or even Australia what their favourite type of pizza is? It would be costly and time-consuming for you to do. Instead of asking everyone, you could just ask some people in Australia. This, however, throws up additional questions, such as 'how many people do you ask?', 'how do I pick the right people?' and 'is the answer I get what all of Australia thinks?' All these questions open up the world of **samples**.

Statistical terms

- The **population** is the whole group from which information is obtained.
- A **census** is the information collected from the whole population.
- A **parameter** is a qualitative characteristic (e.g. mean, median) of the population.
- A **sample** is part of the population.
- A **survey** is the information collected from the sample.
- A **statistic** is a numerical estimation of a parameter taken from a sample.

A group of people can be considered either a population or a sample depending on the scenario. For example, asking everyone in a class their favourite colour is considered a population if you want to know the favourite colour of the class. However, asking everyone in a class their favourite colour is considered a sample if you want to know the favourite colour of the school.

WORKED EXAMPLE 1 Identifying statistical terms

A class of 25 students is asked their height.

The shortest student is 155 cm, the tallest student is 195 cm and the average height is 180 cm.

Identify the population, parameters and statistics for this sample.

Steps	Working
1 Population is the whole group.	The population is the 25 students of the class.
2 Parameters are qualitative characteristics.	The parameters are minimum height, maximum height and mean height.
3 Statistics are quantitative values of the parameters.	The minimum height is 155 cm, the maximum height is 195 cm and the mean height is 180 cm.

Although performing a census is often seen as asking everyone in the population, sometimes even the best attempts to ask the entire population don't quite get everyone. The Australian Bureau of Statistics conducts a census of the Australian population every 5 years. The 2021 Census had a participation rate of approximately 96%.

Asking everyone in the population is time-consuming and costly. This is one of many reasons that the Australian Census only occurs every 5 years. Sampling allows for a smaller number of people in the population being asked to respond, allowing for quicker and more timely statistics. However, for the statistics of the sample to be representative of the population, samples should be as large as reasonably possible.

Samples

- A **fair sample** is representative of the population and does not favour any subset of the population.
- A **biased sample** is not representative of the population and favours a subset/s of the population.
- A **random sample** is a sample chosen by a chance method to ensure all people in the population have an equal chance of being chosen.

Investigation The Australian Census

Investigate the last Australian Census to see how it was undertaken and the results that come from it.

Investigate:

- how people could undertake the Census
- the time taken for the Australian Bureau of Statistics to process the data
- the cost of undertaking the Census
- how the percentage of 17- to 18-year-olds in Queensland has changed from the 2016 Census to the 2021 Census
- how the percentage of 17- to 18-year-olds in Queensland compared to that of the whole of Australia in 2021
- how the percentage of Queenslanders whose highest education attained was Year 12 compared to that of the whole of Australia in 2021
- any other interesting statistics you found while searching through the Census.

Bias in surveys is something that should be avoided as much as possible. When asking for a person's opinion, avoiding bias may be especially difficult.

Types of bias

- **Completion bias** comes from incomplete surveys. This bias can be avoided by keeping surveys short and focused, as well as avoiding sensitive topics where possible.
- **Design bias** comes from poorly designed questions of the survey, which may confuse the survey taker, lead towards certain responses, or not properly leading to the results the user requires. This bias can be avoided by being thoughtful in preparation of the survey to keep the questions as objective as possible.
- **Interviewer bias** comes from the attitude of the person asking the questions, or differences in the way multiple interviewers ask the questions or respond to queries from the survey taker. This bias can be avoided by ensuring that interviewers are trained to be consistent in their interview technique, or where possible use an online survey.
- **Response bias** comes from people who respond in a way that does not match their views. This may be because they are afraid their opinion may be not socially acceptable, or are unconsciously influenced by those around them to respond a certain way. This bias can be avoided by allowing survey takers privacy to complete the survey as honestly as possible, as well as ensuring that survey responses are deidentified from those taking the survey.
- **Recall bias** comes from an inability of the survey taker to recall information from the past. This bias can be avoided by asking questions focused on recent events.
- **Reporting bias** comes from the way survey results are reported or published. This bias can be avoided by ensuring that, when providing survey results to those reporting on them, the key results are highlighted to avoid errors, misinterpretations or potential misrepresentations.
- **Selection bias** comes from selecting the incorrect sample. This bias can be avoided by using random samples where possible and ensuring that in the preparation phase of the survey the sampling type is clearly defined.

WORKED EXAMPLE 2 Bias in sampling

In each case, **determine** whether the sampling method is fair and, if bias exists, the type of bias.

a 200 people passing through a shopping centre on a Saturday afternoon in Warwick were asked which was their favourite NRL team.

b 1000 Queenslanders were contacted via social media and asked who they would vote for in the upcoming State election.

c A reality TV show asked viewers to vote for the contestant they wanted to eliminate from the show by voting on a poll on the show's subscription webpage.

Steps	Working
a Those interviewed were only available in one location at a particular time.	The sampling method is not fair, there is selection bias and possibly some response bias, as those who don't follow the NRL will not respond or might randomly pick a team.
b Choosing a particular social media platform can ignore certain groups, as well as those who don't use social media at all.	There is response bias and design bias, as well as the possibility of selection bias.
c Those who wish to vote must have a subscription account to be able to do so.	There is a response bias because not all viewers will choose to vote or may not be able to if they don't subscribe to the webpage. There may also be some recall bias in remembering reasons for voting for a particular person to eliminate.

EXERCISE 9.1 Random samples and bias

ANSWERS p. 414

Mastery

1 WORKED EXAMPLE 1 Identify the population, parameters and statistics for each situation.

a 35 people entering a cinema were asked how much they had spent at the snack and sweets counter. The smallest amount was $6.50 and the largest $37.95, with an average of $22.43.

b People with names starting with 'D' on the electoral roll in Gympie were sent a survey about takeaway food consumption. Of the 123 replies, 47 said they had eaten McDonalds; 35 Red Rooster, 38 KFC, 19 Subway, 48 Domino's Pizza and 73 other outlets.

c Students at a bus station were asked what brand of scientific calculator they use.

36 said Sharp, 42 said TI, 39 said Casio, 14 said HP and 17 gave other brand names.

d The kinds of vehicles going along the Bruce Highway were noted. In a 30-minute period, 120 cars, 40 semitrailers, 60 smaller trucks, 18 motorbikes, 29 taxis, 32 buses, 17 utes and 32 vans were observed.

e The price of E10 petrol was checked each day over a month at the service stations in and around Cairns. Prices varied from 163.9 cents/L to 187.9 cents/L, with a median price of 172.3 cents/L.

2 WORKED EXAMPLE 2 State whether the sampling method is fair and the types of any bias in question **1**.

3 A school surveys its students at the beginning of the year to determine the amount of part-time work they do.

a What is the population for this survey?

b What are the parameters?

c What problems might this survey have in determining the parameters?

4 In each of the following cases, state whether or not the sampling method is fair. If it is biased, state the kind(s) of bias.

a A hardware store wants to know if it would be worthwhile opening for longer hours. A survey placed on the counter asks customers to tick the times they are likely to shop in the store from a list.

b A Year 12 student goes to Year 7 classes to investigate the amount of pocket money that students of this age receive from their parents. Students in the class are asked to tell them how much pocket money they received last week as the Year 12 student goes around the class and writes down the responses.

c A TV reality show eliminates one contestant each week by having viewers vote on who to eliminate each week. Only one vote from each viewer is accepted.

d A council needs to establish a new landfill site for rubbish as the old site is almost full. Some councillors urge the council to use 'people power' to decide the new site by asking residents close to each of the possible sites for their opinion of the best site. It is proposed that people attending public meetings near each site will be asked to vote on the suitability of the nearby site.

e A food critic goes to different restaurants and tries a selection of dishes on the menu before rating the restaurant 1 to 10 on ambience, presentation, quality of cooking and variety of the menu. The critic is very well known in the area and their opinion is valued highly. Reviews of three restaurants are published the following week.

Exam practice

5 © QCAA 2022 1Q10 TECH-FREE 81% A survey plans to draw conclusions based on a random sample of 1% of Queensland's adult population. To be regarded as a random sample, every

A adult in the population will be placed in an alphabetical list and every 100th person will be selected for the sample.

B adult in the population can choose to participate until the sample size has been reached.

C subgroup within the population will be represented in a similar proportion in the sample.

D adult in the population will have an equal chance of being selected for the sample.

6 TECH-FREE A person exiting a polling location is asked who they voted for in the Queensland election. The person doesn't know the interviewer and fears they will be judged for their vote, so they lie and say they voted for a candidate other than who they actually voted for. This is an example of

A recall bias. **B** response bias.

C interviewer bias. **D** reporting bias.

7 TECH-FREE A person is asked how many years it had been since they used a payphone. The person was not able to remember the last time that they used a payphone. This is an example of

A recall bias. **B** response bias.

C interviewer bias. **D** reporting bias.

8 TECH-FREE A person is taking an online survey, but some of the questions are vague and confusing. They answer the questions that they can understand and leave the remaining questions blank, hence submitting an incomplete survey. Which of the following is least likely to be a bias in this scenario?

A design bias **B** response bias

C interviewer bias **D** completion bias

9 TECH-FREE (2 marks) The editor of a health and fitness magazine thinks that the weight-and-height tables for Australians are out of date. The editor asks staff to put a survey form in the next issue asking subscribers to send in their current weight, height and age so that new tables can be compiled. Ten people who send in replies will be randomly selected to win a free annual subscription to the magazine. Evaluate the reasonableness of the claim that this is a fair sample.

10 TECH-FREE (2 marks) A high school hall is set up for Year 12 examinations with students seated alphabetically from front to back in rows of 25, with 11 desks in each row. Some desks are vacant due to the illness of some students. Students are selected to provide feedback on the conduct of the test. Every 15th student is chosen, starting with the 3rd desk in the front row and working across to the right, then back to the left in the next row, and so on until 10 students are selected. Is this type of random sampling fair? Justify your decision.

9.2 Random sampling

Using random sampling ensures that every person in the population has an equal chance of people selected. There are many different ways that this random sampling can be achieved.

Random sampling methods

- In **simple random sampling** each person in the population has an equal chance of being selected for the survey.
- **Stratified random sampling** uses groups (or strata) with particular traits within the population. Random sampling is performed from within each stratum (the singular of strata) to maintain their proportion to the population.
- In **cluster sampling,** the population is broken up into cluster groups that are representative of the population. Groups are selected at random and all individuals within the selected groups are surveyed.
- In **systematic sampling** every nth individual from the population is surveyed. The value of n is determined by the survey makers.

WORKED EXAMPLE 3 Stratified random sampling

A company employs 43 assembly workers, 10 office staff and 3 supervisors. The company wants to survey 15 staff members while keeping the proportions of the survey group as close to the proportions of the staff population as possible.

Determine the number of staff to be surveyed from each group of workers.

Steps	Working
1 Determine the type of random sampling to be used.	Stratified random sampling will be used.
2 Calculate the proportion of each group of staff.	Total workers: 56 Assembly workers: $\frac{43}{56}$ Office staff: $\frac{10}{56}$ Supervisors: $\frac{3}{56}$
3 Calculate size of survey group from each strata. If the total of a stratified sample is incorrect, round up the highest rounded down value(s) or vice-versa to get the correct total.	Survey size: 15 Assembly workers: $\frac{43}{56} \times 15 = 11.517...$ Office staff: $\frac{10}{56} \times 15 = 2.678...$ Supervisors: $\frac{3}{56} \times 15 = 0.803...$
4 Interpret the results.	The total is 12 + 3 + 1 = 16. To get 15, round down the smallest rounded up figure. There should be 11 assembly workers, 3 office staff and 1 supervisor.

Video Stratified sampling

Systematic sampling is best used when individuals within the population already have some numerical value attached to them.

WORKED EXAMPLE 4 Application of systematic sampling

Each student in the Year 12 cohort of your school has an ID number of which the last 4 digits are unique to them. These numbers start at 2103 and end at 2345.

The Head of Year wants to survey 10 students from the Year 12 cohort by using systemic sampling.

Determine the ID numbers of the students to be surveyed.

Steps	Working
1 Calculate the total number of students.	Total number $= 2345 - 2103 + 1$ $= 243$
2 Determine the nth repeat from the previous student.	Repeat $= \frac{243}{10} \approx 25$
3 Use a random number generator on your calculator to determine starting value.	Starting value is 2112.
4 Select students to be surveyed.	Students to be surveyed are 2112, 2137, 2162, 2187, 2212, 2237, 2262, 2287, 2312, 2337.

Technology can be used to automate the process of selecting numerical values in random samples. Although **random numbers** from a calculator or computer are technically pseudo-random, there is enough randomness for our uses.

TI-Nspire
Sampling and proportions

TECHNOLOGY Random sampling

Use your graphics calculator to generate 10 random ID numbers from 2103 to 2345.

TI-84 Plus CE



1 From the main menu, press math.



2 Scroll across to choose the **PROB** menu, then scroll down to select **5: randInt**, then press enter.

Casio fx-CG50 AU



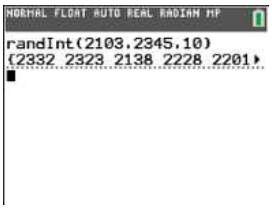
1 From the Main Menu, press **1** to select Run-Matrix.



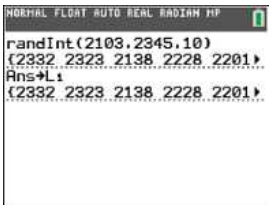
2 Press OPTN, then press F6, then press F3 to select **PROB**.



3 Insert the lower limit, followed by the upper limit and the number of integers, then scroll down and press enter to select **Paste**.

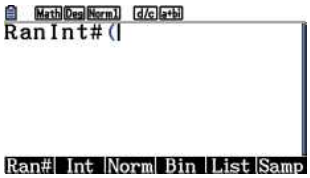


4 Press enter again to perform the operation.

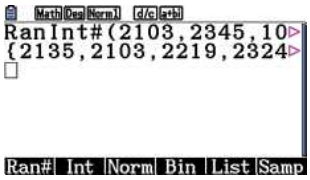


The samples are random, so you could get repeated numbers in the list. If this happens, generate another random number to replace the duplicate.

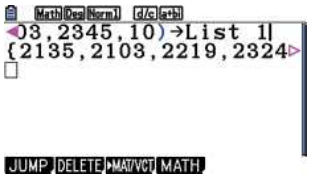
5 Press STO→ 2nd 1 enter to store the list in the variable L1.



3 Press F4, then press F2 to insert RanInt#



4 Insert the lower limit, followed by the upper limit and the number of integers, then close the brackets.



5 By adding the → symbol after closing the brackets, the list can be assigned as a list.

Press the → symbol, then press SHIFT, then press **1**, then press **1**, then press EXE to store the list as List 1 under the STATS menu option.

Non-random sampling methods do exist and are mostly used by survey makers to save time and money.

Non-random sampling methods

- **Convenience samples** are chosen from an easily accessible group, e.g. students within your class.
- **Judgement (purposive) samples** are chosen using the judgement of the survey makers/interviewers to ensure a representative sample, e.g. selecting people with a range of differing political views.
- **Quota samples** use groups similar to those in stratified random sampling; however, individuals are chosen from within these groups by convenience, e.g. the first 5 males and first 5 females to pass the interviewer.
- **Self-selection samples** use volunteers for the sample, e.g. answering a survey at the end of a phone call.

EXERCISE 9.2 Random sampling

ANSWERS p. 415

Recap

1 A student in your class wants to know what everyone's screentime hours were for last week. From the class of 22 students, the minimum was 7 hours, while the maximum was 28 hours with a mean of 22.3 hours. The parameters for this survey are

A 7 hours, 22.3 hours, 28 hours.

B minimum screentime, mean screentime and maximum screentime.

C minimum screentime of 22 hours, mean screen time of 22.3 hours and maximum screentime of 28 hours.

D class of 22 students.

2 A person taking a survey is unsure what a particular question is asking. The interviewer tells them to just select option A 'Always' if they don't know what a question means. This is an example of

A reporting bias.

B interviewer bias.

C completion bias.

D recall bias.

Mastery

3 WORKED EXAMPLE 3 Use stratified sampling to calculate how many of each group should be chosen to obtain the specified sample.

a A sample of 10 swimsuit-wearers from 15 men in board shorts, 10 men in briefs, 25 women in bikinis and 7 women in a one-piece

b A sample of 20 chocolates from 180 soft-centred, 140 hard-centred, 85 liquid-centred and 108 nutty-centred chocolates

c A sample of 16 from 5 15-year-olds, 35 16-year-olds, 10 17-year-olds and 3 18-year-olds

d Sample of 10 from 130 Year 7s, 120 Year 8s, 150 Year 9s, 130 Year 10s, 80 Year 11s and 90 Year 12s

4 A survey of members of a veteran and vintage car-owners club was conducted. The club had 21 members with pre-1925 cars, 42 with cars from 1925 to 1934, 17 with cars from 1935 to 1944, 62 with cars from 1945 to 1954 and 160 with cars from 1955 onwards.

For a survey of 30 members, how many should be chosen from each year group?

5 Biotech Industries wishes to form a staff social committee consisting of 18 members. It is decided that the method of stratified random sampling will be used for selecting the committee members. The employment details of the firm are given in the table.

	Administrative staff	Factory workers
Males	11	73
Females	24	52

a How many from each group should be selected to represent all groups fairly?

b How many from each group should be selected if no distinction is made between males and females?

6 WORKED EXAMPLE 4 Use systematic sampling to select a sample of 4 pies from a batch of 50, starting at number 17.

7 Use systematic sampling to select 10 students from the enrolments at a school. Start from enrolment number 5130 of the current enrolments, which run from 4928 to 5672 inclusive.

8 West St, Toowoomba has street addresses that go from 1 to 673. Use systematic sampling to select 20 addresses in West St, starting from number 301.

9 Use your graphics calculator to choose each random sample.

- **a** 5 of the numbers from 100 to 199 inclusive
- **b** 8 from IDs that start at 1695 and go up to 3725 inclusive
- **c** 10 cards that are numbered 1 to 100
- **d** 7 from 600 to 5000 inclusive

10 State the kind of sampling that is used in each of the following and list some advantages and disadvantages of the method.

- **a** Standing in the Queen St Mall next to an ice cream stall and asking passers-by which kind of ice cream they prefer until 20 men and 20 women have replied
- **b** Asking people in your neighbourhood how many people live in their house
- **c** Choosing 8 streets at random from the Brisbane area and asking each of the people who live in the street their opinion about the State Government
- **d** Sending surveys to 500 people asking for their opinions and to return the survey in a pre-paid envelope
- **e** Surveying students in Year 12 about their ambitions for when they leave school at the end of the year by choosing a sample of 12 students: 4 studying Essential Maths, 4 studying General Maths and 4 studying Maths Methods

Exam practice

11 People who choose to complete a survey that they find on a website is an example of a

- **A** quota sample.
- **B** convenience sample.
- **C** judgement sample.
- **D** self-selection sample.

12 Asking all students in your class to complete a survey in order to represent the entire school is an example of a

- **A** quota sample.
- **B** convenience sample.
- **C** judgement sample.
- **D** self-selection sample.

13 (1 mark) According to the Australian Bureau of Statistics, females aged 15–19 accounted for 2.92% of the Australian population in 2023. In a stratified sample of 1000 people, how many 15- to 19-year-old females should be surveyed?

14 (1 mark) A survey of 89 people was stratified and 5 males aged 15–19 were asked to complete the survey. Males aged 15–19 account for 3.12% of the Australian population (Source: ABS). Determine if the correct number of males aged 15–19 was surveyed.

15 (9 marks) TECH-FREE Many research companies use the electoral roll (available from the Australian Electoral Commission) to choose samples of people to be surveyed. The electoral roll for a federal electorate normally has about 85 000 people on the roll and includes their addresses. There are many pages in the roll for each electorate. Suggest 3 different ways in which the electoral roll could be used to select samples. Give at least one advantage and one disadvantage of each method.

16 (12 marks) This table shows the Australian population in 2024. Use stratified random sampling to determine how many should be chosen from each state and territory to make a sample of 500

a if participants are selected regardless of sex *[5 marks]*

b if males and females are selected in proportion. *[7 marks]*

State	Males	Females	Participants
New South Wales	4 212 478	4 257 119	8 469 597
Victoria	3 441 098	3 518 136	6 959 234
Queensland	2 753 555	2 806 897	5 560 452
South Australia	926 262	947 557	1 873 819
Western Australia	1 482 018	1 469 584	2 951 602
Tasmania	285 072	290 588	575 660
Northern Territory	128 860	125 403	254 263
Australian Capital Territory	232 960	239 843	472 803
Australia	13 464 931	13 657 480	27 122 411

Source: ABS (Australian Bureau of Statistics) (2024) Rebasing of Australia's population estimates using the 2021 Census, ABS website, accessed December 2024

17 (3 marks) A TV travel show ran an online poll to choose whether one of the presenters would visit Bali, Singapore or Nepal in an upcoming episode. What kind of survey is this and what advantages and disadvantages does it have?

Worksheet
Sample proportions

9.3 Sample proportions

One of the statistics that can come from a sample is called the **sample proportion**. This is the proportion of items in a sample that have a particular feature or behave a particular way.

Sample proportions

- A sample proportion is the number of times a characteristic is observed within a sample divided by the sample size of the survey.
- The symbol used for the sample proportion is $\hat{p}$ (read as 'p-hat').
- The number of times the characteristic is observed is defined as x and the sample size is defined by n. Hence $\hat{p} = \frac{x}{n}$.
- $x \leq n$, hence $0 \leq \hat{p} \leq 1$.

WORKED EXAMPLE 5 Sample proportions

A school has 95 students who study Mathematical Methods in Year 12. The school also has 22 students who study Specialist Mathematics.

Determine the sample proportion of Mathematical Methods students also doing Specialist Mathematics.

Steps	Working
1 Write the formula.	$\hat{p} = \frac{x}{n}$
2 Substitute values.	$\hat{p} = \frac{22}{95}$ $= 0.2316$
3 Evaluate the answer.	The sample proportion of Mathematical Methods students also doing Specialist Mathematics is approximately 0.2316.

Exam hack

Don't simplify a sample proportion written as a fraction unless requested. That way, you can refer to the denominator value of a sample proportion as n.

The sample proportion should be similar to the population proportion. However, since this is a sample, a difference between the sample proportion and the population proportion is to be expected. As the sample size approaches the population size, the difference between these values should become smaller.

Sample proportion and population

The sample proportion $\hat{p}$ is an estimator of the population proportion p.

WORKED EXAMPLE 6 Evaluating reasonableness of a sample proportion

In a Queensland town, a dentist has seen 48 patients this week. 9 of these patients have dentures.

a **Determine** the sample proportion of denture patients at this dental practice.

b **Evaluate** the reasonableness of the claim that this sample proportion is an accurate proportion for the whole of Queensland.

Steps	Working
a **1** Write the formula.	$\hat{p} = \frac{x}{n}$
2 Substitute values.	$\hat{p} = \frac{9}{48}$ $= 0.1875$
3 Evaluate the answer.	The sample proportion of denture patients at this practice is 0.1875.
b Evaluate the reasonableness of the claim.	The claim is not very reasonable. The population of the town may not be representative of the population of Queensland and it is possible that there is a higher proportion of older patients at this practice due to an overall older population of the town.

EXERCISE 9.3 Sample proportions

ANSWERS p. 416

Recap

1 A family is walking through a shopping centre when an interviewer asks a question of the mum and dad. They are asked who they would vote for in the upcoming election. Mum said she'd vote for party B, while dad said he'd vote for party A. The mum is asked some more questions, but the dad is not.

When asked why the dad wasn't asked more questions, the interviewer says 'We already have enough responses from people who'd vote for party A.'

This is an example of

A quota sampling.
B convenience sampling.
C judgement sampling.
D self-selection sampling.

2 An example of a systematic sample of 5 numbers from 1 to 30, starting from 2 is

A 2, 4, 6, 8, 10
B 2, 7, 12, 17, 22
C 2, 8, 14, 20, 26
D 5, 20

Mastery

3 WORKED EXAMPLE 5 Of 150 Year 12 students at a suburban Brisbane school, 79 had dark hair. Determine the sample proportion of dark hair.

4 A coin was tossed 28 times and heads came up 20 times. Determine the sample proportion of heads.

5 WORKED EXAMPLE 6 A caravan park in Emerald was almost fully occupied in May. The drivers of 47 of the 65 motor homes and caravans in the park were over 55 years old.

a Estimate the probability of the driver of a motorhome or caravan being over 55 correct to 2 decimal places.

b Evaluate the reasonableness of the claim that this is an accurate proportion for all motorhomes or caravans in Queensland.

6 Of the 45 cars for sale in a Southport second-hand car yard, 28 were priced over $12 000.

a Estimate the probability of second-hand cars costing over $12 000.

b Evaluate the reasonableness of the claim that this is an accurate proportion for all second-hand cars available for sale in Southport.

Exam practice

7 TECH-FREE A marble factory has counted that 234 marbles are coloured red out of the 2691 marbles made today. The sample proportion of red marbles is

A 0.0870 **B** 0.1304 **C** 11.5 **D** 234

8 TECH-FREE A survey of Year 7 students found that 28 of the 184 Year 7 students preferred chocolate ice cream. The sample proportion of not chocolate ice cream is

A 0.1522 **B** 0.8478 **C** 1.1795 **D** 6.5714

9 (1 mark) A new car dealer sold 210 new cars of which 89 were white. Determine the sample proportion of white cars.

10 (4 marks) From a sample of 40 Australian Year 12 students, 9 were found to have heights of 180 cm or greater. Only one of the tall students was female.

a Estimate the probability of an Australian Year 12 student being 180 cm tall or taller. *[2 marks]*

b 19 of the 40 students were male. Estimate the probability of a male Australian Year 12 student being 180 cm or taller. *[1 mark]*

c Estimate the probability of a female Australian Year 12 student being 180 cm or taller. *[1 mark]*

11 (4 marks) Of the 47 streams and creeks tested around Brisbane, 5 were found to have high levels of algae in the water. Estimate the probability of streams and creeks across Brisbane having high levels of algae in the water. Evaluate the reasonableness of the claim that this is an accurate proportion for all streams and creeks across Queensland. If it is not reasonable, suggest a way in which the survey process could be improved.

12 (4 marks) Of the 250 Year 12 students at Sunnybank State High School, 91 had been to at least one other high school. Estimate the probability of students changing high schools before the end of Year 12 and discuss any concerns with the estimate.

9.4 Variation of samples

The sample proportion $\hat{p}$ is an estimator of the population proportion p.

But how good is $\hat{p}$ as that estimator? It depends on the size of sample taken.

Investigation — Changing sample size and its effect on $\hat{p}$

A die is rolled 20 times. The face values of each roll are shown in Table 1 below.

Table 1: Results of 20 rolls of the die

1	2	4	6	1
4	5	5	6	2
2	2	1	6	6
4	6	1	5	6

1 Take 5 different random samples of size 2 from the population above (Table 1) and note the sample proportion of even numbers in each sample.

Use a table similar to the one below to record your results:

Table 2: Sample proportion of even numbers (Samples of size 2)

$n = 2$	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5
Values	[5, 6]	[6, 2]	[1, 6]	[4, 4]	[2, 2]
$\hat{p}$	0.5	1	0.5	1	1

2 Repeat the process for 5 different random samples of size 5 from the population above (Table 1) and note the sample proportion of even numbers in each sample.

3 Repeat the process again for 5 different random samples of size 10 from the population above (Table 1) and note the sample proportion of even numbers in each sample.

4 What do you notice about the spread of $\hat{p}$ values as the sample size increases?

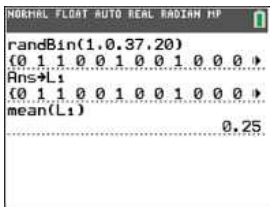
Variation of $\hat{p}$

As n increases and approaches the population size, the variation in the $\hat{p}$ values reduces.

WORKED EXAMPLE 7 — Sample proportions from binomial distribution

Use your graphics calculator to generate 3 sample proportions of samples of 20 from a population whose probability of having a particular property is 0.37.

TI-84 Plus CE

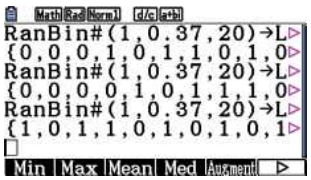


1 Press math, choose **PROB** and **7: randBin**.

Put in n: 1, p: 0.37, repetitions: 20 then select **Paste**.

Use list operations to find the mean of L_1; press 2nd stat, choose **MATH**, **3: mean(** and put in L_1.

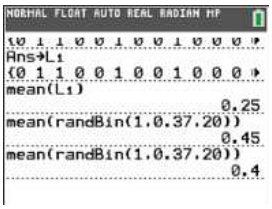
Casio fx-CG50 AU



1 Choose the Run-Matrix menu and press OPTN. Choose **PROB**, **RAND** and **Bin**, put in 1, 0.37, 20, close the brackets and store in List 1.

Use list operations to find the mean of List 1; press OPTN, choose **LIST**, **Mean** and put in List 1.

TI-Nspire
Sampling and proportions



2 Repeat twice. You can find the mean of the list immediately instead of storing the list.



2 Repeat twice.

Due to the use of random numbers, the values you get will most likely be different.

The 3 sample proportions are shown on the screen.

EXERCISE 9.4 Variation of samples

ANSWERS p. 416

Recap

1 Of the 24 students in a Maths Methods class, 21 achieved a C or better. Determine the simplified sample proportion of students who passed.

2 The sample proportion for Year 12 students achieving an A in Maths Methods from a survey of 600 high school students across Brisbane is 0.2467. The number of Year 12 students from the survey who achieved an A is

A 148 **B** 247 **C** 600 **D** 2432

Mastery

3 WORKED EXAMPLE 7

a Generate 10 sample proportions of samples of 30 from a population whose probability of having a particular property is 0.3.

b Generate 10 sample proportions of samples of 80 from a population whose probability of having a particular property is 0.3.

c Comment on the variation of the sample proportions for samples of 30 and samples of 80.

4 Generate 2 random samples of 25 items from a binomial distribution with $n = 30$ trials and probability $p = 0.7$ and draw histograms of the samples.

a Find the means and standard deviations of the samples.

b Compare the samples.

5 a Generate 2 sample proportions of samples of 40 from a population whose probability of having a particular property is 0.65.

b Compare the sample proportions.

6 a Generate 2 random samples of 9 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$, and find the means and standard deviations of the samples.

b Generate 2 random samples of 64 items from a binomial distribution with $n = 15$ trials and probability $p = 0.6$ and find the means and standard deviations of the samples.

c Comment on the variation of the means and standard deviations for samples of 9 and samples of 64 items.

7 **a** Generate 2 random samples of 9 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the means and standard deviations of the samples.

b Generate 2 random samples of 81 items from a normal distribution with mean $\mu = 40$ and standard deviation $\sigma = 8$ and find the means and standard deviations of the samples.

c Comment on the variation of the means and standard deviations for samples of 9 and samples of 81 items.

Exam practice

8 TECH-FREE The mean sample proportions for 2 sample distributions are both approximately 0.32. Distribution A had a sample size of 40, while distribution B had a sample size of 10. The variability of the 2 distributions are

A different, as sample B will have a smaller variability than sample A.

B exactly the same.

C different, as sample A will have a smaller variability than sample B.

D unable to be determined from the information given.

9 TECH-FREE The mean sample proportions for 2 sample distributions are both 0.32. Distribution A has a much lower variability than distribution B. The sample size of A is

A smaller than that for B.

B larger than that for B.

C exactly the same as that for B.

D much smaller than that for B.

10 TECH-FREE (4 marks) Two random samples of 20 values were taken from a Bernoulli distribution with $p = 0.6$.

A: 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1, 0, 1, 1, 1

B: 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 1, 0, 0, 1

a Show the samples as a side-by-side column graph. *[2 marks]*

b Compare the samples. *[2 marks]*

11 TECH-FREE (4 marks) Two random samples of 20 were taken from a binomial distribution with $n = 8$ and $p = 0.2$.

A: 2, 2, 0, 1, 1, 2, 0, 3, 2, 1, 0, 2, 3, 2, 0, 3, 0, 0, 2, 1

B: 1, 2, 0, 3, 2, 2, 3, 1, 3, 2, 0, 2, 1, 0, 4, 2, 1, 1, 2, 1

a Show the samples as side-by-side histograms. *[2 marks]*

b Compare the samples. *[2 marks]*

12 (4 marks) Two random samples were taken from a normal distribution with mean 50 and standard deviation 8. Each sample has 25 values.

A: 50.4, 62.3, 49, 45.3, 59.1, 45.1, 50.2, 49.7, 40, 41.6, 38.5, 46.6, 47.5, 47.1, 63.2, 55.4, 43, 52.1, 53.3, 54.9, 58.4, 34.2, 54.4, 37.1, 56.9

B: 37.3, 49.3, 35.9, 45.7, 53.5, 40.2, 46.7, 44.8, 52.8, 41.7, 59.4, 48.7, 50.2, 39.4, 50.3, 41.6, 43.4, 41.9, 46.1, 47.8, 45.1, 53.3, 44.9, 44.5, 50.6

a Show the samples as histograms with class widths of 5. *[2 marks]*

b Compare the samples. *[2 marks]*

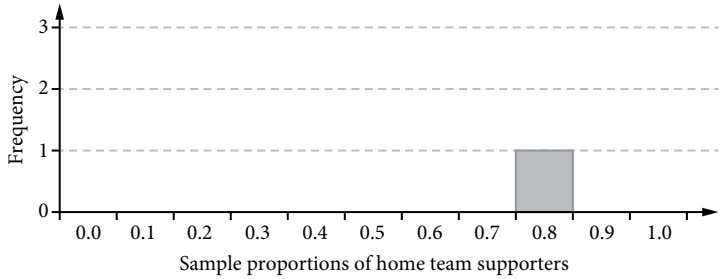
13 © QCAA 2024 1Q14 **TECH-FREE** (5 marks) At a particular game at a local sporting venue, 60% of spectators support the home team and the remainder support the away team.

A researcher asked six groups of 10 spectators which team they supported. Each spectator was recorded as either H (supports home team) or A (supports away team). The results were:

Group 1:	H	A	H	H	H	A	H	H	H	H
Group 2:	A	A	H	A	A	H	A	A	A	A
Group 3:	H	A	A	H	H	A	A	A	H	H
Group 4:	H	H	H	H	H	A	H	H	H	H
Group 5:	A	A	H	H	A	H	H	A	H	A
Group 6:	A	H	A	A	A	A	A	H	A	A

a The researcher would like to see the distribution of the sample proportions of home team supporters obtained by entering the information into a column graph. The sample proportion for group 1 is shown.

Copy and complete the column graph by including the sample proportions for the remaining five groups. *[3 marks]*



b If the researcher had interviewed more groups, each containing 100 spectators, describe two ways the distribution of the resulting sample proportions would be expected to differ from the distribution shown in part **a**. *[2 marks]*

Worksheets
Sample proportion calculations
Sample proportion probabilities

9.5 Sample size and normality

Two samples from the same population may not have the same sample proportion. In fact, every random sample from the same population is likely to have a different sample proportion. So, what happens when we plot these sample proportions on a histogram?

Investigation Sample distributions

Use an online die roller and perform the following simulation:

1. Roll a die 10 times and note the number of 2s that occur.
2. Repeat step 1 until you have a total of 20 samples.
3. Create a histogram for the number of 2s that occur from your 20 samples with frequency on the vertical axis and the observed value x on the horizontal axis.
4. Describe the shape of the histogram.

Repeat the simulation, but this time for a total of 50 samples. Then repeat the simulation again for 100 samples.

Describe the shape of the histogram and how it has changed from 20 samples to 50 samples and then to 100 samples.

Instead of using the observed value x has the horizontal axis, it can be replaced with $\hat{p}$. This means that the vertical axis changes from frequency to relative frequency. Does this change to the axes affect the shape of the histograms?

As the number of samples increases, the distribution of sample proportions becomes more like a normal distribution.

Distribution of sample proportions

The distribution of sample proportions is $p \sim N\left(p, \frac{p(1-p)}{n}\right)$, with

- $E(p) = p$
- $Var(p) = \frac{p(1-p)}{n}$
- $SD(p) = \sqrt{\frac{p(1-p)}{n}}$

where p is the probability and n is the sample size.

In most cases you won't know the value of p, so you use $\hat{p}$ as an approximation to p.

This means that the z-scores of sample proportions p approximate a standard normal distribution.

The z-scores are given by $\frac{\hat{p} - p}{SD(\hat{p})}$. Thus $\frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}$ approximates a standard normal distribution.

WORKED EXAMPLE 8 Simulating sample proportions with varying p values

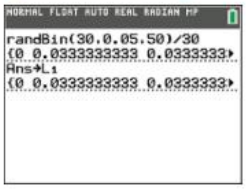
Simulate histograms with 50 random samples of 30 items.

Perform the simulations with your graphics calculator for samples with

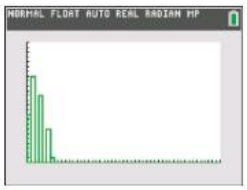
a $\hat{p} = 0.05$ **b** $\hat{p} = 0.1$ **c** $\hat{p} = 0.5$

TI-84 Plus CE

a



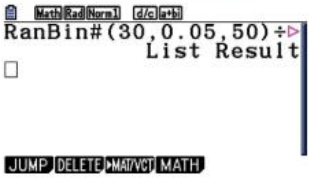
1 Press math, choose **PROB**, **7: randBin(** and complete with 30, 0.05,50) ÷ 30 and store in L_1.



2 Use stat plot to set Plot 1 On, Type  , **XList L1**, **Freq: 1**. Press 2nd mode to quit the plot menu.

3 Press window to set the graph parameters with Xmin=0, Xmax=1, Xscl 0.02, Ymin=0, Ymax=25, Yscl 1 and then press graph to see it.

Casio fx-CG50 AU



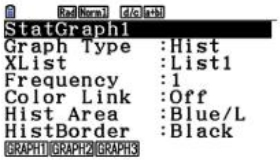
1 Use the Run-Matrix menu and press OPTN.

Choose PROB, RAND and Bin. Complete with 30,0.005,50) ÷ 30 and store in List 1.

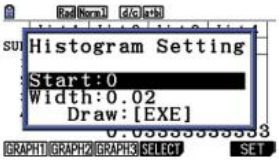
press EXIT, press MENU and choose Statistics.

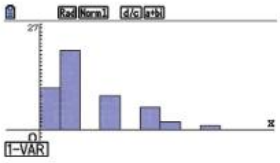
Choose GRAPH, SET and GRAPH1.

Choose Graph Type: Hist, XList: List 1.



2 Press EXIT and use select to check that StatGraph2 and StatGraph3 are Off.

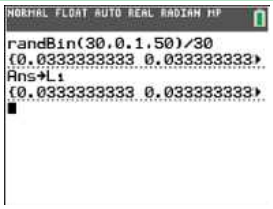




3 Press EXIT, make Start 0 and Width 0.02 and press EXE to see the graph.

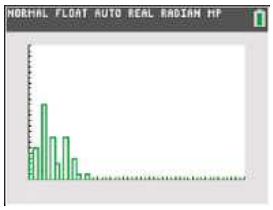
TI-Nspire
Sampling and proportions

b

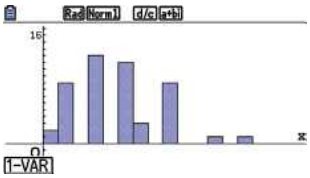


1 Press quit to close the graph.

2 Use **randBin(** again but choose $p = 0.1$. Store again in L_1.



3 Press graph to see the graph. (Leave the window and graphs settings as they were in the previous question.)

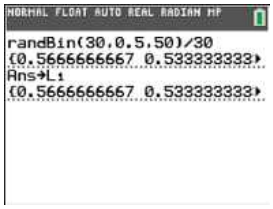


1 Press MENU and go back to the Run-Matrix menu.

2 Use **RanBin#** again, but choose $p = 0.1$. Store again in List 1.

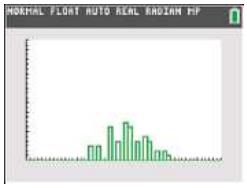
3 Go back to the Statistics menu, leave the settings the same and look at Graph1 again.

c

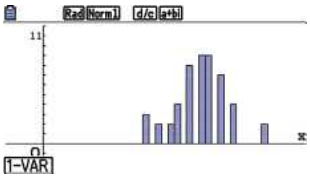


1 Press quit to close the graph.

2 Use **randBin(** again but choose $p = 0.5$. Store again in L_1.



3 Press graph to see the graph. (Leave the window and graphs settings as they were in the previous question.)



1 Press MENU and go back to the Run-Matrix menu.

2 Use **RanBin#** again but choose $p = 0.5$. Store again in List 1.

3 Go back to the Statistics menu, leave the settings the same and look at Graph1 again.

Worked example 8 illustrates that the histograms look more like a normal distribution the closer that $\hat{p}$ gets to 0.5. This makes sense since a $\hat{p}$ value of 0.5 would make the histogram most look symmetrical, which is a key feature of the normal distribution.

There is, however, a point where using the normal distribution to describe the distribution of sample proportions is not appropriate.

Normality check

When deciding whether the distribution of sample proportions can be described as normally distributed, these 2 conditions must be met:

- $n\hat{p} > 5$
- $n(1 - \hat{p}) > 5$

Exam hack

Some questions will tell you that normality already exists, hence you don't need to check. If a question does not mention that normality exists, it should be the first thing you check.

WORKED EXAMPLE 9 Checking for normality

One card is selected from 100 well-shuffled decks of cards. The number of aces is noted.

This process is repeated many times.

Check for normality to **determine** whether probabilities involving the sample proportion can be approximated using the normal distribution.

Steps	Working
1 State the parameters.	$p = \frac{1}{13}$ and $n = 100$
2 Check for normality.	$n\hat{p} > 5$ $100 \times \frac{1}{13} \approx 7.7 > 5$ $n(1 - \hat{p}) > 5$ $100 \times \frac{12}{13} \approx 92.3 > 5$
3 Write the result.	Since both conditions are met, the sample proportion can be approximated using the normal distribution.

EXERCISE 9.5 Sample size and normality

ANSWERS p. 418

Recap

1 The mean sample proportions for 2 sample distributions are both 0.4. Distribution A had a sample size of 50, while distribution B had a sample size of 120. The variability of the 2 distributions are

A different, as B will have a smaller variability than A.

B exactly the same.

C different, as A will have a smaller variability than B.

D unable to be determined from the information given.

2 An example of a random sample of size 10 taken from a binomial distribution with $n = 5$ and $p = 0.25$ is

A 0, 2, 2, 3, 2, 3, 10, 1, 4

B 0, 2, 5, 1, 0, 3, 3, 3, 6, 4

C 2, 2, 0, 3, 0, 0, 1, 2, 1, 1

D 2, 2, 4, 1, 2

Mastery

3 WORKED EXAMPLE 8 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.2$ and $n = 20$

b $p = 0.2$ and $n = 50$

c $p = 0.2$ and $n = 100$

d Compare the sampling distributions.

4 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.9$ and $n = 20$

b $p = 0.9$ and $n = 50$

c $p = 0.9$ and $n = 100$

d Compare the sampling distributions

5 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.9$ and $n = 30$

b $p = 0.7$ and $n = 30$

c $p = 0.5$ and $n = 30$

6 Simulate 100 samples of sample proportions with each pair of values given and draw a histogram of the sampling distribution.

a $p = 0.1$ and $n = 50$

b $p = 0.3$ and $n = 50$

c $p = 0.5$ and $n = 50$

7 Find the mean, variance and standard deviation of sample proportions with

a $\hat{p} = 0.2$ and $n = 50$ **b** $\hat{p} = 0.7$ and $n = 25$

c $\hat{p} = 0.81$ and $n = 120$ **d** $\hat{p} = 0.22$ and $n = 80$

8 WORKED EXAMPLE 9 A die is rolled 50 times. The number of 2s is noted. This process is repeated many times. Check for normality to determine whether probabilities involving the sample proportion can be approximated using the normal distribution.

9 A bag contains 5 blue marbles, 10 red marbles and 25 yellow marbles. A marble is pulled from the bag, its colour noted and replaced in the bag 100 times, with the number of blue marbles noted. This process is repeated many times. Check for normality to determine whether probabilities involving the sample proportion can be approximated using the normal distribution.

Exam practice

10 TECH-FREE Simulations of 100 sample proportions are performed for differing p values.
The simulation that will look most like a normal distribution would be when

A $p = 0.40$ **B** $p = 0.55$ **C** $p = 0.56$ **D** $p = 0.60$

11 TECH-FREE Simulations of n sample proportions are performed for $p = 0.3$. The simulation will look most like a normal distribution when

A $n = 25$ **b** $n = 50$ **C** $n = 75$ **D** $n = 100$

12 A survey has an expected sample proportion of 0.36 and a standard deviation of 0.0537.
The sample size of this survey is

A $n = 4$ **B** $n = 4.3$ **C** $n = 80$ **D** $n = 100$

13 (1 mark) Estimate the standard deviation for a sample of size 50 whose sample proportion is $\frac{2}{11}$.

14 (4 marks) A normal die is tossed. Samples of 45 tosses are taken and in each case the proportion of times that the number is less than 3 is calculated. What is the mean proportion and standard deviation of the sample proportion when this experiment is repeated multiple times?

15 (4 marks) 45% of Canadian high school students catch a bus to school. Samples of 200 students from high schools across Canada are questioned to determine the proportion travelling to school by bus.
What is the expected proportion and standard deviation of the sample proportion?

16 (4 marks) A normal pair of dice are thrown, and the total is noted. Samples of 30 such throws are performed and the proportion of times the total is more than 9 is calculated for each sample.
What is the mean and standard deviation of the sample proportions for totals of more than 9?

17 (4 marks) A card is cut from a well-shuffled deck. This is done 100 times and the number of times that a picture card (jack, king, queen) appears is noted in each case. This is repeated many times.
What is the mean and standard deviation of the sample proportions for picture cards?

18 (2 marks) CF A weighted coin was tossed 25 times, and the proportion of heads was noted. This was done a total of 40 times. The mean number of heads was found to be 15.02, and the standard deviation of the sample proportion of heads was 0.1. Estimate the probability of heads for this coin.

19 (4 marks) CF A sample has a standard deviation of 0.02 with a sample size of 50.
Determine all possible sample proportions for this sample.

EXAM QUESTION ANALYSIS

© QCAA 2020 2Q13 (6 marks)

An online retailer claims that 90% of all orders are shipped within 12 hours of being received.

On a particular day, 121 orders were received and 102 orders were shipped within 12 hours.

a State the sample proportion of orders shipped within 12 hours. *[1 mark]*

The distribution of the sample proportion of all orders that are shipped within 12 hours of being received on any day is approximately normal.

b Assuming the online retailer's claim is true, find the probability that, in a random sample of 121, less than 85% of all orders are shipped within 12 hours. *[3 marks]*

c Use the result from part **b** to evaluate the reasonableness of the online retailer's claim. *[2 marks]*

Reading the question

- Sample proportion information given
- Use claim percentage to determine probability
- Evaluate reasonableness of claim – is the claim reasonable and why is/isn't it reasonable?

Thinking about the question

- Check normality to determine if normal distribution can be used.
- Use calculator to determine probability (Normal CDF).
- Use probability as evidence in evaluating reasonableness of claim.

Worked solution (✓ = 1 mark)

a $\hat{p} = \frac{102}{121} = \mathbf{0.84}$ ✓correctly determines the sample proportion

b Normality: $n\hat{p} = 102 > 5$

$n(1 - \hat{p}) = 19 > 5$, $\therefore$ normal distribution can be used

Using normal distribution, $\boldsymbol{\mu = 0.9}$ ✓correctly identifies the mean of the population

$\sigma = \sqrt{\frac{0.9 \times 0.1}{121}}$ ✓determines expression for standard deviation of the normal distribution

Using GDC ($\hat{p} < 0.85$) = **0.0334** ✓determines probability of producing sample proportion or less

c There is 3.34% chance of shipping less than 85% of orders when the population parameter is 0.90.

✓identifies meaning of the probability of producing the sample proportion

Observing a sample proportion of 0.843 (or even lower) would have occurred by chance less than 3.34% of the time if the retailer's claim is true. Therefore, we suspect that the retailer's claim is dubious.

✓evaluates reasonableness of the retailer's claim

Video Exam question analysis: Sampling and proportions

9 Chapter summary

Random samples and bias

- The **population** is the whole group from which information is obtained.
- A **census** is the information collected from the whole population.
- A **parameter** is a qualitative characteristic (e.g. mean, median) of the population.
- A **sample** is part of the population.
- A **survey** is the information collected from the sample.
- A **statistic** is a numerical estimation of a parameter taken from a sample.
- A **fair sample** is representative of the population and does not favour any subset of the population.
- A **biased sample** is not representative of the population and favours a subset/s of the population.
- A **random sample** is a sample chosen by a chance method to ensure all people in the population have an equal chance of being chosen.
- **Completion bias** comes from incomplete surveys. This bias can be avoided by keeping surveys short and focused, as well as avoiding sensitive topics where possible.
- **Design bias** comes from poorly designed questions of the survey which may confuse the survey taker, lead towards certain responses or not properly leading to the results the user requires. This bias can be avoided by being thoughtful in preparation of the survey to keep the questions as objective as possible.
- **Interviewer bias** comes from attitudes of the person asking the questions, or differences in the way multiple interviewers ask the questions or respond to queries from the survey taker. This bias can be avoided by ensuring that interviewers are trained to be consistent in their interview technique, or where possible by using an online survey.
- **Response bias** comes from people who respond in a way that does not match their views. This may be because they are afraid their opinion may be not socially acceptable or are unconsciously influenced by those around them to respond a certain way. This bias can be avoided by allowing survey takers privacy to complete the survey as honestly as possible, as well as ensuring that survey responses are deidentified from those taking the survey.
- **Recall bias** comes from an inability of the survey taker to recall information from the past. This bias can be avoided by asking questions focused on recent events.
- **Reporting bias** comes from the way survey results are reported or published. This bias can be avoided by ensuring that when providing survey results to those reporting on them the key results are highlighted to avoid errors or misinterpretations.
- **Selection bias** comes from selecting the incorrect sample. This bias can be avoided by using random samples where possible and ensuring that in the preparation phase of the survey the sampling type is clearly defined.

Random sampling

- In **simple random sampling** each person in the population has an equal chance of being selected for the survey.
- **Stratified random sampling** uses groups (or strata) with particular traits within the population. Random sampling is performed from within each stratum (the singular of strata) to maintain their proportion to the population.
- In **cluster sampling** the population is broken up into cluster groups that are representative of the population. Groups are selected at random and all individuals within the selected groups are surveyed.
- In **systematic sampling** every nth individual from the population is surveyed. The value of n is determined by the survey makers.
- **Convenience samples** are chosen from an easily accessible group, e.g. students within your class.
- **Judgement (purposive) samples** are chosen using the judgement of the survey makers or interviewers to ensure a representative sample, e.g. selecting people with a range of differing political views.
- **Quota samples** use groups similar to stratified random sampling; however, individuals are chosen from within these groups by convenience, e.g. the first 5 males and first 5 females to pass the interviewer.
- **Self-selection samples** use volunteers for the sample, e.g. answering a survey at the end of a phone call.

Sample proportions

- A **sample proportion** is the number of times a characteristic is observed within a sample divided by the sample size of the survey.
- The symbol used for the sample proportion is $\hat{p}$.
- The number of times the characteristic is observed is defined as x while the sample size is defined by n. Hence $\hat{p} = \frac{x}{n}$.
- $x \leq n$, hence $0 \leq \hat{p} \leq 1$.
- The sample proportion $\hat{p}$ is an estimator of the population proportion p.

Variation of samples

- As n increases and approaches the population size, the variation in the $\hat{p}$ values reduces.

Sample size and normality

- The distribution of sample proportions is $\hat{p} \sim N\left(p, \frac{p(1-p)}{n}\right)$, with

  $E(\hat{p}) = p$

  $Var(\hat{p}) = \frac{p(1-p)}{n}$

  $\text{SD}(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$

  where p is the probability and n is the sample size.
- In most cases you won't know the value of p, so you use $\hat{p}$ as an approximation to p.
- When deciding whether the distribution of sample proportions can be described as normally distributed, 2 conditions must be met:

  $n\hat{p} > 5$

  $n(1 - \hat{p}) > 5$

Worksheet
Maths Methods formula sheet

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (4 marks): 4 multiple choice questions

Section 2 (16 marks): 4 short response questions

Total: 20 marks

Section 1 4 multiple choice questions

4 marks

1 Information collected from the whole population is known as a

A statistic **B** parameter **C** census **D** survey

2 Samples for which the proportion of a group within the population is maintained is known as

A simple random sampling **B** stratified random sampling

C cluster sampling **D** systemic sampling

3 The derivative of $y = \ln\left(\frac{6}{x}\right)$ is

A $-\frac{36}{x^3}$ **B** $-\frac{6}{x^2}$ **C** $-\frac{1}{x}$ **D** $\frac{1}{x}$

4 If $y = x^3 + 3x^2 - 5x + 2$, then $f''(1)$ is

a 1 **b** 1 **c** 4 **d** 12

Section 2 4 short response questions

16 marks

5 (4 marks) 64 out of the 213 students at Charleville State High School have requested at least 1 subject change in the first semester of this year. Estimate the probability of students changing subjects from any Queensland high school and discuss the accuracy of this estimate.

6 (4 marks) A card is cut from a well-shuffled deck. This is done 50 times and the number of times that a jack appears is noted in each case.

a Show that the conditions for normality are not met. *[2 marks]*

b Determine the minimum sample size needed for a normality approximation to be useful. *[2 marks]*

7 (4 marks) Determine the derivative of $\ln\left(3x \times e^{(7x+1)^2}\right)$.

8 (4 marks) For $y = e^x(\sin(x) + \cos(x))$, use the result of $\frac{dy}{dx}$ to determine $\int e^x \cos(x)\,dx$.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (4 marks): 4 multiple choice questions

Section 2 (16 marks): 4 short response questions

Total: 20 marks



Worksheet
Maths Methods formula sheet

Section 1 4 multiple choice questions

4 marks

1 The mean sample proportions for 2 sample distributions are both approximately 0.64. Distribution A had a sample size of 50, while distribution B had a sample size of 100. The variability of the 2 distribiutions are

A exactly the same

B different, as B will have a smaller variability than A

C different, as A will have a smaller variability than B

D unable to be determined from the information given

2 Simulations of 100 sample proportions are performed for differing $\hat{p}$ values.

The simulation will look most like a normal distribution when

A $\hat{p} = 0.45$ **B** $\hat{p} = 0.54$ **C** $\hat{p} = 0.55$ **D** $\hat{p} = 0.56$

3 The smallest angle in a triangle with sides 3 cm, 10 cm and 12 cm is

A 11.72° **B** 42.60° **C** 63.61° **D** 125.69°

4 The z-score for a normal distribution with mean 24, variance 6 and x value of 28 is

A −1.6330 **B** −0.6667 **C** 0.6667 **D** 1.6330

Section 2 4 short response questions

16 marks

5 (4 marks) A stratified random sample of 1600 people is to be surveyed about their views on funding for the Australian Institute of Sport.

Using the estimated Australian population table below, calculate the number of people from each state or territory if the only stratification is by state or territory.

State	Males	Females	Persons
New South Wales	4 212 478	4 257 119	8 469 597
Victoria	3 441 098	3 518 136	6 959 234
Queensland	2 753 555	2 806 897	5 560 452
South Australia	926 262	947 557	1 873 819
Western Australia	1 482 018	1 469 584	2 951 602
Tasmania	285 072	290 588	575 660
Northern Territory	128 860	125 403	254 263
Australian Capital Territory	232 960	239 843	472 803
Australia	13 464 931	13 657 480	27 122 411

Source: ABS (Australian Bureau of Statistics) (2024) Rebasing of Australia's population estimates using the 2021 Census, ABS website, accessed December 2024

6 (4 marks) Two random samples of 10 were taken from a binomial distribution with $n = 6$ and $p = 0.45$.

A: 1, 5, 2, 3, 1, 1, 4, 2, 3, 2

B: 3, 2, 3, 3, 3, 3, 1, 1, 5, 1

a Find the mean and standard deviation of each sample. *[2 marks]*

b Compare the samples. *[2 marks]*

7 (4 marks) Determine the area between $y = x^2 - 6x + 4$ and $y = 2 + 2x - x^2$.

8 (4 marks) A darts player throws a scoring dart 95% of the time. Determine the number of throws needed if the probability of missing at most one scoring shot is less than 10%.

CHAPTER

10 CONFIDENCE INTERVALS FOR PROPORTIONS



Syllabus coverage
Nelson MindTap chapter resources
Terminology
10.1 Interval estimates of sample proportions
10.2 Confidence intervals
10.3 Margins of error and confidence levels
10.4 Variations in confidence levels
10.5 Problems involving confidence intervals
Exam question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

Prior learning
Confidence intervals for proportions

Syllabus coverage

UNIT 4, TOPIC 5: INTERVAL ESTIMATES FOR PROPORTIONS

Confidence intervals for proportions

- Understand the concept of an interval estimate for a parameter associated with a random variable.
- Understand and use the approximate confidence interval, $\left(\hat{p}-z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$, as an interval estimate for p, the population proportion, where z is the appropriate quantile for the standard normal distribution.
- Understand and use the approximate margin of error, $z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
- Understand and use the relationship between margin of error, level of confidence and sample size.
- Understand that there are variations in confidence levels between samples and that most, but not all, confidence intervals contain p.
- Model and solve problems that involve interval estimates for proportions, with and without technology.

Mathematical Methods 2025 v1.2 General senior syllabus, p. 30,
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Videos (2):
10.1 Mean and variance of sample proportions
Exam question analysis Confidence intervals for proportions

Prior learning (1):
10.1 Confidence intervals for proportions

Worksheets (8):
10.1 Calculations with sample proportions • Simple interval estimates
10.2 Sample proportion confidence intervals • Margins of error for standard normal variables
10.5 Sample sizes • Sample proportions assignment • Sample proportions problems
Cumulative exams Maths Methods formula sheet

Puzzle (1):
10.5 Confidence intervals crossword

TI-Nspire CX II instructions (1):
10.1, 10.2, 10.3, 10.4 Confidence intervals for proportions

Nelson MindTap
To access resources above, visit **cengage.com.au/nelsonmindtap**

Terminology

confidence interval
confidence level
interval estimate
margin of error
point estimate

10.1 Interval estimates of sample proportions

A normal distribution can be converted into a standard normal distribution using z-scores.

The distribution of sample proportions (for $np > 5$ and $nq > 5$) is approximately normally distributed.

Distribution of z-scores of sample proportions

$$z = \frac{x - \mu}{\sigma}$$

$\dfrac{\hat{p} - p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$ approximates a standard normal distribution provided $np > 5$ and $nq > 5$.

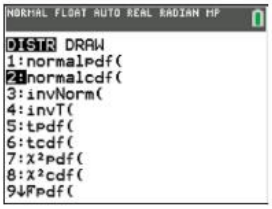
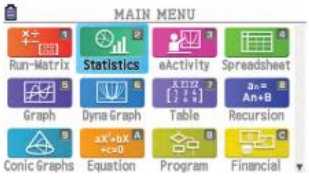
This value best approximates a standard normal distribution as $p \to 0.5$ and $n \to \infty$.

Worksheets
Calculations with sample proportions

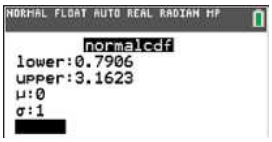
Simple interval estimates

WORKED EXAMPLE 1 Calculating probabilities involving proportions

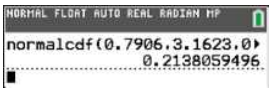
Calculate the probability that $\hat{p}$ is between 0.45 and 0.6 for $n = 60$ and $p = 0.4$.

Steps	Working
1 Check for normality.	$np > 5$ $60 \times 0.4 = 24 > 5$ $n(1-p) > 5$ $60 \times 0.6 = 36 > 5$ Therefore, normal distribution can be used.
2 Calculate the mean and standard deviation.	$E(\hat{p}) = p = 0.4$ $\text{SD}(\hat{p}) = \sqrt{\dfrac{p(1-p)}{n}} = \sqrt{\dfrac{0.4 \times 0.6}{60}} = 0.0632$
3 Determine z-scores for the boundaries.	For lower bound, $\dfrac{\hat{p} - p}{\sqrt{\dfrac{p(1-p)}{n}}} = \dfrac{0.45 - 0.4}{0.0632} = 0.7906$ For upper bound, $\dfrac{\hat{p} - p}{\sqrt{\dfrac{p(1-p)}{n}}} = \dfrac{0.6 - 0.4}{0.0632} = 3.1623$
4 Use graphics calculator to solve for the probability.	**TI-84 Plus CE**  1 Press 2nd vars to open the **DISTR** menu. Choose **2: normalcdf(**.
	Casio fx-CG50 AU  1 From the Main menu, select **2: Statistics**.

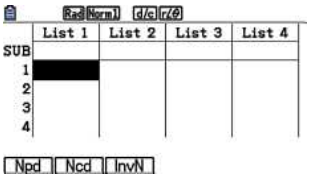
TI-Nspire
Confidence intervals for proportions



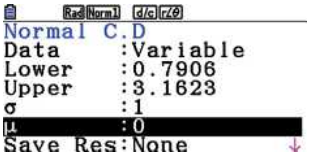
2 Enter the values and press enter twice.



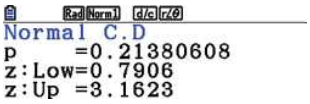
Probability will be shown.



2 Press F5 (DIST), then press F1 (NORM) and then press F2 (Ncd)



3 Enter the values and press EXE



Probability will be shown.

5 Write the answer. The probability is approximately 0.2138.

The probability found in the example above can also be found without finding *z*-scores first.

TI-Nspire
Confidence intervals for proportions

TECHNOLOGY Calculating probabilities involving proportions

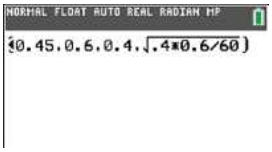
TI-84 Plus CE



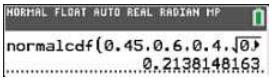
1 Press 2nd vars to open the **DISTR** menu. Choose **2: normalcdf(**.



2 Enter the values as shown above.

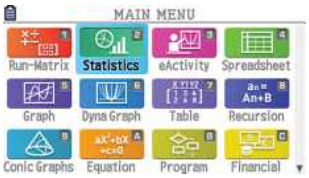


3 Press enter twice.

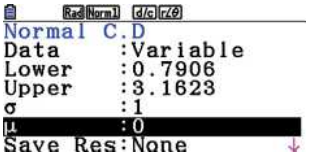


Probability will be shown.

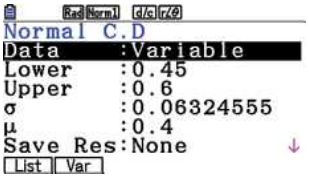
Casio fx-CG50 AU



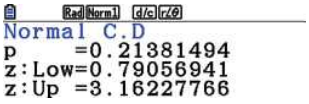
1 From the Main menu, select **2: Statistics**.



2 Press F5 (DIST), then press F1 (NORM) and then press F2 (Ncd).



3 Enter the values as shown above and press EXE.



Probability will be shown.

The sample proportion resulting from a survey is a **point estimate**, an estimate of the population proportion from the survey.

An estimate of the interval within which the population proportion lies is known as an **interval estimate**.

Estimates

- A **point estimate** is the best estimate of a population proportion.
- An **interval estimate** is the best estimate of the interval that the population proportion lies within.

It is not always possible to know the population proportion. In this case, use the sample proportion as the best estimate of p within the normal distribution.

Estimated mean, variance and standard deviation of a sample proportion

- $E(\hat{p}) = \hat{p}$
- $Var(\hat{p}) = \dfrac{\hat{p}(1-\hat{p})}{n}$
- $\text{SD}(\hat{p}) = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

where n is the sample size.

WORKED EXAMPLE 2 Determining mean, variance and standard deviation involving sample proportions

From a sample of 80 surfers, 15 said they preferred to use longboards.

a **Determine** the probability and variance of the sampling distribution of surfers preferring longboards.

b **Determine** the interval within ±1 standard deviation of the probability.

Steps	Working
a 1 Use sample proportion to estimate p.	$p \approx \hat{p} = \dfrac{15}{80} = 0.1875$
2 Use $\hat{p}$ to determine variance.	$Var(\hat{p}) = \dfrac{\hat{p}(1-\hat{p})}{n} = \dfrac{0.1875 \times 0.8125}{80} = 0.0019$
3 Write the answer.	From the sample taken, the probability of surfers preferring longboards is approximately 0.1875, with a variance of approximately 0.0019.
b 1 Use $Var(\hat{p})$ to determine $\text{SD}(\hat{p})$.	$\text{SD}(\hat{p}) = \sqrt{Var(\hat{p})} = \sqrt{0.0019} = 0.0436$
2 Determine the interval.	$\text{Interval} = (\hat{p} - \text{SD}(\hat{p}), \hat{p} + \text{SD}(\hat{p}))$ $= (0.1875 - 0.0436, 0.1875 + 0.0436)$ $= (0.1439, 0.2311)$
3 Write the answer.	The interval that is within ±1 standard deviation of the probability is 0.1439 to 0.2311.

Video: Mean and variance of sample proportions

EXERCISE 10.1 Interval estimates of sample proportions

ANSWERS p. 419

Mastery

1 WORKED EXAMPLE 1 Calculate the percentage of values of $\hat{p}$ that would lie between 0.5 and 0.6 for samples with $n = 30$ and $p = 0.7$.

2 Calculate the proportion of values of $\hat{p}$ that would lie between 0.6 and 0.7 for samples with $n = 300$ and $p = 0.72$.

3 Calculate the probability that $\hat{p}$ would lie between 0.48 and 0.52 for samples with $n = 45$ and $p = 0.25$.

4 Calculate the percentage of sample proportions that would be expected to lie between 0.4 and 0.6 for samples with $n = 40$ and $p = 0.55$.

5 Of 80 people interviewed at Roma St Station, 57 said they could use chopsticks. Estimate the probability that Queensland residents can use chopsticks, and find the variance and standard deviation of the sampling distribution.

6 From 40 tosses, a coin lands with heads up 18 times. Estimate the probability of the coin landing with heads up, and find the variance and standard deviation of the sampling distribution.

7 The protractors of 50 students were checked for accuracy, and 16 were found to be inaccurate in measuring an angle of 60° (more than 1° out). Estimate the probability of students' protractors being inaccurate, and find the variance and standard deviation of the sampling distribution.

8 WORKED EXAMPLE 2 From a group of 50 computer professionals, 32 said they used the Linux operating system on their home computers. Determine the interval of a computer professional using Linux on their home computer to within one standard deviation of the probability.

9 From a class of 28 Year 12 students, only 19 had completed all their homework the previous night. Use this information to determine the interval of a Year 12 student completing all their homework, to within 1.5 standard deviations of the probability.

10 A group of 32 parents waiting for interviews at a high school parent–teacher evening were discussing the habits of teenagers and 20 of them agreed that 'teenagers take too long in the bathroom'.

Determine the interval that parents of high school students think teenagers take too long in the bathroom, to within 1.6 standard deviations of the probability.

Exam practice

11 (4 marks) State whether each sentence describes a point or interval estimate.

- **a** The mean income is about $47 000. *[1 mark]*
- **b** The median house price in suburbs is between $350 000 and $720 000. *[1 mark]*
- **c** The variance is between 800 and 1500. *[1 mark]*
- **d** The probability of dark hair is about 0.6. *[1 mark]*

12 (2 marks) A poll of 500 voters found that 44% supported a particular political party. What is the standard deviation of the sampling distribution?

13 (4 marks) A sample showed 30% support for a proposal. The estimated variance of the sampling distribution was 0.005. How many people were surveyed?

14 (4 marks) CF The probability of an adult having fair hair is 0.21. What is the probability that a sample of 400 adults has between 70 and 75 with fair hair?

15 (8 marks) CF About 20% of Australians are Queenslanders and of these 49% live in Brisbane.

- **a** Of 300 Australians, what is the probability that between 50 and 60 are Queenslanders? *[4 marks]*
- **b** Of 300 Queenslanders, what is the probability that between 200 and 250 live in Brisbane? *[4 marks]*

16 (4 marks) CF About 6.5% of Brisbane students travel to school by train. What is the probability that between 50 and 60 students from a random sample of 1000 Brisbane students travel to school by train?

17 (4 marks) CF The probability of an Australian having fair hair is about 0.21. What is the probability that there are between 10 and 20 fair-haired people in a random sample of 80 Australians?

18 © QCAA 2024 2Q16 (4 marks) CF At council meetings in a particular town, new proposals are only discussed if more than 80% of the community are in favour of the proposal.

To discover community opinion on a new bus route proposal, the council conducted several surveys, each with a sample size of 120. The distribution of the sample proportions from the surveys had a standard deviation of 0.04.

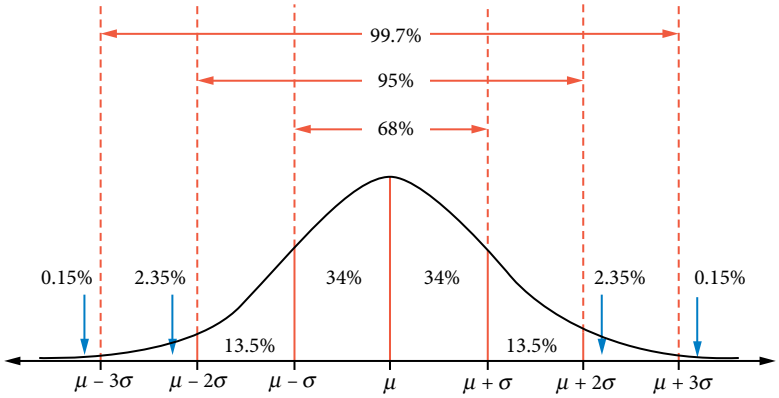
Make a justified decision as to whether the new bus route proposal would be discussed at a council meeting.

10.2 Confidence intervals

Worksheets
Sample proportion confidence intervals

Margins of error for standard normal variables

This diagram of the normal distribution shows the areas (as percentages) under the curve for different intervals of standard deviations from the mean.



For example, the area of the normal distribution that is within ±1 standard deviation of the mean is approximately 68%, while the area of the normal distribution that is within ±2 standard deviations of the mean is approximately 95%.

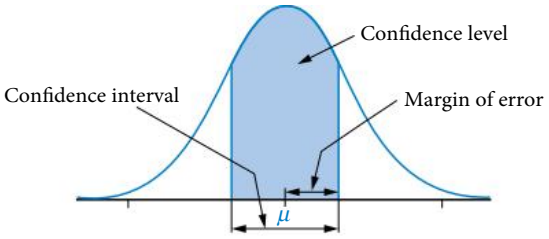
This means you have a 95% level of confidence that the population proportion p lies within ±2 standard deviations of $\hat{p}$.

This is called a **confidence interval**.

Confidence intervals

- A **confidence interval** is an interval estimate involving the sample proportion $\hat{p}$ and a level of confidence the population proportion lies within the interval.
- The **confidence level** is the percentage of values that lie within the interval. This is usually written as α%.
- The **margin of error** is the distance from the mean value ($\hat{p}$) to the end values of the interval.

The diagram below shows visually how all of these terms are linked.



The confidence interval is the distance between the end points of the area. The margin of error is half the width of the confidence interval, since the area is symmetrical about the mean. The confidence level is the shaded area.

TI-Nspire
Confidence intervals for proportions

WORKED EXAMPLE 3 Determining confidence level

Determine the confidence level that produces a margin of error of 1.5 standard deviations either side of the mean.

Steps	Working
1 Since no information given about mean and standard deviation, use the standard normal distribution.	$-1.5 < z < 1.5$
2 Use your technology instructions in Worked example 1 to determine the area.	**TI-84 Plus CE** NORMAL FLOAT AUTO REAL DEGREE MP normalcdf(-1.5,1.5,0,1) 0.8663855426 **Casio fx-CG50 AU** Rad Norm1 d/c r∠θ Normal C.D p =0.86638559 z:Low=-1.5 z:Up =1.5
3 Write the answer.	The confidence level is 86.64%

The equation for the values of a confidence interval comes from the z-score equation $z_\alpha = \dfrac{x-\mu}{\sigma}$.

Since the end points are equidistant from the mean and the z-scores are both positive and negative, the equation is $\pm z_\alpha = \dfrac{x-\mu}{\sigma}$. When this equation is rearranged to isolate x, you get $x = \mu \pm z_\alpha\sigma$, or the interval $(\mu - z_\alpha\sigma, \mu + z_\alpha\sigma)$.

Replacing these values with what you saw in the previous section, you get the interval

$$\left(\hat{p} - z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right).$$

Confidence intervals

The equation for an α% confidence interval for the sample proportion $\hat{p}$ is

$$\left(\hat{p} - z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$$

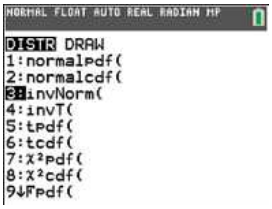
where n is the sample size and z_α is the z-score that is associated with an α% confidence level.

You can work out the margin of error and confidence interval for any confidence level using your calculator.

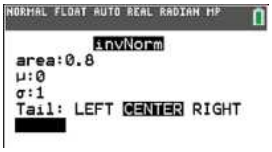
TECHNOLOGY Confidence interval and margin of error

Determine the confidence interval and hence the margin of error for an 80% confidence level for a standard normal variable.

TI-84 Plus CE



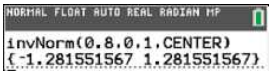
1 Press 2nd vars to open the **DISTR** menu. Choose **2: normalcdf(**.



2 Enter the values as shown above.

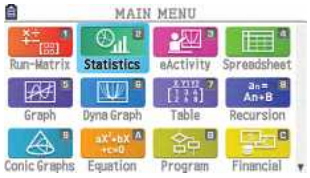


3 Press enter twice.

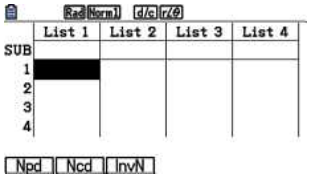


The values appear.

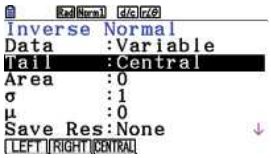
Casio fx-CG50 AU



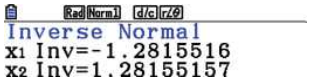
1 From the Main menu, select **2: Statistics**.



2 Press F5 (DIST), then press F1 (NORM) and then press F3 (InvN).



3 Enter the values as shown above. Then press EXE.



The values appear on a new screen.

The confidence interval for an 80% confidence level for a standard normal variable is (–1.2816, 1.2816) and the margin of error is 1.28.

TI-Nspire Confidence intervals for proportions

EXERCISE 10.2 Confidence intervals

ANSWERS p. 419

Recap

1 A survey is undertaken in which 100 staff at a high school are asked if they preferred receiving chocolates or flowers as an end of year gift from students. 84 staff responded that they preferred receiving chocolates. The standard deviation of the sampling distribution of staff who prefer to receive chocolates as an end of year gift is

A 0.0013 **B** 0.0054 **C** 0.0367 **D** 0.0733

2 From the survey in question **1**, the interval within ±2 standard deviations of the sample proportion is

A (0.6963, 0.9837) **B** (0.7681, 0.9119)

C (0.8035, 0.8765) **D** (0.7667, 0.9134)

Mastery

3 WORKED EXAMPLE 3 Determine the confidence level for each margin of error in standard normal variables.

a margin of error of 2.5
b margin of error of 3
c margin of error of 1
d margin of error of 2.7

4 Determine the margin of error needed in standard normal variables to give each confidence level or area.

a $\alpha = 0.98$
b 85% confidence
c $\alpha = 99\%$
d $\alpha = 0.75$

5 Determine the confidence levels for each confidence interval in standard normal variables.

a $-2.5 \leq z \leq 2.5$
b $-1.5 \leq z \leq 1.5$
c $-1 \leq z \leq 1$
d $-3 \leq z \leq 3$

6 Determine the confidence intervals in standard normal variables needed to give each confidence level or area.

a $\alpha = 0.9$
b $\alpha = 0.995$
c 95% confidence
d 99% confidence

7 11 out of 40 Year 12 students said that they have a part-time job.

a What is the sample proportion for Year 12 students having a part-time job?
b Estimate the standard deviation of the sampling distribution.
c Estimate the 95% confidence interval for the probability of Year 12 students having part-time jobs.
d What does the 95% confidence interval mean in this case?

8 11 out of 40 Year 12 students said that they have a part-time job.

a What is the sample proportion for Year 12 students not having a part-time job?
b Estimate the standard deviation of the sampling distribution.
c Estimate the 95% confidence interval for the probability of Year 12 students not having part-time jobs.
d What does the 95% confidence interval mean in this case?

Exam practice

9 © QCAA 2022 2Q4 79% The distribution for a sample proportion $\hat{p}$ has a mean of 0.15 and a standard deviation of 0.0345.

The sample size is

A 10
B 14
C 107
D 116

10 © QCAA 2021 2Q3 73% A random sample of people were surveyed about the most important factor when deciding where to shop. The results appear in the table.

Factor	Percentage (%)
Price	40
Quality of merchandise	30
Service	15
Shopping environment	15

If the sample size was 1200, the approximate 95% confidence interval for the proportion of people who identified price as the most important factor is

A (0.395, 0.405)
B (0.386, 0.414)
C (0.377, 0.423)
D (0.372, 0.428)

11 © QCAA 2020 1Q10 TECH-FREE Two types of material (A and B) are being tested for their ability to withstand different temperatures. A random selection of both materials was subjected to extreme temperature changes and then classified according to their condition after they were removed from the testing facility. The results are shown in the table.

	Material		
	A	B	Total
Broke completely	25	43	68
Showed defects	35	38	73
Remained intact	35	24	59
Total	**95**	**105**	**200**

An approximate 95% confidence interval for the probability that material A will break completely or show defects is given by

$$\left(c-1.96\sqrt{\frac{c(1-c)}{n}}, c+1.96\sqrt{\frac{c(1-c)}{n}}\right)$$

The values of c and n are

A $\frac{60}{95}$ and 95 **B** $\frac{60}{200}$ and 95 **C** $\frac{140}{200}$ and 95 **D** $\frac{60}{200}$ and 200

12 © QCAA 2023 2Q11 (4 marks) A researcher found that 17 out of 50 randomly selected people had used public transport in the past week.

a Determine the sample proportion of people who had used public transport in the past week. *[1 mark]*

b Determine the approximate 95% confidence interval for the proportion of people who had used public transport in the last week. *[2 marks]*

c Someone claims that: *50% of people us public transport each week*. Use your answer from part **b** to explain whether the data can or cannot support this claim. *[1 mark]*

13 (2 marks) Evaluate the reasonableness of the claim that you can have a small margin of error and a high confidence level for a standard normal variable.

14 (2 marks) Determine the confidence level produced by a margin of error of 1.8 in a standard normal variable.

15 (2 marks) Determine the confidence interval needed to give a confidence level of 68% for a standard normal variable.

16 (4 marks) A survey of 500 Australians renting houses in the outer suburbs found that 32% of their income was paid in rent. Estimate the 85% confidence interval for the proportion of income used for rent in the outer suburbs in Australia.

17 (3 marks) TECH-FREE CF 10% of the packets of biscuits sold at a supermarket have at least one broken biscuit. 15 samples of 30 packets are checked. How many times would you expect the 90% confidence interval for broken biscuits to include the value 0.1?

10.3 Margins of error and confidence levels

The margin of error is the distance from the mean value ($\hat{p}$) to the end values of the interval.

Margin of error

The equation for the margin of error (E) is

$$E = z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

where n is the sample size and z_α is the z-score that is associated with an α% confidence level.

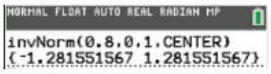
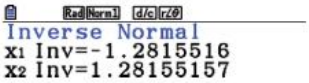
This also means that the confidence interval can be simplified to $(\hat{p} - E, \hat{p} + E)$.

WORKED EXAMPLE 4 Determining margins of error

A sample of 70 has a sample proportion of 0.6.

a **Determine** the margin of error for a confidence level of 80%.

b **Determine** the confidence interval for the sample proportion.

Steps	Working
a 1 Use your graphics calculator to determine the z_α to be used.	**TI-84 Plus CE**  Therefore, $z_\alpha = 1.2816$ **Casio fx-CG50 AU** 
2 Determine the margin of error.	$E = z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ $= 1.2816 \times \sqrt{\frac{0.6 \times 0.4}{70}}$ $= 0.0750$
3 Write the answer.	Therefore, the margin of error is approximately 0.0750.
b 1 Determine the confidence interval.	Interval $= (\hat{p} - E,\ \hat{p} + E)$ $= (0.6 - 0.0750, 0.6 + 0.0750)$ $= (0.5250, 0.6750)$
2 Write the answer.	The 80% confidence interval is approximately $(0.5250, 0.6750)$.

TI-Nspire
Confidence intervals for proportions

You can also determine the margin of error from the confidence interval.

Margin of error from a confidence interval

Given that (lower, upper) $= (\hat{p} - E, \hat{p} + E)$, then:

- lower $= \hat{p} - E$, hence $E = \hat{p} -$ lower
- upper $= \hat{p} + E$, hence $E =$ upper $- \hat{p}$
- upper $-$ lower $= 2E$, hence $E = \frac{\text{upper} - \text{lower}}{2}$

Investigation — Margins of error and confidence levels

Use your graphics calculator to determine the margin of error values for the confidence levels below. For each of these calculations, use a sample proportion of 0.5 and a sample size of 100.

a 50% confidence level **b** 75% confidence level **c** 80% confidence level
d 90% confidence level **e** 95% confidence level **f** 99% confidence level

What do you notice about the relationship between the margin of error and the confidence level?

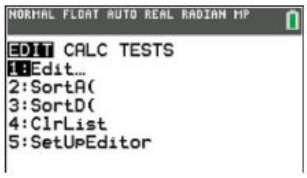
Assuming the $\hat{p}$ and n values remain constant, the margin of error increases as the confidence level increases. To be more confident that the population proportion p lies within an interval, you must accept a broader interval of values.

You can use your graphics calculator as well to find a confidence interval and margin of error without finding the z-score first.

TECHNOLOGY — Confidence intervals and margins or error without z-scores

Determine the 95% confidence interval for a sample proportion of 0.46 and a sample size of 100 and hence find the margin of error.

TI-84 Plus CE



1 Press stat to open the statistics menu.

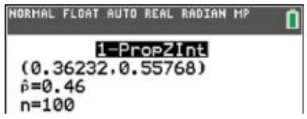


2 Arrow right to **TESTS** and then arrow down to **A: 1-PropZInt**.



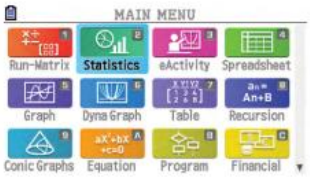
3 Enter the values as shown above. Then go down and press enter.

Recall from Chapter 9 that $\hat{p} = \frac{x}{n}$.

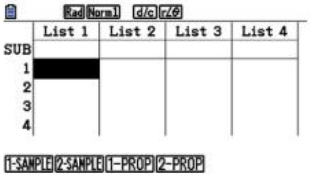


The values appear on a new screen.

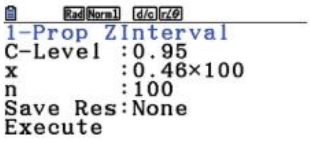
Casio fx-CG50 AU



1 From the Main Menu, select **2: Statistics**.

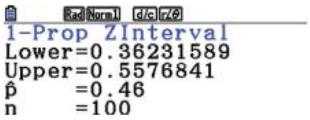


2 Press F4 (INTR), then press F1 (Z) and then press F3 (1-PROP).



3 Enter the values as shown above. Then press EXE.

Recall from Chapter 9 that $\hat{p} = \frac{x}{n}$.



The values appear on a new screen.

The confidence interval is (0.3623, 0.5577).

The margin of error $E = \text{upper} - \hat{p}$ or $\hat{p} - \text{lower}$ or $\frac{\text{upper} - \text{lower}}{2}$

$= 0.5577 - 0.46$ or $0.46 - 0.3623$ or $\frac{0.5577 - 0.3623}{2}$

$= 0.0977$

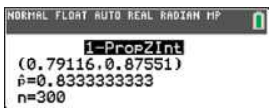
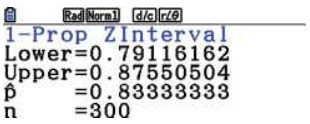
TI-Nspire
Confidence intervals for proportions

TI-Nspire
Confidence intervals for proportions

WORKED EXAMPLE 5 Applying confidence intervals

300 carrot seeds were moistened and placed in a warm place. 5 days later, 250 had germinated.

Estimate the 95% confidence interval for the germination rate.

Steps	Working
1 Identify the necessary components to use a 1-Prop interval on your graphics calculator.	$x = 250$ $n = 300$ C-level = 0.95
2 Enter the values into the graphics calculator.	**TI-84 Plus CE**  **Casio fx-CG50 AU** 
3 Write the answer.	The 95% confidence interval is (0.7912, 0.8755).

EXERCISE 10.3 Margins of error and confidence levels

ANSWERS p. 419

Recap

1 The confidence interval for a sample proportion of 0.38, a sample size of 200 and a confidence level of 90% is

A (0.1444, 0.2356) **B** (0.2671, 0.4929)

C (0.3002, 0.4598) **D** (0.3235, 0.4365)

2 The confidence level for the confidence interval $-1.75 < z < 1.75$ for a standard normal variable is

A 0.7704% **B** 0.9199% **C** 77.04% **D** 91.99%

Mastery

3 WORKED EXAMPLE 4 Determine the approximate margin of error for each confidence level for the stated values of $\hat{p}$ and n.

a $\hat{p} = 0.8, n = 40, \alpha = 90$ **b** $\hat{p} = 0.5, n = 60, \alpha = 95$

c $\hat{p} = 0.45, n = 140, \alpha = 99$ **d** $\hat{p} = 0.65, n = 100, \alpha = 80$

4 Determine the confidence interval for each value of $\hat{p}$, n and α.

a $\hat{p} = 0.1, n = 60, \alpha = 85$ **b** $\hat{p} = 0.75, n = 40, \alpha = 95$

c $\hat{p} = 0.32, n = 55, \alpha = 98$ **d** $\hat{p} = 0.61, n = 70, \alpha = 90$

5 35 of 50 men crossing Eagle St in Brisbane in February were wearing shorts.

a Estimate the probability of men wearing shorts in the city in summer.

b Estimate the standard deviation of the sampling distribution.

c Estimate the 95% confidence interval for the probability of men wearing shorts in the city in summer.

6 WORKED EXAMPLE 5 The songs on popular radio stations in 'drive-time' were checked over several weeks. Out of 140 songs, 125 were of less than 3 minutes duration. Estimate the 90% confidence interval for songs on drive-time radio being less than 3 minutes long.

Exam practice

7 The sample proportion for the 95% confidence interval (0.4567, 0.6789) is

A 0.1055 **B** 0.1111 **C** 0.2222 **D** 0.5678

8 The margin of error for the 95% confidence interval (0.4567, 0.6789) is

A 0.1055 **B** 0.1111 **C** 0.2222 **D** 0.5678

9 TECH-FREE In a survey of 80 people, 53 said they ate the recommended 2 serves of fruit every day. For a 95% confidence interval, the margin of error for people eating the recommended 2 serves of fruit every day is

A $0.95\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$ **B** $\dfrac{1.96}{2}\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$

C $1.96\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$ **D** $95\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$

10 In a survey of 80 people, 53 said they ate the recommended 2 serves of fruit every day. For a 95% confidence interval, the margin of error for people not eating the recommended 2 serves of fruit every day is

A $0.95\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$ **B** $\dfrac{1.96}{2}\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$

C $1.96\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$ **D** $95\sqrt{\dfrac{\frac{53}{80}\left(1-\frac{53}{80}\right)}{80}}$

11 (3 marks) A survey of 178 Queensland students found that 87 came to school by car. Estimate the 95% confidence interval for the probability of Queensland students coming to school by car.

12 (3 marks) A flour mill produced 2 kg packets of flour. Of 100 packets checked at the end of the run, 6 were found to be underweight. Estimate the 95% confidence interval for the proportion of underweight packets.

13 (4 marks) A survey of 1200 voters commissioned by a political party found that, on a two-party preferred basis, 48.5% intended to vote for the party. Estimate the 90% margin of error for this survey.

14 (4 marks) CF Estimate the 90% error margins for $n = 100$ for $\hat{p} = 0.1$, 0.3, 0.5, 0.7 and 0.9 and discuss the similarities and differences between the margins.

15 (4 marks) CF Determine the confidence level given the margin of error is 0.05, the sample proportion is 0.45 and the sample size is 100.

16 © QCAA 2024 1Q3 TECH-FREE 66% A sample of size n can be used to obtain a sample proportion $\hat{p}$. An approximate margin of error for the population proportion can be obtained using the formula

$$E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

If the level of confidence is increased from 95% to 99%, then

A the associated z-value would decrease, so E would increase.

B the associated z-value would increase, so E would increase.

C the associated z-value would decrease, so E would decrease.

D the associated z-value would increase, so E would decrease.

10.4 Variations in confidence levels

A 95% confidence interval means that if different people undertook the same survey and developed their own confidence intervals, approximately 95% of the intervals developed should contain the population proportion. This is not a guarantee, however, due to the random nature of the survey. The more confidence intervals created, the more likely that the 95% will be an accurate percentage.

Investigation Dice simulation of confidence intervals

Use a die to simulate sample proportions and confidence intervals. The probability of the event (rolling a 5 or 6) for the simulation is $p = \frac{1}{3}$ and the sample size $n = 20$.

- Roll a normal die and note whether it lands with 5 or 6.
- Repeat the roll another 19 times and find the sample proportion of high numbers (5 or 6).
- Calculate the approximate 80% confidence interval for your 20 rolls.
- Do another 20 rolls and find the new sample proportion and 80% confidence interval.
- Do this whole procedure a total of 5 times.
- How many of your 5 simulations of the confidence interval contained $p = \frac{1}{3}$?
- Collect the results from other students in your class. Compare the total number of confidence intervals and the number of times that the confidence intervals contained the probability and thus the percentage of times the confidence intervals contained $p = \frac{1}{3}$.

Do the results align with what you expected?

Variations in confidence levels

As the number of different sample proportions and their associated α% confidence intervals from a survey increases, the probability that the confidence intervals contain the population proportion approaches α%.

You can use your graphics calculator to perform a simulation.

WORKED EXAMPLE 6 Simulating confidence intervals

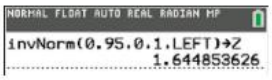
A property has a population proportion of 0.6.

a Simulate 10 samples of 40 items and estimate the 90% confidence interval from each sample proportion.

b **Comment** on how many times p is within the confidence interval.

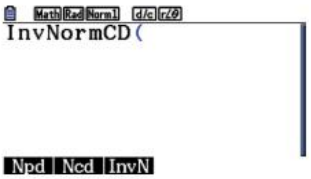
Steps	Working
a To get the positive z-score only, determine the left-tail area up to the same z-score.	0.9 area is distributed equally above and below the mean, hence 0.45 area exists above the mean. Left-tail area = 0.5 + 0.45 = 0.95

TI-84 Plus CE



1 Press 2nd vars to open the **DISTR** menu.

Casio fx-CG50 AU

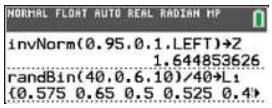


1 From the Main Menu, select **1: Run-Matrix.**

TI-Nspire
Confidence intervals for proportions

2 Arrow down to select **3: invNorm(**.

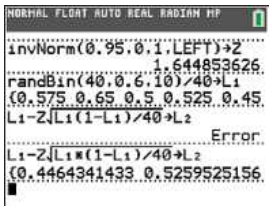
3 Enter the data as shown, then press STO→ alpha 2 and press enter to store the z-score.



4 Press math, arrow right to select **PROB** and select **7: randBin(**.

5 Enter values as shown and press enter.

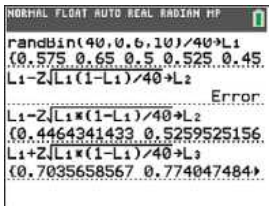
6 Press STO→, then press 2nd 1 and press enter to store the values as List 1.



7 Determine the lower limits of the confidence interval by entering the equation shown, with L_1 representing the sample proportion and Z representing the z-score.

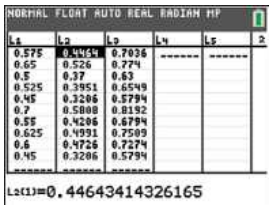
8 Press STO→, then press 2nd 2 and press enter to store the values as List 2.

Ensure there is a multiply sign between L_1 and $(1 - L_1)$, otherwise an error will occur.



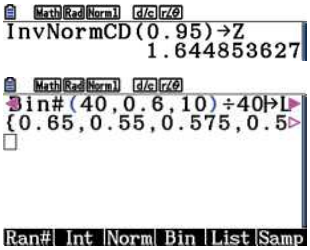
9 Repeat the step above for the upper limit, entering the equation seen below.

10 Press STO→, then press 2nd 3 and press enter to store the values as List 3.



11 Press stat then enter to show the stored data.

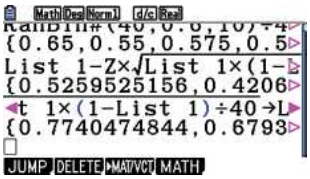
2 Press OPTN, then select F5 (STAT), F3 (DIST), F1 (NORM) and then F3 InvN.



3 Enter the data as shown, then press → ALPHA 0 and press EXE to store the z-score.

4 Press EXIT three times, then press F5 F3 (PROB) F4 (RAND) F4 (Bin).

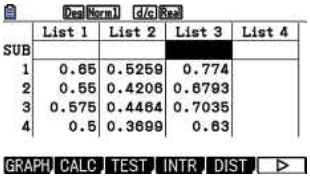
5 Enter values as shown, then press → SHIFT 1 and press EXE to store as List 1.



6 Determine the lower limits of the confidence interval by entering the equation shown, with List 1 representing the sample proportion and Z representing the z-score.

7 Repeat the step above for the upper limit, entering the equation seen below.

> Do not use the button to insert the fraction inside the surd. An Invalid Type error will occur.



8 Press MENU then select **2: Statistics** to show the stored data.

b Comment on the results.

From the TI-84 simulation, $\frac{9}{10}$ sample proportions contained the 0.6 population proportion.

From the Casio simulation, $\frac{10}{10}$ sample proportions contained the 0.6 population proportion.

EXERCISE 10.4 Variations in confidence levels

ANSWERS p. 420

Recap

1 The sample proportion for the 90% confidence interval (0.6942, 0.8888) is

A 0.0876 **B** 0.0973 **C** 0.1946 **D** 0.7915

2 The margin of error for the 90% confidence interval (0.6942, 0.8888) is

A 0.0876 **B** 0.0973 **C** 0.1946 **D** 0.7915

Mastery

3 WORKED EXAMPLE 6 A property has a population proportion of $p = 0.3$.

a Simulate 10 samples of 20 items and estimate the 75% confidence interval for each sample proportion.

b Comment on how many times p is within the confidence interval.

4 A property has a population proportion of $p = 0.8$.

a Simulate 10 samples of 30 items and estimate the 90% confidence interval from each sample proportion.

b Comment on how many times p is within the confidence interval.

5 A property has a population proportion of $p = 0.4$.

a Simulate 10 samples of 20 items and estimate the 60% confidence interval from each sample proportion.

b Comment on how many times p is within the confidence interval.

Exam practice

6 A simulation is performed for 10 samples of 30 items with a population proportion of 0.6, with estimates of the sample proportions using an 80% confidence level. What effect does repeating the simulation for 100 samples instead of 10 samples have on the percentage of confidence intervals that contain p?

A A negligible effect exists.

B The percentage is more likely to be closer to 80%.

C The percentage is less likely to be closer to 80%.

D The percentage for 10 samples will match the percentage for 100 samples.

7 (5 marks) A property has a population proportion of $p = 0.7$.

a Simulate 10 samples of 40 items and estimate the 85% confidence interval for each sample proportion. *[4 marks]*

b Comment on how many times p is within the confidence interval. *[1 mark]*

8 (5 marks) A property has a population proportion of $p = 0.2$.

a Simulate 10 samples of 20 items and estimate the 80% confidence interval for each sample proportion. *[4 marks]*

b Comment on how many times p is within the confidence interval. *[1 mark]*

10.5 Problems involving confidence intervals

Determining the sample size

When the margin of error equation $E = z_\alpha \times \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is rearranged to isolate n, the equation becomes $n = \left(\frac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$.

WORKED EXAMPLE 7 Determining the sample size 1

The probability of an Australian Year 12 student driving to school is estimated to be about 6%.

Determine the sample size of Australian Year 12 students needed to establish a 95% confidence level with a 0.5% margin of error.

Steps	Working
1 State the formula and the parameters to use.	$n = \left(\frac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$ $z_\alpha = 1.96, E = 0.005, \hat{p} = 0.06$
2 Substitute values.	$n = \left(\frac{1.96}{0.005}\right)^2 \times 0.06 \times 0.94$ $= 8666.65$
3 Write the answer.	The sample size required for a 95% confidence level with a margin of error of 0.5% for a 6% sample proportion is 8667 students.

Exam Hack

When solving for n, always round up to the next whole number. Remember, the larger the sample size is, the more accurate the interval.

Using a small margin of error is often impractical. To survey a large number of people takes time and is costly. Before doing a survey, those involved need to work out how accurate they need it to be. They have to balance this with how much time and effort they are prepared to spend.

WORKED EXAMPLE 8 Determining the sample size 2

The probability of an Australian Year 12 student driving to school is estimated to be about 6%.

Determine the sample size of Australian Year 12 students to establish a 95% confidence level with a 5% margin of error.

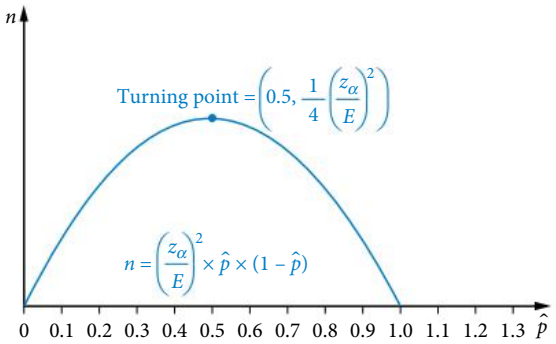
Steps	Working
1 State the formula and the parameters to use.	$n = \left(\frac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$ $z_\alpha = 1.96, E = 0.05, \hat{p} = 0.06$
2 Substitute values.	$n = \left(\frac{1.96}{0.05}\right)^2 \times 0.06 \times 0.94$ $= 86.67$
3 Write the answer.	The sample size required for a 95% confidence level with a margin of error of 5% for a 6% sample proportion is 87 students.

Worksheets
Sample sizes
Sample proportions assignment
Sample proportions problems
Puzzle
Confidence intervals crossword

A factor of 10 increase in the margin of error reduces the sample size by a factor of 100 or 10^2.

If a survey has never been performed for the question being asked, how do you come up with a sample proportion?

The equation $n = \left(\frac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$ with n on the y-axis and $\hat{p}$ on the x-axis is a parabola with a maximum turning point.



The maximum value is at $\hat{p} = 0.5$.

Determining the sample size with no sample proportion

When no sample proportion is available, use $\hat{p} = 0.5$. This will maximise the n value for the given z-score that is associated with an α% confidence interval and the margin of error.

WORKED EXAMPLE 9 Determining the sample size with no sample proportion

A student wants to determine the proportion of cars on the road that are electric vehicles (EVs) to within 10% of the correct value with a 98% confidence level.

Determine how many vehicles the student should note the EV status of.

Steps	Working
1 State the formula and the parameters to use.	$n = \left(\frac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$ $z_\alpha = 2.3263, E = 0.1$ Since the student has not performed the survey before, they should use $\hat{p} = 0.5$.
2 Substitute values.	$n = \left(\frac{2.3263}{0.1}\right)^2 \times 0.5 \times 0.5$ $= 135.29$
3 Write the answer	The sample size required for a 98% confidence level with a margin of error of 10% for an unknown sample proportion is 136 cars.

EXERCISE 10.5 Problems involving confidence intervals

ANSWERS p. 420

Recap

1 A simulation is performed of 10 samples of 50 items with a population proportion of 0.55, with estimates of the sample proportions using an 95% confidence level. What effect does repeating the simulation for 12 samples instead of 10 samples have on the percentage of confidence intervals that will contain p?

A A negligible effect exists.

B The percentage is more likely to be closer to 95%.

C The percentage is less likely to be closer to 95%.

D The percentage for 10 samples will match the percentage for 12 samples.

2 A simulation is performed of 100 samples of 50 items with a population proportion of 0.55, with estimates of the sample proportions using an 95% confidence level. What effect does repeating the simulation for 1000 samples instead of 100 samples have on the percentage of confidence intervals that will contain p?

A A negligible effect exists.

B The percentage is more likely to be closer to 95%.

C The percentage is less likely to be closer to 95%.

D The percentage for 100 samples will match the percentage for 1000 samples.

Mastery

3 WORKED EXAMPLE 7 What sample size would be needed to establish a margin of error of 0.03 at a 98% confidence level for the probability of a property occurring in a proportion of about 0.3 in a population?

4 What sample size would be needed to establish a margin of error of 0.003 at a 98% confidence level for the probability of a property occurring in a proportion of about 0.3 in a population?

5 WORKED EXAMPLE 8 The probability of a particular property in a population is unknown. What sample size should be used to establish the probability with a margin of error of 5% at a confidence level of 90%?

6 What sample size should be used to establish a 98% confidence interval with a margin of error of 5% for a property that occurs about 70% of the time?

7 It is generally assumed that the chances of a baby being a girl or boy are 50–50. How many births would need to be checked to establish the true proportion in Australia to within 0.1% at a confidence level of 95%?

8 WORKED EXAMPLE 9 The probability of a property is completely unknown. What sample size should you use to determine a confidence interval to a level of confidence of 95% with a margin of error of 8%?

9 The probability of a property is completely unknown. What sample size should you use to determine a confidence interval to a level of confidence of 99% with a margin of error of 8%?

10 An advertising company wants to conduct a small survey of consumers to establish a baseline for the proportion who were aware of a particular brand of ice cream before a marketing campaign. How many consumers should they target to get a result accurate to within 3% at a confidence level of 95%?

Exam practice

11 © QCAA 2023 1Q8 TECH-FREE 64% A sample of size n was used to estimate a population proportion. An approximate margin of error of 3% was calculated using $z = 1.96$.

Given the sample proportion was 0.6, determine n.

A $n = \dfrac{\left(\dfrac{0.03}{1.96}\right)^2}{0.24}$ **B** $n = \dfrac{0.24}{\left(\dfrac{0.03}{1.96}\right)^2}$ **C** $n = \dfrac{\left(\dfrac{0.03}{1.96}\right)^2}{2.4}$ **D** $n = \dfrac{2.4}{\left(\dfrac{0.03}{1.96}\right)^2}$

12 A company wants to survey whether people like their new website design. The company wants a 98% confidence level with a 3% margin of error.

The sample size for this survey should be

A 376 **B** 1068 **C** 1504 **D** 6013

13 TECH-FREE A survey is to be undertaken and for a 95% confidence level, given margin of error E and $\hat{p}$ value of 0.5, the sample size is found to be n. It is decided that the margin of error is too big and it will be decreased by a factor of 5. The sample size n will

A increase by a factor of 5. **B** decrease by a factor of 25.

C increase by a factor of 25. **D** decrease by a factor of 5.

14 TECH-FREE A survey is undertaken with no known sample proportion, so the confidence interval is found using $\hat{p} = 0.5$. Once the confidence interval was found, $\hat{p}$ was determined to actually be 0.35.

Which of the following will occur if the confidence interval was recalculated with the same sample size and $\hat{p} = 0.35$?

A Confidence level and margin of error will both remain the same.

B Confidence level will remain the same and the margin of error will increase.

C Confidence level will decrease and the margin of error will increase.

D Confidence level will remain the same and the margin of error will decrease.

15 (4 marks) The tourist information centre at the town square said that about 2 in every 5 of the people coming in asked about local attractions. How many enquiries would need to be noted to establish the proportion to within 2% at a level of confidence of 99%?

16 (4 marks) Suppose you were deciding whether to have a live band play at the Year 12 formal. How many students should you survey to establish whether to have a band at a margin of error of 10% and confidence level of 90%?

17 (4 marks) A poll of voting intentions for 300 voters claimed that the vote for a new political party would be 15% ± 3%. What level of confidence has been used in the survey?

18 (4 marks) The estimated probability and 95% margin of error from a survey were given as 0.31 and 0.058. What was the sample size?

19 (4 marks) CF A survey of 50 people claimed that 71% ± 5% of the population could roll their tongues. What is the level of confidence for this result?

20 (5 marks) CF A regular surfer at Kirra Beach said that about 1 in 5 waves give great rides. Another surfer standing at the lookout watching the waves sees that they are coming about every 11 seconds.

How long would you need to watch to determine the proportion of good waves to an accuracy of 10% at a 95% level of confidence?

21 © QCAA 2024 1Q15 TECH-FREE (4 marks) A survey was conducted to understand whether people support a new policy.

Using a z-score of 2, the approximate confidence interval for the population proportion of people who support the policy was calculated as $\left(\frac{3}{10}, \frac{7}{10}\right)$.

a Determine the margin of error. *[1 mark]*

b Determine the number of people surveyed. *[3 marks]*

Video
Exam question analysis: Confidence intervals for proportions

EXAM QUESTION ANALYSIS

© QCAA 2022 2Q14 (8 marks)

Ravi randomly sampled 200 different pet owners in Brisbane and found that 50 celebrate their pet's birthday.

a Determine an approximate 95% confidence interval for the proportion of Brisbane pet owners who celebrate their pet's birthday. *[2 marks]*

Two of Ravi's friends also randomly sampled Brisbane pet owners. The results are shown in the table.

Friend's name	Number sampled	Number who celebrate their pet's birthday
Khadija	100	26
Tim	150	34

Khadija suggested a more precise estimate for the proportion of Brisbane pet owners who celebrate their pet's birthday could be obtained by combining their results.

b Using all available data, determine a 95% confidence interval for the proportion of Brisbane pet owners who celebrate their pet's birthday. *[2 marks]*

c Use the results from parts **a** and **b** to evaluate the reasonableness of Khadija's suggestion. *[2 marks]*

The proportion of all Brisbane pet owners who celebrate their pet's birthday is 0.24.

d Using the normal approximation, determine the probability that in a randomly selected sample of size 200, more than 30% of pet owners celebrate their pet's birthday. *[2 marks]*

Reading the question

- 200 is sample size, 50 is observed value.
- Sample size is all sample sizes from the three friends, similar for observed values.
- Evaluate the reasonableness; decide if reasonable or unreasonable, then provide evidence to justify decision.
- Solve for $P(x > 0.3)$ in a normal distribution.

Thinking about the question

- Identify elements necessary for a 95% confidence interval.
- Use 1-Prop on calculator to determine confidence interval.
- Increase in n usually means a more accurate interval.
- Normal distribution needs μ and σ to determine probability.

Worked solution (✓ = 1 mark)

a $\hat{p} = \frac{50}{200} = 0.25$, $n = 200$, $z = 1.96$

✓ correctly identifies all of the information required to establish the confidence interval

From graphics calculator, interval is **(0.19, 0.31)**

✓ correctly determines the confidence interval

b For combined results,

$\hat{p} = \frac{110}{450}$, $n = 450$, $z = 1.96$

✓ correctly determines n and $\hat{p}$ for the combined sample

From graphics calculator, interval is **(0.2047, 0.2842)**

✓ correctly determines the confidence interval

c Khadija's suggestion is reasonable.

✓ evaluates the reasonableness of Khadija's suggestion

The margin of error in part **a** is 0.06, while the margin of error in part **b** is 0.0398.

Since n is larger in part b, the margin of error is reduced and hence provides a more precise estimate.

✓ identifies changed width of confidence interval

d For the normal distribution,

$\mu = 0.24$, $\sigma = \sqrt{\frac{0.24 \times 0.76}{200}} = 0.0302$

✓ correctly determines the mean and standard deviation of the normal distribution

From graphics calculator, $P(\hat{p} > 0.3) = 0.0235$

✓ determines the probability

10 Chapter summary

Interval estimates of sample proportions

- Given some number of samples each of size n from a population with probability p, $\dfrac{\hat{p}-p}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$ approximates a standard normal distribution.
- This value best approximates a standard normal distribution as $p \to 0.5$ and $n \to \infty$.
- A **point estimate** is the best estimate of a population proportion.
- An **interval estimate** is the best estimate of the interval that the population proportion lies within.
- $E(\hat{p}) = \hat{p}$

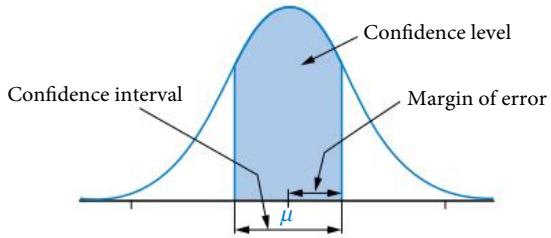
  $Var(\hat{p}) = \dfrac{\hat{p}(1-\hat{p})}{n}$

  $SD(\hat{p}) = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$

  where n is the sample size.

Confidence intervals

- A **confidence interval** is an interval estimate involving the sample proportion $\hat{p}$ and a level of confidence the population proportion lies within the interval.
- The **confidence level** is the percentage of values that lie within the interval. This is usually written as α%.
- The **margin of error** is the distance from the mean value, $\hat{p}$, to the end values of the interval.



- The equation for an α% confidence interval for the sample proportion $\hat{p}$ is

  $\left(\hat{p} - z_\alpha \times \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_\alpha \times \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$

  where n is the sample size and z_α is the z-score that is associated with an α% confidence interval.

Margin of error and confidence levels

- The equation for the margin of error (E) is $E = z_\alpha \times \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$, where n is the sample size and z_α is the z-score that is associated with an α% confidence interval.
- This also means that the confidence interval can be simplified to be $(\hat{p} - E, \hat{p} + E)$.
- Given that (lower, upper) $= (\hat{p} - E, \hat{p} + E)$, then
  - lower $= \hat{p} - E$, hence $E = \hat{p} -$ lower
  - upper $= \hat{p} + E$, hence $E =$ upper $- \hat{p}$
  - upper $-$ lower $= 2E$, hence

    $E = \dfrac{\text{upper} - \text{lower}}{2}$

Variations in confidence levels

- As the number of samples and their associated α% confidence intervals from a survey increases, the percentage of confidence intervals that contain the population proportion approaches α%.

Applying confidence intervals

- When the margin of error equation $E = z_\alpha \times \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ is rearranged to isolate n, the equation becomes $n = \left(\dfrac{z_\alpha}{E}\right)^2 \times \hat{p} \times (1-\hat{p})$.
- When no sample proportion is available, use $\hat{p} = 0.5$. This will maximise the n value for the given z-score that is associated with an α% confidence interval and the margin of error.

Cumulative examination 1

Technology-free

Perusal time: 2 minutes Working time: 30 minutes

TECH-FREE *Calculators are not permitted.*

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.
Section 1 (4 marks): 4 multiple choice questions
Section 2 (16 marks): 4 short response questions
Total: 20 marks

Section 1 4 multiple choice questions

4 marks



Worksheet
Maths Methods formula sheet

1 © QCAA 2022 1Q8 In a survey, 80 respondents exercised daily, while 120 did not. When calculating the approximate 95% confidence interval for the proportion of people who exercise daily, the margin of error is

A $1.96\sqrt{\frac{0.4(1-0.4)}{200}}$

B $0.95\sqrt{\frac{0.4(1-0.4)}{200}}$

C $1.96\sqrt{\frac{0.67(1-0.67)}{200}}$

D $0.95\sqrt{\frac{0.67(1-0.67)}{200}}$

2 © QCAA 2020S 1Q10 If the sample size is decreased, but the sample proportion and the confidence level remain the same, the width of the confidence interval

A will increase.

B will decrease.

C remains unchanged.

D may increase or decrease.

3 The derivative of $f(x) = \sin^2(3x-1)$ is

A $2\sin(3x-1) \times \cos(3x-1)$

B $6\sin(3x-1) \times \cos(3x-1)$

C $2\sin(3) \times \cos(3)$

D $\cos 2(3x-1)$

4 A survey of 150 Year 12 students in a Brisbane high school found that 60 students have their provisional driver's licence while a further 25 had their learner's permit. The sample proportion of Year 12 students in a Brisbane high school who do not have their provisional driver's licence is

A $\frac{2}{25}$

B $\frac{1}{6}$

C $\frac{2}{5}$

D $\frac{3}{5}$

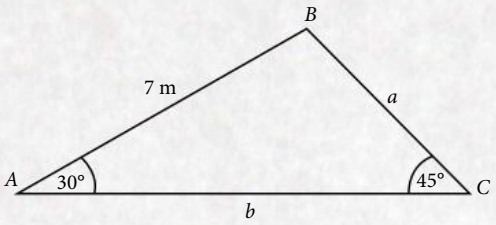
Section 2 4 short response questions

16 marks

5 (3 marks) A normal distribution is known to have mean of 10.

Given that $P(X > 16) = 0.25$, determine $P(X > 4)$.

6 (4 marks) Using the diagram below, determine the exact values of side a and angle B.



7 (5 marks) Use the second derivative to determine the location and nature of all stationary points in the function $f(x) = 0.5x^4 - 4x^2 - 1$.

8 © QCAA 2021 1Q19 (4 marks) A firm aims to have 95% confidence in estimating the proportion of office workers who respond to an email in less than an hour to within ±0.05.

A survey has never been undertaken before, so no past data is available.

The firm believes that if the proportion is 0.5, then this will result in the largest variability in the sample proportion.

Based on this, determine the sample size needed using the approximate value of $z = 2$ for the 95% confidence interval.

Justify the choice of 0.5 for the proportion.

Cumulative examination 2

Technology-active

Perusal time: 2 minutes Working time: 30 minutes

Calculators are permitted.

Questions worth more than one mark require mathematical reasoning and/or working to be shown to support answers.

Section 1 (3 marks): 3 multiple choice questions

Section 2 (17 marks): 3 short response questions

Total: 20 marks



Worksheet
Maths Methods formula sheet

Section 1 3 multiple choice questions

3 marks

1 © QCAA 2023 2Q7 The distribution of a certain sample proportion has a mean of 0.70 and a standard deviation of 0.02. Determine the sample size.

A 525 **B** 750 **C** 1750 **D** 2500

2 © QCAA 2020S 2Q10 A survey found that 142 out of 200 people aged 30 to 39 have some form of tertiary qualification.

The approximate 95% confidence interval for the proportion of people aged 30 to 39 who have some form of tertiary qualification for this survey is

A $(0.62, 0.80)$ **B** $(0.63, 0.79)$ **C** $(0.65, 0.77)$ **D** $(0.66, 0.76)$

3 The area between the curves $f(x) = x^3 - 3x^2 + x + 6$ and $g(x) = 1.5x + 4$ is

A −4.0510 **B** −1.8695 **C** 2.1815 **D** 6.2325

Section 2 3 short response questions

17 marks

4 © QCAA 2020S 2Q15 (8 marks) Suppose the proportion of Australians who supported the removal of single-use plastic bags from supermarkets is 64%.

a Using the normal approximation, determine the probability that, in a randomly selected sample of size 100, more than 70% of those surveyed supported the removal of the single-use plastic bags. *[2 marks]*

b Determine the size of the sample required for the survey to achieve a margin of error of 4% in an approximate 95% confidence interval for this proportion. *[2 marks]*

c Identify the effect that halving the margin of error has on the sample size obtained in part **b**. *[2 marks]*

d Determine the probability that in a randomly selected sample size of 25, the sample proportion is equal to the population proportion. *[2 marks]*

5 © QCAA 2020 2Q17 (4 marks) In a survey of 326 lecturers, 303 said that on at least one occasion a mobile phone had rung in a lecture they were giving.

Determine the sample size required to conduct a follow-up survey that provides 95% confidence that this one-point estimate is correct to within ±0.02 of the population proportion.

6 © QCAA 2020 2Q20 (5 marks) Assuming the approximate normality of sample proportions ($\hat{p}_1$ and $\hat{p}_2$) and based on two independent samples, the approximate confidence interval for the difference of two proportions is given by

$$\left(\hat{p}_1-\hat{p}_2-z\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}},\ \hat{p}_1-\hat{p}_2+z\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1}+\frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}\right)$$

If the approximate confidence interval for the difference between the two proportions does not contain 0, this provides evidence that the two proportions are not equal.

The data in the table shows the observed frequencies for two drink preferences for independent samples of people who live in Town A and Town B.

Town	Tea	Coffee	Total
A	111	105	216
B	150	107	257

Using the approximate 99% confidence interval for the difference of the two proportions, determine if there is evidence to conclude that drink preference is associated with the town where the person lives.

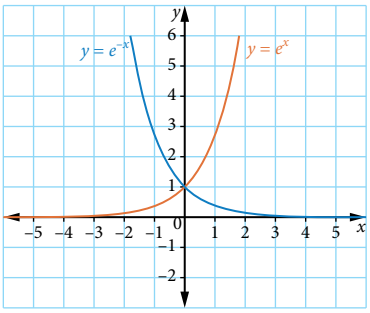
Answers

CHAPTER 1

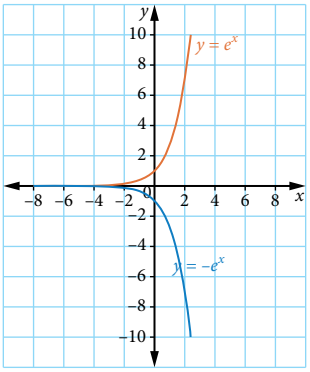
EXERCISE 1.1

1 **a** 4.482 **b** 7.389 **c** 0.050
d 0.368 **e** 0.271 **f** −14.778

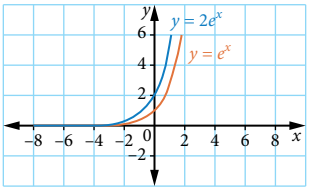
2 **a**



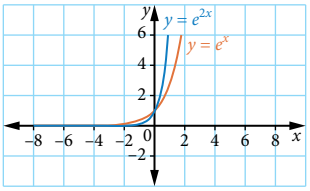
b



c

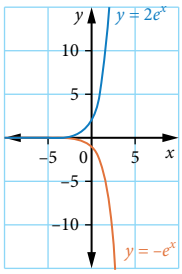


d



3 $y = e^2x - e^2$ **4** 1.92
5 $y = e^4x - 3e^4$ **6** C
7 C **8** B **9** 0.199

10



EXERCISE 1.2

1 **a** 12.182 **b** 0.135 **c** 20.086

2 $y = e^5x - 4e^5$

3 **a** $3 + e^x$ **b** $\dfrac{xe^x - e^x}{x^2}$

c $x^3e^x + 4x^2e^x + 2xe^x + x^{-2}e^x - x^{-3}e^x - 3x^{-4}e^x$

d $9x^2 + 4e^x$ **e** $9e^x$

f $-e^x$ **g** $e^x + 2x$

h $6x^2 - 6x + 5 - e^x$ **i** $3e^x(e^x + 1)^2$

j $-9e^x(5 - e^x)^8$ **k** $12e^x(2e^x - 3)^5$

l $4(e^x + x)^3(e^x + 1)$

4 **a** $e^x + xe^x$ **b** $-\dfrac{2e^x}{(e^x - 1)^2}$

c $(x^2 + 5x + 4)e^x$ **d** $\dfrac{4e^x(x^2 - 2x + 1) - 10x}{(x^2 + 1)^2}$

e $\dfrac{2e^x(x - 1)}{(e^x + x)^2}$ **f** $2x^2e^x(x + 3)$

g $(x^2 + 3x + 2)\,e^x$ **h** $e^x(x^2 - 2)$

i $e^x(x^4 + 4x^3)$ **j** $2xe^{x^2}(x^2 + 1)$

k $\dfrac{e^{x^2}(2x - 1)}{e^x}$ **l** $\dfrac{2e^x}{(1 - e^x)^2}$

m $e^x(2 + x)$ **n** $\left(\dfrac{2}{x} - \dfrac{1}{3x^2}\right)e^{6x}$

o $\left(\dfrac{2}{x^3} - \dfrac{6}{5x^4}\right)e^{5x}$ **p** $(2 - x)e^{-x}$

5 **a** $2xe^{x^2}$ **b** $-2xe^{-x^2+4}$ **c** $e^{2x} + e^{x+1}$

d $20e^{4x-1}$ **e** $6e^{6x-2}$ **f** $-5e^{7-5x}$

g $8e^{4x+1}$ **h** $-8e^{6-2x}$ **i** $-27e^{9x+4}$

j $8e^{6-4x}$ **k** $-80e^{6-8x}$ **l** $-10e^{2x+4}$

m $-42e^{-7x+1}$ **n** $\dfrac{2}{3}e^{\frac{x}{3}+1}$ **o** $-\dfrac{3}{5}e^{2-\frac{x}{5}}$

p $-\dfrac{4}{5}e^{\frac{x}{5}+4}$ **q** $5e^x$ **r** $\dfrac{-4}{x^2}e^{\frac{1}{x}}$

s $2e^{2x-1} - 8xe^{4x^2}$ **t** $-32e^{-8x-1}$ **u** $10e^{10x-5}$

v $3(x-1)^2e^{(x-1)^3}$ or $(3x^2 - 6x + 3)e^{(x^3-3x^2+3x-1)}$

6 **a** $2(x+1)e^{x^2+2x}$ **b** $(2x-3)e^{x^2-3x-2}$

c $2(x-2)e^{x^2-4x}$ **d** $-6e^{2-6x}$

e $(10x - 3)e^{6-3x+5x^2}$ **f** $(3x^2 + 4)e^{x^3+4x}$

g $3(8x-5)e^{4x^2-5x}$ **h** $10(5x+1)e^{1-2x-5x^2}$

i $4(2x + 1)e^{(2x+1)^2}$ **j** $-5(4-x)^4e^{(4-x)^5}$

k $-\dfrac{2e^{\frac{1}{(x+5)^2}}}{(x+5)^3}$ **l** $\dfrac{e^{\sqrt{x-4}}}{2\sqrt{x-4}}$ **m** $\dfrac{e^{(x+4)^{\frac{1}{3}}}}{3(x+4)^{\frac{2}{3}}}$

n $4(x^3 + 6x^2 + 8x)e^{x^4+8x^3+16x^2}$
or $2(x^2 + 4x)(2x + 4)e^{(x^2+4x)^2}$

o $3(1+2x^2)e^{x^2+4}$ **p** $2xe^{x-4}(x+2)$

q $2e^{2x}(1-2e+e^2)$, $2(e^x-e^{x+1})^2$ or $2e^{2x}(1-e)^2$

r $4x^2e^x(x+3)$

7 a e^x+4 **b** $\dfrac{e^x(x-1)}{2x^2}$

c $2e^x(e^x+2)$ **d** $3(e^x+3x)^2(e^x+3)$

e $2e^x(x+1)$ **f** $\dfrac{-4e^x}{(e^x-2)^2}$

g $2e^x(x^2-2)$ **h** $4xe^{x^2}(x^2+1)$

i $(2-x)e^{-x}$ **j** $2e^x(e^x+2)$

8 a $4xe^{x^2}$ **b** $-3e^{8-3x}$

c $8e^{2x+1}$ **d** $2e^{4-2x}$

e $\dfrac{-e^{\frac{x}{4}+5}}{2}$ **f** $-12e^{-6x-1}$

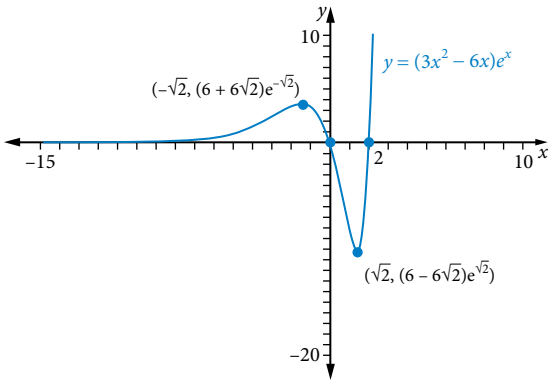
g $4(x-1)e^{x^2-2x}$ **h** $3(4x+3)e^{-2x^2-3x+1}$

i $-\dfrac{e^{\frac{1}{4(x+1)^2}}}{2(x+1)^3}$ **j** $\dfrac{e^{\sqrt{x-4}}}{\sqrt{x-4}}$

k $-\dfrac{2e^{\frac{1}{x}}}{x^2}$

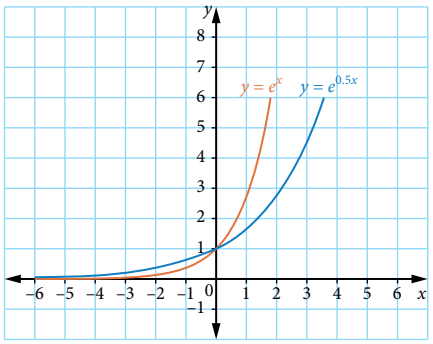
9 D **10** D **11** D

12



EXERCISE 1.3

1



2 B

3 a $e^{3.86} \approx 47.5$ **b** $e^{-0.46} \approx 0.63$

c $e^{7.648} \approx 2096$ **d** $e^{9.22} \approx 10\,100$

e $e^{-7.62} \approx 0.000\,489$

4 a $\ln(0.002\,77) = -5.89$ **b** $\ln(109) = 4.69$

c $\ln(1.4) = 0.37$ **d** $\ln(13\,000) = 9.47$

e $\ln(0.1) = -2.3$

5 a 3.61 **b** −2.59 **c** 6.21

d 0.751 **e** −9.41

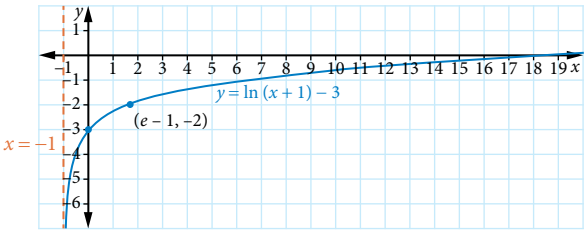
6 a 47.0 **b** 0.001 14 **c** 5.37

d 0.387 **e** 1.70

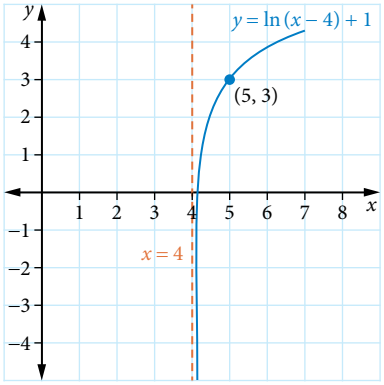
7 a $\ln(12xy^2)$ **b** $\ln(6p)$ **c** $\ln\left(\dfrac{1}{q}+\dfrac{1}{p}\right)$

d $\ln(ex^2y)$ **e** 1 **f** $\ln\left(\dfrac{m}{2n}-\dfrac{n}{2m}\right)$

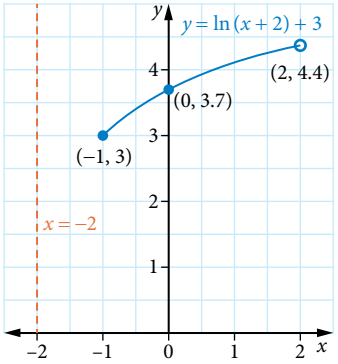
8 Domain is $x > -1$, range is R



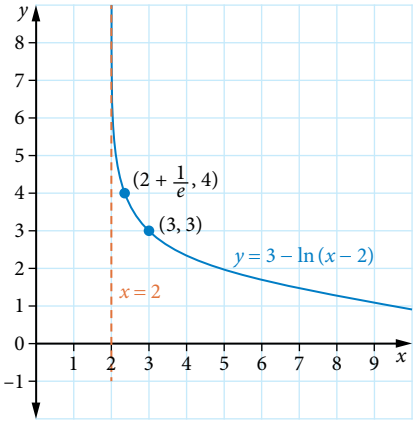
9 Domain is $x > 4$, range is R



10 Range is $3 \le y < 3 + \ln(4)$



11 Domain is $x > 2$, range is R

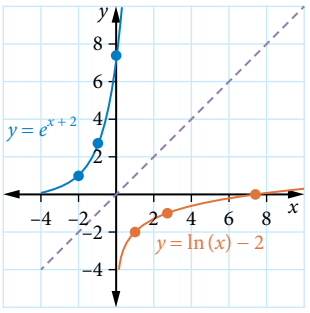


9780170484985

12 **a** $x = -2$ **b** $x = 4$
c $x = -1$ **d** $x = \frac{1}{2}$

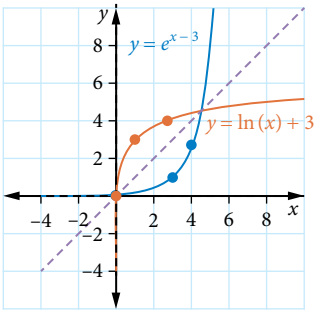
13 **a** $y = \ln(x) - 2$

b



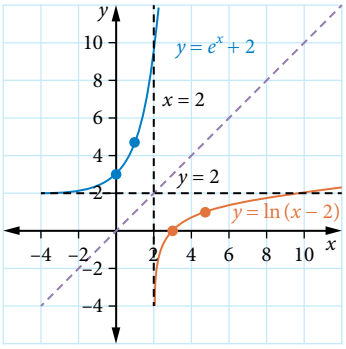
14 **a** $y = \ln(x) + 3$

b



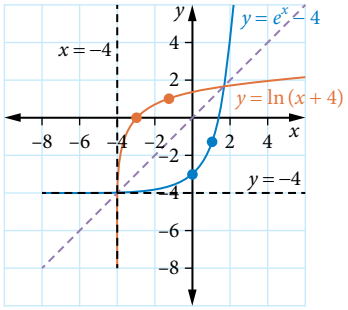
15 **a** $y = \ln(x - 2)$

b



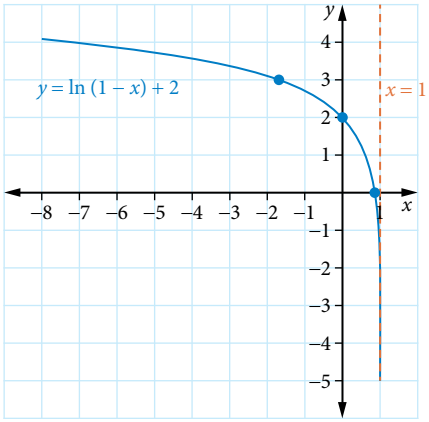
16 **a** $y = \ln(x + 4)$

b



17 A **18** B **19** D

20 Domain is $x < 1$, range is R.



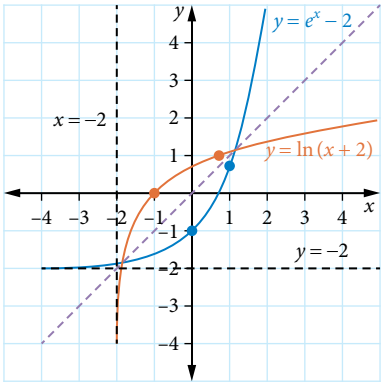
21 $x_P = 12$

EXERCISE 1.4

1 **a** $a \approx 19.7$ **b** $b \approx 0.0488$

2 **a** $e^x - 2$

b



3 **a** $x = 5$ **b** $x = 0$ **c** $x = 2$
d $x = 2$ **e** $x = 3$

4 **a** $x = e^3$ **b** $x = 3$ **c** $x = 1$ or $x = -1$
d $x = 3$ **e** $x = 3$

5 **a** $x \approx \pm 1.731$ **b** $x \approx 2.015$
c $x \approx 2.254$ **d** $x \approx 0.693$ and $x \approx 1.099$
e $x \approx 1.319$ **f** $x \approx -0.510$, $x \approx 4.510$

6 **a** $x \approx 0.474$ **b** $x \approx 16.558$
c $x \approx 0.878$ **d** $x \approx 2.194$

7 A **8** B **9** A

10 **a** $x \approx 1.897$ **b** $x \approx 0.154$, $x \approx 8.068$
c $x \approx 1.130$ **d** $x \approx -0.257$

11 **a** $x = -3$
b $x = 6$
c $x^2 = 1$, so $x = 1$ or $x = -1$
d $x = \frac{1}{4}$

12 A

EXERCISE 1.5

1 $x = 3$ **2** $x \approx -4.381$, $x \approx 1.381$

3 **a** 2 **b** 0.5 **c** 0.25
d 0.05 **e** 100

4 **a** $\frac{7}{x}$ **b** $\frac{5}{x}$ **c** $\frac{1}{x} + 1 + 2x$
d $18x^2 - \frac{7}{x}$ **e** $1 - 2x + \frac{9}{x}$

5 **a** $\frac{1}{x}$ **b** $\frac{1}{x}$ **c** $\frac{1}{x}$

d $\frac{4}{x}$ **e** $\frac{3}{x}$

6 **a** $\frac{8}{x}$ **b** $\frac{3(2x+3)}{x^2+3x+2}$ **c** $\frac{3}{3x-1}$

d $\frac{5}{x}$ **e** $\frac{18}{3x-4}$

7 **a** $\ln(x)+1$

b $2x\ln(x^2)+2x$

c $(4x+3)\ln(2x+1)+2x+2$

d $2x\ln(x)+x+\ln(x^2)+2$

e $\frac{x\left[\ln(x^2+5)+2\right]}{\sqrt{x^2+5}}$

8 **a** $\frac{2x(x+3)-(x^2+3)\ln(x^2+3)}{(x+3)^2(x^2+3)}$

b $\frac{(x-3)\ln(x-3)-(x+3)\ln(x+3)}{(x+3)(x-3)[\ln(x-3)]^2}$

c $\frac{x\left[2\ln(x)-1\right]}{[\ln(x)]^2}$

d $\frac{2(x+1)^2-2(x^2+2x+5)\ln(x^2+2x+5)}{(x^2+2x+5)(x+1)^3}$

e $\frac{3(3x+1)-3\sqrt{3x-1}\ln\left(\sqrt{3x-1}\right)}{2(3x+1)\left(\sqrt{3x-1}\right)^3\sqrt{3x+1}}$

9 **a** 1

b $\ln(x)+2x+1$

c $e^x\ln(x)+\frac{e^x}{x}=\frac{e^x\left[x\ln(x)+1\right]}{x}$

d $2e^{2x+1}\ln(2x+1)+\frac{2e^{2x+1}}{2x+1}=$

$\frac{2e^{2x+1}[(2x+1)\ln(2x+1)+1]}{2x+1}$

e $12e^{4x+3}-3e^{3x}\ln(4x+3)-\frac{4e^{3x}}{4x+3}$

10 **a** $\frac{6x+2}{3x^2+2x+4}$ **b** $\frac{7x^2+14x+5}{7x^2+5}$

c $\frac{1}{x\ln(x)}$ **d** $\frac{5}{x}$ **e** $\frac{15}{x}$

11 **a** $\frac{3}{x-5}$ **b** $\frac{20}{4x+3}$

c $\frac{6(3x+2)}{3x^2+4x+2}$ **d** $\frac{10}{x+1}$

e $\frac{4(4x+3)}{(2x+1)(x+1)}$

12 **a** $\frac{2(2x-1)(3x-1)}{4x^3-5x^2+2x-3}$ **b** $\frac{5x^4+12x^3-2}{x(x^4+3x^3-2)}$

c $\frac{28x^3+9x^2-4x+5}{7x^4+3x^3-2x^2+5x-6}$

13 **a** $\frac{5x-1}{(x+3)(x-5)}$

b $\frac{6\ln(x-5)}{x+3}+\frac{6\ln(x+3)}{x-5}$

$=\frac{6[(x-5)\ln(x-5)+(x+3)\ln(x+3)]}{(x+3)(x-5)}$

c $\frac{2}{3(x+3)\ln(x-5)}-\frac{2\ln(x+3)}{3(x-5)[\ln(x-5)]^2}$

$=\frac{2[(x-5)\ln(x-5)-(x+3)\ln(x+3)]}{3(x+3)(x-5)[\ln(x-5)]^2}$

d $\frac{-x-19}{(x+3)(x-5)}$

14 **a** $\frac{2x-3}{2(x^2-3x+4)}$ **b** $\frac{6}{4x-5}$

c $\frac{6\sqrt{\ln(4x-5)}}{4x-5}$ **d** $\frac{1}{x[3\ln(x)]^{\frac{2}{3}}}$

15 **a** $-\frac{1}{x}$ **b** $-\frac{1}{x[\ln(x)]^2}$

c $\frac{-4(2x-3)}{x^2-3x+5}$ **d** $\frac{-(3x^2-6x+5)}{2(x^3-3x^2+5x+4)}$

16 $2(3x+1)\ln(x^3+x^2+x-5)+\frac{(3x^2+2x+1)^2}{x^3+x^2+x-5}$

17 $(3x^2+2x+1)\ln(3x^2+2x+1)+\frac{2(3x+1)(x^3+x^2+x-5)}{3x^2+2x+1}$

18 **a** $\frac{4}{x}$ **b** $\frac{1-3\ln(x)}{x^4}$

19 B **20** C

21 **a** $\frac{2}{(2x+5)}$ **b** $x=-2$ **c** $m=2$

22 $\frac{8[(x^2+3)\ln(x^2+3)-x(2x+5)\ln(2x+5)]}{3(2x+5)(x^2+3)[\ln(x^2+3)]^2}$

23 $\frac{(3x+1)(3x-1)}{6x^3-2x+5}$ **24** $f'(x)=\frac{5x+12}{x(x+4)}$

25 $g'(5)=0.25$ **26** $f'(x)=\frac{na}{ax+b}$

EXERCISE 1.6

1 $\frac{1}{15}$ **2** $f'(x)=\frac{3x^2+6x}{x^3+3x^2-8}$

3 **a** 200 g

b **i** 197.6 g **ii** 177.4 g

iii 60.2 g

c **i** 2.4 g/year **ii** 2.1 g/year

iii 0.7 g/year

4 **a** 200 **b** 492

c 74 sales/week **d** 181 sales/week

5 **a** 502

b **i** 504 **ii** 1307

c $0.2e^{0.1t}$

d **i** 0.4 fish/month **ii** 81 fish/month

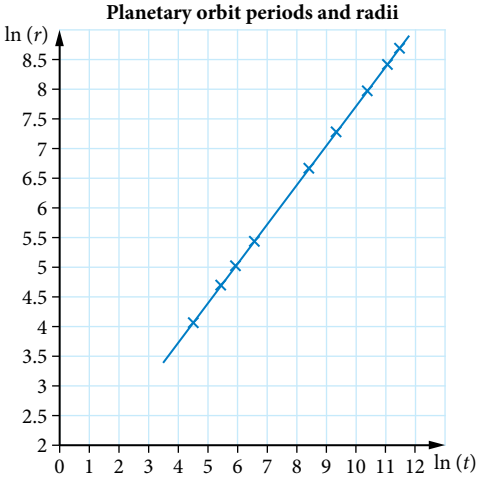
6 **a** **i** 74°C **ii** 27°C

b **i** −18 °C/min **ii** −7 °C/min

7 $r\approx 120c^{1.5}$

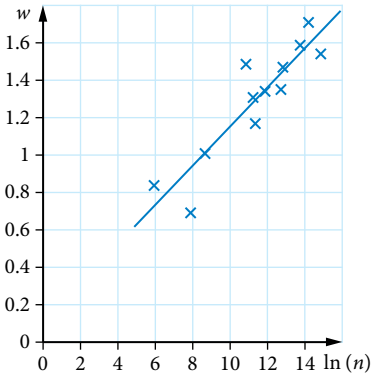
8 $r \approx 2.9\sqrt[3]{T^2}$ (*see* Kepler's laws)

Planetary orbit periods and radii



9 $w \approx 0.1 \ln(n) + 0.1$, using a rough line

Walking speed



10 **a** 80 000 hectares

b **i** 78 416 hectares

ii 65 498 hectares

iii 10 827 hectares

c **i** −1568 hectares/year

ii −1310 hectares/year

iii −217 hectares/year

11 **a** 85

b **i** 127 **ii** 261 **iii** 70 449

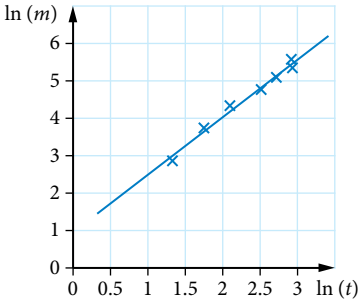
c **i** 10 people/day

ii 21 people/day

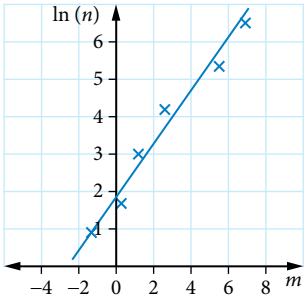
iii 5636 people/day

d ≈0.88 days or 12.5 weeks

12 $m = 2.5t^{1.5}$



13 $n = 6.5e^{0.7m}$



14 approx. 36 months

15 **a** 16%

b **i** approx. 23 years

ii approx. 37 years

iii approx. 46 years

16 **a** approx. 27.5 hours

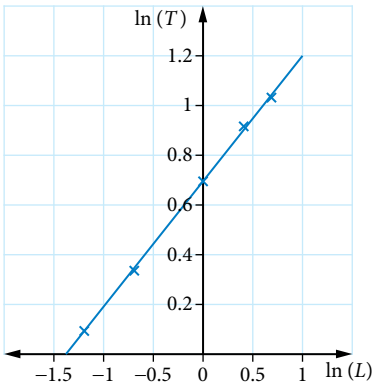
b **i** 0.465 units/h

ii 0.372 units/h

iii 0.298 units/h

17 $T \approx 2L^{0.5} = 2\sqrt{L}$

Pendulum length and period



18 B **19** A

20 **a** 15 °/min **b** 4.3 °/min **c** 1.2 °/min

21 approx. 13.8 years

22 **a** $A = 9 \times 10^5$ **b** 449 443 termites

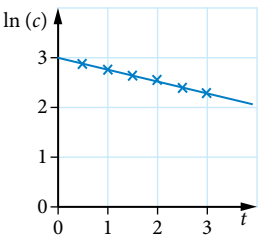
c 2.3749 months **d** $\dfrac{dN}{dt} = \dfrac{(9\times10)^5}{e^t(2+e^{-t})^2}$

e 1506 termites/month

23 approx. 13 months

24

Antibiotic concentration

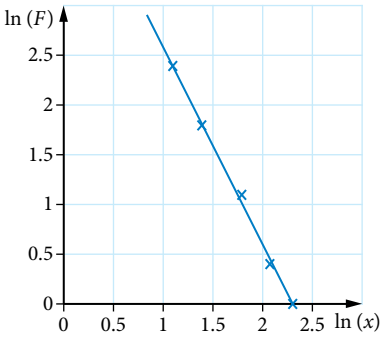


$c \approx 20e^{-0.24}$, every 5 or 6 hours

25 Yes, the total number of customers will be the same every day in the future, i.e. 50.

26 $s \approx 9.1 \ln(t+1) + 20$; After another 6 weeks, she should reach her target.

27 Magnetic force



$F \approx 98x^{-2}$, the force is inversely proportional to the square of the separation.

28 $I \approx 2.2\ln(d+1)+2$, about 37 days training to find all 10 bags

CUMULATIVE EXAMINATION 1

1 B **2** D **3** A

4 $\dfrac{2}{(x+1)^2}$ **5** $x=-1$ and $x=-5$

6 $\log_{10}(x^2-9)=\log_{10}(9x-29)$, $x^2-9x+20=0$, $x=4, 5$

7 $A=100$, $k=\dfrac{-1}{2}$, $\dfrac{1}{7}=e^{-0.5t}$, $t=-2\ln\left(\dfrac{1}{7}\right)$, $t=\ln(49)$, 1 mark for logical steps

CUMULATIVE EXAMINATION 2

1 A **2** C

3 C **4** 3255

5 $P(\text{all 6 correct})=0.0244\%$

6 a $b=\dfrac{1}{2}$, $A=60e$

b (2, 120) is the maximum point of the function, so it is reasonable

7 approx. 50 000

CHAPTER 2

EXERCISE 2.1

1 a $6\cos(x)$ **b** $-3\cos(x)$

c $-\cos(x)$ **d** $4\cos(4\theta)$

e $7\cos(-7\theta)$ **f** $2\cos(8x)$

g $\dfrac{25}{4}\cos(10t)$ **h** $\dfrac{\pi}{30}\cos(6x°)$

2 a $(3\alpha^2)\cos(\alpha^3)$

b $(8\alpha)\cos(4\alpha^2)$

c $(6x^2)\cos(2x^3-7)$

d $3(2y-5)\cos(y^2-5y)$

e $2(3t^2-6t+1)\cos(t^3-3t^2+t-1)$

f $\dfrac{3}{2\sqrt{x}}\cos(\sqrt{x})$

g $\dfrac{1-3x^2}{2\sqrt{x-x^3}}\cos(\sqrt{x-x^3})$

h $\dfrac{3\cos(3x)}{\sqrt{\sin(3x)}}$

3 a $12\cos(4t)$ **b** $-15\cos(3p)$

c $\dfrac{7\pi}{90}\cos(2\theta°)$ **d** $-\cos(4-x)$

e $3\cos(3v-5)$ **f** $-5\cos(3-5t)$

g $8\cos(2t+3)$ **h** $6\cos(m-4)$

4 a $[9(3x-2)^2]\cos[(3x-2)^3]$

b $[30(5t+4)^5]\cos[(5t+4)^6]$

c $[-2(2m-7)^{-2}]\cos[(2m-7)^{-1}]$

d $[3(5-v)^{-4}]\cos[(5-v)^{-3}]$

e $-60\sin^3(3n+1)\times\cos(3n+1)$

f $-60\sin^2(6-5t)\times\cos(6-5t)$

g $-40(2x+7)^{-5}\cos[(2x+7)^{-4}]$

h $-24\sin(4x+5)\times\cos(4x+5)$

5 a $3[\sin(t)]^2\times\cos(t)$

b $40[\sin(4t)]^4\times\cos(4t)$

c $2\sin(x)\cos(x)+2\cos(x)$

d $-6\cos(x)\times[5-2\sin(x)]^2$

e $40\sin(5x-3)\times\cos(5x-3)$

f $2[1-3\sin(t)+\sin(3t)][-3\cos(t)+3\cos(3t)]$

g $24g^2\times\sin^3(2g^3)\times\cos(2g^3)$

h $-120(1-3f)^{-3}\times\sin^4[(1-3f)^{-2}]\cos[(1-3f)^{-2}]$

6 A **7** D **8** $-\dfrac{\cos(x)}{\sin^2(x)}$

9 $\left(\dfrac{\pi}{2}, 4\right)$ **10** $x=\dfrac{\pi}{3k^2}$

11 B

EXERCISE 2.2

1 B **2** B

3 a $-4\sin(x)$ **b** $5\sin(x)$

c $2\sin(x)$ **d** $-3\sin(3a)$

e $6\sin(-6\theta)$ **f** $-4\sin(12x)$

g $-\dfrac{50}{6}\sin(10t)$ **h** $-5\sin(5m)$

4 a $-3x^2\times\sin(x^3)$

b $-20n^3\times\sin(5n^4)$

c $-(10x)\sin(5x^2-9)$

d $-6(4y-5)\sin(2y^2-5y-5)$

e $-4(9t^2+8t+3)\sin(3t^3+4t^2+3t+2)$

f $\left(-\dfrac{5}{2\sqrt{x}}\right)\sin(\sqrt{x})$

g $\dfrac{5x^4-2x}{2\sqrt{x^5-x^2}}\sin(\sqrt{x^5-x^2})$

h $\dfrac{-9\sin(3x)}{2\sqrt{3\cos(3x)}}$

5 a $-24\sin(3t)$ **b** $8\sin(4x)$

c $-\dfrac{\pi}{15}\sin(3\theta°)$ **d** $-3\sin(m-5)$

e $-5\sin(5v-3)$ **f** $3\sin(5-3t)$

g $-24\sin(4t+7)$ **h** $2\sin(7-2x)$

6 a $-8(4x-5)\sin[(4x-5)^2]$

b $-15(3t+2)^4\times\sin[(3t+2)^5]$

c $(m-3)^{-2}\times\sin[(m-3)^{-1}]$

d $-4(4-v)^{-5}\times\sin[(4-v)^{-4}]$

e $36(2n+5)^5\times\sin[(2n+5)^6]$

f $48(5-4t)^3\times\sin[(5-4t)^4]$

g $72(3x+1)^{-4}\times\sin[(3x+1)^{-3}]$

h $-70(7x+3)^{-6}\times\sin[(7x+3)^{-5}]$

7 a $-4[\cos(\alpha)]^3\times\sin(\alpha)$

b $-24[\cos(2t)]^3\times\sin(2t)$

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c $-2\cos(x)\sin(x) - 3\sin(x)$

d $20[6 - 5\cos(x)]^3 \times \sin(x)$

e $-24\cos(3x - 5) \times \sin(3x - 5)$

f $2[3 - \cos(2t) + \cos(t)] \times [2\sin(2t) - \sin(t)]$

g $90[\cos[(1 - f)^{-3}]]^5 \times [(1 - f)^{-4}]\sin[(1 - f)^{-3}]$

h $(-60g^4)[\cos(4g^5)]^2 \times \sin(4g^5)$

8 D **9** B **10** D

11 $\left(\frac{2\pi}{3}, 2.5\right)$

12 a $t = \frac{3\pi}{2}$

b $v(t) = -\frac{4}{3}\sin\left(\frac{t}{3}\right)$

c $t = 0$

d $a(t) = -\frac{4}{9}\cos\left(\frac{t}{3}\right)$

e At $t = \frac{3\pi}{2}$, the particle is back at the origin, but also has no acceleration. The particle is moving with the largest speed in the negative direction.

13 $a = -k^2x$

EXERCISE 2.3

1 A **2** D

3 a $\frac{9}{\cos^2(x)}$ **b** $\frac{-2}{\cos^2(x)}$

c $\frac{-1}{\cos^2(x)}$ **d** $\frac{5}{\cos^2(5m)}$

e $\frac{4x^3}{\cos^2(x^4)}$ **f** $\frac{6n}{\cos^2(3n^2)}$

g $\frac{8x}{\cos^2(4x^2 - 25)}$ **h** $\frac{7 \times (6x - 6)}{\cos^2(3x^2 - 6x + 3)}$

4 a $\frac{-4}{\cos^2(-4\theta)}$ **b** $\frac{4}{\cos^2(10x)}$

c $\frac{20}{3\cos^2(15t)}$ **d** $\frac{\pi}{60\cos^2(3m°)}$

e $\frac{8 \times (12t^2 - 6t + 5)}{\cos^2(4t^3 - 3t^2 + 5t - 1)}$ **f** $\frac{3}{\sqrt{x}\cos^2(\sqrt{x})}$

g $\frac{6x^5 - 4x^3}{2\sqrt{x^6 - x^4}\cos^2(\sqrt{x^6 - x^4})}$

h $\frac{8}{\sqrt{8\tan(2x)}\cos^2(x)}$

5 $y = -\frac{3\sqrt{3}}{2}x + \sqrt{3}\pi - \frac{3}{2}$

6 $43.77\,\text{km h}^{-1}$

7 D **8** D **9** A **10** C

11 a $h(t) = -7\cos\left(\frac{\pi t}{5}\right) - 1$

b $h'(12) = 4.1830\,\text{mm s}^{-1}$ upwards

12 a $h(t) = 0.3\sin(6\pi t)$

b $h'(t) = 1.8\pi\cos(6\pi t)$

13 $y = \frac{x}{2} - \frac{\pi}{4}$

14 The acute angle between the two curves is 70.53°.

15 $x = -\frac{2}{5.1}\cos(5.1t)$. Greatest angle is 7.51°.

EXERCISE 2.4

1 A **2** D

3 $\frac{\cos[\ln(x)]}{x} - \frac{\cos(x)}{\sin(x)} + e^x \times \cos(e^x) + \cos(x) \times e^{\sin(x)}$

4 a $[15(5x - 6)^2]e^{(5x-6)^3}$

b $\frac{10}{(5x + 4)}$

c $[7(y - 2)^6] \times \cos[(y - 2)^7]$

d $-60[\sin(5q + 8)]^3 \times \cos(5q + 8)$

e $105\sin[7(7 - 5a)^3] \times (7 - 5a)^2$

f $-30(5g^3 + 3) \times \sin[\sin 6(5g^3 + 3)] \times \sin(5g^3 + 3)$

5 $x(t) = 8\sin(\pi t)$. Speed when vertical is $8\pi\,\text{cm s}^{-1}$ or $25.16\,\text{cm s}^{-1}$.

6 A **7** C

8 $f'(x) = 6e^{2x+1}$ **9** $\frac{dy}{dx} = \frac{\cos(x)}{2\sqrt{\sin(x)}}$

10 $y' = 3e^x \times (e^x + 1)^2$ **11** $0.57°\,\text{s}^{-1}$

12 $x(t) = 4.4\cos(420\pi t)$; $x'(2) = 0\,\text{mm s}^{-1}$

13 A

EXERCISE 2.5

1 C **2** C

3 a $5[(6x + 4)(e^{4x-2} + 4) + (4e^{4x-2})(3x^2 + 4x - 3)]\cos[(3x^2 + 4x - 3)(e^{4x-2} + 4]$

b $-16e^{-2x} \times \cos(5x - 3) + 8e^{-2x} \times -5\sin(5x - 3)$

c $[e^{3x} \times \cos(4x) \times (\cos(x) \times \ln(2x) + \frac{\sin(x)}{x})] + [\sin(x) \times \ln(2x) \times (3e^{3x} \times \cos(4x) + e^{3x} \times -4\sin(4x))]$

d $4e^{6-3x}[(10x + 3)\cos(5x^2 + 3x - 2) - 3\sin(5x^2 + 3x - 2)]$

4 a $2x \times \sin(3x) + x^2 \times 3\cos(3x)$

b $(2c + 1) \times \sin(2c - 3) + (c^2 + c - 1) \times 2\cos(2c - 3)$

c $(10m - 3) \times \sin(4m - 3) + (5m^2 - 3m + 2) \times 4\cos(4m - 3)$

d $\cos(x - 3) \times \sin(x + 3) + \cos(x + 3) \times \sin(x - 3)$

5 a $3x^2 \times \cos(4x) + x^3 \times -4\sin(4x)$

b $(2m + 3) \times \cos(2m - 3) + (m^2 + 3m - 4) \times -2\sin(2m - 3)$

c $(15x^2 - 8x) \times \cos(3x - 5) + (5x^3 - 4x^2 + 3) \times -3\cos(3x - 5)$

d $-2\sin(2x + 1) \times \cos(3x + 1) + -3\sin(3x + 1) \times \cos(2x + 1)$

6 a $\sin(3x) \times 3x^2 + 3\cos(3x) \times x^3$

b $e^x \times (x^2 - 1) + e^x \times 2x$

c $e^{3x} \times 4x + 2x^2 \times 3e^{3x}$

d $\cos(2x) \times 6x^2 + 2x^3 \times -2\sin(2x)$

e $\sqrt{3x^2 - 1} \times 2e^{2x} + e^{2x} \times \frac{6x}{2\sqrt{3x^2 - 1}}$

f $\cos^3(2x) \times 2x + x^2 \times 3\cos^2(2x) \times -2\sin(2x)$

7 a $(6x - 2) \times e^{\sin(2x+1)} + (3x^2 - 2x - 1) \times 2\cos(2x + 1) \times e^{\sin(2x+1)} + 10x \times \ln[x \times \sin(x^2 + 1)] + 5x^2 \times \frac{\sin(x^2 + 1) + 2x^2 \times \cos(x^2 + 1)}{x \times \sin(x^2 + 1)}$

b $(2\sin[\ln(2x - 3)] + \cos[\ln(2x + 3)]) \times (7e^{-0.4x^2} - 5.6xe^{-0.4x^2}) + 7xe^{-0.4x^2} \times (2\cos[\ln(2x - 3)] \times \frac{2}{2x - 3} - \sin[\ln(2x + 3)] \times \frac{2}{2x + 3})$

8 A **9** B

10 a $x(t) = 2e^{\frac{\ln(2)}{20}t} \times \cos\left(\frac{2\pi t}{5}\right)$

b 3.29 mm s^{-1} in a downwards direction

11 $x = 1$ and $x = -1.5$

12 approx. 2 minutes

13 D

EXERCISE 2.6

1 C **2** B

3 a $\frac{\sin(4-x) \times (3x^2 - 6x) + (x^3 - 3x^2 + 4) \times \cos(4-x)}{\sin^2(4-x)}$

b $\frac{\sin(3x+2) \times -10\cos(1-2x) - 5\sin(1-2x) \times 3\cos(3x+2)}{\sin^2(3x+2)}$

c $\frac{\cos(5-3x) \times (4x - 9x^2) - (2x^2 - 3x^3 + 2) \times 3\sin(5-3x)}{\cos^2(5-3x)}$

d $\frac{\cos(3+x) \times 3\sin(2-x) + 3\cos(2-x) \times \sin(3+x)}{\cos^2(3+x)}$

4 a $\frac{(3x+2) \times -5\sin(5x-3) - \cos(5x-3) \times 3}{(3x+2)^2}$

b $\frac{(3x-5)^3 \times 20\sin(3-4x) - 5\cos(3-4x) \times 9(3x-5)^2}{(3x-5)^6}$

c $\frac{(x+4) \times 3\cos(3x-5) - \sin(3x-5)}{(x+4)^2}$

d $\frac{(4x-1)^2 \times 4\cos(2x+3) - 2\sin(2x+3) \times 8(4x-1)}{(4x-1)^4}$

5 a $\frac{189(q-2)^2(2q^9+17)}{(2q+3)^3}\cos\left(\frac{27(q-2)^3}{(2q+3)^2}\right)$

b $\frac{-e^x(2\cos(x) + e^x)}{(1 + e^x\cos(x))^2}$

c $\frac{9\left[\sin\left((3x-2)^4\right)\right]\left[8(3x-2)^3\cos\left((3x-2)^4\right)\cos\left(x^3 - x^2 + 5\right) - x(3x-2)\sin\left((3x-2)^4\right)\right)\sin\left(x^3 - x^2 + 5\right)\right]}{5\cos^4\left(x^3 - x^2 + 5\right)}$

d $\frac{\tan(1+e^{-x}) \times \left[(\cos(1-e^x) \times \cos(1+e^{-2x}) \times -2e^{-2x}) + (\sin(1+e^{-2x}) \times \sin(1-e^x) \times e^x)\right] - \sin(1+e^{-2x})\cos(1-e^x) \times \left(\frac{-e^{-x}}{\cos^2(1+e^{-x})}\right)}{\tan^2(1+e^{-x})}$

6 C **7** A **8** D **9** D

10 $\frac{dy}{dx} = \frac{x\cos(x) - 2\sin(x)}{x^3}$

11 $g'(e) = 0$

12 max. = $(-1.732, 2.527)$, min. at $(-0.161, -0.525)$

CUMULATIVE EXAMINATION 1

1 A **2** A **3** D **4** C

5 $y = 1 + \frac{3}{e}x$

6 a $f'\left(\frac{\pi}{4}\right) = 0$ **b** $f'\left(\frac{\pi}{4}\right) = -2e^{\frac{\pi}{4}}$

7 $x = \frac{1}{2}\ln\left(\frac{7}{3}\right)$

CUMULATIVE EXAMINATION 2

1 B **2** D **3** C

4 a $w(t) = 1.1\cos\left(\frac{\pi}{6}t\right) + 7.4$

b 1.1512 hours or 69.07 minutes

5 $f'(2) = 9.8103$

6 $A = 20, b = \frac{e}{4}$

7 Speed when parachute is deployed is 47.2536 m s^{-1}.
Car comes to a stop after another 3.7093 s.

CHAPTER 3

EXERCISE 3.1

1 a $f'(x) = 5x^4 - 9x^2 + 7, f''(x) = 20x^3 - 18x$

b $f'(x) = 2\cos(2x), f''(x) = -4\sin(2x)$

c $y' = -8e^{5-4x}, y'' = 32e^{5-4x}$

d $y' = \dfrac{5}{5x-7}, y'' = \dfrac{-25}{(5x-7)^2}$

e $g'(x) = -12(2-3x)^3, g''(x) = 108(2-3x)^2$

f $y' = 12e^{3x+1}, y'' = 36e^{3x+1}$

2 a 156 **b** $\dfrac{9}{4}$

c $8\sqrt{3}$ **d** $-\dfrac{115}{484}$

e 482.05 **f** $\dfrac{-9\sqrt{2}}{2}$

3 a $f'(x) = -30(1-6x)^4, f''(x) = 720(1-6x)^3$

b $f'(t) = \dfrac{-3}{(t-1)^2}, f''(t) = \dfrac{6}{(t-1)^3}$

c $g'(w) = \dfrac{w}{\sqrt{w^2+1}}, g''(w) = \dfrac{1}{(w^2+1)^{\frac{3}{2}}}$

d $f'(x) = 2x^2\cos(2x) + 2x\sin(2x)$,
$f''(x) = 8x\cos(2x) + (2-4x^2)\sin(2x)$

e $h'(m) = \dfrac{-1}{m^2}, h''(m) = \dfrac{2}{m^3}$

f $g'(p) = \dfrac{3}{p} + 1, g''(p) = -\dfrac{3}{p^2}$

4 $f'(2) = \dfrac{17}{4}, f''(-3) = \dfrac{56}{27}$

5 $g''(x) = 40x^3$

6 $y'(e) = 4, y''(1) = 2$

7 $h''(x) = 2\cos(x) - x\sin(x)$

8 $x = \dfrac{11}{24}$

9 $f'(x) = \dfrac{4}{\cos^2(4x)}, f''(x) = \dfrac{32\sin(4x)}{\cos^3(4x)}$

10 $a = 2$ and $b = 3$

11 a $6x - 6$ **b** -12 **c** $(1, -2)$

EXERCISE 3.2

1 B

2 $f'(2) = 9x^2 - \dfrac{1}{2}, f''(x) = 18x$

3 a increasing for $1 < x < 3$ and $7 < x < 8$;
decreasing for $3 < x < 7$

b increasing for $1 < x < 3$ and $6 < x < 7$;
decreasing for $0 < x < 1$ and $3 < x < 6$

4 a concave upwards for $q < x < r$, $r < x < s$ and $t < x < u$

b concave downwards for $p < x < q$, $s < x < t$ and $u < x < v$

5 a $x > 1.5$ **b** $x < 1.5$

6 Point of inflection at $x = 1$ because the graph of $f(x)$ changes from concave upwards to concave downwards.
Point of inflection at $x = 7$ because the graph of $f(x)$ changes from concave downwards to concave upwards.

7 point of inflection at $(2, 0)$

8 non-stationary point of inflection at $(2, -13)$

9 Points of inflection at $(-2\pi, -2\pi)$, $(-\pi, -\pi)$, $(0, 0)$, (π, π) and $(2\pi, 2\pi)$; concave upwards for $-\pi < t < 0$ and $\pi < t < 2\pi$.
Concave downwards for $-2\pi < t < -\pi$ and $0 < t < \pi$

10 A **11** D **12** B

13 C **14** D **15** $4 < x < 5$

16 $y''(x) = 0$ at $x = 0$. But the concavity does not change on either side of $x = 0$. $y''(-1) = 12 > 0$ and $y''(1) = 12 > 0$.
It is not a point of inflection it is a minimum.

17 $a = 6$

EXERCISE 3.3

1 C

2 Points of inflection at $(-0.71, 1.41)$, $(0, 3)$ and $(0.71, 4.59)$.
Concave downwards for $-2 < x < -0.71$ and $0 < x < 0.71$.
Concave upwards for $-0.71 < x < 0$ and $0.71 < x < 2$.

3 a minimum at $(1.5, 1.25)$

b maximum at $(1, 5)$, minimums at $(3, -27)$ and $(0, 0)$

c minimum at $(0, 1)$

d none

4 a point of horizontal inflection on a rising curve at $(1, 1)$

b maximum at $(0, -2)$, minimum at $(2, 6)$

c point of horizontal inflection on a falling curve at $(4, 5)$

d maximum at $\left(2, \dfrac{4}{e^2}\right)$, minimum at $(0, 0)$

5 point of horizontal inflection on a falling curve at $(0, 6)$, minimum at $\left(1\dfrac{1}{2}, 4\dfrac{5}{16}\right)$

6 points of horizontal inflection on a rising curve at $\left(-\dfrac{3\pi}{2}, -\dfrac{3\pi}{2}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

7 B

8 a $f'(x) = 0, x = 1$

b $(e^{-2}, 4e^{-2})$

c $p = -1$

9 a $a = -4$ **b** minimum

10 $a = 2, b = -6$

11 no solution for $f'(x) = 0$

12 increasing when $t > \dfrac{1}{3\sqrt{e}}$; decreasing when $0 < t < \dfrac{1}{3\sqrt{e}}$

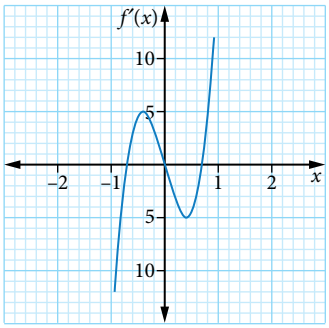
13 C

EXERCISE 3.4

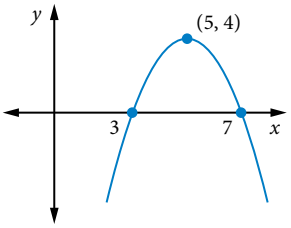
1 D

2 point of horizontal inflection at $(0, 10)$, minimum at $(3, -17)$

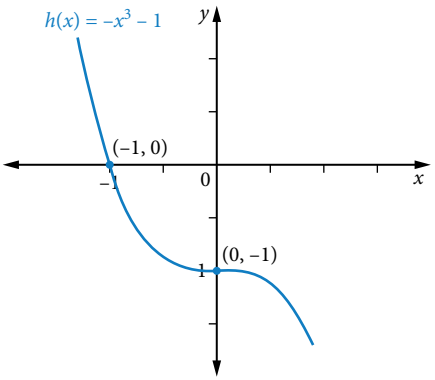
3



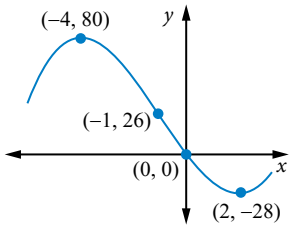
4 a $g(x) = 10x - x^2 - 21$



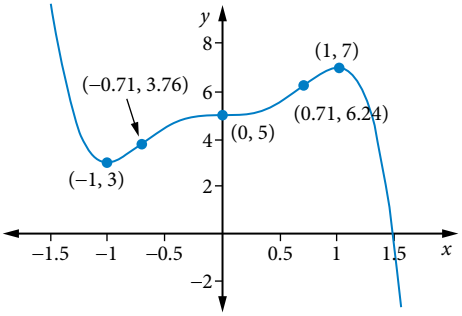
b $h(x) = -x^3 - 1$



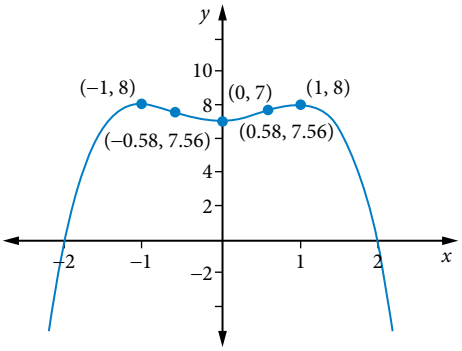
c $f(x) = x^3 + 3x^2 - 24x$



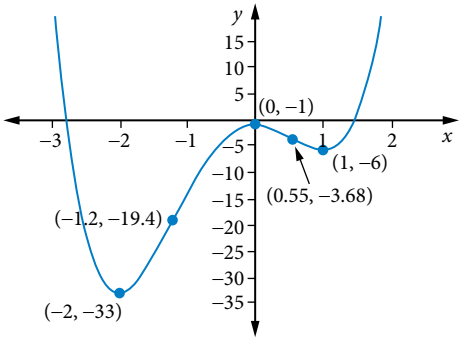
d $f(x) = 5 + 5x^3 - 3x^5$



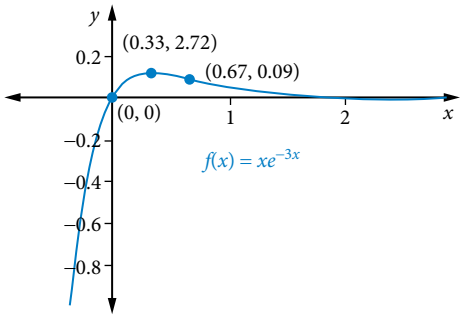
e $f(x) = 7 + 2x^2 - x^4$



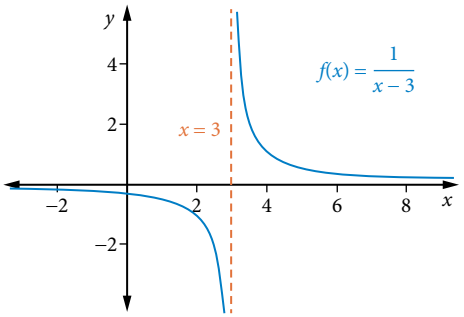
f $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$



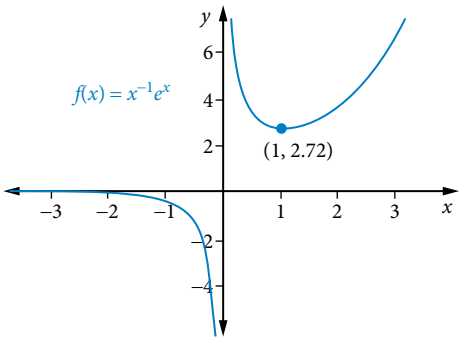
5 a



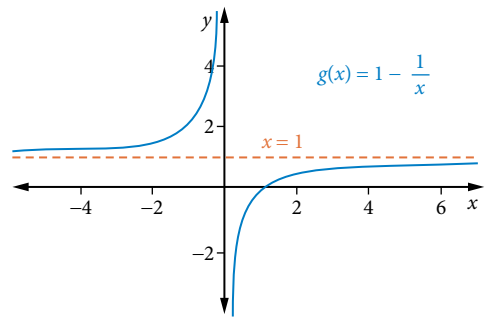
b



c

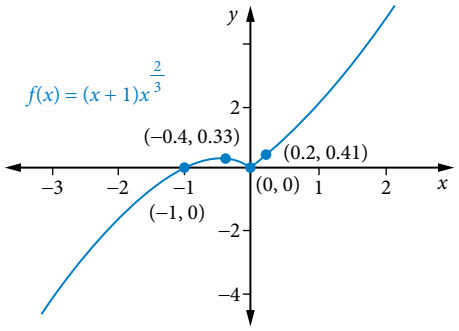


d

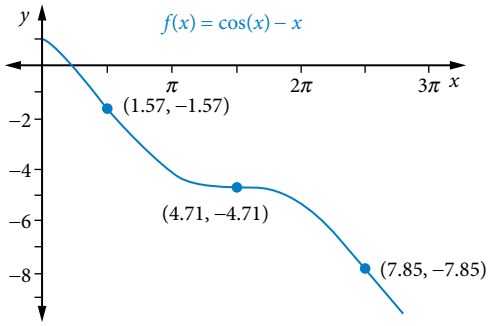


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e



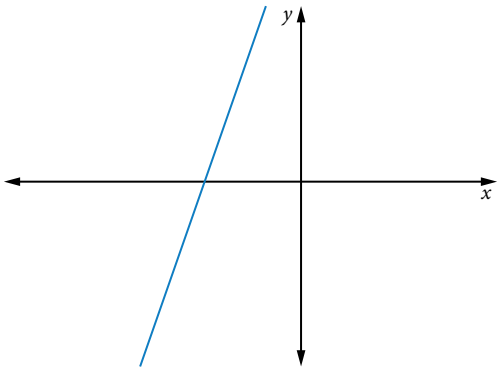
f



6 A **7** A

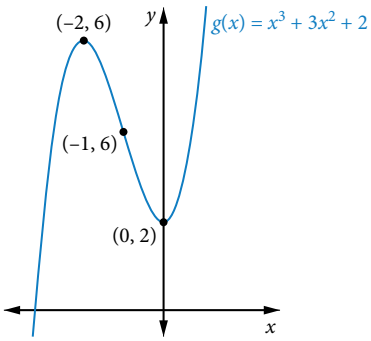
8 C **9** B

10

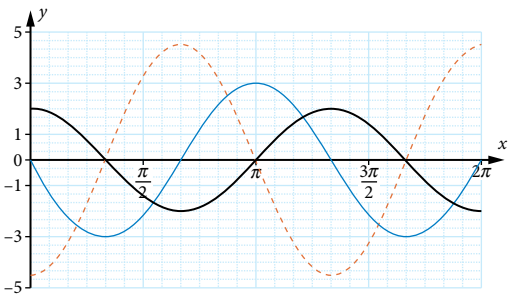


11 Diagram 1

12



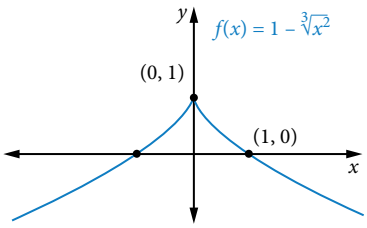
13



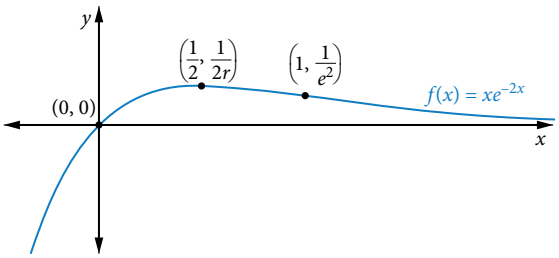
14 global minimum = −54, global maximum = 27

15 **a** As there is no solution for $f'(x) = 0$, there are no stationary points.

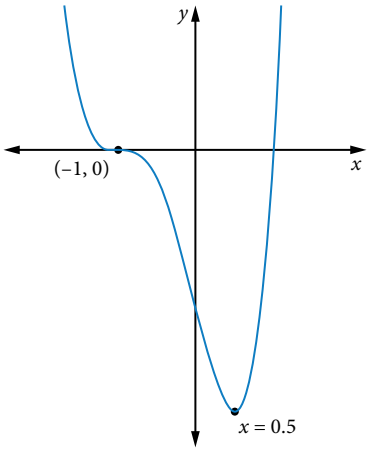
b



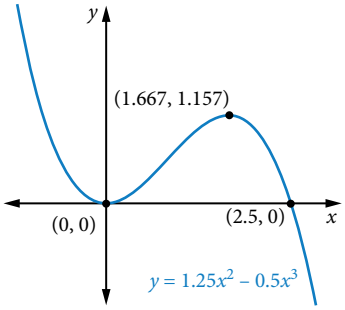
16



17



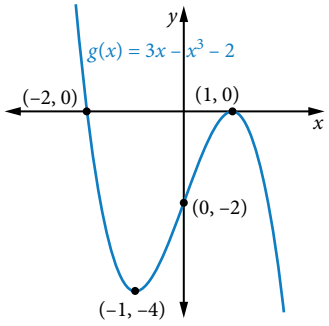
18 $a = 0, b = 0, c = \frac{5}{4}, d = -\frac{1}{2}$



EXERCISE 3.5

1 D

2



3 **a** $a = 6t - 8$ **b** $-8\,\text{m s}^{-2}$

c $10\,\text{m s}^{-2}$ **d** $\frac{14}{3}\text{s}$

4 **a** 40 m

b $v = 17 - 10t$, $a = -10$

c The acceleration is constant and is approximately $10\,\text{m s}^{-2}$ downwards.

d 1.7 s

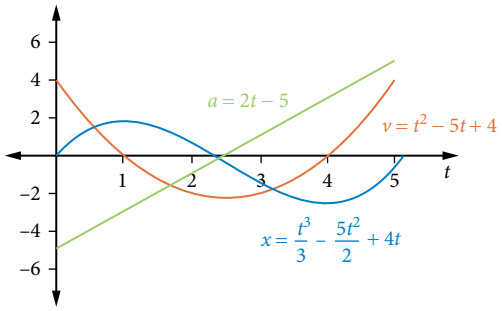
5 $\frac{176}{343}\,\text{m s}^{-2}$

6 about $4.85\,\text{cm s}^{-2}$

7 **a** $v = t^2 - 5t + 4$, $a = 2t - 5$

b $v = -2\,\text{m s}^{-1}$, $a = 1\,\text{m s}^{-2}$

c



d $0 < t < 1$ and $2.5 < t < 4$

e $1 < t < 2.5$ and $4 < t < 5$

8 **a** $h(2) = 6\,\text{m}$, $v(2) = -1\,\text{m s}^{-1}$, $a(2) = -8\,\text{m s}^{-2}$

b The ball is moving downwards at a speed of $1\,\text{m s}^{-1}$. It is moving with an acceleration of $8\,\text{m s}^{-2}$. The ball is increasing in speed.

9 A **10** C

11 **a** $t = 2$ and $t = 6$

b $2 < t < 4$ and $t > 6$

c $0 < t < 2$ and $4 < t < 6$

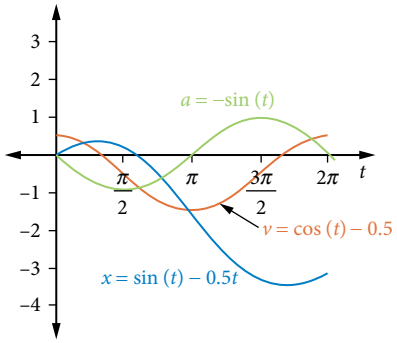
12 **a** $a(t) = 2\,\text{m s}^{-2}$

b 1 s

c $s(1) = 0\,\text{m}$

13 **a** $v \approx -1.15\,\text{m s}^{-1}$, $a = 0.76\,\text{m s}^{-2}$

b



c $0 < t < 1.05$ and $\pi < t < 5.24$

d $1.05 < t < \pi$ and $5.24 < t < 2\pi$

14 $0.597\,\text{m s}^{-2}$

15 0 m and $\approx 2.48\,\text{m}$

16 $p = -0.5$, $q = -7$ and $r = 15$

EXERCISE 3.6

1 D **2** $a(t) = 24t + 32$

3 **a** $P(x) = -50 + 2.7x - 0.001x^2$

b 1350 units

4 **a** $A = x(200 - 2x)$ **b** $5000\,\text{m}^2$

5 **a** $A = x\sqrt{100 - x^2}$ **b** $50\,\text{cm}^2$

6 **a** $h = \frac{200\,000}{1200x}$ **b** $C(x) = \frac{4000}{3x} + 2x$

c 26 plates

7 **a** $t = \frac{d}{x}$ hours

b $P(x) = \frac{250\,000d}{x} + dx^2$

c $50\,\text{km h}^{-1}$

8 Both numbers are 6.

9 $h = 3\sqrt{3} \approx 3.46\,\text{cm}$ **10** A

11 1 hour 20 minutes

12 15 tonnes per month

13 Absorption is at a maximum at 10:50 am. **14** $3.464\,\text{m}^2$

15 **a** Proof. See worked solutions. **b** 28 m

16 radius $\approx 1.89\,\text{m}$, height $\approx 2.67\,\text{m}$, for volume $\approx 9.93\,\text{m}^3$

17 Area = 12, $x = 1$ **18** $P(2, 2)$, $2\sqrt{2}$

19 Area = 12

20 **a** \$12 200

b The cable should be laid directly from the data hub underwater to the village.

21 about 15 m

22 **a** $w \approx 8.08\,\text{cm}$ and $d \approx 11.43\,\text{cm}$

b The rectangular beam of maximum strength is approximately 9.01% stronger than the square beam.

23 **a** $a = 2000$, $b = 8000$

b $7 \leq t < 9$

c $C'(7) = -906.899 = 907\,\text{kWh}$

CUMULATIVE EXAMINATION 1

1 D **2** A **3** C

4 **a** $f'(x) = 6e^{2x+1}$ **b** 0

5 **a** $(2, 4e^2)$

b $f''(x) < 0$, therefore maximum

c $x = \frac{4 \pm \sqrt{8}}{2}$

6 $y = 4x + 2$

CUMULATIVE EXAMINATION 2

1 D **2** B **3** C

4 **a** Proof. See worked solutions.

b $r \approx 4.301\,\text{cm}$

5 $f'(x) = \frac{5(x-2)}{x(x-5)}$

6 The claim is incorrect as the maximum total population occurs on one occasion during the year and does not exceed 25 000 (the maximum is 24 031).

7 about $15.302\,\text{m s}^{-2}$

CHAPTER 4

EXERCISE 4.1

1 **a** continuous **b** discrete

c discrete **d** continuous

e discrete **f** continuous

g discrete **h** continuous

2 a No, the sum of the values is not 1.
b Yes, all values are non-negative and add up to 1.
c No, not all the values are positive.

3 a Yes all values are positive and add up to 1.
b All the values are $\frac{1}{4}$, so it would be easier to write $f(x) = \frac{1}{4}$

4 a 0.12 **b** 0.76

5 a 0.34 **b** 0.3433

6 a about 0.135 **b** about 0.191
c about 0.225

7 D **8** B

9 $f(x) = \frac{11 - x}{66}$, $f(x) = \frac{30 - x}{275}$ or $f(x) = \frac{22 - 2x}{132}$, etc for $0 < x < 10$

10 a 57.4% **b** 0.6%

EXERCISE 4.2

1 a no, $\sum p(x) \neq 1$
b yes, $\sum p(x) = 1$ and $p(x) \geq 0$ for all values
c no, $p(2) < 0$

2 0.12

3 a $P(p) = \frac{1}{8}$, $p \in \{2, 3, 5, 7, 11, 13, 17, 19\}$
b uniform
c $\frac{3}{8}$
d 0

4 a $P(1) = \frac{1}{10}$, $P(2) = \frac{3}{20}$, $P(3) = \frac{1}{5}$, $P(4) = \frac{1}{4}$, $P(5) = \frac{3}{20}$, $P(6) = \frac{1}{10}$, $P(7) = \frac{1}{20}$
b not uniform **c** $\frac{17}{20}$ **d** $\frac{7}{10}$

5 a $p(n) = \frac{2^n}{3^{n+1}}$ **b** $\frac{2^5}{3^6} \approx 0.0439$ **c** $\frac{2^8}{3^9} \approx 0.0130$

6 a $p(m) = \frac{1}{8}$, $m \in Z^+$, $0 \leq m \leq 7$
b uniform **c** 1 **d** $\frac{3}{8}$

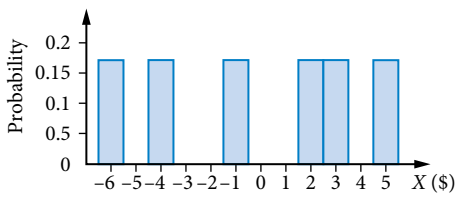
7 a $p(n) = \frac{1}{12}$, $x = 0, 7, 8$; $p(n) = \frac{1}{8}$, $n = \in Z^+$, $1 \leq n \leq 6$
b not uniform **c** $\frac{11}{12}$ **d** $\frac{3}{8}$

8 C **9** B **10** A

11 a

x	−6	−4	−1	2	3	5
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b



12 a

x	−5	5	20
$P(X = x)$	$\frac{5}{9}$	$\frac{5}{12}$	$\frac{1}{36}$

b $\frac{4}{9}$

13 a

P	5	10	15	20	25	50
$P(P = p)$	$\frac{3}{10}$	$\frac{4}{15}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{1}{15}$

b $\frac{2}{3}$

14 13 times

15 B

EXERCISE 4.3

1 $p(x) = \frac{1}{9}$, a uniform discrete random variable

2 a

q	3	4	5	6	7	8	9	10	11	12
$p(q)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{21}$	$\frac{2}{21}$	$\frac{1}{21}$

b $\frac{13}{21}$

3 15 **4** 9.625

5 a 5.5 **b** 4 **c** −1 **d** 3

6 22.9

7 a 57.5 **b** 17.25

8 8.25, about 2.87 **9** 8.25, about 2.87

10 about 26.23, and about 5.12

11 a 37.25, about 6.10 **b** 10, about 3.16
c 8.25, about 2.87

12 D **13** C **14** B

15 2.53 **16** −$3

17 about 2.92, about 1.71

18 about 1.96, about 1.4

EXERCISE 4.4

1 12.8

2 variance about 11.53, SD about 3.40

3 a no, $E(X) < 0$ **b** $15

4 a 40% **b** $12

5 −$9.30 and 93%

6 a female −$390, about 59.1%; male −$250, about 27.8%
b Men are more likely to die, so their insurance costs more.
c Some of the costs are fixed (e.g. administration), so these costs are the same for men and women.

7 About 5.26% for all bets because of the different odds

8 C **9** B **10** 60%

11 about 7.69%

EXERCISE 4.5

1 The expected value is 0, so it is fair, and it makes no difference if they take it in turns.

2 4%

3 **a**, **b**, **d**, **e** and **f** are Bernoulli distributions.

4 **a** **i** success = odd number

ii $p = \frac{1}{2}$

b **i** success = flu injection

ii unknown

d **i** success = head

ii $p = \frac{1}{2}$

e **i** success = ace

ii $p = \frac{1}{13}$

f **i** success = multiple of 3

ii $p = \frac{1}{3}$

5 **a** $\frac{2}{5}$ **b** $\frac{3}{15}$ **c** $\frac{2}{5}$

d $\frac{1}{3}$ **e** $\frac{1}{4}$

6 **a**

x	0	1
$P(X = x)$	0.8	0.2

b $E(X) = 0.2$

c $\text{SD}(X) = 0.4$

7 **a** $\frac{3}{16}$ **b** 0.24 **c** $\frac{6}{25}$

d $\frac{33}{196}$ **e** 0.21

8 **a** 0.4714 **b** 0.4583 **c** 0.4989

d 0.4996 **e** 0.4330

9 D **10** A **11** C

12 **a**

y	0	1
$P(Y = y)$	0.72	0.28

b $E(Y) = 0.28$, $Var(Y) = 0.2016$

13 **a**

y	0	1
$P(Y = y)$	0.21	0.79

b $\text{SD}(X) = 0.4073$

14 **a** 0.6 **b** 0.6

15 **a** 0.2356 **b** 0.62

EXERCISE 4.6

1 **a**, **c** and **d** result in a Bernoulli distribution

2 **a**

y	0	1
$P(Y = x)$	0.6	0.4

b 0.4 **c** 0.4899

3 **a**, **b** and **d** are geometric distributions.

4 about 0.0804

5 **a** 0.128

b 0.081 92

c about 0.041 94

6 0.0625

7 **a** 0.1029 **b** ≈ 0.3724 **c** ≈ 0.8319

8 **a** 0.1024 **b** 0.262

9 B **10** A **11** C

12 **a** 0.28 **b** 0.3249

c 0.3732 **d** 0.1045

13 **a** $\frac{1}{6}$ **b** about 0.096 45

c about 0.4213 **d** about 0.4019

14 **a** about 0.011 72 **b** about 0.2499

15 **a** 0.032 **b** 4.096×10^{-7}

c 0.9984 **d** 0.0016

EXERCISE 4.7

1 **a** 0.0074 **b** 0.9993

2 about 0.5905

3 **a** binomial

b binomial

c not binomial as the number of trials is not fixed

d not binomial as there are more than two outcomes

e binomial

f not binomial as the probability of success changes from trial to trial

g binomial

h binomial

4 **a** 0.3125 **b** 0.0413 **c** 0.1318

d 0.1499 **e** 0.0037 **f** 0.0208

5 **a** $p = 0.5, q = 0.5, n = 12$

b $p = 0.1, q = 0.9, n = 15$

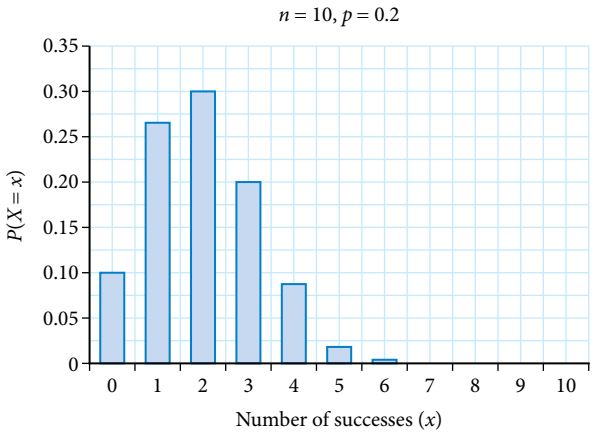
c $p = 0.6, q = 0.4, n = 7$

d $p = 0.23, q = 0.77, n = 25$

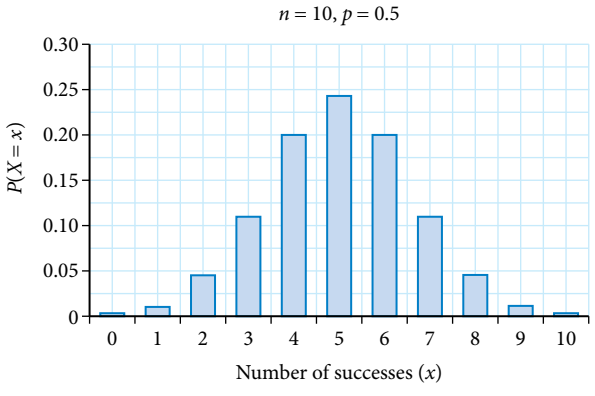
e $p = 0.47, q = 0.53, n = 8$

f $p = 0.67, q = 0.33, n = 20$

6 **a**

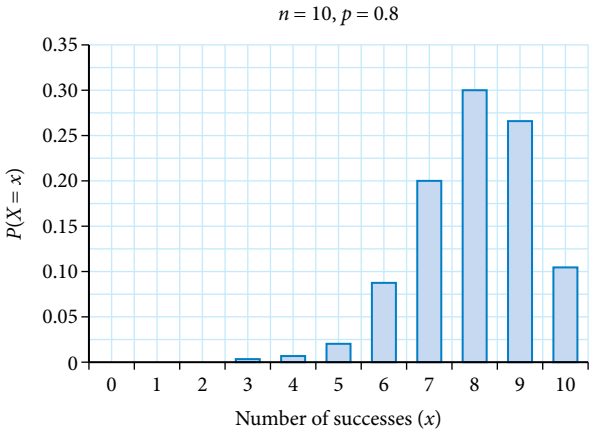


b



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c



7 **a** The distribution is positively skewed.

b The distribution is symmetrical.

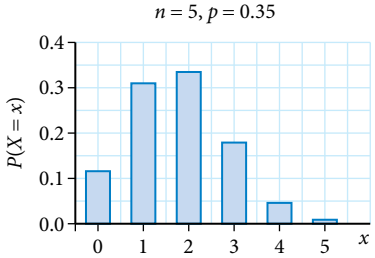
c The distribution is negatively skewed.

8 **a** 5 **b** 0.35

c

x	0	1	2	3	4	5
$P(X = x)$	0.1160	0.3124	0.3364	0.1811	0.0488	0.0053

d The distribution is positively skewed.

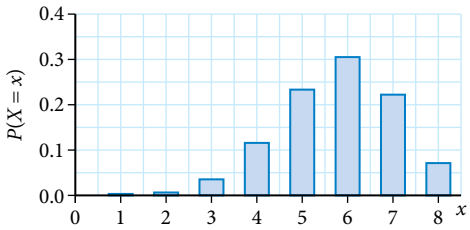


9 **a** 8 **b** 0.72

c

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0	0.0008	0.0070	0.0360	0.1156	0.2379	0.3058	0.2247	0.0722

d The distribution is negatively skewed.



10 **a** 0.0102 **b** 0.3456 **c** 0.0778

11 **a** 0.2241 **b** 0.0024 **c** 0.000 03

12 0.0543

13 **a** 0.1062 **b** 0.2156

c 0.2010 **d** 0.1444

14 **a** 0.1510 **b** 0.0003

c 0.0391 **d** 0.1512

e 0.0553

15 **a** 0.0273 **b** 0.1398

c 0.2509 **d** 0.2291

16 **a** 0.4557 **b** 0.0410

c 0.9959 **d** 0.5443

17 **a** 0.4096 **b** 0.1808

c 0.5904 **d** 0.4096

18 **a** 0.3177 **b** 0.8740

c 0.0288 **d** 0.6471

19 **a** 0.9764 **b** 1.0

c 0.0446 **d** 0.9554

20 B **21** D

22 **a** There are only two outcomes of each trial, The probabilities of each outcome do not change.

b $\frac{13}{125}$

23 **a** 0.9598

b 0.4269

c 0.000 000 000 034

d 0.9989

24 **a** 0.0100 **b** 0.0389

c 0.3893 **d** 0.6983

25 **a** 8 **b** 0.38

c

x	0	1	2	3	4	5	6	7	8
$P(X = x)$	0.0218	0.1071	0.2297	0.2815	0.2157	0.1058	0.0324	0.0057	0.0004

26 0.0735

27 **a** 0.9957 **b** 0.0782

c 0.9994 **d** 0.0043

28 0.6672

29 **a** $\frac{1}{27}$

b **i** $\frac{4}{9}$ **ii** $\frac{19}{27}$

EXERCISE 4.8

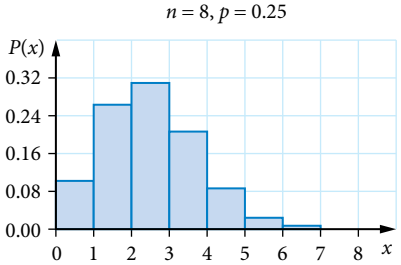
1 **b** and **c** are binomial experiments

2 0.0305

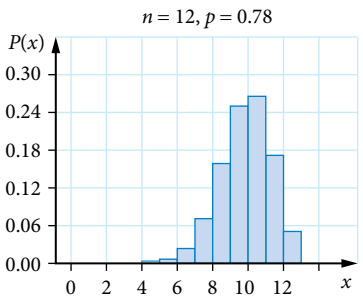
3 **a** 0.9 **b** 0.3 **c** 0.5

d 0.7 **e** 0.1

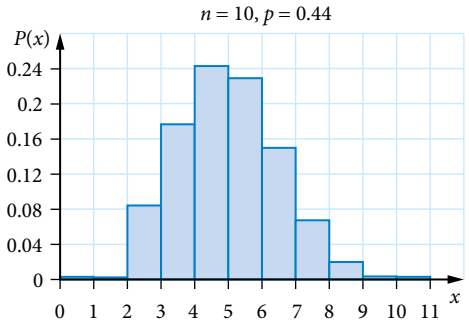
4 The graph is positively skewed.



5 The graph is negatively skewed.



6 The graph is positively skewed.



7 **a** 57 **b** 35.34

8 **a** $\mu = 3.9$ and $\sigma \approx 1.699$

b $\mu = 32.8$ and $\sigma \approx 2.430$

c $\mu = 37.4$ and $\sigma \approx 4.968$

d $\mu = 14.96$ and $\sigma \approx 3.524$

9 **a** 4 or 5 **b** about 71.78%

c about 94.53%

10 **a** 48 or 49 **b** about 95.64%

11 **a** $p = 0.38$ **b** $n = 20$

12 **a** $p = 0.66$ **b** $n = 35$

13 A **14** C **15** D

16 **a** Each trial results in two outcomes only,

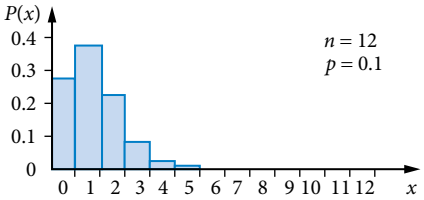
b $\mu = 4.05$, $\sigma \approx 1.92$

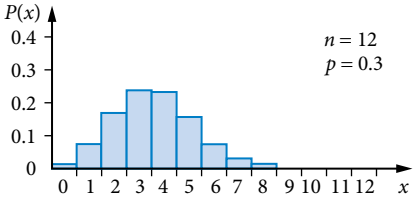
c 0.41

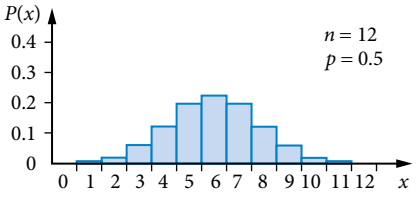
17 0.75

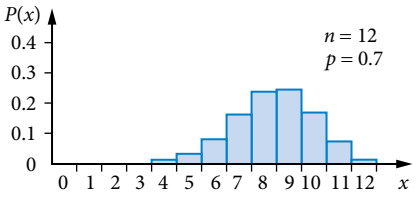
18 2 car parts

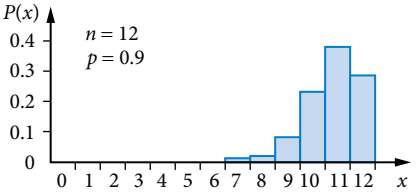
19 **a**











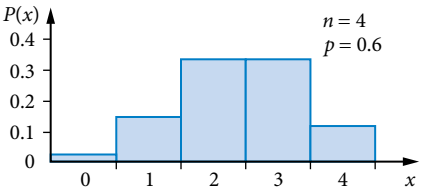
b The smaller the value of p, the more the graph is skewed to the right. For $p = 0.5$ the graph is symmetrical and lowest. The larger the value of p, the more it is skewed to the left.

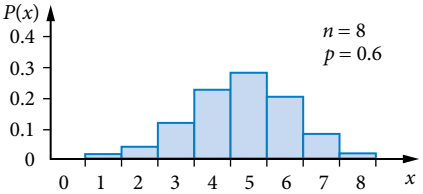
20 **a** 0.45 **b** 12

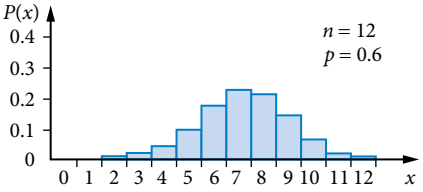
21 16

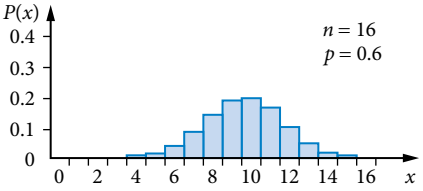
EXERCISE 4.9

1 **a**









b The graphs are skewed to the left. As n increases, they become lower, more symmetrical and smoother.

2 **a** 42 **b** 0.7076

3 **a** 0.1730 **b** 0.8665

c 0.7564 **d** 0.9871

4 **a** 0.0041 **b** 0.8237

c 0.5785 **d** 0.9976

5 **a** 0.7500 **b** 9

c 0.7393 **d** 0.9597

6 **a** about 15 **b** 0.6783

7 **a** 0.5391 **b** 6 times

8 **a** 0.3277 **b** 14 widgets

9 **a** 0.6010 **b** 7

c 2.13 **d** 0.7600

10 about 5 (4.6) **11** 625

12 **a** 1 **b** $\sigma \approx 0.97$

13 **a** 0.0002 **b** 17, 1.579 **c** 0.9781

9780170484985

14 **a** 4 **b** 3.68 **c** 1.918

15 **a** 168 **b** about 10.04

16 **a** about 2 (1.6) **b** 0.96 **c** 0.98

17 **a** 0.2725 **b** 0.1702

c 34 **d** 19

18 4 desks **19** 9 shots

20 B **21** C **22** D

23 **a** 0.0244 **b** 0.5846 **c** 0.6416

24 **a** 2 **b** 0.7414

25 **a** 0.9962 **b** 2 games

26 **a** 0.2275 **b** $p = 0.65$

27 $\frac{48}{625}$

28 $p = 0.6$, $n = 50$, $P(20 \leq X \leq 30) = 0.5521$

29 26.39%

CUMULATIVE EXAMINATION 1

1 A **2** D **3** A

4 **a** $\frac{4}{5}$ **b** $\frac{4}{25}$ **c** $\frac{2}{5}$

5 $5 < x < 6$ **6** $y = -3x + 10$

7 **a** hit $= \frac{1}{4}$, miss $= \frac{3}{4}$

b mean $= 5$, variance $= 3\frac{3}{4}$

CUMULATIVE EXAMINATION 2

1 C **2** C **3** B

4 **a** 1 **b** 0.95

c $n = 20$, $p = 0.05$, $P(\text{at most 1 underweight}) = 0.735$

5 $f'(x) = \frac{5(x-2)}{x(x-5)}$

6 It is concave down.

7 $P(\text{Sum} < 7) = \frac{5}{54}$

CHAPTER 5

EXERCISE 5.1

1 $\frac{1}{4}x^4 - 2x^5 - 2x^3 + 12x + c$

2 **a** $4x^2 - 7x + c$ **b** $4x + c$

c $7u - \frac{u^4}{4} + \frac{u^3}{3} + c$ **d** $\frac{1}{2}z^2 - \frac{1}{4}z^4 + 8z + c$

3 $\frac{4x^6}{3} + x^3 + c$

4 **a** $0.35t^2 - \frac{2t^3}{15} + 0.6t + c$ **b** $\frac{4x^3}{3} + 12x^2 + 44x + c$

c $\frac{4x^{\frac{5}{2}}}{5} + c = \frac{4}{5}\sqrt{x^5} + c$ **d** $g^4 + g^3 - 5g^2 + c$

e $\frac{5}{4}x^4 + \frac{5}{2x^2} + c$ **f** $\frac{3x^2}{4} - \frac{7x^3}{6} + c$

5 **a** $F(x) = \frac{x^3}{3} - 3x^2 + 9x + c$

b $y = \frac{15\sqrt[3]{x^7}}{7} + c$

c $v = \frac{49}{30}t^3 - 24.5t^2 + 122.5t + c$

d $G(x) = \frac{10x^3}{3} + \frac{7x^2}{2} - 12x + c$

6 $f(x) = -\frac{3}{28x^4} + c$

7 $x(t) = 0.2t^3 - t^2 + 4t - 5.2$

8 **a** $\frac{1}{20}(5x + 8)^4 + c$ **b** $-\frac{1}{2(2x - 11)} + c$

c $\frac{-3}{8}\sqrt[3]{(25 - 2x)^4} + c$ **d** $\frac{1}{36}(3x + 6)^4$

9 C **10** D **11** C

12 $2x^3 + 2x^2 + x + c$

13 $\frac{1}{2(1-x)^2} + c$

14 $\frac{x^3}{3} - \frac{1}{x} - 2x + c$

15 $-\frac{1}{25}(32 - 5x)^5$

16 $y = 2(x + 4)\sqrt{x + 4} - 54$

17 -3

18 $f(x) = g(x) + 4x + c$

$\therefore f(0) = g(0) + 4(0) + c$

$\therefore 3 = 2 + c \Rightarrow c = 1$

$f(x) = g(x) + 4x + 1$

19 D

EXERCISE 5.2

1 D

2 $h(x) = \frac{2}{3}x^6 + 3x + \frac{1}{3}$

3 **a** $-3e^{-4x} + c$ **b** $e^{3x+1} + c$

c $\frac{1}{2}e^{2x+1} + e^{-x} + c$ **d** $2e^{5x} + 3x^2 + c$

e $-\frac{5}{2}e^{-2x} + 4e^{\frac{x}{4}} + c$ **f** $\frac{2}{3}x^3 - 2e^{3x} + c$

4 **a** $\frac{1}{2}e^{2x} + 2e^x + x + c$ **b** $2e^{2x} - 12e^x + 9x + c$

c $x - \frac{1}{e^x} + c$ **d** $e^x + 2x - \frac{1}{e^x} + c$

5 $F(x) = 5e^{2x+3} + 2$

6 **a** $4\ln(x) + c$ **b** $\frac{7}{9}\ln(x) + c$

c $\frac{5}{2}\ln(2x + 3) + c$ **d** $-\frac{2}{3}\ln(5 - 3x) + c$

e $\frac{1}{3}\ln(x) + c$ **f** $\frac{2}{3}\ln(9x + 2) + c$

7 **a** $\ln(x + 4) + c$ **b** $\frac{1}{4}\ln(4x + 3) + c$

c $\frac{1}{2}x^2 + 2x - 3\ln(x) + c$ **d** $\frac{3}{4}x^2 - 4x + 6\ln(x) + c$

8 **a** $\ln(2x + 5x^3) + c$ **b** $4\ln(3x^3 + 6x^2 + 1) + c$

9 $y = -3\ln(4 - 3x) + 9$

10 **a** $\frac{dy}{dx} = 4x^3e^{x^4}$ **b** $\frac{1}{4}e^{x^4} + c$

11 $3\ln(x^2 - 21) + c$

12 $y = \frac{1}{4}x^4 - \frac{3}{2}x^2 + 3\ln(x) + \frac{1}{2x^2} + c$

13 C **14** C

15 $\frac{3e^{4x}}{4} + c$ **16** $\frac{1}{2}e^x - 2e^{-x} + 2x + c$

17 $\frac{2}{5}\ln(5x+6) + c$ **18** $\frac{1}{2}\ln(2x^3 + 4x^2 - 5) + c$

19 $y = \frac{1}{3}x^3 - \ln(x) + 2$ **20** $\frac{1}{2}\left(e^{x^2+3}\right) + c$

21 $\frac{1}{2}x^2\ln(3x) - \frac{1}{4}x^2 + c$

22 $\frac{2}{5}e^{x^5-3} + c$ **23** $2e^x - x + c$

24 $y = \frac{1}{4}e^{4x} + \frac{3}{4}e^4$

EXERCISE 5.3

1 $\frac{1}{4}e^{4x} - \frac{2}{3}e^{3x} + \frac{1}{2}e^{2x} + c$

2 $y = \frac{7}{2}\ln\left(\frac{2-x}{4}\right) + 5$

3 **a** $-2\cos(5x) + c$

b $2\sin\left(\frac{x}{3}\right) + c$

c $4\cos(2x+3) + c$

d $-\frac{8}{5}\cos\left(\frac{1}{2}(5x-6)\right) + c$

4 **a** $\frac{1}{8}\cos(7-8x) + c$

b $-2\cos\left(\frac{1}{2}x\right) - 4\sin\left(\frac{1}{2}x\right) + c$

c $x + \frac{1}{2}x^2 + \sin(x-5) + c$

d $\frac{1}{2}e^{4x} + \frac{1}{2}\cos(2x) + c$

e $\frac{2}{5}\sin(5x) - 9\cos\left(\frac{1}{3}x\right) + c$

f $x + \frac{3}{2}x^2 - \frac{1}{2}\cos(2x+3) + c$

5 $\int 8x\cos(x)\,dx = 8x\sin(x) + 8\cos(x) + c$

6 $f(x) = -\frac{1}{4}\cos(4x) + \frac{1}{2}\sin(2x) + \frac{21}{4}$

7 $-\frac{2}{3x} - \frac{2}{3\pi}\sin\left(\frac{3\pi x}{2}\right) + c$

8 $y = \frac{1}{3}\sin(3x) + 4x + c$

9 B **10** A

11 $\frac{1}{5}\sin(5x-6) + c$

12 $-\frac{5}{4}\cos(4x) - 32\sin\left(\frac{x}{4}\right) + c$

13 $\frac{1}{8}\sin(4x+5) + 2e^{-x} + c$

14 $2e^{3x} - 3\sin(3x) + c$

15 $\int x\sin(4x)\,dx = \frac{1}{16}\sin(4x) - \frac{1}{4}x\cos(4x) + c$

16 $a = 30.910\ F(x) = 7.727\sin\left(4x + \frac{\pi}{4}\right) + 12.464$

17 $1.378\ \text{m s}^{-1}$

EXERCISE 5.4

1 D

2 $\frac{1}{3}(\sin(3x) - \cos(3x)) + c$

3 $v(t) = \frac{3t^2}{2} - t$

4 **a** $x = \frac{4}{3}t^3 - 5t^2 - 6t + 27$

b $9\frac{2}{3}$ cm closer to the origin

c $x = 0$ (the origin)

5 **a** $-3\ \text{m s}^{-1}$ ($3\ \text{m s}^{-1}$ to the left)

b $x = 6t + 3e^{-3t} - 7$

6 $v = 8\ \text{m s}^{-1}$, $x = 8\frac{1}{6}$ m to the right of the origin.

7 **a** $v = 3t^2 - 18t + 24$ **b** $x = t^3 - 9t^2 + 24t + 4$

8 **a** $9.6\ \text{m s}^{-1}$ downwards **b** about 1.02 s

c about 25.1 m **d** about 5.9 m

9 **a** $a(t) = 3\cos(t)$

b $x(t) = \frac{t}{\pi} - 3\cos(t) + \frac{31}{6}$

10 **a** $1\frac{1}{3}$ m **b** $a(t) = 2t - 1$

11 **a** $v = \frac{1}{16}\sin(4t) + \frac{1}{2}$

b $x = -\frac{1}{64}\cos(4t) + t + \frac{65}{64}$

12 **a** $v = 1.75t^2 + 0.6t$ **b** $181\ \text{m s}^{-1}$

c $x = \frac{7}{12}t^3 + 0.3t^2$ **d** 613 m

13 **a** $-12.65\ \text{cm s}^{-1}$

b -180.25 cm (180.25 cm to the left of the origin)

14 $\frac{154}{3} - \frac{10\sqrt{5}}{3}$ m in the positive direction from which it started.

15 6 m – You need to find the times at which the particle changes direction, i.e. when $v(t) = 0$.

EXERCISE 5.5

1 $x(t) = \frac{-1}{3}t^3 + t^2 - \frac{4}{3}$

2 $v(2) = 18\ \text{m s}^{-1}$, $x(2)$ = about 42.7 m to the right of the origin

3 $\frac{2}{5}e^{x^5} + c$

4 $y = -4\ln(9-4x) + 4\ln(5) - 5$

5 $\frac{5}{3}\sqrt{x^2 - 8} + c$

6 **a** $h(t) = 30\ln(t+1) + 10$ **b** 72.38 cm

7 **a** $2.6\ \text{m s}^{-1}$ ($2.6\ \text{m s}^{-1}$ upwards)

b about 3.27 s

c about 87.24 m

d 84.6 m above the ground

8 $\frac{(3x^4+1)^3}{36} + c$

9 $-2\cos(\sqrt{x}) + c$

10 **a** $P(t) = \frac{5t^3}{3} + t^2 + 20t + 30$

b $t = 4.55$ months

11 $y = \frac{8}{3}x^3 - 2x^2 + 7x - 8\frac{1}{3}$, $y = -51\frac{2}{3}$ when $x = -2$

12 $d = 9\sin(6.3t)$ cm; since at $t = 0$, $d = 0$, the constant of integration, c, is zero.

13 about 30.63 m

14 The particle is about 2.4 cm from its starting point and is moving to the right, away from the starting point.

15 $v = -62.5L\cos(0.016t)$ and $d = -3906.25L\sin(0.016t)$, since the constants of integration are both 0. The acceleration and displacement of the molecules are in opposite directions, so $d = -3906.25a$.

16 **a** 2250 m **b** at about 42.9 s

17 $t = 2(\sqrt{7} - 2)$

18 11 m (= 5 + 5 + 1)

19 The ratio is $\frac{450\,000\pi}{112\,500\pi} = 4$

20 41.77 minutes

CUMULATIVE EXAMINATION 1

1 A **2** D **3** D

4 **a** π m s^{-2} **b** 0.5 m s^{-1} **c** $\left(0.5 + \frac{2}{\pi}\right)$ m

5 $a = 2$, $b = -2$, $c = 1$

6 $y = -2e^{-\frac{2\pi}{3}}x - e^{-\frac{2\pi}{3}}\left(1 + \frac{2\pi}{3}\right) + e^{\frac{2\pi}{3}}$

CUMULATIVE EXAMINATION 2

1 A **2** B **3** C

4 **a** 5 **b** 2 **c** 10

5 $x = 9.292$ [ln (−1.292) is not defined]

6 about 116 minutes

CHAPTER 6

EXERCISE 6.1

1 **a** 18.375 units^2 **b** 0.494 units^2

c 1.634 units^2 **d** 3.952 units^2

2 **a** 20.66 units^2 **b** 3.3 units^2

3 27.92 units^2

4 **a** 132.08 units^2 **b** 0.512 units^2

c 0.977 units^2 **d** 9.356 units^2

5 A **6** A **7** B **8** C

9 When n is larger and h is smaller.

10 area $= \frac{\sqrt{2}}{\sqrt{e}}$ units^2

11 36.5 units^2

EXERCISE 6.2

1 5.02 units^2 **2** 72 units^2

3 **a** 508.5 **b** $\frac{2}{5}$

c $2(e^6 - e^{-3}) \approx 806.76$ **d** $\frac{1}{\sqrt{2}} \approx 0.71$

e $2\ln(19) \approx 5.89$ **f** $225 - 2\ln(10) \approx 220.39$

4 **a** 5.997×10^{10} **b** 215.417

c $\pi \approx 3.142$ **d** 150.913

5 **a** 48 **b** 2

c $\frac{1}{3}(e^{12} - e^3) \approx 54\,245$ **d** 480

e 0.07 **f** $\ln(6) \approx 1.79$

6 1.666 666 667, 9, 10.666 666 67

$$\int_{-1}^{0}(3 + 2x - x^2)dx + \int_{0}^{3}(3 + 2x - x^2)dx$$
$$= \int_{-1}^{3}(3 + 2x - x^2)dx$$

7 28 **8** −18 **9** D **10** B

11 D **12** D **13** $\frac{1}{6}$ **14** $10\frac{2}{3}$

15 $\int_{-3}^{3}(4x - x^2)dx$ **16** $\frac{1}{5}(e^{40} - e^{15}) \approx 4.71 \times 10^{16}$

17 The function is not defined at the limit $x = 0$ and $x = 0$ is the domain of the definite integral, so the definite integral is undefined.

18 10 **19** $b = 5$ **20** B

EXERCISE 6.3

1 $\frac{5}{6}$ **2** 15 **3** $43\frac{1}{2}$ units^2

4 **a** $7\frac{1}{2}$ units^2 **b** 4.457 units^2

c 74 units^2 **d** 0.275 units^2

5 $73\frac{5}{6}$ units^2

6 **a** $-\int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$

b $\int_{-9}^{-6} g(x)\,dx - \int_{-6}^{-2} g(x)\,dx$

c $-\int_{-5}^{4} h(x)\,dx$

d $\int_{-4}^{-1} k(x)\,dx - \int_{-1}^{3} k(x)\,dx$

e $-\int_{-5}^{-2} m(x)\,dx + \int_{-2}^{2} m(x)\,dx$

f $\int_1^3 p(x)\,dx - \int_3^5 p(x)\,dx + \int_5^6 p(x)\,dx$

7 **a** 14.91 units^2 **b** 94.03 units^2

8 32 units^2 **9** $49\frac{1}{3}$ units^2 **10** C **11** B

12 **a** $10\,\text{m}^2$ **b** $10\frac{2}{3}\,\text{m}^2$

13 $2e^1 + \frac{2}{e^1} - 4$ units^2

14 $A = e^1 - \int_0^1 e^x dx = 1$ units^2

15 **a** $x = 0,\ \pi,\ 2\pi$

b $\int_0^{\pi} e^x \sin(x)\,dx + \left|\int_{\pi}^{2\pi} e^x \sin(x)\,dx\right|$

c 291 units^2

16 8.5 units^2 **17** 4 units^2 **18** C

EXERCISE 6.4

1 39 units^2 **2** 6 units^2

3 **a** 9 units^2 **b** 36 units^2

4 **a** $58\frac{2}{3}$ units^2 **b** $63\frac{1}{3}$ units^2

5 **a** 9 units^2 **b** 8.20417... units^2

c 2.217 units^2 **d** 2.299 units^2

6 $78\frac{1}{12}$ units^2 **7** D **8** A

9 **a** $x = 0, 3$ **b** 4.5 units^2

10 **a** $A(1.44140, 4.31203), B(3.47247, 6.16818)$

b $\int_{1.44140}^{3.47247} (10\cos(x + 10) - (x^2 - 4x + 8))\,dx$

c 7.647 02 units^2

11 18 units^2 **12** $21\frac{1}{12}$ units^2 **13** 15 units^2

EXERCISE 6.5

1 $20\frac{5}{6}$ units^2 **2** $57\frac{1}{6}$ units^2

3 **a** −63.21°C **b** −4.58°C per minute

c $T(t) = 100e^{-0.1t} - 10$

4 **a** \$768.66 **b** \$92 432

c $C(x) = 2x^3 - 57x^2 + 780x + 15$ (in thousands of dollars), with x in thousands

d $P(x) = 40x - 2x^3 + 57x^2 - 15$ (in thousands of dollars), with x in thousands

e about \$7.611 million, 19 345 items

5 **a** $v = 15 - 9.8t$ **b** 1.53 s

c $h = 2 + 15t - 4.9t^2$ **d** 13.48 m

e 12.56 m

6 about \$5600 **7** A

8 **a** $C(x) = 800 + 100x - 2x^2$ **b** \$1600

9 **a** 498.1 litres per hour **b** −477.46 L

c $V(t) = \frac{300}{\pi}\sin\left(\frac{\pi t}{6}\right) + 2000$

10 The marginal cost (\$3.30 per drive) is cheaper than the average cost (\$3.80 per drive). The average cost per drive must reduce as more drives are made, at least up to 800 drives.

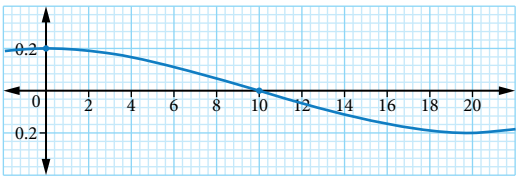
11 $a = 2$ and $b = 5$

12 \$95 751 for 96 dresses

13 **a** $s(t) = 2\ln(6 - 5\cos(2t)) + 6$

b 7.21227 m

14 **a** $a(t) = \frac{\pi^2}{50}\cos\left(\frac{\pi t}{20}\right)$



b For $0 < t < 10$, the acceleration function $a(t)$ is positive. This means the velocity is increasing, so the elevator is speeding up as it ascends.

For $10 < t < 20$, the acceleration function $a(t)$ is negative. This indicates that the velocity is decreasing, so the elevator is slowing down as it approaches the fifth floor.

c $x(t) = -8\cos\left(\frac{t\pi}{20}\right) + 8$, 16 metres

15 22.3477 cm min^{-1}

16 about \$968 for the shade cloth, but this does not include labour, stitching of seams, suspension cable and so on

17 about 39 h

18 $a = 8$, $m = 4$

19 about \$1150

20 about 2 kg (calculation 1.996)

21 approx. 194.09 m, using a 4-interval trapezoidal approximation; a more exact calculation gives approx. 192.61 m. Other answers are possible depending on calculation method.

22 The vertical distance of 5.8278 m is not between 7.5 m and 8.5 m, so the prediction is not reasonable.

23 17.24 km from the point O

CUMULATIVE EXAMINATION 1

1 A **2** C **3** D

4 **a** $x = 0$ and $x = 4$ **b** $\frac{32}{3}$ units^2

5 **a** $V = e^{0.25t} - 1$ **b** 35 litres **c** 1.12 litres

6 **a** $P(\text{hit}) = \frac{1}{4}, P(\text{miss}) = \frac{3}{4}$

b mean = 5, variance = 3.75

CUMULATIVE EXAMINATION 2

1 D **2** A **3** D

4 **a** $\int_0^1 \frac{3}{2}(1 - x^2)\,dx = 1$

b 0.367

c 0.059 tonnes^2

5 **a** 1.94 m **b** 11.5162 m^2 **c** 3.44%

6 7034 flying foxes at 9:58 pm

CHAPTER 7

EXERCISE 7.1

1 **a** $a = 13.7$, $b = 13.4$ **b** $h = 17.2$, $k = 13.9$

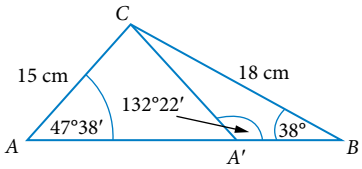
c $\alpha = 31.7°$, $\beta = 96.3°$ **d** $x = 27.1°$, $\theta = 17.4°$

e $x = 38.5$, $y = 51.0$ **f** $\sigma = 17.5°$, $\phi = 138.9°$

9780170484985

2 **a** $X = 77°$, $Y = 38°$, $x = 17.4$ m
b $R = 91.9°$, $r = 84.7$ cm, $p = 49.0$ cm
c $E = 27°$, $D = 42°$, $e = 6.08$ km
d $A = 37.2°$, $a = 178.4$ cm, $b = 244.3$ cm

3 $A = 47.63°$ or $132.37°$



4 $D = 21.13°$ or $158.87°$, $e = 61.27$ cm or 5.30 cm
5 75.82 mm
6 $Y = 42.0°$, $Z = 100.3°$, $y = 2.52$ cm
or $Y = 62.6°$, $Z = 79.7°$, $y = 3.33$ cm
7 **a** 6.5 m **b** 3.19 km
8 C **9** B **10** 31.4 cm **11** 111.4°
12 6.69 km
13 Their height is 2992 m above the hill, so the plane's altitude is really about 3397 m; the altimeter is reading about 1200 m too high.
14 1965 m **15** B

EXERCISE 7.2

1 $B \approx 59.4°$ or $120.6°$
2 $N \approx 22.1°$, $K \approx 122.9°$, $k \approx 42.5$ m
3 **a** $a = 7$ m **b** $B = 22.3°$ **c** $c = 21.4$ m
d $D = 78.0°$ **e** $E = 134.4°$ **f** $f = 43.8$ cm
4 **a** $A = 21.4°$, $G = 37.6°$, $d = 7$ m
b $U = 56.7°$, $V = 68.3°$, $w = 44.1$ cm
c $X = 33.4°$, $Y = 107.3°$, $Z = 39.3°$
d $A = 23.2°$, $B = 112.0°$, $C = 44.8°$
e $M = 81.7°$, $N = 56.0°$, $l = 10.5$ km
f $R = 5.2°$, $F = 6.6°$, $e = 341$ mm
5 $X = 24.15°$ **6** $P = 90°$ **7** 27 m
8 26.4° and 36.3°
9 120°T or S 60° E
10 C **11** B **12** 28.59 cm
13 16 km at 277°T

EXERCISE 7.3

1 28.1 m **2** 56.6°
3 **a** 141.7 m^2 **b** 10.6 cm^2
c 177.45 cm^2 **d** 37.0 mm^2
4 54.38 cm^2
5 **a** 70.99 cm^2 **b** 739.38 cm^2
c 1434.72 m^2 **d** 1443.79 m^2
6 **a** 170 m^2 **b** 473 cm^2
c 139 982 m^2 **d** 348 mm^2
e 1599 cm^2
7 36.1 cm^2, 62.4 cm^2, 29.73 cm^2, 52.6 cm^2
8 B **9** $\frac{9}{2}$ m^2 **10** 58.58°
11 370.2 m^2

EXERCISE 7.4

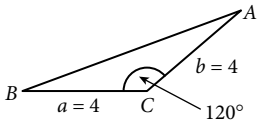
1 74.1 cm^2 **2** 3537 m^2
3 **a** 3.85 m **b** 17 mm **c** 50.25°
d 9.2 cm **e** 57.5 m **f** 10.8 m
4 60.7 m, 76.12°
5 713 m **6** 155 m **7** 4.31 km
8 361 km, 94.63°T
9 54 m **10** 91.1 km **11** 208.17 m
12 16 m

EXERCISE 7.5

1 116 cm **2** 17.1 m
3 **a** 72.6° **b** 54.4°
4 233.27°T or S 53.27° W; 401 m
5 76°T or N 76° E at 12.78°
6 3.16 m, 161.57°T or S 18.43° E
7 77.1 m
8 AYZ 51.34°, AZW 59.04°
9 81.5 m
10 163 m **11** 15.3° **12** 150 m
13 $h = \sqrt{20} > 4$, does meet requirements

CUMULATIVE EXAMINATION 1

1 D **2** C **3** B
4 **a** 0.5, 0.25 **b** Sample B larger variance, more variability.
5 **a**



b $4\sqrt{3}$ cm^2
6 $\sqrt{50}$; $f''(50) > 0$, so area is minimised

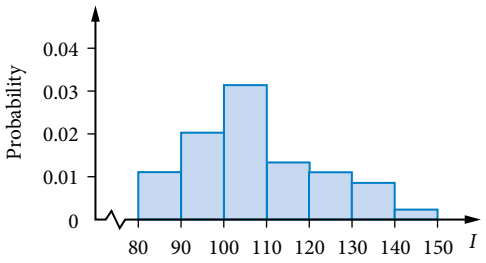
CUMULATIVE EXAMINATION 2

1 A **2** B **3** D
4 263°30′T **5** 150 000 **6** 208.1363 kg

CHAPTER 8

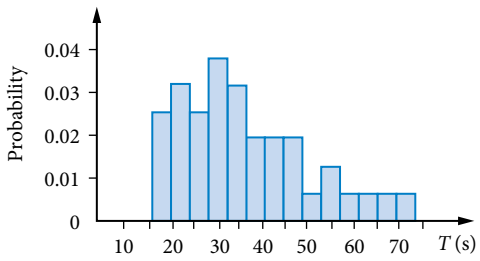
EXERCISE 8.1

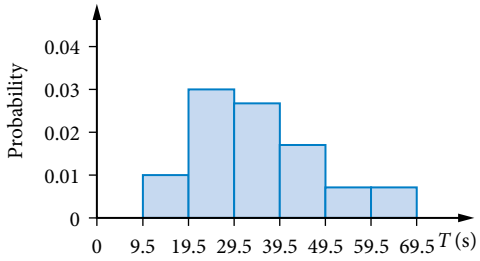
1 **a**



b The total area is 1.

2 a



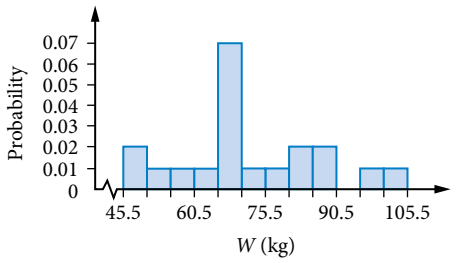


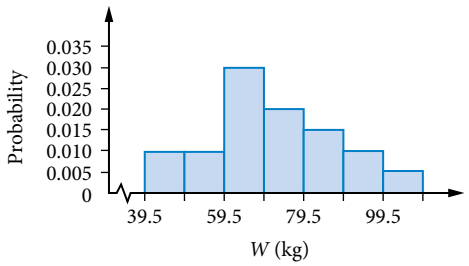
b Total area of each is 1.

3 ≈ 0.1239

4 0.23375

5 a



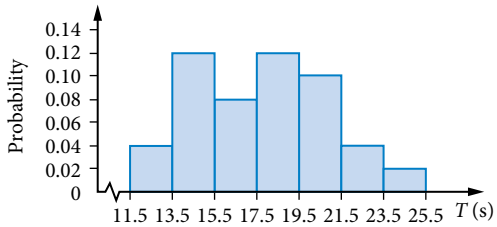


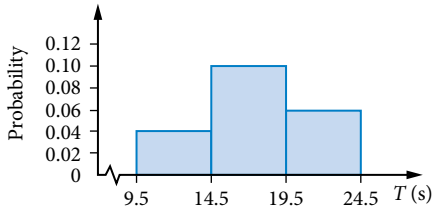
b The total area of each histogram is 1.

c 0.295

6 C **7** A **8** C

9

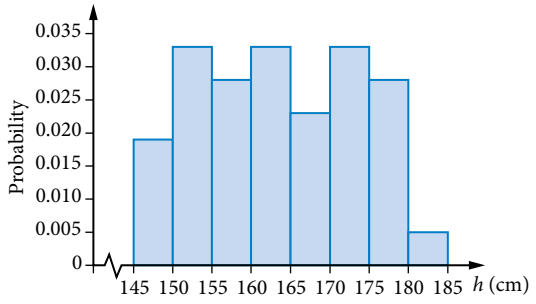




b The area of each histogram is 1.

c 0.204

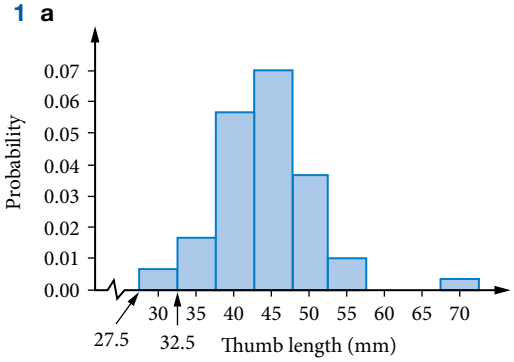
10 a



b 1 **c** ≈ 0.28

EXERCISE 8.2

1 a



b area = 1 **c** 0.57

2 **a**, **c**, **e**, **f**, **g** and **h** are pdfs

3 **a** 1.5 **b** 24 **c** $\frac{1}{18}$

d $\frac{1}{6}$ **e** 4 **f** 0.1

4 **a** $f(x) = x^4 - 1$ for $0 < x \leq 1$

b **i** $\frac{1}{81}$ **ii** 0.1350 **iii** 0.9375

5 $f(x) = 2 - \frac{2}{x}$ for $1 < x < 2$

6 **a** $f(x) = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right)$ for $-1 \leq x \leq 1$

b **i** 0.8438 **ii** 0.6875 **iii** 0.1563

7 **a** $k = \frac{4}{11}$

b $f(x) = \frac{4}{11}\left(\frac{x^4}{4} - x + \frac{3}{4}\right)$ for $1 \leq x \leq 2$

c **i** 0.1875 **ii** 0.5725 **iii** 0.8125

8 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** $\frac{5}{8}$

9 **a** $f(x) \geq 0$ for $x \geq 0$ and $\int_0^\infty f(x)\,dx = 1$

b **i** 0.2835 **ii** 0.4119

10 **a** $f(x) \geq 0$ for $0 \leq x \leq 1$ and $\int_0^1 f(x)\,dx = 1$

b **i** $\frac{17}{81}$ **ii** $\frac{47}{162}$

11 **a** $p(x) = \frac{1}{10}$ **b** $p(x) = \frac{1}{6}$

c $p(x) = \frac{1}{13}$ **d** $p(x) = \frac{1}{90}$

e $p(x) = \frac{1}{400}$

12 **a** $p(x) = \frac{1}{18}$ for $0 \le x \le 18$ **b** $\frac{2}{3}$

13 0.5833

14 **a** $f(x) = \begin{cases} \frac{x}{36} & \text{for } 0 \le x \le 6 \\ \frac{12 - x}{36} & \text{for } 6 \le x \le 12 \end{cases}$

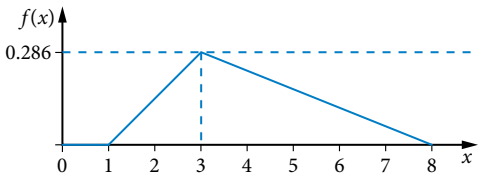
b $f(x) = \begin{cases} \frac{x - 3}{36} & \text{for } 3 \le x \le 9 \\ \frac{15 - x}{36} & \text{for } 9 \le x \le 15 \end{cases}$

c $f(x) = \begin{cases} \frac{x - 8}{26} & \text{for } 8 \le x \le 12 \\ \frac{2(21 - x)}{117} & \text{for } 12 \le x \le 21 \end{cases}$

d $f(x) = \begin{cases} \frac{x - 5}{48} & \text{for } 5 \le x \le 13 \\ \frac{17 - x}{24} & \text{for } 13 \le x \le 17 \end{cases}$

e $f(x) = \begin{cases} \frac{x - 7}{56} & \text{for } 7 \le x \le 15 \\ \frac{21 - x}{42} & \text{for } 15 \le x \le 21 \end{cases}$

15 **a**



b $f(x) = \begin{cases} \frac{x - 1}{7} & \text{for } 1 \le x \le 3 \\ \frac{16 - 2x}{35} & \text{for } 3 \le x \le 8 \end{cases}$

c **i** $\frac{1}{14}$ **ii** $\frac{9}{35}$ **iii** $\frac{3}{7}$

16 **a** $f(x) = \begin{cases} \frac{x - 2}{18} & \text{for } 2 \le x \le 5 \\ \frac{14 - x}{54} & \text{for } 5 \le x \le 14 \end{cases}$

b **i** $\frac{1}{9}$ **ii** $\frac{49}{108}$ **iii** $\frac{101}{108}$

17 D **18** B **19** B

20 **a** $\frac{1}{8}$ **b** 0.9844

21 **a** $f(d) = \frac{1}{7.4} = \frac{5}{37}$ for $0 \le d \le 7.4$ **b** 0.5946

22 $k \approx 1.815$

23 **a** **i** 0.5625 **ii** 0.0347

b $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{x^2}{4} & \text{for } 0 \le x \le 2 \\ 1 & \text{for } x > 2 \end{cases}$

24 **a** $f(x) = \frac{1}{24}$ for $[0, 24]$ **b** about 0.1146

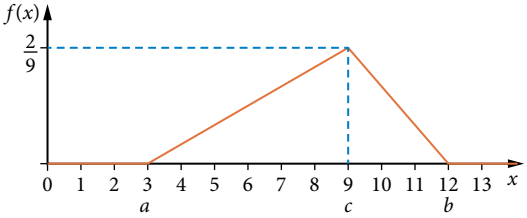
25 $\frac{3}{5}$ **26** $\frac{7}{9}$ **27** $\frac{56}{65}$

28 **a** 0.245 **b** 0.031 25

EXERCISE 8.3

1 **a** and **c** are pdfs.

2 **a**



b $f(x) = \begin{cases} \frac{1}{27}(x - 3) & \text{for } 3 \le x \le 9 \\ \frac{2}{27}(12 - x) & \text{for } 9 \le x \le 12 \end{cases}$

c **i** $\frac{1}{6}$ **ii** $\frac{1}{27}$ **iii** $\frac{37}{54}$

3 **a** $p(x) = \frac{1}{17}$ for $5 \le x \le 22$ **b** $E(X) = 13.5$

4 $E(X) = 28$ **5** $E(X) = 55.5$

6 **a** $f(x) = \begin{cases} \frac{1}{56}(x - 4) & \text{for } 4 \le x \le 12 \\ \frac{1}{42}(18 - x) & \text{for } 12 \le x \le 18 \end{cases}$

b $E(X) = 11\frac{1}{3}$

7 $E(X) = 4$

8 **a** $E(X) = 9$ **b** $E(X) = 36\frac{2}{3}$

c $E(X) = 32\frac{1}{3}$ **d** $E(X) = 38$

9 $E(X) = \frac{1}{4}$ **10** $E(X) = \frac{1}{3}$

11 **a** $E(X) = 4$ **b** $E(X) = \frac{3}{4}$

c $E(X) = 1\frac{1}{5}$

12 **a** $p(x) = \frac{1}{16}$ **b** $E(X) = 16$

c $Var(X) = 21\frac{1}{3}$, $SD(X) = \frac{8\sqrt{3}}{3}$

13 **a** $E(X) = 10$, $Var(X) = \frac{100}{3}$, $SD(X) = \frac{10\sqrt{3}}{3}$

b $E(X) = 7.5$, $Var(X) = \frac{27}{4}$, $SD(X) = \frac{3\sqrt{3}}{2}$

c $E(X) = 25$, $Var(X) = \frac{100}{3}$, $SD(X) = \frac{10\sqrt{3}}{3}$

d $E(X) = 45$, $Var(X) = \frac{625}{3}$, $SD(X) = \frac{25\sqrt{3}}{3}$

e $E(X) = 1.5$, $Var(X) = \frac{1}{3}$, $SD(X) = \frac{\sqrt{3}}{3}$

14 **a** $f(x) = \begin{cases} \frac{1}{64}(x - 6) & \text{for } 6 \le x \le 14 \\ \frac{1}{64}(22 - x) & \text{for } 14 \le x \le 22 \end{cases}$

b $E(X) = 14$ **c** $Var(X) = 10\frac{2}{3}$, $SD(X) = \frac{4\sqrt{6}}{3}$

ANSWERS

15 a $f(x) = \begin{cases} \frac{25}{36}(x - 0.4) & \text{for } 0.4 \leq x \leq 1.6 \\ \frac{25}{36}(2.8 - x) & \text{for } 1.6 \leq x \leq 2.8 \end{cases}$

b $E(X) = 1.6$ **c** $Var(X) = 0.24$

16 a $f(x) = \begin{cases} \frac{1}{63}(x - 8) & \text{for } 8 \leq x \leq 15 \\ \frac{1}{99}(26 - x) & \text{for } 15 \leq x \leq 26 \end{cases}$

b $E(X) = 16\frac{1}{3}$ **c** $Var(X) = 13\frac{13}{18}\ (\approx 13.72)$

17 $E(X) = -2\frac{2}{3}$, $Var(X) = \frac{2}{9}$

18 $E(X) = 2\ln(2)$, $Var(X) \approx 0.0782$

19 $E(X) = \frac{3}{4}$, $Var(X) = \frac{3}{80}$

20 D **21** C **22** A

23 $Var(X) \approx 20.17$ and $SD(X) \approx 4.49$

24 $p(r) = \frac{d}{dr}\left(\frac{\pi r^2}{\pi \times 15^2}\right) \approx 0.00889r$,

$E(R) = \int_0^{15} r \times p(r)\,dr = 10\,\text{cm}$

25 $\sigma = \frac{5\sqrt{6}}{3}$ **26** $E(X) = 15.5$ **27** $E(X) = 12$

EXERCISE 8.4

1 a 15 **b** 6

2 $E(X) = \frac{1}{2}$, $Var(X) = \frac{1}{4}$

3 a Distribution A **b** Distribution B

4 a $0.5319e^{-0.888889(x-3)^2}$

b $0.07979e^{-0.02(x-20)^2}$

c $0.05249e^{-0.008657(x-34.5)^2}$

d $0.01247e^{-0.000488(x-128)^2}$

e $0.9974e^{-3.125(x-1.7)^2}$

5 a $\mu = 12$ and $\sigma = 3$ **b** $\mu = 2$ and $\sigma = \frac{1}{2}$

c $\mu = 2$ and $\sigma = 1$ **d** $\mu = 20$ and $\sigma = 6$

e $\mu = 18.5$ and $\sigma = 4.6$ **f** $\mu = 1.2$ and $\sigma = 0.35$

6 a 0.1056 **b** 0.7039

c 0.4078 **d** 0.1341

7 a 0.0740 **b** 0.8464 **c** 0.0025

d 0.7122 **e** 0.9509

8 a i 97 and 133 **ii** 79 and 151 **iii** 61 and 169

b i 0.6827 **ii** 0.9545 **iii** 0.9973

c i 0.0062 **ii** 0.0062

9 a 0.1423 **b** 0.4745

c 0.3437 **d** 0.7706

10 a 0.815 **b** 0.475 **c** 0.34

11 D **12** A **13** B **14** A

15 a 0.0044 **b** 0.7410

16 a $f(x) = \begin{cases} \frac{10}{x^2}, & 5 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$

b $\frac{4}{7}$ **c** 6.93 minutes

17 a 73 and 97 **b** 61 and 109 **c** 49 and 121

18 a $\mu = 127$ cm and $\sigma = 21$ cm **b** 0.5178

19 D

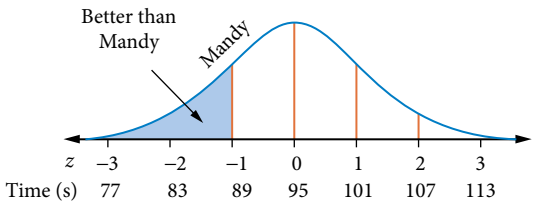
EXERCISE 8.5

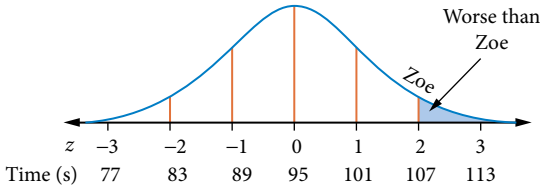
1 a C **b** A

2 a $p(x) = 0.079\,788e^{-0.02(x-20)^2}$

b $\mu = 20$, $\sigma = \frac{3}{4}$

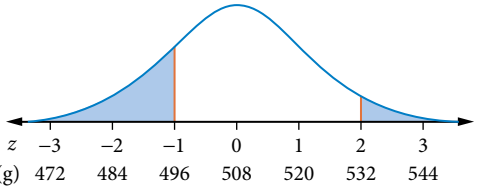
3 a–e Z Mandy $= -1$, Z Zoe $= 2$





f about 0.16 **g** about 97 or 98

4 a, b



c about 185

5 a 2 **b** −1 **c** 1.5

d −3 **e** 3.25 **f** −1.5

6 a 58 **b** 24 **c** 66

d 33 **e** 80 **f** 20

7 a −0.7647 **b** −0.2941 **c** 0.1765

d 0.6176 **e** −1.2647 **f** 1

g −1.0588 **h** 0.1471 **i** −0.1176

j 0.008 824

8 a 2.0542 **b** −0.8625 **c** −0.7792

d 0.425 **e** 0.15 **f** −1.8042

g −1.1333 **h** 1.6958 **i** 2.9292

j −2.8417

9 **a** 0.2030 **b** 0.4898 **c** 0.0222
d 0.9738 **e** 0.8238 **f** 0.1478

10 **a** 0.34 **b** 0.475 **c** 0.68
d 0.815 **e** 0.95

11 **a** 0.3106 **b** 0.4608
c 0.3059 **d** 0.4792

12 **a** 0.1423 **b** 0.0158
c 0.0250 **d** 0.9292

13 **a** 0.0959 **b** 0.2666
c 0.2756 **d** 0.8997

14 B **15** B **16** C **17** B **18** A

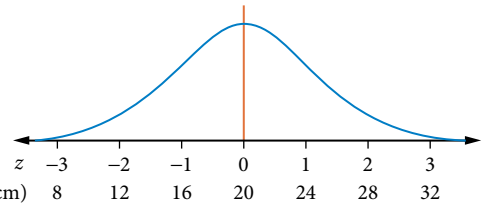
19 **a** 0.841 **b** minimum is 196 cm
c Student in School B ranked higher.

20 $z_E \approx 0.375$ and $z_M \approx 0.4$, so he did better in Maths.

21 $z_{IQ} \approx 0.67$ and $z_H \approx 0.33$, so her IQ is more unusual than her height.

EXERCISE 8.6

1 **a**



b 0.1908
c the second fish (using z-scores)

2 **a** −1.8 **b** 56.5 cm

3 **a** 14.93 **b** 8.25

4 **a** 42.78 **b** 57.71

5 **a** 110.39 **b** 89.88

6 **a** −2.99 **b** −1.41 **c** −2.18
d 1.17 **e** 1.08 **f** 2.47

7 **a** 1.131 **b** 2.473 **c** 0.53
d 0.458 **e** −0.919 **f** 0.174

8 **a** 0.8819 **b** 0.7947 **c** 0.9894
d 0.9452 **e** 0.3159

9 **a** −0.2293 **b** 1.3141 **c** −0.0075
d 0.6259 **e** 1.4985 **f** 0.2671

10 **a** −0.1257 **b** 1.1750 **c** 0.0753
d 0.6745 **e** 0

11 **a** 54.02 **b** 64.99 **c** 62.62
d 49.91 **e** 70.96

12 **a** 106.58 **b** 146.51 **c** 95.55
d 128.30 **e** 131.48

13 $\sigma = 15$ **14** $\mu = 32$ **15** $\mu = 36.5$

16 B **17** C **18** A

19 $\sigma \approx 2.8$ **20** $\mu \approx 126.4$

21 **a** $\sigma \approx 5.4$ **b** $\mu \approx 12.4$

22 $\mu \approx 38.5$, $\sigma \approx 11.3$ **23** $\mu \approx 70.3$, $\sigma \approx 14.7$

24 0.6875 **25** 587 mL

EXERCISE 8.7

1 **a** 26.60 **b** 21.07

2 **a** 57.65 **b** 61.42 **c** 55.66

3 **a** 0.1393 **b** 0.2863 **c** 0.0078

4 **a** ≈ 0.7170 **b** ≈ 0.1764

5 **a** 0.0668 **b** 0.0228 **c** 0.5865

6 **a** 0.0062 **b** 0.9330
c 0.0228 **d** 0.0606

7 **a** 0.1587 **b** 0.5393
c 0.2839 **d** 0.0395

8 90.34

9 **a** 132.86 **b** 0.9412

10 The z-scores for English and Physics are 0.375 and 0.5 respectively. Assuming the classes are of equal ability, Antoine appeared to do better in Physics, as this score is further above the mean score.

11 0.2525

12 **a** about 3 **b** about 31 **c** $18.88

13 First player; $P(\text{Player 1} > 37) \approx 0.0912$, $P(\text{Player 2} > 37) \approx 0.0548$

14 **a** about 5.72% **b** about 5.72% **c** about 0.42%

15 2 push-ups, using z-scores

16 **a** 0.2525 **b** 0.0599 **c** 0.4191

17 D **18** C **19** D

20 **a** 0.2525 **b** 0.4950
c 0.2525 **d** 18.11 kg

21 $k = -1.5$

22 median ≈ -0.299

23 $\mu \approx 19.09$, $\sigma \approx 0.4798$

24 a diameter of 12 cm as $|z_{\text{diameter}}| > |z_{\text{mass}}|$

25 **a** 0.2023 **b** 387.32 g

26 **a** 1.51%
b It would be unusual because of the relatively large z-score 2.17.

27 0.8697

28 Tyres that wear out by 35 500 km will be replaced free of charge.

29 No, only about 1.4% of mufflers would need to be replaced for free.

30 total = \$1133 < \$1500, so suggestion not reasonable

CUMULATIVE EXAMINATION 1

1 D **2** C **3** B

4 8 m^2, $AC = 4.96$ m

5 $a = \frac{3}{5}, b = \frac{6}{5}$ **6** 1.6

CUMULATIVE EXAMINATION 2

1 C **2** B **3** B

4 **a** 0.3085 **b** 14 561 km

5 3.28 hours **6** 373 units **7** 109 grams

CHAPTER 9

EXERCISE 9.1

1 a Population: 35 cinema attendees who bought from the snack and sweets counter.

Parameters: Smallest amount spent at snack and sweet counter, largest amount spent at snack and sweet counter, average amount spent at snack and sweet counter.

Statistics: Smallest amount spent at snack and sweet counter is $6.50, largest amount spent at snack and sweet counter is $37.95, average amount spent at snack and sweet counter is $22.43.

b Population: 123 people with names starting with 'D' on the electoral roll in Gympie.

Parameters: number of people whose names start with D on electoral roll in Gympie who had eaten McDonald's, number of people whose names start with D on electoral roll in Gympie who had eaten Red Rooster, number of people whose names start with D on electoral roll in Gympie who had eaten KFC, number of people whose names start with D on electoral roll in Gympie who had eaten Subway, number of people whose names start with D on electoral roll in Gympie who had eaten Domino's Pizza, number of people whose names start with D on electoral roll in Gympie who ate from other outlets.

Statistics: Number of people whose names start with D on electoral roll in Gympie who had eaten McDonald's is 47. Number of people whose names start with D on electoral roll in Gympie who had eaten Red Rooster is 35. Number of people whose names start with D on electoral roll in Gympie who had eaten KFC is 38. Number of people whose names start with D on electoral roll in Gympie who had eaten Subway is 19. Number of people whose names start with D on electoral roll in Gympie who had eaten Domino's Pizza is 48. Number of people whose names start with D on electoral roll in Gympie who ate from other outlets is 73.

c Population: 148 students at a bus station.

Parameters: Number of students at bus station who use Sharp brand scientific calculators, number of students at bus station who use TI brand scientific calculators, number of students at bus station who use Casio brand scientific calculators, number of students at bus station who use HP brand scientific calculators, number of students at bus station who use other brands of scientific calculators.

Statistics: Number of students at bus station who use Sharp brand scientific calculators is 36. Number of students at bus station who use TI brand scientific calculators is 42. Number of students at bus station who use Casio brand scientific calculators is 39. Number of students at bus station who use HP brand scientific calculators is 14. Number of students at bus station who use other brand scientific calculators is 17.

e Population: 348 vehicles travelling on Bruce Highway in a 30-minute period.

Parameters: Number of cars on the Bruce Highway during a 30-minute period, number of semitrailers on the Bruce Highway during a 30-minute period, number of smaller trucks on the Bruce Highway during a 30-minute period, number of motorbikes on the Bruce Highway during a 30-minute period, number of taxis on the Bruce Highway during a 30-minute period, number of buses on the Bruce Highway during a 30-minute period, number of utes on the Bruce Highway during a 30-minute period, number of vans on the Bruce Highway during a 30-minute period.

Statistics: Number of cars on the Bruce Highway during a 30-minute period is 120, number of semitrailers on the Bruce Highway during a 30-minute period is 40, number of smaller trucks on the Bruce Highway during a 30-minute period is 60, number of motorbikes on the Bruce Highway during a 30-minute period is 18, number of taxis on the Bruce Highway during a 30-minute period is 29, number of buses on the Bruce Highway during a 30-minute period is 32, number of utes on the Bruce Highway during a 30-minute period is 17, number of vans on the Bruce Highway during a 30-minute period is 32.

e Population: E10 petrol pumps at service stations in and around Cairns in a one month period.

Parameters: Lowest E10 petrol price in and around Cairns, Highest E10 petrol price in and around Cairns, Median E10 petrol price in and around Cairns.

Statistics: Lowest E10 petrol price in and around Cairns is 163.9c/L. Highest E10 petrol price in and around Cairns is 187.9c/L. Median E10 petrol price in and around Cairns is 172.3c/L.

2 a Fairness: Fair (no group favoured)

Bias: Response bias

b Fairness: Not fair (only using 1 letter of the alphabet)

Bias: Recall bias, selection bias, completion bias, response bias

c Fairness: Not fair (not random)

Bias: Selection bias

d Fairness: Fair (all vehicles on highway are categorised)

Bias: Interviewer bias

e Fairness: Fair (all service stations checked for Cairns area, but not anywhere else)

Bias: Design bias

3 a Population: Students in the school

b Parameters: Number of students who have a part-time job, number of hours worked by those students with a part-time job.

c Recall bias (students may not be able to recall the number of hours they work), design bias (number of hours may vary for some students per time period, meaning responses may not be accurate representation).

4 a Fairness: Reasonably fair (all customers have the ability to respond if they wish); however, it is unfair that only people who are able to visit the store during its current hours are able to participate in the survey. If the store is only open (e.g.) 9-5, then we would be very interested in knowing about would-be customers who can only get there after 5 pm, but these people can't participate in the survey.
Bias: Response bias from people who are not interested in responding to the survey, or that people who come in on a regular basis may respond multiple times

b Fairness: Fair (all Year 7 students within the school population are asked the question)
Bias: Response bias from students who may not want to disclose low pocket money amount in front of friends, recall bias from students who can't remember how much pocket money they received

c Fairness: Not fair (only viewers can vote)
Bias: Selection bias from people invested in the reality TV show, and possible manipulation by friends of the contestants, design bias from the method of voting

d Fairness: Not fair (attendance at meetings not likely to be tracked, hence people from outside of the local area may attend)
Bias: Selection bias from not considering the views of the people who live outside the area of the potential sites, as well as from people who may have strong opinions but are unable to attend the meetings

e Fairness: Not fair (reporter chooses which reviews to publish)
Bias: Reporting bias from selecting the three restaurants they wish to publish each week, selection bias as the reviewer decides which restaurants to go to each week, response bias in that if restaurants know the reviewer is coming they will ensure that they are treated better than other customers to ensure a good review

5 D **6** B **7** A **8** C

9 Not reasonable to say this is a fair sample. Data is only coming from readers of the magazine, so there is selection bias. There is design bias as people are self-reporting values at a single point in time. There may be response bias as well, given respondents may report their data differently from what it actually is to affect the new tables in a particular way.

10 The result is somewhat fair. Although the systematic random sampling ensures a degree of fairness, this may not be representative of the population in terms of certain strata. It may also be possible that information from some rows will be ignored although they have valid information to give (e.g. air-con is too cold/hot in this row that is felt less in other rows).

EXERCISE 9.2

1 C **2** B

3 a 3 who wear board shorts, 2 who wear briefs, 4 who wear bikinis and 1 who wears a one-piece

b 7 soft-centred, 6 hard-centred, 3 liquid-centred and 4 nutty-centred (hard-centred is rounded up as it had the highest decimal value)

c 1 15-year-old, 11 16-year-olds, 3 17-year-olds and 1 18-year-old (15-year-olds is rounded down as it is the most over-represented based on its decimal value)

d 2 Year 7s, 2 Year 8s, 2 Year 9s, 2 Year 10s, 1 Year 11 and 1 Year 12

4 2 pre-1925 cars, 4 1925–1934 cars, 2 1935–1944 cars, 6 1945–1954 cars and 16 post-1954 cars

5 a 1 male admin staff, 3 female admin staff, 8 male factory workers and 6 female factory workers

b 4 admin staff and 14 factory workers

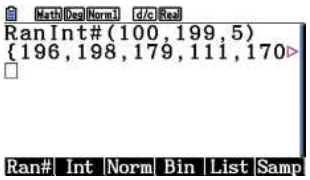
6 17, 30, 43, 6

7 5130, 5205, 5280, 5355, 5430, 5505, 5580, 5655, 4986, 5061

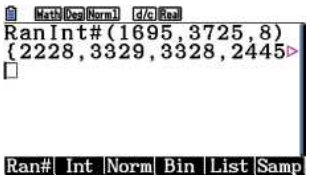
8 301, 355, 369, 403, 437, 471, 505, 539, 573, 607, 641, 2, 36, 70, 104, 138, 172, 206, 240, 274

9 Answers will vary due to random sampling.

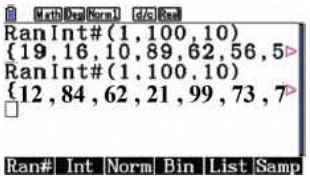
a 196, 198, 179, 111, 170



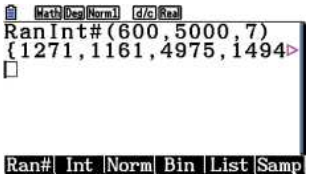
b 2228, 3329, 3328, 2445, 2930, 1928, 3472, 2876



c 19, 16, 10, 89, 62, 56, 57, 9, 77,12



d 1271, 1161, 4975, 1494, 625, 1890, 2038



10 a Sampling: Quota sampling
Advantages: Stratified, quick to complete
Disadvantages: Not random, could get a large group who have similar tastes

b Sampling: Convenience sampling
Advantages: Quick and cheap to complete
Disadvantages: Unable to extrapolate beyond your neighbourhood

c Sampling: Cluster sampling
Advantages: Allows for some randomness due to street selection, cheaper and quicker than stratified sampling
Disadvantages: Biased towards Brisbane only

d Sampling: Self-selection and judgement sampling
Advantages: Easy for interviewees to complete and return
Disadvantages: May get low return rate

e Sampling: Judgement sampling
Advantages: Easy to administer
Disadvantages: May not have proportional representation of the population

11 D **12** B **13** 29

14 Number of males is incorrect, it should have been 3.

15 Answers will vary.
Sample method 1: Systematic random sampling, using row number
Advantages: Unbiased process, assuming we want to study the population of eligible voters
Disadvantages: Can take time
Sample method 2: Pick 1 random person from each page
Advantages: Faster than Method 1 while maintaining some randomness
Disadvantages: May not have proportional representation of the population
Sample method 3: Ask all people on a randomly selected page
Advantages: Quick
Disadvantages: May not have proportional representation of the population

16

State	b Males	b Females	a Participants
NSW	78	79	156
Vic.	64	65	128
Qld	51	52	102
SA	17	18	35
WA	27	27	54
Tas.	5	5	11
NT	2	2	5
ACT	4	4	9
Australia	**248**	**252**	**500**

Values in the male and female cells for a particular state don't necessarily match those in the participants cell, since each value is compared to the overall population.

17 Sampling: Self-selection sampling
Advantages: People who truly have an opinion will express it.
Disadvantages: May be prone to a campaign of voting by an organisation/bots voting for a particular place.

EXERCISE 9.3

1 A **2** C

3 $\frac{79}{150}$ or 0.5267 **4** $\frac{20}{28}$ or 0.7143

5 a $\frac{47}{65}$ or 0.7231

b This is not a reasonable claim. This particular caravan park in Emerald is not representative of the entire state.

6 a $\frac{28}{45}$

b This is not a reasonable claim. This car yard does not take into consideration other car yards in Southport, nor sales made online or privately by people living in Southport.

7 A **8** B

9 $\frac{89}{210}$ or 0.4238

10 a $\frac{9}{40}$ or 0.225

b $\frac{8}{19}$ or 0.4211

c $\frac{1}{21}$ or 0.0476

11 $\frac{5}{47}$. This claim is not reasonable, since creeks outside of Brisbane have not been considered. To improve the survey process, creeks across all of Queensland should be considered, through a random sampling selection process.

12 $\frac{91}{250}$. This value may be representative of what happens in urban schools, where there are many other high schools in nearby suburbs. However, this may be less representative in regional and rural towns where there are fewer High School options. A concern with this estimate is that the students who had been to more than one school had probably changed schools before Year 12. Students are less likely to change during Year 12 than in previous year levels.

EXERCISE 9.4

Note: Answers in this exercise involving random samples will vary.

1 $\frac{21}{24}$ or $\frac{7}{8}$ **2** A

Specific solutions to Q3–Q7 will vary but the conclusions should be the same.

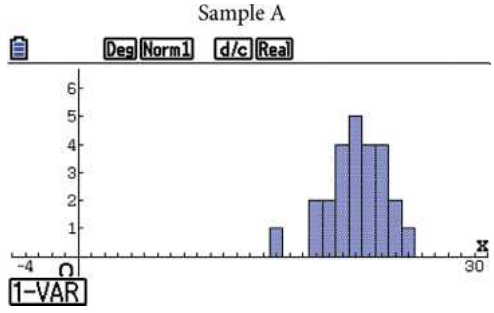
3 a Sample A sample proportions: 0.27, 0.43, 0.27, 0.2, 0.37, 0.33, 0.47, 0.17, 0.4, 0.47

b Sample B sample proportions: 0.36, 0.35, 0.25, 0.39, 0.31, 0.39, 0.29, 0.25, 0.3, 0.35.

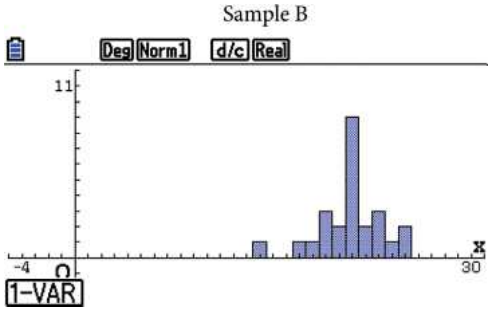
c Sample proportions are all about 0.3; however, sample B has less variation than sample A.

4 a $\mu_A = 21.08$, $\sigma_A = 2.1894$; $\mu_B = 20.88$, $\sigma_B = 2.3886$

Sample A

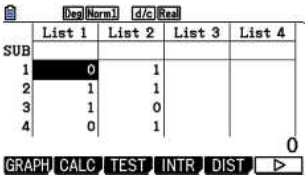


Sample B



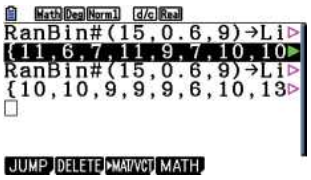
b Although sample A has a larger mean, it also has a smaller standard deviation, leading to less variability. But it can be seen that the means and standard deviations of the two samples are similar.

5 a



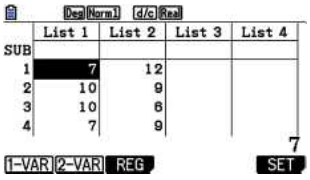
b Sample proportions are 0.725 and 0.6.

6 a



$\mu_A = 9$, $\sigma_A = 1.7638$; $\mu_B = 9.44$, $\sigma_B = 1.7069$

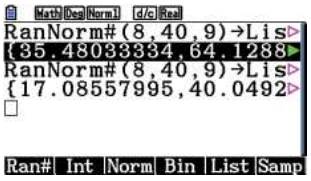
b



$\mu_A = 8.9375$, $\sigma_A = 1.5799$; $\mu_B = 8.5625$, $\sigma_B = 1.7309$

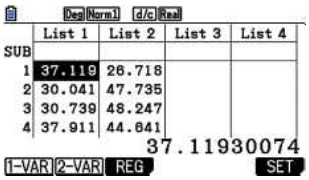
c For this set of data, the means are lower and there is less variation for sample size 64 than for sample size 9.

7 a



$\mu_A = 40.1513$, $\sigma_A = 9.1480$; $\mu_B = 37.9806$, $\sigma_B = 9.0132$

b

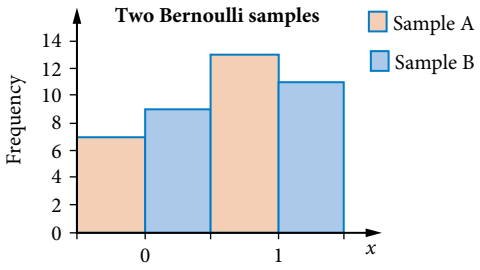


$\mu_A = 40.6406$, $\sigma_A = 7.3928$; $\mu_B = 42.0825$, $\sigma_B = 7.4549$

c For this set of data, the means are lower for sample size 9 than for sample size 81. There is less variation in the samples of size 81 than for sample size 9.

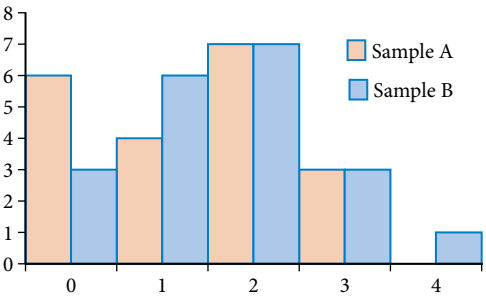
8 C **9** B

10 a



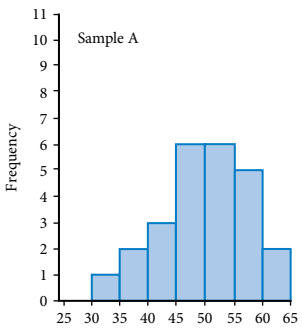
b $\mu_A = 0.65$, $\sigma_A = 0.4770$; $\mu_B = 0.55$, $\sigma_B = 0.4975$
Sample A has a larger mean as well as a small spread than sample B due to the lower standard deviation.

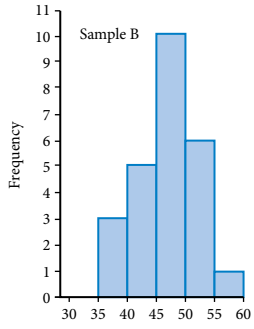
11 a



b $\mu_A = 1.35$, $\sigma_A = 1.0618$; $\mu_B = 1.65$, $\sigma_B = 1.0618$
Sample B has a larger mean than sample A, but they have the same variability.

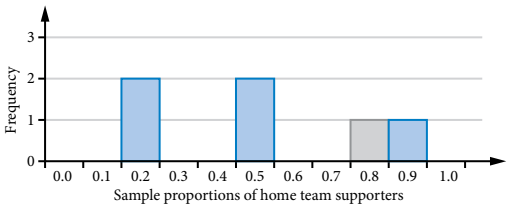
12 a





b $\mu_A = 49.412$, $\sigma_A = 7.6083$; $\mu_B = 46.204$, $\sigma_B = 5.4245$
Sample A has a larger mean than sample B, but sample B has lower variability than sample A because of a lower standard deviation.

13 a



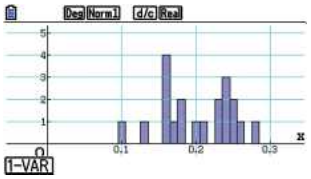
b It will become more symmetrical and there will be less variability, so the graph will look narrower.

EXERCISE 9.5

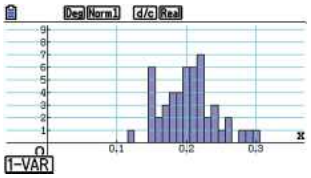
1 A **2** C

Specific solutions to Q3–Q6 will vary but the conclusions should be the same.

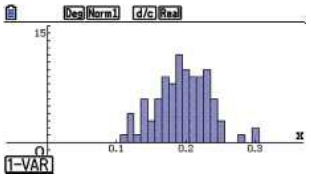
3 a $n = 20$



b $n = 50$

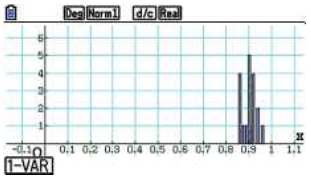


c $n = 100$

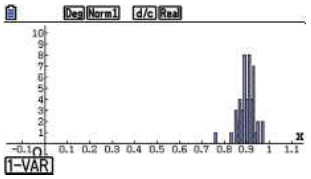


d As the value of n increases, the distribution more closely resembles a normal distribution.

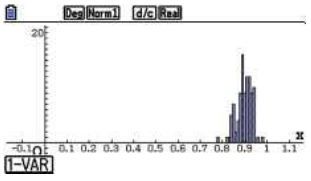
4 a $n = 20$



b $n = 50$

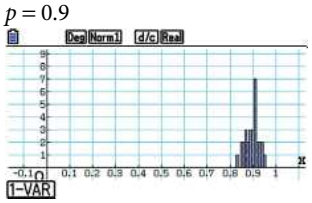


c $n = 100$

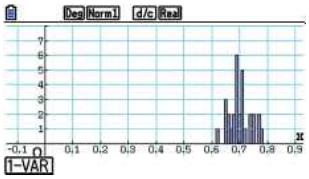


d As the value of n increases, the distribution more closely resembles a normal distribution.

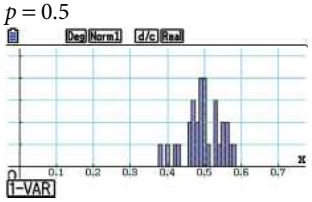
5 a $p = 0.9$



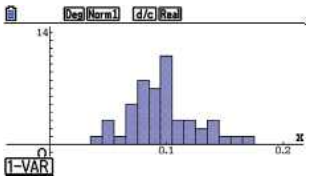
b $p = 0.7$



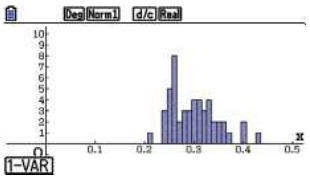
c $p = 0.5$



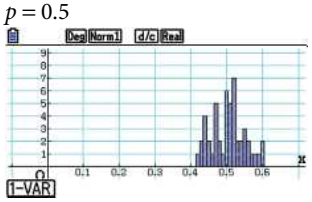
6 a $p = 0.1$



b $p = 0.3$



c $p = 0.5$



7 a $E(\hat{p}) = 0.2$, $Var(\hat{p}) = 0.0032$, $SD(\hat{p}) = 0.0566$

b $E(\hat{p}) = 0.7$, $Var(\hat{p}) = 0.0084$, $SD(\hat{p}) = 0.0917$

c $E(\hat{p}) = 0.81$, $Var(\hat{p}) = 0.0013$, $SD(\hat{p}) = 0.0358$

d $E(\hat{p}) = 0.22$, $Var(\hat{p}) = 0.0021$, $SD(\hat{p}) = 0.0463$

8 $np > 5$: $50 \times \frac{1}{6} \approx 8.33 > 5$

$n(1-p) > 5$: $50 \times \frac{5}{6} \approx 41.67 > 5$

Hence normality can be assumed.

9 $np > 5$: $100 \times \frac{5}{40} \approx 12.5 > 5$

$n(1-p) > 5$: $100 \times \frac{35}{40} \approx 87.5 > 5$

Hence normality can be assumed.

10 B **11** D **12** C

9780170484985

13 $\text{SD}(\hat{p}) = 0.0545$

14 $E(\hat{p}) = \frac{1}{3}$, $\text{SD}(\hat{p}) = 0.0703$

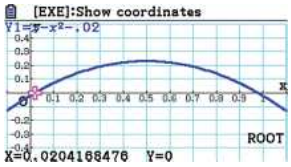
15 $E(\hat{p}) = 0.45$, $\text{SD}(\hat{p}) = 0.0352$

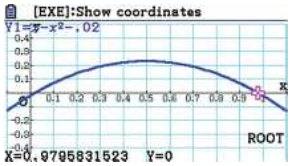
16 $E(\hat{p}) = \frac{1}{6}$, $\text{SD}(\hat{p}) = 0.0680$

17 $E(\hat{p}) = \frac{12}{52}$, $\text{SD}(\hat{p}) = 0.0421$

18 $p \approx \frac{15.02}{25} \approx 0.6$

19 $\hat{p} = 0.0204$ or 0.9796





CUMULATIVE EXAMINATION 1

1 C **2** B **3** C **4** D

5 $\frac{64}{213}$. This estimate is not very accurate, as data has only come from one school in a town in Queensland. Using data from high schools across Queensland will give a more accurate estimate.

6 a $np > 5$: $50 \times \frac{1}{13} = 3\frac{11}{13}$, hence $np \not> 5$
Hence normality cannot be assumed since at least one of the tests fails.

b Minimum sample size is 66.

7 $98x + 14 + \frac{1}{x}$

8 $\int e^x \times \cos(x) = \frac{1}{2}e^x(\sin(x) + \cos(x))$

CUMULATIVE EXAMINATION 2

1 B **2** B **3** A **4** C

5 NSW: 500, Vic: 410, Qld.: 328, SA: 111, WA: 174, Tas. 34, NT: 15, ACT: 28

6 a $\mu_A = 2.4$, $\sigma_A = 1.2806$; $\mu_B = 2.5$, $\sigma_B = 1.2042$

b The samples have similar means, with sample B slightly larger. Sample B has a slightly smaller standard deviation than sample A.

7 13.8564 units^2

8 At least 77 throws are needed.

CHAPTER 10

EXERCISE 10.1

1 0.1076 **2** 0.2202

3 0.0002 **4** 70.92%

5 $E(\hat{p}) = \frac{57}{80}$ or 0.7125, $Var(\hat{p}) = 0.0026$, $\text{SD}(\hat{p}) = 0.0506$

6 $E(\hat{p}) = \frac{18}{40}$ or 0.45, $Var(\hat{p}) = 0.0062$, $\text{SD}(\hat{p}) = 0.0787$

7 $E(\hat{p}) = \frac{16}{50}$ or 0.32, $Var(\hat{p}) = 0.0044$, $\text{SD}(\hat{p}) = 0.0660$

8 (0.5721, 0.7079) **9** (0.5462, 0.8110)

10 (0.4881, 0.7619)

11 a point estimate **b** interval estimate
c interval estimate **d** point estimate

12 0.0222 **13** 42 people **14** 0.0918

15 a 0.4255 **b** 4.6676×10^{-10}, or almost 0

16 0.2335 **17** 0.7791

18 Both proportions (26% and 74%) are less than 80% so the proposal will not be discussed.

EXERCISE 10.2

1 C **2** D

3 a 98.76% **b** 99.73%
c 68.27% **d** 99.31%

4 a 2.3263 **b** 1.4395
c 2.5758 **d** 1.1503

5 a 98.76% **b** 86.64%
c 68.27% **d** 99.73%

6 a 1.6449 **b** 2.8070
c 1.9600 **d** 2.5758

7 a $\frac{11}{40}$ **b** 0.0706 **c** (0.1366, 0.4134)
d The proportion of Year 12 students having a part-time job has a 95% probability of being in the interval (0.1366, 0.4134).

8 a $\frac{29}{40}$ **b** 0.0706 **c** (0.5866, 0.8634)
d The proportion of Year 12 students not having a part-time job has a 95% probability of being in the interval (0.5866, 0.8634).

9 C **10** D **11** A

12 a $\frac{17}{50}$ or 0.34 **b** (0.2087, 0.4713)
c From part **b** we can be 95% confident that the proportion of people using public transport is approximately between 21% and 47%; 50% is outside of this range. This claim is not supported.

13 This claim is unreasonable. A high confidence level is indicative of a larger margin of error, not a smaller one.

14 92.81% **15** (–0.9945, 0.9945)

16 (0.2900, 0.3500)

17 A 90% confidence interval has a 90% probability of containing 0.1 on any sample taken.
Hence $0.9 \times 15 = 13.5$.
The 90% confidence interval for broken biscuits would be expected to include the value 0.1 13–14 times in this scenario.

EXERCISE 10.3

1 D **2** D

3 a 0.1040 **b** 0.1265
c 0.1083 **d** 0.0611

4 a (0.0442, 0.1558) **b** (0.6158, 0.8842)
c (0.1737, 0.4663) **d** (0.5141, 0.7059)

5 a $\frac{35}{50}$ or 0.7 **b** 0.0648
c (0.5730, 0.8270)

6 (0.8499, 0.9359)

7 D **8** B **9** C **10** C

11 (0.4153, 0.5622) **12** (0.0135, 0.1065)

13 0.0237 or 2.37%

14

$\hat{p}$	E
0.1	0.0493
0.3	0.0754
0.5	0.0822
0.7	0.0754
0.9	0.0493

Margin of error is symmetrical about 0.5, e.g. E for $\hat{p} = 0.1$ is the same as E for $\hat{p} = 0.9$, both of which are 0.4 away from 0.5. Margin of error is at its highest at $\hat{p} = 0.5$ and decreases as $\hat{p}$ approaches 0 and 1.

15 68.51%

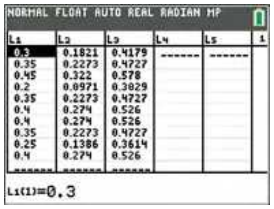
16 B

EXERCISE 10.4

1 D **2** B

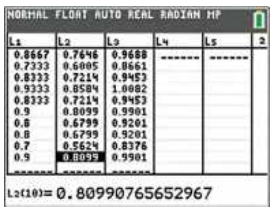
Specific solutions to Q3–5, 8 and 9 will vary but the conclusion is that as the number of sample proportions increases the probability that the confidence intervals contain the population proportion approaches α%.

3 a



b $\frac{9}{10}$ samples contain the population proportion

4 a



b $\frac{7}{10}$ samples contain the population proportion

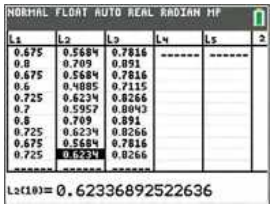
5 a



b $\frac{6}{10}$ samples contain the population proportion

6 B

7 a



b $\frac{8}{10}$ samples contain the population proportion

8 a



b $\frac{7}{10}$ samples contain the population proportion

EXERCISE 10.5

1 B **2** B **3** 1263 **4** 126 240

5 271 **6** 455 **7** 960 365 **8** 151

9 260 **10** 1068 **11** B **12** C

13 C **14** D **15** 3982 **16** 68

17 85.44% **18** 245

19 56.41% **20** about 11.3 minutes

21 a $E = \frac{1}{5}$ **b** 25

CUMULATIVE EXAMINATION 1

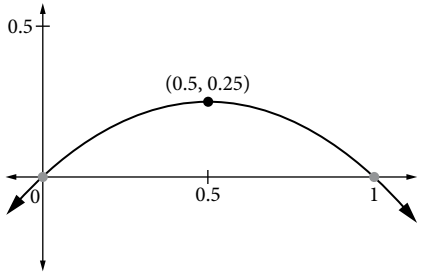
1 A **2** A **3** B **4** D

5 $P(X > 4) = 0.75$

6 $a = \frac{7\sqrt{2}}{2}$, $B = 105°$

7 Minimum turning point at $(-2, -9)$, maximum turning point at $(0, -1)$ and minimum turning point at $(2, -9)$.

8 $n = 400$



The largest sample size will result when $\hat{p}(1 - \hat{p})$ is maximised in the numerator, therefore generating the largest n value.

CUMULATIVE EXAMINATION 2

1 A **2** C **3** D

4 a $P(\hat{p} > 0.7) = 0.11$

b 554

c Halving the margin of error results in a sample size that is four times as large.

d 0.16

5 630 lecturers

6 This interval contains zero; therefore, there is no evidence in the data to say that the two proportions are different, i.e. preference to drink tea or coffee does not depend on where the person lives.

Glossary and index

acceleration (a) The rate of change of velocity: $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}$. (p. 90)

ambiguous case The possibility of two triangles from a case where two sides and the non-enclosed angle are known, usually arising in an application of the sine rule. (p. 246)

angle of depression The angle between the horizontal and the line of sight of a point at a distance below the observer. (p. 243)

angle of elevation The angle between the horizontal and the line of sight of a point at a distance above the observer. (p. 243)

anti-derivative The anti-derivative of the function $f(x)$ is $F(x) + c$, where $\frac{d}{dx}[F(x)] = f(x)$. (p. 166)

anti-differentiation The reverse of the process of differentiation. (p. 166)

average value (of a function) The average value of a function over the interval $[a, b]$ is given by $\frac{1}{b-a}\int_a^b f(x)\,dx$. (p. 225)

bearing A direction on the Earth given by the angle measured clockwise from north. (p. 243)

Bernoulli event An event that has only two possible outcomes: success and failure. (p. 132)

Bernoulli probability distribution The probability distribution for a Bernoulli event. (p. 132)

Bernoulli random variable The variable, X, which is given the values of 1 for success and 0 for failure. (p. 132)

Bernoulli trial An independent trial with a fixed probability of success. (p. 135)

biased sample A sample not typical of the population from which it is drawn. (p. 334)

binomial distribution The probability distribution for the number of successes and failures in n independent Bernoulli trials given by $P(X = x) = \binom{n}{x} p^x q^{n-x}$, where $x = 0, 1, 2, \ldots, n$ and $\binom{n}{x} = \frac{n!}{x!(n-x)!}$ (p. 138)

binomial experiment The repetition of fixed number (n) Bernoulli trials. (p. 138)

binomial random variable The number of successes for n trials of a binomial experiment. (p. 138)

census A survey of the entire population. (p. 333)

centred rectangles In approximation of the area under a curve, rectangles with their top centre touching the curve of the function. (p. 199) *See also* **left rectangles, right rectangles**.

chain rule If $y = f(u)$ and $u = g(x)$, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, the formula for determining the derivative of the composition of two functions $f(g(x))$. (p. 52)

closed interval Part of the number line including the endpoints; $a \le x \le b$ or $[a, b]$. (p. 280)

cluster sample A random sample in which all members of randomly chosen groups (clusters) in a population are selected. (p. 337)

completion bias Bias resulting from incomplete surveys. (p. 334)

composite function A function of a function, such as $f[g(x)]$; the value of the inner function $g(x)$, is the input of the outer function $f(x)$. (p. 52)

concave down (downwards) A curve with its centre of curvature below the curve; the function $f(x)$ is concave downwards where $f''(x) < 0$. (p. 72)

concave up (upwards) A curve with its centre of curvature above the curve; the function $f(x)$ is concave upwards where $f''(x) > 0$. (p. 72)

concavity The way a graph or curve bends. (p. 71)

confidence interval An interval symmetric about the mean containing a specific proportion of the values of a random variable. (p. 365)

confidence level The proportion of values in a confidence interval. (p. 365)

constant of integration A constant that is added to the indefinite integral of a given function. (p. 166)

continuous random variable A random variable where the data can take any value from a continuum. (p. 275)

convenience sample A non-random sample using conveniently available members of a population. (p. 339)

cosine In a right-angled triangle, $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$; in the unit circle $\cos(\theta) = x$. (p. 243)

cosine rule The rule $a^2 = b^2 + c^2 - 2bc\cos(A)$ for a triangle ABC. (p. 251)

cumulative distribution function (cdf) The integral of the probability distribution function of a random variable X from $-\infty$ to the given value; $\text{cdf}(x) = \int_{-\infty}^{x} f(x)\,dx$. (p. 282)

decile A tenth of an ordered set of values. The 3rd decile has $\frac{3}{10}$ of the values below it. *See also* **fractile**, **quartile**, **quantile**, **percentile**. (p. 312)

decreasing A function is decreasing where it has a negative gradient. $\frac{dy}{dx} < 0$. (p. 71) *See also* **increasing, stationary**.

definite integral The area between the graph of a function and the x-axis. (p. 207)

derivative The gradient of the tangent to a function $f(x)$ given by $f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. (p. 3)

design bias Bias resulting from poor design of a survey. (p. 334)

difference function The difference between two functions. (p. 219)

difference quotient The ratio $\frac{f(x+h)-f(x)}{h}$, which is the slope of the secant through the points $(x, f(x))$ and $f(x+h, f(x+h))$. As $h \to 0$, this quotient approaches the gradient of the tangent of $f(x)$ at $(x+h, f(x+h))$, or the derivative. (p. 41)

differentiate Find the derivative of a function. (p. 21)

discrete probability function (distribution) The probability function $P(X = x)$ or $p(x)$ of a discrete random variable X. (p. 116)

discrete random variable A random variable having only discrete values. (p. 116)

displacement (s) The change in position of a particle. (p. 90)

Euler's number e An irrational number $e = 2.718\,281\,828\,459\ldots$; the base of the natural logarithms; given by $1 + \frac{1}{1} + \frac{1}{1\times 2} + \frac{1}{1\times 2\times 3} + \frac{1}{1\times 2\times 3\times 4} + \ldots$ or $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n$. (p. 3)

expected value [$E(X)$ or μ] The mean value of a random variable given by $E(X) = \mu = \sum xp(x)$ for the discrete random variable X, or $E(X) = \mu = \int_{-\infty}^{\infty} xp(x)\,dx$ for the continuous random variable X defined on the interval $[a, b]$. In many cases $[a, b] = [-\infty, \infty]$. (pp. 124, 291)

exponential decay A function whose rate of decrease is proportional to its current value; $f(x) = Ae^{kx}$, where $k < 0$. (p. 24)

exponential function, the The exponential function of base e, $y = e^x$. (p. 3)

exponential growth A function whose rate of increase is proportional to its current value; $f(x) = Ae^{kx}$, where $k > 0$. (p. 24)

fair game A (gambling) game with an expected value of 0, including any entry fee as negative. (p. 129)

fair sample A sample that is typical of the population. (p. 334)

first derivative The derivative of a function; $f'(x) = \frac{dy}{dx}$. (p. 68)

fractile *See* **quantile**.

fundamental theorem of calculus The theorem that the definite integral of a function between two values is the difference of the values of an indefinite integral at those values: $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F'(x) = f(x)$. (p. 208)

geometric probability distribution The distribution of x failures before the first success in a series of Bernoulli trials given by $P(X = x) = q^x p$. Also defined as the number of trials needed *until* the first success is achieved, given by $P(X = k) = q^{k-1}p$ for $k = 1, 2, 3, \ldots$ (p. 136)

global maximum The greatest value of a function over its entire domain. (p. 76) *See also* **local maximum**.

global minimum The least value of a function over its entire domain. (p. 76) *See also* **local minimum**.

horizontal point of inflection A stationary point that is also a point of inflection. A point of horizontal inflection *on a rising curve* changes from concave downwards to concave upwards while a point of horizontal inflection *on a falling curve* changes from concave upwards to concave downwards. (p. 76)

house percentage The expected amount a casino or other gambling institution receives in the long run, expressed as a percentage of the total amount bet. (p. 130)

increasing A function is increasing where it has a positive gradient. $\frac{dy}{dx} > 0$. (p. 71) *See also* **decreasing**, **stationary**

indefinite integral A function $F(x)$ whose derivative is a given function $f(x)$; $F'(x) = f(x)$. Also called the anti-derivative or primitive of the given function. The indefinite integral of $f(x)$ is written as $F(x) = \int f(x)dx$. (pp. 166, 208)

integral *See* **definite integral, indefinite integral**

integrand The expression being integrated. (p. 166)

integration The process of finding the anti-derivative (integral) of a function. (p. 166)

integration by recognition The process of finding the integral of a function by recalling a derivative. (p. 175)

interval estimate An interval that estimates the value of a variable. (p. 363)

interviewer bias Bias arising from conscious or unconscious attitudes of the interviewer affecting the responses to questions. (p. 334)

inverse cumulative distribution function The value that corresponds to a given area under the normal or standard normal distribution curve. (p. 310)

inverse function For a function $y = f(x)$, the inverse function $f^{-1}(x)$, if it exists, reverses the effect of $f(x)$; $f^{-1}[f(x)] = x$ for all x in the domain of $f(x)$. (p. 11)

irrational number A real number that is not rational; a number that cannot be expressed as the ratio of two integers. (p. 3)

judgment sample or purposive sample A non-random sample chosen by the surveyor based on their judgement. (p. 339)

left rectangles In approximation of the area under a curve, rectangles with their top left corner touching the curve of the function. (p. 199) *See also* **centred rectangles**, **right rectangles**

limits (of a definite integral) The start and end values of the between which a definite integral is to be evaluated. (p. 208)

limit of a function The value that a function 'approaches' as the independent variable (input) approaches some value. (p. 208)

ln Logarithm to the base e; the natural logarithm; $b = \ln(a)$ is equivalent to $a = e^b$. (p. 10)

local maximum The greatest value of a function over an interval, particularly a short interval including the point with this value. (p. 71) *See also* **global maximum**.

local minimum The smallest value of a function over an interval, particularly a short interval, including the point with this value. (p. 71) *See also* **global minimum**.

log graph The graph of a logarithmic function. (p. 27)

logarithmic equation An equation that contains a logarithmic expression and constants. (p. 11)

logarithmic scale A nonlinear scale used when there is a large range in the values of quantities. It is based on orders of magnitude, rather than a standard linear scale. (p. 28)

log–log graph The graph of ordered pairs produced by the logarithms of two data variables. (p. 27)

margin of error The difference between value of the mean and the extremes of a confidence interval. (p. 365)

marginal change The change in a function (or quantity) from a unit change of the independent variable; approximately equal to the derivative. (p. 225)

marginal rate of change The instantaneous marginal change of a function (or quantity). It is the derivative of the function. (p. 225)

median The middle score of a distribution – the second quartile. (p. 291)

natural logarithm A logarithm to the base e; $\ln(x) = \log_e(x)$; $b = \ln(a)$ is equivalent to $a = e^b$. (p. 10)

negatively skewed A distribution with more than 50% of the values below the mean; having a 'tail' on the left. (p. 148)

non-random sample Sample that is not random; not every member of the population has an equal chance of being chosen. (p. 342)

normal distribution The distribution with probability density function $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, where the mean = μ and the standard deviation = σ. (p. 299)

odds For a bet or game, the ratio of the amount won to the amount bet (or entry fee). (p. 129)

open interval Part of the number line, not including the endpoints; $a < x < b$ or (a, b). (p. 280)

optimisation The process of maximising or minimising a quantity. (p. 95)

parameter The value of a variable for the population. (p. 333)

percentile A value that divides a set of values into hundredths. The 23rd percentile has 23% of the values below it. (p. 312) *See also* **fractile**, **quantile**.

point estimate A single estimate of the value of a variable. (pp. 117, 363)

point of inflection The point where a curve changes concavity, from concave upwards to concave downwards or the other way around. If P is a point of inflection, then $f''(x) = 0$ at P. (p. 73) *See also* **horizontal point of inflection.**

population The complete group from which information is desired. (p. 333)

positively skewed A distribution with more than 50% of the values above the mean; having a 'tail' on the right. (p. 148)

primitive *See* **indefinite integral**.

probability density function (pdf) A function of a continuous random variable X whose area over an interval gives the probability of X over that interval. So $P(a \le X \le b) = \int_a^b p(x)\,dx$. (p. 279)

probability distribution *See* **probability function**.

probability function The function $p(x)$ of a discrete random variable X such that $P(X = x) = p(x)$. (p. 116)

product rule The formula $\frac{d(uv)}{dx} = \frac{du}{dx} \times \frac{dv}{dx}$ or $f'(x) = g'(x)h(x) + g(x)h'(x)$ for the derivative of a product function $f(x) = g(x)h(x)$. (p. 55)

purposive sample *See* **judgement sample**.

quantile A value that divides a probability distribution into groups. For a continuous random variable X, a quantile t_α is the value such that $P(X < t_\alpha) = \alpha$, where $0 < \alpha < 1$. (p. 312)

quartile A value that divides a set of values into quarters. The 3rd quartile has $\frac{3}{4}$ of the values below it. (p. 312) *See also* **decile**, **quantile**, **percentile**. (p. 312)

quota sample A non-random sample of particular numbers of groups within a population, similar to stratified sampling, but with individuals chosen by convenience. (p. 339)

quotient rule The formula $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{du}{dx}}{v^2}$ or $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{\left[h(x)\right]^2}$ for the derivative of a quotient function $f(x) = \frac{g(x)}{h(x)}$. (p. 58)

radian An angle measure; the angle at the centre of a circle subtended by an arc equal in length to the radius. (p. 41)

random number A value chosen from a set of numbers so that every number has an equal chance of being chosen. (p. 338)

random sample A sample in which every member of the population has an equal chance of being selected. (p. 334)

random variable A variable whose value is determined by chance. (p. 116) *See also* **continuous random variable**, **discrete random variable**.

recall bias Bias arising from faulty memory of the subject, particularly for past events. (p. 334)

reporting bias Bias arising from the way that results of a survey are reported or published. (p. 334)

response bias Bias arising from the desire of people to conform to social acceptability or the perceived position of an interviewer. (p. 334)

right rectangles In approximation of the area under a curve, rectangles with their top right corner touching the curve of the function. (p. 199) *See also* **centred rectangles**, **left rectangles**.

sample Part of the population. (p. 333)

sample proportion The ratio of the number of times x, a characteristic, occurs in a sample to the sample size, n, given by $\hat{p} = \frac{x}{n}$. (p. 342)

sampling distribution The distribution of values of means obtained from samples of the same size. (p. 354)

second derivative The derivative of the derivative of a function; $f''(x) = \frac{d^2y}{dx^2} = \frac{d\left[f'(x)\right]}{dx}$. (p. 68)

second derivative test Test used to determine if a stationary point on a curve is a local maximum or local minimum. (p. 77)

selection bias Bias arising from a non-random method of sample selection. (p. 334)

self-selection sample A non-random sample of volunteers. (p. 339)

signed area A positive or negative area between a function and the x-axis; positive above the axis and negative below; the definite integral of the function. (p. 212)

simple random sampling A random sample from the entire population without the use of subgroups. (p. 337)

sine In a right-angled triangle, $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$; in the unit circle $\sin(\theta) = y$. (p. 243)

sine rule The rule $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{c}{\sin(C)}$ for a triangle ABC. (p. 244)

sinusoidal Based on a sine or cosine function, such as the height of the tide. (p. 48)

skewness The measure of the symmetry of a probability distribution. (p. 148)

solve a triangle Find all the angles and sides of a triangle. (p. 245)

standard deviation (σ) A measure of the dispersion in a distribution; the square root of the variance; $\text{SD}(X) = \sigma = \sqrt{Var(X)}$. (p. 125)

standard normal distribution The distribution with probability density function $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. (p. 304)

standard score *See* ***z*-score**.

stationary (point) A function has a **stationary point** when the gradient is zero. $\dfrac{dy}{dx} = 0$. (p. 71)

See also **decreasing**, **increasing**.

statistic The value of a variable for a sample. (p. 333)

stratified random sample A sample that uses groups within the population. Random samples are taken from all groups in proportion to their sizes. (p. 337)

survey Collection of information from a sample. (p. 333)

systematic sample A random sample in which every nth member of a numbered population is selected to get a sample of the desired size. (p. 337)

tangent In a right-angled triangle, $\tan(\theta) = \dfrac{\text{opposite}}{\text{adjacent}}$; in the unit circle $\tan(\theta) = \dfrac{y}{x}$. (p. 243)

total change (or net change) The amount by which the quantity has changed over time; the integral of the marginal change from $x = 0$. (p. 224)

transcendental number A real number that is not the solution of an algebraic equation with integer coefficients. For example, e, π. (p. 3)

trapezoidal rule An approximation for the definite integral of a function, obtained by adding the areas of trapeziums under the curve. (p. 202)

triangular distribution A probability distribution, $f(x)$, with minimum value (a), peak value (c), and maximum value (b) defined as:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{2(x-a)}{(b-a)(c-a)} & \text{for } a \le x \le c \\ \dfrac{2(b-x)}{(b-a)(b-c)} & \text{for } c \le x \le b \\ 0 & \text{for } x > b \end{cases}$$

(p. 286)

turning point The exact location (point) where a curve changes between rising and falling; a maximum or minimum point. (p. 76)

uniform discrete random variable A discrete random variable whose probability function is a constant, whose distribution is uniform, such as the outcome of rolling a die. (p. 119)

uniform distribution A probability distribution that is constant over the domain of the variable. The uniform continuous distribution $p(x)$ is defined as:

$$p(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

A discrete uniform variable has the value $\dfrac{1}{n}$, where n is the number of values. (p. 285)

uniform variable A variable that has a constant value. (p. 119)

variance The value $Var(X) = \int_a^b p(x)(x-\mu)^2 dx$ for the continuous random variable X defined on the interval $[a, b]$. In many cases $[a, b] = [-\infty, \infty]$. (p. 125)

velocity (v) The rate of change of position or displacement: $v = \dfrac{dx}{dt} = \dfrac{ds}{dt}$ (p. 90)

***z*-score, *z* value** The number of standard deviations of a value from the mean; $z = \dfrac{x-\mu}{\sigma}$ (p. 304)

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