CAMBRIDGE

MATHEMATICAL METHODS
UNITS 1 & 2

SECOND EDITION

CAMBRIDGE SENIOR MATHEMATICS
FOR QUEENSLAND

MICHAEL EVANS | KAY LIPSON | DOUGLAS WALLACE
DAVID GREENWOOD

INCLUDES INTERACTIVE TEXTBOOK POWERED BY CAMBRIDGE HOTMATHS

CAMBRIDGE
UNIVERSITY PRESS & ASSESSMENT

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press & Assessment is a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

First published 2018
Second Edition 2024
20 19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1

Cover designed by Sardine Design
Text designed by diacriTech
Typeset by diacriTech
Printed in China by C & C Offset Printing Co. Ltd.

A catalogue record for this book is available from the National Library of Australia at www.nla.gov.au

ISBN 978-1-009-53719-3

Additional resources for this publication at www.cambridge.edu.au/GO

Cambridge University Press & Assessment acknowledges the Aboriginal and Torres Strait Islander peoples of this nation. We acknowledge the traditional custodians of the lands on which our company is located and where we conduct our business. We pay our respects to ancestors and Elders, past and present. Cambridge University Press & Assessment is committed to honouring Aboriginal and Torres Strait Islander peoples' unique cultural and spiritual relationships to the land, waters and seas and their rich contribution to society.

Contents

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

UNIT 1

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

 ISBN 978-1-009-53719-3 Cambridge University Press

UNIT 2

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

About the lead author and consultants

About the lead author

Michael Evans was a consultant to ACARA on the writing of the Australian Curriculum from which the Queensland syllabus has evolved. He is a consultant with the Australian Mathematical Sciences Institute, and is coordinating author of the ICE-EM 7–10 series also published by Cambridge.

He has also been active in the Australian Mathematics Trust, being involved with the writing of enrichment material and competition questions. He has many years' experience as a Chief Examiner and Chairperson of examination panels.

About the consultants

Noel Covill is Head of Mathematics at St Joseph's College, Gregory Terrace

Trevor Redmond is Head of Mathematics at Somerville House, South Brisbane

Fiona Steele is Head of Mathematics & Digital Technology at St Michael's College, Merrimac

Joel Speranza is currently working with the Toowoomba Catholic Schools Office, delivering Specialist Mathematics hybridly to students from eight different regional schools. He also creates supporting material for the senior mathematics curriculum through the website Maths Video Australia.

Introduction and overview

Cambridge Senior Mathematics for Queensland Mathematical Methods Units 1 & 2 provides complete and aligned coverage of the QCAA syllabus to be implemented in Year 11 from 2025. Its four components – the print book, downloadable PDF textbook, online Interactive Textbook (ITB) and Online Teaching Resource (OTS) – contain a huge range of resources, including worked solutions and revision of Year 10 material, available to schools in a single package at one convenient price (the OTS is included with class adoptions, conditions apply). There are no extra subscriptions or per-student charges to pay.

New features in the second edition:

- **Learning intentions** complemented by a Skills Checklist at the end of each chapter that allows you to check your understanding and tick off your achievements,
- **Technology-free and technology-active** short-response and multiple-choice questions are clearly labelled in Chapter review sections and Unit Revision chapters.
- **The problem-solving and modelling task (Appendix C):** There are three appendices at the end of the book. Appendix C is written by consultant Joel Speranza, providing advice on how to complete problem-solving and modelling tasks (PSMTs). This is supported by video resources accessed through QR codes and in the Interactive Textbook.
  Appendix A provides further material for Topic 2 of Unit 1 of this course and together with Appendix B provides a preparation for the study of the probability and statistics components of Mathematical Methods Units 3 & 4.

The Second Edition also features significantly revised and updated material from the first edition, including:

Review of Years 9 and 10: The first two chapters can be considered as a review of material covered in Years 9 and 10. The topics covered in these chapters are essential knowledge before proceeding with the rest of the book. You may choose to complete these chapters prior to the beginning of Year 11.

Degree of difficulty classification of questions: in the exercises, questions are classified as simple familiar SF, complex familiar CF, or complex unfamiliar CU questions and are indicated by a strip along the margin. The revision chapters described below also contain model questions for each of these categories, and tests are also provided in the teacher resources, made up of such categorised model questions.

Three revision chapters of material covered in the course: The first two of these chapters each cover an entire unit, and the last revision chapter contains questions revising the whole book. Each is divided into technology-free and technology-active short-response and multiple-choice questions, followed by problem-solving and modelling questions and investigations.

The problem-solving and modelling questions are multi-part questions where the students undertake modelling or problem solving with an indication of the path to be taken. These can be used in introducing the techniques of problem-solving and modelling. The investigation section is less structured and the student will have to make decisions about the pathway they will follow. Answers for them are not given in the textbook, but model solutions are in the Online Teaching Suite.

Calculator guidance: Throughout the book there is guidance for the use of the TI-Nspire CX non-CAS and the Casio fx-CG20AU and fx-CG50AU graphics calculators for the solution of problems. Guidance on the TI-84Plus CE is included in the Interactive Textbook, accessed via icons next to the TI-Nspire boxes. There are also online guides for the general use of each of these calculators.

Assessment: Examination practice questions and assessment tasks are provided in the revision chapters and the Online Teaching Suite. Check the updates there for developments as more guidance is published closer to implementation of the new assessment.

Interactive Textbook (ITB)

The Interactive Textbook (ITB) is an online HTML version of the print textbook powered by the HOTmaths platform, included with the print book or available as a separate purchase.

Updated and revised for the new syllabus, the Interactive Textbook includes:

- **Video demonstrations** of all worked examples
- **Quick quizzes** containing auto-marked multiple-choice questions have been thoroughly updated and revised, enabling students to check their understanding.
- A **success criteria** checklist at the end of each chapter with linked questions and examples available for download
- Comprehensive **worked solutions** for all questions are provided in the Interactive Textbook as an option that teacher can choose to enable for their students.
- **Downloadable skillsheets** can be used for homework or in class to focus on a single skill or small set of related skills.
- **Definitions** pop up for key terms in the text, and are also provided in a dictionary.

The Online Teaching Suite (OTS)

The Online Teaching Suite is automatically enabled with a teacher account and is integrated with the teacher's copy of the Interactive Textbook. All the teacher resources are in one place for easy access. The features include:

- A teacher's view of a student's working and self-assessment which enables them to modify the student's self-assessed marks, and respond where students flag that they had difficulty.

- The **task manager** allowing to direct students on a custom activity sequence based on their scores in measurable activities
- Quickly create customised tests from a bank of multiple-choice questions using the **test generator**. Tests are auto-marked in the Interactive Textbook or can be printed and used for homework or assessment practice.
- An expanded and revised suite of **chapter tests** and **assignments**
- Editable **curriculum grids** and **teaching programs**.
- A brand-new **Exam Generator**, allowing the creation of customised printable and online trial exams (see below for more).

More about the Exam Generator

The Online Teaching Suite, available from 2025, will include a comprehensive bank of QCAA exam questions, augmented by exam-style questions written by experts, to allow teachers to create custom trial exams.

Custom exams can model end-of-year exams, or target specific topics or types of questions that students may be having difficulty with.

Features include:

- Filtering by question-type, topic and degree of difficulty
- Searchable by key words
- Answers provided to teachers
- Worked solutions for all questions
- QCAA marking scheme
- Multiple-choice exams can be auto-marked if completed online, with filterable reports
- All custom exams can be printed and completed under exam-like conditions or used as revision.

Acknowledgements

The authors and publishers wish to thank **Peter Flynn** and **Jacinta Foley** for their expertise and timeliness in preparing the calculator screenshots, instructions and how-to guides for this edition.

Thanks to **Joel Speranza** for the creation of the PSMT appendix and supplementary supporting videos.

The authors and publisher wish to thank the following sources for permission to reproduce material:

Cover: © Getty Images / DuxX

Images: © Getty Images / Benito Catudio, Chapter 1 Opener / Lvcandy, Chapter 2 Opener / ADELART, Chapter 3 Opener / yuanyuan yan, Chapter 4 Opener / Xuanyu Han, Chapter 5 Opener / Adrienne Bresnahan, Chapter 6 Opener / zhengshun tang, Chapter 7 Opener / filo, Chapter 8 Opener / Govindanmarudhai, Chapter 9 Opener / Ali Kahfi, Chapter 10 Opener / Rizky Panuntun, Chapter 11 Opener / fbatista72, Chapter 12 Opener / a-r-t-i-s-t, Chapter 13 Opener / Ralf Hiemisch, Chapter 14 Opener / Xuanyu Han, Chapter 15 Opener / Redlio Designs, Chapter 16 Opener / bgblue, Chapter 17 Opener / Martine Df, Chapter 18 Opener / Govindanmarudhai, Appendix A / enot-poloskun, Appendix B; © Shutterstock.com / 2H p.72; © Alamy / RTImages, 4L p.154.

Every effort has been made to trace and acknowledge copyright. The publisher apologises for any accidental infringement and welcomes information that would redress this situation.

1

Reviewing linear equations

Chapter contents

Many problems may be solved by first translating them into mathematical equations and then solving the equations using algebraic techniques. An equation is solved by finding the value or values of the variables that would make the statement true.

Consider the equation $2x + 11 = 3 - 2x$. If $x = -2$, then

$$\text{LHS} = 2(-2) + 11 = 7 \quad \text{and} \quad \text{RHS} = 3 - 2(-2) = 7$$

The statement is true when $x = -2$. The solution to the equation is therefore $x = -2$. In this case there is no other value of x that would give a true statement.

The equations that we deal with in this chapter are called linear equations since they are related to the equation of a straight line.

Linear equations have either one solution (as shown in the example above), no solutions or infinitely many solutions. The equation $2x + 3 = 2x + 4$ has no solutions, since no value of x makes the statement true. The equation $2(x + 3) = 2x + 6$ has infinitely many solutions, since it is true for all values of x.

We note that the equation $x^2 = 16$ has exactly two solutions, $x = 4$ or $x = -4$, but this equation is not linear.

ISBN 978-1-009-53719-3

1A Linear equations

Learning intentions

- To be able to solve linear equations.

A **linear equation** (in one unknown) is a particular type of polynomial equation in which the variable is to the first power. The following are examples of linear equations:

$$3x - 5 = 11, \qquad 7 - 2t = 8t - 11, \qquad \frac{z-3}{4} + \frac{2z-5}{3} = 11$$

In each of these equations, the variable is to the first power.

The following are examples of non-linear polynomial equations:

$$x^2 - x - 12 = 0 \quad \text{(quadratic)}, \qquad 2x^3 - x = 0 \quad \text{(cubic)}, \qquad x^4 = 16 \quad \text{(quartic)}$$

In each of these equations, the highest power of the variable is not the first power. You have met some of these in your previous study, and they are also in later chapters of this book.

Solving linear equations

It is important when setting out the solution to an equation that each step is written under the previous one with the equals signs aligned. This careful setting out makes the algebra easy to check. Unsystematic methods, such as guess and check, will generally be of limited use for more complicated equations.

It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order.

Linear equations of the form $ax + b = c$

Many linear equations that arise in applications are of the form $ax + b = c$.

Example 1

Solve the equation $3x + 4 = 16$ for x.

Solution	Explanation
$3x + 4 = 16$	
$3x = 12$	Subtract 4 from both sides.
$x = 4$	Divide both sides by 3.
Check: LHS $= 3(4) + 4 = 16$, RHS $= 16$. $\therefore$ the solution is correct.	Once a solution has been found it may be checked by substituting the value back into both sides of the original equation to ensure that the left-hand side (LHS) equals the right-hand side (RHS).

The first three equations in the above example are equivalent equations. Obtaining the second two equations enables us to solve the first equation.

 ISBN 978-1-009-53719-3

Given an equation, an equivalent equation can be formed by:

- adding or subtracting the same number on both sides of the equation
- multiplying or dividing both sides of the equation by the same non-zero number.

Importantly, two equivalent equations have the same solution. By forming suitable equivalent equations, we solve linear equations.

Equations with the unknown on both sides

Group all the terms containing the variable on one side of the equation and the remaining terms on the other side.

Example 2

Solve $4x + 3 = 3x - 5$.

Solution

$$4x + 3 = 3x - 5$$
$$x + 3 = -5$$
$$x = -8$$

Check:

$$\text{LHS} = 4(-8) + 3 = -29$$
$$\text{RHS} = 3(-8) - 5 = -29$$

$\therefore$ the solution is correct.

Explanation

Subtract $3x$ from both sides and then subtract 3 from both sides.

The solution can be checked as previously shown.

Equations containing brackets

A frequently used first step is to remove brackets and then to follow the procedure for solving an equation without brackets.

Example 3

Solve $3(2x + 5) = 27$.

Solution

$$3(2x + 5) = 27$$
$$6x + 15 = 27$$
$$6x = 12$$
$$x = 2$$

Check:

$$\text{LHS} = 3(2 \times 2 + 5) = 27$$
$$\text{RHS} = 27$$

$\therefore$ the solution is correct.

Explanation

We note that since 27 is divisible by 3, the following method is also possible:

$$3(2x + 5) = 27$$
$$2x + 5 = 9$$
$$2x = 4$$
$$x = 2$$

ISBN 978-1-009-53719-3

Equations containing fractions

A frequently used first step is to multiply both sides of the equation by the lowest common multiple of the denominators of the fractions.

Example 4

Solve $\frac{x}{5} - 2 = \frac{x}{3}$.

Solution

$$\begin{aligned} \frac{x}{5} - 2 &= \frac{x}{3} \\ \frac{x}{5} \times 15 - 2 \times 15 &= \frac{x}{3} \times 15 \\ 3x - 30 &= 5x \\ -2x &= 30 \\ x &= -15 \end{aligned}$$

Check: $\text{LHS} = \frac{-15}{5} - 2 = -3 - 2 = -5$

$\text{RHS} = \frac{-15}{3} = -5$

$\therefore$ the solution is correct.

Explanation

The denominators of the fractions are 3 and 5. The lowest common multiple of 3 and 5 is 15.

Multiply both sides of the equation by 15. This means that each term of the LHS and the RHS of the equation is multiplied by 15.

Example 5

Solve $\frac{x-3}{2} - \frac{2x-4}{3} = 5$.

Solution

$$\begin{aligned} \frac{x-3}{2} \times 6 - \frac{2x-4}{3} \times 6 &= 5 \times 6 \\ 3(x-3) - 2(2x-4) &= 30 \\ 3x - 9 - 4x + 8 &= 30 \\ -x &= 31 \\ x &= -31 \end{aligned}$$

Check:

$$\begin{aligned} \text{LHS} &= \frac{-31-3}{2} - \frac{2 \times (-31) - 4}{3} \\ &= \frac{-34}{2} - \frac{-66}{3} = -17 + 22 = 5 \end{aligned}$$

$\text{RHS} = 5$

$\therefore$ the solution is correct.

Explanation

Remember that the vinculum (the line separating the numerator and the denominator) acts as brackets.

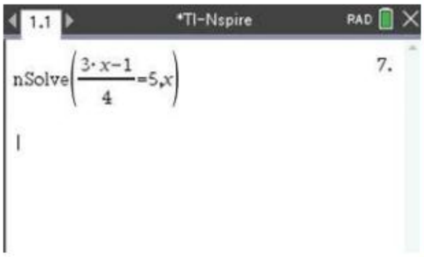
Multiply both sides of the equation by 6, the lowest common multiple of 2 and 3.

 ISBN 978-1-009-53719-3

Using the TI-Nspire CX non-CAS

To solve the linear equation $\frac{3x-1}{4} = 5$:

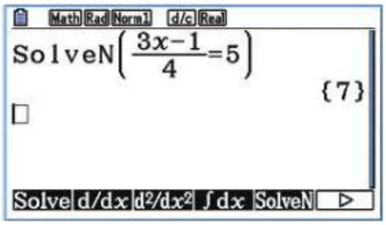
- Open a **Calculator** application.
- Select (menu) > **Algebra** > **Numerical Solve**.
- Enter the equation as shown. (A template for fractions is obtained by pressing (ctrl) (÷).)
- Press (enter) to obtain the solution $x = 7$.



Using the Casio

To solve the linear equation $\frac{3x-1}{4} = 5$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Use the numerical solver **SolveN**: (OPTN) (F4) (F5)
- Enter the equation: (▭/▭) (3) (X,θ,T) (−) (1) ▼ (4) ► (SHIFT) (•) (5) ( )) (EXE)
- Press (EXIT) to reveal the solution $x = 7$.



Note: The appendices of the Interactive Textbook give more details on using your calculator.

Literal equations

An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.

Example 6

Solve $ax + b = cx + d$ for x.

Solution

$$ax + b = cx + d$$

$$ax - cx = d - b$$

$$(a - c)x = d - b$$

$$x = \frac{d-b}{a-c}$$

Explanation

Collect terms in x on the left-hand side and constants on the right-hand side.

Factorise the left-hand side.

Summary 1A

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the variable is to the first power.
- The following steps provide some suggestions for solving a linear equation:
  1. Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  2. Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.

Exercise 1A

SF

1 Solve each of the following equations for x:

a $x + 3 = 6$ **b** $x - 3 = 6$ **c** $3 - x = 2$ **d** $x + 6 = -2$

e $2 - x = -3$ **f** $2x = 4$ **g** $3x = 5$ **h** $-2x = 7$

i $-3x = -7$ **j** $\frac{3x}{4} = 5$ **k** $\frac{-3x}{5} = 2$ **l** $\frac{-5x}{7} = -2$

2 Solve each of the following literal equations for x:

a $x - b = a$ **b** $x + b = a$ **c** $ax = b$ **d** $\frac{x}{a} = b$ **e** $\frac{ax}{b} = c$

Example 1

3 Solve the following linear equations:

a $2y - 4 = 6$ **b** $3t + 2 = 17$ **c** $2y + 5 = 2$ **d** $7x - 9 = 5$

e $2a - 4 = 7$ **f** $3a + 6 = 14$ **g** $\frac{y}{8} - 11 = 6$ **h** $\frac{t}{3} + \frac{1}{6} = \frac{1}{2}$

i $\frac{x}{3} + 5 = 9$ **j** $3 - 5y = 12$ **k** $-3x - 7 = 14$ **l** $14 - 3y = 8$

Example 2

4 Solve the following linear equations:

a $6x - 4 = 3x$ **b** $x - 5 = 4x + 10$ **c** $3x - 2 = 8 - 2x$

CF

5 Solve the following linear equations:

Example 3

a $2(y + 6) = 10$ **b** $2y + 6 = 3(y - 4)$ **c** $2(x + 4) = 7x + 2$

d $5(y - 3) = 2(2y + 4)$ **e** $x - 6 = 2(x - 3)$ **f** $\frac{y + 2}{3} = 4$

Example 4

g $\frac{x}{2} + \frac{x}{3} = 10$ **h** $x + 4 = \frac{3}{2}x$ **i** $\frac{7x + 3}{2} = \frac{9x - 8}{4}$

Example 5

j $\frac{2(1 - 2x)}{3} - 2x = -\frac{2}{5} + \frac{4(2 - 3x)}{3}$ **k** $\frac{4y - 5}{2} - \frac{2y - 1}{6} = y$

Example 6

SF

6 Solve the following literal equations for x:

a $ax + b = 0$ **b** $cx + d = e$ **c** $a(x + b) = c$ **d** $ax + b = cx$

e $\frac{x}{a} + \frac{x}{b} = 1$ **f** $\frac{a}{x} + \frac{b}{x} = 1$ **g** $ax - b = cx - d$ **h** $\frac{ax + c}{b} = d$

7 Solve each of the following for x, giving answers in exact form:

a $0.2x + 6 = 2.4$ **b** $0.6(2.8 - x) = 48.6$ **c** $\frac{2x + 12}{7} = 6.5$

d $0.5x - 4 = 10$ **e** $\frac{1}{4}(x - 10) = 6$ **f** $6.4x + 2 = 3.2 - 4x$

CF

8 Solve $\frac{b - cx}{a} + \frac{a - cx}{b} + 2 = 0$ for x.

9 Solve $\frac{a}{x + a} + \frac{b}{x - b} = \frac{a + b}{x + c}$ for x.

1B Constructing linear equations

Learning intentions

- To be able to construct a linear equation from a worded problem.

As stated earlier, many problems can be solved by translating them into mathematical language and using an appropriate mathematical technique to determine the solution. By representing the unknown quantity in a problem with a symbol and constructing an equation from the information, the value of the unknown can be found by solving the equation. Before constructing the equation, each symbol and what it stands for (including the units) should be stated.

Example 7

A chef uses the following rule for cooking a turkey:
'Allow 30 minutes for each kilogram weight of turkey and then add an extra 15 minutes.'

If the chef forgot to weigh a turkey before cooking it, but knew that it had taken 3 hours to cook, calculate how much it weighed.

Solution

Let the weight of the turkey be x kilograms.
Then the time taken is $(30x + 15)$ minutes.

$$\therefore \quad 30x + 15 = 180$$
$$30x = 165$$
$$x = 5.5$$

The turkey weighed 5.5 kilograms.

Explanation

Assign a variable to the quantity that is to be found. In this example, the weight of the turkey is x kilograms.

Determine, in terms of x, the time to cook the turkey. Then form the equation. Note that 3 hours is 180 minutes.

State the solution to the problem in words.

Example 8

Determine the area of a rectangle whose perimeter is 1.08 m, if it is 8 cm longer than it is wide.

Solution

Let length $= \ell$ cm.
Then width $= (\ell - 8)$ cm.

$$\begin{aligned} \text{Perimeter} &= 2 \times \text{length} + 2 \times \text{width} \\ &= 2\ell + 2(\ell - 8) \\ &= 4\ell - 16 \text{ cm} \end{aligned}$$

$$\text{Perimeter} = 108 \text{ cm}$$

$$\therefore \quad 4\ell - 16 = 108$$
$$4\ell = 124$$
$$\ell = 31 \text{ cm}$$

The length is 31 cm and the width is 23 cm.

Therefore the area is $31 \times 23 = 713$ cm^2.

Explanation

We know that

$$\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}$$

and that the width is 8 cm less than the length. Let ℓ cm be the length. Then the width is $(\ell - 8)$ cm.

Determine the perimeter in terms of ℓ.
Determine the length and width, and hence determine the area.

Example 9

Adam normally takes 5 hours to travel between Higett and Logett. One day he increases his speed by 4 km/h and determines the journey from Higett to Logett takes half an hour less than the normal time. Determine his normal speed.

Solution

Let x km/h be his normal speed.

The distance from Higett to Logett is $x \times 5 = 5x$ kilometres.

Adam's new speed is $(x + 4)$ km/h.

Hence
$$(x + 4) \times \frac{9}{2} = 5x$$
$$9(x + 4) = 10x$$
$$9x + 36 = 10x$$
$$36 = x$$

His normal speed is 36 km/h.

Explanation

In problems such as this, the speed is the average speed.

We note that

$$\text{distance} = \text{speed} \times \text{time}$$

Summary 1B

Steps for solving a word problem with a linear equation:

- Read the question carefully and write down the known information clearly.
- Identify the unknown quantity that is to be found.
- Assign a variable to this quantity.
- Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
- Solve the equation.
- Write a sentence answering the initial question.

Exercise 1B

1 For each of the following, write an equation using the variable x, then solve the equation for x: SF

- **a** A number plus two is equal to six.
- **b** A number multiplied by three is equal to ten.
- **c** Six is added to a number multiplied by three and the result is twenty-two.
- **d** Five is subtracted from a number multiplied by three and the result is fifteen.
- **e** Three is added to a number. If the result of this is multiplied by six, then fifty-six is obtained.
- **f** Five is added to a number and the result divided by four gives twenty-three.

 ISBN 978-1-009-53719-3 Cambridge University Press

CF

2 \$48 is divided among three students, A, B and C. If B receives three times as much as A, and C receives twice as much as A, how much does each receive?

3 The sum of two numbers is 42, and one number is twice the other. Determine the two numbers.

Example 7

4 A chef uses the following rule for cooking food on a spit: 'Allow 20 minutes for each kilogram weight and then add an extra 20 minutes.' If the chef forgot to weigh the food before cooking it but knew that it had taken 3 hours to cook, calculate how much it weighed.

Example 8

5 Determine the area of a rectangle whose perimeter is 4.8 m, if it is 0.5 m longer than it is wide.

6 Determine three consecutive whole numbers with a sum of 150.

7 Determine four consecutive odd numbers with a sum of 80.

8 Two tanks contain equal amounts of water. They are connected by a pipe and 3000 litres of water is pumped from one tank to the other. One tank then contains 6 times as much water as the other. How many litres of water did each tank contain originally?

9 A 120-page book has p lines to a page. If the number of lines were reduced by three on each page, the number of pages would need to be increased by 20 to give the same amount of writing space. How many lines were there on each page originally?

Example 9

10 A rower travels upstream at 6 km/h and back to the starting place at 10 km/h. The total journey takes 48 minutes. How far upstream did the rower go?

11 A shopkeeper buys a crate of eggs at \$1.50 per dozen. He buys another crate, containing 3 dozen more than the first crate, at \$2.00 per dozen. He sells them all for \$2.50 a dozen and makes \$15 profit. How many dozens were there in each of the crates?

Example 9

12 Jess walked for 45 minutes at 3 km/h and then ran for half an hour at x km/h. At the end of that time she was 6 km from the starting point. Determine the value of x.

13 A man travels from A to B at 4 km/h and from B to A at 6 km/h. The total journey takes 45 minutes. Determine the distance travelled.

14 A boy is 24 years younger than his father. In two years' time the sum of their ages will be 40. Determine the present ages of father and son.

 ISBN 978-1-009-53719-3

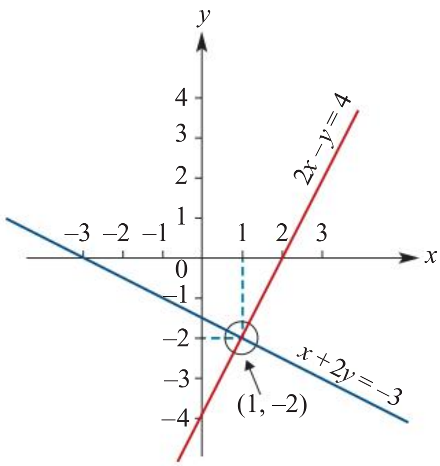
1C Simultaneous equations

Learning intentions

- To be able to solve linear simultaneous equations in two unknowns.

A linear equation that contains two unknowns, e.g. $2y + 3x = 10$, does not have a single solution. Such an equation actually expresses a relationship between pairs of numbers, x and y, that satisfy the equation. If all possible pairs of numbers (x, y) that satisfy the equation are represented graphically, the result is a straight line; hence the name **linear relation**.

If the graphs of two such equations are drawn on the same set of axes, and they are non-parallel, the lines will intersect at one point only. Hence there is one pair of numbers that will satisfy both equations simultaneously.



The intersection point of two straight lines can be found graphically; however, the accuracy of the solution will depend on the accuracy of the graphs.

Alternatively, the intersection point may be found algebraically by solving the pair of simultaneous equations. We shall consider two techniques for solving simultaneous equations.

Example 10

Solve the equations $2x - y = 4$ and $x + 2y = -3$.

Solution

Method 1: Substitution

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

Using one of the two equations, express one variable in terms of the other variable.

From equation (2), we get $x = -3 - 2y$.

Substitute in equation (1):

$$\begin{aligned} 2(-3 - 2y) - y &= 4 \\ -6 - 4y - y &= 4 \\ -5y &= 10 \\ y &= -2 \end{aligned}$$

Then substitute this expression into the other equation (reducing it to an equation in one variable, y). Solve the equation for y.

Substitute the value of y into (2):

$$\begin{aligned} x + 2(-2) &= -3 \\ x &= 1 \end{aligned}$$

Substitute this value for y in one of the equations to determine the other variable, x.

Check in (1): $\text{LHS} = 2(1) - (-2) = 4$

$\text{RHS} = 4$

A check can be carried out with the other equation.

Explanation

Method 2: Elimination

$$2x - y = 4 \qquad (1)$$
$$x + 2y = -3 \qquad (2)$$

To eliminate x, multiply equation (2) by 2 and subtract the result from equation (1).

When we multiply equation (2) by 2, the pair of equations becomes:

$$2x - y = 4 \qquad (1)$$
$$2x + 4y = -6 \qquad (2')$$

Subtract (2′) from (1):

$$-5y = 10$$
$$y = -2$$

Now substitute for y in equation (2) to determine x, and check as in the substitution method.

If one of the variables has the same coefficient in the two equations, we can eliminate that variable by subtracting one equation from the other.

It may be necessary to multiply one of the equations by a constant to make the coefficients of x or y the same in the two equations.

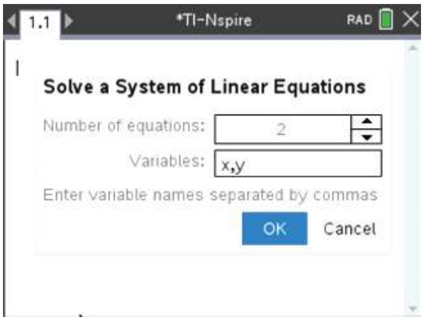
Note: This example shows that the point $(1, -2)$ is the point of intersection of the graphs of the two linear relations.

Using the TI-Nspire CX non-CAS

Method 1: Using a Calculator application

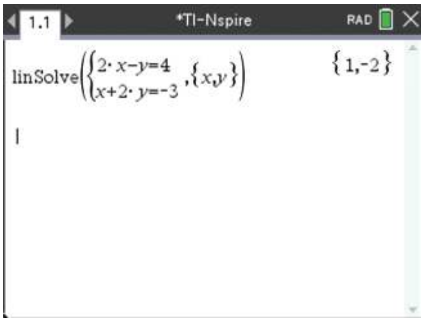
Simultaneous linear equations can be solved in a **Calculator** application.

- Use (menu) > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen.



- Enter the equations as shown to give the solution to the simultaneous equations $2x - y = 4$ and $x + 2y = -3$.
- Hence the solution is $x = 1$ and $y = -2$.

Note: The solution can also be found using linSolve($2x - y = 4$ and $x + 2y = -3, x, y$).

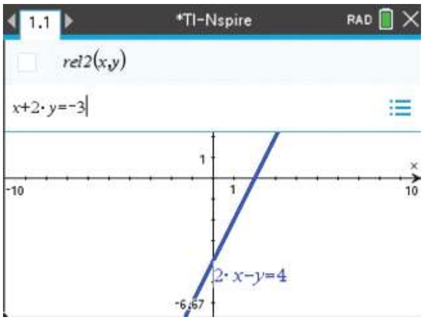


Method 2: Using a Graphs application

Simultaneous linear equations can also be solved graphically in a **Graphs** application.

Entering the equations:

- Equations of the form $a \cdot x + b \cdot y = c$ can be entered directly by using menu > **Graph Entry/Edit** > **Relation.**
- Enter the equations as shown.



Alternatively:

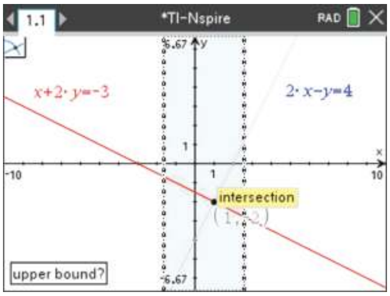
- The equations can be rearranged to make y the subject. The equations in this form are $f1(x) = 2x - 4$ and $f2(x) = \dfrac{-3 - x}{2}$.
- Enter these in the default function entry line.

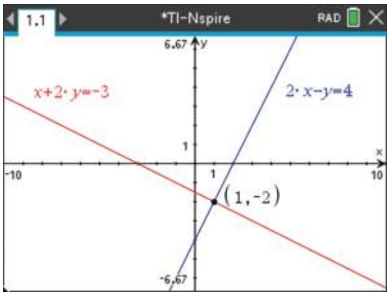
Notes:

- If the entry line is not visible, press tab or double click in an open area. Pressing enter will hide the entry line.
- Equations of the form $ax + by = c$ can also be entered directly using menu > **Graph Entry/Edit** > **Equation Templates** > **Line** > **Line Standard a ∗ x + b ∗ y = c**.

Finding the intersection point:

- Use menu > **Analyze Graph** > **Intersection**.
- Use the touchpad to move the cursor to the left of the intersection point (lower bound), click, move to the right of the intersection point (upper bound) and click again.
- The intersection point's coordinates will appear on the screen.





Note: You can also determine the intersection point using menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**. Use the touchpad to move the cursor to select each of the two graphs. Press esc to exit the **Intersection Point(s)** tool.

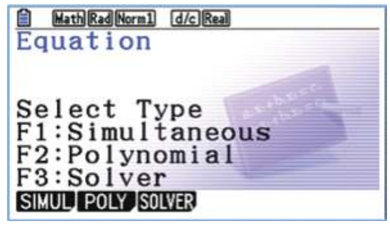
 ISBN 978-1-009-53719-3 Cambridge University Press

Using the Casio

Simultaneous linear equations can be solved in **Equation** mode or in **Graph** mode.

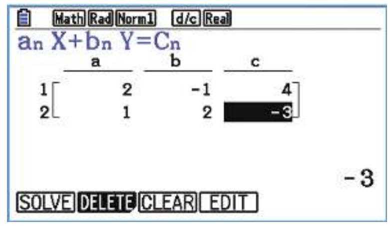
Method 1: Using Equation mode

- Press MENU and then select **Equation** mode by pressing ALPHA X,θ,T.
- Select **Simultaneous** F1.
- Select **Number of unknowns?** F1.
  Choose (2).

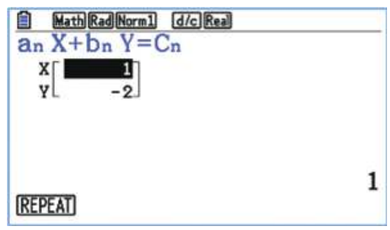


- Enter the coefficients of the two equations $2x - y = 4$ and $x + 2y = -3$ as shown:

  2 EXE (−) 1 EXE 4 EXE
  1 EXE 2 EXE (−) 3 EXE



- Select **Solve** F1.
- Hence the solution is $x = 1$ and $y = -2$.



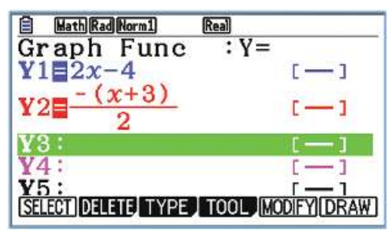
Method 2: Using Graph mode

- Press MENU 5 to select **Graph** mode.
- Transpose $2x - y = 4$ to make y the subject, and enter the equation in $Y1$:

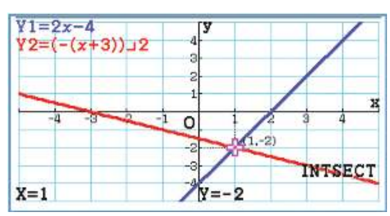
  2 X,θ,T − 4 EXE

- Transpose $x + 2y = -3$ to make y the subject, and enter the equation in $Y2$:

  ▭ (−) (X,θ,T + 3) ▼ 2 EXE



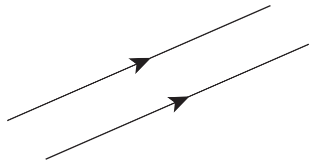
- Select **Draw** F6.
- Go to the **G-Solve** menu SHIFT F5 and select **Intersection** F5.

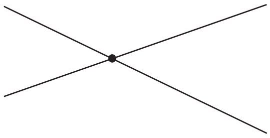


 ISBN 978-1-009-53719-3

The geometry of simultaneous equations

Two distinct straight lines are either parallel or meet at a point.





There are three cases for a system of two linear equations with two variables.

	Example	Solutions	Geometry
Case 1	$2x + y = 5$ $x - y = 4$	Unique solution: $x = 3,\ y = -1$	Two lines meeting at a point
Case 2	$2x + y = 5$ $2x + y = 7$	No solutions	Distinct parallel lines
Case 3	$2x + y = 5$ $4x + 2y = 10$	Infinitely many solutions	Two copies of the same line

This is further discussed in Chapter 2.

Summary 1C

We have two methods for solving simultaneous linear equations in two variables by hand.

1 Substitution

Make one of the variables the subject in one of the equations and substitute for that variable in the other equation.

For example, for the equations:

$$3x + y = 6 \qquad (1)$$
$$5x + 3y = -2 \qquad (2)$$

- From (1), we have $y = 6 - 3x$.
- Substitute this into (2), and solve the resulting equation $5x + 3(6 - 3x) = -2$ for x.
- Substitute this value for x back into one of the original equations to determine the value of the other variable, y.
- Check that your solution satisfies the original equations.

 ISBN 978-1-009-53719-3 Cambridge University Press

2 Elimination

In this method, one of the variables is eliminated. For example, with the same equations considered above:

$$3x + y = 6 \qquad (1)$$
$$5x + 3y = -2 \qquad (2)$$

- Decide which variable to eliminate. We will eliminate y.
- If necessary, multiply both sides of one or both equations by a number to obtain the same or opposite coefficient for the variable to be eliminated. With this example, multiply (1) by 3:

$$9x + 3y = 18 \qquad (1')$$

- Add or subtract the equations to eliminate the chosen variable. In this example, subtract (2) from (1′).
- Solve the resulting equation, and then proceed as with the substitution method.

Exercise 1C

SF

1 Solve each of the following pairs of simultaneous equations by the substitution method:

a $y = 2x + 1$
$y = 3x + 2$

b $y = 5x - 4$
$y = 3x + 6$

c $y = 2 - 3x$
$y = 5x + 10$

d $y - 4 = 3x$
$y - 5x + 6 = 0$

e $y - 4x = 3$
$2y - 5x + 6 = 0$

f $y - 4x = 6$
$2y - 3x = 4$

2 Solve each of the following pairs of simultaneous equations by the elimination method:

a $x + y = 6$
$x - y = 10$

b $y - x = 5$
$x + y = 3$

c $x - 2y = 6$
$x + 6y = 10$

Example 10

3 Solve each of the following pairs of simultaneous linear equations by either the substitution or the elimination method:

a $2x - 3y = 7$
$y = 5 - 3x$

b $2x - 5y = 10$
$4x + 3y = 7$

c $2m - 1 = n$
$2n + m = 8$

d $7x - 6y = 20$
$3x + 4y = 2$

e $3s - 1 = t$
$5s + 2t = 20$

f $4x - 3y = 1$
$4y - 5x = 2$

g $15x - 4y = 6$
$9x - 2y = 5$

h $2p + 5q = -3$
$7p - 2q = 9$

i $2x - 4y = -12$
$2y + 3x - 2 = 0$

4 For each of the following pairs of simultaneous linear equations, state whether there is one, none or infinitely many solutions:

a $3x + y = 6$
$6x + 2y = 7$

b $3x + y = 6$
$6x + 2y = 12$

c $3x + y = 6$
$6x - 2y = 7$

d $3x - y = 6$
$6x + 2y = 7$

ISBN 978-1-009-53719-3

1D Constructing simultaneous linear equations

Learning intentions

- To be able to construct simultaneous linear equations from a worded problem.

Problems involving two unknowns can often be solved by using simultaneous equations with two variables. The following examples show how this may be done.

Example 11

The sum of two numbers is 24 and their difference is 96. Determine the two numbers.

Solution

Let x and y be the two numbers with $x > y$. Then

$$x + y = 24 \qquad (1)$$
$$x - y = 96 \qquad (2)$$

Add equations (1) and (2):

$$2x = 120$$
$$x = 60$$

Substitute in equation (1):

$$60 + y = 24$$
$$y = -36$$

The two numbers are 60 and -36.

Explanation

The problem can also be solved by eliminating x. Subtracting (2) from (1) gives $2y = -72$ and hence $y = -36$.

The problem can also be solved by substitution. From (1), we have $y = 24 - x$. Substitute in (2).

The values found for x and y have to make each of the equations true. The equation which has not been used in the final substitution is the one to use for the check.

Example 12

3 kg of jam and 2 kg of butter cost \$29, and 6 kg of jam and 3 kg of butter cost \$54.

Determine the cost per kilogram of jam and butter.

Solution

Let the cost of 1 kg of jam be x dollars and the cost of 1 kg of butter be y dollars.

Then $\quad 3x + 2y = 29 \qquad (1)$

and $\quad 6x + 3y = 54 \qquad (2)$

Multiply (1) by 2: $\quad 6x + 4y = 58 \qquad (1')$

Subtract (1′) from (2): $\quad -y = -4$

$$y = 4$$

Substitute in (2): $\quad 6x + 3(4) = 54$

$$6x = 42$$
$$x = 7$$

Jam costs \$7 per kg and butter \$4 per kg.

Explanation

The unknowns are the cost per kilogram of jam and the cost per kilogram of butter.

Three kilograms of jam and two kilograms of butter cost \$29.

Six kilograms of jam and three kilograms of butter cost \$54.

Check in the original problem:

3 kg of jam = \$21 and 2 kg of butter = \$8
Total = \$29

6 kg of jam = \$42 and 3 kg of butter = \$12
Total = \$54

 ISBN 978-1-009-53719-3

Summary 1D

Steps for solving a word problem with simultaneous linear equations:

- Read the question carefully and write down the known information clearly.
- Identify the two unknown quantities that are to be found.
- Assign variables to these two quantities.
- Form expressions in terms of x and y (or other suitable variables) and use the other relevant information to form the two equations.
- Solve the system of equations.
- Write a sentence answering the initial question.

Skill-sheet

Exercise 1D

CF

1 Determine two numbers whose sum is 138 and whose difference is 88.

2 Determine two numbers whose sum is 36 and whose difference is 9.

3 Six stools and four chairs cost \$58, while five stools and two chairs cost \$35.

a How much do ten stools and four chairs cost?
b How much do four stools cost?
c How much does one stool cost?

4 A belt and a wallet cost \$42, while seven belts and four wallets cost \$213.

a How much do four belts and four wallets cost?
b How much do three belts cost?
c How much does one belt cost?

Use simultaneous equations to solve the following.

Example 11

5 Determine a pair of numbers whose sum is 45 and whose difference is 11.

6 In four years' time a mother will be three times as old as her son. Four years ago she was five times as old as her son. Determine their present ages.

7 A party was organised for thirty people at which they could have either a hamburger or a pizza. If there were five times as many hamburgers as pizzas, calculate the number of each.

8 Two children had 110 marbles between them. After one child had lost half her marbles and the other had lost 20 they had an equal number. How many marbles did each child start with and how many did they finish with?

 ISBN 978-1-009-53719-3

CF

9 One hundred and fifty tickets were sold for a basketball match and \$560 was the total amount collected. Adult tickets were sold at \$4.00 each and child tickets were sold at \$1.50 each. How many adult tickets and how many child tickets were sold?

10 The sum of the numerator and denominator of a fraction expressed in simplest form is 17. If 3 is added to the numerator, the value of the fraction will be 1. What is the fraction?

11 Linda thinks of a two-digit number. The sum of the digits is 8. If she reverses the digits, the new number is 36 greater than her original number. What was Linda's original number?

12 Tickets to a musical cost \$30 for adults and \$12 for children. At one particular performance 960 people attended and \$19 080 was collected in ticket sales. Determine the number of adults and the number of children who attended the performance.

CU

13 An investor received \$1400 interest per annum from a sum of money, with part of it invested at 10% and the remainder at 7% simple interest. This investor found that if she interchanged the amounts she had invested she could increase her return by \$90 per annum. Calculate the total amount invested.

Example 12

14 A shopkeeper sold his entire stock of shirts and ties in a sale for \$10 000. The shirts were priced at 3 for \$100 and the ties \$20 each. If he had sold only half the shirts and two-thirds of the ties he would have received \$6000. How many of each did he sell in the sale?

15 A tent manufacturer produces two models, the Outback and the Bush Walker. From earlier sales records it is known that 20 per cent more of the Outback model is sold than the Bush Walker. A profit of \$200 is made on each Outback sold, but \$350 is made on each Bush Walker. If during the next year a profit of \$177 000 is planned, how many of each model must be sold?

16 Oz Jeans has factories in Brisney and Sydbane. At the Brisney factory, fixed costs are \$28 000 per month and the cost of producing each pair of jeans is \$30. At the Sydbane factory, fixed costs are \$35 200 per month and the cost of producing each pair of jeans is \$24. During the next month Oz Jeans must manufacture 6000 pairs of jeans. Calculate the production order for each factory, if the total manufacturing costs for each factory are to be the same.

17 A tea wholesaler blends together three types of tea that normally sell for \$10, \$11 and \$12 per kilogram so as to obtain 100 kilograms of tea worth \$11.20 per kilogram. If the same amounts of the two higher priced teas are used, calculate how much of each type must be used in the blend.

 ISBN 978-1-009-53719-3

1E Solving linear inequalities

Learning intentions

- To be able to solve linear inequalities.

An **inequality** is a mathematical statement that contains an inequality symbol rather than an equals sign: for example, $2x + 1 < 4$. When you solve the inequality $2x + 1 < 4$, you answer the question:

'Which numbers x satisfy the property that $2x + 1$ is less than 4?'

You will find that your answers can be described using a number line. This is a good way to represent the solution, as there are infinitely many numbers that satisfy an inequality such as $2x + 1 < 4$. For example:

$$2(1) + 1 = 3 < 4, \quad 2(0) + 1 = 1 < 4, \quad 2\left(\frac{1}{2}\right) + 1 = 2 < 4, \quad 2(-1) + 1 = -1 < 4$$

To solve linear inequalities, proceed exactly as for equations with the following exception:

- When multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed.

Example 13

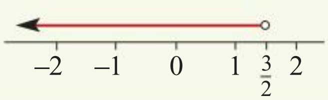
Solve the inequality $2x + 1 < 4$.

Solution

$$2x + 1 < 4$$

$$2x < 3$$

$$x < \frac{3}{2}$$



Explanation

Subtract 1 from both sides.

Divide both sides by 2.

The solution can be represented on a real number line.

Note: In a number-line diagram, the 'endpoint' of an interval is indicated with a closed circle if the point is included and with an open circle if it is not.

Example 14

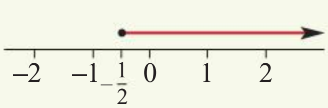
Solve the inequality $3 - 2x \le 4$.

Solution

$$3 - 2x \le 4$$

$$-2x \le 1$$

$$x \ge -\frac{1}{2}$$



Explanation

Subtract 3 from both sides.

Divide both sides by -2. Note that the inequality symbol is reversed.

ISBN 978-1-009-53719-3

Example 15

Solve the inequality $\dfrac{2x+3}{5} > \dfrac{3-4x}{3} + 2$.

Solution

$$\frac{2x+3}{5} > \frac{3-4x}{3} + 2$$

$$3(2x+3) > 5(3-4x) + 30$$

$$3(2x+3) - 5(3-4x) > 30$$

$$6x + 9 - 15 + 20x > 30$$

$$26x - 6 > 30$$

$$x > \frac{36}{26}$$

$$\therefore \quad x > \frac{18}{13}$$

Explanation

Multiply both sides by 15, the lowest common denominator of 5 and 3.

Collect the terms containing x on the left-hand side of the inequality.

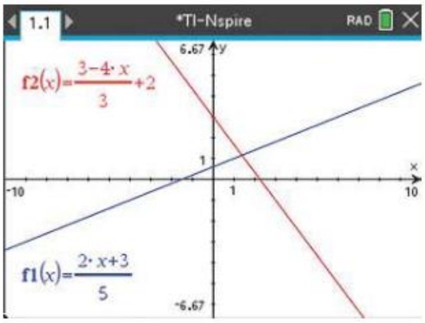
TI-84

Using the TI-Nspire CX non-CAS

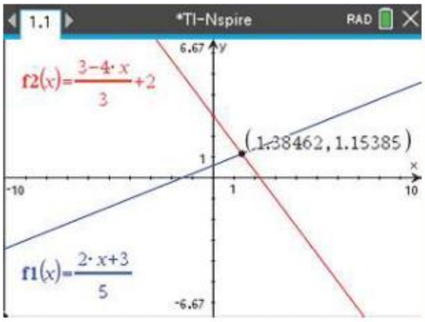
The inequality can be solved graphically in a **Graphs** application.

Method 1: Finding an intersection point

- Plot the graphs of $f1(x) = \dfrac{2x+3}{5}$ and $f2(x) = \dfrac{3-4x}{3} + 2$.



- Use (menu) > **Analyze Graph** > **Intersection** to determine the intersection point.
- Hence $f1(x) > f2(x)$ when $x > 1.385$ (value correct to 3 decimal places).



Note: The display digits can be changed using (menu) > **Settings** > **Display Digits**. Solving graphically will only give answers as whole numbers or decimals.

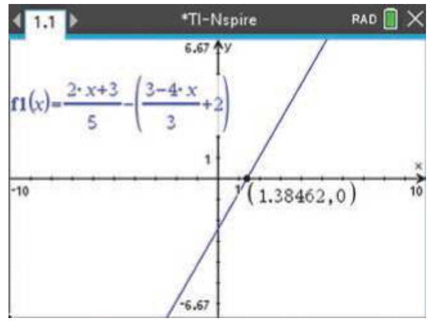
ISBN 978-1-009-53719-3

Method 2: Finding a zero

- Plot the graph of

$$f1(x) = \frac{2x+3}{5} - \left(\frac{3-4x}{3} + 2\right)$$

- Use menu > **Analyze Graph** > **Zero** to determine the intersection with the x-axis.
- Hence $f1(x) > 0$ when $x > 1.385$ (value correct to 3 decimal places).



Using the Casio

Method 1: Finding an intersection point

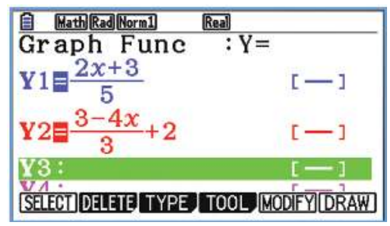
To solve the inequality:

- Press MENU 5 to select **Graph** mode.
- Enter the equation $y = \frac{2x+3}{5}$ in $Y1$:

  ▭ 2 X,θ,T + 3 ▼ 5 EXE

- Enter the equation $y = \frac{3-4x}{3} + 2$ in $Y2$:

  ▭ 3 − 4 X,θ,T ▼ 3 ► + 2 EXE

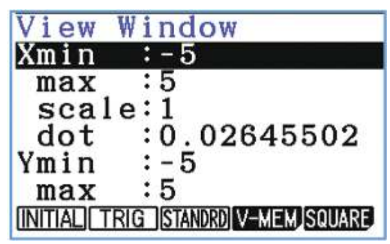


- Select **Draw** F6.
- Adjust the View Window:

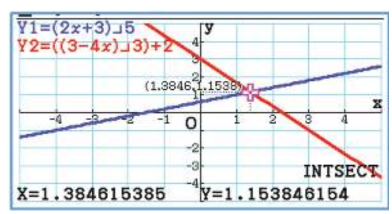
  SHIFT F3

  (−) 5 EXE 5 EXE 1 EXE ▼

  (−) 5 EXE 5 EXE 1 EXE

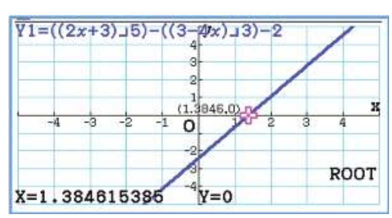


- To return to the graph, press EXIT.
- Go to the **G-Solve** menu SHIFT F5 and select **Intersection** F5.
- Hence $Y1 > Y2$ when $x > 1.385$, correct to three decimal places.



Method 2: Finding a zero

- Plot the graph of $y = \frac{2x+3}{5} - \frac{3-4x}{3} - 2$.
- To determine the x-axis intercept, go to the **G-Solve** menu SHIFT F5 and select **Root** F1.
- Hence $y > 0$ when $x > 1.385$, correct to three decimal places.



Summary 1E

- We can add or subtract the same number on both sides of an inequality, and the resulting inequality is equivalent to the original.
- We can multiply or divide both sides of an inequality by a positive number, and the resulting inequality is equivalent to the original.
- If we multiply or divide both sides of an inequality by a negative number, then we must reverse the inequality sign so that the resulting inequality is equivalent.

Skill-sheet

Exercise 1E

SF

1 Solve each of the following inequalities for x:

Example 13, 14

a $x + 3 < 4$ **b** $x - 5 > 8$ **c** $2x \geq 6$

d $\frac{x}{3} \leq 4$ **e** $-x \geq 6$ **f** $-2x < -6$

g $6 - 2x > 10$ **h** $\frac{-3x}{4} \leq 6$ **i** $4x - 4 \leq 2$

Example 14, 15

2 Solve for x in each of the following and show the solutions on a real number line:

a $4x + 3 < 11$ **b** $3x + 5 < x + 3$ **c** $\frac{1}{2}(x + 1) - x > 1$

CF

d $\frac{1}{6}(x + 3) \geq 1$ **e** $\frac{2}{3}(2x - 5) < 2$ **f** $\frac{3x - 1}{4} - \frac{2x + 3}{2} < -2$

g $\frac{4x - 3}{2} - \frac{3x - 3}{3} < 3$ **h** $\frac{1 - 7x}{-2} \geq 10$ **i** $\frac{5x - 2}{3} - \frac{2 - x}{3} > -1$

3 **a** For which real numbers x is $2x + 1$ a positive number?

b For which real numbers x is $100 - 50x$ a positive number?

c For which real numbers x is $100 + 20x$ a positive number?

4 In a certain country it costs $1 to send a letter weighing less than 20 g. A sheet of paper weighs 3 g. Write a suitable inequality and hence state the maximum number of pages that can be sent for $1. (Ignore the weight of the envelope in this question.)

5 A student receives marks of 66 and 72 on two tests. What is the lowest mark she can obtain on a third test to have an average for the three tests greater than or equal to 75?

ISBN 978-1-009-53719-3

1F Using and transposing formulas

Learning intentions

- To be able to substitute in a formula and transpose a formula.

An equation containing symbols that states a relationship between two or more quantities is called a **formula**. An example of a formula is $A = \ell w$ (area = length × width). The value of A, called the subject of the formula, can be found by substituting in given values of ℓ and w.

Example 16

Determine the area of a rectangle with length (ℓ) 10 cm and width (w) 4 cm.

Solution

$A = \ell w$

$A = 10 \times 4$

$A = 40 \text{ cm}^2$

Explanation

Substitute $\ell = 10$ and $w = 4$.

Sometimes we wish to rewrite a formula to make a different symbol the subject of the formula. This process is called **transposing** the formula. The techniques for transposing formulas include those used for solving linear equations detailed in Section 1A.

Example 17

Transpose the formula $v = u + at$ to make a the subject.

Solution

$v = u + at$

$v - u = at$

$a = \dfrac{v - u}{t}$

Explanation

Subtract u from both sides.

Divide both sides by t and reorder to place the subject of the equation on the left.

If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the given values for the other variables and then solve the resulting equation, or we can first transpose the formula and then substitute the given values.

Example 18

Evaluate p if $2(p + q) - r = z$, and $q = 2$, $r = -3$ and $z = 11$.

Solution

Method 1: Substituting then solving

$2(p + 2) - (-3) = 11$

$2p + 4 + 3 = 11$

$2p = 4$

$p = 2$

Explanation

First substitute $q = 2$, $r = -3$ and $z = 11$.

Then solve for p.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Method 2: Transposing then substituting

$$\begin{aligned} 2(p+q)-r &= z && \text{First solve for } p. \\ 2(p+q) &= z+r \\ p+q &= \frac{z+r}{2} \\ p &= \frac{z+r}{2}-q \\ \therefore\quad p &= \frac{11+(-3)}{2}-2 && \text{Substitute } q=2,\ r=-3 \text{ and } z=11. \\ p &= 2 \end{aligned}$$

Example 19

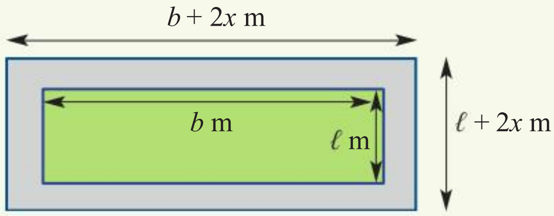
A path x metres wide surrounds a rectangular lawn. The lawn is ℓ metres long and b metres wide. The total area of the path is A m^2.

a Determine A in terms of ℓ, b and x.

b Determine b in terms of ℓ, A and x.

Solution

a



The area of the path is

$$\begin{aligned} A &= (b+2x)(\ell+2x)-b\ell \\ &= b\ell+2x\ell+2xb+4x^2-b\ell \\ \therefore\quad A &= 2x\ell+2xb+4x^2 \end{aligned}$$

b $A-(2x\ell+4x^2)=2xb$

Therefore

$$b=\frac{A-(2x\ell+4x^2)}{2x}$$

ISBN 978-1-009-53719-3

Example 20

For each of the following, make c the subject of the formula:

a $e = \sqrt{3c - 7a}$

b $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{c-2}$

Solution

a
$$e = \sqrt{3c - 7a}$$

Square both sides of the equation:

$$e^2 = 3c - 7a$$

Therefore

$$3c = e^2 + 7a$$

$$c = \frac{e^2 + 7a}{3}$$

b
$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c-2}$$

Establish common denominator on the left-hand side of the equation:

$$\frac{b-a}{ab} = \frac{1}{c-2}$$

Take the reciprocal of both sides:

$$\frac{ab}{b-a} = c - 2$$

Therefore $c = \dfrac{ab}{b-a} + 2$

Summary 1F

- A formula relates different quantities: for example, the formula $A = \pi r^2$ relates the radius r with the area A of the circle.
- The variable on the left is called the subject of the formula: for example, in the formula $A = \pi r^2$, the subject is A.
- To calculate the value of a variable which is not the subject of a formula:

  Method 1 Substitute the values for the known variables, then solve the resulting equation for the unknown variable.

  Method 2 Rearrange to make the required variable the subject, then substitute values.

Skill-sheet

Exercise 1F

SF

1 For each of the following, determine the value of the letter in parentheses:

a $c = ab,\ a = 6,\ b = 3 \quad (c)$

b $r = p + q,\ p = 12,\ q = -3 \quad (r)$

c $c = ab,\ a = 6,\ c = 18 \quad (b)$

d $r = p + q,\ p = 15,\ r = -3 \quad (q)$

e $c = \sqrt{a},\ a = 9 \quad (c)$

f $c = \sqrt{a},\ c = 9 \quad (a)$

g $p = \dfrac{u}{v},\ u = 10,\ v = 2 \quad (p)$

h $p = \dfrac{u}{v},\ p = 10,\ v = 2 \quad (u)$

 ISBN 978-1-009-53719-3

SF

2 For each of the following, construct a formula using the given symbols:

a S, the sum of three numbers a, b and c

b P, the product of two numbers x and y

c the cost, \$$C$, of five CDs which each cost \$$p$

d the total cost, \$$T$, of d chairs which cost \$$p$ each and c tables which cost \$$q$ each

e the time, T, in minutes, of a train journey that takes a hours and b minutes

Example 16

3 Determine the values of the following:

a $E = IR$, when $I = 5$ and $R = 3$

b $C = pd$, when $p = 3.14$ and $d = 10$

c $P = \dfrac{RT}{V}$, when $R = 60$, $T = 150$ and $V = 9$

d $I = \dfrac{E}{R}$, when $E = 240$ and $R = 20$

e $A = \pi r\ell$, when $\pi = 3.14$, $r = 5$ and $\ell = 20$

f $S = 90(2n - 4)$, when $n = 6$

Example 17

4 For each of the following, make the symbol indicated the subject of the formula:

a $PV = c$; V **b** $F = ma$; a **c** $I = Prt$; P

d $w = H + Cr$; r **e** $S = P(1 + rt)$; t **f** $V = \dfrac{2R}{R - r}$; r

Example 18

5 Determine the value of the unknown symbol in each of the following:

a $D = \dfrac{T + 2}{P}$, when $D = 10$, $P = 5$

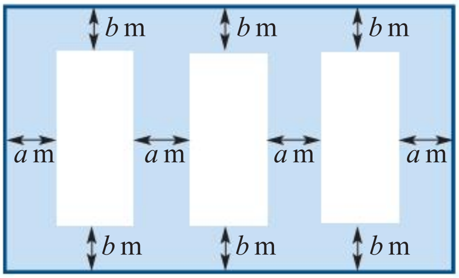
b $A = \frac{1}{2}bh$, when $A = 40$, $h = 10$

c $V = \frac{1}{3}\pi r^2 h$, when $\pi = 3.14$, $V = 100$, $r = 5$ (to two decimal places)

d $A = \frac{1}{2}h(a + b)$, when $A = 50$, $h = 5$, $a = 10$

CF

6 The diagram represents the brick wall of a dwelling with three windows. Each of the windows is h m high and w m wide. Other dimensions are as shown.



a Determine the length of the wall.

b Determine the height of the wall.

c Determine the total area of the three windows.

d Determine the total area of brickwork.

 ISBN 978-1-009-53719-3

CF

7 A lampshade has a metal frame consisting of two circular hoops of radii p cm and q cm joined by four straight struts of length h cm. The total length of metal is T cm.

a **i** Find an expression for T in terms of p, q and h.

ii Determine the value of T when $p = 20$, $q = 24$ and $h = 28$.

b The area of the material covering the frame is A cm^2, where $A = \pi h(p + q)$. Determine an expression for p in terms of A, h, q and π.

SF

8 Determine the value of the unknown symbol in each of the following:

a $P = \dfrac{T - M}{D}$, $P = 6$, $T = 8$, $M = 4$

b $H = \dfrac{a}{3} + \dfrac{a}{b}$, $H = 5$ and $a = 6$

c $a = \dfrac{90(2n - 4)}{n}$, $a = 6$

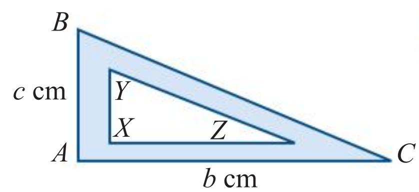
d $R = \dfrac{r}{a} + \dfrac{r}{3}$, $a = 2$ and $R = 4$

Example 19

CF

9 Right-angled triangles XYZ and ABC are similar.

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = k$$



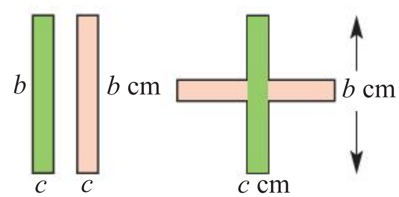
If $AB = c$ cm and $AC = b$ cm, determine:

a the area, D cm^2, of the shaded region in terms of c, b and k

b k in terms of D, b and c

c the value of k if $D = 2$, $b = 3$ and $c = 4$.

10 Two rectangles each with dimensions c cm $\times$ b cm are used to form a cross as shown. The arms of the cross are all of equal length.



a Determine the perimeter, P cm, of the cross in terms of b and c.

b Determine the area, A cm^2, of the cross in terms of b and c.

c Determine b in terms of A and c.

Example 20

11 For each of the following, make the symbol in brackets the subject of the formula:

a $a = \sqrt{a + 2b}$ $\quad (b)$

b $\dfrac{a + x}{a - x} = \dfrac{b - y}{b + y}$ $\quad (x)$

c $px = \sqrt{3q - r^2}$ $\quad (r)$

d $\dfrac{x}{y} = \sqrt{1 - \dfrac{v^2}{u^2}}$ $\quad (v)$

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Chapter summary

- A **linear equation** is one in which the variable is to the first power.
- It is often helpful to look at how the equation has been constructed so that the steps necessary to 'undo' the equation can be identified. It is most important that the steps taken to solve the equation are done in the correct order.
- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**: for example, $ax + b = c$.
- The two methods for solving simultaneous linear equations are **substitution** and **elimination**.
- An **inequality** is a mathematical statement that contains an inequality symbol rather than an equals sign: for example, $2x + 1 < 4$.
- To solve linear inequalities, proceed exactly as for equations except that, when multiplying or dividing both sides by a negative number, the 'direction' of the inequality symbol is reversed.
- An equation containing symbols that states a relationship between two or more quantities is called a **formula**. An example of a formula is $A = \ell w$ (area = length × width). The subject of this formula is A.
- If we wish to evaluate an unknown that is not the subject of the formula, we can either substitute the given values for the other variables and then solve the resulting equation, or we can first transpose the formula and then substitute the given values.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

1A **1** I can solve linear equations of the form $ax + b = c$. ☐

See Example 1 and Question 3

1A **2** I can solve linear equations with the unknown on both sides. ☐

See Example 2 and Question 4

1A **3** I can solve linear equations which involve brackets. ☐

See Example 3 and Question 5

1A **4** I can solve linear equations which involve fractions. ☐

See Example 4, Example 5 and Question 5

1A **5** I can solve linear literal equations. ☐

See Example 6 and Question 6

1B **6** I can construct linear equations from suitable different contexts. ☐

See Example 7, Example 8, Example 9 and Questions 4, 5 and 10

1C **7** I can solve simultaneous linear equations by substitution and elimination. ☐

See Example 10 and Question 3

1D **8** I can construct and solve simultaneous linear equations from different contexts. ☐

See Example 11, Example 12 and Questions 5 and 14

1E **9** I can solve linear inequalities. ☐

See Example 13, Example 14, Example 15 and Questions 1 and 2

1F **10** I can use formulas and transpose them. ☐

See Example 16, Example 17, Example 18, Example 19, Example 20 and Questions 3, 4, 5 and 9

Short-response questions

Technology-free short-response questions

SF

1 Solve each of the following equations for x:

a $2x + 6 = 8$ **b** $3 - 2x = 6$ **c** $2x + 5 = 3 - x$

d $\dfrac{3 - x}{5} = 6$ **e** $\dfrac{x}{3} = 4$ **f** $\dfrac{13x}{4} - 1 = 10$

g $3(2x + 1) = 5(1 - 2x)$ **h** $\dfrac{3x + 2}{5} + \dfrac{3 - x}{2} = 5$

2 Solve each of the following for t:

a $a - t = b$ **b** $\dfrac{at + b}{c} = d$ **c** $a(t - c) = d$

d $\dfrac{a - t}{b - t} = c$ **e** $\dfrac{at + b}{ct - b} = 1$ **f** $\dfrac{1}{at + c} = d$

3 Solve each of the following inequalities for x:

a $2 - 3x > 0$ **b** $\dfrac{3 - 2x}{5} \geq 60$

c $3(58x - 24) + 10 < 70$ **d** $\dfrac{3 - 2x}{5} - \dfrac{x - 7}{6} \leq 2$

4 Make x the subject of the formula $z = \frac{1}{2}x - 3t$. Determine x when $z = 4$ and $t = -3$.

5 Solve each of the following pairs of simultaneous equations for a and b:

a $4a - b = 11$

$3a + 2b = 6$

b $a = 2b + 11$

$4a - 3b = 11$

 ISBN 978-1-009-53719-3

6 A number d is equal to the square of a number e plus twice a number f.

a Determine a formula for d in terms of e and f.

b Make f the subject of the formula.

c Determine f when $d = 10$ and $e = 3$.

7 The surface area of a sphere of radius r is given by the formula $A = 4\pi r^2$. Calculate the surface area of a sphere of radius 10 cm. Give your answer in terms of π.

8 The volume of metal in a tube is given by the formula $V = \pi\ell[r^2 - (r - t)^2]$, where ℓ is the length of the tube, r is the radius of the outside surface and t is the thickness of the material. Determine V when:

a $\ell = 100$, $r = 5$ and $t = 0.2$

b $\ell = 50$, $r = 10$ and $t = 0.5$

9 For each of the following, make the variable in brackets the subject of the formula:

a $A = \pi rs \quad (r)$

b $T = P(1 + rw) \quad (w)$

c $v = \sqrt{\dfrac{n - p}{r}} \quad (r)$

d $ac = b^2 + bx \quad (x)$

10 Let $s = \left(\dfrac{u + v}{2}\right)t$.

a Determine the value of s if $u = 10$, $v = 20$ and $t = 5$.

b Determine the value of t if $u = 10$, $v = 20$ and $s = 120$.

11 The volume, V cm^3, of a cylinder is given by $V = \pi r^2 h$, where r cm is the radius and h cm is the height. Determine the radius of the cylinder if the volume of the cylinder is 500π cm^3 and the height is 10 cm.

12 A rope of length 205 m is cut into 10 pieces of one length and 5 pieces of another length. The total length of three of the first 10 lengths exceeds that of two of the second length by 2 m. Determine the lengths of the pieces.

13 If I add one to the numerator of a fraction $\dfrac{m}{n}$ it simplifies to $\dfrac{1}{5}$. If I subtract one from the denominator it simplifies to $\dfrac{1}{7}$. Determine the fraction $\dfrac{m}{n}$.

14 Mr Adonis earns \$7200 more than Mr Apollo, and Ms Aphrodite earns \$4000 less than Mr Apollo. If the total of the three incomes is \$303 200, determine the income of each person.

15 A motorist travelled a total distance of 424 km, and had an average speed of 80 km/h on highways and 24 km/h while passing through towns. If the journey took six hours, determine how long the motorist spent travelling on highways.

16 The formula for converting degrees Celsius, C, to degrees Fahrenheit, F, is $F = \dfrac{9}{5}C + 32$.

a Convert 30°F to Celsius.

b Convert 30°C to Fahrenheit.

c If x°C = x°F determine x.

d If $(x + 10)$°C = x°F determine x.

e If $2x$°C = x°F determine the value of x.

f If k°F = $(-3k)$°C determine k.

 ISBN 978-1-009-53719-3

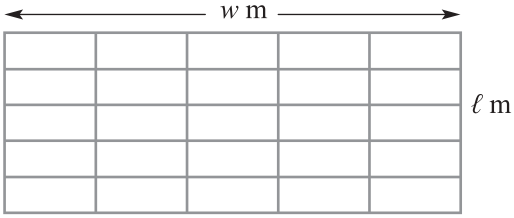
CU

17 For a spherical mirror of radius r cm, $\frac{1}{v} + \frac{1}{u} = \frac{2}{r}$, where u cm is the distance from the mirror to the object and v cm is the distance from the mirror to the image. The magnification is given by $m = \frac{v - r}{r - u}$.

a Determine r in terms of v and u from the first formula.

b Determine m in terms of v and u only.

18 The diagram shows a section of wire mesh w metres in width and ℓ metres in length.



a Determine an expression in terms of w and ℓ for the total length of wire required for the mesh.

b **i** If $w = 3\ell$, determine an expression in terms of w for the total length of wire required.

ii If the length of wire used is 100 m, determine the value of w and the value of ℓ.

c The total length of wire, L m, required for another type of rectangular mesh of dimensions x m by y m, is given by the formula $L = 6x + 8y$.

i Determine y in terms of x and L.

ii Determine y if $L = 200$ and $x = 4$.

d A third type of mesh can also be used to cover a rectangular region of dimensions x m by y m. In this case, the type of mesh introduced in part c requires 100 m of wire and so $6x + 8y = 100$. This third type of mesh requires 80 m and this gives the equation $3x + 2y = 40$. Determine the values of x and y.

Technology-active short-response questions

SF

19 For each of the following, determine the value of the letter in parentheses:

a $c = ab, \ a = 6.2, \ b = 3.01 \quad (c)$

b $r = p + q, \ p = 7.3, \ q = -3.87 \quad (r)$

c $c = ab, \ a = 6.2, \ c = 47.12 \quad (b)$

d $r = p + q, \ p = 15.6, \ r = -7.9 \quad (q)$

e $c = \sqrt{a}, \ a = 16.4025 \quad (c)$

f $c = \sqrt{a}, \ c = 26.7 \quad (a)$

g $p = \frac{u}{v}, \ u = 9.1, \ v = 3.5 \quad (p)$

h $p = \frac{u}{v}, \ p = 23, \ v = 2.8 \quad (u)$

20 Solve the following linear equations:

a $2z - 6.2 = 6.12$

b $0.3x + 6 = 7.86$

c $2y + 5 = 27.4$

d $17x - 9.2 = 51.32$

e $\frac{y}{8} - 56 = 6.2$

f $\frac{z}{3} + 0.84 = 0.652$

g $3.6 - 5.4y = -8.82$

h $-3x - 13 = 14$

Review

SF

21 Solve each of the following pairs of simultaneous linear equations by either the substitution or the elimination method:

a $2x - 5y = 0$
$y = 0.5 - 0.2x$

b $2x - 5y = 10$
$4.5x - 2.2y = 8$

c $0.8x - 6y = 20$
$0.3x + 0.4y = 2$

CF

22 The mean coefficient of linear expansion, α, of a metal rod over the temperature range t_1°C to t_2°C is defined by the formula

$$\alpha = \frac{\ell_2 - \ell_1}{\ell_1(t_2 - t_1)}$$

where ℓ_1 cm is the length of the rod at t_1°C and ℓ_2 cm is the length of the rod at t_2°C. Calculate the value of ℓ_2 correct to two decimal places if $\ell_1 = 435, t_1 = 15, t_2 = 95$ and $\alpha = 0.000012$.

23 The value A of an investment of \$$P$ after n years at r% per annum is given by the formula

$$A = P\left(1 + \frac{r}{100}\right)^n$$

Determine A when $P = 5342, r = 5.5$ and $n = 6$. Give your answer to the nearest dollar.

24 Prove that the lines with equations $2y - x = 2$, $y + x = 7$ and $y - 2x = -5$ meet at the one point.

CU

25 Tom leaves town A and travels towards town B at a constant speed of u km/h. At the same time, Julie leaves town B and travels towards town A at a constant speed of v km/h. Town B is d km from town A.

a How far has each travelled after t hours?

b By considering that the sum of their distances travelled must be d km when they meet, determine:

  i the time it takes for them to meet

  ii their distance from town A when they meet.

c If $u = 30$, $v = 50$ and $d = 100$, determine the time it takes for them to meet and their distance from town A.

26 Xiu travels from town A to town B at u km/h and then returns at v km/h. Town A is d km from town B.

a Determine the average speed at which Xiu travels for the complete journey, in terms of u and v. Remember that

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

b If it takes T hours to travel from A to B, determine the time taken:

  i for the return trip from B to A, in terms of T, u and v

  ii for the entire trip, in terms of T, u and v.

 ISBN 978-1-009-53719-3

Multiple-choice questions

Technology-free multiple-choice questions

1 The solution of the linear equation $3x - 7 = 11$ is

A $\frac{11}{3}$ **B** $\frac{-3}{4}$ **C** 6 **D** -6

2 If $\frac{x}{3} + \frac{1}{3} = 2$ then $x =$

A $\frac{2}{3}$ **B** $\frac{7}{3}$ **C** 5 **D** 7

3 The solution of the equation $x - 8 = 3x - 16$ is

A $x = \frac{11}{3}$ **B** $x = 4$ **C** $x = 2$ **D** $x = -2$

4 The statement that 7 is 11 times the result of subtracting 2 from x can be written as

A $11(x - 2) = 7$ **B** $7 = 11(x + 2)$

C $\frac{11}{x - 2} = 7$ **D** $\frac{7}{11} - 2 = x$

5 The solution of the simultaneous equations $2x - y = 10$ and $x + 2y = 0$ is

A $x = 2$ and $y = -3$ **B** $x = 4$ and $y = -2$

C $x = 6$ and $y = 2$ **D** $x = 1$ and $y = -8$

6 I bought x CDs for \$$a$ and y DVDs for \$$b$. The average price paid, in dollars, is

A $\frac{a + b}{y + x}$ **B** $\frac{xa + yb}{y + x}$ **C** $\frac{y + x}{xa + yb}$ **D** $\frac{a}{x} + \frac{b}{y}$

7 The solution of the equation $\frac{x + 1}{4} - \frac{2x - 1}{6} = x$ is

A $x = \frac{5}{13}$ **B** $x = 1$ **C** $x = -\frac{1}{5}$ **D** $x = -1$

8 The values of z that satisfy the inequality $\frac{72 + 15z}{3} > 4$ are

A $z > -4$ **B** $z = -4$ **C** $z < 4$ **D** $z < -4$

9 If $A = \frac{hw + k}{w}$ then

A $w = \frac{k}{A - h}$ **B** $w = \frac{A - 2k}{2h}$

C $w = \frac{3Ah}{2} - k$ **D** $w = \frac{2}{3}h(A + k)$

 ISBN 978-1-009-53719-3 Cambridge University Press

10 Bronwyn walks one lap of an oval at 2.5 km/h and then jogs another eight laps of the oval at 5 km/h. If it takes her 30 minutes in total, how long in metres is each lap?

A 250 m **B** 300 m **C** 350 m **D** 400 m

11 Which of the following equations has no solutions?

A $2(x + 3) = 2x + 6$ **B** $3x - 4 = 5 - x$

C $3x + 6 = 2x + 6$ **D** $2x + 4 = 2x + 6$

12 Which of the following equations is true for all values of x?

A $5(x + 3) = 5x + 15$ **B** $3x - 4 = 10 - x$

C $7x + 6 = -2x + 6$ **D** $2x + 4 = 2x + 6$

Technology-active multiple-choice questions

13 A boat travels 30 km up a river in the same time it takes to travel 47 km down the same river. If the speed of the current of water is 5.5 km/h, then the speed of the boat in still water correct to one decimal place is?

A 26.8 km/h **B** 22.9 km/h **C** 24.9 km/h **D** 31.8 km/h

14 Charles and Lois paddled downstream at an average speed of 12.8 km/h. They turned around and paddled back upstream at an average rate of 4.2 km/h. The total trip took one hour twenty minutes. After how much time, correct to the nearest minute, did Charles and Lois turn around?

A 18 minutes **B** 19 minutes **C** 20 minutes **D** 21 minutes

15 The volume of a sphere V cm^3 is given by the formula $V = \frac{4}{3}\pi r^3$ where r cm is the radius of the sphere. If $V = 27$, the value of r correct to two decimal places is

A 1.62 **B** 1.86 **C** 2.21 **D** 2.62

16 The cost \$$C$ of renting a truck is given by the formula $C = 4.35x + 200$ where x is the number of kilometres travelled. A person is charged \$1248 for the rent of a truck. The number of kilometres travelled (given to the nearest kilometre) is

A 229 **B** 239 **C** 240 **D** 241

17 The shorter sides of a parallelogram are each $\frac{2}{3}$ of the length of a longer side. The perimeter of the parallelogram is 170 cm. The length of each of the longer sides is x cm. The value of x is

A 51 **B** 45 **C** 43 **D** 41

2

Review of coordinate geometry and linear relations

Chapter contents

The number plane (Cartesian plane) is divided into four quadrants by two perpendicular axes. These axes intersect at a point called the origin. The position of any point in the plane can be represented by an **ordered pair** of numbers (x, y), called the **coordinates** of the point. Given the coordinates of two points, we can determine the equation of the straight line through the two points, the distance between the two points and the midpoint of the line segment joining the points. These are the beginning ideas of coordinate geometry. The topic of calculus, which is introduced later in this book, builds on these ideas.

A **relation** is defined as a set of ordered pairs in the form (x, y). Sometimes we can give a rule relating the x-value to the y-value of each ordered pair, such as $y = 2x + 1$, and this is a more convenient way of describing the relation. A relation may also be represented graphically on a set of axes. If the resulting graph is a straight line, then the relation is called a **linear relation**.

2A Distance and midpoints

Learning intentions

- ▶ To be able to determine the distance between two points.
- ▶ To be able to determine the coordinates of the midpoint of a line segment.

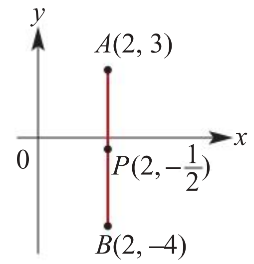
Midpoint of a line segment

Finding the midpoint of a line segment parallel to an axis is a simple special case, and it is useful in obtaining the more general result.

A line segment parallel to an axis

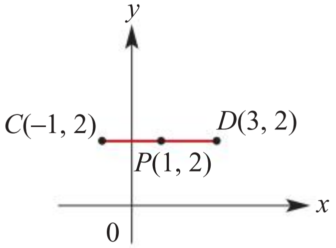
The midpoint of the line segment AB with endpoints $A(2, 3)$ and $B(2, -4)$ is the point P with coordinates $(2, -\frac{1}{2})$.

Note that $-\frac{1}{2}$ is the average of 3 and -4. The line through A and B is parallel to the y-axis.



Similarly for the line segment CD with endpoints $C(-1, 2)$ and $D(3, 2)$, the midpoint is the point P with coordinates $(1, 2)$.

Note that 1 is the average of -1 and 3. The line through C and D is parallel to the x-axis.

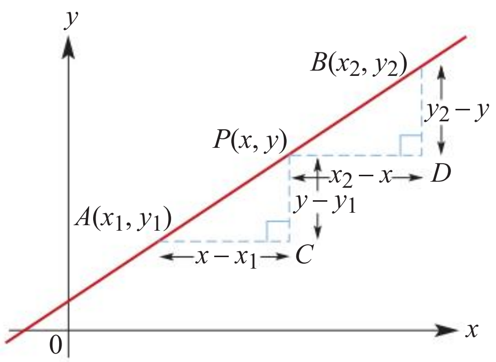


A line segment not parallel to one of the axes

Let $P(x, y)$ be the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, where the line through A and B is not parallel to either axis.

Let points C and D be chosen so that AC and PD are parallel to the x-axis, and PC and BD are parallel to the y-axis.

The triangles APC and PBD are congruent (AAS). Hence



$$AC = PD \qquad \text{and} \qquad PC = BD$$

$$\therefore \quad x - x_1 = x_2 - x \qquad\qquad y - y_1 = y_2 - y$$

$$2x = x_1 + x_2 \qquad\qquad 2y = y_1 + y_2$$

$$x = \frac{x_1 + x_2}{2} \qquad\qquad y = \frac{y_1 + y_2}{2}$$

We have proved the following result.

The coordinates of the midpoint P of the line segment AB joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

That is, we take the average of the x-coordinates and the average of the y-coordinates.

Example 1

Determine the midpoint of the line segment joining $A(2, 6)$ with $B(-3, -4)$.

Solution

The midpoint of line segment AB has coordinates

$$\left(\frac{2 + (-3)}{2}, \frac{6 + (-4)}{2}\right) = \left(-\frac{1}{2}, 1\right)$$

Explanation

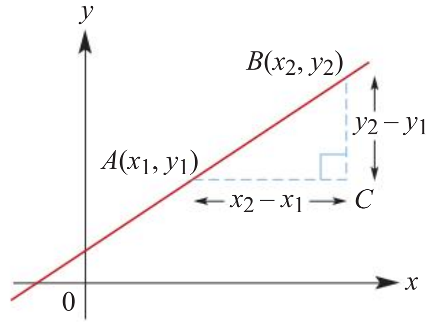
The coordinates of the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

The distance between two points

The distance between given points $A(x_1, y_1)$ and $B(x_2, y_2)$ can be found by applying Pythagoras' theorem to the triangle ABC:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \end{aligned}$$



Therefore, the distance between the two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

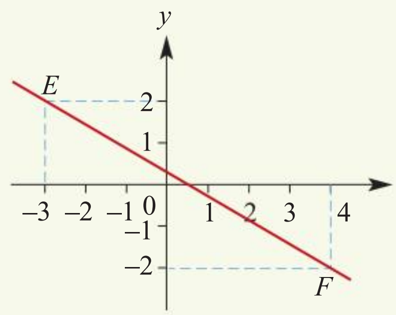
Example 2

Calculate the distance EF if E is $(-3, 2)$ and F is $(4, -2)$.

Solution

$$\begin{aligned} EF &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(4 - (-3))^2 + (-2 - 2)^2} \\ &= \sqrt{7^2 + (-4)^2} \\ &= \sqrt{65} \\ &= 8.06 \quad \text{(to two decimal places)} \end{aligned}$$

Explanation



 ISBN 978-1-009-53719-3

Summary 2A

- The coordinates of the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ are
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
- The distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Skill-sheet

Exercise 2A

SF

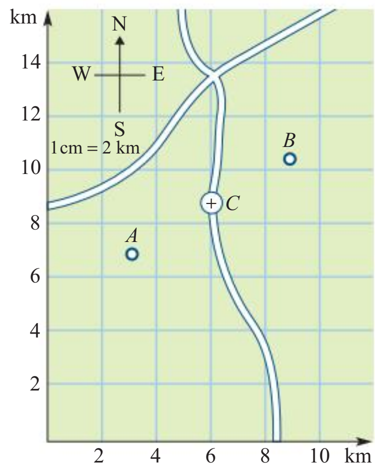
1 Determine the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

Example 1

a $A(2, 12)$, $B(8, 4)$ **b** $A(-3, 5)$, $B(4, -4)$
c $A(-1.6, 3.4)$, $B(4.8, -2)$ **d** $A(3.6, -2.8)$, $B(-5, 4.5)$

2 Determine the midpoints of each of the sides of a triangle ABC, where A is $(1, 1)$, B is $(5, 5)$ and C is $(11, 2)$.

3 The secretary of a motocross club wants to organise two meetings on the same weekend. One is a hill climb starting from point $A(3.1, 7.1)$ and the other is a circuit event with the start at $B(8.9, 10.5)$, as shown on the map. Only one ambulance can be provided. The ambulance can be called up by radio, so it is decided to keep it at C, halfway between A and B. What are the coordinates of C?



4 If M is the midpoint of XY, determine the coordinates of Y when X and M have the following coordinates:

a $X(-4, 2)$, $M(0, 3)$ **b** $X(-1, -3)$, $M(0.5, -1.6)$
c $X(6, -3)$, $M(2, 1)$ **d** $X(4, -3)$, $M(0, -3)$

5 Determine the coordinates of the midpoint of the line segment joining $(1, 4)$ and (a, b), in terms of a and b. If $(5, -1)$ is the midpoint, determine the values of a and b.

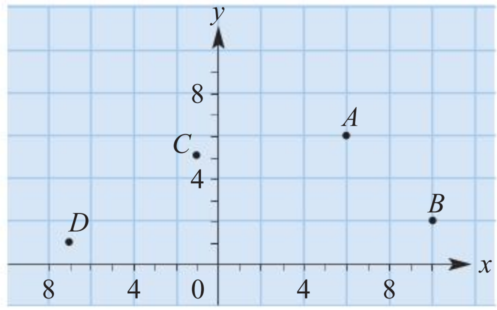
Example 2

6 Determine the distance between each of the following (correct to two decimal places):

a $(3, 6)$ and $(-4, 5)$ **b** $(4, 1)$ and $(5, -3)$
c $(-2, -3)$ and $(-5, -8)$ **d** $(6, 4)$ and $(-7, 4)$

7 Calculate the perimeter of a triangle with vertices $(-3, -4)$, $(1, 5)$ and $(7, -2)$.

8 The diagram shows the four points $A(6, 6)$, $B(10, 2)$, $C(-1, 5)$ and $D(-7, 1)$. If the midpoint of AB is P and the midpoint of CD is M, calculate the distance PM to two decimal places.



9 There is an off-shore oil drilling platform in Bass Strait situated at $D(0, 6)$, where 1 unit = 5 km. Pipes for this oil drill come ashore at $M(-6, 1)$ and $N(3, -1)$. Assuming the pipelines are straight, which is the shorter, DM or DN?

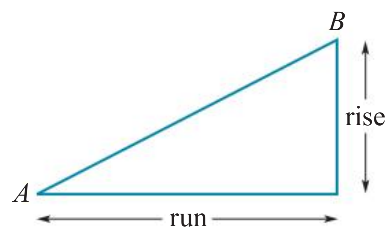
2B The gradient of a straight line

Learning intentions

- To be able to determine the gradient of a straight line.
- To be able to determine the angle a straight line makes with the x-axis.

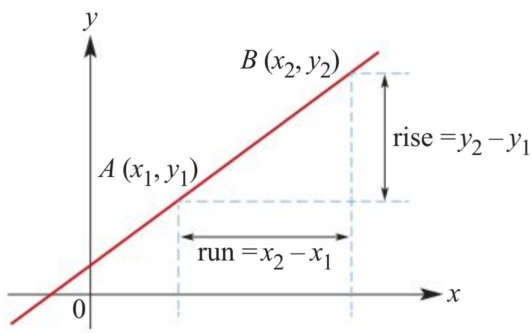
Through any two points it is only possible to draw a single straight line. Therefore a straight line is defined by any two points on the line.

In coordinate geometry the standard way to define the **gradient of a line segment** AB is $\dfrac{\text{rise}}{\text{run}}$ where:



- rise is the change in the y-values as you move from A to B
- run is the change in the x-values as you move from A to B.

The **gradient of a line** is defined to be the gradient of any segment within the line. This definition depends on the fact that any two segments of a line have the same gradient. Hence given any two points on the line, $A(x_1, y_1)$ and $B(x_2, y_2)$, the gradient of the line can be found. The symbol used for gradient is m.



$$\text{Gradient } m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

Note that since

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

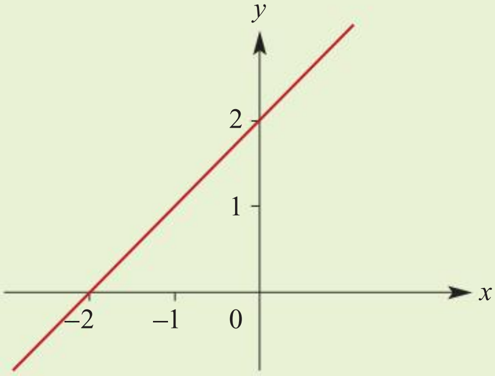
it does not matter which point we take as the first and which point we take as the second.

 ISBN 978-1-009-53719-3

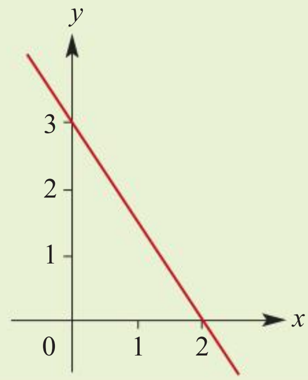
Example 3

Determine the gradient of each line:

a



b



Solution

a Gradient $m = \dfrac{2-0}{0-(-2)}$

$= \dfrac{2}{2}$

$= 1$

Explanation

Let $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, 2)$.

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

b Gradient $m = \dfrac{0-3}{2-0}$

$= -\dfrac{3}{2}$

Let $(x_1, y_1) = (0, 3)$ and $(x_2, y_2) = (2, 0)$.

Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Notes:

- The gradient of a line that slopes upwards from left to right is **positive**, as illustrated in Example 3a.
- The gradient of a line that slopes downwards from left to right is **negative**, as illustrated in Example 3b.
- The gradient of a **horizontal line** (parallel to the x-axis) is zero, since $y_2 - y_1 = 0$.
- The gradient of a **vertical line** (parallel to the y-axis) is undefined, since $x_2 - x_1 = 0$.

Example 4

Determine the gradient of the line that passes through the points $(1, 6)$ and $(-3, 7)$.

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{7-6}{-3-1}$$

$$= -\frac{1}{4}$$

Explanation

The gradient can also be found using

$$m = \frac{y_1 - y_2}{x_1 - x_2}$$

$$= \frac{6-7}{1-(-3)} = -\frac{1}{4}$$

 ISBN 978-1-009-53719-3

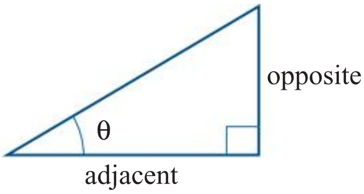
The tangent of the angle of slope

We will look first at the case when the gradient is positive and then when the gradient is negative.

Positive gradient

From Year 10 you will be familiar with the trigonometric ratio

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



Consider a straight line with positive gradient. The line forms an acute angle, θ, with the positive direction of the x-axis.

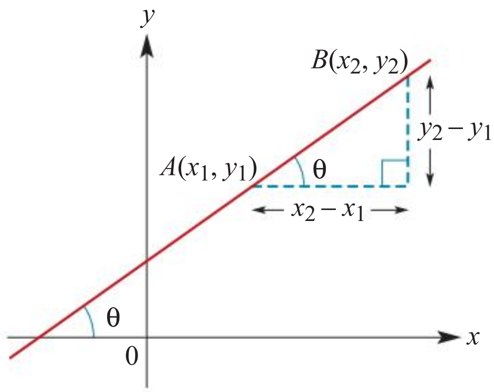
The gradient, m, of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2)$$

From the diagram, it follows that

$$m = \tan\theta$$

where θ is the angle that the line makes with the positive direction of the x-axis.



Example 5

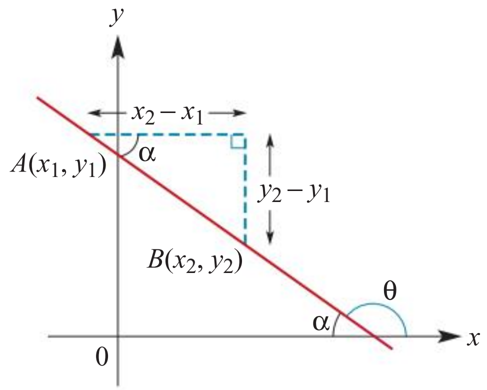
Determine the gradient of the line passing through the points (3, 2) and (5, 7) and the angle θ that the line makes with the positive direction of the x-axis.

Solution	Explanation
$m = \frac{7-2}{5-3}$ $= \frac{5}{2}$	The gradient is given by $m = \frac{y_2 - y_1}{x_2 - x_1}$
$\tan\theta = \frac{5}{2}$	The gradient is positive and so the angle θ is acute.
$\therefore\ \theta = 68.1986\ldots^\circ$ $= 68.20^\circ$ correct to two decimal places.	The angle can be found with a calculator using inverse tan.

 ISBN 978-1-009-53719-3 Cambridge University Press

Negative gradient

Now consider a line with negative gradient. The line forms an acute angle α with the negative direction of the x-axis, and an obtuse angle θ with the positive direction of the x-axis. We have $\theta = 180° - \alpha$.



From the diagram, we see that the gradient satisfies

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{x_2 - x_1} = -\tan\alpha$$

From your work on trigonometric functions in Year 10 you may recall that

$$\tan\theta = \tan(180° - \alpha) = -\tan\alpha$$

Thus the gradient satisfies

$$\boxed{m = -\tan\alpha = \tan\theta}$$

Example 6

Determine the gradient of the line passing through the points $(5, -3)$ and $(-1, 5)$ and the angle θ that the line makes with the positive direction of the x-axis.

Solution

$$m = \frac{5 - (-3)}{-1 - 5}$$
$$= -\frac{4}{3}$$

$$\tan\theta = -\frac{4}{3}$$

$$\therefore \quad \theta = 180° - (53.130\ldots°)$$
$$= 126.87°$$

correct to two decimal places.

Explanation

The gradient is negative and so the angle θ between the line and the positive direction of the x-axis is obtuse.

You first use your calculator to determine the adjacent supplementary angle α, which is acute. You do this by finding the inverse tangent of $\frac{4}{3}$. The magnitude of this angle is $53.130\ldots°$.

You subtract this magnitude from 180° to obtain θ.

Summary 2B

- The gradient of a line segment AB joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

  $$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{provided } x_1 \neq x_2.$$

- The gradient of a line is defined as the gradient of any line segment of that line.
- A horizontal line has gradient zero.
- A vertical line does not have a gradient.

 ISBN 978-1-009-53719-3

- Let θ be the angle that a line makes with the positive direction of the x-axis.
  - The gradient m is positive if and only if θ is acute.
  - The gradient m is negative if and only if θ is obtuse.
  - If θ is acute or obtuse, then $m = \tan\theta$.
  - If $\theta = 0°$, then $m = \tan 0° = 0$.
  - If $\theta = 90°$, then the gradient is not defined.

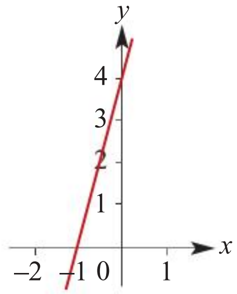
Exercise 2B

SF

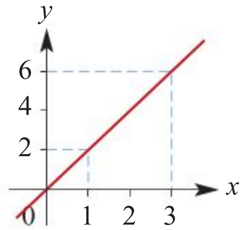
Example 3

1 Calculate the gradient of each of the following lines:

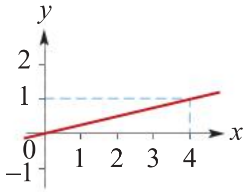
a



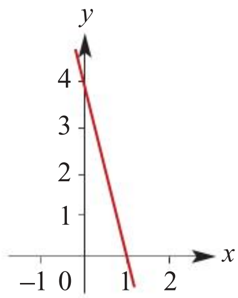
b



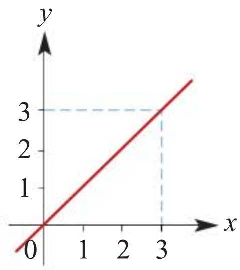
c



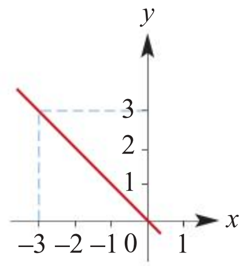
d



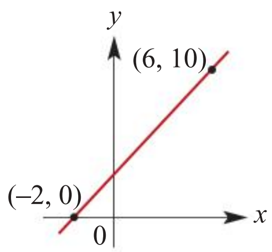
e



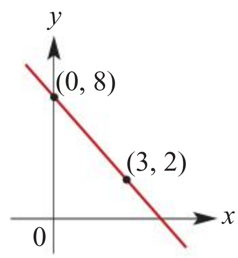
f



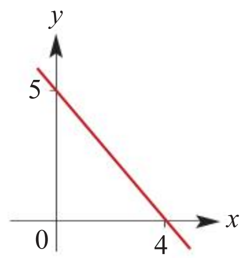
g



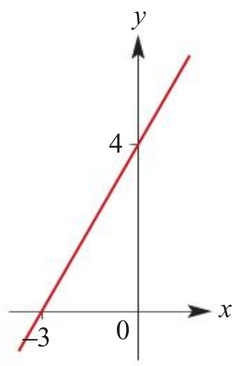
h



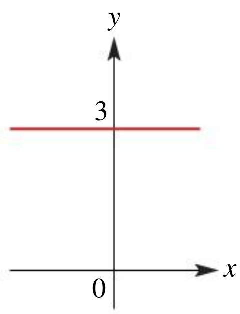
i



j



k



 ISBN 978-1-009-53719-3

SF

2 Sketch a graph of a line with gradient 1.

3 Sketch a graph of a line with gradient 0 which passes through the point $(1, 6)$.

Example 4

4 For each of the following, determine the gradient of the line that passes through the two points with the given coordinates:

a $(6, 3),\ (2, 4)$
b $(-3, 4),\ (1, -6)$
c $(6, 7),\ (11, -3)$
d $(5, 8),\ (6, 0)$
e $(6, 0),\ (-6, 0)$
f $(0, -6),\ (-6, 0)$
g $(3, 9),\ (4, 16)$
h $(5, 25),\ (6, 36)$
i $(-5, 25),\ (-8, 64)$
j $(1, 1),\ (10, 100)$
k $(1, 1),\ (10, 1000)$
l $(5, 125),\ (4, 64)$

5 a Determine the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(3a, 6a)$.
b Determine the gradient of the straight line that passes through the points with coordinates $(5a, 2a)$ and $(5b, 2b)$.

6 a A line has gradient 6 and passes through the points with coordinates $(-1, 6)$ and $(7, a)$. Determine the value of a.
b A line has gradient -6 and passes through the points with coordinates $(1, 6)$ and $(b, 7)$. Determine the value of b.

7 Determine the angle, correct to two decimal places, that the lines joining the given points make with the positive direction of the x-axis:

a $(0, 3),\ (-3, 0)$
b $(0, -4),\ (4, 0)$
c $(0, 2),\ (-4, 0)$
d $(0, -5),\ (-5, 0)$

Example 5, 6

8 Determine the angle, correct to two decimal places, that the lines joining the given points make with the positive direction of the x-axis:

a $(-4, -2),\ (6, 8)$
b $(2, 6),\ (-2, 4)$
c $(-3, 4),\ (6, 1)$
d $(-4, -3),\ (2, 4)$
e $(3b, a),\ (3a, b)$
f $(c, b),\ (b, c)$

9 Determine the gradient of a straight line which is:

a inclined at an angle of 45° to the positive direction of the x-axis
b inclined at an angle of 135° to the positive direction of the x-axis
c inclined at an angle of 60° to the positive direction of the x-axis
d inclined at an angle of 120° to the positive direction of the x-axis.

2C The equation of a straight line

Learning intentions

- To be able to determine the equation of a straight line.

In this section we discuss different ways of determining the equation of a straight line. In general two independent pieces of information are required. The following given information is considered:

- gradient and y-axis intercept
- gradient and a point
- two points.

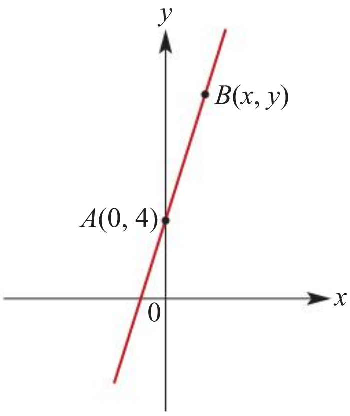
Sketching straight lines given the equation is discussed in Section 2D.

Gradient–intercept form of the equation of a straight line

We first consider an example before stating the general result. The argument in the general case is exactly the same.

The line $y = 2x + 4$

Consider the line with gradient 2 and y-axis intercept 4. This line passes through the point $A(0, 4)$. Let $B(x, y)$ be any other point on the line.



Gradient of line segment $AB = \dfrac{y-4}{x-0} = \dfrac{y-4}{x}$

We know that the gradient of the line is 2. Therefore

$$\frac{y-4}{x} = 2$$

$$y - 4 = 2x$$

$$y = 2x + 4$$

So the coordinates (x, y) satisfy the equation $y = 2x + 4$.

Conversely, if a point $B(x, y)$ in the plane satisfies $y = 2x + 4$, then

$$\frac{y-4}{x} = 2$$

Thus we know that the gradient of the line segment joining point B to the point $A(0, 4)$ is 2. Therefore the line through $A(0, 4)$ and $B(x, y)$ has gradient 2 and y-axis intercept 4.

The line $y = mx + c$

In the same way as for the line $y = 2x + 4$, we can show that:

- The line with gradient m and y-axis intercept c has equation $y = mx + c$.
- Conversely, the line with equation $y = mx + c$ has gradient m and y-axis intercept c.

This is called the **gradient–intercept form**.

 ISBN 978-1-009-53719-3

Example 7

Determine the gradient and y-axis intercept of the line $y = 3x - 4$.

Solution

The gradient is 3 and the y-axis intercept is -4.

Explanation

$m = 3$ and $c = -4$

Example 8

Determine the equation of the line with gradient -3 and y-axis intercept 5.

Solution

$y = -3x + 5$

Explanation

$y = mx + c$

Example 9

State the gradient and y-axis intercept of the line $3y + 6x = 9$.

Solution

$$3y + 6x = 9$$
$$3y = 9 - 6x$$
$$y = \frac{9 - 6x}{3}$$
$$y = 3 - 2x$$
$$\text{i.e.}\quad y = -2x + 3$$

Therefore $m = -2$ and $c = 3$.

Explanation

Rearrange the equation $3y + 6x = 9$ into gradient–intercept form.

Now the gradient and y-axis intercept can be read directly from the equation.

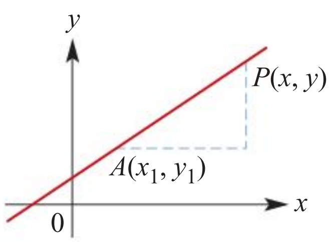
Point–gradient form of the equation of a straight line

If $A(x_1, y_1)$ is a point on a line with gradient m and $P(x, y)$ is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m$$

and so we have

$$y - y_1 = m(x - x_1)$$



The **point–gradient form** of the equation of a straight line is

$$y - y_1 = m(x - x_1)$$

where (x_1, y_1) is a point on the line and m is the gradient.

 ISBN 978-1-009-53719-3

Example 10

Determine the equation of the line which passes through the point $(-1, 3)$ and has gradient 4.

Solution

Method 1

$(x_1, y_1) = (-1, 3)$ and $m = 4$.

We use the equation $y - y_1 = m(x - x_1)$. (It is not necessary to work from first principles every time.)

The equation is

$$y - 3 = 4(x - (-1))$$
$$y = 4(x + 1) + 3$$
$$= 4x + 4 + 3$$
$$= 4x + 7$$

Rearrange to make y the subject and write the equation in the form $y = mx + c$.

Method 2

Since $m = 4$, the equation is of the form $y = 4x + c$.

We can also use the equation $y = mx + c$ and determine the value of c. The gradient is 4.

When $x = -1$, $y = 3$.

The point $(-1, 3)$ lies on the line.

Therefore

$$3 = 4 \times (-1) + c$$
$$7 = c$$

Solve for c.

The equation is $y = 4x + 7$.

Example 11

Determine the equation of the line that passes through the point $(3, 2)$ and has a gradient of -2.

Solution

$$y - 2 = -2(x - 3)$$
$$y - 2 = -2x + 6$$
$$y = -2x + 8$$

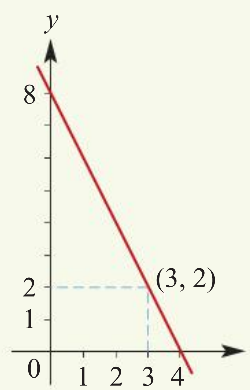
The equation is

$$y = -2x + 8$$

which could also be expressed as

$$2x + y - 8 = 0$$

Explanation



 ISBN 978-1-009-53719-3 Cambridge University Press

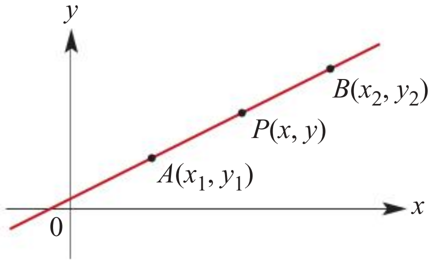
A line through two points

To determine the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first determine the gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

and then use the point–gradient form

$$y - y_1 = m(x - x_1)$$



We can also determine the equation directly by taking the point $P(x, y)$ and noting that

$$\frac{y - y_1}{x - x_1} = m$$

Example 12

Determine the equation of the straight line passing through the points $(1, -2)$ and $(3, 2)$.

Solution

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2 - (-2)}{3 - 1} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

$$\therefore \quad 2 = \frac{y - (-2)}{x - 1}$$

$$2x - 2 = y + 2$$

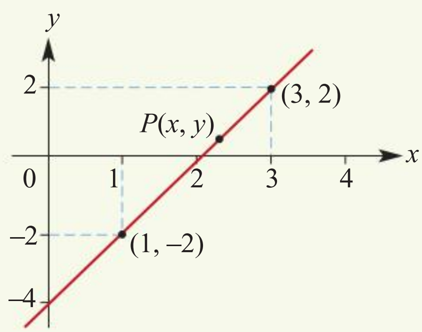
$$\therefore \quad y = 2x - 4$$

Explanation

First determine the gradient m and then use

$$\frac{y - y_1}{x - x_1} = m$$

Choose $(x_1, y_1) = (1, -2)$.



Example 13

Determine the equation of the straight line with y-axis intercept -3 which passes through the point with coordinates $(1, 10)$.

Solution

The gradient is

$$m = \frac{10 - (-3)}{1 - 0} = 13$$

Therefore the equation is $y = 13x - 3$.

Explanation

Determine the gradient using $(x_1, y_1) = (0, -3)$ and $(x_2, y_2) = (1, 10)$.

The general equation of a line with y-axis intercept -3 is $y = mx - 3$.

Two intercepts

A special case of finding the equation of a line given the coordinates of two points is when the intercept with each axis is known, that is, finding the line through $(a, 0)$ and $(0, b)$, where $a, b \neq 0$.

In this case, the gradient is

$$m = \frac{b - 0}{0 - a} = -\frac{b}{a}$$

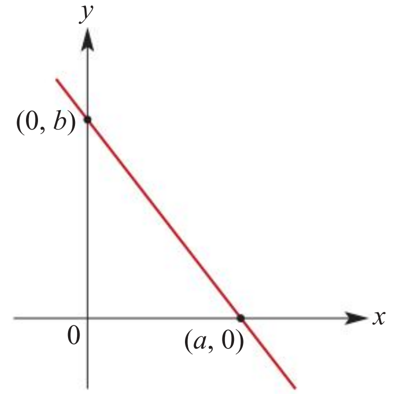
Thus the equation of the line is

$$y - 0 = -\frac{b}{a}(x - a)$$

Multiplying both sides of the equation by a gives

$$ay + bx = ab$$

Dividing both sides of the equation by ab gives the following:



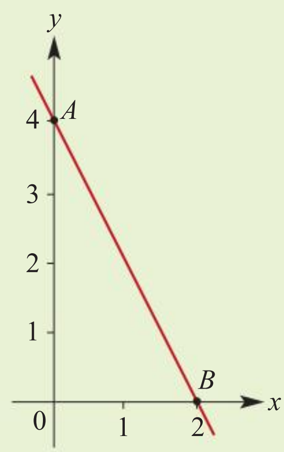
The **intercept form** of the equation of a straight line is

$$\frac{x}{a} + \frac{y}{b} = 1$$

where a and b are the x-axis intercept and y-axis intercept respectively.

Example 14

Determine the equation of the line shown in the graph.



Solution

The intercept form of the equation is

$$\frac{x}{2} + \frac{y}{4} = 1$$

Multiply both sides by 4:

$$2x + y = 4$$

The equation of the line is $y = -2x + 4$.

Explanation

The coordinates of A and B are $(0, 4)$ and $(2, 0)$.

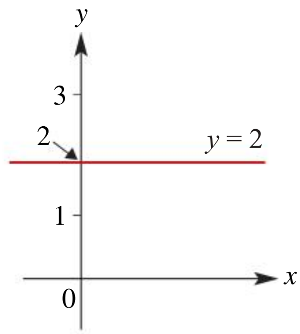
 ISBN 978-1-009-53719-3

Vertical and horizontal lines

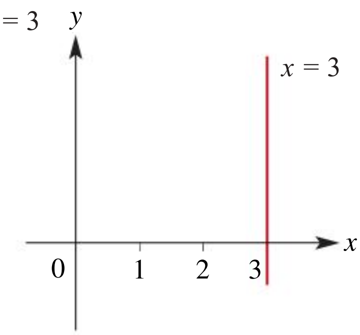
If a line is **horizontal**, then its gradient $m = 0$ and its equation is simply $\boldsymbol{y = c}$, where c is the y-axis intercept.

If a line is **vertical**, then its gradient is undefined and its equation is $\boldsymbol{x = a}$, where a is the x-axis intercept.

Equation $y = 2$



Equation $x = 3$



Note that the equation of a vertical line is not of the form $y = mx + c$.

General form of the equation of a straight line

We have seen that all points on the line through two given points satisfy an equation of the form $mx + ny + p = 0$, with m and n not both 0. Conversely, any 'linear equation' $mx + ny + p = 0$ is the equation of a (straight) line. This is called the **general form** of the equation of a line.

Summary 2C

- **Gradient–intercept form:** The line with gradient m and y-axis intercept c has equation $y = mx + c$. Conversely, the line with equation $y = mx + c$ has gradient m and y-axis intercept c.
- **Point–gradient form:** If (x_1, y_1) is a point on a line with gradient m and (x, y) is any other point on the line, then
$$\frac{y - y_1}{x - x_1} = m$$
which can be written as
$$y - y_1 = m(x - x_1)$$
- **Two points:** To determine the equation of the line through two given points (x_1, y_1) and (x_2, y_2), first determine the gradient
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
and then use the point–gradient form
$$y - y_1 = m(x - x_1)$$

 ISBN 978-1-009-53719-3

- **Intercept form:** If a line has x-axis intercept a and y-axis intercept b, the equation of the line is
$$\frac{x}{a} + \frac{y}{b} = 1$$
- **Horizontal line:** A line parallel to the x-axis through the point (a, c) has equation $y = c$.
- **Vertical line:** A line parallel to the y-axis through the point (a, c) has equation $x = a$.
- **General form:** Every straight line satisfies an equation of the form $mx + ny + p = 0$, with m and n not both 0. Conversely, any 'linear equation' $mx + ny + p = 0$ is the equation of a straight line.

Skill-sheet

Exercise 2C

SF

1 State the gradient and y-axis intercept of the graph of each equation:

Example 7

a $y = 3x + 6$ **b** $y = -6x + 7$ **c** $y = 3x - 6$ **d** $y = -x - 4$

Example 8

2 **a** Determine the equation of the straight line with gradient 3 and y-axis intercept 5.
b Determine the equation of the straight line with gradient -4 and y-axis intercept 6.
c Determine the equation of the straight line with gradient 3 and y-axis intercept -4.

Example 9

3 State the gradient and y-axis intercept of the graph of each equation:
a $3x - y = 6$ **b** $4x - 2y = 8$ **c** $5x - 10y = 20$ **d** $2x - 6y = 10$

4 Express in gradient–intercept form and hence state the gradient and y-axis intercept of each of the following linear relations:
a $2x - y = 9$ **b** $3x + 4y = 10$ **c** $-x - 3y = 6$ **d** $5x - 2y = 4$

Example 10

5 **a** Determine the equation of the straight line that has gradient 3 and passes through the point with coordinates $(6, 7)$.

Example 11

b Determine the equation of the straight line that has gradient -2 and passes through the point with coordinates $(1, 7)$.

Example 12

6 Determine the equations of the straight lines passing through the following pairs of points. (Express your answer in gradient–intercept form.)
a $(-1, 4)$, $(2, 3)$ **b** $(0, 4)$, $(5, -3)$ **c** $(3, -2)$, $(4, -4)$ **d** $(5, -2)$, $(8, 9)$

Example 13

7 For the straight line that has y-axis intercept 6 and passes through the point with coordinates $(1, 8)$, determine:
a the gradient **b** the equation

8 Determine the equation of the straight line that passes through the point $(1, 6)$ and has gradient:
a 2 **b** -2

 ISBN 978-1-009-53719-3 Cambridge University Press

SF

9 Write, in the form $y = mx + c$, the equations of the lines which have the given gradient and pass through the given point:

a $m = 2; \quad (-1, 4)$ **b** $m = -2; \quad (0, 4)$ **c** $m = -5; \quad (3, 0)$

Example 14

10 Determine equations defining the lines which pass through the following pairs of points:

a $(0, 4), \ (6, 0)$ **b** $(-3, 0), \ (0, -6)$

c $(0, 4), \ (4, 0)$ **d** $(2, 0), \ (0, 3)$

11 Determine the equations, in the form $y = mx + c$, of the lines which pass through the following pairs of points:

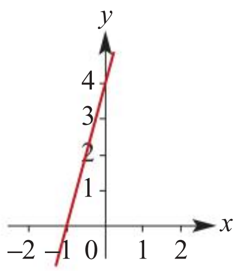
a $(0, 4), \ (3, 6)$ **b** $(1, 0), \ (4, 2)$

c $(-3, 0), \ (3, 3)$ **d** $(-2, 3), \ (4, 0)$

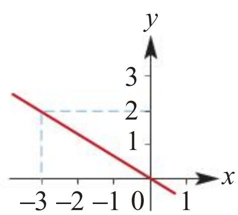
e $(-1.5, 2), \ (4.5, 8)$ **f** $(-3, 1.75), \ (4.5, -2)$

12 Determine the equation of each of the following lines:

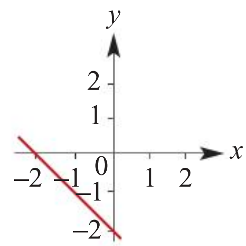
a



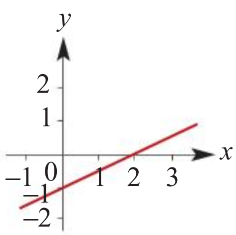
b



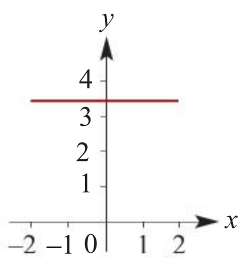
c



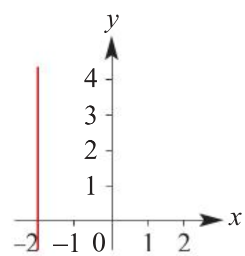
d



e



f



13 Do the points $P(1, -3)$, $Q(2, 1)$ and $R(2\frac{1}{2}, 3)$ lie on the same straight line?

14 For which of the following does the line pass through the origin?

a $y + x = 1$ **b** $y + 2x = 2(x + 1)$ **c** $x + y = 0$ **d** $x - y = 1$

15 **a** Determine the equation of the line that is parallel to the y-axis and passes through the point with coordinates $(4, 7)$.

b Determine the equation of the line that is parallel to the x-axis and passes through the point with coordinates $(-4, 11)$.

c Determine the equation of the line that is parallel to the y-axis and passes through the point with coordinates $(11, -7)$.

d Determine the equation of the line that is parallel to the x-axis and passes through the point with coordinates $(5, -1)$.

 ISBN 978-1-009-53719-3 Cambridge University Press

2D Graphing straight lines

Learning intentions

- To be able to sketch the graph of a straight line.

In the previous section we discussed methods of finding the equation of a straight line given suitable information. In this section we look at sketching a straight line from an equation. To sketch the graph we need to derive the coordinates of two points on the line. A convenient way to sketch graphs of straight lines is to plot the two axis intercepts.

Example 15

Sketch the graph of $2x + 4y = 10$.

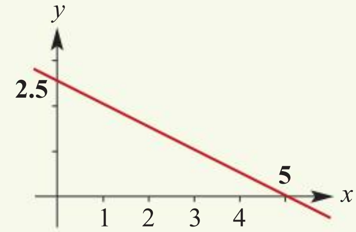
Solution

x-axis intercept ($y = 0$): $\quad 2x + 4(0) = 10$

$$x = 5$$

y-axis intercept ($x = 0$): $\quad 2(0) + 4y = 10$

$$y = 2.5$$



Example 16

Sketch the graph of $y = 2x - 6$ by first finding the intercepts.

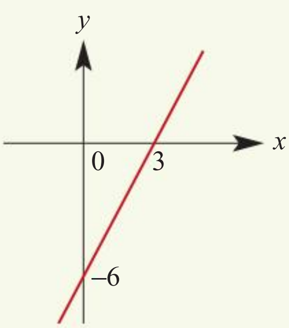
Solution

x-axis intercept ($y = 0$): $\quad 0 = 2x - 6$

$$x = 3$$

y-axis intercept ($x = 0$): $\quad y = 2(0) - 6$

$$y = -6$$



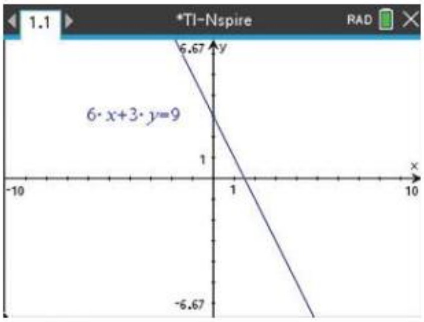
Note: You can also obtain the y-axis intercept directly from the equation.

TI-84

Using the TI-Nspire CX non-CAS

To plot the graph of $6x + 3y = 9$:

- Open a **Graphs** application: press on and select the **Graphs** icon, or use ctrl I and select **Add Graphs**.
- Equations of the form $a \cdot x + b \cdot y = c$ can be entered directly using menu > **Graph Entry/Edit** > **Relation**. Enter as $6x + 3y = 9$.

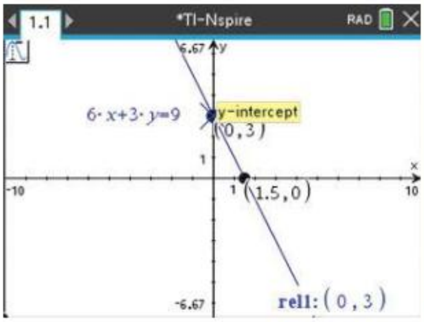


Note: The window settings (menu > **Window/Zoom** > **Window Settings**) will have to be changed if the axis intercepts do not appear on the screen.

 ISBN 978-1-009-53719-3

To determine the axis intercepts:

- Select menu > **Trace** > **Graph Trace**.
- Move the cursor to an axis and press enter. Then move to the other axis and press enter.
- Press esc to exit the **Graph Trace** tool.

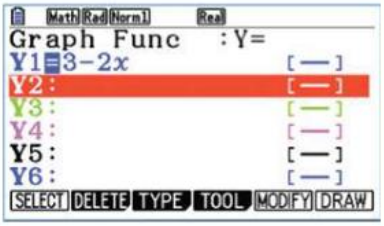


Using the Casio

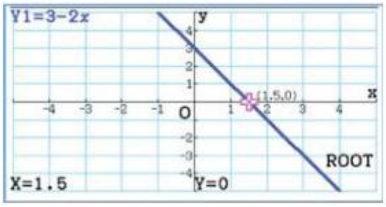
To plot the graph of $6x + 3y = 9$:

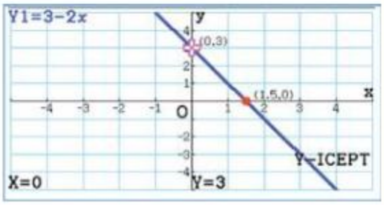
- Press MENU 5 to select **Graph** mode.
- Transpose $6x + 3y = 9$ to make y the subject, and enter the equation in $Y1$:

  3 − 2 X,θ,T EXE



- Select **Draw** F6.
- To determine the x-axis intercept, go to **G-Solve** SHIFT F5 and select **Root** F1.
- To determine the y-axis intercept, go to **G-Solve** SHIFT F5 and select **y-Intercept** F4.





From Section 2B we know that the gradient of a line is the tangent of the angle of slope (that is, the angle formed by the line with the positive direction of the x-axis).

Example 17

For each of the following lines, determine the magnitude of the angle θ (correct to two decimal places) that the line makes with the positive direction of the x-axis:

a $y = 2x + 3$ **b** $3y = 3x - 6$

c $y = -0.3x + 1.5$

Solution

a

$$y = 2x + 3$$

Gradient $= 2$

Hence $\tan\theta = 2$

Therefore $\theta = 63.43^\circ$ correct to two decimal places

b

$$3y = 3x - 6$$

$$y = x - 2$$

Gradient $= 1$

Hence $\tan\theta = 1$

Therefore $\theta = 45^\circ$

c $y = -0.3x + 1.5$

Gradient $= -0.3$

Hence $\tan\theta = -0.3$

Therefore $\theta = (180 - 16.699\ldots)^\circ$

$= 163.30^\circ$ correct to two decimal places

Summary 2D

- The most practical way to sketch a straight line is to plot two points known to be on the required line and draw the line through them.
  - Two important points are the intercept with the x-axis and the intercept with the y-axis. These are the best two points to use in order to sketch a line that does not pass through the origin and is not parallel to one of the axes.
  - To sketch a line with equation of the form $y = mx$, plot one other point on the line besides the origin.
- The gradient m of a line $y = mx + c$ is equal to $\tan\theta$, where θ is the angle of slope measured between the line and the positive direction of the x-axis.

Exercise 2D

SF

1 For each of the following, give the coordinates of the axis intercepts:

a $x + y = 4$ **b** $x - y = 4$ **c** $-x - y = 6$ **d** $y - x = 8$

Example 15

2 Sketch the graphs of each of the following linear relations:

a $2x - 3y = 12$ **b** $x - 4y = 8$ **c** $-3x + 4y = 24$

d $-5x + 2y = 20$ **e** $4x - 3y = 15$ **f** $7x - 2y = 15$

Example 16

3 For each of the following, sketch the graph by first determining the axis intercepts:

a $y = x - 1$ **b** $y = x + 2$ **c** $y = 2x - 4$

4 Sketch the graphs of each of the following by first determining the axis intercepts:

a $y = 2x - 10$ **b** $y = 3x - 9$ **c** $y = 5x + 10$ **d** $y = -2x + 10$

5 Sketch the graphs of each of the following:

a $y = x + 2$ **b** $y = -x + 2$ **c** $y = 2x + 1$ **d** $y = -2x + 1$

6 Sketch the graphs of each of the following:

a $x + y = 1$ **b** $x - y = 1$ **c** $y - x = 1$ **d** $-x - y = 1$

7 Sketch the graphs of each of the following:

a $y = x + 3$ **b** $y = 3x + 1$ **c** $y = 4 - \frac{1}{2}x$ **d** $y = 3x - 2$

e $4y + 2x = 12$ **f** $3x + 6y = 12$ **g** $4y - 6x = 24$ **h** $8x - 3y = 24$

 ISBN 978-1-009-53719-3

SF

8 Sketch the graphs of each of the following:

a $y = 3$ **b** $x = -2$ **c** $y = -2$ **d** $x = 5$

Example 17

9 Determine the magnitude of the angle, correct to two decimal places, made by each of the following with the positive direction of the x-axis:

a $y = x$ **b** $y = -x$ **c** $y = x + 1$ **d** $x + y = 1$
e $y = 2x$ **f** $y = -2x$

10 Determine the magnitude of the angle, correct to two decimal places, made by each of the following with the positive direction of the x-axis:

a $y = 3x + 2$ **b** $2y = -2x + 1$ **c** $2y - 2x = 6$ **d** $3y + x = 7$

11 A straight line has equation $y = 3x - 4$. The points with coordinates $(0, a)$, $(b, 0)$, $(1, d)$ and $(e, 10)$ lie on the line. Determine the values of a, b, d and e.

2E Parallel and perpendicular lines

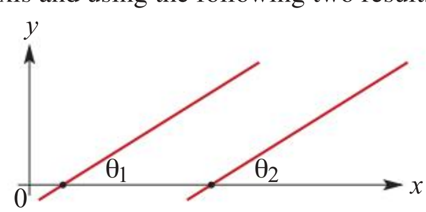
Learning intentions

- To be able to determine if two lines are parallel.
- To be able to determine if two lines are perpendicular.

Parallel lines

- Two non-vertical lines are **parallel** if they have the same gradient.
- Conversely, if two non-vertical lines are parallel, then they have the same gradient.

This is easily proved through considering the angles of inclination of such lines to the positive direction of the x-axis and using the following two results:



1 Two non-vertical lines are parallel if and only if the corresponding angles θ_1 and θ_2 formed by the lines and the positive direction of the x-axis are equal.

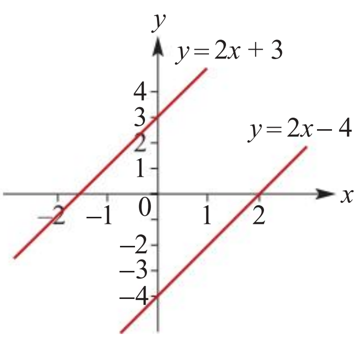
2 If two angles θ_1 and θ_2 are acute, obtuse or zero, then $\tan\theta_1 = \tan\theta_2$ implies $\theta_1 = \theta_2$.

For example, consider the lines

$y = 2x + 3$

$y = 2x - 4$

Both lines have gradient 2, and so they are parallel.



Perpendicular lines

We prove that two lines are **perpendicular** if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical).

> Two lines with gradients m_1 and m_2 (both non-zero) are perpendicular if and only if $m_1 m_2 = -1$.

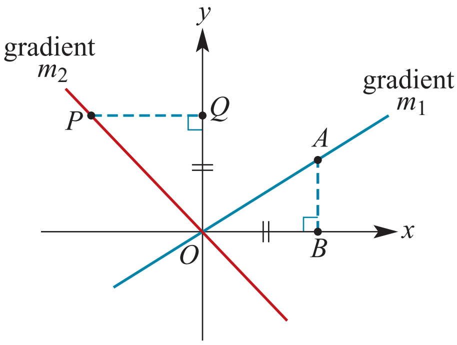
Proof Initially we consider the case where the two lines intersect at the origin.

Step 1 Draw two lines passing through the origin with one of the lines having positive gradient, m_1, and the other negative gradient, m_2. Form right-angled triangles OPQ and OAB with $OQ = OB$.

$$\text{Gradient } m_1 = \frac{AB}{BO}$$

$$\text{Gradient } m_2 = -\frac{OQ}{PQ}$$

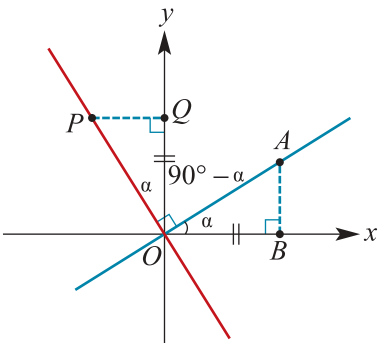
$$\begin{aligned}\text{Product } m_1 m_2 &= -\frac{OQ}{PQ} \times \frac{AB}{BO}\\ &= -\frac{OQ}{PQ} \times \frac{AB}{OQ}\\ &= -\frac{AB}{PQ}\end{aligned}$$



Step 2 We now prove: *If two lines passing through the origin are perpendicular, then the product of their gradients is* -1.

If the lines are perpendicular, then $\angle POQ = \angle AOB$. Therefore triangles OPQ and OAB are congruent. So $PQ = AB$ and therefore the product of the gradients is

$$m_1 m_2 = -\frac{AB}{PQ} = -\frac{AB}{AB} = -1$$



Step 3 We next prove the converse: *For two lines passing through the origin, if the product of their gradients is* -1 *then the lines are perpendicular.*

If the product $m_1 m_2 = -1$, then $AB = PQ$, which implies that the triangles OAB and OPQ are congruent. Therefore $\angle POQ = \angle AOB$ and so $\angle AOP = 90°$.

Step 4 If we are given two lines anywhere in the plane, we can draw lines through the origin parallel to the original two lines. The slopes of the new lines are the same. So the result holds for lines that do not necessarily pass through the origin.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Example 18

Determine the equation of the straight line which passes through $(1, 2)$ and is:

a parallel to the line with equation $2x - y = 4$

b perpendicular to the line with equation $2x - y = 4$.

Solution

The equation $2x - y = 4$ is equivalent to $y = 2x - 4$. The line $y = 2x - 4$ has gradient 2.

a The required line passes through $(1, 2)$ and has gradient 2. Hence

$$y - 2 = 2(x - 1)$$

Therefore the line has equation $y = 2x$.

b The required line passes through $(1, 2)$ and has gradient $-\frac{1}{2}$. Hence

$$y - 2 = -\frac{1}{2}(x - 1)$$

Therefore $2y - 4 = -x + 1$ and equivalently $2y + x = 5$.

The line has equation $2y + x = 5$.

Explanation

a A line parallel to $y = 2x - 4$ has gradient 2.

We use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (1, 2)$ and $m = 2$.

b A line perpendicular to a line with gradient m has gradient $-\frac{1}{m}$ (provided $m \neq 0$).

We use $y - y_1 = m(x - x_1)$ where $(x_1, y_1) = (1, 2)$ and $m = -\frac{1}{2}$.

Example 19

The coordinates of the vertices of a triangle ABC are $A(0, -1)$, $B(2, 3)$ and $C(3, -2\frac{1}{2})$. Show that the side AB is perpendicular to the side AC.

Solution

Let m_1 be the gradient of the line AB and let m_2 be the gradient of the line AC.

$$m_1 = \frac{3 - (-1)}{2 - 0} = 2$$

$$m_2 = \frac{-2\frac{1}{2} - (-1)}{3 - 0} = \frac{-1\frac{1}{2}}{3} = -\frac{1}{2}$$

Since $m_1 \times m_2 = 2 \times (-\frac{1}{2}) = -1$, the lines AB and AC are perpendicular to each other.

Explanation

We use the fact that two lines with gradients m_1 and m_2 (both non-zero) are perpendicular if and only if $m_1 m_2 = -1$.

We show the product of the gradients is -1. Hence the lines are perpendicular.

 ISBN 978-1-009-53719-3

Summary 2E

- Two non-vertical lines are **parallel** if they have the same gradient. Conversely, if two non-vertical lines are parallel, then they have the same gradient.
- Two lines are **perpendicular** if the product of their gradients is -1 (or if one is horizontal and the other vertical). Conversely, if two lines are perpendicular, then the product of their gradients is -1 (or one is horizontal and the other vertical).

Skill-sheet

Exercise 2E

SF

1 Determine the equation of the straight line which passes through $(4, -2)$ and is:

Example 18

a parallel to the line with equation $y = 2x + 1$
b perpendicular to the line with equation $y = 2x + 1$
c parallel to the line with equation $y = -2x + 1$
d perpendicular to the line with equation $y = -2x + 1$
e parallel to the line with equation $2x - 3y = 4$
f perpendicular to the line with equation $2x - 3y = 4$
g parallel to the line with equation $x + 3y = 5$
h perpendicular to the line with equation $x + 3y = -4$.

2 For which of the following pairs of equations are the corresponding lines parallel to each other? Sketch graphs to show the pairs of non-parallel lines.

a $2y = 6x + 4; \quad y = 3x + 4$
b $x = 4 - y; \quad 2x + 2y = 6$
c $3y - 2x = 12; \quad y + \frac{1}{3} = \frac{2}{3}x$
d $4y - 3x = 4; \quad 3y = 4x - 3$

3 Determine the equation of the line:

a perpendicular to the line $x = 3$ and which passes through the point $(3, 4)$
b perpendicular to the line $y = 3$ and which passes through the point $(2, 3)$
c perpendicular to the line $x = -2$ and which passes through the point $(-2, 4)$
d perpendicular to the line $y = -4$ and which passes through the point $(3, -4)$.

4 Determine the equation of the straight line which passes through the point $(1, 4)$ and is perpendicular to the line with equation $y = -\frac{1}{2}x + 6$.

5 Points A and B have coordinates $(1, 5)$ and $(-3, 7)$ respectively. Determine the coordinates of the midpoint M of the line segment AB and determine the equation of the line which passes through the point M and is perpendicular to the line AB.

Example 19

6 If the points A, B and C have the coordinates $A(5, 2)$, $B(2, -3)$ and $C(-8, 3)$, show that the triangle ABC is a right-angled triangle.

7 Given the points $A(3, 7)$, $B(6, 1)$ and $C(20, 8)$, prove that AB is perpendicular to BC.

 ISBN 978-1-009-53719-3

8 Show that $RSTU$ is a rectangle if the coordinates of the vertices are respectively $R(2, 6)$, $S(6, 4)$, $T(2, -4)$ and $U(-2, -2)$.

CF

9 Given that the lines $4x - 3y = 10$ and $4x - \ell y = m$ are perpendicular and intersect at the point $(4, 2)$, determine the values of ℓ and m.

10 The line $y = 2x + 3$ intersects the y-axis at A. The points B and C on this line, distinct from A, are such that $AB = BC$. The line through B perpendicular to AC passes through the point $D(-1, 6)$. Determine:

a the equation of BD

b the coordinates of B

c the coordinates of C.

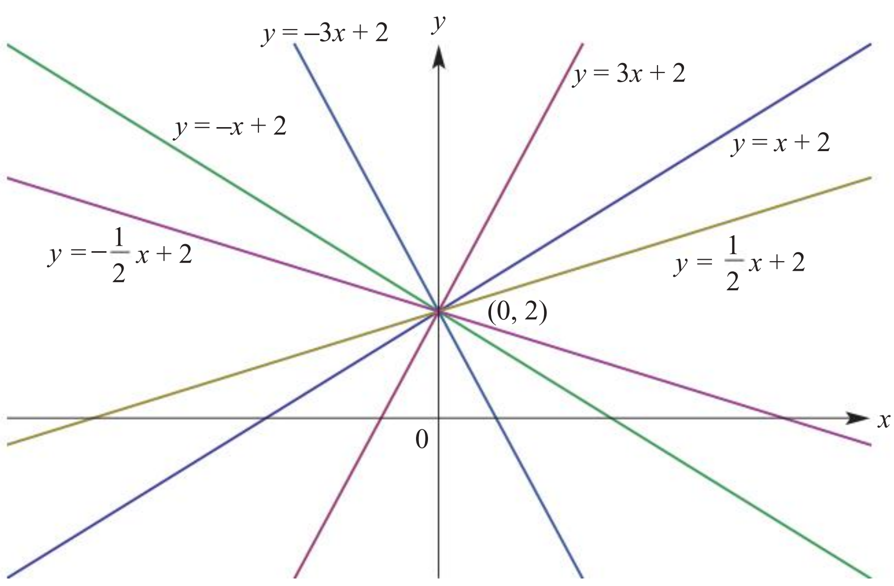
2F Families of straight lines

Learning intentions

► To be able to consider families of straight lines determined by a parameter.

Here are three families of straight lines:

- $y = mx$, where the gradient m of the lines varies – the graphs are the straight lines through the origin.
- $y = 3x + c$, where the y-axis intercept c of the lines varies – the graphs are the straight lines with gradient 3.
- $y = mx + 2$, where the gradient m of the lines varies – the graphs are the straight lines with y-axis intercept 2. Some graphs in this family are illustrated below.



The variable m is called a **parameter**. We will use parameters to explore other families of graphs in later chapters of this book.

Example 20

Determine the value of m if the line $y = mx + 2$ passes through the point $(3, 11)$.

Solution

We can write

$$11 = 3m + 2$$

Therefore $3m = 9$ and hence $m = 3$. It is the line $y = 3x + 2$.

Example 21

A family of lines have equations of the form $y = mx + 2$, where m is a negative number.

a Determine the x-axis intercept of a line in this family in terms of m.

b For which values of m is the x-axis intercept greater than 3?

c Determine the equation of the line perpendicular to the line $y = mx + 2$ at the point $(0, 2)$.

Solution

a When $y = 0$:

$$mx + 2 = 0$$
$$mx = -2$$
$$x = -\frac{2}{m}$$

The x-axis intercept is $-\frac{2}{m}$.

b

$$-\frac{2}{m} > 3$$
$$-2 < 3m$$
$$-\frac{2}{3} < m$$

Therefore the x-axis intercept is greater than 3 for $-\frac{2}{3} < m < 0$.

c The equation is $y - 2 = -\frac{1}{m}x$ and the gradient–intercept form is $y = -\frac{1}{m}x + 2$.

Explanation

a To determine the x-axis intercept put $y = 0$.

b Multiply both sides of the inequality by m. Since m is negative, the inequality sign is reversed. Then divide both sides by 3.

c The perpendicular line has gradient $-\frac{1}{m}$ and passes through $(0, 2)$.

Summary 2F

Families of straight lines can be described through the use of a parameter. For example:

- All the non-vertical lines passing through $(0, 2)$ have equation of the form $y = mx + 2$, where m is a real number. (If $m = 0$, the line is horizontal.)
- All the lines with gradient 3 have equation of the form $y = 3x + c$, where c is a real number.
- All the lines with x-axis intercept 4 have equation of the form $\frac{x}{4} + \frac{y}{b} = 1$, where b is a non-zero real number.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 2F

Example 20

1 The line with equation $y = mx - 3$, where m is a positive number, passes through the point $(2, 7)$. Determine the value of m.

2 The line with equation $y = 2x + c$, where c is a number, passes through the point $(3, 11)$. Determine the value of c.

3 A line has equation $y = mx + 3$, where $m \neq 0$. A second line is perpendicular to this line and passes through the point $(0, 3)$.

a Determine the equation of the second line in terms of m.

b Determine the value of m if the second line passes through the point $(1, -4)$.

4 The line with equation $y = mx + 2$, where m is a positive number, passes through the point $(3, 8)$. Determine the value of m.

SF

Example 21

5 A family of lines have equations of the form $y = mx - 3$, where m is a positive number.

a Determine the x-axis intercept of a line in this family in terms of m.

b Determine the value of m if the line passes through the point $(5, 6)$.

c For which values of m is the x-axis intercept less than or equal to 1?

d Determine the equation of the line perpendicular to the line $y = mx - 3$ at the point $(0, -3)$.

6 A family of lines have equations of the form $y = 2x + c$, where c is a number.

a Determine the x-axis intercept of a line in this family in terms of c.

b Determine the value of c if the line passes through the point $(5, 6)$.

c For which values of c is the x-axis intercept less than or equal to 1?

d Determine the equation of the line perpendicular to the line $y = 2x + c$ at the point $(0, c)$.

7 A family of lines have equations of the form $\frac{x}{a} - \frac{y}{12} = 4$, where a is a non-zero number.

a Determine the x-axis intercept of a line in this family in terms of a.

b Determine the gradient of the line in terms of a.

c Determine the value of a if the gradient is

i 2 **ii** -2

8 A family of lines have equations of the form $y = -2x + c$, where c is a positive number.

a Determine the x-axis intercept of a line in this family in terms of c.

b Determine the value of c if the line passes through the point $(1, 7)$.

c For which values of c is the x-axis intercept less than or equal to 1?

d Determine the equation of the line perpendicular to $y = -2x + c$ at the point $(0, c)$.

e Let O denote the origin, and let A and B be the x-axis and y-axis intercepts of $y = -2x + c$.

i If the midpoint M of the line segment AB is $(3, 6)$, determine the value of c.

ii If the triangle AOB has area 4, determine the value of c.

iii If the midpoint M of AB is such that $OM = 2\sqrt{5}$, determine the value of c.

CF

 ISBN 978-1-009-53719-3

9 A family of lines have equations of the form $3x + by = 12$, where b is a non-zero number.

a Determine the y-axis intercept of a line in this family in terms of b.

b Determine the gradient of the line in terms of b.

c Determine the value of b if the gradient is

i 1

ii -2

d Determine the equation of the line perpendicular to the line $3x + by = 12$ at the point $(4, 0)$.

2G Linear models

Learning intentions

- To be able to use a linear relation to model.

There are many practical situations where a linear relation can be used.

Example 22

Austcom's rates for local calls from private telephones consist of a quarterly rental fee of \$50 plus 25c for every call. Construct a cost function that describes the quarterly telephone bill and sketch the linear graph for this.

Solution

Let

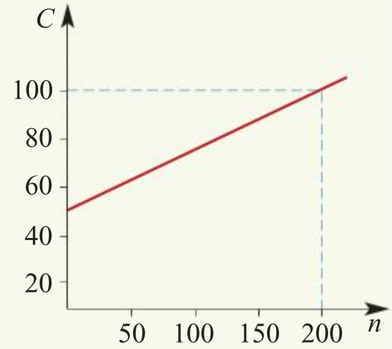
C = cost (\$) of quarterly telephone bill

n = number of calls

Then $C = 0.25n + 50$

The number of calls is counted in whole numbers including zero.

Draw the graph of the function $C = 0.25n + 50$ where n is a whole number including zero.



Note: The graph should be a series of discrete points rather than a continuous line, because n is a whole number. With the scale used it is not practical to show it correctly.

An important linear relation is the relation between distance travelled and time taken when an object is travelling with constant speed. If a car travels at 40 km/h, the relationship between distance travelled (s kilometres) and time taken (t hours) is

$$s = 40t \qquad (\text{for } t \geq 0)$$

The graph of s against t is a straight-line graph through the origin. The gradient of this graph is 40.

 ISBN 978-1-009-53719-3

Example 23

A car starts from point A on a highway 10 kilometres past the Corumbilla post office. The car travels at a constant speed of 90 km/h towards picnic stop B, which is 120 kilometres further on from A. Let t hours be the time after the car leaves point A.

a Determine an expression for the distance d_1 of the car from the post office at time t hours.

b Determine an expression for the distance d_2 of the car from point B at time t hours.

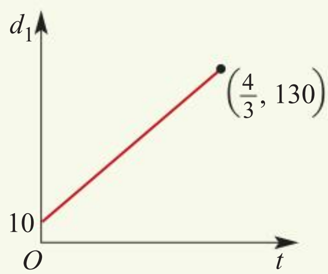
c On separate sets of axes, sketch the graphs of d_1 against t and d_2 against t and state the gradient of each graph.

Solution

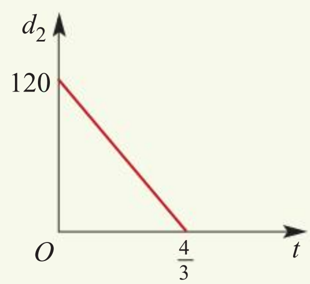
a At time t the distance of the car from the post office is $10 + 90t$ kilometres.

b At time t the distance of the car from B is $120 - 90t$ kilometres.

c



Gradient = 90



Gradient = −90

Skill-sheet

Exercise 2G

Example 22

1 The weekly wage, $\$w$, of a vacuum cleaner salesperson consists of a fixed sum of \$350 plus \$20 for each cleaner sold. If n cleaners are sold per week, construct a rule that describes the weekly wage of the salesperson.

CF

Example 23

2 **a** A train moves at 50 km/h in a straight line away from a town. Give a rule for the distance, d_1 km, from the town at time t hours after leaving the town.

b A train has stopped at a siding 80 km from the town and then moves at 40 km/h in a straight line away from the siding towards the town. Give a rule for the distance, d_2 km, from the town at time t hours after leaving the siding.

c On separate sets of axes, sketch the graphs of d_1 against t $(0 \le t \le 4)$ and d_2 against t and state the gradient of each graph.

3 **a** An initially empty container is being filled with water at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the filling of the container starts.

b A container contains 10 litres of water. Water is then poured in at a rate of 5 litres per minute. Give a rule for the volume, V litres, of water in the container at time t minutes after the pouring starts.

4 The reservoir feeding an intravenous drip contains 500 mL of a saline solution. The drip releases the solution into a patient at the rate of 2.5 mL/minute.

a Construct a rule which relates the amount of solution left in the reservoir, v mL, to time, t minutes.

b State the possible values of t and v.

c Sketch the graph of the relation.

CF

5 The cost (\$$C$) of hiring a taxi consists of two elements, a fixed flagfall and an amount that varies with the number (n) of kilometres travelled. If the flagfall is \$2.60 and the cost per kilometre is \$1.50, determine a rule which gives C in terms of n.

6 A car rental company charges \$85, plus an additional amount of 24c per kilometre.

a Write a rule to determine the total charge \$$C$ for hiring a car and travelling x kilometres.

b What would be the cost to travel 250 kilometres?

7 Two towns A and B are 200 km apart. A man leaves town A and walks at a speed of 5 km/h towards town B. Determine the distance of the man from town B at time t hours after leaving town A.

8 The following table shows the extension of a spring when weights are attached to it.

x, extension (cm)	0	1	2	3	4	5	6
w, weight (g)	50	50.2	50.4	50.6	50.8	51.0	51.2

a Sketch a graph to show the relationship between x and w.

b Write a rule that describes the graph, expressing w in terms of x.

c What will be the extension if $w = 52.5$ g?

9 A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$47 for printing 800 sheets.

a Assuming the relationship is linear, determine a formula, for the charge, \$$C$, in terms of number of sheets printed, n.

b How much would they charge for printing 1000 sheets?

10 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

CU

a Determine a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.

b Was there a sum already on the register when counting began?

c If so, how much?

 ISBN 978-1-009-53719-3

2H Simultaneous linear equations

Learning intentions

- To be able to consider the geometry of simultaneous equations with two unknowns.

In this section we revisit the geometry of simultaneous equations, first introduced in Section 1C, and also make use of parameters to explore these properties. Finally we consider some of the many applications of simultaneous equations.

The geometry of simultaneous equations

There are three possible outcomes when considering a system of two simultaneous linear equations in two unknowns:

- There is a unique solution. (Lines intersect at a point.)
- There are infinitely many solutions. (Lines coincide.)
- There is no solution. (Lines are parallel.)

Example 24

Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

Solution

First write the equations in the form $y = mx + c$. They become

$$y = -\frac{2}{3}x + 2 \qquad \text{and} \qquad y = -\frac{2}{3}x + 4$$

Each of the lines has gradient $-\frac{2}{3}$. The y-axis intercepts are 2 and 4 respectively. The equations have no solution as they correspond to parallel lines and they are different lines.

Example 25

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

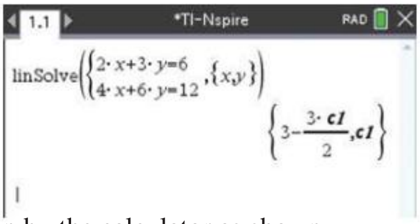
Solution

The two lines coincide, and so the solutions are all points on this line. We make use of a third variable λ as the parameter. If $y = \lambda$, then $x = \dfrac{6 - 3\lambda}{2}$. The points on the line are all points of the form $\left(\dfrac{6 - 3\lambda}{2}, \lambda\right)$.

Using the TI-Nspire CX non-CAS

Simultaneous linear equations can be solved in a **Calculator** application.

- Use menu > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen.



The solution to this system of linear equations is given by the calculator as shown. The parameter $c1$ takes the place of λ.

Using the Casio

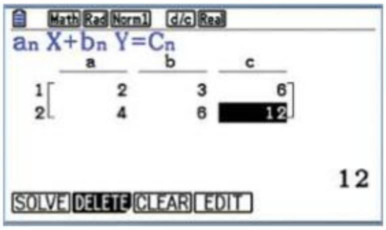
To solve the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$:

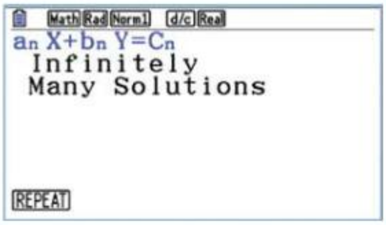
- Press MENU ALPHA X,θ,T to select **Equation** mode. Then select **Simultaneous** F1. Select **Number of Unknowns?**. Choose (2).
- Enter the coefficients of the equations as shown:

  2 EXE 3 EXE 6 EXE

  4 EXE 6 EXE 1 2 EXE
- Select **Solve** F1.

There are infinitely many solutions, so the two lines coincide. The solutions are all points on this line.





Example 26

The family of lines $y = mx + 2$ with varying gradient m all pass through the point $(0, 2)$.

a For what values of m does the line $y = mx + 2$ not intersect the line $y = 5x - 3$?

b For what values of m does the line $y = mx + 2$ intersect the line $y = 5x - 3$?

c If the line $y = mx + 2$ intersects the line $y = 5x - 3$ at the point $(5, 22)$, determine the value of m.

Solution

a The y-axis intercept of $y = mx + 2$ is 2 and the y-axis intercept of $y = 5x - 3$ is -3. The lines will not intersect if they are parallel, that is, if they have the same gradient. So $m = 5$.

b The lines intersect when $m \neq 5$.

c If $(5, 22)$ lies on the line $y = mx + 2$, then

$$22 = 5m + 2$$
$$20 = 5m$$
$$m = 4$$

Thus the lines intersect at $(5, 22)$ when $m = 4$.

Example 27

The lines $y = x + k$ and $y = mx + 4$ intersect at $(1, 3)$. Determine the values of m and k.

Solution

When $x = 1$, $y = 3$.

So $3 = 1 + k$ and $3 = m + 4$.

Hence $k = 2$ and $m = -1$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 28

The lines $(m-2)x + y = 2$ and $mx + 2y = k$ intersect at $(2, 8)$. Determine the values of m and k.

Solution

$$(m-2)x + y = 2 \qquad (1)$$
$$mx + 2y = k \qquad (2)$$

When $x = 2$, $y = 8$. Substituting these values into equations (1) and (2), we have the equations

$$2(m-2) + 8 = 2 \qquad (3)$$
$$2m + 16 = k \qquad (4)$$

From (3), we have $2m - 4 + 8 = 2$. Therefore $m = -1$.

From (4), we have $2(-1) + 16 = k$. Therefore $k = 14$.

Example 29

Consider the simultaneous linear equations $(m-2)x + y = 2$ and $mx + 2y = k$. Determine the values of m and k such that the system of equations has:

a no solution **b** infinitely many solutions **c** a unique solution.

Solution

$$(m-2)x + y = 2 \qquad (1)$$
$$mx + 2y = k \qquad (2)$$

We know that for *no solution* or *infinitely many solutions*, the corresponding lines are parallel or coincide. If the corresponding lines are parallel or coincide, the gradients are the same.

Gradient of line (1) $= 2 - m$

Gradient of line (2) $= -\dfrac{m}{2}$

Hence
$$2 - m = -\frac{m}{2}$$
$$4 - 2m = -m$$
$$m = 4$$

Substitute $m = 4$ in equations (1) and (2). We have

$$2x + y = 2$$
$$4x + 2y = k$$

a There is no solution if $m = 4$ and $k \neq 4$.

b If $m = 4$ and $k = 4$, there are infinitely many solutions as the equations are equivalent.

c The solution is unique if $m \neq 4$ and k is any real number.

 ISBN 978-1-009-53719-3

Applications of simultaneous equations

Example 30

There are two possible methods for paying gas bills:

Method A A fixed charge of \$25 per quarter + 50c per unit of gas used

Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

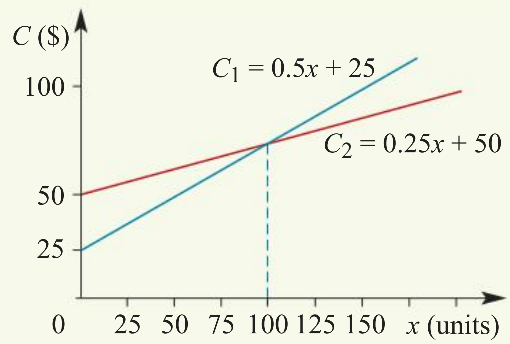
Let C_1 = charge (\$) using method A

C_2 = charge (\$) using method B

x = number of units of gas used

Then $C_1 = 25 + 0.5x$

$C_2 = 50 + 0.25x$



From the graph we see that method B is cheaper if the number of units exceeds 100.

The solution can be obtained by solving simultaneous linear equations:

$$\begin{aligned} C_1 &= C_2 \\ 25 + 0.5x &= 50 + 0.25x \\ 0.25x &= 25 \\ x &= 100 \end{aligned}$$

Example 31

Robyn and Cheryl race over 100 metres. Robyn runs so that it takes a seconds to run 1 metre, and Cheryl runs so that it takes b seconds to run 1 metre. Cheryl wins the race by 1 second. The next day they again race over 100 metres but Cheryl gives Robyn a 5-metre start so that Robyn runs 95 metres. Cheryl wins this race by 0.4 seconds. Determine the values of a and b and the speed at which Robyn runs.

Solution

For the first race: Time for Robyn − time for Cheryl = 1 s.

$$100a - 100b = 1 \qquad (1)$$

For the second race: Time for Robyn − time for Cheryl = 0.4 s.

$$95a - 100b = 0.4 \qquad (2)$$

Subtract (2) from (1). This gives $5a = 0.6$ and therefore $a = 0.12$.

Substitute in (1) to determine $b = 0.11$.

Robyn's speed $= \dfrac{1}{0.12} = \dfrac{25}{3}$ m/s.

 ISBN 978-1-009-53719-3

Summary 2H

- There are three cases for a system of two linear equations with two variables:
  - unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
  - infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
  - no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.
- There are many applications of simultaneous linear equations with two variables. The problems often arise by working with two quantities both changing at a constant but often different rate.

Skill-sheet

Exercise 2H

SF

Example 24

1 Explain why the following simultaneous equations have no solution:

$$x + y = 6$$
$$2x + 2y = 13$$

CF

Example 25

2 The simultaneous equations $x + y = 6$ and $2x + 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 26

3 The family of lines $y = mx + 6$ with varying gradient m all pass through the point $(0, 6)$.

a For what values of m does the line $y = mx + 6$ not intersect the line $y = 4x - 5$?

b For what values of m does the line $y = mx + 6$ intersect the line $y = 4x - 5$?

c If the line $y = mx + 6$ intersects the line $y = 4x - 5$ at the point $(5, 15)$, determine the value of m.

Example 27

4 The lines $y = 2x + k$ and $y = mx - 4$ intersect at $(2, 6)$. Determine the values of m and k.

Example 28

5 The lines $(m - 2)x + y = 4$ and $mx + 3y = k$ intersect at $(2, 8)$. Determine the values of m and k.

Example 29

6 Determine the value of m for which the simultaneous equations $mx - y = 5$ and $3x + y = 6$ have no solution.

7 Determine the value of m for which the pair of simultaneous equations $3x + my = 5$ and $(m + 2)x + 5y = m$ have:

a infinitely many solutions

b no solutions.

Example 30

8 Two bicycle hire companies have different charges. Company A charges \$$C$, according to the rule $C = 10t + 20$, where t is the time in hours for which a bicycle is hired. Company B charges \$$C$, according to the rule $C = 8t + 30$.

a Sketch each of the graphs on the same set of axes.

b Determine the time, t, for which the charge of both companies is the same.

 ISBN 978-1-009-53719-3

Example 31

9 John and Michael race over 50 metres. John runs so that it takes a seconds to run 1 metre and Michael runs so that it takes b seconds to run 1 metre. Michael wins the race by 1 second. The next day they again race over 50 metres but Michael gives John a 3-metre start so that John runs 47 metres. Michael wins this race by 0.1 seconds. Determine the values of a and b and the speed at which Michael runs.

10 The distances, d_A km and d_B km, of cyclists A and B travelling along a straight road from a town hall step are given respectively by $d_A = 10t + 15$ and $d_B = 20t + 5$, where t is the time in hours after 1 p.m.

a Sketch each of the graphs on the one set of axes.

b Determine the time at which the two cyclists are at the same distance from the town hall step.

11 A helicopter can be hired for \$210 per day plus a distance charge of \$1.60 per km or, alternatively, at a fixed charge of \$330 per day for an unlimited distance.

a For each of the methods of hiring, determine an expression for cost, \$$C$, in terms of the distance travelled, x km.

b On one set of axes, draw the graph of cost versus distance travelled for each of the methods.

c Determine for what distances the fixed-charge method is cheaper.

12 Three power boats in a 500 km handicap race leave at 5-hourly intervals. Boat A leaves first and has a speed for the race of 20 km/h. Boat B leaves 5 hours later and travels at an average speed of 25 km/h. Boat C leaves last, 5 hours after B, and completes the race at a speed of 40 km/h.

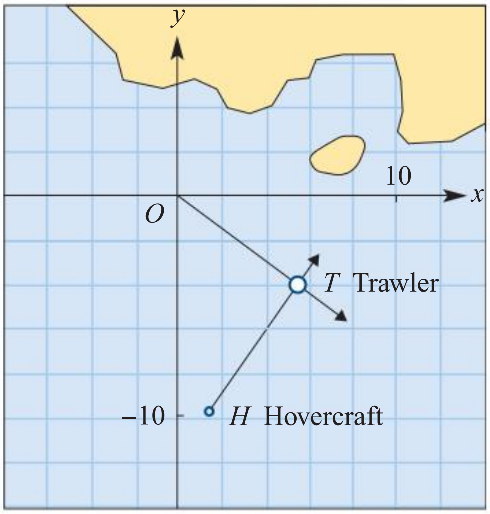
a Draw a graph of each boat's journey on the same set of axes.

b Use your graphs to determine the winner of the race.

c Check your answer algebraically.

d Write a short description of what happened to each boat in the race.

13 If the line OT has the equation $y = -\frac{3}{4}x$ and the line HT has the equation $y = \frac{3}{2}x - 12$, determine the point over which both craft would pass.



Scale: 1 unit = 1 nautical mile

14 A school wishes to take some of its students on an excursion. If they travel by public transport, it will cost the school \$2.80 per student. Alternatively, the school can hire a bus at a cost of \$54 for the day plus a charge of \$1 per student. CF

a For each mode of transport, write an expression for the cost (\$$C$) of transport in terms of the number of students (x).

b On one set of axes, draw the graph of cost, \$$C$, versus number of students, x, for each mode of transport.

c Determine for how many students it will be more economical to hire the bus.

15 Anne and Maureen live in towns that are 57 km apart. Anne sets out at 9 a.m. one day to ride her bike to Maureen's town at a constant speed of 20 km/h. At the same time Maureen sets out to ride to Anne's town at a constant speed of 18 km/h. CU

a Write down a rule for the distance, d km, that each of them is from Anne's place at a time t minutes after 9 a.m.

b On the same set of axes, draw graphs of the distance, d km, versus time, t minutes after 9 a.m., for each cyclist.

c Determine the time at which they will meet.

d How far has each of them travelled when they meet?

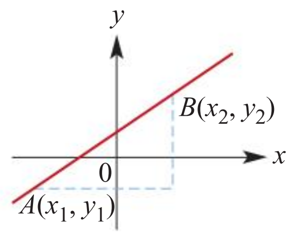
Chapter summary

- The **midpoint** of the line segment joining (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

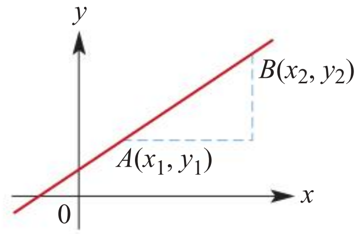
- The **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



- The **gradient** of a straight line joining two points:

$$\text{Gradient } m = \frac{y_2 - y_1}{x_2 - x_1}$$



- For a line with gradient m, the **angle of slope** (θ) can be found using

$$m = \tan\theta$$

where θ is the angle the line makes with the positive direction of the x-axis.

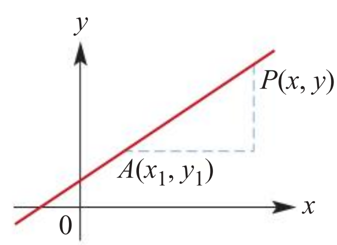
- The gradient–intercept form of the equation of a straight line is

$$y = mx + c$$

where m is the gradient and c is the y-axis intercept.

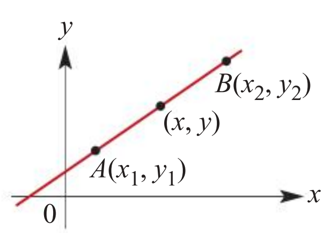
- The equation of a line passing through a given point (x_1, y_1) and having gradient m is

$$y - y_1 = m(x - x_1)$$



- The equation of a line passing through two given points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = m(x - x_1) \quad \text{where} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$



- Two straight lines are **perpendicular** to each other if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical):

$$m_1 m_2 = -1$$

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

2A **1** I can determine the midpoint of a line segment. ☐

See Example 1 and Question 1

2A **2** I can determine the distance between two points. ☐

See Example 2 and Question 6

2B **3** I can determine the gradient of a line given the coordinates of its intercepts with the axes. ☐

See Example 3 and Question 1

2B **4** I can determine the gradient of a line given the coordinates of two points. ☐

See Example 4 and Question 4

2B **5** I can determine the angle a line makes with the positive direction of the x-axis. ☐

See Example 5, Example 6 and Question 8

2C **6** I can determine the y-axis intercept and the gradient of a line given an equation of the form $y = mx + c$. ☐

See Example 7 and Question 1

2C **7** I can determine the equation of a line given the gradient and y-axis intercept. ☐

See Example 8 and Question 2

2C **8** I can determine the gradient and y-axis intercept of a line given an equation of the form $ax + by = c$. ☐

See Example 9 and Question 3

2C **9** I can determine the equation of a line given the gradient and the coordinates of one point. ☐

See Example 10, Example 11 and Question 5

2C **10** I can determine the equation of a line given the coordinates of two points. ☐

See Example 12, Example 13, Example 14 and Questions 6, 7 and 10

2D **11** I can sketch the graph of a straight line given its equation. ☐

See Example 15, Example 16 and Questions 2 and 3

2D **12** I can determine the angle a line makes with the positive direction of the x-axis given its equation. ☐

See Example 17 and Question 9

 ISBN 978-1-009-53719-3

Review

2E **13** I can determine the equation of a line parallel or perpendicular to a given line when given a point on the other line. ☐

See Example 18, Example 19 and Questions 1 and 6

2F **14** I can work with families of lines determined by a parameter. ☐

See Example 20, Example 21 and Questions 1 and 5

2G **15** I can use linear equations to model suitable real life situations. ☐

See Example 22, Example 23 and Questions 1 and 2

2H **16** I can determine when simultaneous equations with two variables have two, infinitely many or no solutions. ☐

See Example 24, Example 25, Example 26, Example 27, Example 28, Example 29 and Questions 1, 2, 3, 4, 5 and 6

2H **17** I can use simultaneous equations to solve problems. ☐

See Example 30, Example 31 and Questions 8 and 9

Short-response questions

Technology-free short-response questions

SF

1 Determine the length and the coordinates of the midpoint of the line segment joining each of the following pairs of points:

a $A(1, 2)$ and $B(5, 2)$ **b** $A(-4, -2)$ and $B(3, -7)$ **c** $A(3, 4)$ and $B(7, 1)$

2 Determine the gradients of the lines joining each of the following pairs of points:

a $(4, 3)$ and $(8, 12)$ **b** $(-3, 4)$ and $(8, -6)$ **c** $(2, 1)$ and $(2, 9)$
d $(0, a)$ and $(a, 0)$ **e** $(0, 0)$ and (a, b) **f** $(0, b)$ and $(a, 0)$

3 Determine the equation of the straight line of gradient 4 which passes through the point with coordinates:

a $(0, 0)$ **b** $(0, 5)$ **c** $(1, 6)$ **d** $(3, 7)$

4 **a** The point $(1, a)$ lies on the line with equation $y = 3x - 5$. Determine the value of a.
b The point $(b, 15)$ lies on the line with equation $y = 3x - 5$. Determine the value of b.

5 Determine the equation of the straight line joining the points $(-5, 2)$ and $(3, -4)$.

6 Determine the equation of the straight line of gradient $-\frac{2}{3}$ which passes through $(-4, 1)$.

7 Write down the equation of the straight line that:

a passes through $(5, 11)$ and is parallel to the x-axis
b passes through $(0, -10)$ and is parallel to the line with equation $y = 6x + 3$
c passes through the point $(0, -1)$ and is perpendicular to the line with equation $3x - 2y + 5 = 0$.

 ISBN 978-1-009-53719-3

Review

SF

8 Determine the equation of the straight line which passes through the point $(2, 3)$ and is inclined at $45°$ to the positive direction of the x-axis.

9 Determine the equation of the straight line which passes through the point $(-2, 3)$ and makes an angle of $135°$ with the positive direction of the x-axis.

10 Determine the equation of the straight line passing through the point $(4, 8)$ and which is perpendicular to the line with equation $y = -3x + 2$.

11 A straight line has equation $y = 2x + 1$. The points with coordinates $(0, a)$, $(b, 0)$, $(2, d)$ and $(e, 7)$ lie on this line. Determine the values of a, b, d and e.

12 Sketch the graph of each of the following by first determining axis intercepts. Clearly label each axis intercept.

a $y = 2x - 8$ b $3x + y = 6$ c $3x + y + 6 = 0$

d $y - 2x - 8 = 0$ e $y = -6x + 6$ f $2x + 5y + 10 = 0$

13 a Determine the equation of the line parallel to the x-axis passing through the point $(8, -9)$. Sketch the graph.

b Determine the equation of the line parallel to the y-axis passing through the point $(8, -9)$. Sketch the graph.

c Determine the equation of the straight line that passes through the origin and has gradient:

i 3 ii -3

Sketch the graphs of each of these.

14 A car travels at a constant 60 km/h.

a Write down the formula for the distance d km travelled by the car in t hours.

b What is the gradient of the graph of d against t?

15 The weekly wage $\$S$ of a car salesperson consists of a fixed sum of \$800 plus \$500 for each car sold. If n cars are sold in a week, construct a formula that describes the weekly wage of the salesperson in terms of n.

CF

16 A family of straight lines satisfy the rule $y = ax + 2$.

a Determine the equation of the straight line in this family for which $y = 6$ when $x = 2$.

b i Determine the x-axis intercept of the line with equation $y = ax + 2$.

ii If $a < 0$, determine the values of a for which the x-axis intercept is greater than 1.

c Determine the coordinates of the point of intersection of the line with equation $y = x + 3$ and the line with equation $y = ax + 2$, given that $a \neq 1$.

17 The cost of hiring a motor cruiser consists of a down payment of \$500 and a running charge of \$100 per day, or part of a day. The cost of fuel is \$27.50 per day. There is also a charge of \$50 for filling the freshwater tanks and charging the batteries. Food for a cruise of n days costs \$62.50 per day.

a Give a formula for C, the total cost in dollars of hiring the cruiser for n days (all costs to be included).

 ISBN 978-1-009-53719-3

Review

CF

b For how many days can a cruiser be hired if the cost of a cruise is to be no more than \$3000?

c A rival company has a fixed rate of \$300 per day. For how many days would it be cheaper to hire from this company?

18 The cost of fitting a new plug and cable for an electric drill is \$$C$, when the length of the cable is x metres and $C = 4.5 + 1.8x$.

a What meaning could be given for the constant term 4.5?

b What could be the meaning of the coefficient 1.8?

c What would be the gradient of the graph of C against x?

d What length of cable would give a total cost of \$24.50?

19 The profit made on a single journey of an Easyride bus tour is \$$P$, when there are x empty seats and $P = 1020 - 24x$.

a What do you think is the meaning of the constant term 1020?

b What is the least number of empty seats which would result in a loss on a single journey?

c Suggest a meaning for the coefficient 24.

CU

20 The diagram shows a quadrilateral. Angle BAD is a right angle and C lies on the perpendicular bisector of AB. The equation of the line through points B and C is $3y = 4x - 14$. Determine:

a the equation of the line AD

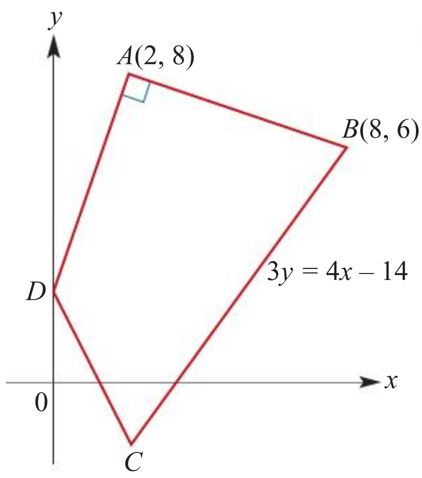
b the coordinates of D

c the equation of the perpendicular bisector of AB

d the coordinates of C

e the area of triangle ADC

f the area of the quadrilateral $ABCD$.



Technology-active short-response questions

SF

21 Determine the length of the line segment joining each of the following pairs of points, giving your answer correct to two decimal places:

a $A(1, 1)$ and $B(6, 6)$ **b** $A(-4, -2)$ and $B(5, -7)$ **c** $A(2, 4)$ and $B(11, 2)$

22 For each of the following lines, determine the magnitude of the angle θ, correct to two decimal places, that the line makes with the positive direction of the x-axis:

a $y = -0.2x + 4$ **b** $y = 0.4x - 5$ **c** $y = 1.2x + 3$

 ISBN 978-1-009-53719-3

Review

23 Solve each of the following sets of simultaneous equations giving your solutions correct to two decimal places. SF

a $3x - 4.6y = 11.2$
$6.4x + 3.3y = -4.64$

b $-8.9x - 3.6y = 21.8$
$0.4x - 7.8y = -56.83$

c $9.23x - 4.65y = 7.86$
$1.02x + 0.4y = 6.32$

24 Determine the equation of the line passing through the points $(4, -5)$ and $(1, 6)$ and the angle θ, correct to two decimal places, that the line makes with the positive direction of the x-axis. CF

25 A quarterly electricity bill shows the following charges:

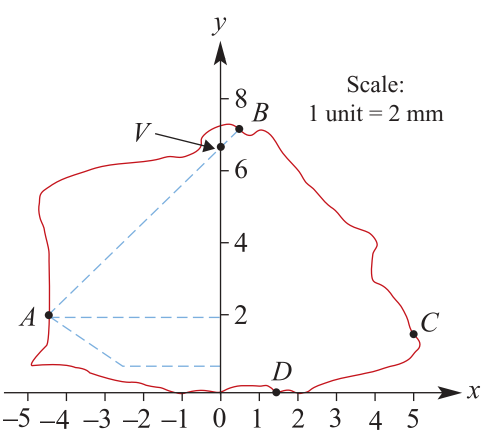
- For the first 50 kWh (kilowatt hours): 9.10c per kWh
- For the next 150 kWh: 5.80c per kWh
- Thereafter: 3.56c per kWh

a Write down a formula relating cost, $\$C$, to n, the number of kWh of electricity used:
  - **i** for the first 50 kWh
  - **ii** for the next 150 kWh
  - **iii** for more than 200 kWh.

b Draw a graph of C against n. Use the graph, or otherwise, to determine the charges for:
  - **i** 30 kWh
  - **ii** 90 kWh
  - **iii** 300 kWh

c How much electricity could be used for a cost of \$20?

26 The diagram shows the side view of a rough, uncut diamond fixed in position on a computer-controlled cutting machine. The diamond is held at the points $A(-4.5, 2)$, $B(0.25, 7)$, $C(5, 1.5)$ and $D(1.5, 0)$.



a If a straight cut is made joining A and B, determine the y-coordinate of the point V at which the cut will cross the vertical axis.

b determine the equation of the line joining V and C.

c Would the cuts AB and VC be equally inclined to the vertical axis? Explain your answer.

27 A construction company estimates that for every 1% of air left in concrete as it is being laid the strength of the hardened concrete decreases by 7%. Let x represent the percentage of air in the concrete (by volume), and the strength of the concrete be s units, where $s = 100$ when $x = 0$.

a Write a formula for s in terms of x.

b Sketch a graph of s against x.

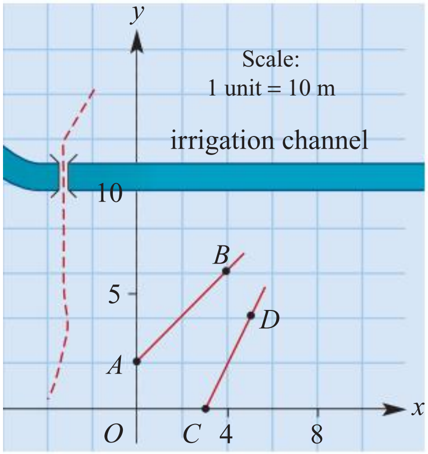
c Calculate how much air can be allowed to remain in the concrete for a strength of at least 95%.

d Estimate how much air the concrete will contain at 0% strength.

e Is the model sensible at 0% strength?

f State the possible values of x.

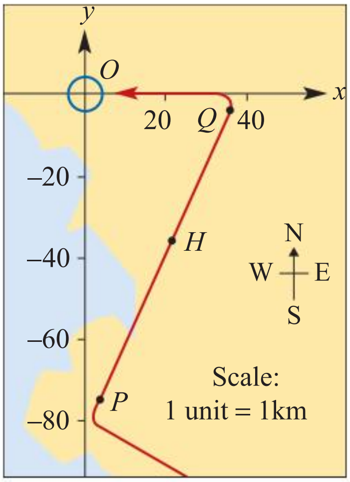
28 The diagram shows a plan view of a paddock over which a Cartesian framework has been superimposed. From an observation point O, a rabbit has been spotted first at $A(0, 2)$ and then at $B(4, 6)$. A fox is seen at $C(3, 0)$ and later at $D(5, 4)$.



a Determine the equations of the lines AB and CD.

b Assuming that both the rabbit and the fox were running along straight lines, calculate whether the fox's path would cross the rabbit's track before the irrigation channel.

29 A new light beacon is proposed at $P(4, -75)$ for air traffic flying into an airport located at $O(0, 0)$. It is intended that the aircraft should follow a course over beacons at P and $Q(36, -4)$, turning at Q towards the runway at O.



a Would a direct line from P to Q pass directly over a hospital located at $H(20, -36)$?

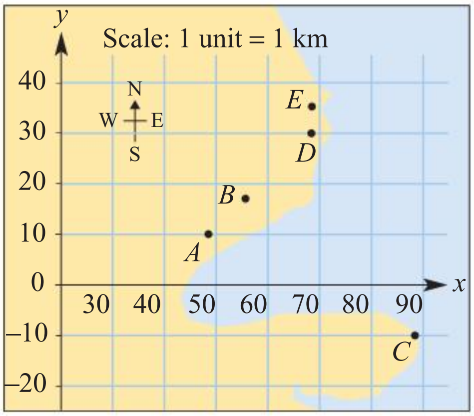
b If not, state how far east or west of H the aircraft would be when the y-coordinate of an aircraft's flight path is -36.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

CU

30 The map shows an area where it is proposed to construct a new airport. It is thought that the main runway of the airport will have one end of its centre line at $A(48, 10)$, but the position of the other end of this line, B, has not been decided. There is a light aircraft airport at $E(68, 35)$ and a radio beacon at $C(88, -10)$.



a What is the equation that will define the new runway if aircraft coming in to land from the east must be on the extended central line of the new runway when they are 5 km due south of E?

b If B is to be 8 km to the east of A, what will be its coordinates?

c A marker beacon is to be built at $D(68, 30)$ and it is proposed that several auxiliary beacons should be placed on the line CD. What is the equation of the line CD?

d If one of the auxiliary beacons is to be placed due east of A, what are the coordinates of its position?

31 An electricity supply authority is offering customers a choice of two methods of paying electricity bills. Method 1 involves payment annually and method 2 involves payment each quarter (that is, every three months). The charges for each method are as follows:

Method 1 – per year		Method 2 – per quarter	
Fixed charge	\$100	Fixed charge	\$27.50
Price per unit	\$0.08125	Price per unit	\$0.075

a Suppose a customer used 1560 units of electricity in a year. Calculate which is the cheaper method of payment.

b Copy and then complete the following table:

	Number of units of electricity			
	0	1000	2000	3000
Cost (\$) calculated by method 1				
Cost (\$) calculated by method 2				

c Use these values to plot graphs of the costs for each method of paying for electricity. Clearly indicate the approximate number of units of electricity for which the cost is the same for both methods of payment.

d If C_1 is the cost by method 1, C_2 is the cost by method 2, and x is the number of units of electricity used in a year, write down the two formulas which show the cost of x units calculated by each method. Use these formulas to calculate the exact number of units for which the cost is the same for both methods.

ISBN 978-1-009-53719-3

Multiple-choice questions

Technology-free multiple-choice questions

1 The coordinates of the midpoint of AB, where A has coordinates $(4, 12)$ and B has coordinates $(6, 2)$, are

A $(4.5, 8)$ **B** $(5, 8)$ **C** $(5, 7)$ **D** $(1, 5)$

2 If $(6, 3)$ is the midpoint of the line segment joining the points $(-4, y)$ and $(x, -6)$, then the value of $x + y$ is

A 16 **B** 20 **C** -10 **D** 28

3 The gradient of the line passing through the points $(5, -8)$ and $(6, -10)$ is

A -2 **B** $\frac{1}{2}$ **C** $-\frac{1}{18}$ **D** $\frac{3}{2}$

4 The gradient of the line passing through points $(4a, 2a)$ and $(9a, -3a)$ is

A $-5a$ **B** 1 **C** -5 **D** -1

5 The equation of the straight line with gradient 3 that passes through the point $(1, 9)$ is

A $y = 3x + 9$ **B** $y = 3x + 6$ **C** $y = -\frac{1}{3}x + 1$ **D** $y = -\frac{1}{3}x + 6$

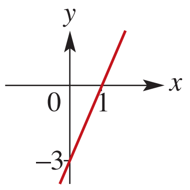
6 A straight line passes through the points $(2, -6)$ and $(-2, -14)$. The equation of the line is

A $y = \frac{1}{2}x - 7$ **B** $y = \frac{1}{2}x - 10$ **C** $y = 2x - 10$ **D** $y = -\frac{1}{2}x - 8$

7 The line with equation $y = 2x - 6$ passes through the point $(a, 2)$. The value of a is

A 4 **B** 5 **C** -4 **D** -2

8 The relation with graph as shown has rule



A $y = -\frac{1}{3}x - 3$
B $y = \frac{1}{3}x - 3$
C $y = 3x + 3$
D $y = 3x - 3$

9 If two lines $5x - y + 7 = 0$ and $ax + 2y - 11 = 0$ are parallel, then a equals

A 5 **B** -10 **C** 10 **D** $-\frac{1}{2}$

10 The cost ($\$C$) of hiring a car is given by the formula $C = 2.5x + 65$, where x is the number of kilometres travelled. A person is charged \$750 for the hire of the car. The number of kilometres travelled was

A 145 **B** 160 **C** 200 **D** 274

11 The solution of the two simultaneous equations $2ax + 2by = 3$ and $3ax - 2by = 7$ for x and y is

A $x = 2,\ y = \dfrac{3 - 4a}{2b}$ **B** $x = \dfrac{2}{a},\ y = -\dfrac{1}{2b}$ **C** $x = 0,\ y = 0$ **D** $x = 3a,\ y = 7b$

Technology-active multiple-choice questions

12 The magnitude of the angle that the line with equation $y = 2.8x + 3$ makes with the positive direction of the x-axis is closest to

A 35° **B** 48° **C** 52° **D** 70°

13 The distance between the points with coordinates $(4.6, 7)$ and $(-3.5, 9)$ is closest to

A 6.75 **B** 8.34 **C** 9.21 **D** 9.87

14 A straight line makes an angle of 124° with the positive direction with x-axis. The gradient of the line is closest to:

A 1.52 **B** −1.48 **C** 2.23 **D** −2.13

15 A straight line with equation $-150x + 76y = 230$ crosses the x-axis at $(a, 0)$. The value of a is closest to:

A −1.52 **B** 2.35 **C** 3.03 **D** −1.53

 ISBN 978-1-009-53719-3

3

Sets of numbers and the arithmetic of surds

Chapter contents

In the first section of this chapter elementary notation for sets is introduced. The use of sets in this course is largely informal and often not explicitly identified but the ideas are present in the study of functions and used in the study of probability.

In the final two sections of the chapter numbers such as $\sqrt{2}$, $\sqrt{77}$ are considered. These are examples of what are called **surds**. You use surds when you are dealing with exact solutions in problems. For example the solutions of the equation $x^2 = 2$ are $x = \sqrt{2}$ or $x = -\sqrt{2}$. Calculators may give an approximate values for $\sqrt{2}$ as $x = 1.41421$. This decimal approximation can be continued to any number of decimal places but it is not exact.

In the previous chapter the distance between two points was considered. The distance between the points $A(0, 0)$ and $B(1, 1)$ is $\sqrt{2}$. This is the exact distance between the two points derived by an application of Pythagoras' theorem.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

3A Set notation and sets of numbers

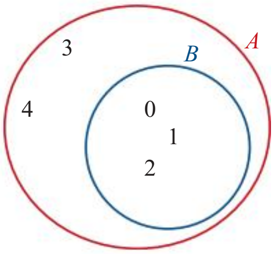
Learning intentions

- To be able to read and use the notation of sets
- To be able to identify and describe subsets of the real numbers.
- To be able to represent subsets of the real numbers on a number line.

Set notation

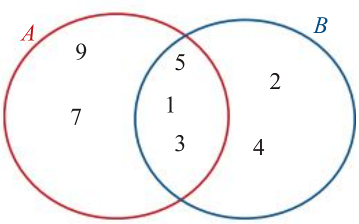
Set notation is used widely in mathematics and in this book where appropriate. This section summarises all the set notation which may be useful in your study of functions and relations. We will discuss set notation again in Chapter 9 for our study of probability.

- A **set** is a collection of objects.
- The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'. For example: $2 \in$ set of even numbers.
- If x is **not an element** of A, we write $x \notin A$. For example: $2 \notin$ set of odd numbers.
- Set B is called a **subset** of set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.



For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then B is a subset of A, as illustrated in the diagram opposite. This diagram is called a **Venn diagram**. Venn diagrams are revisited in Chapter 9.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.



For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$A \cap B = \{1, 3, 5\}$ and $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, determine:

a $A \cap B$ **b** $A \cup B$

Solution	Explanation
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

Note: In Example 1, we have $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

 ISBN 978-1-009-53719-3

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

Example 2

If $A = \{\text{positive even numbers less than } 17\}$ and
$B = \{\text{positive natural numbers which are multiples of 3 and less than } 17\}$
list the elements in each of the following sets.

a A **b** B **c** $A \cap B$ **d** $A \cup B$

Note: An **even** natural number is a natural number that is a multiple of 2.
That is, $\{2, 4, 6, \dots\}$.

Solution

a $A = \{2, 4, 6, 8, 10, 12, 14, 16\}$

b $B = \{3, 6, 9, 12, 15\}$

c $A \cap B = \{6, 12\}$

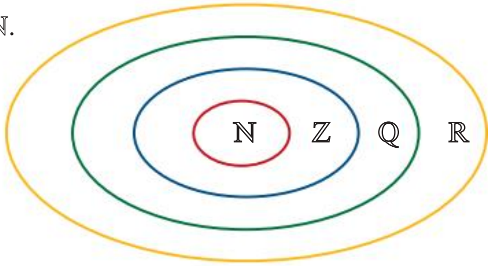
d $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16\}$

The numbers of the form $\dfrac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.
The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).
A subset of the irrational numbers called 'surds' are discussed in the following two sections.
$\sqrt{2}$ is a an example of a surd.

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{x : \_\_\_\}$. For example:

- $\{x : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x : x \geq 3\}$ is the set of all real numbers greater than or equal to 3
- $\{x : x \neq 0\}$ is the set of all real numbers other than 0.

The following are subsets of the real numbers for which we have special notation:

Positive real numbers $\mathbb{R}^+ = \{x : x > 0\}$ Sometimes, we just write $x > 0$

Negative real numbers $\mathbb{R}^- = \{x : x < 0\}$ Sometimes, we just write $x < 0$

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. You can stipulate an interval by simply using inequality signs. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \text{ or without set notation } a < x < b$$
$$[a, b] = \{ x : a \le x \le b \} \text{ or without set notation } a \le x \le b$$
$$(a, b] = \{ x : a < x \le b \} \text{ or without set notation } a < x \le b$$
$$[a, b) = \{ x : a \le x < b \} \text{ or without set notation } a \le x < b$$
$$(a, \infty) = \{ x : a < x \} \text{ or without set notation } x > a$$
$$[a, \infty) = \{ x : a \le x \} \text{ or without set notation } x \ge a$$
$$(-\infty, b) = \{ x : x < b \} \text{ or without set notation } x < b$$
$$(-\infty, b] = \{ x : x \le b \} \text{ or without set notation } x \le b$$

Intervals may be represented by diagrams as shown in Example 3.

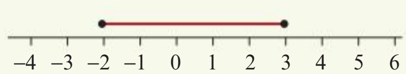
Example 3

Illustrate each of the following intervals of real numbers:

a $-2 \le x \le 3$, interval notation $[-2, 3]$
b $-3 < x \le 4$, interval notation $(-3, 4]$
c $x \le 5$, interval notation $(-\infty, 5]$
d $-2 < x < 4$, interval notation $(-2, 4)$
e $x > -3$, interval notation $(-3, \infty)$

Solution

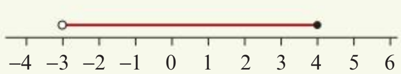
a $-2 \le x \le 3$, interval notation $[-2, 3]$



Explanation

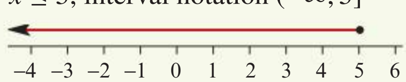
The square brackets indicate that the endpoints are included; this is shown with closed circles.

b $-3 < x \le 4$, interval notation $(-3, 4]$



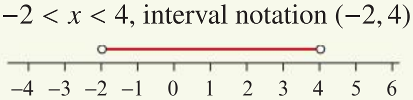
The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c $x \le 5$, interval notation $(-\infty, 5]$



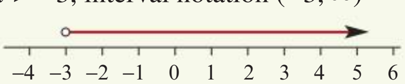
The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

d $-2 < x < 4$, interval notation $(-2, 4)$



Both brackets are round; the endpoints are not included.

e $x > -3$, interval notation $(-3, \infty)$



The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

 ISBN 978-1-009-53719-3 Cambridge University Press

Note: The 'closed' circle (●) indicates that the number is included.
The 'open' circle (○) indicates that the number is not included.

Summary 3A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- **Intersection** The set $A \cap B$ contains the elements in common to A and B.
- **Union** The set $A \cup B$ contains the elements that are in A or in B (or in both).
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Skill-sheet

Exercise 3A

SF

Example 1

1 For $A = \{7, 11, 13, 19\}$ and $B = \{3, 4, 5, 6, 7, 13\}$, determine:

a $A \cap B$ **b** $A \cup B$

Example 2

2 If A = {prime numbers less than 14} and B = {positive odd numbers less than 15} list the elements in each of the following sets

a A **b** B **c** $A \cap B$ **d** $A \cup B$

Note: A **prime** number is a natural number greater than one that is divisible only by itself and one. That is, $\{2, 3, 5, \dots\}$.

3 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$ and $C = \{1, 7, 11, 25, 30\}$, determine:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

4 For $A = \{11, 14, 17, 23, 25, 36, 37\}$, $B = \{11, 17, 25, 30, 32\}$ and $C = \{11, 14, 15, 25, 60\}$, determine:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

5 If $A = \{-1, \frac{3}{4}, 1, \sqrt{2}, \frac{11}{8}, \pi\}$ and $\mathbb{Q}$ is the set of rationals then determine $A \cap \mathbb{Q}$.

 ISBN 978-1-009-53719-3

6 In each part, list the elements of $A \cap B \cap C$ and $A \cup B \cup C$:

a $A = \{1, 3, 5, 7, 9, 11\}, B = \{1, 2, 3, 4, 5, 6\}, C = \{7, 8, 9, 10, 11, 12\}$

b $A = \{0, 4, 8, 12, 16\}, B = \{0, 3, 6, 9, 12, 15\}, C = \{0, 10, 20\}$

7 If $B = \{-1, \frac{3}{4}, 1, \frac{11}{8}, \sqrt{3}, 2, \pi, 100\}$ and $\mathbb{Z}$ is the set of integers then determine $B \cap \mathbb{Z}$.

Example 3

8 Illustrate each of the following intervals on a number line:

a $-3 \le x < 4$ (written with interval notation $[-3, 4)$)

b $x \le 3$ (written with interval notation $(-\infty, 3]$)

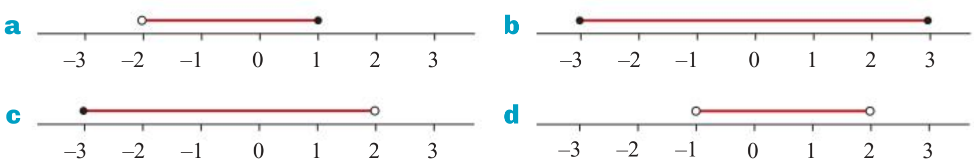
c $-2 \le x \le -1$ (written with interval notation $[-2, -1]$)

d $x > -2$ (written with interval notation $(-2, \infty)$)

e $-2 < x < 3$ (written with interval notation $(-2, 3)$)

f $-2 < x \le 4$ (written with interval notation$(-2, 4]$)

9 Describe each of the following subsets of the real number line using either simply inequality signs without set notation or interval notation $[a, b)$, (a, b), etc.



10 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{x : -1 \le x \le 2\}$ **b** $\{x : -4 < x \le 2\}$ **c** $\{y : 0 < y < \sqrt{2}\}$

d $\{y : -3 \le y < 2\}$ **e** $\{x : x > -1\}$ **f** $\{x : x \le -2\}$

g $\mathbb{R}$ **h** $\mathbb{R}^+ \cup \{0\}$ **i** $\mathbb{R}^- \cup \{0\}$

11 Let $A = \{x : -2 \le x \le 8\}$ and $B = \{x : 3 < x < 11\}$. Determine:

a $A \cap B$ **b** $A \cup B$

12 For $B = \{7, 11, 25, 30, 32\}$, determine:

a $(-2, 10] \cap B$ **b** $(3, \infty) \cap B$ **c** $(2, \infty) \cup B$ **d** $(25, \infty) \cap B$

13 For each of the following, use one number line on which to represent the sets:

a $[-2, 5]$, $[3, 4]$, $[-2, 5] \cap [3, 4]$ **b** $[3, \infty)$, $(-\infty, 7]$, $[3, \infty) \cap (-\infty, 7]$

14 Illustrate each of these sets on a number line:

a $[-3, 2] \cup [4, 8]$ **b** $(-\infty, 2] \cup [4, \infty)$ **c** $(-\infty, -3) \cup (0, \infty)$

d $(-5, -2] \cup (2, 6]$ **e** $(-\infty, 2) \cup (2, \infty)$ **f** $(-\infty, -3) \cup (-3, \infty)$

15 Describe each of the following intersections of intervals as simply as possible:

a $(-\infty, -3) \cap (-6, \infty)$ **b** $(-\infty, 1) \cap (4, \infty)$ **c** $(-\infty, 0] \cap [-6, \infty)$

d $[-3, 2] \cap [-1, 8]$ **e** $[-3, 1] \cap [1, 8]$ **f** $(-\infty, -1] \cap (-10, \infty)$

 ISBN 978-1-009-53719-3

3B Surds

Learning intentions

- To be able to apply the rules for working with surds.

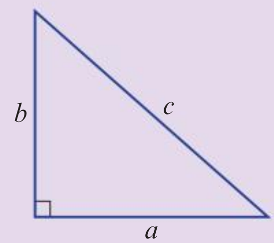
We begin by recalling Pythagoras' theorem which you have met in previous years.

Pythagoras' theorem and its converse

- In any right-angled triangle, the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides.

  $$c^2 = a^2 + b^2$$

- A triangle with side lengths a, b, c which satisfy $c^2 = a^2 + b^2$ is a right-angled triangle. The right angle is opposite the side of length c.



From Pythagoras' theorem we have:

If $a = b = 1$, then $c^2 = 2$ and $c = \sqrt{2}$

If $a = 1, b = \sqrt{2}$, then $c^2 = 1^2 + (\sqrt{2})^2 = 3$ and $c = \sqrt{3}$

If $a = 2, b = 3\sqrt{2}$, then $c^2 = 2^2 + (3\sqrt{2})^2$

$$= 4 + 9 \times 2$$

$$= 22$$

Therefore, $c = \sqrt{22}$

We have used properties of square roots in the above which you may be familiar with. We discuss these on the following page.

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the non-negative square root.

For example:

- $\sqrt{7}, \sqrt{24}, \sqrt{\frac{9}{7}}, \sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}, \sqrt{16}, \sqrt{\frac{9}{4}}$ are *not* surds

Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$$x = \frac{1 + \sqrt{5}}{2} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0.$$

We return to the study of quadratic functions in the following chapter.

Some well-known values of trigonometric functions involve surds.

For example: $\sin 60° = \dfrac{\sqrt{3}}{2}$ and $\sin 45° = \dfrac{\sqrt{2}}{2}$

We consider and apply exact values of trigonometric functions in Chapter 8.

Exact solutions are often required in Mathematical Methods Units 3 & 4.

 ISBN 978-1-009-53719-3

Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b:

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\dfrac{7}{9}} = \dfrac{\sqrt{7}}{\sqrt{9}} = \dfrac{\sqrt{7}}{3}$
- $(\sqrt{a}\,)^2 = a$
- $(\sqrt{a^2}\,) = a$

Properties of surds

Multiplication of surds

Using the above properties we can simplify expressions involving square roots.

Example 4

Write each of the following in the form $m\sqrt{n}$ where m and n are integers.

a $3\sqrt{6} \times 4\sqrt{5}$ **b** $-7\sqrt{3} \times 4\sqrt{2}$

Solution	Explanation
a $3\sqrt{6} \times 4\sqrt{5} = 3 \times 4 \times \sqrt{6} \times \sqrt{5}$ $= 12\sqrt{30}$	Property of square roots: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
b $-7\sqrt{3} \times 4\sqrt{2} = -7 \times 2 \times \sqrt{3} \times \sqrt{2}$ $= -28\sqrt{6}$	Property of square roots: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Example 5

Simplify each of the following:

a $(\sqrt{5}\,)^2$ **b** $(3\sqrt{5}\,)^3$

Solution	Explanation
a $(\sqrt{5}\,)^2 = \sqrt{5} \times \sqrt{5}$ $= 5$	$(\sqrt{a}\,)^2 = a$
b $(3\sqrt{5}\,)^3 = (3\sqrt{5}\,)^2 \times 3\sqrt{5}$ $= 9 \times 5 \times 3\sqrt{5}$ $= 135\sqrt{5}$	$a^3 = a^2 \times a$ $(3\sqrt{5}\,)^2 = (3\sqrt{5}\,) \times (3\sqrt{5}\,)$ $= 3 \times 3 \times \sqrt{5} \times \sqrt{5}$ $= 9 \times 5$ $= 45$

 ISBN 978-1-009-53719-3

Simplest form

If possible, a factor which is the square of a rational number is 'taken out' of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 6

Write each of the following in simplest form:

a $\sqrt{72}$ **b** $\sqrt{28}$ **c** $\sqrt{\dfrac{700}{144}}$ **d** $\sqrt{\dfrac{99}{64}}$

Solution

a
$$\begin{aligned}\sqrt{72} &= \sqrt{36 \times 2}\\ &= \sqrt{6^2 \times 2}\\ &= \sqrt{6^2} \times \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}\sqrt{28} &= \sqrt{4 \times 7}\\ &= \sqrt{2^2} \times \sqrt{7}\\ &= 2\sqrt{7}\end{aligned}$$

c
$$\begin{aligned}\sqrt{\frac{700}{144}} &= \frac{\sqrt{700}}{\sqrt{144}}\\ &= \frac{\sqrt{7 \times 100}}{\sqrt{9 \times 16}}\\ &= \frac{10\sqrt{7}}{12} = \frac{5\sqrt{7}}{6}\end{aligned}$$

d
$$\begin{aligned}\sqrt{\frac{99}{64}} &= \frac{\sqrt{99}}{\sqrt{64}}\\ &= \frac{\sqrt{9 \times 11}}{8}\\ &= \frac{3\sqrt{11}}{8}\end{aligned}$$

Like surds

Surds which have the same 'irrational factor' are called **like surds**.

For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

The sum or difference of two like surds can be simplified:

- $m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}$
- $m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$

Example 7

Express each of the following in simplest form:

a $3\sqrt{2} + 5\sqrt{2} - \sqrt{2}$ **b** $7\sqrt{5} + 4\sqrt{3} - 2\sqrt{5} + \sqrt{3}$

Solution

a
$$\begin{aligned}3\sqrt{2} + 5\sqrt{2} - \sqrt{2} &= (3 + 5 - 1)\sqrt{2}\\ &= 7\sqrt{2}\end{aligned}$$

b
$$\begin{aligned}&7\sqrt{5} + 4\sqrt{3} - 2\sqrt{5} + \sqrt{3}\\ &= (7 - 2)\sqrt{5} + (4 + 1)\sqrt{3}\\ &= 5\sqrt{5} + 5\sqrt{3}\end{aligned}$$

 ISBN 978-1-009-53719-3

Example 8

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$ **b** $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$ **c** $3\sqrt{28} - 7\sqrt{63}$

Solution

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

$= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3}$

$= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3}$

$= 2\sqrt{3}$

b $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$

$= 8\sqrt{2} - 6\sqrt{2}$

$= 2\sqrt{2}$

c $3\sqrt{28} - 7\sqrt{63} = 3\sqrt{7 \times 4} - 7\sqrt{7 \times 9}$

$= 6\sqrt{7} - 21\sqrt{7}$

$= -15\sqrt{7}$

Example 9

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3 - \sqrt{2})^2$ **b** $(3 - \sqrt{2})(1 + \sqrt{2})$

Solution

a $(3 - \sqrt{2})^2$

$= (3 - \sqrt{2})(3 - \sqrt{2})$

$= 3(3 - \sqrt{2}) - \sqrt{2}(3 - \sqrt{2})$

$= 9 - 3\sqrt{2} - 3\sqrt{2} + 2$

$= 11 - 6\sqrt{2}$

b $(3 - \sqrt{2})(1 + \sqrt{2})$

$= 3(1 + \sqrt{2}) - \sqrt{2}(1 + \sqrt{2})$

$= 3 + 3\sqrt{2} - \sqrt{2} - 2$

$= 1 + 2\sqrt{2}$

Summary 3B

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

ISBN 978-1-009-53719-3

Skill-sheet

Exercise 3B

SF

Example 4

1 Write each of the following in the form $m\sqrt{n}$ where m and n are integers:

a $7\sqrt{3} \times 4\sqrt{2}$ **b** $-6\sqrt{7} \times 4\sqrt{11}$ **c** $4\sqrt{2} \times 7\sqrt{5}$

Example 5

2 Evaluate each of the following:

a $(\sqrt{11})^2$ **b** $(5\sqrt{2})^2$ **c** $(2\sqrt{11})^2$ **d** $(7\sqrt{3})^2$

Example 6

3 Express each of the following in terms of the simplest possible surds:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{27}$ **d** $\sqrt{50}$
e $\sqrt{45}$ **f** $\sqrt{1210}$ **g** $\sqrt{98}$ **h** $\sqrt{108}$
i $\sqrt{25}$ **j** $\sqrt{75}$ **k** $\sqrt{512}$

Example 6

4 Express each of the following in terms of the simplest possible surds:

a $\sqrt{\dfrac{24}{25}}$ **b** $\sqrt{\dfrac{63}{64}}$ **c** $\sqrt{\dfrac{48}{49}}$ **d** $\sqrt{\dfrac{80}{81}}$

Example 7

5 Simplify each of the following:

a $5\sqrt{3} + 11\sqrt{3} - 2\sqrt{3}$ **b** $7\sqrt{5} + 2\sqrt{5} - 4\sqrt{5}$
c $6\sqrt{2} + 4\sqrt{2} - \sqrt{13}$ **d** $4\sqrt{11} - 2\sqrt{11} - 5\sqrt{13} + 11\sqrt{13}$
e $6\sqrt{7} + 3\sqrt{17} + 4\sqrt{6} - 5\sqrt{17}$ **f** $4\sqrt{3} - 3\sqrt{6} - \sqrt{6} + 5\sqrt{3}$

Example 8

6 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$ **b** $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$ **d** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$ **f** $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

7 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$ **b** $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$ **d** $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$ **f** $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 9

8 Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(5 - \sqrt{2})^2$ **b** $(7 - \sqrt{2})(1 + \sqrt{2})$ **c** $(3 - \sqrt{2})(3 + \sqrt{2})$
d $(2 - \sqrt{2})^2$ **e** $(11 - \sqrt{5})(1 + \sqrt{5})$ **f** $(2 - \sqrt{2})(2 + \sqrt{2})$
g $(\sqrt{2} + \sqrt{3})^2$ **h** $(3\sqrt{2} - \sqrt{11})(4\sqrt{2} + \sqrt{11})$ **i** $(2\sqrt{5} - \sqrt{2})(2\sqrt{5} + \sqrt{2})$

9 **a** Express $(4\sqrt{5} - 1)(\sqrt{5} + 3)$ in the form $p + q\sqrt{5}$ where p and q are integers.

b Show that $\dfrac{\sqrt{75} - \sqrt{27}}{\sqrt{3}}$ is an integer and determine its value.

c Simplify $(3\sqrt{2} - 1)^2 + (3 + \sqrt{2})^2$

d Simplify $(\sqrt{3} + \sqrt{12})^2$

 ISBN 978-1-009-53719-3

10 Expand and simplify each of the following:

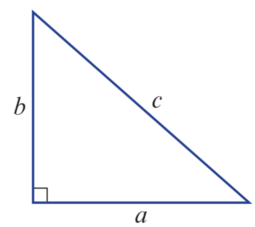
a $(2\sqrt{a} - 1)^2$ **b** $(\sqrt{x+1} + \sqrt{x+2})^2$

11 Since $8 = \sqrt{64}$ and $3\sqrt{7} = \sqrt{63}$, it is easy to see that $8 > 3\sqrt{7}$. Using the same idea, order these numbers from smallest to largest: $7,\ 3\sqrt{5},\ 5\sqrt{2},\ 4\sqrt{3}$.

12 For real numbers a and b, we have $a > b$ if and only if $a - b > 0$. Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$ **b** $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

13 The following refer to the sides of length a, b and c of a right angled triangle as shown. In each case determine the value of c as a surd in simplest form:



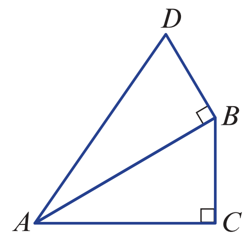
a $a = 8, b = 12$ **b** $a = 5, b = 10$

c $a = 2\sqrt{3}, b = 3\sqrt{3}$ **d** $a = 3\sqrt{3}, b = 3\sqrt{2}$

e $a = 9, b = 3\sqrt{5}$ **f** $a = 7\sqrt{2}, b = 3\sqrt{7}$

14 A right-angled triangle has a hypotenuse of length $2\sqrt{5}$ cm and one of the shorter sides has a length of $3\sqrt{2}$ cm. Determine the length of the other side.

15 Triangles ABC and ADB are right-angled triangles with the right-angles as shown. If AC has length 2 and CB and BD each have length 1 determine the exact length of:

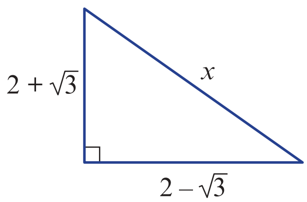


a AB

b AD

16 A rectangle has length 7 cm and width 14 cm. Determine the exact length of the diagonal.

17 A right-angled triangle is shown with side lengths $2 - \sqrt{3}$, $2 + \sqrt{3}$ and hypotenuse of length x. Determine:



a the value of x

b the perimeter of the triangle

c the area of the triangle.

18 If $x = \sqrt{2}$ evaluate each of the following:

a $x^2 - 2x + 3\sqrt{2}$ **b** $x^3 - 3x$ **c** $2x^2 - 5x - 4$

19 If $x = \sqrt{3} - 2$, express each of the following in simplest form:

a $(x - \sqrt{3})^2$ **b** $x^2 + 1$ **c** $(x - 1)^2$

 ISBN 978-1-009-53719-3 Cambridge University Press

3C Rationalising the denominator

Learning intentions

- To be able to rationalise the denominator of surd expressions.

A neat form for surds is to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers. For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

Example 10

Rationalise the denominator of each of the following:

a $\dfrac{1}{2\sqrt{7}}$ **b** $\dfrac{\sqrt{3}}{\sqrt{2}}$ **c** $\dfrac{5\sqrt{3}}{7\sqrt{5}}$

Solution

a $\dfrac{1}{2\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{14}$ **b** $\dfrac{\sqrt{3}}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{6}}{2}$ **c** $\dfrac{5\sqrt{3}}{7\sqrt{5}} \times \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{15}}{7}$

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.

Example 11

Rationalise the denominator of each of the following:

a $\dfrac{1}{2 - \sqrt{3}}$ **b** $\dfrac{1}{\sqrt{3} - \sqrt{6}}$ **c** $\dfrac{3 + \sqrt{8}}{3 - \sqrt{8}}$

Solution

a
$$\frac{1}{2 - \sqrt{3}} \times \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{4 - 3} = 2 + \sqrt{3}$$

b
$$\frac{1}{\sqrt{3} - \sqrt{6}} \times \frac{\sqrt{3} + \sqrt{6}}{\sqrt{3} + \sqrt{6}} = \frac{\sqrt{3} + \sqrt{6}}{3 - 6} = \frac{-1}{3}(\sqrt{3} + \sqrt{6})$$

c
$$\frac{3 + \sqrt{8}}{3 - \sqrt{8}} = \frac{3 + 2\sqrt{2}}{3 - 2\sqrt{2}} \times \frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} = \frac{9 + 12\sqrt{2} + 8}{9 - 8} = 17 + 12\sqrt{2}$$

Skill-sheet

Exercise 3C

Example 10

1 Express each of the following with rational denominators:

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{7}{\sqrt{7}}$ **c** $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{2}{\sqrt{3}}$ **e** $\dfrac{3}{4\sqrt{6}}$

f $\dfrac{1}{2\sqrt{2}}$ **g** $\dfrac{\sqrt{3}}{\sqrt{11}}$ **h** $\dfrac{5}{2\sqrt{7}}$ **i** $\dfrac{3}{\sqrt{12}}$ **j** $\dfrac{\sqrt{2}}{3\sqrt{10}}$

SF

 ISBN 978-1-009-53719-3

Example 11

2 Express each of the following with rational denominators:

a $\dfrac{1}{2+\sqrt{2}}$ **b** $\dfrac{1}{\sqrt{2}+1}$ **c** $\dfrac{1}{2-\sqrt{3}}$ **d** $\dfrac{1}{4-\sqrt{10}}$ **e** $\dfrac{2}{\sqrt{6}+2}$

f $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **g** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$ **h** $\dfrac{1}{3-2\sqrt{2}}$ **i** $\dfrac{3\sqrt{2}+\sqrt{5}}{3\sqrt{2}-\sqrt{5}}$ **j** $\dfrac{7+2\sqrt{11}}{7-2\sqrt{11}}$

3 Express each of the following in the form $a+b\sqrt{c}$:

a $\dfrac{2}{3-2\sqrt{2}}$ **b** $(\sqrt{5}+2)^2$ **c** $(1+\sqrt{2})(3-2\sqrt{2})$ **d** $(\sqrt{3}-1)^2$

e $\sqrt{\dfrac{1}{3}}-\dfrac{1}{\sqrt{27}}$ **f** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ **g** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ **h** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$

4 Let $x=\sqrt{2}-1$ and $y=\sqrt{5}-2$. Determine the exact value of:

a $x+y$ **b** $\dfrac{1}{x}+\dfrac{1}{y}$ **c** $\dfrac{1}{x}-\dfrac{1}{y}$

5 Solve each of the following equations for x, writing your answers in exact form with surds in simplest form and rational denominator when appropriate:

a $\sqrt{2}x=2$ **b** $\dfrac{x}{\sqrt{2}}=\sqrt{8}$ **c** $\dfrac{x}{\sqrt{2}}=\sqrt{18}$

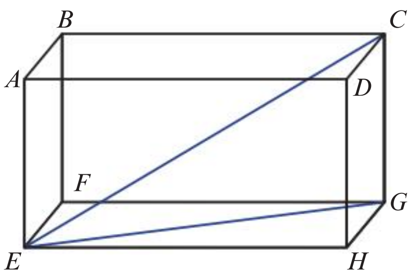
d $(\sqrt{5}-2)x=4$ **e** $(\sqrt{3}-\sqrt{2})x=2$ **f** $\dfrac{x}{\sqrt{6}-\sqrt{5}}=\sqrt{6}+\sqrt{5}$

g $(\sqrt{7}-\sqrt{6})x=1$ **h** $(\sqrt{5}-\sqrt{3})x=1$ **i** $(\sqrt{5}-\sqrt{2})x=1$

6 If $x=\sqrt{3}$, express in simplest form with rational denominator, each of the following:

a $\dfrac{1}{x}$ **b** $\dfrac{1}{x+2}$ **c** $\dfrac{1}{x+\sqrt{2}}$

7 A diagram of a rectangular prism is shown opposite. Determine the lengths of EG and EC given that:



a $EH=2$, $HG=\dfrac{1}{2}$ and $GC=\dfrac{3}{2}$

b $EH=2-\sqrt{3}$, $HG=2+\sqrt{3}$ and $GC=7\sqrt{2}$

c $EH=\dfrac{2}{\sqrt{3}}$, $HG=\dfrac{6}{\sqrt{5}}$ and $GC=6\sqrt{2}$

d $EH=\dfrac{11}{5-\sqrt{3}}$, $HG=\dfrac{11}{5+\sqrt{3}}$ and $GC=\sqrt{5}$

8 **a** Use the result that $(a+b\sqrt{2})^2=a^2+2ab\sqrt{2}+2b^2$ to write $\sqrt{3+2\sqrt{2}}$ in the form $m+n\sqrt{2}$.

b Use the result that $(a-b\sqrt{3})^2=a^2-2ab\sqrt{3}+3b^2$ to write $\sqrt{4-2\sqrt{3}}$ in the form $m+n\sqrt{3}$.

c Write each of the following in the form $m+n\sqrt{c}$ where m, n and c are integers

i $\sqrt{\frac{1}{3+2\sqrt{2}}}$ **ii** $\sqrt{\frac{4}{4-2\sqrt{3}}}$

SF

CF

Review

Chapter summary

Sets

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- **Intersection** The set $A \cap B$ contains the elements in common to A and B.
- **Union** The set $A \cup B$ contains the elements that are in A or in B (or in both).
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Surds

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order *n*** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

3A **1** I can determine the intersection and union of two sets. ☐

See Example 1, Example 2 and Questions 1 and 2

3A **2** I can illustrate intervals on the real number line. ☐

See Example 3 and Question 8

3B **3** I can multiply two surds together, for example, $3\sqrt{7} \times 5\sqrt{11}$. ☐

See Example 4 and Question 1

3B **4** I can expand powers of surds using the results that $\sqrt{a^2} = a$ and $(\sqrt{a})^2 = a$ for positive a. ☐

See Example 5 and Question 2

3B **5** I can write surds in simplest form. ☐

See Example 6 and Questions 3 and 4

3B **6** I can add like surds. ☐

See Example 7, Example 8 and Questions 5 and 6

3B **7** I can expand a product of binomial factors and collect like terms. ☐

See Example 9 and Question 8

3C **8** I can rationalise the denominator of surd expressions. ☐

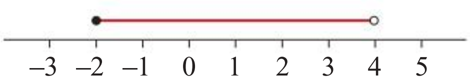
See Example 10, Example 11 and Questions 1 and 2

Short-resonse questions

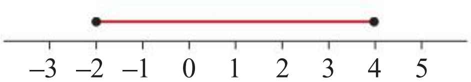
SF

1 Describe each of the following using either inequality signs or interval notation:

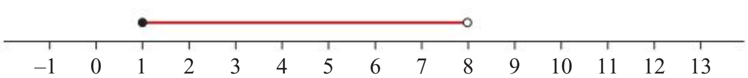
a



b

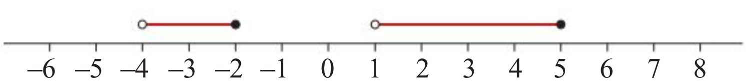


c

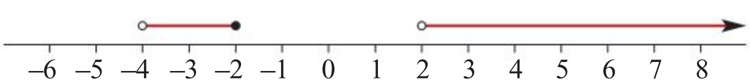


d

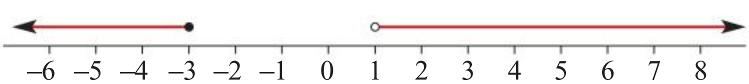
e



f



g



 ISBN 978-1-009-53719-3

Review

SF

2 Simplify each of the following:

a $4\sqrt{28} + 3\sqrt{175} - 5\sqrt{63}$ **b** $\sqrt{800} - \sqrt{288} + \sqrt{363} - \sqrt{242}$

c $3\sqrt{45} + 2\sqrt{5} - \sqrt{80}$ **d** $\sqrt{28} + 2\sqrt{63} - 5\sqrt{7}$

e $\sqrt{44} + \sqrt{275} - 4\sqrt{11}$ **f** $3\sqrt{6} - \sqrt{150} + 11\sqrt{6}$

3 Simplify each of the following:

a $\left(\sqrt{6}\right)^6$ **b** $\sqrt{18} \times \sqrt{50}$ **c** $\dfrac{24}{\sqrt{6}}$

d $(\sqrt{3} + \sqrt{27})^2$ **e** $(2 + \sqrt{10})(\sqrt{5} + \sqrt{20})$ **f** $2\sqrt{3} \times \sqrt{6}$

g $(2 + \sqrt{3})^2$ **h** $(2\sqrt{2} - 3)(3\sqrt{2} - 1)$ **i** $\dfrac{\sqrt{2} - 1}{\sqrt{2} + 1}$

j $\dfrac{\sqrt{48} - \sqrt{75}}{\sqrt{3}}$ **k** $(7 - 2\sqrt{5})(7 + 2\sqrt{5})$ **l** $\sqrt{2}(2\sqrt{3} + 1) - \sqrt{2}(\sqrt{3} - 1)$

4 Determine the values of a and b such that $\dfrac{2}{3 - \sqrt{2}} = \dfrac{1}{7}\left(a + b\sqrt{2}\right)$

CF

5 Given that $(5\sqrt{2} - a)(3\sqrt{2} + a) = b\sqrt{2} - 6$ where a and b are positive integers, determine the value of a and b.

6 A right-angled triangle has hypotenuse of length $7\sqrt{5}$ cm and the two other sides have lengths $5\sqrt{2}$ cm and x cm. Determine the value of x.

7 Simplify $\sqrt{112} - \sqrt{63} + \dfrac{224}{\sqrt{28}}$

8 A rectangle has sides of length $3 + \sqrt{2}$ and $1 + 2\sqrt{2}$. Determine:

a the perimeter of the rectangle **b** the area of the rectangle

9 A rectangular prism has a square base with edges of length $(2 - \sqrt{3})$ m. The volume of the prism is $(2\sqrt{3} - 3)$ m^3. Determine the height of the prism in the form $(a + b\sqrt{3})$ m where a and b are integers.

10 Determine the the distance between the points A and B, expressing your answer as a surd in simplest form:

a $A(1, 3), B(4, 5)$ **b** $A(2, 6), B(8, 10)$ **c** $A(7, 2), B(4, 11)$ **d** $A(3, 5), B(5, 13)$

11 If $x = \dfrac{\sqrt{5} + 1}{2}$ show that $1 + \dfrac{1}{x} = x$.

12 Write $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{6}} + \dfrac{1}{\sqrt{8}} + \dfrac{1}{\sqrt{12}}$ in simplest form with rational denominators.

 ISBN 978-1-009-53719-3

Review

Multiple-choice questions

1 $\sqrt{20}+\sqrt{45}$ is equal to

A $5\sqrt{15}$ **B** $13\sqrt{5}$ **C** $5\sqrt{5}$ **D** $\sqrt{65}$

2 $6\sqrt{45}+3\sqrt{20}$ is equal to

A $9\sqrt{65}$ **B** $9\sqrt{5}$ **C** $4\sqrt{5}$ **D** $24\sqrt{5}$

3 $\dfrac{9}{4-\sqrt{7}}$ may be written as $a+b\sqrt{7}$ where a and b are integers. The values of a and b are

A $a=1, b=4$ **B** $a=3, b=4$ **C** $a=4, b=1$ **D** $a=9, b=4$

4 $(4\sqrt{5}-1)(\sqrt{5}+3)$ can be written in the form $p+q\sqrt{5}$ where p and q are integers. The values of p and q are

A $p=2, q=3$ **B** $p=11, q=12$ **C** $p=3, q=2$ **D** $p=17, q=11$

5 $\dfrac{3}{\sqrt{2}}+\dfrac{3}{2+\sqrt{2}}$ is equal to

A $\dfrac{6-\sqrt{3}}{2}$ **B** $\dfrac{6-3\sqrt{3}}{2}$ **C** 3 **D** 6

6 If $(\sqrt{5}-1)\sqrt{k}=\sqrt{2}$, then k is equal to

A $\dfrac{3-\sqrt{5}}{4}$ **B** $\dfrac{3+\sqrt{5}}{4}$ **C** $3+\sqrt{5}$ **D** $6+\sqrt{5}$

7 $\sqrt{288}+\sqrt{200}-\sqrt{392}$ is equal to

A $30\sqrt{2}$ **B** $\dfrac{3+\sqrt{5}}{4}$ **C** $8\sqrt{2}$ **D** $16\sqrt{2}$

8 $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}-\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ is equal to

A 0 **B** $\dfrac{1}{2}$ **C** 3 **D** $2\sqrt{3}$

9 If $x+\dfrac{1}{x}=1+\sqrt{3}$ then $x^2+\dfrac{1}{x^2}$ is equal to

A 1 **B** 4 **C** $1+2\sqrt{3}$ **D** $2+2\sqrt{3}$

10 If $x=3+2\sqrt{2}$, then the value of $\sqrt{x}-\dfrac{1}{\sqrt{x}}$ is

A 1 **B** 2 **C** $2\sqrt{2}$ **D** $3\sqrt{3}$

11 The expression $\dfrac{\sqrt{6}+1}{\sqrt{3}+\sqrt{2}}$ can be written in the form $a\sqrt{3}+b\sqrt{2}$. The values of a and b are

A $a=2, b=1$ **B** $a=-1, b=2$ **C** $a=1, b=-1$ **D** $a=6, b=-6$

 ISBN 978-1-009-53719-3 Cambridge University Press

4 Quadratics

Chapter contents

A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and $a_0, a_1, \ldots, a_n$ are numbers called **coefficients**.

The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient. For example:

- $y = 2x + 3$ is a polynomial function of degree 1
- $y = 2x^2 + 3x - 2$ is a polynomial function of degree 2.

This chapter deals with polynomials of degree 2. These are called **quadratic polynomials**.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

4A Expanding and collecting like terms

Learning intentions

- To be able to expand and collect like terms when dealing with algebraic expressions.

In order to sketch graphs of quadratics, we need to determine the x-axis intercepts (if they exist), and to do this we need to solve quadratic equations. As an introduction to the methods of solving quadratic equations, the first two sections of this chapter review the basic algebraic processes of expansion and factorisation.

An algebraic expression is the sum of its **terms**. For example:

- The terms of the linear expression $3x - 1$ are $3x$ and -1.
- The terms of the quadratic expression $-2x^2 + 3x - 4$ are $-2x^2$, $3x$ and -4.

Example 1

Simplify $2(x - 5) - 3(x + 5)$ by first expanding.

Solution	Explanation
$2(x - 5) - 3(x + 5) = 2x - 10 - 3x - 15$	Expand each bracket.
$= 2x - 3x - 10 - 15$	Collect like terms.
$= -x - 25$	

Example 2

Expand $2x(3x - 2) + 3x(x - 2)$.

Solution

$$\begin{aligned} 2x(3x - 2) + 3x(x - 2) &= 6x^2 - 4x + 3x^2 - 6x \\ &= 9x^2 - 10x \end{aligned}$$

For expansions of the type $(a + b)(c + d)$, proceed as follows:

$$\begin{aligned} (a + b)(c + d) &= a(c + d) + b(c + d) \\ &= ac + ad + bc + bd \end{aligned}$$

Example 3

Expand the following:

a $(x + 3)(2x - 3)$ **b** $(x - 3)(2x - 2\sqrt{2})$

Solution	Explanation
a $(x + 3)(2x - 3)$	Each term in the second pair of brackets is multiplied by each term in the first.
$= x(2x - 3) + 3(2x - 3)$	
$= 2x^2 - 3x + 6x - 9$	
$= 2x^2 + 3x - 9$	

 ISBN 978-1-009-53719-3

b $(x-3)(2x-2\sqrt{2})$

$= x(2x-2\sqrt{2}) - 3(2x-2\sqrt{2})$

$= 2x^2 - 2\sqrt{2}x - 6x + 6\sqrt{2}$

$= 2x^2 - (2\sqrt{2}+6)x + 6\sqrt{2}$

Be careful with negative signs.

You can also complete binomial expansions with a table; this emphasises the terms.

	x	-3
$2x$	$2x^2$	$-6x$
$-2\sqrt{2}$	$-2\sqrt{2}x$	$6\sqrt{2}$

You add the terms to complete the expansion.

Example 4

Expand $(2x-1)(3x^2+2x+4)$.

Solution

$$\begin{aligned}(2x-1)(3x^2+2x+4) &= 2x(3x^2+2x+4) - 1(3x^2+2x+4)\\ &= 6x^3+4x^2+8x-3x^2-2x-4\\ &= 6x^3+x^2+6x-4\end{aligned}$$

Perfect squares

Consider the expansion of a perfect square, $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Thus the general result can be stated as:

$$(a+b)^2 = a^2+2ab+b^2$$

That is, to expand $(a+b)^2$ take the sum of the squares of the terms and add twice the product of the terms.

Example 5

Expand $(3x-2)^2$.

Solution

$$\begin{aligned}(3x-2)^2 &= (3x)^2+2(3x)(-2)+(-2)^2\\ &= 9x^2-12x+4\end{aligned}$$

Explanation

Use the expansion $(a+b)^2 = a^2+2ab+b^2$.
Here $a=3x$ and $b=-2$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Difference of two squares

Consider the expansion of $(a + b)(a - b)$:

$$\begin{aligned}(a + b)(a - b) &= a(a - b) + b(a - b)\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

Thus the expansion of the difference of two squares has been obtained:

$$(a + b)(a - b) = a^2 - b^2$$

Example 6

a Expand $(2x - 4)(2x + 4)$.

b Expand $(x - 2\sqrt{7})(x + 2\sqrt{7})$.

Solution

a $$\begin{aligned}(2x - 4)(2x + 4) &= (2x)^2 - (4)^2\\ &= 4x^2 - 16\end{aligned}$$

b $$\begin{aligned}(x - 2\sqrt{7})(x + 2\sqrt{7}) &= x^2 - (2\sqrt{7})^2\\ &= x^2 - 28\end{aligned}$$

Example 7

Expand $(2a - b + c)(2a - b - c)$.

Solution

$$\begin{aligned}(2a - b + c)(2a - b - c) &= ((2a - b) + c)((2a - b) - c)\\ &= (2a - b)^2 - c^2\\ &= 4a^2 - 4ab + b^2 - c^2\end{aligned}$$

Summary 4A

- A **polynomial function** has a rule of the type

  $$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and $a_0, a_1, \ldots, a_n$ are numbers called **coefficients**.
- The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient.
- A polynomial function of degree 2 is called a **quadratic function**. The general rule is of the form $y = ax^2 + bx + c$, where $a \neq 0$.
- General binomial expansion:

  $$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$

- Perfect square expansion:

  $$(a + b)^2 = a^2 + 2ab + b^2$$

- Difference of two squares expansion:

  $$(a + b)(a - b) = a^2 - b^2$$

ISBN 978-1-009-53719-3

Exercise 4A

SF

1 Expand each of the following:

a $2(x-4)$ **b** $-2(x-4)$ **c** $3(2x-4)$
d $-3(4-2x)$ **e** $x(x-1)$ **f** $2x(x-5)$

2 Collect like terms in each of the following:

a $2x+4x+1$ **b** $2x-6+x$
c $3x+1-2x$ **d** $-x+2x-3+4x$

Example 1

3 Simplify each of the following by expanding and collecting like terms:

a $8(2x-3)-2(x+4)$ **b** $2x(x-4)-3x$
c $4(2-3x)+4(6-x)$ **d** $4-3(5-2x)$

Example 2

4 Simplify each of the following by expanding and collecting like terms:

a $2x(x-4)-3x$ **b** $2x(x-5)+x(x-5)$ **c** $2x(-10-3x)$
d $3x(2-3x+2x^2)$ **e** $3x-2x(2-x)$ **f** $3(4x-2)-6x$

Example 3

5 Simplify each of the following by expanding and collecting like terms:

a $(3x-7)(2x+4)$ **b** $(x-10)(x-12)$ **c** $(3x-1)(12x+4)$
d $(4x-5)(2x-3)$ **e** $(x-\sqrt{3})(x-2)$ **f** $(2x-\sqrt{5})(x+\sqrt{5})$
g $(3x-2\sqrt{7})(x+\sqrt{7})$ **h** $(5x-3)(x+2\sqrt{2})$ **i** $(\sqrt{5}x-3)(\sqrt{5}x-32\sqrt{2})$

Example 4

6 Simplify each of the following by expanding and collecting like terms:

a $(2x-3)(3x^2+2x-4)$ **b** $(x-1)(x^2+x+1)$
c $(6-2x-3x^2)(4-2x)$ **d** $(5x-3)(x+2)-(2x-3)(x+3)$
e $(2x+3)(3x-2)-(4x+2)(4x-2)$

Example 5

7 Simplify each of the following by expanding and collecting like terms:

a $(x-4)^2$ **b** $(2x-3)^2$ **c** $(6-2x)^2$
d $\left(x-\dfrac{1}{2}\right)^2$ **e** $(x-\sqrt{5})^2$ **f** $(x-2\sqrt{3})^2$

Example 6

8 Simplify each of the following by expanding and collecting like terms:

a $(x-3)(x+3)$ **b** $(2x-4)(2x+4)$
c $(9x-11)(9x+11)$ **d** $(2x-3)(2x+3)$
e $(2x+5)(2x-5)$ **f** $(x-\sqrt{5})(x+\sqrt{5})$
g $(2x+3\sqrt{3})(2x-3\sqrt{3})$ **h** $(\sqrt{3}x-\sqrt{7})(\sqrt{3}x+\sqrt{7})$

Example 7

9 Simplify each of the following by expanding and collecting like terms:

a $(x-y+z)(x-y-z)$ **b** $(2a-b+c)(2a-b-c)$
c $(3w-4z+u)(3w+4z-u)$ **d** $(2a-\sqrt{5}b+c)(2a+\sqrt{5}b+c)$

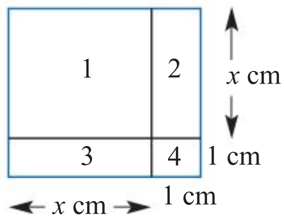
Mathematical Methods Units 1 & 2   ISBN 978-1-009-53719-3

10 Determine the area of each of the following by:

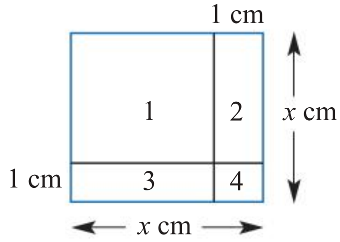
i adding the areas of the four 'non-overlapping' rectangles (two of which are squares)

ii multiplying length by width of the undivided square (boundary in blue).

a



b



4B Factorising

Learning intentions

- To be able to factorise algebraic expressions including quadratic expressions.

Four different types of factorisation will be considered.

Factorisation using common factors

If each term in an algebraic expression to be factorised contains a **common factor**, then this common factor is a factor of the entire expression. To determine the other factor, divide each term by the common factor. The common factor is placed outside the brackets. This process is known as 'taking the common factor outside the brackets'.

Example 8

a Factorise $9x^2 + 81x$. **b** Factorise $2a^2 - 8ax^2$.

Solution

a $9x^2 + 81x = 9x \times x + 9x \times 9$
$= 9x(x + 9)$

b $2a^2 - 8ax^2 = 2a \times a - 2a \times 4x^2$
$= 2a(a - 4x^2)$

Note: The answers can be checked by expanding.

Explanation

The common factor $9x$ is 'taken out' of the brackets.

The common factor $2a$ is 'taken out' of the brackets.

In general, take out as many common factors as possible.

Example 9

Factorise $7x^2y - 35xy^2$.

Solution

$7x^2y - 35xy^2 = 7xy(x - 5y)$

Explanation

The common factor $7xy$ is 'taken out' of the brackets.

Grouping of terms

This method can be used for expressions containing four terms.

Example 10

Factorise $x^3 + 4x^2 - 3x - 12$.

Solution

$$\begin{aligned} x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12) \\ &= x^2(x + 4) - 3(x + 4) \\ &= (x^2 - 3)(x + 4) \end{aligned}$$

Explanation

The terms in this expression can be grouped as shown.

The common factor $(x + 4)$ is 'taken out' of the brackets.

Difference of two squares

You will recall the following identity from the previous section:

$$(a + b)(a - b) = a^2 - b^2$$

We can now use the result the other way in order to factorise:

$$a^2 - b^2 = (a + b)(a - b)$$

Example 11

a Factorise $3x^2 - 75$.

b Factorise $9x^2 - 36$.

Solution

a $$\begin{aligned} 3x^2 - 75 &= 3(x^2 - 25) \\ &= 3(x + 5)(x - 5) \end{aligned}$$

b $$\begin{aligned} 9x^2 - 36 &= 9(x^2 - 4) \\ &= 9(x - 2)(x + 2) \end{aligned}$$

Explanation

a First 'take out' the common factor 3.

Use the difference of squares identity.

b First 'take out' the common factor 9.

Use the difference of squares identity.

Example 12

Factorise $(x - y)^2 - 16y^2$.

Solution

$$\begin{aligned} (x - y)^2 - 16y^2 &= (x - y)^2 - (4y)^2 \\ &= (x - y + 4y)(x - y - 4y) \\ &= (x + 3y)(x - 5y) \end{aligned}$$

Explanation

Use the difference of squares identity

$$a^2 - b^2 = (a + b)(a - b)$$

with $a = (x - y)$ and $b = 4y$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Factorising quadratic polynomials

A quadratic polynomial is an expression of the form $ax^2 + bx + c$ with $a \neq 0$. We have seen in the previous section that we can expand a product of two binomial factors to obtain a quadratic expression. For example:

$$\begin{aligned}(x+2)(x-4) &= x(x-4) + 2(x-4)\\ &= x^2 - 4x + 2x - 8\\ &= x^2 - 2x - 8\end{aligned}$$

We want to be able to reverse this process. That is, we want to start from the expanded expression and obtain the factorised form. We have already done this for expressions that are differences of squares. We now turn our attention to the general case.

Example 13

Factorise $x^2 - 2x - 8$.

Solution

Using the method described in the explanation opposite, we can factorise without any further setting out:

$$x^2 - 2x - 8 = (x-4)(x+2)$$

Alternatively, we can reverse the process we used for expanding:

$$\begin{aligned}x^2 - 2x - 8 &= x^2 - 4x + 2x - 8\\ &= x(x-4) + 2(x-4)\\ &= (x-4)(x+2)\end{aligned}$$

Explanation

We want

$$\begin{aligned}x^2 - 2x - 8 &= (x+a)(x+b)\\ &= x^2 + (a+b)x + ab\end{aligned}$$

The values of a and b are such that $ab = -8$ and $a + b = -2$.

Values of a and b which satisfy these two conditions are $a = -4$ and $b = 2$.

A quadratic polynomial is called a **monic quadratic polynomial** if the coefficient of x^2 is 1. The quadratic polynomial $x^2 - 2x - 8$ factorised in the previous example is monic.

Factorising non-monic quadratic polynomials involves a slightly different approach. We need to consider all possible combinations of factors of the x^2 term and the constant term. The next example and the following discussion give two methods.

Example 14

Factorise $6x^2 - 13x - 15$.

Solution

There are several combinations of factors of $6x^2$ and -15 to consider. Only one combination is correct.

$$6x^2 - 13x - 15 = (6x+5)(x-3)$$

Factors of $6x^2$	Factors of -15	'Cross-products' add to give $-13x$
$6x$	$+5$	$+5x$
x	-3	$-18x$
		$-13x$

ISBN 978-1-009-53719-3

Here is a second method for factorising $6x^2 - 13x - 15$ which still requires some trial and error but is more systematic. It is the reverse process of expanding $(x - 3)(6x + 5)$.

We let

$$ax^2 + bx + c = (\alpha x + \gamma)(\beta x + \delta)$$

Expanding the right-hand side gives

$$ax^2 + bx + c = \alpha\beta x^2 + (\gamma\beta + \alpha\delta)x + \gamma\delta$$

Note that $ac = \alpha\beta\gamma\delta$ and $b = \gamma\beta + \alpha\delta$.

We now apply this to factorising $6x^2 - 13x - 15$.

First we look for two numbers that multiply together to give ac and add to give b. That is, we look for two numbers whose product is $6 \times (-15) = -90$ and whose sum is -13.

The two numbers are -18 and 5. We write:

$$\begin{aligned} 6x^2 - 13x - 15 &= 6x^2 - 18x + 5x - 15 \\ &= 6x(x - 3) + 5(x - 3) \\ &= (x - 3)(6x + 5) \end{aligned}$$

Example 15

Factorise $8x^2 + 2x - 15$.

Solution

$$\begin{aligned} 8x^2 + 2x - 15 &= 8x^2 + 12x - 10x - 15 \\ &= 4x(2x + 3) - 5(2x + 3) \\ &= (4x - 5)(2x + 3) \end{aligned}$$

Explanation

Here $ac = 8 \times (-15) = -120$ and $b = 2$.

We look for two numbers whose product is -120 and whose sum is 2.

The two numbers are 12 and -10. So we write $2x = 12x - 10x$.

It is sometimes possible to take out a common factor first to simplify the factorisation.

Example 16

Factorise $2x^2 + 6x - 20$.

Solution

$$\begin{aligned} 2x^2 + 6x - 20 &= 2(x^2 + 3x - 10) \\ &= 2(x + 5)(x - 2) \end{aligned}$$

Explanation

The common factor 2 is ‘taken out’ first.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 17

Factorise $(x+1)^2 - 2(x+1) - 3$.

Solution

$$\begin{aligned}(x+1)^2 - 2(x+1) - 3 &= a^2 - 2a - 3\\ &= (a-3)(a+1)\\ &= (x+1-3)(x+1+1)\\ &= (x-2)(x+2)\end{aligned}$$

Explanation

The substitution $a = x + 1$ makes it easier to recognise the required factorisation.

Summary 4B

- Difference of two squares identity: $a^2 - b^2 = (a+b)(a-b)$.
- Factorisation of monic quadratics: To factorise a quadratic of the form $x^2 + bx + c$, determine two numbers whose sum is the coefficient of x and whose product is the constant term.
- Factorisation of general quadratics: To factorise a quadratic of the form $ax^2 + bx + c$, determine two numbers e and f whose product is ac and whose sum is b. Split the middle term bx as $ex + fx$ and then factorise by grouping.

Exercise 4B

SF

1 Factorise each of the following:

a $2x + 4$ **b** $4a - 8$ **c** $6 - 3x$
d $2x - 10$ **e** $18x + 12$ **f** $24 - 16x$

Example 8

2 Factorise:

a $4x^2 - 2xy$ **b** $8ax + 32xy$ **c** $6ab - 12b$
d $6xy + 14x^2y$ **e** $x^2 + 2x$ **f** $5x^2 - 15x$
g $-4x^2 - 16x$ **h** $7x + 49x^2$ **i** $2x - x^2$

Example 9

3 Factorise:

a $6x^3y^2 + 12y^2x^2$ **b** $7x^2y - 6y^2x$ **c** $8x^2y^2 + 6y^2x$

Example 10

4 Factorise:

a $x^3 + 5x^2 + x + 5$ **b** $xy + 2x + 3y + 6$ **c** $x^2y^2 - x^2 - y^2 + 1$
d $ax + ay + bx + by$ **e** $a^3 - 3a^2 + a - 3$ **f** $2ab - 12a - 5b + 30$
g $2x^2 - 2x + 5x - 5$ **h** $x^3 - 4x + 2x^2 - 8$ **i** $x^3 - bx^2 - a^2x + a^2b$

Example 11

5 Factorise:

a $x^2 - 36$ **b** $x^2 - 81$ **c** $x^2 - a^2$ **d** $4x^2 - 81$
e $9x^2 - 16$ **f** $25x^2 - y^2$ **g** $3x^2 - 48$ **h** $2x^2 - 98$
i $3ax^2 - 27a$ **j** $a^2 - 7$ **k** $2a^2 - 5$ **l** $x^2 - 12$

 ISBN 978-1-009-53719-3

Example 12 **6** Factorise:

a $(x-2)^2-16$ **b** $25-(2+x)^2$ **c** $3(x+1)^2-12$
d $(x-2)^2-(x+3)^2$ **e** $(2x-3)^2-(2x+3)^2$ **f** $(2x-1)^2-(3x+6)^2$

Example 13 **7** Factorise:

a $x^2-7x-18$ **b** $y^2-19y+48$ **c** $a^2-14a+24$
d $a^2+18a+81$ **e** $x^2-5x-24$ **f** $x^2-2x-120$

Example 14, 15 **8** Factorise:

a $3x^2-7x+2$ **b** $6x^2+7x+2$ **c** $5x^2+23x+12$
d $2x^2+9x+4$ **e** $6x^2-19x+10$ **f** $6x^2-7x-3$
g $12x^2-17x+6$ **h** $5x^2-4x-12$ **i** $5x^3-16x^2+12x$

Example 16 **9** Factorise:

a $3y^2-12y-36$ **b** $2x^2-18x+28$ **c** $4x^2-36x+72$
d $3x^2+15x+18$ **e** $ax^2+7ax+12a$ **f** $48x-24x^2+3x^3$

Example 17 **10** Factorise:

a $(x-1)^2+4(x-1)+3$
b $2(x-1)^2+5(x-1)-3$
c $(2x+1)^2+7(2x+1)+12$

SF

CF

4C Quadratic equations

Learning intentions

▶ To be able to solve quadratic equations by simple factorisation.

This section looks at the solution of quadratic equations by simple factorisation. There are three steps to solving a quadratic equation by factorisation:

Step 1 Write the equation in the form $ax^2+bx+c=0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the result that $mn=0$ implies $m=0$ or $n=0$ (or both); this is known as the **null factor theorem**.

For example, to solve the equation $x^2-x=12$:

$$\begin{aligned} x^2-x&=12 \\ x^2-x-12&=0 && \text{(Step 1)} \\ (x-4)(x+3)&=0 && \text{(Step 2)} \\ \therefore\quad x-4=0 \quad &\text{or} \quad x+3=0 && \text{(Step 3)} \\ x=4 \quad &\text{or} \quad x=-3 \end{aligned}$$

In the simplest cases, the first two steps may have been done already.

 ISBN 978-1-009-53719-3

Example 18

Solve $x^2 + 11x + 24 = 0$.

Solution

x^2	$+24$	$+11x$
x	$+3$	$+3x$
x	$+8$	$+8x$
		$+11x$

Factorising gives

$$x^2 + 11x + 24 = 0$$
$$(x + 3)(x + 8) = 0$$
$$\therefore \quad x + 3 = 0 \quad \text{or} \quad x + 8 = 0$$
$$x = -3 \quad \text{or} \quad x = -8$$

Explanation

The quadratic can also be factorised in the following way:

$$\begin{aligned} x^2 + 11x + 24 &= x^2 + 8x + 3x + 24 \\ &= x(x + 8) + 3(x + 8) \\ &= (x + 8)(x + 3) \end{aligned}$$

Note: We can check the answer for this example by substituting into the equation:

$$(-3)^2 + 11(-3) + 24 = 0$$
$$(-8)^2 + 11(-8) + 24 = 0$$

Example 19

Solve $2x^2 + 5x - 12 = 0$.

Solution

$2x^2$	-12	$+5x$
$2x$	-3	$-3x$
x	$+4$	$+8x$
		$+5x$

Factorising gives

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$
$$\therefore \quad 2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -4$$

Explanation

The quadratic can also be factorised in the following way:

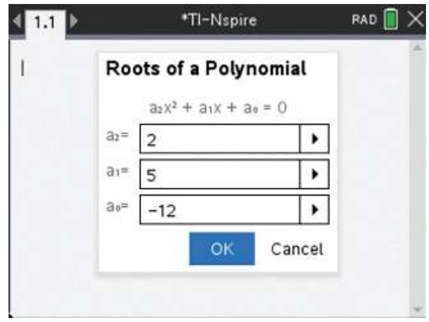
$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 + 8x - 3x - 12 \\ &= 2x(x + 4) - 3(x + 4) \\ &= (2x - 3)(x + 4) \end{aligned}$$

 ISBN 978-1-009-53719-3

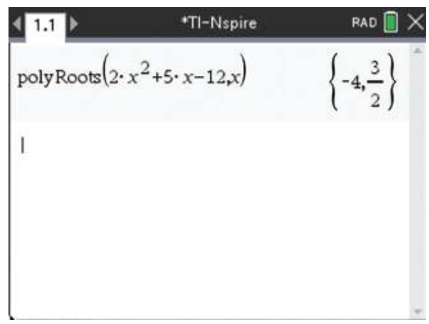
Using the TI-Nspire CX non-CAS

To solve the quadratic equation $2x^2 + 5x - 12 = 0$, use a **Calculator** application.

- Select (menu) > **Algebra** > **Polynomial Tools** > **Find Roots of Polynomial**.
- Select OK in the initial pop-up screen, and then complete as shown opposite.



- Hence $x = -4$ or $x = \frac{3}{2}$.



Note: You can also use (menu) > **Algebra** > **Numerical Solve**. But you need to be careful, as quadratic equations can have two solutions. You may need to try several guess values using the syntax: nSolve(*equation*, x = *guess value*).

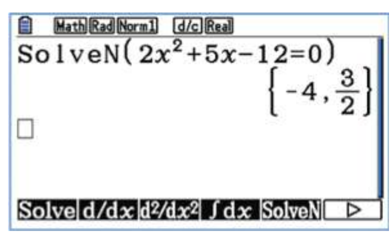
Using the Casio

To solve the quadratic equation $2x^2 + 5x - 12 = 0$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** (OPTN) (F4), then **SolveN** (F5).
- Enter the equation:

  (2) (X,θ,T) (x^2) (+) (5) (X,θ,T) (−) (1) (2)
  (SHIFT) (•) (0) ( )) (EXE)

- Press (EXIT).
- Hence $x = -4$ or $x = \frac{3}{2}$.



 ISBN 978-1-009-53719-3 Cambridge University Press

Applications of quadratic equations

Problems involving the solution of quadratic equations arise in many situations. We will meet more such problems in Section 4L.

Example 20

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Calculate the length and width of the rectangle.

Solution

Let x cm be the length of the rectangle and y cm the width.

Then $2(x + y) = 20$ and thus $y = 10 - x$.

The area is 24 cm^2 and therefore $xy = x(10 - x) = 24$.

i.e. $10x - x^2 = 24$

$$x^2 - 10x + 24 = 0$$

$$(x - 6)(x - 4) = 0$$

Thus the length is 6 cm or 4 cm. The width is 4 cm or 6 cm.

Summary 4C

To solve a quadratic equation by factorisation:

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factorise the quadratic polynomial.

Step 3 Use the result that $mn = 0$ implies $m = 0$ or $n = 0$ (or both).

Skill-sheet

Exercise 4C

SF

1 Solve each of the following for x:

a $(x - 2)(x - 3) = 0$ **b** $x(2x - 4) = 0$ **c** $(x - 4)(2x - 6) = 0$

d $(3 - x)(x - 4) = 0$ **e** $(2x - 6)(x + 4) = 0$ **f** $2x(x - 1) = 0$

g $(5 - 2x)(6 - x) = 0$ **h** $x^2 = 16$

2 Use a calculator to solve each of the following equations. Give your answer correct to two decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 - 4x - 3 = 0$ **c** $-2x^2 - 4x + 3 = 0$

Example 18

3 Solve for x in each of the following:

a $x^2 - x - 72 = 0$ **b** $x^2 - 6x + 8 = 0$ **c** $x^2 - 8x - 33 = 0$

d $x(x + 12) = 64$ **e** $x^2 + 5x - 14 = 0$ **f** $x^2 = 5x + 24$

 ISBN 978-1-009-53719-3

Example 19

4 Solve for x in each of the following:

a $2x^2 + 5x + 3 = 0$ **b** $4x^2 - 8x + 3 = 0$ **c** $6x^2 + 13x + 6 = 0$

d $2x^2 - x = 6$ **e** $6x^2 + 15 = 23x$ **f** $2x^2 - 3x - 9 = 0$

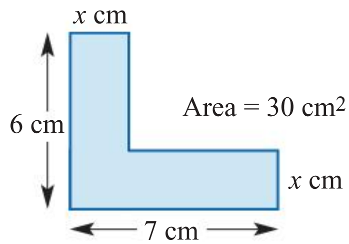
g $10x^2 - 11x + 3 = 0$ **h** $12x^2 + x = 6$ **i** $4x^2 + 1 = 4x$

j $x(x + 4) = 5$ **k** $\frac{1}{7}x^2 = \frac{3}{7}x$ **l** $x^2 + 8x = -15$

m $5x^2 = 11x - 2$

SF

5 Calculate the value of x.



CF

6 The bending moment, M, of a simple beam used in bridge construction is given by the formula

$$M = \frac{w\ell}{2}x - \frac{w}{2}x^2$$

If $\ell = 13$ m, $w = 16$ kg/m and $M = 288$ kg m, calculate the value of x.

7 The height, h metres, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 metres.

8 A polygon with n sides has $\frac{n(n-3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

9 For a particular electric train, the tractive 'resistance' R at speed v km/h is given by $R = 1.6 + 0.03v + 0.003v^2$. Determine v when the tractive resistance is 10.6.

Example 20

10 The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate the length and width of the rectangle.

11 The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the altitude.

12 Tickets for a concert are available at two prices. The more expensive ticket is \$30 more than the cheaper one. Determine the cost of each type of ticket if a group can buy 10 more of the cheaper tickets than the expensive ones for \$1800.

13 The members of a club hire a bus for \$2100. Seven members withdraw from the club and the remaining members have to pay \$10 more each to cover the cost. How many members originally agreed to go on the bus?

 ISBN 978-1-009-53719-3

4D Graphing quadratics

Learning intentions

- To be able to graph quadratic relations which are simple transformations of $y = x^2$.

A quadratic polynomial function is defined by the general rule

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$. This is called **polynomial form**.

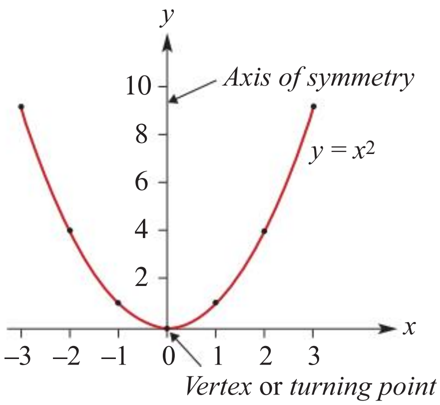
The parabola $y = x^2$

The simplest quadratic function is $y = x^2$. If a table of values is constructed for $y = x^2$ for $-3 \leq x \leq 3$, these points can be plotted and then connected to produce a continuous curve.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

Features of the graph of $y = x^2$:

- The graph is called a **parabola**.
- The possible y-values are all positive real numbers and 0. (This is called the **range** of the quadratic and is discussed in a more general context in Chapter 6.)
- The graph is symmetrical about the y-axis. The line about which the graph is symmetrical is called the **axis of symmetry**.
- The graph has a **vertex** or **turning point** at the origin (0, 0).
- The minimum value of y is 0 and it occurs at the turning point.



Transformations of $y = x^2$

By a process called **completing the square** (to be discussed in Section 4E), all quadratics in polynomial form $y = ax^2 + bx + c$ may be transposed into what will be called the **turning point form**:

$$y = a(x - h)^2 + k$$

We first consider the effect of changing the value of a for our basic graph of $y = x^2$.

We then consider the effect of changing h and k for graphs of the form $y = ax^2$. Graphs of the form $y = a(x - h)^2 + k$ are formed by **translating** the graph of $y = ax^2$. The graph of $y = a(x - h)^2 + k$ is exactly the same shape as $y = ax^2$. All of these graphs are indeed congruent to $y = ax^2$ and each other.

Graphs of $y = ax^2$

We first consider graphs of the form $y = ax^2$. In this case both $h = 0$ and $k = 0$. In the basic graph of $y = x^2$, the value of a is 1.

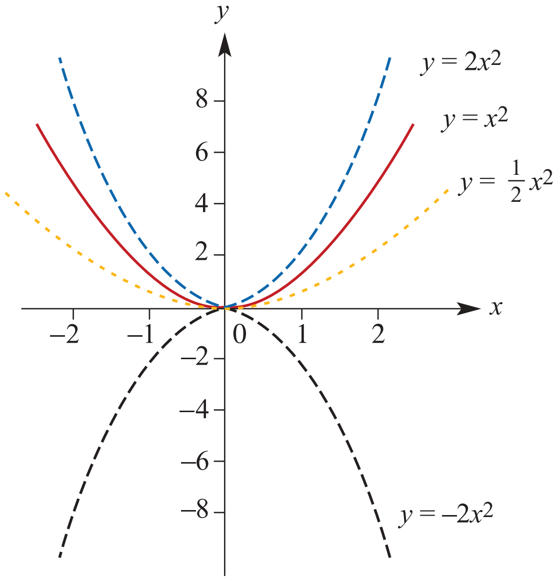
The following graphs are shown on the same set of axes:

$y = x^2$

$y = 2x^2 \qquad (a = 2)$

$y = \frac{1}{2}x^2 \qquad (a = \frac{1}{2})$

$y = -2x^2 \qquad (a = -2)$



If $a > 1$, the graph is 'narrower'. If $0 < a < 1$, the graph is 'broader'. The transformation which produces the graph of $y = 2x^2$ from the graph of $y = x^2$ is called a **dilation of factor 2 from the x-axis**.

When a is negative, the graph is reflected in the x-axis. The transformation which produces the graph of $y = -x^2$ from the graph of $y = x^2$ is called a **reflection in the x-axis**.

Graphs of $y = x^2 + k$

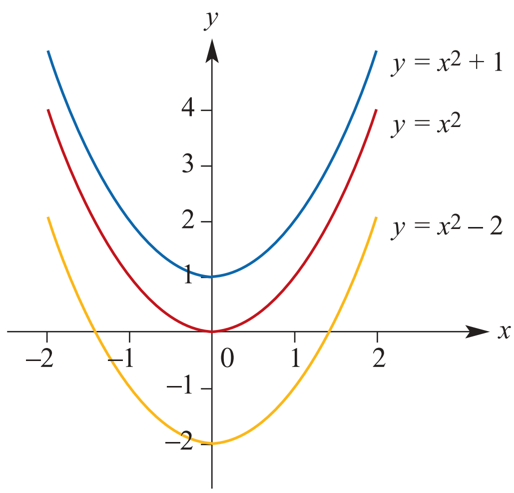
On this set of axes are the graphs of

$y = x^2$

$y = x^2 - 2 \qquad (k = -2)$

$y = x^2 + 1 \qquad (k = 1)$

As can be seen, changing k moves the basic graph of $y = x^2$ in a vertical direction.



- When $k = -2$ the graph is **translated** 2 units in the negative direction of the y-axis. The vertex is now $(0, -2)$ and the range is now all real numbers greater than or equal to -2.
- When $k = 1$ the graph is **translated** 1 unit in the positive direction of the y-axis. The vertex is now $(0, 1)$ and the range is now all real numbers greater than or equal to 1.

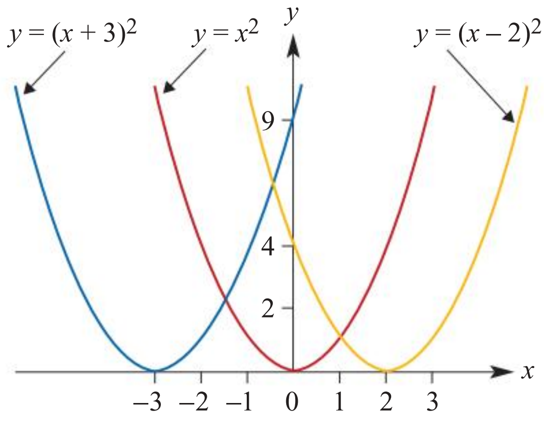
All other features of the graph are unchanged. The axis of symmetry is still the y-axis.

 ISBN 978-1-009-53719-3

Graphs of $y = (x - h)^2$

On this set of axes are the graphs of

$$y = x^2$$
$$y = (x-2)^2 \qquad (h = 2)$$
$$y = (x+3)^2 \qquad (h = -3)$$



As can be seen, changing h moves the graph in a horizontal direction.

- When $h = 2$ the graph is **translated** 2 units in the positive direction of the x-axis. The vertex is now $(2, 0)$ and the axis of symmetry is now the line $x = 2$.
- When $h = -3$ the graph is **translated** 3 units in the negative direction of the x-axis. The vertex is now $(-3, 0)$ and the axis of symmetry is now the line $x = -3$.

In both cases, the range is unchanged and is still all non-negative real numbers.

Examples of transformations

By combining dilations, reflections and translations, we can sketch the graph of any quadratic expressed in the form $y = a(x - h)^2 + k$:

- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 21

Sketch the graph of $y = x^2 - 3$.

Solution

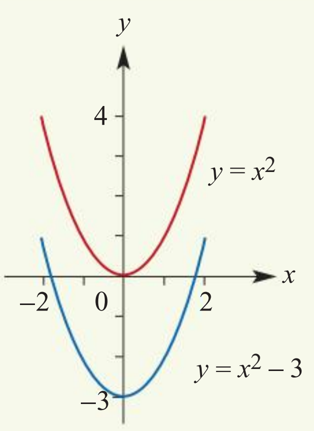
The graph of $y = x^2 - 3$ is obtained from the graph of $y = x^2$ by translating 3 units in the negative direction of the y-axis.

The vertex is now at $(0, -3)$. The axis of symmetry is the line with equation $x = 0$.

To determine the x-axis intercepts, let $y = 0$:

$$0 = x^2 - 3$$
$$x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

Hence the x-axis intercepts are $\pm\sqrt{3}$.



ISBN 978-1-009-53719-3

Example 22

Sketch the graph of $y = -(x + 1)^2$.

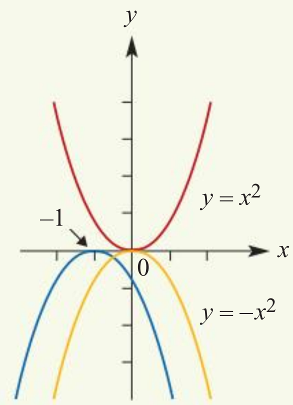
Solution

The graph of $y = -(x + 1)^2$ is obtained from the graph of $y = x^2$ by a reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis.

The vertex is now at $(-1, 0)$.

The axis of symmetry is the line with equation $x = -1$.

The x-axis intercept is -1.



Example 23

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

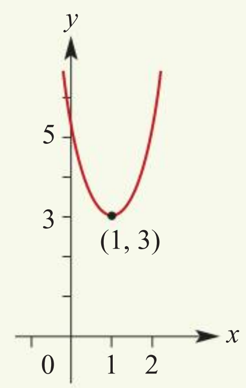
The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range will be $y \geq 3$.

To add further detail to our graph, we can determine the axis intercepts:



y-axis intercept

When $x = 0$, $y = 2(0 - 1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$\begin{aligned} 0 &= 2(x-1)^2 + 3 \\ -3 &= 2(x-1)^2 \\ -\tfrac{3}{2} &= (x-1)^2 \end{aligned}$$

As the square root of a negative number is not a real number, this equation has no real solutions.

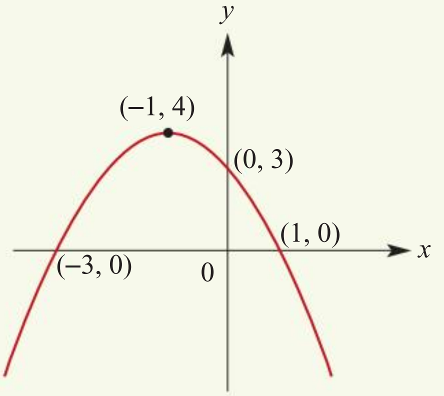
ISBN 978-1-009-53719-3

Example 24

Sketch the graph of $y = -(x+1)^2 + 4$.

Solution

The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.



y-axis intercept

When $x = 0$,

$$\begin{aligned} y &= -(0+1)^2 + 4 \\ &= 3 \end{aligned}$$

$\therefore$ the y-axis intercept is 3.

x-axis intercepts

When $y = 0$,

$$\begin{aligned} -(x+1)^2 + 4 &= 0 \\ (x+1)^2 &= 4 \\ x + 1 &= \pm 2 \\ x &= \pm 2 - 1 \end{aligned}$$

$\therefore$ the x-axis intercepts are 1 and -3.

Summary 4D

- The graph of $y = x^2$ is called a parabola. The vertex (or turning point) is the point $(0, 0)$ and the axis of symmetry is the y-axis.
- The graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ in the x-axis.
- For $y = ax^2$ and $a > 1$, the graph is 'narrower' than the graph of $y = x^2$.
- For $y = ax^2$ and $0 < a < 1$, the graph is 'broader' than the graph of $y = x^2$.
- All quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x-h)^2 + k$.
- The graph of $y = a(x-h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$.
  - The vertex (or turning point) is the point (h, k).
  - The axis of symmetry is $x = h$.
  - If h and k are positive numbers, then the graph of $y = a(x-h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
  - Similar results hold for different combinations of h and k positive and negative.

 ISBN 978-1-009-53719-3 Cambridge University Press

Exercise 4D

SF

Sketch the graph of each of the following quadratics by first finding:

- **i** the coordinates of the turning point
- **ii** the axis of symmetry
- **iii** the x-axis intercepts (if any).

Example 21

1 **a** $y = x^2 - 4$ **b** $y = x^2 + 2$ **c** $y = -x^2 + 3$
d $y = -2x^2 + 5$ **e** $y = -x^2 + 4$ **f** $y = 3x^2 - 9$

Example 22

2 **a** $y = (x-2)^2$ **b** $y = (x+3)^2$ **c** $y = -(x+1)^2$ **d** $y = -\frac{1}{2}(x-4)^2$

Example 23, 24

3 **a** $y = (x-2)^2 + 1$ **b** $y = (x-2)^2 - 1$ **c** $y = (x-1)^2 + 2$
d $y = (x+1)^2 - 1$ **e** $y = -(x-3)^2 + 1$ **f** $y = (x+2)^2 - 4$
g $y = 2(x+2)^2 - 18$ **h** $y = -3(x-4)^2 + 3$ **i** $y = -\frac{1}{2}(x+5)^2 - 2$
j $y = 3(x+2)^2 - 12$ **k** $y = -4(x-2)^2 + 8$ **l** $y = \frac{1}{3}(x-1)^2 - 3$

4E Completing the square and turning points

Learning intentions

- To be able to complete the square of a quadratic expression.

To sketch the graph of a quadratic using the techniques from the previous section, the quadratic must be expressed in **turning point form**. This can be done using two different but related methods: by completing the square and by using the equation of the axis of symmetry.

Completing the square

To transpose a quadratic in polynomial form we can **complete the square**.

Consider the expansion of a perfect square:

$$(x+a)^2 = x^2 + 2ax + a^2$$

The last term of the expansion is the square of half the coefficient of the middle term.

Now consider the quadratic polynomial

$$x^2 + 2x - 3$$

This is not a perfect square. However, by adding and subtracting a new term, we can form a perfect square as part of a new expression for the same polynomial.

We have that

$$x^2 + 2x + 1 = (x+1)^2$$

which is a perfect square. In order to keep our original quadratic 'intact', we both add and subtract the 'correct' new term. For example:

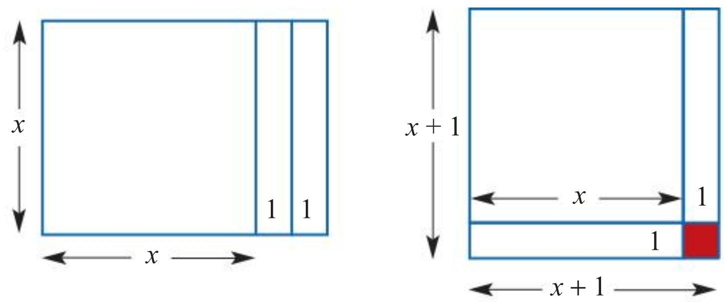
$$\begin{aligned} x^2 + 2x - 3 &= (x^2 + 2x + 1) - 1 - 3 \\ &= (x+1)^2 - 4 \end{aligned}$$

Hence the quadratic $y = x^2 + 2x - 3$ is expressed in turning point form as $y = (x + 1)^2 - 4$, and so the **vertex** (turning point) of its graph is the point with coordinates $(-1, -4)$.

In the above example, the coefficient of x^2 was 1. If the coefficient is not 1, this coefficient must first be 'factored out' before proceeding to complete the square.

A geometric representation of completing the square

Completing the square for $x^2 + 2x$ is represented in the following diagrams. The diagram on the left shows $x^2 + 2x$. The small rectangle to the right is moved to the 'base' of the x by x square. The red square of area 1 unit is added. Thus $x^2 + 2x + 1 = (x + 1)^2$.



Solving equations by completing the square

The process of completing the square can also be used for the solution of equations.

Example 25

Solve each of the following equations for x by first completing the square:

a $x^2 - 3x + 1 = 0$

b $2x^2 - 3x - 1 = 0$

Solution

a Completing the square:

$$x^2 - 3x + 1 = 0$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$$

Therefore $x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

and so $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$

Explanation

$\frac{1}{2} \times (-3) = -\frac{3}{2}$

We add and subtract $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ on the left-hand side of the equation.

This gives an equivalent expression to the expression of the left-hand side.

Solve the equation as shown.

b Completing the square:

$$2x^2 - 3x - 1 = 0$$

$$2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$$

$$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Therefore $x - \dfrac{3}{4} = \pm\dfrac{\sqrt{17}}{4}$

and so $x = \dfrac{3}{4} \pm \dfrac{\sqrt{17}}{4} = \dfrac{3 \pm \sqrt{17}}{4}$

Divide both sides by 2 before completing the square.

$\dfrac{1}{2} \times \left(-\dfrac{3}{2}\right) = -\dfrac{3}{4}$

We add and subtract $\left(-\dfrac{3}{4}\right)^2 = \dfrac{9}{16}$ on the left-hand side of the equation.

Sketching the graph of a quadratic polynomial after completing the square

Completing the square enables the quadratic rule to be written in turning point form. We have seen that this can be used to sketch the graphs of quadratic polynomials.

Example 26

Determine the coordinates of the vertex by completing the square and hence sketch the graph of $y = -2x^2 + 6x - 8$.

Solution

Take out -2 as a common factor and then complete the square:

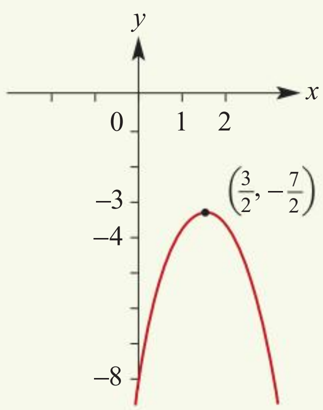
$$y = -2x^2 + 6x - 8$$

$$= -2(x^2 - 3x + 4)$$

$$= -2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4\right)$$

$$= -2\left(\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}\right)$$

$$\therefore \quad y = -2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$$



Therefore the vertex is $\left(\dfrac{3}{2}, -\dfrac{7}{2}\right)$ and the axis of symmetry is $x = \dfrac{3}{2}$.

The y-axis intercept is -8.

The graph has maximum value of $-\dfrac{7}{2}$, and so there are no x-axis intercepts.

 ISBN 978-1-009-53719-3

The equation for the axis of symmetry of a parabola

We first complete the square for $y = ax^2 + bx + c$:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \qquad \text{completing the square} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

Axis of symmetry of a parabola

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\dfrac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\dfrac{b}{2a}$. Substitute this value into the quadratic polynomial to determine the y-coordinate of the turning point.

Example 27

Use the axis of symmetry to determine the turning point of the graph and hence express in turning point form:

a $y = x^2 - 4x + 3$

b $y = -2x^2 + 12x - 7$

Solution

a The x-coordinate of the turning point is 2.

When $x = 2$, $y = 4 - 8 + 3 = -1$.

The coordinates of the turning point are $(2, -1)$.

Hence the equation is $y = (x - 2)^2 - 1$.

b The x-coordinate of the turning point is 3.

When $x = 3$, $y = -2 \times 3^2 + 12 \times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$.

Hence the equation is $y = -2(x - 3)^2 + 11$.

Explanation

a Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\dfrac{-4}{2}\right) = 2$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.

b Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\dfrac{12}{-4}\right) = 3$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

ISBN 978-1-009-53719-3

Summary 4E

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though some have no solutions.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\frac{b}{2a}$.
- To convert the quadratic function $y = ax^2 + bx + c$ into turning point form using the axis of symmetry:
  1. The x-coordinate h of the vertex of the parabola is $-\frac{b}{2a}$.
  2. Determine the y-coordinate k of the vertex by substituting in $y = ax^2 + bx + c$.
  3. Substitute these values for h and k in $y = a(x - h)^2 + k$.

Skill-sheet

Exercise 4E

SF

1 Expand each of the following:

a $(x - 1)^2$ **b** $(x + 2)^2$ **c** $(x - 3)^2$ **d** $(-x + 3)^2$

e $(-x - 2)^2$ **f** $(x - 5)^2$ **g** $\left(x - \frac{1}{2}\right)^2$ **h** $\left(x - \frac{3}{2}\right)^2$

2 Factorise each of the following:

a $x^2 - 4x + 4$ **b** $x^2 - 12x + 36$ **c** $-x^2 + 4x - 4$ **d** $2x^2 - 8x + 8$

e $-2x^2 + 12x - 18$ **f** $x^2 - x + \frac{1}{4}$ **g** $x^2 - 3x + \frac{9}{4}$ **h** $x^2 + 5x + \frac{25}{4}$

Example 25

3 Solve each of the following equations for x by first completing the square:

a $x^2 - 2x - 1 = 0$ **b** $x^2 - 4x - 2 = 0$ **c** $x^2 - 6x + 2 = 0$

d $x^2 - 5x + 2 = 0$ **e** $2x^2 - 4x + 1 = 0$ **f** $3x^2 - 5x - 2 = 0$

g $x^2 + 2x + k = 0$ **h** $kx^2 + 2x + k = 0$ **i** $x^2 - 3kx + 1 = 0$

Example 26

4 Express each of the following in the form $y = a(x - h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x + 1$ **c** $y = x^2 - 3x + 1$

5 Express each of the following in the form $y = a(x - h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = 2x^2 - 2x - 5$ **b** $y = 4x^2 + 8x + 8$ **c** $y = 3x^2 - 6x - 4$

Example 27

6 Express each of the following in the form $y = a(x - h)^2 + k$ using the axis of symmetry. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y = x^2 - 8x + 12$ **b** $y = x^2 - x - 2$ **c** $y = 2x^2 + 4x - 2$

d $y = -x^2 + 4x + 1$ **e** $y = -2x^2 - 12x - 12$ **f** $y = 3x^2 - 6x + 12$

 ISBN 978-1-009-53719-3

4F Graphing quadratics in polynomial form

Learning intentions

- To be able to graph a quadratic polynomial written in polynomial form.

It is not always essential to convert a quadratic to turning point form in order to sketch its graph. We can sometimes determine the x- and y-axis intercepts and the axis of symmetry from polynomial form by other methods and use these details to sketch the graph.

Step 1 **Determine the y-axis intercept**

Let $x = 0$. For the general quadratic $y = ax^2 + bx + c$, this gives

$y = a(0)^2 + b(0) + c$

$y = c$

Hence the y-axis intercept is always equal to c.

Step 2 **Determine the x-axis intercepts**

Let $y = 0$. In general, this gives

$$0 = ax^2 + bx + c$$

In order to solve such an equation it is necessary to factorise the right-hand side and then use the **null factor theorem**.

Step 3 **Determine the equation of the axis of symmetry**

Once the x-axis intercepts have been found, the equation of the axis of symmetry can be found by using the symmetry properties of the parabola. The axis of symmetry is the perpendicular bisector of the line segment joining the x-axis intercepts.

Step 4 **Determine the coordinates of the turning point**

The axis of symmetry gives the x-coordinate of the turning point. Substitute this into the quadratic polynomial to obtain the y-coordinate.

Example 28

Determine the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 4x$.

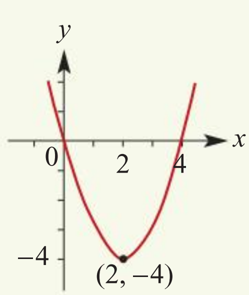
Solution

Step 1 $c = 0$. Therefore the y-axis intercept is 0.

Step 2 Let $y = 0$. Then

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$
$$\therefore \quad x = 0 \text{ or } x = 4$$

The x-axis intercepts are 0 and 4.



Step 3 The axis of symmetry is the line with equation $x = \frac{0 + 4}{2}$, that is, $x = 2$.

Step 4 When $x = 2$, $y = (2)^2 - 4(2) = -4$. The turning point has coordinates $(2, -4)$.

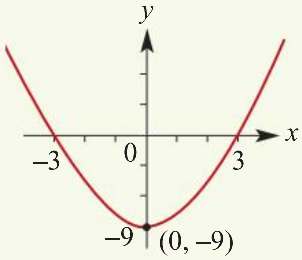
 ISBN 978-1-009-53719-3 Cambridge University Press

Example 29

Determine the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 9$.

Solution

Step 1 $c = -9$. Therefore the y-axis intercept is -9.



Step 2 Let $y = 0$. Then

$$0 = x^2 - 9$$
$$0 = (x+3)(x-3)$$
$$\therefore \quad x = -3 \text{ or } x = 3$$

The x-axis intercepts are -3 and 3.

Step 3 The axis of symmetry is the line with equation

$x = \dfrac{-3+3}{2}$, that is, $x = 0$.

Step 4 When $x = 0$, $y = (0)^2 - 9$

$= -9$

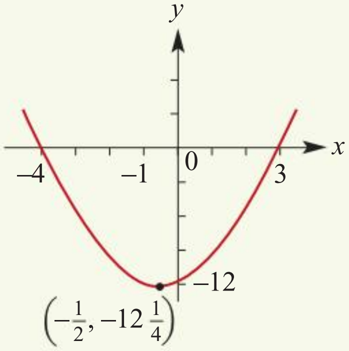
The turning point has coordinates $(0, -9)$.

Example 30

Determine the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.



Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x+4)(x-3)$$
$$\therefore \quad x = -4 \text{ or } x = 3$$

The x-axis intercepts are -4 and 3.

Step 3 The axis of symmetry is the line with equation

$$x = \frac{-4+3}{2} = -\frac{1}{2}$$

Step 4 When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

$= -12\frac{1}{4}$

The turning point has coordinates $(-\frac{1}{2}, -12\frac{1}{4})$.

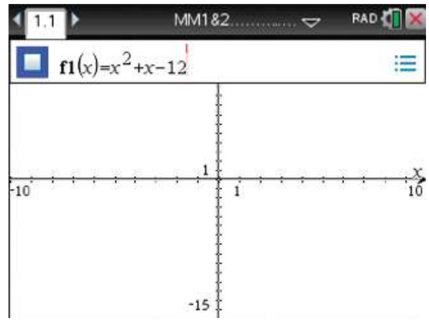
 ISBN 978-1-009-53719-3

TI-84

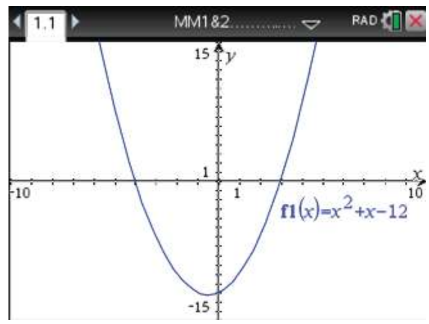
Using the TI-Nspire CX non-CAS

To graph the quadratic function with rule
$y = x^2 + x - 12$:

- Enter the rule in the entry line of a **Graphs** application as shown, and press enter.



- Using menu > **Window/Zoom** > **Window Settings**, select the window settings $-10 \le x \le 10$ and $-15 \le y \le 15$ to obtain the graph shown.

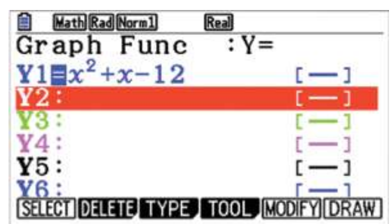


Using the Casio

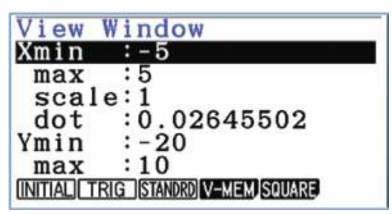
To graph the quadratic function with rule
$y = x^2 + x - 12$:

- Press MENU 5 to select **Graph** mode.
- Enter the expression $x^2 + x - 12$ in $Y1$:

  X,θ,T x^2 + X,θ,T − 1 2 EXE



- Select **Draw** F6.
- Adjust the View Window SHIFT F3 if required.



- To determine the x-axis intercepts, go to **G-Solve** and select **Root**:

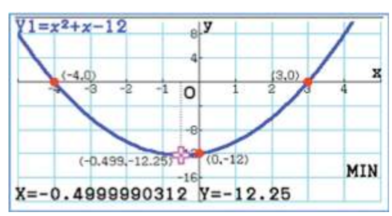
  SHIFT F5 F1 EXE ► EXE

- To determine the y-axis intercept, go to **G-Solve** and select **y-Intercept**:

  SHIFT F5 F4 EXE

- To determine the turning point, go to **G-Solve** and select **Minimum**:

  SHIFT F5 F3 EXE



 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 4F

Steps for sketching the graph of a quadratic function given in polynomial form:

Step 1 Determine the y-axis intercept.

Step 2 Determine the x-axis intercepts.

Step 3 Determine the equation of the axis of symmetry.

Step 4 Determine the coordinates of the turning point.

Exercise 4F

SF

1 **a** A parabola has x-axis intercepts 4 and 10. State the x-coordinate of the vertex.

b A parabola has x-axis intercepts 6 and 8. State the x-coordinate of the vertex.

c A parabola has x-axis intercepts -6 and 8. State the x-coordinate of the vertex.

2 **a** A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at 6. Determine the other x-axis intercept.

b A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at -4. Determine the other x-axis intercept.

c A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at the origin. Determine the other x-axis intercept.

Example 28, 29

3 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 - 1$ **b** $y = x^2 + 6x$ **c** $y = 25 - x^2$

d $y = x^2 - 4$ **e** $y = 2x^2 + 3x$ **f** $y = 2x^2 - 4x$

g $y = -2x^2 - 3x$ **h** $y = x^2 + 1$

Example 30

4 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 + 3x - 10$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 + 2x - 3$

d $y = x^2 + 4x + 3$ **e** $y = 2x^2 - x - 1$ **f** $y = 6 - x - x^2$

g $y = -x^2 - 5x - 6$ **h** $y = x^2 - 5x - 24$

4G Solving quadratic inequalities

Learning intentions

- To be able to solve a quadratic inequality.

In Chapter 1 we looked at solving linear inequalities. The situation is a little more complex for quadratic inequalities. We suggest one possible approach. To solve a quadratic inequality (for example, $x^2 + x - 12 > 0$):

 ISBN 978-1-009-53719-3 Cambridge University Press

Step 1 Solve the corresponding equation (for example, $x^2 + x - 12 = 0$).

Step 2 Sketch the graph of the quadratic polynomial (for example, $y = x^2 + x - 12$).

Step 3 Use the graph to determine the set of x-values which satisfy the inequality.

Example 31

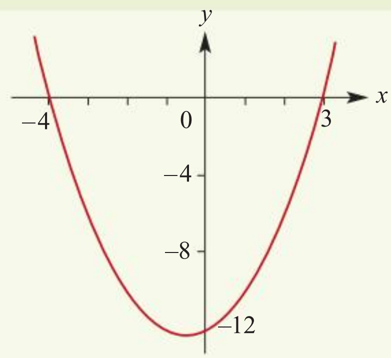
Solve $x^2 + x - 12 > 0$.

Solution

Step 1 Solve the equation

$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$\therefore \quad x = -4 \text{ or } x = 3$$

Step 2 Sketch the graph of the quadratic $y = x^2 + x - 12$.

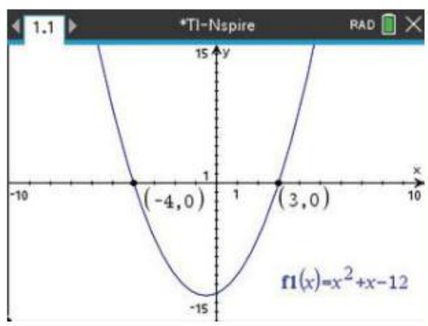


Step 3 From the graph it can be seen that $x^2 + x - 12 > 0$ when $x < -4$ or $x > 3$.

Using the TI-Nspire CX non-CAS

To solve the inequality $x^2 + x - 12 > 0$:

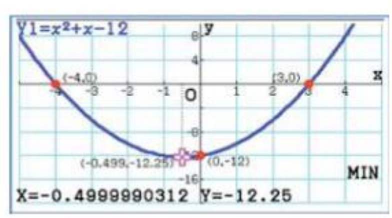
- In a **Graphs** application, plot the graph of $f1(x) = x^2 + x - 12$.
- Use (menu) > **Analyze Graph** > **Zero** to determine the intersections with the x-axis.
- From the graph, we see that $f1(x) > 0$ when $x < -4$ or $x > 3$.



Using the Casio

To solve the inequality $x^2 + x - 12 > 0$:

- Plot the graph of $y = x^2 + x - 12$ and determine the x-axis intercepts, as shown in Section 4F.
- From the graph, we see that $y > 0$ when $x < -4$ or $x > 3$.



 ISBN 978-1-009-53719-3

Summary 4G

When solving quadratic inequalities of the form $ax^2 + bx + c \le 0$ (or with $\ge$, $>$ or $<$), it is best to sketch the graph of $y = ax^2 + bx + c$.

Skill-sheet

Exercise 4G

Example 31

SF

1 **a** Solve the equation $x^2 - 2x - 8 = 0$.
b Sketch the graph of $y = x^2 - 2x - 8$.
c Solve the inequality $x^2 - 2x - 8 \le 0$.
d Solve the inequality $x^2 - 2x - 8 > 0$.

2 Solve each of the following inequalities:

a $(x - 3)(x + 2) \ge 0$
b $(x + 4)(x + 3) < 0$
c $(2x - 1)(x + 4) \le 0$
d $(x - 6)(2x - 4) > 0$
e $(2x - 6)(2x - 4) < 0$
f $(7 - 2x)(2x - 3) \ge 0$
g $(2x + 7)(2x - 4) < 0$
h $(3x + 6)(2x - 5) \le 0$
i $(5 - 2x)(5 + x) < 0$
j $(7 - 2x)(x + 2) \ge 0$
k $(7 - 2x)(5x - 2) < 0$
l $(11 - 2x)(5 - 2x) \ge 0$

3 Solve each of the following inequalities:

a $(5 - x)(5 + x) < 0$
b $4 - 9y^2 \ge 0$
c $16 - y^2 < 0$
d $36 - 25x^2 \ge 0$
e $1 - 16y^2 \le 0$
f $25 - 36y^2 < 0$

4 Solve each of the following inequalities:

a $x^2 + 2x - 8 \ge 0$
b $x^2 - 5x - 24 < 0$
c $x^2 - 4x - 12 \le 0$
d $2x^2 - 3x - 9 > 0$
e $6x^2 + 13x < -6$
f $-x^2 - 5x - 6 \ge 0$
g $12x^2 + x > 6$
h $10x^2 - 11x \le -3$
i $x(x - 1) \le 20$
j $4 + 5p - p^2 \ge 0$
k $3 + 2y - y^2 < 0$
l $x^2 + 3x \ge -2$

5 Solve each of the following inequalities:

a $x^2 + 3x - 5 \ge 0$
b $x^2 - 5x + 2 < 0$
c $2x^2 - 3x - 1 \le 0$
d $8 - 3x - x^2 > 0$
e $2x^2 + 7x + 1 < 0$
f $2x^2 - 8x + 5 \ge 0$

6 Explain why $(x - 3)^2 \ge 0$ for all x.

7 Explain why $-(x - 1)^2 \le 0$ for all x.

CF

8 Complete the square for $x^2 + 2x + 7$ and hence show that $x^2 + 2x + 7 \ge 6$ for all x.

9 Complete the square for $-x^2 - 2x - 7$ and hence show that $-x^2 - 2x - 7 \le -6$ for all x.

 ISBN 978-1-009-53719-3

4H The general quadratic formula

Learning intentions

- To be able to use the quadratic formula to solve quadratic equations.

Not all quadratics can be factorised by inspection, and it is often difficult to determine the x-axis intercepts this way. There is a general formula for determining the solutions of a quadratic equation in polynomial form. This formula comes from 'completing the square' for the general quadratic. In Section 4E we showed that

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

We can use this to solve the general quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \end{aligned}$$

Now divide both sides by a:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic formula provides an alternative method for solving quadratic equations to 'completing the square', but it is probably not as useful for curve sketching as 'completing the square', which gives the turning point coordinates directly.

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation $x = -\frac{b}{2a}$.

Also, from the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

This will be further explored in the next section.

Example 32

Solve each of the following equations for x by using the quadratic formula:

a $x^2 - x - 1 = 0$

b $x^2 - 2kx - 3 = 0$

Solution

a $x^2 - x - 1 = 0$

Here $a = 1$, $b = -1$ and $c = -1$.

The formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

b $x^2 - 2kx - 3 = 0$

Here $a = 1$, $b = -2k$ and $c = -3$.

The formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$
$$= \frac{2k \pm \sqrt{4k^2 + 12}}{2}$$
$$= k \pm \sqrt{k^2 + 3}$$

Note that $k^2 + 3 \geq 0$ for all values of k, since $k^2 \geq 0$.

Example 33

Sketch the graph of $y = -3x^2 - 12x - 7$. Use the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

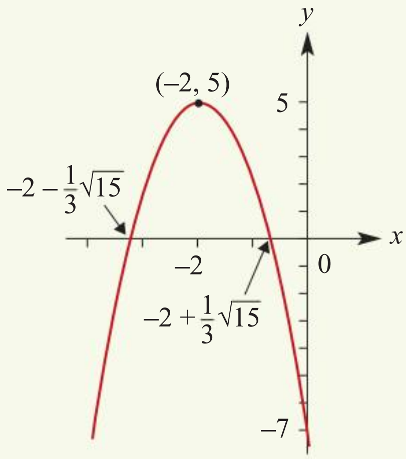
Axis of symmetry $x = -\frac{b}{2a} = -\left(\frac{-12}{2 \times (-3)}\right) = -2$

Turning point

When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$. The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$$
$$= \frac{12 \pm \sqrt{60}}{-6}$$
$$= \frac{12 \pm 2\sqrt{15}}{-6}$$



The x-axis intercepts are $-2 - \frac{1}{3}\sqrt{15}$ and $-2 + \frac{1}{3}\sqrt{15}$.

ISBN 978-1-009-53719-3

Summary 4H

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Exercise 4H

SF

1 For each of the following, the coefficients a, b and c of a quadratic $y = ax^2 + bx + c$ are given. Determine:

i $b^2 - 4ac$ **ii** $\sqrt{b^2 - 4ac}$ in simplest surd form

a $a = 2$, $b = 4$ and $c = -3$
b $a = 1$, $b = 10$ and $c = 18$
c $a = 1$, $b = 10$ and $c = -18$
d $a = -1$, $b = 6$ and $c = 15$
e $a = 1$, $b = 9$ and $c = -27$

2 Simplify each of the following:

a $\dfrac{2 + 2\sqrt{5}}{2}$ **b** $\dfrac{9 - 3\sqrt{5}}{6}$ **c** $\dfrac{5 + 5\sqrt{5}}{10}$ **d** $\dfrac{6 + 12\sqrt{2}}{6}$

Example 32

3 Solve each of the following for x. Give exact answers.

a $x^2 + 6x = 4$
b $x^2 - 7x - 3 = 0$
c $2x^2 - 5x + 2 = 0$
d $2x^2 + 4x - 7 = 0$
e $2x^2 + 8x = 1$
f $5x^2 - 10x = 1$
g $-2x^2 + 4x - 1 = 0$
h $2x^2 + x = 3$
i $2.5x^2 + 3x + 0.3 = 0$
j $-0.6x^2 - 1.3x = 0.1$
k $2kx^2 - 4x + k = 0$
l $2(1 - k)x^2 - 4kx + k = 0$

Example 33

4 Sketch the graphs of the following parabolas. Use the quadratic formula to determine the x-axis intercepts (if they exist) and the axis of symmetry and, hence, the turning point.

a $y = x^2 + 5x - 1$
b $y = 2x^2 - 3x - 1$
c $y = -x^2 - 3x + 1$
d $y + 4 = x^2 + 2x$
e $y = 4x^2 + 5x + 1$
f $y = -3x^2 + 4x - 2$
g $y = -x^2 + 5x + 6$
h $y = 4x^2 - 3x + 2$
i $y = 3x^2 - x - 4$

 ISBN 978-1-009-53719-3

4I The discriminant

Learning intentions

- ► To be able to use the discriminant to investigate the number of solutions and their nature for a quadratic equation.

In the previous section we found that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root sign is called the **discriminant**. We write

$$\Delta = b^2 - 4ac$$

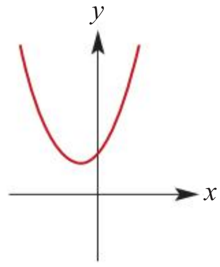
The number of x-axis intercepts

There are three different possibilities for the number of x-axis intercepts of a parabola:

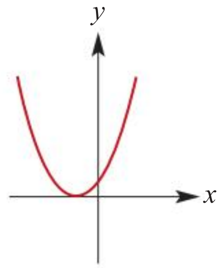
- zero – the graph is either all above or all below the x-axis
- one – the graph touches the x-axis and the turning point is the x-axis intercept
- two – the graph crosses the x-axis.

For a parabola $y = ax^2 + bx + c$, we can use the discriminant $\Delta = b^2 - 4ac$ to determine when each of these three situations occur.

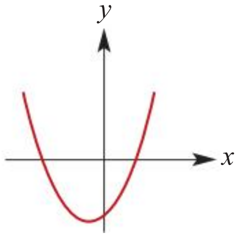
- If the discriminant $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions and the corresponding parabola will have no x-axis intercepts.



- If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



- If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



 ISBN 978-1-009-53719-3

Example 34

Determine the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

a Discriminant $\Delta = b^2 - 4ac$

$$= (-6)^2 - (4 \times 1 \times 8)$$
$$= 4$$

As $\Delta > 0$, the graph intersects the x-axis at two distinct points, i.e. there are two distinct solutions of the equation $x^2 - 6x + 8 = 0$.

b $\Delta = b^2 - 4ac$

$$= (-8)^2 - (4 \times 1 \times 16)$$
$$= 0$$

As $\Delta = 0$, the graph touches the x-axis, i.e. there is one solution of the equation $x^2 - 8x + 16 = 0$.

c $\Delta = b^2 - 4ac$

$$= (-3)^2 - (4 \times 2 \times 4)$$
$$= -23$$

As $\Delta < 0$, the graph does not intersect the x-axis, i.e. there are no real solutions for the equation $2x^2 - 3x + 4 = 0$.

Example 35

Determine the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$$\Delta = 0$$

i.e. $4m^2 - 36 = 0$

$$m^2 = 9$$

$\therefore$ $m = \pm 3$

b For no solution:

$$\Delta < 0$$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$$-3 < m < 3$$

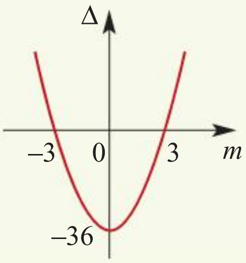
c For two distinct solutions:

$$\Delta > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$$m > 3 \quad \text{or} \quad m < -3$$



ISBN 978-1-009-53719-3

The nature of the solutions of a quadratic equation

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation $ax^2 + bx + c = 0$.

For a, b and c rational numbers with $a \neq 0$:

- If $\Delta = b^2 - 4ac$ is a perfect square and $\Delta \neq 0$, then the quadratic equation has two rational solutions.
- If $\Delta = b^2 - 4ac = 0$, then the quadratic equation has one rational solution.
- If $\Delta = b^2 - 4ac$ is not a perfect square and $\Delta > 0$, then the quadratic equation has two irrational solutions.

Example 36

Show that the solutions of the equation $3x^2 + (m - 3)x - m = 0$ are rational for all rational values of m.

Solution

$$\begin{aligned}\Delta &= (m-3)^2 - 4 \times 3 \times (-m)\\ &= m^2 - 6m + 9 + 12m\\ &= m^2 + 6m + 9\\ &= (m+3)^2 \geq 0 \quad \text{for all } m\end{aligned}$$

Furthermore, for any rational m, $(m + 3)$ is also rational and therefore Δ is the square of a rational number.

Summary 4I

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$ where a, b and c are rational numbers and $a \neq 0$:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Skill-sheet

Exercise 4I

SF

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 + 3x - 4$

d $2x^2 + 3x - 4$ **e** $-2x^2 + 3x + 4$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Example 34

2 Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis:

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$

d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - x - 1$

SF

3 By examining the discriminant, determine the number of distinct solutions of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $10x^2 - x - 3 = 0$

d $2x^2 + 8x - 7 = 0$ **e** $3x^2 - 8x - 7 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of distinct solutions for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$

d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 2 = 0$

Example 35

5 Determine the values of m for which each of the following equations:

CF

i has no solutions **ii** has one solution **iii** has two distinct solutions.

a $x^2 - 4mx + 20 = 0$ **b** $mx^2 - 3mx + 3 = 0$

c $5x^2 - 5mx - m = 0$ **d** $x^2 + 4mx - 4(m - 2) = 0$

Example 36

6 For m and n rational numbers show that $mx^2 + (2m + n)x + 2n = 0$ has rational solutions.

7 Determine the values of p for which the equation $px^2 + 2(p + 2)x + p + 7 = 0$ has no solution.

8 Determine the values of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has one solution.

9 Determine the value(s) of p for which:

a $px^2 - 6x + p = 0$ has one solution **b** $2x^2 - 4x + 3 = p$ has two solutions

c $3x^2 = 2x + p - 1$ has two solutions **d** $x^2 - 2x + 2 = p$ has two solutions.

10 Determine the values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis.

11 Show that the equation $(p^2 + 1)x^2 + 2pqx + q^2 = 0$ has no real solution for any values of p and q $(q \neq 0)$.

12 a Determine the discriminant of $x^2 + 4mx + 24m - 44$.

b Show the equation $x^2 + 4mx + 24m - 44 = 0$ has a solution for all values of m.

13 a Determine the discriminant of $4mx^2 + 4(m - 1)x + m - 2$.

b Show the equation $4mx^2 + 4(m - 1)x + m - 2 = 0$ has a solution for all non-zero values of m.

14 Determine the discriminant of the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number, and hence show that the equation has rational solution(s).

15 Determine the discriminant of the equation $x^2 - (m + 2n)x + 2mn = 0$, where m and n are rational numbers, and hence show that the equation has rational solution(s).

16 If both a and c are positive, what can be said about the graph of $y = ax^2 + bx - c$?

17 If a is negative and c is positive, what can be said about the graph of $y = ax^2 + bx + c$?

4J Solving simultaneous linear and quadratic equations

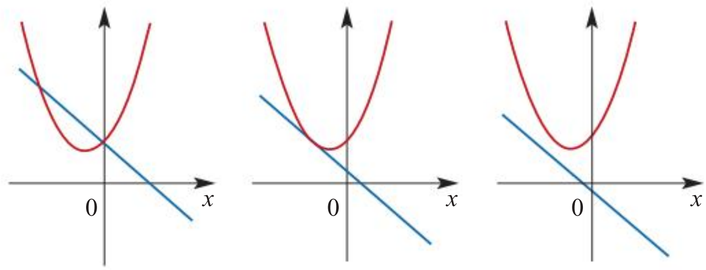
Learning intentions

- To be able to solve simultaneous linear and quadratic equations.

As discussed in Section 2H, when solving simultaneous linear equations we are actually determining the point of intersection of the two corresponding linear graphs.

If we wish to determine the point or points of intersection between a straight line and a parabola, we can solve the equations simultaneously.

It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola we may get two, one or zero points of intersection.



Two points of intersection   One point of intersection   No point of intersection

If there is one point of intersection between the parabola and the straight line, then the line is a **tangent** to the parabola.

As we usually have the quadratic equation written with y as the subject, it is necessary to have the linear equation written with y as the subject. Then the linear expression for y can be substituted into the quadratic equation.

Example 37

Determine the points of intersection of the line with equation $y = -2x + 4$ and the parabola with equation $y = x^2 - 8x + 12$.

Solution

At the point of intersection:

$$x^2 - 8x + 12 = -2x + 4$$
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$

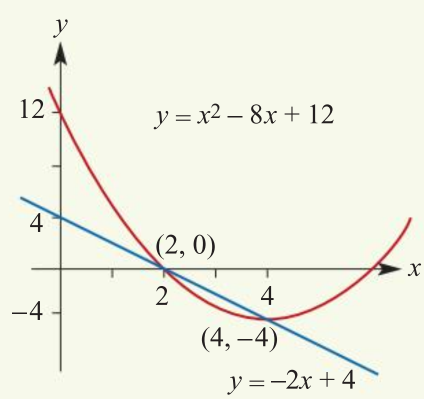
Hence $x = 2$ or $x = 4$.

When $x = 2$, $y = -2(2) + 4 = 0$.

When $x = 4$, $y = -2(4) + 4 = -4$.

Therefore the points of intersection are $(2, 0)$ and $(4, -4)$.

The result can be shown graphically.

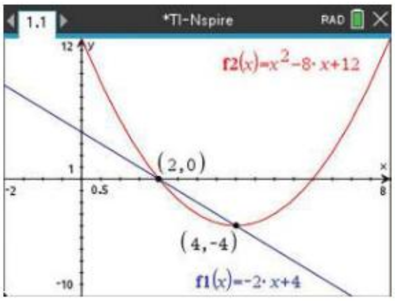


 ISBN 978-1-009-53719-3 Cambridge University Press

TI-84

Using the TI-Nspire CX non-CAS

To determine the intersection points of the two graphs:

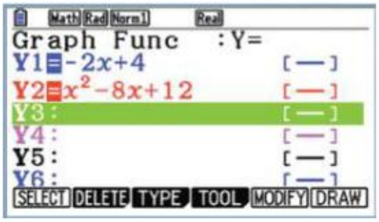


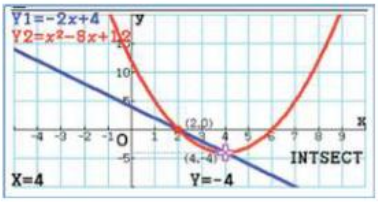
- In a **Graphs** application, plot the graphs of $f1(x) = -2x + 4$ and $f2(x) = x^2 - 8x + 12$.
- Adjust the **Window Settings** as required.
- Use (menu) > **Analyze Graph** > **Intersection** to determine the intersection points.

Note: Alternatively, using (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)** will determine both intersection points at once.

Using the Casio

To determine the intersection points of the two graphs:





- Press (MENU) (5) to select **Graph** mode.
- Enter $y = -2x + 4$ in $Y1$:

  ((−)) (2) (X,θ,T) (+) (4) (EXE)
- Enter $y = x^2 - 8x + 12$ in $Y2$:

  (X,θ,T) (x^2) (−) (8) (X,θ,T) (+) (1) (2) (EXE)
- Select **Draw** (F6).
- Adjust the View Window (SHIFT) (F3) if required.
- Go to the **G-Solve** menu and select **Intersection**:

  (SHIFT) (F5) (F5) (EXE) ► (EXE)

Example 38

Prove that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

Solution

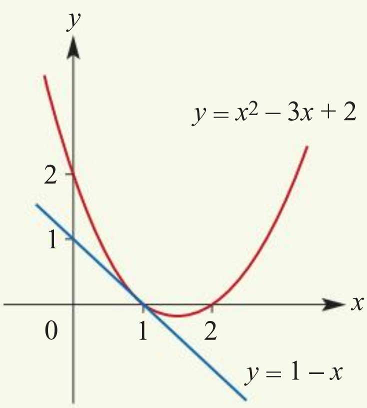
At the point of intersection:

$$x^2 - 3x + 2 = 1 - x$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$

Therefore $x = 1$ and $y = 1 - 1 = 0$.

The straight line just touches the parabola at $(1, 0)$.

This can be illustrated graphically.



ISBN 978-1-009-53719-3

Summary 4J

To determine the points of intersection of a straight line $y = mx + c_2$ and a parabola $y = ax^2 + bx + c_1$:

- Form the quadratic equation

$$ax^2 + bx + c_1 = mx + c_2$$

- Rearrange the equation so that the right-hand side is zero:

$$ax^2 + (b - m)x + (c_1 - c_2) = 0$$

- Solve the equation for x and substitute these x-values into the equation of the line to determine the corresponding y-values.

The discriminant applied to the second equation, $ax^2 + (b - m)x + (c_1 - c_2) = 0$, can be used to determine the number of intersection points:

- If $\Delta > 0$, there are two intersection points.
- If $\Delta = 0$, there is one intersection point.
- If $\Delta < 0$, there are no intersection points.

Exercise 4J

SF

Example 37

1 **a** Determine the points of intersection of the line with equation $y = x - 2$ and the parabola with equation $y = x^2 - x - 6$.

b Determine the points of intersection of the line with equation $x + y = 6$ and the parabola with equation $y = x^2$.

c Determine the points of intersection of the line with equation $5x + 4y = 21$ and the parabola with equation $y = x^2$.

d Determine the points of intersection of the line with equation $y = 2x + 1$ and the parabola with equation $y = x^2 - x + 3$.

2 Solve each of the following pairs of equations:

a $y = x^2 + 2x - 8$, $y = 2 - x$

b $y = x^2 - x - 3$, $y = 4x - 7$

c $y = x^2 + x - 5$, $y = -x - 2$

d $y = x^2 + 6x + 6$, $y = 2x + 3$

e $y = 6 - x - x^2$, $y = -2x - 2$

f $y = x^2 + x + 6$, $y = 6x + 8$

Example 38

3 Prove that, for each of the following pairs of equations, the straight line meets the parabola only once:

a $y = x^2 - 6x + 8$, $y = -2x + 4$

b $y = x^2 - 2x + 6$, $y = 4x - 3$

c $y = 2x^2 + 11x + 10$, $y = 3x + 2$

d $y = x^2 + 7x + 4$, $y = -x - 12$

 ISBN 978-1-009-53719-3

SF

4 Solve each of the following pairs of equations:

a $y = x^2 - 6x$
$y = 8 + x$

b $y = 3x^2 + 9x$
$y = 32 - x$

c $y = 5x^2 + 9x$
$y = 12 - 2x$

d $y = -3x^2 + 32x$
$y = 32 - 3x$

e $y = 2x^2 - 12$
$y = 3(x - 4)$

f $y = 11x^2$
$y = 21 - 6x$

CF

5 **a** Determine the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$. **Hint:** Consider the discriminant of the resulting quadratic.

b **i** Sketch the parabola with equation $y = -2x^2 - 6x + 2$.

ii Determine the values of m for which the straight line $y = mx + 6$ is tangent to the parabola. **Hint:** Use the discriminant of the resulting quadratic.

6 **a** Determine the value of c such that the line with equation $y = 2x + c$ is tangent to the parabola with equation $y = x^2 + 3x$.

b Determine the possible values of c such that the line with equation $y = 2x + c$ twice intersects the parabola with equation $y = x^2 + 3x$.

7 Determine the value(s) of a such that the line with equation $y = x$ is tangent to the parabola with equation $y = x^2 + ax + 1$.

8 Determine the value of b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

9 Determine the equation of the straight line(s) which pass through the point $(1, -2)$ and is (are) tangent to the parabola with equation $y = x^2$.

4K Families of quadratic polynomial functions

Learning intentions

- To be able to investigate families of quadratic equations by using parameters.

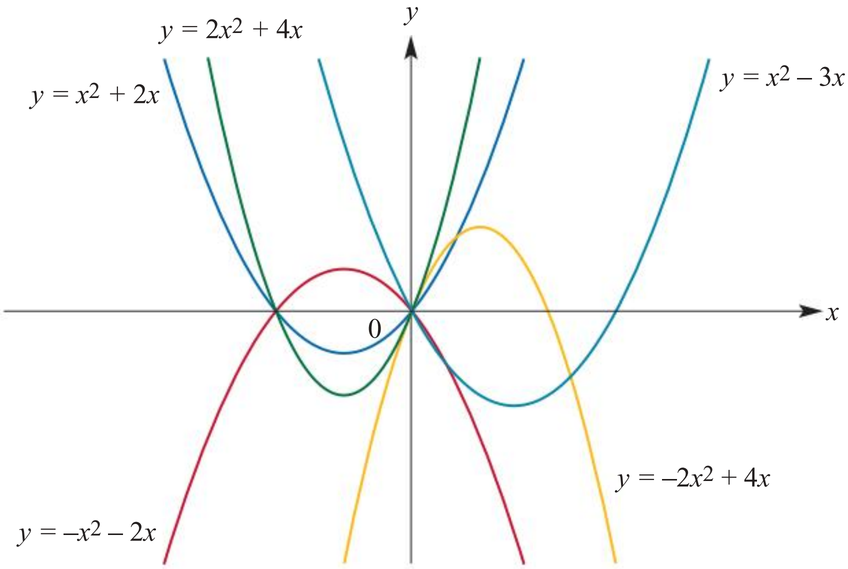
In Chapter 2 we considered the information that is necessary to determine the equation of a straight line and we also studied families of straight lines. In this section these two ideas are extended for our study of quadratic polynomials.

Families of quadratics

Here are some examples of families of quadratic polynomial functions:

$y = ax^2,\ a \neq 0$	The parabolas with their vertices at the origin.
$y = a(x - 2)^2 + 3,\ a \neq 0$	The parabolas with turning point at $(2, 3)$.
$y = a(x - 2)(x + 5),\ a \neq 0$	The parabolas with x-axis intercepts 2 and -5.
$y = a(x - h)(x - 2),\ a \neq 0$	The parabolas with x-axis intercept 2.
$y = ax^2 + bx,\ a \neq 0$ and $b \neq 0$	The parabolas with two x-axis intercepts, one of which is the origin.

The letters a, b and h used to define these families are called parameters. Varying the values of the parameters produces different parabolas. For example, for $y = ax^2 + bx$ some possible curves are shown below.



Example 39

A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(1, 7)$ and $(2, 10)$, determine the values of a and c.

Solution

When $x = 1$, $y = 7$ and when $x = 2$, $y = 10$.

$$7 = a + c \qquad (1)$$
$$10 = 4a + c \qquad (2)$$

Subtract (1) from (2):
$3 = 3a$ and hence $a = 1$.

Substitute in (1):
$7 = 1 + c$ and therefore $c = 6$.

The equation is $y = x^2 + 6$.

Explanation

Substitute $x = 1$, $y = 7$ in the equation $y = ax^2 + c$ to obtain (1).

Substitute $x = 2$, $y = 10$ in the equation $y = ax^2 + c$ to obtain (2).

Example 40

A family of parabolas have rules of the form $y = ax^2 + bx + 2$, where $a \neq 0$.

a For a parabola in this family with its turning point on the x-axis, determine a in terms of b.

b If the turning point is at $(4, 0)$, determine the values of a and b.

 ISBN 978-1-009-53719-3 Cambridge University Press

Solution

a The discriminant $\Delta = b^2 - 8a$.

We have $\Delta = 0$ and therefore $a = \dfrac{b^2}{8}$.

b We have $-\dfrac{b}{2a} = 4$, which implies $b = -8a$.

From part a, we have $a = \dfrac{b^2}{8}$.

Hence $a = \dfrac{64a^2}{8} = 8a^2$.

Thus $a(1 - 8a) = 0$ and, since $a \neq 0$, $a = \frac{1}{8}$.

Substituting for a in $b = -8a$ gives $b = -1$.

Explanation

The discriminant of $ax^2 + bx + c$ is $\Delta = b^2 - 4ac$. In this case $c = 2$.

The discriminant $\Delta = 0$ since the parabola touches the x-axis at its turning point.

The axis of symmetry has equation $x = -\dfrac{b}{2a}$.

Determining quadratic rules

At the beginning of this section we looked at different families of quadratic polynomial functions. We now consider three important such families which can be used as a basis for determining a quadratic rule from given information. These are certainly not the only useful forms. You will see others in the worked examples.

1. $y = a(x - e)(x - f)$ — This can be used if two x-axis intercepts and the coordinates of one other point are known.
2. $y = a(x - h)^2 + k$ — This can be used if the coordinates of the turning point and one other point are known.
3. $y = ax^2 + bx + c$ — This can be used if the coordinates of three points on the parabola are known.

Example 41

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Determine the rule for this parabola.

Solution

$y = a(x + 3)(x - 4)$

When $x = 1$, $y = 24$.

Therefore
$$\begin{aligned} 24 &= a(1 + 3)(1 - 4) \\ 24 &= -12a \\ a &= -2 \end{aligned}$$

The rule is $y = -2(x + 3)(x - 4)$.

Explanation

Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

ISBN 978-1-009-53719-3

Example 42

The coordinates of the turning point of a parabola are $(2, 6)$ and the parabola passes through the point $(3, 3)$. Determine the rule for this parabola.

Solution

$y = a(x - 2)^2 + 6$

When $x = 3$, $y = 3$.

Therefore

$$3 = a(3 - 2)^2 + 6$$
$$3 = a + 6$$
$$a = -3$$

The rule is $y = -3(x - 2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x - h)^2 + k$.

Example 43

A parabola passes through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$. Determine the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

$$4 = a + b + c \quad (1)$$
$$5 = c \quad (2)$$
$$10 = a - b + c \quad (3)$$

Substitute from equation (2) into equations (1) and (3):

$$-1 = a + b \quad (1')$$
$$5 = a - b \quad (3')$$

Add (1′) and (3′):

$$4 = 2a$$
$$a = 2$$

Using equation (1′), we obtain $b = -3$.

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore substitute values into the polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

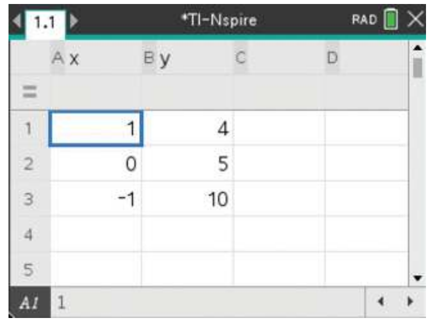
 ISBN 978-1-009-53719-3

TI-84

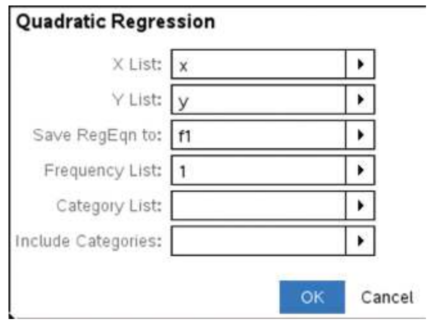
Using the TI-Nspire CX non-CAS

To determine the rule for the parabola passing through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$:

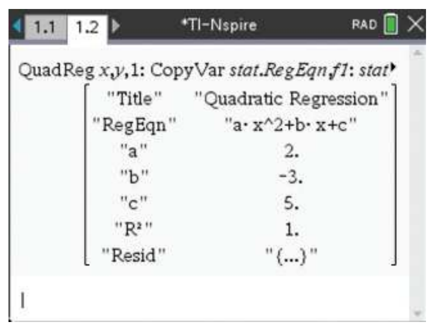
- Open a **Lists & Spreadsheet** application.
- Enter the coordinates in lists named x and y.



- Insert a **Calculator** page (ctrl I > **Add Calculator**).
- Use menu > **Statistics** > **Stat Calculations** > **Quadratic Regression**.
- Complete the pop-up screen as shown. Select OK to display the values of a, b and c.



- Hence the rule is $y = 2x^2 - 3x + 5$.



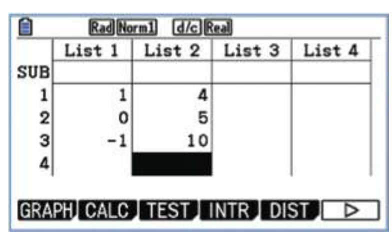
Using the Casio

To determine the rule for the parabola passing through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$:

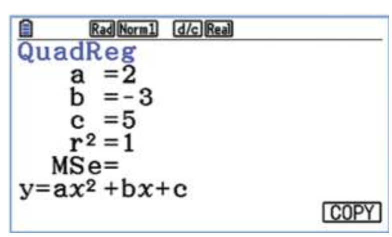
- Press MENU 2 to select **Statistics** mode.
- Enter the x-coordinates in List1 and the y-coordinates in List2:

  1 EXE 0 EXE (−) 1 EXE ►

  4 EXE 5 EXE 1 0 EXE



- Select Quadratic Regression by going to **Calculation** F2, **Regression** F3, then **X²** F3.
- Hence the rule is $y = 2x^2 - 3x + 5$.

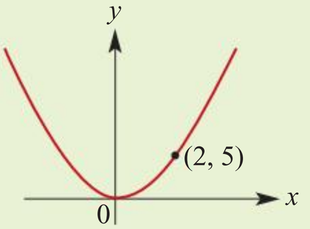


 ISBN 978-1-009-53719-3

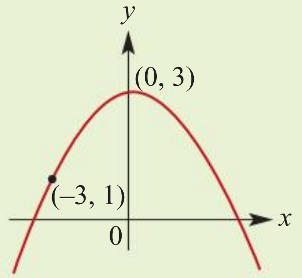
Example 44

Determine the quadratic rule for each of the following parabolas:

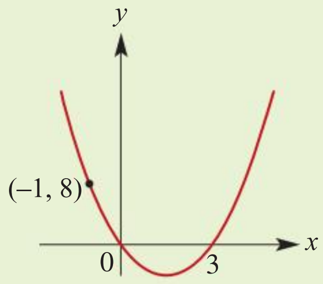
a



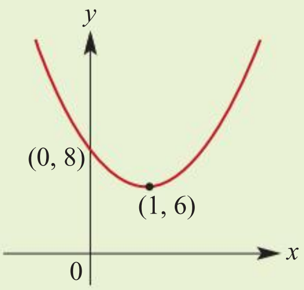
b



c



d



Solution

a This is of the form $y = ax^2$

For (2, 5): $5 = 4a$

$\therefore \quad a = \dfrac{5}{4}$

Hence the rule is $y = \dfrac{5}{4}x^2$

b This is of the form $y = ax^2 + c$

For (0, 3): $3 = a(0) + c$

$\therefore \quad c = 3$

For (−3, 1): $1 = a(-3)^2 + 3$

$1 = 9a + 3$

$\therefore \quad a = -\dfrac{2}{9}$

Hence the rule is $y = -\dfrac{2}{9}x^2 + 3$

c This is of the form $y = ax(x - 3)$

For (−1, 8): $8 = -a(-1 - 3)$

$8 = 4a$

$\therefore \quad a = 2$

Hence the rule is $y = 2x(x - 3)$

$y = 2x^2 - 6x$

d This is of the form $y = a(x - 1)^2 + 6$

For (0, 8): $8 = a + 6$

$\therefore \quad a = 2$

Hence the rule is $y = 2(x - 1)^2 + 6$

$y = 2(x^2 - 2x + 1) + 6$

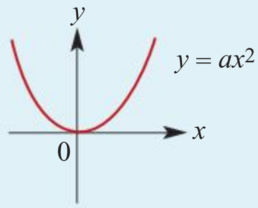
$y = 2x^2 - 4x + 8$

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 4K

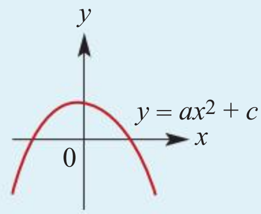
To determine a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



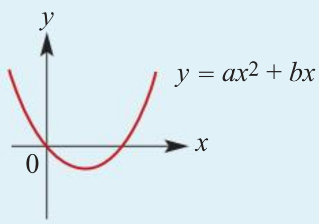
One point is needed to determine a.

ii



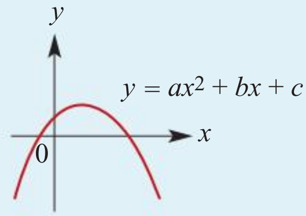
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Skill-sheet

Exercise 4K

Example 39

1 A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(-1, 2)$ and $(0, 6)$, determine the values of a and c.

CF

Example 40

2 A family of parabolas have rules of the form $y = ax^2 + bx + 4$, where $a \neq 0$.

a determine the discriminant of the quadratic polynomial $ax^2 + bx + 4$.

b For a parabola in this family with its turning point on the x-axis, determine a in terms of b.

c If the turning point is at $(-4, 0)$, determine the values of a and b.

Example 41

3 **a** A parabola has x-axis intercepts -2 and 6 and it passes through the point $(1, -30)$. Determine the rule for this parabola.

Example 42

b The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(3, -46)$. Determine the rule for this parabola.

Example 43

c A parabola passes through the points $(1, -2)$, $(0, -3)$ and $(-1, -6)$. Determine the rule for this parabola.

4 A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with coordinates $(2, 8)$. Determine the value of a.

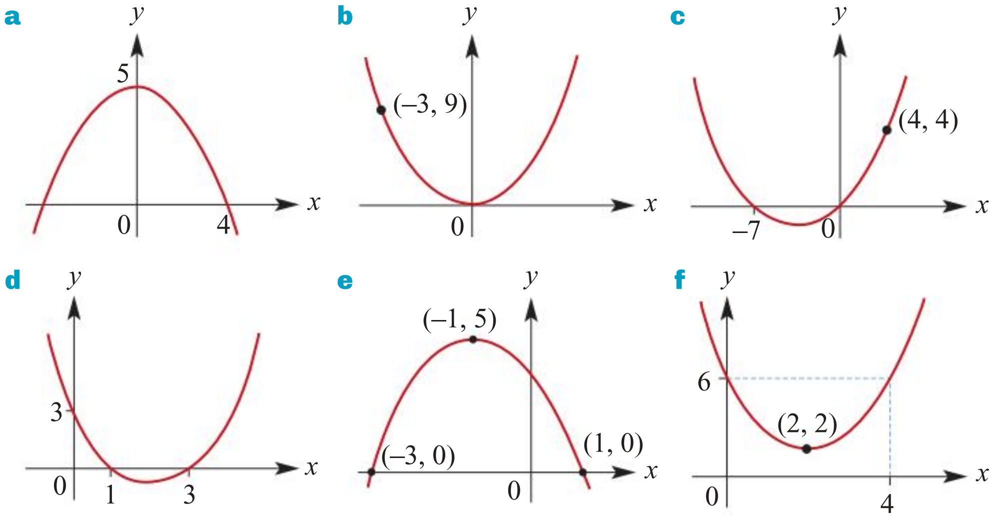
 ISBN 978-1-009-53719-3

5 A quadratic rule for a particular parabola is of the form $y = ax^2 + bx$. The parabola passes through the point with coordinates $(-1, 4)$ and one of its x-axis intercepts is 6.Determine the values of a and b.

6 A quadratic rule for a particular parabola is of the form $y = a(x - b)^2 + c$. The parabola has vertex $(1, 6)$ and passes through the point with coordinates $(2, 4)$. Determine the values of a, b and c.

Example 44

7 Determine the equation of each of the following parabolas:

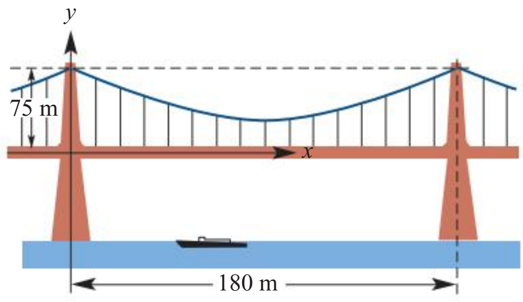


8 A parabola has vertex with coordinates $(-1, 3)$ and passes through the point with coordinates $(3, 8)$. Determine the equation for the parabola.

9 A parabola has x-axis intercepts 6 and -3 and passes through the point $(1, 10)$. Determine the equation of the parabola.

10 A parabola has vertex with coordinates $(-1, 3)$ and y-axis intercept 4. Determine the equation for the parabola.

11 Assuming that the suspension cable shown in the diagram forms a parabola, determine the rule which describes its shape. The minimum height of the cable above the roadway is 30 m.



12 A parabola has the same shape as $y = 2x^2$, but its turning point is $(1, -2)$. Write its equation.

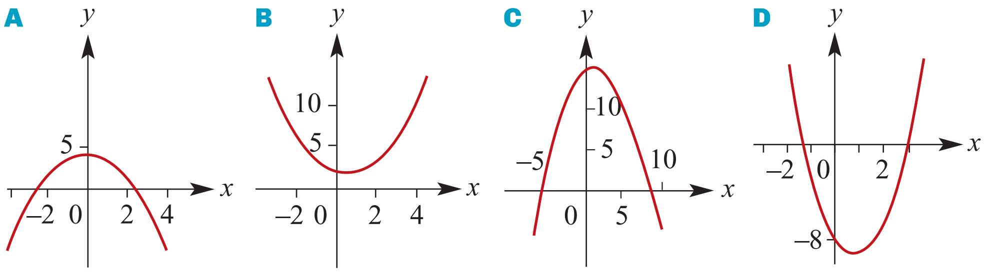
13 A parabola has its vertex at $(1, -2)$ and passes through the point $(3, 2)$. Write its equation.

 ISBN 978-1-009-53719-3

14 Which of the curves could be most nearly defined by each of the following?

a $y = \frac{1}{3}(x + 4)(8 - x)$ **b** $y = x^2 - x + 2$

c $y = -10 + 2(x - 1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$



15 A family of parabolas satisfies the rule $y = ax^2 + 2x + a$.

a Express $ax^2 + 2x + a$ in the form $a(x + b)^2 + c$ for real numbers b and c.

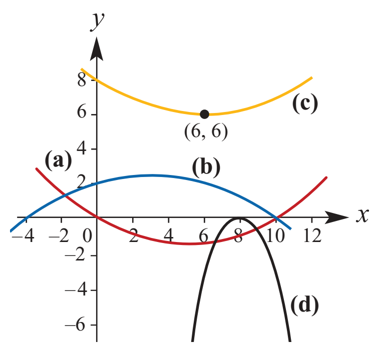
b Give the coordinates of the turning point of the graph of $y = ax^2 + 2x + a$ in terms of a.

c For which values of a is $ax^2 + 2x + a$ a perfect square?

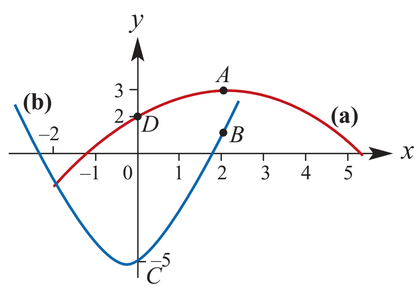
d For which values of a does the graph of $y = ax^2 + 2x + a$ have two x-axis intercepts?

16 A parabola has its vertex at $(2, 2)$ and passes through $(4, -6)$. Write its equation.

17 Write down four quadratic rules that have graphs similar to those in the diagram.



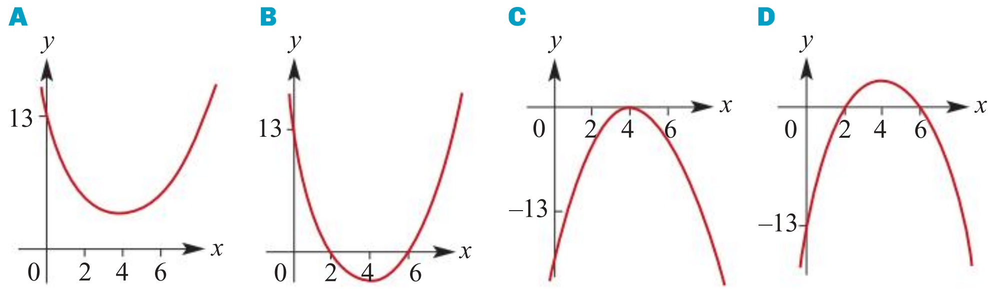
18 Determine quadratic expressions which could represent the two curves in this diagram, given that the coefficient of x is 1 in each case. The labelled points are $A(2, 3)$, $B(2, 1)$, $C(0, -5)$ and $D(0, 2)$.



19 The rate of rainfall during a storm t hours after it began was 3 mm per hour when $t = 5$, 6 mm per hour when $t = 9$ and 5 mm per hour when $t = 13$. Assuming that a quadratic model applies, determine an expression for the rate of rainfall, r mm per hour, in terms of t.

20 **a** Which of the graphs shown below could represent the equation $y = (x-4)^2 - 3$?

b Which graph could represent $y = 3 - (x-4)^2$?



21 Determine the equation of the quadratic which passes through the points with coordinates:

a $(-2,-1),\ (1,2),\ (3,-16)$ **b** $(-1,-2),\ (1,-4),\ (3,10)$ **c** $(-3,5),\ (3,20),\ (5,57)$

4L Quadratic models

In this section it is shown how quadratics can be used to solve worded problems, including problems which involve determining the maximum or minimum value of a quadratic polynomial that has been used to model a 'practical' situation.

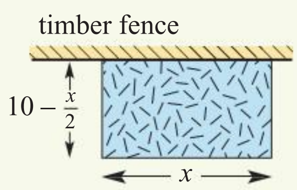
Learning intentions

- To be able to model situations by using a quadratic polynomial.

Example 45

Jenny wishes to fence off a rectangular vegetable garden in her backyard. She has 20 m of fencing wire which she will use to fence three sides of the garden, with the existing timber fence forming the fourth side. Calculate the maximum area she can enclose.

Solution



Let A = area of the rectangular garden

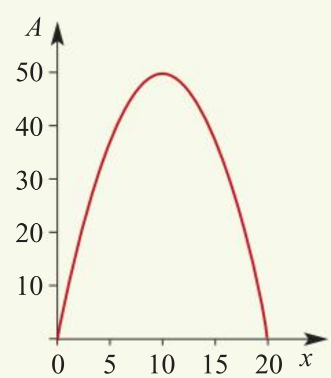
x = length of the garden

Then width $= \dfrac{20-x}{2} = 10 - \dfrac{x}{2}$

Therefore

$$A = x\left(10 - \frac{x}{2}\right) = 10x - \frac{x^2}{2}$$
$$= -\tfrac{1}{2}(x^2 - 20x + 100 - 100)$$
(completing the square)
$$= -\tfrac{1}{2}(x^2 - 20x + 100) + 50$$
$$= -\tfrac{1}{2}(x-10)^2 + 50$$

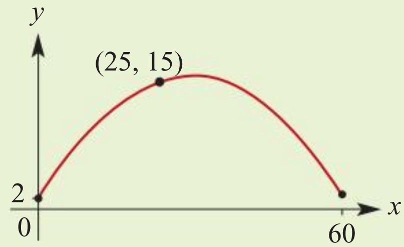
Hence the maximum area is 50 m^2 when $x = 10$.



 ISBN 978-1-009-53719-3

Example 46

A cricket ball is thrown by a fielder. It leaves his hand at a height of 2 metres above the ground and the wicketkeeper takes the ball 60 metres away, again at a height of 2 metres. It is known that after the ball has gone 25 metres it is 15 metres above the ground. The path of the cricket ball is a parabola with equation $y = ax^2 + bx + c$.



a Determine the values of a, b and c.

b Determine the maximum height of the ball above the ground.

c Determine the height of the ball when it is 5 metres horizontally before it hits the wicketkeeper's gloves.

Solution

a The data can be used to obtain three equations:

$$2 = c \qquad (1)$$

$$15 = (25)^2a + 25b + c \qquad (2)$$

$$2 = (60)^2a + 60b + c \qquad (3)$$

Substitute equation (1) in equations (2) and (3):

$$13 = 625a + 25b \qquad (2')$$

$$0 = 3600a + 60b \qquad (3')$$

Simplify (3′) by dividing both sides by 60:

$$0 = 60a + b \qquad (3')$$

Multiply this by 25 and subtract from equation (2′):

$$13 = -875a$$

$$\therefore \quad a = -\frac{13}{875} \quad \text{and} \quad b = \frac{156}{175}$$

The path of the ball has equation

$$y = -\frac{13}{875}x^2 + \frac{156}{175}x + 2$$

b The maximum height occurs when $x = 30$ and $y = \frac{538}{35}$.

$\therefore$ maximum height is $\frac{538}{35}$ m.

c When $x = 55$, $y = \frac{213}{35}$.

$\therefore$ height of the ball is $\frac{213}{35}$ m.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 4L

Example 45

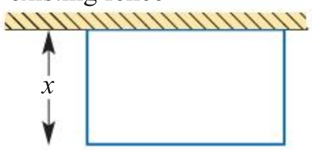
CF

1 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence.

a If the width of the yard is x m and the area inside the yard is A m^2, write down the rule connecting A and x.

b Sketch the graph of A against x.

c Determine the maximum area that can be formed for the yard.



2 A rectangle has a perimeter of 20 m. Let x m be the length of one side. Determine a formula for the area A of the rectangle in terms of x. Hence determine the maximum area A.

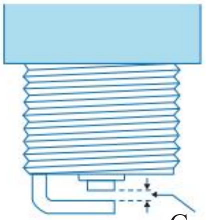
3 The efficiency rating, E, of a particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$.

a Sketch the graph of E against x for $0 \le x \le 1$.

b What values of x give a zero efficiency rating?

c What value of x gives the maximum efficiency rating?

d Use the graph, or otherwise, to determine the values of x between which the efficiency rating is 70 or more.



4 A piece of wire 68 cm in length is bent into the shape of a rectangle.

a If x cm is the length of the rectangle and A cm^2 is the area enclosed by the rectangular shape, write down a formula which connects A and x.

b Sketch the graph of A against x for suitable x-values.

c Use your graph to determine the maximum area formed.

5 A piece of wire 80 cm long is to be cut into two pieces. One piece is to be bent into a square and the other into a rectangle four times as long as it is wide.

a Let x cm be the length of a side of the square and y cm be the width of the rectangle. Write a formula connecting y and x.

b Let A cm^2 be the sum of the areas of the square and the rectangle.

i Determine a formula for A in terms of x.

ii Determine the length of both pieces of wire if A is to be a minimum.

6 A construction firm has won a contract to build cable-car pylons at various positions on the side of a mountain. Because of difficulties associated with construction in alpine areas, the construction firm will be paid an extra amount $\$C$ for each pylon, given by the formula $C = 240h + 100h^2$, where h is the height in km above sea level.

a Sketch the graph of C as a function of h. Comment on the possible values of h.

b Does C have a maximum value?

c What is the value of C for a pylon built at an altitude of 2500 m?

 ISBN 978-1-009-53719-3 Cambridge University Press

7 A tug-o-war team produces a tension in a rope described by the rule

$$T = 290(8t - 0.5t^2 - 1.4) \text{ units}$$

where t is the number of seconds after commencing the pull.

a Sketch a graph of T against t, stating the practical domain.

b What is the greatest tension produced during a 'heave'?

Example 46

8 A cricketer struck a cricket ball such that its height, d metres, after it had travelled x metres horizontally was given by the rule $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2$, $x \geq 0$.

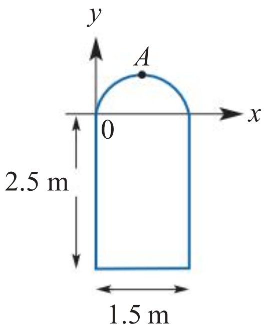
a Use a calculator to graph d against x for values of x ranging from 0 to 30.

b **i** What was the maximum height reached by the ball?

ii If a fielder caught the ball when it was 2 m above the ground, how far was the ball from where it was hit?

iii At what height was the ball when it was struck?

9 An arch on the top of a door is parabolic in shape. The point A is 3.1 m above the bottom of the door. The equation $y = ax^2 + bx + c$ can be used to describe the arch. Determine the values of a, b and c.



10 It is known that the daily spending of a government department follows a quadratic model. Let t be the number of days after 1 January and s be the spending in hundreds of thousands of dollars on a particular day, where $s = at^2 + bt + c$.

t	30	150	300
s	7.2	12.5	6

a Determine the values of a, b and c.

b Sketch the graph for $0 \leq t \leq 360$. (Use a calculator.)

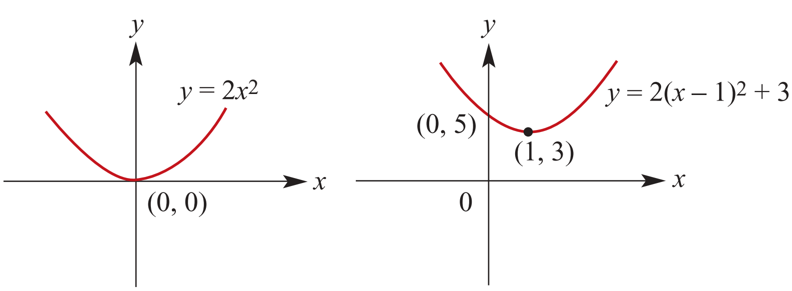
c Determine an estimate for the spending when:

i $t = 180$

ii $t = 350$

Chapter summary

- The general expression for a quadratic function is $y = ax^2 + bx + c$.
- Methods for **factorising**:
  - Taking out a common factor
    e.g. $9x^3 + 27x^2 = 9x^2(x + 3)$
  - Grouping of terms
    e.g. $$\begin{aligned} x^3 + 4x^2 - 3x - 12 &= (x^3 + 4x^2) - (3x + 12) \\ &= x^2(x + 4) - 3(x + 4) \\ &= (x^2 - 3)(x + 4) \end{aligned}$$
  - Difference of two squares: $x^2 - a^2 = (x + a)(x - a)$
    e.g. $16x^2 - 49 = (4x - 7)(4x + 7)$
  - Factorising quadratic expressions
    e.g. $x^2 + 2x - 8 = (x + 4)(x - 2)$
    $6x^2 - 13x - 15 = (6x + 5)(x - 3)$
- The graph of a quadratic function may be sketched by first expressing the rule in **turning point** form, $y = a(x - h)^2 + k$. The graph can then be obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis (for h, k positive).
  e.g. for $y = 2(x - 1)^2 + 3$



- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the general **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The following steps can be used to sketch the graph of a quadratic function in **polynomial** form, $y = ax^2 + bx + c$:
  - If $a > 0$, the function has a minimum value.
  - If $a < 0$, the function has a maximum value.
  - The value of c gives the y-axis intercept.
  - The equation of the axis of symmetry is $x = -\dfrac{b}{2a}$.
  - The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$.

 ISBN 978-1-009-53719-3

- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two distinct solutions.
  - If $\Delta = 0$, the quadratic equation has one solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.
- To determine a quadratic rule to fit given points, choose an appropriate form. For example:

$y = a(x - e)(x - f)$	This can be used if two x-axis intercepts and the coordinates of one other point are known.
$y = a(x - h)^2 + k$	This can be used if the coordinates of the turning point and one other point are known.
$y = ax^2 + bx + c$	This can be used if the coordinates of three points on the parabola are known.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

4A **1** I can expand and collect like terms for expressions such as $5x(2x - 3) + 3x(5x - 2)$. ☐

See Example 1, Example 2 and Questions 3 and 4

4A **2** I can expand and collect like terms for expressions such as $(x - 5)(2x - \sqrt{3})$ and $(x - 5)(x^2 - \sqrt{3}x + 4)$. ☐

See Example 3, Example 4 and Questions 5 and 6

4A **3** I can expand and collect like terms for perfect squares of the form $(a + b)^2$. ☐

See Example 5 and Question 7

4A **4** I can recognise a difference of squares. That is, $(a + b)(a - b) = a^2 - b^2$. ☐

See Example 6, Example 7 and Questions 8 and 9

4B **5** I can factorise an expression by taking a common factor outside the brackets. ☐

See Example 8, Example 9 and Questions 2 and 3

4B **6** I can factorise suitable expressions by first grouping terms. ☐

See Example 10 and Question 4

4B **7** I can factorise an expression by recognising a difference of perfect squares. ☐

See Example 11, Example 12 and Questions 5 and 6

 ISBN 978-1-009-53719-3

4B **8** I can factorise suitable monic quadratic polynomials. ☐

See Example 13 and Question 7

4B **9** I can factorise suitable non-monic quadratic polynomials. ☐

See Example 14, Example 15 and Question 8

4B **10** I can factorise suitable non-monic quadratic polynomials by first taking out a common factor. ☐

See Example 16 and Question 9

4B **11** I can factorise suitable non-monic quadratic polynomials by first making a suitable substitution. ☐

See Example 17 and Question 10

4C **12** I can solve quadratic equations. ☐

See Example 18, Example 19 and Questions 3 and 4

4C **13** I can apply quadratic equations to solve problems posed in a context. ☐

See Example 20 and Question 10

4D **14** I can graph quadratic equations of the form $y = a(x - h)^2 + k$. ☐

See Example 21, Example 22, Example 23, Example 24 and Questions 1, 2 and 3

4E **15** I can solve quadratic equations by first completing the square. ☐

See Example 25 and Question 3

4E **16** I can graph a polynomial with quadratic rule by first completing the square. ☐

See Example 26 and Question 4

4E **17** I can use the equation of the axis of symmetry of a parabola to help sketch the graph of that parabola. ☐

See Example 27 and Question 6

4F **18** I can graph a polynomial with quadratic rule by determining the coordinates of the points of intersection with each axis, the equation of the axis of symmetry and the coordinates of the turning point. ☐

See Example 28, Example 29, Example 30 and Questions 3 and 4

4G **19** I can solve a quadratic inequality. ☐

See Example 31, and Question 1

4H **20** I can use the quadratic formula to solve an equation. ☐

See Example 32, Example 33 and Questions 3 and 4

Review

4I **21** I can use the discriminant to determine how many solutions a quadratic equation has. ☐

See Example 34, Example 35 and Question 5

4J **22** I can determine the coordinates of the points of intersection of a straight line and a parabola. ☐

See Example 37 and Question 1

4K **23** I can determine the equation of a parabola given sufficient information about the parabola. ☐

See Example 41, Example 42, Example 43, Example 44, and Questions 3 and 7

4L **24** I can model suitable real-world situations with a quadratic polynomial. ☐

See Example 45, Example 46 and Questions 1 and 8

Short-response questions

Technology-free short-response questions

SF

1 Express each of the following in the form $(ax + b)^2$:

a $x^2 + 9x + \frac{81}{4}$ **b** $x^2 + 18x + 81$ **c** $x^2 - \frac{4}{5}x + \frac{4}{25}$

d $x^2 + 2bx + b^2$ **e** $9x^2 - 6x + 1$ **f** $25x^2 + 20x + 4$

2 Expand each of the following products:

a $-3(x - 2)$ **b** $-a(x - a)$

c $(7a - b)(7a + b)$ **d** $(x + 3)(x - 4)$

e $(2x + 3)(x - 4)$ **f** $(x + y)(x - y)$

g $(a - b)(a^2 + ab + b^2)$ **h** $(2x + 2y)(3x + y)$

i $(3a + 1)(a - 2)$ **j** $(x + y)^2 - (x - y)^2$

k $u(v + 2) + 2v(1 - u)$ **l** $(3x + 2)(x - 4) + (4 - x)(6x - 1)$

3 Express each of the following as a product of factors:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$

d $4 - x^2$ **e** $au + 2av + 3aw$ **f** $4a^2b^2 - 9a^4$

g $1 - 36x^2a^2$ **h** $x^2 + x - 12$ **i** $x^2 + x - 2$

j $2x^2 + 3x - 2$ **k** $6x^2 + 7x + 2$ **l** $3x^2 - 8x - 3$

m $3x^2 + x - 2$ **n** $6a^2 - a - 2$ **o** $6x^2 - 7x + 2$

4 Solve each of the following equations for x by first factorising:

a $x^2 - 2x - 15 = 0$ **b** $x^2 - 9x = 0$ **c** $2x^2 - 10x + 12 = 0$

d $x^2 - 24x - 25 = 0$ **e** $3x^2 + 15x + 18 = 0$ **f** $x^2 - 12x + 36 = 0$

g $2x^2 - 5x - 3 = 0$ **h** $12x^2 - 8x - 15 = 0$ **i** $5x^2 + 7x - 12 = 0$

Review

SF

5 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = -2x^2 + 3$ **c** $y = 2(x-2)^2 + 3$
d $y = 2(x+2)^2 + 3$ **e** $y = 2(x-4)^2 - 3$ **f** $y = 9 - 4x^2$
g $y = 3(x-2)^2$ **h** $y = 2(2-x)^2 + 3$

6 Express in the form $y = a(x-h)^2 + k$ and hence sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 6x$ **c** $y = x^2 - 8x + 4$
d $y = 2x^2 + 8x - 4$ **e** $y = -3x^2 - 12x + 9$ **f** $y = -x^2 + 4x + 5$

7 For each of the following, determine

i the axis intercepts
ii the axis of symmetry
iii the turning point

and hence sketch the graph:

a $y = x^2 - 7x + 6$ **b** $y = -x^2 - x + 12$ **c** $y = -x^2 + 5x + 14$
d $y = x^2 - 10x + 16$ **e** $y = 2x^2 + x - 15$ **f** $y = 6x^2 - 13x - 5$
g $y = 9x^2 - 16$ **h** $y = 4x^2 - 25$

8 Solve the following quadratic inequalities:

a $x^2 > x$ **b** $(x+2)^2 \le 34$ **c** $3x^2 + 5x - 2 \le 0$
d $-2x^2 + 13x \ge 15$

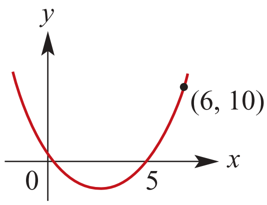
9 Use the quadratic formula to solve each of the following:

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 9x + 12 = 0$ **c** $x^2 - 4x + 2 = 0$
d $2x^2 + 7x + 2 = 0$ **e** $2x^2 + 7x + 4 = 0$ **f** $3x^2 + 9x - 1 = 0$

CF

10 Determine the value(s) of p that will make the quadratic $(5p-1)x^2 - 4x + (2p-1)$ a perfect square.

11 Determine the equation of the quadratic, the graph of which is shown.



12 A parabola has the same shape as $y = 3x^2$ but its vertex is at $(5, 2)$. Determine the equation corresponding to this parabola.

13 Determine the possible values of m if $(2m-3)x^2 + (5m-1)x + (3m-2) = 0$ has two solutions.

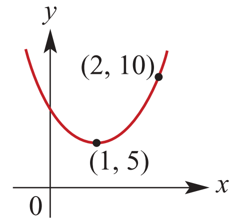
 ISBN 978-1-009-53719-3

Review

CF

14 Two numbers have a sum of 30. Determine the maximum value of the product of such numbers.

15 Determine the rule of the quadratic function which describes the graph.



16 Determine the coordinates of the points of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$

b $y = 8x + 11$ and $y = 2x^2$

c $y = 3x^2 + 7x$ and $y = 2$

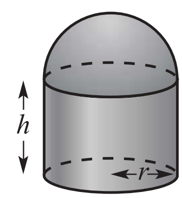
d $y = 2x^2$ and $y = 2 - 3x$

17 **a** A parabola has x-axis intercepts -4 and 1 and it passes through the point $(-1, -12)$. Determine the rule for this parabola.

b The coordinates of the turning point of a parabola are $(-1, 3)$ and the parabola passes through the point $(1, -5)$. Determine the rule for this parabola.

c A parabola passes through the points $(1, -3)$, $(0, -3)$ and $(-1, 1)$. Determine the rule for this parabola.

18 The surface area, S, of a cylindrical tank with a hemispherical top is given by the formula $S = ar^2 + brh$, where $a = 9.42$ and $b = 6.28$. What is the radius of a tank of height 6 m which has a surface area of 125.6 m^2?



19 **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution?

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions?

20 Consider the family of quadratics with rules of the form $y = x^2 + bx$, where b is a non-zero real number.

a Determine the x-axis intercepts.

b Determine the coordinates of the vertex of the parabola.

c **i** Determine the coordinates of the points of intersection of the graph of $y = x^2 + bx$ with the line $y = x$, in terms of b.

ii For what value(s) of b is there one point of intersection?

iii For what value(s) of b are there two points of intersection?

Technology-active short-response questions

SF

21 Solve each of the quadratic equations, giving the solutions correct to three decimal places.

a $-2.6x^2 - 3.76x + 9.21 = 0$

b $3.06x^2 - 2.816x - 8.26 = 0$

ISBN 978-1-009-53719-3

Review

CF

22 The parabola with equation $y = ax^2 + bx + c$ passes through the points with coordinates $(-1.5, 5.2)$, $(3.98, -3.54)$ and $(6, 4.32)$. Determine the values of a, b and c correct to three decimal places.

23 A particle is projected from a point 2 metres above a stretch of horizontal land so that its height y metres above the ground after it has travelled x metres horizontally is defined by the equation $y = 2 + 1.731x - 0.024x^2$. For the following give your answers correct to two decimal places.

a How far does the particle go horizontally before it hits the ground?

b How high does the particle go?

c How far has the particle gone horizontally before it first reaches a height of 5 metres above the ground?

d How far has the particle gone horizontally before it again is at a height of 2 metres above the ground?

24 A stone is thrown vertically upwards from the edge of a cliff which is 25 m high. The vertical height y m above the base of the cliff after t seconds is given by $y = 25 + 21.8t - 4.91t^2$. For the following give your answers correct to two decimal places.

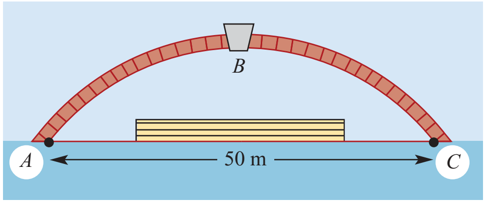
a How far in metres above the edge of the cliff does the stone go?

b How long after being thrown does it take for the stone to hit the ground at the base of the cliff?

c For how long is the stone above the point where it was thrown?

CU

25 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. The line AC is the water level and B is the highest point of the bridge.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

26 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other a rectangular shape in which the length is twice its width.

a If x cm is the side length of the square, write down the dimensions of the rectangle in terms of x.

 ISBN 978-1-009-53719-3

CU

b Formulate a rule for A, the combined area of the square and rectangle in cm^2, in terms of x.

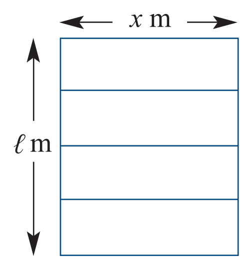
c Determine the lengths of the two pieces if the sum of the areas is to be a minimum.

27 Water is pumped into an empty metal tank at a steady rate of 0.2 litres/min. After 1 hour the depth of water in the tank is 5 cm; after 5 hours the depth is 10 cm.

a If the volume of water in the tank is V litres when the depth is x cm and there is a quadratic relationship between V and x, write down a rule for V in terms of x.

b It is known that the maximum possible depth of water in the tank is 20 cm. For how long, from the beginning, can water be pumped into the tank at the same rate without overflowing?

28 100 m of angle steel is used to make a rectangular frame with three crossbars as shown in the figure.



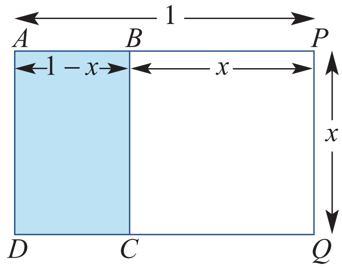
a If the width of the frame is x m, determine an expression for ℓ, the length of the frame in metres, in terms of x.

b The frame is to be covered by light aluminium sheeting. If the area of this sheeting is A m^2, formulate a rule connecting A and x.

c Sketch a graph of A against x, stating the axis intercepts and the turning point.

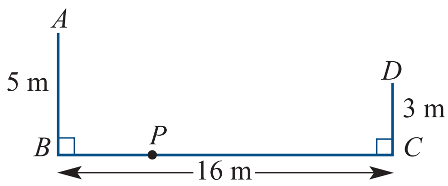
d What is the maximum area and the value of x which gives this area?

29 A shape which has been of interest to architects and artists over the centuries is the 'golden rectangle'. Many have thought that it has the perfect proportions for buildings. The rectangle is such that, if a square is drawn on one of the longer sides, then the new rectangle is similar to the original. Let the length of $AP = 1$ unit; then $AB = 1 - x$ units and $\dfrac{AP}{AD} = \dfrac{AD}{AB}$.



Determine the value of x. (This value is the reciprocal of the 'golden ratio'.)

30 The point P is x m from B along the line BC.



a Determine distance PA in terms of x.

b **i** Determine distance PC in terms of x.

ii Determine distance PD in terms of x.

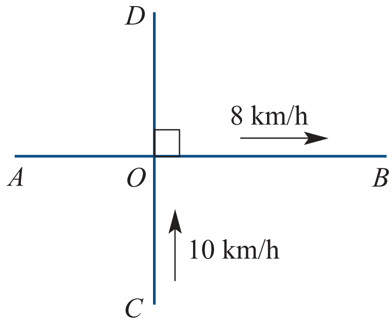
c Determine x if $PA = PD$.

d Determine x if $PA = 2PD$. (Answer correct to three decimal places.)

e Determine x if $PA = 3PD$. (Answer correct to three decimal places.)

 ISBN 978-1-009-53719-3 Cambridge University Press

31 AB and CD are crossroads. A jogger runs along road AB at a speed of 8 km/h and passes O at 1 p.m. Another runner is moving along road CD. The second runner is moving at 10 km/h and passes O at 1:30 p.m.



a Let y km be their distance apart t hours after 1 p.m.

i Determine an expression for y in terms of t.

ii Plot the graph of y against t on a calculator.

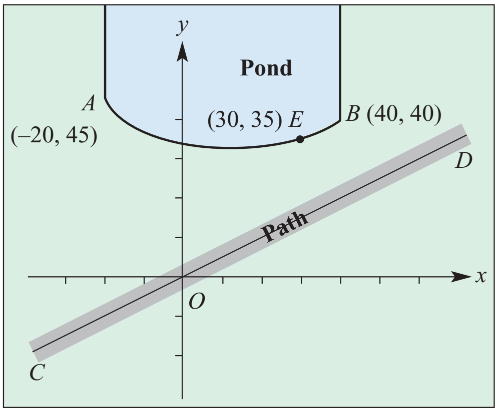
iii Determine the time(s) when the runners are 4 km apart. (Use a calculator.)

iv Determine the time at which the runners are closest and their distance apart at this time.

b Determine the exact value(s) of t for which:

i $y = 5$ **ii** $y = 6$

32 A path cuts across a park. Its centreline can be described by the equation $y = \frac{x}{2}$, where the origin is at a point O in the park. The path starts at a point $C(-30, -15)$ and finishes at a point $D(60, 30)$.



a How long is the path?

One boundary of the pond in the park is parabolic in shape. The boundary passes through the points $A(-20, 45)$, $B(40, 40)$ and $E(30, 35)$. The equation of the parabola is of the form $y = ax^2 + bx + c$.

b **i** Determine the equation of the parabola.

ii Determine the coordinates of the vertex of the parabola.

c On the one set of axes sketch the graphs of $y = \frac{x}{2}$ and the parabola. (Use a calculator to help.)

d Consider the rule $y = (ax^2 + bx + c) - \frac{1}{2}x$, where a, b and c have been determined in part b i.

i What does this expression determine?

ii Determine the minimum value of this expression and the value of x for which this occurs.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Multiple-choice questions

Technology-free multiple-choice questions

1 The linear factors of $12x^2 + 7x - 12$ are

A $4x - 3$ and $3x + 4$ **B** $3x - 2$ and $4x + 6$
C $3x + 2$ and $4x - 6$ **D** $6x + 4$ and $2x - 3$

2 The solutions of the equation $x^2 - 5x - 14 = 0$ are

A $x = -7, x = 2$ **B** $x = -2, x = 7$
C $x = -2, x = -7$ **D** $x = -2$ only

3 For $y = 8 + 2x - x^2$, the maximum value of y is

A $5\frac{1}{4}$ **B** 9 **C** $9\frac{1}{2}$ **D** 10

4 If the graph of $y = 2x^2 - kx + 3$ touches the x-axis, then the possible values of k are

A $k = 1$ **B** $k = -3$ or $k = -\frac{1}{2}$
C $k = 1$ or $k = 3$ **D** $k = 2\sqrt{6}$ or $k = -2\sqrt{6}$

5 The solutions of the equation $x^2 - 56 = x$ are

A $x = -7$ or 8 **B** $x = 7$ or 8
C $x = -9$ or 6 **D** $x = 9$ or -6

6 The value of the discriminant of $x^2 + 3x - 10$ is

A -5 **B** 49 **C** 7 **D** -2

7 The coordinates of the turning point of the graph with equation $y = 3x^2 + 6x - 1$ are

A $(-\frac{1}{3}, 2)$ **B** $(-\frac{1}{3}, -4)$ **C** $(1, -4)$ **D** $(-1, -4)$

8 The quadratic $5x^2 - 10x - 2$ in turning point form $a(x - h)^2 + k$, by completing the square, is

A $(5x - 1)^2 - 5$ **B** $5(x - 1)^2 - 5$
C $5(x + 1)^2 - 2$ **D** $5(x - 1)^2 - 7$

9 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two solutions is (are)

A $m = 3$ **B** $m = 0$ **C** $m > -3$ **D** $m < -3$

10 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(6x - 4y)(x + 2y)$
C $(3x - 2y)(2x + 4y)$ **D** $(6x + y)(x - 8y)$

11 The turning point of a quadratic with rule $y = x^2 - ax$ has coordinates

A $\left(\frac{a}{2}, \frac{-a^2}{4}\right)$ **B** $(a, 0)$ **C** $\left(a, \frac{-a^2}{2}\right)$ **D** $\left(\frac{-a}{2}, \frac{-a^2}{4}\right)$

 ISBN 978-1-009-53719-3

12 The solution of the inequality $x^2 > b^2$, where $b < 0$, is

A $x > b$ **B** $b < x < -b$ **C** $-b < x < b$ **D** $x < b$ or $x > -b$

13 The quadratic equation $x^2 - 2ax + b = 0$, where a and b are positive constants, has one solution when

A $b = \sqrt{a}$ and $b = -\sqrt{a}$ **B** $b = 1$ and $a \neq 1$
C $a = \sqrt{b}$ or $a = -\sqrt{b}$ **D** $b = a = 2$

Technology-active multiple-choice questions

14 The sum of the solutions, correct to two decimal places, of the quadratic equation $2.1x^2 - 3.5x - 2.7 = 0$ is

A -1.21 **B** 1.67 **C** 2.85 **D** -1.66

15 A helicopter climbs vertically from the top of a 110 m building so that its height above ground after t seconds is given by $h = 110 + 54t - 5.6t^2$. The maximum height above the ground, in metres, correct to two decimal places, it reaches is

A 178 m **B** 224 m **C** 240.18 m **D** 296.78 m

16 A parabola passes through the points with coordinates (3.6, 4.8), (4.4, 8.04) and (6.3, 1.24). The equation of the parabola, giving values correct to three decimal places, is

A $y = 1.123x^2 + 26.655x + 53.541$ **B** $y = -0.821x^2 - 26.655x + 54.577$
C $y = -2.826x^2 + 26.654x - 54.536$ **D** $y = -2.821x^2 - 26.654x - 53.54$

17 The cost, c, in dollars per hour of running a certain steamboat is modelled by the function $c = 1.7v^2 - 13.6v + 166.4$, where v is the speed in kilometres per hour. The minimum cost which can be achieved is

A \$4.25 per hour **B** \$111.32 per hour
C \$128.22 per hour **D** \$139.20 per hour

18 The line $y = x + k$ touches the parabola $y = 9.61x^2 - 15.12x + 1.76$ at exactly one point. The value of k is

A -5.6 **B** -5.0 **C** -6.7 **D** 3.2

19 A group of n people agreed to buy a racehorse for a total price of \$119 000. They were each to put in an equal amount. Three of the people dropped out and an additional cost of \$1050 was reallocated to each of the remaining people. How much did each of the remaining people have to contribute?

A \$8500 **B** \$7933.33 **C** \$5950 **D** \$7000

5

Graphs of relations

Chapter contents

In Chapter 2, we looked at linear graphs, sketching them and determining their rules given sufficient information. All linear graphs can be considered as transformations of $y = x$. The features we concentrated on for linear graphs were the x-axis intercept, the y-axis intercept and the gradient.

In Chapter 4, we considered quadratics written in 'turning point form' and sketched their graphs by using transformations of the graph of the basic quadratic $y = x^2$. The features we concentrated on for graphs of quadratic polynomials were the x-axis intercepts, the y-axis intercept and the coordinates of the turning point (vertex).

In this chapter, we study some other common algebraic relations, and develop methods similar to those used in Chapter 4 to sketch the graphs of these relations. The relations in this chapter have different types of key features. For example, we introduce asymptotes for graphs of rectangular hyperbolas, and the coordinates of the centre and the length of the radius are key features in the study of circles.

5A Rectangular hyperbolas

Learning intentions

- To be able to sketch the graph of $y = x^{-1}$ and simple transformations of this graph.

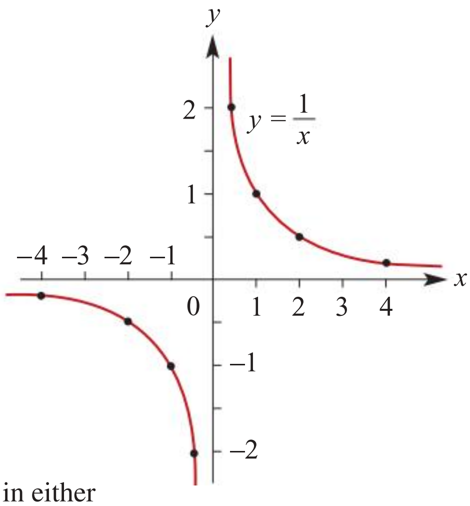
Consider the rule

$$y = \frac{1}{x} = x^{-1} \qquad \text{for } x \neq 0$$

We can construct a table of values for $y = \frac{1}{x}$ for values of x between -4 and 4 as follows:

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

We can plot these points and then connect the dots to produce a continuous curve. A graph of this type is an example of a **rectangular hyperbola**.



Note that y is undefined when $x = 0$, and that there is no x-value that will produce the value $y = 0$.

Asymptotes

There are two lines associated with this graph that help to describe its shape.

Horizontal asymptote

From the graph we see that, as x approaches infinity in either direction, the value of y approaches zero. The following notation will be used to state this:

- As $x \to \infty$, $y \to 0^+$. This is read: 'As x approaches infinity, y approaches 0 from the positive side.'
- As $x \to -\infty$, $y \to 0^-$. This is read: 'As x approaches negative infinity, y approaches 0 from the negative side.'

The graph approaches the x-axis (the line $y = 0$) but does not cross this line. The line $y = 0$ is a **horizontal asymptote**.

Vertical asymptote

As x approaches zero from either direction, the magnitude of y becomes very large. The following notation will be used to state this:

- As $x \to 0^+$, $y \to \infty$. This is read: 'As x approaches zero from the positive side, y approaches infinity.'
- As $x \to 0^-$, $y \to -\infty$. This is read: 'As x approaches zero from the negative side, y approaches negative infinity.'

The graph approaches the y-axis (the line $x = 0$) but does not cross this line. The line $x = 0$ is a **vertical asymptote**.

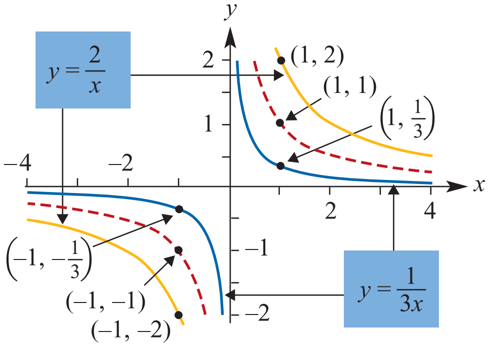
 ISBN 978-1-009-53719-3 Cambridge University Press

Dilations from an axis

The diagram on the right shows the graphs of

$$y = \frac{1}{x}, \quad y = \frac{2}{x} \quad \text{and} \quad y = \frac{1}{3x}$$

The asymptotes are the x-axis and the y-axis, and they have equations $y = 0$ and $x = 0$ respectively.

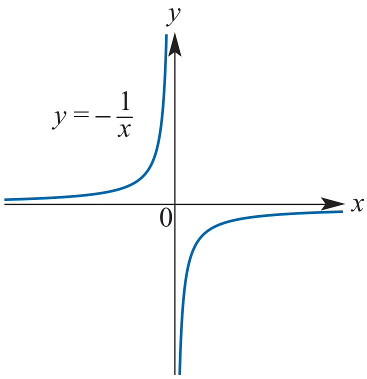


As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same 'shape' and asymptotes as the graph of $y = \frac{1}{x}$, but they have been 'stretched'.

The transformation that takes the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2}{x}$ is called the **dilation** of factor 2 from the x-axis. For example, the point $(1, 1)$ on the graph of $y = \frac{1}{x}$ is taken to the point $(1, 2)$ on the graph of $y = \frac{2}{x}$. Dilations will be considered formally in Chapter 6.

Reflection in the x-axis

When the graph of $y = \frac{1}{x}$ is reflected in the x-axis, the result is the graph of $y = -\frac{1}{x}$.



The asymptotes are still the two axes, that is, the lines $x = 0$ and $y = 0$.

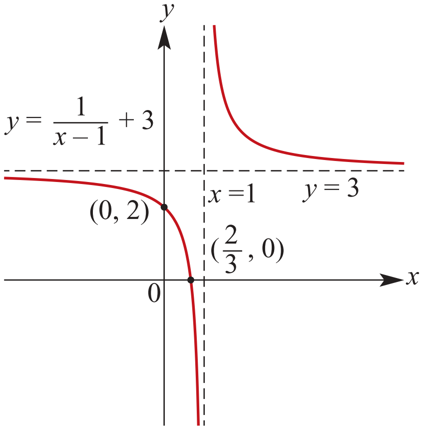
Similarly, $y = -\frac{2}{x}$ is the reflection of $y = \frac{2}{x}$ in the x-axis.

Reflecting in the y-axis gives the same result for these two graphs.

Translations

Now let us consider the graph of $y = \frac{1}{x - 1} + 3$.

The basic graph of $y = \frac{1}{x}$ has been translated 1 unit to the right and 3 units up.



Asymptotes The equation of the vertical asymptote is now $x = 1$, and the equation of the horizontal asymptote is now $y = 3$.

Intercepts with the axes The graph now has x-axis and y-axis intercepts. These can be calculated in the usual way to add further detail to the graph.

 ISBN 978-1-009-53719-3

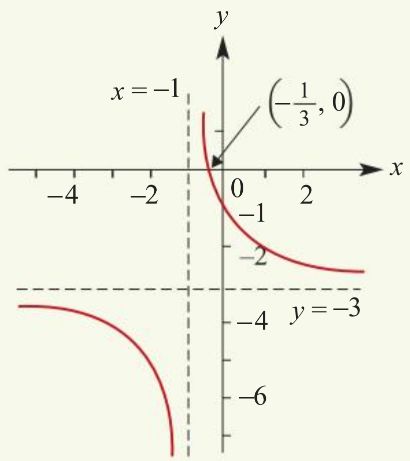
Sketching rectangular hyperbolas

Using dilations, reflections and translations, we are now able to sketch the graphs of all rectangular hyperbolas of the form $y = \dfrac{a}{x-h} + k$.

Example 1

Sketch the graph of $y = \dfrac{2}{x+1} - 3$.

Solution



Explanation

The graph of $y = \dfrac{2}{x}$ has been translated 1 unit to the left and 3 units down. The asymptotes have equations $x = -1$ and $y = -3$.

When $x = 0$, $y = \dfrac{2}{0+1} - 3 = -1$.

$\therefore$ the y-axis intercept is -1.

When $y = 0$,

$$0 = \frac{2}{x+1} - 3$$

$$3 = \frac{2}{x+1}$$

$$3(x+1) = 2$$

$$x = -\frac{1}{3}$$

$\therefore$ the x-axis intercept is $-\frac{1}{3}$.

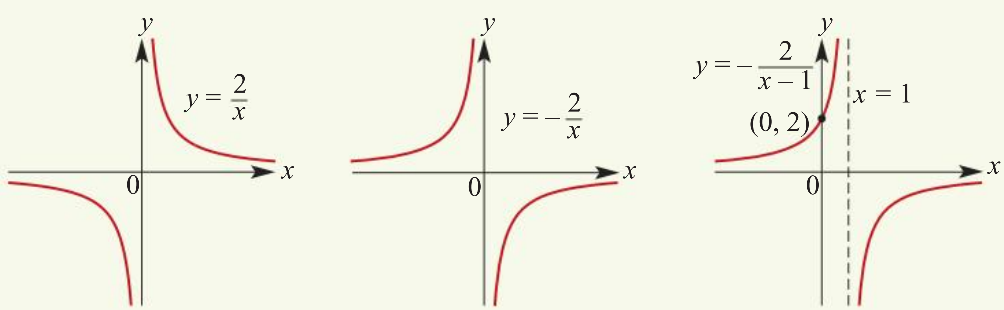
Example 2

Sketch the graph of $y = \dfrac{-2}{x-1}$.

Solution

The graph of $y = -\dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{2}{x}$ by reflection in the x-axis.

This graph is then translated 1 unit to the right to obtain the graph of $y = \dfrac{-2}{x-1}$.



ISBN 978-1-009-53719-3

Summary 5A

- For $a > 0$, a dilation of factor a from the x-axis transforms the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{a}{x}$.
- A reflection in the x-axis transforms the graph of $y = \frac{a}{x}$ to the graph of $y = -\frac{a}{x}$.
- For $h, k \geq 0$, a translation of h to the right and k upwards transforms the graph of $y = \frac{a}{x}$ to the graph of $y = \frac{a}{x-h} + k$.
- A rectangular hyperbola with rule of the form $y = \frac{a}{x-h} + k$ has:
  - vertical asymptote $x = h$
  - horizontal asymptote $y = k$.

Exercise 5A

Example 1, 2

SF

1 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{1}{2x}$ **d** $y = \frac{-3}{x}$

e $y = \frac{1}{x} + 2$ **f** $y = \frac{1}{x} - 3$ **g** $y = \frac{2}{x} - 4$ **h** $y = \frac{-1}{2x} + 5$

i $y = \frac{1}{x-1}$ **j** $y = \frac{-1}{x+2}$ **k** $y = \frac{1}{x+1} + 3$ **l** $y = \frac{-2}{x-3} - 4$

2 Write down the values of the asymptotes for each of the graphs in Question 1.

3 **a** We can write $y = \frac{1}{3x+6}$ as $y = \frac{1}{3(x+2)}$.
Sketch the graph of $y = \frac{1}{3x}$ and hence the graph of $y = \frac{1}{3x+6}$.

b We can write $y = \frac{3}{2x+4}$ as $y = \frac{3}{2(x+2)}$.
Sketch the graph of $y = \frac{3}{2x}$ and hence the graph of $y = \frac{3}{2x+4}$.

c We can write $y = \frac{-1}{2x+4}$ as $y = -\frac{1}{2(x+2)}$.
Sketch the graph of $y = -\frac{1}{2x}$ and hence the graph of $y = \frac{-1}{2x+4}$.

d We can write $y = \frac{1}{2x+1}$ as $y = \frac{1}{2(x+\frac{1}{2})}$.
Sketch the graph of $y = \frac{1}{2x}$ and hence the graph of $y = \frac{1}{2x+1}$.

4 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{3x+1}$ **b** $y = \frac{1}{3x+1} - 1$ **c** $y = \frac{-1}{3x+1} - 1$ **d** $y = \frac{-2}{3x+1}$

e $y = \frac{-2}{3x+1} - 4$ **f** $y = \frac{-2}{3x+1} + 3$ **g** $y = \frac{2}{3x+2} - 1$ **h** $y = \frac{3}{3x+4} - 1$

 ISBN 978-1-009-53719-3 Cambridge University Press

CF

5 Show that $\dfrac{x+3}{x-1} = \dfrac{4}{x-1} + 1$ and hence sketch the graph of $y = \dfrac{x+3}{x-1}$.

6 Show that $\dfrac{2x+3}{x+1} = \dfrac{1}{x+1} + 2$ and hence sketch the graph of $y = \dfrac{2x+3}{x+1}$.

7 Show that $\dfrac{3-2x}{x-2} = -\dfrac{1}{x-2} - 2$ and hence sketch the graph of $y = \dfrac{3-2x}{x-2}$.

5B The graph of $y^2 = x$

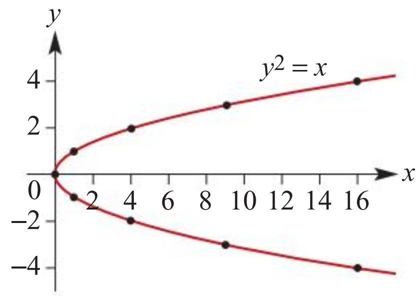
Learning intentions

▶ To be able to sketch the graph of $y^2 = x$ and simple transformations of this graph.

Now consider the rule $y^2 = x$. We can construct a table of values for y between -4 and 4:

y	-4	-3	-2	-1	0	1	2	3	4
x	16	9	4	1	0	1	4	9	16

We plot these points with y against x and then connect the dots to produce a continuous curve.



The graph of $y^2 = x$ is a parabola. It can be obtained from the graph of $y = x^2$ by a reflection in the line $y = x$.

The vertex of the parabola is at $(0, 0)$, and the axis of symmetry is the x-axis.

The transformations considered in the previous section can be applied to the graph of $y^2 = x$. All graphs of the form

$$(y-k)^2 = a^2(x-h)$$

will have the same basic parabola shape. The vertex of the parabola will be at the point (h, k), and the axis of symmetry will be the line $y = k$.

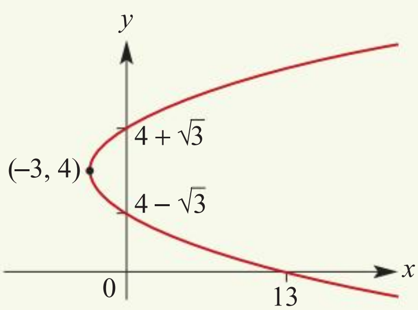
Example 3

Sketch the graph of:

a $(y-4)^2 = x+3$ **b** $y^2 + 2y = 2x + 3$ **c** $y^2 + 2y = -x - 1$

Solution

a



Explanation

The graph of $y^2 = x$ is translated 3 units to the left and 4 units up. The vertex is $(-3, 4)$.

When $x = 0$, $(y-4)^2 = 3$

$y - 4 = \pm\sqrt{3}$

$y = 4 \pm \sqrt{3}$

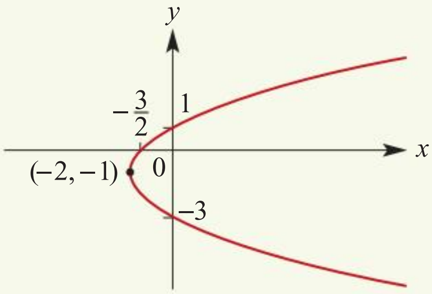
When $y = 0$, $16 = x + 3$

$x = 13$

 ISBN 978-1-009-53719-3 Cambridge University Press

b Complete the square:

$$y^2 + 2y = 2x + 3$$
$$y^2 + 2y + 1 = 2x + 4$$
$$(y + 1)^2 = 2(x + 2)$$



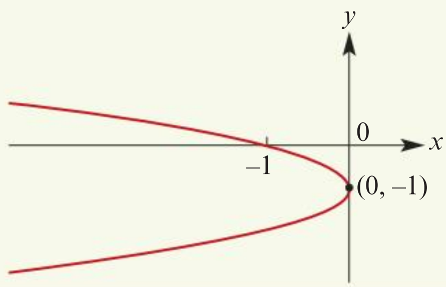
The graph of $(y + 1)^2 = 2(x + 2)$ is obtained from the graph of $y^2 = x$ by a dilation of factor $\sqrt{2}$ from the x-axis and then a translation 2 units to the left and 1 unit down.

The vertex has coordinates $(-2, -1)$.

When $x = 0$, $(y + 1)^2 = 4$
$$y + 1 = \pm 2$$
$$y = -1 \pm 2$$
$$y = 1 \text{ or } y = -3$$

When $y = 0$, $1 = 2x + 4$
$$x = -\frac{3}{2}$$

c Completing the square as in part **b** gives $(y + 1)^2 = -x$



The graph of $(y + 1)^2 = -x$ is obtained from the graph of $y^2 = -x$ by a translation of 1 unit down.

The vertex has coordinates $(0, -1)$.

When $y = 0$, $x = -1$.

Summary 5B

- All graphs of the form $(y - k)^2 = a(x - h) + c$ will have the same basic shape as the graph of $y^2 = x$. The graph will have a vertex at (h, k).

Exercise 5B

Example 3

1 Sketch the graph of each of the following relations, showing all important features:

a $(y - 2)^2 = x - 3$ **b** $(y + 2)^2 = x + 4$ **c** $y^2 = 2x$
d $y^2 = 2(x + 5)$ **e** $(y - 4)^2 = 2(x + 3)$ **f** $(y + 4)^2 = 2x$
g $(y + 3)^2 = 2x - 4$ **h** $y^2 = \frac{x}{2}$ **i** $y^2 + 4y = 2x + 4$
j $y^2 + 6y - 2x + 3 = 0$ **k** $y^2 + y - x = 0$ **l** $y^2 + 7y - 5x + 3 = 0$
m $y^2 = -x$ **n** $y^2 + 2y - x = 0$ **o** $(y - 2)^2 = -x$

SF

2 Sketch the graph of each of the following with the help of your calculator. (Instructions are given in the calculator appendices of the Interactive Textbook.)

a $y^2 = -4(x + 5)$ **b** $y^2 + 2y + 3x + 12 = 0$ **c** $(y - 5)^2 = 7 - 4x$

5C The graph of $y = \sqrt{x}$

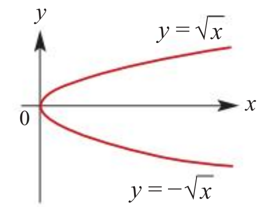
Learning intentions

- To be able to sketch the graph of $y = \sqrt{x}$ and simple transformations of this graph.

The rule

$$y = \sqrt{x} = x^{\frac{1}{2}} \qquad \text{for } x \geq 0$$

corresponds to the upper part of the graph shown opposite. It is one arm of the parabola $y^2 = x$.

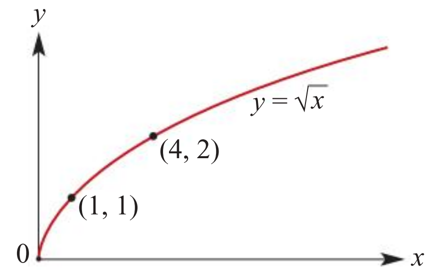


Coordinates of points on the graph of $y = \sqrt{x}$ include (0, 0), (1, 1), (4, 2) and (9, 3).

All graphs of the form

$$y = a\sqrt{x - h} + k$$

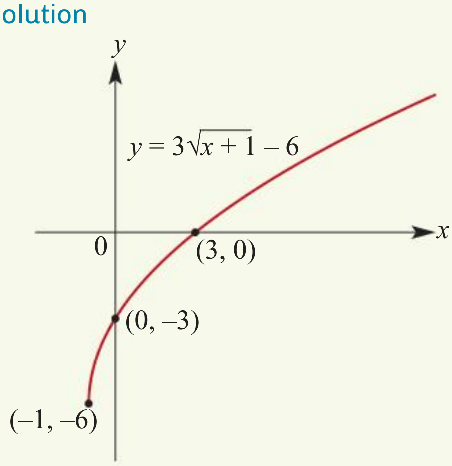
will have the same basic shape as the graph of $y = \sqrt{x}$.



Example 4

Sketch the graph of $y = 3\sqrt{x + 1} - 6$.

Solution



When $x = 0$, $y = -3$

When $y = 0$, $3\sqrt{x + 1} - 6 = 0$

$$3\sqrt{x + 1} = 6$$
$$\sqrt{x + 1} = 2$$
$$x + 1 = 4$$
$$x = 3$$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 3 and then translating 1 unit to the left and 6 units down.

The rule is defined for $x \geq -1$.

The set of values the rule can take (the range) is all numbers greater than or equal to -6, i.e. $y \geq -6$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 5

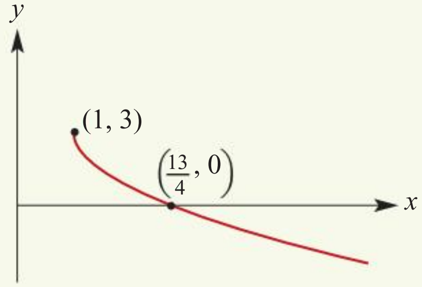
Sketch the graph of

a $y = -2\sqrt{x-1} + 3$.

b $y = -3\sqrt{2x+4} + 4$.

Solution

a



When $y = 0$: $\quad -2\sqrt{x-1} + 3 = 0$

$$2\sqrt{x-1} = 3$$

Square both sides: $\quad 4(x-1) = 9$

Therefore $x = \dfrac{9}{4} + 1 = \dfrac{13}{4}$

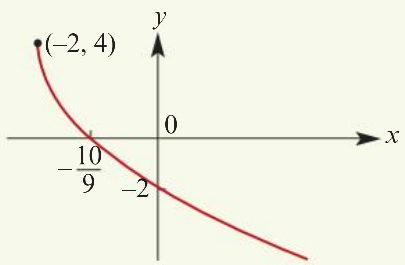
Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 2, reflecting this in the x-axis and then translating it 1 unit to the right and 3 units up.

The rule is defined for $x \geq 1$.

The set of values the rule can take (the range) is all numbers less than or equal to 3, i.e. $y \leq 3$.

b



When $y = 0$: $\quad -3\sqrt{2x+4} + 4 = 0$

$$\sqrt{2x+4} = \frac{4}{3}$$

$$\therefore x = -\frac{10}{9}$$

Note that,
$-3\sqrt{2x+4} + 4 = -3\sqrt{2}\sqrt{x+2} + 4$.

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor $3\sqrt{2}$, reflecting this in the x-axis and then translating it 2 units to the left and 4 units up.

The rule is defined for $x \geq -2$.

The set of values the rule can take (the range) is all numbers less than or equal to 4, i.e. $y \leq 4$.

The graph of $y = \sqrt{-x}$

The rule

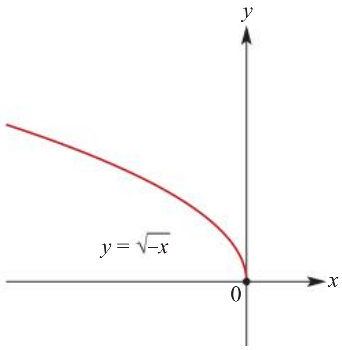
$$y = \sqrt{-x} \qquad \text{for } x \leq 0$$

yields a graph which is the reflection of the graph of $y = \sqrt{x}$ in the y-axis.

All graphs of the form

$$y = a\sqrt{-(x-h)} + k$$

will have the same basic shape as the graph of $y = \sqrt{-x}$.



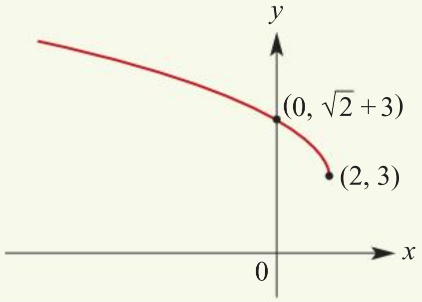
ISBN 978-1-009-53719-3

Example 6

Sketch the graph of $y = \sqrt{2 - x} + 3$.

Note: $\sqrt{2 - x} = \sqrt{-(x - 2)}$

Solution



Explanation

We can write the rule as

$$y = \sqrt{-(x - 2)} + 3$$

The rule is defined for $x \le 2$. The set of values the rule can take (the range) is all numbers greater than or equal to 3, i.e. $y \ge 3$. When $x = 0$, $y = \sqrt{2} + 3$.

Summary 5C

- All graphs of the form $y = a\sqrt{x - h} + k$ will have the same basic shape as the graph of $y = \sqrt{x}$. The graph will have endpoint (h, k).
- The graph of $y = \sqrt{-x}$ is the reflection in the y-axis of the graph of $y = \sqrt{x}$.

Exercise 5C

SF

Example 4

1 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = 2\sqrt{x} + 3$ **b** $y = \sqrt{x - 2} + 3$ **c** $y = \sqrt{x - 2} - 3$

d $y = \sqrt{x + 2} + 1$ **e** $y = -\sqrt{x + 2} + 3$ **f** $y = 2\sqrt{x + 2} - 3$

Example 5, 6

2 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = -\sqrt{x - 2} + 3$ **b** $y = \sqrt{-(x - 4)} - 2$

c $y = -2\sqrt{-(x + 4)} - 1$ **d** $y = 2\sqrt{3 - x}$

e $y = -2\sqrt{3 - x}$ **f** $y = 4\sqrt{3 - x} - 4$

3 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = \sqrt{3x}$ **b** $y = \sqrt{3(x - 1)}$ **c** $y = -\sqrt{2x}$

d $y = \sqrt{2(3 - x)}$ **e** $y = -2\sqrt{4(2 - x)}$ **f** $y = 4\sqrt{2(3 - x)} - 4$

 ISBN 978-1-009-53719-3

5D Circles

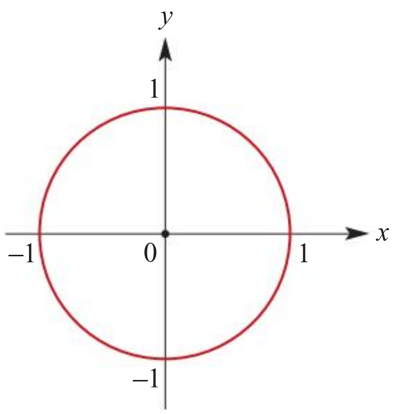
Learning intentions

- To be able to sketch the graph of $x^2 + y^2 = 1$ and simple transformations of this graph.

Consider a circle in the coordinate plane with centre the origin and radius r. If $P(x, y)$ is a point on the circle, its distance from the origin is r and so by Pythagoras' theorem $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ in the plane satisfies the equation $x^2 + y^2 = r^2$, its distance from the origin is r, so it lies on a circle with centre the origin and radius r.

To the right is the graph of the circle with equation $x^2 + y^2 = 1$.



All circles can be considered as being transformations of this basic graph.

As has been seen with other graphs, the basic graph may be translated horizontally and vertically.

The equation for a circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

where the **centre** of the circle is the point (h, k) and the **radius** is r.

If the radius and the coordinates of the centre of the circle are given, the equation of the circle can be determined.

Example 7

Write down the equation of the circle with

a centre $(-3, 5)$ and radius 2.

b centre $(4, -5)$ and radius $\sqrt{3}$.

Solution

a If the radius is 2 and the centre is the point $(-3, 5)$, then the equation will be

$$(x - (-3))^2 + (y - 5)^2 = 4$$
$$(x + 3)^2 + (y - 5)^2 = 4$$

b If the radius is $\sqrt{3}$ and the centre is the point $(4, -5)$, then the equation will be

$$(x - 4)^2 + (y - (-5))^2 = (\sqrt{3})^2$$
$$(x - 4)^2 + (y + 5)^2 = 3$$

 ISBN 978-1-009-53719-3

If the equation of the circle is given, the radius and the centre of the circle can be determined and the graph sketched.

Example 8

Determine the centre and radius of the circle $(x-1)^2 + (y-2)^2 = 4$.

Solution

The equation

$(x-1)^2 + (y-2)^2 = 4$

defines a circle of radius 2 with centre at $(1, 2)$.

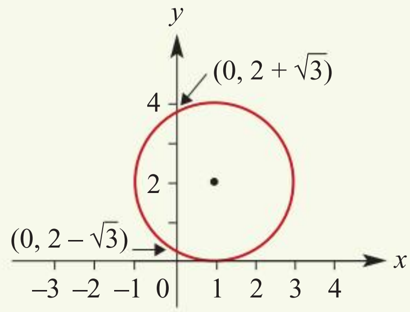
Explanation

We can sketch the circle with a little extra work.

When $x = 0$,

$$1 + (y-2)^2 = 4$$
$$(y-2)^2 = 3$$

Hence $y = 2 \pm \sqrt{3}$



Example 9

a Sketch the graph of the circle $(x+1)^2 + (y+4)^2 = 9$.

b Determine the coordinates of the points of the intersection of the circle $(x+1)^2 + (y+1)^2 = 9$ with each axis.

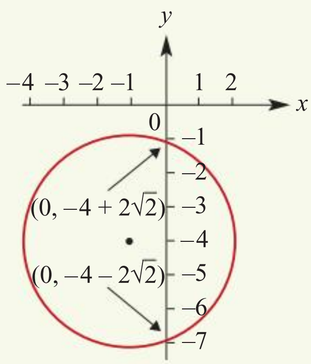
Solution

a When $x = 0$,

$$1 + (y+4)^2 = 9$$
$$(y+4)^2 = 8$$

Hence $y = -4 \pm \sqrt{8}$

$= -4 \pm 2\sqrt{2}$



b When $x = 0$, $1 + (y+1)^2 = 9$

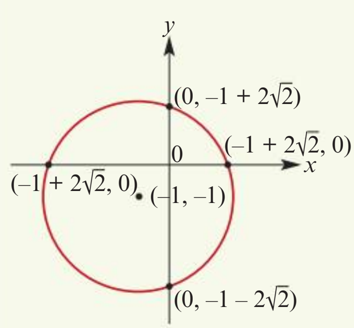
$y = -1 \pm 2\sqrt{2}$

When $y = 0$, $1 + (x+1)^2 = 9$

$x = -1 \pm 2\sqrt{2}$

Explanation

The circle has radius 3 and centre $(-1, -4)$. The y-axis intercepts can be found in the usual way.



ISBN 978-1-009-53719-3

The equation of a circle may not always be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Expanding the general equation of a circle gives

$$\begin{aligned}(x - h)^2 + (y - k)^2 &= r^2\\ x^2 - 2hx + h^2 + y^2 - 2ky + k^2 &= r^2\\ x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 &= 0\end{aligned}$$

Let $c = h^2 + k^2 - r^2$.

Then we obtain an alternative form for the equation of a circle:

The **general form** for the equation of a circle is

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

You will note that there is some similarity with the general form of a straight line, $ax + by + c = 0$.

Notice that in the general form of the circle equation, the coefficients of x^2 and y^2 are both 1 and there is no xy term.

In order to sketch a circle with equation expressed in this form, the equation can be converted to the 'centre–radius' form by completing the square for both x and y.

Example 10

Determine the radius and the coordinates of the centre of the circle with equation

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

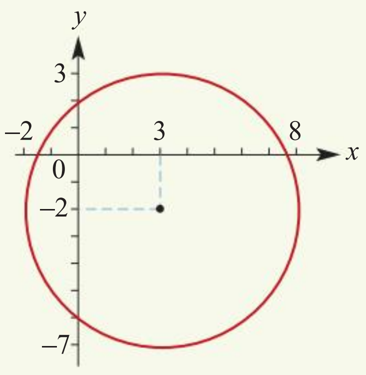
and hence sketch the graph.

Solution

By completing the square for both x and y we have

$$\begin{aligned}x^2 + y^2 - 6x + 4y - 12 &= 0\\ (x^2 - 6x + 9) - 9 + (y^2 + 4y + 4) - 4 - 12 &= 0\\ (x^2 - 6x + 9) + (y^2 + 4y + 4) &= 25\\ (x - 3)^2 + (y + 2)^2 &= 5^2\end{aligned}$$

The radius is 5 and the centre is at $(3, -2)$.



 ISBN 978-1-009-53719-3 Cambridge University Press

Semicircles

Transposing the equation of the circle $x^2 + y^2 = r^2$ to make y the subject, we have

$$y^2 = r^2 - x^2$$

$$y = \pm\sqrt{r^2 - x^2}$$

We can now consider two separate rules

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

which correspond to the top half and bottom half of the circle respectively.

Similarly, solving for x will give you the semicircles to the left and right of the y-axis:

$$x = \pm\sqrt{r^2 - y^2}$$

The separate rules are

$$x = +\sqrt{r^2 - y^2} \quad \text{and} \quad x = -\sqrt{r^2 - y^2}$$

Example 11

Sketch the graphs of:

a $y = +\sqrt{4 - x^2}$

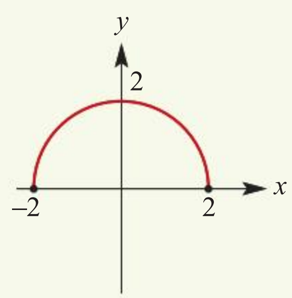
b $y = -\sqrt{4 - x^2}$

c $x = -\sqrt{4 - y^2}$

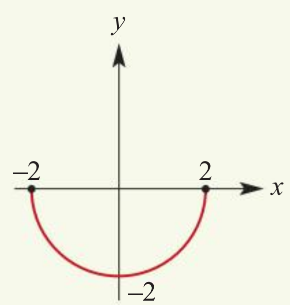
d $x = +\sqrt{4 - y^2}$

Solution

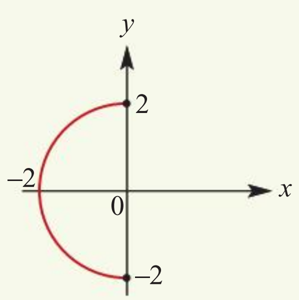
a $y = +\sqrt{4 - x^2}$



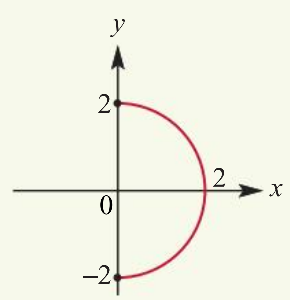
b $y = -\sqrt{4 - x^2}$



c $x = -\sqrt{4 - y^2}$



d $x = +\sqrt{4 - y^2}$



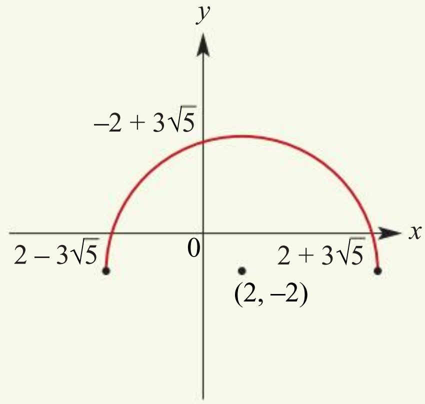
ISBN 978-1-009-53719-3

Cambridge University Press

Example 12

Sketch the graph of $y = -2 + \sqrt{49 - (x - 2)^2}$.

Solution



When $x = 0$,

$$y = -2 + \sqrt{45}$$
$$= -2 + 3\sqrt{5}$$

When $y = 0$,

$$-2 + \sqrt{49 - (x - 2)^2} = 0$$
$$\sqrt{49 - (x - 2)^2} = 2$$
$$49 - (x - 2)^2 = 4$$
$$(x - 2)^2 = 45$$
$$x = 2 \pm 3\sqrt{5}$$

Explanation

It is a semicircle of the circle

$$(x - 2)^2 + (y + 2)^2 = 49$$

The centre is at the point $(2, -2)$ and the radius is 7.

It is the semicircle $y = \sqrt{49 - x^2}$ translated 2 units the right and 2 units down.

In the usual way, we determine the x-axis intercepts and the y-axis intercept.

Summary 5D

- The equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

- The **general form** for the equation of a circle is

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

- The two separate rules for semicircles with their base on the x-axis are

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

  They correspond to the top half and bottom half of the circle respectively.
- The two separate rules for semicircles with their base on the y-axis are

$$x = +\sqrt{r^2 - y^2} \quad \text{and} \quad x = -\sqrt{r^2 - y^2}$$

  They correspond to the right half and left half of the circle respectively.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 5D

SF

1 Write down the equation of each of the following circles, with centre at $C(h, k)$ and radius r:

Example 7

a $C(0, 0),\ r = 3$ **b** $C(0, 0),\ r = 4$ **c** $C(1, 3),\ r = 5$

d $C(2, -4),\ r = 3$ **e** $C(-3, 4),\ r = \frac{5}{2}$ **f** $C(-5, -6),\ r = 4.6$

Example 8

2 Determine the centre, C, and the radius, r, of the following circles:

a $(x - 1)^2 + (y - 3)^2 = 4$ **b** $(x - 2)^2 + (y + 4)^2 = 5$

c $(x + 3)^2 + (y - 2)^2 = 9$ **d** $(x + 5)^2 + (y - 4)^2 = 8$

Example 9

3 Sketch the graphs of each of the following:

a $x^2 + y^2 = 64$ **b** $x^2 + (y - 4)^2 = 9$

c $(x + 2)^2 + y^2 = 25$ **d** $(x + 1)^2 + (y - 4)^2 - 169 = 0$

e $(2x - 3)^2 + (2y - 5)^2 = 36$ **f** $(x + 5)^2 + (y - 5)^2 = 36$

Example 10

4 Determine the centre, C, and the radius, r, of the following circles:

a $x^2 + y^2 - 6y - 16 = 0$ **b** $x^2 + y^2 - 8x + 12y + 10 = 0$

c $x^2 + y^2 - 6x + 4y + 9 = 0$ **d** $x^2 + y^2 + 4x - 6y - 12 = 0$

e $x^2 + y^2 - 8x + 4y + 1 = 0$ **f** $x^2 + y^2 - x + 4y + 2 = 0$

Example 10

5 Sketch the graphs of each of the following:

a $x^2 + y^2 - 6y - 16 = 0$

b $x^2 + y^2 + 4x - 6y - 3 = 0$

c $x^2 + y^2 - 8x + 22y + 27 = 0$

Example 11

6 Sketch the graphs of each of the following:

a $y = +\sqrt{9 - x^2}$ **b** $x = +\sqrt{9 - y^2}$ **c** $y = -\sqrt{16 - x^2}$

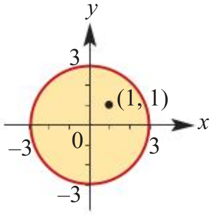
d $y = -\sqrt{25 - x^2}$ **e** $x = -\sqrt{49 - y^2}$ **f** $x = \sqrt{\frac{25}{4} - y^2}$

Example 12

7 Sketch the graphs of each of the following:

a $y = \sqrt{36 - (x - 2)^2}$ **b** $y - 2 = \sqrt{4 - (x + 2)^2}$

8 The graph of $x^2 + y^2 \le 9$ is as shown. Note that $(1, 1)$ satisfies $1^2 + 1^2 \le 9$. The coordinates of every point in the shaded region satisfy the inequality.



Sketch the graphs of each of the following. Use a dotted line when indicating that the boundary is not included.

a $x^2 + y^2 \le 4$ **b** $x^2 + y^2 > 1$ **c** $x^2 + y^2 \le 5$

d $x^2 + y^2 > 9$ **e** $x^2 + y^2 \ge 6$ **f** $x^2 + y^2 < 8$

 ISBN 978-1-009-53719-3

5E Determining rules

Learning intentions

- To be able to determine the rules of the graphs discussed in this chapter.

In Chapters 2 and 4 we looked at some sufficient conditions for determining the rules for straight lines and parabolas. For straight lines these included:

- the coordinates of two points
- the gradient and a point.

For parabolas these included:

- the coordinates of three points
- the coordinates of the vertex and the coordinates of one other point.

In this section we are looking at some sufficient conditions for determining the rules for the graphs of this chapter.

Example 13

a The rectangular hyperbola $y = \frac{a}{x} + 8$ passes through the point $(-2, 6)$. Determine the value of a.

b The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(2, 7)$ and $(-1, 1)$. Determine the values of a and k.

Solution

a When $x = -2$, $y = 6$. Hence

$$\begin{aligned} 6 &= \frac{a}{-2} + 8 \\ -2 &= \frac{a}{-2} \\ a &= 4 \end{aligned}$$

The equation is $y = \frac{4}{x} + 8$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x-h} + k$$

In this case $h = 0$ and $k = 8$.

Solution

b When $x = 2$, $y = 7$. When $x = -1$, $y = 1$. So we have the equations

$$\begin{aligned} 7 &= \frac{a}{2} + k \qquad (1) \\ 1 &= -a + k \qquad (2) \end{aligned}$$

Subtract (2) from (1):

$$6 = \frac{a}{2} + a \qquad (3)$$

Multiply both sides of equation (3) by 2:

$$\begin{aligned} 12 &= a + 2a \\ a &= 4 \end{aligned}$$

From equation (2): $k = 5$.

The equation is $y = \frac{4}{x} + 5$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x-h} + k$$

In this case $h = 0$ and the values of a and k are unknown.

Simultaneous equations need to be formed and then solved.

ISBN 978-1-009-53719-3

Example 14

A graph which has rule $y = a\sqrt{x-h}$ passes through the points $(4, 2)$ and $(7, 4)$. Determine the values of a and h.

Solution

When $x = 4$, $y = 2$. When $x = 7$, $y = 4$.

We have the equations

$$2 = a\sqrt{4-h} \qquad (1)$$

$$4 = a\sqrt{7-h} \qquad (2)$$

Divide (2) by (1):

$$2 = \frac{\sqrt{7-h}}{\sqrt{4-h}} \qquad (3)$$

Multiply both sides of equation (3) by $\sqrt{4-h}$:

$$2\sqrt{4-h} = \sqrt{7-h}$$

Square both sides of the equation:

$$\begin{aligned} 4(4-h) &= 7-h \\ 16-4h &= 7-h \\ 3h &= 9 \\ h &= 3 \end{aligned}$$

Substitute in (1) to determine $a = 2$.

The required equation is $y = 2\sqrt{x-3}$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = a\sqrt{x-h} + k$$

In this case $k = 0$ and the values of a and h are unknown.

Simultaneous equations need to be formed and then solved. Note that $h \neq 4$ from equation (1).

Example 15

Determine the equation of the circle whose centre is at the point $(1, -1)$ and which passes through the point $(4, 3)$.

Solution

Let r be the length of the radius. Then

$$\begin{aligned} r &= \sqrt{(4-1)^2 + (3-(-1))^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

Hence the equation of the circle is

$$(x-1)^2 + (y+1)^2 = 25$$

Explanation

We use the centre–radius form for the equation of a circle:

$$(x-h)^2 + (y-k)^2 = r^2$$

The centre is $(1, -1)$. We need to determine the radius.

 ISBN 978-1-009-53719-3 Cambridge University Press

Skill-sheet

Exercise 5E

SF

Example 13a

1 The rectangular hyperbola $y = \frac{a}{x} + 3$ passes through the point $(1, 8)$. Determine the value of a.

2 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x-h} + k$$

has vertical asymptote $x = 3$, horizontal asymptote $y = 4$ and passes through the point $(0, 6)$. Determine the values of a, h and k.

Example 13b

3 The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(1, 8)$ and $(-1, 7)$. Determine the values of a and k.

4 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x-h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = -4$ and passes through the point $(0, 4)$. Determine the values of a, h and k.

5 A graph which has rule $y = a\sqrt{x}$ passes through the point $(2, 8)$. Determine the value of a.

Example 14

6 A graph which has rule $y = a\sqrt{x-h}$ passes through the points $(1, 2)$ and $(10, 4)$. Determine the values of a and h.

Example 15

7 Determine the equation of the circle whose centre is at the point $(2, 1)$ and which passes through the point $(4, -3)$.

8 Determine the equation of the circle whose centre is at the point $(-2, 3)$ and which passes through the point $(-3, 3)$.

9 Determine the equation of the circle whose centre is at the point $(-2, 3)$ and which passes through the point $(2, 3)$.

10 Determine the equation of the circle with centre $(2, -3)$ which touches the x-axis .

CF

11 Determine the equation of the circle whose centre lies on the line $y = 4$ and which passes through the points $(2, 0)$ and $(6, 0)$.

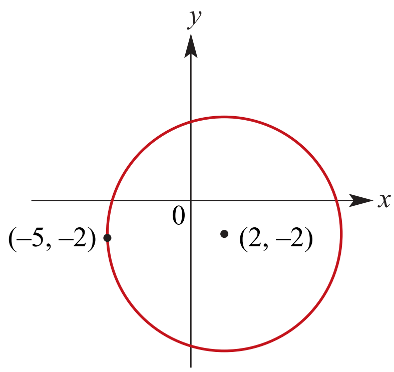
12 Determine the equations of the circles which touch the x-axis, have radius 5 and pass through the point $(0, 8)$.

13 Determine the equation of a circle which passes through the points $A(0, 2)$, $B(2, 0)$ and $C(-4, 0)$.

 ISBN 978-1-009-53719-3

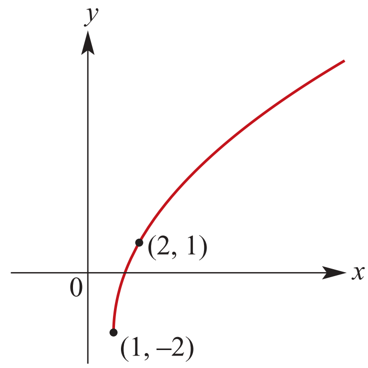
14 Determine the rule for each of the following graphs. The general form of the rule is given for each graph.

a $(x-h)^2 + (y-k)^2 = r^2$

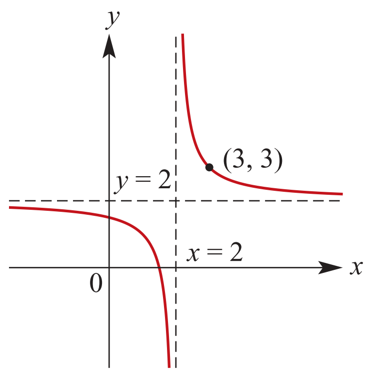


Centre at $(2, -2)$

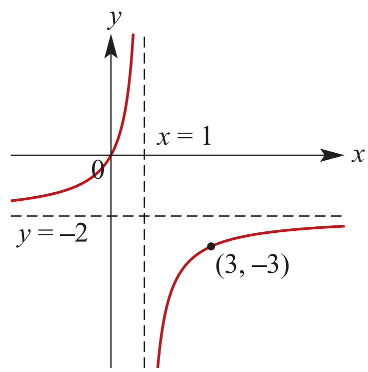
b $y = a\sqrt{x-h} + k$



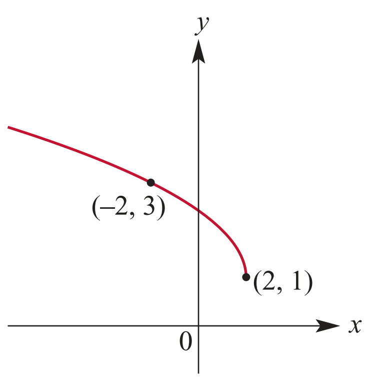
c $y = \dfrac{a}{x-h} + k$



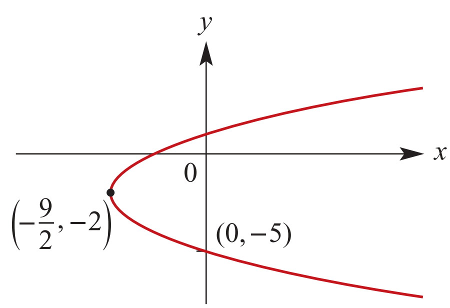
d $y = \dfrac{a}{x-h} + k$



e $y = a\sqrt{h-x} + k$

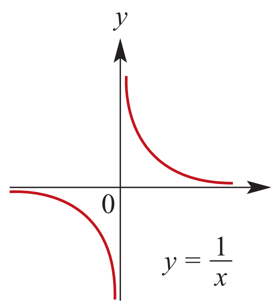


f $(y-k)^2 = b(x-h)$

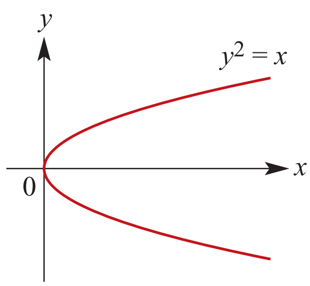


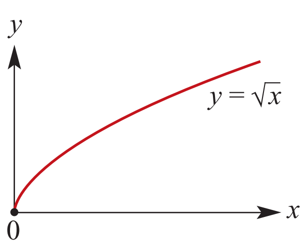
Chapter summary

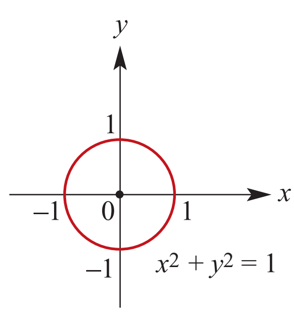
- The standard graphs:



Rectangular hyperbola

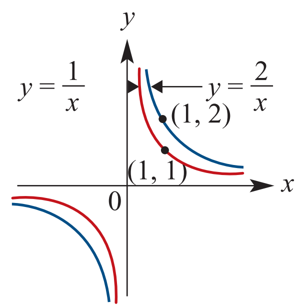


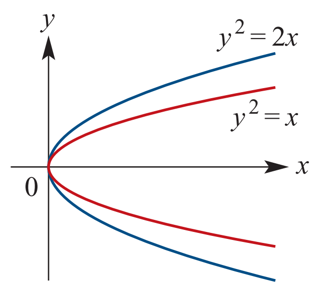


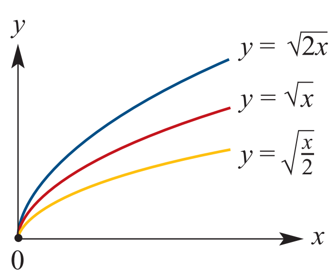


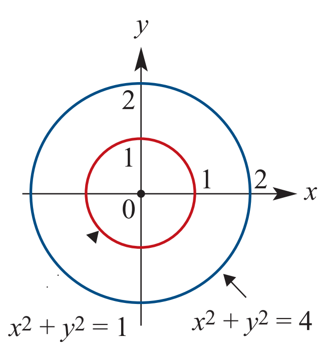
Circle

- Dilations of these graphs:

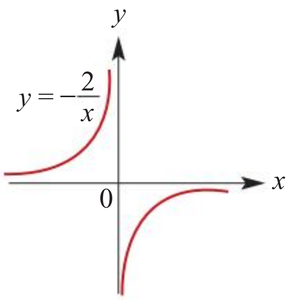


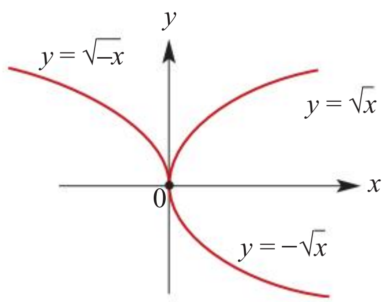




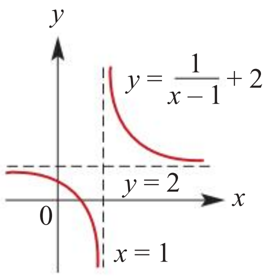


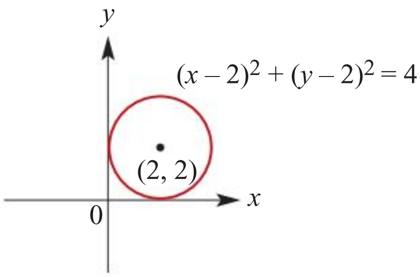
- Reflections in the axes:





- Translations of graphs:





- Equation for a circle with centre at (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

Alternative form:

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

where $c = h^2 + k^2 - r^2$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

5A **1** I can sketch and analyse graphs with rule of the form $y = \dfrac{a}{x - h} + k$. ☐

See Example 1, Example 2 and Question 1

5B **2** I can sketch and analyse graphs with rule of the form $(y - k)^2 = a(x - h) + c$. ☐

See Example 3 and Question 1

5C **3** I can sketch and analyse graphs with rule of the form $y = a\sqrt{x - h} + k$. ☐

See Example 4, Example 5 and Questions 1 and 2.

5C **4** I can sketch and analyse graphs with rule of the form $y = a\sqrt{h - x} + k$. ☐

See Example 6 and Question 2.

Review

5D **5** I can sketch and analyse graphs of circles given the coordinates of the centre and the radius. ☐

See Example 7, Example 8, Example 9 and Questions 1, 2 and 3

5D **6** I can determine the centre and radius of circle where the equation of the circle is given in the form $x^2 + y^2 - 2hx - 2ky + c = 0$. ☐

See Example 10, and Questions 4 and 5

5D **7** I can sketch and analyse graphs of semicircles. ☐

See Example 11, Example 12 and Questions 6 and 7

5E **8** I can determine rules for graphs considered in this chapter given sufficient information. ☐

See Example 13, Example 14, Example 15 and Questions 13, 14 and 15

Short-response questions

Technology-free short-response questions

1 Sketch the graphs of each of the following: SF

a $y = \dfrac{-3}{x}$ **b** $y = \dfrac{1}{x-1}$ **c** $y = \dfrac{2}{x+1} + 1$ **d** $y = \dfrac{-1}{x-1}$

e $y = \dfrac{4}{2-x} + 3$ **f** $y^2 = x - 2$ **g** $(y-1)^2 = x + 1$ **h** $y = 2\sqrt{x} + 2$

i $y = 2\sqrt{x-3} + 2$ **j** $y = -2\sqrt{x+2} + 2$ **k** $y^2 = 4(x-2)$

l $(y-1)^2 = 16(x-2)$ **m** $(y-4)^2 = -4(x-1)$

2 By completing the square, write each of the following equations in the form $(x-a)^2 + (y-b)^2 = r^2$:

a $x^2 + y^2 - 6x + 4y - 12 = 0$ **b** $x^2 + y^2 - 3x + 5y - 4 = 0$

c $2x^2 + 2y^2 - x + y - 4 = 0$ **d** $x^2 + y^2 + 4x - 6y = 0$

e $x^2 + y^2 = 6(x + y)$ **f** $x^2 + y^2 = 4x - 6y$

3 For the circle $x^2 + y^2 - 4x + 6y = 14$, determine the equation of the diameter which passes through the origin. CF

4 For the circle $x^2 + y^2 - 3x + 2y = 26$, determine the equations of the diameters which cut the x-axis at an angle of 45°.

5 Determine the equation of the circle with centre C and radius r for each of the following and sketch the graph: SF

a $C(3, 4),\ r = 5$ **b** $C(-1, 0),\ r = 1$

c $C(4, 4),\ r = 2$ **d** $C(\frac{1}{2}, -\frac{1}{3}),\ r = \frac{1}{6}$

 ISBN 978-1-009-53719-3

6 The equation of a circle is $x^2 + y^2 + 4x - 6y = 23$. Determine the centre and radius. SF

7 Determine the length cut off on the x-axis and y-axis by the circle $x^2 + y^2 - 2x - 4y = 20$. CF

8 Sketch the graphs of the following semicircles: SF

a $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{16 - (x + 1)^2}$

c $y - 2 = -\sqrt{1 - x^2}$ **d** $y + 3 = \sqrt{4 - (x + 2)^2}$

9 A rectangular hyperbola with rule of the form CF

$$y = \frac{a}{x - h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = 5$ and passes through the point $(0, 8)$. Determine the values of a, h and k.

10 For each of the following, write the equation in the form $(y - k)^2 = a(x - h)$ by completing the square, and state the coordinates of the vertex, the x-axis intercept and the y-axis intercepts of the graph of the equation:

a $y^2 + 4y = x + 2$ **b** $y^2 + 6y + 2x + 4 = 0$ **c** $2y^2 + 8y - 5x + 6 = 0$

11 Show that the line $x + 4y = 4$ touches the hyperbola $y = \dfrac{1}{x}$ at only one point.

12 Consider the hyperbola $y = \dfrac{12}{x}$. The points $P(2, 6)$ and $Q(-4, -3)$ lie on the hyperbola. The line PQ cuts the y-axis at B and the x-axis at A.

a Determine the coordinates of points A and B.

b Prove that $PB = AQ$.

Technology-active short-response questions

13 The variables x and y are connected by a rule of the form $y = \dfrac{k}{x}$. A table of values, with two missing values, constructed from this rule is shown here. SF

x	2	4	6	a
y	b	1	0.667	1.414

Determine the value of k and the values of a and b. Give the value of a correct to two decimal places.

14 The maximum speed of yachts is given by a formula of the form $y = k\sqrt{x}$, where y knots is the maximum speed and x m the length. If a yacht 20 m long can maintain a maximum speed of 15 knots, determine the maximum speed of a yacht 15 m long. Give your answer to the nearest knot.

15 **a** The air in a tube occupies 43.5 cm^3 and the pressure is 2.8 kg/cm^2. The volume of air is given by a formula of the form $V = \dfrac{k}{P}$, where k is a positive constant and P is the pressure. Determine the formula connecting V and P.

b Calculate the pressure when the volume is decreased to 12.7 cm^3.

Review

Review

SF

16 A vehicle moves past a post at a speed of 5 m/s. The vehicle is accelerating and the speed v m/s of the vehicle s m past the post is given by the equation $v = \sqrt{25 + s}$.

a Find the speed, correct to two decimal places, of the vehicle when it has gone 10 m.

b How many metres does the vehicle go before it reaches a speed of 12 m/s?

17 The weight (w kg) which a beam that is supported at each end will carry without breaking is given by a formula of the form $w = \dfrac{k}{d}$ where k is a positive constant and d is the distance between supports. A beam which measures 6 m between supports will just carry a load of 500 kg.

a Determine the formula connecting w and d.

b What weight could the beam carry if the distance between the supports were 5 m?

c What weight could the beam carry if the distance between the supports were 9 m?

CF

18 A graph which has rule $y = -2\sqrt{x - h} + k$ passes through the points $(12, 0)$ and $(39, -6)$. Determine the values of h and k.

19 A graph which has rule $y = \dfrac{a}{x - 6} + k$ passes through the points $(0, 34)$ and $(12, 26)$. Determine the values of a and k.

20 A circle passes through the points $(-5, 1)$, $(6, 1)$ and $(3, 5)$. Determine the equation of the circle.

CU

21 Consider the parabola with equation $(y - 2)^2 = 3x - 2$.

a Determine the coordinates of the points of intersection of this parabola with the line $y = x$.

b Determine, in terms of a, the coordinates of the points of intersection of this parabola with the line $y = x + a$.

c **i** Determine the value of a for which the line $y = x + a$ just touches the parabola.

ii Determine the values of a for which the line $y = x + a$ does not intersect theparabola.

22 **a** Show that the line $y = x + 1$ touches the parabola $y^2 = 4x$ at the point $(1, 2)$.

b Determine the coordinates of the points of intersection, P and Q, of the line $y = x + 1$ and the parabola $y^2 = -4x$.

c Determine the distance between points P and Q.

d Determine the midpoint of the line segment PQ.

23 The line with equation $y = mx$ is tangent to the circle with centre $(10, 0)$ and radius 5 at the point $P(x, y)$.

a Determine the equation of the circle.

b Show that the x-coordinate of the point P satisfies the equation $(1 + m^2)x^2 - 20x + 75 = 0$.

c Use the discriminant for this equation to determine the exact value of m.

d Determine the coordinates of P. (There are two such points.)

e Determine the distance of P from the origin.

ISBN 978-1-009-53719-3

Review CU

24 A circle has its centre at the origin and radius 4.

a Determine the equation of this circle.

b Two lines which pass through the point $(8, 0)$ are tangents to this circle.

i Show that the equations of these tangents are of the form $y = mx - 8m$.

ii Use techniques similar to those used in Question 23 to determine the value of m and, hence, the equations of the tangents.

25 A circle has centre at the origin and radius 5. The point $P(3, 4)$ lies on the circle.

a Determine the gradient of the line segment OP, where O is the origin.

b Determine the gradient of the tangent to the circle at P.

c Determine the equation of the tangent at P.

d If the tangent crosses the x-axis at A and the y-axis at B, determine the length of line segment AB.

26 Let $P(x_1, y_1)$ be a point on the circle with equation $x^2 + y^2 = a^2$.

a **i** Give the gradient of the line segment OP, where O is the origin.

ii Give the gradient of the tangent to the circle at P.

b Show that the equation of the tangent at $P(x_1, y_1)$ is $x_1x + y_1y = a^2$.

c If $x_1 = y_1$ and $a = 4$, determine the equations of the possible tangents.

27 An equilateral triangle ABC circumscribes the circle with equation $x^2 + y^2 = a^2$. The side BC of the triangle has equation $x = -a$.

a Determine the equations of AB and AC.

b Determine the equation of the circle circumscribing triangle ABC.

28 Consider the curve with equation $y = \sqrt{x - b} + c$.

a Show that if the curve meets the line with equation $y = x$ at the point (a, a), then a satisfies the equation $a^2 - (2c + 1)a + c^2 + b = 0$.

b **i** If the line with equation $y = x$ is a tangent to the curve, show that $c = \dfrac{4b - 1}{4}$.

ii Sketch the graph of $y = \sqrt{x} - \frac{1}{4}$ and determine the coordinates of the point on the graph at which the line with equation $y = x$ is a tangent.

c Determine the values of k for which the line with equation $y = x + k$:

i meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ twice

ii meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ once

iii does not meet the curve with equation $y = \sqrt{x} - \frac{1}{4}$.

29 For the curve with equation $y = \sqrt{x} - 1$ and the straight line with equation $y = kx$, determine the values of k such that:

a the line meets the curve twice

b the line meets the curve once.

 ISBN 978-1-009-53719-3

Review

CU

30 The line $px + qy = 1$ touches the parabola $y^2 = 4a(a - x)$, where p, q and a are non-zero real numbers.

a Show that $a = \dfrac{p}{p^2 + q^2}$.

b For each of the following values of p and q, determine the value of a and the coordinates of the point where the line touches the parabola:

i $p = 1$ and $q = 1$ **ii** $p = 3$ and $q = 4$

Now assume that $a = 1$ and consider the parabola $y^2 = -4(x - 1)$.

c By using simultaneous equations in p and q, determine the equation of the line $px + qy = 1$ that touches this parabola at:

i the point $(0, 2)$ **ii** the point $(-3, -4)$

31 The line $px + qy = 1$ touches the hyperbola $y = \dfrac{a}{x}$, where p, q and a are non-zero real numbers.

a Show that $a = \dfrac{1}{4pq}$.

b For each of the following values of p and q, determine the value of a and the coordinates of the point where the line touches the hyperbola:

i $p = 1$ and $q = 1$ **ii** $p = 2$ and $q = 2$

Now assume that $a = 1$ and consider the hyperbola $y = \dfrac{1}{x}$.

c By using simultaneous equations in p and q, determine the equation of the line $px + qy = 1$ that touches this hyperbola at:

i the point $(2, \frac{1}{2})$ **ii** the point $(-2, -\frac{1}{2})$

Multiple-choice questions

Technology-free multiple-choice questions

1 The circle with equation $(x - a)^2 + (y - b)^2 = 36$ has its centre on the x-axis and passes through the point with coordinates $(6, 6)$. The values of a and b are

A $a = 0$ and $b = 0$ **B** $a = 2$ and $b = 0$

C $a = -6$ and $b = 0$ **D** $a = 6$ and $b = 0$

2 The equations of the asymptotes of the graph of $y = 5 - \dfrac{1}{3x - 5}$ are

A $y = 5,\ x = \dfrac{5}{3}$ **B** $x = 5,\ y = \dfrac{5}{3}$

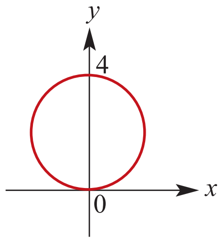
C $y = 5,\ x = \dfrac{3}{5}$ **D** $x = 5,\ y = -\dfrac{5}{3}$

3 For the rule $y = \dfrac{5}{x} + 3$, when $x = \dfrac{a}{2}$, $y =$

A $\dfrac{5a}{2} + 3$ **B** $\dfrac{20a + 3}{a}$ **C** $\dfrac{5}{a} + 3$ **D** $\dfrac{10}{a} + 3$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

4 If the y-axis is an axis of symmetry and the circle passes through the origin and $(0, 4)$, the equation of the circle shown is



A $x^2 + (y - 2)^2 = 4$ **B** $(x + 2)^2 + y^2 = 4$

C $x^2 + (y + 2)^2 = 4$ **D** $x^2 + y^2 - 2 = 4$

5 The equations of the asymptotes of the graph of $y = \frac{2}{x + 2} - 4$ are

A $x = -2,\ y = -4$ **B** $x = -2,\ y = -2$

C $x = 2,\ y = 4$ **D** $x = -4,\ y = -2$

6 For the circle with equation $(x - 5)^2 + (y + 2)^2 = 9$, the coordinates of the centre and the radius are

A $(-5, 2)$ and 9 **B** $(5, -2)$ and 9

C $(5, -2)$ and 3 **D** $(-2, 5)$ and 3

7 For the rule $y = -2\sqrt{x} + 3$, where $x \geq 0$, the range of possible y-values is

A $y > -3$ **B** $y \geq -3$ **C** $y \leq 3$ **D** $y > 3$

8 The equation of the circle which has a diameter with endpoints at $(-2, 8)$ and $(6, 8)$ is

A $(x + 2)^2 + (y + 8)^2 = 64$ **B** $(x - 2)^2 + (y - 8)^2 = 16$

C $(x - 2)^2 + (y + 8)^2 = 4$ **D** $(x - 2)^2 + (y + 8)^2 = 16$

9 Which of the following is the equation for the graph of a circle?

A $y^2 = 16 + x^2$ **B** $x + y^2 = 16$

C $x^2 - y^2 = 16$ **D** $y^2 = 16 - x^2$

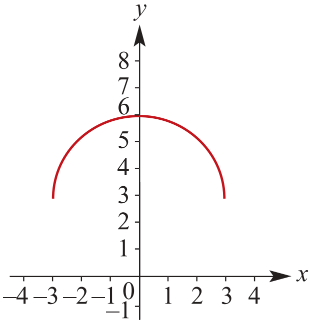
10 The equation of the semicircle shown is

A $y = \sqrt{9 - x^2} + 3$

B $y = \sqrt{9 + x^2} - 3$

C $y = -\sqrt{9 - x^2} - 3$

D $y = -\sqrt{9 - x^2} + 3$



11 The graph of $(y + a)^2 = -3(x - 2)$, where $a > 0$, has

A vertex at $(2, -a)$ and y-axis intercepts at $(0, -a - \sqrt{6})$ and $(0, -a + \sqrt{6})$

B vertex at $(-2, a)$ and y-axis intercepts at $(0, -a - \sqrt{3})$ and $(0, -a + \sqrt{3})$

C vertex at $(2, -a)$ and y-axis intercepts at $(0, a - \sqrt{6})$ and $(0, a + \sqrt{6})$

D vertex at $(-2, a)$ and y-axis intercepts at $(0, -a)$ and $(0, a)$

Review

12 The graph of $y - b = -\sqrt{a - x}$, where $a > 0$ and $b > 0$, has

A endpoint (b, a) and y-values $y \le a$ **B** endpoint (a, b) and y-values $y \le b$

C endpoint (a, b) and y-values $y \ge a + b$ **D** endpoint $(a, -b)$ and y-values $y \le -b$

13 The rule $(y - 2)^2 = -3x + 1$ is defined for

A $x \le \frac{1}{3}$ **B** $x \ge 3$ **C** $x \le 2$ **D** $x \ge \frac{1}{3}$

Technology-active multiple-choice questions

14 The graph of $y = -3\sqrt{3x + 5} + 7$ passes through the points $(a, 0)$ and $(0, b)$. The values of a and b correct to two decimal places are

A $a = 0.15, b = 0.29$ **B** $a = 0.29, b = 0.15$

C $a = 0.13, b = 0.30$ **D** $a = 0.31, b = 0.15$

15 The straight line $y = x + 1$ crosses the rectangular hyperbola $y = \frac{1}{x + 1}$ at points A and B. The distance AB is equal to

A $2\sqrt{2}$ **B** 8 **C** 4 **D** $\sqrt{2} + 1$

16 The straight line $y = x + a$ touches the semicircle with equation $y = \sqrt{4x - x^2}$. The value of a is

A $-2 + 2\sqrt{2}$ **B** 1 **C** -2 **D** $-2 - 2\sqrt{2}$

17 The speed, v m/s, at time t seconds, of a small stone falling is given by the formula $v = k\sqrt{s}$ where s is the distance the stone has fallen at time t seconds and k is a real constant. The stone has a speed of 7 m/s after falling 2.5 m. Its speed after falling 10 m is

A 7 m/s **B** 28 m/s **C** 14 m/s **D** $7\sqrt{10}$ m/s

18 A curve passes through the points with coordinates $(2, -2)$, $(4, 2)$ and $\left(6, \frac{2}{3}\right)$. A possible equation of the curve is

A $y = 2x^2 - 4$ **B** $y = \sqrt{2x}$ **C** $y = \frac{2}{x - 3}$ **D** $y = \frac{2}{x} + 1$

19 A curve has a rule of the form $y = a\sqrt{x - h}$. The points with coordinates $(13, 9)$ and $(20, 12)$ lie on the curve. The values of a and h are

A $a = 3, h = 4$ **B** $a = 4, h = 3$ **C** $a = 5, h = 4$ **D** $a = 3, h = 5$

20 The circle with equation $x^2 + y^2 - 6x - 10y + c = 0$ does not intersect (or touch) either coordinate axis and the point $(1, 4)$ lies inside the circle. The range of possible values of c are

A $25 < c < 29$ **B** $c > 9$ **C** $c > 254$ **D** $c < 29$

 ISBN 978-1-009-53719-3

6

Functions and relations

Chapter contents

In Chapters 2 and 4 we looked at linear polynomials and quadratic polynomials, and in Chapter 5 we studied rectangular hyperbolas, square-root graphs, circles and semicircles. These are all examples of relations. You will meet them all again in this chapter, but using a new notation which will be carried through into the following chapters of this book.

We have studied transformations in the earlier chapters of this book without any systematic consideration of transformations of the points of the plane in general. In this chapter we offer the opportunity to develop techniques for both applying and identifying transformations.

 ISBN 978-1-009-53719-3 Cambridge University Press

6A Relations, domain and range

Learning intentions

► To be able to represent a **relation** and identify the **domain** and **range** of a relation.

In previous chapters we have looked at how to sketch the graphs of various mathematical relations. We will now look at this aspect of representing relations in a more formal way. An introduction to set notation was given in Chapter 3.

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs. The following are examples of relations:
  a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$
  b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$
- Every relation finds two sets:
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The domain is the set $\{1, 2, 3, 4\}$ and the range is the set $\{2, 3, 4, 5\}$.

Representing relations

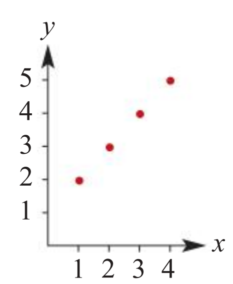
Graphing relations

We can represent a relation as a graph on a set of Cartesian axes. The domain is the set $\{1, 2, 3, 4\}$ and the range is the set $\{2, 3, 4, 5\}$.

On the right is the graph of the relation

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

Note that we only graph the individual points of this relation.

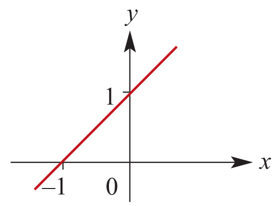


Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

If the domain of the relation is the set of real numbers, $\mathbb{R}$, then there are infinitely many points. For example, the graph of

$$\{(x, y) : y = x + 1,\ x \in \mathbb{R}\}$$

is a continuous straight line. The range is $\mathbb{R}$.

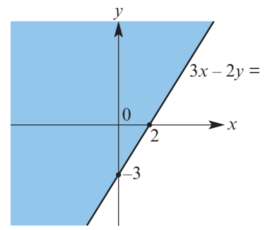


Another example of a relation is one defined by an inequality. For example,

$$\{(x, y) : 3x - 2y \leq 6,\ x \in \mathbb{R}\}$$

is a region of the real plane.

The domain is $\mathbb{R}$ and the range is $\mathbb{R}$.



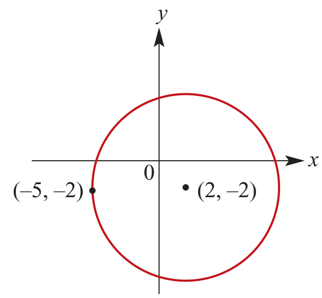
A circle is a further example of a relation.

$$\{(x, y) : (x - 2)^2 + (y + 2)^2 = 49,\ x \in \mathbb{R}\}$$

The domain is $[-5, 9]$ and the range is $[-9, 5]$.

Without interval notation:

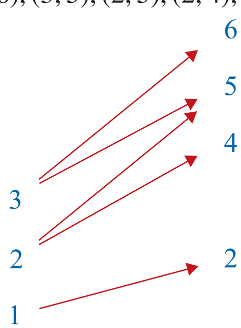
Domain: $-5 \leq x \leq 9$, Range: $-9 \leq y \leq 5$



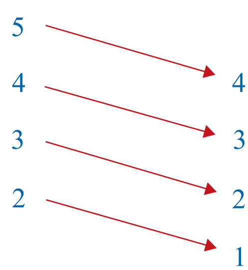
Arrow diagrams

A relation may also be represented by an arrow diagram.

This diagram represents the relation $\{(3, 6), (3, 5), (2, 5), (2, 4), (1, 2)\}$:



This diagram represents the relation $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$:



- A relation may be written as:
  - a listed set of ordered pairs (not always convenient or possible)
  - a rule with a specified or implied domain.
- A relation may be represented by a graph or an arrow diagram.

Often set notation is not used when describing a relation. For example:

- $\{(x, y) : y = x^2\}$ is written as $y = x^2$
- $\{(x, y) : y = x + 1,\ x \geq 0\}$ is written as $y = x + 1, x \geq 0$.

This has been the case in your previous considerations of relations.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Example 1

Sketch a graph of each of the following relations and state its domain and range:

a $\{(-2,-1),(-1,-1),(-1,1),(0,1),(1,-1)\}$

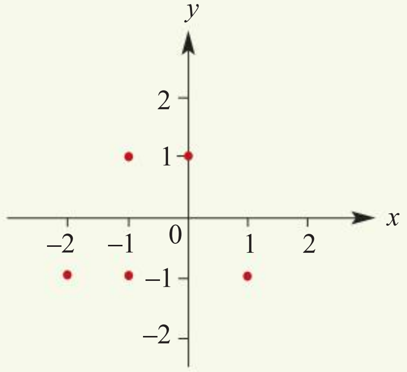
b $x^2+y^2=1$

c $2x+3y=6,\ x\ge 0$

d $y=2x-1,\ x\in[-1,2]$

Solution

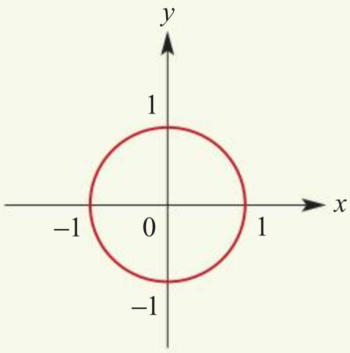
a



Domain $=\{-2,-1,0,1\}$

Range $=\{-1,1\}$

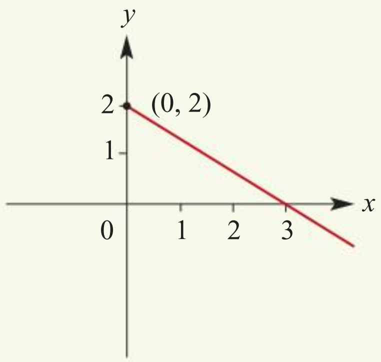
b



$-1\le x\le 1 \quad \therefore$ Domain $=[-1,1]$

$-1\le y\le 1 \quad \therefore$ Range $=[-1,1]$

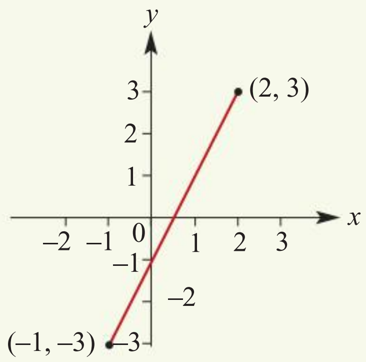
c



$x\ge 0 \quad \therefore$ Domain $=[0,\infty)$

$y\le 2 \quad \therefore$ Range $=(-\infty,2]$

d



$-1\le x\le 2 \quad \therefore$ Domain $=[-1,2]$

$-3\le y\le 3 \quad \therefore$ Range $=[-3,3]$

In order to determine the range of a relation it is necessary to consider the graph. This strategy is used in the following examples.

 ISBN 978-1-009-53719-3

Example 2

For each of the following, complete the square, sketch the graph and state the range. The domain is $\mathbb{R}$.

a $y = x^2 - 4x + 5$

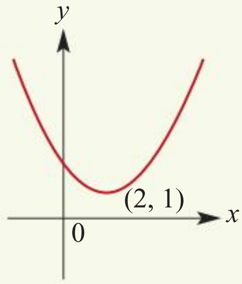
b $y = -x^2 + 4x - 5$

Solution

a $y = x^2 - 4x + 5$

$= (x - 2)^2 + 1$

The range is $[1, \infty)$. Without interval notation: $y \geq 1$



Explanation

Complete the square:

$$x^2 - 4x + 5 = x^2 - 4x + 4 - 4 + 5$$
$$= (x - 2)^2 + 1$$

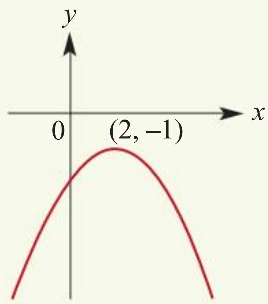
The vertex is at $(2, 1)$.

The minimum y-value is 1.

b $y = -x^2 + 4x - 5$

$= -(x - 2)^2 - 1$

The range is $(-\infty, -1]$. Without interval notation: $y \leq -1$



Complete the square:

$$-x^2 + 4x - 5 = -(x^2 - 4x + 5)$$
$$= -(x^2 - 4x + 4 + 1)$$
$$= -((x - 2)^2 + 1)$$
$$= -(x - 2)^2 - 1$$

The vertex is at $(2, -1)$.

The maximum y-value is -1.

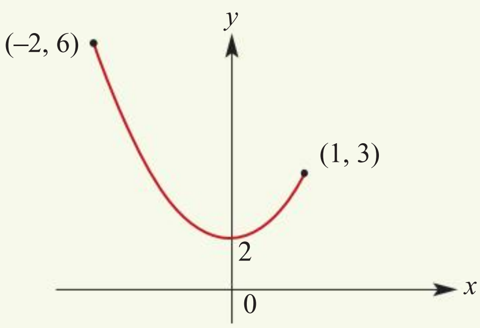
Example 3

Sketch the graph of the relation defined by

$$y = x^2 + 2 \text{ for } -2 \leq x \leq 1$$

and state the range.

Solution



The range is $[2, 6]$ or without interval notation: $2 \leq y \leq 6$

Explanation

Note that the range is not determined by considering the endpoints alone. The minimum y-value is 2, not 3.

 ISBN 978-1-009-53719-3

Implied (maximal) domain

When the rule for a relation is written and no domain is stipulated, then it is understood that the domain taken is the largest for which the rule has meaning. This domain is called the **maximal** or **implied domain**.

For example, the implied domain of $y = x^2$ is $\mathbb{R}$, and the implied domain of $x^2 + y^2 = 1$ is $-1 \le x \le 1$, that is, $[-1, 1]$. This concept is considered again in Section 6B.

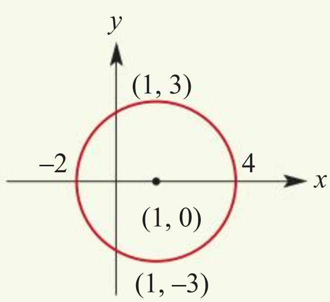
Example 4

For each of the following relations, state the implied domain and the range:

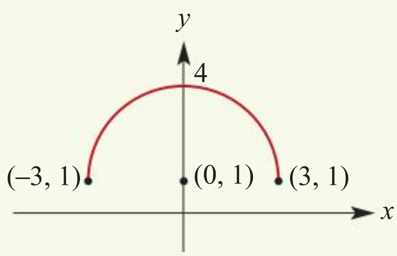
a $(x-1)^2 + y^2 = 9$ **b** $y = \sqrt{9 - x^2} + 1$

Solution

a This relation is a circle with centre $(1, 0)$ and radius 3.
The implied domain is $-2 \le x \le 4$, that is $[-2, 4]$.
The range is $-3 \le x \le 3$, that is $[-3, 3]$.



b This relation is a semicircle with centre $(0, 1)$ and radius 3.
The implied domain is $-3 \le x \le 3$, that is $[-3, 3]$
The range is $1 \le x \le 4$, that is $[1, 4]$.



Summary 6A

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example:
$$\{(x, y) : y = x + 1,\ x \ge 0\}$$
- For a relation described by a rule with y in terms of x, the domain is the x-values and the range is the y-values.
- The **maximal** or **implied domain** is the largest domain for which the rule of the relation has meaning.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 6A

SF

Example 1

1 Sketch a graph of each of the following relations and state its domain and range:

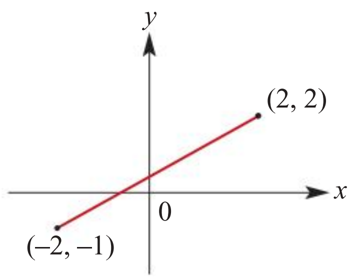
a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ **b** $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

c $x^2 + y^2 = 4$ **d** $3x + 2y = 12, \ x \geq 0$

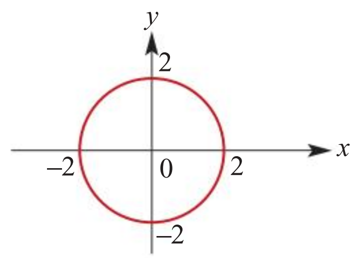
e $x - y = 4, \ -1 \leq x \leq 2$ **f** $y = 2x + 3, \ -4 \leq x \leq 1$

2 State the domain and range for the relations represented by each of the following graphs:

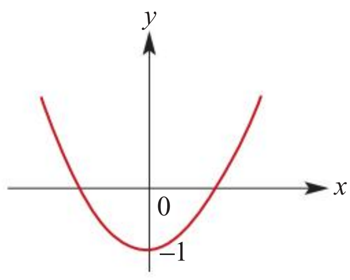
a



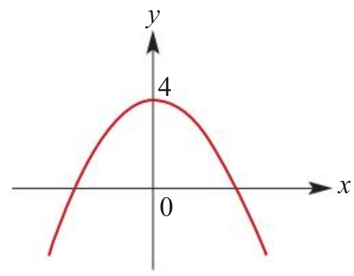
b



c



d



3 Draw an arrow diagram to represent each of the following relations:

a $\{(3, 6), (3, 4), (2, 3), (2, 2), (1, 3)\}$ **b** $\{(1, 2), (1, 3), (2, 2), (2, 4), (3, 3)\}$

c $\{(1, 2), (2, 3), (3, 4), (4, 1), (5, 2)\}$ **d** $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

Example 2

4 For each of the following, complete the square, sketch the graph and state the range:

a $y = x^2 + 6x + 10$ **b** $y = -x^2 - 4x - 6$ **c** $y = 2x^2 - 4x + 6$

5 Sketch the graphs of each of the following and state the range of each:

a $y = x + 1$ for $x \geq 2$ **b** $y = -x + 1$ for $x \geq 2$

c $y = 2x + 1$ for $x \geq -4$ **d** $y = 3x + 2$ for $x < 3$

e $y = x + 1$ for $x \leq 3$ **f** $y = -3x - 1$ for $-2 \leq x \leq 6$

g $y = -3x - 1$ for $-5 \leq x \leq -1$ **h** $y = 5x - 1$ for $-2 < x < 4$

Example 3

6 Sketch the graphs of each of the following and state the range of each:

a $y = x^2 + 3, \ x \in [-1, 1]$ **b** $y = x^2 + 4, \ x \in [-2, 1]$

c $y = x^2 - 4, \ x \in [-1, 2]$ **d** $y = 2x^2 + 1, \ x \in [-2, 3]$

 ISBN 978-1-009-53719-3

SF

7 Sketch the graphs of each of the following relations, stating the range of each:

a $y = x^2 + 1$ **b** $y = x^2 + 2x + 1$ **c** $y = 4 - x^2, \; -2 \le x \le 2$
d $y = x^2 + 2x + 3$ **e** $y = -x^2 + 2x + 3$ **f** $y = x^2 - 2, \; x \in [-1, 2]$
g $y = 2x^2 - 3x + 6$ **h** $y = 6 - 3x + x^2$

Example 4

8 Sketch the graphs of each of the following relations, stating the implied domain and range of each:

a $x^2 + y^2 = 9$ **b** $(x-2)^2 + (y-3)^2 = 16$
c $(2x-1)^2 + (2y-4)^2 = 1$ **d** $y = \sqrt{25 - x^2}$
e $y = -\sqrt{25 - x^2}$ **f** $y = -\sqrt{25 - (x-2)^2}$

9 Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $y = \dfrac{2}{2x-5} + 3$ **b** $y = \sqrt{2x - 5}$ **c** $y = \sqrt{5 - 2x}$
d $y = \sqrt{4 - (x-5)^2}$ **e** $y = (4 - 2x)^{\frac{1}{2}}$

10 Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $x = y^2$ **b** $x = y^2 + 1$ **c** $x = y^2 - 1$ **d** $x = (y-3)^2 + 1$

6B Functions

Learning intentions

- To be able to identify when a relation is a **function** and use function notation.

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 5

Which of the following sets of ordered pairs defines a function?

a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$
b $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ is a function, because for each x-value there is only one y-value.

b $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$ is *not* a function, because there is an x-value with two different y-values. The relation contains two ordered pairs, $(-4, 1)$ and $(-4, -1)$, with the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

 ISBN 978-1-009-53719-3

Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

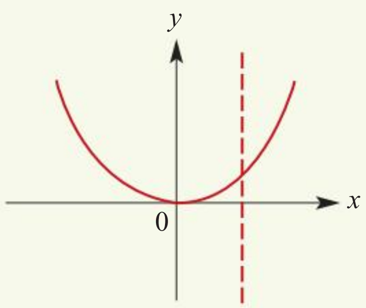
Example 6

a Is $y = x^2$ a function? State the maximal domain and range of $y = x^2$.

b Is $x^2 + y^2 = 4$ a function? State the maximal domain and range of $x^2 + y^2 = 4$.

Solution

a



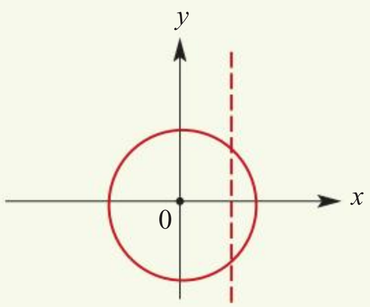
The vertical-line test shows that $y = x^2$ is a function. The maximal domain is $\mathbb{R}$ and the range is $[0, \infty)$. Without using interval notation the range is $y \geq 0$.

Explanation

For each x-value there is only one y-value.

The ordered pairs of the relation are all of the form (a, a^2).

b



The vertical-line test shows that $x^2 + y^2 = 4$ is *not* a function. The maximal domain is $[-2, 2]$ and the range is $[-2, 2]$.

Explanation

Note that $(\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ are ordered pairs of the relation.

There is an x-value with more than one y-value.

Function notation

Functions are usually denoted with lowercase letters such as f, g, h.

If f is a function, then for each x in the domain of f there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

Since the y-value obtained is a *function* of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.

ISBN 978-1-009-53719-3

For example, instead of $y = 2x + 1$ we can write $f(x) = 2x + 1$. Then $f(2)$ means the y-value obtained when $x = 2$.

e.g. $f(2) = 2(2) + 1 = 5$

$f(-4) = 2(-4) + 1 = -7$

$f(a) = 2a + 1$

By incorporating this notation, we have an alternative way of writing functions:

- For the function with rule $y = x^2$ and domain $\mathbb{R}$, we write $f(x) = x^2$, $x \in \mathbb{R}$.
- For the function with rule $y = 2x - 1$ and domain $[0, 4]$, we write $f(x) = 2x - 1$, $x \in [0, 4]$.

If the domain is $\mathbb{R}$, we often just write the rule. For example: $f(x) = x^2$.

Example 7

If $f(x) = 2x^2 + x$, determine:

a $f(3)$ **b** $f(-2)$ **c** $f(x - 1)$ **d** $f\left(\frac{1}{a}\right)$, $a \neq 0$

Solution

a $f(3) = 2(3)^2 + 3 = 21$

b $f(-2) = 2(-2)^2 - 2 = 6$

c $f(x - 1) = 2(x - 1)^2 + x - 1$
$= 2(x^2 - 2x + 1) + x - 1$
$= 2x^2 - 3x + 1$

d $f\left(\frac{1}{a}\right) = 2\left(\frac{1}{a}\right)^2 + \frac{1}{a}$
$= \frac{2}{a^2} + \frac{1}{a}$
$= \frac{2 + a}{a^2}$

Example 8

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Determine the value of $f(2)$ and $f(t)$.

b Find the value of x for which $f(x) = 6$.

c Find the value of x for which $f(x) = 0$.

d For what values of t is $f(t) = t$?

e For what values of x is $f(x) \geq x$?

f For what values of x is $f(x) \leq 3x$?

Solution

a $f(2) = 2(2) - 4 = 0$
$f(t) = 2t - 4$

b $f(x) = 6$
$2x - 4 = 6$
$2x = 10$
$\therefore\ x = 5$

c $f(x) = 0$
$2x - 4 = 0$
$2x = 4$
$\therefore\ x = 2$

d $f(t) = t$
$2t - 4 = t$
$t - 4 = 0$
$\therefore\ t = 4$

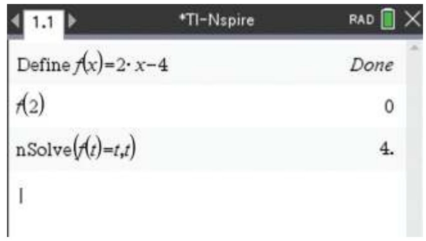
e $f(x) \geq x$
$2x - 4 \geq x$
$x - 4 \geq 0$
$\therefore\ x \geq 4$

f $f(x) \leq 3x$
$2x - 4 \leq 3x$
$-4 \leq x$
$\therefore\ x \geq -4$

 ISBN 978-1-009-53719-3

Using the TI-Nspire CX non-CAS

- Use (menu) > **Actions** > **Define** to define the function $f(x) = 2x - 4$.
- Determine $f(2)$ as shown.
- Use (menu) > **Algebra** > **Numerical Solve** to solve the equation $f(t) = t$.



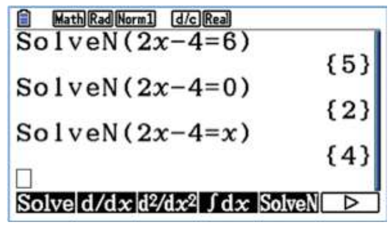
Using the Casio

Using the numerical solver

For the function $f(x) = 2x - 4$:

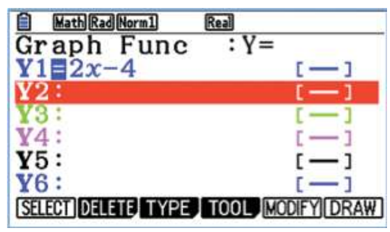
- In **Run-Matrix** mode, select the numerical solver (**Calculation** (OPTN) (F4), **SolveN** (F5)).
- To solve $f(x) = 6$, enter the equation $2x - 4 = 6$.

Repeat to solve $f(x) = 0$ and $f(x) = x$ as shown.



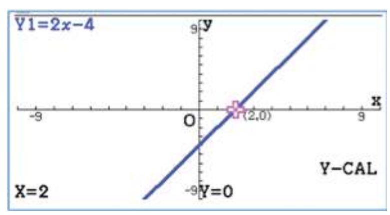
Using Graph mode

- In **Graph** mode, enter the rule for the function $f(x) = 2x - 4$ in $Y1$ as shown.
- Select **Draw** (F6).



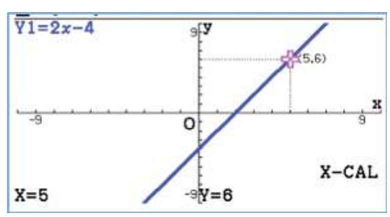
To determine $f(2)$:

- Go to the **G-Solve** menu (SHIFT) (F5) and select **y-Cal** (F6) (F1).
- Enter the x-value 2.
- Press (EXE) to obtain the answer $f(2) = 0$.



To determine the value of x for which $f(x) = 6$:

- Go to the **G-Solve** menu (SHIFT) (F5) and select **x-Cal** (F6) (F2).
- Enter the y-value 6.
- Press (EXE) to obtain the answer $x = 5$.



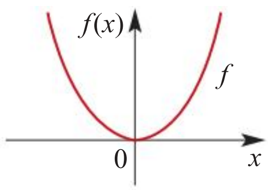
A **constant function** has a rule of the form $f(x) = a$. For example, let $f(x) = 7$; the domain is $\mathbb{R}$ and the range is $\{7\}$.

A **linear function** has a rule of the form $f(x) = mx + c$. For example, let $f(x) = 3x + 1$; the domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

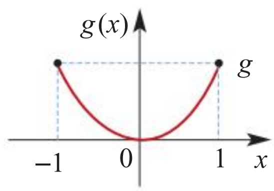
 ISBN 978-1-009-53719-3

Restriction of a function

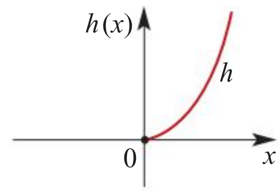
Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$



$g(x) = x^2, \ -1 \le x \le 1$



$h(x) = x^2, \ x \ge 0$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they all have the same rule. They are different because they are defined for different domains. We say that g and h are **restrictions** of f, since their domains are subsets of the domain of f.

Example 9

Sketch the graph of each of the following functions and state its range:

a $f(x) = x, \ -1 \le x \le 2$

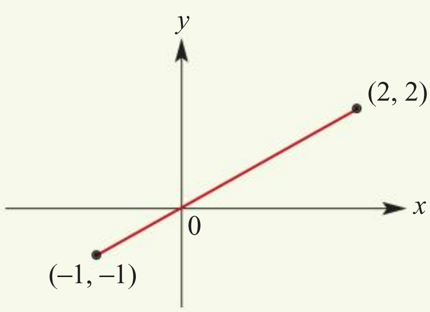
b $f(x) = x^2 + x, \ -1 \le x \le 1$

c $f(x) = \dfrac{1}{x}, \ 0 < x \le 2$

d $f(x) = x^2 - 2x + 8, \ x \in \mathbb{R}$

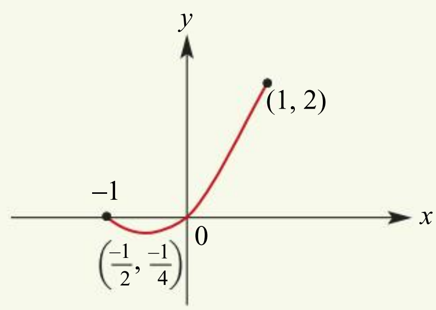
Solution

a



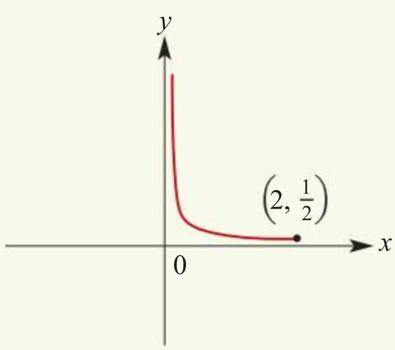
Range is $-1 \le y \le 2$, that is $[-1, 2]$

b



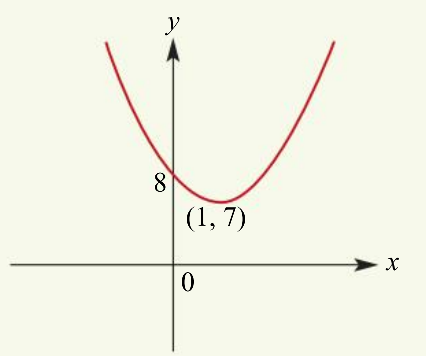
Range is $-\frac{1}{4} \le y \le 2$, that is $[-\frac{1}{4}, 2]$

c



Range is $y \ge \dfrac{1}{2}$, that is $[\frac{1}{2}, \infty)$

d



$f(x) = x^2 - 2x + 8 = (x-1)^2 + 7$

Range is $y \ge 7$. That is, $[7, \infty)$

 ISBN 978-1-009-53719-3

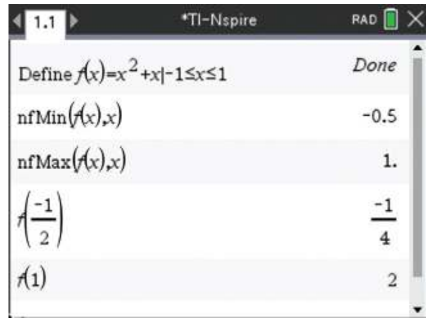
Using the TI-Nspire CX non-CAS

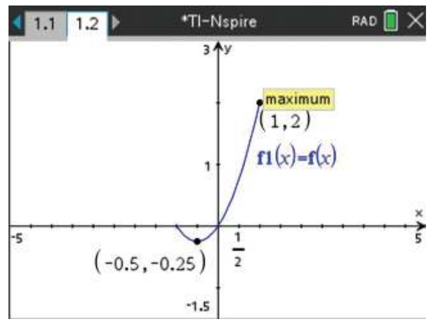
- In a **Calculator** application, use (menu) > **Actions** > **Define** to define the function $f(x) = x^2 + x, \ -1 \le x \le 1$.

Note: The 'with' or 'given' symbol | and the inequality signs can be accessed using (ctrl) (=).

- Use (menu) > **Calculus** > **Numerical Function Minimum** and (menu) > **Calculus** > **Numerical Function Maximum** to help determine the range of this restricted function. The range is $[-\frac{1}{4}, 2]$.
- The graph of $y = f(x)$ is plotted by entering $f1(x) = f(x)$ in a **Graphs** application.
- Use (menu) > **Analyze Graph** > **Minimum or Maximum** to show the key points.

Note: You can also enter the restricted function directly in the function entry line in the **Graphs** application if preferred.



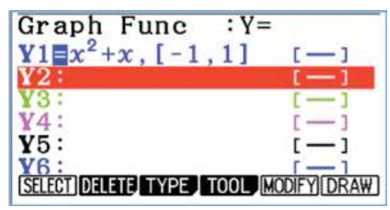


Using the Casio

- Press (MENU) (5) to select **Graph** mode.
- Enter the rule $y = x^2 + x$ and the domain $[-1, 1]$ in $Y1$:

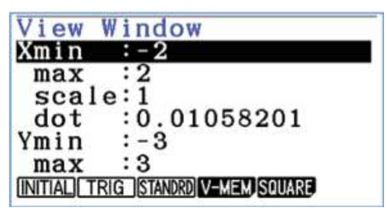
  (X,θ,T) (x^2) (+) (X,θ,T) (,)
  (SHIFT) (+) ((−)) (1) (,) (1) (SHIFT) (−) (EXE)

Note: Use (SHIFT) (+) for the left square bracket [.
Use (SHIFT) (−) for the right square bracket].

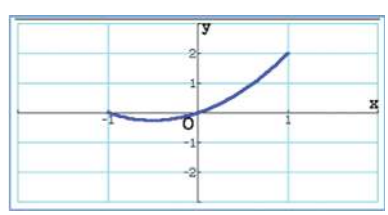


- Adjust the View Window:

  (SHIFT) (F3)
  ((−)) (2) (EXE) (2) (EXE) ▼ ▼
  ((−)) (3) (EXE) (3) (EXE) (EXIT)



- Select **Draw** (F6) to view the graph.



Note: When defining a restricted function, always use square brackets to specify the domain (not round brackets).

 ISBN 978-1-009-53719-3 Cambridge University Press

Implied (maximal) domain

We considered implied domains for relations in Section 6B. We recall our definition but this time we do so for functions in particular.

The **implied (maximal) domain** of a function is the set of all real numbers for which the rule of the function has meaning.

For example:

$f(x) = 3x^2 - 2x$ has implied domain $\mathbb{R}$

$g(x) = \sqrt{x}$ has implied domain $[0, \infty)$

Example 10

State the implied domain, sketch the graph and determine the corresponding range of each of the following:

a $f(x) = \sqrt{2x - 5}$ **b** $g(x) = \dfrac{1}{2x - 5}$ **c** $h(x) = 4 - x^2$

Solution

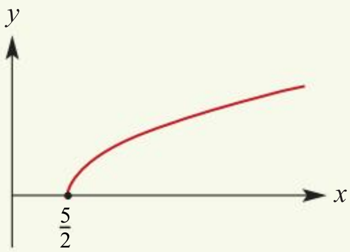
a For $f(x)$ to be defined, we need

$$2x - 5 \geq 0$$

$$\therefore \quad x \geq \frac{5}{2}$$

Hence the implied domain is $[\frac{5}{2}, \infty)$.

The range of the function is $[0, \infty)$.



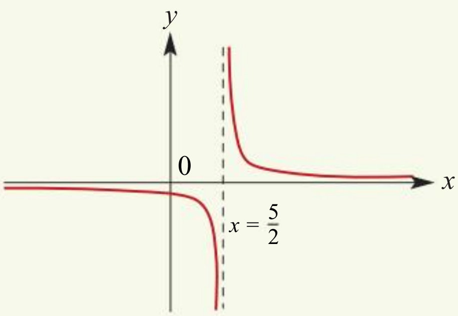
b For $g(x)$ to be defined, we need

$$2x - 5 \neq 0$$

$$\therefore \quad x \neq \frac{5}{2}$$

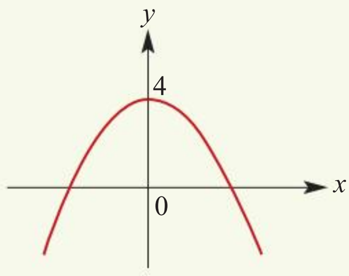
Hence the implied domain is $\mathbb{R} \setminus \{\frac{5}{2}\}$.

The range is $\mathbb{R} \setminus \{0\}$.



c Since $h(x) = 4 - x^2$ is defined for all x, the implied domain is $\mathbb{R}$.

The range is $(-\infty, 4]$.



 ISBN 978-1-009-53719-3

Summary 6B

- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- **Vertical-line test** If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- Functions are usually denoted with lowercase letters such as f, g, h.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y.
- Since the y-value obtained is a function of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- A **restriction** of a function has the same rule but a 'smaller' domain.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning. We refer to the **implied (maximal) domain** of a function, because the domain is implied by the rule.

Skill-sheet

Exercise 6B

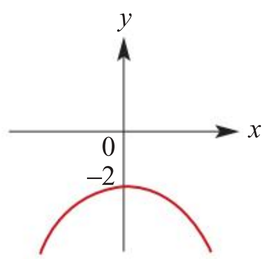
SF

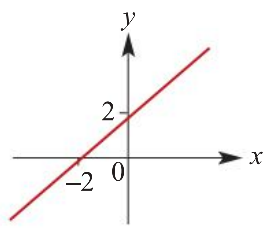
Example 5

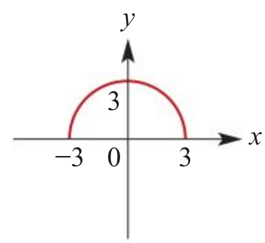
1 Which of the following relations are functions? State the domain and range for each:

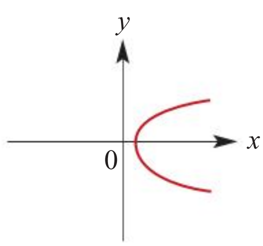
a $\{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4)\}$
b $\{(-2, -1), (-1, -2), (0, 2), (1, 4), (2, -5)\}$
c $\{(0, 1), (0, 2), (-1, 2), (3, 4), (5, 6)\}$
d $\{(1, 3), (2, 3), (4, 3), (5, 3), (6, 3)\}$

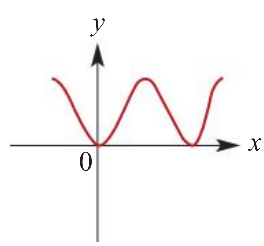
2 Each of the following is the graph of a relation. State which are the graph of a function.

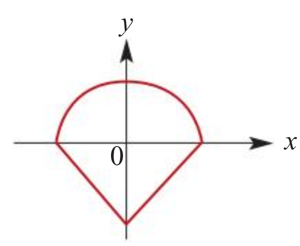
a


b


c


d


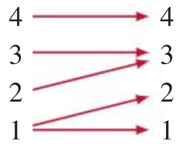
e


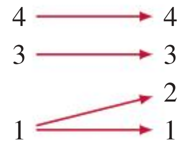
f


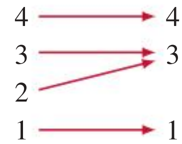
 ISBN 978-1-009-53719-3 Cambridge University Press

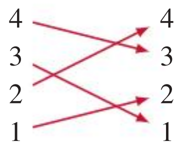
SF

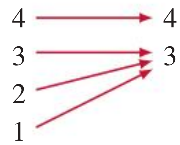
3 Which of the following arrow diagrams represent functions?

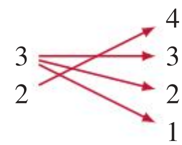
a


b


c


d


e


f


Example 6

4 Sketch the graph of each of the following relations, then state the range of each and specify whether the relation is a function or not:

a $y = x^2,\ x \in [0, 4]$ **b** $x^2 + y^2 = 4,\ x \in [0, 2]$ **c** $2x + 8y = 16,\ x \in [0, \infty)$

d $y = \sqrt{x},\ x \in \mathbb{R}^+$ **e** $y = \dfrac{1}{x},\ x \in \mathbb{R}^+$ **f** $y = x^2,\ x \in [-1, 4]$

g $y^2 = x,\ x \geq 0$

5 Which of the following relations are functions? State the domain and range for each:

a $\{(x, -2) : x \in \mathbb{R}\}$ **b** $\{(3, y) : y \in \mathbb{Z}\}$ **c** $y = -x + 3$

d $y = x^2 + 5$ **e** $\{(x, y) : x^2 + y^2 = 9\}$

Example 7

6 **a** Given that $f(x) = 2x - 3$, determine:

i $f(0)$ **ii** $f(4)$ **iii** $f(-1)$ **iv** $f(x - 1)$ **v** $f\left(\dfrac{1}{a}\right)$

b Given that $g(x) = \dfrac{4}{x}$, determine:

i $g(1)$ **ii** $g(-1)$ **iii** $g(3)$ **iv** $g(2)$

c Given that $g(x) = (x - 2)^2$, determine:

i $g(4)$ **ii** $g(-4)$ **iii** $g(8)$ **iv** $g(a)$

d Given that $f(x) = 1 - \dfrac{1}{x}$, determine:

i $f(1)$ **ii** $f(1 + a)$ **iii** $f(1 - a)$ **iv** $f\left(\dfrac{1}{a}\right)$

Example 8

7 Consider the function defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$.

a Determine the value of $f(2)$ and $f(t)$. **b** Find the value of x for which $f(x) = 6$.

c Find the value of x for which $f(x) = 0$. **d** For what values of t is $f(t) = t$?

e For what values of x is $f(x) \geq x$? **f** For what values of x is $f(x) \leq 3x$?

8 Determine the value(s) of x for which the function has the given value:

a $f(x) = 5x - 2,\ f(x) = 3$ **b** $f(x) = \dfrac{1}{x},\ f(x) = 6$

c $f(x) = x^2,\ f(x) = 9$ **d** $f(x) = (x + 1)(x - 4),\ f(x) = 0$

e $f(x) = x^2 - 2x,\ f(x) = 3$ **f** $f(x) = x^2 - x - 6,\ f(x) = 0$

 ISBN 978-1-009-53719-3

SF

9 Let $g(x) = x^2 + 2x$ and $h(x) = 2x^3 - x^2 + 6$.

a Evaluate $g(-1)$, $g(2)$ and $g(-2)$.

b Evaluate $h(-1)$, $h(2)$ and $h(-2)$.

c Express the following in terms of x:

i $g(-3x)$ **ii** $g(x-5)$ **iii** $h(-2x)$ **iv** $g(x+2)$ **v** $h(x^2)$

10 Consider the function $f(x) = 2x^2 - 3$. Determine:

a $f(2)$, $f(-4)$ **b** the range of f

11 Consider the function $f(x) = 3x + 1$. Determine:

a the image of 2 **b** the pre-image of 7 **c** $\{ x : f(x) = 2x \}$

12 Consider the function $f(x) = 3x^2 + 2$. Determine:

a the image of 0 **b** the pre-image(s) of 5 **c** $\{ x : f(x) = 11 \}$

13 Consider the functions $f(x) = 7x + 6$ and $g(x) = 2x + 1$. Determine:

a $\{ x : f(x) = g(x) \}$ **b** $\{ x : f(x) > g(x) \}$ **c** $\{ x : f(x) = 0 \}$

Example 9

14 Sketch the graphs of each of the following functions and state the range of each:

a $f(x) = x^2, \ -1 \le x \le 2$ **b** $f(x) = x^2 + 2x, \ -2 \le x \le 2$

c $f(x) = \dfrac{1}{x}, \ 0 < x \le 3$ **d** $f(x) = x^2 - 2x + 3, \ x \in \mathbb{R}$

e $f(x) = x^2 - 4x + 6, \ 1 < x < 6$ **f** $f(x) = x^2 - 2x + 1, \ -3 \le x \le 6$

Example 10

15 For each of the following, determine the implied domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$ **b** $y = 2\sqrt{x}$ **c** $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$ **e** $y = \dfrac{1}{\sqrt{x}}$ **f** $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$ **h** $y = \sqrt{2x - 1}$ **i** $y = \sqrt{3 - 2x}$

j $y = \dfrac{1}{2x - 1}$ **k** $y = \dfrac{1}{(2x-1)^2}$ **l** $y = \dfrac{1}{2x - 1} + 2$

16 For each of the following, state the implied domain and range:

a $f(x) = \sqrt{x - 4}$ **b** $f(x) = \sqrt{4 - x}$ **c** $f(x) = 2\sqrt{x - 2} + 3$

d $f(x) = \dfrac{1}{x - 4}$ **e** $f(x) = \dfrac{1}{x - 4} + 3$ **f** $f(x) = \dfrac{3}{x + 2} - 3$

17 Each of the following is the rule of a function. In each case write down the maximal domain and the range:

a $f(x) = 3x + 4$ **b** $g(x) = x^2 + 2$

c $y = -\sqrt{16 - x^2}$ **d** $y = \dfrac{1}{x + 2}$

ISBN 978-1-009-53719-3

6C Piecewise-defined functions

Learning intentions

- To be able to work with **piecewise-defined functions**.

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 11

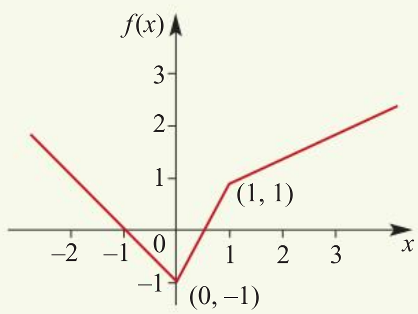
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

Explanation

- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$.

Example 12

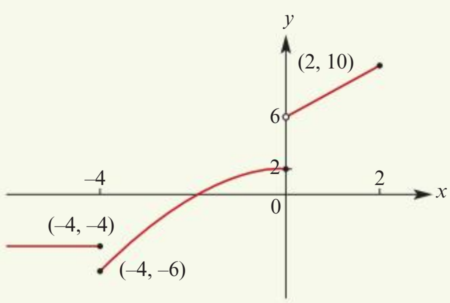
a Sketch the graph of the function:

$$f(x) = \begin{cases} -4 & \text{for } x < -4 \\ -\frac{1}{2}x^2 + 2 & \text{for } -4 < x \le 0 \\ 2x + 6 & \text{for } 0 < x \le 2 \end{cases}$$

b State the range of the function.

Solution

a



b The range is $(-6, 2] \cup (6, 10]$.

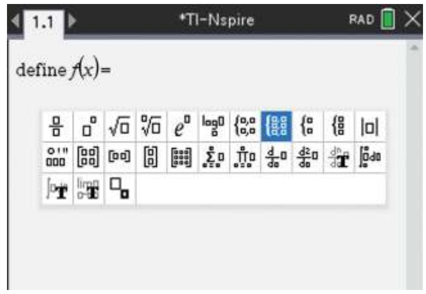
Explanation

- The graph of $y = -4$ is sketched for $x < -4$.
- The graph of $y = -\frac{1}{2}x^2 + 2$ is sketched for $-4 < x \le 0$.
- The graph of $y = 2x + 6$ is sketched for $0 < x \le 2$.

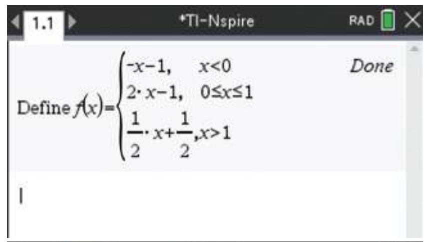
 ISBN 978-1-009-53719-3

Using the TI-Nspire CX non-CAS

- In a **Calculator** application, use `menu` > **Actions** > **Define** as shown.
- Select the piecewise template from the 2D-template palette.
- In the pop-up screen, select OK for a function definition with 3 pieces.

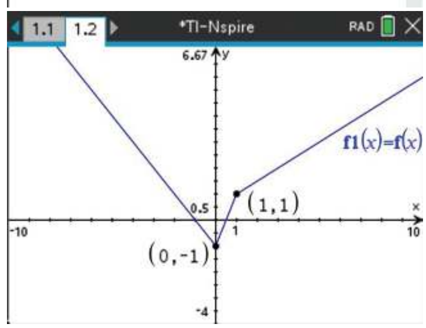


- Complete the template as shown.



- In a **Graphs** application, plot the graph of $f1(x) = f(x)$.

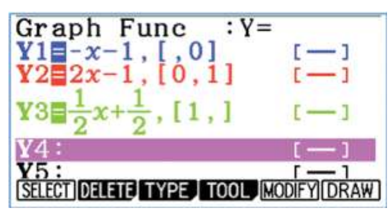
Note: You can also enter the piecewise-defined function directly in the function entry line if preferred.



Using the Casio

- Press `MENU` `5` to select **Graph** mode.
- Enter the first rule, $y = -x - 1$ for $x < 0$, in $Y1$:

  `(−)` `X,θ,T` `−` `1` `,`
  `SHIFT` `+` `,` `0` `SHIFT` `−` `EXE`



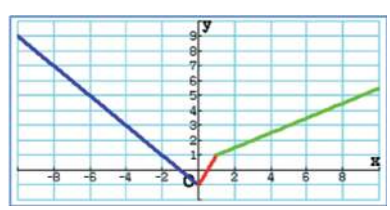
- Enter the second rule, $y = 2x - 1$ for $0 \le x \le 1$, in $Y2$:

  `2` `X,θ,T` `−` `1` `,` `SHIFT` `+` `0` `,` `1` `SHIFT` `−` `EXE`

- Enter the third rule, $y = \frac{1}{2}x + \frac{1}{2}$ for $x > 1$, in $Y3$:

  `▭/▭` `1` ▼ `2` ► `X,θ,T` `+` `▭/▭` `1` ▼ `2` ► `,` `SHIFT` `+` `1` `,` `SHIFT` `−` `EXE`

- Adjust the View Window `SHIFT` `F3` for $-10 \le x \le 10$ and $-2 \le y \le 10$.
- Select **Draw** `F6` to view the graph.



ISBN 978-1-009-53719-3

Cambridge University Press

Exercise 6C

Example 11

1 Sketch the graph of each of the following functions and state its range:

a $h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

b $h(x) = \begin{cases} x - 1, & x \geq 1 \\ 1 - x, & x < 1 \end{cases}$

c $h(x) = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}$

d $h(x) = \begin{cases} 1 + x, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$

e $h(x) = \begin{cases} x, & x \geq 1 \\ 2 - x, & x < 1 \end{cases}$

2 **a** Sketch the graph of the function:

$$f(x) = \begin{cases} \frac{2}{3}x + 3, & x < 0 \\ x + 3, & 0 \leq x \leq 1 \\ -2x + 6, & x > 1 \end{cases}$$

b What is the range of f?

3 Sketch the graph of the following functions :

a $g(x) = \begin{cases} -x - 3, & x < 1 \\ x - 5, & 1 \leq x \leq 5 \\ 3x - 15, & x > 5 \end{cases}$

b $f(x) = \begin{cases} x + 3, & x < -3 \\ x^2 - 9, & -3 \leq x \leq 3 \\ x - 3, & x > 3 \end{cases}$

4 Sketch the graph of the following functions and state their range:

a $h(x) = \begin{cases} x^2 + 1, & x \geq 0 \\ 1 - x, & x < 0 \end{cases}$

b $f(x) = \begin{cases} \frac{1}{x}, & x > 1 \\ x, & x \leq 1 \end{cases}$

5 Sketch the graph of each of the following and state the range:

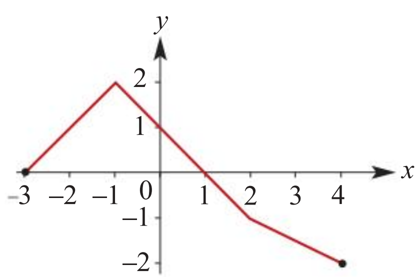
a $f(x) = \begin{cases} x + 3 & \text{if } x < -3 \\ x & \text{if } x > 2 \end{cases}$

b $f(x) = \begin{cases} \dfrac{1}{x + 3} & \text{if } x < -3 \\ x^2 & \text{if } x > 0 \end{cases}$

c $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ x & \text{if } 1 < x < 3 \\ -x & \text{if } x \geq 4 \end{cases}$

d $f(x) = \begin{cases} \dfrac{1}{x + 3} & \text{if } x < -3 \\ (x + 2)^2 & \text{if } x > -1 \end{cases}$

6 Specify the function represented by this graph:



6D Applying function notation

Learning intentions

- To be able to apply function notation to solve solve problems.

The first five chapters of this book involve functions without using function notation. This section presents further questions which arise from the first five chapters of this book but where function notation can now be used.

Example 13

The volume of a sphere of radius r is determined by the function with rule $V(r) = \frac{4}{3}\pi r^3$.

State the practical domain of the function V and determine $V(5)$.

Solution	Explanation
The practical domain is $(0, \infty)$.	
$V(5) = \frac{4}{3} \times \pi \times 5^3 = \frac{500\pi}{3}$	A sphere of radius 5 has volume $\frac{500\pi}{3}$ cubic units.

Note: Since V is expressed as a function of r, we say that V is the **dependent variable** and r is the **independent variable**.

Example 14

A function has rule $f(x) = ax + b$, where a and b are constants. If $f(1) = 7$ and $f(5) = 19$, determine the values of a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$,

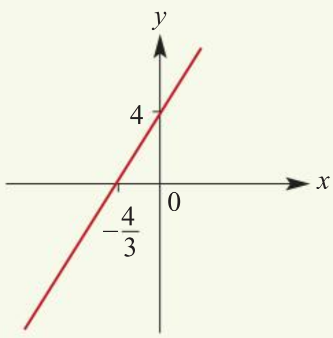
$$7 = a + b \qquad (1)$$

and $$19 = 5a + b \qquad (2)$$

Subtract (1) from (2):

$$12 = 4a$$

Thus $a = 3$ and substituting in (1) gives $b = 4$. Hence $f(x) = 3x + 4$.



Example 15

Determine the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

Since 4 and -2 are solutions to the quadratic equation $f(x) = 0$, we have

$$f(x) = k(x - 4)(x + 2)$$

Since $f(0) = 16$, we obtain

$$16 = k(-4)(2)$$

$$\therefore \quad k = -2$$

 ISBN 978-1-009-53719-3

Hence $f(x) = -2(x-4)(x+2)$
$= -2(x^2 - 2x - 8)$
$= -2x^2 + 4x + 16$

Skill-sheet

Exercise 6D

Example 13

1 A metal bar is L cm long when its temperature is T°C. The quantities L and T are approximately related by the formula $L = 0.002T + 25$.

a L is a function of T and the rule can be written $L(T) = 0.002T + 25$. State a possible practical domain for the function.

b Determine:

i $L(30)$ **ii** $L(16)$ **iii** $L(100)$ **iv** $L(500)$

SF

Example 14

2 If $f(x) = a + bx$ with $f(4) = -1$ and $f(8) = 1$:

a determine a and b

b solve the equation $f(x) = 0$.

CF

3 Determine a linear function f such that $f(0) = 7$ and whose graph is parallel to that of the function with rule $g(x) = 2 - 5x$.

4 If f is a linear function such that $f(-5) = -12$ and $f(7) = 6$:

a **i** determine $f(0)$ **ii** determine $f(1)$

b solve the equation $f(x) = 0$.

Example 15

5 Determine the quadratic function f such that $f(2) = f(4) = 0$ and 7 is the greatest value of $f(x)$.

6 Write $f(x) = x^2 - 6x + 16$ in the form $f(x) = (x-3)^2 + p$ and hence state the range of f.

7 Let $f(x) = ax^2 + bx + c$. Determine a, b and c if $f(0) = 2$, $f(4) = 0$ and $f(5) = 0$.

8 Determine two quadratic functions f and g such that $f(1) = 0$, $g(1) = 0$ and $f(0) = 10$, $g(0) = 10$ and both have a maximum value of 18.

9 **a** Determine the set of values of k for which $f(x) = 3x^2 - 5x - k$ is greater than 1 for all real x.

b Show that, for all k, the minimum value of $f(x)$ occurs when $x = \frac{5}{6}$. Determine k if this minimum value is zero.

CU

10 Define a function f with domain $\mathbb{N}$ recursively by $f(1) = 4$ and $f(n+1) = f(n) + 1$ for all $n \in \mathbb{N}$.

a Give an explicit rule for $f(n)$ in terms of n.

b What is the range of f?

 ISBN 978-1-009-53719-3

6E Functions and modelling exercises

Learning intentions

- To be able to apply function notation to model practical situations.

In the following examples we see how function notation can be used when applying mathematics in 'real' situations.

Example 16

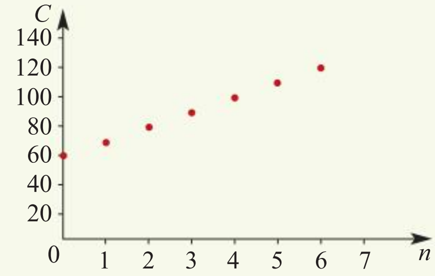
A book club has a membership fee of \$60.00 and each book purchased is \$10.00. Construct a cost function that can be used to determine the cost of different numbers of books, then sketch its graph.

Solution

Let $C(n)$ denote the cost (in dollars) when n books are purchased. Then

$$C(n) = 60 + 10n$$

The domain of this function is $\mathbb{N} \cup \{0\}$, the set of non-negative integers, and its graph will be as shown.



Note: The graph of this function consists of discrete points. Sometimes to simplify the situation we represent such functions by a continuous line. Strictly, this is not mathematically correct, but it may aid our understanding of the situation.

Example 17

The following table shows the Younanistan Post rates for sending letters.

Mass, m (g)	Cost, C (\$)
Up to 50 g	\$0.70
Over 50 g up to 100 g	\$1.15
Over 100 g up to 250 g	\$1.70
Over 250 g up to 500 g	\$3.00

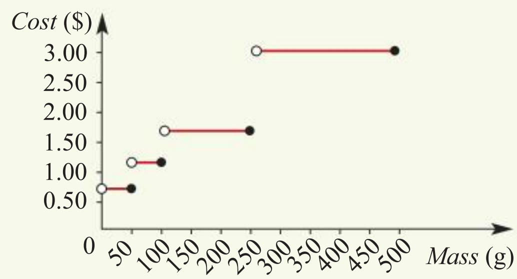
Sketch a graph of the cost function, C, giving its domain and range and the rules that define it.

Solution

$$C(m) = \begin{cases} 0.70 & \text{for } 0 < m \le 50 \\ 1.15 & \text{for } 50 < m \le 100 \\ 1.70 & \text{for } 100 < m \le 250 \\ 3.00 & \text{for } 250 < m \le 500 \end{cases}$$

Domain $= (0, 500]$

Range $= \{0.70, 1.15, 1.70, 3.00\}$



Example 18

A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of its length, ℓ. By sketching a graph, determine the maximum area that can be fenced.

Solution

Let ℓ = length of the enclosure

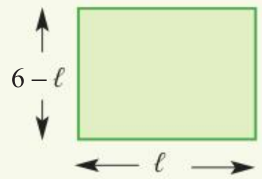
Then width $= \dfrac{12 - 2\ell}{2} = 6 - \ell$

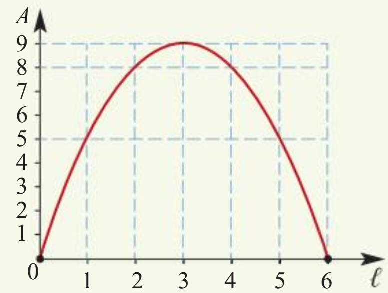
The area is

$$A(\ell) = \ell(6 - \ell)$$
$$= 6\ell - \ell^2$$

The domain of A is the interval $[0, 6]$.

The maximum area is 9 m^2 and occurs when $\ell = 3$ m, i.e. when the enclosure is a square.





Exercise 6E

Example 16 **1** Austel's rates for local calls from private telephones consist of a quarterly rental fee of \$45 plus 15c for every call. Construct a cost function that describes the quarterly telephone bill.

Example 17 **2** Suppose that Younanistan Post charges the following rates for airmail letters to Africa: \$1.20 up to 20 g; \$2.00 over 20 g and up to 50 g; \$3.00 over 50 g and up to 150 g.

a Write a cost function, C (\$), in terms of the mass, m (g), for letters up to 150 g.

b Sketch the graph of the function, stating the domain and range.

3 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 \$64 per day + 25 cents per kilometre

Method 2 \$89 per day with unlimited travel

a Write a rule for each method if C_1 is the cost, in dollars, using method 1 for x kilometres travelled, and C_2 is the cost using method 2.

b Draw a graph of each rule on the same axes.

c Determine, from the graph, the distance which must be travelled per day if method 2 is cheaper than method 1.

SF

CF

 ISBN 978-1-009-53719-3

Example 18

4 A piece of wire 100 cm long is bent to form a rectangle of width x cm.

a Determine the length of the rectangle in terms of x.

b Determine the rule $A(x)$ for the function A that gives the area of the rectangle in cm^2.

c Determine the allowable values for x.

d Determine the maximum possible area of the rectangle and the value of x for which this maximum occurs.

5 The table gives the rates charged by MD Couriers for delivering parcels within Brisbane.

Weight, w kg	Cost
$w < 1$	\$3.50
$1 \le w < 2.5$	\$6.50
$2.5 \le w < 5$	\$12.00
$5 \le w < 10$	\$20.00

a Sketch a graph representing the information given in the table.

b Determine the cost of sending a parcel that weighs:

i 2 kg **ii** 4 kg **iii** 5 kg

c Alexa has two items to send to the same destination: one item weighs 2.6 kg and the other weighs 2.3 kg. Should she send the two items separately, or package them together? Justify your answer.

6 Two taxi services use the following different systems for charging for a journey:

Gold Taxi Initial charge of \$10, plus a charge of 50 cents for each 200 m travelled

Purple Taxi Flat fee of \$25 for travelling up to 20 km, plus a charge of \$1 for each kilometre travelled beyond 20 km

a Let $G(d)$ be the cost (in dollars) of a journey of d km in a Gold Taxi. Show that

$$G(d) = 2.5d + 10 \quad \text{for } d \ge 0$$

b Let $P(d)$ be the cost (in dollars) of a journey of d km in a Purple Taxi. Show that

$$P(d) = \begin{cases} 25 & \text{for } 0 \le d \le 20 \\ d + 5 & \text{for } d > 20 \end{cases}$$

c On the same coordinate axes, sketch graphs to represent $G(d)$ and $P(d)$.

d Determine the cost of a journey of:

i 7 km in a Gold Taxi **ii** 12 km in a Purple Taxi

e Art wants to travel a distance of 15 km. Which taxi service will be cheaper?

f Determine the distances for which a Purple Taxi is the cheaper option.

7 A gardener charges \$40 per hour for work that takes up to 2 hours and then charges \$30 per hour for work after the first 2 hours, up to a maximum of 6 hours.

a Write the rule for a function that describes the total fee, \$$F$, for work that takes the gardener t hours. (Assume a continuous model.)

b Sketch a graph of your function.

c Determine the fee for work that takes the gardener:

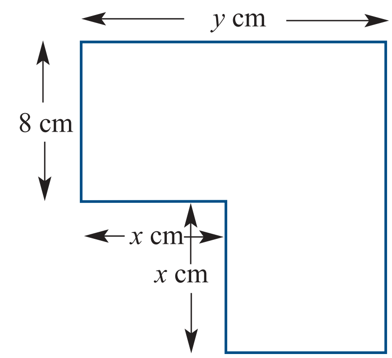
i 1.5 hours **ii** 2.5 hours **iii** 3.5 hours

d What is the effective hourly rate for work that takes 4 hours?

 ISBN 978-1-009-53719-3

CU

8 Assume that angles that look like right angles are right angles.



a **i** Determine an expression for the area A in terms of x and y.

ii Determine an expression for the perimeter P in terms of x and y.

b **i** If $P = 64$ cm, determine A in terms of x.

ii Determine the allowable values for x.

iii Sketch the graph of A against x for these values.

iv What is the maximum area?

9 After taking a medication, the concentration of the drug in a patient's bloodstream first increases and then gradually decreases. When a second medication is taken, the concentration starts to increase again. The concentration of the drug, C mg/L, in the patient's bloodstream over a 12-hour period is modelled by the function

$$C(t) = \begin{cases} -5(t-3)^2 + 45 & \text{for } 0 \le t < 3 \\ -(t-3)^2 + 45 & \text{for } 3 \le t < 9 \\ -5(t-12)^2 + 54 & \text{for } 9 \le t \le 12 \end{cases}$$

a Determine the concentration of the drug 2 hours after the first medication was taken.

b What was the maximum concentration of the drug during the first 3 hours?

c Determine the concentration of the drug 6 hours after the first medication was taken.

d When did the patient take the second medication, and what was the concentration of the drug at that time?

e What was the maximum concentration of the drug during the 12-hour period?

10 Sections of the rail of a ride at an amusement park are modelled by the function

$$h(x) = \begin{cases} -0.2(x-5)^2 + 8 & \text{for } 0 \le x < 7 \\ -0.8x + 12.8 & \text{for } 7 \le x < 11 \\ 0.2(x-13)^2 + 3.2 & \text{for } 11 \le x \le 16 \end{cases}$$

where $h(x)$ m is the height of the rail above the ground at a horizontal distance of x m from the start of the ride.

a Determine the height at the start of the ride.

b Determine the height at each of the following horizontal distances from the start of the ride:

i 5 m **ii** 10 m **iii** 15 m

c Determine the horizontal distance(s) from the start of the ride when the height is 6 m.

 ISBN 978-1-009-53719-3 Cambridge University Press

6F Fitting data

Learning intentions

- To be able to model data with functions of the form $y = kx^n$ where $k > 0$ and $n \neq 0$

In Section 4K of the chapter on quadratics we used **quadratic regression** with a graphics calculator to determine the equation of parabola passing through points. In the example in that section only 3 points were used (these points are not collinear). If more than 3 points are used the information returned by the calculator will define the parabola that best fits the given data. We will not discuss what 'best fits' means but comment that the calculator returns a value of R^2, which is a measurement of the fit. If the fit is perfect the calculator returns $R^2 = 1$.

In the following examples **power regression** is employed. We will use other forms of regression in Chapter 8, Trigonometric functions and Chapter 11, Exponential functions.

Example 19

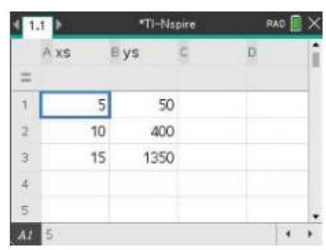
The following data was collected, recording the number of calls, $N(D)$, made to a company D days after the start of an advertising campaign.

Days (D)	5	10	15
Number of calls ($N(D)$)	50	400	1350

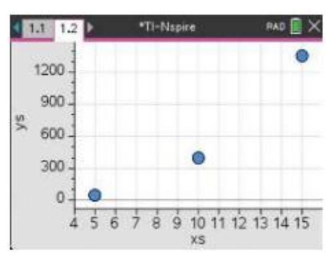
Determine a relationship between N and D using a calculator.

Using the TI-Nspire CX non-CAS

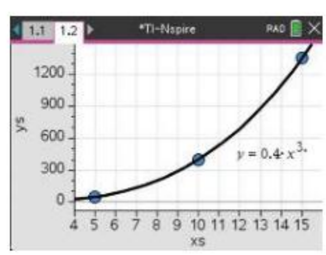
- In a **Lists & Spreadsheet** page, enter the data in lists *xs* and *ys* as shown.



- Insert a **Data & Statistics** page (ctrl I > **Add Data & Statistics**).
- Click on the 'Click to add variable' box and select *xs* from the dropdown menu.
- Repeat for the y-axis and select *ys*.



- Use menu > **Analyze** > **Regression** > **Show Power**.
- This gives $y = 0.4x^3$ and so the required relationship is $N = 0.4D^3$.

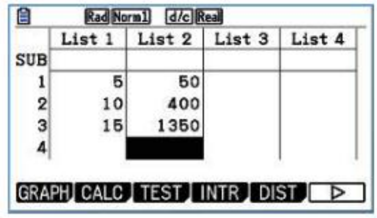


ISBN 978-1-009-53719-3

Using the Casio

- Press MENU 2 to select **Statistics** mode.
- If required, clear any contents in List1 and List2:

  F6 F4 F1 ► F4 F1 ◄ F6 F6



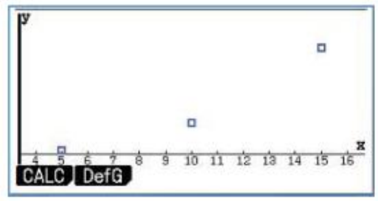
- Enter the days in List1 and the number of calls in List2:

  5 EXE 1 0 EXE 1 5 EXE ►

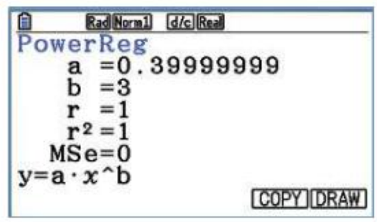
  5 0 EXE 4 0 0 EXE 1 3 5 0

  EXE

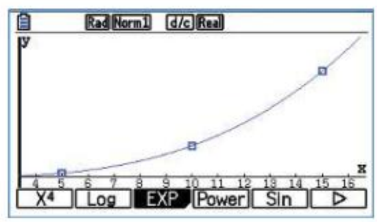
- Graph the data by selecting **Graph** F1, then **Graph1** F1.



- For power regression, select **Calculation** F1, then **Power** F6 F4.
- This gives $y = 0.4x^3$ and so the required relationship is $N = 0.4D^3$.



- Select **Draw** F6 to view the graph.



Example 20

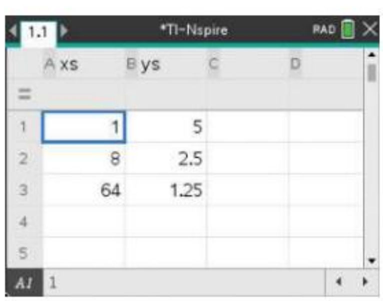
Using the data in the table, establish a rule connecting y and x.

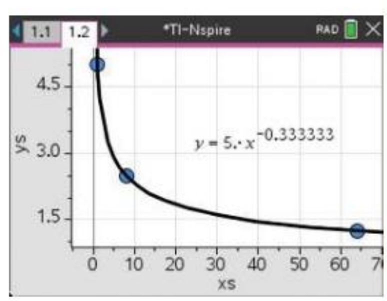
x	1	8	64
y	5	2.5	1.25

Using the TI-Nspire CX non-CAS

The rule is

$$y = 5x^{-\frac{1}{3}} = \frac{5}{\sqrt[3]{x}}$$

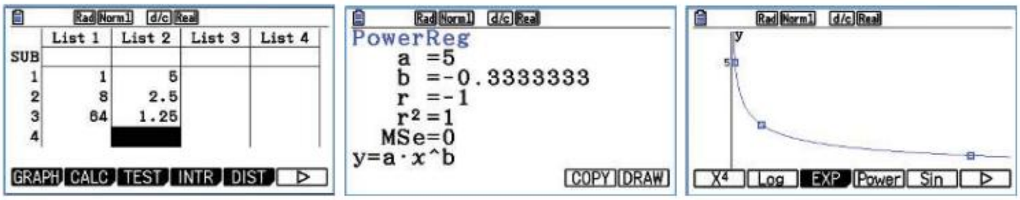




 ISBN 978-1-009-53719-3

Using the Casio

The rule is $y = 5x^{-\frac{1}{3}} = \dfrac{5}{\sqrt[3]{x}}$



Exercise 6F

Example 19, 20

1 Given that $f(x) = ax^b$ for each of the following, use your calculator's power regression function to establish the values of a and b:

a

x	4.00	8.00	12.00	16.00
y	0.50	0.71	0.87	1.00

b

x	1	5	10	15
y	2.00	14.95	35.57	59.04

c

x	1	10	100	1000
y	3.50	8.79	22.08	55.47

d

x	10	20	30	40
y	46.42	73.68	96.55	116.96

e

x	1	2	3	4
y	2.00	0.35	0.13	0.06

f

x	1	3	5	7
y	3.20	2.06	1.68	1.47

2 The concentration of antibodies (C) in an animal's bloodstream depends on the time (t hours) after which the animal is injected with an antigen. The relationship can be modelled by a rule of the form $C = at^b$. The following data is collected.

t	1	2	3	4
C	100	114.87	124.57	131.95

a Determine values for a and b.

b Determine the concentration after 10 hours.

3 The level of infestation (I) of a pest in a crop depends on the time (t days) after which the crop is sprayed with an insecticide. The relationship can be modelled by a rule of the form $I = at^b$, for $t \geq 1$. The following data is collected.

t	1	2	3	4
I	1500	1061	866	750

a Determine values for a and b.

b Determine the level of infestation after 10 days.

6F

 ISBN 978-1-009-53719-3

Note: *Sections* 6G, 6H, 6I *and* 6J *offer a systematic approach to transformation.*

6G Translations of functions

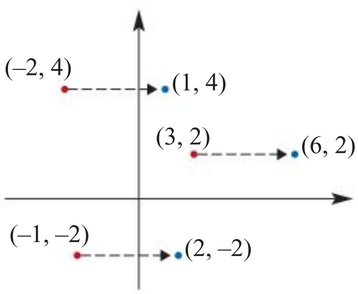
Learning intentions

- To be able to use a notation for considering **translations** of the plane.

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$. This translation is a transformation of the plane. Each point in the plane is mapped to a unique second point. Furthermore, every point in the plane is an image of another point under this translation.

Notation

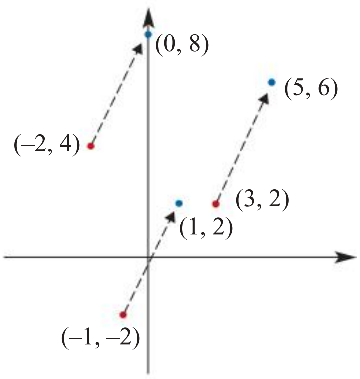


The translation 3 units to the right can be written $(x, y) \to (x + 3, y)$. This reads as '(x, y) maps to $(x + 3, y)$'.

For example, $(-1, -2) \to (-1 + 3, -2)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written for this translation:

$$x' = x + 3 \quad \text{and} \quad y' = y$$



As another example, consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up). This can be described by the rule $(x, y) \to (x + 2, y + 4)$.

For example, $(3, 2) \to (3 + 2, 2 + 4)$.

The following can be written for this translation:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \to (x + h, y + k)$$

or $\quad x' = x + h \quad$ and $\quad y' = y + k$

where h and k are positive numbers.

 ISBN 978-1-009-53719-3 Cambridge University Press

A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

$$(x, y) \to (x - h, y - k)$$

or $\quad x' = x - h \quad$ and $\quad y' = y - k$

where h and k are positive numbers.

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a particular translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(a - h, b - k) \to (a, b)$ under this translation.)

Applying translations to sketch graphs

We look at a particular example.

Translate the set of points defined by the function

$$\{ (x, y) : y = x^2 \}$$

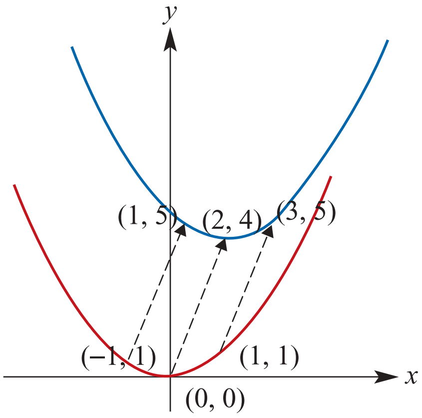
by the translation defined by the rule

$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.



This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.

Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$\Leftrightarrow \quad f(a) = b$

$\Leftrightarrow \quad f(a + h - h) = b$

$\Leftrightarrow \quad f((a + h) - h) = (b + k) - k$

$\Leftrightarrow \quad (a + h, b + k)$ is a point on the graph of $y - k = f(x - h)$

Note: The double arrows indicate that the steps are reversible.

 ISBN 978-1-009-53719-3

Example 21

Determine the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = f(x)$ is mapped to the graph of

$y' + 2 = f(x' - 3)$

i.e. $y = \frac{1}{x}$ is mapped to

$$y' + 2 = \frac{1}{x' - 3}$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of

$$y = \frac{1}{x - 2}$$

note that it is of the form $y = f(x - 2)$ where $f(x) = \frac{1}{x}$. That is, the graph of $y = \frac{1}{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x - 2) = (x - 2)^2$$

$$f(x) = \sqrt{x} \qquad f(x - 2) = \sqrt{x - 2}$$

Summary 6G

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

 ISBN 978-1-009-53719-3

Exercise 6G

SF

1 Determine the image of the point $(-3, 4)$ after a mapping of a translation:

a of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b of 2 units in the negative direction of the x-axis and 4 units in the positive direction of the y-axis

c of 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

d defined by the rule $(x, y) \to (x - 4, y + 5)$

e defined by the rule $(x, y) \to (x - 2, y - 1)$.

Example 21

2 In each of the following, determine the rule for the image of the graph of $y = f(x)$ under the given translation:

a $f(x) = \frac{1}{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis

b $f(x) = \frac{1}{x}$ under a translation 4 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

c $f(x) = x^2$ under a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

d $f(x) = x^2$ under a translation 4 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

e $f(x) = \sqrt{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis.

3 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$

d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

4 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$

d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

5 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1) + 2$ **b** $y = f(x - 3) + 1$ **c** $y = f(x + 3) - 5$

d $y = f(x + 1) - 3$ **e** $y + 2 = f(x + 1)$ **f** $y = f(x - 5) - 1$

6H Dilations and reflections

Learning intentions

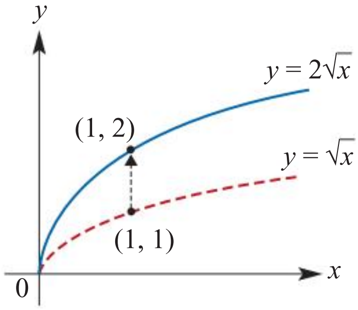
- ▶ To be able to use a notation for considering dilations and reflections of the plane.

Dilation from the x-axis

A dilation of factor 2 from the x-axis can be defined by the rule $(x, y) \to (x, 2y)$.

Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.



- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \frac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\frac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.

For b a positive constant, a dilation of factor b from the x-axis is described by the rule

$$(x, y) \to (x, by)$$

or $\quad x' = x \quad$ and $\quad y' = by$

For the graph of $y = f(x)$, the following two processes yield the same result:

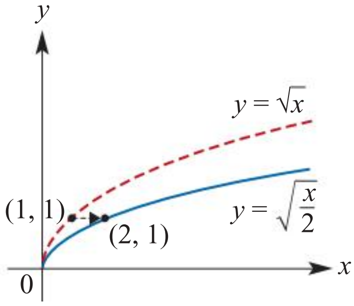
- Applying the dilation from the x-axis $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
- Replacing y with $\frac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.

Dilation from the y-axis

A dilation of factor 2 from the y-axis can be defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = 2x$ and $y' = y$, and thus $x = \frac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



 ISBN 978-1-009-53719-3 Cambridge University Press

For a a positive constant, a dilation of factor a from the y-axis is described by the rule

$$(x, y) \to (ax, y)$$

or $\quad x' = ax \quad$ and $\quad y' = y$

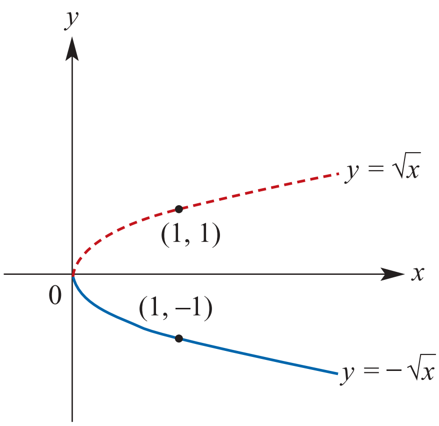
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation from the y-axis $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
- Replacing x with $\dfrac{x}{a}$ in the equation to obtain $y = f\left(\dfrac{x}{a}\right)$ and graphing the result.

Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



A reflection in the x-axis is described by the rule

$$(x, y) \to (x, -y)$$

or $\quad x' = x \quad$ and $\quad y' = -y$

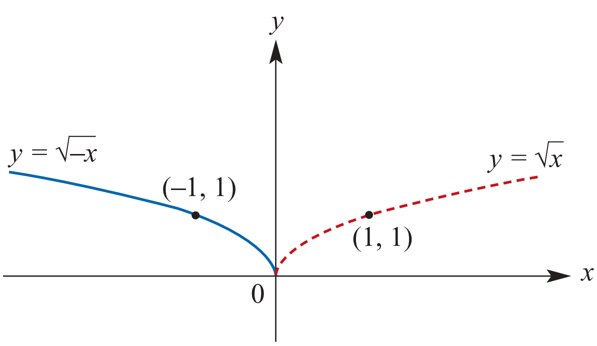
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the x-axis $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



 ISBN 978-1-009-53719-3

A reflection in the y-axis is described by the rule

$$(x, y) \to (-x, y)$$

or $\quad x' = -x \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the y-axis $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Example 22

Determine the rule of the image when the graph of $y = x^2$ is dilated by a factor of 3:

a from the y-axis **b** from the x-axis.

Solution

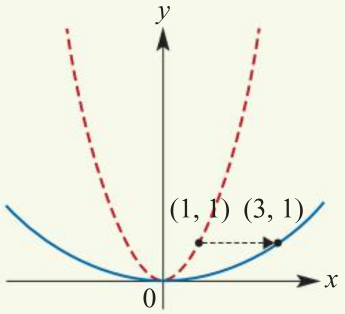
a $(x, y) \to (3x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 3x$, $y' = y$.

Rearranging gives $x = \frac{x'}{3}$, $y = y'$.

Therefore $y = x^2$ becomes $y' = \left(\frac{x'}{3}\right)^2$.

The rule of the transformed function is $y = \frac{x^2}{9}$.



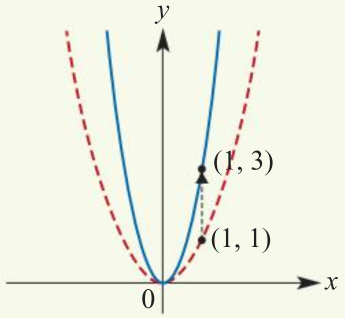
b $(x, y) \to (x, 3y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 3y$.

Rearranging gives $x = x'$, $y = \frac{y'}{3}$.

Therefore $y = x^2$ becomes $\frac{y'}{3} = (x')^2$.

The rule of the transformed function is $y = 3x^2$.



Applying dilations and reflections to sketch graphs

In order to sketch the graph of $y = \sqrt{\frac{x}{2}}$, note that it is of the form $y = f\left(\frac{x}{2}\right)$ where $f(x) = \sqrt{x}$. This is the graph of $y = \sqrt{x}$ dilated by factor 2 from the y-axis.

Examples of other functions under this dilation are:

$$f(x) = x^2 \qquad f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

$$f(x) = \frac{1}{x} \qquad f\left(\frac{x}{2}\right) = \frac{1}{\frac{x}{2}} = \frac{2}{x}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

It should be noted that each of these functions formed by a dilation of factor 2 from the y-axis can also be formed by a dilation from the x-axis. This result is not true in general, as will be seen when new functions are introduced in Chapters 8, 11 and 12.

- For the graph of $y = \sqrt{\dfrac{x}{2}}$, we can write $y = \dfrac{1}{\sqrt{2}}\sqrt{x} = \dfrac{1}{\sqrt{2}}f(x)$, where $f(x) = \sqrt{x}$. That is, it is formed by a dilation of factor $\frac{1}{\sqrt{2}}$ from the x-axis.
- For the graph of $y = \dfrac{x^2}{4}$, we can write $y = \frac{1}{4}x^2 = \frac{1}{4}f(x)$, where $f(x) = x^2$. That is, it is formed by a dilation of factor $\frac{1}{4}$ from the x-axis.
- For the graph of $y = \dfrac{2}{x}$, we can write $y = 2f(x)$, where $f(x) = \dfrac{1}{x}$. That is, it is formed by a dilation of factor 2 from the x-axis.

Summary 6H

For the graph of $y = f(x)$, we have the following four pairs of equivalent processes:

1. - Applying the **dilation from the x-axis** $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
   - Replacing y with $\dfrac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.
2. - Applying the **dilation from the y-axis** $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
   - Replacing x with $\dfrac{x}{a}$ in the equation to obtain $y = f\left(\dfrac{x}{a}\right)$ and graphing the result.
3. - Applying the **reflection in the x-axis** $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
   - Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.
4. - Applying the **reflection in the y-axis** $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
   - Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Exercise 6H

SF

1 Determine the image of the point $(-2, -3)$ after:

a a reflection in the x-axis **b** a reflection in the y-axis

c a dilation of factor 4 from the x-axis **d** a dilation of factor 4 from the y-axis.

Example 22

2 Write down the equation of the image obtained when the graph of each of the functions below is transformed by:

i a dilation of factor $\frac{1}{2}$ from the y-axis **ii** a dilation of factor 5 from the y-axis

iii a dilation of factor $\frac{2}{3}$ from the x-axis **iv** a dilation of factor 4 from the x-axis

v a reflection in the x-axis **vi** a reflection in the y-axis.

a $y = x^2$ **b** $y = \dfrac{1}{x}$ **c** $y = \sqrt{x}$

3 Sketch the graph of each of the following:

a $y = 3\sqrt{x}$ **b** $y = -\dfrac{1}{x}$ **c** $y = \dfrac{3}{x}$ **d** $y = \dfrac{1}{2x}$ **e** $y = \sqrt{3x}$ **f** $y = \dfrac{3}{2x}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

6I Combinations of transformations

Learning intentions

- To be able to use notation for considering combinations of transformations of the plane.

In this section, we look at sequences of transformations. For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 23

Determine the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

The image has equation $\dfrac{y'}{-2} = \sqrt{x'}$ and hence $y' = -2\sqrt{x'}$.

b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

The image has equation $\dfrac{y' + 3}{2} = \sqrt{x' - 2}$ and hence $y' = 2\sqrt{x' - 2} - 3$.

Summary 6I

- Given a sequence of transformations, we can determine the rule for transforming points of the plane. For example, the sequence
  - a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
  - followed by a reflection in the y-axis

  can be described by the rule $(x, y) \to (x + 2, y - 3) \to (-x - 2, y - 3)$.

 ISBN 978-1-009-53719-3

- Using the rule, we can then determine the image of a graph under the sequence of transformations. For example, if the rule is

$$(x, y) \to (-x - 2, y - 3)$$

then $x' = -x - 2$ and $y' = y - 3$. This gives $x = -x' - 2$ and $y = y' + 3$. Therefore the graph of $y = f(x)$ maps to $y' + 3 = f(-x' - 2)$. That is, the graph of $y = f(x)$ maps to the graph of $y = f(-x - 2) - 3$.

Skill-sheet

Exercise 6I

Example 23

SF

1 Determine the equation of the image of the graph of $y = \sqrt{x}$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 from the x-axis

b a translation of 3 units in the negative direction of the x-axis followed by a reflection in the x-axis

c a reflection in the x-axis followed by a dilation of factor 3 from the x-axis

d a reflection in the x-axis followed by a dilation of factor 2 from the y-axis

e a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

f a dilation of factor 2 from the y-axis followed by a translation of 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.

2 Repeat Question 1 for $y = \dfrac{1}{x}$.

6J Determining transformations

Learning intentions

- ▶ To be able to determine a transformation which takes a graph to its image.

The method that has been used to determine the effect of transformations can be reversed to determine the sequence of transformations used to take a graph to its image.

For example, to determine a sequence of transformations which maps $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution of Example 22a:

- $y = \sqrt{x}$ maps to $\dfrac{y'}{-2} = \sqrt{x'}$.
- We can write $x = x'$ and $y = \dfrac{y'}{-2}$, and so $x' = x$ and $y' = -2y$.
- The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, if you recognise the form of the image. For the combinations of transformations in this course, it is often simpler to do this.

Example 24

a Determine a sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = 2(x-2)^2 + 3$.

b Determine a sequence of transformations which takes the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{5x-2}$.

Solution

a $y = x^2$ maps to $y' = 2(x'-2)^2 + 3$. Rearranging this equation gives

$$\frac{y'-3}{2} = (x'-2)^2$$

We choose to write $y = \dfrac{y'-3}{2}$ and $x = x' - 2$.

Solving for x' and y' gives

$$x' = x + 2 \quad \text{and} \quad y' = 2y + 3$$

The transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b We have $y' = \sqrt{5x'-2}$ and $y = \sqrt{x}$. We choose to write $y = y'$ and $x = 5x' - 2$. Hence

$$x' = \frac{x+2}{5} = \frac{x}{5} + \frac{2}{5} \quad \text{and} \quad y' = y$$

The transformation is a dilation of factor $\frac{1}{5}$ from the y-axis followed by a translation of $\frac{2}{5}$ units in the positive direction of the x-axis.

Note: You may be able to determine these answers by inspection, without any working out, if you recognise the form of the image.

Example 25

a Determine a sequence of transformations which takes the graph of $y = \dfrac{3}{x-1} + 6$ to the graph of $y = \dfrac{1}{x}$.

b Determine a sequence of transformations which takes the graph of $y = (5x-1)^2 + 6$ to the graph of $y = x^2$.

Solution

a Write $\dfrac{y-6}{3} = \dfrac{1}{x-1}$ and $y' = \dfrac{1}{x'}$. The points (x, y) satisfying $\dfrac{y-6}{3} = \dfrac{1}{x-1}$ are mapped to the points (x', y') satisfying $y' = \dfrac{1}{x'}$.

Hence we choose to write

$$y' = \frac{y-6}{3} \quad \text{and} \quad x' = x - 1$$

One transformation is a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis followed by a dilation of factor $\frac{1}{3}$ from the x-axis.

 ISBN 978-1-009-53719-3

b Write $y - 6 = (5x - 1)^2$ and $y' = (x')^2$. The points (x, y) satisfying $y - 6 = (5x - 1)^2$ are mapped to the points (x', y') satisfying $y' = (x')^2$.
Hence we choose to write

$$y' = y - 6 \quad \text{and} \quad x' = 5x - 1$$

One transformation is a dilation of factor 5 from the y-axis followed by a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis.

We note that the transformations we found are far from being the only possible answers. In fact there are infinitely many choices.

Summary 6J

The notation developed in this chapter can be used to help determine the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x' - 3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 from the x-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x' - 3)$. The one we chose is conventional.

Skill-sheet

Exercise 6J

Example 24

1 For each of the following, determine a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 2(x - 1)^2 + 3$ **ii** $y = -(x + 1)^2 + 2$ **iii** $y = (2x + 1)^2 - 2$

b the graph of $y = \dfrac{1}{x}$ to the graph of

i $y = \dfrac{2}{x + 3}$ **ii** $y = \dfrac{1}{x + 3} + 2$ **iii** $y = \dfrac{1}{x - 3} - 2$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x + 3} + 2$ **ii** $y = 2\sqrt{3x}$ **iii** $y = -\sqrt{x} + 2$

CF

Example 25

2 For each of the following, determine a sequence of transformations that takes:

a the graph of $y = (3x + 2)^2 + 5$ to the graph of $y = x^2$

b the graph of $y = -3(3x + 1)^2 + 7$ to the graph of $y = x^2$

c the graph of $y = 2\sqrt{4 - x}$ to the graph of $y = \sqrt{x}$

d the graph of $y = 2\sqrt{4 - x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

 ISBN 978-1-009-53719-3

Review

Chapter summary

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- For a function f and an element x of the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f, and the element x is called a **pre-image** of y.
- The **implied domain** (or **maximal domain**) of a function is the largest subset of $\mathbb{R}$ for which the rule is defined.
- For a function f with domain D, a new function g may be defined with domain $A \subseteq D$ and rule given by $g(x) = f(x)$ for all $x \in A$. The function g is called a **restriction** of f.

Transformations of the graphs of functions

Mapping	Rule	Image of $y = f(x)$
Reflection in the x-axis	$x' = x,\ y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x,\ y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax,\ y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x,\ y' = by$	$y = bf(x)$
Translation	$x' = x + h,\ y' = y + k$	$y - k = f(x - h)$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

6A **1** I can determine the domain and range of a relation. ☐

See Example 1 and Question 1

6A **2** I can determine the range of a quadratic function. ☐

See Example 2 and Question 4

6A **3** I can determine the range of a function with restricted domain. ☐

See Example 3 and Question 6

6A **4** I can determine the implied domain of a function. ☐

See Example 4 and Question 8

 ISBN 978-1-009-53719-3

6B	5	I can determine when a relation is a function and use the vertical line test.	☐
		See Example 5, Example 6 and Questions 1 and 4	
6B	6	I can use function notation.	☐
		See Example 7, Example 8 and Questions 6 and 7	
6B	7	I can sketch the graphs of functions with restricted domain and determine their range.	☐
		See Example 9 and Question 14	
6B	8	I can determine the impled domain of a function.	☐
		See Example 10 and Question 15	
6C	9	I can sketch the graph of a piecewise-defined function.	☐
		See Example 11, Example 12 and Question 1	
6D	10	I can apply functions in different contexts.	☐
		See Example 13 and Question 1	
6D	11	I can determine the rule for linear and quadratic functions given sufficient information.	☐
		See Example 14, Example 15 and Questions 2 and 5	
6E	12	I can use functions to model in different practical situations.	☐
		See Example 16, Example 17, Example 18 and Questions 1, 2 and 4	
6F	13	I can use a graphics calculator to fit suitable points by power regression.	☐
		See Example 19, Example 20 and Question 1	
6G	14	I can determine the equation of the image of the graph of a function after translation.	☐
		See Example 21 and Question 2	
6H	15	I can determine the equation of the image of the graph of a function after dilations and reflections in an axis.	☐
		See Example 22 and Question 2	
6I	16	I can determine the equation of the image of the graph of a function after a combination of translations, dilations and reflection in an axis.	☐
		See Example 23 and Question 1	
6J	17	I can determine the sequence of transformations which takes a graph of a function to its image given the equation of both the original function and its image.	☐
		See Example 24, Example 25 and Questions 1 and 2	

Review

SF

Short-response questions

Technology-free short-response questions

1 If f is the function with rule $f(x) = 2 - 6x$, determine:

a $f(3)$ **b** $f(-4)$ **c** the value of x for which f maps x to 6.

2 If f is a function with rule $f(x) = 6 - x$ and domain $-1 \le x \le 6$:

a sketch the graph of f **b** state the range of f.

3 Sketch the graphs of each of the following, stating the range of each:

a $3x + y = 6$ **b** $y = 3x - 2, \ -1 \le x \le 2$

c $y = x^2, \ x \in [-2, 2]$ **d** $y = 9 - x^2$

e $y = x^2 + 4x + 6$ **f** $\{(1, 2), (3, 4), (2, -6)\}$

g $f(x) = (x - 2)^2$ **h** $f(x) = \dfrac{1}{x} + 2, \ x \ne 0$

i $(x - \frac{1}{2})^2 + (y + 2)^2 = 9$ **j** $f(x) = x, \ -1 \le x \le 3$

4 The function f has rule $f(x) = \dfrac{a}{x} + b$ such that $f(1) = \dfrac{3}{2}$ and $f(2) = 9$.

a Determine the values of a and b. **b** State the implied domain of f.

5 Given that $f(x) = 2x - x^2$ for $0 \le x \le 2$:

a sketch the graph **b** state the range.

6 Given that $f(x) = ax + b$, $f(5) = 10$ and $f(1) = -2$, determine the values of a and b.

7 Given that $f(x) = ax^2 + bx + c$, $f(0) = 0$, $f(4) = 0$ and $f(-2) = -6$, determine the values of a, b and c.

8 State the implied (maximal) domain for each of the following:

a $y = \dfrac{1}{x - 2}$ **b** $f(x) = \sqrt{x - 2}$ **c** $y = \sqrt{25 - x^2}$

d $f(x) = \dfrac{1}{2x - 1}$ **e** $g(x) = \sqrt{100 - x^2}$ **f** $h(x) = \sqrt{4 - x}$

9 Sketch the graphs of each of the following:

a $f(x) = \begin{cases} 3x - 1, & x \ge 0 \\ x^2, & -3 \le x < 0 \\ 9, & x < -3 \end{cases}$ **b** $h(x) = \begin{cases} 1 - 2x, & x \ge 0 \\ x^2, & -3 \le x < 0 \\ -x^2, & x < -3 \end{cases}$

10 Sketch the graph of each of the following and state the range:

a $f(x) = (x - 1)^2$ for $0 \le x \le 3$ **b** $f(x) = (x + 2)^2 + 1$ for $-4 \le x \le 1$

c $f(x) = \dfrac{1}{2x - 1}$ for $1 \le x \le 5$ **d** $f(x) = -x^2 + 3$ for $-2 \le x \le 3$

 ISBN 978-1-009-53719-3

11 State the maximal domain and range of each of the following:

a $f(x) = \sqrt{x-1}$ **b** $f(x) = \sqrt{1-x}$ **c** $f(x) = 1 - \sqrt{x}$

12 State the maximal domain and range of each of the following:

a $f(x) = \dfrac{2}{x-1}$ **b** $f(x) = \dfrac{2}{x+1}$ **c** $f(x) = \dfrac{2}{x-1} + 3$

13 State the maximal domain and range of each of the following:

a $f(x) = \sqrt{1-x^2}$ **b** $f(x) = \sqrt{9-x^2}$ **c** $f(x) = \sqrt{1-x^2} + 3$

14 If $f(x) = 2x + 5$, determine:

a $f(p)$ **b** $f(p+h)$

c $f(p+h) - f(p)$ **d** $f(p+1) - f(p)$

15 If $f(x) = 3 - 2x$, determine $f(p+1) - f(p)$.

16 State the range of each of the following:

a $f(x) = -2x^2 + x - 2$ **b** $f(x) = 2x^2 - x + 4$

c $f(x) = -x^2 + 6x + 11$ **d** $g(x) = -2x^2 + 8x - 5$

17 Let $f(x) = 5 - 3x$ for $x \in [-1, 6]$.

a Sketch the graph of f. **b** State the range of f.

18 Let $f(x) = (x-2)^2$ for $x \in [-1, 8]$.

a Sketch the graph of f. **b** State the range of f.

19 State the implied domain and range of each of the following relations:

a $x^2 + y^2 = 9$ **b** $(x-2)^2 + y^2 = 1$ **c** $(2x-1)^2 + (2y-1)^2 = 1$

d $(x-4)^2 + y^2 = 25$ **e** $(y-2)^2 + x^2 = 16$

20 The domain of the function f is $\{1, 2, 3, 4\}$. Determine the range of f if:

a $f(x) = 2x$ **b** $f(x) = 5 - x$ **c** $f(x) = x^2 - 4$ **d** $f(x) = \sqrt{x}$

21 Sketch the graph of each of the following, labelling asymptotes and axis intercepts:

a $y = \dfrac{1}{x} + 3$ **b** $y = \sqrt{x-2}$ **c** $y = \dfrac{1}{x-1}$

d $y = \dfrac{1}{x} - 4$ **e** $y = \dfrac{1}{x+2}$ **f** $y = \dfrac{1}{x-3}$

g $f(x) = \dfrac{1}{x-1} + 1$ **h** $f(x) = \dfrac{1}{x-2} + 2$

22 For each of the following, determine a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 3(x+1)^2 + 2$ **ii** $y = -2(x+2)^2 + 3$ **iii** $y = (3x+1)^2 - 1$

 ISBN 978-1-009-53719-3

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{4}{x+2}$ **ii** $y = \frac{1}{x+6} - 12$ **iii** $y = \frac{4}{x-3} - 5$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x-4} + 2$ **ii** $y = 2\sqrt{2x}$ **iii** $y = -2\sqrt{x} + 3$

SF

23 A parking meter is designed to accept 200 twenty-cent coins.

a Write a rule which gives the number of hours parking, P, in terms of the number n of twenty-cent coins inserted, when the cost of parking is 20c for each half hour.

b Sketch the function, stating the domain and range.

CF

24 The Exhibition Centre hires a graphics company to produce a poster for an exhibit. The graphics company charges \$1000 and an additional \$5 for each poster produced.

a **i** Write down the rule for a function, $C(n)$, which describes the cost to the Exhibition Centre of obtaining n posters.

ii Sketch the graph of C against n. (Use a continuous model.)

b The Exhibition Centre is going to sell the posters for \$15 each.

i Write down the rule for a function, $P(n)$, which describes the profit when the Exhibition Centre prints and sells n posters.

ii Sketch the graph of this function. (Use a continuous model.)

Technology-active short-response questions

CF

25 The surface area of a sphere is defined by the function $S(r) = 4\pi r^2$ where r is the radius of the sphere and the domain of the function is the positive real numbers. Determine the radius of the sphere, correct to two decimal places, which has a surface area of 45 cm^2.

26 Let $f(x) = \sqrt{x} + x^2 - 4$ for $x \geq 0$. Solve each of the following equations for x. Give your answers correct to three decimal places.

a $f(x) = 0$ **b** $f(x) = x$ **c** $f(x) = 1$

27 Let $f(x) = x^2 + \frac{1}{x}$ for $x \neq 0$ and $g(x) = f(x-2)$ for $x \neq 2$. With your calculator look at the graph of $h(x) = f(x) - g(x)$ for $x \neq 0, 2$. Solve the equation $f(x) = g(x)$. Give your answer correct to two decimal places.

28 The following table records the fin length ℓ cm and the weight $w(\ell)$ kg of a certain type of fish.

ℓ cm	5.2	8.5	11.5	14.3	16.8	19.2	21.3	23.3	25	26.7
$w(\ell)$ kg	2	8	21	38	69	117	148	190	264	293

Assuming w is related to ℓ by a rule of the form $w = a \times \ell^b$ use your calculator to determine values of a and b. Give values correct to two decimal places.

29 An Easyride coach leaves town X and maintains a constant speed of 80 km/h for 4 hours, stops at town Y for $\frac{3}{4}$ hours before travelling for a further $2\frac{1}{2}$ hours at 80 km/h to its destination at town Z. A second coach leaves town Z that the first coach leaves X, and runs express to town X, completing its journey in $5\frac{1}{2}$ hours.

- **a** Construct functions that describe the distance, d km, from X of each coach at time t, stating the domain, range and rule for each.
- **b** Calculate the distance, from X, at which the two coaches pass each other.

30 An article depreciates by 5% of its original cost each year. If the original cost was \$8000, determine an expression for the value, V, of the item n years after purchase.

31 The organisers of a sporting event know that, on average, 50 000 people will visit the venue each day. They are presently charging \$15.00 for an admission ticket. Each time in the past when they have raised the admission price, an average of 2500 fewer people have come to the venue for each \$1.00 increase in ticket price. Let x represent a potential increase in price.

- **a** Write the rule for a function which gives the revenue, R, in terms of x.
- **b** Sketch the graph of R against x.
- **c** Determine the price which will maximise the revenue.

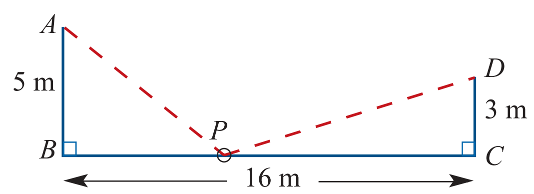
32 A thin wire of length a cm is bent to form the perimeter of a pentagon $ABCDE$ in which $BCDE$ is a rectangle and ABE is an equilateral triangle. Let x cm be the length of CD and let $A(x)$ be the area of the pentagon.

- **a** Determine $A(x)$ in terms of x.
- **b** State the allowable values for x.
- **c** Show that the maximum area is $\dfrac{a^2}{4(6-\sqrt{3})}$ cm^2.

33 The Year 12 students at a school want to get custom-designed rugby tops. The school has 148 Year 12 students. The company that produces the rugby tops charges the following rates: \$45 for the design, \$24 per top for the first 100 orders, and \$20 per top for any further orders over 100.

- **a** Write the rule for a function that describes the total cost, \$$C$, of ordering n tops.
- **b** Determine the total cost of ordering a top for every Year 12 student.
- **c** If the total cost is to be shared by all the Year 12 students, then how much (to the nearest dollar) must each student pay for their top?

34 Let P be a point between B and C on the line BC.
Let $d(x)$ be the distance $(PA + PD)$ m, where x is the distance of P from B.



- **a** **i** Determine an expression for $d(x)$.
  - **ii** Determine the allowable values of x.

b **i** Use a calculator to plot the graph of $y = d(x)$ for a suitable window setting.
ii Determine the value of x if $d(x) = 20$ (correct to two decimal places).
iii Determine the values of x for which $d(x) = 19$ (correct to two decimal places).

c **i** Determine the minimum value of $d(x)$ and the value of x for which this occurs.
ii State the range of the function.

CU

35 Let $f(x) = x^2$.

a Determine the value of k such that the line $y = x$ is tangent to the graph of $y = f(x) + k$.

b Determine the value of h such that the line $y = x$ is tangent to the graph of $y = f(x - h)$.

36 **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. determine the possible values of h if $f(1 + h) = 8$.

b The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Determine the possible values of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quadratic with equation $y = ax^2 + bx$ has a vertex with coordinates $(1, 8)$. Determine the values of a and b.

37 A quadratic function g has rule $g(x) = x^2 + 4x - 6$.

a Determine the value of k for which the equation $g(x) + k = 0$ has only one solution.

b Determine the values of h for which the equation $g(x - h) = 0$ has:

i two positive solutions
ii two negative solutions
iii one positive and one negative solution.

Multiple-choice questions

Technology-free multiple-choice questions

1 For $f(x) = 10x^2 + 2$, $f(2a)$ equals

A $40a^2 + 2$ **B** $2a^2 + 2a$ **C** $100a^2 + 2$ **D** $10a^2 + 2a$

2 The maximal domain of the function f with rule $f(x) = \sqrt{3x + 5}$ is

A $\left(-\frac{5}{3}, \infty\right)$ **B** $(5, \infty)$ **C** $[-5, \infty)$ **D** $\left[-\frac{5}{3}, \infty\right)$

3 The maximal domain of the function f with rule $f(x) = \sqrt{6 - 2x}$ is

A $[3, \infty)$ **B** $(-\infty, 2)$ **C** $(-\infty, 3]$ **D** $[6, \infty)$

4 The range of the relation $x^2 + y^2 > 9$ when $x, y \in \mathbb{R}$ is

A $\mathbb{R}$ **B** $(-\infty, 0]$
C $(3, \infty) \cup (-\infty, -3)$ **D** $(-\infty, 0)$

5 The range of the function with rule $f(x) = x^2 + 1$ for $x \in (-1, 2]$ is

A $(2, 5]$ **B** $(1, 5]$ **C** $[0, 5]$ **D** $[1, 5]$

6 Let $f(x) = 3 - x$ for $x \in (a, b]$. The range of f is

A $(3 - a, 3 - b]$ **B** $(3 - b, 3 - a)$ **C** $(3 - b, 3 - a]$ **D** $[3 - b, 3 - a)$

7 The graph of $y = \frac{2}{x} + 3$ is reflected in the x-axis and then in the y-axis. The equation of the final image is

A $y = -\frac{2}{x} - 3$ **B** $y = \frac{2}{x} + 3$

C $y = \frac{2}{x} - 3$ **D** $y = 2x - 3$

8 For the function with rule $f(x) = x^2$, where $x \in [-1, 5)$, the range is

A $[0, \infty)$ **B** $[0, 25)$ **C** $[1, 25]$ **D** $[0, 5]$

9 Which of the following rules does *not* describe a function?

A $y = \sqrt{4 - x^2}$ **B** $y = 3, x > 0$

C $x = 3$ **D** $y = 3x$

10 Let $f(x) = x^2 - 3x + 4$. Then $\frac{f(3 + h) - f(3 - h)}{2h}$ is equal to

A $-2h$ **B** 3 **C** 6 **D** $3h$

11 Let $f(x) = x^3 - 27$. Then $\frac{f(3 + h) - f(3 - h)}{2h}$ is equal to

A $27h$ **B** $h^2 + 27$ **C** 27 **D** $h^2 - 27$

12 The point $P(1, 7)$ is translated 3 units in the positive direction of the y-axis and then reflected in the x-axis. The coordinates of the final image of P are

A $(-1, -7)$ **B** $(1, -10)$ **C** $(0, 7)$ **D** $(1, 0)$

13 The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated 4 units in the positive direction of the y-axis and then reflected in the y-axis. The coordinates of the final image of P are

A $(-4, -1)$ **B** $(-4, -7)$ **C** $(-4, 1)$ **D** $(4, -7)$

14 A transformation of the plane is defined by $(x, y) \to (3x - 1, y + 2)$. If $(a, b) \to (8, 8)$, then

A $a = 3,\ b = 6$ **B** $a = 7,\ b = 6$

C $a = 6,\ b = 3$ **D** $a = 23,\ b = 10$

15 A transformation of the plane is defined by $(x, y) \to (3x - 1, 2y + 2)$. If $(a, b) \to (a, b)$, then

A $a = \frac{1}{2},\ b = -2$ **B** $a = -1,\ b = 2$

C $a = 5,\ b = -2$ **D** $a = -\frac{1}{2},\ b = 2$

 ISBN 978-1-009-53719-3

Technology-active multiple-choice questions

16 Let $f(x) = 2\sqrt{x} - 4$ for $x \geq 0$. The solution of the equation $f(2x - 1) = 0$ is

A 2.415 **B** 1.414 **C** 2.5 **D** 5.414

17 The function with rule $f(x) = 2(x - 3)^2 + 4$ has domain $1 \leq x \leq 5$. The range of the function is

A $y \geq 5$ **B** $4 \leq y \leq 12$ **C** $5 \leq y \leq 12$ **D** $3 \leq y \leq 4$

18 The graphs of $f(x) = x^2 - 4x$ and $g(x) = \sqrt{x + 4} + 2$ have one point of intersection on the line $y = x$. The coordinates of this point of intersection are

A $(0.79, 0.79)$ **B** $(4, 4)$ **C** $(5, 5)$ **D** $(-0.8, -0.8)$

19 Let $f(x) = -2\sqrt{2 - x} + 3$. The graph of $y = f(x)$ crosses the x-axis at the point

A $(0, 0)$ **B** $(0.75, 0)$ **C** $(-0.5, 0)$ **D** $(-0.25, 0)$

20 A quadratic function has rule $f(x) = 2(x - 3)^2 + 6$. The quadratic function g with rule $g(x) = f(x - 4) + 3$ has vertex at the point with coordinates

A $(-7, 3)$ **B** $(-1, 9)$
C $(7, 9)$ **D** $(0, 10)$

21 Let $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ x & \text{if } 1 < x < 3 \\ -x & \text{if } x \geq 4 \end{cases}$. The number of solutions of the equation $f(x) = 3$ is

A 0 **B** 1 **C** 2 **D** 3

22 Let $f(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ x & \text{if } 1 < x < 3 \\ -x & \text{if } x \geq 4 \end{cases}$. The number of solutions of the equation $f(x) = -5$ is

A 0 **B** 1 **C** 2 **D** 3

23 Let $g(x) = \begin{cases} 2x^2 & \text{if } 0 \leq x \leq 1 \\ x^2 + 1 & \text{if } 1 < x < 3 \\ -x^2 - 2 & \text{if } x \geq 3 \end{cases}$. The range of g is

A $0 \leq y < 10$ or $y \leq -11$ **B** $y \leq 10$
C $y \geq -11$ **D** $y \leq -11$ or $y \geq 0$

24 Let $g(x) = \begin{cases} \dfrac{1}{x - 2} & \text{if } 0 \leq x < 2 \\ \sqrt{x - 2} + 3 & \text{if } 2 \leq x < 3 \\ -\sqrt{x - 3} & \text{if } x \geq 3 \end{cases}$. The range of g is

A $3 \leq y < 4$ or $y \leq 0$ **B** $y \leq 3$
C $-\frac{1}{2} \leq y \leq 3$ **D** $y \leq 0$ or $y \geq 3$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

7

The binomial expansion and cubic functions

Chapter contents

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

Similarly, a polynomial function of degree 4 is called a **quartic function** with general rule

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In Chapter 4 it was shown that all quadratic functions can be written in 'turning point form' and that the graph of a quadratic has one basic form, the parabola.

This is not true of cubic or quartic functions. There is a range of different graph 'shapes' for cubic and quartic functions, depending on the values of the coefficients (a, b, c, d and e).

 ISBN 978-1-009-53719-3 Cambridge University Press

7A The language of polynomials

Learning intentions

- To be able to **add, subtract** and **multiply polynomials**.
- To be able to identify **coefficients** and the **degree** of a polynomial.

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 1

Let $P(x) = x^4 - 3x^3 - 2$. Determine:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 2

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, determine the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$$\begin{aligned} P(1) &= 2(1)^4 - (1)^3 + 2c + 6 \\ &= 2 - 1 + 2c + 6 \\ &= 7 + 2c \end{aligned}$$

Since $P(1) = 21$, $7 + 2c = 21$

$\therefore c = 7$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.

First determine $Q(-1)$ and $Q(2)$ in terms of a and b.

$$\begin{aligned} Q(-1) &= 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20 \\ &= 2 + 1 + a - b + 20 \\ &= 23 + a - b \end{aligned}$$

$$\begin{aligned} Q(2) &= 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20 \\ &= 128 - 8 + 4a + 2b + 20 \\ &= 140 + 4a + 2b \end{aligned}$$

Since $Q(-1) = Q(2) = 0$, this gives

Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.

$$23 + a - b = 0 \qquad (1)$$

$$140 + 4a + 2b = 0 \qquad (2)$$

Divide (2) by 2:

$$70 + 2a + b = 0 \qquad (3)$$

Add (1) and (3):

$$\begin{aligned} 93 + 3a &= 0 \\ a &= -31 \end{aligned}$$

Substitute in (1):

$$b = -8$$

Hence $a = -31$ and $b = -8$.

The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 3

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Determine:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a $P(x) + Q(x)$

$= x^3 - 6x + 3 + x^2 - 3x + 1$

$= x^3 + x^2 - 6x - 3x + 3 + 1$

$= x^3 + x^2 - 9x + 4$

b $P(x) - Q(x)$

$= x^3 - 6x + 3 - (x^2 - 3x + 1)$

$= x^3 - 6x + 3 - x^2 + 3x - 1$

$= x^3 - x^2 - 6x + 3x + 3 - 1$

$= x^3 - x^2 - 3x + 2$

c $P(x)Q(x) = (x^3 - 6x + 3)(x^2 - 3x + 1)$

$= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1)$

$= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3$

$= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3$

$= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

$$\deg(f \times g) = \deg(f) + \deg(g)$$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 4

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Determine the values of b, c and r.

Solution

$(x - 2)(x^2 + bx + c) + r$

$= x(x^2 + bx + c) - 2(x^2 + bx + c) + r$

$= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r$

$= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r$

Explanation

We first expand the brackets of

$$(x - 2)(x^2 + bx + c) + r$$

ISBN 978-1-009-53719-3

We have

$x^3 + 3x^2 + 2x + 1 = x^3 + (b-2)x^2 + (c-2b)x - 2c + r$

for all real numbers x. This implies

$$b - 2 = 3 \qquad \therefore\ b = 5$$
$$c - 2b = 2 \qquad \therefore\ c = 2b + 2 = 12$$
$$-2c + r = 1 \qquad \therefore\ r = 2c + 1 = 25$$

We now equate coefficients: first the coefficients of x^2, then the coefficients of x, and finally the constants.

Hence $b = 5$, $c = 12$ and $r = 25$. This means that

$$P(x) = (x-2)(x^2 + 5x + 12) + 25$$

Substitute the values for b, c and r into $(x-2)(x^2 + bx + c) + r$.

Summary 7A

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Skill-sheet

Exercise 7A

SF

Example 1

1 Let $P(x) = x^3 - 3x^2 - 2x + 1$. Determine:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

2 Let $P(x) = 8x^3 - 4x^2 - 2x + 1$. Determine:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{1}{2}\right)$

3 Let $P(x) = x^3 + 4x^2 - 2x + 6$. Determine:

a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 2

4 **a** Let $P(x) = x^3 + 5x^2 - ax - 20$. If $P(2) = 0$, determine the value of a.

b Let $P(x) = 2x^3 + ax^2 - 5x - 7$. If $P(3) = 68$, determine the value of a.

c Let $P(x) = x^4 + x^3 - 2x + c$. If $P(1) = 6$, determine the value of c.

d Let $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$. If $P(-1) = P(2) = 0$, find the values of a and b.

e Let $P(x) = x^5 - 3x^4 + ax^3 + bx^2 + 24x - 36$. If $P(3) = P(1) = 0$, determine the values of a and b.

 ISBN 978-1-009-53719-3

Example 3

5 Let $f(x) = x^3 - 2x^2 + x$, $g(x) = 2 - 3x$ and $h(x) = x^2 + x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$

d $3f(x)$ **e** $f(x)g(x)$ **f** $g(x)h(x)$

g $f(x) + g(x) + h(x)$ **h** $f(x)h(x)$

6 Expand each of the following products and collect like terms:

a $(x - 2)(x^2 - 2x + 3)$ **b** $(x - 4)(x^2 - 2x + 3)$ **c** $(x - 1)(2x^2 - 3x - 4)$

d $(x - 2)(x^2 + bx + c)$ **e** $(2x + 1)(x^2 - 4x - 3)$

7 Expand each of the following products and collect like terms:

a $(x - 2)(x - 1)(x - 3)$ **b** $(2x - 3)(x - 1)(x + 1)$ **c** $(x - 4)(2x - 1)(3 - x)$

d $(2 - x)(2x + 1)(2x - 4)$ **e** $(x - 5)^2(x - 2)$

Example 4

8 It is known that $x^3 - 7x^2 + 4x + 12 = (x + 1)(x^2 + bx + c)$ for all values of x, for suitable values of b and c.

a Expand $(x + 1)(x^2 + bx + c)$ and collect like terms.

b Determine b and c by equating coefficients.

c Hence write $x^3 - 7x^2 + 4x + 12$ as a product of three linear factors.

SF

7B Choosing r objects from n objects

Learning intentions

- To be able to use $\binom{n}{r}$ to determine the number of ways r objects can be taken from a set of n distinct objects, where order doesn't matter.
- To be able to use **Pascal's triangle** to evaluate $\binom{n}{r}$.

The contents of this section are discussed more fully in Appendix A.

Permutations and factorial notation

In how many ways can six people line up in a queue?

We can draw a diagram for this situation, with each box indicating the number of choices for each position.

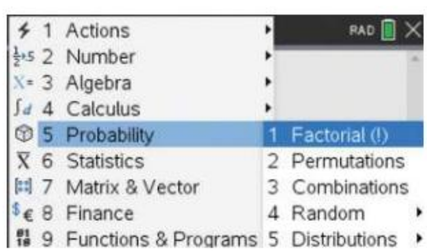
$$\boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1}$$

So the answer is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways. The notation for this is 6!. This is read as six **factorial**. You can use your calculator to calculate factorials.

TI-84

Using the TI-Nspire CX non-CAS

- Enter 6 in a calculator menu, press menu and select **Probability** and then **Factorial(!)**.



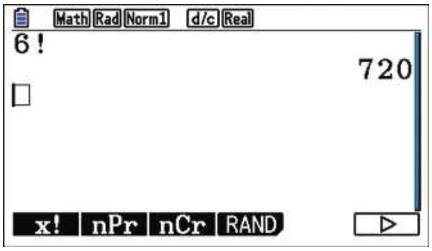
- The factorial symbol can also be accessed using ?!► or using the symbols palette.

 ISBN 978-1-009-53719-3 Cambridge University Press

Using the Casio

Evaluate 6! as follows:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to the **Probability** menu OPTN F6 F3.
- Enter 6, select **x!** F1 and press EXE.



The number of ways to arrange n different objects in a row is $n!$.

If there are six people in a race, in how many ways can the first three places be filled? We can draw a box diagram for this situation, with each box indicating the number of choices for each position.

$$\boxed{6}\ \boxed{5}\ \boxed{4}$$

So the answer is $6 \times 5 \times 4 = 120$ ways.

A **permutation** is an arrangement of elements chosen from a certain set. For example, consider the set $\{a, b, c, d\}$. Some of the permutations of three letters taken from this set include

$$abc, \qquad bac, \qquad cab, \qquad abd$$

Altogether, there are $4 \times 3 \times 2 = 24$ such permutations.

The symbol ${}^nP_r$ is used to denote the number of permutations of r distinct objects chosen from n objects. We see that

$${}^nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}.$$

The notation $\binom{n}{r}$

Let us return to the set of four letters $\{a, b, c, d\}$, but this time we are only interested in choosing three of the letters with no concern for order. The three letters a, b, c can be arranged in six ways:

$$abc, \qquad acb, \qquad bac, \qquad bca, \qquad cab, \qquad cba.$$

Similarly, any group of three letters can be arranged in six ways. The number of permutations of three letters chosen from four letters is ${}^4P_3 = 4 \times 3 \times 2 = 24$. Therefore the number of ways of choosing three letters from four letters is $\dfrac{24}{6} = 4$. We list them here:

$$\{a, b, c\}, \qquad \{a, b, d\}, \qquad \{a, c, d\} \qquad \{b, c, d\}$$

Remember that each set of three elements can be arranged in six ways.

We denote the number of ways of choosing r objects from n objects by $\dbinom{n}{r}$, which is read as 'n **choose** r'.

 ISBN 978-1-009-53719-3

In the example above, we found that

$$\binom{4}{3} = \frac{{}^4P_3}{3!} = \frac{24}{6} = 4.$$

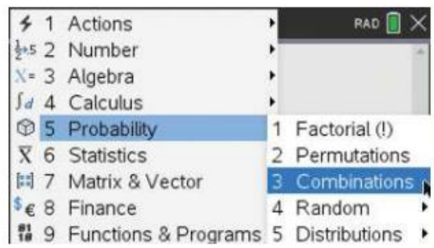
In general, we have

$$\binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}.$$

You can use a calculator to calculate $\binom{n}{r}$. On the calculator the notation is ${}^nC_r$.

Using the TI-Nspire CX non-CAS

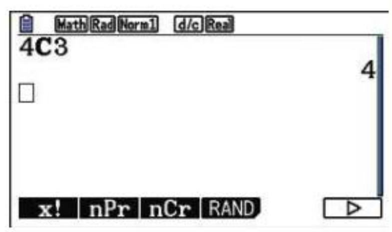
- Press (menu) and select **Probability** and then **Combinations**.



Using the Casio

To evaluate $\binom{4}{3}$:

- Press (MENU) (1) to select **Run-Matrix** mode.
- Go to the **Probability** menu (OPTN) (F6) (F3).
- Enter 4, select **nCr** (F3), enter 3 and press (EXE).



Example 5

Consider a group of six students. In how many ways can a group of:

a two students be selected

b four students be selected?

Solution

a
$$\binom{6}{2} = \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!\,4!} = \frac{6 \cdot 5 \cdot 4!}{2! \cdot 4!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

b
$$\binom{6}{4} = \frac{6!}{4!\,(6-4)!} = \frac{6!}{4!\,2!} = \frac{6 \cdot 5 \cdot 4!}{4! \cdot 2!} = \frac{6 \cdot 5}{2 \cdot 1} = 15$$

 ISBN 978-1-009-53719-3

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore $\binom{6}{2} = \binom{6}{4}$.

More generally:

$$\binom{n}{r} = \binom{n}{n-r}$$

Pascal's triangle

Consider the possibilities when choosing from a small group of objects. In the first instance, consider when the group has only one object, A. We can choose either no objects or one object from this group, each in only one way:

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

Next, consider when the group has two objects, A and B. We can choose either no objects, one object or two objects from this group:

$$\binom{2}{0} = 1 \qquad \binom{2}{1} = 2 \qquad \binom{2}{2} = 1$$

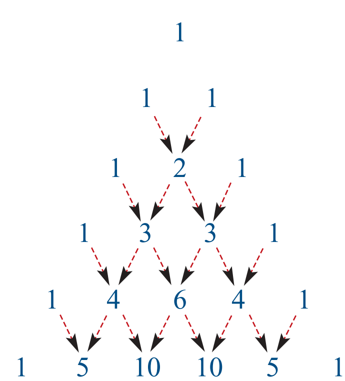
Choosing from three objects we have:

$$\binom{3}{0} = 1 \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3} = 1$$

We can continue in this way to create the following pattern of numbers. We also add a top line for a group of 0 objects.

$n = 0$:	$\binom{0}{0}$	1
$n = 1$:	$\binom{1}{0}$ $\binom{1}{1}$	1 1
$n = 2$:	$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$	1 2 1
$n = 3$:	$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$	1 3 3 1
$n = 4$:	$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$	1 4 6 4 1
$n = 5$:	$\binom{5}{0}$ $\binom{5}{1}$ $\binom{5}{2}$ $\binom{5}{3}$ $\binom{5}{4}$ $\binom{5}{5}$	1 5 10 10 5 1

This triangle of numbers is usually called **Pascal's triangle** after the French mathematician Blaise Pascal, who made ingenious use of it in his studies of probability.



The pattern can be continued indefinitely by noting:

- The first and last numbers in every row are always 1.
- The number in any position in a row is the sum of the two numbers in the row above which are to the left and right of it, as shown.

This is proved by the following result.

Pascal's identity

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $0 < r < n$

Proof Using the formula for $\binom{n}{r}$, we determine

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(n-r)!\,(r-1)!} + \frac{(n-1)!}{(n-1-r)!\,r!}\\ &= \frac{(n-1)!}{(n-r-1)!\,(r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right)\\ &= \frac{(n-1)!}{(n-r-1)!\,(r-1)!}\left(\frac{n}{(n-r)r}\right)\\ &= \frac{n!}{(n-r)!\,r!}\\ &= \binom{n}{r}\end{aligned}$$

Summary 7B

- **Number of ways to choose r objects from n objects**

$$\binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!r!}.$$

- **Pascal's triangle**

$$\begin{array}{ccccccccccc} &&&&&1&&&&& \\ &&&&1&&1&&&& \\ &&&1&&2&&1&&& \\ &&1&&3&&3&&1&& \\ &1&&4&&6&&4&&1& \\ 1&&5&&10&&10&&5&&1 \end{array}$$

- **Pascal's identity**

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $0 < r < n$

Exercise 7B

1 Evaluate $\binom{5}{r}$ for $r = 0, 1, 2, 3, 4, 5$.

SF

2 Evaluate each of the following without the use of your calculator:

a $\binom{7}{1}$ **b** $\binom{6}{5}$ **c** $\binom{12}{10}$ **d** $\binom{8}{5}$ **e** $\binom{100}{99}$ **f** $\binom{1000}{998}$

3 Simplify each of the following:

a $\binom{n}{1}$ **b** $\binom{n}{2}$ **c** $\binom{n}{n-1}$

SF

4 Solve the equation $\binom{n}{2} = 45$

Example 5

5 Consider a group of seven students. In how many ways can a group of:

a three students be selected

b four students be selected?

6 A playlist contains eight of Danielle's favourite songs. How many ways can she:

a arrange three songs in a list

b select three songs for a list?

7 How many ways can five cards be selected from a deck of 52 playing cards?

8 How many subsets of $\{1, 2, 3, \ldots, 10\}$ contain exactly:

a 1 element

b 2 elements

c 8 elements

d 9 elements?

9 Determine the number of selections of four letters that can be made from the word SPHERICAL (order does not matter).

10 Determine the number of selections of four letters that can be made from the word CHEMISTRY (order does not matter).

11 In how many ways can a team of 7 players be selected from a group of 10 players?

12 A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

13 The rows of Pascal's triangle can be numbered $0, 1, 2, \cdots, n$ as shown above.

a List the entries for the row $n = 6$.

b List the entries for the row $n = 7$.

14 From Using Pascal's identity explain why

$$\binom{12}{6} = 2 \times \binom{11}{5}$$

15 The row of Pascal's triangle containing the elements $\binom{10}{k}$, $0 \le k \le 10$ is

$$1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$$

Use this to write the row with elements $\binom{11}{k}$, $0 \le k \le 11$

16 A committee of 5 people is to be chosen from a group of 6 men and 4 women. How many committees are possible if:

a there are no restrictions?

b one particular person must be chosen on the committee?

c one particular woman must be excluded from the committee?

17 Prove that $\binom{n}{r} = \dfrac{n-r+1}{r}\binom{n}{r-1}$.

18 If $\binom{n}{3} = \dfrac{2}{3}\binom{n}{2}$, determine n.

 ISBN 978-1-009-53719-3 Cambridge University Press

7C The binomial theorem

Learning intentions

- To be able to use the **binomial theorem** to expand expressions.

We know that $(a+b)^2 = a^2 + 2ab + b^2$.

This is called an **identity**; it is true for all a and b. If we multiply both sides of this identity by $(a+b)$, then we obtain

$$\begin{aligned}(a+b)^3 = (a+b)(a+b)^2 &= (a+b)(a^2+2ab+b^2)\\ &= a(a^2+2ab+b^2) + b(a^2+2ab+b^2)\\ &= a^3 + 2a^2b + ab^2 + ba^2 + 2ab^2 + b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

So we have a new identity:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

We can continue to build identities in this way:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Notice that

- the expansion of $(a+b)^2$ has three terms and in each term the sum of the indices is 2
- the expansion of $(a+b)^3$ has four terms and in each term the sum of the indices is 3
- the expansion of $(a+b)^4$ has five terms and in each term the sum of the indices is 4.

We are now able to apply the techniques of this chapter to obtain the general result that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

This may look very complicated, but by substituting in $n = 3$ and $n = 4$ we obtain the results we know:

$$\begin{aligned}(a+b)^3 &= a^3 + \binom{3}{1}a^{3-1}b + \binom{3}{2}a^{3-2}b^2 + \binom{3}{3}a^{3-3}b^3\\ &= a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3\\ &= a^3 + 3a^2b + 3ab^2 + b^3\\ (a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\end{aligned}$$

Before proceeding further, we note that the coefficients can be found from Pascal's triangle. We present the triangle in this form so the coefficient of the b^i term can easily be read off.

n	b^0	b^1	b^2	b^3	b^4	b^5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

 ISBN 978-1-009-53719-3 Cambridge University Press

From this we could read off the coefficients of $(a+b)^5$, for example:

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Using combinations, we can now see why this identity holds without having to multiply out the left-hand side.

If we expand $(a+b)^5$, we know that the terms will be of the form

$$c_0a^5 \qquad c_1a^4b \qquad c_2a^3b^2 \qquad c_3a^2b^3 \qquad c_4ab^4 \qquad c_5b^5$$

where the c_i are the coefficients. Writing

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

we can use our knowledge of combinations to determine these coefficients.

For determining the coefficient c_1 of a^4b, the relevant terms when multiplying out are

$$aaaab \qquad aaaba \qquad aabaa \qquad abaaa \qquad baaaa$$

Each of these terms comes from choosing either a or b from each of the five brackets. There are $\binom{5}{1}$ ways of choosing exactly one b from the five brackets. Therefore the coefficient of a^4b is $c_1 = \binom{5}{1} = 5$.

Similarly, there are $\binom{5}{2}$ ways of choosing exactly two bs from the five brackets. Therefore the coefficient of a^3b^2 is $c_2 = \binom{5}{2} = 10$.

Continuing in this way, we can see that the remaining coefficients are $c_3 = \binom{5}{3} = 10$, $c_4 = \binom{5}{4} = 5$ and $c_5 = \binom{5}{5} = 1$.

This argument in the special case of $(a+b)^5$ can be extended to $(a+b)^n$ to give the following general result.

Binomial theorem

For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

Example 6

Use the binomial theorem to expand $(2-3x)^4$.

Solution

We know $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Let $a = 2$ and $b = -3x$. Then

$$\begin{aligned}(2+(-3x))^4 &= 2^4 + 4 \times 2^3(-3x) + 6 \times 2^2(-3x)^2 + 4 \times 2(-3x)^3 + (-3x)^4\\ &= 16 + 4 \times 8 \times (-3)x + 6 \times 4 \times (-3)^2x^2 + 4 \times 2 \times (-3)^3x^3 + (-3)^4x^4\\ &= 16 - 96x + 216x^2 - 216x^3 + 81x^4\end{aligned}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 7

Use the binomial theorem to expand $(2 - 3x^2)^4$.

Solution

We know $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. Let $a = 2$ and $b = -3x^2$. Then

$$\begin{aligned}(2 + (-3x^2))^4 &= 2^4 + 4 \times 2^3(-3x^2) + 6 \times 2^2(-3x^2)^2 + 4 \times 2(-3x^2)^3 + (-3x^2)^4 \\ &= 16 + 4 \times 8 \times (-3)x^2 + 6 \times 4 \times (-3)^2x^4 + 4 \times 2 \times (-3)^3x^6 + (-3)^4x^8 \\ &= 16 - 96x^2 + 216x^4 - 216x^6 + 81x^8\end{aligned}$$

Using the formula for the general term in a binomial expansion, we can determine a particular coefficient without needing to complete the entire expansion. This is shown in Example 8.

General term in a binomial expansion

The general term in the expansion of $(a+b)^n$ is

$$\binom{n}{r} a^{n-r}b^r, \quad \text{where } 0 \le r \le n$$

Example 8

For the expansion of $(x - 3y^2)^6$, determine the general term and determine the coefficient of x^3y^6.

Solution

The general term is

$$\binom{6}{r} x^{6-r}(-3y^2)^r = \binom{6}{r} \times (-3)^r x^{6-r} y^{2r}, \quad \text{where } 0 \le r \le 6$$

To determine the coefficient of x^3y^6, we want $6 - r = 3$ and $2r = 6$. Therefore $r = 3$.

Thus the coefficient of x^3y^6 is $\binom{6}{3} \times (-3)^3 = -540$.

Summary 7C

- **Pascal's triangle**

n	b^0	b^1	b^2	b^3	b^4	b^5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

 ISBN 978-1-009-53719-3

- **Binomial theorem**

  For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

- **General term in a binomial expansion**

  The general term in the expansion of $(a+b)^n$ is

$$\binom{n}{r}a^{n-r}b^r, \quad \text{where } 0 \le r \le n$$

Exercise 7C

SF

Example 6

1 Expand each of the following using the binomial theorem:

a $(x+2)^4$ **b** $(2x+1)^4$ **c** $(2x-3)^4$ **d** $(3x-1)^3$
e $(1-2x)^4$ **f** $(1-2x)^5$ **g** $(1-3x)^5$ **h** $(3-2x)^4$

Example 7

2 Expand and simplify each of the following using the binomial theorem:

a $(2x^2+y)^4$ **b** $(5x^2+3y)^4$ **c** $(2x+y^2)^5$ **d** $(x^3-y^4)^3$

Example 8

3 **a** Determine the coefficient of x^8 in the expansion of $(x-2)^{10}$.
b Determine the coefficient of x^5 in the expansion of $(x-2)^8$.
c Determine the coefficient of x^3 in the expansion of $(x-3)^6$.
d Determine the coefficient of x^8 in the expansion of $(x-4)^{10}$.

4 Determine in ascending powers of x the first four terms of $(1+4x)^6$.

5 Determine in ascending powers of x the first four terms of $(1-3x)^7$.

6 Determine in ascending powers of x the first four terms of $(1+x^2)^8$.

7 State the number of terms in the expansion of $(x+2y)^7$ and give the first and last terms. (Assume that the terms are ordered by increasing powers of y.)

8 Write down an expression for the indicated term:
a in the expansion of $(x+y)^{35}$, the term containing x^{20}
b in the expansion of $(5x-3y)^{12}$, the term containing x^3

CF

9 Determine the value of c if, in the expansion of $(cx+2)^3$, the coefficient of x is 24.

10 Determine the general term in the expansion of $(x^2-y)^5$.

11 Determine the value of k if, in the expansion of $(5x-k)^4$, the constant term is 81.

12 In the expansion of $(2+3x)^n$ the coefficients of x^3 and x^4 are in the ratio $8:15$. Determine the value of n.

13 In the expansion of $(1+2x)^n$ the coefficients of x^3 and x^4 are in the ratio $1:3$. Determine the value of n.

 ISBN 978-1-009-53719-3

7D Solving cubic equations

Learning intentions

- To be able to solve equations involving **factorised cubic polynomials.**

In order to solve a cubic equation, the first step is often to factorise. We start with an example of a cubic already written in factorised form.

Example 9

Solve $(x-2)(x+1)(x+3)=0$.

Solution

Using the null factor theorem, $(x-2)(x+1)(x+3)=0$ implies

$$x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x+3=0$$

Thus the solutions are $x=2, -1$ and -3.

Explanation

In this example, the cubic has already been factorised.

In the following example, a common factor of x is first taken out.

Example 10

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$

b $x^3 + 2x^2 - 10x = 0$

Solution

a
$$\begin{aligned} 2x^3 - x^2 - x &= 0 \\ x(2x^2 - x - 1) &= 0 \\ x(2x+1)(x-1) &= 0 \end{aligned}$$
$\therefore\ x = 0$ or $x = -\frac{1}{2}$ or $x = 1$

b
$$\begin{aligned} x^3 + 2x^2 - 10x &= 0 \\ x(x^2 + 2x - 10) &= 0 \\ x(x^2 + 2x + 1 - 11) &= 0 \\ x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) &= 0 \end{aligned}$$
$\therefore\ x = 0$ or $x = -1 + \sqrt{11}$ or $x = -1 - \sqrt{11}$

In the following example, grouping is used to factorise.

Example 11

Solve each of the following equations for x:

a $x^3 - 4x^2 - 11x + 44 = 0$

b $x^3 - ax^2 - 11x + 11a = 0$

Solution

a $x^3 - 4x^2 - 11x + 44 = 0$

$x^2(x-4) - 11(x-4) = 0$

Therefore $(x-4)(x^2-11) = 0$

Hence $x = 4$ or $x = \pm\sqrt{11}$

b $x^3 - ax^2 - 11x + 11a = 0$

$x^2(x-a) - 11(x-a) = 0$

Therefore $(x-a)(x^2-11) = 0$

Hence $x = a$ or $x = \pm\sqrt{11}$

ISBN 978-1-009-53719-3

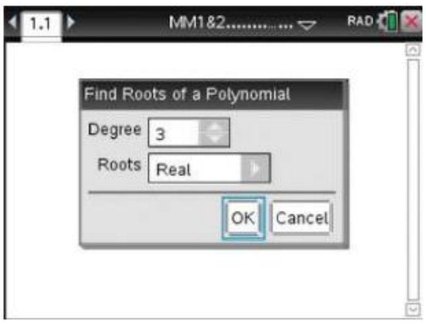
Example 12

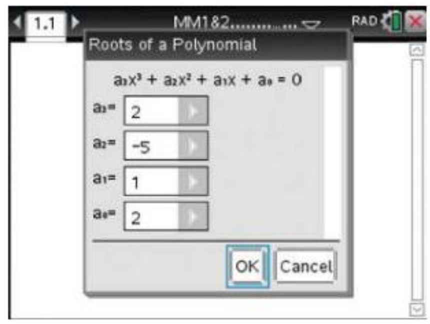
Use your calculator to solve the equation $2x^3 - 5x^2 + x + 2 = 0$.

Using the TI-Nspire CX non-CAS

To solve the equation $2x^3 - 5x^2 + x + 2 = 0$:

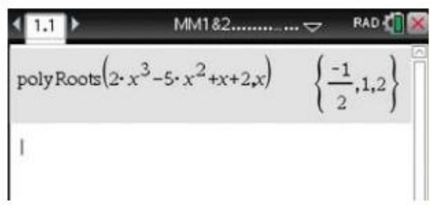
- Use menu > **Algebra** > **Polynomial Tools** > **Find Roots of Polynomial** and edit the pop-up screens as shown.





- The solutions are $x = -\frac{1}{2}$, $x = 1$ and $x = 2$.

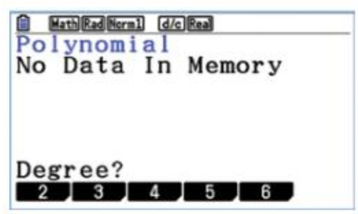
Hint: The **polyRoots** command can also be entered directly from the keypad.



Using the Casio

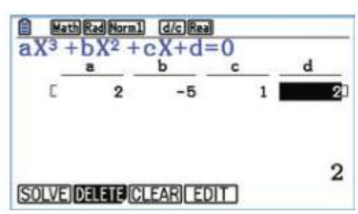
To solve the equation $2x^3 - 5x^2 + x + 2 = 0$:

- Press MENU and then select **Equation** mode by pressing ALPHA X,θ,T.
- Select **Polynomial** F2.
- Choose degree 3 by pressing F2.

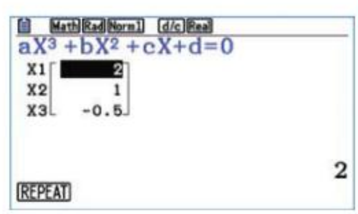


- Enter the coefficients of the equation:

  2 EXE (–) 5 EXE 1 EXE 2 EXE



- Select **Solve** F1.
- The solutions are $x = 2$, $x = 1$ and $x = -\frac{1}{2}$.



 ISBN 978-1-009-53719-3

Summary 7D

Cubic polynomial equations can be solved by first using an appropriate factorisation technique. Factorisation may involve:

- extracting a simple common factor
- using the factor theorem
- polynomial division or equating coefficients
- sum or difference of two cubes
- using the quadratic formula to complete the factorisation.

Exercise 7D

SF

Example 9 **1** Solve each of the following:

a $(x-1)(x+2)(x-4)=0$ **b** $(x-4)^2(x-6)=0$

c $(2x-1)(x-3)(3x+2)=0$ **d** $x(x+3)(2x-5)=0$

Example 10 **2** Solve each of the following:

a $x^3-2x^2-8x=0$ **b** $x^3+2x^2-11x=0$

c $x^3-3x^2-40x=0$ **d** $x^3+2x^2-16x=0$

Example 11 **3** Use grouping to solve each of the following:

a $x^3-x^2+x-1=0$ **b** $x^3+x^2+x+1=0$

c $x^3-5x^2-10x+50=0$ **d** $x^3-ax^2-16x+16a=0$

Example 12 **4** Using your calculator solve each of the following for x:

a $x^3-19x+30=0$ **b** $3x^3-4x^2-13x-6=0$

c $x^3-x^2-2x+2=0$ **d** $5x^3+12x^2-36x-16=0$

e $6x^3-5x^2-2x+1=0$ **f** $2x^3-3x^2-29x-30=0$

Example 12 **5** Using your calculator solve each of the following for x:

a $x^3+x^2-24x+36=0$ **b** $6x^3+13x^2-4=0$

c $x^3-x^2-2x-12=0$ **d** $2x^3+3x^2+7x+6=0$

e $x^3-x^2-5x-3=0$ **f** $x^3+x^2-11x-3=0$

6 Solve each of the following equations for x:

a $2x^3=16x$ **b** $2(x-1)^3=32$ **c** $x^3+8=0$

d $2x^3+250=0$ **e** $1000=\dfrac{1}{x^3}$

7 Factorise each of the following cubic expressions, using your calculator:

a $2x^3-22x^2-250x+2574$ **b** $2x^3+27x^2+52x-33$

c $2x^3-9x^2-242x+1089$ **d** $2x^3+51x^2+304x-165$

7E Cubic functions of the form $f(x) = a(x - h)^3 + k$

Learning intentions

- ► To be able to recognise and sketch the graphs of a **cubic function** of the form $y = a(x - h)^3 + k$.

In Chapter 4 we saw that all quadratic functions can be written in 'turning point form' and that the graphs of all quadratics have one basic form, the parabola. This is not true of cubic functions.

Let us first consider those cubics that are of the form

$$f(x) = a(x - h)^3 + k$$

The graphs of these functions can be formed by simple transformations of the graph of $f(x) = x^3$.

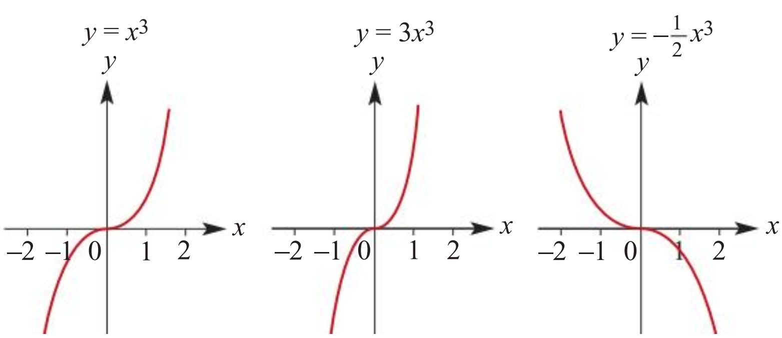
For example, the graph of $f(x) = (x - 1)^3 + 3$ is obtained from the graph of $f(x) = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

Transformations of the graph of $f(x) = x^3$

Dilations from an axis and reflections in an axis

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is reflected in an axis. Note that reflecting in the x-axis and reflecting in the y-axis result in the same graph. This is because $(-x)^3 = -x^3$.

For example:



The implied **domain** of all cubics is $\mathbb{R}$ and the **range** is also $\mathbb{R}$.

Point of inflection

The significant feature of the graph of a cubic of this form is the **point of inflection** (a point of zero gradient). This will be discussed fully in Chapter 15, but for the moment we note that it is the 'flat point' of the graph.

The point of inflection of $y = x^3$ is at the origin $(0, 0)$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Vertical translations

By adding or subtracting a constant term to $y = x^3$, the graph moves either 'up' or 'down'.

The graph of $y = x^3 + k$ is the basic graph moved k units up (for $k > 0$). The point of inflection becomes $(0, k)$. In this case, the graph of $y = x^3$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^3$ is simply the basic graph moved h units to the 'right' (for $h > 0$). The point of inflection is at $(h, 0)$. In this case, the graph of $y = x^3$ is translated h units in the positive direction of the x-axis.

General form

For the graph of a cubic function of the form

$$y = a(x - h)^3 + k$$

the point of inflection is at (h, k).

When sketching cubic graphs of the form $y = a(x - h)^3 + k$, first identify the point of inflection. To add further detail to the graph, determine the x-axis and y-axis intercepts.

Example 13

Sketch the graph of the function $y = (x - 2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units to the right and 4 units up.

Point of inflection is $(2, 4)$.

x-axis intercept:

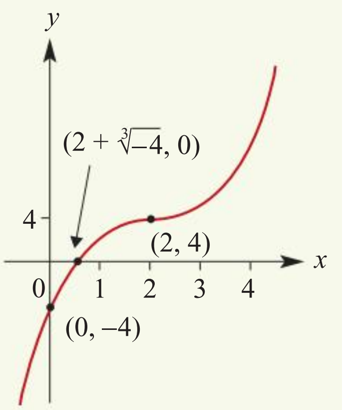
Let $y = 0$

$$\begin{aligned} 0 &= (x - 2)^3 + 4 \\ -4 &= (x - 2)^3 \\ \sqrt[3]{-4} &= x - 2 \\ x &= 2 + \sqrt[3]{-4} \\ &\approx 0.413 \end{aligned}$$

y-axis intercept:

Let $x = 0$

$$\begin{aligned} y &= (0 - 2)^3 + 4 \\ y &= -8 + 4 \\ y &= -4 \end{aligned}$$



 ISBN 978-1-009-53719-3

Summary 7E

- The graph of $y = a(x - h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants).
- The implied domain of all cubic functions is $\mathbb{R}$.

Exercise 7E

SF

Example 13

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x + 2)^3 - 1$ **b** $y = (x - 1)^3 - 1$

c $y = (x + 3)^3 + 2$ **d** $y = (x - 2)^3 + 5$

e $y = (x + 2)^3 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^3 + 3$ **b** $y = 2(x - 3)^3 + 2$

c $3y = x^3 - 5$ **d** $y = 3 - x^3$

e $y = (3 - x)^3$ **f** $y = -2(x + 1)^3 + 1$

g $y = \frac{1}{2}(x - 3)^3 + 2$

7F Graphs of factorised cubic functions

Learning intentions

▶ To be able to recognise and sketch the graphs of a **cubic function** of the form $y = (x - x_1)(x - x_2)(x - x_3)$

The general cubic function written in **polynomial form** is

$$y = ax^3 + bx^2 + cx + d$$

There is a variety of graph shapes for cubic functions, depending on the values of the coefficients. The graph of a cubic function is not necessarily a simple transformation (dilations, translations, reflections) of the graph of $y = x^3$.

All cubics have at least one x-axis intercept. We have seen that cubic functions of the form $f(x) = a(x - h)^3 + k$ have only one x-axis intercept, but these are not the only cubic functions with one x-axis intercept. Some cubic functions have two and others have three.

The y-axis intercept is easily found by letting $x = 0$, and it is the point $(0, d)$.

When sketching the graphs of cubics which are not of the form $f(x) = a(x - h)^3 + k$, begin by determining the x-axis intercepts.

In the following example, the cubic is in factored form already.

 ISBN 978-1-009-53719-3

Example 14

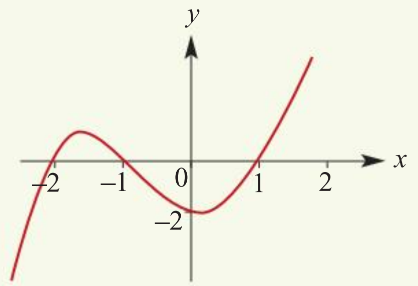
Sketch the graph of $y = (x - 1)(x + 2)(x + 1)$. Do not give coordinates of turning points.

Solution

To determine the x-axis intercepts, let $y = 0$.

Then $0 = (x - 1)(x + 2)(x + 1)$
$\therefore x - 1 = 0$ or $x + 2 = 0$ or $x + 1 = 0$
$\therefore$ x-axis intercepts are 1, -1 and -2.

To determine the y-axis intercept, let $x = 0$.
Then $y = (0 - 1)(0 + 2)(0 + 1) = -2$.
$\therefore$ y-axis intercept is -2.



Check the following by substituting values:

- When $x > 1$, $y > 0$.
- When $-1 < x < 1$, $y < 0$.
- When $-2 < x < -1$, $y > 0$.
- When $x < -2$, $y < 0$.

(Notice how the sign of the y-value changes from one side of an x-axis intercept to the other.) Finally, consider what happens to the graph 'beyond' the x-axis intercepts:

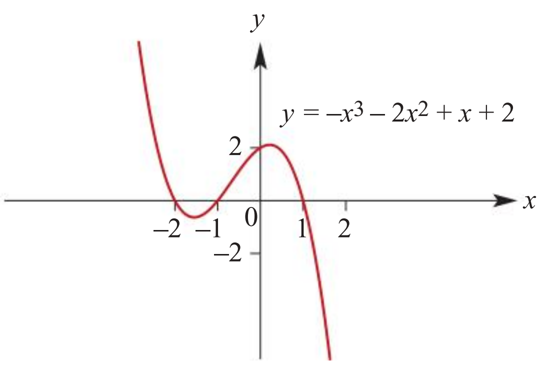
- For $x > 1$, $y > 0$ and as x increases y increases.
- For $x < -2$, $y < 0$ and as x decreases y decreases.

The polynomial form of the cubic in this example is $y = x^3 + 2x^2 - x - 2$. The coefficient of x^3 is positive. We now see what happens when the coefficient of x^3 is negative.

The graph of the cubic function

$$y = -x^3 - 2x^2 + x + 2$$

is the reflection in the x-axis of the graph of the cubic function considered in Example 14.



- When $x > 1$, $y < 0$.
- When $-1 < x < 1$, $y > 0$.
- When $-2 < x < -1$, $y < 0$.
- When $x < -2$, $y > 0$.

- For $x > 1$, $y < 0$ and as x increases y decreases.
- For $x < -2$, $y > 0$ and as x decreases y increases.

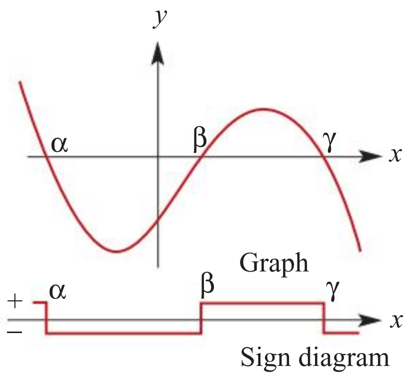
At this stage the location of the turning points is unspecified. However, it is important to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. How to determine the exact values of the coordinates of the turning points will be shown later in this book.

 ISBN 978-1-009-53719-3

Sign diagrams

A sign diagram is a number-line diagram which shows when an expression is positive or negative.

The following is a sign diagram for a cubic function, the graph of which is also shown.



Using a sign diagram requires that the factors, and the x-axis intercepts, be found. The y-axis intercept and sign diagram can then be used to complete the graph.

This procedure is shown in Example 15.

Example 15

Sketch the graph of $y = (x + 1)(x - 2)(x + 3)$.

Solution

.

The x-axis intercepts are -1, 2 and -3.

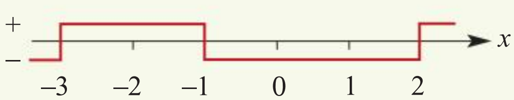
When $x < -3$, y is negative.

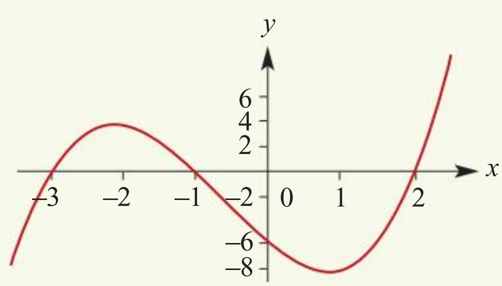
When $-3 < x < -1$, y is positive.

When $-1 < x < 2$, y is negative.

When $x > 2$, y is positive.

This gives the sign diagram.





ISBN 978-1-009-53719-3

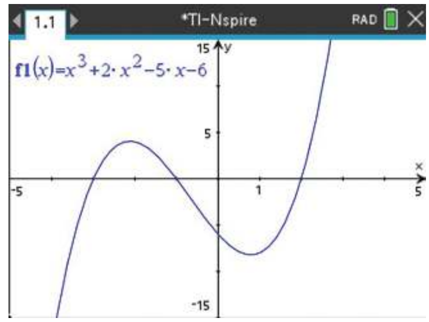
Cambridge University Press

TI-84

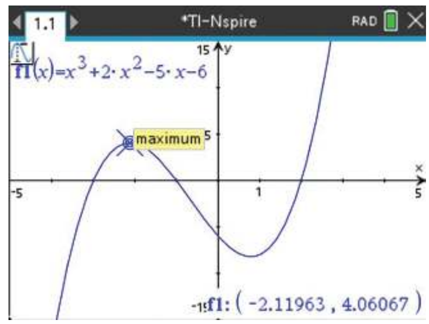
Using the TI-Nspire CX non-CAS

To add detail to the graph, the coordinates of the turning points can be found with a calculator.

- Enter $f1(x) = x^3 + 2x^2 - 5x - 6$ in a **Graphs** application.
- Choose a suitable window (menu > **Window/Zoom** > **Window Settings**).



- Use menu > **Analyze Graph** > **Maximum**.
- Move the cursor to the left of the point (lower bound), click, move to the right of the point (upper bound) and click to display the coordinates.
- Use menu > **Analyze Graph** for other points of interest.



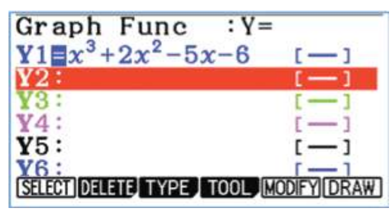
Note: Alternatively, use menu > **Trace** > **Graph Trace** to determine the coordinates of the two turning points. A label will appear near a turning point to indicate that the calculator has found a local maximum or a local minimum.

Using the Casio

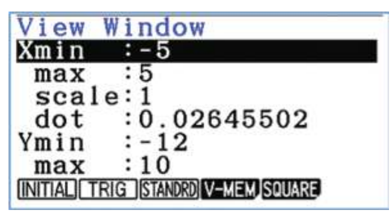
To add detail to the graph, the coordinates of the turning points can be found with a calculator.

- Press MENU 5 to select **Graph** mode.
- Enter the equation $y = x^3 + 2x^2 - 5x - 6$ in $Y1$:

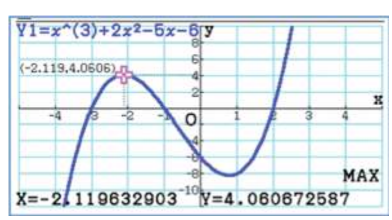
  X,θ,T ∧ 3 ▶ + 2 X,θ,T x^2
  − 5 X,θ,T − 6 EXE



- Adjust the View Window SHIFT F3 for $-5 \le x \le 5$ and $-12 \le y \le 10$ with y-scale 2.
- Select **Draw** F6.



- To determine the local maximum, go to the **G-Solve** menu SHIFT F5 and select **Maximum** F2.
- For the local minimum, go to **G-Solve** and select **Minimum** F3.



 ISBN 978-1-009-53719-3

Repeated factors

The polynomial function $f(x) = (x - 1)^2(x + 3)$ has a **repeated factor**. In this case $(x - 1)$ is repeated. Since the repeated factor is squared, it is easy to see that the sign of the y-value is the same 'close in' on either side of the corresponding x-axis intercept.

If the factorised cubic has a repeated factor and another linear factor, there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 16

Sketch the graph of $y = x^2(x - 1)$.

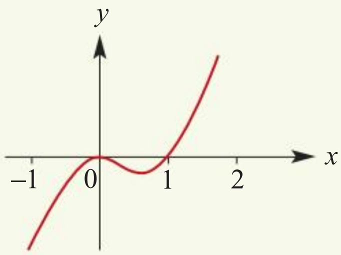
Solution

To determine the x-axis intercepts, let $y = 0$.

Then $x^2(x - 1) = 0$.

Thus the x-axis intercepts are at $x = 0$ and $x = 1$.
Because the repeated factor is x^2, there is also a turning point at $x = 0$.

The y-axis intercept (letting $x = 0$) is at $y = 0$.



Cubics with one x-axis intercept

Cubics of the form $y = (x - a)^3$ have only one x-axis intercept. Some other cubics also have only one x-axis intercept because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 17

Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

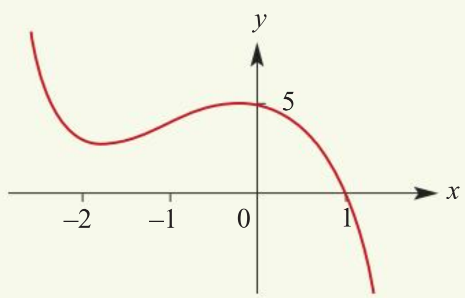
Solution

To determine the x-axis intercept, let $y = 0$.

First, we note that the factor $x^2 + 4x + 5$ cannot be factorised further:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(5) \\ &= -4\end{aligned}$$

$\therefore$ there are no further linear factors.



Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

$\therefore$ x-axis intercept is $x = 1$.

To determine the y-axis intercept, let $x = 0$. Then $y = -(0 - 1)(0^2 + 4(0) + 5) = 5$.

A calculator can be used to determine the turning points $(0, 5)$ and $(-1.82, 2.91)$, where the coordinates of the second point are given to two decimal places.

 ISBN 978-1-009-53719-3

Summary 7F

- The graph of a cubic function can have one, two or three x-axis intercepts.
- If a cubic can be written as the product of three linear factors, $y = a(x-\alpha)(x-\beta)(x-\gamma)$, then its graph can be sketched by following these steps:
  - Determine the y-axis intercept.
  - Determine the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to the power 2, the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

Exercise 7F

Example 14

1 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = x(x-1)(x-3)$ **b** $y = (x-1)(x+1)(x+2)$
c $y = (2x-1)(x-2)(x+3)$ **d** $y = (x-1)(x-2)(x-3)$

Example 15

2 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = x^3 - 9x$ **b** $y = (x-3)^2(x+2)$ **c** $y = (1-x)(x^2-3)$
d $y = (x-3)(x+1)(3x+2)$ **e** $y = (x-1)((2x+1)(3x-1)$ **f** $y = (x-3)(x-2)(2x+1)$

Example 16

3 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.)

a $y = (x-1)(x-2)^2$ **b** $y = x^2(x-4)$ **c** $y = 2(x+1)^2(x-3)$
d $y = x^3 + x^2$ **e** $y = (2x-1)^2(x-1)$ **f** $y = (x-1)^2(x-3)$

Example 17

4 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the axis intercepts. (Do not determine coordinates of turning points.) Use your calculator to help sketch each of them.

a $y = (x-1)(x^2+1)$ (Note: There is no turning point or 'flat point' of this cubic.)
b $y = (x^2+2)(x-4)$ (Note: There are two turning points.)

5 Sketch the graph for each of the following, using a calculator to determine the coordinates of axis intercepts and local maximum and local minimum values:

a $y = -4x^3 - 12x^2 + 37x - 15$ **b** $y = -4x^3 + 19x - 15$
c $y = -4x^3 + 0.8x^2 + 19.8x - 18$ **d** $y = 2x^3 + 11x^2 + 15x$
e $y = 2x^3 + 6x^2$ **f** $y = 2x^3 + 6x^2 + 6$

6 The graph of f, where $f(x) = x^3 - x^2 - 5x - 3 = (x+1)^2(x-3)$, cuts the x-axis at one point and touches it at another. Determine the values of x at these points.

SF

CF

7G Families of cubic polynomial functions

Learning intentions

- To be able to determine the rule of a cubic function given sufficient information.

In Chapter 2 we considered the information that is necessary to determine the equation of a straight line. In Chapter 4 this was considered for quadratic functions, and in Chapter 5 for rectangular hyperbolas, circles and other types of relations.

Here are some examples of families of cubic polynomial functions:

$y = ax^3,\ a > 0$	The cubic graphs that are dilations from the x-axis of $y = x^3$.
$y = a(x - h)^3 + k,\ a \neq 0$	The cubic graphs that are translations of $y = ax^3$.
$y = a(x - 2)(x + 5)(x - 4),\ a \neq 0$	The cubic graphs with x-axis intercepts 2, −5 and 4.
$y = ax^3 + bx^2 + cx,\ a \neq 0$	The cubic graphs that pass through the origin.

Recall that in this context we call a, b, c, h and k parameters.

Determining rules for cubic polynomial functions

The method used for determining the equation from the graph of a cubic will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x - h)^3 + k$ and the point of inflection (h, k) is given, then only one other point needs to be known in order to determine the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

Example 18

A cubic function has rule of the form $y = a(x - 2)^3 + 2$. The point (3, 10) is on the graph of the function. Determine the value of a.

Solution

$y = a(x - 2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$10 = a(3 - 2)^3 + 2$$
$$8 = a \times 1^3$$
$$a = 8$$

Explanation

In each of these problems we substitute in the given values to determine the unknown.

The coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 19

A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(1, 16)$ and $(2, 30)$ are on the graph of the function. Determine the values of a and b.

Solution

Since $f(1) = 16$ and $f(2) = 30$, we obtain the simultaneous equations

$$16 = a + b \qquad (1)$$

$$30 = a(2)^3 + 2b \qquad (2)$$

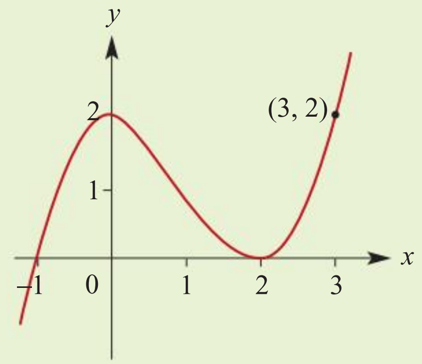
Multiply (1) by 2 and subtract from (2). This gives $-2 = 6a$ and hence $a = -\frac{1}{3}$.

Substitute in (1) to determine $b = \frac{49}{3}$.

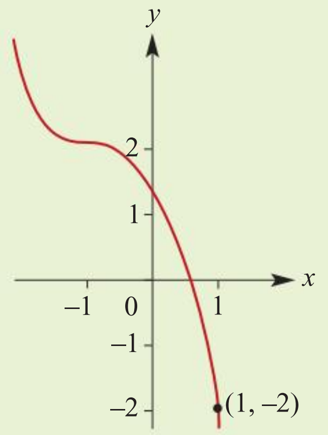
Example 20

Determine the rule for the cubic function shown in each of the following graphs:

a



b



Solution

a $y = a(x + 1)(x - 2)^2$

Put $(3, 2)$ into the equation:

$$2 = a(4)(1)$$

$$\tfrac{1}{2} = a$$

The rule is $y = \frac{1}{2}(x + 1)(x - 2)^2$.

b $y = a(x + 1)^3 + 2$

To determine a, put the known point $(1, -2)$ into the equation:

$$-2 = a(2)^3 + 2$$

$$-4 = 8a$$

$$-\tfrac{1}{2} = a$$

The rule is $y = -\frac{1}{2}(x + 1)^3 + 2$.

Explanation

The x-axis intercepts are -1 and 2, and the graph touches the x-axis at 2. So the cubic has a repeated factor $(x - 2)$.

Therefore the form of the rule appears to be $y = a(x + 1)(x - 2)^2$.

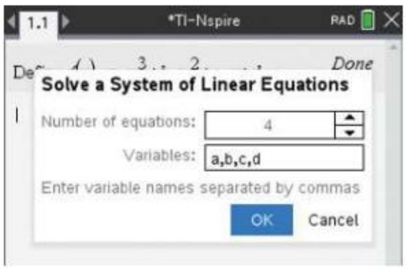
This graph appears to be of the form $y = a(x - h)^3 + k$. The point of inflection is at $(-1, 2)$. Therefore $h = -1$ and $k = 2$.

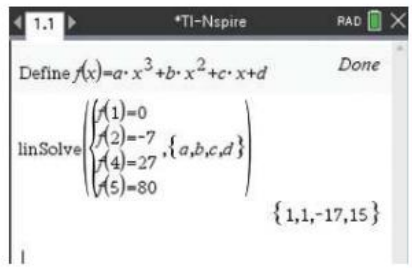
Example 21

A cubic function f has rule $f(x) = ax^3 + bx^2 + cx + d$. If $f(1) = 0$, $f(2) = -7$, $f(4) = 27$ and $f(5) = 80$. Determine the values of a, b, c and d.

Using the TI-Nspire CX non-CAS

- Define the function $f(x) = ax^3 + bx^2 + cx + d$.
- Use (menu) > **Algebra** > **Solve System of Linear Equations**.
- Complete the pop-up screen and enter the equations as shown to determine a, b, c and d.





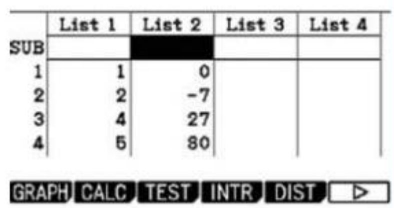
Note: You must enter $a \times x^3 + b \times x^2 + c \times x + d$ using multiplication signs. Otherwise the calculator will interpret ax, bx and cx as variables.

Using the Casio

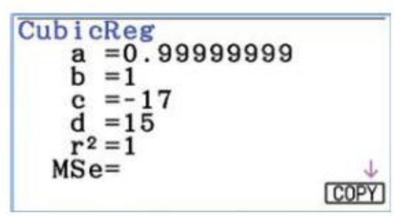
- Press (MENU) (2) to select **Statistics** mode.
- Enter the x-values in List1 and the corresponding y-values in List2:

  (1) (EXE) (2) (EXE) (4) (EXE) (5) (EXE) ▶

  (0) (EXE) ((–)) (7) (EXE) (2) (7) (EXE) (8) (0) (EXE)



- For cubic regression, go to **Calculation** (F2), **Regression** (F3), then $\mathbf{X^3}$ (F4).



- Hence $a = 1$, $b = 1$, $c = -17$ and $d = 15$.

Summary 7G

The rule of a cubic function can be determined if:

- the coordinates of four points on the graph are known
- the form of the function is known to be $f(x) = a(x - \alpha)^2(x - \beta)$, and α and β and the coordinates of one other point are known
- the form of the function is known to be $f(x) = a(x - h)^3 + k$, and the coordinates of the inflection point (h, k) and one other point are known.

There are other sets of information which can be used to find the rule of a cubic function and more of these will be given in Chapter 15.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 7G

Example 18

1 **a** A cubic function has rule of the form $y = a(x-3)^3 + 1$. The point $(4, 12)$ is on the graph of the function. Determine the value of a.

b A cubic function has rule of the form $y = a(x-2)(x+3)(x-1)$. The point $(3, 24)$ is on the graph of the function. Determine the value of a.

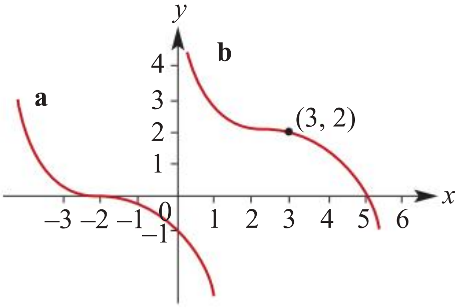
Example 19

c A cubic function has rule of the form $y = ax^3 + bx$. The points $(1, 16)$ and $(2, 40)$ are on the graph of the function. Determine the values of a and b.

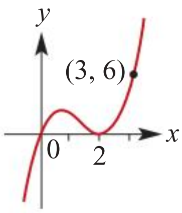
SF

Example 20

2 The graphs shown are similar to the basic curve $y = -x^3$. Determine possible cubic functions which define each of the curves.



3 Determine the equation of the cubic function for which the graph is shown.



CF

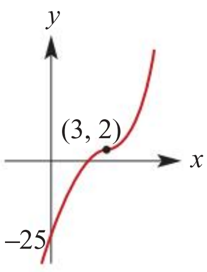
4 Determine a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

5 The graph of a cubic function has x-axis intercepts 1, 3 and -1 and y-axis intercept -6. Determine the rule for this cubic function.

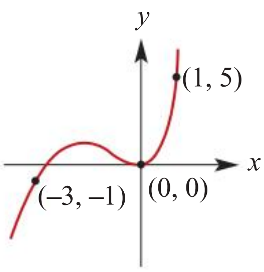
6 A cubic function f has rule $f(x) = (x-3)(x^2+a)$ and $f(6) = 216$. Find the value of a.

7 The graphs below have equations of the form shown. In each case, determine the equation.

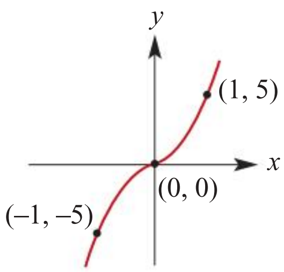
a $y = a(x-h)^3 + k$



b $y = ax^3 + bx^2$

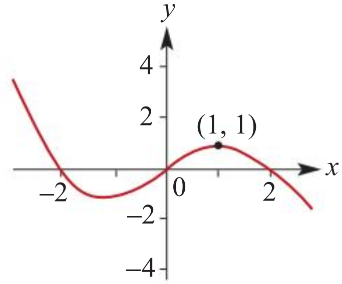


c $y = ax^3$

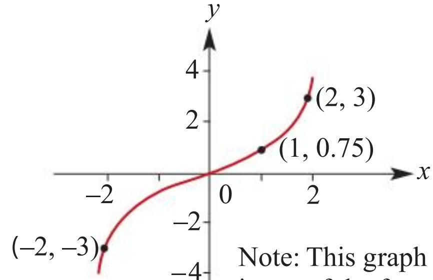


8 Determine the expressions which define the following cubic curves:

a



b



Note: This graph is not of the form $y = a(x - h)3 + k$.

Example 21

9 For each of the following, use a calculator to determine the values of a, b, c, d in the cubic equation $y = ax^3 + bx^2 + cx + d$, given that the following points lie on its graph:

a $(0, 270)$, $(1, 312)$, $(2, 230)$, $(3, 0)$
b $(-2, -406)$, $(0, 26)$, $(1, 50)$, $(2, -22)$
c $(-2, -32)$, $(2, 8)$, $(3, 23)$, $(8, 428)$
d $(1, -1)$, $(2, 10)$, $(3, 45)$, $(4, 116)$
e $(-3, -74)$, $(-2, -23)$, $(-1, -2)$, $(1, -2)$
f $(-3, -47)$, $(-2, -15)$, $(1, -3)$, $(2, -7)$
g $(-4, 25)$, $(-3, 7)$, $(-2, 1)$, $(1, -5)$

7H Applications of polynomial functions

Learning intentions

- To be able to apply cubic functions to solving problems.

In this section, we use a calculator to determine maximum and minimum values of restricted polynomial functions.

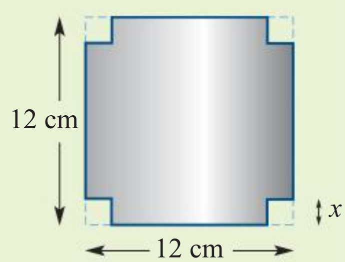
Example 22

A square sheet of tin measures 12 cm × 12 cm.

Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Determine:

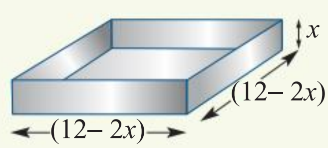
a the values of x for which the volume is 100 cm^3

b the maximum volume.



Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.



The volume of the box is

$$V = x(12 - 2x)^2, \quad 0 \le x \le 6$$

which is a cubic model. We complete the solution using a calculator as follows.

ISBN 978-1-009-53719-3

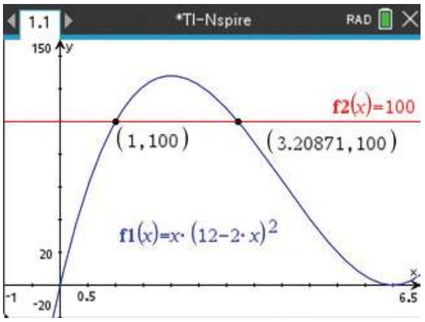
TI-84

Using the TI-Nspire CX non-CAS

Plot the graph of $V = x(12 - 2x)^2$.

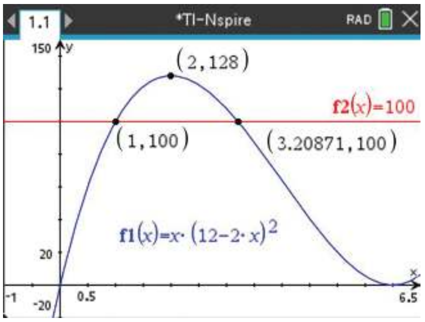
a To determine the values of x for which $V = 100$, plot the graph of $V = 100$ on the same screen and then determine the intersection points using menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

The values are $x = 1$ cm and $x = 3.21$ cm, correct to two decimal places.



b To determine the maximum volume, use either menu > **Trace** > **Graph Trace** or menu > **Analyze Graph** > **Maximum**.

The maximum volume is 128 cm^3.



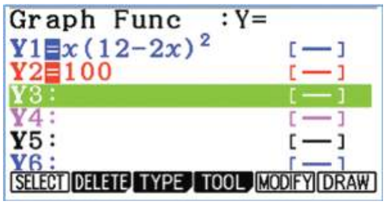
Note: You must enter $x \times (12 - 2x)^2$ using the multiplication sign.

Using the Casio

- In **Graph** mode, enter $V = x(12 - 2x)^2$ in $Y1$ and enter $V = 100$ in $Y2$:

  X,θ,T (1 2 − 2 X,θ,T) x^2 EXE

  1 0 0 EXE



- Adjust the View Window SHIFT F3 for $-2 \le x \le 8$ and $-20 \le y \le 160$ with y-scale 20.
- Select **Draw** F6 to view the graph.

a To determine the values of x for which $V = 100$, go to **G-Solve** and select **Intersection**:

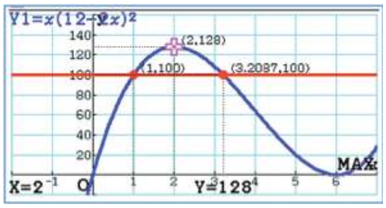
  SHIFT F5 F5 EXE ► EXE

The values are $x = 1$ cm and $x = 3.21$ cm, correct to two decimal places.

b To determine the maximum volume, go **G-Solve** and select **Maximum**:

  SHIFT F5 F2 EXE EXE

The maximum volume is 128 cm^3.



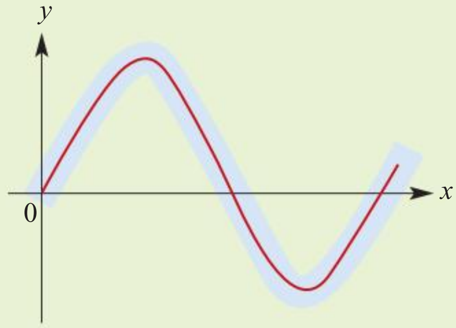
 ISBN 978-1-009-53719-3

Example 23

It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes through the points $(0, 0)$, $(100, 22)$, $(150, -10)$, $(200, -20)$.

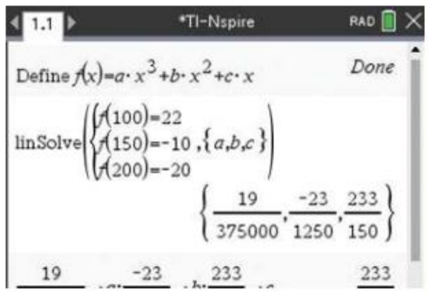
a Determine the equation of the cubic function.

b Determine the maximum deviation of the graph from the x-axis for $x \in [0, 250]$.

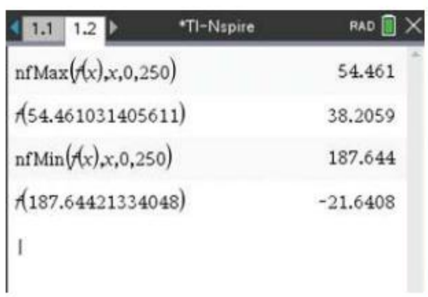


Using the TI-Nspire CX non-CAS

- **Define** $f(x) = ax^3 + bx^2 + cx$.
- Solve using the **Solve System of Linear Equations** command. Enter the equations using the function notation $f(100) = 22$, $f(150) = -10$ and $f(200) = -20$ as shown.
- **Store** these values as a, b and c respectively. To use the store command, press (ctrl) (var).



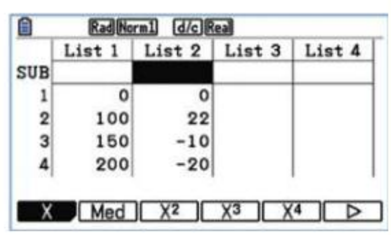
- Use **nfMax( )** from (menu) > **Calculus** > **Numerical Function Maximum** to determine where f obtains its maximum value.
- Use **nfMin( )** from (menu) > **Calculus** > **Numerical Function Minimum** to determine where f obtains its minimum value.



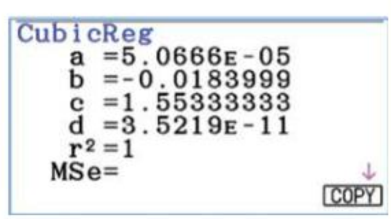
The maximum deviation is 38.21 metres.

Using the Casio

- Press (MENU) (2) to select **Statistics** mode.
- Enter the x-coordinates in List1 and the corresponding y-coordinates in List2.
- For cubic regression, go to **Calculation** (F2), **Regression** (F3), then $\mathbf{X^3}$ (F4).

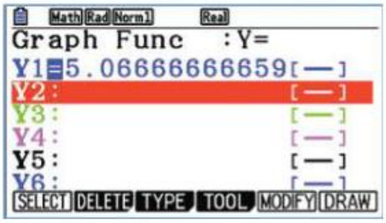


- Correct to three significant figures, the equation is $y = 0.0000507x^3 - 0.0184x^2 + 1.55x$.
- Select **Copy** (F6) (EXE) to copy this equation.

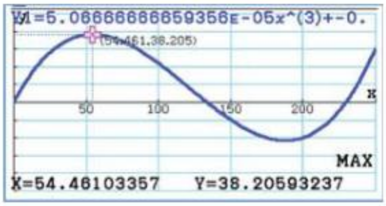


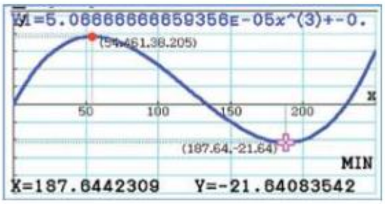
ISBN 978-1-009-53719-3

- Press (MENU) (5) to change to **Graph** mode.
- Press ◄ (EXE) to paste the copied equation into $Y1$.
- Adjust the View Window (SHIFT) (F3) for:
  - $0 \le x \le 250$ with x-scale 50
  - $-50 \le y \le 50$ with y-scale 10.
- Select **Draw** (F6) to view the graph.



- Go to **G-Solve** (SHIFT) (F5) to locate the maximum and minimum points.



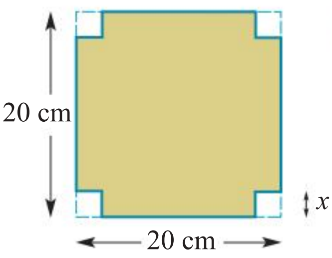


The maximum deviation is 38.21 metres.

Exercise 7H

Example 22, 23

1 A square sheet of cardboard has edges of length 20 cm. Four equal squares of edge length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.



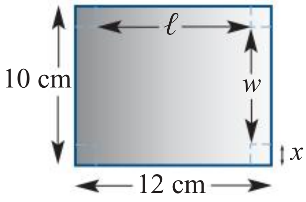
a Determine the length of each edge of the base of the box in terms of x.

b Determine the volume, V cm^3, of the box in terms of x.

c Determine the volume of the box when $x = 5$.

d Determine the values of x for which the volume is 500 cm^3.

2 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from the corners of the sheet as shown in the diagram.



a If the edge of each cut-out square is x cm, express ℓ and w in terms of x.

b Write down a rule for the volume, V cm^3, of the open tray in terms of x.

c Use a calculator to help draw the graph of V against x for suitable values of x.

d Determine the value of V when $x = 1$.

e Determine the values of x for which $V = 50$.

f Determine the maximum volume of the box and the value of x for which this occurs.

CF

 ISBN 978-1-009-53719-3 Cambridge University Press

3 The outside surface area of an open box is 75 cm^2. The base is a square with each edge x cm. Let h cm be the height of the box.

a Determine the surface area of the box in terms of x and h.

b Hence, determine h in terms of x.

c Determine V in terms of x if V cm^3 is the volume of the box.

d Determine V when:

i $x = 2$ **ii** $x = 5$ **iii** $x = 8$

e Given that $V = 59$ when $x = 4$, determine the other value of x for which $V = 59$.

4 In an upright triangular prism, the triangular base has sides of length $5x$ cm, $12x$ cm and $13x$ cm. The height of the prism is h cm. The sum of the lengths of all of its edges is 180 cm.

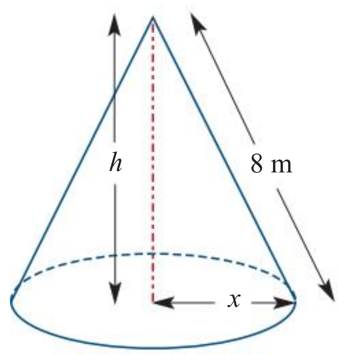
a Determine h in terms of x.

b Determine V in terms of x where V cm^3 is the volume of the prism.

c Determine V when $x = 3$.

d Determine the values of x for which $V = 1200$.

5 The diagram shows a conical heap of gravel. The slant height of the heap is 8 m, the radius of the base x m, and the height h m.



a Express x in terms of h.

b Construct a function which expresses V, the volume of the heap in m^3, in terms of h.

c Use a calculator to help draw the graph of V against h.

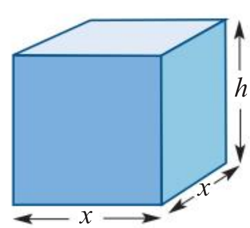
d State the domain for the function.

e Determine the value of V when $h = 4$.

f Determine the values of h for which $V = 150$.

g Determine the maximum volume of the cone and the corresponding value of h.

6 The diagram shows a rectangular prism with a square cross-section. Measurements are in centimetres.



a If the sum of the dimensions, length plus width plus height, is 160 cm, express the height, h, in terms of x.

b Write down an expression for the volume, V cm^3, of the prism in terms of x.

c State the domain.

d Use a calculator to help draw the graph of V against x.

e Determine the value(s) of x for which $V = 50\,000$.

f Determine the maximum volume of the prism.

 ISBN 978-1-009-53719-3 Cambridge University Press

7I Quartic and other polynomial functions

Learning intentions

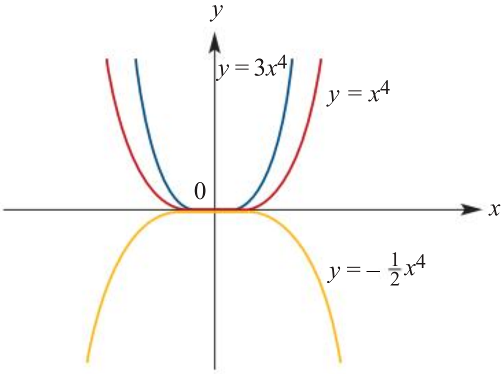
- To be able to recognise properties of polynomials of any degree.

In this section we look at polynomial functions of degree 4 and greater. These functions are not listed in the syllabus for Unit 1 of Mathematical Methods but some knowledge of them is required for Unit 2.

Quartic functions of the form $f(x) = a(x - h)^4 + k$

The graph of $f(x) = (x - 1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is inverted.



The significant feature of the graph of a quartic of this form is the **turning point** (a point of zero gradient). The turning point of $y = x^4$ is at the origin $(0, 0)$.

For the graph of a quartic function of the form

$$y = a(x - h)^4 + k$$

the turning point is at (h, k). When sketching quartic graphs of the form $y = a(x - h)^4 + k$, first identify the turning point. To add further detail to the graph, determine the x-axis and y-axis intercepts.

Example 24

Sketch the graph of the function $y = (x - 2)^4 - 1$.

Solution

Turning point is $(2, -1)$.

x-axis intercepts:

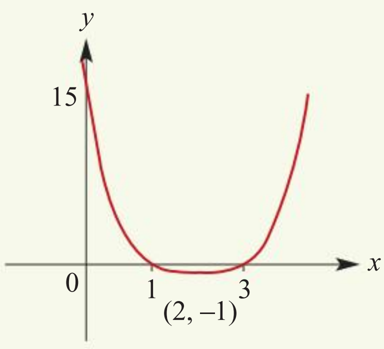
Let $y = 0$

$$\begin{aligned} 0 &= (x-2)^4 - 1 \\ 1 &= (x-2)^4 \\ \pm\sqrt[4]{1} &= x - 2 \\ x &= 2 + 1 \text{ or } x = 2 - 1 \\ x &= 3 \text{ or } x = 1 \end{aligned}$$

y-axis intercept:

Let $x = 0$

$$\begin{aligned} y &= (0-2)^4 - 1 \\ &= 16 - 1 \\ &= 15 \end{aligned}$$



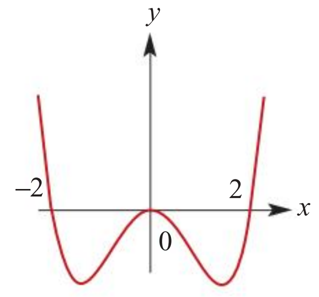
 ISBN 978-1-009-53719-3 Cambridge University Press

The implied **domain** of all quartics is $\mathbb{R}$, but unlike cubics the range is not $\mathbb{R}$.

Other quartic functions

The techniques for graphing quartic functions in general are very similar to those employed for cubic functions. A calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

The graph of $y = 2x^4 - 8x^2$ is shown.



Example 25

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a $x^4 - 8x = 0$

Factorise to obtain

$x(x^3 - 8) = 0$

$\therefore\ x = 0$ or $x^3 - 8 = 0$

Thus $x = 0$ or $x = 2$.

b $2x^4 - 8x^2 = 0$

Factorise to obtain

$2x^2(x^2 - 4) = 0$

$\therefore\ 2x^2 = 0$ or $x^2 - 4 = 0$

Thus $x = 0$ or $x = 2$ or $x = -2$.

c $x^4 - 2x^3 - 24x^2 = 0$

Factorise to obtain $x^2(x^2 - 2x - 24) = 0$

$\therefore\ x^2 = 0$ or $x^2 - 2x - 24 = 0$

i.e. $x = 0$ or $(x - 6)(x + 4) = 0$

Thus $x = 0$ or $x = 6$ or $x = -4$.

Odd and even polynomials

In this subsection we look briefly at odd and even polynomial functions. Knowing that a function is even or that it is odd is very helpful when sketching its graph.

- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis. That is, the graph appears the same after reflection in the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number. In this book we focus on the cases where r is a positive integer or $r = -1$ or $r = \frac{1}{2}$.

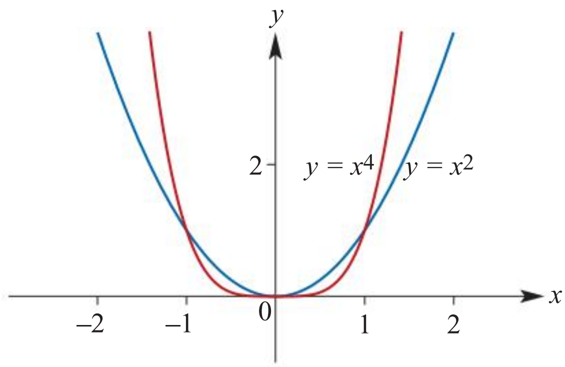
 ISBN 978-1-009-53719-3 Cambridge University Press

Even-degree power functions

The functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of even-degree power functions.

The following are properties of all even-degree power functions:

- $f(-x) = (x)$ for all x
- $f(0) = 0$
- As $x \to \pm\infty$, $y \to \infty$.



Note that, if m and n are positive even integers with $m > n$, then:

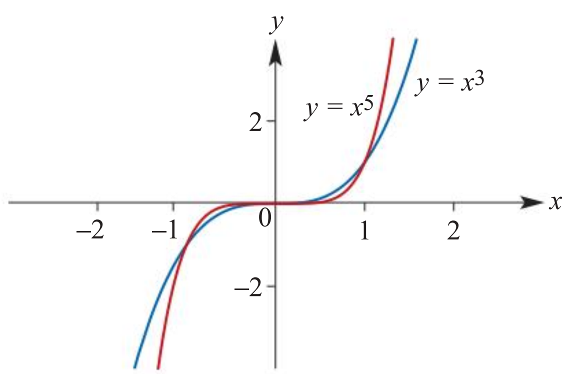
- $x^m > x^n$ for $x > 1$ or $x < -1$
- $x^m < x^n$ for $-1 < x < 1$ with $x \neq 0$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

Odd-degree power functions

The functions with rules $f(x) = x^3$ and $f(x) = x^5$ are examples of odd-degree power functions.

The following are properties of all odd-degree power functions:

- $f(-x) = -f(x)$ for all x
- $f(0) = 0$
- As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.



Note that, if m and n are positive odd integers with $m > n$, then:

- $x^m > x^n$ for $x > 1$ or $-1 < x < 0$
- $x^m < x^n$ for $x < -1$ or $0 < x < 1$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

Example 26

State whether each of the following polynomials is even or odd:

a $f(x) = 6x^4 - 3x^2$

b $g(x) = 3x^5 - x^3 + x$

c $h(x) = x^6 - 3x^2 + 2$

d $m(x) = x^7 - 4x$

Solution

a $f(-x) = 6(-x)^4 - 3(-x)^2$

$= 6x^4 - 3x^2$

$= f(x)$

The function is even.

b $g(-x) = 3(-x)^5 - (-x)^3 + (-x)$

$= -3x^5 + x^3 - x$

$= -g(x)$

The function is odd.

 ISBN 978-1-009-53719-3

c $h(-x) = (-x)^6 - 3(-x)^2 + 2$
$= x^6 - 3x^2 + 2$
$= h(x)$
The function is even.

d $m(-x) = (-x)^7 - 4(-x)$
$= -x^7 + 4x$
$= -m(x)$
The function is odd.

The results of the example are not surprising since:

- The sum of two even functions is even, and any constant multiple of an even function is even.
- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.

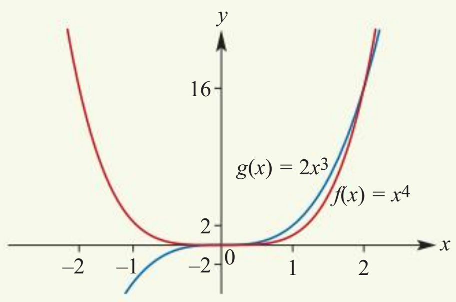
Not every polynomial is even or odd. For example, the polynomial $f(x) = x^2 + x$ is neither.

Example 27

a On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 2x^3$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

Solution

a



b
$$f(x) = g(x)$$
$$x^4 = 2x^3$$
$$x^4 - 2x^3 = 0$$
$$x^3(x - 2) = 0$$
$$x = 0 \text{ or } x = 2$$

c $f(x) \le g(x)$
$x^4 \le 2x^3$
From the graphs and part b, $f(x) \le g(x)$ for $x \in [0, 2]$.

Summary 7I

- The graph of $y = a(x - h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (for h and k positive constants).
- The implied domain of all quartic functions is $\mathbb{R}$.
- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

 ISBN 978-1-009-53719-3

Exercise 7I

Example 24

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x + 2)^4 - 1$ **b** $y = (x - 1)^4 - 1$ **c** $y = (x + 3)^4 + 2$
d $y = (x - 2)^4 + 5$ **e** $y = (x + 2)^4 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^4 + 3$ **b** $y = 2(x - 3)^4 + 2$ **c** $y = x^4 - 16$
d $y = 16 - x^4$ **e** $y = (3 - x)^4$ **f** $y = -2(x + 1)^4 + 1$

Example 25

3 Solve each of the following equations for x:

a $x^4 - 27x = 0$ **b** $(x^2 - x - 2)(x^2 - 2x - 15) = 0$
c $x^4 + 8x = 0$ **d** $x^4 - 6x^3 = 0$
e $x^4 - 9x^2 = 0$ **f** $81 - x^4 = 0$
g $x^4 - 16x^2 = 0$ **h** $x^4 - 7x^3 + 12x^2 = 0$
i $x^4 - 9x^3 + 20x^2 = 0$ **j** $(x^2 - 4)(x^2 - 9) = 0$
k $(x - 4)(x^2 + 2x + 8) = 0$ **l** $(x + 4)(x^2 + 2x - 8) = 0$

4 Use a calculator to help draw the graph of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points to be given correct to two decimal places.)

a $y = x^4 - 125x$ **b** $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$ **d** $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$ **f** $y = 16 - x^4$
g $y = x^4 - 81x^2$ **h** $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$ **j** $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$ **l** $y = (x + 4)(x^2 + 2x - 35)$

Example 26

5 State whether each of the following polynomials is even or odd:

a $f(x) = 5x^6 - 3x^2$ **b** $g(x) = 7x^{11} - x^3 + 2x$
c $h(x) = x^4 - 3x^2 + 2$ **d** $m(x) = x^5 - 4x^3$

Example 27

6 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 2x^2$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

7 **a** On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 9x^2$.
b Solve the equation $f(x) = g(x)$. **c** Solve the inequality $f(x) \le g(x)$.

8 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 4x$.
b Solve the equation $f(x) = g(x)$.
c Solve the inequality $f(x) \le g(x)$.

SF

CF

7J Division of polynomials (Optional)

Learning intentions

- To be able to divide a polynomial by a polynomial of lesser degree.

In order to sketch the graphs of many cubic and quartic functions (as well as higher degree polynomials) it is often necessary to determine the x-axis intercepts. As with quadratics, determining x-axis intercepts can be done by factorising and then solving the resulting equation using the null factor theorem.

All cubics will have at least one x-axis intercept. Some will have two and others three.

We shall first look at the techniques for dividing one polynomial by another. One process for division of polynomials is exactly the same as the long division process for numbers.

Long division with positive integers

We show the process for $274 \div 13$.

$$\begin{array}{r|l} & 21 \\ \cline{2-2} 13 & 274 \\ & \underline{26} \\ & 14 \\ & \underline{13} \\ & 1 \end{array}$$

We have

$$274 = 13 \times 21 + 1$$

Here 274 is the **dividend**, 13 the **divisor**, 21 the **quotient** and 1 the **remainder**.

When we divide the number p by d we obtain two integers, q the quotient and r the remainder, such that

$$p = dq + r \quad \text{and} \quad 0 \leq r < d$$

For example,

$$27 = 4 \times 6 + 3$$

If $r = 0$, then d is a **factor** of p. For example, $24 = 4 \times 6$.

Long division with polynomials

The process for dividing a polynomial by a linear polynomial follows very similar steps. For example, $(x^2 + 7x + 11) \div (x - 2)$ gives

$$\begin{array}{r|l} & x + 9 \\ \cline{2-2} x - 2 & x^2 + 7x + 11 \\ & \underline{x^2 - 2x} \\ & \phantom{x^2 +{}} 9x + 11 \\ & \phantom{x^2 +{}} \underline{9x - 18} \\ & \phantom{x^2 + 9x +{}} 29 \end{array}$$

Divide x^2 by x. This gives x.

Multiply $x - 2$ by x and subtract from $x^2 + 7x + 11$.

This leaves $9x + 11$. Now x into $9x$ goes 9 times.

Multiply $x - 2$ by 9 and subtract from $9x + 11$.

This leaves 29 remainder.

 ISBN 978-1-009-53719-3 Cambridge University Press

Thus $(x^2 + 7x + 11) \div (x - 2) = x + 9$ with remainder 29. We write

$$x^2 + 7x + 11 = (x - 2)(x + 9) + 29$$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$. We can also write the result as

$$\frac{x^2 + 7x + 11}{x - 2} = x + 9 + \frac{29}{x - 2}$$

In this example:

- $x^2 + 7x + 11$ is the dividend
- $x - 2$ is the divisor
- $x + 9$ is the quotient
- 29 is the remainder.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

Note: If $R(x) = 0$, then $D(x)$ is a **factor** of $P(x)$. For example, let $P(x) = x^2 + 6x + 8$ and $D(x) = x + 2$. Then $P(x) = (x + 2)(x + 4) = D(x)(x + 4) + 0$.

Example 28

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r|l} & x^2 - x - 12 \\ \hline x + 2 & x^3 + x^2 - 14x - 24 \\ & x^3 + 2x^2 \\ \hline & \quad -x^2 - 14x - 24 \\ & \quad -x^2 - 2x \\ \hline & \qquad -12x - 24 \\ & \qquad -12x - 24 \\ \hline & \qquad\qquad 0 \end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

Example 29

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

$$
\begin{array}{r|l}
 & 3x^2 + 6x + 13 \\
\cline{2-2}
x - 2 & 3x^3 + 0x^2 + x - 3 \\
 & 3x^3 - 6x^2 \\
\cline{2-2}
 & \quad 6x^2 + x - 3 \\
 & \quad 6x^2 - 12x \\
\cline{2-2}
 & \qquad 13x - 3 \\
 & \qquad 13x - 26 \\
\cline{2-2}
 & \qquad\qquad 23
\end{array}
$$

Explanation

Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

- Divide x, from $x - 2$, into $3x^3$ to get $3x^2$.
- Multiply $3x^2$ by $x - 2$ to give $3x^3 - 6x^2$.
- Subtract from $3x^3 + 0x^2 + x - 3$, leaving $6x^2 + x - 3$.
- Now divide x, from $x - 2$, into $6x^2$ to get $6x$.
- Multiply $6x$ by $x - 2$ to give $6x^2 - 12x$.
- Subtract from $6x^2 + x - 3$, leaving $13x - 3$.
- Divide x into $13x$ to get 13.
- Multiply 13 by $x - 2$ to give $13x - 26$.
- Subtract from $13x - 3$, leaving remainder of 23.

From this example, we have

$$3x^3 + x - 3 = (x - 2)(3x^2 + 6x + 13) + 23$$

Alternatively, we can write

$$\frac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

Equating coefficients to divide

We will briefly outline how to carry out divisions by equating coefficients as shown in the first section of this chapter.

To divide $x^3 - 7x^2 + 5x - 4$ by $x - 3$, first write the identity

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 + bx + c) + r$$

We first determine b, then c and finally r by equating coefficients of the left-hand side and right-hand side of this identity.

x^2 term Left-hand side: $-7x^2$. Right-hand side: $-3x^2 + bx^2 = (-3 + b)x^2$.
Therefore $-3 + b = -7$. Hence $b = -4$.

x term Left-hand side: $5x$. Right-hand side: $12x + cx = (12 + c)x$.
Therefore $12 + c = 5$. Hence $c = -7$.

constant term Left-hand side: -4. Right-hand side: $21 + r$.
Therefore $21 + r = -4$. Hence $r = -25$.

So we can write $x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 - 4x - 7) - 25$.

We do the following example using this method. You can see how the long division has the same arithmetic steps.

 ISBN 978-1-009-53719-3

Example 30

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r|l}
 & \frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\ \hline
2x + 1 & 3x^3 + 2x^2 - x - 2 \\
 & 3x^3 + \frac{3}{2}x^2 \\ \cline{2-2}
 & \qquad \frac{1}{2}x^2 - x - 2 \\
 & \qquad \frac{1}{2}x^2 + \frac{1}{4}x \\ \cline{2-2}
 & \qquad\qquad -\frac{5}{4}x - 2 \\
 & \qquad\qquad -\frac{5}{4}x - \frac{5}{8} \\ \cline{2-2}
 & \qquad\qquad\qquad -1\frac{3}{8}
\end{array}
$$

Explanation

For the alternative method, write the identity

$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:

$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:

$2 = a + 2b$. Therefore $b = \frac{1}{2}(2 - \frac{3}{2}) = \frac{1}{4}$.

Equate coefficients of x:

$-1 = 2c + b$. Therefore $c = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8}$.

Equate constant terms:

$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

Dividing by a non-linear polynomial

To divide by a non-linear polynomial, use the same technique as for a linear polynomial.

Example 31

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$
\begin{array}{r|l}
 & 3x - 2 \\ \hline
x^2 + 0x - 1 & 3x^3 - 2x^2 + 3x - 4 \\
 & 3x^3 + 0x^2 - 3x \\ \cline{2-2}
 & \qquad -2x^2 + 6x - 4 \\
 & \qquad -2x^2 + 0x + 2 \\ \cline{2-2}
 & \qquad\qquad\quad 6x - 6
\end{array}
$$

$\therefore\ 3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

Summary 7J

- When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
- Two methods for dividing polynomials are long division and equating coefficients.

 ISBN 978-1-009-53719-3

Exercise 7J

SF

Example 28

1 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + x^2 - 2x + 3,\ x - 1$
b $2x^3 + x^2 - 4x + 3,\ x + 1$
c $3x^3 - 4x^2 + 2x + 1,\ x + 2$
d $2x^3 - 3x^2 + x - 2,\ x - 3$

Example 29

2 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 3x - 4,\ x + 1$
b $2x^3 + 17x + 15,\ x + 4$
c $x^3 + 4x^2 + 2,\ x + 3$
d $x^3 - 3x^2 + 6x,\ x - 2$

3 For each of the following, divide the polynomial by the accompanying linear expression and hence show that the linear expression is a factor of the polynomial:

a $x^3 - x^2 + 3x + 5,\ x + 1$
b $2x^3 + 6x^2 - 14x - 24,\ x + 4$
c $x^3 - 5x^2 + 18,\ x - 3$
d $3x^3 - 7x^2 - 4x + 12,\ x - 2$

4 Determine the quotient and remainder when the first polynomial is divided by the second:

a $x^3 + 2x^2 - 3x + 1,\ x + 2$
b $x^3 - 3x^2 + 5x - 4,\ x - 5$
c $2x^3 - x^2 - 3x - 7,\ x + 1$
d $5x^3 - 3x + 7,\ x - 4$

Example 30

5 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 6x^2 + 8x + 11,\ 2x + 5$
b $2x^3 + 5x^2 - 4x - 5,\ 2x + 1$
c $2x^3 + 3x^2 - 32x + 15,\ 2x - 1$
d $x^3 - 3x^2 + 1,\ 3x - 1$

6 **a** Write $\dfrac{x^3 + 2x^2 + 5x + 1}{x - 1}$ in the form $P(x) + \dfrac{a}{x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

b Write $\dfrac{2x^3 - 2x^2 + 5x + 3}{2x - 1}$ in the form $P(x) + \dfrac{a}{2x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

Example 31

7 For each of the following, divide the polynomial $P(x)$ by the polynomial $D(x)$:

a $P(x) = 2x^3 - 6x^2 - 4x + 12,\ D(x) = x^2 - 2$
b $P(x) = x^3 - 6x^2 + x - 8,\ D(x) = x^2 + 1$
c $P(x) = 2x^3 - 6x^2 - 4x + 54,\ D(x) = x^2 - 2$
d $P(x) = x^4 - 2x^3 - 7x^2 + 7x + 5,\ D(x) = x^2 + 2x - 1$
e $P(x) = x^4 - x^3 + 7x + 2,\ D(x) = x^2 + 2x - 1$
f $P(x) = 2x^4 + x^3 + 13x + 10,\ D(x) = 2x^2 - x + 4$

ISBN 978-1-009-53719-3

7K Factorisation of polynomials (Optional)

Learning intentions

- To be able to use the **factor theorem** and **remainder theorem**.

Remainder theorem

Since the aim of factorising a cubic is usually to solve an equation or to determine the x-axis intercepts of a graph, the first step is to establish whether a particular linear expression is a factor of the given cubic or not. It is possible to do this without actually doing the division process.

Let $P(x) = x^3 + 3x^2 + 2x + 1$.

Divide $P(x)$ by $x - 2$:

$$\begin{array}{r|l} & x^2 + 5x + 12 \\ \hline x - 2 & x^3 + 3x^2 + 2x + 1 \\ & \underline{x^3 - 2x^2} \\ & \qquad 5x^2 + 2x + 1 \\ & \qquad \underline{5x^2 - 10x} \\ & \qquad\qquad 12x + 1 \\ & \qquad\qquad \underline{12x - 24} \\ & \qquad\qquad\qquad 25 \end{array}$$

The remainder is 25.

$$\begin{aligned} \text{Now } P(2) &= (2)^3 + 3(2)^2 + 2(2) + 1 \\ &= 8 + 12 + 4 + 1 \\ &= 25 \end{aligned}$$

The example suggests that, when $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$. This is in fact true, and the result is called the **remainder theorem**.

It is proved as follows. Suppose that, when the polynomial $P(x)$ is divided by $x - \alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x - \alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$\therefore \quad P(\alpha) = (\alpha - \alpha)Q(\alpha) + R \qquad \therefore \quad R = P(\alpha)$$

i.e. the remainder when $P(x)$ is divided by $x - \alpha$ is equal to $P(\alpha)$. We therefore have

$$P(x) = (x - \alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\dfrac{\alpha}{\beta}\right)$.

 ISBN 978-1-009-53719-3

Example 32

Use the remainder theorem to determine the value of the remainder when:

a $P(x) = x^3 - 3x^2 + 2x + 6$ is divided by $x - 2$

b $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution

a $P(2) = (2)^3 - 3(2)^2 + 2(2) + 6$

$= 8 - 12 + 4 + 6$

$= 6$

The remainder is 6.

Explanation

We apply the remainder theorem by evaluating $P(2)$.

Solution

b $P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4$

$= -\frac{1}{8} + 1 + 4$

$= \frac{39}{8}$

The remainder is $\frac{39}{8}$.

Explanation

We apply the remainder theorem by evaluating $P(-\frac{1}{2})$.

Note: It is not necessary to perform polynomial division to determine the remainder.

Example 33

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$, the remainder is 4. Determine the value of a.

Solution

$P(2) = 8 + 4 + a = 4$

Therefore $a = -8$.

Explanation

We apply the remainder theorem to form a linear equation in a.

Factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

 ISBN 978-1-009-53719-3

Example 34

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence determine the other linear factors.

Solution

Let $P(x) = x^3 - 4x^2 + x + 6$

Then $P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$
$= 0$

Thus $x + 1$ is a factor (by the factor theorem).

Divide by $x + 1$ to determine the other factor:

$$
\begin{array}{r|l}
 & x^2 - 5x + 6 \\ \hline
x + 1 & x^3 - 4x^2 + x + 6 \\
 & \underline{x^3 + x^2} \\
 & \quad -5x^2 + x + 6 \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad\quad 6x + 6 \\
 & \qquad\quad \underline{6x + 6} \\
 & \qquad\qquad\ 0
\end{array}
$$

$\therefore\ x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$
$= (x + 1)(x - 3)(x - 2)$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Explanation

We can use the factor theorem to determine one factor, and then divide to determine the other two linear factors.

Here is an alternative method:

Once we have found that $x + 1$ is a factor, we know that we can write

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$$

By equating constant terms, we have $6 = 1 \times c$. Hence $c = 6$.

By equating coefficients of x^2, we have $-4 = 1 + b$. Hence $b = -5$.

$$\therefore\quad x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

Thinking about the numbers involved in the process of factorisation gives us a way of searching for factors.

For example, consider the polynomial $x^3 - 2x^2 - 5x + 6$. Assume that this polynomial has a linear factor $x - \alpha$, where α is an integer.

Then we can write

$$\begin{aligned} x^3 - 2x^2 - 5x + 6 &= (x - \alpha)(x^2 + bx + c) \\ &= x^3 - (\alpha - b)x^2 - (\alpha b - c)x - \alpha c \end{aligned}$$

By considering the constant term, it can be seen that $\alpha c = -6$. Therefore α divides 6. (Since α is an integer, it follows that b and c are too.)

Thus only the factors of 6 need be considered (i.e. $\pm1, \pm2, \pm3, \pm6$).

Try these in turn until a value for α makes $P(\alpha) = 0$. This process is completed in the following example.

 ISBN 978-1-009-53719-3

Example 35

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

$P(1) = 1 - 2 - 5 + 6 = 0$

$\therefore\ x - 1$ is a factor.

Now divide to determine the other factors:

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x - 1 & x^3 - 2x^2 - 5x + 6 \\ & \underline{x^3 - x^2\phantom{{}-5x+6}} \\ & -x^2 - 5x + 6 \\ & \underline{-x^2 + x\phantom{{}+6}} \\ & -6x + 6 \\ & \underline{-6x + 6} \\ & 0 \end{array}$$

$$\begin{aligned} \therefore\ x^3 - 2x^2 - 5x + 6 &= (x - 1)(x^2 - x - 6) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

Explanation

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

We evaluate the first option, $P(1)$, which in fact equals 0. If $P(1)$ did not equal 0, we would try the other factors of 6 in turn until a zero result is found.

Note: For some cubics, the quadratic factor may not be able to be factorised. Such a cubic has only one linear factor. We considered the consequences for its graph in Section 7F.

Special cases: sums and differences of cubes

Example 36

Factorise $x^3 - 27$.

Solution

Let $\quad P(x) = x^3 - 27$

Then $\quad P(3) = 27 - 27 = 0$

$\therefore\ x - 3$ is a factor.

Divide to determine the other factor:

$$\begin{array}{r|l} & x^2 + 3x + 9 \\ \hline x - 3 & x^3 + 0x^2 + 0x - 27 \\ & \underline{x^3 - 3x^2\phantom{{}+0x-27}} \\ & 3x^2 + 0x - 27 \\ & \underline{3x^2 - 9x\phantom{{}-27}} \\ & 9x - 27 \\ & \underline{9x - 27} \\ & 0 \end{array}$$

$\therefore\ x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

Explanation

The division can also be performed using the method of equating coefficients:

Let $x^3 - 27 = (x - 3)(x^2 + bx + c)$.

Equating constant terms gives $c = 9$.

Equating coefficients of x^2 gives $-3 + b = 0$, and so $b = 3$.

 ISBN 978-1-009-53719-3 Cambridge University Press

In general, if $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 37

Factorise $8x^3 + 64$.

Solution

$$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

Summary 7K

- **Remainder theorem** When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- **Factor theorem**
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic function can be factorised using the factor theorem to determine the first linear factor and then using polynomial division or the method of equating coefficients to complete the process.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Skill-sheet

Exercise 7K

SF

Example 32

1 Without dividing, determine the remainder when the first polynomial is divided by the second:

a $x^3 - x^2 - 3x + 1,\ x - 1$

b $x^3 - 3x^2 + 4x - 1,\ x + 2$

c $2x^3 - 2x^2 + 3x + 1,\ x - 2$

d $x^3 - 2x + 3,\ x + 1$

e $x^3 + 2x - 5,\ x - 2$

f $2x^3 + 3x^2 + 3x - 2,\ x + 2$

g $6 - 5x + 9x^2 + 10x^3,\ 2x + 3$

h $10x^3 - 3x^2 + 4x - 1,\ 2x + 1$

i $108x^3 - 27x^2 - 1,\ 3x + 1$

 ISBN 978-1-009-53719-3 Cambridge University Press

SF

Example 33

2 Determine the value of a for each of the following:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$

b $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$

c $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$

d $x^3 + x^2 + ax + 8$ has remainder 0 when divided by $x - 1$

Example 34

3 Without dividing, show that the first polynomial is exactly divisible by the second polynomial:

a $x^3 - x^2 + x - 1,\ x - 1$

b $x^3 + 3x^2 - x - 3,\ x - 1$

c $2x^3 - 3x^2 - 11x + 6,\ x + 2$

d $2x^3 - 13x^2 + 27x - 18,\ 2x - 3$

4 Determine the value of m if the first polynomial is exactly divisible by the second:

a $x^3 - 4x^2 + x + m,\ x - 3$

b $2x^3 - 3x^2 - (m + 1)x - 30,\ x - 5$

c $x^3 - (m + 1)x^2 - x + 30,\ x + 3$

Example 35

5 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$

b $x^3 + 3x^2 + 3x + 1$

c $6x^3 - 13x^2 + 13x - 6$

d $x^3 - 21x + 20$

e $2x^3 + 3x^2 - 1$

f $x^3 - x^2 - x + 1$

g $4x^3 + 3x - 38$

h $4x^3 + 4x^2 - 11x - 6$

6 Determine the remainder when $(1 + x)^4$ is divided by $x + 2$.

Example 36, 37

7 Factorise each of the following:

a $x^3 - 1$

b $x^3 + 64$

c $27x^3 - 1$

d $64x^3 - 125$

e $1 - 125x^3$

f $8 + 27x^3$

g $64m^3 - 27n^3$

h $27b^3 + 8a^3$

8 Factorise each of the following:

a $x^3 + x^2 - x + 2$

b $3x^3 - 7x^2 + 4$

c $x^3 - 4x^2 + x + 6$

d $6x^3 + 17x^2 - 4x - 3$

CF

9 Determine the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

10 **a** Show that, for any constant a and any natural number n, $x - a$ is a factor of $x^n - a^n$.

b Determine conditions (if any) on n that are required in order that:

i $x + a$ is a factor of $x^n + a^n$

ii $x + a$ is a factor of $x^n - a^n$.

11 The polynomial $P(x)$ has a remainder of 2 when divided by $x - 1$ and a remainder of 3 when divided by $x - 2$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$, i.e. $P(x)$ can be written as $P(x) = (x - 1)(x - 2)Q(x) + ax + b$.

a Determine the values of a and b.

b **i** Given that $P(x)$ is a cubic polynomial with coefficient of x^3 being 1, and -1 is a solution of the equation $P(x) = 0$, determine $P(x)$.

ii Show that the equation $P(x) = 0$ has no other real solutions.

 ISBN 978-1-009-53719-3

Review

Chapter summary

- **Number of ways to choose r objects from n objects**

$$\binom{n}{r} = \frac{^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$$

- **Pascal's identity**

$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$ for $0 < r < n$

- **Binomial theorem**

For each positive integer n,

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

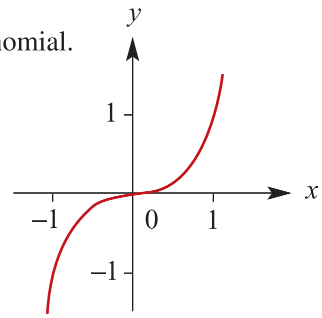
- **General term in a binomial expansion**

The general term in the expansion of $(a+b)^n$ is

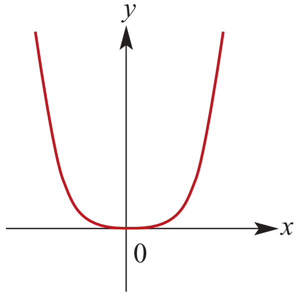
$$\binom{n}{r}a^{n-r}b^r, \quad \text{where } 0 \le r \le n$$

Polynomial and Cubic functions

- The sum, difference and product of two polynomials is a polynomial.
- The basic shape of the curve defined by $y = x^3$ is shown in the graph.



- The graph of $y = a(x-h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants). The point of inflection is at (h, k).
- The basic shape of the curve defined by $y = x^4$ is shown in the graph.

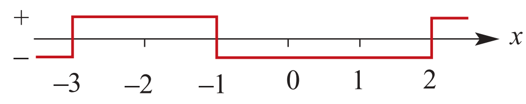


- The graph of $y = a(x-h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants). The turning point is at (h, k).
- Sign diagrams assist in sketching graphs of cubic functions.

e.g.
$$\begin{aligned} y &= x^3 + 2x^2 - 5x - 6 \\ &= (x+1)(x-2)(x+3) \end{aligned}$$

 ISBN 978-1-009-53719-3

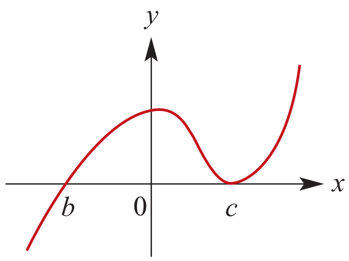
When $x < -3$, y is negative.
When $-3 < x < -1$, y is positive.
When $-1 < x < 2$, y is negative.
When $x > 2$, y is positive.



- To sketch the graph of a cubic function $y = ax^3 + bx^2 + cx + d$, the steps are:
  - **i** Use the factor theorem and division to determine the x-axis intercepts.
  - **ii** The y-axis intercept is d.
  - **iii** Draw a sign diagram.
- Sums and differences of cubes:
  $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
  $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

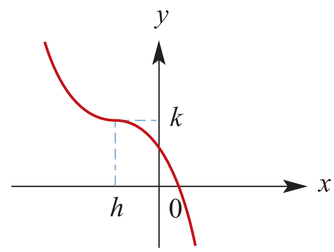
  $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
- Determining equations for given cubic graphs. The following may assist:

i



Form: $y = a(x - b)(x - c)^2$
Assume b and c are known, substitute another known point to calculate a.

ii



Form: $y = a(x - h)^3 + k$
Substitute known values to determine a.

  Alternatively, use the general form $y = ax^3 + bx^2 + cx + d$ and the known points to determine a, b, c and d.
- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

The factor theorem and remainder theorem

- Division of one polynomial by another does not always result in a polynomial.
- **Remainder theorem** When $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$.
  e.g. When $P(x) = x^3 + 3x^2 + 2x + 1$ is divided by $x - 2$, the remainder is
  $$P(2) = (2)^3 + 3(2)^2 + 2(2) + 1 = 25$$
- **Factor theorem** For a polynomial $P(x)$, if $P(\alpha) = 0$, then $x - \alpha$ is a factor. Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.
  e.g. For $P(x) = x^3 - 4x^2 + x + 6$,
  $$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$
  and so $x + 1$ is a factor of $P(x)$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

- 7A **1** I can evaluate polynomials for a given value.

  See Example 1 and Question 1

- 7A **2** I can determine values of unknown coefficients of a polynomial given sufficient values of the polynomial.

  See Example 2 and Question 4

- 7A **3** I can add, subtract and multiply polynomials.

  See Example 3 and Question 5

- 7A **4** I can use equating coefficients to express a polynomial in a different form.

  See Example 4 and Question 8

- 7B **5** I can determine the numbers of ways of selecting r objects from n objects.

  See Example 5 and Question 5

- 7C **6** I can use the binomial theorem to expand brackets of the form $(a + b)^n$.

  See Example 6, Example 7 and Questions 1 and 2

- 7C **7** I can determine the general term of a binomial expansion.

  See Example 8 and Question 3

- 7D **8** I can solve a factorised cubic equation.

  See Example 9 and Question 1

- 7D **9** I can solve a cubic equation for which there is a common factor of x.

  See Example 10 and Question 2

- 7D **10** I can solve a cubic equation which can be factorised by grouping.

  See Example 11 and Question 3

- 7D **11** I can use a calculator to solve a cubic equation.

  See Example 12 and Questions 4 and 5

- 7E **12** I can solve cubic equations of the form $a(x - h)^3 + k = 0$ and sketch the graphs of polynomials with rules of the form $y = a(x - h)^3 + k$.

  See Example 13 and Question 1

7F **13** I can sketch the graph of factorised cubic polynomials of the form $y = (x - x_1)(x - x_2)(x - x_3)$. ☐

See Example 14, Example 15 and Questions 1 and 2

7F **14** I can sketch the graph of factorised cubic polynomials with a repeated factor $y = (x - x_1)^2(x - x_2)$. ☐

See Example 16 and Question 3

7F **15** I can sketch the graph of factorised cubic polynomials with only one x-axis intercept. ☐

See Example 17 and Question 4

7G **16** I can determine the rule for a cubic function given sufficient information about the graph of the function. ☐

See Example 18, Example 19, Example 20, Example 21 and Questions 1, 2 and 9

7H **17** I can use a calculator to determine the maximum and minimum values of restricted cubic functions. ☐

See Example 22, Example 23 and Question 1

7I **18** I can sketch the graph of quartics of the form $y = a(x - h)^4 + k$. ☐

See Example 24 and Question 1

7I **19** I can solve quartic equations where the quartic expression is easily factorised ☐

See Example 25 and Question 3

7I **20** I can determine whether a polynomial is even or odd ☐

See Example 26 and Question 5

7J **21** I can divide a polynomial by a polynomial of lesser degree ☐

See Example 28, Example 29, Example 30, Example 31 and Questions 1, 2, 5 and 7

7K **22** I can use the remainder theorem and factor theorem ☐

See Example 32, Example 33, Example 34, Example 35, Example 36, Example 37 and Questions 1, 2, 3, 5 and 7

 ISBN 978-1-009-53719-3

Review

Technology-free questions

Technology-free short-response questions

SF

1 Sketch the graph of each of the following:

a $y = (x-1)^3 - 2$ **b** $y = (2x-1)^3 + 1$ **c** $y = 3(x-1)^3 - 1$

d $y = -3x^3$ **e** $y = -3x^3 + 1$ **f** $y = -3(x-2)^3 + 1$

g $y = 4(x+2)^3 - 3$ **h** $y = 1 - 3(x+2)^3$

2 Sketch the graph of each of the following:

a $y = (x-1)^4$ **b** $y = (2x-1)^4 + 1$ **c** $y = (x-1)^4 - 1$

d $y = -2x^4$ **e** $y = -3x^4 + 1$ **f** $y = -(x-2)^4 + 1$

g $y = 2(x+1)^4 - 3$ **h** $y = 1 - 2(x+2)^4$

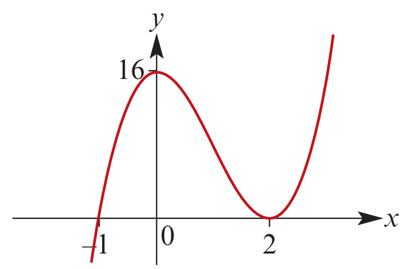
3 Solve each of the following equations:

a $x^3 - 16x = 0$ **b** $(x-2)^3 - 16(x-2) = 0$

c $(x+2)^3 - 16(x+2) = 0$

CF

4 The graph of the cubic function f is shown to the right and $f(-1) = f(2) = 0$ and $f(0) = 16$ and the graph touches the x-axis at $x = 2$. The rule of the function is $f(x) = ax^3 + bx^2 + cx + d$. Determine the values of a, b, c and d.



5 The graph of the function $f(x) = x^3 + ax^2 + bx + 3$ touches the x-axis at the point $(1, 0)$ and passes through the x-axis at the point $(-3, 0)$.

a Determine the values of a and b.

b Let $g(x) = f(x+1)$. Solve the equation $g(x) = 0$.

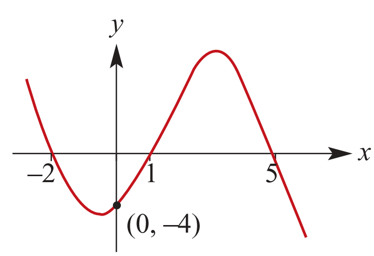
6 Let $P(x) = x^3 + ax^2 - 10x + b$. If $P(-4) = P(3) = 0$, determine the values of a and b.

7 Draw a sign diagram for each of the following and hence sketch the graph:

a $y = (x+2)(3-x)(x+4)$ **b** $y = (x-2)(x+3)(x-4)$

c $y = (x+2)(2x-1)(3x+2)$ **d** $y = (x-3)(x-2)(x+6)$

8 Determine the rule of the cubic for which the graph is shown.



CF

9 Determine a cubic function whose graph touches the x-axis at $x = -4$, passes through the origin and has a value of 10 when $x = 5$.

10 For each of the following, determine a sequence of transformations that takes the graph of $y = x^3$ to the graph of:

a $y = 2(x-1)^3 + 3$ **b** $y = -(x+1)^3 + 2$
c $y = (2x+1)^3 - 2$

11 Let $f(x) = x^3 - kx^2 + 2kx - k - 1$. It can be shown that $f(x) = (x-1)(x^2 + ax + b)$. By equating coefficients determine a and b in terms of k.

You should attempt the following questions only if you have completed sections 7J and 7K.

SF

12 Let $f(x) = 2x^3 + ax^2 - bx + 3$. When $f(x)$ is divided by $x - 2$ the remainder is 15 and $f(1) = 0$.

a Calculate the values of a and b.
b Determine the other two factors of $f(x)$.

13 **a** Use the factor theorem to show that $2x - 3$ and $x + 2$ are factors of $6x^3 + 5x^2 - 17x - 6$. Determine the other factor.
b Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.
c Solve the equation $x^3 + x^2 - 11x - 3 = 8$.

14 **a** Show that $3x - 1$ is a factor of $3x^3 + 2x^2 - 19x + 6$.
b Determine the factors of $3x^3 + 2x^2 - 19x + 6$.

15 Without actually dividing, determine the remainder when the first polynomial is divided by the second:

a $x^3 + 4x^2 - 5x + 1,\ x + 6$ **b** $2x^3 - 3x^2 + 2x + 4,\ x - 2$
c $3x^3 + 2x + 4,\ 3x - 1$

Technology-active short-response questions

SF

16 Solve each of the following equations for x:

a $2x^3 + 3x^2 = 11x + 6$ **b** $x^2(5 - 2x) = 4$ **c** $x^3 + 4x + 12 = 7x^2$

17 The graph of a cubic function f passes through the points $(-3, -37)$, $(-2, -5)$, $(1, 7)$, $(2, 3)$. Determine the rule for f.

CF

18 The volume of a cylinder is given by $V = \pi r^2 h$. It is given that $h + r = 6$.

a Write $V(r)$ in terms of r.
b State the values that r can have.
c Determine $V(3)$.
d Determine the values of r for which $V(r) = 27\pi$.
e Use your calculator to determine the maximum possible volume of the cylinder.

Review

CF

19 If $81x^2$ is one of the terms in the expansion of $(1 + cx)^3$, determine c.

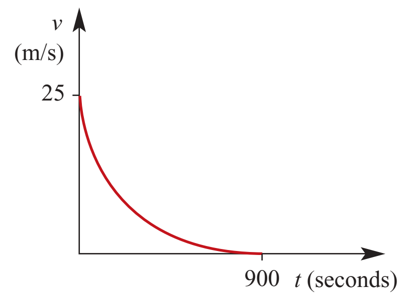
20 Given that $(1 + ax)^n = 1 - 12x + 63x^2 + \cdots$ determine the values of a and n.

21 Given that $(1 + ax)^n = 1 + 12x + 64x^2 + \cdots$ determine the values of a and n.

CU

22 There is a proposal to provide a quicker, more efficient and more environmentally friendly system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can only be driven for a limited time before requiring recharging.

The graph shows the speed v m/s that the taxi will maintain if it is driven at constant speed in such a way that it uses all its energy up in t seconds. The curve is a section of a parabola which touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.



a Construct a rule for v in terms of t.

b If s metres is the distance that a taxi can travel before running out of electrical energy, write down a rule connecting s and t.

c Use a calculator to help draw the graph of s against t.

d Originally the power sources were to be located at 2 km intervals. However there is a further proposal to place them at 3.5 km intervals. Is this new distance feasible?

e With the power sources at 2 km intervals, use your graph to determine approximately both the maximum and minimum speeds recommended for drivers. Explain your answer.

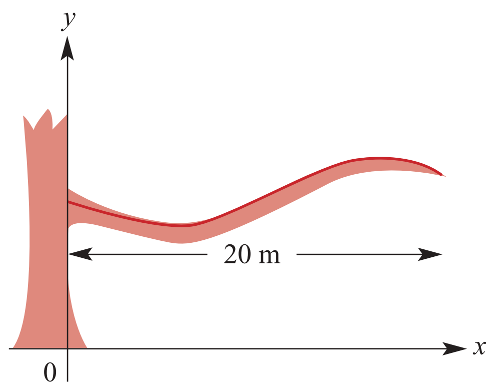
23 It is found that the shape of a branch of a eucalyptus tree can be modelled by a cubic function. The coordinates of several points on the branch are (0, 15.8), (10, 14.5), (15, 15.6), (20, 15).

a The rule for the function is of the form $y = ax^3 + bx^2 + cx + d$. Determine the values of a, b, c and d.

b Determine the coordinates of the point on the branch that is:

i closest to the ground

ii furthest from the ground.

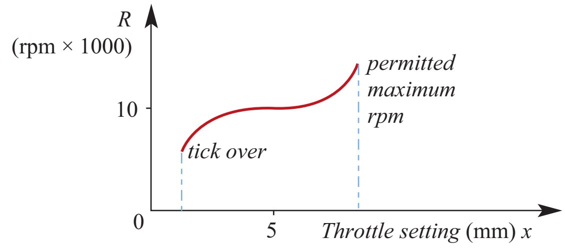


24 The figure shows part of a cubic graph that represents the relationship between the engine speed, $R \times 1000$ rpm, and the throttle setting, x mm from the closed position, for a new engine. It can be seen from the graph that the engine has a ‘flat spot’ where an increase in x has very little effect on R.

 ISBN 978-1-009-53719-3

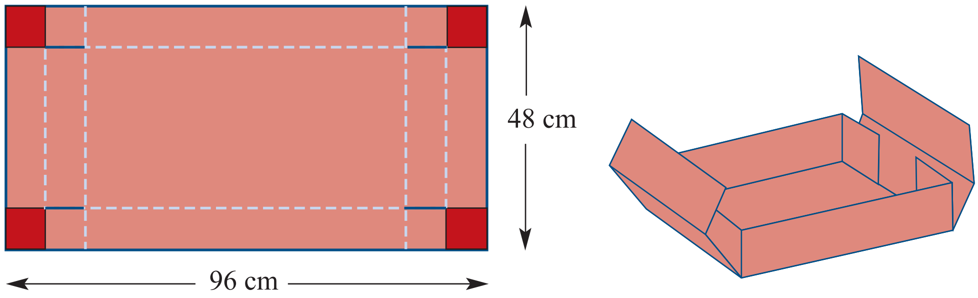
a Develop a cubic expression for R in terms of x of the form $R = a(x - h)^3 + k$, determining k but not a.

b Determine a, if when the graph is extended, it passes through the origin.



c In a proposed modification to the design, the 'flat spot' will occur when $x = 7$ mm. The speed of the engine in this case will be 12 000 rpm when $x = 7$ mm. Assuming that a cubic model still applies and that $R = 0$ when $x = 0$, write down an expression for R as a function of x.

25 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.



a Determine an expression for V, the volume of the box formed.

b Plot a graph of V against x on your calculator.

i What is the domain of the function V?

ii Using your calculator, determine the maximum volume of the box and the value of x for which this occurs (approximate values required).

c Determine the volume of the box when $x = 10$.

d It is decided that $0 \le x \le 5$. Determine the maximum volume possible.

e If $5 \le x \le 15$, what is the minimum volume of the box?

Multiple-choice questions

Technology-free multiple choice questions

1 If $P(x) = x^3 + 3x^2 + x - 3$, then $P(-2)$ equals

A -1 **B** -25 **C** 3 **D** -5

2 If $a > b > c$ and $P(x) = (x - a)^2(x - b)(x - c)$, then $P(x) < 0$ for $x \in$

A $x < b$ **B** $x < c$ **C** $c < x < b$ **D** $b < x < a$

 ISBN 978-1-009-53719-3

Review

3 The image of the graph of $y = x^3$ under a dilation of factor 2 from the y-axis followed by a reflection in the y-axis and then a translation of 4 units in the negative direction of the y-axis is

A $y = -\dfrac{x^3}{8} - 4$ **B** $y = -8x^3 - 4$

C $y = -\dfrac{x^3}{2} + 4$ **D** $y = \dfrac{x^3}{8} + 4$

4 The equation $x^3 + 5x - 10 = 0$ has only one solution. This solution lies between

A -1 and 0 **B** 0 and 1 **C** 1 and 2 **D** 2 and 8

5 Let $P(x) = x^4 + ax^2 - 4$. If $P(\sqrt{2}) = 0$ then the value of a is

A 0 **B** -2 **C** -3 **D** 3

6 Let $P(x) = x^3 + ax^2 + bx - 9$. If $P(1) = 0$ and $P(-3) = 0$, the values of a and b are

A $a = -1,\ b = 3$ **B** $a = 5,\ b = 3$

C $a = -5,\ b = -3$ **D** $a = 0,\ b = 0$

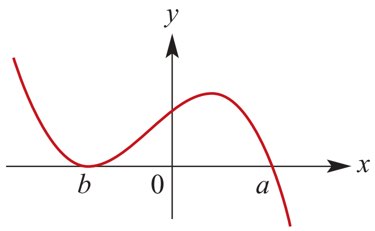
7 Let $P(x) = ax^3 + 2x^2 + 5$. If $P(-1) = 0$, the value of a is

A 7 **B** -1 **C** 3 **D** -7

8 Let $P(x) = x^3 + 2x^2 - 5x + d$. If $P(2) = 10$ then the value of d is

A 4 **B** -10 **C** -4 **D** 3

9 The diagram shows part of the graph of a polynomial function.



A possible equation for the rule of the function is

A $y = (x - a)^2(x - b)$ **B** $y = -(x + b)^2(x - a)$

C $y = (x - b)^2(a - x)$ **D** $y = (x + b)^2(a - x)$

10 The coefficient of $x^{10}y^4$ in the expansion of $(x + y)^{14}$ is

A 1000 **B** 1004 **C** 1001 **D** 3003

11 The expansion of $(1 - x)^5$ is

A $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$ **B** $1 + 5x - 10x^2 + 10x^3 - 5x^4 + x^5$

C $1 - x^5$ **D** $1 - 5x^5$

 ISBN 978-1-009-53719-3 Cambridge University Press

12 If the expansion of $(3 - x)^6$ is written with the terms ordered by increasing powers of x, then the fifth term is

A $-135x^4$ **B** $235x^5$ **C** $135x^4$ **D** $-1215x^4$

13 If the expansion of $(2 + cx)^4$ includes the term $64x^3$, then the value of c is

A 2 **B** 3 **C** 4 **D** 5

14 The solutions of the polynomial equation $ax^3 + bx^2 + cx + d = 0$ are α, β and γ. The product of the three solutions $\alpha\beta\gamma$ is equal to

A $-\dfrac{b}{a}$ **B** $-d$ **C** $-\dfrac{d}{a}$ **D** $-\dfrac{c}{a}$

15 Let $P(x) = x^3 + ax^2 + bx + c = 0$. Also $P(6) = P(-4) = P(2) = 0$. The polynomial $P(x) =$

A $x^3 - 4x^2 - 20x + 48$ **B** $x^3 + 4x^2 - 20x - 48$

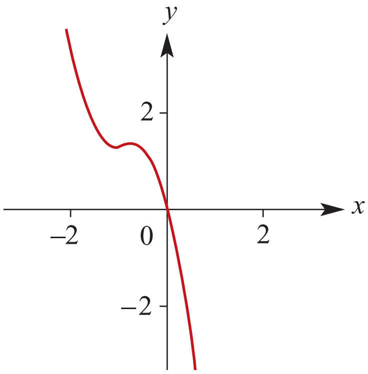
C $x^3 - 48$ **D** $x^3 + 48$

16 Let $P(x) = x^3 + ax^2 + bx + c = 0$. Also $P(6) = P(1 + \sqrt{2}) = P(1 - \sqrt{2}) = 0$. The polynomial $P(x) =$

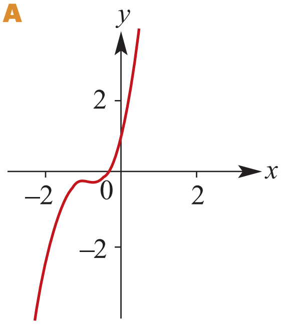
A $x^3 + 6x - 1$ **B** $x^3 + \sqrt{2}x^2 - 6$

C $x^3 - 4x^2 - 13x + 6$ **D** $x^3 - 8x^2 + 11x + 6$

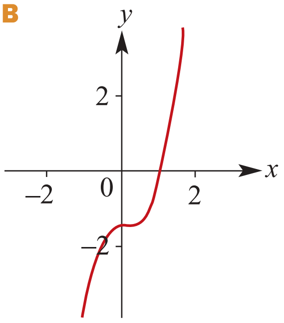
17 The graph of $y = f(x)$ is shown on the right. Which one of the following could be the graph of $y = 1 - f(x)$?



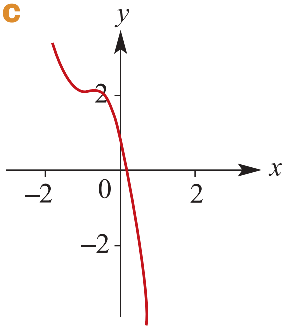
A



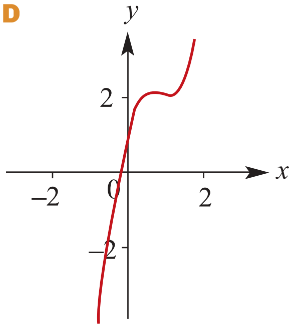
B



C



D



 ISBN 978-1-009-53719-3

Review

Technology-active multiple choice questions

18 The one positive solution of the equation $x^3 - 5x = 3$ is closest to

A 0 **B** 1 **C** 2 **D** 2.5

19 The cubic function f with rule $f(x) = ax^3 + bx^2 + cx + d$ satisfies

$$f(0) = -5, f(1) = -2, f(2) = 7, f(3) = 34.$$

The values of a, b, c and d are:

A $a = 2, b = -3, c = 4, d = -5$
B $a = -2, b = -4, c = 3, d = -5$
C $a = 1, b = -2, c = -3, d = -5$
D $a = 1, b = 3, c = -4, d = -5$

20 The cubic function P with rule $P(x) = x^3 - ax^2 - bx + c$ satisfies

$$P(0) = 18, P(1) = 12, P(3) = 0.$$

The rule for P is

A $P(x) = (x - 3)^2(x - 2)$
B $P(x) = (x - 3)^2(x + 2)$
C $P(x) = (x - 3)(x + 2)^2$
D $P(x) = (x + 3)(x + 2)(x - 1)$

21 Let $f(x) = 2(x - 1)^2(x + 2) + 1$. The equation $f(x) = 0$ has exactly one solution. Let $g(x) = f(x) + d$. A value of d such that the equation $g(x) = 0$ has exactly two solutions is

A 0 **B** 1 **C** −2 **D** −1

22 Let $f(x) = 2x^3 + 5x^2 + 9x - 6$. Then $f(x)$ is positive for:

A $-3 < x < \frac{1}{2}$ **B** $x > \frac{1}{2}$ **C** $x < -\frac{1}{2}$ **D** $-4 < x < 0$

23 Let $f(x) = 2x^3 + 3x^2 - 1$ for $-3 \le x \le 1$. The range of f is

A $-28 \le x \le 4$ **B** $x \ge -28$ **C** $0 \le x \le 4$ **D** $-1 \le x \le 4$

24 Let $f(x) = \begin{cases} x^3 + 4x & x > 4 \\ x^3 - 2x + 4 & -2 < x \le 4 \\ 2x^2 - 1 & x \le -2 \end{cases}$.

The number of solutions of the equation $f(x) = 6$ is

A 0 **B** 1 **C** 2 **D** 3

25 Let $P(x) = (x + 2y)^7$ and $Q(x) = (x + y)^7$. Then the coefficient of x^4y^3 in the polynomial $P(x) - Q(x)$ is

A 63 **B** 245 **C** 525 **D** 127

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

8

Trigonometric functions

Chapter contents

Following on from our study of polynomial, exponential and logarithmic functions, we meet a further three important functions in this chapter. Again we use the notation developed in Chapter 6 for describing functions and their properties.

You have studied trigonometry in earlier years, mainly for angles between 0° and 90°. In this chapter we see how the trigonometry you have studied may be extended to form three new functions: sine, cosine and tangent. We will see that the first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

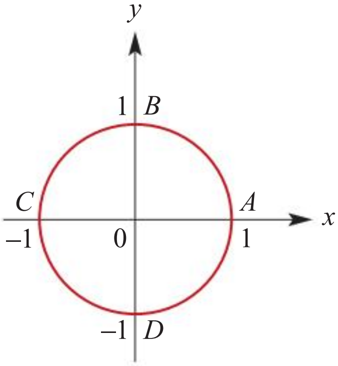
 ISBN 978-1-009-53719-3

8A Measuring angles in degrees and radians

Learning intentions

- To be able to use **radians** and **degrees** for the measurement of angles.
- To be able to convert radians to degrees, and vice versa.

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



$$\begin{aligned}\text{The circumference of the unit circle} &= 2\pi \times 1\\ &= 2\pi \text{ units}\end{aligned}$$

Thus, the distance in an anticlockwise direction around the circle from

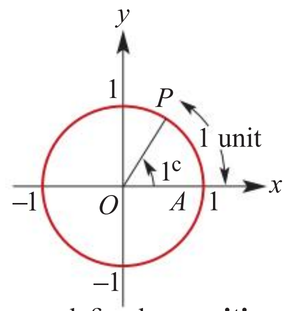
$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

a Convert 30° to radians. **b** Convert $\frac{\pi^c}{4}$ to degrees.

Solution

a $$1^\circ = \frac{\pi^c}{180}$$

$$\therefore \quad 30^\circ = \frac{30 \times \pi}{180} = \frac{\pi^c}{6}$$

Explanation

To convert from degrees to radians, multiply by $\frac{\pi}{180}$ and then simplify by cancelling.

ISBN 978-1-009-53719-3

b $1^c = \dfrac{180^\circ}{\pi}$

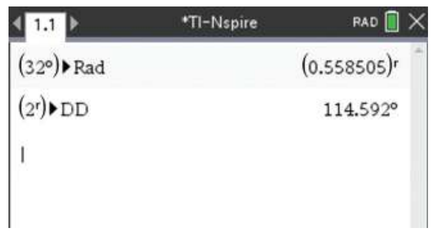
$\therefore \quad \dfrac{\pi^c}{4} = \dfrac{\pi \times 180}{4 \times \pi} = 45^\circ$

To convert from radians to degrees, multiply by $\dfrac{180}{\pi}$ and then simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\dfrac{\pi}{4}$ rather than $\dfrac{\pi^c}{4}$.

Using the TI-Nspire CX non-CAS

To convert 32 degrees to radians, enter 32° ▸ **Rad** as shown.



- The degree symbol ° is found in the symbols palette (ctrl 📖) or the catalog (📖 4).
- The ▸ **Rad** command can be found in the catalog (📖 1 R).

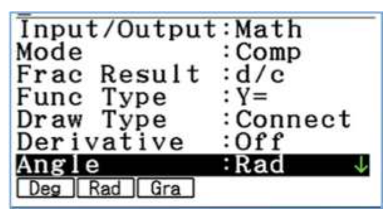
To convert 2 radians to degrees, enter 2^r ▸ **DD** as shown.

- The radian symbol ^{r} is found in the symbols palette (ctrl 📖) or the catalog (📖 4).
- The ▸ **DD** command can be found in the catalog (📖 1 D).

Note: If the calculator is in radian mode, you can convert 32° to radians by simply entering 32° then enter. If the calculator is in degree mode, enter 2^r then enter.

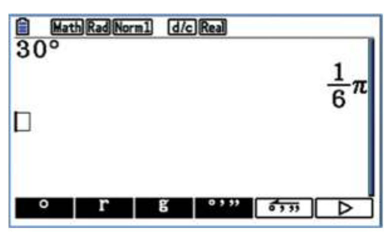
Using the Casio

To change the angle setting:



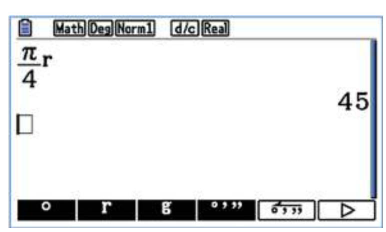
- In **Run-Matrix** mode, go to the set-up screen by pressing SHIFT MENU.
- Use the cursor key ▼ to move down to **Angle**.
- Press F1 to change to **Degrees**, or press F2 to change to **Radians**. Then press EXIT.

To convert 30° to radians:



- In **Run-Matrix** mode, ensure that the angle setting is Radians.
- Enter 30°:
  3 0 OPTN F6 F5 F1 EXE

To convert $\frac{\pi}{4}$ radians to degrees:



- In **Run-Matrix** mode, ensure that the angle setting is Degrees.
- Enter $\frac{\pi}{4}$ radians:
  ▭ SHIFT EXP ▼ 4 ▶ OPTN F6 F5 F2 EXE

 ISBN 978-1-009-53719-3

Summary 8A

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\frac{\pi}{180}$
  - radians to degrees, multiply by $\frac{180}{\pi}$.

Exercise 8A

SF

Example 1a

1 Express the following angles in radian measure in terms of π:

a $60°$ **b** $144°$ **c** $240°$ **d** $330°$ **e** $420°$ **f** $480°$

Example 1b

2 Express, in degrees, the angles with the following radian measures:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{6}$ **d** 0.9π

e $\frac{5\pi}{9}$ **f** $\frac{9\pi}{5}$ **g** $\frac{11\pi}{9}$ **h** 1.8π

3 Use a calculator to convert the following angles from radians to degrees:

a 0.6 **b** 1.89 **c** 2.9 **d** 4.31

e 3.72 **f** 5.18 **g** 4.73 **h** 6.00

4 Use a calculator to express the following in radian measure:

a $38°$ **b** $73°$ **c** $107°$ **d** $161°$

e $84.1°$ **f** $228°$ **g** $136.4°$ **h** $329°$

5 Express, in degrees, the angles with the following radian measures:

a $-\frac{\pi}{3}$ **b** -4π **c** -3π **d** $-\pi$

e $\frac{5\pi}{3}$ **f** $\frac{-11\pi}{6}$ **g** $\frac{23\pi}{6}$ **h** $\frac{-23\pi}{6}$

6 Express each of the following in radian measure in terms of π:

a $-360°$ **b** $-540°$ **c** $-240°$ **d** $-720°$ **e** $-330°$ **f** $-210°$

7 **a** On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{4}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{4}$

b On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{3}$ **ii** $-\frac{\pi}{3}$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{4\pi}{3}$

c On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{6}$ **ii** $-\frac{7\pi}{6}$ **iii** $\frac{13\pi}{6}$ **iv** $\frac{17\pi}{6}$

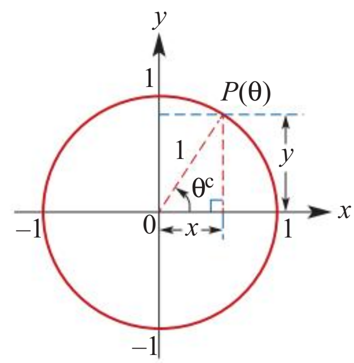
 ISBN 978-1-009-53719-3

8B Defining trigonometric functions: sine and cosine

Learning intentions

- To be able to work with the definitions of the trigonometric functions sine and cosine.

Consider the unit circle. The position of point P on the circle can be described by relating the angle θ to the Cartesian coordinates x and y.



The point P on the circle corresponding to an angle θ is written $P(\theta)$.

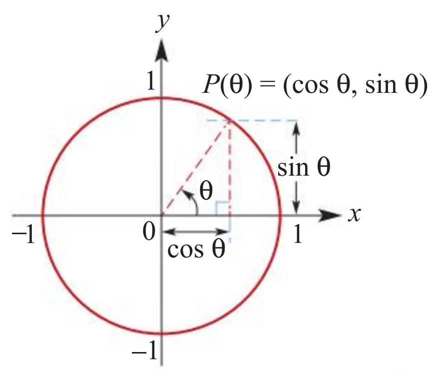
The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:

The x-coordinate of $P(\theta)$ is given by $\quad x = \text{cosine}\ \theta, \quad$ for $\theta \in \mathbb{R}$.

The y-coordinate of $P(\theta)$ is given by $\quad y = \text{sine}\ \theta, \quad$ for $\theta \in \mathbb{R}$.

These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$
$$y = \sin\theta$$



Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.

Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

Example 2

Evaluate $\sin\pi$ and $\cos\pi$.

Solution

After moving through an angle of π, the position is $P(\pi) = (-1, 0)$.

$\therefore\ \cos\pi = -1 \quad$ and $\quad \sin\pi = 0$

Example 3

Evaluate $\sin\left(-\dfrac{3\pi}{2}\right)$ and $\cos\left(-\dfrac{\pi}{2}\right)$.

Solution	Explanation
$\sin\left(-\dfrac{3\pi}{2}\right) = 1$	The point $P\left(-\dfrac{3\pi}{2}\right)$ has coordinates $(0, 1)$.
$\cos\left(-\dfrac{\pi}{2}\right) = 0$	The point $P\left(-\dfrac{\pi}{2}\right)$ has coordinates $(0, -1)$.

 ISBN 978-1-009-53719-3

Example 4

Evaluate $\sin\left(\frac{5\pi}{2}\right)$ and $\sin\left(\frac{7\pi}{2}\right)$.

Solution

$$\sin\left(\frac{5\pi}{2}\right) = \sin\left(2\tfrac{1}{2}\pi\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(3\tfrac{1}{2}\pi\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Example 5

Evaluate $\sin\left(\frac{9\pi}{2}\right)$ and $\cos(27\pi)$.

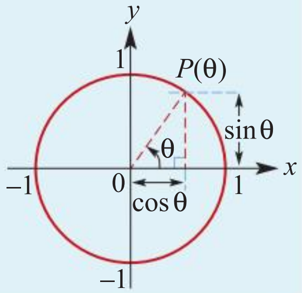
Solution

$$\sin\left(\frac{9\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\cos(27\pi) = \cos(26\pi + \pi) = \cos\pi = -1$$

Summary 8B

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$
- If an angle is an odd multiple of $\frac{\pi}{2}$, then its sine is either 1 or -1, and its cosine is zero.
- If an angle is a multiple of π, then its sine is zero, and its cosine is either 1 or -1.



Exercise 8B

Example 3, 4

1 For each of the following angles, t, determine the values of $\sin t$ and $\cos t$:

a $t = 0$ **b** $t = \frac{3\pi}{2}$ **c** $t = -\frac{3\pi}{2}$ **d** $t = \frac{5\pi}{2}$

Example 4, 5

e $t = -3\pi$ **f** $t = \frac{9\pi}{2}$ **g** $t = \frac{7\pi}{2}$ **h** $t = 4\pi$

2 Evaluate using your calculator. (Check that your calculator is in radian mode.)

a $\sin 1.9$ **b** $\sin 2.3$ **c** $\sin 4.1$ **d** $\cos 0.3$

e $\cos 2.1$ **f** $\cos(-1.6)$ **g** $\sin(-2.1)$ **h** $\sin(-3.8)$

Example 5

3 For each of the following angles, θ, determine the values of $\sin\theta$ and $\cos\theta$:

a $\theta = 27\pi$ **b** $\theta = -\frac{5\pi}{2}$ **c** $\theta = \frac{27\pi}{2}$ **d** $\theta = -\frac{9\pi}{2}$

e $\theta = \frac{11\pi}{2}$ **f** $\theta = 57\pi$ **g** $\theta = 211\pi$ **h** $\theta = -53\pi$

SF

 ISBN 978-1-009-53719-3

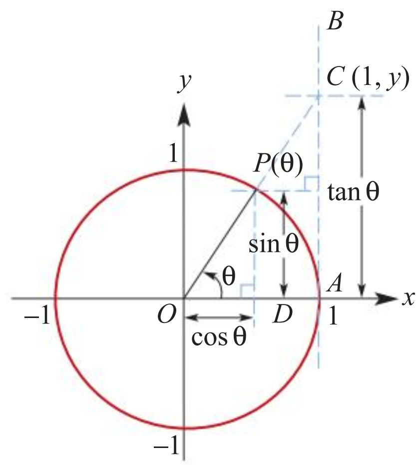
8C Another trigonometric function: tangent

Learning intentions

- To be able to work with the definition of the trigonometric function tangent.

Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the extension of OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$.

Hence $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

Example 6

Evaluate using a calculator. (Give answers to two decimal places.)

a $\tan 1.3$ **b** $\tan 1.9$ **c** $\tan(-2.8)$ **d** $\tan 59^\circ$

Solution	Explanation
a $\tan 1.3 = 3.60$	Don't forget that your calculator must be in radian mode.
b $\tan 1.9 = -2.93$	$\cos 1.9$ is negative.
c $\tan(-2.8) = 0.36$	Both $\cos(-2.8)$ and $\sin(-2.8)$ are negative, so tan is positive.
d $\tan 59^\circ = 1.66$	Calculate in degree mode.

Exercise 8C

SF

1 Evaluate:

a $\tan\pi$ **b** $\tan(-\pi)$ **c** $\tan\left(\frac{7\pi}{2}\right)$ **d** $\tan(-2\pi)$ **e** $\tan\left(\frac{5\pi}{2}\right)$ **f** $\tan\left(-\frac{\pi}{2}\right)$

Example 6

2 Use a calculator to determine correct to two decimal places:

a $\tan 1.6$ **b** $\tan(-1.2)$ **c** $\tan 136^\circ$ **d** $\tan(-54^\circ)$
e $\tan 3.9$ **f** $\tan(-2.5)$ **g** $\tan 239^\circ$

3 For each of the following values of θ, determine $\tan\theta$:

a $\theta = 180^\circ$ **b** $\theta = 360^\circ$ **c** $\theta = 0$
d $\theta = -180^\circ$ **e** $\theta = -540^\circ$ **f** $\theta = 720^\circ$

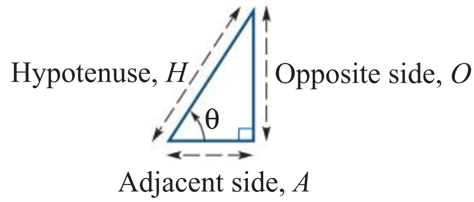
 ISBN 978-1-009-53719-3

8D Reviewing trigonometric ratios

Learning intentions

- To review the trigonometric ratios

For right-angled triangles:

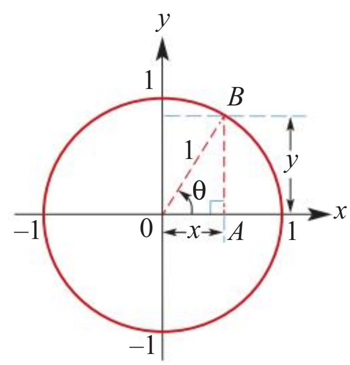


$$\sin\theta = \frac{O}{H}$$

$$\cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A}$$

Applying these trigonometric ratios to the right-angled triangle, OAB, in the unit circle:



$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

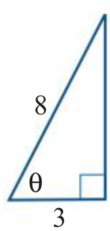
$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

For $0 < \theta < \frac{\pi}{2}$, the functions sin, cos and tan defined by the trigonometric ratios agree with the trigonometric functions introduced in this chapter.

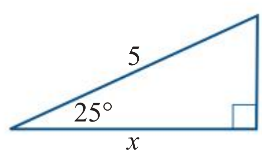
Exercise 8D

1 Determine the value of the pronumeral for the following, giving angles in degrees: SF

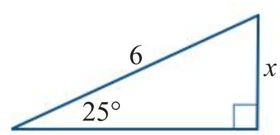
a



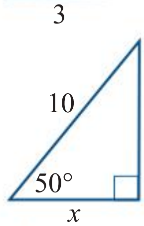
b



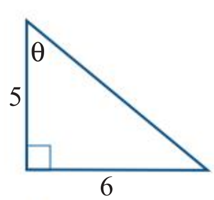
c



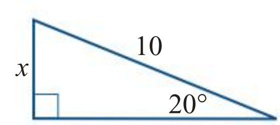
d



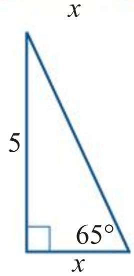
e



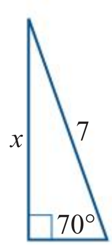
f



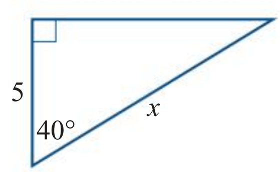
g



h



i



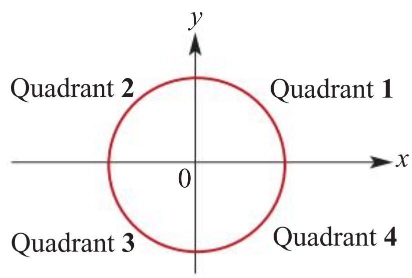
 ISBN 978-1-009-53719-3 Cambridge University Press

8E Symmetry properties of trigonometric functions

Learning intentions

- To be able to work with the symmetry properties of the trigonometric functions.

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.



Using symmetry, we can determine relationships between the trigonometric functions for angles in different quadrants.

Quadrant 2

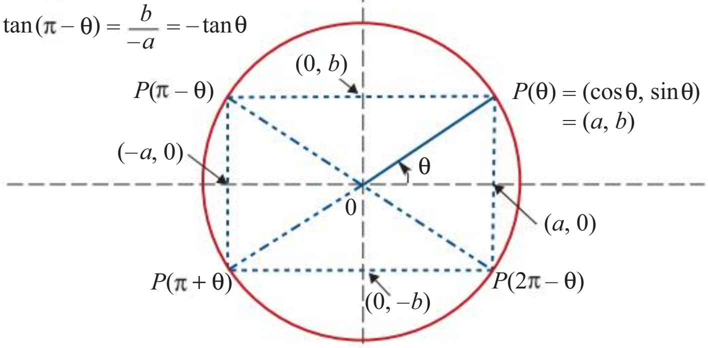
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

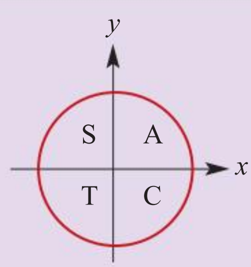
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of trigonometric functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



 ISBN 978-1-009-53719-3

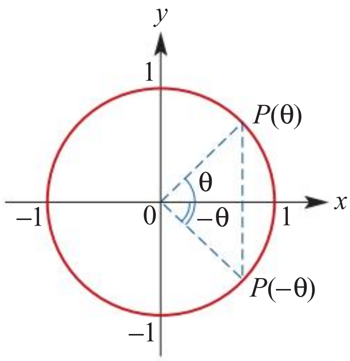
Negative angles

By symmetry:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Example 7

If $\sin x = 0.6$, determine the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $\sin(\pi - x)$
$= \sin x$
$= 0.6$

b $\sin(\pi + x)$
$= -\sin x$
$= -0.6$

c $\sin(2\pi - x)$
$= -\sin x$
$= -0.6$

d $\sin(-x)$
$= -\sin x$
$= -0.6$

Example 8

If $\cos x^\circ = 0.8$, determine the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ$
$= -\cos x^\circ$
$= -0.8$

b $\cos(180 + x)^\circ$
$= -\cos x^\circ$
$= -0.8$

c $\cos(360 - x)^\circ$
$= \cos x^\circ$
$= 0.8$

d $\cos(-x)^\circ$
$= \cos x^\circ$
$= 0.8$

Exercise 8E

SF

Example 7 **1** If $\sin\theta = 0.42$, $\cos x = 0.7$ and $\tan\alpha = 0.38$, write down the values of:

a $\sin(\pi + \theta)$ **b** $\cos(\pi - x)$ **c** $\sin(2\pi - \theta)$ **d** $\tan(\pi - \alpha)$
e $\sin(\pi - \theta)$ **f** $\tan(2\pi - \alpha)$ **g** $\cos(\pi + x)$ **h** $\cos(2\pi - x)$

Example 8 **2** If $\sin x^\circ = 0.7$, $\cos\theta^\circ = 0.6$ and $\tan\alpha^\circ = 0.4$, write down the values of:

a $\sin(180 + x)^\circ$ **b** $\cos(180 + \theta)^\circ$
c $\tan(360 - \alpha)^\circ$ **d** $\cos(180 - \theta)^\circ$
e $\sin(360 - x)^\circ$ **f** $\sin(-x)^\circ$
g $\tan(360 + \alpha)^\circ$ **h** $\cos(-\theta)^\circ$

 ISBN 978-1-009-53719-3

SF

3 **a** If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\frac{\pi}{2} < x < \pi$, determine the value of x.

b If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\pi < x < \frac{3\pi}{2}$, determine the value of x.

c If $\cos x = \cos\left(\frac{\pi}{6}\right)$ and $\frac{3\pi}{2} < x < 2\pi$, determine the value of x.

4 For the diagram shown, write down the values of:

a $a = \cos(\pi - \theta)$

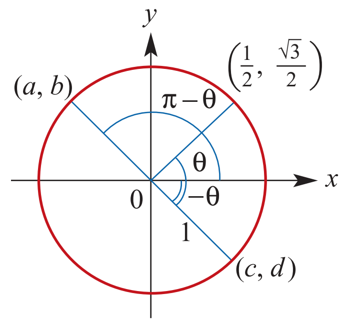
b $b = \sin(\pi - \theta)$

c $c = \cos(-\theta)$

d $d = \sin(-\theta)$

e $\tan(\pi - \theta)$

f $\tan(-\theta)$



5 For the diagram shown, write down the values of:

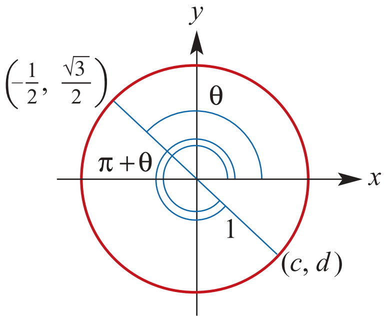
a $d = \sin(\pi + \theta)$

b $c = \cos(\pi + \theta)$

c $\tan(\pi + \theta)$

d $\sin(2\pi - \theta)$

e $\cos(2\pi - \theta)$

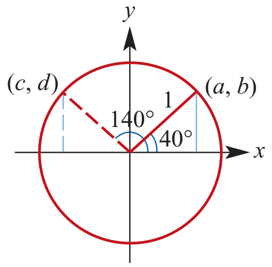


6 **a** For the diagram shown, use your calculator to determine a and b correct to four decimal places.

b Hence determine the values of c and d.

c **i** Use your calculator to determine $\cos 140°$ and $\sin 140°$.

ii Write $\cos 140°$ in terms of $\cos 40°$.



7 **a** If $\sin x° = \sin 60°$ and $90° < x° < 180°$, determine the value of x.

b If $\sin x° = -\sin 60°$ and $180° < x° < 270°$, determine the value of x.

c If $\sin x° = -\sin 60°$ and $-90° < x° < 0°$, determine the value of x.

d If $\cos x° = -\cos 60°$ and $90° < x° < 180°$, determine the value of x.

e If $\cos x° = -\cos 60°$ and $180° < x° < 270°$, determine the value of x.

f If $\cos x° = \cos 60°$ and $270° < x° < 360°$, determine the value of x.

 ISBN 978-1-009-53719-3 Cambridge University Press

8F Exact values of trigonometric functions

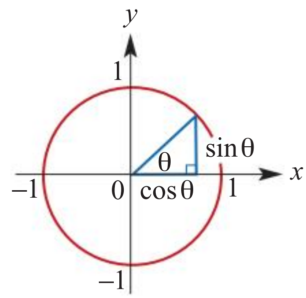
Learning intentions

- To be able to work with exact values of the trigonometric functions.

A calculator can be used to determine the values of the trigonometric functions for different values of θ. For many values of θ, the calculator gives an approximation. We now consider some values of θ such that sin, cos and tan can be calculated exactly.

Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:



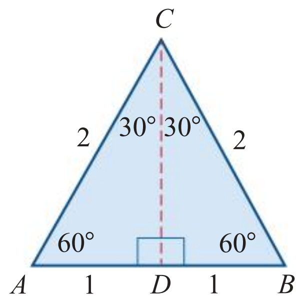
$\sin 0^\circ = 0 \qquad \sin 90^\circ = 1$

$\cos 0^\circ = 1 \qquad \cos 90^\circ = 0$

$\tan 0^\circ = 0 \qquad \tan 90^\circ$ is undefined

Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units. In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.



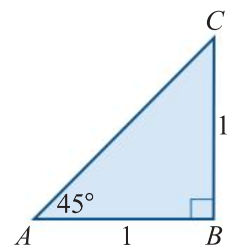
$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.



$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$

Example 9

Evaluate:

a $\cos 150^\circ$

b $\sin 690^\circ$

Solution

a
$$\cos 150^\circ = \cos(180 - 30)^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

b
$$\sin 690^\circ = \sin(2 \times 360 - 30)^\circ = \sin(-30^\circ) = -\frac{1}{2}$$

ISBN 978-1-009-53719-3

Example 10

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$

Solution

a $\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$ (by symmetry)

$= -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$ (by symmetry)

$= -\frac{1}{2}$

Summary 8F

As an aid to memory, the exact values for trigonometric functions can be tabulated.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined

Exercise 8F

Example 9

1 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 120° **b** 135° **c** 210° **d** 240° **e** 315°
f 390° **g** 420° **h** −135° **i** −300° **j** −60°

Example 10

2 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(\frac{5\pi}{6}\right)$

d $\sin\left(\frac{7\pi}{6}\right)$ **e** $\cos\left(\frac{5\pi}{4}\right)$ **f** $\tan\left(\frac{4\pi}{3}\right)$

g $\sin\left(\frac{5\pi}{3}\right)$ **h** $\cos\left(\frac{7\pi}{4}\right)$ **i** $\tan\left(\frac{11\pi}{6}\right)$

SF

 ISBN 978-1-009-53719-3 Cambridge University Press

3 Write down the exact values of:

a $\sin\left(-\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{11\pi}{4}\right)$ **c** $\tan\left(\frac{13\pi}{6}\right)$ **d** $\tan\left(\frac{15\pi}{6}\right)$

e $\cos\left(\frac{14\pi}{4}\right)$ **f** $\cos\left(-\frac{3\pi}{4}\right)$ **g** $\sin\left(\frac{11\pi}{4}\right)$ **h** $\cos\left(-\frac{21\pi}{3}\right)$

SF

8G Graphs of sine and cosine

Learning intentions

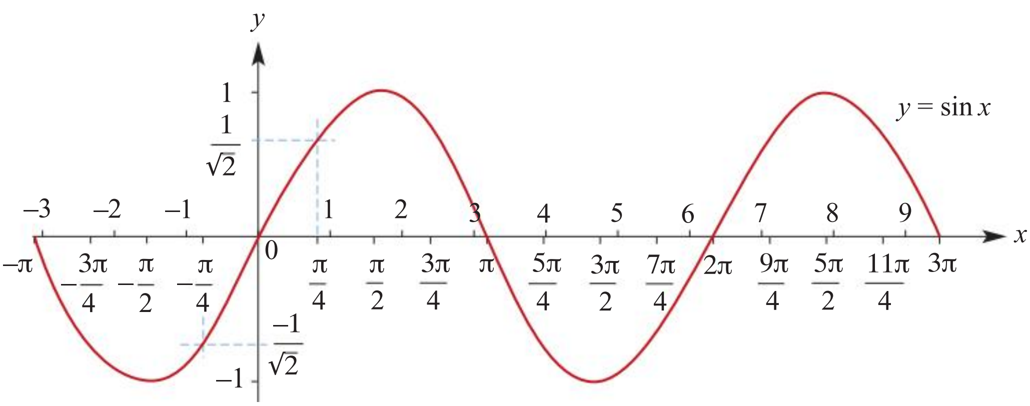
- To be able to sketch the graphs of the trigonometric functions.

Graph of the sine function

A table of exact values for $y = \sin x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

A calculator can be used to plot the graph of $y = \sin x$ (for $-\pi \le x \le 3\pi$). Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength). Thus $\sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

 ISBN 978-1-009-53719-3

Graph of the cosine function

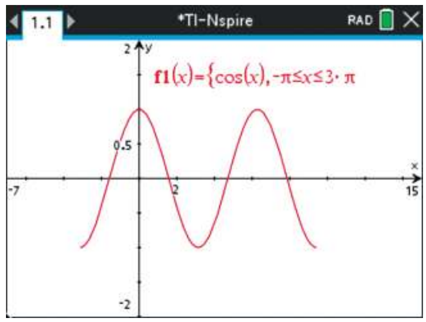
A table of exact values for $y = \cos x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1

TI-84

Using the TI-Nspire CX non-CAS

- A graph of $y = \cos x$ for $-\pi \le x \le 3\pi$ can be plotted in a **Graphs** application by entering $f1(x) = \cos(x) \mid -\pi \le x \le 3\pi$.
- Change the window to suit (menu > **Window/Zoom** > **Window Settings**).



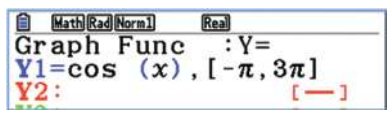
Using the Casio

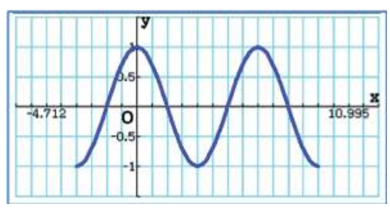
To plot the graph of $y = \cos x$ for $-\pi \le x \le 3\pi$:

- In **Graph** mode, enter the function rule $y = \cos(x)$ and domain $[-\pi, 3\pi]$ in $Y1$:

  cos (X,θ,T) ,
  SHIFT + (−) SHIFT ×10^x (π) ,
  3 ×10^x (π) SHIFT − EXE

- Adjust the View Window SHIFT F3 for:
  - $-2\pi \le x \le 4\pi$ with x-scale $\pi \div 4$
  - $-1.5 \le y \le 1.5$ with y-scale 0.5.
- Select **Draw** F6 to view the graph.





Observations from the graph of $y = \cos x$

- The period is 2π and the amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

 ISBN 978-1-009-53719-3 Cambridge University Press

Sketch graphs of $y = a\sin(bt)$ and $y = a\cos(bt)$

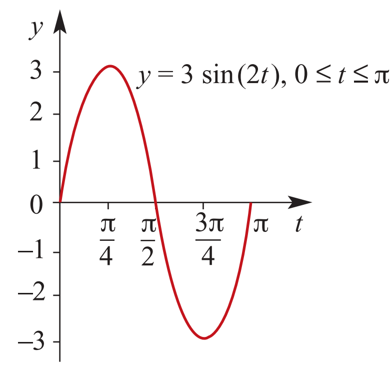
The graphs of functions of the forms $y = a\sin(bt)$ and $y = a\cos(bt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and b are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{2}$ from the y-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	-3	0

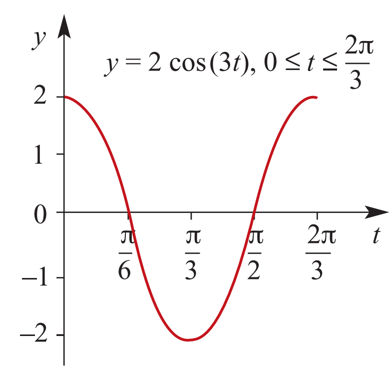


We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 from the t-axis and a dilation of factor $\frac{1}{3}$ from the y-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	-2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and b positive:

Function	Amplitude	Period
$y = a\sin(bt)$	a	$\frac{2\pi}{b}$
$y = a\cos(bt)$	a	$\frac{2\pi}{b}$

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 11

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$ **b** $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$ **c** $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(bt)$ and $y = a\cos(bt)$

In general, for a and b positive numbers, the graph of $y = a\sin(bt)$ (or $y = a\cos(bt)$) is obtained from the graph of $y = \sin t$ (or $y = \cos t$) by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{b}$ from the y-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{b}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(bt)$ and $g(t) = a\cos(bt)$:

- The period is $\frac{2\pi}{b}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

Example 12

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$

b $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 from the x-axis and a dilation of factor 2 from the y-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 13

Sketch the graph of each of the following functions:

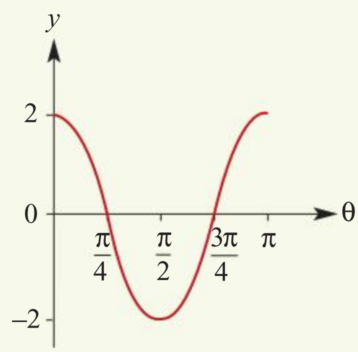
a $y = 2\cos(2\theta)$

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a $y = 2\cos(2\theta)$



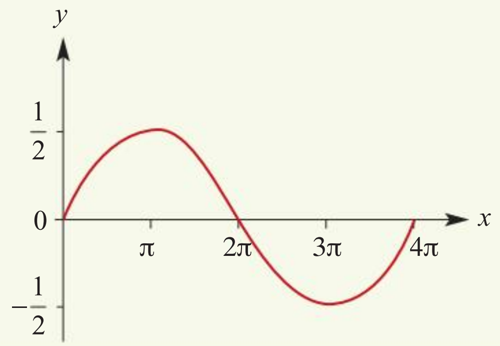
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$



The amplitude is $\frac{1}{2}$.

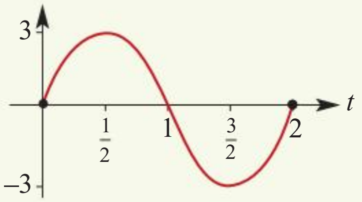
The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin(\frac{x}{2})$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 2 from the y-axis.

Example 14

Sketch the graph of the function with rule $f(t) = 3\sin(\pi t)$, where $t \in [0, 2]$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{\pi}$ from the y-axis.

 ISBN 978-1-009-53719-3

Transformations: reflection in the horizontal axis

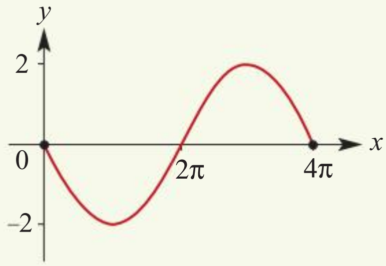
Example 15

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

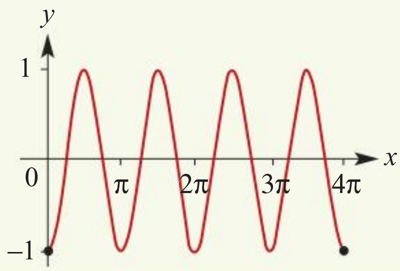


Explanation

The graph of $f(x) = -2\sin(\frac{x}{2})$ is obtained from the graph of $y = 2\sin(\frac{x}{2})$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Transformations: reflection in the vertical axis

Remember that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

Hence, when reflected in the y-axis:

- The graph of $y = \cos x$ transforms onto itself. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = \cos x$ by the observation above.)
- The graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = -\sin x$ by the observation above.)

Summary 8G

For positive numbers a and b, the graphs of $y = a\sin(bt)$, $y = -a\sin(bt)$, $y = a\cos(bt)$ and $y = -a\cos(bt)$ all have the following properties:

- The period is $\frac{2\pi}{b}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 8G

Example 11

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $2\sin\theta$ **b** $3\sin(2\theta)$ **c** $\frac{1}{2}\cos(3\theta)$

d $3\sin\left(\frac{1}{2}\theta\right)$ **e** $4\cos(3\theta)$ **f** $-\frac{1}{2}\sin(4\theta)$

g $-2\cos\left(\frac{1}{2}\theta\right)$ **h** $2\cos(\pi t)$ **i** $-3\sin\left(\frac{\pi t}{2}\right)$

Example 12

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin x$ **b** $g(x) = \sin(5x)$

c $g(x) = \sin\left(\frac{x}{3}\right)$ **d** $g(x) = 2\sin(5x)$

Example 13

3 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 3\sin(2x)$ **b** $y = 2\cos(3\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$

d $y = \frac{1}{2}\cos(3x)$ **e** $y = 4\sin(3x)$ **f** $y = 5\cos(2x)$

g $y = -3\cos\left(\frac{\theta}{2}\right)$ **h** $y = 2\cos(4\theta)$ **i** $y = -2\sin\left(\frac{\theta}{3}\right)$

Example 14

4 Sketch the graph of:

a $f(t) = 2\cos(\pi t),\ t \in [0, 2]$ **b** $f(t) = 3\sin(2\pi t),\ t \in [0, 2]$

Example 15

5 Sketch the graph of:

a $f(x) = \sin(2x)$ for $x \in [-2\pi, 2\pi]$ **b** $f(x) = 2\sin\left(\frac{x}{3}\right)$ for $x \in [-6\pi, 6\pi]$

c $f(x) = 2\cos(3x)$ for $x \in [0, 2\pi]$ **d** $f(x) = -2\sin(3x)$ for $x \in [0, 2\pi]$

6 Sketch the graph of the function with rule $f(x) = \frac{5}{2}\cos\left(\frac{2x}{3}\right)$, where $x \in [0, 2\pi]$.
Hint: For the endpoints, determine $f(0)$ and $f(2\pi)$.

7 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = -\sin(5x)$ **b** $g(x) = \sin(-x)$ **c** $g(x) = 2\sin\left(\frac{x}{3}\right)$

d $g(x) = -4\sin\left(\frac{x}{2}\right)$ **e** $g(x) = 2\sin\left(\frac{-x}{3}\right)$

8 **a** On the one set of coordinate axes, sketch the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ for $0 \le x \le 2\pi$.

b By inspection from these graphs, state the values of x for which $\sin x = \cos x$.

SF

CF

8H Solution of trigonometric equations

Learning intentions

- To be able to solve trigonometric equations of the form $a\sin(bt) = d$ and $a\cos(bt) = d$.

Solving equations of the form $\sin t = d$ and $\cos t = d$

First we look at the techniques for solving equations of the form $\sin t = d$ and $\cos t = d$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 16

Determine all solutions to the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

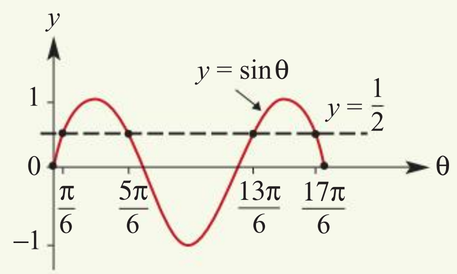
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

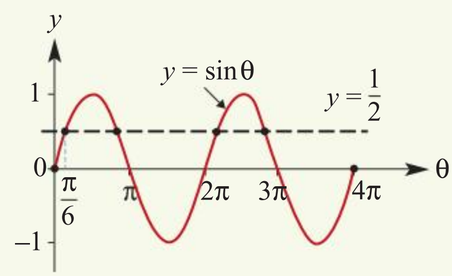
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



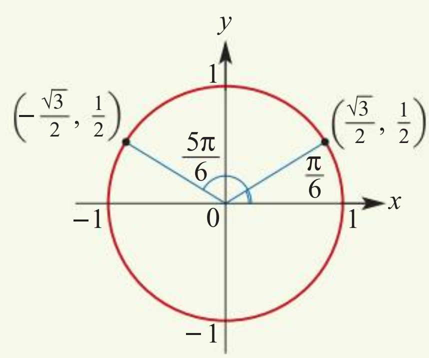
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

 ISBN 978-1-009-53719-3

Example 17

Determine two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$

b $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

Solution

a First solve the equation $\sin\alpha = 0.3$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$. Use your calculator to determine the solution $\alpha = 0.30469\ldots$

Now the value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of $y = \sin x$):

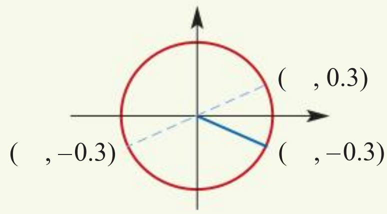
3rd quadrant: $x = \pi + 0.30469\ldots$

$= 3.446$ (to 3 d.p.)

4th quadrant: $x = 2\pi - 0.30469\ldots$

$= 5.978$ (to 3 d.p.)

$\therefore$ If $\sin x = -0.3$, then $x = 3.446$ or $x = 5.978$.



b First solve the equation $\cos\alpha^\circ = 0.7$ for $\alpha^\circ \in [0^\circ, 90^\circ]$. Use your calculator to determine the solution $\alpha^\circ = 45.57^\circ$.

Now the value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

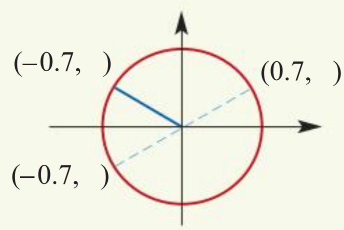
2nd quadrant: $x^\circ = 180^\circ - 45.57^\circ$

$= 134.43^\circ$

3rd quadrant: $x^\circ = 180^\circ + 45.57^\circ$

$= 225.57^\circ$

$\therefore$ If $\cos x^\circ = -0.7$, then $x^\circ = 134.43^\circ$ or $x^\circ = 225.57^\circ$.



Example 18

Determine all the values of θ° between 0° and 360° for which:

a $\cos\theta^\circ = \dfrac{\sqrt{3}}{2}$

b $\sin\theta^\circ = -\dfrac{1}{2}$

c $\cos\theta^\circ - \dfrac{1}{\sqrt{2}} = 0$

Solution

a $\cos\theta^\circ = \dfrac{\sqrt{3}}{2}$

$\theta^\circ = 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$

$\theta^\circ = 30^\circ$ or $\theta^\circ = 330^\circ$

b $\sin\theta^\circ = -\dfrac{1}{2}$

$\theta^\circ = 180^\circ + 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$

$\theta^\circ = 210^\circ$ or $\theta^\circ = 330^\circ$

Explanation

$\cos\theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

$\cos(360^\circ - \theta^\circ) = \cos\theta^\circ$

$\sin\theta^\circ$ is negative, and so $P(\theta^\circ)$ lies in the 3rd or 4th quadrant.

$\sin(180^\circ + \theta^\circ) = -\sin\theta^\circ$

$\sin(360^\circ - \theta^\circ) = -\sin\theta^\circ$

 ISBN 978-1-009-53719-3

c $\cos\theta° - \dfrac{1}{\sqrt{2}} = 0$

$\therefore \quad \cos\theta° = \dfrac{1}{\sqrt{2}}$

$\theta° = 45° \quad \text{or} \quad \theta° = 360° - 45°$

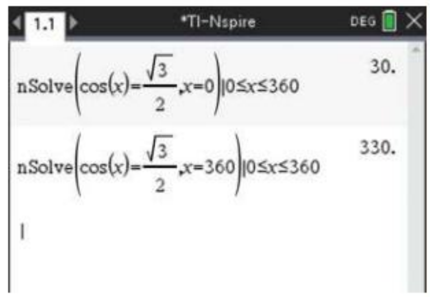
$\theta° = 45° \quad \text{or} \quad \theta° = 315°$

$\cos\theta°$ is positive, and so $P(\theta°)$ lies in the 1st or 4th quadrant.

Using the TI-Nspire CX non-CAS

For Example 18a, make sure the calculator is in degree mode and use (menu) > **Algebra** > **Numerical Solve** as shown.

Note: Two different guess values are used to determine the two solutions. It is advisable to plot the graph over the given domain to determine the number of solutions.

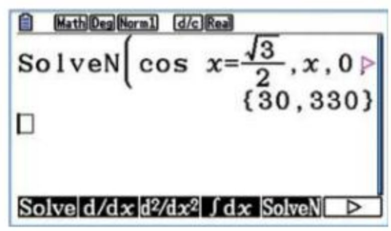


Using the Casio

For Example 18a:

- In **Run-Matrix** mode, ensure that the angle setting is Degrees.
- Go to the numerical solver **SolveN** (OPTN) (F4) (F5).
- Complete the equation and domain by entering:

$$\cos x = \frac{\sqrt{3}}{2}, x, 0, 360)$$



Solving equations of the form $a\sin(bt) = d$ and $a\cos(bt) = d$

The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 19

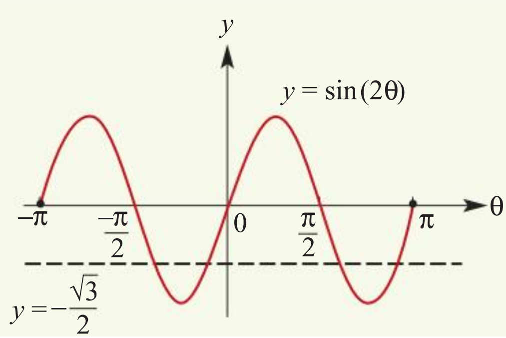
Solve the equation $\sin(2\theta) = -\dfrac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.

To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have $x = 2\theta \in [-2\pi, 2\pi]$.



ISBN 978-1-009-53719-3

Now consider the equation

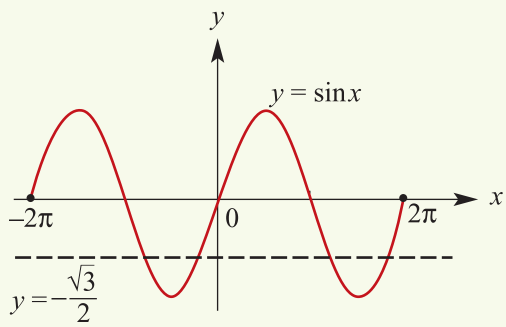
$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$

The 1st quadrant solution to the equation $\sin\alpha = \frac{\sqrt{3}}{2}$ is $\alpha = \frac{\pi}{3}$.

Using symmetry, the solutions to $\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$

i.e. $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$



The other two solutions (obtained by subtracting 2π) are $x = \frac{4\pi}{3} - 2\pi$ and $x = \frac{5\pi}{3} - 2\pi$.

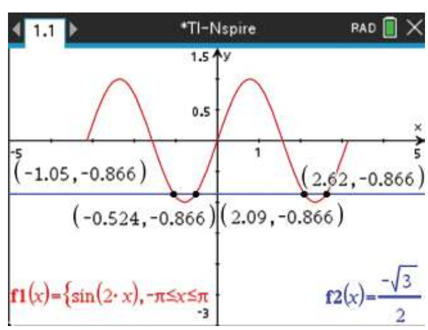
$\therefore$ The required solutions for x are $-\frac{2\pi}{3}$, $-\frac{\pi}{3}$, $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\frac{\pi}{3}$, $-\frac{\pi}{6}$, $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

TI-84

Using the TI-Nspire CX non-CAS

- Ensure that the calculator is in radian mode.
- In a **Graphs** application, plot the graphs of:
  - $f1(x) = \sin(2x) \mid -\pi \le x \le \pi$
  - $f2(x) = -\frac{\sqrt{3}}{2}$
- Use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- The solutions are -1.05, -0.524, 2.09 and 2.62, correct to three significant figures.

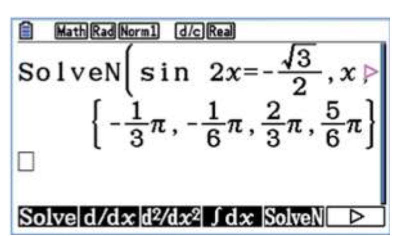


Note: You can also use (menu) > **Algebra** > **Numerical Solve**, but you will need to try several guess values to obtain all four solutions.

Using the Casio

- In **Run-Matrix** mode, ensure that the angle setting is Radians.
- Go to the numerical solver **SolveN** (OPTN) (F4) (F5).
- Complete the equation and domain by entering:

$$\sin 2x = -\frac{\sqrt{3}}{2}, x, -\pi, \pi)$$



 ISBN 978-1-009-53719-3

Summary 8H

- For solving equations of the form $\sin t = d$ and $\cos t = d$:
  - First determine the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(bt) = d$ and $a\cos(bt) = d$:
  - First substitute $x = bt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

  For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Skill-sheet

Exercise 8H

SF

Example 16

1 Determine all the values of x between 0 and 4π for which:

a $\cos x = \dfrac{1}{2}$ **b** $\sin x = \dfrac{1}{\sqrt{2}}$ **c** $\sin x = \dfrac{\sqrt{3}}{2}$

Example 17

2 Determine, correct to two decimal places, all the values of x between 0 and 2π for which:

a $\sin x = 0.8$ **b** $\cos x = -0.4$ **c** $\sin x = -0.35$

d $\sin x = 0.4$ **e** $\cos x = -0.7$ **f** $\cos x = -0.2$

CF

Example 18

3 Without using a calculator, determine all the values of $\theta°$ between 0° and 360° for each of the following:

a $\cos\theta° = -\dfrac{\sqrt{3}}{2}$ **b** $\sin\theta° = \dfrac{1}{2}$ **c** $\cos\theta° = -\dfrac{1}{2}$

d $2\cos(\theta°) + 1 = 0$ **e** $2\sin\theta° = \sqrt{3}$ **f** $\sqrt{2}\sin(\theta°) - 1 = 0$

4 Without using a calculator, determine all the values of x between 0 and 2π for each of the following:

a $2\cos x = \sqrt{3}$ **b** $\sqrt{2}\sin(x) + 1 = 0$ **c** $\sqrt{2}\cos(x) - 1 = 0$

5 Determine all the values of x between $-\pi$ and π for which:

a $\cos x = -\dfrac{1}{\sqrt{2}}$ **b** $\sin x = \dfrac{\sqrt{3}}{2}$ **c** $\cos x = -\dfrac{1}{2}$

6 **a** Sketch the graph of $f(x) = \cos x$ for $-2\pi \le x \le 2\pi$.

b On the graph, mark the points with y-coordinate $\frac{1}{2}$ and give the associated x-values.

c On the graph, mark the points with y-coordinate $-\frac{1}{2}$ and give the associated x-values.

Example 19

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\frac{1}{2}$ **b** $\cos(2\theta) = \frac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \frac{1}{2}$

d $\sin(3\theta) = -\frac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\frac{1}{\sqrt{2}}$

8 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

8I Sketch graphs of $y = a \sin b(t \pm h)$ and $y = a \cos b(t \pm h)$

Learning intentions

- To be able to sketch translations of the graphs of the form $f(t) = a\sin(bt)$ and $g(t) = a\cos(bt)$ in the direction of the t-axis.

In this section, we consider translations of graphs of functions of the form $f(t) = a\sin(bt)$ and $g(t) = a\cos(bt)$ in the direction of the t-axis.

When a translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis is applied to the graph of $y = f(t)$, the resulting image has equation $y = f\left(t - \frac{\pi}{4}\right)$.

For example, the graph of $f(t) = 3\sin(2t)$ is mapped to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

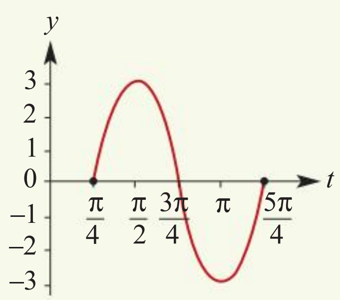
Example 20

On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

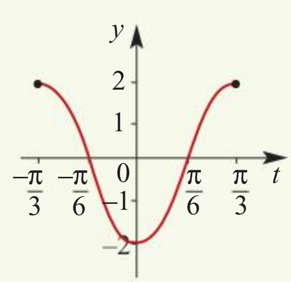
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, $\frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, $-\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π.



b The range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.



Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ is the same shape as $y = 3\sin(2t)$, but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$ is the same shape as $y = 2\cos(3t)$, but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm h$ is to translate the graph parallel to the t-axis. (Here $\pm h$ is called the phase.)

 ISBN 978-1-009-53719-3

Note: To determine the sequence of transformations, the techniques of Chapter 6 can also be used. The graph of $y = \sin t$ is transformed to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

Write the second equation as $\frac{y'}{3} = \sin 2\left(t' - \frac{\pi}{4}\right)$.

From this it can be seen that $y = \frac{y'}{3}$ and $t = 2\left(t' - \frac{\pi}{4}\right)$. Thus $y' = 3y$ and $t' = \frac{t}{2} + \frac{\pi}{4}$.

Hence the sequence of transformations is:

- dilation of factor 3 from the t-axis
- dilation of factor $\frac{1}{2}$ from the y-axis
- translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis.

Alternatively, we can determine this sequence by observing that the graph of $y = f(t)$ is transformed to the graph of $y = 3f\left(2\left(t - \frac{\pi}{4}\right)\right)$, where $f(t) = \sin t$.

Summary 8I

The graphs of $y = a\sin b(t \pm h)$ and $y = a\cos b(t \pm h)$ are translations of the graphs of $y = a\sin(bt)$ and $y = a\cos(bt)$ respectively.

The graphs are translated $\mp h$ units parallel to the t-axis, where $\pm h$ is called the phase.

Exercise 8I

Example 20

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin(2x)$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin(2x)$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

2 For the function with rule $f(x) = \cos\left(x - \frac{\pi}{3}\right)$ and domain $[0, 2\pi]$:

a determine $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

3 For the function with rule $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$ and domain $[0, 2\pi]$:

a determine $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

4 For the function with rule $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$ and domain $[-\pi, \pi]$:

a determine $f(-\pi)$, $f(\pi)$ **b** sketch the graph of f.

CF

 ISBN 978-1-009-53719-3

5 Determine the equation of the image of $y = \sin x$ for each of the following transformations:

a dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis

8J Sketch graphs of $y = a \sin b(t \pm h) \pm k$ and $y = a \cos b(t \pm h) \pm k$

Learning intentions

- To be able to sketch translations of the graphs considered in the previous section in the direction of the y-axis.

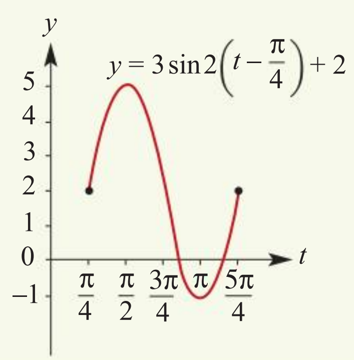
Example 21

Sketch each of the following graphs. Use a calculator to help establish the shape.

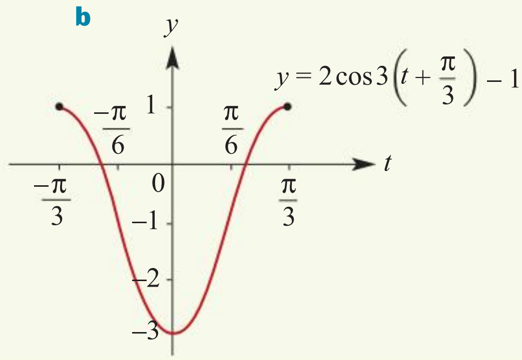
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2, \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ is the same shape as the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, but is translated 2 units in the positive direction of the y-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ is the same shape as the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, but is translated 1 unit in the negative direction of the y-axis.

In general, the effect of $\pm k$ is to translate the graph $\pm k$ units parallel to the y-axis.

 ISBN 978-1-009-53719-3

Determining axis intercepts

Example 22

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, the equation $\sqrt{2}\sin(x) + 1 = 0$ must be solved.

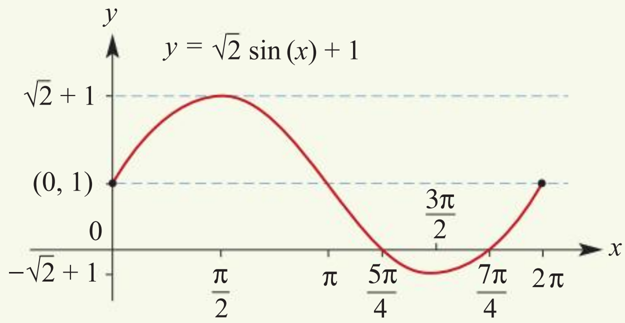
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



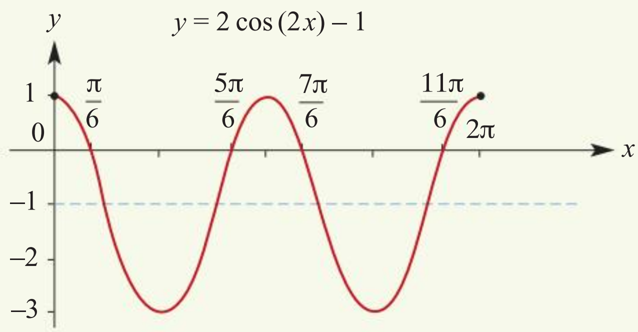
b $2\cos(2x) - 1 = 0$

$$\therefore \quad \cos(2x) = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The x-axis intercepts are $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



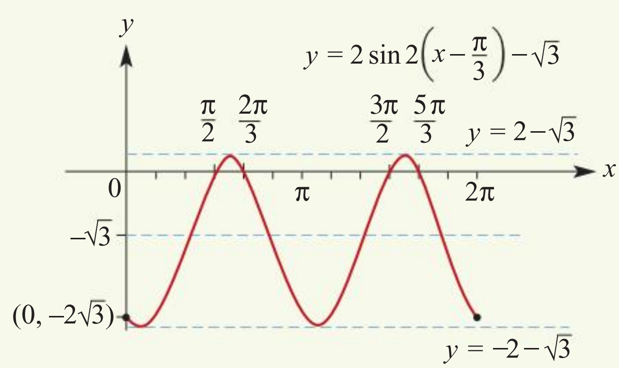
c $\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$$

$$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$$

The x-axis intercepts are $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 8J

Example 21, 22

1 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

2 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

3 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$ **c** $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

CF

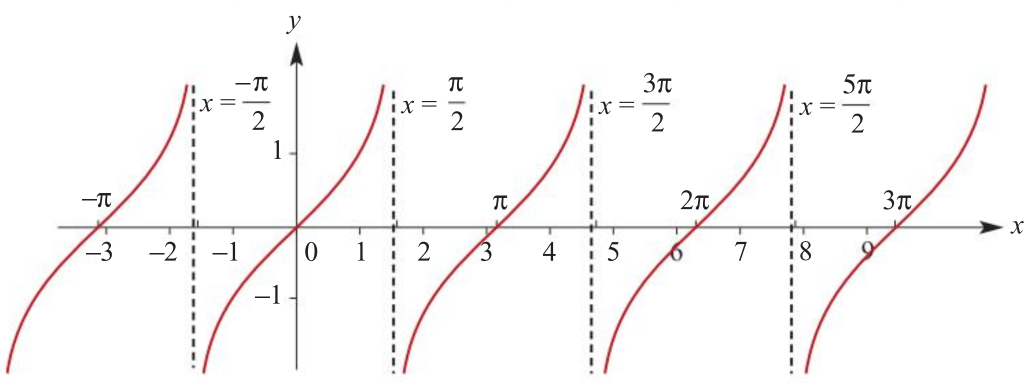
8K The tangent function

Learning intentions

► To be able to sketch the graph of $f(x) = \tan(x)$ and transformations of this graph.

A table of values for $y = \tan x$ is given below. Use a calculator to check these values and plot the graph of $y = \tan x$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0



Note: The lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ and $x = \frac{5\pi}{2}$ are asymptotes.

Observations from the graph of $y = \tan x$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The equations of the asymptotes are of the form $x = \frac{(2k+1)\pi}{2}$, where k is an integer.
- The x-axis intercepts occur when $\sin x = 0$, which is for $x = k\pi$, where k is an integer.

 ISBN 978-1-009-53719-3

Transformations of $y = \tan x$

Consider a dilation of factor $\frac{1}{2}$ from the y-axis and a dilation of factor 3 from the x-axis:

$$(x, y) \to (\tfrac{1}{2}x, 3y)$$

If the image of (x, y) under the transformation is (x', y'), then $x' = \frac{1}{2}x$ and $y' = 3y$. Hence $x = 2x'$ and $y = \dfrac{y'}{3}$.

Thus the graph of $y = \tan x$ is transformed to the graph of $\dfrac{y'}{3} = \tan(2x')$. That is, it is transformed to the graph of $y = 3\tan(2x)$. The period of the graph will be $\dfrac{\pi}{2}$.

Graph of $y = a\tan(bt)$

In general, for a and b positive numbers, the graph of $y = a\tan(bt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a dilation of factor $\dfrac{1}{b}$ from the y-axis.

The following are important properties of the function $f(t) = a\tan(bt)$:

- The period is $\dfrac{\pi}{b}$.
- The range is $\mathbb{R}$.
- The asymptotes have equations $t = \dfrac{(2k+1)\pi}{2b}$, where k is an integer.
- The t-axis intercepts are $t = \dfrac{k\pi}{b}$, where k is an integer.

Example 23

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$

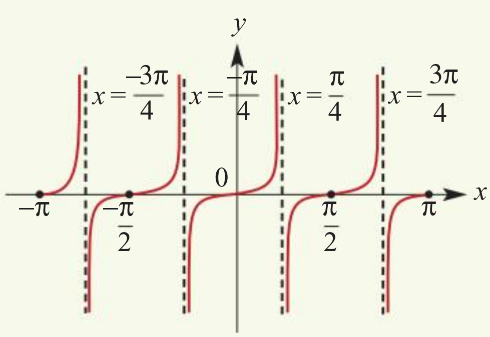
b $y = -2\tan(3x)$

Solution

a Period $= \dfrac{\pi}{b} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}, \ k \in \mathbb{Z}$

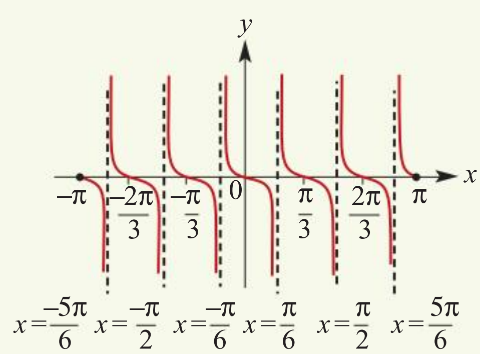
Axis intercepts: $x = \dfrac{k\pi}{2}, \ k \in \mathbb{Z}$



b Period $= \dfrac{\pi}{b} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}, \ k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{3}, \ k \in \mathbb{Z}$



 ISBN 978-1-009-53719-3

Solution of equations

The techniques for solving equations of the form

$$a \tan(bt) = d$$

are similar to those for solving equations of the form $a \sin(bt) = d$ and $a \cos(bt) = d$ discussed in Section 8H. An important difference is that the period of tan is π. The method for obtaining further solutions is to add and subtract multiples of π.

Example 24

Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\tan x = -1$

b $\tan(2x) = \sqrt{3}$

c $2\tan(3x) = 0$

Solution

a $\tan x = -1$

$$\therefore \quad x = \frac{3\pi}{4} \text{ or } \frac{-\pi}{4}$$

Explanation: Since $\tan x$ is negative, the point $P(x)$ lies in the 2nd or 4th quadrant. Solutions are required for $x \in [-\pi, \pi]$.

Use $\tan(\pi - x) = -\tan x$ and $\tan(-x) = -\tan x$.

b $\tan(2x) = \sqrt{3}$

Let $a = 2x$. The equation becomes

$$\tan a = \sqrt{3}$$

$$\therefore \quad a = \frac{\pi}{3} \text{ or } \frac{4\pi}{3} \text{ or } \frac{-2\pi}{3} \text{ or } \frac{-5\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{4\pi}{6} \text{ or } \frac{-2\pi}{6} \text{ or } \frac{-5\pi}{6}$$

$$= \frac{\pi}{6} \text{ or } \frac{2\pi}{3} \text{ or } \frac{-\pi}{3} \text{ or } \frac{-5\pi}{6}$$

Explanation: Consider solutions for $a \in [-2\pi, 2\pi]$.

Since $\tan a$ is positive, the point $P(a)$ lies in the 1st or 3rd quadrant.

Use $\tan(\pi + x) = \tan x$.

Subtract π from each of the first two solutions to obtain the second two.

c $2\tan(3x) = 0$

$$\tan(3x) = 0$$

$$3x = -3\pi,\ -2\pi,\ -\pi,\ 0,\ \pi,\ 2\pi \text{ or } 3\pi$$

$$\therefore \ x = -\pi,\ \frac{-2\pi}{3},\ \frac{-\pi}{3},\ 0,\ \frac{\pi}{3},\ \frac{2\pi}{3} \text{ or } \pi$$

Explanation: The solutions for $3x$ are to be in the interval $[-3\pi, 3\pi]$.

 ISBN 978-1-009-53719-3

Example 25

Sketch the graph of $y = \tan(2x) + 1$ for $x \in [-\pi, \pi]$.

Solution

The graph of $y = \tan(2x) + 1$ is obtained from the graph of $y = \tan(2x)$ by a translation of 1 unit in the positive direction of the y-axis.

For the y-axis intercept, let $x = 0$. Then $y = \tan 0 + 1 = 1$.

For the x-axis intercepts, consider

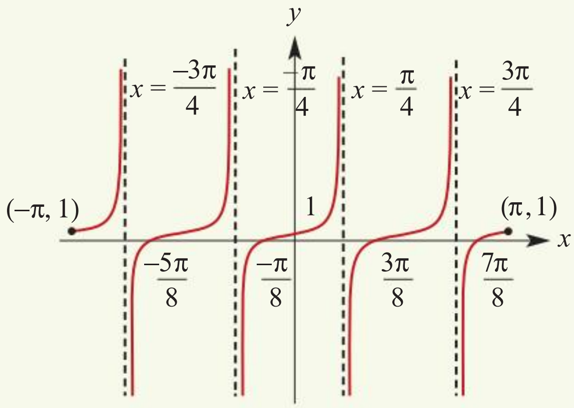
$$\tan(2x) + 1 = 0$$
$$\tan(2x) = -1$$

Hence $2x = \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{7\pi}{4}$ or $\frac{-5\pi}{4}$

and so $x = \frac{3\pi}{8}, \frac{-\pi}{8}, \frac{7\pi}{8}$ or $\frac{-5\pi}{8}$

The asymptotes are the same as those for $y = \tan(2x)$,

that is, $x = \frac{(2k+1)\pi}{4}, k \in \mathbb{Z}$



Exercise 8K

SF

1 For each of the following, state the period:

a $y = \tan(4x)$ **b** $y = \tan\left(\frac{2x}{3}\right)$ **c** $y = -3\tan(2x)$

CF

Example 23 **2** Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -\tan x$

Example 24 **3** Solve each of the following equations for $x \in [-\pi, \pi]$:

a $2\tan(2x) = 2$ **b** $3\tan(3x) = \sqrt{3}$

c $2\tan(2x) = 2\sqrt{3}$ **d** $3\tan(3x) = -\sqrt{3}$

Example 25 **4** Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(x) + \sqrt{3}$ **b** $y = 2\tan(x) + 2$

c $y = 3\tan(x) - 3$

 ISBN 978-1-009-53719-3 Cambridge University Press

8L Further symmetry properties and the Pythagorean identity

Learning intentions

- To be able to use the complementary relationships and the Pythagorean identity for the trigonometric functions.

Complementary relationships

From the diagram to the right,

$$\sin\left(\frac{\pi}{2} - \theta\right) = a$$

and, since $a = \cos\theta$,

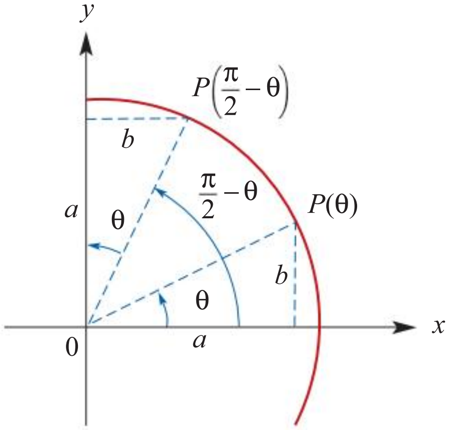
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

From the same diagram,

$$\cos\left(\frac{\pi}{2} - \theta\right) = b$$

and, since $b = \sin\theta$,

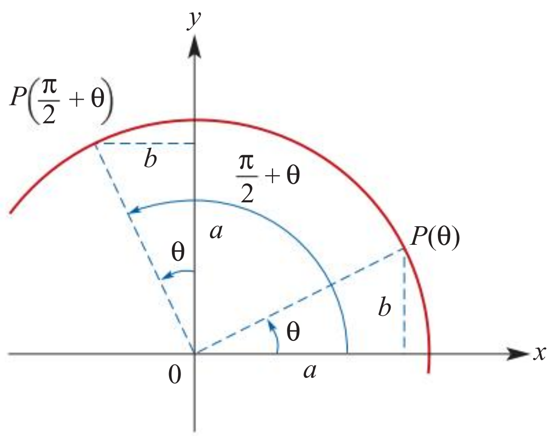
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 26

If $\sin\theta = 0.3$ and $\cos\alpha = 0.8$, determine the values of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.3$

c $\sin(-\theta) = -\sin\theta$
$= -0.3$

 ISBN 978-1-009-53719-3

The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle. By Pythagoras' theorem,

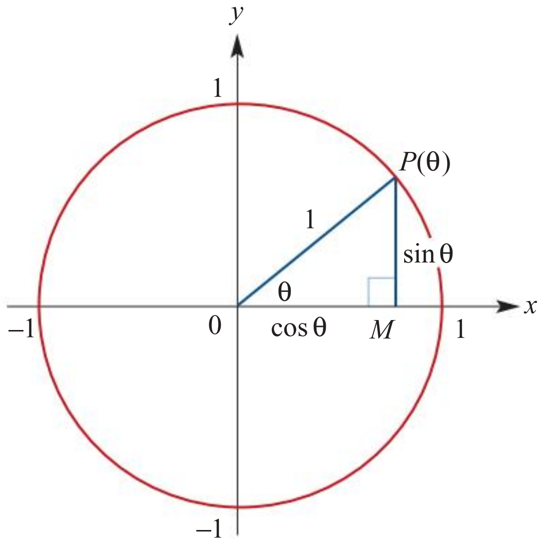
$$OP^2 = OM^2 + MP^2$$
$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$.

$$\therefore \quad 1 = \cos^2\theta + \sin^2\theta$$

As this is true for all values of θ, it is called an identity. In particular, this is called the **Pythagorean identity**:

$$\cos^2\theta + \sin^2\theta = 1$$



Example 27

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, determine:

a $\cos x$

b $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x + \frac{9}{25} = 1$$
$$\cos^2 x = 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part a, we have

$$\tan x = \frac{\sin x}{\cos x}$$
$$= \frac{3}{5} \div \left(-\frac{4}{5}\right)$$
$$= \frac{3}{5} \times \left(-\frac{5}{4}\right)$$
$$= -\frac{3}{4}$$

Summary 8L

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

ISBN 978-1-009-53719-3

Exercise 8L

Example 26

1 If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, determine the values of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2}+\alpha\right)$ **c** $\tan(-\theta)$ **d** $\cos\left(\frac{\pi}{2}-x\right)$

e $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2}-\theta\right)$ **g** $\cos\left(\frac{\pi}{2}+x\right)$ **h** $\sin\left(\frac{\pi}{2}-\alpha\right)$

i $\sin\left(\frac{3\pi}{2}+\alpha\right)$ **j** $\cos\left(\frac{3\pi}{2}-x\right)$

SF

2 **a** Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\left(\frac{\pi}{6}\right)$, determine the value of θ.

b Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\left(\frac{\pi}{6}\right)$, determine the value of θ.

c Given that $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \sin\left(\frac{\pi}{12}\right)$, determine the value of θ.

d Given that $0 < \theta < \frac{\pi}{2}$ and $\sin\theta = \cos\left(\frac{3\pi}{7}\right)$, determine the value of θ.

Example 27

3 Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, determine $\sin x$ and $\tan x$.

CF

4 Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, determine $\cos x$ and $\tan x$.

5 Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, determine $\sin x$ and $\tan x$.

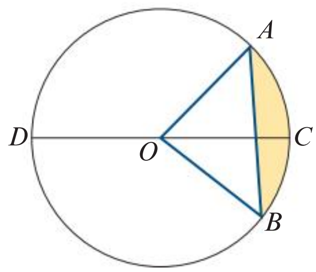
8M Circle mensuration

Learning intentions

- To be able to determine the length of an arc, the length of a chord, the area of a sector and the area of a segment for a circle.

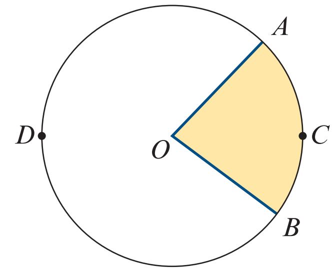
Terminology

In the diagram, the circle has centre O.

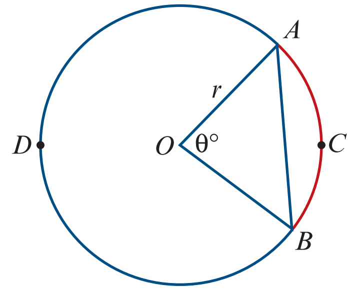


- **Chords** A **chord** of a circle is a line segment with endpoints on the circle; e.g. line segment AB in the diagram. A chord passing through the centre of the circle is called a **diameter**; e.g. line segment CD in the diagram.
- **Arcs** Any two points on a circle divide the circle into arcs. The shorter arc is called the **minor arc** and the longer is the **major arc**. In the diagram, arc ACB is a minor arc and arc ADB is a major arc. The arcs DAC and DBC are called **semicircular arcs**.
- **Segments** Every chord divides the interior of a circle into two regions called segments. The smaller is called the **minor segment** and the larger is the **major segment**. In the above diagram, the minor segment has been shaded.

- **Sectors** Two radii and an arc define a region called a sector. In this diagram, with circle centre O, the shaded region is a **minor sector** and the unshaded region is a **major sector**.



Arc length



The circle in the diagram has centre O and radius r. The arc ACB and the corresponding chord AB are said to **subtend** the angle $\angle AOB$ at the centre of the circle.

The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The length ℓ of arc ACB will be the same fraction of the circumference of the circle, $2\pi r$.

Length of an arc using degrees

$$\ell = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{\pi r\theta}{180} \qquad \text{(where } \theta \text{ is measured in degrees)}$$

The formula for arc length is simpler if we measure the angle in radians.

We recall that, in the unit circle, an arc of length θ units subtends an angle of θ radians at the centre. A circle of radius r is similar to the unit circle, with similarity factor r, and therefore an arc of length $r\theta$ units subtends an angle of θ radians at the centre.

Length of an arc using radians

$$\ell = r\theta \qquad \text{(where } \theta \text{ is measured in radians)}$$

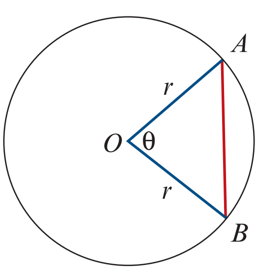
Note: As there are 2π radians in a circle, the arc length is $\ell = \dfrac{\theta}{2\pi} \times 2\pi r = r\theta$.

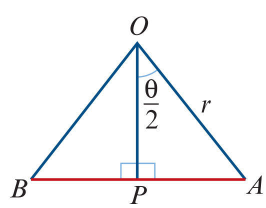
Chord length

In triangle OAP:

$$AP = r\sin\left(\frac{\theta}{2}\right)$$

$$\therefore \quad \boxed{AB = 2r\sin\left(\frac{\theta}{2}\right)}$$

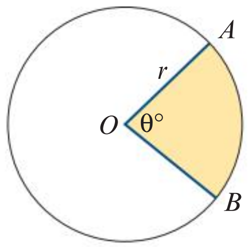




ISBN 978-1-009-53719-3

Area of a sector

The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The area of the sector will be the same fraction of the area of the circle, πr^2.



Using degrees: Area of sector $= \dfrac{\pi r^2 \theta}{360}$

Using radians: Area of sector $= \dfrac{1}{2} r^2 \theta$

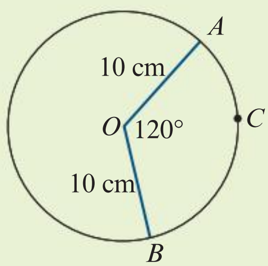
Example 28

The circle shown has centre O and radius length 10 cm.
The angle subtended at O by arc ACB has magnitude $120°$.
Determine:

a **i** the exact length of the chord AB
ii the exact length of the arc ACB

b the exact area of the minor sector AOB

c the magnitude of angle AOC, in degrees, if the minor arc AC has length 4 cm.



Solution

a **i** Chord length $= 2r\sin\left(\dfrac{\theta}{2}\right)$

$= 20\sin 60°$ since $r = 10$ and $\theta = 120°$

$= 20 \times \dfrac{\sqrt{3}}{2}$

$= 10\sqrt{3}$

Length of chord is $10\sqrt{3}$ cm.

ii Arc length $\ell = r\theta$ using radians

$= 10 \times \dfrac{2\pi}{3}$ since $r = 10$ and $\theta = \dfrac{2\pi}{3}$

$= \dfrac{20\pi}{3}$

Length of arc is $\dfrac{20\pi}{3}$ cm.

b Area of sector $= \dfrac{1}{2} r^2 \theta$ using radians

$= \dfrac{1}{2} \times 10^2 \times \dfrac{2\pi}{3}$ since $r = 10$ and $\theta = \dfrac{2\pi}{3}$

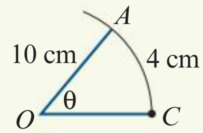
$= \dfrac{100\pi}{3}$

Area of minor sector AOB is $\dfrac{100\pi}{3}$ cm^2.

c Using radians: $\ell = r\theta$

$4 = 10\theta$

$\therefore \quad \theta = \dfrac{4}{10}$



Convert to degrees: $\angle AOC = 0.4 \times \dfrac{180}{\pi}$

$= 22.92°$ (correct to two decimal places)

Area of a segment

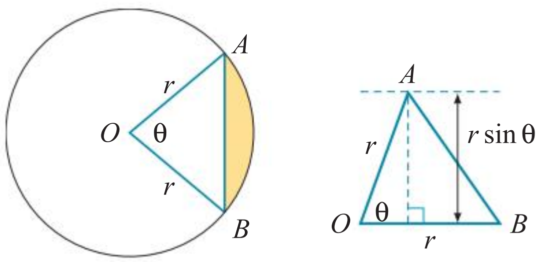
The area of the shaded segment is found by subtracting the area of $\triangle AOB$ from the area of the minor sector OAB.

Using degrees:

$$\text{Area of segment} = \frac{\pi r^2 \theta}{360} - \frac{1}{2} r^2 \sin\theta$$

Using radians:

$$\text{Area of segment} = \frac{1}{2} r^2 \theta - \frac{1}{2} r^2 \sin\theta$$



Example 29

A circle, with centre O and radius length 20 cm, has a chord AB that is 10 cm from the centre of the circle. Calculate the area of the minor segment formed by this chord.

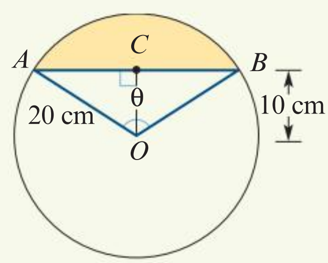
Solution

The area of the segment is $\frac{1}{2} r^2(\theta - \sin\theta)$. We know $r = 20$, but we need to determine θ.

In $\triangle OCB$: $\cos\left(\dfrac{\theta}{2}\right) = \dfrac{10}{20}$

$\dfrac{\theta}{2} = \dfrac{\pi}{3}$

$\therefore \quad \theta = \dfrac{2\pi}{3}$



$$\begin{aligned}
\text{Area of segment} &= \frac{1}{2} \times 20^2 \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) \\
&= 200\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \\
&= 200\left(\frac{4\pi - 3\sqrt{3}}{6}\right) \\
&= \frac{100(4\pi - 3\sqrt{3})}{3}\ \text{cm}^2
\end{aligned}$$

 ISBN 978-1-009-53719-3

Summary 8M

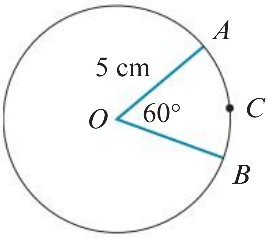
- **Circle mensuration formulas with θ in radians**
  - Arc length $= r\theta$
  - Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$
  - Area of sector $= \frac{1}{2}r^2\theta$
  - Area of segment $= \frac{1}{2}r^2(\theta - \sin\theta)$
- **Circle mensuration formulas with θ in degrees**
  - Arc length $= \frac{\pi r\theta}{180}$
  - Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$
  - Area of sector $= \frac{\pi r^2\theta}{360}$
  - Area of segment $= \frac{1}{2}r^2\left(\frac{\pi\theta}{180} - \sin\theta\right)$

Skill-sheet

Exercise 8M

Example 28

1 The circle shown has centre O and radius 5 cm. The angle subtended by arc ACB has magnitude 60°. Determine:



a **i** the exact length of the chord AB

ii the exact length of the arc ACB

b the exact area of the minor sector AOB

c the magnitude of angle AOC, in degrees, if the minor arc AC has length 3 cm.

SF

2 Determine the length of an arc which subtends an angle of magnitude 105° at the centre of a circle of radius length 25 cm.

3 Determine the magnitude, in degrees, of the angle subtended at the centre of a circle of radius length 30 cm by:

a an arc of length 50 cm

b a chord of length 50 cm.

Example 29

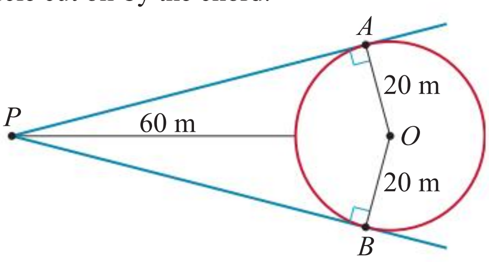
CF

4 A chord of length 6 cm is drawn in a circle of radius 7 cm. Determine:

a the length of the minor arc cut off by the chord

b the area of the smaller region inside the circle cut off by the chord.

5 A person stands on level ground 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate:



a the circumference of the tank

b the percentage of the circumference that is visible to the person.

Note: In the diagram, the point O is the centre of the circle and $\angle PAO = \angle PBO = 90°$.

6 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m.?

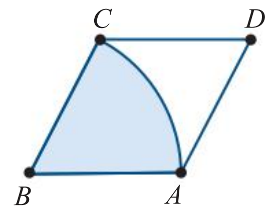
b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m.?

 ISBN 978-1-009-53719-3

CF

7 A sector of a circle has perimeter 32 cm and area 63 cm^2. Determine the radius length and the magnitude of the angle subtended at the centre of the two possible sectors.

8 The diagram shows a rhombus $ABCD$ with each side of length 4 cm. An arc AC, centre B, has length 5 cm. Calculate the area of the white region ACD.



8N Applications of trigonometric functions

Learning intentions

- To be able to use trigonometric functions to model.

Example 30

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

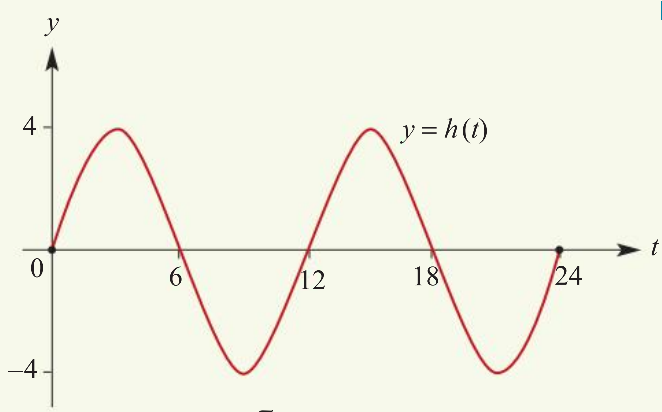
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$:

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\sin\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, \ 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$

The water is $2\sqrt{3}$ metres below the mean height at 8 a.m.

 ISBN 978-1-009-53719-3

e We first consider $4\sin\left(\frac{\pi}{6}t\right) = 1$.

Thus $\sin\left(\frac{\pi}{6}t\right) = \frac{1}{4}$

$\therefore \quad \frac{\pi}{6}t = 0.2526,\ 2.889,\ 6.5358,\ 9.172$

$\therefore \quad t = 0.4824,\ 5.5176,\ 12.4824,\ 17.5173$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.
Thus the boat can pass across the harbour bar between 00:29 and 05:31, and between 12:29 and 17:31.

Skill-sheet

Exercise 8N

Example 30

1 It is suggested that the height, $h(t)$ metres, of the tide above mean sea level during a particular day at Seabreak is given approximately by the rule

$$h(t) = 5\sin\left(\frac{\pi}{6}t\right)$$

where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \leq t \leq 24$.
b When was high tide?
c What was the height of the high tide?
d What was the height of the tide at 2 a.m.?
e What was the height of the tide at 2 p.m.?
f A boat can only cross the harbour bar when the tide is at least 2.5 metres above mean sea level. When could the boat cross the harbour bar on this day?

CF

2 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 3 + 2\sin(3t)$.

a Determine its greatest distance from O.
b Determine its least distance from O.
c Determine the times at which it is 5 metres from O for $0 \leq t \leq 5$.
d Determine the times at which it is 3 metres from O for $0 \leq t \leq 3$.
e Describe the motion of the particle.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 5 + 2\sin(2\pi t)$.

a Determine its greatest distance from O.
b Determine its least distance from O.
c Determine the times at which it is 7 metres from O for $0 \leq t \leq 5$.
d Determine the times at which it is 6 metres from O for $0 \leq t \leq 3$.
e Describe the motion of the particle.

 ISBN 978-1-009-53719-3

CF

4 A particle moves in a vertical circle of radius 10 m. The height, $h(t)$ m, of the particle above the ground at time t seconds is given by the function

$$h(t) = 10\sin\left(\frac{\pi t}{3}\right) + 10 \quad \text{where } t \geq 0$$

a Determine the height of the particle above the ground for:

i $t = 0$ **ii** $t = 1$ **iii** $t = 2$ **iv** $t = 4$ **v** $t = 5$

b Determine the period of the motion of the particle.

c Determine the greatest height of the particle above the ground.

d Determine the first four times that the particle is at a height 15 m above the ground.

e Determine the first four times that the particle is at a height 5 m above the ground.

5 The temperature, T°C, in a town over a day is modelled by the function with rule

$$T = 17 - 8\cos\left(\frac{\pi t}{12}\right)$$

where t is the time in hours after midnight, $0 \leq t \leq 24$.

a What is the temperature at midnight?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

d Sketch the graph for the temperatures over a day.

CU

6 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.

a Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.

b Determine the values of t for which $D(t) \geq 8.5$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Determine, correct to one decimal place, the largest value of w which satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D metres, where $D = p + q\cos(rt)°$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and determine the values of p and q.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Determine how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

8 For each of the following, construct a formula involving a trigonometric function which could be used to model the situation described:

a Water depths in a canal vary between a minimum of 3 metres and a maximum of 6 metres over a 24-hour period.

b At a certain town just south of the Arctic circle, the number of hours of daylight varies between 2 and 22 hours during a 365-day year.

 ISBN 978-1-009-53719-3 Cambridge University Press

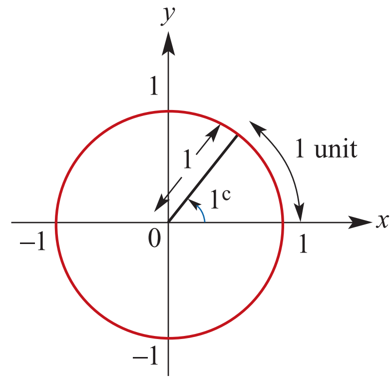
Review

Chapter summary

- **Definition of a radian**

  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

  $$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



- **Sine and cosine**

  x-coordinate of $P(\theta)$ on unit circle:

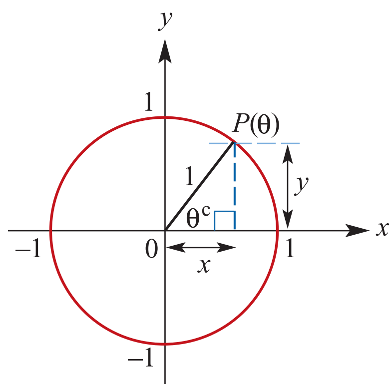
  $$x = \text{cosine } \theta, \quad \theta \in \mathbb{R}$$

  y-coordinate of $P(\theta)$ on unit circle:

  $$y = \text{sine } \theta, \quad \theta \in \mathbb{R}$$

  Abbreviated to

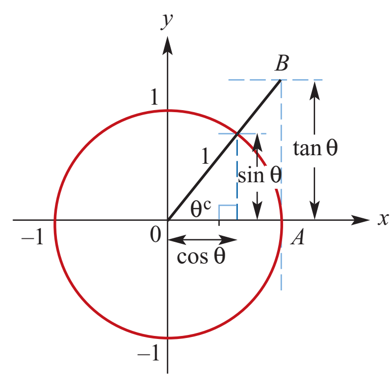
  $$x = \cos\theta$$
  $$y = \sin\theta$$



- **Tangent**

  If the tangent to the unit circle at A is drawn, then the y-coordinate of B is called tangent θ (abbreviated to $\tan\theta$). By using similar triangles:

  $$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

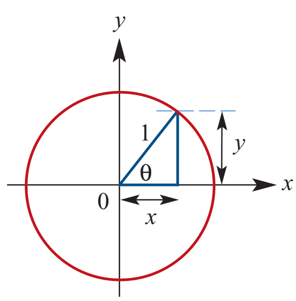


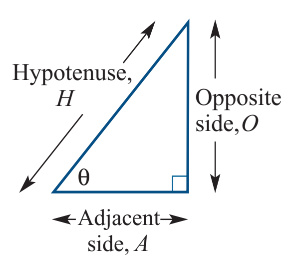
- **Trigonometric functions and trigonometric ratios**

  $$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

  $$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

  $$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



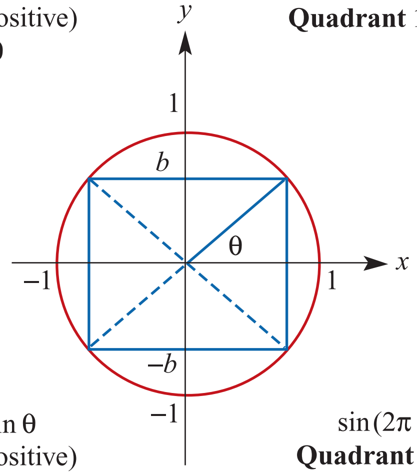


 ISBN 978-1-009-53719-3 Cambridge University Press

■ Symmetry properties of trigonometric functions

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

■ Further symmetry properties

Negative angles:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

■ Exact values of trigonometric functions

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

■ Pythagorean identity

$\cos^2\theta + \sin^2\theta = 1$

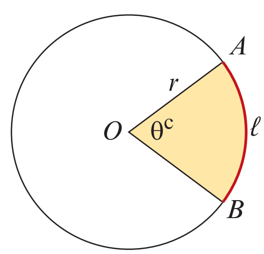
■ Circles

- Length of minor arc AB (red curve) is given by

$$\ell = r\theta$$

- Area of the shaded sector is given by

$$\text{Area} = \frac{1}{2}r^2\theta$$

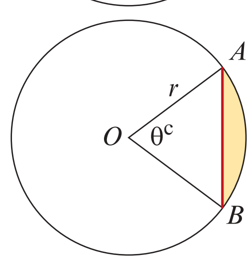


- Length of chord AB (red line) is given by

$$\ell = 2r\sin\left(\frac{\theta}{2}\right)$$

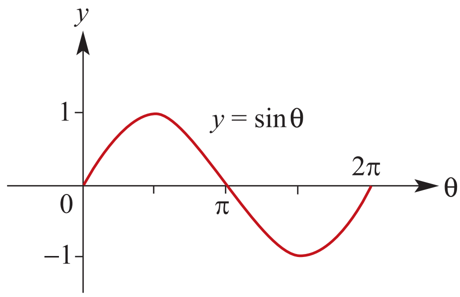
- Area of the shaded segment is given by

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

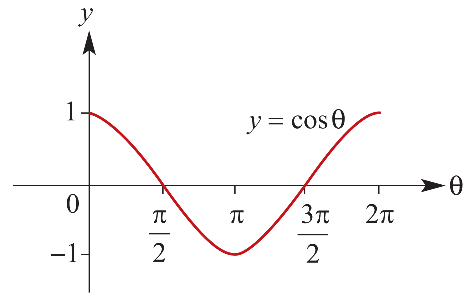


 ISBN 978-1-009-53719-3 Cambridge University Press

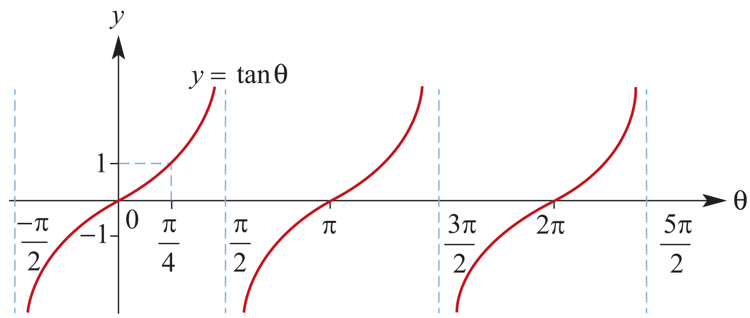
Graphs of trigonometric functions



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π

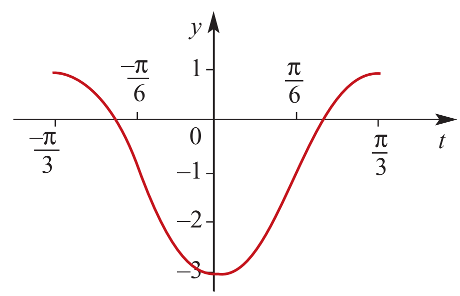


Amplitude is undefined
Period = π

Transformations

Graphs of trigonometric functions of the type
$y = a \sin b(t \pm h) \pm k$ and $y = a \cos b(t \pm h) \pm k$

e.g. $y = 2 \cos 3\left(t + \frac{\pi}{3}\right) - 1$



- Amplitude, $a = 2$
- Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
- The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.

Solution of trigonometric equations

Solution of equations of the type $\sin x^\circ = a$ and $\cos x^\circ = a$

e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

$$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

 ISBN 978-1-009-53719-3

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

8A **1** I can change radians to degrees and degrees to radians. ☐

See Example 1 and Questions 1 and 2

8B **2** I can evaluate sine and cosine functions for multiples of $\frac{\pi}{2}$. ☐

See Example 2, Example 3, Example 4, Example 5 and Questions 1 and 3

8C **3** I can evaluate tangent functions. ☐

See Example 6 and Questions 1, 2 and 3

8D **4** I can use the trigonometric ratios to solve for unknown sides and angles in right-angled triangles. ☐

See Question 1

8E **5** I can use the symmetry properties of the trigonometric functions to evaluate these functions for suitable values. ☐

See Example 7, Example 8 and Questions 1 and 2

8F **6** I can use exact values for $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), $\frac{\pi}{3}$ (60°) to evaluate trigonometric functions. ☐

See Example 9, Example 10 and Questions 1 and 2

8G **7** I can determine the period and amplitude of functions of the form $y = a\sin(bt)$ and $y = a\cos(bt)$. ☐

See Example 11, Example 12 and Questions 1 and 2

8G **8** I can sketch and analyse graphs of the form $y = a\sin(bt)$ and $y = a\cos(bt)$. ☐

See Example 13, Example 14, Example 15 and Questions 3, 4 and 5

8H **9** I can solve equations of the form $\sin(t) = m$ and $\cos(t) = m$ for a given domain of t. ☐

See Example 16, Example 17, Example 18 and Questions 1, 2 and 3

8H **10** I can solve equations of the form $a\sin(bt) = m$ and $a\cos(bt) = m$ for a given domain of t. ☐

See Example 19 and Questions 7 and 8

8I **11** I can sketch and analyse graphs of the form $a\sin(b(t-h))$ and $a\cos(b(t-h))$. ☐

See Example 20 and Question 1

 ISBN 978-1-009-53719-3

Review

8J **12** I can sketch and analyse graphs of the form $a\sin(b(t-h))+k$ and $a\cos(b(t-h))+k$. ☐

See Example 21, Example 22 and Question 1

8K **13** I can sketch and analyse graphs of the form $a\tan b(t-h))$. ☐

See Example 23, Example 25 and Questions 2 and 4

8K **14** I can solve equations of the form $a\tan(bt)=m$ for a given domain of t. ☐

See Example 24 and Question 3

8L **15** I can use symmetry properties and the Pythagorean identity to evaluate trigonometric functions. ☐

See Example 26, Example 27 and Questions 1 and 3

8M **16** I can determine the lengths of arcs and chords of a circle. ☐

See Example 28 and Question 1

8M **17** I can determine the area of sectors and segments of a circle. ☐

See Example 28, Example 29 and Questions 1 and 4

8N **18** I can apply trigonometric functions to solve application problems. ☐

See Example 30 and Question 1

Short-response questions

Technology-free short-response questions

SF

1 Convert each of the following to radian measure in terms of π:

a $330°$ **b** $810°$ **c** $1080°$ **d** $1035°$ **e** $135°$
f $405°$ **g** $390°$ **h** $420°$ **i** $80°$

2 Convert each of the following to degree measure:

a $\frac{5\pi^c}{6}$ **b** $\frac{7\pi^c}{4}$ **c** $\frac{11\pi^c}{4}$ **d** $\frac{3\pi^c}{12}$ **e** $\frac{15\pi^c}{2}$
f $-\frac{3\pi^c}{4}$ **g** $-\frac{\pi^c}{4}$ **h** $-\frac{11\pi^c}{4}$ **i** $-\frac{23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin\left(\frac{11\pi}{4}\right)$ **b** $\cos\left(-\frac{7\pi}{4}\right)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\cos\left(-\frac{7\pi}{6}\right)$
e $\cos\left(\frac{13\pi}{6}\right)$ **f** $\sin\left(\frac{23\pi}{6}\right)$ **g** $\cos\left(-\frac{23}{3}\pi\right)$ **h** $\sin\left(-\frac{17}{4}\pi\right)$

ISBN 978-1-009-53719-3

4 State the amplitude and period of each of the following:

a $2\sin\left(\frac{\theta}{2}\right)$ **b** $-3\sin(4\theta)$ **c** $\frac{1}{2}\sin(3\theta)$

d $-3\cos(2x)$ **e** $-4\sin\left(\frac{x}{3}\right)$ **f** $\frac{2}{3}\sin\left(\frac{2x}{3}\right)$

5 Sketch the graph of each of the following (showing one cycle):

a $y = 2\sin(2x)$ **b** $y = -3\cos\left(\frac{x}{3}\right)$ **c** $y = -2\sin(3x)$

d $y = 2\sin\left(\frac{x}{3}\right)$ **e** $y = \sin\left(x - \frac{\pi}{4}\right)$ **f** $y = \sin\left(x + \frac{2\pi}{3}\right)$

g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\cos\left(x + \frac{\pi}{6}\right)$

6 Solve each of the following equations for θ:

a $\sin\theta = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$ **b** $\sin(2\theta) = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$

c $\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$ **d** $\sin\left(\theta + \frac{\pi}{3}\right) = -1,\ \theta \in [0, 2\pi]$

e $\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$

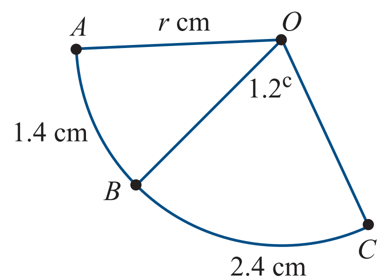
7 The area of a sector of a circle with radius 6 cm is 33 cm^2. Calculate the angle of the sector.

8 A sector of a circle has an arc length of 30 cm. If the radius of the circle is 12 cm, determine the area of the sector.

9 The diagram shows part of a circle with centre O and radius r cm. The length of arc AB is 1.4 cm and the length of arc BC is 2.4 cm. The magnitude of $\angle BOC$ is 1.2 radians. Calculate:

a the value of r

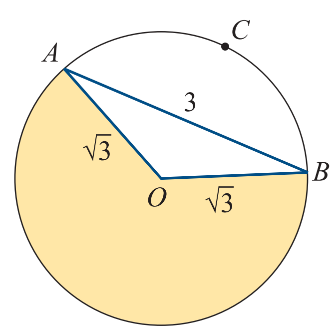
b the magnitude of $\angle AOB$ in radians.



10 The diagram shows a circle of radius length $\sqrt{3}$ cm and a chord AB of length 3 cm. Calculate:

a the length of arc ACB

b the area of the shaded region.



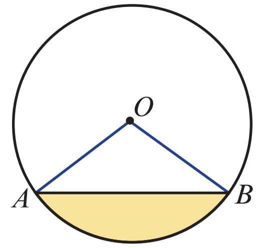
 ISBN 978-1-009-53719-3

Review

CF

Technology-active short-response questions

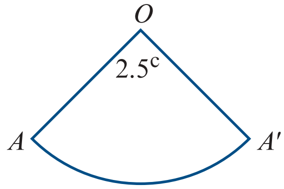
11 In a circle, centre O, radius r, AB is a chord such that $\angle AOB = \theta$ with $0 < \theta < \pi$. The area of the shaded segment is $\frac{1}{4}$ of the area of the circle.



a From the given information write down an equation for θ.

b Determine the value of θ correct to one decimal place.

12 The angle of a sector of a circle, centre O and radius length 12 cm, has magnitude 2.5 radians. The sector is folded so that OA and OA' are joined to form a cone. Calculate:

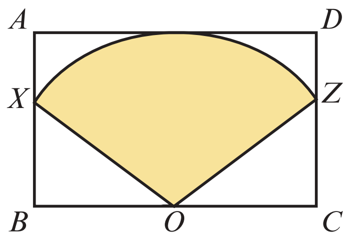


a the base radius length of the cone

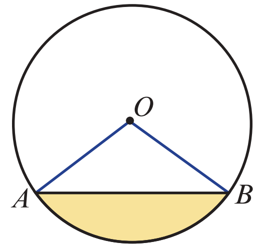
b the curved surface area of the cone

c the shortest distance between two points diametrically opposite on the edge of the base.

13 The diagram opposite shows the rectangular windscreen, $ABCD$ of a vehicle. The width of the windscreen is $1\frac{1}{2}$ times the height. A windscreen wiper is pivoted at O, the midpoint of BC, and wipes the shaded area which is a sector of the circle centred at O. Determine the percentage of the window which is wiped. Give your answer correct to the nearest percent.



14 In a circle, centre O, radius r, AB is a chord such that $\angle AOB = \theta$ with $0 < \theta < \pi$. If the area of triangle AOB is equal to the area of the shaded segment determine the value of θ. (Give your answer correct to one decimal place.)



15 The number of hours of daylight at a point on the Antarctic Circle is modelled by the function

$$d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$$

where t is the number of months which have elapsed since 1 January.

a **i** Determine d on 21 June ($t \approx 5.7$).

ii Determine d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

 ISBN 978-1-009-53719-3 Cambridge University Press

16 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule

$$A = 21 - 3\cos\left(\frac{\pi t}{12}\right), \quad \text{for } 0 \le t \le 24$$

The temperature, B°C, outside the house at the same time is given by

$$B = 22 - 5\cos\left(\frac{\pi t}{12}\right), \quad \text{for } 0 \le t \le 24$$

- **a** Determine the temperature inside the house at 8 a.m.
- **b** Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.
- **c** Sketch the graph of D for $0 \le t \le 24$.
- **d** Determine when the inside temperature is less than the outside temperature.

17 At a certain time of the year the depth of water, d metres, in the harbour at Bunk Island is given by the rule

$$d = 3 + 1.8\cos\left(\frac{\pi}{6}t\right)$$

where t is the time in hours after 3 a.m.

- **a** Sketch the graph of d against t for a 24-hour period from 3 a.m. to 3 a.m.
- **b** At what time(s) does high tide occur for $t \in [0, 24]$?
- **c** At what time(s) does low tide occur for $t \in [0, 24]$?

A ferry operates between Main Beach and Bunk Island. It takes 50 minutes to go from Main Beach to Bunk Island. The ferry only runs between 8 a.m. and 8 p.m., and is only able to enter the harbour at Bunk Island if the depth of water is at least 2 metres.

- **d** What is the earliest time the ferry should leave Main Beach so that it arrives at Bunk Island and can immediately enter the harbour?
- **e** The time to go from Bunk Island to Main Beach is also 50 minutes. The minimum time the ferry takes at Bunk Island harbour is 5 minutes. The minimum time at Main Beach is also 5 minutes.
  - **i** What is the latest time the ferry can leave Main Beach to complete a round trip in 105 minutes?
  - **ii** How many complete round trips can the ferry make in a day?

Multiple-choice questions

Technology-free multiple-choice questions

1 The minimum value of $3 - 10\cos(2x)$ is

A -17 **B** -23 **C** -7 **D** -10

2 The range of the function given by $f(x) = 4\sin\left(2x - \frac{\pi}{2}\right)$, for $x \in [0, 2\pi]$, is

A $[0, 4]$ **B** $[-4, 0]$ **C** $[0, 8]$ **D** $[-4, 4]$

Review

3 The period of the graph of $y = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is

A 3 **B** 4π **C** $\pi + 4$ **D** 2π

4 The graph of $y = \sin x$ is dilated by factor $\frac{1}{2}$ from the y-axis and translated $\frac{\pi}{4}$ units in the positive direction of the x-axis. The equation of the image is

A $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **B** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

C $y = \sin\left(2x - \frac{\pi}{4}\right)$ **D** $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

5 Let $f(x) = a\sin(bx) + c$, where a, b and c are positive constants. The period of the function f is

A b **B** $\frac{2\pi}{a}$ **C** $\frac{2\pi}{b}$ **D** $\frac{b}{2\pi}$

6 One cycle of the graph of $y = \tan(ax)$ has vertical asymptotes at $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$. A possible value of a is

A π **B** $\frac{\pi}{6}$ **C** $\frac{1}{3}$ **D** 3

7 The equation $3\sin(x) + 1 = b$, where b is a positive real number, has one solution in the interval $[0, 2\pi]$. The value of b is

A 1.5 **B** 2 **C** 3 **D** 4

8 The number of solutions of the equation $b = a\sin x$, where $x \in [-2\pi, 2\pi]$ and a and b are positive real numbers with $a > b$, is

A 3 **B** 4 **C** 5 **D** 6

9 An arc PQ of length 7 mm subtends an angle of 18° at the centre O of the circle. The radius of the circle is

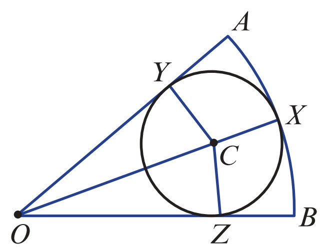
A $\frac{\pi}{70}$ mm **B** $\frac{35}{\pi}$ mm **C** $\frac{70}{\pi}$ mm **D** $\frac{126}{\pi}$ mm

10 The depth of water, in metres, in a harbour at a certain point at time t hours is given by $D(t) = 8 + 2\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$. The depth of the water is first 9 m at

A $t = 1$ **B** $t = 2$ **C** $t = 3$ **D** $t = 4$

11 The circle with centre C of radius 1 is inscribed in the sector of a circle, centre O, of radius 3. We know that $\angle OYC = \angle OZC = 90°$. The length of arc AB is

A π **B** $\frac{\pi}{2}$ **C** $\frac{2\pi}{3}$ **D** $\frac{3\pi}{4}$



12 If $\sin\theta = a$ and $\frac{\pi}{2} < \theta < \pi$ then $\tan\theta =$

A $\frac{a}{1+a^2}$ **B** $-\sqrt{1-a^2}$ **C** $\frac{\sqrt{1-a^2}}{a}$ **D** $-\frac{a}{\sqrt{1-a^2}}$

Technology-active multiple-choice questions

13 In a right-angled triangle, the two shorter side lengths are 3 cm and 4 cm. To the nearest degree, the value of the smallest angle is:

A $1°$ **B** $23°$ **C** $37°$ **D** $53°$

14 For the function $f(x) = -5\sin(0.6x) + 4$, for all real numbers x,which of the following statements is false?

A The range of f is $-1 \le y \le 9$
B The period of f is $\frac{10\pi}{3}$
C The amplitude of f is 5
D $f\left(\frac{4\pi}{9}\right) = \frac{7}{25}$

15 The number of solutions of the equation $\sin x = \frac{x}{10}$ is:

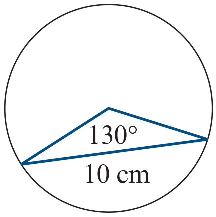
A 1 **B** 3 **C** 5 **D** 7

16 The area of a segment of a circle of radius 8 cm is 20 cm^2. Let θ be the angle subtended by the arc of the segment at the centre of the circle. The area of this segment in terms of θ is $32(\theta - \sin\theta)$. Correct to two decimal places, the approximate value of θ in degrees is:

A $1.62°$ **B** $1.60°$ **C** $93.02°$ **D** $93.03°$

17 The length of the radius of the circle shown, correct to two decimal places, is:

A 5.52 cm **B** 8.36 cm
C 9.01 cm **D** 12.18 cm



18 The period of the function $f(x) = \cos\left(\frac{x}{3}\right) + \sin(2x)$ is:

A π **B** 2π **C** 3π **D** 6π

19 A chord of length 5 cm is drawn in a circle of radius 6 cm. The area of the smaller region inside the circle cut off by the chord, correct to one decimal place, is:

A 1.8 cm^2 **B** 2.3 cm^2 **C** 3.9 cm^2 **D** 13.6 cm^2

9

Probability

Chapter contents

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

However, whether through our prior observations or through a theoretical understanding of the circumstances, we are often able to assign a numerical value, or **probability**, to each possible outcome of an experiment. This probability, which will take a value between 0 and 1, gives us an indication as to how likely we are to observe the outcome. A probability of 0 means that the outcome is impossible and a probability of 1 means that it is certain. Generally, the probability will be somewhere in between, with a higher value meaning that the outcome is more likely.

9A Sets and Venn diagrams

Learning intentions

- To be able to work with **set notation** and use Venn diagrams.

We will start our discussion of probability with a review of sets and set notation. The basic definitions were given in Section 3A.

Sets, elements and subsets

A set is a collection of objects. Some examples of sets are given in the following table.

Set	Description
$A = \{\text{walk, ride, drive, bus}\}$	The methods of transport used by students to get to school today
$B = \{1, 2, 3, 4, 5, 6\}$	The set of natural numbers from 1 to 6
$C = \{ t : 0 < t < 1 \}$	The set of all real numbers strictly between 0 and 1

The objects that are in a set are called the **elements** of the set.

- If x is an element of the set A, we write $x \in A$.
- If x is not an element of the set A, we write $x \notin A$.

For example, for the set A defined in the table above, we have walk $\in A$ and swim $\notin A$.

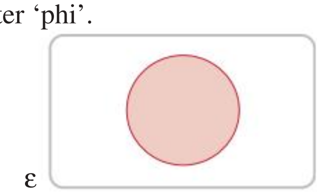
Two sets are **equal** if they have exactly the same elements – they do not have to be listed in the same order. Thus $D = \{1, 2, 3\}$ and $E = \{3, 2, 1\}$ are equal sets.

Set B is a **subset** of set A if every element of B is also an element of A; we write $B \subseteq A$. For example, if $A = \{1, 2, 3, \dots, 20\}$ and $B = \{1, 2, 3, 4, 5, 6\}$, then $B \subseteq A$.

We have special notation for two sets:

- The **universal set**, denoted ε, is the set of all elements which are being considered. The symbol ε is the Greek letter 'epsilon'.
- The **empty set**, denoted $\varnothing$, is the set with no elements. This is different from $\{0\}$, which is a set containing one element, 0. The symbol $\varnothing$ is the Greek letter 'phi'.

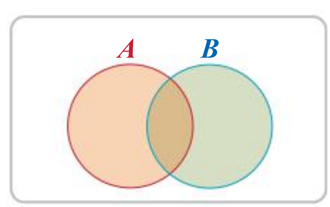
Sets, and the relationships between sets, can be illustrated clearly by using **Venn diagrams**. The universal set ε is usually shown as a rectangle, and a subset as a circle.



Operations on sets

Union of two sets

The **union** of sets A and B, denoted $A \cup B$, is the set of all elements in A or B (or both). This is shown on a Venn diagram by shading both sets A and B.



Union

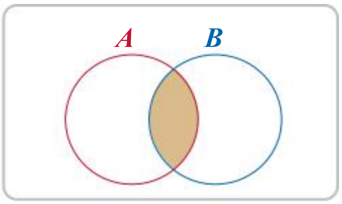
For example, if A is the set of students in a school who play hockey, and B the set of students who play tennis, then the union of A and B is the set of students who play either hockey or tennis or both.

 ISBN 978-1-009-53719-3 Cambridge University Press

Intersection of two sets

The **intersection** of A and B, denoted $A \cap B$, is the set of elements that are in both A and B. This is shown on a Venn diagram by shading only the area contained in both A and B.

For example, the intersection of the two sets previously described is the set of students who play both hockey and tennis.



Intersection

Example 1

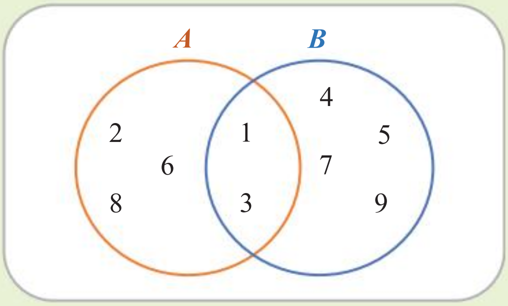
This Venn diagram represents the sets

$$A = \{1, 2, 3, 6, 8\}$$
$$B = \{1, 3, 4, 5, 7, 9\}$$

Determine:

a the union of A and B

b the intersection of A and B



Solution

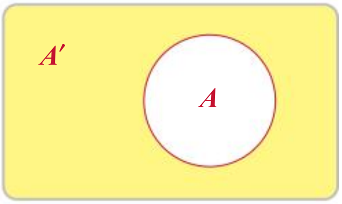
a $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

b $A \cap B = \{1, 3\}$

Complement of a set

The **complement** of A, denoted A', is the set of all elements of ε that are not in A. This is shown on a Venn diagram by shading only the area outside A.

The complement of the set of students in a school who play hockey is the set of students in that school who do not play hockey.

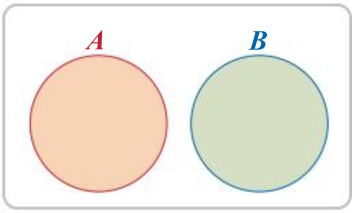


Complement

Mutually exclusive sets

Two sets A and B are said to be **mutually exclusive** or **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$. The Venn diagram opposite shows two sets that are mutually exclusive.

If A is the set of girls in a school who play hockey and B is the set of boys who play hockey, then A and B are mutually exclusive, as no student can belong to both sets.



Mutually exclusive sets

ISBN 978-1-009-53719-3

The number of elements in a set

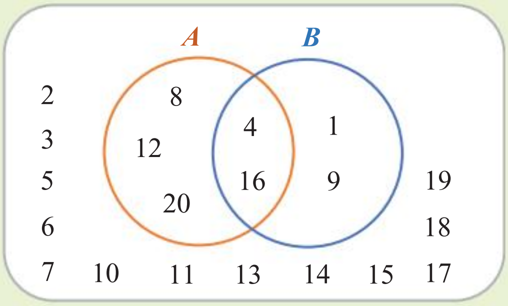
The number of elements in a set A is usually denoted $n(A)$. For example, if $A = \{2, 4, 6\}$, then $n(A) = 3$.

Example 2

Let ε be the set of natural numbers less than or equal to 20. That is, $\varepsilon = \{1, 2, 3, \dots, 20\}$.

Let $A = \{4, 8, 12, 16, 20\}$

$B = \{1, 4, 9, 16\}$



Determine each of the following:

a A' **b** $n(A)$

c $n(A')$ **d** $n(A \cup B)$

e $n((A \cup B)')$

Solution

a $A' = \{1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19\}$

b $n(A) = 5$

c $n(A') = n(\varepsilon) - n(A) = 20 - 5 = 15$

d $n(A \cup B) = 7$

e $n((A \cup B)') = n(\varepsilon) - n(A \cup B) = 20 - 7 = 13$

Venn diagrams can be used to help us solve practical problems involving sets.

Example 3

A group of 50 students were asked what they did on the weekends. A total of 35 said they went to football matches, the movies or both. Of the 22 who went to football matches, 12 said they also went to the movies. Show this information on a Venn diagram.

a How many students went to the movies but not to football matches?

b How many went neither to football matches nor to the movies?

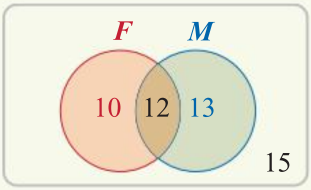
Solution

Let F denote the set of students who attend football matches and M denote the set of students who attend movies.

We are given that $n(F \cup M) = 35$, $n(F) = 22$ and $n(F \cap M) = 12$. Use this information to fill in the Venn diagram.

a Students who go to the movies but not to football matches are found in the region $F' \cap M$, and from the diagram $n(F' \cap M) = 13$.

b Those who attend neither are found in the region $F' \cap M'$, and from the diagram $n(F' \cap M') = 15$.



$n = 50$

ISBN 978-1-009-53719-3

Summary 9A

Set notation

$x \in A$	x is an element of A	$A \cap B$	A intersect B
$x \notin A$	x is not an element of A	$A \cup B$	A union B
$B \subseteq A$	B is a subset of A	A'	A complement
$\varnothing$	the empty set	$n(A)$	the number of elements in A
ε	the universal set		

Skill-sheet

Exercise 9A

SF

1 Write down each of the following sets:

a the set of letters used in the word INTERNET

b the set of vowels used in the word MATHEMATICS

c the set of consonants used in the word PROBABILITY

2 Let $A = \{1, 2, 3, 4, 5\}$. Insert the correct symbol $\in$ or $\subseteq$ in each of the following:

a $2 \ \square \ A$ **b** $\{1, 2, 5\} \ \square \ A$ **c** $3 \ \square \ A$ **d** $\{3\} \ \square \ A$ **e** $A \ \square \ A$

3 Write down each of the following sets:

a the possible outcomes when a coin is tossed

b the possible outcomes when a six-sided dice is rolled

4 Which of the following is the empty set?

a the set of odd numbers divisible by 2 **b** the set of prime numbers divisible by 2

5 State whether each of the following is true or false:

a $\{1, 7, 9\} = \{9, 7, 1\}$ **b** $\{4, 9, 6, 2\} \neq \{6, 2, 4, 9\}$ **c** $\{a, b, c, c, d\} = \{a, b, c\}$

d $\{2, 3, 3, 4, 4\} = \{2, 3, 4\}$ **e** $\{5, 4\} \neq \{5, 4, 4, 5\}$ **f** $\{8, 3\} = \{3, 3, 8\}$

g {vowels in the word STUDENT} = {vowels in the word TRUE}

h {letters in the word ELEMENT} = {E, L, M, N, T}

Example 1

6 Write down the union and intersection of each of the following pairs of sets:

a $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 3, 5, 7, 9\}$

b $M = \{a, b, c, d, e\}$, $N = \{a, c, e, f, g\}$

c X = {letters in the word MOBILE}, Y = {letters in the word TABLET}

7 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$.
Show these sets on a Venn diagram and use your diagram to determine:

a $A \cup B$ **b** $A \cap B$ **c** A' **d** $A \cap B'$ **e** $(A \cap B)'$ **f** $(A \cup B)'$

8 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, A = {multiples of four}, B = {even numbers}.
Show these sets on a Venn diagram and use your diagram to determine:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

 ISBN 978-1-009-53719-3

SF

9 ε = {letters in the word MATHEMATICS},
A = {letters in the word ATTIC}, B = {letters in the word TASTE}
Show ε, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

10 Determine the number of elements in each of the following sets:

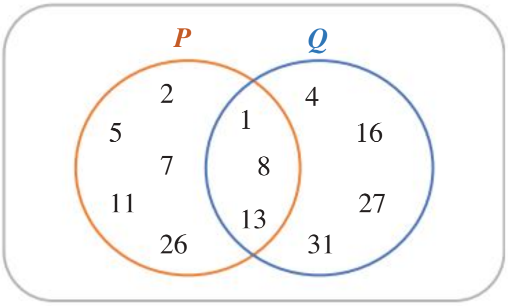
a the set of months in a year
b the set of even numbers between 0 and 20 inclusive
c the set of letters in the word CALCULUS

11 If $A = \{5, 7, 8, 9\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 4, 6, 8, 10\}$, determine:

a $n(A) + n(B)$ **b** $n(A \cup B)$ **c** $n(A \cap B)$
d $n(A \cup B) + n(A \cap B)$ **e** $n(B \cup C)$ **f** $n(B) + n(C) - n(B \cap C)$

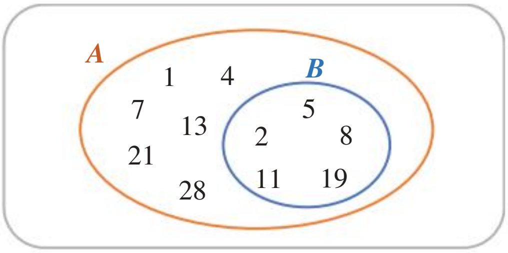
Example 2

12 From the given Venn diagram, determine each of the following:



a P **b** P'
c Q **d** $P \cup Q$
e $P \cap Q$ **f** $n(P)$
g $n(P')$ **h** $n(Q)$
i $n(P \cup Q)$ **j** $n((P \cup Q)')$
k $n(P \cap Q)$

13 From the given Venn diagram, determine each of the following sets:



a A **b** B
c $A \cup B$ **d** $A \cap B$
e $A \cap B'$

CF

14 Construct a Venn diagram to represent sets A and B such that $n(A) = 15$, $n(B) = 25$ and $n(A \cup B) = 37$. Use your Venn diagram to determine $n(A \cap B)$.

Example 3

15 In a group of 200 people, 110 can speak French, 95 can speak German and 30 can speak neither. Use a Venn diagram to determine how many people in the group can speak:

a both French and German **b** either French or German, but not both.

16 Two sets X and Y are such that X has 25 elements, Y has 30 elements and $X \cap Y$ has 15 elements. Use a Venn diagram to determine the number of elements in $X \cup Y$.

17 Of the Year 11 students at a certain school, 50 students play basketball, 25 play football and 12 play both. How many play at least one of the two games?

18 In the most recent exams at a certain school, 70 girls received As in either chemistry or mathematics or both. Out of these, 40 girls received an A in mathematics, and 10 girls in both chemistry and mathematics. How many girls received an A in chemistry only?

 ISBN 978-1-009-53719-3 Cambridge University Press

CF

19 In a group of 100 people, 42 regularly jog, 30 regularly cycle, and eight regularly jog and cycle. Determine the number of people in the group who like:

a jogging only **b** cycling only **c** at least one of them **d** neither of them.

20 In a group of students, 23 study a language, 19 study mathematics, eight study both a language and mathematics, and two study neither mathematics nor a language.

a How many students are in the group?

b How many students in the group study mathematics but not a language?

c How many students in the group study a language but not mathematics?

9B Sample spaces and probability

Learning intentions

- To be able to use set language and notation for outcomes, samples spaces and events.

We now look at two essential components of probability: a sample space, which is the set of all possible outcomes of an experiment, and a set of probabilities, one for each outcome.

Sample spaces

Suppose we toss a coin to see whether a head (H) or a tail (T) appears uppermost. The toss of the coin can be termed a single **trial** of a **random experiment**. The word ‘random’ is used here because, while the outcome observed must be either a head or a tail, on a particular toss we don’t know which will be observed. However, we do know that the outcome observed will be one of a known set of possible outcomes, and the set of all possible outcomes is called the **sample space** for the experiment.

Set notation can be used in listing all the elements in the sample space. For example, the sample space for the tossing of a coin would be written as

$$\{H, T\}$$

where H indicates head and T indicates tail. Throughout this chapter, we use ε to denote the sample space.

The following table gives some examples of random experiments and their sample spaces.

Random experiment	Sample space
The number observed when a dice is rolled	$\varepsilon = \{1, 2, 3, 4, 5, 6\}$
The number of brown eggs in a carton of 12 eggs	$\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed	$\varepsilon = \{HH, HT, TH, TT\}$
The number of calls to your phone in the next two hours	$\varepsilon = \{0, 1, 2, 3, 4, \dots\}$
The time, in hours, it takes to complete your homework	$\varepsilon = \{\, t : t \geq 0 \,\}$

 ISBN 978-1-009-53719-3 Cambridge University Press

Events

An **event** is a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes. For example, when rolling a dice, the event of interest may be 'getting a six', which consists of just one outcome and is described by the set $\{6\}$. However, the event 'getting an odd number' can be achieved by rolling 1, 3 or 5 and is described by the set $\{1, 3, 5\}$.

It is convenient to use set notation to list the elements of the event. In general we use capital letters, $A, B, C, \dots$, to denote events.

The following table lists the experiments described earlier and gives the sample space and an example of an event for each one.

Sample space	An event
The number observed when a dice is rolled $\varepsilon = \{1, 2, 3, 4, 5, 6\}$	'An even number' $= \{2, 4, 6\}$
The number of brown eggs in a carton of 12 eggs $\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$	'More than half brown' $= \{7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed $\varepsilon = \{HH, HT, TH, TT\}$	'Two heads' $= \{HH\}$
The number of calls to your phone in the next two hours $\varepsilon = \{0, 1, 2, 3, 4, \dots\}$	'Fewer than two phone calls' $= \{0, 1\}$
The time, in hours, it takes to complete your homework $\varepsilon = \{\, t : t \geq 0 \,\}$	'More than two hours' $= \{\, t : t > 2 \,\}$

Note: Both a sample space and an event can be discrete or continuous, finite or infinite.

Example 4

A bag contains seven marbles numbered from 1 to 7 and a marble is withdrawn.

a Give the sample space for this experiment.

b List the outcomes (elements) of the event 'a marble with an odd number is withdrawn'.

Solution	Explanation
a $\{1, 2, 3, 4, 5, 6, 7\}$	Any number from 1 to 7 could be observed.
b $\{1, 3, 5, 7\}$	This set contains the odd numbers in the sample space.

 ISBN 978-1-009-53719-3 Cambridge University Press

Determining probabilities for equally likely outcomes

There are many situations for which we can develop a simple model that can be used to assign a probability to an event. The most obvious of these is when it is reasonable to assume that all of the outcomes are equally likely, such as when a dice is rolled.

Probability of an outcome

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$.

Since six outcomes are possible when rolling a dice, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$$

Probability of an event

When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the dice. Then $A = \{2, 4, 6\}$ and so

$$\begin{aligned} P(A) &= P(2) + P(4) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2} \end{aligned}$$

Since the outcomes are equally likely, we can calculate this more easily as

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$P(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

Of course, there are many situations where the outcomes are not equally likely. For example, it has been established worldwide that the probability of a male birth is in fact 0.51, not 0.5. However, in many situations the assumption of equally likely is justified, and allows us to assign probabilities reasonably.

 ISBN 978-1-009-53719-3

Example 5

Suppose a number is drawn at random from the numbers 7, 8, 9, 10, 11, 12, 13, 14. What is the probability of choosing a prime number?

Solution

Let A be the event the chosen number is prime. Then

$$P(A) = \frac{3}{8}$$

Explanation

Since the number is drawn at random, we can assume each number is equally likely to be drawn.

$A = \{7, 11, 13\}$, $n(A) = 3$, $n(\varepsilon) = 8$

Example 6

Suppose that a card is drawn from a pack of 52 playing cards, and that each card has equal likelihood of being drawn. Determine:

a the probability that the card is black

b the probability that the card is a king

c the probability that the card is a black king.

Solution

a $P(\text{black card}) = \dfrac{26}{52} = \dfrac{1}{2}$

b $P(\text{king}) = \dfrac{4}{52} = \dfrac{1}{13}$

c $P(\text{black king}) = \dfrac{2}{52} = \dfrac{1}{26}$

Explanation

There are 52 cards in a pack and 26 are black.

There are 52 cards in a pack and 4 are kings.

There are 52 cards in a pack and 2 are black kings.

The following rules of probability hold for finite sample spaces:

- $P(A) \geq 0$, for any event A.
- The sum of the probabilities of all the outcomes of a random experiment must equal 1.

The second of these two rules can be used to determine probabilities as follows.

Example 7

A random experiment may result in 1, 2, 3 or 4. If $P(1) = \dfrac{1}{13}$, $P(2) = \dfrac{2}{13}$ and $P(3) = \dfrac{3}{13}$, calculate the probability of obtaining a 4.

Solution

$$P(4) = 1 - \left(\frac{1}{13} + \frac{2}{13} + \frac{3}{13}\right)$$
$$= 1 - \frac{6}{13} = \frac{7}{13}$$

Explanation

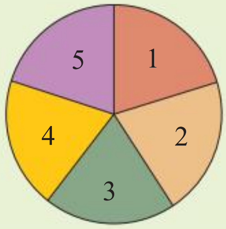
The sum of the probabilities is 1.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 8

For each spinner, determine the probability that each of the possible outcomes is observed:

a



b

Solution

a $P(1) = P(2) = P(3) = P(4) = P(5) = \frac{1}{5}$

b $P(1) = P(2) = P(3) = \frac{1}{8} = 0.125$

$P(4) = \frac{2}{8} = \frac{1}{4} = 0.25$

$P(5) = \frac{3}{8} = 0.375$

Note that in both these cases
$P(1) + P(2) + P(3) + P(4) + P(5) = 1$

Explanation

On spinner a, there are five equally likely outcomes.

Since there are 8 equal segments, we assume each has a probability of $\frac{1}{8}$.

The results 1, 2 and 3 appear once.
The result 4 appears twice.
The result 5 appears three times.

Complementary events

When two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of event A is the event A', which consists of all the outcomes in ε that are not in A. Since the sum of the probabilities is 1, we can write

$$P(A') = 1 - P(A)$$

Example 9

A card is drawn at random from a pack of 52 cards. What is the probability that the card is:

a not a heart

b not an ace?

Solution

a Let H be the event a heart is drawn.

Then
$$\begin{aligned} P(H') &= 1 - P(H) \\ &= 1 - \frac{13}{52} \\ &= 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

b Let A be the event an ace is drawn.

Then
$$\begin{aligned} P(A') &= 1 - P(A) \\ &= 1 - \frac{4}{52} \\ &= 1 - \frac{1}{13} = \frac{12}{13} \end{aligned}$$

 ISBN 978-1-009-53719-3

Combining our knowledge of the rules of probability enables us to solve more complex problems.

Example 10

A random experiment may result in outcomes A, B, C, D or E, where A, B, C, D are equally likely and E is twice as likely as A. Determine:

a $P(E)$ **b** $P(B')$

Solution

a Let $P(A) = P(B) = P(C) = P(D) = x$.
Then $P(E) = 2x$.

$$x + x + x + x + 2x = 1$$
$$6x = 1$$
$$x = \frac{1}{6}$$

Thus $P(E) = 2x = \frac{1}{3}$

b $P(B') = 1 - P(B)$
$$= 1 - \frac{1}{6} = \frac{5}{6}$$

Explanation

Summarise the information in the question in terms of one unknown.

The sum of the probabilities is 1.

Since B' is the complement of B, the probabilities will add to 1.

Summary 9B

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $P(A)$.
- Rules of probability for finite sample spaces:
  - $P(A) \geq 0$, for each event A.
  - The sum of the probabilities of all the outcomes of a random experiment must be equal to 1.
- **Equally likely outcomes** If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$P(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- If two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of any event A is denoted A' and we can write

$$P(A') = 1 - P(A)$$

Exercise 9B

Example 4

SF

1 List the sample space for the toss of a coin.

2 List the sample space for the outcomes when a dice is rolled.

3 List the sample spaces for the following experiments:
  a the number of aces in a hand of five cards
  b the number of female children in a family with six children
  c the number of female students on a committee of three students chosen from a class of 10 male and 10 female students

4 List the sample spaces for the following experiments:
  a the number of cars which pass through a particular intersection in a day
  b the number of people on board a bus licensed to carry 40 passengers
  c the number of times a dice is rolled before a six is observed

5 List the outcomes associated with the following events:
  a 'an even number' when a dice is rolled
  b 'more than two female students' when three students are chosen for a committee from a class of 10 male and 10 female students
  c 'more than four aces' when five cards are dealt from a standard pack of 52 cards

Example 5

6 A number is drawn at random from the set $\{1, 2, 3, \ldots, 20\}$. What is the probability that the number is:
  a divisible by 2 b divisible by 3 c divisible by both 2 and 3?

7 A bag has 15 marbles numbered $1, 2, 3, \ldots, 15$. If one marble is drawn at random from the bag, what is the probability that the number on the marble is:
  a less than 5 b greater than or equal to 6
  c a number from 5 to 8 inclusive?

Example 6

8 A card is drawn at random from a pack of 52 cards. Determine the probability that the card is:
  a a club b red
  c a picture card (jack, queen, king) d a red picture card.

9 A card is drawn at random from a pack of 52 cards. Determine the probability that the card is:
  a less than 10 b less than or equal to 10
  c an even number d an ace

10 Suppose that in a certain city the same number of people were born on each of the 365 days of the year, and that nobody was born on 29 February. Determine the probability that the birthday of a person selected at random:

a is 29 November **b** is in November

c falls between 15 January and 15 February, not including either day

d is in the first three months of the year.

11 One letter is drawn at random from the nine letters in the word AUSTRALIA. Determine the probability that the letter is:

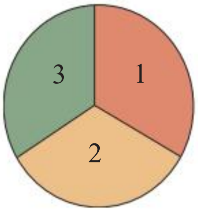
a a T **b** an A **c** a vowel **d** a consonant.

Example 7 **12** A random experiment results in 1, 2, 3, 4, 5 or 6. If $P(1) = \frac{1}{12}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{8}$, $P(5) = \frac{1}{6}$ and $P(6) = \frac{1}{8}$, determine the probability of obtaining a 4.

13 A random experiment results in 1, 2, 3 or 4. If $P(1) = 0.2$, $P(3) = 0.1$ and $P(4) = 0.3$, determine $P(2)$.

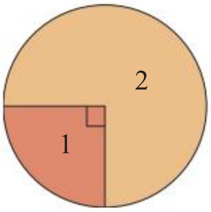
Example 8 **14** For each spinner, what is the chance of the pointer stopping in region 1?

a



b

c



Example 9 **15** Assume that the probability of a baby being born on a certain day is the same for each day of the week. Determine the probability that a randomly chosen person was born:

a on a Wednesday **b** not on the weekend.

16 A card is drawn at random from a pack of 52 cards. Determine the probability that the card is:

a not a club **b** not red

c not a picture card (jack, queen, king) **d** not a red picture card.

Example 10 **17** A random experiment results in 1, 2, 3 or 4. If 1, 2 and 3 are equally likely to occur, and 4 is twice as likely to occur as 3, determine the probability of each of the possible outcomes.

18 For a particular biased six-sided dice it is known that the numbers 2, 3, 4 and 5 are equally likely to occur, that the number 6 occurs twice as often as the number 2, and that the number 1 occurs half as often as the number 2.

a Determine the probability of each of the possible outcomes.

b Determine the probability that the number observed is not a 6.

SF

CF

9C Estimating probabilities

Learning intentions

- To be able to use relative frequencies to estimate probabilities.

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of estimating probability are required.

Subjective probabilities

Sometimes, the probability is assigned a value just on the basis of experience. For example, a sports journalist may suggest that Australia has a 60% chance of winning the next Ashes series relying on their own judgement. Another journalist might well assign this probability an entirely different value. Such probabilities are called subjective probabilities, and whether or not they are accurate estimates of the true probability would be open to dispute.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment leading to the event of interest many times and recording the results. This information can then be used to estimate the chances of it happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

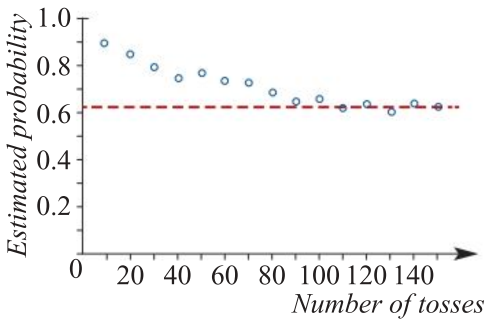
Suppose, for example, that we are interested in determining the probability that a drawing pin will land 'point up' when it is tossed. Since a drawing pin is not symmetrical, the assumption of equally likely outcomes cannot be used to determine probabilities.

Our strategy to estimate this probability is to toss the drawing pin many times and count the number of times it lands point up. We can then calculate the relative frequency:

$$\text{Relative frequency of 'point up'} = \frac{\text{number of times drawing pin landed 'point up'}}{\text{number of trials}}$$

This proportion, or relative frequency, is an estimate of the probability of a drawing pin landing with the point up.

The graph opposite shows the results of one experiment where a drawing pin is tossed 150 times, with the probability of the drawing pin landing point up estimated every 10 throws.



From the graph it may be seen that, as the number of trials (repetitions of the experiment) increases, the estimated probability converges to a value and then stays fairly stable.

In general, when the same experiment is repeated several times, the relative frequency of a particular event is called a point estimate of the probability of that event.

 ISBN 978-1-009-53719-3 Cambridge University Press

If the same experiment is repeated many, many times, then the relative frequency of the event will stabilise to a constant value. This limiting value of the relative frequency is considered to be the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $P(A)$. That is,

$$P(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials.}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and will be a topic of your later studies in mathematics. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 11

In order to investigate the probability that a drawing pin lands point up, Katia decides to toss it 50 times and to count the number of favourable outcomes, which turns out to be 33. Mikki repeats the experiment, but she tosses the same drawing pin 100 times and counts 62 favourable outcomes.

a What is Katia's estimate of the probability of the drawing pin landing point up?

b What is Mikki's estimate?

c Which of these is the preferred estimate of the probability from these experiments?

d Based on the information available, what would be the preferred estimate of the probability?

Solution

a From Katia's information: $P(\text{point up}) \approx \dfrac{33}{50} = 0.66$

b From Mikki's information: $P(\text{point up}) \approx \dfrac{62}{100} = 0.62$

c Since Mikki has estimated the probability from a larger number of trials, her estimate would be preferred to Katia's.

d Based on the information available, the preferred estimate of the probability would be found by combining the data from both experiments, and so maximising the number of trials. In total, 95 favourable outcomes were observed in 150 tosses, and this gives a 'best' estimate of the probability of $\dfrac{95}{150} = 0.63$.

 ISBN 978-1-009-53719-3

Probability can be considered as the proportion of times that an event will occur in the long run. This interpretation also defines the minimum and maximum values of probability as 0 (the event never occurs) and 1 (the event always occurs), and confirms that the sum of the probabilities for all possible outcomes will equal 1.

Probabilities from random samples

Another way to estimate a probability is to take a random sample.

For example, suppose that we are interested in the probability that a fish selected from a certain lake is a juvenile. We could randomly choose a sample of 50 fish from the lake, and determine how many of these are juvenile. This proportion of juvenile fish in the sample can then be used as an estimate for the proportion of juvenile fish in the lake.

Example 12

A researcher is interested in estimating the probability that a parent intends to have their child immunised against a certain disease.

a A sample of 200 parents is chosen randomly and 183 say that they will have their child immunised. Based on this sample, estimate the probability that a parent intends to have their child immunised.

b In an additional sample of 300 parents, 87% intend to have their child immunised. By combining both samples, estimate the probability that a parent intends to have their child immunised.

Solution

a From the first sample: $P(\text{immunise}) \approx \dfrac{183}{200} = 0.915$

b From both samples:

$$P(\text{immunise}) \approx \frac{183 + 0.87 \times 300}{200 + 300}$$

$$= \frac{444}{500} = 0.888$$

Random sampling is studied further in Mathematical Methods Units 3 & 4.

Simulation

The word simulate means to pretend or to imitate. In statistics, simulation is a way to model a random experiment, such that simulated outcomes closely match real-world outcomes. Simulation does not involve repeating the actual experiment. Instead, more complex probabilities can be estimated via multiple trials of an experiment which approximates the actual experiment, but can be carried out quickly and easily. A more detailed discussion of simulation is found in Section 9I.

Probabilities from area

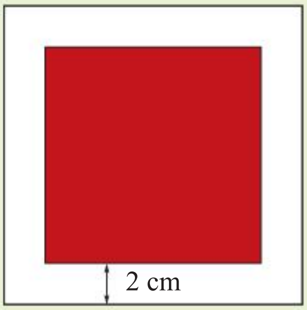
We previoulsy used the model of equally likely outcomes to determine probabilities. We counted both the outcomes in the event and the outcomes in the sample space, and used the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that two events represented by equal areas in the region are equally likely.

Example 13

Suppose that a square dartboard consists of a red square drawn inside a larger white square of side length 12 cm, as shown.

If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the red area? (Ignore the possibility that it might land on the line or miss the board altogether!)



Solution

$$P(\text{landing in the red area}) = \frac{\text{area of red square}}{\text{area of dartboard}} = \frac{64}{144} = \frac{4}{9}$$

Explanation

There are really only two outcomes for this experiment: landing in the red area or landing in the white area.

Assume that the probability of landing in an area is proportional to the size of the area.

Summary 9C

- When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$P(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \quad \text{for a large number of trials}$$

- Whichever method of estimating probability is used, the rules of probability hold:
  - $P(A) \geq 0$, for each event A
  - $P(\varepsilon) = 1$
  - The sum of the probabilities of all the outcomes of a random experiment equals 1.

Skill-sheet

Exercise 9C

Example 11

1 Estimate the probability of the event specified occurring, using the data given:

a P(head) if a coin is tossed 100 times and 34 heads observed

b P(ten) if a spinner is spun 200 times and lands on the 'ten' 20 times

c P(two heads) if two coins are tossed 150 times and two heads are observed on 40 occasions

d P(three sixes) if three dice are rolled 200 times and three sixes observed only once

SF

2 A student decides to toss two coins and notes the results.

a Do you think relative frequencies obtained from 20 trials would make for a good estimate of the probabilities?

b Perform the experiment 20 times and estimate P(two heads), P(one head) and P(no heads).

c Combine your results with those of your friends, so that you have results from at least 100 trials. Use these results to again estimate the probabilities.

d Do you think the data from 100 trials give better estimates of the probabilities?

e How many trials would you need to determine the probabilities exactly?

CF

3 Two misshapen six-sided dice were used for the following experiment. The first dice was thrown 500 times and 78 sixes were observed. The second dice was thrown 700 times and 102 sixes were observed. If you wished to throw a six, which dice would you choose to throw, and why?

4 A bowl contains 340 red and 60 black balls.

a State the proportion of red balls in the bowl.

b A random sample of 60 balls is taken from the bowl and is found to have 48 red balls. Determine the proportion of red balls in the sample.

c Another random sample of 60 balls is taken from the bowl and is found to have 54 red balls. Determine the proportion of red balls in the sample.

d What is the expected number of red balls in a sample of 60?

Example 12

5 In a survey of 2000 people, 890 indicated that they regularly use social media to keep in touch with friends. What is an estimate for the probability that the next person surveyed also uses social media?

SF

6 A random sample of students from a certain town were asked how they travel to school. The table on the right gives the number of students using each method of transport and each student only uses one method. Estimate the probability that a student in this town:

Method of transport	Number of students
Walk	17
Bicycle	22
Bus	38
Car	23

a walks to school

b does not catch the bus to school

c either walks or bicycles to school.

SF

7 A randomly chosen sample of 400 factory workers were classified as to whether or not they smoked and how many days of sick leave they used in the last year.

Days of sick leave	Smoker	Non-smoker	Total
Less than 10	34	260	294
10 or more	78	28	106
Total	112	288	400

Use the data in the table to estimate the probability that a worker in that factory will:

a both smoke and use 10 or more days of sick leave

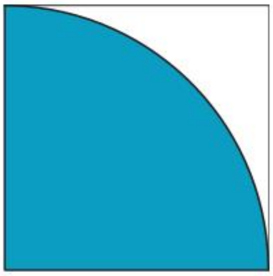
b smoke

c use 10 or more days of sick leave.

CF

Example 13

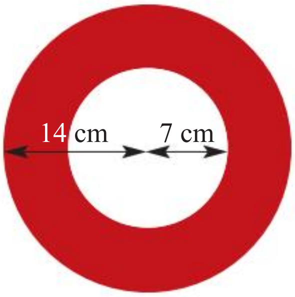
8 A square of side length 1 metre contains a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.



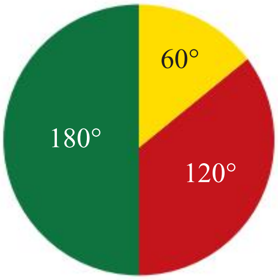
a What proportion of the square is blue?

b If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, determine the probability of it hitting the blue region.

9 A dart is thrown at random onto a board that has the shape of a circle as shown. Calculate the probability that the dart will hit the shaded region.



10 A spinner is as shown in the diagram.
Determine the probability that when spun the pointer will land on:

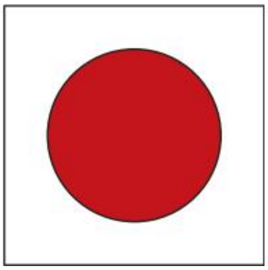


a the red section

b the yellow section

c any section except the yellow section.

11 In a sideshow at a fete, a dart is thrown at a square with side length 1 metre. The circle shown has a radius of 0.4 metres. The dart is equally likely to hit any point on the square. Determine the probability that the dart will hit:



a the shaded part of the square

b the unshaded part of the square.

 ISBN 978-1-009-53719-3 Cambridge University Press

9D Multi-stage experiments

Learning intentions

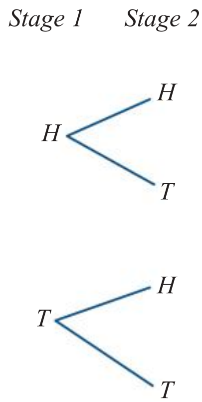
- To be able to systematically list the outcomes of a multistage experiment

Often we are interested in probabilities arising from more complex **multi-stage experiments**, that is, experiments which can be considered to take place in more than one stage.

For example, when considering the outcomes from tossing two coins (or tossing one coin twice) we should consider the possible outcomes in two stages:

- the outcome from coin 1
- followed by the outcome from coin 2.

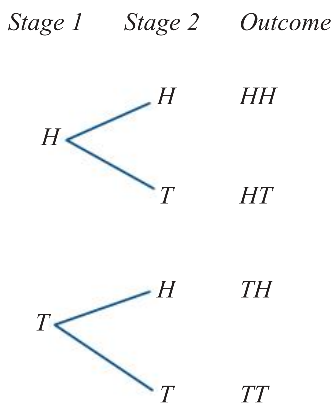
In such cases, it is helpful to list the elements of the sample space systematically by means of a **tree diagram** as shown.



Each path along the branches of the tree gives an outcome, which we determine by reading along the branches, making sure we maintain the order of the outcome at each stage carefully.

Collecting together all the outcomes, we see that the sample space is

$$\varepsilon = \{HH, HT, TH, TT\}$$



When the outcomes of the multi-stage experiment are equally likely, we can still determine the probability of an event occurring by dividing the number of outcomes in the event by the number of outcomes in the sample space.

Example 14

Determine the probability that when a fair coin is tossed twice:

a one head is observed

b at least one head is observed

c both heads or both tails are observed.

Solution	Explanation
a $P(\text{one head}) = \frac{2}{4} = \frac{1}{2}$	List the outcomes of the event: 'one head' $= \{HT, TH\}$. There are 2 outcomes in the event and 4 in the sample space (see tree diagram).

ISBN 978-1-009-53719-3

b $P(\text{at least one head}) = \frac{3}{4}$	List the outcomes of the event: 'at least one head' $= \{HH, HT, TH\}$. There are 3 outcomes in the event and 4 in the sample space.
c $P(\text{both heads or both tails})$ $= \frac{2}{4} = \frac{1}{2}$	List the outcomes of the event: 'both heads or both tails' $= \{HH, TT\}$. There are 2 outcomes in the event and 4 in the sample space.

When listing the outcomes for a two-stage experiment, it can also be convenient to display the sample space in a table.

For example, when rolling two dice (or a single dice twice) there is the possibility of $\{1, 2, 3, 4, 5, 6\}$ on dice 1 (or the first roll), and $\{1, 2, 3, 4, 5, 6\}$ on dice 2 (or the second roll). So the sample space for this experiment can be written as:

		dice 2					
		1	2	3	4	5	6
dice 1	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Example 15

Determine the probability that when two fair dice are rolled:

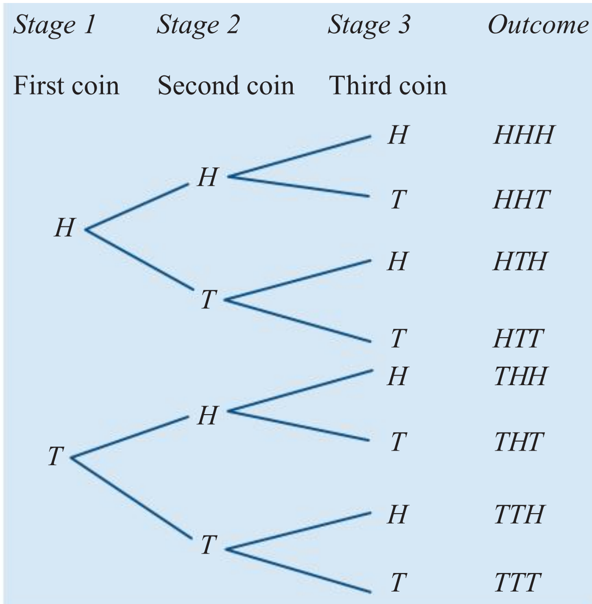
a the same number shows on both dice (a double)

b the sum of the two numbers shown is greater than 10.

Solution	Explanation
a $P(\text{double}) = \frac{6}{36} = \frac{1}{6}$	'double' $= \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$. There are 6 outcomes in the event and 36 in the sample space.
b $P(\text{sum} > 10) = \frac{3}{36} = \frac{1}{12}$	'sum is greater than 10' $= \{(5, 6), (6, 5), (6, 6)\}$. There are 3 outcomes in the event and 36 in the sample space.

 ISBN 978-1-009-53719-3

When the experiment involves more than two stages, it is best to use a **tree diagram** to determine all of the possible outcomes. Suppose, for example, that three coins are tossed and the outcomes noted. The three-stage tree diagram for listing the sample space for this experiment is as follows:



Thus the required sample space is

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Example 16

Determine the probability that when a coin is tossed three times:

a one head is observed

b at least one head is observed

c the second toss results in a head

d all heads or all tails are observed.

Solution	Explanation
a $P(\text{one head}) = \frac{3}{8}$	'one head' $= \{HTT, THT, TTH\}$. There are 3 outcomes in the event and 8 in the sample space.
b $P(\text{at least one head}) = \frac{7}{8}$	'at least one head' $= \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$. There are 7 outcomes in the event and 8 in the sample space.

c $P(\text{second toss is a head})$ $= \frac{4}{8} = \frac{1}{2}$	'second toss is a head' $= \{HHH, HHT, THH, THT\}$. There are 4 outcomes in the event and 8 in the sample space.
d $P(\text{all heads or all tails})$ $= \frac{2}{8} = \frac{1}{4}$	'all heads or all tails' $= \{HHH, TTT\}$. There are 2 outcomes in the event and 8 in the sample space.

Summary 9D

The sample space for a two-stage experiment can be displayed using a tree diagram or a table. If an experiment involves more than two stages, then a tree diagram should be used.

Exercise 9D

SF

Example 14

1 Two fair coins are tossed. Use the sample space for this experiment to determine the probability of observing:

a no heads **b** more than one tail.

2 A fair coin is tossed twice. Determine the probability that:

a the first toss is a head **b** the second toss is a head **c** both tosses are heads.

Example 15

3 Two regular dice are rolled. Use the sample space for this experiment to determine the probability that the sum of the numbers showing is:

a even **b** 3 **c** less than 6.

4 Two regular dice are rolled. Use the sample space for this experiment to determine the probability that the sum of the numbers showing is:

a equal to 10 **b** odd **c** less than or equal to 7.

Example 16

5 A fair coin is tossed three times. Use the sample space for this experiment to determine the probability that:

a exactly one tail is observed **b** exactly two tails are observed
c exactly three tails are observed **d** no tails are observed.

6 A fair coin is tossed three times. Use the sample space for this experiment to determine the probability that:

a the third toss is a head
b the second and third tosses are heads
c at least one head and one tail are observed.

 ISBN 978-1-009-53719-3

SF

7 An experiment consists of rolling a dice and tossing a coin. Use a tree diagram to list the sample space for the experiment. Determine the probability of obtaining a head and an even number.

8 Two coins are tossed and a dice is rolled.

a Draw a tree diagram to show all the possible outcomes.

b Determine the probability of observing:

i two heads and a 6

ii one head, one tail and an even number

iii two tails and an odd number

iv an odd number on the dice.

9 Madison has a choice of two entrees (soup or salad), three main courses (fish, chicken or steak) and three desserts (ice-cream, lemon tart or cheese).

a Draw a tree diagram to show all her possible dinner combinations.

b If Madison chooses all three courses, and is equally likely to choose any of the options at each course, determine the probability that:

i she chooses soup, fish and lemon tart

ii she chooses fish

iii she chooses salad and chicken

iv she doesn't have the lemon tart.

c Suppose Madison has the choice to omit the entree and/or the dessert course altogether. Determine the probability that:

i she chooses soup, fish and lemon tart

ii she chooses all three courses

iii she chooses only two courses

iv she has only the main course.

10 A bag contains five balls, numbered 1 to 5. A ball is chosen at random, the number noted and the ball replaced. A second ball is then chosen at random and its number noted.

a Draw up a table of ordered pairs to show the sample space for the experiment.

b Determine the probability that:

i the sum of the two numbers is 5

ii the two numbers are different

iii the second number is two more than the first.

 ISBN 978-1-009-53719-3

9E The addition rule

Learning intentions

- To be able to use the **addition rule** to calculate probabilities.

Venn diagrams were introduced in the first section of this chapter. In this section we use Venn diagrams to determine probabilities.

Example 17

A group of 50 students were asked what they did on the weekends. A total of 35 said they went to football matches, the movies or both. Of the 22 who went to football matches, 12 said they also went to the movies.

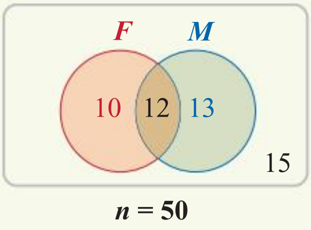
What is the probability that a student chosen at random from this group:

a went to the movies but not to football matches

b went neither to football matches nor to the movies?

Solution

Let F denote the set of students who attend football matches and M denote the set of students who attend movies. Use the given information to fill in the Venn diagram.

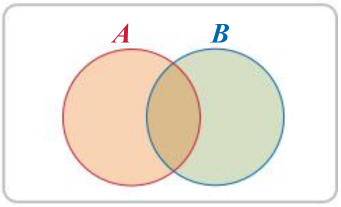


To determine each probability, divide the size of the event by the size of the sample space:

a $P(F' \cap M) = \dfrac{n(F' \cap M)}{n(\varepsilon)} = \dfrac{13}{50}$

b $P(F' \cap M') = \dfrac{n(F' \cap M')}{n(\varepsilon)} = \dfrac{15}{50} = \dfrac{3}{10}$

Venn diagrams can be used to illustrate a very important rule that will enable us to calculate probabilities for more complex events. If A and B are two events in a sample space ε and $A \cap B \neq \emptyset$, then the relationship between them can be represented by a Venn diagram, as shown.



From the Venn diagram we can see that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(As the intersection has been counted twice, in both $n(A)$ and $n(B)$, we must subtract it.)

Dividing through by $n(\varepsilon)$ gives

$$\frac{n(A \cup B)}{n(\varepsilon)} = \frac{n(A)}{n(\varepsilon)} + \frac{n(B)}{n(\varepsilon)} - \frac{n(A \cap B)}{n(\varepsilon)}$$

Now, if each of the outcomes in ε is equally likely to occur, then each term in this expression is equal to the probability of that event occurring. This can be rewritten as:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This is called the **addition rule** for combining probabilities.

 ISBN 978-1-009-53719-3 Cambridge University Press

Addition rule

The probability of A or B or both occurring can be calculated using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

This rule also applies when the outcomes are not equally likely; it is always true.

This rule can be used to help solve more complex problems in probability.

Example 18

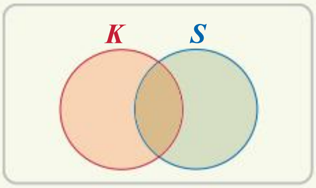
If one card is chosen at random from a well-shuffled deck, what is the probability that the card is a king or a spade?

Solution

Let event K be 'a king'. Then $K = \{K\clubsuit, K\heartsuit, K\spadesuit, K\diamondsuit\}$ and $n(K) = 4$.

Let event S be 'a spade'. Then $S = \{A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit, 9\spadesuit, \dots, 2\spadesuit\}$ and $n(S) = 13$.

The event 'a king or a spade' corresponds to the union of sets K and S. We have



$$P(K) = \frac{4}{52}, \quad P(S) = \frac{13}{52}, \quad P(K \cap S) = \frac{1}{52}$$

and so, using the addition rule, we find

$$P(K \cup S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = 0.3077 \quad \text{(correct to 4 decimal places)}$$

Summary 9E

- For any two events A and B, the **addition rule** can be applied:

  $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If the two events A and B are mutually exclusive, then $P(A \cap B) = 0$ and therefore $P(A \cup B) = P(A) + P(B)$.

Exercise 9E

Example 17 SF

1 In a survey of 100 university students, a market research company found that 70 students owned smartphones, 45 owned cars and 35 owned smartphones and cars. What is the probability that a student chosen at random from this group owns:

a neither a car nor a smartphone

b either a car or a smartphone, but not both?

2 Let $\varepsilon = \{1, 2, 3, 4, 5, 6\}$, where the outcomes are equally likely. If $A = \{2, 4, 6\}$ and $B = \{3\}$, determine:

a $P(A \cup B)$ **b** $P(A \cap B)$ **c** $P(A')$ **d** $P(B')$

 ISBN 978-1-009-53719-3

Example 18

SF

3 Let $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, where the outcomes are equally likely. If A is the event 'an even number' and B is the event 'a multiple of three', determine:

a $P(A)$ **b** $P(B)$ **c** $P(A \cap B)$ and hence $P(A \cup B)$.

4 In Sam's class at school many of the students are on the swimming team. There are 18 students in the class. Seven of the students swim freestyle and four swim backstroke. Two students swim both freestyle and backstroke. Draw a Venn diagram and use it to determine the probability that a student randomly chosen from Sam's class:

a swims freestyle **b** swims backstroke
c swims freestyle and backstroke **d** is on the swimming team.

5 Suppose that A is the set of factors of 12, and B is the set of prime numbers less than 10. If a number is chosen at random from the integers from 1 to 20, draw a Venn diagram and use it to determine:

a $P(A)$ **b** $P(B)$ **c** $P(A \cap B)$ **d** $P(A \cup B)$

6 Suppose $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.2$. Determine $P(A \cup B)$.

7 Suppose $P(A) = 0.35$, $P(B) = 0.24$ and $P(A \cap B) = 0.12$. Determine $P(A \cup B)$.

8 Suppose $P(A) = 0.28$, $P(B) = 0.45$ and $A \subseteq B$. Determine:

a $P(A \cap B)$ **b** $P(A \cup B)$

9 Suppose $P(A) = 0.58$, $P(B) = 0.45$ and $B \subseteq A$. Determine:

a $P(A \cap B)$ **b** $P(A \cup B)$

10 Suppose $P(A) = 0.3$, $P(B) = 0.4$ and $A \cap B = \varnothing$. Determine:

a $P(A \cap B)$ **b** $P(A \cup B)$

11 Suppose $P(A) = 0.08$, $P(B) = 0.15$, and A and B are disjoint. Determine:

a $P(A \cap B)$ **b** $P(A \cup B)$

12 Suppose $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cup B) = 0.5$. Determine $P(A \cap B)$.

13 Suppose $P(A) = 0.24$, $P(B) = 0.44$ and $P(A \cup B) = 0.63$. Determine $P(A \cap B)$.

14 Suppose $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B') = 0.2$. Determine $P(A \cup B')$.

15 Suppose that in a certain school the probability that a student plays soccer is 0.18, the probability that a student plays tennis is 0.25, and probability that a student plays both soccer and tennis is 0.11. Determine the probability that a student plays either or both of these sports.

16 Suppose that the probability that a student studies Chinese is 0.22, the probability that a student studies French is 0.35, and the probability that a student studies both languages is 0.14. Determine the probability that a student in that school studies:

a at least one of these languages **b** exactly one of these languages.

 ISBN 978-1-009-53719-3 Cambridge University Press

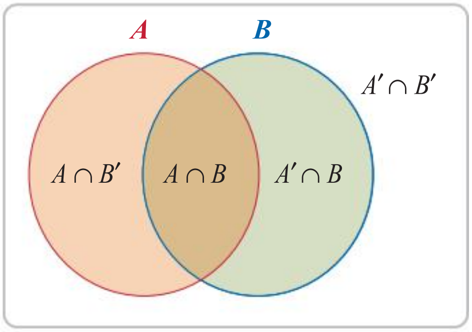
9F Probability tables

Learning intentions

- To be able to use probability tables to calculate probabilities.

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically.

Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions:

- $A \cap B$
- $A \cap B'$
- $A' \cap B$
- $A' \cap B'$

These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

In a probability table, the entries give the probabilities of each of these events occurring.

		Column 1	Column 2
		B	B'
Row 1	A	$P(A \cap B)$	$P(A \cap B')$
Row 2	A'	$P(A' \cap B)$	$P(A' \cap B')$

Further, from the Venn diagram we can see that set A is the union of the part of set A that intersects with set B and the part of set A that does not intersect with set B. That is,

$$A = (A \cap B) \cup (A \cap B')$$

The sets $A \cap B$ and $A \cap B'$ are mutually exclusive, so

$$P(A \cap B) + P(A \cap B') = P(A) \qquad \text{(row 1)}$$

and thus summing the probabilities in row 1 gives $P(A)$. Similarly:

$$P(A' \cap B) + P(A' \cap B') = P(A') \qquad \text{(row 2)}$$

$$P(A \cap B) + P(A' \cap B) = P(B) \qquad \text{(column 1)}$$

$$P(A \cap B') + P(A' \cap B') = P(B') \qquad \text{(column 2)}$$

Finally, since $P(A) + P(A') = 1$ and $P(B) + P(B') = 1$, the totals for both column 3 and row 3 are equal to 1.

 ISBN 978-1-009-53719-3 Cambridge University Press

The completed table is as follows.

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$P(A \cap B)$	$P(A \cap B')$	$P(A)$
Row 2	A'	$P(A' \cap B)$	$P(A' \cap B')$	$P(A')$
Row 3		$P(B)$	$P(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next two examples.

Example 19

If A and B are events such that $P(A) = 0.7$, $P(A \cap B) = 0.4$ and $P(A' \cap B) = 0.2$, determine:

a $P(A \cap B')$ **b** $P(B)$ **c** $P(A' \cap B')$ **d** $P(A \cup B)$

Solution

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$P(A \cap B) = 0.4$	$P(A \cap B')$	$P(A) = 0.7$
Row 2	A'	$P(A' \cap B) = 0.2$	$P(A' \cap B')$	$P(A')$
Row 3		$P(B)$	$P(B')$	1

The given information has been entered in the table in red.

a From row 1: $P(A \cap B') = P(A) - P(A \cap B) = 0.7 - 0.4 = 0.3$

b From column 1: $P(B) = P(A \cap B) + P(A' \cap B) = 0.4 + 0.2 = 0.6$

c From column 3: $P(A') = 1 - P(A) = 1 - 0.7 = 0.3$
From row 2: $P(A' \cap B') = 0.3 - 0.2 = 0.1$

d Using the addition rule:
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.7 + 0.6 - 0.4 \\ &= 0.9 \end{aligned}$$

The completed table is shown below.

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$P(A \cap B) = 0.4$	$P(A \cap B') = 0.3$	$P(A) = 0.7$
Row 2	A'	$P(A' \cap B) = 0.2$	$P(A' \cap B') = 0.1$	$P(A') = 0.3$
Row 3		$P(B) = 0.6$	$P(B') = 0.4$	1

Example 20

Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to determine:

a the probability that a student selected at random is not Australian by birth

b the probability that a student selected at random is not Australian by birth and does not participate in sport.

Solution

The information in the question may be entered into a table as shown. We use A to represent 'Australian by birth' and S to represent 'participates in sport'.

	S	S'	
A	0.53		0.71
A'			
	0.65		1

All the empty cells in the table may now be filled in by subtraction.

In column 1: $P(A' \cap S) = 0.65 - 0.53 = 0.12$

In column 3: $P(A') = 1 - 0.71 = 0.29$

In row 1: $P(A \cap S') = 0.71 - 0.53 = 0.18$

In row 3: $P(S') = 1 - 0.65 = 0.35$

In row 2: $P(A' \cap S') = 0.29 - 0.12 = 0.17$

	S	S'	
A	0.53	0.18	0.71
A'	0.12	0.17	0.29
	0.65	0.35	1

a The probability that a student selected at random is not Australian by birth is given by $P(A') = 0.29$.

b The probability that a student selected at random is not Australian by birth and does not participate in sport is given by $P(A' \cap S') = 0.17$.

Exercise 9F

Example 19

1 If A and B are events such that $P(A) = 0.6$, $P(A \cap B) = 0.4$ and $P(A' \cap B) = 0.1$, determine:

a $P(A \cap B')$ **b** $P(B)$ **c** $P(A' \cap B')$ **d** $P(A \cup B)$

2 If A and B are events such that $P(A') = 0.25$, $P(A' \cap B) = 0.12$ and $P(B) = 0.52$, determine:

a $P(A)$ **b** $P(A \cap B)$ **c** $P(A \cup B)$ **d** $P(B')$

3 If C and D are events such that $P(C \cup D) = 0.85$, $P(C) = 0.45$ and $P(D') = 0.37$, determine:

a $P(D)$ **b** $P(C \cap D)$ **c** $P(C \cap D')$ **d** $P(C' \cup D')$

4 If E and F are events such that $P(E \cup F) = 0.7$, $P(E \cap F) = 0.15$ and $P(E') = 0.55$, determine:

a $P(E)$ **b** $P(F)$ **c** $P(E' \cap F)$ **d** $P(E' \cup F)$

SF

 ISBN 978-1-009-53719-3

SF

5 If A and B are events such that $P(A) = 0.8$, $P(B) = 0.7$ and $P(A' \cap B') = 0.1$, determine:

a $P(A \cup B)$ **b** $P(A \cap B)$ **c** $P(A' \cap B)$ **d** $P(A \cup B')$

CF

Example 20

6 In a recent survey of senior citizens, it was found that 85% favoured giving greater powers of arrest to police, 60% favoured longer sentences for convicted persons, and 50% favoured both propositions.

a What percentage favoured at least one of the two propositions?

b What percentage favoured neither proposition?

7 Suppose a card is selected at random from an ordinary deck of 52 playing cards.
Let A = event a picture card is selected (i.e. jack, queen or king)
C = event a heart is selected

a List the outcomes corresponding to events A and C.

b Determine the following probabilities and express your results in words:

i $P(A)$ **ii** $P(C)$ **iii** $P(A \cap C)$ **iv** $P(A \cup C)$ **v** $P(A \cup C')$

8 The following information applies to a particular class:

- The probability that a student's name begins with M is $\frac{3}{10}$.
- The probability that a student does not study French is $\frac{7}{15}$.
- The probability that a student studies French and has a name beginning with M is $\frac{1}{6}$.

Determine the probability that a student chosen at random from this class:

a studies French

b has a name which does not begin with M

c has a name which does begin with M, but does not study French

d has a name which does not begin with M and does not study French.

9 A frame is chosen at random from a shop where picture frames are sold. It is known that in this shop:

- the probability that the frame is made of wood is 0.72
- the probability that the frame is freestanding is 0.65
- the probability that the frame is not made of wood and is not freestanding is 0.2.

Determine the probability that the randomly chosen frame:

a is made of wood or is freestanding **b** is made of wood and is freestanding

c is not made of wood **d** is not made of wood but is freestanding.

10 A book is chosen at random from a bookshop. It is known that in this bookshop:

- the probability that the book is a hardback but not a novel is 0.05
- the probability that the book is not hardback but is a novel is 0.12
- the probability that the book is not a novel is 0.19.

Determine the probability that the randomly chosen book is:

a a novel **b** a hardback novel

c a hardback **d** a novel or a hardback.

 ISBN 978-1-009-53719-3

11 At a school camp consisting of 60 students, sailing was offered as an activity one morning, and bushwalking in the afternoon. Every student attended at least one activity. If 32 students went sailing and 40 students went bushwalking, determine the probability that a student chosen at random:

a undertook neither of these activities
b has sailed or bushwalked
c has sailed and bushwalked
d has sailed but not bushwalked.

12 At a barbecue attended by 50 people, hamburgers and sausages were available. It was found that 35 hamburgers and 38 sausages were eaten, and six people were noted to have eaten neither a hamburger nor a sausage. If no person ate more than one hamburger or one sausage, determine the probability that a person chosen at random ate:

a a hamburger or a sausage
b a hamburger and a sausage
c only one serve of food
d only a hamburger.

9G Conditional probability

Learning intentions

- To be able to solve problems involving **conditional probability**.

We are often interested in calculating the probability of one event in the light of whether another event has or has not already occurred. For example, consider tossing a coin twice. What is the probability that the second toss shows a head, if we know that the first toss shows a head? Is the probability the same as if the first toss was a tail?

Suppose that we define event A as 'the second toss is a head', and event B as 'the first toss is a head'. Then the probability that the second toss shows a head, given that the first toss shows a head, is written $P(A \mid B)$ and is an example of conditional probability.

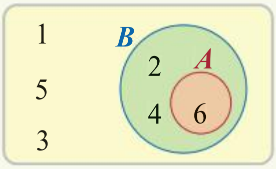
The probability of an event A occurring when it is known that some event B has occurred is called **conditional probability** and is written $P(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Example 21

Suppose we roll a fair dice and define event A as 'rolling a six' and event B as 'rolling an even number'. What is the probability of rolling a six given the information that an even number was rolled?

Solution

The events A and B can be shown on a Venn diagram. We know that event B has already occurred so we know that the outcome was 2, 4 or 6. Thus



P(six is rolled given an even number is rolled)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}} = \frac{n(A)}{n(B)} = \frac{1}{3}$$

 ISBN 978-1-009-53719-3

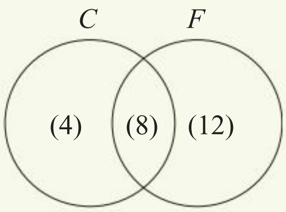
Example 22

In Stephen's class 12 students study Chinese, 20 study French, and 8 study both Chinese and French.

a Given that a student in his class studies Chinese (C), what is the probability that they also study French (F)?

b Given that a student in his class studies French, what is the probability that they also study Chinese?

Solution



a $P(F \mid C) = \dfrac{8}{12} = \dfrac{2}{3}$

b $P(C \mid F) = \dfrac{8}{20} = \dfrac{2}{5}$

Explanation

Display the information in the question in a Venn diagram. The numbers in brackets indicate the number of elements in each region.

If we know that the student studies Chinese, the sample space is restricted to those 12 students. From the Venn diagram we can see that 8 of these students also study French.

If we know that the student studies French, the sample space is restricted to those 20 students. From the Venn diagram we can see that 8 of these students also study Chinese.

This example clearly demonstrates that, in general, $P(A \mid B) \neq P(B \mid A)$. So care needs to be taken when determining conditional probabilities.

Conditional probabilities can also be calculated from a table, as shown in Example 23.

Example 23

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected at random from these 500. Given that the selected person is less than 25 years of age, what is the probability that they regularly use social media?

Solution

$P(\text{Yes} \mid \text{Age} < 25) = \dfrac{200}{240} = \dfrac{5}{6}$

Explanation

If we know the person is less than 25 years old, then the sample space is restricted to those 240 people. Of these, 200 regularly use social media.

 ISBN 978-1-009-53719-3

Note that, in Example 23,

$$P(\text{Yes} \cap \text{Age} < 25) = \frac{200}{500} \qquad \text{and} \qquad P(\text{Age} < 25) = \frac{240}{500}$$

Hence we have

$$\frac{P(\text{Yes} \cap \text{Age} < 25)}{P(\text{Age} < 25)} = \frac{\frac{200}{500}}{\frac{240}{500}} = \frac{200}{240} = \frac{5}{6}$$

which is equal to the conditional probability $P(\text{Yes} \mid \text{Age} < 25)$.

This illustrates a general principle which is always true.

The conditional probability of an event A, given that event B has already occurred, is given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

Example 24

Given that for two events A and B, $P(A) = 0.7$, $P(B) = 0.3$ and $P(B \mid A) = 0.4$, determine:

a $P(A \cap B)$

b $P(A \mid B)$

Solution

a $P(A \cap B) = P(B \mid A) \times P(A)$

$= 0.4 \times 0.7 = 0.28$

b $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.28}{0.3} = \dfrac{14}{15}$

Example 25

In a particular school 55% of the students are male and 45% are female. Of the male students 13% say mathematics is their favourite subject, while of the female students 18% prefer mathematics. Determine the probability that:

a a student chosen at random prefers mathematics and is female

b a student chosen at random prefers mathematics and is male.

Solution

Let us use M to represent male, F for female, and P for prefers mathematics. Then

$$P(M) = 0.55, \quad P(F) = 0.45, \quad P(P \mid M) = 0.13, \quad P(P \mid F) = 0.18$$

We can use the multiplication rule to determine the required probabilities:

a The event 'prefers mathematics and is female' is represented by $P \cap F$, with

$$P(P \cap F) = P(P \mid F) \times P(F) = 0.18 \times 0.45 = 0.081$$

b The event 'prefers mathematics and is male' is represented by $P \cap M$, with

$$P(P \cap M) = P(P \mid M) \times P(M) = 0.13 \times 0.55 = 0.0715$$

 ISBN 978-1-009-53719-3

The law of total probability

As has already been seen, the tree diagram is an efficient way of listing a multi-stage sample space. If the probabilities associated with each stage are also added to the tree diagram, it becomes a very useful way of calculating the probability for each outcome. The probabilities at each stage are conditional probabilities that the particular path will be followed and the multiplication rule says that the probability of reaching the end of a given branch is the product of the probabilities associated with each segment of that branch.

Example 26

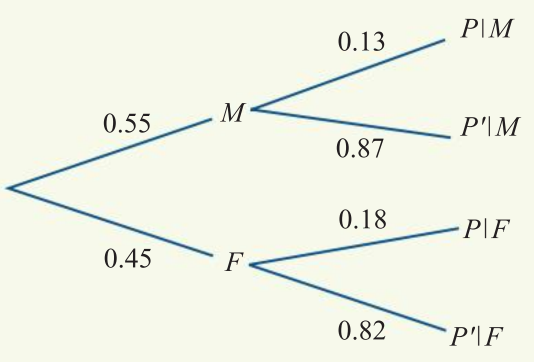
Using the information from Example 25, construct a tree diagram and use it to determine:

a the probability that a student selected is female and does not prefer mathematics

b the overall percentage of students who prefer mathematics.

Solution

The situation described can be represented by a tree diagram as follows:



$P(P \cap M) = 0.55 \times 0.13 = 0.0715$

$P(P' \cap M) = 0.55 \times 0.87 = 0.4785$

$P(P \cap F) = 0.45 \times 0.18 = 0.081$

$P(P' \cap F) = 0.45 \times 0.82 = 0.369$

a To determine the probability that a student is female and does not prefer mathematics we multiply along the appropriate branches thus:

$$P(F \cap P') = P(F) \times P(P' \mid F) = 0.45 \times 0.82 = 0.369$$

b Now, to determine the overall percentage of students who prefer mathematics we recall that:

$$P = (P \cap F) \cup (P \cap M)$$

Since $P \cap F$ and $P \cap M$ are mutually exclusive,

$$P(P) = P(P \cap F) + P(P \cap M) = 0.081 + 0.0715 = 0.1525$$

Thus 15.25% of all students prefer mathematics.

The solution to part b of Example 26 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

 ISBN 978-1-009-53719-3

A further example of the use of the law of total probability is given in the following example.

Example 27

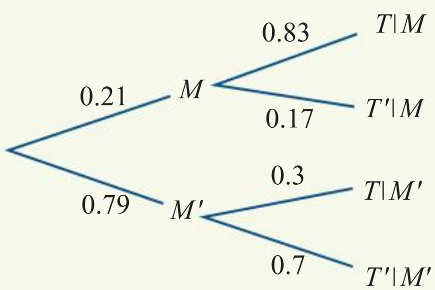
In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, determine the probability that it rains:

a on both Monday and Tuesday **b** on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$P(T \cap M) = 0.21 \times 0.83 = 0.1743$

$P(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$P(T \cap M') = 0.79 \times 0.3 = 0.237$

$P(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$P(T \cap M) = 0.21 \times 0.83 = 0.1743$$

b The probability that it rains on Tuesday is given by

$$P(T) = P(T \cap M) + P(T \cap M') = 0.1743 + 0.237 = 0.4113$$

Summary 9G

- The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $P(A \mid B)$.
- In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$P(A \cap B) = P(A \mid B) \times P(B)$$

- The **law of total probability** states that, in the case of two events A and B,

$$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 9G

SF

Example 21

1 Suppose that a fair dice is rolled, and event A is defined as 'rolling a six' and event B as 'rolling a number greater than 2'. Determine $P(A \mid B)$.

2 Suppose that a fair dice is rolled, and event A is defined as 'rolling a three' and event B as 'rolling an odd number'. Draw a Venn diagram and use it to determine $P(A \mid B)$.

Example 22

3 Suppose that a card is drawn from a pack of 52 cards, and event A is defined as 'selecting an ace' and event B as 'selecting a club'. Draw a Venn diagram and use it to determine the probability that the card drawn is an ace, given that it is a club.

4 In Sam's class 12 students play violin, 12 play piano and 7 play both. Draw a Venn diagram and use it to determine the probability that a randomly chosen student plays the violin given that they play the piano.

5 Two dice are rolled and the outcomes observed. Given that the dice both show the same number, what is the probability that it's a 'double six'?

6 In Annabelle's class 17 students own an iPhone, 7 own an iPad, and 4 own both.

a What is the probability that a student owns an iPad, given that they own an iPhone?

b What is the probability that a student owns an iPhone, given that they own an iPad?

CF

Example 23

7 A group of 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

One person is chosen at random from the group. Given that this person is male, what is the probability that they think private individuals should be allowed to carry guns?

8 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

One student is selected at random from these 500. Determine:

a the probability that the student prefers sport

b the probability that the student prefers sport, given that they are male.

ISBN 978-1-009-53719-3

CF

9 The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	Probable cause: Speed	Alcohol	Reckless driving	Other	Total
Fatal	42	61	22	12	137
Non-fatal	88	185	98	60	431
Total	130	246	120	72	568

Use the table to estimate:

a the probability that speed is the cause of the accident

b the probability that the accident is fatal

c the probability that the accident is fatal, given that speed is the cause

d the probability that the accident is fatal, given that alcohol is the cause.

SF

Example 24 **10** Given that for two events A and B, $P(A) = 0.6$, $P(B) = 0.3$ and $P(B \mid A) = 0.1$, determine:

a $P(A \cap B)$ **b** $P(A \mid B)$

11 For events A and B:

a if $P(A) = 0.7$ and $P(A \cap B) = 0.4$, determine $P(B \mid A)$

b if $P(A \mid B) = 0.6$ and $P(B) = 0.5$, determine $P(A \cap B)$

c if $P(A \mid B) = 0.44$ and $P(A \cap B) = 0.3$, determine $P(B)$.

12 In a random experiment $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.7$. Determine:

a $P(A \cap B)$ **b** $P(A \mid B)$ **c** $P(B \mid A)$

13 In a random experiment $P(A) = 0.6$, $P(B) = 0.54$ and $P(A \cap B') = 0.4$. Determine:

a $P(A \cap B)$ **b** $P(A \mid B)$ **c** $P(B \mid A)$

14 In a random experiment $P(A) = 0.4$, $P(A \mid B) = 0.6$ and $P(B) = 0.5$. Determine:

a $P(A \cap B)$ **b** $P(B \mid A)$

CF

Example 25 **15** The current football fixture has the local team playing at home for 60% of its matches. When it plays at home, the team wins 80% of the time. When it plays away, the team wins only 40% of the time. What percentage of its games does the team play away and win?

16 The probability that a car will need an oil change is 0.15, the probability that it needs a new oil filter is 0.08, and the probability that both the oil and the filter need changing is 0.03. Given that the oil has to be changed, what is the probability that a new oil filter is also needed?

 ISBN 978-1-009-53719-3

CF

17 A card is selected from a pack of 52 playing cards. The card is replaced and a second card chosen. Determine the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d one card is red and the other is black

18 A card is selected from a pack of 52 playing cards and **not** replaced. Then a second card is chosen. Determine the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d one card is red and the other is black

19 A person is chosen at random from the employees of a large company. Let W be the event that the person chosen is a woman, and let A be the event that the person chosen is 25 years or older. Suppose the probability of selecting a woman is $P(W) = 0.652$ and the probability of a woman being 25 years or older is $P(A \mid W) = 0.354$. Determine the probability that a randomly chosen employee is a woman aged 25 years or older.

20 In a class of 28 students there are 15 girls. Of the students in the class, six girls and eight boys play basketball. A student is chosen at random from the class. If G represents the event that a girl student is chosen and B represents the event that the student chosen plays basketball, determine:

a $P(G)$ **b** $P(B)$ **c** $P(B')$ **d** $P(B \mid G)$
e $P(G \mid B)$ **f** $P(B \mid G')$ **g** $P(B' \cap G')$ **h** $P(B \cap G)$

21 In a recent survey it was found that 85% of the population eats red meat. Of those who eat red meat, 60% preferred lamb. A person is chosen at random from the population. If R represents the event that the person eats red meat and L represents the event that the person prefers lamb, determine:

a $P(R)$ **b** $P(L \mid R)$ **c** $P(L \cap R)$ **d** $P(L)$

Example 26, 27

22 In a senior college, 25% of the Year 11 students and 40% of the Year 12 students would prefer not to wear school uniform. This particular college has 320 Year 11 students and 280 Year 12 students. Determine the probability that a randomly chosen student is in Year 11 and is in favour of wearing school uniform. What is the overall percentage of students who are in favour of wearing school uniform?

 ISBN 978-1-009-53719-3 Cambridge University Press

CU

23 At a certain school it was found that 35% of the 500 boys and 40% of the 400 girls enjoyed bushwalking. One student from the school is chosen at random. Let G represent the event that the student is a girl, and B represent the event that the student enjoys bushwalking.

a Determine, correct to two decimal places:

i $P(G)$ **ii** $P(B \mid G)$ **iii** $P(B \mid G')$

iv $P(B \cap G)$ **v** $P(B \cap G')$

b Determine $P(B)$.

c Hence determine:

i $P(G \mid B)$ **ii** $P(G \mid B')$

24 In a factory two machines produce a particular circuit board. The older machine produces 480 boards each day, of which an average of 12% are defective. The newer machine produces 620 boards each day, of which an average of 5% are defective.

A board is chosen at random and checked. Let N represent the event that the board comes from the newer machine, and D represent the event that the board is defective.

a Determine, correct to two decimal places:

i $P(N)$ **ii** $P(D \mid N)$ **iii** $P(D \mid N')$

iv $P(D \cap N)$ **v** $P(D \cap N')$

b Determine $P(D)$.

c Hence determine $P(N \mid D)$, correct to two decimal places.

25 Jane has three bags of lollies. In bag 1 there are three mints and three toffees, in bag 2 there are three mints and two toffees, and in bag 3 there are two mints and one toffee. Jane selects a bag at random, and then selects a lolly at random. Determine:

a the probability she chooses a mint from bag 1

b the probability she chooses a mint

c the probability that Jane chose bag 1, given that she selects a mint.

26 Assuming a finite sample space, describe the relationship between events A and B if:

a $P(A \mid B) = 1$ **b** $P(A \mid B) = 0$ **c** $P(A \mid B) = \dfrac{P(A)}{P(B)}$

 ISBN 978-1-009-53719-3

9H Independent events

Learning intentions

- To be able to solve problems involving independent events.

Let us again consider the question of the probability that the second toss shows a head, given that the first toss shows a head, when tossing a coin twice. If we define A as the event 'the second toss is a head' and B as the event 'the first toss is a head', then what is $P(A \mid B)$? Using the definition of conditional probability:

$$\begin{aligned} P(A \mid B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(\text{both tosses show heads})}{P(\text{first toss shows a head})} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \end{aligned}$$

That is, the probability of the second toss showing a head does not seem to be affected by the outcome of the first toss. This is an example of independent events.

Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$P(A \mid B) = P(A)$$

If $P(B) \neq 0$, then the multiplication rule of probability gives

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Thus, when events A and B are independent, we can equate the two expressions for $P(A \mid B)$ to obtain

$$P(A) = \frac{P(A \cap B)}{P(B)}$$

and therefore $P(A \cap B) = P(A) \times P(B)$

In fact, we can also use this final equation as a test for independence:

Events A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B)$$

Notes:

- For events A and B with $P(A) \neq 0$ and $P(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:
  - $P(A \mid B) = P(A)$
  - $P(B \mid A) = P(B)$
  - $P(A \cap B) = P(A) \times P(B)$
- In the special case that $P(A) = 0$ or $P(B) = 0$, the condition $P(A \cap B) = P(A) \times P(B)$ holds since both sides are zero, and so we say that A and B are independent.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 28

An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{4, 6, 8\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

Solution

$P(A) = \frac{1}{2},\ P(B) = \frac{1}{2},\ P(C) = \frac{3}{8}$

a $P(A \cap B) = \frac{1}{4}$ since $A \cap B = \{1, 3\}$

$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore\ P(A) \times P(B) = P(A \cap B)$

Thus A and B are independent since these two probabilities are equal.

b $P(A \cap C) = \frac{1}{8}$ since $A \cap C = \{4\}$

$P(A) \times P(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore\ P(A) \times P(C) \neq P(A \cap C)$

Thus A and C are not independent since these two probabilities are not equal.

c $P(B \cap C) = 0$ since $B \cap C = \varnothing$

$P(B) \times P(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore\ P(B) \times P(C) \neq P(B \cap C)$

Thus B and C are not independent since these two probabilities are not equal.

The concept of mathematical independence is sometimes confused with that of physical independence. If two events are physically independent, then they are also mathematically independent, but the converse is not necessarily true. The following example illustrates this.

Example 29

Suppose we roll a dice twice and define the following events:

A = the first roll shows a 4

B = the second roll shows a 4

C = the sum of the numbers showing is at least 10

a Are A and B independent events?

b What about A and C?

Solution

a Since A and B are physically independent, they must also be mathematically independent, but we can also check this directly.

We have

$$P(A) \times P(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

 ISBN 978-1-009-53719-3

If we write the sample space as ordered pairs, in which the first entry is the result of the first throw and the second is the result of the second throw, then

$$\varepsilon = \{(1,1), (1,2), (1,3), \dots, (6,5), (6,6)\}$$

and $n(\varepsilon) = 36$

The event $A \cap B$ corresponds to the outcome $(4, 4)$, and so $n(A \cap B) = 1$. Thus

$$P(A \cap B) = \frac{1}{36} = P(A) \times P(B)$$

and so A and B are independent.

b We have

$$C = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$$

so $n(C) = 6$

Thus

$$P(A) \times P(C) = \frac{1}{6} \times \frac{6}{36} = \frac{1}{36}$$

The event $A \cap C$ corresponds to the outcome $(4, 6)$, and so $n(A \cap C) = 1$. Thus

$$P(A \cap C) = \frac{1}{36} = P(A) \times P(C)$$

This means that A and C are also independent events.

Example 30

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Is the regular use of social media independent of the respondent's age?

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

Solution

From the table:

$$P(\text{Age} < 25 \cap \text{Yes}) = \frac{200}{500} = 0.4$$

$$P(\text{Age} < 25) \times P(\text{Yes}) = \frac{240}{500} \times \frac{300}{500} = 0.48 \times 0.6 = 0.288$$

Hence

$$P(\text{Age} < 25 \cap \text{Yes}) \neq P(\text{Age} < 25) \times P(\text{Yes})$$

and therefore these events are not independent.

 ISBN 978-1-009-53719-3

Knowing that events are independent means that we can determine the probability of their intersection by multiplying together their individual probabilities. This is illustrated in the following example.

Example 31

Suppose that the probability that a family in a certain town owns a television set (T) is 0.75, and the probability that a family owns a station wagon (S) is 0.25. If these events are independent, determine the following probabilities:

a A family chosen at random owns both a television set and a station wagon.

b A family chosen at random owns at least one of these items.

Solution

a The event 'owns both a television set and a station wagon' is represented by $T \cap S$, with

$$\begin{aligned} P(T \cap S) &= P(T) \times P(S) && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.75 \times 0.25 = 0.1875 \end{aligned}$$

b The event 'owns at least one of these items' is represented by $T \cup S$, with

$$\begin{aligned} P(T \cup S) &= P(T) + P(S) - P(T \cap S) && \text{(from the addition rule)} \\ &= 0.75 + 0.25 - 0.75 \times 0.25 && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.8125 \end{aligned}$$

Confusion often arises between independent and mutually exclusive events. That two events A and B are mutually exclusive means that $A \cap B = \varnothing$ and hence that $P(A \cap B) = 0$. Thus, if two events are independent, they cannot also be mutually exclusive, unless the probability of at least one of the events is zero.

Summary 9H

- The probability of an event A occurring when it is known that some event B has already occurred is called **conditional probability** and is written $P(A \mid B)$, where

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

- Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$P(A \mid B) = P(A)$$

- Events A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B)$$

 ISBN 978-1-009-53719-3

Exercise 9H

SF

1 Two events A and B are such that $P(A) = 0.3$, $P(B) = 0.1$ and $P(A \cap B) = 0.1$. Are A and B independent?

Example 28

2 An experiment consists of drawing a number at random from $\{1, 2, 3, \ldots, 12\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9, 11\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?

b Are A and C independent?

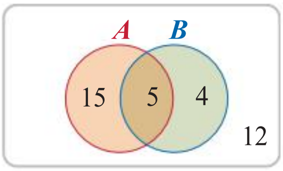
c Are B and C independent?

Example 29

3 A dice is thrown and the number uppermost is recorded. Events A and B are defined as 'an even number' and 'a square number' respectively. Show that A and B are independent.

CF

4 Events A and B are as shown in the Venn diagram. Show that A and B are independent.



Example 30

5 A group of 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

Is support for private individuals carrying guns independent of sex?

6 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the following table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

Is preference for playing sport or listening to music independent of sex?

 ISBN 978-1-009-53719-3 Cambridge University Press

7 An analysis of traffic accidents in a certain city classified the accident as serious or minor, as well as whether the driver was speeding or not.

Type of accident	Speeding		Total
	Yes	No	
Serious	42	61	103
Minor	88	185	273
Total	130	246	376

Is the seriousness of the accident independent of speed?

8 If A and B are independent events with $P(A) = 0.6$ and $P(B) = 0.7$, determine:

a $P(A \mid B)$ **b** $P(A \cap B)$ **c** $P(A \cup B)$

9 If A and B are independent events with $P(A) = 0.5$ and $P(B) = 0.2$, determine $P(A \cup B)$.

Example 31 **10** Two people decide to marry. The probability that each will have a specific blood group is as follows:

Blood group	O	A	B	AB
Probability	0.5	0.35	0.1	0.05

If the blood groups of each of the people are independent, determine the probability that:

a person 1 is group A **b** person 1 is group A and person 2 is group B

c both are group A **d** person 2 is group AB and person 1 is group O.

11 The 165 subjects volunteering for a medical study are classified by sex and blood pressure (high (H), normal (N) and low (L)).

	H	N	L
M	88	22	10
F	11	22	12
Total	99	44	22

If a subject is selected at random, determine:

a $P(N)$ **b** $P(F \cap H)$ **c** $P(F \cup H)$

d $P(F \mid L)$ **e** $P(L \mid F)$

Are F and L independent? Explain.

12 The probability that Billy watches a certain television show is 0.4, and the probability that her partner watches the same show is 0.5. The probability that Billy watches a show, given that her partner watches it is 0.7. Determine the probability that:

a both Billy and her partner watch the show

b her partner watches the show given that Billy watches it.

 ISBN 978-1-009-53719-3

CU

13 The 65 middle managers in a company are classified by age and income as follows:

Income	Age 30–39 (T)	40–49 (F)	50–69 (S)
Low (L)	13	4	1
Moderate (M)	8	10	3
High (H)	2	16	8
Total	23	30	12

A middle manager is selected at random from the company. Determine:

a $P(L)$ **b** $P(S)$ **c** $P(T)$ **d** $P(M)$
e $P(L \cap F)$ **f** $P(T \cap M)$ **g** $P(L \mid F)$ **h** $P(T \mid M)$

Is income independent of age? Explain your answer.

14 A consumer research organisation has studied the services provided by the 150 TV repair persons in a certain city and their findings are summarised in the following table.

	Good service (G)	Poor service (G')
Factory trained (F)	48	16
Not factory trained (F')	24	62

a One of the TV repairers is randomly selected. Calculate the following probabilities:
  i $P(G \mid F)$, the probability that a factory-trained repairer is one who gives good service
  ii $P(G \cap F)$, the probability that the repairer is giving good service and is factory trained
  iii $P(G \cup F)$, the probability that the repairer is giving good service or is factory trained or both.

b Are events G and F independent?

c Are the events G and F mutually exclusive?

9I Solving probability problems using simulation

Learning intentions

▶ To be able to use simulation to solve probability problems

Simulation is a very powerful and widely used procedure which enables us to determine approximate answers to difficult probability questions. It is a technique which imitates the operation of the real-world system being investigated. Some problems are not able to be solved directly and simulation allows a solution to be obtained where otherwise none would be possible. In this section some specific probability problems are looked at which may be solved by using simulation, a valuable and legitimate tool for the statistician.

Suppose, for example, we would like to know the probability that a family of five children will include at least four girls. There are ways of determining this probability exactly which we will cover later. What we can do now, however, is estimate the probability by simulation.

Firstly, we need to make some assumptions so we can decide on a suitable model for the simulation. We will assume:

- There is a probability of 0.5 of each child being female.
- The sex of each child is independent of the sex of the other children. That is, the probability of a female child is always 0.5.

Since the probability of a female child is 0.5, a suitable simulation model would be tossing a fair coin. Let a head represent a female child and a tail a male child. A trial consists of tossing the coin five times to represent one complete family of five children, and the result of the trial is the number of heads obtained in the trial.

To estimate the required probability, several trials need to be conducted. How many trials are needed to estimate the probability? As we have already noted in Section 9C, the more repetitions of an experiment the better the estimate of the probability. Initially about 50 trials could be considered.

An example of the results that might be obtained from 10 trials is:

Trial number	Simulation results	Number of heads
1	$T\,H\,H\,T\,T$	2
2	$H\,H\,H\,T\,H$	4
3	$H\,H\,H\,T\,H$	4
4	$H\,T\,T\,T\,H$	2
5	$H\,T\,H\,H\,H$	4
6	$H\,T\,T\,T\,H$	2
7	$T\,T\,H\,H\,H$	3
8	$H\,T\,H\,H\,T$	3
9	$T\,T\,T\,H\,H$	2
10	$H\,H\,T\,T\,T$	2

Continuing in this way, the following results were obtained for 50 trials:

Number of heads	0	1	2	3	4	5
Number of times obtained	1	8	17	13	10	1

The results in the table can be used to estimate the required probability. Since at least four heads were obtained in 11 trials, estimate the probability of at least four female children as $\frac{11}{50}$ or 0.22. Of course, since this probability has been estimated experimentally, repeating the simulations would give a slightly different result, but we would expect to obtain approximately this value most of the time.

Simulation is also widely used to estimate the values of other quantities which are of interest in a probability problem. We may wish to know the average result, the largest result, the number of trials required to achieve a certain result, and so on. An example of this type of problem is given in Example 32.

Example 32

A pizza shop is giving away football cards with each pizza bought. There are six different cards available, and a fan decides to continue buying the pizzas until all six are obtained. How many pizzas will need to be bought, on average, to obtain the complete set of cards?

Solution

As there are more than two outcomes of interest, a coin is not a suitable simulation model, but a fair six-sided dice could be used. Each of the six different cards is represented by one of the six sides of the dice. Rolling the dice and observing the outcome is equivalent to buying a pizza and noting which card was obtained. This simulation model is based on the following assumptions:

- The six cards all occur with equal frequency.
- The card obtained with one pizza is independent of the cards obtained with the other pizzas.

A trial would consist of rolling the dice until all of the six numbers 1, 2, 3, 4, 5 and 6 have been observed, and the result of the trial is the number of rolls necessary to do this. The results of one trial are shown:

5 2 5 2 2 2 3 3 1 2 6 3 5 4

In this instance, 14 pizzas were bought before the whole set was obtained. Of course, we would not expect to buy 14 pizzas every time – this is just the result from one trial. To obtain an appropriate estimate, we would need to conduct several trials.

The following is an example of the results that might be obtained from 50 trials. In each case the number listed represents the number of pizzas that were bought to obtain a complete set of football cards:

14	8	12	11	16	8	8	11	15	26	14	20	11	13	35
23	19	14	10	10	20	9	10	14	29	13	7	15	15	22
9	10	14	16	14	17	12	10	24	13	19	27	31	11	9
16	21	22	8	9										

To estimate the number of pizzas that need to be bought, the average of the numbers obtained in these simulations is calculated. Thus we estimate that, in order to collect the complete set of cards, it would be necessary to buy approximately

$$\frac{14 + 8 + 12 + 11 + 16 + \cdots + 16 + 21 + 22 + 8 + 9}{50} \approx 15 \text{ pizzas}$$

In practice there are situations where coins and dice may not be useful. Other methods of simulation need to be adopted to deal with a wide range of situations. Suppose we wished to determine how many pizzas would need to be bought, on average, to obtain a complete set of eight souvenirs. This time we need to generate random numbers from 1 to 8 and a six-sided dice would no longer be appropriate, but there are other methods that could be used.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

We could construct a spinner with eight equal sections marked from 1 to 8, or we could mark eight balls from 1 to 8 and draw them (with replacement) from a bowl, or one of a number of other methods. Generally, when we wish to simulate we use random number generators on a calculator or computer.

Summary 9I

- Simulation is a simple and legitimate method for finding solutions to problems when an exact solution is difficult, or impossible, to find.
- In order to use simulation to solve a problem, a clear statement of the problem and the underlying assumptions must be made.
- A model must be selected to generate outcomes for a simulation. Possible choices for physical simulation models are coins, dice and spinners. Random number tables, calculators and computers may also be used.
- Each trial should be defined and repeated several times (at least 50).
- The results from all the trials should be recorded and summarised appropriately to provide an answer to a problem.

Exercise 9I

CU

1 Use simulation to estimate the probability that a family with three children have all boys.

2 A teacher gives her class a test consisting of five 'true or false' questions. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least three correct.

3 A teacher gives the class a test consisting of 10 multiple-choice questions, each with five alternatives. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least five correct.

4 Use simulation to estimate the number of pizzas we would need to buy if the number of football cards described in Example 32 was extended to 10.

5 Eight players are entered into a tennis tournament.
In round one, every player plays (four matches).
In round two, the four winners from round one play (two matches).
In round three, the two winners from round two play (one match).

- **a** Suppose Shaun has a 50% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.
- **b** Suppose he has a 70% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.

 ISBN 978-1-009-53719-3

Chapter summary

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le P(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $P(\varepsilon) = 1$
  - $P(\varnothing) = 0$
  - $P(A') = 1 - P(A)$, where A' is the complement of A
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$.
- If events A and B are mutually exclusive, then $P(A \cap B) = 0$ and $P(A \cup B) = P(A) + P(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

  $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad \text{if } P(B) \neq 0$$

  giving $\quad P(A \cap B) = P(A \mid B) \times P(B) \quad$ (the **multiplication rule**)
- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

  $$P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$$
- Two events A and B are **independent** if

  $$P(A \mid B) = P(A)$$

  so whether or not B has occurred has no effect on the probability of A occurring.
- Events A and B are independent if and only if $P(A \cap B) = P(A) \times P(B)$.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

9A **1** I can list the elements of a set, and use set language and notation. ☐

See Questions 1, 2, 3, 4 and 5

9A **2** I can determine the union and intersection of sets, and represent these on a Venn diagram. ☐

See Example 1 and Question 6

 ISBN 978-1-009-53719-3

9A **3** I can determine the number of elements in a set, and in the union and intersection of sets.

See Example 2, and Question 12

9A **4** I can use my knowledge of sets and Venn diagrams to solve everyday problems.

See Example 3, and Question 15

9B **5** I can list the sets representing the sample space and associated events for a simple random experiment.

See Example 4, and Questions 3 and 5

9B **6** I can determine the probability of an event when outcomes are equally likely.

See Example 5, Example 6, and Questions 6 and 8

9B **7** I can use the knowledge that the sum of the probabilities of all outcomes must equal 1 to determine probabilities.

See Example 7, Example 8, and Questions 12 and 14

9B **8** I can determine the probability of an event using the rule $P(A) = 1 - P(A')$.

See Example 9 and Question 15

9C **9** I can use relative frequency obtained from data to estimate probability.

See Example 11 and Question 1

9C **10** I can data from a random sample to estimate probability.

See Example 12 and Question 5

9C **11** I can my knowledge of area to estimate probability.

See Example 13 and Question 8

9D **12** I can list the sample space for a multi-stage experiment and use it to determine probabilities.

See Example 14, Example 15, and Questions 1 and 3

9D **13** I can use a tree diagram to list the sample space for a multi-stage experiment and use it to determine probabilities.

See Example 16 and Question 5

9E **14** I can use the addition rule to determine probabilities.

See Example 18 and Question 3

9F **15** I can construct probability tables and use them to determine probabilities.

See Example 19, Example 20, and Questions 1 and 6

 ISBN 978-1-009-53719-3 Cambridge University Press

9G **16** I can understand the notion of conditional probability. ☐

See Example 21, Example 22, and Questions 1 and 3

9G **17** I can use relative frequencies obtained from data as point estimates of conditional probability. ☐

See Example 23 and Question 7

9G **18** I can use relative frequencies obtained from data as point estimates of conditional probabilities. ☐

See Example 23 and Question 7

9G **19** I can use the formula $P(A \cap B) = P(A|B)P(B)$ to solve problems. ☐

See Example 24, Example 25, and Questions 10 and 15

9G **20** I can solve problems using the knowledge that $P(A) = P(A|B)P(B) + P(A|B')P(B')$. ☐

See Example 26, Example 27 and Question 22

9H **21** I can understand and use the notion of independence of events. ☐

See Example 28, Example 29, and Questions 2 and 3

9H **22** I can use point estimates of conditional probability obtained from data to indicate possible independence of events. ☐

See Example 31 and Questions 5 and 10

9I **23** I can use technology to model and solve problems that involve probability. ☐

See Example 32 and Question 1

Short-response questions

Technology-free short-response questions

SF

1 Let ε be the set of letters of the alphabet. If C is the set of vowels and D is the set of letters in the word UNIVERSITY, determine:

a $C \cup D$ **b** $C \cap D$ **c** $C \cap D'$ **d** $C' \cap D$

2 In a group of 200 school children, 63 attend dance classes, 124 attend swimming classes and 48 attend both. Determine the number of children in the group who attend:

a dance classes only **b** swimming classes only

c at least one of these classes **d** neither of these classes.

 ISBN 978-1-009-53719-3

Review

SF

3 Two six-sided dice are tossed. Determine the probability that:

a the sum of the values of the uppermost faces is 7 **b** the sum is not 7.

4 The probability that a computer chip is operational is 0.993. What is the probability that it is not operational?

5 A whole number between 1 and 300 (inclusive) is chosen at random. Determine the probability that the number is:

a divisible by 3 **b** divisible by 4 **c** divisible by 3 or by 4.

CF

6 A drawer contains 30 red socks and 20 blue socks.

a If a sock is chosen at random, its colour noted, the sock replaced and a second sock withdrawn, what is the probability that both socks are red?

b If replacement doesn't take place, what is the probability that both socks are red?

7 Box A contains five pieces of paper numbered 1, 3, 5, 7, 9.
Box B contains three pieces of paper numbered 1, 4, 9.
One piece of paper is removed at random from each box. Determine the probability that the two numbers obtained have a sum that is divisible by 3.

8 A three-digit number is formed by arranging the digits 1, 5 and 6 in a random order.

a List the sample space.

b Determine the probability of getting a number larger than 400.

c What is the probability that an even number is obtained?

9 A letter is chosen at random from the twelve letters of the word STATISTICIAN.

a What is the probability that it is a vowel?

b What is the probability that it is a T?

10 Ivan and Joe are chess players. In any game the probabilities of Ivan beating Joe, Joe beating Ivan or the game resulting in a draw are 0.6, 0.1 or 0.3 respectively. They play a series of three games. Calculate the probability that:

a they win alternate games, with Ivan winning the first game

b all three games are drawn

c exactly two of the games are drawn

d Joe does not win a game.

11 A dice with two red faces and four blue faces is thrown three times. Each face is equally likely to face upward. Determine the probability of obtaining the following:

a three red faces

b a blue on the first, a red on the second and a blue on the third

c exactly one red face

d at least two blue faces

12 If $P(A) = 0.6$ and $P(B) = 0.5$, can A and B be mutually exclusive? Why or why not?

CF

13 Events A and B are such that $P(A) = 0.6$, $P(B) = 0.5$ and $P(A' \cap B) = 0.4$. Construct a probability table and use it to determine:

a $P(A \cap B')$ **b** $P(A' \cap B')$ **c** $P(A \cup B)$

14 In Minh's class 18 students study mathematics, 14 study music, and 7 study both mathematics and music.

a Given that a student in his class studies mathematics, what is the probability that they also study music?

b Given that a student in his class studies music, what is the probability that they also study mathematics?

15 Given $P(B) = \frac{1}{3}$, $P(A \mid B) = \frac{2}{3}$ and $P(A \mid B') = \frac{3}{7}$, determine:

a $P(A \cap B')$ **b** $P(A)$ **c** $P(B' \mid A)$

16 A group of executives is classified according to body weight and incidence of hypertension. The proportion of the various categories is as shown.

	Overweight	Normal weight	Underweight
Hypertensive	0.10	0.08	0.02
Not hypertensive	0.15	0.45	0.20

a What is the probability that a person selected at random from this group will have hypertension?

b A person, selected at random from this group, is found to be overweight. What is the probability that this person is also hypertensive?

17 Given an experiment such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.2$, determine:

a $P(A \cup B)$ **b** $P(A' \cap B')$ **c** $P(A \mid B)$ **d** $P(B \mid A)$

Technology-active short-response questions

SF

18 The following data was obtained from a survey of 2000 randomly chosen people from a certain state.

Number of hours per week spent exercising

	Male	Female
Less than 5	398	494
From 5 to less than 15	271	370
15 or more	217	250

Use the data in the table to estimate the probability that:

a a female from that state will exercise for less than 5 hours per week

b a female from that state will exercise for less than 15 hours per week

c a person from that state will exercise for at least 15 hours per week.

 ISBN 978-1-009-53719-3 Cambridge University Press

19 To have a stage production ready for opening night there are three tasks which must be done and, as the same people are involved in each task, these must be done in sequence. The following probabilities are estimated for the duration of the activities: CF

Task	6 days	7 days	8 days
Build scenery	0.3	0.3	0.4
Paint scenery	0.6	0.3	0.1
Print programs	0.4	0.4	0.2

a What is the probability that the building and painting of the scenery will together take exactly 15 days?

b What is the probability that all three tasks will together take exactly 22 days?

20 Two bowls each contain eight pieces of fruit. In bowl A there are five oranges and three apples; in bowl B there is one orange and seven apples. CU

a One bowl is chosen at random and from it two pieces of fruit are chosen at random without replacement. If both pieces of fruit are apples, determine the probability that bowl A was chosen.

b One bowl is chosen at random and from it two pieces of fruit are chosen at random, the first piece of fruit being replaced before the second is chosen. If both pieces of fruit are apples, determine the probability that bowl A was chosen.

21 Rachel is a keen runner. She is supposed to attend running training five days per week. Rachel finds that if she runs one day, the probability that she will run again the next day is $\frac{4}{5}$, and if she does not run one day, the probability that she will not run the next day is $\frac{3}{4}$. Suppose that Rachel runs one day, what is the probability that she runs exactly twice in the next three days?

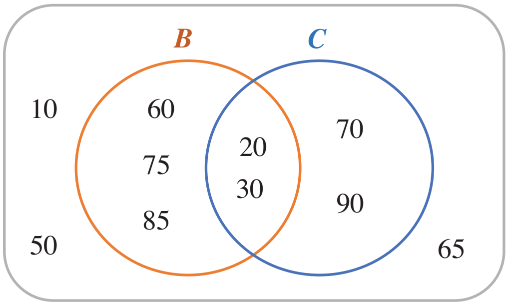
 ISBN 978-1-009-53719-3

Multiple-choice questions

Technology-free multiple-choice questions

1 From the diagram, the set $(B \cup C)'$ is equal to

A $\{20, 30, 60, 70, 75, 85, 90\}$
B $\{20, 30\}$
C $\{10, 50, 60, 65, 75, 85\}$
D $\{10, 50, 65\}$



2 If A and B are sets such that $A \cap B = A \cup B$, then

A $A = \emptyset$ **B** $B = \emptyset$ **C** $A = B$ **D** $A \neq B$ and $A \subseteq B$

3 Sets A and B are contained in a universal set ε with 40 elements. If $n(A) = 13$, $n(B) = 13$ and $n(A \cap B) = 8$, then the number of elements in $A \cup B$ is

A 13 **B** 18 **C** 26 **D** 34

4 If the probability of Chris scoring 50 or more marks in the exam is 0.7, then the probability he scores less than 50 marks is

A 0 **B** 0.3 **C** 0.4 **D** 0.7

5 A spinner is coloured red, yellow, blue and green. When spun the probability that it lands on red is 0.1, yellow is 0.2 and blue is 0.4. What is the probability that it lands on green?

A 0.1 **B** 0.2 **C** 0.3 **D** 0.4

6 Phillip is making a sign, and has cut the letters of the word THEATRETTE out of wood and placed them in his toolbox. If a letter is selected at random from the toolbox, then the probability that it is a T is

A $\frac{2}{5}$ **B** $\frac{3}{10}$ **C** $\frac{1}{5}$ **D** $\frac{1}{6}$

7 Of a group of 25 people in a restaurant, three chose a vegetarian meal, five chose fish, ten chose beef and the rest chose chicken for their main course. What is the probability that a randomly chosen diner chose chicken?

A $\frac{3}{25}$ **B** $\frac{6}{25}$ **C** $\frac{7}{25}$ **D** $\frac{2}{5}$

8 Suppose that a card is chosen at random from a well-shuffled deck of 52 playing cards. What is the probability that the card is a spade or a jack?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{17}{52}$ **D** $\frac{4}{13}$

 ISBN 978-1-009-53719-3

9 An experiment consists of tossing a coin and then rolling a fair six-sided dice. What is the probability of observing a head and a 'six'?

A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{35}$ **D** $\frac{1}{12}$

10 If A and B are events such that $P(A) = 0.35$, $P(A \cap B) = 0.18$ and $P(B) = 0.38$, then $P(A \cup B)$ is equal to

A 0.55 **B** 0.73 **C** 0.15 **D** 0.21

11 If A and B are events such that $P(A) = 0.47$, $P(B) = 0.28$ and $B \subseteq A$, then $P(A \cup B)$ is equal to

A 0.47 **B** 0.75 **C** 0.62 **D** 0.13

12 The following information applies to a particular class:

- The probability that a student studies mathematics is $\frac{2}{3}$.
- The probability that a student studies German is $\frac{3}{10}$.
- The probability that a student studies mathematics and does not study German is $\frac{7}{15}$.

The probability that a randomly chosen student does not study either mathematics or German is

A $\frac{4}{5}$ **B** $\frac{7}{30}$ **C** $\frac{7}{15}$ **D** $\frac{7}{10}$

13 In Imogen's class 15 students play tennis, 14 play basketball and 7 play both. If everyone in the class plays at least one of these sports, the probability that a randomly chosen student plays basketball, given that they play tennis, is

A $\frac{14}{15}$ **B** $\frac{7}{15}$ **C** $\frac{7}{29}$ **D** $\frac{15}{29}$

14 If for two events A and B, $P(A) = \frac{3}{8}$, $P(B) = \frac{4}{7}$ and $P(A \cap B) = \frac{8}{21}$, then $P(A \mid B)$ is equal to

A $\frac{3}{8}$ **B** $\frac{3}{14}$ **C** $\frac{2}{3}$ **D** $\frac{21}{32}$

The following information relates to Questions 15 *and* 16

The probability that Miller goes to the gym on Monday is 0.6. If he goes to the gym on Monday, then the probability that he will go again on Tuesday is 0.7. If he doesn't go to the gym on Monday, then the probability that Miller will go on Tuesday is only 0.4.

15 The probability that Miller goes to the gym on both Monday and Tuesday is

A 0.36 **B** 0.24 **C** 0.42 **D** 0.16

16 The probability that Miller goes to the gym on Tuesday is

A 0.58 **B** 0.42 **C** 0.16 **D** 0.84

 ISBN 978-1-009-53719-3

Technology-active multiple-choice questions

17 A square has side length of 4 metres. Inside the square is a circle of radius 1.5 metres. If a dart thrown at the square is equally likely to land at any point inside the square, then the probability that it will land outside the circle is closest to

A 0.442 **B** 0.295 **C** 0.558 **D** 0.375

The following information relates to Questions 18 *and* 19

The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	Probable cause				Total
	Speed	Alcohol	Reckless driving	Other	
Fatal	42	61	22	12	137
Non-fatal	88	185	98	60	431
Total	130	246	120	72	568

18 The probability that the accident is not fatal, given that reckless driving is the cause, is closest to

A 0.82 **B** 0.17 **C** 0.21 **D** 0.23

19 Given that the accident is fatal, the probability that its cause is alcohol is closest to

A 0.25 **B** 0.11 **C** 0.45 **D** 0.55

20 If A and B are independent events such that $P(A) = 0.35$ and $P(B) = 0.46$, then $P(A \cup B)$ is equal to

A 0.810 **B** 0.649 **C** 0.161 **D** 0.110

21 The primary cooling unit in a nuclear power plant has a reliability of 0.95. There is also a back-up cooling unit to substitute for the primary unit when it fails. The reliability of the back-up unit is 0.85. The cooling system of the plant is considered reliable if either one of the systems is working. Assuming that the two systems are independent, the reliability of the cooling system of the power plant is

A 0.95 **B** 0.85 **C** 0.8075 **D** 0.9925

22 Two events A and B are independent events, where $P(A) = \frac{P(B)}{2}$ and $P(A \cup B) = 0.88$. $P(A)$ is closest to

A 0.3 **B** 0.4 **C** 0.5 **D** 0.8

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

10 Revision of Unit 1

10A Short-response questions

Technology-free short-response questions

SF

1 Consider the points $A(5, 2)$ and $B(-6, 8)$ in the Cartesian plane.

a Determine the midpoint of the line segment AB.

b Determine the gradient of the line AB.

2 Let $P(x) = 6x^3 + 5ax - 12a$. If $P(a) = -4$ determine the possible values of a.

3 **a** Express $x^2 + 8x + 9$ in the form $(x + a)^2 + b$.

b For what value of x is $x^2 + 8x + 9$ a minimum?

4 Let $f(x) = 6 - x$ for $-3 \le x \le 5$. Determine the range of the function f.

5 Determine the equation of the circle with centre $(2, -3)$ and radius 5.

6 Given that $f(x) = x^2 - 5$ for all real numbers x, state the range of f.

7 Given that $g(x) = x^2 - 3x + 5$ for $0 \le x \le 5$, state the range of g.

8 Determine the equation of the image of $y^2 = x$ under the sequence of transformations:

- a dilation of factor 2 from the x-axis
- followed by a translation of 3 units in the negative direction of the y-axis.

9 For each of the following relations, state the implied (maximal) domain and range:

a $y^2 = 1 - x$

b $(x - 2)^2 + y^2 = 9$

10 Solve the equation $4 + 2\sin x = 2$ for $-4\pi \le x \le 4\pi$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

SF

11 Sketch the graph of the function $f(x) = -2\sin(x)$ for $-2\pi \le x \le \pi$.

12 Given that $\cos\theta = 0.7$, determine:

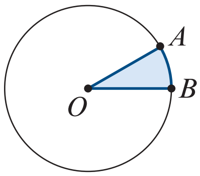
a $\cos(\pi - \theta)$ **b** $\cos(-\theta)$ **c** $\cos(\pi + \theta)$ **d** $\cos(2\pi - \theta)$

13 Consider the function with rule $f(x) = 2\cos(3x) + 1$ for $x \in \mathbb{R}$.

a State the period and amplitude of the function.

b Determine the range of the function.

14 Consider the shaded sector AOB, where $\angle AOB = 30°$ and the radius OA is 8 cm.



a Determine the length of the arc AB.

b Determine the area of the sector AOB.

15 The points $A(-2, 1)$, $B(3, -4)$ and $C(5, 7)$ are the vertices of triangle ABC.

a Let M and N be the midpoints of line segments AB and AC respectively. Determine the coordinates of M and N.

b Show that MN is parallel to BC.

16 Let $P(x) = 8x^3 + 4x - 3$.

a If $P(x) = (2x + 1)(4x^2 - 2x + 3) + R$ for all x, then determine the value of R .

b If $P(x) = (2 - x)(-8x^2 - 16x + c) + 69$ for all x, then determine the value of c .

c Write $P(x + 1)$ in the form $ax^3 + bx^2 + cx + d$.

17 Write each of the following in turning point form:

a $y = x^2 + 4x - 9$ **b** $y = x^2 - 3x - 11$ **c** $y = 2x^2 - 3x + 11$

18 Determine the coordinates of the points of intersection of the graphs for each of the following pairs of functions:

a $y = 4x + 1$ and $y = x^2 + 3x - 9$ **b** $y = 2x + 2$ and $y = x^2 - 2x + 6$

c $y = -3x + 2$ and $y = x^2 + 5x + 18$

19 State the maximal (implied) domain of each of the following functions:

a $f(x) = \dfrac{3}{2x - 5} - 7$ **b** $g(x) = \sqrt{5 - x} - 4$ **c** $f(x) = x^2 - 4x - 7$

20 Determine the range of each of the following functions:

a $f(x) = x^2,\ x \in [1, 2]$ **b** $h(x) = 2 - x,\ x \in [-1, 2]$

c $g(x) = x^2 - 4,\ x < 0$ **d** $f(x) = \sqrt{2 - x} + 3,\ x \in (-\infty, 2]$

e $f(x) = (x - 2)^3 + 8$

 ISBN 978-1-009-53719-3 Cambridge University Press

SF

21 Convert to radians:

a $60°$ **b** $270°$ **c** $140°$

22 Evaluate:

a $\sin\left(-\frac{\pi}{2}\right)$ **b** $\cos\left(\frac{3\pi}{2}\right)$ **c** $\tan(3\pi)$ **d** $\tan\left(-\frac{\pi}{2}\right)$

23 Determine:

a $\sin(2\pi - \theta)$ if $\sin\theta = 0.3$

b $\cos(-\theta)$ if $\cos\theta = -0.5$

c $\tan(\pi + \theta)$ if $\tan\theta = 1.6$

d $\sin(\pi + \theta)$ if $\sin\theta = 0.6$

e $\sin\left(\frac{\pi}{2} - \theta\right)$ if $\cos\theta = 0.1$

f $\cos\theta$ if $\sin\theta = \frac{3}{5}$ and $0 < \theta < \frac{\pi}{2}$

24 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{5\pi}{6}\right)$ **c** $\tan\left(\frac{-\pi}{4}\right)$

d $\sin\left(\frac{-7\pi}{6}\right)$ **e** $\cos\left(\frac{-7\pi}{4}\right)$ **f** $\tan\left(\frac{5\pi}{3}\right)$

25 Consider the function $f(x) = 2\cos\left(\frac{x}{2}\right)$.

a State the period and amplitude.

b Sketch a graph for one complete cycle.

c Give the sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = f(x)$.

26 Sketch $f(x) = -3\sin(2\pi x)$ for $x \in [0, 2]$.

27 Expand and simplify $(1 - 2x)^4$.

28 Solve these equations for $\theta \in [-2\pi, 2\pi]$:

a $\cos\theta = -\frac{\sqrt{3}}{2}$ **b** $\sqrt{2}\sin\theta = 1$ **c** $\sin(2\theta) = -\frac{1}{2}$ **d** $\tan\theta = -\sqrt{3}$

29 Let the universal set ε be the set of all natural numbers less than 20, and define three subsets of ε by $A = \{\text{multiples of } 5\}$, $B = \{\text{odd numbers}\}$ and $C = \{\text{multiples of } 7\}$. Determine:

a $A \cup C$ **b** $B \cap C'$ **c** $B' \cap C'$

30 Of the 30 students in a class, 18 students are learning Indonesian, and eight are learning both Indonesian and French. If every student in the class is learning at least one of these languages, how many students are learning French?

31 A six-sided die is rolled twice. Calculate the probability that:

a the sum of the numbers showing is 5

b the sum of the numbers showing is not 5.

 ISBN 978-1-009-53719-3

SF

32 A number is formed by arranging the digits 3, 4 and 8 in random order.

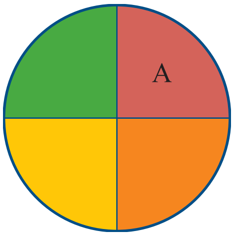
a List the sample space.

b Calculate the probability that a number less than 500 is obtained.

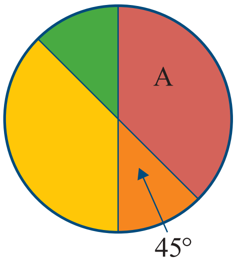
c What is the probability that the number obtained is even?

33 Consider the following spinners. In each case, what is the chance of the pointer stopping in region A?

a



b



34 Let $\varepsilon = \{1, 2, 3, \dots, 15\}$, $A = \{1, 2, 3, \dots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12\}$. Show these sets on a Venn diagram and use your diagram to determine:

a A' **b** B' **c** $A \cup B$

d $A \cap B$ **e** $A' \cap B'$

35 Construct a Venn diagram to represent sets C and D such that $n(C) = 10$, $n(D) = 25$ and $n(C \cap D) = 6$. Use your Venn diagram to determine $n(C \cup D)$.

36 A number is chosen at random from the set $\{1, 2, 3, \dots, 12\}$. What is the probability that the number is:

a divisible by 2

b divisible by 3

c divisible by both 2 and 3?

37 A bag contains 20 balls numbered $1, 2, 3, \dots, 20$. One ball is drawn at random from the bag. Determine the probability that the number on the ball is:

a less than 6

b greater than or equal to 7

c a number from 7 to 12 inclusive.

38 If A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.5$, determine:

a $P(A \mid B)$ **b** $P(A \cap B)$ **c** $P(A \cup B)$

CF

39 If $g(x) = 3x^2 - 4$, then determine:

a $g(2a)$ **b** $g(a - 1)$ **c** $g(a + 1) - g(a - 1)$

40 Determine the equation of the parabola that passes through the points $A(1, 6)$, $B(0, 2)$ and $C(-3, 8)$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

CF

41 Let $f(x) = 4 - 5x$ and $g(x) = 7 + 2x$.

a Is it true that $f(2) + f(3) = f(2 + 3)$?

b Solve the equation $f(x) = g(x)$ for x.

c Solve the inequality $f(x) \geq g(x)$.

d Determine the value of k for which $f(2k) = g(3k)$.

42 Solve the simultaneous equations:

$$x + y = 5$$
$$(x + 1)^2 + (y + 1)^2 = 25$$

43 Consider the graph of $y = \dfrac{4x - 7}{x + 1}$.

a Rewrite the equation of the graph as $y = \dfrac{a}{x + 1} + b$, for real numbers a and b.

b Determine the coordinates of the points of intersection of the graph with the axes.

c Determine the equations of the asymptotes of the graph.

44 The line ℓ_1 passes through the points $A(2, 1)$ and $B(0, 5)$.

a Determine the equation of the line ℓ_2 that is perpendicular to ℓ_1 and passes through B.

b The point $C(c, d)$ lies on the line ℓ_2 and the gradient of AC is $\dfrac{4}{3}$. Determine the values of c and d.

45 Solve the equation $2\cos^2 x - \cos x = 1$ for $0 \leq x \leq 2\pi$.

46 Determine the values of k for which the quadratic equation $-x^2 + kx + k + 1 = 0$ has:

a one solution **b** two solutions **c** two negative solutions.

47 A line passes through the point $(2, 3)$ and has gradient m, where m is a negative number.

a Determine the equation of the line.

b Determine the coordinates of the points of intersection of the line with the axes.

c Determine the values of m for which the triangle enclosed by the line and the two axes has area 16.

48 Determine the coordinates of the points of intersection of the circle $x^2 + y^2 - 7x + 8y + 12 = 0$ with the axes.

49 The expression $c + bx - x^2$ has its greatest value 5 when $x = 2$. Determine the values of b and c.

50 An open rectangular cardboard box has a height of 5 cm, and its length is 6 cm greater than its width.

a If the width of the box is x cm, write down expressions for:

i its length **ii** the area of its base

iii the area of each of its four other sides **iv** the total surface area.

b If the total surface area is 180 cm^2, determine the value of x.

51 A graph has a rule of the form $y = a\sqrt{x-h} + k$ and passes through the three points $(4, 6)$, $(7, 8)$ and $(12, 10)$. Determine the values of a, h and k.

52 Show that the points $A(0, -5)$, $B(-1, 2)$, $C(4, 7)$ and $D(5, 0)$ form a rhombus $ABCD$.

53 Solve each of the following inequalities:

a $x^2 + 3x - 5 > 0$ **b** $2x^2 - 5x - 5 \geq 0$

54 If $5x^3 - 3x^2 + ax + 7 = (x+2)Q_1(x) + R$ and $4x^3 + ax^2 + 7x - 4 = (x+2)Q_2(x) + 2R$, determine the values of a and R.

55 At Big Bob's Burger Bar, a Big Bob burger costs \$$b$ and a regular fries costs \$$f$. Three Big Bob burgers and two regular fries cost \$18.20.

a Write an equation to show this.

b If one Big Bob burger costs \$4.20, what is the cost of one regular fries?

56 $4x + ky = 7$ and $y = 3 - 4x$ are the equations of two lines. Determine k if:

a the lines are parallel **b** the lines are perpendicular.

57 Line ℓ_1 has an x-axis intercept at $(5, 0)$ and a y-axis intercept at $(0, -2)$. Line ℓ_2 is perpendicular to ℓ_1 and passes through the point $(1, 6)$.

a Determine the gradient of line ℓ_1.

b Determine the equation of line ℓ_2 expressing your answer in the form $ax + by + c = 0$, where a, b and c are integers and $a > 0$.

58 A family of parabolas have rules of the form $y = ax^2 + 2x + a$, for $a \neq 0$.

a Express $ax^2 + 2x + a$ in the form $a(x+b)^2 + c$ for real numbers b and c.

b Give the coordinates of the vertex of the graph of $y = ax^2 + 2x + a$ in terms of a.

c For which values of a is $ax^2 + 2x + a$ a perfect square?

d For which values of a does the graph of $y = ax^2 + 2x + a$ have two x-axis intercepts?

59 Sketch $y = 2\sin 2\left(x - \dfrac{\pi}{4}\right) + 1$ for $x \in [0, 2\pi]$ and determine the x-axis intercepts.

60 Sketch the graph of $y = \dfrac{1}{2}\tan(2x)$ for $x \in [-\pi, \pi]$.

61 For each of the following, write down the first two terms and the last two terms in the expansion, without simplifying the coefficients. (Assume that the terms are ordered by increasing powers of y.)

a $(2x + 3y)^{35}$ **b** $(2 + y^3)^{20}$

62 Determine the value of k if, in the expansion of $(2 + kx)^5$, the coefficient of x^3 is 1080.

63 A swimming pool initially contains 45 000 litres of water. At 12 p.m., an inlet valve is opened letting 40 litres of water per minute into the pool. Assume no water evaporates.

a Determine an expression for the volume, V litres, in the pool m minutes after 12 p.m.

b When will the pool reach its maximum capacity of 55 000 litres?

c Sketch the graph of V against m. (Use the appropriate domain.)

Revision

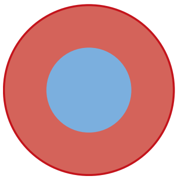
CF

64 For a particular biased six-sided die, it is known that the numbers 1, 2, 3 and 5 are equally likely to occur, that the number 4 occurs four times as often as the number 2, and that the number 6 occurs half as often as the number 2.

a Determine the probability of each of the possible outcomes.

b Determine the probability that the number observed is not a 4.

65 The dartboard shown consists of two concentric circular regions. The radius of the inner circle is 10 cm, and the radius of the outer circle is 20 cm. If Mike throws a dart, it is equally likely to hit any part of the dartboard, and it hits the dartboard every time. Determine the probability of the dart hitting the blue region.



66 Suppose that in a certain school the probability that a student plays basketball is 0.3, the probability that a student plays hockey is 0.4, and probability that a student plays both basketball and hockey is 0.1.

a Determine the probability that a student plays either or both of these sports.

b Determine the probability that a student plays hockey, given that they play basketball.

67 At a school holiday program for 60 students, painting was offered as an additional activity one morning, and music as an additional activity in the afternoon. Every student went to at least one additional activity, with 45 students attending painting and 30 students attending music. Construct a probability table, and use it to determine the probability that a student chosen at random:

a went to music **b** went to painting

c went to painting but not music **d** went to music but not painting.

68 In a certain town, the probability that it will be sunny on any Saturday is 0.6. If it is sunny on Saturday, then the probability that it is sunny on Sunday is 0.8. If it is not sunny on Saturday, then the probability that it is sunny on Sunday is 0.2. Determine the probability that:

a it is sunny all weekend **b** it is sunny on Sunday.

69 For two events A and B such that $P(A) = 0.5$, $P(B) = 0.2$ and $P(B \mid A) = 0.1$, determine:

a $P(A \cap B)$ **b** $P(A \mid B)$

70 In a group of students, 23 study a language, 19 study mathematics, eight study both a language and mathematics, and two study neither mathematics nor a language.

a Construct a Venn diagram and use it to determine:

i the total number of students in the group

ii the number of students who study mathematics but not a language

iii the number of students who study a language but not mathematics.

b Determine the probability that a student selected at random from the group:

i studies mathematics **ii** studies a language but not mathematics.

CF

71 For a particular biased six-sided die it is known that the numbers 1, 2 and 3 are equally likely to occur, that the numbers 4 and 5 each occur twice as often as the number 2, and that the number 6 occurs half as often as the number 2.

a Determine the probability of each of the possible outcomes.

b Determine the probability that the number observed is not a 6.

72 Two cards are selected, without replacement, from a pack of 52 playing cards. Determine the probability that:

a both cards are hearts

b both cards are red

c the first card is red and the second card is black

d the second card is red.

CU

73 **a** Sketch the graph of $y = 1 + \dfrac{1}{2+x}$, where $x \neq -2$.

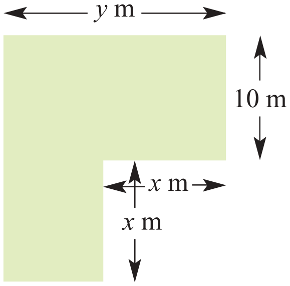
b The graph crosses the y-axis at A and the x-axis at B. Give the coordinates of A and B.

c Determine the equation of line AB.

d Determine the coordinates of the midpoint M of AB.

e Determine the equation of the straight line passing through M perpendicular to AB.

74 A region is enclosed as shown.



a Determine expressions for:

i A (m^2), the area of the region in terms of x and y

ii P (m), the perimeter of the region in terms of x and y.

b **i** If the perimeter is 100 m, determine A in terms of x.

ii What is the maximum area possible?

iii State the possible values for x.

iv Sketch the graph of A against x for these values.

75 The point $A(3, -3)$ lies on the circle with equation $x^2 + y^2 - 12x - 2y + 12 = 0$.

a Determine the coordinates of the point B if AB is a diameter of the circle.

b Determine the value of k, where k is a positive number, if $C(2, k)$ is a point on the circle.

c Show that $\triangle ABC$ is an isosceles right-angled triangle.

76 A rectangular block is 2 cm wider than it is high and twice as long as it is wide. Let x cm be the height of the block.

a Determine an expression for the total surface area, A cm^2, in terms of x.

b Determine A if:

i $x = 1$ **ii** $x = 2$

c Determine the value of x if $A = 190$.

d Sketch the graph of A against x for suitable values of x.

e Determine an expression for the volume, V cm^3, of the block in terms of x.

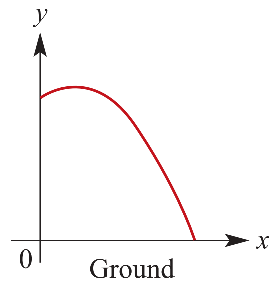
f If the volume of the block is 150 cm^3, determine the value of x.

g If the volume of the block is 1000 cm^3, determine the value of x.

77 A stone is projected from a balcony as shown. The path is described by the equation

$$y = -\frac{1}{10}(x + 10)(x - 20), \quad x \geq 0$$

where y metres is the height above the ground when the stone has travelled x metres horizontally.



a What is the height of the stone at the point of the projection?

b What is the horizontal distance travelled before the stone hits the ground?

c What is the maximum height reached by the stone?

78 A tank of capacity 1000 litres is initially empty. Water flows into the tank at 20 litres per minute for 10 minutes, and then the rate is decreased to 15 litres per minute. The water continues to flow in at this rate until the tank is filled.

a How much water is in the tank after 10 minutes?

The volume of water in the tank can be described by the rule

$$V = \begin{cases} at & 0 \leq t \leq 10 \\ bt + c & 10 < t \leq d \end{cases}$$

b Determine the values of a, b, c and d.

c Sketch the graph of V against t.

79 A family of functions have rules of the form $f(x) = \sqrt{a - x}$, where a is a positive real number.

a State the maximal domain of f.

b Determine the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

c For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(1, 1)$?

d For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(2, 2)$?

e For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (c, c), where c is a positive real number?

80 Let A, B and C be events, with $0 < P(C) < 1$. Prove each of the following:

a If $A \subseteq B$, then $P(A \mid C) \leq P(B \mid C)$.

b $P(A' \mid C) = 1 - P(A \mid C)$

c $P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$

d $P(A \mid C') = \dfrac{P(A) - P(A \cap C)}{1 - P(C)}$

CU

81 **a** Prove that, if events A and B are independent, then:

i the events A and B' are independent

ii the events A' and B are independent.

b For independent events A and B such that $P(A) = \frac{1}{2}$ and $P(A \cup B) = \frac{2}{3}$, determine:

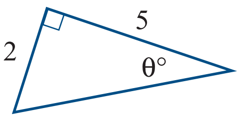
i $P(B)$ **ii** $P(A \mid B)$ **iii** $P(B' \mid A)$

Technology-active short-response questions

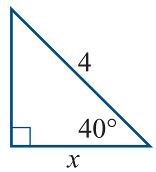
SF

82 Determine the value of each pronumeral to two decimal places. Give the angle in degrees.

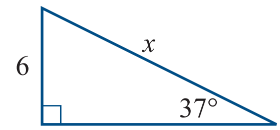
a



b



c



83 A sector cut from a circle of radius 5 cm has a perimeter of 16 cm. Determine the area of this sector.

84 Let $P(x) = 3x^3 + x^2 + px + 24$. Given that $P(-4) = 0$ determine the value of p and hence factorise the expression completely.

85 Using a calculator, evaluate each of the following correct to four significant figures:

a $\sin 28^\circ$ **b** $\cos 143^\circ$ **c** $\tan 297^\circ$

d $\sin 67^\circ$ **e** $\sin 1.83^c$ **f** $\tan 2.37^c$

86 Solve the equation $x^3 - 4x^2 + x + 6 = 0$ for x.

87 The table shows the time T s for a pendulum of length ℓ cm to swing up and back.

ℓ cm	7	12	17	22	32	42	52	62
T s	0.5	0.714	0.852	0.962	1.162	1.342	1.5	1.614

Use power regression with your calculator to determine T in terms of ℓ.

88 A circle of radius 8 cm contains a sector of perimeter 32 cm. Determine the area of the sector.

89 A random sample of students were asked to name their favourite fast food. The results are shown in the table. Estimate (to two decimal places) the probability that a student in this town:

a prefers pizza

b does not prefer hamburgers

c either prefers chicken or prefers fish and chips.

Fast-food	Students
Fish and chips	23
Hamburgers	57
Chicken	43
Pizza	85

 ISBN 978-1-009-53719-3

Revision

SF

90 Suppose that $P(A) = 0.25$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$. Determine $P(A \cup B)$.

91 Suppose that $P(A) = 0.3$, $P(B) = 0.35$ and $P(A \cup B) = 0.45$. Determine $P(A \cap B)$.

92 **a** Complete the probability table on the right by filling in the missing entries.

	B	B'	
A	0.3		0.65
A'	0.24		
			1

b Use the table to write down the values of:

i $P(A \cap B')$ **ii** $P(A)$ **iii** $P(B')$

93 If A and B are events such that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.15$, determine:

a $P(A \mid B)$ **b** $P(B \mid A)$

94 For independent events C and D such that $P(C) = 0.3$ and $P(D) = 0.4$, determine:

a $P(C \cap D)$ **b** $P(C \mid D)$

CF

95 Let $P(x) = 2x^3 - 9x^2 + ax + b$.

a If $P(-1) = P(2) = 3$ determine the values of a and b.

b Solve the equation $P(x) = 3$ for x.

c Solve the equation $P(3x) = 3$ for x.

96 A particle is moving along a straight line. Its distance, x metres, from a point O on the line at time t seconds is given by $x = 5 - 2\sin(3t)$.

a Determine the particle's greatest distance from O.

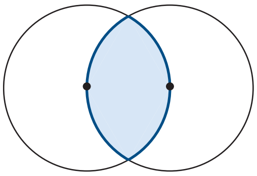
b Determine its least distance from O.

c Determine the times at which it is 5 metres from O for $0 \le t \le 3$.

d Determine the times at which it is 3 metres from O for $0 \le t \le 3$.

e Describe the motion of the particle.

97 Two circles of radius 12 cm overlap as shown in the diagram. Determine the area of the shaded region.



98 Let $f(x) = 3\sin(2\pi x)$ and $g(x) = 3\sin\left(\dfrac{\pi x}{2}\right)$.

a **i** Write down the minimum and maximum values of f.

ii Write down the period of g.

b How many solutions of the equation $f(x) = g(x)$ are there for $0 \le x \le 2$?

 ISBN 978-1-009-53719-3

99 The annual sales figures of a soft-drink manufacturing company for the years 2010 to 2015 can be modelled by the quadratic function $S(t) = (1.411t^2 + 7.411t + 55)$ million dollars, where $t = 0$ corresponds to 2010, and $t = 1$ corresponds to 2011, and so on.

a Use your calculator to plot the graph of the function S.

b Use this model to give the sales figures for 2010, 2011 and 2015.

c In which year did sales reach 100 million dollars?

d Use this model to predict the sales in 2020.

e Explain why there may be problems using this model to predict the future sales figures of the company.

100 A cleaner charges \$30 per hour for work that takes up to 2 hours and then charges \$25 per hour for work after the first 2 hours, up to a maximum of 8 hours.

a Write the rule for a function that describes the total fee, $\$F$, for work that takes the cleaner t hours. (Assume a continuous model.)

b Sketch a graph of your function.

c Determine the fee for work that takes the cleaner:

i 1.5 hours **ii** 2 hours **iii** 4.5 hours

101 The general equation of the circle can be written as $x^2 + y^2 + bx + cy + d = 0$. A circle passes through the points with coordinates $(-4, 5)$, $(-2, 7)$ and $(4, -3)$.

a Write three simultaneous equations in b, c and d.

b Determine the equation of the circle.

102 Two dice are rolled. Determine the probability that the two numbers showing on the dice are the same, given that their sum is 8.

103 At a certain ski resort, the probability of snow falling on a Saturday during the ski season is 0.45. If it snows on Saturday, then the probability of snow on Sunday is 0.55. If it does not snow on Saturday, then the probability of snow on Sunday is 0.30. Determine the probability that during a given week in the ski season:

a it snows on both Saturday and Sunday

b it snows on Sunday.

104 A randomly chosen sample of 500 students were classified according to whether or not they regularly participate in a physical activity (such as sport or dance) and their performance on a fitness test.

Level of fitness	Physically active	Not physically active
Below average	37	85
Average	115	73
Above average	155	35

Use the data in the table to estimate the probability that a randomly chosen student:

a is both physically active and of above-average fitness

b is physically active

c is of below-average fitness.

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

CF

105 Samira chooses an ice-cream at random from an ice-cream shop. It is known that in this shop:

- 70% of ice-creams are vanilla
- 45% of ice-creams are chocolate-coated
- 30% of ice-creams are vanilla and not chocolate-coated.

Determine the probability that Samira's randomly chosen ice-cream:

a is not vanilla but is chocolate-coated
b is neither vanilla nor chocolate-coated
c is either vanilla or chocolate-coated or both.

106 A jar contains four red marbles, five white marbles and three green marbles. Two marbles are drawn from the jar at random, one after the other, without replacement. Consider the following events:

- R_1 the first marble is red
- W_1 the first marble is white
- G_1 the first marble is green
- R_2 the second marble is red
- W_2 the second marble is white
- G_2 the second marble is green.

Determine:

a $P(R_2 \mid R_1)$ **b** $P(G_2 \mid R_1)$ **c** $P(W_2 \mid R_1)$ **d** $P(G_1 \cap G_2)$
e $P(R_2)$ **f** $P(G_2)$ **g** $P(W_1 \cap R_2)$ **h** $P(R_1 \mid R_2)$

107 In a certain town, 40% of people have brown hair, 25% have brown eyes and 15% have both brown hair and brown eyes. A person is selected at random from the town.

a If the person has brown hair, what is the probability that this person also has brown eyes?
b If the person has brown eyes, what is the probability that this person does not have brown hair?
c What is the probability that this person has neither brown hair nor brown eyes?

108 The heights (measured to the nearest centimetre) of 56 plants grown under experimental conditions are given in the following table.

Note: The table gives the number of plants for which the measured height lies in a given interval.

Interval	(0, 10]	(10, 20]	(20, 30]	(30, 40]	(40, 50]	(50, 60]	(60, 70]
Number of plants	1	2	4	6	13	22	8

a If one plant is selected at random, determine the probability that its height is:

i greater than 50 cm
ii either greater than 50 cm or less than or equal to 30 cm
iii greater than 40 cm, given that it is greater than 30 cm.

CF

b The probability that a randomly selected plant produces a flower is $\frac{6}{7}$ and the probability that it will suffer from a disease is $\frac{1}{4}$. Assume independence.

i Determine the probability that a single plant selected at random produces a flower but does not suffer from disease.

ii Determine the probability that a single plant selected at random produces a flower, does not suffer from disease and has a height greater than 50 cm.

CU

109 A circle passes through the origin. It has equation $x^2 + y^2 + bx + cy = 0$. The circle also passes through the point $(4, 4)$.

a Determine c in terms of b.

b Determine the y-axis intercepts in terms of b.

c For what value of b does the circle touch the y-axis without crossing it?

110 A particular plastic plate manufactured at a factory sells at \$11.50. The cost of production consists of an initial cost of \$3500 and then \$10.50 a plate. Let x be the number of plates produced.

a Let \$$C$ be the cost of production of x plates. Write an expression for C in terms of x.

b Let \$$I$ be the income from selling x plates. Write an expression for I in terms of x.

c On the one set of axes sketch the graphs of I against x and C against x.

d How many plates must be sold for the income to equal the cost of production?

e Let $P = I - C$. Sketch the graph of P against x. What does P represent?

f How many plates must be sold for a profit of \$2000 to be made?

111 From a piece of wire 42 cm long, a length $10x$ cm is cut off and bent into a rectangle whose length is one and a half times its width. The remainder is bent to form a square.

a State the area of the rectangle in terms of x.

b State the area of the square in terms of x.

c State the possible values for x.

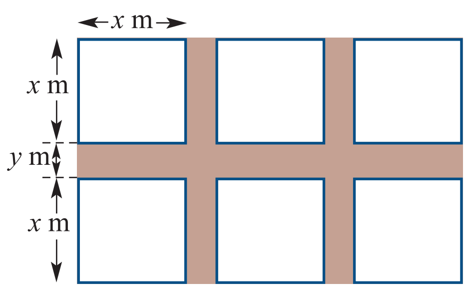
d Determine an expression for the total area, A cm^2, of the square and the rectangle.

e Sketch the graph of A against x.

f What is the maximum possible total area?

g If the combined area of the rectangle and the square is 63 cm^2, find their dimensions.

112 The diagram shows a window with six equal square panes divided by equally thick timber framing.



a Write down the total area of the whole window in terms of x and y.

b Show that the total area of the dividing wood is $7xy + 2y^2$.

c The total area of glass is 1.5 m^2 and the total area of the dividing wood is 1 m^2.

i Determine x. **ii** Determine y.

113 The height of the tide, h metres, at a harbour at any time during a 24-hour period is given by the rule $h(t) = 10 + 4\sin(15t)^\circ$, where t is measured in hours.

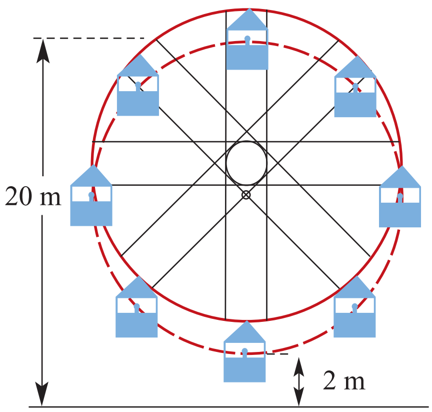
a Sketch the graph of h against t for $0 \le t \le 24$.

b Determine the times at which $h = 13$ during the 24-hour period.

c A boat can leave the harbour when the height of the tide is at 11 metres or more. State the times during the 24 hours when the boat can leave the harbour.

114 For a ride on a Ferris wheel, the height above the ground, h metres, of a person at time t seconds is given by

$$h(t) = 11 + 9\cos\left(\frac{\pi}{30}(t - 10)\right)$$



a How long does it take for the Ferris wheel to make one complete revolution?

b Sketch the graph of h against t for one revolution.

c State the range of the function.

d At what time(s) is the person at a height of 2 metres above the ground?

e At what time(s) is the person at a height of 15.5 metres above the ground?

115 Two points A and C have coordinates $(1, 3)$ and $(7, 7)$.

a Determine the equation of the perpendicular bisector of AC.

b B is a point on the y-axis which is equidistant from A and C, and $ABCD$ is a rhombus. Determine:

i the coordinates of B **ii** the coordinates of D

c Determine the area of the rhombus $ABCD$.

d Calculate the perpendicular distance of A from BC correct to two decimal places.

116 The voltage, V, in a circuit after t seconds is given by $V = 120\cos(60\pi t)$.

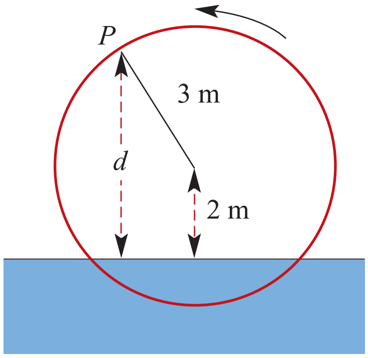
a Sketch the graph of V against t for one cycle.

b Determine the first time the voltage is 60.

c Determine all times at which the voltage is maximised.

117 The figure shows a waterwheel rotating at 4 revolutions per minute. The distance, d in metres, of a point P from the surface of the water as a function of time, t in seconds, can be modelled by a rule of the form

$$d = a + b \sin c(t - h)$$



a Using seconds as the unit of time, determine:

i the period **ii** the amplitude **iii** c

b If $d = 0$ when $t = 0$ and P is moving upward at this time, determine h.

c Sketch the graph of d against t for $0 \le t \le 30$.

118 A fountain is 5 metres high. Water is coming vertically out of the fountain. The function which describes the height of a drop of water above the ground t seconds after leaving the spout of the fountain is given by $h(t) = -4.9t^2 + 30t + 5$.

a How high is the drop of water after 3 seconds?

b When will the drop be back at the height of the spout?

c Sketch the graph of the height of the drop above the ground against time.

d If there is a little wind, the drop will not travel along a vertical line but its time in the air will not be affected and h will still describe its height above the ground. How long will it take for a drop of water to hit the ground?

119 An open rectangular cardboard box is 7 cm high and its length is 5 cm greater than its breadth, which is x cm.

a Determine the length of the box in terms of x.

b Determine the volume (V cm^3) of the box in terms of x.

c Determine the surface area (S cm^2) in terms of x.

d Sketch the graphs of S against x and V against x for a suitable domain on the one set of axes.

e For what value of x is $V = S$?

f If the total surface area is 500 cm^2, determine x.

120 Two machines in a factory produce a particular component for a car.

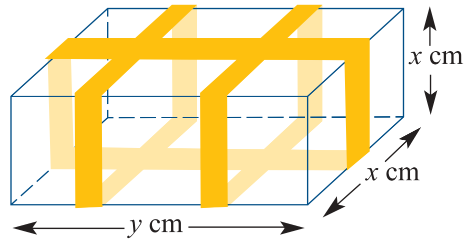
- Machine A produces 1000 components each day, with an average of 5% defective.
- Machine B produces 1500 components each day, with an average of 2% defective.

a Determine the overall percentage of defective components produced each day.

b A component is chosen at random from the combined output of the two machines and is found to be defective. What is the probability that this component was produced by Machine A?

 ISBN 978-1-009-53719-3

121 A parcel in the form of a rectangular block is held together by three pieces of tape as shown.



The parcel has square ends with side length x cm. The length of the parcel is y cm.

The total length of tape is 500 cm.

a **i** Determine y in terms of x.

ii Determine V, the volume of the parcel in cm^3, in terms of x.

b Draw the graph of V against x for suitable values of x.

c State the domain of this function.

d Determine the values of x for which $V = 25\,000$.

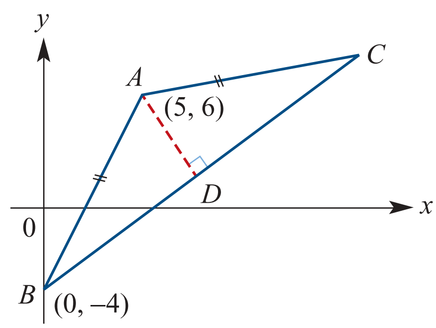
e Find the maximum volume of the parcel and the corresponding values of x and y.

122 **a** A train travels 300 km at a constant speed of V km/h. If the train had travelled 5 km/h faster, the journey would have taken two hours less. Determine the speed of the train travelling at the slower speed.

b A tank can be filled by two taps A and B in $33\frac{1}{3}$ minutes (33 minutes 20 seconds) when they are running together. Tap A running by itself fills the tank in 15 minutes less than tap B. Determine the time taken for each tap running by itself to fill the tank.

c A hall can be paved with 200 square tiles of a certain size. If each tile were 1 cm longer and wider it would take 128 tiles. Determine the length of each tile.

123 ABC is an isosceles triangle. The coordinates of A and B are $(5, 6)$ and $(0, -4)$ respectively. Given that the gradient of BC is $\frac{3}{4}$ and D is the midpoint of BC, determine:



a the equation of BC

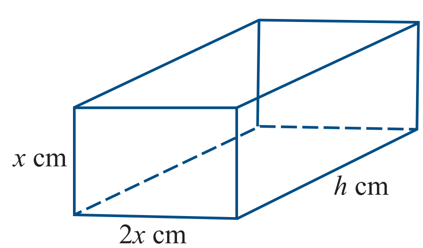
b the equation of AD

c the coordinates of D

d the length of the perpendicular AD

e the area of triangle ABC.

124 A piece of wire 400 cm long is used to make the 12 edges of a cuboid with dimensions as shown.



a Determine h in terms of x.

b Determine the volume, V cm^3, in terms of x.

c State the possible values for x.

d Plot the graph of V against x on a calculator for the domain determined in part c.

e State the values of x (correct to 3 decimal places) which will result in a volume of:

i 30 000 cm^3 **ii** 20 000 cm^3.

f State the maximum volume (correct to 3 decimal places) and the corresponding value of x.

g The cuboid is covered in paper.

i Determine the surface area, S cm^2, of the cuboid in terms of x.

ii Determine the maximum value of S and the value of x for which this occurs.

h Determine the values of x for which $S = V$.

125 Jac can buy x kilograms of apples for \$60. For the same amount of money, he can buy 3 kg more bananas than apples.

a If the price of each type of fruit is reduced by \$1 per kilogram, then Jac can buy 5 kg more bananas than apples for \$60.

i How many kilograms of apples can he buy for \$60 at the original price?

ii How many kilograms of bananas can he buy for \$60 at the original price?

b Repeat part a using the following information instead: If the price of each type of fruit is reduced by \$2 per kilogram, then Jac can buy 10 kg more bananas than apples for \$60.

c Discuss your solutions to parts a and b, and determine how many kilograms of each type of fruit Jac can buy for \$60 at the reduced prices.

126 The temperature, T_1°C, of a room in a house at time t hours after midnight is modelled by the rule

$$T_1 = 8\sin\left(\frac{\pi}{12}(t + 13)\right) + 14 \quad \text{for } 0 \le t \le 24$$

a Determine the temperature of the room at:

i midnight **ii** midday.

b At what time of the day is the room warmest?

c At what times of the day is the temperature of the room greater than 20°C?

The temperature, T_2°C, of a second room in the house at time t hours after midnight is modelled by the rule

$$T_2 = 5\sin\left(\frac{\pi}{12}(t + 11)\right) + 15 \quad \text{for } 0 \le t \le 24$$

d Arty the cat always likes to be in the warmer of the two rooms. At what times of the day is Arty likely to be found in the second room?

127 A bag contains four red balls and six blue balls. One ball is drawn at random from the bag. If it is blue, then it is replaced in the bag; but if it is red, then it is not replaced. A second ball is then drawn at random from the bag.

Let A be the event ‘the first ball is red’, and let B be the event ‘the second ball is red’. Determine:

a $P(A)$

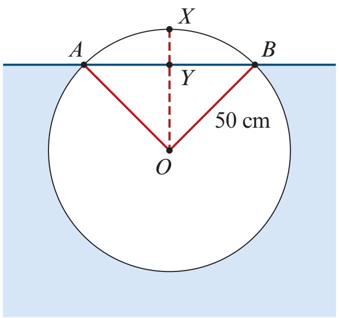
b $P(B \mid A)$

c $P(B)$

d $P(A \text{ or } B \text{ but not both})$

Revision

128 The diagram shows the cross-section of a log of radius 50 cm floating in water. The circle has centre O, the highest point of the log is X, and the points A, Y and B are level with the surface of the water.



a If the distance XY is 8 cm, determine:

  i the magnitude of $\angle AOB$ in radians

  ii the area of the cross-section below the water surface

  iii the percentage of the volume of the log below the water surface.

b If 90% of the log (by volume) is below the water surface, how far above the water surface is the highest point of the log?

129 A series of three rugby games is to be played between Australia and England. Suppose that there are no drawn games and that:

- the probability that Australia will win the first game is 0.66
- if Australia win a game, the probability that they will win the next game is 0.57
- if Australia lose a game, the probability that they will lose the next game is 0.53.

Determine the probability, correct to three decimal places, that:

a Australia win all three games

b England win all three games

c Australia win the second game

d Australia win the third game

e Australia win the series (that is, win at least two of the three games).

130 A mathematics teacher sets a multiple-choice test for his class, where each question has five alternatives. A student who has studied for the test will select the correct alternative for a question with probability 0.7. A student who has not studied will guess and therefore will select the correct alternative for a question with probability 0.2.

a What is the probability that a student who has studied for the test will:

  i answer the first two questions on the test correctly

  ii answer at least one of the first two questions on the test correctly?

b Suppose the probability that a randomly chosen student from the class will answer the first question correctly is 0.6. What is the probability that a student who answers the first question correctly has studied for the test?

131 It is known that, among all patients displaying certain symptoms, the probability that a patient has a particular disease is 0.005. A test for the disease has been developed. The test result is positive for 90% of patients who have the disease and for 5% of patients who do not have the disease.

a The test is given to a particular patient displaying the symptoms, and it gives a positive result. Determine the probability that this patient has the disease.

b Comment on your answer to part a.

c Repeat your calculations using different probabilities for the disease and different percentages of positive test results. Comment on your findings.

CU

132 Pam runs a factory that produces potato chips. Each crate of potatoes that is delivered to the factory is tested for quality using the following strategy:

> A random sample of three potatoes is taken from the crate. If one or more of the potatoes in the sample is unacceptable, then the whole crate is rejected. Otherwise, the crate is accepted.

a Suppose that 3% of potatoes from Paul's Potato Farm are unacceptable. Pam takes a random sample of three potatoes from a crate delivered by Paul's Potato Farm.

i Determine the probability that exactly one potato in the sample is unacceptable.

ii Determine the probability that exactly two potatoes in the sample are unacceptable.

iii Determine the probability that all three potatoes in the sample are unacceptable.

iv Hence, determine the probability that the crate will be rejected.

b Suppose that 2% of potatoes from Pete's Potato Farm are unacceptable. If the probability that a randomly selected crate comes from Paul's farm is 0.7 and the probability that it comes from Pete's farm is 0.3, what is the probability that it will be rejected?

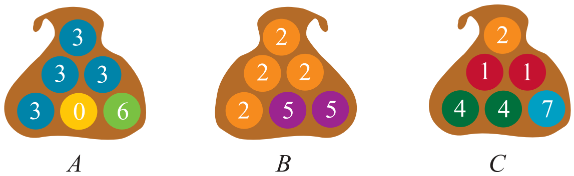
c Suppose that Pam wants to reject only 6.5% of the crates that are delivered to her factory. What percentage of her deliveries should she get from Pete's farm?

d Paul delivers a crate of 100 potatoes containing exactly three unacceptable potatoes. If Pam randomly selects three potatoes without replacement from the crate, determine:

i the probability that the sample will contain exactly one unacceptable potato

ii the probability that the crate will be rejected.

133 Three people, A, B and C, are each given a bag containing six numbered balls as shown in the diagram below, and each of them draws a ball from the bag that they were given.



a Calculate the probability that B draws a higher number than C.

b Calculate the probability that A draws a higher number than both B and C.

134 Of the employees in a large factory, $\frac{1}{8}$ travel to work by bus, $\frac{3}{8}$ by train, and the remainder by car. Those travelling by bus have a probability of $\frac{1}{3}$ of being late, those by train will be late with probability $\frac{1}{5}$, and those by car will be late with probability $\frac{3}{4}$.

a Draw and complete a tree diagram and calculate the probability that an employee chosen at random will be late.

b If an employee is late, calculate the probability that he or she travelled by car.

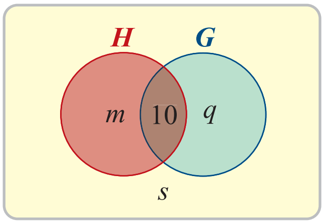
 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

CU

135 Of the 100 students at a particular level in a school, 40 take history, 45 take geography and 10 take both subjects.

a A Venn diagram interpreting this situation is as shown:



m is the number of students who take history only.

q is the number of students who take geography only.

s is the number of students who take neither subject.

i Determine the values of m, q and s.

ii Determine the number of students who take either history or geography but not both.

b If a student is chosen at random, determine the probability that he or she takes history but not geography.

c If a student is chosen randomly from the students who do not take history, determine the probability that he or she takes geography.

136 Suppose that in a game of chance bag A contains balls numbered 1 and 2, bag B contains balls numbered 3, 4 and 5, and bag C contains balls numbered 3 and 5.

a David chooses one bag at random and then draws one ball randomly from the bag.

i Determine the probability that the ball drawn is a 4.

ii Determine the probability that the ball drawn is a 3.

b After David has drawn a ball in the way described in part a, he puts it back into the original bag. Next, Sally chooses one bag at random and then draws one ball randomly from the bag.
Determine the probability that:

i the balls drawn by David and Sally are both numbered 4

ii the balls drawn by David and Sally are both numbered 3 and from the same bag.

137 A, B and C fire one shot each at a target. The probability that A will hit the target is $\frac{1}{5}$, the probability that B will hit the target is $\frac{1}{4}$, and the probability that C will hit the target is $\frac{1}{3}$. If they fire together, calculate the probability that:

a all three shots hit the target

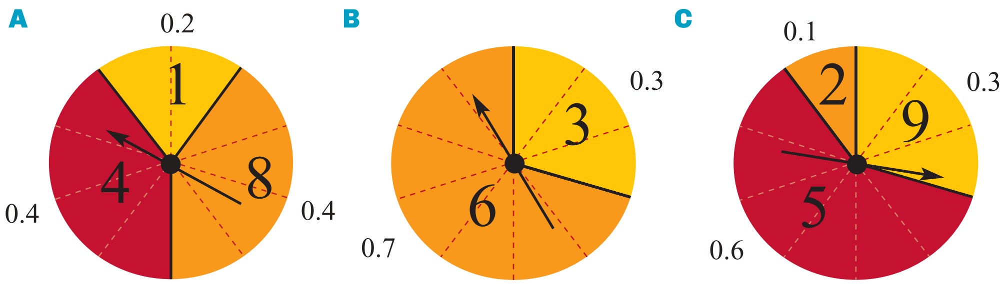
b only C's shot hits the target

c at least one shot hits the target

d given that only one shot hits the target, it is the shot by C.

 ISBN 978-1-009-53719-3

138 Players A, B and C each have a disc and a spinning arrow which when spun comes to rest with its head pointing towards one sector. The probability of the arrowhead stopping at any sector and the score recorded by the player is shown in the diagram. For example, player A scores 1, 4 or 8 with probabilities 0.2, 0.4 or 0.4 respectively.



If A, B and C all spin their arrows simultaneously, determine the probability (assuming independence) that:

a the total of the scores is 23

b B scores more than C

c C scores more than A.

139 The journey from town A to town D passes through towns B and C. There are three roads from A to B, four from B to C and five from C to D.

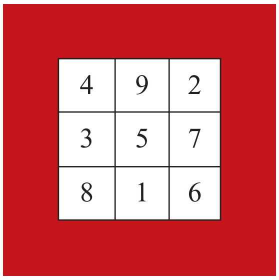
a Determine the number of different routes that can be taken from A to D.

There are road works on one of the roads between A and B, on two roads between B and C and on three roads between C and D.

b Determine the number of different routes that can be taken from A to D without encountering any road works.

c Calculate the probability that a traveller from A to D, picking a route at random, encounters road works at each stage of their journey.

140 In a game at a fair, darts are thrown at a board. The outer square is of side length 100 cm and each of the nine inner squares is of side length 20 cm.



a Calculate, in cm^2, the area of:

i the larger outer square

ii one of the inner squares

iii the shaded region.

The number shown in each region is the score obtained by a dart hitting that region. A dart hitting the shaded region scores 0. Assume that all darts hit the board and that each point on the board is equally likely to be hit.

b Determine the probability that with one dart the score will be:

i 7 **ii** at least 7 **iii** 0

c Each turn consists of throwing three darts. Calculate the probability of scoring:

i a total of 18 with the first two darts **ii** a total of 24 with the three darts.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

10B Multiple-choice questions

Technology-free multiple-choice questions

1 The turning point of a quadratic with rule $y = x^2 - ax$ has coordinates

A $(0, a)$ **B** $\left(\frac{a}{2}, \frac{-a^2}{4}\right)$ **C** $(a, 0)$ **D** $\left(a, \frac{-a^2}{2}\right)$

2 The quadratic equation $x^2 - 2ax + b = 0$, where a and b are positive constants, has one solution when

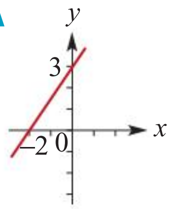
A $b = a$ and $a \neq 1$ **B** $b = \sqrt{a}$ or $b = -\sqrt{a}$
C $b = 1$ and $a \neq 1$ **D** $a = \sqrt{b}$

3 The simultaneous equations $(m - 2)x + 3y = 6$ and $2x + (m + 2)y = m$ have a unique solution when

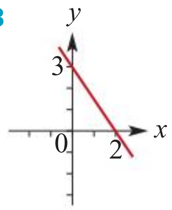
A $m \neq 0$ **B** $m \neq 1$ and $m \neq -1$
C $m \neq \sqrt{10}$ and $m \neq -\sqrt{10}$ **D** $m > 1$ or $m < -1$

4 The graph of $3x - 2y = -6$ is

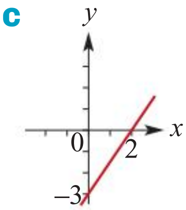
A



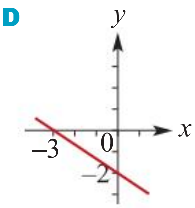
B



C



D



5 Which of the following points lies on both the line with equation $y = 3x - 1$ and the line with equation $4x + 2y = 8$?

A $(0, 0)$ **B** $(1, 2)$ **C** $(0, 4)$ **D** $(2, 0)$

6 The factors of $x^3 - 8$ are

A $(x + 2)(x^2 - 2x + 4)$ **B** $(x - 2)^3$
C $(x^2 - 4)(x + 2)$ **D** $(x - 2)(x^2 + 2x + 4)$

7 The linear factors of $2x^2 - 5x - 12$ are

A $(2x + 1)$ and $(x - 12)$ **B** $(2x - 1)$ and $(x + 12)$
C $(2x + 3)$ and $(x - 4)$ **D** $2(2x - 3)$ and $(x + 2)$

8 If $P(x) = 4x^3 - 5x + 5$ then $P\left(\frac{-3}{2}\right)$ is equal to

A 4 **B** 11 **C** -1 **D** 1

9 The equation $x^2 + y^2 + 6x - 2y + 6 = 0$ corresponds to a circle of radius

A 4 **B** $\sqrt{6}$ **C** 2 **D** 6

10 The gradient of the line with equation $2x + 4y - 6 = 0$ is

A $-\frac{1}{2}$ **B** 2 **C** 4 **D** -2

 ISBN 978-1-009-53719-3

11 The equation of the line perpendicular to the line with equation $2x + 4y = 3$ and containing the point $(1, 2)$ is

A $y = -\frac{1}{2}x + \frac{5}{2}$ **B** $2y = x$ **C** $y = 2x - 4$ **D** $y = 2x$

12 Let $P(x) = x^3 + ax^2 - x - 6$, If $P(3) = 0$, then a is equal to

A 2 **B** -2 **C** 1 **D** -1

E -3

13 The linear factors of $x^3 + 8x^2 + 9x - 18$ are

A $x - 1,\ x + 3,\ x + 6$ **B** $x - 1,\ x - 3,\ x - 6$

C $x + 1,\ x - 3,\ x + 6$ **D** $x + 1,\ x + 3,\ x - 1$

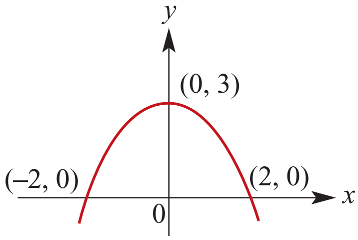
14 The parabola shown has equation

A $y = -2(x - 2)^2$

B $3y = 4(x + 2)(x - 2)$

C $3y = 2(x + 2)(x - 2)$

D $4y = -3(x + 2)(x - 2)$



15 The line with equation $y = mx + c$ is perpendicular to the line with equation $y = 4 - 3x$. The value of m is

A 3 **B** $\frac{1}{3}$ **C** -3 **D** $-\frac{1}{3}$ **E** -1

16 If $f(x) = x^2 - 1$, then $f(x - 1)$ is equal to

A 0 **B** $x^2 - 2x$ **C** 1 **D** $x - 1$

17 The graph of $y = x^2 + kx + k + 8$ touches the x-axis. The value of k is

A -4 **B** 8 **C** 12 **D** -4 or 8

18 Let $P(x) = 3x^3 - 4x - k$. If $P(k) = 0$, then $P(-k)$ is

A $2k$ **B** k **C** $-2k$ **D** $-k$

19 The turning point of the graph of the quadratic function whose rule is $y = a(x - b)^2 + c$ is the point with coordinates

A $\left(\frac{b}{a}, c\right)$ **B** (b, c) **C** (c, b) **D** $(-b, c)$

20 If the graphs of $y = 3 + 4x - x^2$ and $y = k$ have only one common point, then k is

A -1 **B** 1 **C** 4 **D** 7

21 The coordinates of the midpoint of the line segment with endpoints $(12, 7)$ and $(1, 5)$ are

A $(9, 12)$ **B** $(15, 2)$ **C** $(7\frac{1}{2}, 1)$ **D** $(6\frac{1}{2}, 6)$

 ISBN 978-1-009-53719-3 Cambridge University Press

22 M is the midpoint of XY. The points M and Y have coordinates $(7, -3)$ and $(5, 4)$ respectively. The point X has coordinates

A $(6, \frac{1}{2})$ **B** $(4, -14)$ **C** $(1, 0)$ **D** $(9, -10)$

23 The range of the function defined by $y = x^2 + 1$ for $-2 \le x \le 1$ is

A $-3 \le y \le 1$ **B** $2 \le y \le 5$ **C** $1 \le y \le 5$ **D** $y > 1$

24 The equation $x^3 + 2x - 8 = 0$ has only one solution. This solution lies between

A -2 and -1 **B** -1 and 0 **C** 0 and 1 **D** 1 and 2

25 If $f(x) = x(x - 2)$, then $f(-3)$ is equal to

A -8 **B** 2 **C** 8 **D** 15

26 The circle $x^2 + y^2 - 11x - 10y + 24 = 0$ cuts the y-axis at M and N. The distance between M and N is

A 2 **B** 5 **C** 10 **D** 11

27 The length of the line segment joining the points with coordinates $(-4, -3)$ and $(-5, -10)$ is

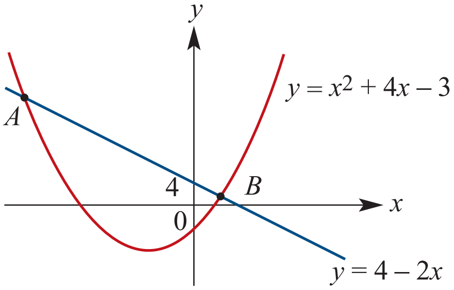
A $2\sqrt{5}$ **B** $5\sqrt{2}$ **C** $5\sqrt{10}$ **D** $25\sqrt{2}$

28 If $a^2 - 5a = 7$ where $a > 0$ and $b^2 - 5b = 7$ where $b \ne a$ then $a + b$ is equal to

A 5 **B** $5\sqrt{2}$ **C** $2\sqrt{53}$ **D** $\sqrt{53}$

29 The straight line intersects the parabola at points A and B. The length of line segment AB is

A $\sqrt{48}$ **B** $\sqrt{60}$
C $\sqrt{320}$ **D** $\sqrt{180}$



30 Which one of the following ordered pairs is *not* a member of the relation $y \le 2x + 3$?

A $(1, 4)$ **B** $(-1, 1)$ **C** $(\frac{1}{2}, 3\frac{1}{2})$ **D** $(-\frac{1}{2}, 2\frac{1}{2})$

31 If the graph of $y = k + 2x - x^2$ touches the x-axis, the value of k is

A -2 **B** -1 **C** 0 **D** 1

32 $\dfrac{1}{\sqrt{3}} + \dfrac{2}{\sqrt{3} - 3}$ is equal to

A $\dfrac{1}{\sqrt{3} - 3}$ **B** -1 **C** $\dfrac{6}{\sqrt{3}}$ **D** $3\sqrt{3} - 1$

33 If the line with equation $kx + y - 4 = 0$ is perpendicular to the line with equation $x - 2y + 3 = 0$, then k equals

A -2 **B** $\frac{1}{2}$ **C** 2 **D** $-\frac{1}{2}$

34 The simultaneous equations $y = x^2 + k$ and $y = x$ have only one solution. The value of k is

A $\frac{1}{4}$ **B** $-\frac{1}{4}$ **C** 4 **D** -4

35 A circle has its centre at $(-4, 2)$ and touches the y-axis. Its equation is

A $x^2 + y^2 = 20$ **B** $x^2 + y^2 - 8x - 4y = 0$
C $x^2 + y^2 + 8x - 4y + 4 = 0$ **D** $x^2 + y^2 + 8x - 4y + 16 = 0$

36 If two lines $2x - y + 3 = 0$ and $ax + 3y - 1 = 0$ are parallel, then the value of a is

A -6 **B** -2 **C** 2 **D** 3

37 The maximal domain of the function with rule $f(x) = \sqrt{4 - x^2}$ is

A $0 \leq x \leq 2$ **B** $-2 \leq x \leq 2$ **C** $x > 2$ **D** $x < 2$

38 The range of the function defined by $y = 2x^2 + 3x + 4$ for $x \in \mathbb{R}$ is

A $y \leq 4$ **B** $y \geq \dfrac{4}{3}$ **C** $y \geq \dfrac{23}{8}$ **D** $y \geq 1$

39 Let $P(x) = x^3 - kx^2 - 10kx + 25$. Given that $P(2) = 9$, the value of k is

A 25 **B** $-\dfrac{1}{2}$ **C** $\dfrac{7}{4}$ **D** 1

40 If $f(x) = x^2 - 7x + k$ and $f(k) = -9$, then $f(-1)$ equals

A -3 **B** 3 **C** 5 **D** 11

41 $2xy - x^2 - y^2$ is equal to

A $(-x - y)^2$ **B** $(-x + y)^2$ **C** $-(x + y)^2$ **D** $-(x - y)^2$

42 The inequality $x^2 - x - 12 \leq 0$ holds for

A $3 \leq x \leq 4$ **B** $x < 3$ or $x \geq 4$
C $-3 \leq x \leq 4$ **D** $x \leq -4$ or $x \geq -3$

43 If $f(x) = \frac{1}{2}x(x - 1)$, then $f(x) - f(x + 1)$ is equal to

A 1 **B** -2 **C** $-x$ **D** $-2x$

44 Solving the inequality $2x^2 - 2 \leq 0$ gives

A $x \leq 1$ **B** $x \geq -1$
C $-1 \leq x \leq 1$ **D** $x \leq -1$ or $x \geq 1$

45 The range of the function $f(x) = -2[(x - \frac{1}{2})^2 - 3]$ is

A $(-\infty, 6]$ **B** $(-\infty, 3]$ **C** $(-3, \infty)$ **D** $[6, \infty)$

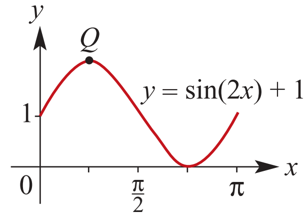
46 An angle is measured as $2x$ radians. The measure of the angle in degrees is

A $\left(\dfrac{\pi x}{90}\right)^\circ$ **B** $\left(\dfrac{90x}{\pi}\right)^\circ$ **C** $\left(\dfrac{\pi}{180x}\right)^\circ$ **D** $\left(\dfrac{360x}{\pi}\right)^\circ$

 ISBN 978-1-009-53719-3 Cambridge University Press

47 The figure shows the graph of $y = \sin(2x) + 1$.

The coordinates of Q are



A $\left(\frac{\pi}{4}, 2\right)$ **B** $\left(\frac{\pi}{12}, 2\right)$

C $(\pi, 1)$ **D** $\left(\frac{\pi}{4}, 1\right)$

48 The smallest value of $1 - 3\cos\theta$ is

A -5 **B** -4 **C** -3 **D** -2

49 A roller coaster is constructed in such a way that any car is y metres above the ground when it is x metres from the starting point, where $y = 16 + 15\sin\left(\frac{\pi}{60}x\right)$.

The height of the car, in metres, when $x = 10$ is

A 23.5 **B** 31 **C** 1 **D** 16

50 $\sin(\pi + \theta) + \cos(\pi + \theta)$ is equal to

A $\sin\theta + \cos\theta$ **B** $-\sin\theta + \cos\theta$

C $\sin\theta - \cos\theta$ **D** $-\sin\theta - \cos\theta$

51 For $x \in [0, \pi]$, which of the following equations has exactly two solutions?

A $\sin x = 0$ **B** $\cos x = 0$

C $\sin x = 1$ **D** $\sin x = -1$

52 The function $f(x) = -2\cos(3x)$ has

A amplitude -2 and period $\frac{2\pi}{3}$ **B** amplitude 2 and period 6π

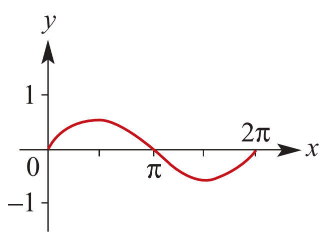
C amplitude 3 and period π **D** amplitude 2 and period $\frac{2\pi}{3}$

53 The minimum value of $2 - 3\sin\theta$ is

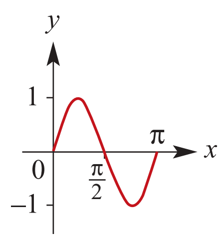
A 2 **B** -1 **C** 1 **D** 0

54 Which of the following is the graph of $y = \sin\left(\frac{x}{2}\right)$ for one cycle?

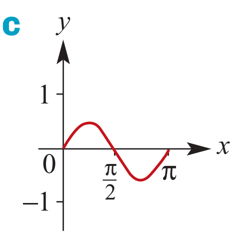
A



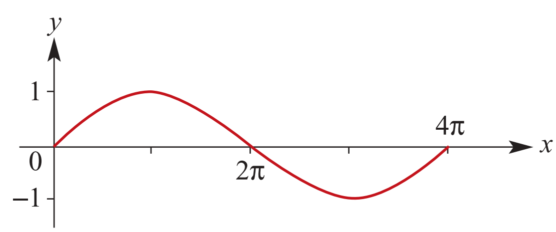
B



C



D



 ISBN 978-1-009-53719-3 Cambridge University Press

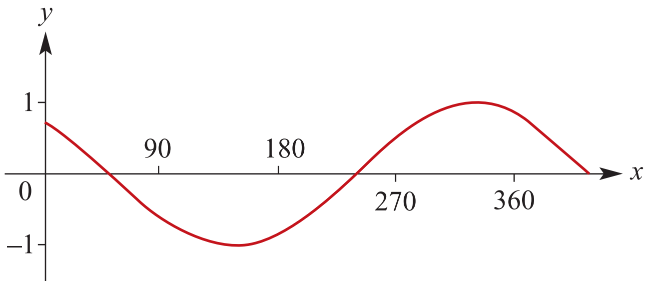
55 The graph shown is of

A $y = \cos(x - 30)°$

B $y = \frac{1}{2}\cos(x + 30)°$

C $y = \cos(x + 30)°$

D $y = \frac{1}{2}\cos(x - 30)°$



56 If $\cos\theta - \sin\theta = \dfrac{1}{4}$, then $\sin\theta\cos\theta$ equals

A $\dfrac{15}{32}$ **B** $\dfrac{1}{16}$ **C** $\dfrac{15}{16}$ **D** $\dfrac{1}{32}$

57 In the expansion of $(2x + 3)^5$, the coefficient of x^3 is

A $2^3 \times 3^2$ **B** $\dbinom{5}{2} \times 2^3$ **C** $5 \times 2^3 \times 3^2$ **D** $\dbinom{5}{2} \times 2^3 \times 3^2$

58 The expansion of $(2 + y)^4$ is

A $1 + 4y + 6y^2 + 4y^3 + y^4$ **B** $8 + 24y + 24y^2 + 8y^3 + y^4$

C $16 + y^4$ **D** $16 + 32y + 24y^2 + 8y^3 + y^4$

59 If -256 is the coefficient of x in the expansion of $(k + x)^4$, then k is equal to

A 3 **B** 4 **C** -3 **D** -4

60 $(A \cap B) \cup (A \cap B')$ is always equal to

A A **B** B **C** $A \cup B$ **D** A'

61 In a group of 100 people, 35 like chocolate ice-cream, 45 like vanilla ice-cream, and 10 like both. How many people in the group like only one of these flavours?

A 40 **B** 60 **C** 70 **D** 80

62 Mary and Ann try to guess the month in which the other was born. The probability that both guess correctly is

A $\dfrac{1}{2}$ **B** $\dfrac{1}{4}$ **C** $\dfrac{1}{24}$ **D** $\dfrac{1}{144}$

63 Bag A contains 2 white and 3 black balls. Bag B contains 3 white and 2 black balls. If one ball is drawn from each bag, the probability that they are of different colours is

A $\dfrac{6}{25}$ **B** $\dfrac{10}{25}$ **C** $\dfrac{13}{25}$ **D** $\dfrac{21}{25}$

64 Two dice are thrown. The probability of getting a sum that is greater than or equal to 12 is

A 0 **B** $\dfrac{1}{6}$ **C** $\dfrac{1}{12}$ **D** $\dfrac{1}{36}$

65 A group consists of four boys and three girls. If two of them are chosen at random (without replacement), the probability that a boy and a girl are chosen is

A $\frac{2}{7}$ **B** $\frac{4}{7}$ **C** $\frac{24}{49}$ **D** $\frac{27}{49}$

66 If X and Y are mutually exclusive events such that $P(X) = P(Y')$, then $P(X \cup Y)$ is

A $P(X)$ **B** $P(Y)$ **C** 0 **D** 1

67 If six fair dice are rolled, the probability of getting at least one 4 is

A $\frac{4}{6}$ **B** $\left(\frac{5}{6}\right)^6$ **C** $1 - \left(\frac{5}{6}\right)^6$ **D** $\frac{1}{6}$

68 A bag contains k red marbles and 1 white marble. Two marbles are drawn without replacement. The probability that both are red is

A $\frac{k}{(k+1)^2}$ **B** $\frac{k-1}{k+1}$ **C** $\frac{k}{k+1}$ **D** $\frac{2k}{k+1}$

69 Six cards labelled 1, 2, 3, 4, 5 and 6 are put into a box. Three cards are then drawn from the box (without replacement). The probability that the three cards are all labelled with odd numbers is

A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{8}$ **D** $\frac{1}{20}$

70 If $P(A \cap B) = \frac{1}{5}$ and $P(B) = \frac{1}{2}$ and $P(B\,|\,A) = \frac{1}{3}$, then

A $P(A\,|\,B) = \frac{3}{5}$ and $P(A) = \frac{1}{5}$ **B** $P(A\,|\,B) = \frac{1}{3}$ and $P(A) = \frac{2}{5}$

C $P(A\,|\,B) = \frac{2}{5}$ and $P(A) = \frac{3}{5}$ **D** $P(A\,|\,B) = \frac{2}{3}$ and $P(A) = \frac{3}{5}$

71 A pair of unbiased dice is thrown. The probability of the sum of the numbers uppermost being 10 is

A $\frac{1}{36}$ **B** $\frac{1}{18}$ **C** $\frac{1}{12}$ **D** $\frac{1}{2}$

72 A box contains twelve red balls and four green balls. A ball is selected at random from the box and not replaced and then a second ball is drawn. The probability that the two balls are both green is equal to

A $\frac{1}{4}$ **B** $\frac{3}{64}$ **C** $\frac{1}{8}$ **D** $\frac{1}{20}$

Technology-active multiple-choice questions

73 If $P(x) = 3.01x^3 + 5.2x^2 - 6.7x - 4$ then $P(3)$ is equal to

A 103.97 **B** −2.49 **C** 67.45 **D** 106.4

74 A child on a swing travels through an arc of length 3 m. If the ropes of the swing are 4 m in length, then the angle which the arc makes at the top of the swing (where the swing is attached to the support) is best approximated by

A 75° **B** 12° **C** 75^c **D** 43°

75 Correct to two decimal places, the area of a sector with an included angle of 60° in a circle of diameter 10 cm is

A 104.72 cm^2 **B** 52.36 cm^2 **C** 13.09 cm^2 **D** 26.16 cm^2

76 Let $f(x) = x^2 - 2x - 24\sqrt{x}$ for $x \geq 0$. The positive solution of the equation $f(3 - 2x) = 0$ is

A 2.415 **B** 1.414 **C** 1.5 **D** 5.414

77 Let $f(x) = \begin{cases} 2x^2 + 3 & \text{if } x \leq 0 \\ x^3 - 1 & \text{if } 1 < x < 3 \\ -x + 6 & \text{if } x \geq 4 \end{cases}$. The number of solutions of the equation $f(x) = 26$ is

A 0 **B** 1 **C** 2 **D** 3

78 Let $g(x) = \begin{cases} 4 - 2x^2 & \text{if } -5 \leq x \leq 1 \\ 3x^2 + 1 & \text{if } 1 < x < 3 \\ -x^3 - 2 & \text{if } x \geq 3 \end{cases}$. The range of g is

A $x < -29$ **B** $x < 28$ **C** $x > -48$ **D** $3 \leq x \leq 28$

79 The number of solutions of the equation $7\sin(2\pi x) + 5 = 4$ for $0 \leq x \leq 3$ is:

A 2 **B** 4 **C** 6 **D** 8

80 The function $f(x) = x^3 - 3x^2 - 8x - 9, \quad -3 \leq x \leq 5$ has range

A $x \leq 1$ **B** $-39 < x < 1$ **C** $-39 \leq x \leq 1$ **D** $-39 \leq x \leq -4.96$

81 A parabola passes through the points with coordinates $(1, 30)$, $(2, 69)$ and $(3, 112)$. The equation of the parabola is

A $y = -2x^2 + 50x + 19$ **B** $y = 4x^2 + 65$
C $y = 2x^2 + 33x - 5$ **D** $y = x^2 + 36x + 50$

82 Which of the following functions is an odd function

A $y = x^3 + 50x + 20$ **B** $y = 4x^3 - 3x$
C $y = 3x^2 - 4x^3$ **D** $y = x^4 + x^2 - 150x$

83 The coefficient of x^8 in the expansion of $(1 + 2x)^{15}$ is

A 823 680 **B** 256
C 6435 **D** 1 647 360

84 A square has side length of 30 cm. Inside the square are two circles, each of radius 5 cm, which do not overlap. If a dart thrown at the square is equally likely to land at any point inside the square, then the probability that it will not land in either of the circles is closest to

A 0.825 **B** 0.913 **C** 0.175 **D** 0.375

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

85 The probability of Bill hitting the bullseye with a single shot is $\frac{1}{2}$. The probability that Charles does the same is $\frac{1}{4}$. Bill has 2 shots and Charles has 4. The probability of both players hitting the bullseye at least once is closest to

A 0.75 **B** 0.513 **C** 0.747 **D** 0.684

The following information relates to Questions 86 *and* 87

The following data was derived from a survey of secondary students.

Intends to go to university	*Current year level* 8	9	10	11	Total
Yes	80	45	56	85	266
No	10	42	29	11	92
Total	90	87	85	96	358

86 The probability that a student does not intend to go to university, given that that they are in year 10, is closest to

A 0.081 **B** 0.315 **C** 0.341 **D** 0.659

87 Given that a student intends to go university, the probability that they are in Year 8 is closest to

A 0.223 **B** 0.301 **C** 0.870 **D** 0.889

88 If A and B are independent events such that $P(A) = 0.22$ and $P(B) = 0.76$, then $P(A \cup B)$ is closest to

A 0.17 **B** 0.76 **C** 0.81 **D** 0.98

89 Ellen is setting up an experiment. She can either use:

- 4 small components, each of which has a probability of 0.01 of failing, or
- 3 medium components, each of which has a probability of 0.015 of failing, or
- 2 large components, each of which has a probability of 0.018 of failing, or
- 1 extra large component, each of which has a probability of 0.04 of failing.

The experiment will fail if any of the individual components fail. Which is option will minimise the probability that her experiment will fail?

A 4 small components **B** 3 medium components
C 2 large components **D** 1 extra large component

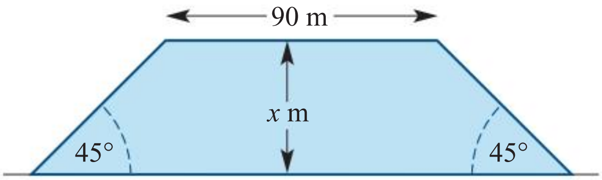
90 Two events A and B are independent events, where $P(A) = \dfrac{P(B)}{2}$ and $P(A \cup B) = 0.5325$. $P(A)$ is closest to

A 0.05 **B** 0.15 **C** 0.21 **D** 0.45

 ISBN 978-1-009-53719-3

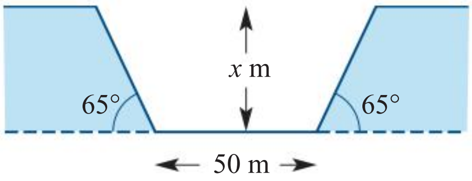
10C Problem-solving and modelling questions

1 The figure shows a section view of a freeway embankment to be built across a flood-prone river flat. The height of the embankment is x m and the width at the top is 90 m.



a Determine a formula, in terms of x, for V, the volume of earth in m^3 required to build a 120 m length of freeway embankment.

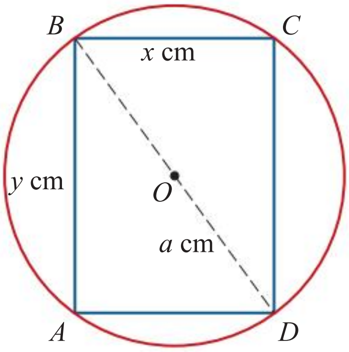
This figure shows another section of the freeway which is to be constructed by cutting through a hillside. The depth of the cutting is x m and the width of the cutting at the base is 50 m.



b Determine a formula for the volume of earth, in m^3, which would have to be excavated to form a straight 100 m section of the cutting.

c If $x = 4$ m, what length of embankment could be constructed from earth taken from the cutting?

2 A rectangle of perimeter b cm is inscribed in a circle of radius a cm. The rectangle has width x cm and length y cm.



a Apply Pythagoras' theorem in triangle BCD to show that $x^2 + y^2 = 4a^2$.

b Form a second equation involving x, y and b.

c Eliminate y from these equations to form a quadratic equation in terms of x, a and b.

d As x, y and $2a$ are the sides of a triangle, $x + y > 2a$. Use this result and apply the discriminant to the quadratic equation formed in part c to show that the rectangle can be inscribed in the circle only if $4a < b \le 4\sqrt{2}a$.

e **i** If $a = 5$ and $b = 24$, determine the values of x and y.

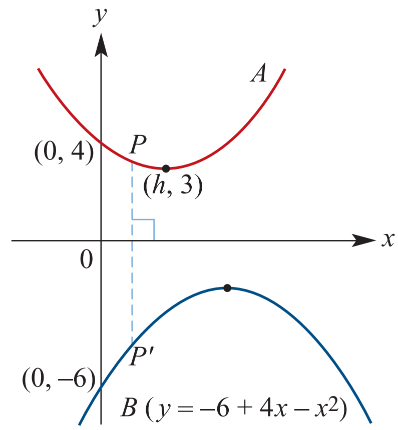
ii If $b = 4\sqrt{2}a$, determine the values of x and y in terms of a.

f If $\dfrac{b}{a} = 5$, determine the values of x and y in terms of a.

g Write a program to solve the quadratic equation found in part c for suitable choices of a and b and state the values of x and y. (Answers correct to two decimal places.)

 ISBN 978-1-009-53719-3 Cambridge University Press

3 **a** The point $(h, 3)$ is the vertex of a parabola A, with equation $y = x^2 + bx + c$ and y-axis intercept 4. Determine the values of b, c and h for $h > 0$.



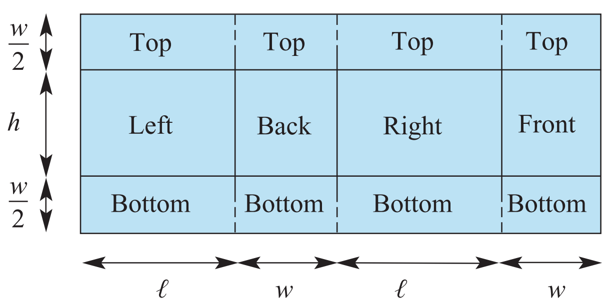
b The equation of curve B is $y = -6 + 4x - x^2$. Let P be a point on curve A, and P' be a point on curve B such that PP' is perpendicular to the x-axis.

- **i** The coordinates of P are $(x, x^2 + bx + c)$. State the coordinates of P' in terms of x.
- **ii** Determine the coordinates of M, the midpoint of PP', in terms of x.
- **iii** Determine the coordinates of M for $x = 0, 1, 2, 3, 4$.
- **iv** Give the equation of the straight line on which all of these points lie. (This is called the locus of the midpoints.)

c Let d be the distance PP'.

- **i** Express d in terms of x.
- **ii** Sketch the graph of d against x.
- **iii** Determine the minimum value of d and the value of x for which this occurs.

4 A net for making a cardboard box with overlapping flaps is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded.



a If $\ell = 35$ cm, $w = 20$ cm and $h = 23$ cm, calculate the area of the net.

b If the area of the net is to remain constant at the value calculated in part a and $\ell = h$, write down an expression for V, the volume of the box in cm^3, as a function of ℓ. (The maximum volume of the box will occur when $\ell = h$).

c Use a calculator to help draw the graph of V against ℓ.

d Determine the value of ℓ when the volume of the box is:

- **i** 14 000 cm^3
- **ii** 10 litres = 10 000 cm^3

e Determine the maximum volume of the box and the value of ℓ for which this occurs.

f Investigate other nets which are possible in forming a box.

 ISBN 978-1-009-53719-3 Cambridge University Press

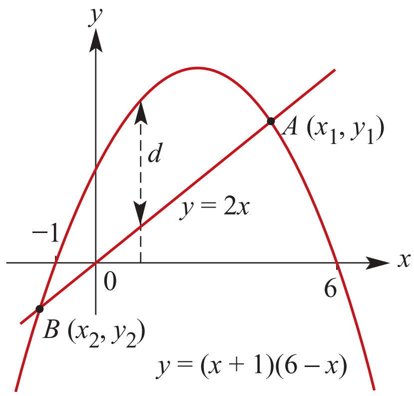
Revision

5 **a** Determine the coordinates of the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

b Let $d(x)$ be the 'vertical' distance between the graphs for $x \in [x_2, x_1]$.

  i Determine $d(x)$ in terms of x.

  ii Use a calculator to plot the graph of $d(x)$ against x for $x \in [x_2, x_1]$, and on the same screen plot the graphs of $y = 2x$ and $y = (x + 1)(6 - x)$.



c **i** State the maximum value of the function defined by $d(x)$ for $x \in [x_2, x_1]$.

  ii State the range of this function.

d Repeat with the graphs $y = 5x$ and $y = (x + 1)(6 - x)$.

6 **a** Determine the equation of the straight line through the points $(1, 2)$ and $(2, 4)$.

b A parabola with equation $y = ax^2 + bx + c$ passes through the points $(1, 2)$ and $(2, 4)$.

  i Determine b and c in terms of a.

  ii Describe the family of parabolas which pass through the points $(1, 2)$ and $(2, 4)$.

  iii Determine the coordinates of the vertex of a parabola in this family in terms of a.

  iv On the one set of axes, sketch the graphs of several parabolas from this family.

c The graph of a cubic function $f(x) = (x - a)^2(x - b)$ passes through the point $(2, 4)$.

  i Determine b in terms of a.

  ii For what values of a is $b > 0$?

  iii Sketch the graph of b against a.

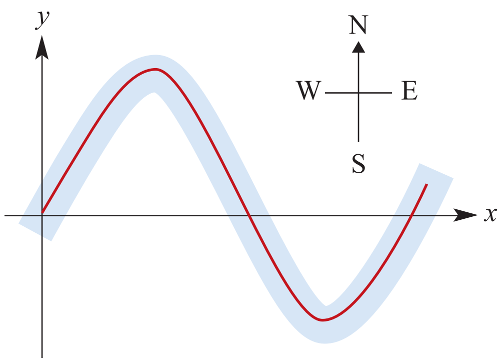
  iv Determine three pairs (a, b), where both a and b are integers, such that the graph of $f(x) = (x - a)^2(x - b)$ passes through the point $(2, 4)$.

7 A section of a path along the edge of a river can be modelled by a cubic function for $x \in [0, 250]$. Measurements are in metres. The following are the coordinates of points on the path: $(0, 0)$, $(100, 33)$, $(50, 57)$ and $(150, -15)$.

a Determine the equation of the graph of the cubic function which passes through these points.

b Determine the largest deviation of the path from the x-axis.

c Determine a piecewise-defined function that approximates this stretch of path, where each component of the function is either a parabola or a straight line.



 ISBN 978-1-009-53719-3 Cambridge University Press

8 The depth of water, D metres, at the end of a pier t hours after low tide is given by the rule $D = p - 2\cos(rt)$, where p and r are suitable constants. At low tide ($t = 0$) the depth is 2 metres; at high tide, which occurs 8 hours later, the depth is 6 metres.

a Show that $r = \frac{\pi}{8}$ and $p = 4$.

b Sketch the graph of $D = 4 - 2\cos\left(\frac{\pi}{8}t\right)$ for $0 \le t \le 16$.

c If the first low tide occurs at 4 a.m., when will the next low tide occur?

d At what times will the depth be equal to 4 metres?

The poles that support the pier stand 7.5 metres above the sea bed.

e How much of a particular pole is exposed at:

i high tide **ii** 2 p.m.?

Over the years mussels have attached themselves to the pole. A particular mussel is attached 4 metres from the top of the pole so that some of the time it is exposed and some of the time it is covered by water.

f For how long will the mussel be covered by water during the time from one low tide to the next?

9 A river gate is used to control the height of water in a river.

- On one side of the gate, the height of the water is subject to tides. The height of the water (in metres) on this side is given by

$$h_1(t) = 18 + 10\sin\left(\frac{\pi}{6}t\right)$$

where t is the time in hours past midnight.

- On the other side of the gate, the height of the water (in metres) is rising according to the rule

$$h_2(t) = 8 + 6t$$

where t is the time in hours past midnight.

a Sketch the graphs of $y = h_1(t)$ and $y = h_2(t)$, for $0 \le t \le 6$, on the one set of axes.

b Determine the time at which $h_1(t) = h_2(t)$.

c When the water levels on both sides of the gate are the same, the gate is opened and the height of the water on both sides is given by $y = h_1(t)$.

i The gate is closed again when $h_1(t)$ reaches its minimum value. At what time does this happen?

ii Water flows in on the non-tidal side so that the height increases by 6 metres every hour. What will be the height of the river t hours after the gates have been closed?

10 **a** Expand and simplify $(1 + 2x)^4$. Hence write down the largest coefficient in the expansion of $(1 + 2x)^4$.

b Show that the general term in the expansion of $(1 + 2x)^4$ is $\binom{4}{k}2^k x^k$ for $0 \le k \le 4$. Hence the coefficient of x^k is given by $a_k = \binom{4}{k}2^k$.

c Determine $\frac{a_{k+1}}{a_k}$ in terms of k. Simplify your answer.

d Solve the inequality $\frac{a_{k+1}}{a_k} > 1$ for k, and use this result to determine the largest coefficient in the expansion of $(1 + 2x)^4$.

e Using the technique from part d, determine the largest coefficient in the expansion of each of the following:

i $(3 + 4x)^{13}$ **ii** $(1 + 5x)^{21}$ **iii** $(7 + 3x)^{25}$

11 In order to choose a team for a quiz, a class is split into three groups. Group A contains two boys and three girls, group B contains one boy and three girls, and group C contains two boys and one girl. An unbiased die is thrown. If a 1, 2 or 3 appears, a random choice will be made from group A. If a 4 or 5 appears, a random choice will be made from group C. If a 6 appears, a random choice will be made from group B.

a Calculate the probability that a boy will be chosen when the choice is made.

b Given that a girl is chosen when the choice is made, calculate the probability that she will be chosen from:

i group A **ii** group B.

12 Eighteen numbers are arranged in three groups of six as follows:

Group A: 0, 6, 6, 6, 6, 10

Group B: 1, 1, 1, 7, 7, 11

Group C: 3, 3, 3, 3, 8, 8

One number is drawn at random from each group. Given that a, b and c denote the numbers drawn from groups A, B and C respectively, calculate the probability that:

a $c < b$

b a is greater than both b and c

c $c > a + b$

13 Sixteen players are entered in a tennis tournament.

- In round one, every player plays (eight matches).
- In round two, the eight winners from round one play (four matches).
- In round three, the four winners from round two play (two matches).
- In round four, the two winners from round three play (one match).

Use simulation to estimate how many matches a player will play, on average:

a if the player has a 50% chance of winning each match

b if the player has a 70% chance of winning each match.

14 **a** Suppose that A_1, A_2 and A_3 are mutually exclusive events whose union is the sample space ε of a random experiment, and that B is an event such that $P(B) \neq 0$. Show that

$$P(A_1 \mid B) = \frac{P(B \mid A_1)P(A_1)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3)}$$

 ISBN 978-1-009-53719-3

b A factory has three machines producing a particular type of item.

Machine number	1	2	3
Proportion of all items that are produced by this machine	35%	45%	20%
Proportion of items from this machine that are defective	2%	1%	3%

One item is drawn at random from the factory's production. Let B be the event that the chosen item is defective, and let A_i be the event that the chosen item was produced by machine i, where $i \in \{1, 2, 3\}$.

i From the information in the table, write down $P(A_1)$, $P(A_2)$ and $P(A_3)$.

ii From the information in the table, write down $P(B \mid A_1)$, $P(B \mid A_2)$ and $P(B \mid A_3)$.

iii Using the formula from part a, calculate $P(A_1 \mid B)$, $P(A_2 \mid B)$ and $P(A_3 \mid B)$. Comment on your results.

15 Appendix A material required. A good bridge player should develop a sense of estimating the probability that their partner may possess certain cards. In bridge, there are four players, A, B, C and D. Each player is dealt 13 cards from an ordinary pack of 52 cards.

a If A has no aces, determine the probability that C has exactly two aces.

b If A and C together have nine hearts, determine the probability that B and D each have two hearts.

16 A landscape designer is planning a new layout for a square garden. The four sides of the garden are described by the straight lines with equations

$$x = 0, \quad y = 0, \quad x = 80 \quad \text{and} \quad y = 80$$

There is a large existing tree in the garden centred at the point $(30, 20)$. The base of this tree is described by the circle with equation

$$(x - 30)^2 + (y - 20)^2 = 5$$

The designer wants to divide the garden into two parts by creating a path across the garden. The designer tries three different functions to model the shape of the new path.

a Piecewise-defined function with straight-line components

i Determine the points on the circle where $x = 31$ and where $x = 29$.

ii Determine the gradient of the diameter of the circle through the point $(31, 18)$.

iii Determine the equation of the tangent ℓ_1 to the circle at the point $(31, 18)$.

iv Determine the coordinates of the point X on ℓ_1 where $x = 30$.

v The point X lies on another tangent to the circle, ℓ_2. Find the equation of ℓ_2.

vi Determine the coordinates of the point of intersection of ℓ_1 with the line $x = 80$ and the point of intersection of ℓ_2 with the line $x = 0$.

vii Write a rule for the function f describing the new path in the form

$$f(x) = \begin{cases} m_1 x + c_1 & \text{if } 0 \le x \le 30 \\ m_2 x + c_2 & \text{if } 30 < x \le 80 \end{cases}$$

given that the path follows the line ℓ_2 for $0 \le x \le 30$ and the path follows the line ℓ_1 for $30 < x \le 80$.

viii Sketch the graph of $y = f(x)$ and the circle on the one set of axes.

ix Repeat this process to define an alternative path across the garden, by finding two new tangents ℓ_3 and ℓ_4 to the circle at points where $x = 32$ and $x = 28$.

x Discuss the advantages and disadvantages of these two paths.

b **Piecewise-defined function with parabolic components** Now consider a function with a rule of the form

$$g(x) = \begin{cases} a_1(x-h)^2 + k & \text{if } 0 \le x \le 30 \\ a_2(x-h)^2 + k & \text{if } 30 < x \le 80 \end{cases}$$

where $X(h, k)$ is the point of intersection of the two tangents ℓ_1 and ℓ_2 from part a.

i Find the value of a_1 if the graph of $y = a_1(x-h)^2 + k$ passes through the origin.

ii Find the value of a_2 if the graph of $y = a_2(x-h)^2 + k$ passes through the point $(80, 60)$.

iii Sketch the graph of $y = g(x)$ and the circle on the one set of axes.

iv Define an alternative path across the garden by using a single parabola that has its vertex at X and passes through the point $(80, 80)$.

c **Cubic function** Use a cubic function with a rule of the form $h(x) = a_3(x-h)^3 + k$ to plan the new path, where $X(h, k)$ is the point of intersection of the two tangents ℓ_1 and ℓ_2 from part a.

10D Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 **Iterating functions** Let $f(x) = 1 + \dfrac{1}{x}$. Starting with the initial value 1, we can keep applying this function to obtain a sequence of values:

$$\begin{aligned} f(1) &= 2 \\ f^{(2)}(1) &= f(f(1)) = f(2) = \tfrac{3}{2} \\ f^{(3)}(1) &= f(f^{(2)}(1)) = f(\tfrac{3}{2}) = \tfrac{5}{3} \end{aligned}$$

Here the notation $f^{(n)}$ indicates that the function f has been applied n times.

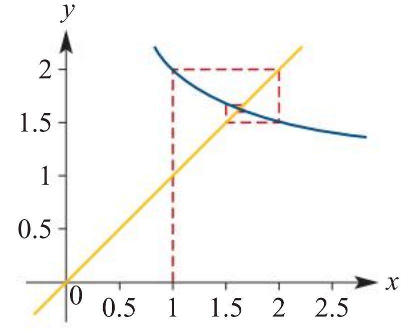
a **i** Using a spreadsheet or a calculator, determine $f(1), f^{(2)}(1), f^{(3)}(1), \dots, f^{(10)}(1)$.

ii Solve the equation $x^2 - x - 1 = 0$. Give solutions correct to three decimal places.

Note that the equation $x^2 - x - 1 = 0$ can be rearranged to

$$1 + \frac{1}{x} = x \qquad \text{i.e. } f(x) = x$$

The 'cobweb' diagram shown may help you to understand this process; the graphs are $y = f(x)$ and $y = x$.



b Repeat this process with $f(x) = a + \frac{1}{x}$ for different values of a.

c Try using this method to solve $x^2 + 3x - 5 = 0$ by iterating the function $f(x) = \frac{5}{x+3}$.

Experiment with using different initial values and using different functions coming from different rearrangements of the equation. Can you obtain both solutions?

d Investigate this method for other quadratic equations.

2 **Flight distance** A plane takes off from Brisbane airport on a sight-seeing flight. The plane travels due west for a certain distance and then travels due east back to Brisbane airport, where it lands. Throughout the flight, the wind is blowing due east at a constant strength. Assume that the plane can fly at a certain speed in still air and that it starts with enough fuel to last a certain number of hours. What is the maximum total distance that the plane can fly?

For example, consider the special case where the plane flies at 300 km/h in still air, it has fuel for 4 hours and the wind speed is 50 km/h. (So the plane travels at 250 km/h on the way out and at 350 km/h on the way back.)

3 **Two shopping centres** Suppose that the attraction of a shopping centre to a potential customer can be modelled as being inversely proportional to the square of the distance to the shopping centre from where the customer lives. For two shopping centres, located at points $S_1(0, 0)$ and $S_2(a, 0)$, the measure of their attraction to a potential customer living at point $P(x, y)$ can be described as

$$A_1 = \frac{k_1}{(S_1P)^2} \quad \text{and} \quad A_2 = \frac{k_2}{(S_2P)^2}$$

where the constants of proportionality k_1 and k_2 are determined by other factors such as the size and quality of each shopping centre. Try to describe the region where the shopping centre at S_1 is preferred to the one at S_2. Start with the special case $k_1 = k_2$.

4 **Parabolas** We start with the basic parabola $y = x^2$.

a The points $A(3, 9)$ and $B(-2, 4)$ lie on the parabola $y = x^2$. Determine the y-axis intercept of the line AB. Find and prove a general result which includes this special case.

b Again consider the points $A(3, 9)$ and $B(-2, 4)$ on the parabola. The line OA joins the origin O to the point A. Draw the line through B parallel to OA, and determine the coordinates of the point where this line crosses the parabola again. Find and prove a general result which includes this special case.

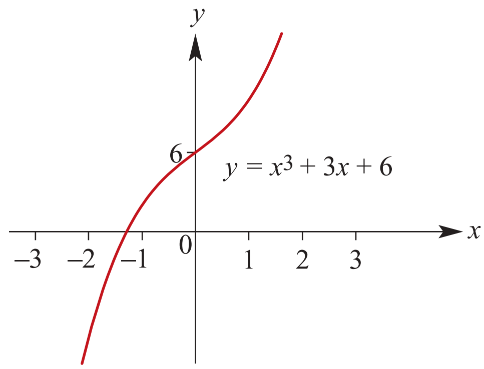
c Again consider $A(3, 9)$. Describe the relationship between the midpoints of all the line segments parallel to OA with both endpoints on the parabola.

d Now consider $C(-4, 16)$ and $D(0, 3)$. Determine the coordinates of the point where the line CD crosses the parabola again. Try to generalise this example.

e Investigate similar results for other parabolas such as $y = \frac{1}{2}x^2$.

5 The bisection method

This method can be used for determining approximate solutions of equations in cases where we do not have an algebraic technique for solving the equation exactly.



For example, let $f(x) = x^3 + 3x + 6$ and consider the equation $f(x) = 0$.

- Since $f(-2) < 0$ and $f(-1) > 0$, the solution to the equation $f(x) = 0$ lies in the interval $[-2, -1]$.
- The midpoint of this interval is -1.5. We calculate $f(-1.5) = -1.875 < 0$. Since $f(-1.5) < 0$ and $f(-1) > 0$, we can now say that the solution lies in the smaller interval $[-1.5, -1]$.

We can continue this process to 'narrow in' on the solution. The general method is described as follows.

Consider an equation $f(x) = 0$ that has one solution α in the interval $[a_1, b_1]$.

- The sign of $f(a_1)$ is the opposite of the sign of $f(b_1)$, because $f(\alpha) = 0$ and so the function must change from being positive to being negative or vice versa.
- Calculate $c_1 = \dfrac{a_1 + b_1}{2}$ and $f(c_1)$.
- Choose the interval $[a_1, c_1]$ if $f(a_1) \times f(c_1) < 0$, and choose the interval $[c_1, b_1]$ otherwise.
- The process is then repeated with the new interval, and then repeatedly until the required accuracy is reached.

a Again consider the cubic function $f(x) = x^3 + 3x + 6$. Using the bisection method for five steps, starting with the interval $[-2, -1]$, we obtain the following table.

	Left endpoint	Right endpoint	Midpoint c_i	$f(c_i)$
Step 1	−2	−1	−1.5	−1.87500000
Step 2	−1.5	−1	−1.25	0.29687500
Step 3	−1.5	−1.25	−1.375	−0.72460938
Step 4	−1.375	−1.25	−1.3125	−0.19848633
Step 5	−1.3125	−1.25	−1.28125	0.05294800

Using a spreadsheet, continue the process to solve the equation $x^3 + 3x + 6 = 0$ correct to two decimal places.

b Use the bisection method to determine approximate solutions of other equations.

Revision

6 **Iterating functions: back to the beginning** As in Question 1, we use the notation $f^{(n)}$ to indicate that the function f has been applied n times.

a For each of the following functions, determine expressions for $f^{(2)}(x)$, $f^{(3)}(x)$, $f^{(4)}(x)$, … and hence evaluate $f^{(1010)}(1010)$:

i $f(x) = \dfrac{1}{1-x}$ **ii** $f(x) = \dfrac{1-x}{1+x}$ **iii** $f(x) = \dfrac{2x-1}{1+x}$

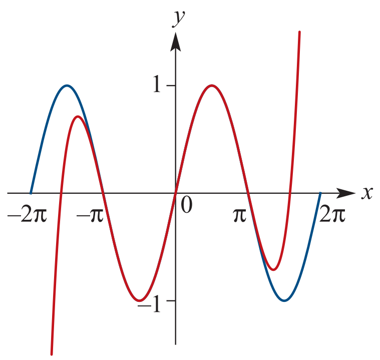
b Work with some functions of your own. They should be of the form $f(x) = \dfrac{ax+b}{cx+d}$. Investigate other properties of this family of functions.

7 **Approximating trigonometric functions** The sine function can be approximated by a sequence of polynomial functions. These are derived from the power series for $\sin x$, which is the infinite sum

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots$$

For example, the graphs of $y = \sin x$ and $y = f(x)$ are shown here, where

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$



a Determine $f(\pi)$ and $f(2\pi)$. Determine some other values of f to understand this approximation better.

b With the help of your calculator, sketch the graph of the cubic approximation

$$g(x) = x - \frac{x^3}{3!}$$

c Using the power series for $\sin x$, determine an infinite sum for each of the following:

i $\sin(-x)$ **ii** $\sin(2x)$ **iii** $\sin(x^2)$

iv $-\sin(x)$ **v** $-\sin(2x)$ **vi** $-\sin(x^2)$

d Leonhard Euler used this power series to prove that $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots = \dfrac{\pi^2}{6}$. Look up the proof.

8 **Lost in the fog** A man is lost in the fog near the edge of a cliff. He takes random steps, either directly towards or directly away from the cliff edge. At each step, the probability that he moves towards the cliff edge is $\frac{1}{3}$ and the probability that he moves away from the cliff edge is $\frac{2}{3}$.

a Suppose that the man is initially standing a distance of 1.5 steps from the cliff edge. What is the probability that he will fall off the cliff if he decides to take n random steps? Consider $n = 1, 2, 3, 4, 5, 6$.

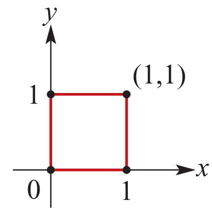
b Investigate other initial positions. For example, suppose that the man is initially standing 0.5 steps from the cliff edge, or 2.5 steps from the cliff edge.

c Investigate other complementary pairs of probabilities for the two directions.

 ISBN 978-1-009-53719-3 Cambridge University Press

9 Points in the plane

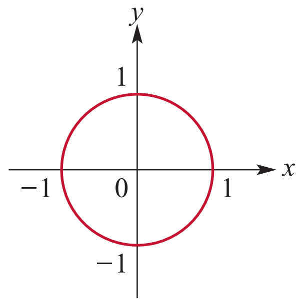
a In the Cartesian plane, the **unit square** is the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Develop a simulation model for randomly selecting a point in the unit square. Use your model to select at random 300 points in the unit square and plot them.



i Using simulation, determine an estimate for the probability that a randomly chosen point in the unit square has a distance of less than $\frac{1}{2}$ from the origin. Compare your estimate with the exact answer.

ii Using simulation, determine an estimate for the 'average' distance between two points chosen at random in the unit square.

b The **unit circle** is the circle with radius 1 centred at the origin.



i Using simulation, determine an estimate for the probability that a randomly chosen point in the unit circle has a distance of less than $\frac{1}{2}$ from the origin. Compare your estimate with the exact answer.

ii Using simulation, determine an estimate for the 'average' distance between two points chosen at random in the unit circle.

c Determine an estimate for the 'average' distance between two points chosen at random in an equilateral triangle of side length 1.

10 Epidemic model Consider the following basic epidemic model. A certain disease is such that, if a person catches the disease, then they are infectious for a period of one day, after which the person has recovered and is immune to the disease. An island has a total population of n people. One person on the island catches the disease and randomly visits one other person on the island during the infectious period. The second person is infected immediately and visits another person at random during the next day (their infectious period). This process continues, with one visit per day, until an infectious person visits an immune person and the disease dies out.

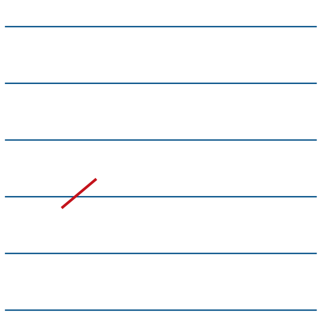
a First consider a population of size $n = 6$. Using an appropriate simulation model, Determine an estimate for the number of people on the island who will catch the disease, on average, before it dies out.

b How does the size of the outbreak vary with the size of the population? Repeat the simulations using populations of size $n = 10, 20, 30, 40, 50$. Compare your answers for the average size of the outbreak.

c How is the size of the outbreak affected by the presence in the population of people who have been vaccinated against the disease? Repeat the simulations for a population of size $n = 30$. But this time consider a second variable, m, the initial number of immune people in the population. Consider $m = 3, 6, 9, 12, 15$. Compare your answers for the average size of the outbreak. Investigate further.

11 **Buffon's needle** The following problem was posed in 1777 by Georges Louis Leclerc, Comte de Buffon:

> Suppose that you drop a short needle on ruled paper – what is then the probability that the needle comes to lie in a position where it crosses one of the lines?



If ℓ is the length of the needle and d is the distance between the lines, then it can be shown that this probability is given by

$$p = \frac{2\ell}{\pi d} \qquad \text{(provided } \ell \leq d\text{)}$$

Perform an experiment as follows. You require a matchstick or similar object. On a large sheet of paper, draw equally spaced parallel lines. (The distance between the lines must be greater than the length of the matchstick.) Place the paper on a flat surface. Now throw the matchstick onto the paper in a random manner, and record whether or not it falls across a line. Repeat this 100 times. Use the results of your experiment to give an estimate for the value of π.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

11

Exponential functions

Chapter contents

The function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1, is called an **exponential function**.

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n one-minute intervals (assuming that all organisms are still alive).

n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus P defines a function which has the rule $P = 2^n$, an exponential function.

In this chapter, many of the concepts introduced in Chapter 6 – such as domain, range and transformations – are used in the context of exponential functions.

There are many applications of exponential functions, and some of these are investigated in the final section of this chapter.

 ISBN 978-1-009-53719-3 Cambridge University Press

11A The index laws

Learning intentions

- To be able to use the rules for manipulating **exponential expressions** with integer exponents.

The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**. In this section we concentrate on indices that are integers. We note that, if n is positive, then $0^n = 0$. But if n is negative or zero, then 0^n is undefined.

Index law 1: Multiplying powers

Index law 1

To **multiply** two powers with the same base, **add** the indices.

$$a^m \times a^n = a^{m+n}$$

If m and n are positive integers,

then $\quad a^m = \underbrace{a \times a \times \cdots \times a}_{m \text{ terms}} \quad$ and $\quad a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}$

$$\begin{aligned} \therefore \quad a^m \times a^n &= (\underbrace{a \times a \times \cdots \times a}_{m \text{ terms}}) \times (\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}) \\ &= \underbrace{a \times a \times \cdots \times a}_{(m+n) \text{ terms}} \\ &= a^{m+n} \end{aligned}$$

Example 1

Simplify each of the following:

a $2^3 \times 2^{12}$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^3$

Solution

a $2^3 \times 2^{12} = 2^{3+12}$
$= 2^{15}$

b $x^2y^3 \times x^4y = x^2 \times x^4 \times y^3 \times y$
$= x^6y^4$

c $2^x \times 2^{x+2} = 2^{x+x+2}$
$= 2^{2x+2}$

d $3a^2b^3 \times 4a^3b^3$
$= 3 \times 4 \times a^2 \times a^3 \times b^3 \times b^3$
$= 12a^5b^6$

Explanation

When multiplying powers with the same base, add the indices.

In part b, the indices of the base-x powers are added, and the indices of the base-y powers are added. Note that $y = y^1$.

In part c, we use $x + x + 2 = 2x + 2$.

In part d, the indices of the base-a powers are added, the indices of the base-b powers are added, and the factors that are numerals are multiplied together.

 ISBN 978-1-009-53719-3 Cambridge University Press

Index law 2: Dividing powers

Index law 2

To **divide** two powers with the same base, **subtract** the indices.

$$a^m \div a^n = a^{m-n}$$

If m and n are positive integers with $m > n$, then

$$a^m \div a^n = \frac{\overbrace{a \times a \times \cdots \times a}^{m \text{ terms}}}{\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}}$$

$$= \underbrace{a \times a \times \cdots \times a}_{(m-n) \text{ terms}} \qquad \text{(by cancelling)}$$

$$= a^{m-n}$$

Example 2

Simplify each of the following:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

Solution

a $\dfrac{x^4y^3}{x^2y^2} = x^{4-2}y^{3-2}$
$= x^2y$

b $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}} = b^{4x+x+1-2x}$
$= b^{3x+1}$

c $\dfrac{16a^5b \times 4a^4b^3}{8ab} = \dfrac{16 \times 4}{8} \times a^{5+4-1} \times b^{1+3-1}$
$= 8a^8b^3$

Explanation

When dividing powers with the same base, subtract the indices.

In part **a**, the indices of the base-x powers are subtracted, and the indices of the base-y powers are subtracted.

In parts **b** and **c**, both index law 1 and index law 2 are used. In part c, the factors that are numerals are grouped together and simplified.

The zero index and negative integer indices

Define $a^0 = 1$ for $a \neq 0$, and define $a^{-n} = \dfrac{1}{a^n}$ for $a \neq 0$.

Index laws 1 and 2 also hold for negative indices m, n. For example:

$$2^4 \times 2^{-2} = \frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2^2 \qquad (\text{i.e. } 2^{4+(-2)})$$

$$2^{-4} \div 2^2 = \frac{1}{2^4} \times \frac{1}{2^2} = \frac{1}{2^4 \times 2^2} = 2^{-6} \qquad (\text{i.e. } 2^{-4-2})$$

The reciprocal of a fraction such as $\frac{2}{3}$ is $\frac{3}{2}$. For fractions, the index -1 means 'the reciprocal of'. For example:

$$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$$

When raising a fraction to other negative indices, take the reciprocal first. For example:

$$\left(\frac{5}{6}\right)^{-2} = \left(\frac{6}{5}\right)^{2} = \frac{36}{25}$$

Example 3

Evaluate each of the following:

a 8^{-2} **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\left(\frac{3}{4}\right)^{-3}$

Solution

a $8^{-2} = \left(\frac{1}{8}\right)^{2} = \frac{1}{64}$

b $\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

c $\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^{3} = \frac{64}{27}$

Index law 3: Raising the power

Consider the following:

$$(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3\times2}$$

$$(4^3)^4 = 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3\times4}$$

$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2\times5}$$

In general, $(a^m)^n = a^{m\times n}$.

Index law 3

To raise a power to another power, **multiply** the indices.

$$(a^m)^n = a^{m\times n}$$

This rule holds for all integers m and n.

Example 4

Simplify each of the following:

a $(a^5)^3$ **b** $\left(\left(\frac{1}{2}\right)^{-3}\right)^2$ **c** $(b^3)^2 \times (b^2)^{-1}$

Solution	Explanation
a $(a^5)^3 = a^{15}$	Index law 3 is used.
b $\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = \left(\frac{1}{2}\right)^{-6} = 2^6 = 64$	Part b can also be completed as follows: $\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = (2^3)^2 = 8^2 = 64$
c $(b^3)^2 \times (b^2)^{-1} = b^6 \times b^{-2} = b^4$	In part c, index law 1 is also used.

 ISBN 978-1-009-53719-3 Cambridge University Press

Index laws 4 and 5: Products and quotients

Index law 4

$$(ab)^n = a^n b^n$$

If n is a positive integer, then

$$\begin{aligned}(ab)^n &= \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{n \text{ terms}} \\ &= (\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}) \times (\underbrace{b \times b \times \cdots \times b}_{n \text{ terms}}) \\ &= a^n b^n\end{aligned}$$

Index law 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

If n is a positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b} = \frac{a^n}{b^n}$$

Example 5

Simplify each of the following:

a $(2a^2b^3)^3 \times (3ab^4)^{-2}$ **b** $\left(\dfrac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3$

Solution

a $$\begin{aligned}(2a^2b^3)^3 \times (3ab^4)^{-2} &= 8a^6b^9 \times 3^{-2}a^{-2}b^{-8} \\ &= \frac{8a^4b}{9}\end{aligned}$$

b $$\begin{aligned}\left(\frac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3 &= \frac{8a^9b^6}{a^3b^3c^6} \times \frac{1}{a^3b^{-3}c^3} \\ &= \frac{8a^3b^6}{c^9}\end{aligned}$$

Explanation

In part a, index law 4 is used, and then laws 3 and 1 and the fact that $3^{-2} = \frac{1}{9}$.

In part b, index law 5 is used. Dividing by a fraction always means multiply by the reciprocal of that fraction.

Working with a negative base

The power $(-a)^n$ can be written as $(-1 \times a)^n = (-1)^n(a)^n$. We note that:

- If n is even, then $(-1)^n = 1$.
- If n is odd, then $(-1)^n = -1$.

Hence, if a is a positive number, then the number $(-a)^n$ is positive when n is even and negative when n is odd.

 ISBN 978-1-009-53719-3

Example 6

Simplify each of the following:

a $(-3)^4$ **b** $(-5a)^3$ **c** $(-2a)^3 \times 3a^2$

Solution

a $(-3)^4 = 81$

b $(-5a)^3 = -125a^3$

c $$\begin{aligned}(-2a)^3 \times 3a^2 &= -8a^3 \times 3a^2 \\ &= -24a^5\end{aligned}$$

Using prime decomposition

Bases that are composite numbers are often best factored into primes before further calculations are undertaken.

Example 7

Simplify the following, expressing the answers in positive-index form:

a $12^n \times 18^{-2n}$ **b** $\dfrac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **c** $\dfrac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a $$\begin{aligned}12^n \times 18^{-2n} &= (3 \times 2^2)^n \times (3^2 \times 2)^{-2n} \\ &= 3^n \times 2^{2n} \times 3^{-4n} \times 2^{-2n} \\ &= 3^{-3n} \times 2^0 \\ &= \frac{1}{3^{3n}}\end{aligned}$$

b $$\begin{aligned}\frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} &= \frac{3^{-3} \times 2^4 \times 3^4 \times 2^{-6} \times 3^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}} \\ &= \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}} \\ &= 3^6\end{aligned}$$

c $$\begin{aligned}\frac{3^{2n} \times 6^n}{8^n \times 3^n} &= \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} \\ &= \frac{3^n \times 3^n}{2^{2n}} \\ &= \left(\frac{3}{2}\right)^{2n}\end{aligned}$$

Explanation

The prime decomposition of 12 is

$$12 = 2^2 \times 3$$

The prime decomposition of 18 is

$$18 = 2 \times 3^2$$

Each number in this question can be expressed using powers of 2 and 3.

Index law 4 is used in each of the parts. For example:

$$\begin{aligned}12^{-3} &= (2^2 \times 3)^{-3} \\ &= (2^2)^{-3} \times 3^{-3} \\ &= 2^{-6} \times 3^{-3}\end{aligned}$$

 ISBN 978-1-009-53719-3

Summary 11A

- The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**.
- **Index laws** The following results hold for all non-zero numbers a and b and all integers m and n:

  1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{mn}$

  4 $(ab)^n = a^n b^n$ **5** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

- For every non-zero number a and positive integer n:
  - $a^0 = 1$
  - $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$
- $0^n = 0$, if n is a positive integer, and 0^0 is undefined.

Skill-sheet

Exercise 11A

SF

1 For each of the following, use the stated rule to give an equivalent expression in simplest form:

a $x^2 \times x^3$ **b** $2 \times x^3 \times x^4 \times 4$ Index law 1

c $\dfrac{x^5}{x^3}$ **d** $\dfrac{4x^6}{2x^3}$ Index law 2

e $(a^3)^2$ **f** $(2^3)^2$ Index law 3

g $(xy)^2$ **h** $(x^2y^3)^2$ Index law 4 (also use law 3 for h)

i $\left(\dfrac{x}{y}\right)^3$ **j** $\left(\dfrac{x^3}{y^2}\right)^2$ Index law 5 (also use law 3 for j)

Example 1

2 Simplify each of the following:

a $3^5 \times 3^{12}$ **b** $x^3y^2 \times x^4y^3$ **c** $3^{x+1} \times 3^{3x+2}$ **d** $5a^3b^2 \times 6a^2b^4$

Example 2

3 Simplify each of the following:

a $\dfrac{x^5y^2}{x^3y}$ **b** $\dfrac{b^{5x} \times b^{2x+1}}{b^{3x}}$ **c** $\dfrac{8a^2b \times 3a^5b^2}{6a^2b^2}$

Example 3

4 Evaluate each of the following:

a 7^{-2} **b** $\left(\dfrac{1}{4}\right)^{-3}$ **c** $\left(\dfrac{5}{2}\right)^{-3}$

Example 4

5 Simplify each of the following:

a $(b^5)^2$ **b** $\left(\left(\dfrac{1}{3}\right)^{-2}\right)^3$ **c** $(b^5)^2 \times (b^2)^{-3}$

ISBN 978-1-009-53719-3

Example 5 **6** Simplify each of the following:

a $(3a^4b^3)^3 \times (4a^2b^4)^{-2}$

b $\left(\dfrac{5a^3b^3}{ab^2c^2}\right)^3 \div (a^2b^{-1}c)^3$

SF

Example 6 **7** Simplify each of the following:

a $(-2)^6$ **b** $(-3a)^3$ **c** $(-2a)^5 \times 3a^{-2}$

Example 7 **8** Simplify the following:

a $36^n \times 12^{-2n}$ **b** $\dfrac{2^{-3} \times 8^4 \times 32^{-3}}{4^{-4} \times 2^{-2}}$ **c** $\dfrac{5^{2n} \times 10^n}{8^n \times 5^n}$

9 Simplify the following:

a $x^3 \times x^4 \times x^2$
b $2^4 \times 4^3 \times 8^2$
c $3^4 \times 9^2 \times 27^3$
d $(q^2p)^3 \times (qp^3)^2$
e $a^2b^{-3} \times (a^3b^2)^3$
f $(2x^3)^2 \times (4x^4)^3$
g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$
h $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

10 Simplify the following:

a $\dfrac{x^3y^5}{xy^2}$
b $\dfrac{16a^5b \times 4a^4b^3}{8ab}$
c $\dfrac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$
d $\dfrac{(-3x^2y^3)^2}{(2xy)^3} \times \dfrac{4x^4y^3}{(xy)^3}$

11 Simplify each of the following, expressing your answer in positive-index form:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\dfrac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\dfrac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\dfrac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

CF

12 Simplify each of the following:

a $3^{4n} \times 9^{2n} \times 27^{3n}$ **b** $\dfrac{2^n \times 8^{n+1}}{32^n}$ **c** $\dfrac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\dfrac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\dfrac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\dfrac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\dfrac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\dfrac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

13 Simplify and evaluate:

a $\dfrac{(8^3)^4}{(2^{12})^2}$ **b** $\dfrac{125^3}{25^2}$ **c** $\dfrac{81^4 \div 27^3}{9^2}$

14 Simplify each of the following, giving your answer with positive integer exponents.

a $\dfrac{x - y^{-1}}{x^{-1} - y}$ **b** $(x^{-1} + y^{-1})(x^{-1} - y^{-1})$ **c** $(x^{-2} + y^{-1})^{-1}$

 ISBN 978-1-009-53719-3

11B Rational indices

Learning intentions

- To be able to use the rules for manipulating **exponential expressions** with rational exponents.

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

Further, the expression a^x can be defined for rational indices, i.e. when $x = \dfrac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition we will always first write the fractional power in simplest form.

Example 8

Evaluate:

a $(-64)^{\frac{1}{3}}$ **b** $9^{-\frac{1}{2}}$ **c** $16^{\frac{5}{2}}$ **d** $64^{-\frac{2}{3}}$

Solution

a $(-64)^{\frac{1}{3}} = -4$

b $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

c $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

d $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(64^{\frac{1}{3}}\right)^2} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

Note: In the previous section, we stated the index laws for m and n integers:

1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{m \times n}$

These laws are applicable for all rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Example 9

Simplify:

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}}$ **b** $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4$

Solution

a
$$\frac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}} = \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^3}$$
$$= \frac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^3}$$
$$= \frac{3^{\frac{3}{4}} \times 2^{\frac{3}{4}}}{2^3}$$
$$= 3^{\frac{3}{4}} \times 2^{\frac{3}{4}-\frac{12}{4}}$$
$$= 3^{\frac{3}{4}} \times 2^{-\frac{9}{4}}$$
$$= \frac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$$

b
$$(x^{-2}y)^{\frac{1}{2}} \times \left(\frac{x}{y^{-3}}\right)^4 = x^{-1}y^{\frac{1}{2}} \times \frac{x^4}{y^{-12}}$$
$$= x^3 \times y^{\frac{25}{2}}$$

Explanation

$\sqrt{6} = \sqrt{3} \times \sqrt{2} = 3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt[4]{2} = 2^{\frac{1}{4}}$

$2^3 = 2^{\frac{12}{4}}$

$\dfrac{x^4}{y^{-12}} = x^4 \times y^{12}$

Summary 11B

- Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a.
- Define $0^{\frac{1}{n}} = 0$, for each $n \in \mathbb{N}$.
- If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a.
- In all three cases we can write:
$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$
- The index laws can be extended to rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Skill-sheet

Exercise 11B

Example 8

1 Evaluate each of the following:

a $125^{\frac{2}{3}}$ **b** $243^{\frac{3}{5}}$ **c** $81^{-\frac{1}{2}}$ **d** $64^{\frac{2}{3}}$

e $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ **f** $32^{-\frac{2}{5}}$ **g** $125^{-\frac{2}{3}}$ **h** $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ **j** $10\,000^{\frac{3}{4}}$ **k** $81^{\frac{3}{4}}$ **l** $\left(\frac{27}{125}\right)^{\frac{1}{3}}$

m $(-8)^{\frac{1}{3}}$ **n** $125^{-\frac{4}{3}}$ **o** $(-32)^{\frac{4}{5}}$ **p** $\left(\frac{1}{49}\right)^{-\frac{3}{2}}$

Example 9

2 Simplify:

a $\sqrt[3]{a^2b} \div \sqrt{ab^3}$ **b** $(a^{-2}b)^3 \times \left(\frac{1}{b^{-3}}\right)^{\frac{1}{2}}$

c $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$ **d** $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$

e $\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2$ **f** $\left(\sqrt[5]{a^2}\right)^{\frac{3}{2}} \times \left(\sqrt[3]{a^5}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ **b** $(x-1)^2\sqrt{x-1}$ **c** $(x^2+1)\sqrt{x^2+1}$

d $(x-1)^3\sqrt{x-1}$ **e** $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1}$ **f** $(5x^2+1)\sqrt[3]{5x^2+1}$

SF

11C Scientific notation

Learning intentions

- To be able to express numbers using **scientific notation**.

Often when dealing with real-world problems, the numbers involved may be very small or very large. For example:

- The distance from Earth to the Sun is approximately 150 000 000 kilometres.
- The mass of an oxygen atom is approximately 0.000 000 000 000 000 000 000 026 grams.

To help deal with such numbers, we can use a more convenient way to express them. This involves expressing the number as a product of a number between 1 and 10 and a power of 10 and is called **standard form** or **scientific notation**. These examples written in standard form are:

- 1.5×10^8 kilometres
- 2.6×10^{-23} grams

Multiplication and division with very small or very large numbers can often be simplified by first converting the numbers into standard form.

 ISBN 978-1-009-53719-3

Example 10

Write each of the following in scientific notation:

a 3 453 000 **b** 0.00675

Solution

a $3\,453\,000 = 3.453 \times 10^6$ **b** $0.00675 = 6.75 \times 10^{-3}$

Example 11

Determine the value of $\dfrac{32\,000\,000 \times 0.000\,004}{16\,000}$.

Solution

$$\begin{aligned}\frac{32\,000\,000 \times 0.000\,004}{16\,000} &= \frac{3.2 \times 10^7 \times 4 \times 10^{-6}}{1.6 \times 10^4}\\ &= \frac{12.8 \times 10^1}{1.6 \times 10^4}\\ &= 8 \times 10^{-3}\\ &= 0.008\end{aligned}$$

Note: Many calculators can display numbers in scientific notation. For example, the number $3\,245\,000 = 3.245 \times 10^6$ may appear as 3.245E6 or 3.245^{06}.

Significant figures

When measurements are made, the result is recorded to a certain number of significant figures. For example, if we say that the length of a piece of ribbon is 156 cm to the nearest centimetre, this means that the length is between 155.5 cm and 156.5 cm. The number 156 is said to be correct to three significant figures. Similarly, we may record π as being 3.1416, correct to five significant figures.

When rounding off to a given number of significant figures, first identify the last significant digit and then:

- if the next digit is 0, 1, 2, 3 or 4, round down
- if the next digit is 5, 6, 7, 8 or 9, round up.

It can help with rounding off if the original number is first written in scientific notation.

So $\pi = 3.141\,592\,653\ldots$ is rounded off to $3,\ 3.1,\ 3.14,\ 3.142,\ 3.1416,\ 3.14159,$ etc. depending on the number of significant figures required.

Writing a number in scientific notation makes it clear how many significant figures have been recorded. For example, it is unclear whether 600 is recorded to one, two or three significant figures. However, when written in scientific notation as $6.00 \times 10^2,\ 6.0 \times 10^2$ or 6×10^2, it is clear how many significant figures are recorded.

Example 12

Evaluate $\dfrac{\sqrt[5]{a}}{b^2}$ if $a = 1.34 \times 10^{-10}$ and $b = 2.7 \times 10^{-8}$.

Solution

$$\frac{\sqrt[5]{a}}{b^2} = \frac{(1.34 \times 10^{-10})^{\frac{1}{5}}}{(2.7 \times 10^{-8})^2}$$

$$= 1.45443\ldots \times 10^{13}$$

$$= 1.45 \times 10^{13} \quad \text{to three significant figures}$$

Summary 11C

- A number is said to be in **scientific notation** (or **standard form**) when it is written as a product of a number between 1 and 10 and an integer power of 10.
  For example: $6547 = 6.547 \times 10^3$ and $0.789 = 7.89 \times 10^{-1}$.
- Writing a number in scientific notation makes it clear how many **significant figures** have been recorded.
- When rounding off to a given number of significant figures, first identify the last significant digit and then:
  - if the next digit is 0, 1, 2, 3 or 4, round down
  - if the next digit is 5, 6, 7, 8 or 9, round up.

Exercise 11C

SF

Example 10

1 Express each of the following numbers in standard form:

a 47.8 **b** 6728 **c** 79.23 **d** 43 580
e 0.0023 **f** 0.000 000 56 **g** 12.000 34 **h** 50 million
i 23 000 000 000 **j** 0.000 000 0013 **k** 165 thousand **l** 0.000 014 567

2 Express each of the following in scientific notation:

a X-rays have a wavelength of 0.000 000 01 cm.
b The mass of a hydrogen atom is 0.000 000 000 000 000 000 000 001 67 g.
c Visible light has wavelength 0.000 05 cm.
d One nautical mile is 1853.18 m.
e A light year is 9 461 000 000 000 km.
f The speed of light is 29 980 000 000 cm/s.

3 Express each of the following as an ordinary number:

a The star Sirius is approximately 8.128×10^{13} km from Earth.
b A single red blood cell contains 2.7×10^8 molecules of haemoglobin.
c The radius of an electron is 2.8×10^{-13} cm.

 ISBN 978-1-009-53719-3

SF

4 Write each of the following in scientific notation, correct to the number of significant figures indicated in the brackets:

a 456.89 (4) **b** 34567.23 (2) **c** 5679.087 (5)
d 0.04536 (2) **e** 0.09045 (2) **f** 4568.234 (5)

Example 11

5 Determine the value of:

a $\dfrac{324\,000 \times 0.000\,0007}{4000}$ **b** $\dfrac{5\,240\,000 \times 0.8}{42\,000\,000}$

Example 12

6 Evaluate the following correct to three significant figures:

a $\dfrac{\sqrt[3]{a}}{b^4}$ if $a = 2 \times 10^9$ and $b = 3.215$ **b** $\dfrac{\sqrt[4]{a}}{4b^4}$ if $a = 2 \times 10^{12}$ and $b = 0.05$

11D Graphs of exponential functions

Learning intentions

- To be able to sketch the graphs of **exponential functions**.

Two types of graphs of exponential functions will be examined.

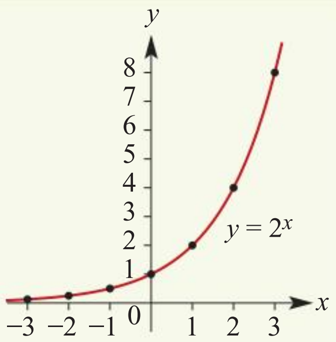
Graph of $y = a^x$ when $a > 1$

Example 13

Plot the graph of $y = 2^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



We can make the following observations about graphs of the form $y = a^x$ where $a > 1$:

- As the magnitude of the negative x-values becomes larger and larger, the y-values get closer and closer to zero, but never reach zero. That is, the graph of $y = a^x$ approaches the x-axis from above. The x-axis is said to be an **asymptote**.

  We can write: As $x \to -\infty$, $y \to 0^+$.

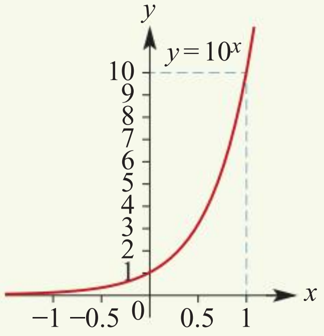
  This is read: As x approaches negative infinity, y approaches 0 from the positive side.
- As the x-values increase, the y-values increase.
- The y-axis intercept is at $(0, 1)$.
- The domain of the function is $\mathbb{R}$ and the range of the function is $\mathbb{R}^+$.

Example 14

Plot the graph of $y = 10^x$ and examine the table of values for $-1 \le x \le 1$. A calculator can be used to obtain approximate values.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	≈ 0.316	1	≈ 3.16	10



- The x-axis is an asymptote.
- The y-axis intercept is at $(0, 1)$.
- As the x-values increase, the y-values increase.
- For a given value of x, this rate of increase for $y = 10^x$ is greater than that for $y = 2^x$.

Note that, for any numbers a and b greater than 1, there is a positive number k with $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$. Using a calculator to solve $2^k = 10$ gives $k = 3.3219\ldots$. Hence $10^x = (2^{3.3219\ldots})^x$ and therefore the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor $\dfrac{1}{k} = \dfrac{1}{3.3219\ldots}$ from the y-axis.

All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis. (This will be discussed again later in the chapter.)

Graph of $y = a^x$ when $0 < a < 1$

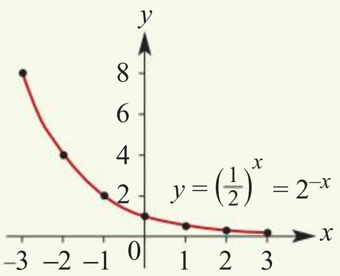
Example 15

Plot the graph of $y = (\frac{1}{2})^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

Note that $y = (\frac{1}{2})^x = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
$y = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



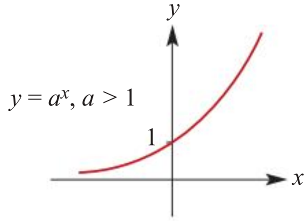
We can make the following observations about graphs of the form $y = a^x$ where $0 < a < 1$:

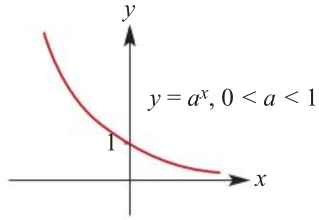
- The x-axis is an asymptote. As the x-values increase, the graph approaches the x-axis from above. This is written: As $x \to \infty$, $y \to 0^+$.
- The y-axis intercept is at $(0, 1)$.
- The domain of the function is $\mathbb{R}$ and the range of the function is $\mathbb{R}^+$.

 ISBN 978-1-009-53719-3

Graphs of $y = a^x$ in general

In general:





In both cases $a > 1$ and $0 < a < 1$, we can write $y = a^x$ as $y = b^{-x}$, where $b = \frac{1}{a}$. The graph of $y = b^{-x}$ is obtained from the graph of $y = b^x$ by a reflection in the y-axis.

Thus, for example, the graph of $y = (\frac{1}{2})^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis, and vice versa. Using function notation: Let $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$. Then

$$g(x) = (\tfrac{1}{2})^x = (2^{-1})^x = 2^{-x} = f(-x)$$

Example 16

Plot the graph of $y = 2^x$ on a calculator and hence determine (correct to three decimal places):

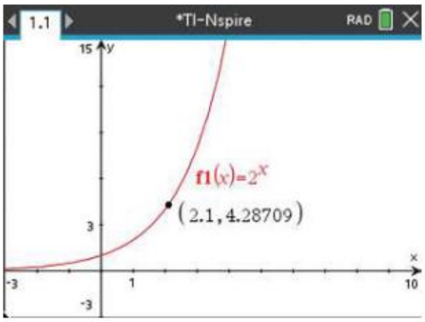
a the value of y when $x = 2.1$ **b** the value of x when $y = 9$.

Using the TI-Nspire CX non-CAS

Plot the graph of $y = 2^x$.

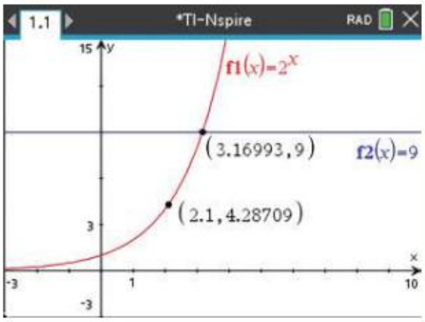
a
- To go to the point with x-coordinate 2.1, use menu > **Trace** > **Graph Trace**, enter 2.1 and press enter.
- Press enter to paste the coordinates to the point.
- Press esc to exit the **Graph Trace** tool.

When $x = 2.1$, $y = 4.287$ (correct to three decimal places).



b
- To determine the value of x for which $y = 9$, plot the graph of $y = 9$ on the same screen and use menu > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Press esc to exit the **Intersection Point(s)** tool.

When $y = 9$, $x = 3.170$ (correct to three decimal places).



Note: Alternatively, determine the intersection point using menu > **Analyze Graph** > **Intersection**.

 ISBN 978-1-009-53719-3 Cambridge University Press

Using the Casio

In **Graph** mode, draw the graph of $y = 2^x$ and set an appropriate View Window.

a To determine the value of y when $x = 2.1$:

- Go to **G-Solve** SHIFT F5, then **y-Cal** F6 F1.
- Enter the x-value 2.1. Press EXE.

When $x = 2.1$, $y = 4.287$ (correct to three decimal places).

b To determine the value of x when $y = 9$:

- Go to **G-Solve** SHIFT F5, then **x-Cal** F6 F2.
- Enter the y-value 9. Press EXE.

When $y = 9$, $x = 3.170$ (correct to three decimal places).

Transformations of exponential graphs

The techniques for transformations that were introduced in earlier chapters are now applied to the graphs of exponential functions.

Example 17

Sketch the graphs of each of the following pairs of functions. For the second function in each pair, state the equation of the asymptote, the y-axis intercept and the range. (The x-axis intercepts need not be given.)

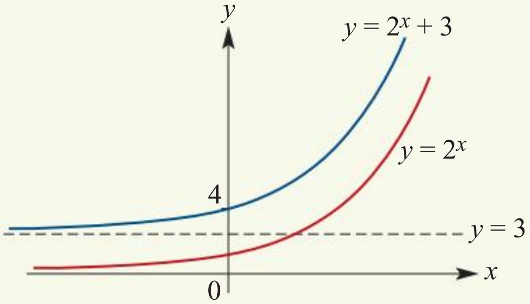
a $f(x) = 2^x$ and $g(x) = 2^x + 3$

b $f(x) = 3^x$ and $g(x) = 2 \times 3^x + 1$

c $f(x) = 3^x$ and $g(x) = -3^x + 2$

Solution

a



Graph of $g(x) = 2^x + 3$:

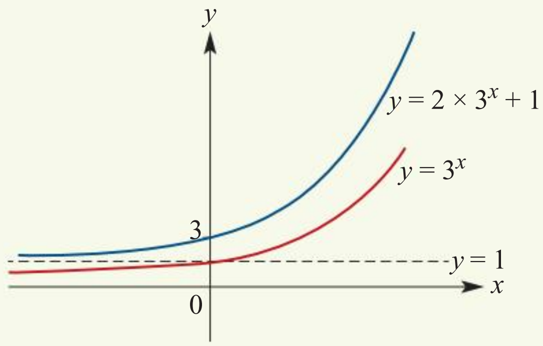
- The asymptote has equation $y = 3$.
- The y-axis intercept is $g(0) = 2^0 + 3 = 4$.
- The range of the function g is $(3, \infty)$.

Explanation

The graph of $y = 2^x + 3$ is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$ is the line with equation $y = 0$, which is transformed to the line with equation $y = 3$.

b



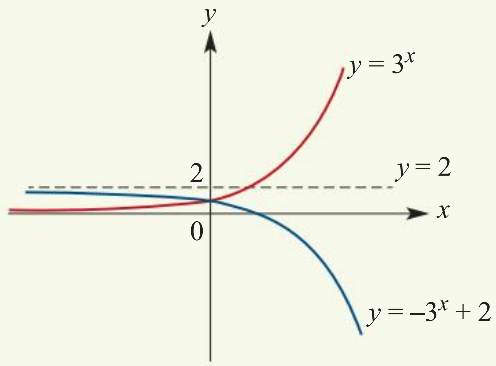
Graph of $g(x) = 2 \times 3^x + 1$:

- The asymptote has equation $y = 1$.
- The y-axis intercept is $g(0) = 2 \times 3^0 + 1 = 3$.
- The range of the function g is $(1, \infty)$.

The graph of $y = 2 \times 3^x + 1$ is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 1$.

c



Graph of $g(x) = -3^x + 2$:

- The asymptote has equation $y = 2$.
- The y-axis intercept is $g(0) = -3^0 + 2 = 1$.
- The range of the function g is $(-\infty, 2)$.

The graph of $y = -3^x + 2$ is obtained by transforming the graph of $y = 3^x$ by a reflection in the x-axis, followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 2$.

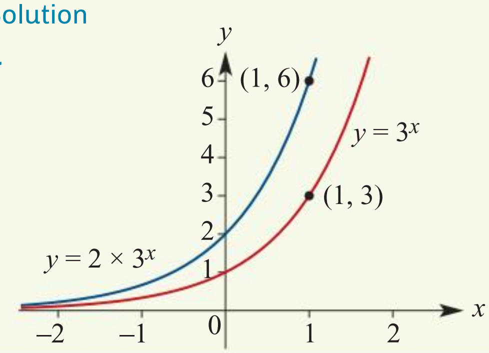
Example 18

Sketch the graph of each of the following:

a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = -3^{2x} + 4$

Solution

a

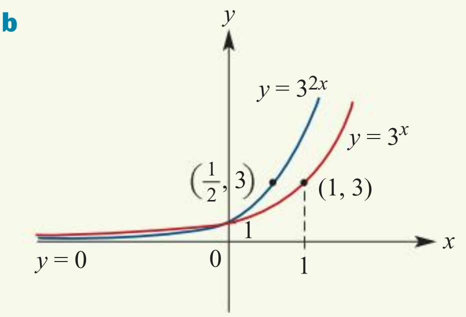


Explanation

The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis.

Both graphs have a horizontal asymptote with equation $y = 0$.

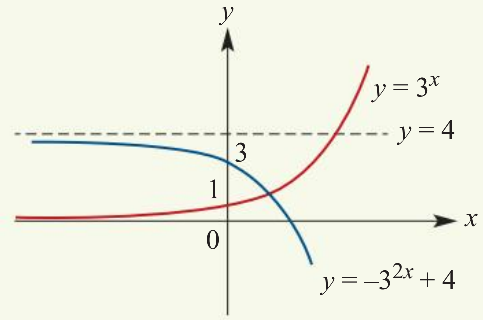
b



The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis.

Both graphs have a horizontal asymptote with equation $y = 0$.

c

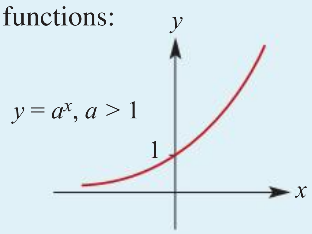


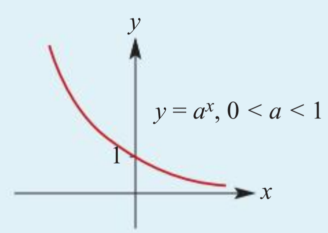
The graph of $y = -3^{2x} + 4$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis and then a translation of 4 units in the positive direction of the y-axis.

The graph of $y = -3^{2x} + 4$ has a horizontal asymptote with equation $y = 4$.

Summary 11D

- Graphs of exponential functions:





- In both cases $a > 1$ and $0 < a < 1$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis. Similarly, all graphs of the form $y = a^x$, where $0 < a < 1$, are related to each other by dilations from the y-axis.
- Let $a > 1$. If $f(x) = a^x$ and $g(x) = \left(\frac{1}{a}\right)^x$, then $g(x) = f(-x)$ and so the graph of $y = g(x)$ is the reflection in the y-axis of the graph of $y = f(x)$.

Exercise 11D

1 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ **b** $y = 2.4^x$ **c** $y = 0.9^x$ **d** $y = 0.5^x$

SF

 ISBN 978-1-009-53719-3 Cambridge University Press

SF

2 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ **b** $y = 5 \times 3^x$ **c** $y = -2 \times 3^x$ **d** $y = -5 \times 3^x$

Example 16

3 Plot the graph of $y = 2^x$ on a graphics calculator and hence determine the solution of the equation $2^x = 14$.

4 Plot the graph of $y = 10^x$ on a graphics calculator and hence determine the solution of the equation $10^x = 6$.

Example 17, 18

5 Sketch the graphs of the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each function. (The x-axis intercepts need not be given.)

a $f(x) = 3 \times 2^x + 2$ **b** $f(x) = 3 \times 2^x - 3$ **c** $f(x) = -3^x - 2$

d $f(x) = -2 \times 3^x + 2$ **e** $f(x) = (\frac{1}{2})^x + 2$ **f** $f(x) = -2 \times 3^x - 2$

6 Sketch the graph of each of the following:

a $y = 2 \times 5^x$ **b** $y = 3^{3x}$ **c** $y = 5^{\frac{x}{2}}$ **d** $y = -3^{2x} - 2$

11E Solving exponential equations and inequalities

Learning intentions

- To be able to solve exponential equations and inequalities.

Solution of equations

One method without using a calculator is to express both sides of the equation as powers with the same base and then equate the indices (since $a^x = a^y$ implies $x = y$, for any $a \in \mathbb{R}^+ \setminus \{1\}$).

Example 19

Determine the value of x for which:

a $4^x = 256$ **b** $3^{x-1} = 81$ **c** $5^{2x-4} = 25^{-x+2}$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore\ x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$\therefore\ x - 1 = 4$$
$$x = 5$$

c
$$\begin{aligned} 5^{2x-4} &= 25^{-x+2} \\ &= (5^2)^{-x+2} \\ &= 5^{-2x+4} \\ \therefore\ 2x - 4 &= -2x + 4 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

 ISBN 978-1-009-53719-3

In the next example, an exponential equation is solved by first solving a quadratic equation.

Example 20

a Solve $9^x = 12 \times 3^x - 27$.

b Solve $3^{2x} = 27 - 6 \times 3^x$.

Solution

a We have $(3^x)^2 = 12 \times 3^x - 27$.

Let $a = 3^x$. The equation becomes

$$\begin{aligned} a^2 &= 12a - 27 \\ a^2 - 12a + 27 &= 0 \\ (a-3)(a-9) &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} a - 3 = 0 \quad &\text{or} \quad a - 9 = 0 \\ a = 3 \quad &\text{or} \quad a = 9 \end{aligned}$$

Hence $3^x = 3^1$ or $3^x = 3^2$

and so $x = 1$ or $x = 2$

b We have $(3^x)^2 = 27 - 6 \times 3^x$.

Let $a = 3^x$. The equation becomes

$$\begin{aligned} a^2 &= 27 - 6a \\ a^2 + 6a - 27 &= 0 \\ (a+9)(a-3) &= 0 \end{aligned}$$

Therefore

$$a = -9 \quad \text{or} \quad a = 3$$

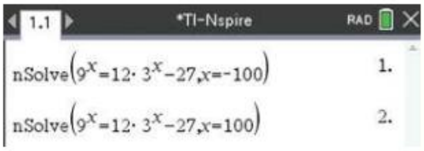
Hence $3^x = -9$ or $3^x = 3^1$

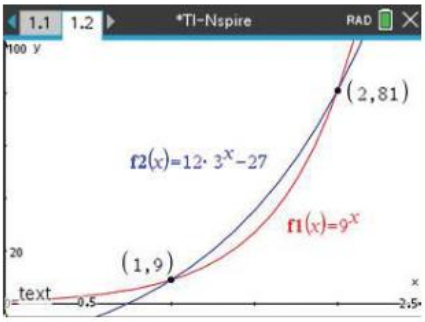
There is only one solution, $x = 1$, since $3^x > 0$ for all values of x.

TI-84

Using the TI-Nspire CX non-CAS

- To solve $9^x = 12 \times 3^x - 27$, use menu > **Algebra** > **Numerical Solve** as shown.
- The solutions are $x = 1$ and $x = 2$.



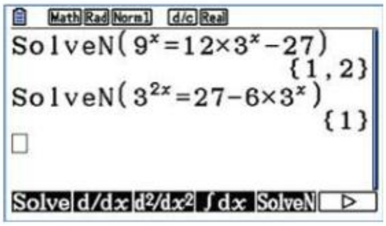


Note: The two different guess values $x = -100$ and $x = 100$ are used here in order to determine the two solutions.

It is advisable to determine the actual number of solutions graphically. In this case, there are only two solutions.

Using the Casio

- In **Run-Matrix** mode, select the numerical solver (**Calculation** OPTN F4, **SolveN** F5).
- Enter the equation and press EXE.



Note: The equations can also be solved graphically. For Example 20a, plot the graphs of $y = 9^x$ and $y = 12 \times 3^x - 27$ and then determine the points of intersection.

ISBN 978-1-009-53719-3

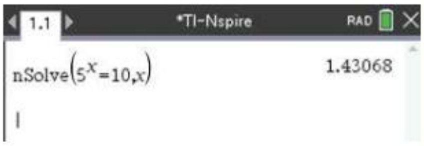
The calculator can be used to obtain approximate answers as shown in the following example. For the equation $5^x = 10$ we can determine an exact solution by hand, but logarithms are involved in the final answer. Logarithms are discussed in the following chapter.

Example 21

Solve $5^x = 10$ correct to two decimal places.

Using the TI-Nspire CX non-CAS

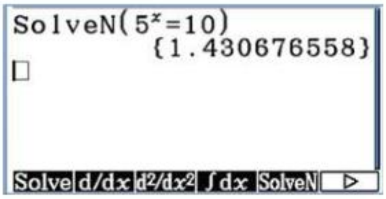
- Use menu > **Algebra** > **Numerical Solve**.
- The solution is $x = 1.43$ (correct to two decimal places).



Note: It is advisable to determine the actual number of solutions graphically. In this case, there is only one solution.

Using the Casio

- In **Run-Matrix** mode, select the numerical solver (**Calculation** OPTN F4, **SolveN** F5).
- Enter the equation and press EXE.
- The solution is $x = 1.43$ (correct to two decimal places).



Solution of inequalities

The following two properties are useful when solving inequalities:

- $a^x > a^y \Leftrightarrow x > y$, when $a \in (1, \infty)$
- $a^x > a^y \Leftrightarrow x < y$, when $a \in (0, 1)$.

Example 22

Solve for x in each of the following:

a $16^x > 2$

b $2^{-3x+1} < \dfrac{1}{16}$

Solution

a
$$\begin{aligned} 2^{4x} &> 2^1 \\ \Leftrightarrow \quad 4x &> 1 \\ \Leftrightarrow \quad x &> \frac{1}{4} \end{aligned}$$

b
$$\begin{aligned} 2^{-3x+1} &< 2^{-4} \\ \Leftrightarrow \quad -3x + 1 &< -4 \\ \Leftrightarrow \quad -3x &< -5 \\ \Leftrightarrow \quad x &> \frac{5}{3} \end{aligned}$$

Note: A calculator can be used to help 'visualise' the inequality. For Example 22a, plot the graphs of $y = 16^x$ and $y = 2$ and then determine the point of intersection.

 ISBN 978-1-009-53719-3

Summary 11E

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in \mathbb{R}^+ \setminus \{1\}$).
  For example: $2^{x+1} = 8 \iff 2^{x+1} = 2^3 \iff x + 1 = 3 \iff x = 2$
- To solve an exponential inequality, first proceed as for an equation and then use the appropriate property:
  - $a^x > a^y \iff x > y$, when $a \in (1, \infty)$
  - $a^x > a^y \iff x < y$, when $a \in (0, 1)$.

  For example: $2^{x+1} > 8 \iff 2^{x+1} > 2^3 \iff x + 1 > 3 \iff x > 2$

Skill-sheet

Exercise 11E

Example 19

1 Solve for x in each of the following:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$
d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$
g $16^x = 256$ **h** $4^{-x} = \frac{1}{64}$ **i** $5^{-x} = \frac{1}{125}$

SF

2 Solve for n in each of the following:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$
d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$
g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$
j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$ **l** $125^{4-n} = 5^{6-2n}$
m $4^{2-n} = \frac{1}{2048}$

CF

3 Solve the following exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^x = 243$ **c** $(27 \times 3^x)^2 = 27^x \times 3^{\frac{1}{2}}$

Example 20

4 Solve for x:

a $4(2^{2x}) = 8(2^x) - 4$ **b** $8(2^{2x}) - 10(2^x) + 2 = 0$
c $3 \times 2^{2x} - 18(2^x) + 24 = 0$ **d** $9^x - 4(3^x) + 3 = 0$

Example 21

5 Use a calculator to solve each of the following, correct to two decimal places:

a $2^x = 5$ **b** $4^x = 6$ **c** $10^x = 18$ **d** $10^x = 56$

SF

Example 22

6 Solve for x in each of the following:

a $7^x > 49$ **b** $8^x > 2$ **c** $25^x \le 5$ **d** $3^{x+1} < 81$
e $9^{2x+1} < 243$ **f** $4^{2x+1} > 64$ **g** $3^{2x-2} \le 81$

CF

 ISBN 978-1-009-53719-3

11F Exponential models and applications

Learning intentions

- To be able to use exponential functions to model practical situations.

Exponential growth and decay

In the following, we consider a variable A that is subject to exponential change. Let A be the quantity at time t. Then $A = A_0 b^t$, where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. Then after t minutes the number of cells, N, is given by the formula

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**. Here we are only dealing with the type of reproduction where the cell divides in two. For most known bacteria that can be cultured, generation times range from about 15 minutes to 1 hour.

Example 23

What is the generation time, in minutes, of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay so that the amount of radioactive material, A, present at time t (in years) is given by the formula

$$A = A_0 2^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

Example 24

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Determine the half-life of radium-226.

Solution

We use the formula $A = A_0 2^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$0.647A_0 = A_0 2^{-1000k}$$
$$0.647 = 2^{-1000k}$$
$$k \approx 0.000628$$

To determine the half-life, we consider when $A = \frac{1}{2}A_0$:

$$A_0 2^{-kt} = \tfrac{1}{2}A_0$$
$$2^{-kt} = 2^{-1}$$
$$-kt = -1$$
$$t = \frac{1}{k} \approx 1591.95$$

The half-life of radium-226 is approximately 1592 years.

Population growth

It is sometimes possible to model population growth through exponential models.

Example 25

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, determine how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

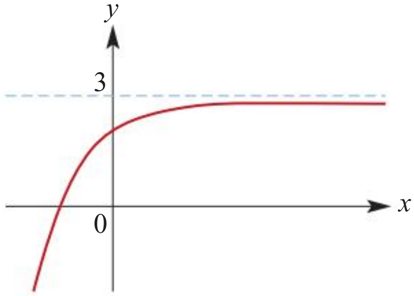
$$G = 10R$$
$$G_0(1.11)^n = 10 \times 10G_0(0.95)^n$$
$$(1.11)^n = 100(0.95)^n$$
$$\therefore \left(\frac{1.11}{0.95}\right)^n = 100$$
$$n \approx 29.6$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

 ISBN 978-1-009-53719-3 Cambridge University Press

Limiting values of exponential functions

Consider the exponential function $f(x) = 3 - 2^{-x}$. We can see from the graph that, as x increases, the value of $f(x)$ gets closer and closer to 3.



In general, for an exponential function f, if there is a real number ℓ such that, as $x \to \infty$, $f(x) \to \ell$, then we say that ℓ is the **limiting value** of f.

There are many applications of exponential functions where the limiting value is significant. For example, if an object is dropped in a liquid, then a possible formula for its speed, v m/s, at time t seconds could be

$$v = 0.5(1 - 3^{-0.4t})$$

It is easily seen that, as $t \to \infty$, $v \to 0.5$. The object's speed gradually approaches 0.5 m/s, but does not ever reach 0.5 m/s.

Example 26

An ice pack is taken out of a freezer, which is kept at -20°C. The temperature, T°C, of the ice pack at time t minutes is given by

$$T = -50 \times (0.98)^t + 30$$

a Determine the limiting value of T as $t \to \infty$. What does this value represent?

b Determine the value of t for which $T = 0$.

c Sketch the graph of T against t.

d Comment on this model.

Solution

a As $t \to \infty$, we have

$$0.98^t \to 0$$

$$\therefore \quad T \to 30$$

The limiting value of T is 30.
The outside temperature is 30°C.

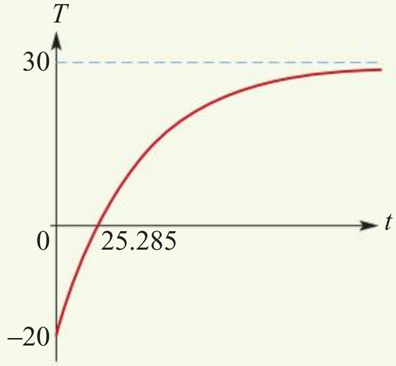
b $-50 \times (0.98)^t + 30 = 0$

$$0.98^t = 0.6$$

$$t \approx 25.285$$

The temperature of the ice pack is 0°C after approximately 25 minutes.

c



d According to the model, the ice pack approaches but never reaches the outside temperature. The temperature of the ice pack passes 29.9°C after about 5 hours. You would probably consider that it has reached outside temperature by this stage.

 ISBN 978-1-009-53719-3 Cambridge University Press

Determining exponential rules

We have looked at determining rules for functions in Chapters 2 to 8. We look at one very useful case for exponential functions.

Example 27

The points $(1, 6)$ and $(5, 96)$ are known to lie on the curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Determine the values of a and b.

Solution

We can write

$$a \times b^1 = 6 \qquad (1)$$
$$a \times b^5 = 96 \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^4 = 16$. Thus $b = 16^{\frac{1}{4}} = 2$, and substituting into equation (1) gives $a = 3$.

In many practical situations, the relationship between variables is exponential.

Example 28

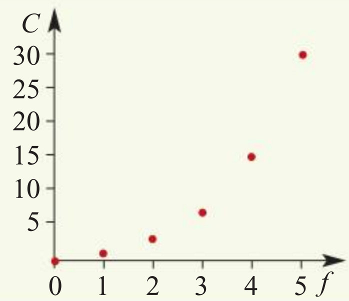
Take a rectangular piece of paper approximately 30 cm × 6 cm. Fold the paper in half, successively, until you have folded it five times. Tabulate the number of times folded, f, and the number of creases in the paper, C.

Solution

Folds, f	0	1	2	3	4	5
Creases, C	0	1	3	7	15	31

The rule connecting C and f is

$$C = 2^f - 1, \quad f \in \{0, 1, 2, \dots\}$$



Fitting data

Example 29

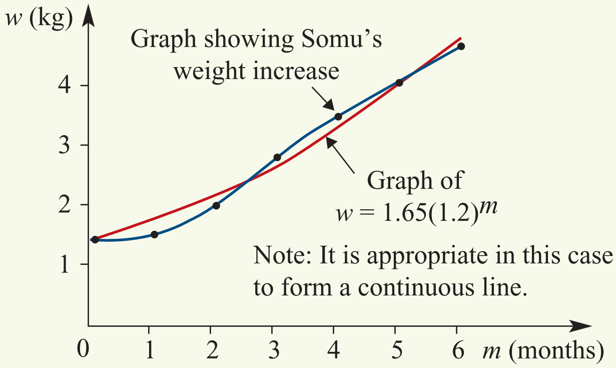
The table below shows the increase in weight of Somu, an orang-utan born at the Eastern Plains Zoo. Draw a graph to show Somu's weight increase for the first six months.

Data values

Months, m	0	1	2	3	4	5	6
Weight (kg), w	1.65	1.7	2.2	3.0	3.7	4.2	4.8

ISBN 978-1-009-53719-3

Solution



The data values are plotted on the same set of axes as the graph of the exponential function $w = 1.65(1.2)^m$, $0 \le m \le 6$. A table of values for this function is shown below.

Values from model

m	0	1	2	3	4	5	6
w	1.65	1.98	2.38	2.85	3.42	4.1	4.93

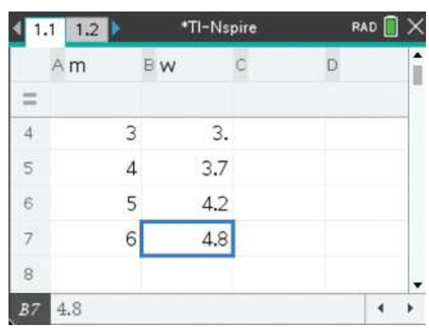
It can be seen from the graphs in Example 29 that the exponential model $w = 1.65(1.2)^m$ approximates the actual weight gain and would be a useful model to predict weight gains for any future orang-utan births at the zoo. This model describes a growth rate of 20% per month for the first 6 months.

This problem can also be attempted with a graphics calculator.

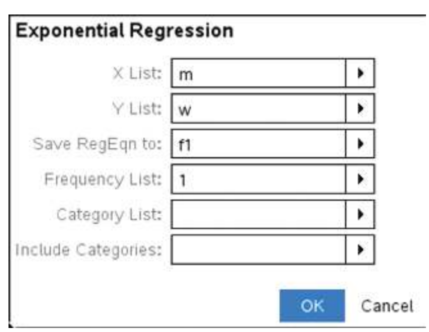
TI-84

Using the TI-Nspire CX non-CAS

- Enter the data either in a **Lists & Spreadsheet** application as shown or in a **Calculator** application as lists.

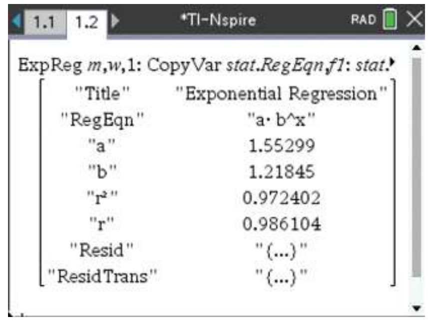


- Insert a **Calculator** page and use (menu) > **Statistics** > **Stat Calculations** > **Exponential Regression**. Complete as shown:
  - Use (tab) to move between fields.
  - Use the selection tool to open a field. Then use the arrows ▲ and ▼ to choose the correct entry, and select this entry using the selection tool.



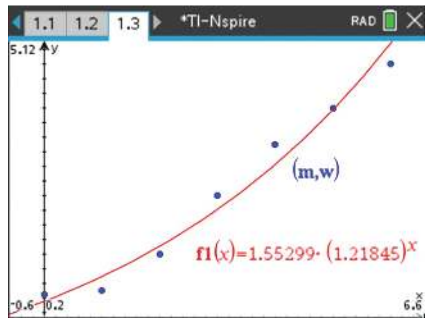
- This now gives the values of a and b, and the equation has been entered in $f1(x)$.

Note: This regression analysis can also be done in the Lists & Spreadsheet application.



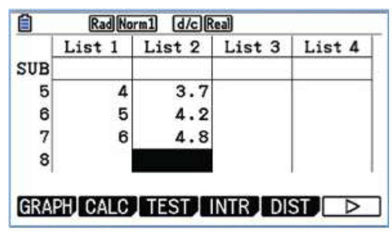
- The curve can be shown in a **Graphs** application together with the scatter plot (menu > **Graph Entry/Edit** > **Scatter Plot**) using an appropriate window (menu > **Window/Zoom** > **Zoom – Data**).

Note: Alternatively, the scatter plot and regression curve can be obtained using the **Data & Statistics** application.

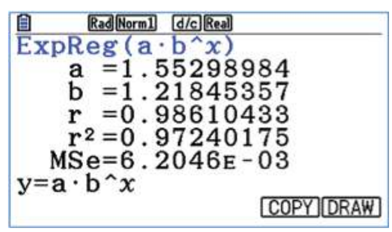


Using the Casio

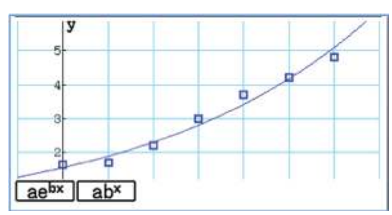
- Press MENU 2 to select **Statistics** mode.
- Enter the months in List1 and the weights in List2.
- Graph the data by selecting **Graph** F1, then **Graph1** F1.



- To fit a curve of the form $y = a \times b^x$ to the data, go to **Calculation** F1, **Exponential** F6 F3, then **ab^x** F2.
- The equation is $y = 1.553 \times 1.218^x$.



- Select **Draw** F6 to view the graph.



 ISBN 978-1-009-53719-3

Summary 11F

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then

$$A = A_0 b^t$$

where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

Skill-sheet

Exercise 11F

Example 23

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

Example 24

2 The half-life of plutonium-239 is 24 000 years. If 10 grams are present now, how long will it take until only 10% of the original sample remains? (Give your answer to the nearest year.)

3 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

4 The pressure, P, in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Determine the pressure at a height of 5 km. (Give your answer to the nearest millibar.)

b Determine the height at which the pressure is 400 millibars. (Give your answer to the nearest metre.)

5 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

6 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 10^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Determine the temperature of the liquid after 15 minutes.

7 Iodine-131 is a radioactive isotope used in the treatment of the thyroid gland. It decays so that, after t days, 1 unit of the isotope is reduced to 0.9174^t units. How many days does it take for the amount to fall to less than 0.2 units?

CF

 ISBN 978-1-009-53719-3

Example 25

8 The populations (in millions), p and q, of two neighbouring American states, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since January 1 1950.

a Plot the graphs of both functions using a calculator.

b Determine when the population of state P is:

i equal to the population of state Q

ii twice the population of state Q.

Example 26

9 A tub of ice-cream is taken out of a freezer, which is kept at $-15°$C. The temperature, T°C, of the ice-cream at time t minutes is given by

$$T = -40 \times (0.98)^t + 25$$

a Determine the limiting value of T as $t \to \infty$. What does this value represent?

b Determine the value of t for which $T = 0$.

c Sketch the graph of T against t.

d Comment on this model.

10 An object is falling such that its speed, V m/s, at time t seconds is given by

$$V = 80(1 - 3^{-0.4t})$$

a Determine the limiting speed, V_∞ m/s, as $t \to \infty$.

b When does the object have speed 40 m/s?

c Sketch the graph of V against t.

11 A company begins an advertising campaign for a new perfume. The percentage, $p(t)$%, of the target market that buys the perfume after t days of the advertising campaign is modelled by

$$p(t) = 100(1 - 3^{-0.05t}), \quad t \ge 1$$

a What is the limiting value of $p(t)$ as $t \to \infty$?

b How many days does the advertising campaign need to run to have 75% of the target market buy the perfume?

c Sketch the graph of $p(t)$ against t.

Example 27

12 Each of the following pairs of points is known to lie on a curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Determine the values of a and b in each case.

a $(1, 15)$ and $(4, 1875)$

b $(2, 1)$ and $\left(5, \frac{1}{8}\right)$

c $\left(1, \frac{15}{2}\right)$ and $\left(\frac{1}{2}, \frac{5\sqrt{6}}{2}\right)$

13 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and five hours later there were 10 000 bacteria.

a Determine the values of a and b.

b Determine, to the nearest minute, when there were 5000 bacteria.

c Determine, to the nearest minute, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

Example 29 **14** Determine an exponential model of the form $y = ab^x$ to fit the following data:

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

15 Determine an exponential model of the form $p = ab^t$ to fit the following data:

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

16 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

CF

17 A sheet of paper 0.2 mm thick is cut in half, and one piece is stacked on top of the other.

CU

a If this process is repeated, complete the following table:

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	
⋮	⋮	⋮
10		

b Write down a formula which shows the relationship between T and n.

c Draw a graph of T against n for $n \leq 10$.

d What would be the total thickness, T, after 30 cuts?

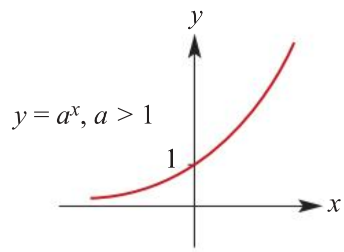
 ISBN 978-1-009-53719-3 Cambridge University Press

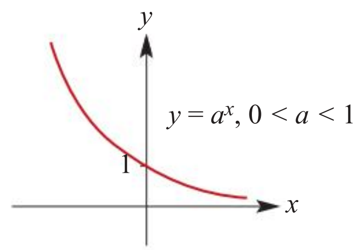
Chapter summary

- **Index laws**
  - To **multiply** two powers with the same base, **add** the indices: $a^m \times a^n = a^{m+n}$
  - To **divide** two powers with the same base, **subtract** the indices: $a^m \div a^n = a^{m-n}$
  - To raise a power to another power, **multiply** the indices: $(a^m)^n = a^{m \times n}$
- Rational indices:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{and} \quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

- A number is expressed in **standard form** or **scientific notation** when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 1.5×10^8
- Graphs of exponential functions:





- For $a \in \mathbb{R}^+ \setminus \{1\}$:

$$\text{if } a^x = a^y, \text{ then } x = y$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

11A **1** I can use the multiplication of powers index law to simplify expressions. ☐

See Example 1, Example 2 and Question 2

11A **2** I can use the division of powers index law to simplify expressions. ☐

See Example 2 and Question 3

11A **3** I can use the raising of powers index law to simplify expressions. ☐

See Example 3 and Question 4

11A **4** I can use prime decomposition to help simplify expressions. ☐

See Example 7 and Question 12

11B **5** I can simplify expressions involving rational powers. ☐

See Example 8, Example 9 and Questions 1 and 2.

11C **6** I can write numbers in scientific notation and undertake computations with numbers in scientific notation. ☐

See Example 10, Example 11 and Questions 1 and 5.

Review

11C **7** I can write numbers correct to a given number of significant figures. ☐

See Example 12 and Question 6

11D **8** I can plot the graphs of simple exponential graphs with and without a calculator. ☐

See Example 13, Example 14, Example 15, Example 16, and Questions 1, 2, 3 and 4

11D **9** I can sketch the graphs of transformations of the basic exponential graph. ☐

See Example 17, Example 18, and Question 5

11E **10** I can solve exponential equations and inequalities with and without a calculator. ☐

See Example 19, Example 20, Example 21, Example 22 and Questions 1, 4, 5 and 6.

11F **11** I can use exponential functions to model practical situations. ☐

See Example 23, Example 24, Example 25, Example 26, Example 27, Example 28 and Questions 1, 2, 8, 9 and 13

Technology-free questions

Technology-free short-response questions

SF

1 Simplify each of the following, expressing your answer with positive index:

a $\dfrac{a^6}{a^2}$ **b** $\dfrac{b^8}{b^{10}}$ **c** $\dfrac{m^3n^4}{m^5n^6}$ **d** $\dfrac{a^3b^2}{(ab^2)^4}$

e $\dfrac{6a^8}{4a^2}$ **f** $\dfrac{10a^7}{6a^9}$ **g** $\dfrac{8(a^3)^2}{(2a)^3}$ **h** $\dfrac{m^{-1}n^2}{(mn^{-2})^3}$

i $(p^{-1}q^{-2})^2$ **j** $\dfrac{(2a^{-4})^3}{5a^{-1}}$ **k** $\dfrac{6a^{-1}}{3a^{-2}}$ **l** $\dfrac{a^4+a^8}{a^2}$

2 Express the product $32 \times 10^{11} \times 12 \times 10^{-5}$ in scientific notation.

3 Evaluate each of the following:

a $25^{-\frac{3}{2}}$ **b** $27^{\frac{4}{3}}$ **c** $(-243)^{\frac{1}{5}}$ **d** $1000^{\frac{5}{3}}$ **e** $625^{\frac{3}{4}}$

4 Simplify each of the following

a $(16x^8y^4)^{\frac{1}{2}}$ **b** $\left(\dfrac{3}{4}\right)^8 \times \left(\dfrac{4}{3}\right)^5$ **c** $\left(\dfrac{5}{7}\right)^5 \times \left(\dfrac{5}{7}\right)^{-5}$

5 Solve each of the following equations for x:

a $2^{2x-1} = 16$ **b** $3^{5x-2} = 27$ **c** $2^{1-x} = 8$

d $2^{x+2} = \dfrac{1}{4}$ **e** $3^{3x-8} = 1$ **f** $5^{2x+1} = \dfrac{1}{5}$

Review

SF

6 Solve for x:

a $16^x = 64$ **b** $8^x = 32$ **c** $27^x = 81$ **d** $25^x = 125$

7 Solve each of the following for x:

a $3^x(3^x - 27) = 0$ **b** $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$ **d** $2^{2x} - 12 \times 2^x + 32 = 0$

8 Sketch the graph of:

a $y = 2 \times 2^x$ **b** $y = -3 \times 2^x$ **c** $y = 5 \times 2^{-x}$

d $y = 2^{-x} + 1$ **e** $y = 2^x - 1$ **f** $y = 2^x + 2$

9 State the range of each of the following exponential functions:

a $f(x) = 3 + 2^{2x}$ **b** $f(x) = 3(4 - 5^{2x})$

c $f(x) = -3(11 - 5^{-2x})$ **d** $f(x) = 6(2 + 7^{-3x})$

CF

10 The human body can produce 2.5 million red blood cells per second. If a person donates 500 mL of blood, how long will it take to replace the red blood cells if a litre of blood contains 5×10^{12} red blood cells?

11 The Sun is approximately 1.5×10^8 km from Earth and a comet is approximately 3×10^6 km from Earth. How many times further from Earth than the comet is the Sun?

12 Solve for x:

a $2^{2x} - 6 \times 2^x + 8 = 0$ **b** $2 \times 3^{2x+1} + 3^x - 1 = 0$ **c** $2 \times 4^{x+1} = 16^{2x}$

13 Simplify each of the following. In your answer use only positive indices. Here $m > 0$.

a $(27m^{-6})^{\frac{1}{3}} \times (64m^2)^{-\frac{1}{2}}$ **b** $(8m^6)^{\frac{1}{3}} \times (6m^2)^{\frac{1}{4}}$ **c** $\dfrac{5^{3x+1} - 5^{3x-1} + 24}{24 \times 5^{3x} + 120}$

14 Simplify $\dfrac{10 \times 5^{n+4} + 125 \times 5^{n+2}}{3 \times 5^{n+2} + 20 \times 5^{n+1}}$

15 Solve each of the following for x:

a $\left(\dfrac{4}{5}\right)^3 \times \left(\dfrac{4}{5}\right)^{-6} = \left(\dfrac{4}{5}\right)^{2x-1}$ **b** $\dfrac{5^{x+1} - 20}{10 + 5^x} = 3$

c $2^x \times 5^x = \dfrac{1}{100}\left(10^{x-1}\right)^4$ **d** $x^{\frac{3}{2}} - 8x^{-\frac{3}{2}} = 7$

16 Solve each pair of equations for x and y:

a $25^x = 125^y$, $16^x \div 8 = 2 \times 4^y$ **b** $3^{y-2} = 27^x$, $4^y \times 64^x = 128$

c $a^{2x} = a^{y-1}$, $b^{2+y} = b^{3x}$ where $a > 0, b > 0$

17 The initial population of a bacteria is 10. The population size at time t hours is modelled by $N(t) = 10 \times 2^{bt}$. The population doubles in 3 hours.

a Determine b.

b Use the model to determine the population after 9 hours.

Review

SF

Technology-active short-response questions

18 Determine approximate solutions to each of the following equations. Give your answers correct to two decimal places.

a $5^{x-3} = 4$ **b** $5 \times 2^{-3x} = 4$ **c** $2^{2x-3} = 5^{2-x}$

19 \$1000 is deposited in a a bank account with an annual compound interest rate of 6%. How long will it take until we have \$5000 in the account?

20 If we take an amount, Q, of a certain medication, the amount of it in our system, t hours after intake is

$$A(t) = Q \times 0.8^t$$

a Approximately what percentage of the medication is in our system 2 hours after taking it?

b How long until we have only 20% left in our system?

c How long until we have only 1% left in our system?

CF

21 The number of cells in a sample at time t hours is

$$N(t) = 50\,000 \times 1.6^{0.5t}.$$

a How many cells are there when $t = 4$?

b How long will it take for the sample to double from the time $t = 4$?

22 The initial population of a bacteria is 10. The population size at time t hours is modelled by

$$N(t) = 10 \times 2^{bt}.$$

The population doubles in 3 hours.

a Determine b.

b Use the model to determine the population after 7 hours. Give your answer in hours correct to one decimal place.

23 **a** Determine an exponential model of the form $y = a \cdot b^x$ to fit the following data:

x	0	2	4	5	10
y	2	5	13	20	200

b Express the model you have found in part a in the form $y = a10^{kx}$.

24 To control an advanced electronic machine, a total of 2187 different switch positions are required. There are two kinds of switches available:

Switch 1 These can be set in 9 different positions.

Switch 2 These can be set in 3 different positions.

If n of switch type 1 and $n + 1$ of switch type 2 are used, calculate the value of n that gives the required total number of switch positions.

CF

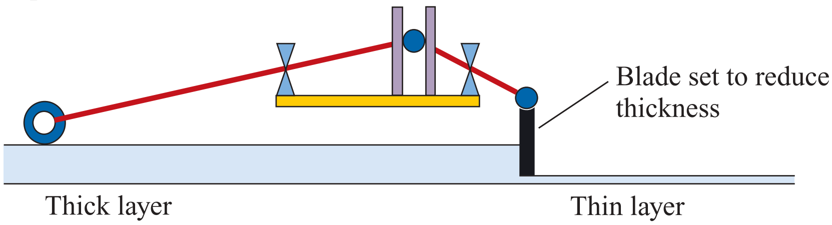
25 The gravitational force between two objects, F N, is given by the formula $F = \dfrac{6.67 \times 10^{-11} m_1 m_2}{r^2}$, where m_1 and m_2 are the masses (in kilograms) of the two objects and r is the distance (in metres) between them.

a What is the gravitational force between two objects each weighing 200 kg if they are 12 m apart? Express the answer in standard form (to two significant figures).

b The gravitational force between a planet and an object 6.4×10^6 m away from the centre of the planet is found to be 2.4×10^4 N. If the object has a mass of 1500 kg, calculate the approximate mass of the planet, giving the answer in standard form (to two significant figures).

CU

26 Research is being carried out to investigate the durability of paints of different thicknesses. The automatic machine shown in the diagram is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting, the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $(n - 1)$ times. At the end of this second stage, express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $(n - 3)$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

27 A hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves after use and then stores the dried tea in an airtight box. He estimates that on each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.
The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

a Write down expressions for the amount of caffeine when the tea leaves are re-used for the first, second, third and nth times.

b Do the same for the amount of tannin remaining.

c Determine the number of times he can re-use the tea leaves if a ‘tea’ containing more than three times as much tannin as caffeine is undrinkable.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

CU

28 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year while the second was losing brightness at the rate of 6% per year.

a Write down expressions for the brightness of each batch after n years.

b A person bought some carpet from the first batch when it was a year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

29 In the year 2000, in an African game park there were approximately 700 wildebeest and their population was increasing at 3% per year. At the same time, in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a calculator to plot the graphs of both functions.

a After how many years were there more wildebeest than zebras in the park?

b It is also estimated that there were 1000 antelope and their numbers were increasing by 50 per year. After how many years was the number of antelope greater than the number of zebras?

Multiple-choice questions

Technology-free multiple-choice questions

1 $8x^3 \div 4x^{-3}$ is equal to:

A 2 **B** $2x^0$ **C** $2x^6$ **D** $2x^{-1}$

2 The expression $\dfrac{a^2b}{(2ab^2)^3} \div \dfrac{ab}{16a^0}$ simplifies to:

A $\dfrac{2}{a^2b^6}$ **B** $\dfrac{2a^2}{b^6}$ **C** $2a^2b^6$ **D** $\dfrac{2}{ab^6}$

3 If $4^x = 10 - 4^{x+1}$, then x is equal to:

A 2 **B** $\dfrac{1}{4}$ **C** $\dfrac{1}{2}$ **D** 1

4 $\dfrac{7^{n+2} - 35 \times 7^{n-1}}{44 \times 7^{n+2}}$ is equal to:

A $\dfrac{1}{44}$ **B** $\dfrac{1}{49}$ **C** $\dfrac{1}{7}$ **D** 7

5 The function with rule $f(x) = 3 \times 2^x - 1$, for $x \in \mathbb{R}$, has range:

A $\mathbb{R}$ **B** $x \neq -1$ **C** $x > -1$ **D** $x > 1$

6 $\sqrt{x\sqrt{x\sqrt{x}}}$ is equal to:

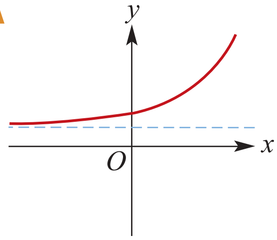
A $x^{\frac{1}{2}}$ **B** $x^{\frac{3}{4}}$ **C** $x^{\frac{5}{8}}$ **D** $x^{\frac{7}{8}}$

 ISBN 978-1-009-53719-3

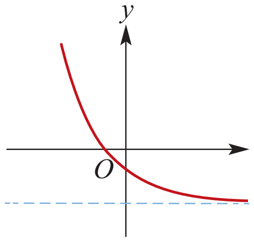
Review

7 Which of the following graphs could be the graph of the function $f(x) = 2^{ax} + b$, where a and b are positive?

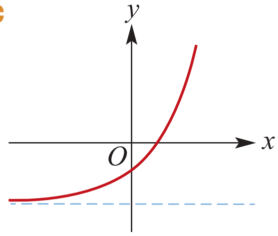
A



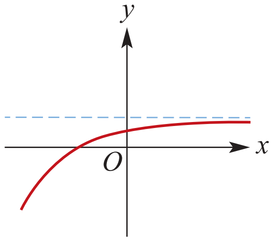
B



C



D



8 The number, $Q(t)$, of small insects alive in a given area after t weeks is given by $Q(t) = 1024 \times 2^{\frac{t}{50}}, t \geq 0$.The time in weeks for the number of insects to double from the number when $t = 0$ is:

A 100 **B** 50 **C** 2 **D** 200

9 The mass, M milligrams of a radioactive substance after time t years is given by the function $M(t) = 210 \times 3^{-0.001t}$, $t \geq 0$. After what time in years is the mass 70 milligrams?

A 10 **B** 300 **C** 1000 **D** 100

10 The expression $\dfrac{2mh}{(3mh^2)^3} \div \dfrac{mh}{81m^2}$ is equal to:

A $\dfrac{6}{mh^6}$ **B** $\dfrac{6m^2}{h^6}$ **C** $6m^2h^6$ **D** $\dfrac{6}{m^2h^6}$

11 The graph of $y = a \times b^x$ passes through the points with coordinates $(3, 30)$ and $(6, 240)$. The values of a and b are:

A $a = 3, b = 6$ **B** $a = 3, b = 24$ **C** $a = \dfrac{15}{4}, b = 2$ **D** $a = \dfrac{30}{7}, b = 2$

Technology-active multiple-choice questions

12 The solution of the equation $3x = 10^{-0.3x}$ is closest to:

A 0.83 **B** 0.28 **C** 0 **D** 0.30

13 The solution of the equation $2^{x-4} - 3 = 0$, correct to three decimal places, is:

A 4.585 **B** 5.584 **C** 3.623 **D** 5.585

 ISBN 978-1-009-53719-3 Cambridge University Press

14 The population on January 1st 2023 for a town A is 1500. The population is said to be increasing at a rate of 2.3% every 5 years. What is the projected population of the town A on Jan 1st 2038? The answers are given to the nearest whole number.

A 1606 **B** 1570 **C** 2967 **D** 3623

15 The half-life of carbon-14 is 5730 years. A cat died in 1850. What percentage of its original carbon-14 remained in 2025 when it was found by scientists?

A 85% **B** 15% **C** 92% **D** 98%

16 When a liquid is placed in a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 2^{-kt}$, where k is a real constant. The temperature of the liquid is initially 75°C and drops to a temperature of 35°C when $t = 8$. The temperature of the liquid when $t = 15$ is:

A 24.97°C **B** 32.67°C **C** 28.65°C **D** 17.97°C

17 The otter population on an island increases by 16% per year. How many full years will it take the population to double?

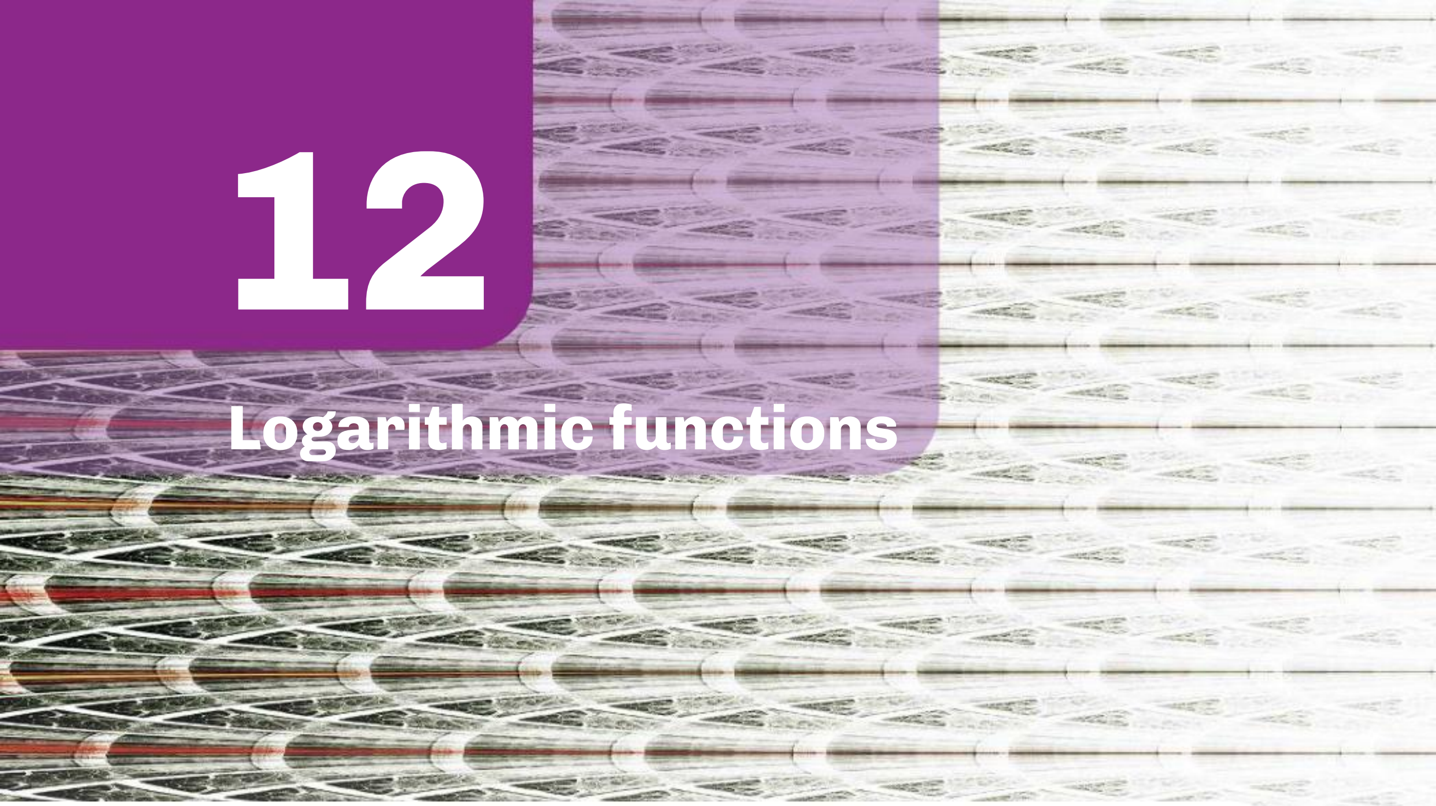
A 3 **B** 4 **C** 5 **D** 6

18 The number of bacteria in the early stages of a growing colony of bacteria can be approximated using the function $N(t) = 450 \times 3.5^{0.065t}$ where t is the time, measured in hours, since the colony started to grow, and $N(t)$ is the number of bacteria in the colony at time t. The time, correct to one decimal place, taken in hours for the population to double is

A 8.5 **B** 6.7 **C** 11.2 **D** 10.5

19 The number of bacteria in the early stages of a first growing colony of bacteria can be approximated using the function $N(t) = 450 \times 3.5^{0.065t}$ The number of bacteria in the early stages of a second colony of bacteria can be approximated using the function $P(t) = 220 \times 3.5^{0.17t}$ where $P(t)$ is the number of bacteria and t is measured in hours. Assume that both colonies start growing at the same time. The time, to the nearest hour, at which the number of bacteria in both colonies will be equal is:

A 3 **B** 4 **C** 5 **D** 6



12 Logarithmic functions

Chapter contents

We define the logarithm of a number in terms of an equivalent index statement. Consider the statement: $2^3 = 8$.
This may be written in an alternative form: $\log_2 8 = 3$.
This is read as ‘the logarithm of 8 to the base 2 is equal to 3’.

> For $a > 1$, the **logarithm function** with base a is defined as follows:
>
> $a^x = y$ is equivalent to $\log_a y = x$

In fact this definition works for any positive number $a \neq 1$ but in this course we restrict our attention to $a > 1$. Similarly:

- $\log_2 16 = 4 \Leftrightarrow 2^4 = 16$
- $\log_3 27 = 3 \Leftrightarrow 3^3 = 27$
- $\log_4 16 = 2 \Leftrightarrow 4^2 = 16$

Before the invention of computers and calculators, logarithms were very important tools for computation. Today, they are used widely in science and you will see some applications in the final section of this chapter. They also have an important role in integral calculus which you will meet in Mathematical Methods units 3 and 4.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

12A Logarithms

Learning intentions

- To be able to evaluate **logarithmic expressions** and use the **logarithm laws**.

We restate the definition.

For a real number $a > 1$, the **logarithm function** with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

The logarithm function with base a is the 'reverse' of the exponential function with base a. We will discuss this in Section 12C.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 1

Without the aid of a calculator, evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

Note: To determine $\log_2 32$, we ask 'What power of 2 gives 32?'
To determine $\log_3 81$, we ask 'What power of 3 gives 81?'

Laws of logarithms

The index laws are used to establish rules for computations with logarithms.

Law 1: Logarithm of a product

The logarithm of a product of two numbers is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using index law 1)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

 ISBN 978-1-009-53719-3

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000\\ &= 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient of two numbers is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using index law 2)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4\\ &= 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p\log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using index law 3)}$$

Hence $\log_a(m^p) = xp = p\log_a m$.

For example: $\log_2 32 = \log_2(2^5) = 5$.

Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example: $\log_a(\frac{1}{2}) = \log_a(2^{-1}) = -\log_a 2$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Law 5

$\log_a 1 = 0$ and $\log_a a = 1$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$. Since $a^1 = a$, we have $\log_a a = 1$.

Finally from the definition of logarithm we can see that for $a > 1$

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = a$ for all $x > 0$

Example 2

Without using a calculator, simplify the following:

$$2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned}
2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right) &= \log_{10}(3^2) + \log_{10} 16 - \log_{10}\left(\frac{6}{5}\right)^2 \\
&= \log_{10} 9 + \log_{10} 16 - \log_{10}\left(\frac{36}{25}\right) \\
&= \log_{10}\left(9 \times 16 \times \frac{25}{36}\right) \\
&= \log_{10} 100 \\
&= 2
\end{aligned}$$

Example 3

Solve each of the following equations for x:

a $\log_5 x = 3$

b $\log_5(2x + 1) = 2$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

Solution

a $\log_5 x = 3 \Leftrightarrow x = 5^3 = 125$

b $\log_5(2x + 1) = 2 \Leftrightarrow 2x + 1 = 5^2$

$$\begin{aligned}
\therefore\ 2x + 1 &= 25 \\
2x &= 24 \\
x &= 12
\end{aligned}$$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

$$\begin{aligned}
\log_2\left(\frac{2x+1}{x-1}\right) &= 4 \\
\therefore\ \frac{2x+1}{x-1} &= 2^4 \\
2x + 1 &= 16(x - 1) \\
17 &= 14x \\
x &= \frac{17}{14}
\end{aligned}$$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

$$\begin{aligned}
\log_3[(x - 1)(x + 1)] &= 1 \\
\log_3(x^2 - 1) &= 1 \\
x^2 - 1 &= 3 \\
x &= \pm 2
\end{aligned}$$

But the original expression is not defined for $x = -2$, and therefore $x = 2$.

ISBN 978-1-009-53719-3

Summary 12A

- For $a > 1$, the logarithm function base a is defined as follows:

  $a^x = y$ is equivalent to $\log_a y = x$

- The expression $\log_a y$ is defined for all positive real numbers y.
- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- **Laws of logarithms**

  1 $\log_a(mn) = \log_a m + \log_a n$

  2 $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

  3 $\log_a(m^p) = p\log_a m$

  4 $\log_a(m^{-1}) = -\log_a m$

  5 $\log_a 1 = 0$ and $\log_a a = 1$

Skill-sheet

Exercise 12A

SF

Example 1

1 Without using a calculator, evaluate the following:

a $\log_2 128$ **b** $\log_3 81$ **c** $\log_5 125$ **d** $\log_{10} 0.1$

2 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $\log_2 10 + \log_2 a$ **b** $\log_{10} 5 + \log_{10} 2$ Law 1

c $\log_2 9 - \log_2 4$ **d** $\log_2 10 - \log_2 5$ Law 2

e $\log_2(a^3)$ **f** $\log_2(8^3)$ Law 3

g $\log_5\left(\dfrac{1}{6}\right)$ **h** $\log_5\left(\dfrac{1}{25}\right)$ Law 4

3 Without using a calculator, evaluate each of the following:

a $\log_3 27$ **b** $\log_5 625$

c $\log_2\left(\dfrac{1}{128}\right)$ **d** $\log_4\left(\dfrac{1}{64}\right)$

e $\log_x(x^4)$ **f** $\log_2 0.125$

g $\log_{10} 10\,000$ **h** $\log_{10} 0.000\,001$

i $-3\log_5 125$ **j** $-4\log_{16} 2$

k $2\log_3 9$ **l** $-4\log_{16} 4$

Example 2

4 Without using a calculator, simplify each of the following:

a $\frac{1}{2}\log_{10} 16 + 2\log_{10} 5$ **b** $\log_2 16 + \log_2 8$

c $\log_2 128 + \log_3 45 - \log_3 5$ **d** $\log_4 32 - \log_9 27$

e $\log_b(b^3) - \log_b \sqrt{b}$ **f** $2\log_x a + \log_x(a^3)$

g $x\log_2 8 + \log_2(8^{1-x})$ **h** $\frac{3}{2}\log_a a - \log_a \sqrt{a}$

Example 3

5 Solve for x:

a $\log_3 9 = x$
b $\log_3 x = 3$
c $\log_5 x = -3$
d $\log_{10} x = \log_{10} 4 + \log_{10} 2$
e $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$
f $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$
g $\log_x 64 = 2$
h $\log_5(2x-3) = 3$
i $\log_3(x+2) - \log_3 2 = 1$
j $\log_x 0.01 = -2$

SF

6 Solve each of the following for x:

a $\log_x\left(\frac{1}{25}\right) = -2$
b $\log_4(2x-1) = 3$
c $\log_4(x+2) - \log_4 6 = 1$
d $\log_4(3x+4) + \log_4 16 = 5$
e $\log_3(x^2 - 3x - 1) = 0$
f $\log_3(x^2 - 3x + 1) = 0$

CF

7 If $\log_{10} x = a$ and $\log_{10} y = c$, express $\log_{10}\left(\frac{100x^3y^{-\frac{1}{2}}}{y^2}\right)$ in terms of a and c.

8 Prove that $\log_{10}\left(\frac{ab^2}{c}\right) + \log_{10}\left(\frac{c^2}{ab}\right) - \log_{10}(bc) = 0$.

9 If $\log_a\left(\frac{11}{3}\right) + \log_a\left(\frac{490}{297}\right) - 2\log_a\left(\frac{7}{9}\right) = \log_a k$, determine k.

10 Solve each of the following equations for x:

a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$
b $\log_{10}(5x) - \log_{10}(3-2x) = 1$
c $3\log_{10}(x-1) = \log_{10} 8$
d $\log_{10}(20x) - \log_{10}(x-8) = 2$
e $2\log_{10} 5 + \log_{10}(x+1) = 1 + \log_{10}(2x+7)$
f $1 + 2\log_{10}(x+1) = \log_{10}(2x+1) + \log_{10}(5x+8)$

12B Solving exponential equations and inequalities with logarithms

Learning intentions

- To be able to solve exponential equations using **logarithmic methods**.

If $a > 1$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations. We can obtain exact solutions to exponential equations in terms of logarithms.

Example 4

Solve for x if $2^x = 11$.

Solution

We immediately see that $x = \log_2 11$.

This can be evaluated with a calculator:

$$x = \log_2 11 \approx 3.45943$$

 ISBN 978-1-009-53719-3

Example 5

Solve $3^{2x-1} = 28$ using logarithms base 3.

Solution

$$\begin{aligned} 2x - 1 &= \log_3 28 \\ 2x &= 1 + \log_3 28 \\ x &= \frac{1 + \log_3 28}{2} \\ &\approx 2.017 \qquad \text{(to three decimal places)} \end{aligned}$$

Example 6

Solve the inequality $0.7^x \geq 0.3$ using logarithms base 10.

Solution

Taking $\log_{10}$ of both sides:

$$\begin{aligned} \log_{10}(0.7^x) &\geq \log_{10} 0.3 \\ x \log_{10} 0.7 &\geq \log_{10} 0.3 \\ x &\leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \qquad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0) \\ x &\leq 3.376 \qquad \text{(to three decimal places)} \end{aligned}$$

Exponential graphs revisited

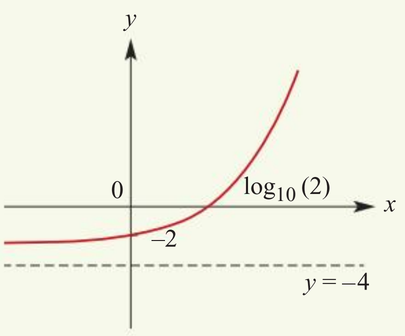
In Section 11D we graphed exponential functions, but often we could not determine the x-axis intercept. Now that we have defined logarithms this can be done.

Example 7

Sketch the graph of $f(x) = 2 \times 10^x - 4$, giving the equation of the asymptote and the axis intercepts.

Solution

- As $x \to -\infty$, $y \to -4^+$. The equation of the horizontal asymptote is $y = -4$.
- The graph crosses the y-axis when $x = 0$, and so the y-axis intercept is given by $f(0) = 2 \times 10^0 - 4 = 2 - 4 = -2$.
- The graph crosses the x-axis when $f(x) = 0$:



$$\begin{aligned} 2 \times 10^x &= 4 \\ 10^x &= 2 \\ x &= \log_{10} 2 \approx 0.3010 \qquad \text{(correct to four decimal places)} \end{aligned}$$

 ISBN 978-1-009-53719-3

Example 8

Giving exact values, determine the coordinates of the intercept with the x-axis and intercept with the y-axis of the graphs of each of the following functions:

a $f(x) = 3 - 5^{x-2}$ **b** $f(x) = 4 - 6^{4x+1}$

Solution

a ***y*-axis intercept**

$$f(0) = 3 - 5^{-2}$$
$$= 3 - \frac{1}{25}$$
$$= \frac{74}{25}$$

Coordinates: $\left(0, \frac{74}{25}\right)$

***x*-axis intercept**

$$3 - 5^{x-2} = 0$$
$$5^{x-2} = 3$$
$$x - 2 = \log_5(3)$$
$$x = 2 + \log_5(3)$$

Coordinates: $(2 + \log_5(3), 0)$

b ***y*-axis intercept**

$$f(0) = 4 - 6^1$$
$$= -2$$

Coordinates: $(0, -2)$

***x*-axis intercept**

$$4 - 6^{4x+1} = 0$$
$$6^{4x+1} = 4$$
$$4x + 1 = \log_6(4)$$
$$4x = \log_6(4) - 1$$
$$x = \frac{1}{4}(\log_6(4) - 1)$$

Coordinates: $\left(\frac{1}{4}(\log_6(4) - 1), 0\right)$

Summary 12B

- If a is a real number greater than one and $x \in \mathbb{R}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities.

  For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \geq 5 \Leftrightarrow x \geq \log_2 5$
- The x-axis intercepts of exponential graphs can now be found.

Skill-sheet

Exercise 12B

SF

Example 4

1 Using logarithms solve each of the following equations giving both the exact answer and the approximate answer correct to two decimal places:

a $2^x = 7$ **b** $2^x = 0.4$ **c** $3^x = 14$

d $4^x = 3$ **e** $2^{-x} = 6$ **f** $0.3^x = 2$

Example 5

2 Using logarithms solve each of the following equations correct to two decimal places:

a $5^{2x-1} = 90$ **b** $3^{x-1} = 10$ **c** $2^{2x+1} = 0.6$

Example 6

3 Solve for x. Give the exact answer and the approximate answer giving values correct to two decimal places if necessary.

a $2^x > 8$ **b** $3^x < 5$ **c** $3^{x-1} \leq 7$

 ISBN 978-1-009-53719-3

SF

Example 7 **4** For each of the following, sketch the graph of $y = f(x)$, giving the equation of the asymptote and the axis intercepts:

a $f(x) = 2^x - 4$ **b** $f(x) = 2 \times 3^x - 6$ **c** $f(x) = 3 \times 10^x - 5$
d $f(x) = -2 \times 10^x + 4$ **e** $f(x) = -3 \times 2^x + 6$ **f** $f(x) = 5 \times 2^x - 6$

Example 8 **5** Giving exact values, determine the coordinates of the intercept with the x-axis and intercept with the y-axis of the graphs of each of the following functions:

a $f(x) = 5 - 6^{x-3}$ **b** $f(x) = 4 - 2^{5x+2}$ **c** $f(x) = -6 + 2^{3-x}$

6 Determine a and k such that the graph of $y = a \times 10^{kx}$ passes through the points $(2, 6)$ and $(5, 20)$.

12C Graphs of logarithmic functions

Learning intentions

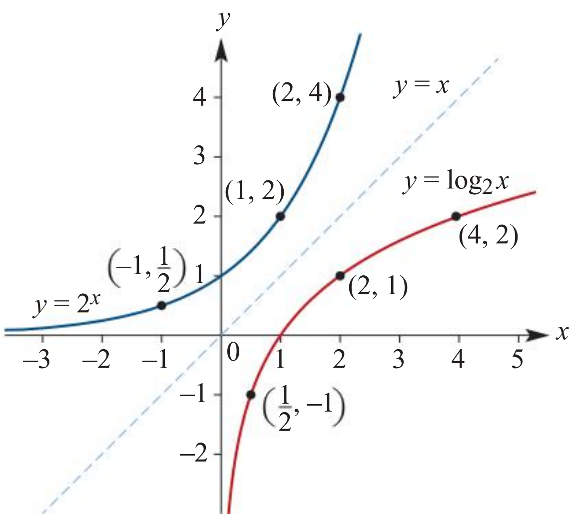
- To be able to sketch graphs of functions of the form $y = \log_a x$ and simple transformations of these.

Graph of $y = \log_a x$ when $a > 1$

$y = \log_2 x$

A table of values for $y = \log_2 x$ is given below, and the graphs of $y = 2^x$ and $y = \log_2 x$ are drawn on the one set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$	−2	−1	0	1	2	3



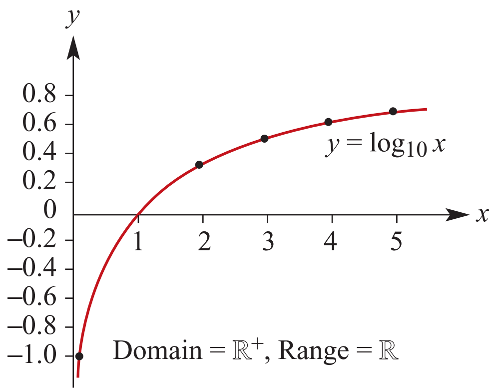
Notes:

- The point $(1, 2)$ is on the graph of $y = 2^x$ and the point $(2, 1)$ is on the graph of $y = \log_2 x$.
- The point $(2, 4)$ is on the graph of $y = 2^x$ and the point $(4, 2)$ is on the graph of $y = \log_2 x$.
- The graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$.

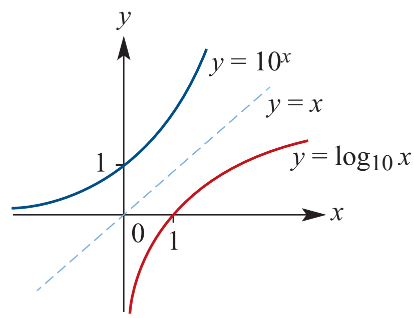
$y = \log_{10} x$

A table of values for $y = \log_{10} x$ is given below (the values are correct to two decimal places). Use your calculator to check these values.

x	0.1	1	2	3	4	5
$y = \log_{10} x$	-1	0	≈ 0.30	≈ 0.48	≈ 0.60	≈ 0.70



Domain = $\mathbb{R}^+$, Range = $\mathbb{R}$



The graph of $y = \log_{10} x$ is the reflection in the line $y = x$ of the graph of $y = 10^x$.

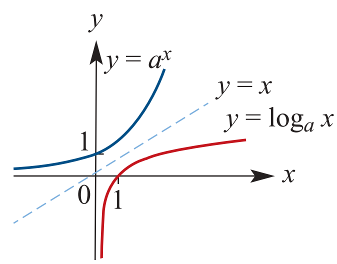
In general

Properties of $y = a^x$, $a > 1$

- domain = $\mathbb{R}$
- range = $\mathbb{R}^+$
- $a^0 = 1$
- as $x \to -\infty$, $y \to 0^+$
- $y = 0$ is an asymptote

Properties of $y = \log_a x$, $a > 1$

- domain = $\mathbb{R}^+$
- range = $\mathbb{R}$
- $\log_a 1 = 0$
- as $x \to 0^+$, $y \to -\infty$
- $x = 0$ is an asymptote



Inverse relationship between logarithms and exponentials

Let $a > 1$. The function with rule $y = \log_a x$, for $x > 0$, and the function with rule $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

For example:

- $\log_2(2^3) = 3$ and $2^{\log_2 3} = 3$
- $\log_{10}(10^5) = 5$ and $10^{\log_{10} 5} = 5$

For $a > 1$:

$f(x) = \log_a x$, $x > 0$ and $g(x) = a^x$ are **inverse functions**. The above relations can be written as

- $f(g(x)) = x$ for all x
- $g(f(x)) = x$ for $x > 0$.

 ISBN 978-1-009-53719-3

Transformations of logarithm graphs

Transformations can be applied to the graphs of logarithm functions. This is shown in the following example.

Example 9

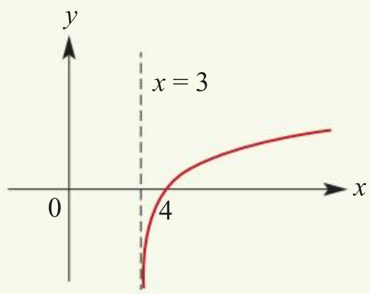
Sketch the graph of each of the following. Give the maximal domain, the equation of the asymptote and the axis intercepts.

a $f(x) = \log_2(x - 3)$ **b** $f(x) = \log_2(x + 2)$ **c** $f(x) = \log_2(3x)$

d $f(x) = -\log_2(x)$ **e** $f(x) = \log_2(-x)$

Solution

a $f(x) = \log_2(x - 3)$



- The maximal domain is $(3, \infty)$.
- Equation of asymptote is $x = 3$.
- x-axis intercept: $\log_2(x - 3) = 0$ implies $x - 3 = 2^0$, i.e. $x = 4$.

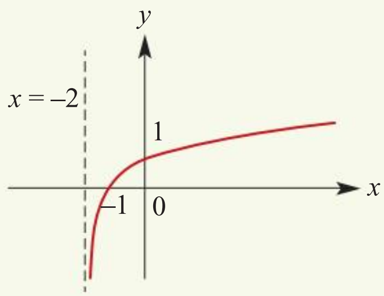
Explanation

The graph of $y = \log_2 x$ is translated 3 units in the positive direction of the x-axis.

Maximal domain: For $f(x) = \log_2(x - 3)$ to be defined, we need $x - 3 > 0$, i.e. $x > 3$.

Asymptote: As $x \to 3^+$, $y \to -\infty$.

b $f(x) = \log_2(x + 2)$



- The maximal domain is $(-2, \infty)$.
- Equation of asymptote is $x = -2$.
- y-axis intercept: $f(0) = \log_2(2) = 1$.
- x-axis intercept: $\log_2(x + 2) = 0$ implies $x + 2 = 2^0$, i.e. $x = -1$.

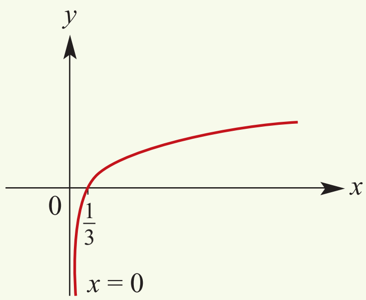
The graph of $y = \log_2 x$ is translated 2 units in the negative direction of the x-axis.

Maximal domain: For $f(x) = \log_2(x + 2)$ to be defined, we need $x + 2 > 0$, i.e. $x > -2$.

Asymptote: As $x \to -2^+$, $y \to -\infty$.

ISBN 978-1-009-53719-3

c $f(x) = \log_2(3x)$



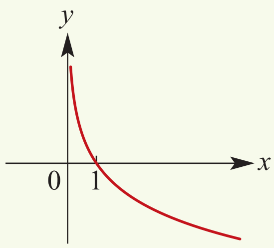
- The maximal domain is $(0, \infty)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $\log_2(3x) = 0$ implies $3x = 2^0$, i.e. $x = \frac{1}{3}$.

The graph of $y = \log_2 x$ is dilated by a factor of $\frac{1}{3}$ from the y-axis.

Maximal domain: For $f(x) = \log_2(3x)$ to be defined, we need $3x > 0$, i.e. $x > 0$.

Asymptote: As $x \to 0^+$, $y \to -\infty$.

d $f(x) = -\log_2(x)$



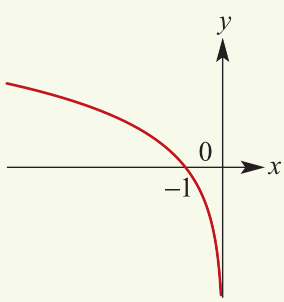
- The maximal domain is $(0, \infty)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $-\log_2(x) = 0$ implies $x = 2^0 = 1$.

The graph of $y = \log_2 x$ is reflected in the x-axis.

Maximal domain: For $f(x) = -\log_2(x)$ to be defined, we need $x > 0$.

Asymptote: As $x \to 0^+$, $y \to \infty$.

e $f(x) = \log_2(-x)$



- The maximal domain is $(-\infty, 0)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $\log_2(-x) = 0$ implies $-x = 2^0$, i.e. $x = -1$.

The graph of $y = \log_2 x$ is reflected in the y-axis.

Maximal domain: For $f(x) = \log_2(-x)$ to be defined, we need $-x > 0$, i.e. $x < 0$.

Asymptote: As $x \to 0^-$, $y \to -\infty$.

Exponential and logarithmic graphs with different bases

It is often useful to know how to go from one base to another.

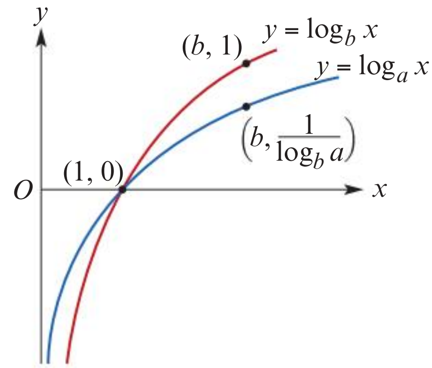
To change the base of $\log_a x$ from a to b (where $a, b > 0$ and $a, b \neq 1$), we use the definition that $y = \log_a x$ implies $a^y = x$. Taking $\log_b$ of both sides:

$$\begin{aligned} \log_b(a^y) &= \log_b x \\ y \log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

Since $y = \log_a x$, this gives:

$$\boxed{\log_a x = \frac{\log_b x}{\log_b a}}$$

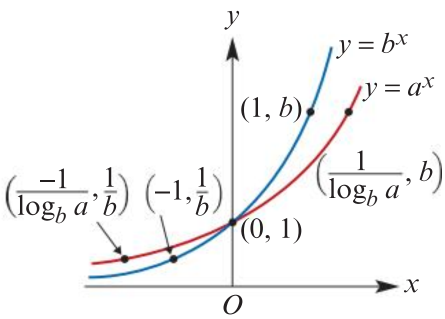
Hence the graph of $y = \log_a x$ can be obtained from the graph of $y = \log_b x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the x-axis.



Using properties of inverses, we can write $a = b^{\log_b a}$. This gives:

$$\boxed{a^x = b^{(\log_b a)x}}$$

Hence the graph of $y = a^x$ can be obtained from the graph of $y = b^x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the y-axis.



Example 10

Determine a transformation that takes the graph of $y = 2^x$ to the graph of $y = 10^x$.

Solution

We can write $10 = 2^{\log_2 10}$ and so

$$\begin{aligned} 10^x &= \left(2^{\log_2 10}\right)^x \\ &= 2^{(\log_2 10)x} \end{aligned}$$

The graph of $y = 10^x$ is the image of the graph of $y = 2^x$ under a dilation of factor $\dfrac{1}{\log_2 10}$ from the y-axis.

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 12C

Let a be a real number greater than one.

- The graph of $y = \log_a x$ is the reflection of the graph of $y = a^x$ in the line $y = x$.
- **Inverse relationship**
  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = a$ for all $x > 0$
- **Change of base**
  - $\log_a x = \dfrac{\log_b x}{\log_b a}$
  - $a^x = b^{(\log_b a)x}$

Exercise 12C

SF

1 Sketch the graph of each of the following and state the domain and range for each:

a $y = \log_{10}(2x)$ **b** $y = 2\log_{10} x$ **c** $y = \log_{10}(\frac{1}{2}x)$

d $y = 2\log_{10}(3x)$ **e** $y = -\log_{10} x$ **f** $y = \log_{10}(-x)$

Example 9

2 Sketch the graph of each of the following. Give the maximal domain, the equation of the asymptote and the axis intercepts.

a $f(x) = \log_2(x - 4)$ **b** $f(x) = \log_2(x + 3)$ **c** $f(x) = \log_2(2x)$

d $f(x) = \log_2(x + 2)$ **e** $f(x) = \log_2\left(\dfrac{x}{3}\right)$ **f** $f(x) = \log_2(-2x)$

3 Sketch the graph of each of the following. Label the axis intercepts and asymptotes. State the implied domain of each function.

a $y = \log_2(2x)$ **b** $y = \log_{10}(x - 5)$

c $y = -\log_{10} x$ **d** $y = \log_{10}(-x)$

e $y = \log_{10}(5 - x)$ **f** $y = 2\log_2(2x) + 2$

g $y = -2\log_2(3x)$ **h** $y = \log_{10}(-x - 5) + 2$

4 Use a calculator to solve each of the following equations correct to two decimal places:

a $2^{-x} = x$ **b** $\log_{10}(x) + x = 0$

5 Use a calculator to plot the graphs of $y = \log_{10}(x^2)$ and $y = 2\log_{10} x$ for $x \in [-10, 10]$, $x \neq 0$.

6 On the one set of axes plot the graphs of $y = \log_{10}(\sqrt{x})$ and $y = \frac{1}{2}\log_{10} x$ for $x \in (0, 10]$.

7 Use a calculator to plot the graphs of $y = \log_{10}(2x) + \log_{10}(3x)$ and $y = \log_{10}(6x^2)$.

Example 10

8 Determine a transformation that takes the graph of $y = 2^x$ to the graph of $y = 3^x$.

9 Determine a transformation that takes the graph of $y = 10^x$ to the graph of $y = 2^x$.

 ISBN 978-1-009-53719-3

12D Transposing equations

Learning intentions

- To be able to transform equations involving exponential and logarithmic functions.

The inverse relationship between logarithms and exponentials can be used when solving and rearranging equations.

Example 11

Rewrite the equation $y = 2\log_2(x) + 3$ with x as the subject.

Solution

$$y = 2\log_2(x) + 3$$

$$\frac{y-3}{2} = \log_2(x)$$

$$\therefore \quad x = 2^{\frac{y-3}{2}}$$

Example 12

Rearrange the equation $x = 2\log_3(y - 1) + 5$ to express y as a function of x.

Solution

$$x = 2\log_3(y-1) + 5$$

$$\frac{x-5}{2} = \log_3(y-1)$$

$$y - 1 = 3^{\frac{x-5}{2}}$$

$$\therefore \quad y = 3^{\frac{x-5}{2}} + 1$$

Example 13

Rewrite the equation $P = A \times 10^{kt}$ with t as the subject.

Solution

$$P = A \times 10^{kt}$$

Take logarithms with base 10 of both sides:

$$\begin{aligned}\log_{10} P &= \log_{10}(A \times 10^{kt})\\ &= \log_{10} A + \log_{10}(10^{kt})\\ &= \log_{10} A + kt\end{aligned}$$

$$\begin{aligned}\therefore \quad t &= \frac{1}{k}(\log_{10} P - \log_{10} A)\\ &= \frac{1}{k}\log_{10}\left(\frac{P}{A}\right)\end{aligned}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 12D

The logarithm laws and the properties of inverse functions are used to transform equations.

Exercise 12D

Example 11 SF

1 For each of the following functions $y = f(x)$, rearrange the rule to show that x can be expressed as a function of y:

a $y = \log_5(2x)$ **b** $y = 3\log_5(2x) + 1$ **c** $y = 2^x + 2$
d $y = 2^{x+2}$ **e** $y = \log_7(2x + 1)$ **f** $y = 4\log_7(3x + 2)$
g $y = \log_{10}(x + 1)$ **h** $y = 2 \times 5^{x-1}$

2 Rewrite the equation $P = A \times 2^{-kt} + b$ with t as the subject.

Example 12, 13

3 For each of the following formulas, make the pronumeral in brackets the subject:

a $y = 2\log_2(x) + 5$ (x) **b** $P = A \times 5^{-6x}$ (x)
c $y = ax^n$ (n) **d** $y = 5 \times 10^x$ (x)
e $y = 5 - 3\log_3(2x)$ (x) **f** $y = 6x^{2n}$ (n)
g $y = \log_4(2x - 1)$ (x) **h** $y = 5(1 - 6^{-x})$ (x)

4 **a** If $a\log_2 7 = 3 - \log_6 14$, determine the value of a, correct to three significant figures.
b If $\log_3 18 = \log_{11} k$, determine the value of k, correct to one decimal place.

CF

5 Prove that if $\log_r p = q$ and $\log_q r = p$, then $\log_q p = pq$.

6 If $u = \log_9 x$, determine in terms of u:

a x **b** $\log_9(3x)$ **c** $\log_x 81$

7 Solve the equation $\log_5 x = 16\log_x 5$.

8 Given that $q^p = 25$, determine $\log_5 q$ in terms of p.

9 **a** Prove that $\log_a x = \dfrac{1}{\log_x a}$.
b Hence solve $\log_x 8 + \log_8 x = \dfrac{13}{6}$ for x, where x is an integer greater than 1.

12E Applications of logarithmic functions

Learning intentions

▶ To be able to use a range of **logarithmic models**.

A **logarithmic scale** is a scale of measurement that uses the logarithm of a quantity. Familiar examples of logarithmic scales include the Richter scale (earthquakes), decibels (noise) and pH (acidity). In this section, we will show why such scales are useful and study these three examples. We will also look at some other applications of logarithmic functions.

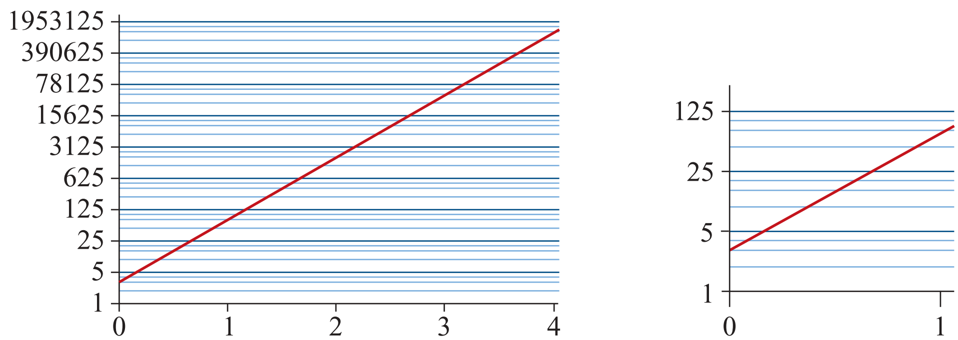
ISBN 978-1-009-53719-3

Charts with logarithmic scales

In the chart below we show an example for which the vertical axis has equally spaced increments that are labelled $1 = 5^0, 5 = 5^1, 25 = 5^2, 125 = 5^3, \ldots$ instead of $0, 1, 2, 3, \ldots$.

The major horizontal grid lines, which are equally spaced, are the integer powers of 5.

The straight line is the result of graphing $y = 3 \times 5^{2x}$ but with the vertical axis having a logarithmic scale (base 5).



The bottom-left corner of the chart is shown 'blown up' on the right.

- The horizontal grid lines between 1 and 5 represent 2, 3 and 4. Their positions are determined by $\log_5 2$, $\log_5 3$ and $\log_5 4$.
- The horizontal grid lines between 5 and 25 represent 10, 15 and 20. Their positions are determined by $\log_5 10$, $\log_5 15$ and $\log_5 20$.
- The horizontal grid lines between 25 and 125 represent 50, 75 and 100. Their positions are determined by $\log_5 50$, $\log_5 75$ and $\log_5 100$.

Notice that there is the same gap between the grid lines for 1 and 2, for 5 and 10, and for 25 and 50. This is because

$$\log_5 2 - \log_5 1 = \log_5 2 \qquad \log_5 10 - \log_5 5 = \log_5 2 \qquad \log_5 50 - \log_5 25 = \log_5 2$$

This is not what we are used to from working with linear scales.

- For a **linear scale**, the change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales to measure temperature (degrees Celsius) and length (metres) and in standard Cartesian graphs.
- For a **logarithmic scale**, the change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1 : 2) would be perceived as the same amount of increase as a change from 6 to 12 (also a ratio of 1 : 2).

The ratio property of logarithmic scales is explained by the logarithm law

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

The difference between $\log_a x$ and $\log_a y$ depends only on the ratio $x : y$.

Presentation of data on a logarithmic scale can be helpful when the data covers a large range of values. The use of the logarithms of the values rather than the actual values reduces a wide range to a more manageable size.

 ISBN 978-1-009-53719-3 Cambridge University Press

Graphing with logarithmic scales

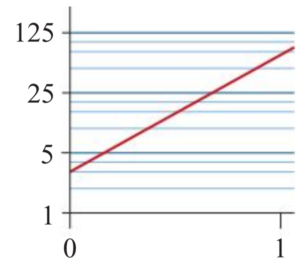
We will see how to sketch the graph of an exponential function using a logarithmic scale for the vertical axis and a linear scale for the horizontal axis.

Let $y = b \times a^{mx}$, where a, b and m are positive real numbers. Then taking logarithms base a of both sides gives:

$$\log_a y = \log_a b + mx$$

So the graph of $\log_a y$ against x is a straight line with gradient m and with ($\log_a y$)-axis intercept at $(0, \log_a b)$.

In the case of the exponential function $y = 3 \times 5^{2x}$, the equation for the straight line becomes $\log_5 y = \log_5 3 + 2x$. On the chart, we have labelled the vertical axis with powers of 5 rather than the logarithm.



This straight line has gradient 2. The line through points $(0.5, 15)$ and $(1, 75)$ on this chart has actual gradient

$$\frac{\log_5 75 - \log_5 15}{1 - 0.5} = 2\log_5\left(\frac{75}{15}\right) = 2\log_5 5 = 2$$

Example 14

Let $y = 4 \times 3^{2x}$, for $x \geq 0$.

a If $\log_3 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_3 y$ against x.

c Sketch the graph of y against x using a log-scale axis labelled in powers of 3.

Solution

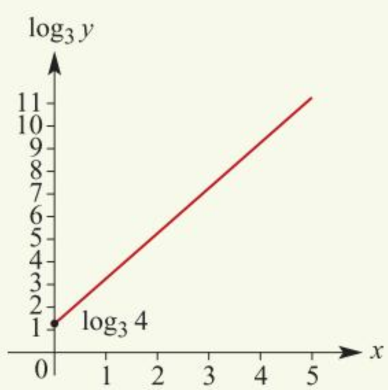
a Take logarithms base 3 of both sides of the equation:

$$\log_3 y = \log_3(4 \times 3^{2x})$$

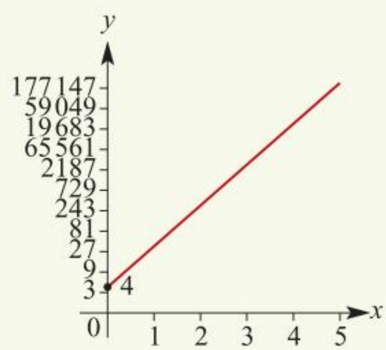
$$\log_3 y = \log_3 4 + \log_3(3^{2x})$$

$$\log_3 y = 2x + \log_3 4 \qquad \text{Therefore } m = 2 \text{ and } c = \log_3 4.$$

b



c



Note: For part **c**, using the logarithmic scale on the vertical axis means that the axis intercept is at $3^{\log_3 4} = 4$.

Applications of logarithmic scales

We now look at three uses of logarithmic scales.

Decibels

A **decibel** is defined as one-tenth of a bel, which is named after Alexander Graham Bell, the inventor of the telephone.

The decibel is a logarithmic scale for measuring 'loudness of noise'. The intensity of a sound in decibels can be defined by

$$dB = 10\log_{10}(P \times 10^{16})$$

where P is the power of the sound in watt/cm^2.

Example 15

A power mower generates a noise of 96 dB and a conversation in a restaurant generates noise of 60 dB. determine the power in watt/cm^2 for each of these.

Solution

- **Power mower**

  We have $96 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\begin{aligned} \log_{10}(P \times 10^{16}) &= 9.6 \\ P \times 10^{16} &= 10^{9.6} \\ P &= 10^{-6.4} \end{aligned}$$

  The power is $10^{-6.4}$ watt/cm^2.

- **Conversation**

  We have $60 = 10\log_{10}(P \times 10^{16})$.
  Hence

  $$\begin{aligned} \log_{10}(P \times 10^{16}) &= 6 \\ P \times 10^{16} &= 10^{6} \\ P &= 10^{-10} \end{aligned}$$

  The power is 10^{-10} watt/cm^2.

Note: The maximum intensity which the ear can tolerate is about 10^{-4} watt/cm^2, which corresponds to a loudness level of about 120 dB.

The Richter scale

Earthquake intensity is often reported on the **Richter scale**. The formula is

$$R = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of the ground motion, T is the period of the seismic wave, and B is a term that allows for the weakening of the seismic wave with increasing distance from the epicentre of the earthquake.

Example 16

Assume that, for a particular earthquake, we have $a = 10$, $T = 1$ and $B = 6.8$. Determine the earthquake's magnitude on the Richter scale.

Solution

$$R = \log_{10}\left(\frac{10}{1}\right) + 6.8 = 7.8$$

 ISBN 978-1-009-53719-3

Example 17

Early in the twentieth century an earthquake in San Francisco registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in South America that was four times stronger. We will take this to mean that the value of $\frac{a}{T}$ for South America is four times that for San Francisco. What was the magnitude of the earthquake in South America? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Solution

For San Francisco:

$$8.3 = \log_{10}\left(\frac{a_1}{T_1}\right) + B \qquad (1)$$

Hence, for South America:

$$\begin{aligned} R &= \log_{10}\left(\frac{a_2}{T_2}\right) + B \\ &= \log_{10}\left(\frac{4a_1}{T_1}\right) + B \\ &= \log_{10} 4 + \log_{10}\left(\frac{a_1}{T_1}\right) + B \\ &= \log_{10} 4 + 8.3 \qquad \text{using equation (1)} \\ &\approx 8.9 \end{aligned}$$

The magnitude was 8.9.

Note: Although the earthquake in South America was four times stronger, the magnitude on the Richter scale only increased by $\log_{10} 4 \approx 0.6$.

The pH scale

The pH scale for measuring the acidity of a solution is logarithmic. The pH of a solution is determined by the concentration of hydronium ions, $[H_3O^+]$, in the solution. (Concentration is measured in moles per litre.) The definition is

$$pH = \log_{10}\left(\frac{1}{[H_3O^+]}\right) = -\log_{10}([H_3O^+])$$

Vinegar has pH around 3, and bananas have pH in the interval $[4.5, 4.7]$. Strong hydrochloric acid solutions are approximately pH 0 and strong sodium hydroxide solutions are approximately pH 14.

Example 18

The pH of blood normally lies in the interval $[7.37, 7.44]$. Determine the range for the concentration of hydronium ions.

 ISBN 978-1-009-53719-3

Solution

For pH 7.37, we have

$$-\log_{10}([H_3O^+]) = 7.37$$
$$\log_{10}([H_3O^+]) = -7.37$$
$$[H_3O^+] = 10^{-7.37} \text{ moles per litre}$$

Now, for pH 7.44, we have

$$\log_{10}([H_3O^+]) = -7.44$$
$$[H_3O^+] = 10^{-7.44} \text{ moles per litre}$$

So the concentration of hydronium ions lies in the interval

$$[10^{-7.44}, 10^{-7.37}]$$

We can write this interval as

$$[3.63 \times 10^{-8}, 4.27 \times 10^{-8}]$$

correct to three significant figures.

Modelling with logarithmic functions

The logarithm function with base a is the 'reverse' of the exponential function with base a:

$$y = a^x \quad \text{is equivalent to} \quad x = \log_a y$$

This enables us to model using a logarithmic function instead of an exponential function when we are interested in the 'other' variable. For example, the exponential model $P = A \times 2^{kt}$ can be rewritten as

$$t = \frac{1}{k}\log_2\left(\frac{P}{A}\right)$$

That is, we can express t in terms of P, instead of expressing P in terms of t.

Summary 12E

- **Linear scale** The change between two values is determined by the difference between the values. That is, a change from 1 to 2 is the same as a change from 6 to 7. We use linear scales for temperature and length.
- **Logarithmic scale** The change between two values is determined by the ratio of the values. That is, a change from 1 to 2 (ratio of 1 : 2) is the same as a change from 6 to 12 (also a ratio of 1 : 2). We use logarithmic scales for noise and acidity.
- Consider the exponential function $y = b \times a^{mx}$, where a, b and m are positive real numbers. Taking logarithms base a of both sides gives

  $$\log_a y = \log_a b + mx$$

  So the graph of $\log_a y$ against x is a straight line with gradient m and with $(\log_a y)$-axis intercept at $(0, \log_a b)$.

 ISBN 978-1-009-53719-3

Exercise 12E

Example 14

1 Let $y = 3 \times 4^{2x}$, for $x \geq 0$.

a If $\log_4 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_4 y$ against x.

c Sketch the graph of y against x using a log-scale axis labelled in powers of 4.

2 Let $y = 2 \times 5^{3x}$, for $x \geq 0$.

a If $\log_5 y = mx + c$, give the values of m and c.

b Sketch the graph of $\log_5 y$ against x.

c Sketch the graph of y against x using a log-scale axis labelled in powers of 5.

Example 15

3 A busy street generates noise of 70 dB and a quiet car generates noise of 50 dB. Determine the power in watt/cm^2 for each of these.

4 Use the formula $dB = 10 \log_{10}(P \times 10^{16})$ to answer the following:

a If P is increased by a factor of 2, what is the effect on dB?

b If P is increased by a factor of 10, what is the effect on dB?

c If dB is increased by a factor of 3, what is the effect on P?

d For what value of P is $dB = 0$?

e For what value of P is $dB = 100$?

5 If $dB_1 - dB_2 = \lambda$, determine P_1 in terms of P_2.

Example 16

6 Determine the magnitude on the Richter scale of an earthquake with $a = 10$, $T = 2$ and $B = 5$.

Example 17

7 An earthquake in Turkey registered 7.3 on the Richter scale. In the same year, an earthquake in Greece had a quarter of this strength. We will take this to mean that the value of $\dfrac{a}{T}$ for Greece is one-quarter that for Turkey. What was the magnitude of the earthquake in Greece? Assume both were measured at the same distance from the epicentre, and so the constant B is the same for both.

Example 18

8 The pH of a soft drink normally lies in the interval $[2.0, 4.0]$. Determine the range for the concentration of hydronium ions.

9 Pure water has pH 7.

a What is the concentration of hydronium ions in water?

b A solution is made by diluting 1 litre of an acid of pH 2 with 1 litre of water. What is the pH of the solution?

c A solution is made by diluting 100 mL of an acid of pH 1.3 with water. If the solution has pH 2.5, how much water was added?

d A solution is made by mixing 500 mL of an acid of pH 2.5 with 400 mL of an acid of pH 3. What is the pH of the solution?

CF

 ISBN 978-1-009-53719-3

CF

10 The energy, E joules, released by an earthquake of magnitude R is given by the formula

$$\log_{10} E = 4.4 + 1.5R$$

a Determine R in terms of E.

b Determine the amount of energy released by an earthquake of magnitude:

i 5 **ii** 6 **iii** 7 **iv** 8 **v** 8.6

c What is the magnitude of an earthquake that releases 10^{13} joules of energy?

11 A siren generates a noise of 100 dB. If you have two sirens, the power is doubled. What is the decibel level of the noise of two sirens?

12 The cost, $\$C$, of manufacturing x units of a particular product is given by

$$C = 1.5x\log_{10} x + 2000 \quad \text{for } 1 \le x \le 3070$$

a Determine the cost when:

i $x = 500$ **ii** $x = 1000$ **iii** $x = 2000$ **iv** $x = 3000$

b Write down a formula that gives the cost per unit in terms of x.

c Determine the cost per unit when:

i $x = 500$ **ii** $x = 1000$ **iii** $x = 2000$ **iv** $x = 3000$

13 When carbon dioxide is absorbed into the bloodstream, it lowers the pH. We can model the pH of the bloodstream using the equation

$$pH = 6.1 + \log_{10}\left(\frac{800}{x}\right)$$

where x is the partial pressure of carbon dioxide in the bloodstream (measured in torr).

a Determine the pH if the partial pressure of carbon dioxide is 50.

b Determine the partial pressure of carbon dioxide if the pH is 7.2.

14 A contagious disease spreads through a flock of 2000 sheep. The time, t days, that it takes for n sheep to become infected is modelled by

$$t = 4\log_3\left(\frac{1999n}{2000 - n}\right)$$

a Determine the number of days that it takes for the disease to spread to:

i 50 sheep **ii** 100 sheep **iii** 1000 sheep **iv** 1500 sheep

b Express n in terms of t.

15 A group of students are given a list of unfamiliar words to memorise. Suppose that the percentage of words that a student remembers, $P\%$, depends on the amount of time spent studying the words, t minutes, according to the rule

$$P = 16\log_3 t + 30 \quad \text{for } 1 \le t \le 122$$

a State the range of the function P.

b What is the percentage of words recalled when the study time is 20 minutes?

c How much study time would a student need in order to recall 80% of the words?

d Determine t in terms of P.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Chapter summary

- $a^x = y$ is equivalent to $\log_a y = x$
- **Laws of logarithms**

  1 $\log_a(mn) = \log_a m + \log_a n$ **2** $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

  3 $\log_a(m^p) = p\log_a m$ **4** $\log_a(m^{-1}) = -\log_a m$ **5** $\log_a 1 = 0$ and $\log_a a = 1$
- **Change of base**
  - $\log_a x = \dfrac{\log_b x}{\log_b a}$
  - $a^x = b^{(\log_b a)x}$
- **Inverses** The function with rule $y = \log_a x$, for $x > 0$, and the function with rule $y = a^x$, for $x \in \mathbb{R}$, are inverse to each other in the following way:
  - $\log_a(a^x) = x$ for all x
  - $a^{\log_a x} = x$ for all positive values of x

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

12A **1** I can evaluate simple logarithm expressions. ☐

See Example 1 and Question 1

12A **2** I can use the logarithm laws to simplify expression. ☐

See Example 2, Example 3 and Questions 1, 2, 3, 4 and 5

12B **3** I can use logarithms to solve exponential equations and inequalities. ☐

See Example 4, Example 5, Example 6, Example 7, Example 8 and Questions 1, 2, 3, 4 and 5

12C **4** I can sketch the graphs of logarithm functions and apply simple transformations. ☐

See Example 9 and Question 2

12C **5** I can determine a transformation which takes one exponential graph to another. ☐

See Example 10 and Question 8

12D **6** I can transform equations involving exponential and logarithmic functions. ☐

See Example 11, Example 12, Example 13 and Questions 1 and 3

12E **7** I can use logarithmic functions to model situations. ☐

See Example 14, Example 15, Example 16, Example 17, Example 18, and Questions 1, 3, 12, 13 and 14

 ISBN 978-1-009-53719-3

Short-response questions

Technology-free short-response questions

SF

1 Use logarithms to solve each of the following equations:

a $2^x = 7$ **b** $2^{2x} = 7$ **c** $10^x = 2$ **d** $10^x = 3.6$
e $10^x = 110$ **f** $10^x = 1010$ **g** $2^{5x} = 100$ **h** $2^x = 0.1$

2 Evaluate each of the following:

a $\log_2 4$ **b** $\log_3 27$ **c** $\log_4 64$ **d** $\log_2\left(\frac{1}{2}\right)$

3 Evaluate each of the following:

a $\log_2 64$ **b** $\log_{10}(10^7)$ **c** $\log_a(a^2)$ **d** $\log_4 1$
e $\log_3 27$ **f** $\log_2\left(\frac{1}{4}\right)$ **g** $\log_{10} 0.001$ **h** $\log_2 16$

4 For each of the following, give an equivalent expression in simplest form:

a $\log_a x + \log_a y$ **b** $\log_2 x + \log_2(x + 3)$ **c** $\log_b(2x) - \log_b(3y)$
d $3\log_a 4 - \log_a 8$ **e** $\log_3\left(\frac{1}{9}\right)$ **f** $\log_3(x^2) + 4\log_3(x)$

5 Solve for x:

a $\log_3 x = 2$ **b** $\log_2 x = 3$ **c** $\log_x 16 = 4$ **d** $\log_5(x - 1) = 2$

6 Express each of the following as a single logarithm:

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$
c $2\log_{10} a - \log_{10} b$ **d** $2\log_{10} a - 3 - \log_{10} 25$
e $\log_{10} x + \log_{10} y - \log_{10} x$ **f** $2\log_{10} a + 3\log_{10} b - \log_{10} c$

7 Sketch the graph of each of the following:

a $y = \log_3(x + 2)$ **b** $y = \log_3(4 - x)$ **c** $y = -\log_3(2x)$

8 Solve for x:

a $\log_2(x - 2) + \log_2(x - 4) = 3$ **b** $\log_2(x + 1) + \log_2(x - 1) = 4$

CF

9 Solve:

a $\log_{10} x + \log_{10}(2x) - \log_{10}(x + 1) = 0.$ **b** $\log_2(x^2) = 4 + \log_2(x + 5)$

10 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

CU

11 **a** Given that $\log_p 7 + \log_p k = 0$, determine k.
b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, determine q.

12 Given $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x + y}$.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Review

13 Suppose that the graph of $\log_2 y$ against x is a straight line that crosses the x-axis at 3 and crosses the $(\log_2 y)$-axis at 2. Determine an equation connecting y and x of the form:

a $y = p \times 2^{kx}$, where p and k are constants

b $y = p \times q^{x}$, where p and q are constants.

CU

Technology-active short-response questions

14 **a** Determine the approximate solution of $\log_{29} x = 1 - \log_{29}(x - 0.4)$. Give your answer correct to one decimal place.

b Solve $\log_{23} x = 1 - \log_{23}(x - 0.4)$. **c** Solve $\log_{10} x = 1 + \log_{10}(x - 0.4)$.

CF

15 The graph of $y = ab^x$ passes through the points $(1, 1)$ and $(2, 5)$.

a Determine the values of a and b.

b Let $b^x = 10^z$.

i Take logarithms of both sides (base 10) to find an expression for z in terms of x.

ii Determine the values of k and a such that the graph of $y = a10^{kx}$ passes through the points $(1, 1)$ and $(2, 5)$.

16
- The value of shares in Company X increased linearly over a two-year period according to the model $x = 0.8 + 0.17t$, where t is the number of months from the beginning of January 2019 and $\$x$ is the value of the shares at time t.
- The value of shares in Company Y increased over the same period of time according to the model $y = 10^{0.03t}$, where $\$y$ is the value of these shares at time t months.
- The value of shares in Company Z increased over the same period according to the model $z = 1.7 \log_{10}(5(t + 1))$, where $\$z$ is the value of the shares at time t months.

CU

Use a calculator to sketch the graphs of the three functions on the one screen.

a Determine the values of the shares in each of the three companies at the end of June 2019.

b Determine the values of the shares in the three companies at the end of September 2020.

c During which months were shares in Company X more valuable than shares in Company Y?

d For how long and during which months were the shares in Company X the most valuable?

17 It is conjectured that the area affected by an earthquake, A km^2, is related to the magnitude of the earthquake on the Richter scale, R, by the formula

$$R = 2.3 \log_{10}(A + 4800) - 7.5 \quad \text{for } 1 \le R \le 8$$

a Make A the subject of this formula.

b Determine the area affected by an earthquake of magnitude 7.

c Determine the magnitude of an earthquake that will affect twice as much area as an earthquake of magnitude 7.

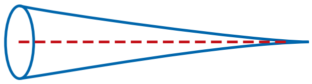
d Assume that the boundary of the affected area is a circle (with centre at the epicentre of the earthquake). Determine the radius of this circle when:

i $R = 5$ **ii** $R = 7$ **iii** $R = 8$

 ISBN 978-1-009-53719-3

18 The process used to manufacture a certain glass fibre means that its cross-sectional area gradually decreases along its length. Let A mm^2 be the cross-sectional area at a distance of d cm from the start of the fibre.

d (cm)	0	1	3	7	15	31
A (mm^2)	3	2.75				



a Using the information in the table, determine the values of constants α and β such that $A = \alpha \log_2(d + 1) + \beta$. Then use your rule for A to fill in the rest of the table.

b What is the theoretical maximum length of a glass fibre that can be manufactured in this way?

c Suppose that such a glass fibre is useable only if its cross-sectional area stays above 1.2 mm^2. What is the maximum length of a glass fibre that meets this restriction? (Answer correct to the nearest centimetre.)

19 There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time t days after 1 January 2000 is given by

$$N_A(t) = 10\,000 + 1000t, \quad 0 \le t \le 15$$

The number of *Cutus pius* alive at time t days after 1 January 2000 is given by

$$N_C(t) = 8000 + 3 \times 2^t, \quad 0 \le t \le 15$$

a With a calculator, plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.

b **i** Determine the coordinates of the point of intersection of the two graphs.

ii At what time is $N_A(t) = N_C(t)$?

iii What is the number of each species of insect at this time?

c **i** Show that $N_A(t) = N_C(t)$ if and only if $t = 3\log_2 10 + \log_2\left(\dfrac{2+t}{3}\right)$.

ii Plot the graphs of $y = x$ and $y = 3\log_2 10 + \log_2\left(\dfrac{2+x}{3}\right)$ and determine the coordinates of the point of intersection.

d It is found by observation that the model for *Cutus pius* does not quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_C(t) = 8000 + c \times 2^t$. Determine the value of c, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

Review

Multiple-choice questions

Technology-free multiple-choice questions

1 If $\log_{10}(x-2) - 3\log_{10}(2x) = 1 - \log_{10} y$, then y is equal to

A $\dfrac{80x^3}{x-2}$ **B** $1 + \dfrac{8x^3}{x-2}$ **C** $\dfrac{60x}{x-2}$ **D** $1 + \dfrac{6x}{x-2}$

2 The expression $\log_2(16 \times 2^{3x})$ is equal to:

A $12x$ **B** 322^{3x} **C** $4 + 3x$ **D** $16 + 2^{3x}$

3 The equation of the asymptote of $y = 3\log_2(5x) + 2$ is

A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 5$

4 The solution of the equation $\log_3(3x - 4) = -1$ is

A $\dfrac{4}{3}$ **B** $-\dfrac{3}{9}$ **C** 2.5 **D** $\dfrac{13}{9}$

5 Which one of the following functions has a graph with a vertical asymptote with equation $x = b$?

A $y = \log_2(x - b)$ **B** $y = \dfrac{1}{x+b}$

C $y = \dfrac{1}{x+b} - b$ **D** $y = 2^x + b$

6 If $\log_2(x+4) - 3\log_2(2x) = 2 - \log_2(16y)$, then y is equal to

A $\dfrac{2x^3}{x+4}$ **B** $2 + \dfrac{16x^3}{x+4}$ **C** $\dfrac{80x}{x+4}$ **D** $2 + \dfrac{4x}{x+4}$

7 The solution of the equation $5 \times 2^{5x} = 10$ is x equals

A $\dfrac{1}{2}$ **B** $\dfrac{1}{5}$ **C** $\dfrac{1}{5}\log_2 10$ **D** $\dfrac{1}{2}\log_2 5$

8 If $x = 5$ is a solution of the equation $\log_{10}(kx - 3) = 2$, then the exact value of k is

A $\dfrac{103}{5}$ **B** $\dfrac{\log_{10} 2 + 3}{5}$ **C** 2 **D** 5

9 $3^{4\log_3(x) + \log_3(4x)}$ is equal to

A $8x$ **B** $x^4 + 4x$ **C** $4x^5$ **D** 3^{8x}

10 Let the rule for a function g be $g(x) = \log_2((x-4)^2)$. For the function g, the maximal domain and range are

A $\mathbb{R}$, $\mathbb{R}$ **B** $(4, \infty)$, $\mathbb{R}^+$ **C** $\mathbb{R} \setminus \{4\}$, $\mathbb{R}$ **D** $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{0\}$

 ISBN 978-1-009-53719-3

11 If $\log_x 8 = \frac{1}{2}$ then x is equal to:

A -8 **B** $\frac{1}{64}$ **C** 64 **D** $2\sqrt{2}$

12 The maximal domain D of the function with rule $f(x) = \log_{10}((x-3)^2) + 6$ is

A $(3, \infty)$ **B** $[3, \infty)$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R}^+$

13 If $\log_7 2 = m$ then $\log_{49} 28$ is equal to

A $2(1+2m)$ **B** $\frac{1+2m}{2}$ **C** $\frac{2}{1+2m}$ **D** $\frac{m}{2}$

14 If $\log_2 x - \frac{3}{\log_2 x} = 2$ then :

A $x = \sqrt{2}$ **B** $x = 1$ or $x = 2$ **C** $x = 4$ or $x = 2$ **D** $x = 8$ or $x = \frac{1}{2}$

Technology-active multiple-choice questions

15 The graphs of $y = -2 + \log_3 x$ and $y = -\log_2 x + 3$ intersect at the point with coordinates (a, b). The values of a and b correct to three decimal places are:

A $a = 8.373, b = -0.066$ **B** $a = 8.351, b = -0.056$

C $a = 8.380, b = -0.1$ **D** $a = 8.380, b = -1$

16 If $\log_{10}(x+1) + \log_{10}(x-2) + \log_{10}(x-3) = 1$ then x is equal to:

A 3 **B** 4 **C** 5 **D** 6

17 The function $v(d) = 480\log_{10}\left(\frac{5d}{8}\right) + 104$ relates $v(d)$, the speed of the wind near the centre of a tornado in kilometres per hour, to d, the distance that the tornado travels, in kilometres. If winds near the centre of tornado reach speeds of 640 km/h, the distance it can travel is approximately:

A 21 km **B** 13 km **C** 210 km **D** 2100 km

18 The formula for the Richter scale is:

$$R = \log_{10}\left(\frac{a}{T}\right) + B$$

where a is the amplitude of the ground motion, T is the period of the seismic wave, and B is a term that allows for the weakening of the seismic wave with increasing distance from the epicentre of the earthquake. If $R = 6.9$, $B = 3.2$ and $T = 1.9$ the value of a correct to the nearest integer is

A 430 **B** 3.7 **C** 9523 **D** 3700

19 The long jump record for students under 16 years of age, in cm, at a particular school can be modelled by $f(x) = 543 + 66\log_3(x+1)$ where x is the number of years since records began to be kept at the school. The record for the long jump 12 years after record started being kept is: (values given to the nearest cm)

A 478 m **B** 587 m **C** 630 m **D** 697 m

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

13

Rates of change

Chapter contents

Throughout this book, we have been looking at situations where there is a relationship between two variables. We have developed and applied the idea that one variable, say y, is a function of another variable, say x.

Furthermore, we have represented such relationships graphically, and investigated key features such as axis intercepts, turning points and asymptotes.

This graphical representation can also be used to see how the relationship is changing.

In applications, how the relationship is changing is of critical importance in establishing how accurately a given rule models the relationship between the variables in question. For example, if x increases, does y also increase, or does it decrease, or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely, etc.?

This chapter serves as an introduction to the ideas of calculus, which we begin to study more formally in the next chapter. In this chapter, we talk about rates of change informally, based on our intuition. Our study of calculus will enable us to talk about these ideas more precisely.

 ISBN 978-1-009-53719-3

13A Recognising relationships

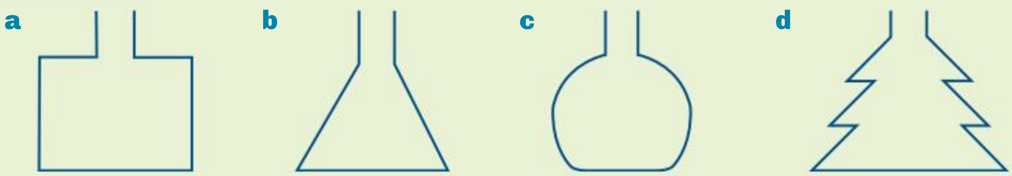
Learning intentions

- To be able to recognise **relationships** between variables.

In previous chapters, we have studied polynomial, exponential, logarithmic and trigonometric functions, and have seen that many real-life situations may be modelled by these functions.

In this first section, we look at several real-life situations involving two variables, and investigate the form of the relationships between the variables through graphs. The algebraic relationship is not established.

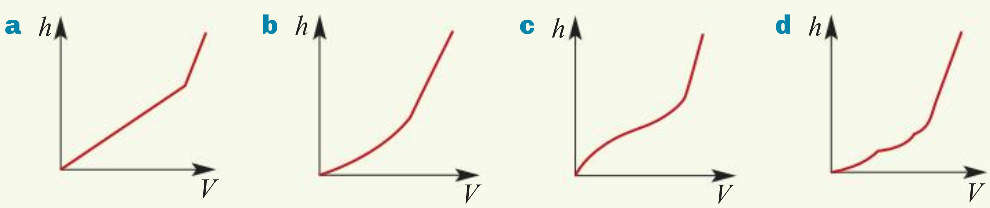
Example 1



Water is being poured steadily into each of these vessels.

Draw a graph that shows the relationship between the height of the water (h) and the volume (V) that has been poured in.

Solution



Example 2

A particle travels in a straight line.

The graph shows the distance, D metres, of the particle from a fixed point O over a period of 20 minutes.

Describe the motion of the particle.

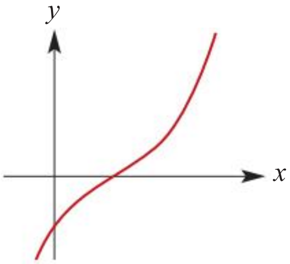
Solution

The particle is initially 5 m from O. It travels away from O for 3 minutes at a constant speed of $\frac{5}{3}$ m/min. It then remains stationary at a distance of 10 m from O for 4 minutes, before returning to O at a speed which is gradually decreasing so that it comes to rest at O at time $t = 20$ minutes.

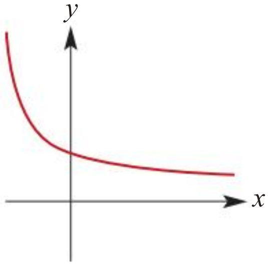
 ISBN 978-1-009-53719-3

By examining the graph representing a function, it can be determined whether the rate of change is positive, negative or neither.

Consider the graph of a function $y = f(x)$.



If the graph shows that y is increasing as x increases, then we can say that the **rate of change** of y with respect to x is **positive**. (The graph 'slopes upwards'.)



If the graph shows that y is decreasing as x increases, then we can say that the **rate of change** of y with respect to x is **negative**. (The graph 'slopes downwards'.)

If y remains the same value as x changes, the corresponding graph is a horizontal line, and we say that the rate of change of y with respect to x is **zero**.

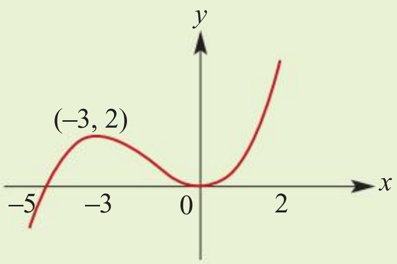
All of this is consistent with the gradient (rate of change) of a linear function, which was discussed in Chapter 2.

Example 3

For the graph shown on the right for $x \in [-5, 2]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is negative

b the rate of change of y with respect to x is positive.



Solution

a The rate of change of y with respect to x is negative for $x \in (-3, 0)$.

b The rate of change of y with respect to x is positive for $x \in [-5, -3) \cup (0, 2]$.

Summary 13A

Consider the graph of a function with rule $y = f(x)$.

- If the graph shows that y is increasing as x increases over some interval, then the **rate of change** of y with respect to x is **positive** for that interval.
- If the graph shows that y is decreasing as x increases over some interval, then the **rate of change** of y with respect to x is **negative** for that interval.
- If y remains the same value as x changes, the corresponding graph is a horizontal line and the rate of change of y with respect to x is **zero**.

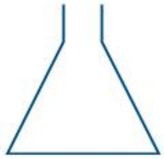
 ISBN 978-1-009-53719-3 Cambridge University Press

Exercise 13A

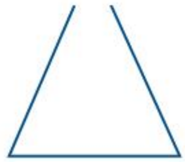
Example 1

1 Water is being poured steadily into these vessels:

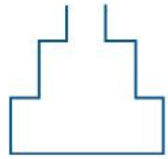
a



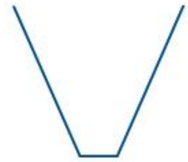
b



c



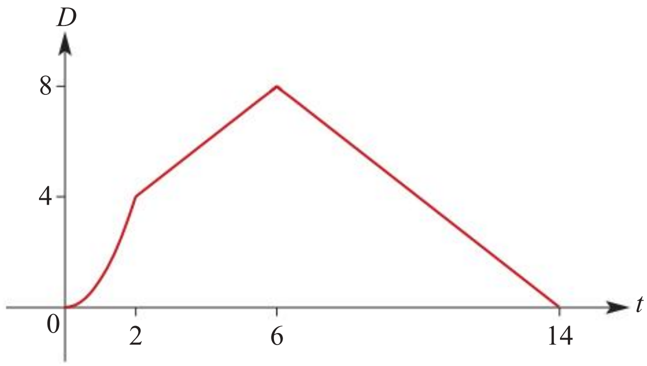
d



For each of the vessels, draw a graph of the height, h, of water in the vessel and the volume, V, that has been poured in. Label the horizontal axis V and the vertical axis h.

Example 2

2 A particle travels in a straight line. The graph shows the distance, D metres, of the particle from a fixed point O on the line over a period of 14 minutes.
Describe the motion of the particle.



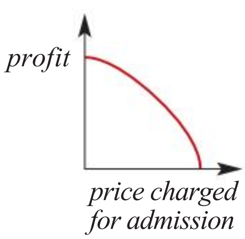
For Questions 3–6 there may be more than one correct answer. Your written explanations are an important part of the exercise.

3 The manager of a theatre wishes to know what effect changing the price of admission will have on the profit she makes.

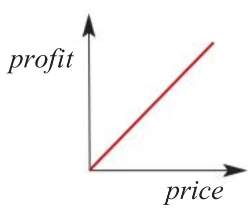
a Which one of the following graphs would show the effect of change?

b Explain your choice, including comments on scales and axes and what the point of intersection of the axes represents.

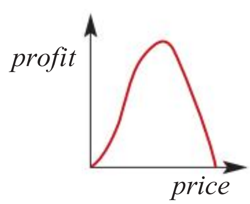
A



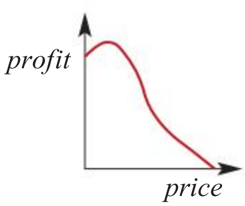
B



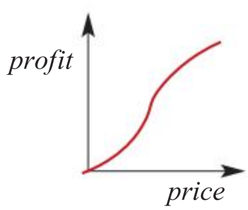
C



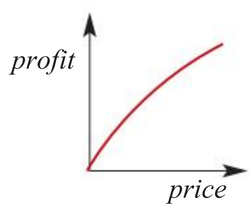
D



E



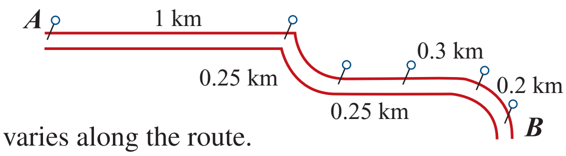
F



4 Sketch a graph to show how the height of a person might vary with age.

 ISBN 978-1-009-53719-3

5 A motorist starts a journey at the point marked A on a country road, drives 2 km along the route shown, and stops at the point marked B. He is able to drive at 100 km/h, but must slow down at corners.



a Explain briefly how the car's speed varies along the route.

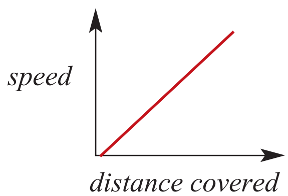
b Sketch a graph showing how the car's speed varies along the route. (Label the vertical axis *car's speed* and the horizontal axis *distance from A*.)

6 An athlete is a competitor in a 10 000 m race. Below are some graphs which could show the relationship between the speed of the runner and the distance covered.

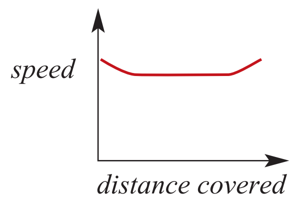
a Explain the meaning of each graph in words.

b Which graph is the most realistic for a winning athlete? If you do not think any of these graphs are realistic, draw your own and explain it fully.

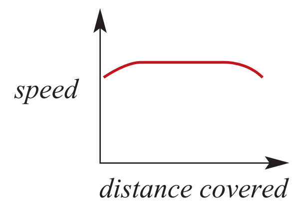
A



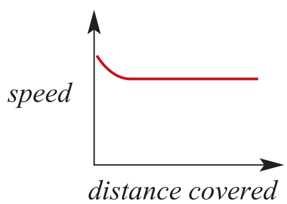
B



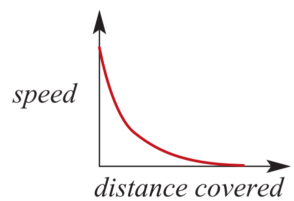
C



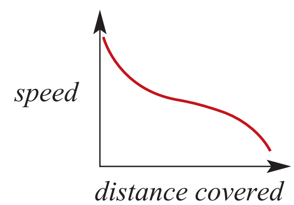
D



E



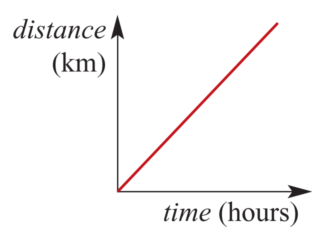
F



7 A sprinter covers 100 metres at a constant speed of 10 m/s. Sketch:

a the distance–time graph **b** the speed–time graph

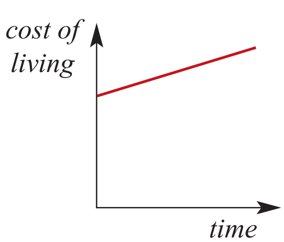
8 The graph relating the distance a car travels to the time taken is a straight line as shown. The graph shows that the car is



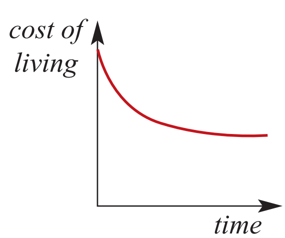
A speeding up **B** slowing down
C travelling uphill **D** travelling at a constant speed
E stationary

9 Which one of these graphs best represents the rate of increase in the cost of living slowing down?

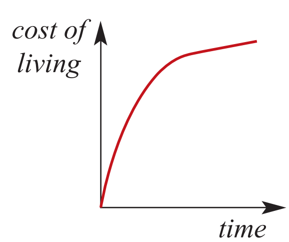
A



B



C

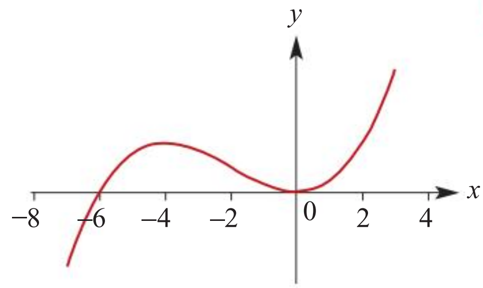


 ISBN 978-1-009-53719-3 Cambridge University Press

Example 3 **10** For the graph shown on the right for $x \in [-7, 3]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is negative

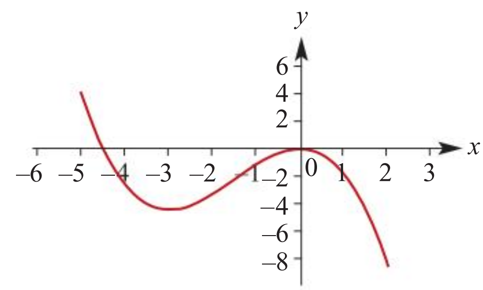
b the rate of change of y with respect to x is positive.



11 For the graph shown on the right for $x \in [-5, 2]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is positive

b the rate of change of y with respect to x is negative.



SF

13B Constant rate of change

Learning intentions

- To be able to calculate constant rates of change.

Any function that is linear will have a **constant rate of change**. That rate of change is simply the gradient of the graph and, given appropriate information, the rate can be calculated from the graph of the function or simply read from the rule of the function if it is stated.

Example 4

A car travels from Copahunga to Charlegum, which is a distance of 150 km, in 2 hours (120 minutes). Assuming the car travels at a constant speed, draw a distance–time graph and calculate the speed.

Solution

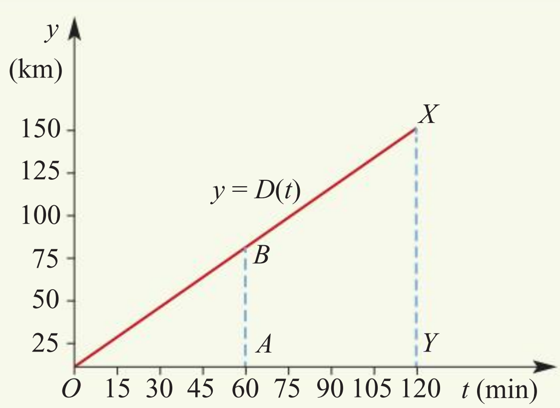
We denote the distance function by D. The graph of $y = D(t)$ is shown.

The rule of the function may be written:

$$D(t) = \frac{150}{120}t = \frac{5}{4}t$$

Note that

$$\frac{XY}{YO} = \frac{BA}{AO} = \frac{5}{4}$$



The gradient of the graph gives the speed. Therefore, the speed of the car is $\frac{5}{4}$ kilometres per minute. This speed may be expressed in kilometres per hour (km/h):

$$\text{Speed} = \frac{5}{4} \times 60 = 75 \text{ km/h}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

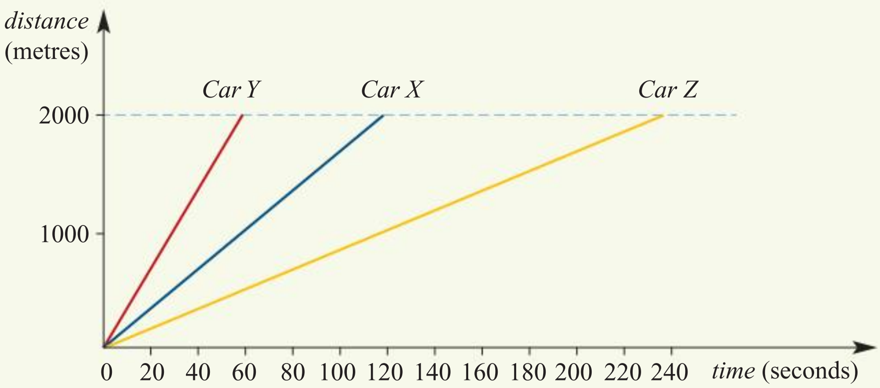
Example 5

Three cars are driven over a 2-kilometre straight track. They are all to go from point A to point B. Each car travels with constant speed. It is not a race as:

- the speed of Car Y is twice that of Car X
- the speed of Car Z is half that of Car X.

Illustrate this situation with a distance–time graph. Assume that Car X travels at 1 km/min.

Solution



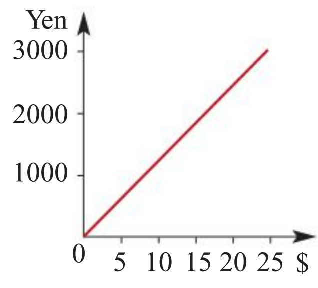
Note: The gradient of the graph for Car X is $\dfrac{2000}{120} = 16\frac{2}{3}$

The gradient of the graph for Car Y is $\dfrac{2000}{60} = 33\frac{1}{3}$

The gradient of the graph for Car Z is $\dfrac{2000}{240} = 8\frac{1}{3}$

An object whose motion can be described by a linear distance–time graph is travelling at a constant speed equal to the gradient of the linear graph.

There are many other examples in which a real-life situation is usefully modelled by a straight-line graph in such a way that the gradient of the graph is meaningful.



In all these situations the gradient of the straight-line graph represents a **rate**.

For example, the graph shown on the right represents the conversion between dollars and yen. The gradient of the graph gives the exchange rate of dollars to yen.

Summary 13B

A linear function $f(x) = mx + c$ has a constant rate of change, m.

Examples include an object travelling with constant speed and currency exchange rates.

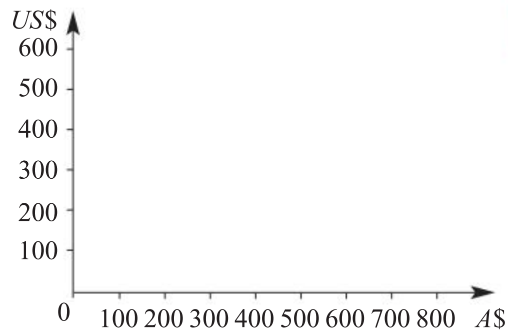
 ISBN 978-1-009-53719-3

Exercise 13B

Example 4 **1** A car travels from Bombay to Pune, a distance of 200 km, at a constant speed. The journey takes 150 minutes. Draw a distance–time graph and calculate the speed.

Example 5 **2** Two cars are driven in a straight line for 5 kilometres. They both go from point A to point B and start at the same time. (They are actually driving parallel to each other.) Each car travels with constant speed. The speed of Car Y is twice that of Car X, and the speed of Car X is 40 km/h. Illustrate this with a distance–time graph.

3 The exchange rate for the Australian dollar in terms of the American dollar was A\$1 = US\$0.75.
Draw a straight-line graph that illustrates this relationship. The axes should be as shown.



4 Determine the speed for each of the following (assume constant speed):

a distance travelled 120 km, time taken 2 hours
b distance travelled 60 m, time taken 20 seconds
c distance travelled 8000 m, time taken 20 minutes
d distance travelled 200 km, time taken 5 hours 40 minutes
e distance travelled 6542 m, time taken 5 minutes 20 seconds

5 Determine the rate of flow from the following taps in litres per minute:

a a tap which fills a 40-litre drum in 5 minutes
b a tap which fills a 600-litre tank in 12 minutes
c a tap which takes 17 minutes 20 seconds to fill a 180-litre tank

6 Water comes out of a tap at the rate of 15 litres per minute.

a Copy and complete this table showing the amount which has come out at time t:

Time in minutes, t	0	0.5	1	1.5	2	3	4	5
Amount in litres, A	0							

b Draw a graph from the table.

7 A worker is paid \$200 for 13 hours work. What is their rate of pay per hour?

8 A spherical balloon is blown up so that its volume is increasing by 8 cm^3 every second. Sketch a graph to show how the volume of the balloon changes with time.

9 Two cars start together and travel with constant speed over a 1-kilometre straight track. Car 1 has speed 60 km/h, and Car 2 travels at three-quarters of this speed. Illustrate this situation with distance–time graphs for both cars on the one set of axes.

 ISBN 978-1-009-53719-3

13C Average rate of change

Learning intentions
- ▶ To be able to calculate average rates of change.

Many moving objects do not travel with constant speed. For example, the speedometer of a car being driven in city traffic rarely stays still for long.

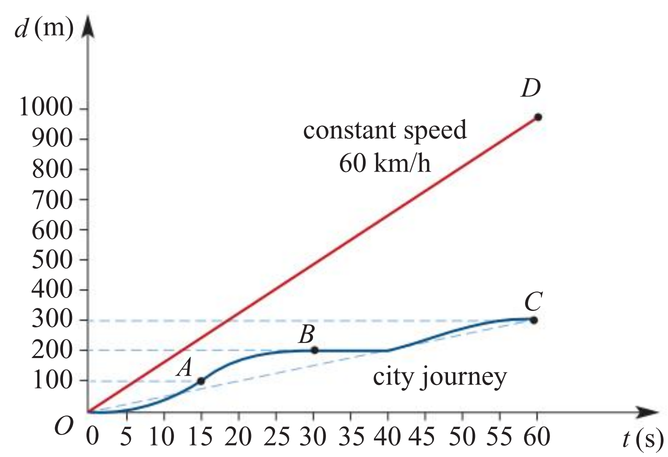
Similarly, not all functions are linear, and so not all functions have a constant rate of change. For a function that is non-linear, the rate of change of the function varies, and may in fact be different for every different point on the graph of the function.

Average speed

We will use a distance–time graph to illustrate the idea of average speed.

The graph below shows the motion of two cars both travelling in a straight line away from a fixed point O, where d is the distance travelled (in metres) at time t (in seconds).

The straight-line graph through D shows a constant speed of 60 km/h. By comparison, the graph through points A, B and C shows a motorist travelling at varying speeds. The motorist accelerates to reach 60 km/h at A before slowing for the lights at B, where there is a 10-second standstill. There is then another short burst of speed before another standstill at C.



Although we do not know the actual speed of the car travelling in the city at any particular time (other than when it is stationary), we can work out the average speed of the car over the full 60 seconds.

The average speed is given by

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{300}{60}$$

which gives an average speed of 5 metres per second. The average speed is the gradient of the line OC.

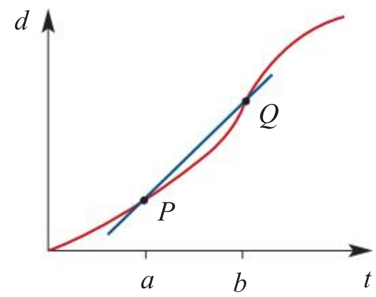
The average speed may also be calculated for any given time interval. For example, for the time interval from $t = 15$ to $t = 30$, the average speed is given by the gradient of the line joining points A and B. This is $\dfrac{100}{15} = 6\frac{2}{3}$ metres per second.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

In general:

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

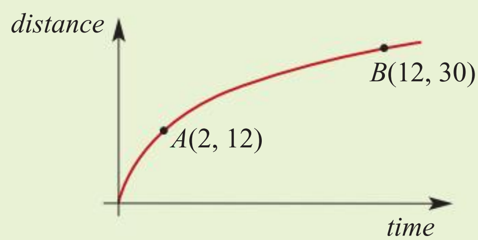
So the average speed of an object for $a \le t \le b$ is given by the gradient of the line passing through points P and Q.



Example 6

The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown.

Determine the average speed of the object in m/s over the interval from $t = 2$ to $t = 12$.



Solution

$$\begin{aligned}\text{Average speed} &= \frac{\text{total distance travelled}}{\text{total time taken}}\\ &= \frac{30-12}{12-2}\\ &= \frac{18}{10}\\ &= 1.8 \text{ m/s}\end{aligned}$$

Average rate of change for a function

The line which passes through two points on a curve is called a **secant**.

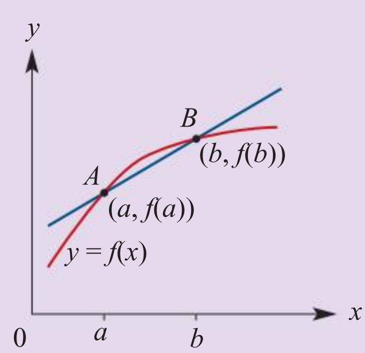
The line segment joining two points on a curve is called a **chord**.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through $A(a, f(a))$ and $B(b, f(b))$ (secant AB).

That is,

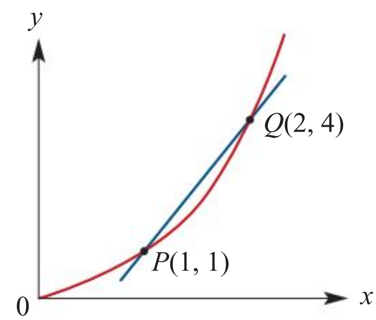
$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



 ISBN 978-1-009-53719-3 Cambridge University Press

For example, for the function with the graph shown, the average rate of change of y with respect to x over the interval $[1, 2]$ is given by the gradient of the secant PQ:

$$\text{gradient} = \frac{4-1}{2-1} = 3$$



Example 7

Determine the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

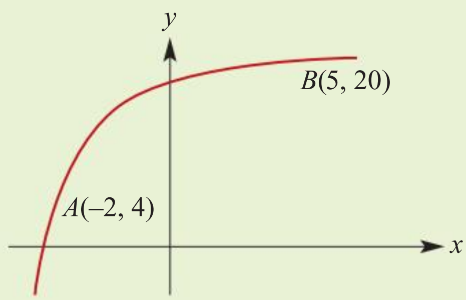
$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

$$\text{Average rate of change} = \frac{20-4}{5-1} = 4$$

Example 8

Determine the average rate of change of the function depicted in the graph for the interval $[-2, 5]$.



Solution

Average rate of change for the interval $[-2, 5]$

$$= \frac{20-4}{5-(-2)}$$

$$= \frac{16}{7}$$

 ISBN 978-1-009-53719-3

Example 9

The air temperature, T°C, at a weather station on a particular evening is modelled by the equation $T = \dfrac{600}{t^2 + 2t + 30}$, where t is the time in hours after 6 p.m.

a Determine the temperature at 6 p.m.

b Determine the temperature at midnight.

c Determine the average rate of change of the air temperature from 6 p.m. until midnight.

Solution

a At 6 p.m., $t = 0$. Hence

$$T = \frac{600}{(0)^2 + 2(0) + 30} = 20°\text{C}$$

b At midnight, $t = 6$. Hence

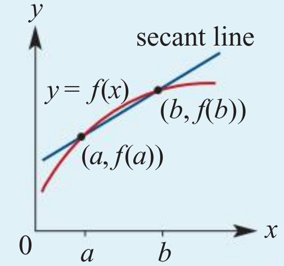
$$T = \frac{600}{(6)^2 + 2(6) + 30} = \frac{100}{13} = 7.69°\text{C} \quad \text{(correct to two decimal places)}$$

c Average rate of change of temperature $= \dfrac{\frac{100}{13} - 20}{6 - 0} = -\dfrac{80}{39} = -2.05$°C per hour

Summary 13C

- The line which passes through two points on a curve is called a **secant**.
- The line segment joining two points on a curve is called a **chord**.
- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.
  That is,

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

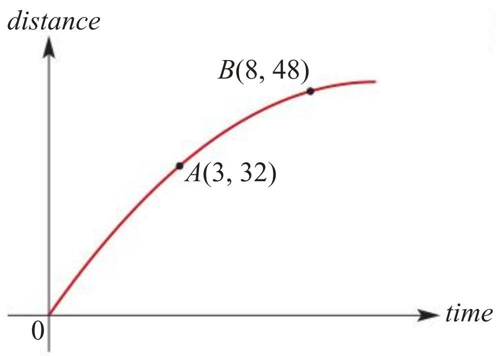


Skill-sheet

Exercise 13C

Example 6

1 The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown. Determine the average speed of the object in m/s over the interval from $t = 3$ to $t = 8$.



SF

Example 7

2 For each function, determine the average rate of change over the stated interval:

a $f(x) = 2x + 5, \quad x \in [0, 3]$

b $f(x) = 3x^2 + 4x - 2, \quad x \in [-1, 2]$

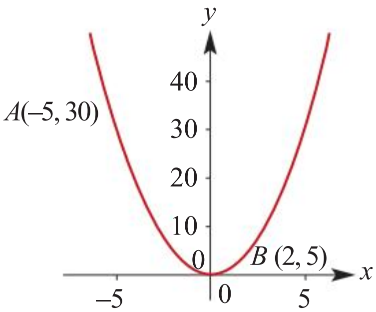
c $f(x) = \dfrac{2}{x-3} + 4, \quad x \in [4, 7]$

d $f(x) = \sqrt{5 - x}, \quad x \in [0, 4]$

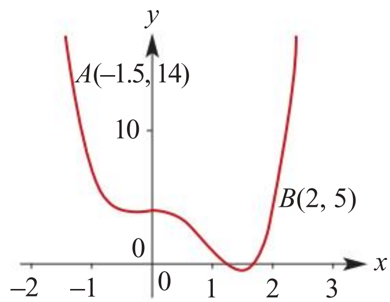
Example 8

3 Determine the average rate of change of y with respect to x from point A to point B for each of the following graphs:

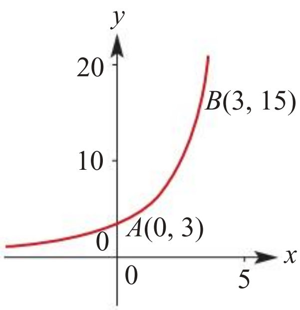
a



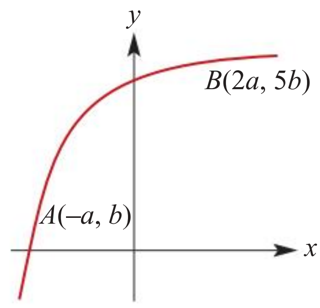
b



c



d



4 The distance (in metres) from a point O of an object t seconds after it starts to move in a straight line is given by the function $S(t) = t^3 + t^2 - 2t + 2$, $t \geq 0$. Determine the average rate of change with respect to time of the distance of the object from O:

a in the first 2 seconds

b in the next 2 seconds.

5 Cara invested \$2000 for three years at an interest rate of 7% p.a. compounded annually.

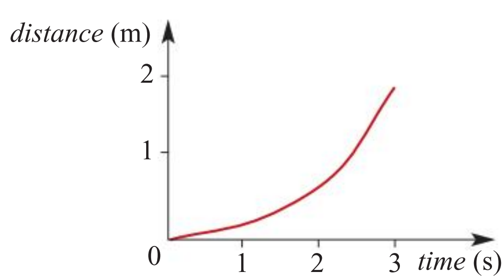
a Determine the value of the investment after three years.

b Determine the average rate of change in the value of the investment over that time.

Example 9

6 The depth, d cm, of water in a bath tub t minutes after the tap is turned on is modelled by the function $d(t) = \dfrac{-300}{t+6} + 50$, $t \geq 0$. Determine the average rate of change of the depth of the water with respect to time over the first 10 minutes after the tap is turned on.

7 Using the information in the graph on the right, the average speed from $t = 0$ to $t = 3$ is



A 2 m/s

B 1 m/s

C $\dfrac{2}{3}$ m/s

D $1\frac{1}{2}$ m/s

SF

CF

ISBN 978-1-009-53719-3

13D Instantaneous rate of change

Learning intentions

- To be able to estimate instantaneous rates of change.

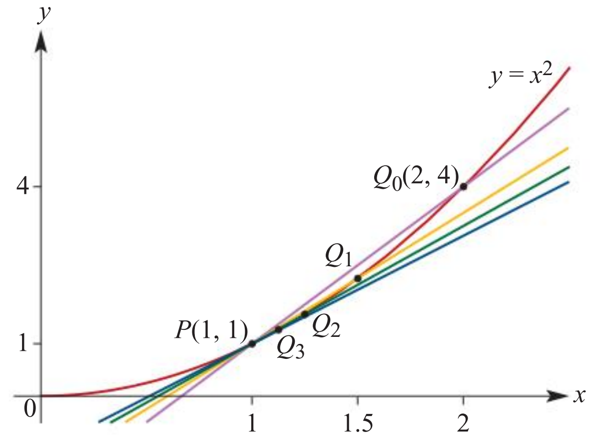
In the previous section, we investigated the average rate of change of a function over a stated interval. We saw that, in general (except for linear functions), the average rate of change of a function over different intervals in the domain of the function is not constant.

In this section, we investigate the idea of instantaneous rate of change.

Tangent line at a point

We talk about the idea of the tangent to a curve at a point informally here. It is a line which has 'the same slope' as the graph at this point. Of course, we don't know exactly what this means, but if we think of a very, very, . . . , very small section of the curve around the point, we can consider it to be a line segment which can be extended out to a straight line. This straight line is what we call the 'tangent line' to the curve at the point.

We can illustrate this idea with a specific example. Part of the graph of $y = x^2$ is shown below. We will determine the tangent line at the point $P(1, 1)$.



We start with the secant PQ_0 passing through $P(1, 1)$ and $Q_0(2, 4)$.

The gradient of PQ_0 is $\dfrac{4-1}{2-1} = 3$, and so the equation of the secant PQ_0 is $y = 3x - 2$.

The points $Q_1, Q_2, Q_3, \dots, Q_n, \dots$ on the curve $y = x^2$ are chosen so that they get closer and closer to P in the following way:

- The x-coordinate of Q_1 is $\dfrac{1}{2}(1 + 2) = \dfrac{3}{2}$.
- The x-coordinate of Q_2 is $\dfrac{1}{2}\left(1 + \dfrac{3}{2}\right) = \dfrac{5}{4}$.
- The x-coordinate of Q_3 is $\dfrac{1}{2}\left(1 + \dfrac{5}{4}\right) = \dfrac{9}{8}$.

We now look at the sequence of secants $PQ_0, PQ_1, PQ_2, PQ_3, \dots, PQ_n, \dots$. The following table shows the gradient and the equation for each secant.

 ISBN 978-1-009-53719-3

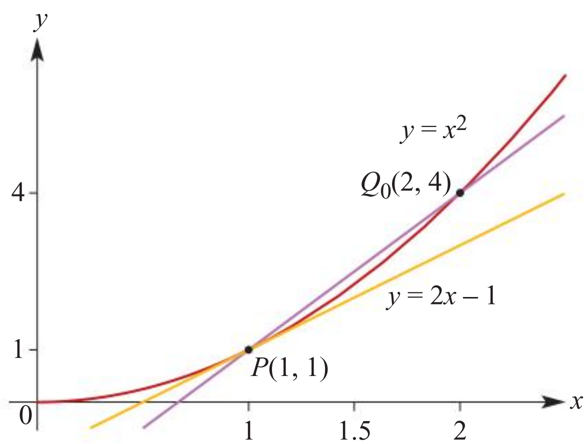
Secants of the curve $y = x^2$ through the point $P(1, 1)$

Step	Endpoint	Secant	Gradient	Equation of secant
0	$Q_0(2, 4)$	PQ_0	3	$y = 3x - 2$
1	$Q_1(\frac{3}{2}, \frac{9}{4})$	PQ_1	$\frac{5}{2}$	$y = \frac{5}{2}x - \frac{3}{2}$
2	$Q_2(\frac{5}{4}, \frac{25}{16})$	PQ_2	$\frac{9}{4}$	$y = \frac{9}{4}x - \frac{5}{4}$
3	$Q_3(\frac{9}{8}, \frac{81}{64})$	PQ_3	$\frac{17}{8}$	$y = \frac{17}{8}x - \frac{9}{8}$
n	$Q_n(1 + 2^{-n}, (1 + 2^{-n})^2)$	PQ_n	$2 + 2^{-n}$	$y = (2 + 2^{-n})x - (1 + 2^{-n})$

The sequence of gradients is $3, \frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \dots, 2 + \frac{1}{2^n}, \dots$

We can see that the gradients get closer and closer to 2. This is particularly evident from the general gradient, $2 + \frac{1}{2^n}$, since as $n \to \infty$, $\frac{1}{2^n} \to 0$.

We can also see that the secants get closer and closer to the line with equation $y = 2x - 1$. This line is the **tangent line** at the point P, and the gradient of the tangent line is the **instantaneous rate of change** of y with respect to x at the point P. We define these two concepts in Chapters 14 and 15.



In the examples and exercises in this chapter, we only require approximations to the instantaneous rate of change.

Example 10

Estimate the instantaneous rate of change of y with respect to x at the point $P(2, 9)$ on the curve $y = x^3 + 1$ by considering the secant PQ, where $Q = (2.01, (2.01)^3 + 1)$.

Solution

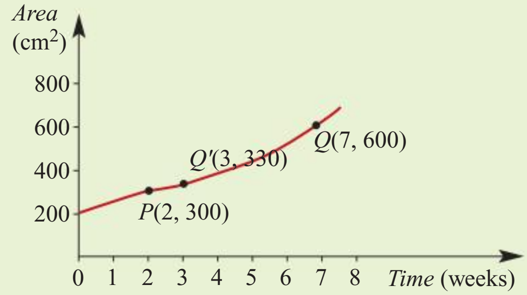
$$\text{Gradient of } PQ = \frac{(2.01)^3 + 1 - 9}{2.01 - 2} = 12.0601$$

Note: An even better approximation can be made by choosing the points $P(2, 9)$ and $Q(2.001, (2.001)^3 + 1)$. Using the approach taken for $y = x^2$ in the discussion above, we would determine that the instantaneous rate of change for this example is 12.

 ISBN 978-1-009-53719-3

Example 11

The graph represents the area covered by a spreading plant. Area is measured in square centimetres and time in weeks.



a Determine the gradient of the secant PQ.

b The point Q' has coordinates $(3, 330)$. Determine the average rate of change of area with respect to time for the interval $[2, 3]$, and hence estimate the instantaneous rate of change of the area of the plant at $t = 2$.

Solution

a $$\text{Gradient of } PQ = \frac{600 - 300}{7 - 2} = \frac{300}{5} = 60$$

The average rate of change of area from $t = 2$ to $t = 7$ is 60 cm^2 per week.

b $$\text{Gradient of } PQ' = \frac{330 - 300}{1} = 30$$

$\therefore$ Gradient at P is approximately 30.

The instantaneous rate of change of the area of the plant with respect to time when $t = 2$ is approximately 30 cm^2 per week.

Example 12

Consider the curve $y = 2^x$.

a Using the secant through the points where $x = 3$ and $x = 3.1$, estimate the instantaneous rate of change of y with respect to x at the point where $x = 3$.

b Repeat for the points where $x = 3$ and $x = 3.001$.

Solution

a When $x = 3$, $y = 8$ and when $x = 3.1$, $y = 8.5742$ (correct to four decimal places).
The gradient of the line through $(3, 8)$ and $(3.1, 8.5742)$ is 5.7419.
Thus an estimate for the instantaneous rate of change of $y = 2^x$ at $x = 3$ is 5.742.

b When $x = 3.001$, $y = 8.005547$.
The gradient of the line through $(3, 8)$ and $(3.001, 8.005547)$ is 5.547.

Note: The true instantaneous rate of change of y with respect to x at $x = 3$ is 5.5452 (correct to four decimal places).

 ISBN 978-1-009-53719-3

Using the graph window of your calculator

The graph of $y = 0.5x^3 - 2x + 1$ is shown. We will investigate the gradient at the point $(0, 1)$.

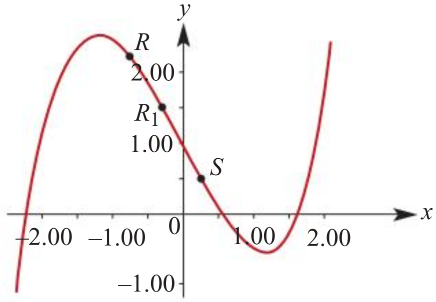
First determine the gradient of the secant RS, where $R = (-0.75, 2.2891)$ and $S = (0.25, 0.5078)$:

gradient of $RS = -1.7813$

(The coordinates of R and S are given to four decimal places.)

Now consider another secant R_1S, where $R_1 = (-0.25, 1.4922)$ and $S = (0.25, 0.5078)$:

gradient of $R_1S = -1.9688$

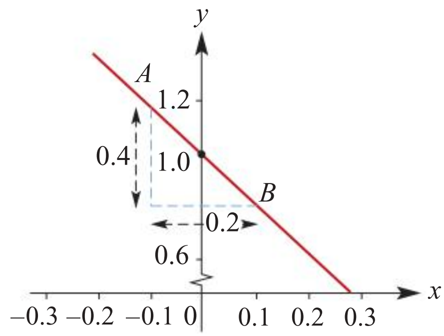


Using a calculator, zoom in on the graph near $x = 0$. As you zoom further in, this section of the curve appears increasingly linear. By assuming that this section of the curve is in fact linear and determining its gradient, we can approximate the gradient of the curve at $x = 0$.

This diagram shows a 'zoomed in' section of the graph around the point $(0, 1)$.

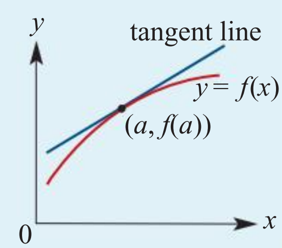
Consider the secant AB where $A = (-0.1, 1.1995)$ and $B = (0.1, 0.8005)$. The gradient of this line is approximately -2.

Therefore we make the approximation that the gradient of the curve $y = 0.5x^3 - 2x + 1$ at the point $(0, 1)$ is -2.



Summary 13D

For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Exercise 13D

Example 10

1 By considering the secant through the points where $x = 1.2$ and $x = 1.3$, estimate the instantaneous rate of change of y with respect to x at the point on the curve $y = x^3 + x^2$ where $x = 1.3$.

SF

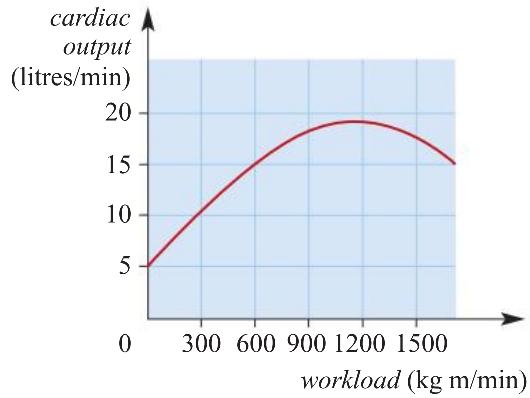
ISBN 978-1-009-53719-3

Example 11

2 Cardiac output is an important factor in athletic endurance. The graph shows a stress-test graph of cardiac output (measured in litres/min of blood) versus workload (measured in kg m/min).

a Estimate the average rate of change of cardiac output with respect to workload as the workload increases from 0 to 1200 kg m/min.

b Estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 450 kg m/min.



Example 12

3 Let $y = 10^x$.

a Determine the average rate at which y changes with respect to x over each of the following intervals:

i $[0, 1]$ **ii** $[0, 0.5]$ **iii** $[0, 0.1]$

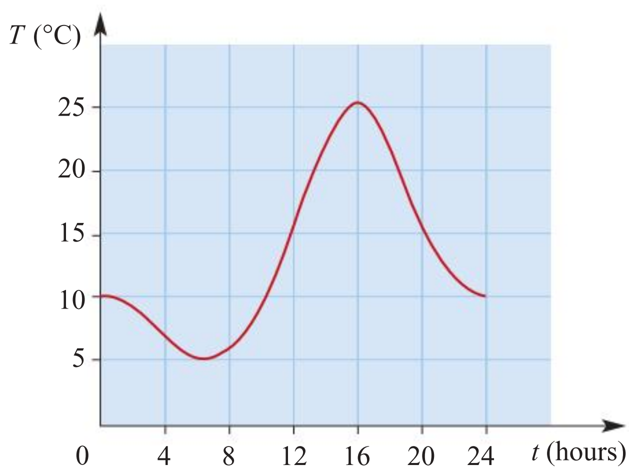
b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

4 Temperature (T°C) varies with time (t hours) over a 24-hour period, as illustrated in the graph.

a Estimate the maximum temperature and the time at which this occurs.

b The temperature rise between 10:00 and 14:00 is approximately linear. Estimate the rate at which the temperature is increasing in this period.

c Estimate the instantaneous rate of change of temperature at $t = 20$.



5 By considering the secant through the points at which $x = 1.2$ and $x = 1.4$, estimate the instantaneous rate of change of y with respect to x of the curve $y = \frac{1}{x}$ at $x = 1.2$.

6 Draw the graph of $y = \sqrt{16 - x^2}$, $-4 \leq x \leq 4$. Use an appropriate technique to determine an estimate of the instantaneous rate of change of y with respect to x at the points:

a $x = 0$ **b** $x = 2$ **c** $x = 3$

 ISBN 978-1-009-53719-3

CF

7 It is known that the straight line $y = 4x - 4$ touches the curve $y = x^2$ at the point $(2, 4)$. Sketch the graphs of both of these functions on the one set of axes.
Determine the instantaneous rate of change of y with respect to x at the point at $(2, 4)$ on the curve $y = x^2$.

8 Water is being collected in a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$.

a Determine the average rate of change of volume with respect to time between times $t = 1$ and $t = 3$.

b Determine an estimate for the instantaneous rate of change of volume with respect to time at $t = 1$.

9 A population of bacteria is growing. The population, P million, after time t minutes is given by $P = 3 \times 2^t$.

a Determine the average rate of change of population between times $t = 2$ and $t = 4$.

b Determine an estimate for the instantaneous rate of change of population with respect to time at $t = 2$.

10 Water is flowing out of a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 5 \times 10^5 - 10^2 \times 2^t$, $0 \le t \le 12$.

a Determine the average rate of change of volume with respect to time between times $t = 0$ and $t = 5$.

b Determine an estimate for the instantaneous rate of change of volume with respect to time when $t = 6$.

c Determine an estimate for the rate of change of volume when $t = 12$.

11 Use the technique of Examples 10 and 12 to estimate the instantaneous rate of change of y with respect to x for each of the following at the stated point:

a $y = x^3 + 2x^2$, $(1, 3)$

b $y = 2x^3 + 3x$, $(1, 5)$

c $y = -x^3 + 3x^2 + 2x$, $(2, 8)$

d $y = 2x^3 - 3x^2 - x + 2$, $(3, 26)$

12 The volume, V, of a cube with edge length x is given by $V = x^3$.

a Determine the average rate at which the volume of the cube changes with respect to x, as x increases from $x = 2$ to $x = 4$.

b Determine an estimate for the instantaneous rate at which V changes with respect to x when $x = 2$.

13 Let $y = 2x^2 - 1$.

a Determine the average rate at which y changes with respect to x over the interval $[1, 4]$.

b Determine an estimate for the instantaneous rate at which y changes with respect to x when $x = 1$.

14 Let $y = \sin x$.

a Determine the average rate at which y changes with respect to x over each of the following intervals:

i $\left[0, \frac{\pi}{2}\right]$ **ii** $\left[0, \frac{\pi}{4}\right]$ **iii** $[0, 0.5]$ **iv** $[0, 0.1]$

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

CF

13E Position and average velocity

Learning intentions

▶ To be able to estimate rates of change when considering motion in a straight line.

One of the key applications of rates of change is in the study of the motion of a particle.

In this section, we consider motion in a straight line. The study of motion in a straight line is continued in Chapter 15.

Position

The **position** of a particle is a specification of its location relative to a reference point.

Consider motion on a straight line with reference point O.

We say that position to the right of O is positive and to the left of O is negative.

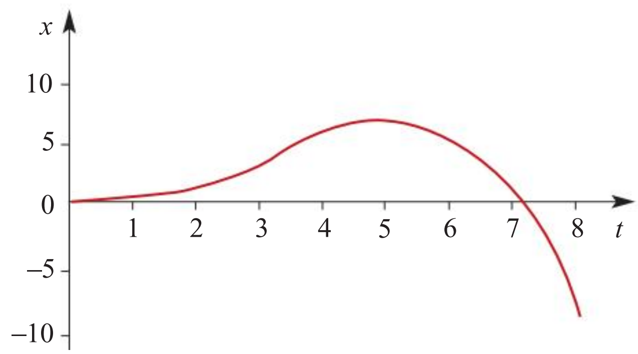
A particle is moving along the straight line. Let x metres denote its position relative to O at time t (where time is measured in seconds).

- At time $t = 0$, $x = 0$.
- At time $t = 5$, $x = 6.25$.
- At time $t = 8$, $x = -8.96$.

At $t = 0$, the particle starts from rest and moves to the right. At $t = 5$, the particle stops and moves back in the opposite direction. Its position–time graph is shown below.

Note that from $t = 0$ until $t = 7.1$, the position is positive, i.e. the particle is to the right of O.

For $t > 7.1$, the position is negative, i.e. the particle is to the left of O.



 ISBN 978-1-009-53719-3

Average velocity

The average velocity of a particle is the average rate of change in position with respect to time:

$$\textbf{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

For the moving particle we have been considering:

- At time $t = 0$, $x = 0$.
- At time $t = 5$, $x = 6.25$.
- At time $t = 8$, $x = -8.96$.

Therefore the particle's average velocity over the time interval $0 \le t \le 5$ is given by

$$\text{average velocity} = \frac{6.25}{5} = 1.25 \text{ metres per second (m/s)}$$

and over the time interval $5 \le t \le 8$,

$$\text{average velocity} = \frac{-8.96 - 6.25}{3} = -5.07 \text{ metres per second (m/s)}$$

Example 13

Let $s(t) = 6t - t^2$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Determine the average velocity for the time interval $[0, 1]$.

b Determine the average velocity for the time interval $[6, 8]$.

Solution

a $s(0) = 0$ and $s(1) = 6 - 1^2 = 5$. Hence

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{5-0}{1-0} = 5\end{aligned}$$

The average velocity for $[0, 1]$ is 5 m/s.

b $s(6) = 0$ and $s(8) = 6 \times 8 - 8^2 = -16$. Hence

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{-16-0}{8-6} = -8\end{aligned}$$

The average velocity for $[6, 8]$ is -8 m/s.

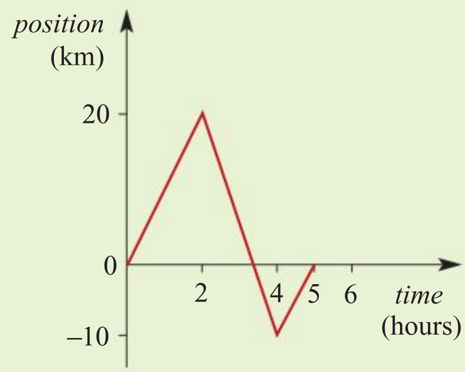
 ISBN 978-1-009-53719-3

Velocity–time graphs

In the next two examples we look at the velocity–time graph obtained from a position–time graph. Each of the position–time graphs is the graph of a piecewise-defined function where each of the components is linear.

Example 14

The graph shown is the position–time graph for the bicycle trip of a boy who lives on a long straight road. The road runs north–south, and north is chosen to be the positive direction.



a Describe his trip.

b Draw the corresponding velocity–time graph.

Solution

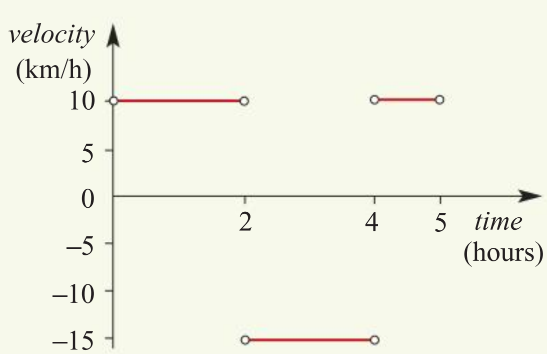
a
- The boy heads north for 2 hours.

  His velocity for this period is $\dfrac{20-0}{2} = 10$ km/h.
- He then turns and rides south for 2 hours.

  His velocity for this period is $\dfrac{-10-20}{2} = -15$ km/h.
- He turns and rides north until he reaches home.

  His velocity for this period is $\dfrac{0-(-10)}{1} = 10$ km/h.

b



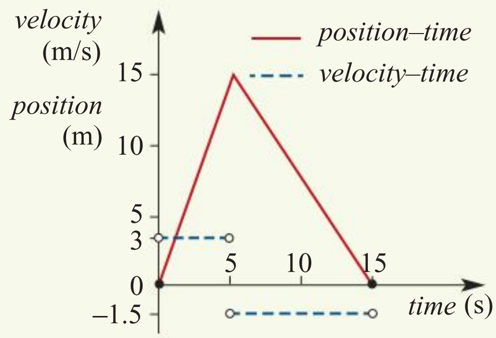
 ISBN 978-1-009-53719-3 Cambridge University Press

Example 15

A particle is moving in a straight line. It was initially at rest at a point O. It moves to the right of O with a constant velocity and reaches a point A, 15 metres from O, after 5 seconds. It then returns to O. The return trip takes 10 seconds. It stops at O.

On the one set of axes draw the position–time graph and the velocity–time graph for the motion.

Solution



Explanation

The gradient of the position–time graph for $0 < t < 5$ is 3.

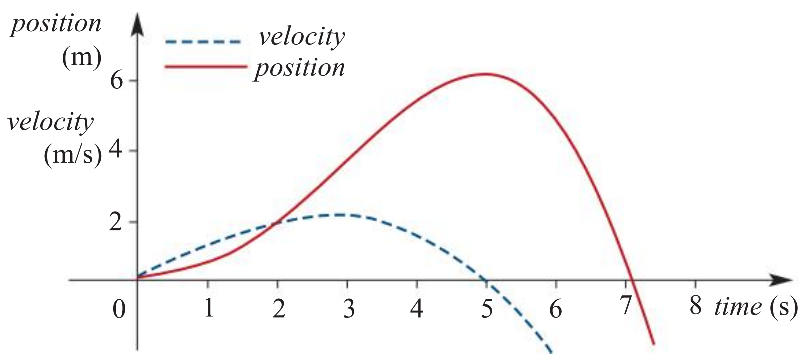
The gradient for $5 < t < 15$ is -1.5.

The gradient of the position–time graph determines the velocity–time graph.

Instantaneous velocity

Instantaneous velocity is the instantaneous rate of change in position with respect to time. It can be thought of as the gradient of the tangent to the position–time graph at a particular point. If we know an object's instantaneous velocity at every moment in time, we can sketch a velocity–time graph for an object moving with non-constant velocity.

In the graph shown below, the position–time graph is the same as the one from the start of this section.



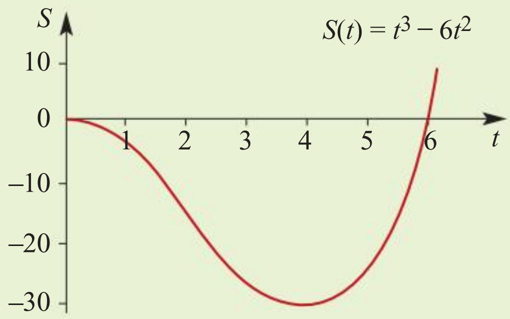
This graph illustrates the relationship between position and velocity. The vertical axis is in both metres per second for velocity and metres for position. In the following, we use velocity to mean instantaneous velocity.

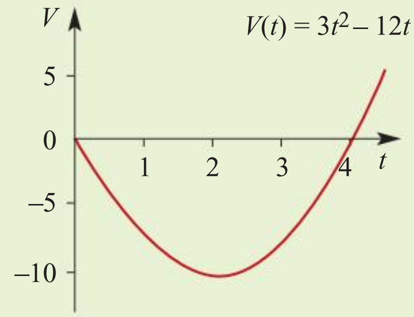
We make the following observations from the graph:

- For $0 < t < 5$, the velocity is positive: the particle is travelling from left to right.
- For $t > 5$, the velocity is negative: the particle is travelling from right to left.
- For $t = 5$, the velocity is zero: the particle is instantaneously at rest.

Example 16

The position of a particle moving in a straight line is given by the function $S(t) = t^3 - 6t^2$, $t \geq 0$. The graph of S against t is shown. The corresponding velocity–time graph is also shown. The function describing the velocity is $V(t) = 3t^2 - 12t$.





a Determine the average velocity of the particle for the intervals:

i $[3.5, 4.5]$ **ii** $[3.9, 4.1]$ **iii** $[3.99, 4.01]$

b From part a, what is the instantaneous velocity when $t = 4$?

c **i** For what values of t is the velocity positive?

ii For what values of t is the velocity negative?

Solution

a **i** Average velocity $= \dfrac{S(4.5) - S(3.5)}{1} = \dfrac{-30.375 + 30.625}{1} = 0.25$

ii Average velocity $= \dfrac{S(4.1) - S(3.9)}{0.2} = \dfrac{-31.939 + 31.941}{0.2} = 0.01$

iii Average velocity $= \dfrac{S(4.01) - S(3.99)}{0.02} = \dfrac{-31.999399 + 31.999401}{0.02} = 0.0001$

b The results of part a suggest that the instantaneous velocity is zero when $t = 4$, and this is consistent with both graphs.

c **i** From the position–time graph, the velocity is positive for $t > 4$.

ii From the position–time graph, the velocity is negative for $0 < t < 4$.

Summary 13E

- The **position** of a particle moving along a straight line specifies its location relative to a reference point.
- The average velocity of a particle is the average rate of change in position with respect to time:

$$\textbf{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

- Instantaneous velocity is the instantaneous rate of change in position with respect to time. It can be thought of as the gradient of the tangent to the position–time graph at a particular point.

Exercise 13E

Example 13

1 Let $s(t) = 6t - 2t^3$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Determine the average velocity for the time interval $[0, 1]$.

b Determine the average velocity for the time interval $[0.8, 1]$.

SF

Example 14

2 The following is the position–time graph for a train travelling on a straight track. Position is measured from the door of the ticket office at Jimbara station.

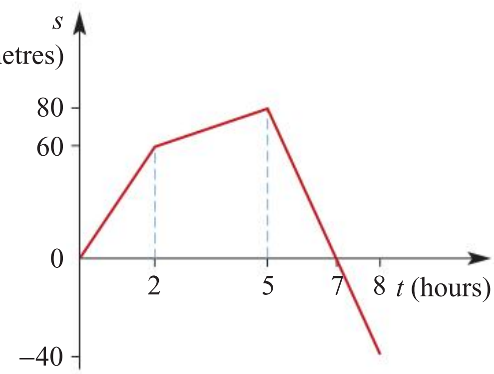
CF

a What was the train's velocity over each of the following time intervals?

i $[0, 2]$

ii $[2, 5]$

iii $[5, 8]$



b Describe the train journey.

c Draw a velocity–time graph for the train's motion for the interval $[0, 8]$.

Example 15

3 The motion of a particle moving in a straight line is described by the following information:

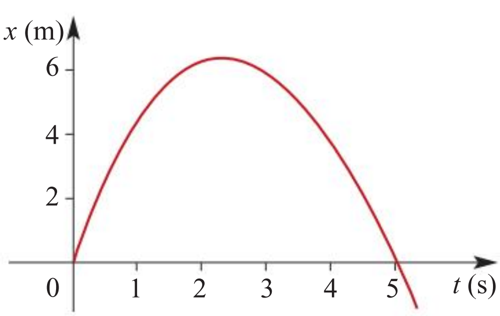
- For the time period $(0, 2)$, velocity $= -3$.
- For the time period $(2, 5)$, velocity $= 3$.
- For the time period $(5, 7)$, velocity $= 4$.

a Draw the velocity–time graph for the interval $[0, 7]$.

b Draw the position–time graph for the interval $[0, 7]$.

Example 16

4 A particle moves along a horizontal straight line. It starts from rest at a point O. The graph is the position–time graph for this motion.



a At what time is the instantaneous velocity zero?

b For which values of t is the instantaneous velocity positive?

c How far from O does the particle go to the right?

d From the time it starts moving, how long does it take to return to O?

e Estimate the instantaneous velocity of the particle at $t = 1$.

CF

5 The table shows the distance, d metres, of a ball from its starting position at time t seconds after being thrown into the air.

t	0	1	2	3	4	5	6
d	0	25	40	45	40	25	0

Using the scales 2 cm = 1 second and 1 cm = 5 metres, draw the graph of d against t. From your graph determine:

a when the ball returns to the starting point

b the average velocity of the ball from $t = 1$ to $t = 2$

c the average velocity of the ball from $t = 1$ to $t = 1.5$

d an estimate of the velocity of the ball when $t = 1$

e an estimate of the velocity of the ball when $t = 4$

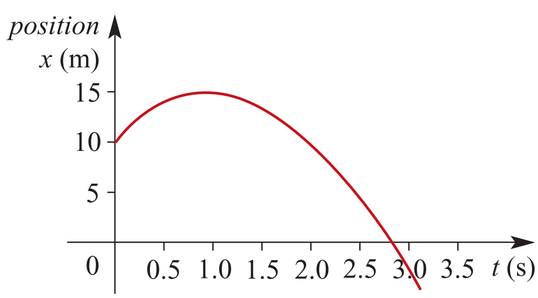
f an estimate of the velocity of the ball when $t = 5$.

6 A stone is thrown vertically upwards from the edge of a platform which is 10 m above the ground. The position–time graph for the motion of the stone is shown. The motion of the stone is in a straight line and the reference point for position is taken as a point at ground level, directly below where the stone was thrown.

a From the graph estimate the instantaneous velocity with which the stone is thrown.

b What is the maximum height reached by the stone?

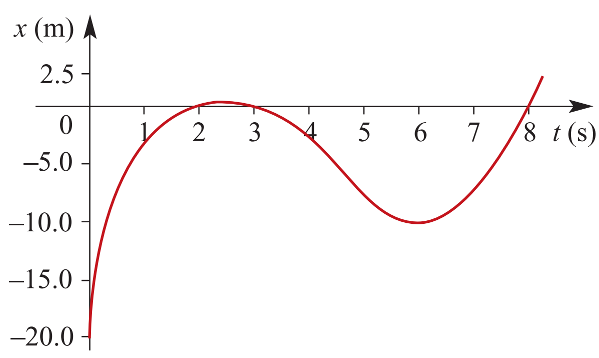
c At what time does the stone reach its maximum height?



d At what time does the stone hit the ground?

e From the graph estimate the instantaneous speed at which the stone hits the ground.

7 A particle is moving in a horizontal straight line. Position is measured from a point O. The particle starts at a point 20 m to the left of O. The position–time graph for the motion of the particle is as shown.



a At which times is the particle at O?

b For which values of t is the particle moving to the right?

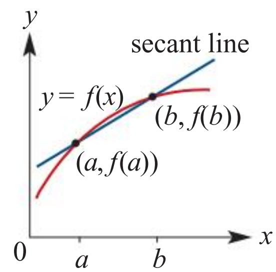
c For which values of t is the particle stationary?

ISBN 978-1-009-53719-3

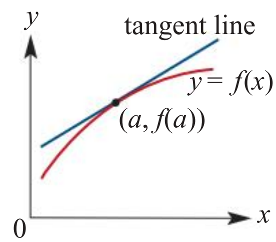
Review

Chapter summary

- Where a real-life situation is modelled by a straight-line graph, the gradient represents the rate of change of one quantity with respect to another.
- Average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$
- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.



- For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



- The **position** of a particle moving along a straight line specifies its location relative to a reference point O.
- The **average velocity** of the particle is the average rate of change in position with respect to time over a given time interval.
- The **instantaneous velocity** is the instantaneous rate of change in position with respect to time at a given moment.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

13A **1** I can interpret a graph relating two variables. ☐

See Example 1, Example 2 and Questions 1 and 2

13A **2** I can recognise where the rate of change for a graph of a function is positive or negative or zero . ☐

See Example 3 and Questions 10

13B **3** I can work with the constant rate of change of linear functions. ☐

See Example 4, Example 5 and Questions 1 and 2

 ISBN 978-1-009-53719-3

Review

13C **4** I can calculate the average speed of a body given the distance travelled and the tine taken. ☐

See Example 6 and Question 1

13C **5** I can calculate the average rate of change of a function over an interval. ☐

See Example 7, Example 8, Example 9 and Questions 2, 3 and 6.

13D **6** I can estimate the instantaneous rate of change of a function $y = f(x)$ at a point by considering the gradient of a secant through two points 'very close' to the given point. ☐

See Example 10, Example 11, Example 12 and Questions 1, 2 and 3.

13E **7** I can calculate the average velocity of a body moving in a straight line. ☐

See Example 13 and Question 1

13E **8** I can draw the velocity-time graph corresponding to a position-time graph when the position-time graph consists of straight line segments. ☐

See Example 14, Example 15 and Questions 2 and 3

13E **9** I can interpret information about motion in a straight line from a given rule for position and velocity-time graphs and position-time graphs. ☐

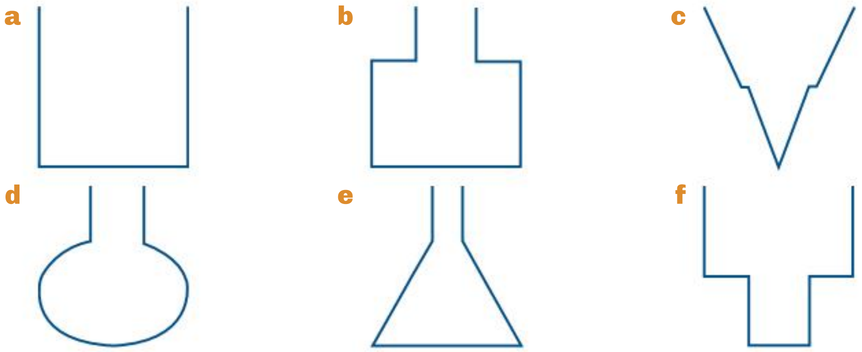
See Example 16 and Question 4

Short-response questions

Technology-free short-response questions

SF

1 A liquid is poured at a constant rate into each of the containers shown below. For each container, draw a graph to show how the depth of the water varies with time.



2 The surface area, s cm^2, of a cube is given by the formula $s = 6x^2$, where x cm is the length of each edge of the cube. Determine the average rate at which the surface area changes with respect to x, as x increases from $x = 2$ to $x = 4$.

ISBN 978-1-009-53719-3

Review

SF

3 **a** A car travels from New Delhi to Agra, a distance of 200 km, in 3 hours. Assuming the car travels at a constant speed, draw a distance–time graph and calculate this constant speed. For the distance–time graph, use minutes as the unit of time.

b A girl walks at a constant speed of 1.5 m/s for 20 seconds. Draw a distance–time graph to illustrate this.

4 A car travels along a road for $6\frac{1}{2}$ hours. The trip can be described in three sections:

Section A Car travels at a constant speed of 40 km/h for 1 hour.

Section B Car travels at a constant speed of 80 km/h for $3\frac{1}{2}$ hours.

Section C Car travels at a constant speed of 90 km/h for 2 hours.

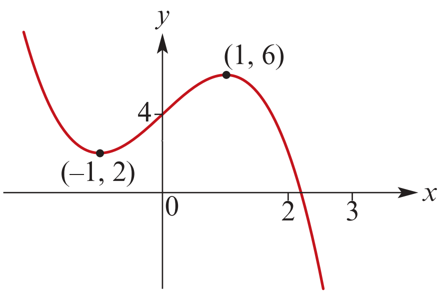
Draw a distance–time graph which illustrates this motion.

5 Let $s(t) = 5t + 5t^2$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres. Determine the average velocity for the time interval $[0, 1]$.

6 Let $y = x^3$. Determine the average rate at which y changes with respect to x over each of the following intervals:

a $[0, 1]$ **b** $[1, 3]$

7 The graph of $y = 4 + 3x - x^3$ is shown to the right.



a Determine the average rate of change for $-1 \le x \le 1$.

b State the intervals for which the rate of change of y with respect to x is

i positive **ii** negative

8 A vending machine in a bus terminus contains cans of soft drink. On a typical day:

- the machine starts one-quarter full
- no drinks are sold between 1 a.m. and 6 a.m.
- the machine is filled at 2 p.m.

Sketch a graph to show how the number of cans in the machine may vary from 6 a.m. until midnight.

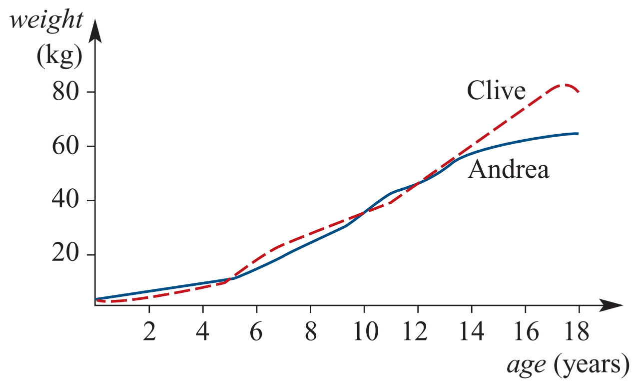
CF

9 In the following, $f(x)$ is the rule for a well-behaved function f.
Assume that, for $y = f(x)$, the average rate of change of y with respect to x is m, over the interval $[a, b]$. Determine the average rate of change of y with respect to x over the same interval $[a, b]$ for:

a $y = f(x) + c$ **b** $y = cf(x)$ **c** $y = -f(x)$

 ISBN 978-1-009-53719-3 Cambridge University Press

10 These graphs compare the weights of two people over the first 18 years of their lives.



a What was the average rate of change of weight with respect to time for Andrea between the ages of 0 and 18?

b What was the average rate of change of weight with respect to time for Clive between the ages of 0 and 18?

c During which periods did Andrea weigh more than Clive?

d During which periods of time was Clive growing more rapidly than Andrea?

Technology-active short-response questions

11 Let $f(x) = \log_2(x)$. Determine the average rate of change of this function over each of the following intervals. Give answers correct to two decimal places.

a $0.5 \leq x \leq 1.5$ **b** $1.5 \leq x \leq 2.5$ **c** $2.5 \leq x \leq 3.5$

12 Let $f(x) = \tan(x)$. Determine the average rate of change of this function over each of the following intervals. Give answers correct to two decimal places.

a $0 \leq x \leq \frac{\pi}{8}$ **b** $0 \leq x \leq \frac{\pi}{4}$ **c** $0 \leq x \leq \frac{3\pi}{8}$

13 Let $s(t) = 4t - 6t^3$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Determine the average velocity for the time interval [0, 1].

b Determine the average velocity for the time interval [0.9, 1].

c Estimate the instantaneous velocity for $t = 1$.

14 The resistance of a copper wire is measured at various temperatures with the following results:

Temperature (°C)	10	20	30	40	50	60
Resistance (ohms)	23.4	23.9	24.4	24.9	25.5	26.0

a Plot the data to see whether the resistance rises approximately linearly with temperature.

b If it does, determine the rate of increase in ohms per degree.

15 **a** $P(a, a^2)$ and $Q(b, b^2)$ are two points on the curve with equation $y = x^2$. Determine the gradient of the line joining the points. (Answer in terms of a and b.)

b Use this result to determine the gradient of the line for points with $a = 1$ and $b = 2$.

c Use this result to determine the gradient of the line if $a = 2$ and $b = 2.01$.

Review

CF

16 A rock is allowed to fall from the top of a high cliff. It falls y metres in t seconds, where $y = 4.9t^2$.

a Determine the average speed of the rock between:

i $t = 0$ and $t = 2$ **ii** $t = 2$ and $t = 4$

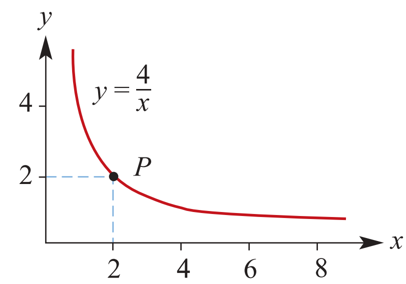
b **i** How far has the rock fallen between $t = 4 - h$ and $t = 4$?

ii What is the average speed between $t = 4 - h$ and $t = 4$?

iii Determine the average speed when $h = 0.2, 0.1, 0.05, 0.01, 0.001$.

17 The figure shows part of the curve with equation $y = \dfrac{4}{x}$ and the point $P(2, 2)$.

a A_1 and A_2 are points on the curve whose x-coordinates are 1.5 and 2.5 respectively. Use your calculator to determine their y-coordinates and hence determine the gradient of A_1A_2.



b Repeat for B_1 and B_2 whose x-coordinates are 1.9 and 2.1 respectively.

c Repeat for C_1 and C_2 whose x-coordinates are 1.99 and 2.01 respectively.

d Repeat for D_1 and D_2 whose x-coordinates are 1.999 and 2.001 respectively.

18 **a** Draw the graph of $y = 10^x$ and determine the gradient of the secant through the points:

i $x = 2.5$ and $x = 2.8$ **ii** $x = 2.6$ and $x = 2.8$

iii $x = 2.7$ and $x = 2.8$ **iv** $x = 2.75$ and $x = 2.8$

b Comment on your result and investigate further.

19 Let $s(t) = 12t - 4t^2$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Determine the average velocity for the time interval $[0, 3]$.

b Determine the average velocity for the time interval $[0, 1.5]$.

c Determine the average velocity for the time interval $[1.5, 3]$.

20 Let $s(t) = \begin{cases} 2t & 0 \le t \le 3 \\ -\dfrac{3t}{2} + \dfrac{21}{2} & 3 < t \le 7 \end{cases}$

be the position function of a particle moving in a straight line, where t is in seconds and s is in metres. State the rule for the velocity function $v(t)$.

21 **a** Use the result that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ to determine an expression for the gradient of the line joining points $P(a, a^3)$ and $Q(b, b^3)$ on the curve with equation $y = x^3$.

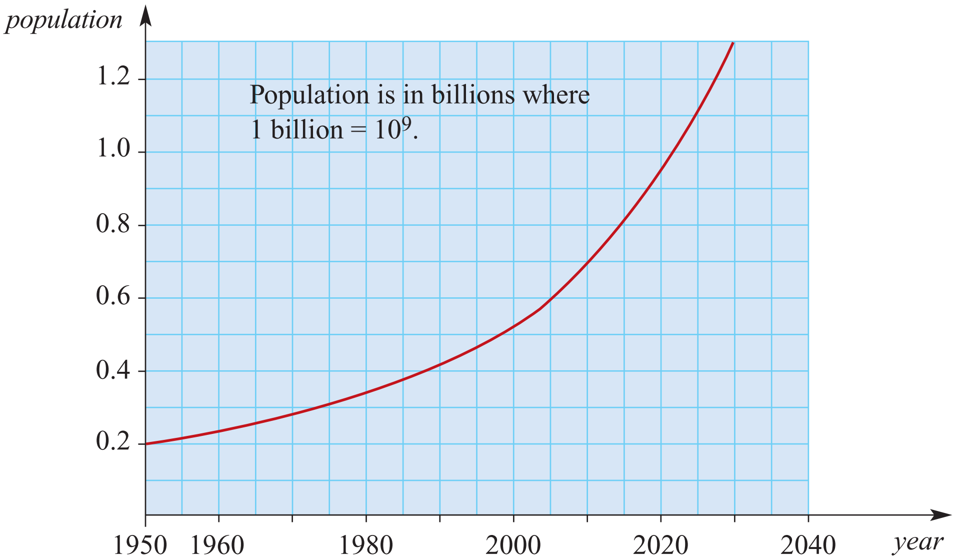
b Determine the gradient of the line for $a = 1$, $b = 2$.

c Determine the gradient of the line for $a = 2$, $b = 2.01$.

d For your expression for the gradient in terms of a and b (from part a), let $a = b$ and write your new expression in simplest terms. Interpret this result.

 ISBN 978-1-009-53719-3

22 The graph below shows growth in the size of the population of Acubaland. As the population grows, the rate of growth also increases.



a From the graph determine the population of Acubaland in:

i 1960 **ii** 2000

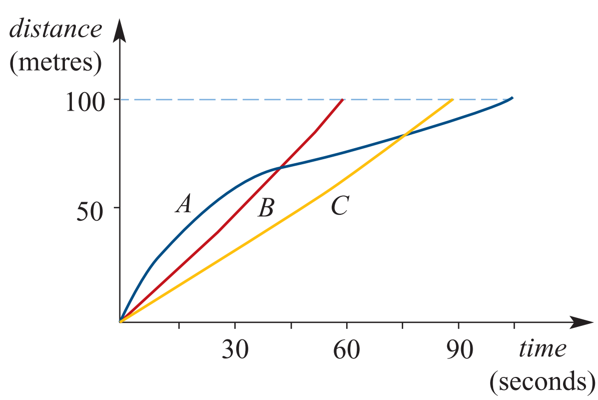
b Calculate the average annual rate of population increase (in billions per year) over the years from 1960 to 2000.

c From the graph estimate the rate of population increase in:

i 1960 **ii** 2000

d How many years do you expect that it will take to double the 2020 population? Explain your reasoning.

23 The rough sketch graph below shows what happens when three swimmers compete in a 100-metre race. (The vertical axis shows distance travelled by a swimmer.)



a Who wins the race?

b Who is in front at the 50 m mark?

c What is the approximate distance separating first and third place when the winner finishes?

d What is the approximate time difference between first and third place?

e What is the average speed of each swimmer?

f Describe the race as if you were a commentator.

Review

Multiple-choice questions

Technology-free multiple-choice questions

1 Postal workers sort 12 000 letters during the normal day shift of 8 hours and, with a reduced workforce during a 2-hour overtime shift, they sort a further 2500 letters. The average rate of letter sorting per hour is

A 1375 letters per hour **B** 1450 letters per hour
C 1300 letters per hour **D** 1400 letters per hour

2 The average rate of change of the function $y = 3 \times 2^x$ over the interval $[0, 2]$ is

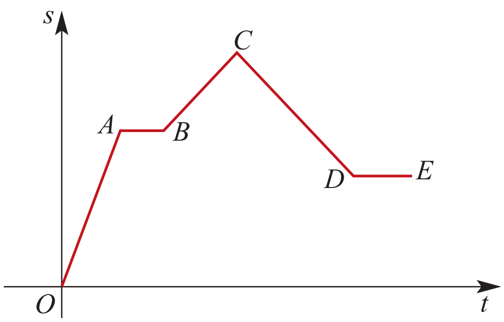
A 9 **B** 4.5 **C** 12 **D** 6

3 Let $f(x) = 5x^2 + 2$. The average rate of change of $f(x)$ with respect to x from $x = 2 - h$ to $x = 2$ is

A $5h^2 - 20h$ **B** $\dfrac{20h - 5h^2}{2}$ **C** $5h - 20$ **D** $20 - 5h$

Questions 4 to 6 refer to the following information:

The graph shows the movement of a vehicle over a period of time. It represents the distance (s) from a fixed point at a given time (t).



4 The line segment OA represents a stage of the movement during which the vehicle is

A speeding up **B** travelling at a constant speed
C slowing down **D** travelling north

5 The line segment AB represents a stage of the movement during which the vehicle is

A slowing down **B** travelling east
C travelling at a constant speed greater than zero
D stationary

6 The section(s) of the graph which represent(s) the vehicle when it is stationary is/are

A at O **B** at A and C **C** between C and D
D between A and B and between D and E

7 The average rate of change of $f(x) = x^3$ over the interval $a \le x \le b$ is

A $\dfrac{b^2 - a^2}{b^2 + a^2}$ **B** $\dfrac{b^3 + a^3}{2}$ **C** $b^2 + ab + a^2$ **D** $\dfrac{b - a}{b^2 + a^2}$

 ISBN 978-1-009-53719-3 Cambridge University Press

Technology-active multiple-choice questions

8 A bushwalker walks 12 km in 2 hours, stops for 45 minutes and then walks a further 8 km in another 1.25 hours. The average walking speed of the bushwalker over the entire walk is

A 10 km/h **B** 9 km/h **C** 5 km/h **D** 4 km/h

9 The population of trout in a trout pond is growing. If the population, P, after t weeks is given by $P = 10 \times 1.1^t$, the average rate of growth of the population during the 5th week is closest to

A 16 trout per week **B** 15 trout per week
C 1.5 trout per week **D** 4 trout per week

10 Given $f(x) = 2x^3 + 3x$, the average rate of change of $f(x)$ with respect to x for the interval $[-2, 2]$ is

A 11 **B** -22 **C** -11 **D** 22

11 The polynomial function with rule $P(x)$ is such that $P(1) = -2.5$ and the average rate of change of P over the interval $1 \le x \le 4$ is 8.2. The value of $P(4)$ is

A 27.1 **B** -22.6 **C** -11.8 **D** 22.1

12 A rock is dropped off the edge of a cliff on the Moon. The distance, $d(t)$, in meters, the rock travels after t seconds can be modelled by the function $d(t) = 0.85t^2$. What was the average speed, in metres per second, of the rock between 4 and 9 seconds after it was dropped?

A 2.21 **B** 11.05 **C** 5.24 **D** 6.5

13 Isa's parents record her height on her birthday every year. Below is a table of her height in centimetres on each of her birthdays for the first five years. What is Isa's average rate of change in height with respect to her age from her 1st birthday to her 5th birthday?

year	0	1	2	3	4	5
height(cm)	54.5	73.7	83.8	94	100.3	114.3

A 15.25 cm/year **B** 11.96 cm/year **C** 10.15 cm/year **D** 12.5 cm/year

14 Let $y = \cos x$. The average rate at which y changes with respect to x for $0 \le x \le \frac{\pi}{2}$, correct to two decimal places, is

A 0.62 **B** -1 **C** -0.54 **D** -0.64

15 Let $f(x) = 2.5x^3 + 3$. The average rate of change of $f(x)$ with respect to x from $x = 1$ to $x = 1 + h$ is

A $0.5(5h^3 + 3h + 3)$ **B** $\dfrac{5(h^2 + 3h + 3)}{2}$
C $\dfrac{h^3 + 3h^2 + 3}{2}$ **D** $\dfrac{h^3 + 3h^2 + 3}{2h}$

 ISBN 978-1-009-53719-3 Cambridge University Press

14

Differentiation of polynomials and power functions

Chapter contents

In the previous chapter, we investigated the rate of change of one quantity with respect to another quantity. In this chapter, we develop a technique for calculating the rate of change for polynomial functions. This technique is part of the area of mathematics called *calculus*.

It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

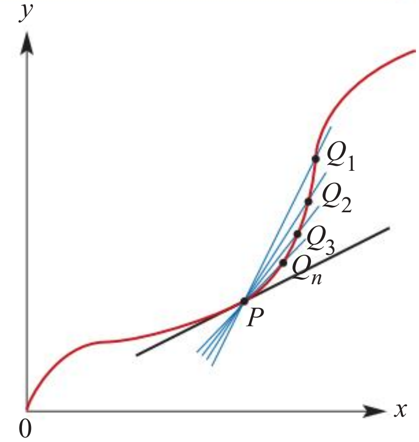
Calculus is now one of the most fundamental areas of mathematics, with applications in a wide range of fields including economics, meteorology and computer graphics.

14A The derivative

Learning intentions

- To be able to determine the gradient of a secant through two points on the curve.
- To be able to differentiate polynomials and power functions by first principles using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

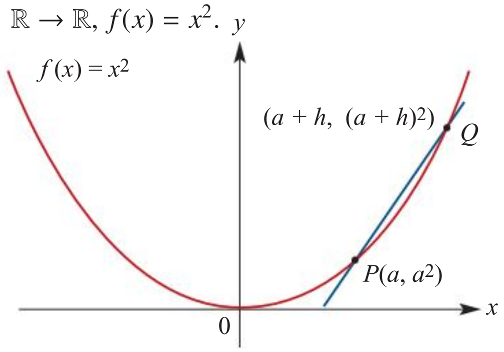


In the previous chapter we considered what happened when we looked at a sequence of secants $PQ_1, PQ_2, \ldots, PQ_n, \ldots$, where the points Q_i get closer and closer to P. The idea of instantaneous rate of change at P was introduced.

In this section we focus our attention on the gradient of the tangent at P.

The tangent to a curve at a point

Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.



The gradient of the secant PQ shown on the graph is

$$\begin{aligned} \text{gradient of } PQ &= \frac{(a+h)^2 - a^2}{a+h-a} \\ &= \frac{a^2 + 2ah + h^2 - a^2}{h} \\ &= 2a + h \end{aligned}$$

The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

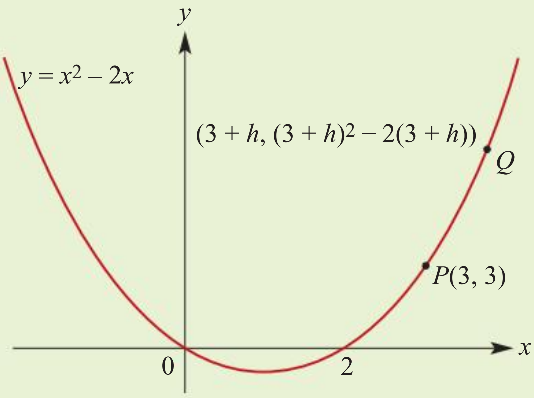
It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 1

By first considering the gradient of the secant PQ, determine the gradient of the tangent line to $y = x^2 - 2x$ at the point P with coordinates $(3, 3)$.



Solution

$$\text{Gradient of } PQ = \frac{(3+h)^2 - 2(3+h) - 3}{3 + h - 3}$$

$$= \frac{9 + 6h + h^2 - 6 - 2h - 3}{h}$$

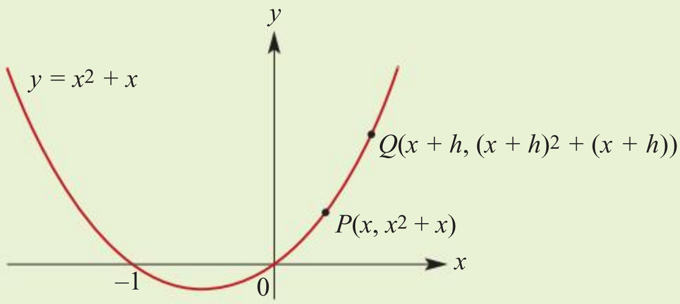
$$= \frac{4h + h^2}{h}$$

$$= 4 + h$$

Now consider the gradient of PQ as h approaches 0. The gradient of the tangent line at the point $P(3, 3)$ is 4.

Example 2

Determine the gradient of the secant PQ and hence determine the derivative of $x^2 + x$.



Solution

$$\text{Gradient of } PQ = \frac{(x+h)^2 + (x+h) - (x^2 + x)}{x + h - x}$$

$$= \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$

$$= \frac{2xh + h^2 + h}{h}$$

$$= 2x + h + 1$$

From this it is seen that the derivative of $x^2 + x$ is $2x + 1$.

 ISBN 978-1-009-53719-3

The expansion of $(a + b)^n$

You are already familiar with the identity

$$(a + b)^2 = a^2 + 2ab + b^2$$

By multiplying both sides by $a + b$, we can obtain new identities:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Limit notation

The notation for the limit of $2x + h + 1$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h + 1)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 3

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, determine the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\text{Gradient of } PQ = \frac{(2 + h)^3 - 8}{2 + h - 2}$$
$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= \frac{12h + 6h^2 + h^3}{h}$$
$$= 12 + 6h + h^2$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

The following example provides practice in determining limits.

Example 4

Determine:

a $\lim_{h \to 0} (22x^2 + 20xh)$ **b** $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$ **c** $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} (22x^2 + 20xh) = 22x^2$

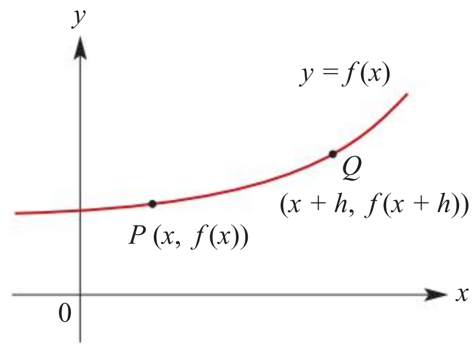
b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h} = \lim_{h \to 0} (3x^2 + 2h)$
$= 3x^2$

c $\lim_{h \to 0} 4 = 4$

Definition of the derivative

In general, consider the graph $y = f(x)$ of a function $f : \mathbb{R} \to \mathbb{R}$.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h}$$



The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.

Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Sections 14E and 14F.

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 5

For $f(x) = x^2 + 2x$, determine $f'(x)$ by first principles.

Solution

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h \to 0} (2x + h + 2) \\ &= 2x + 2 \qquad \therefore \quad f'(x) = 2x + 2 \end{aligned}$$

 ISBN 978-1-009-53719-3

Example 6

For $f(x) = 2 - x^3$, determine $f'(x)$ by first principles.

Solution

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{2 - (x+h)^3 - (2 - x^3)}{h} \\ &= \lim_{h \to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - (2 - x^3)}{h} \\ &= \lim_{h \to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\ &= \lim_{h \to 0} (-3x^2 - 3xh - h^2) \\ &= -3x^2 \end{aligned}$$

$\therefore \quad f'(x) = -3x^2$

Exercise 14A

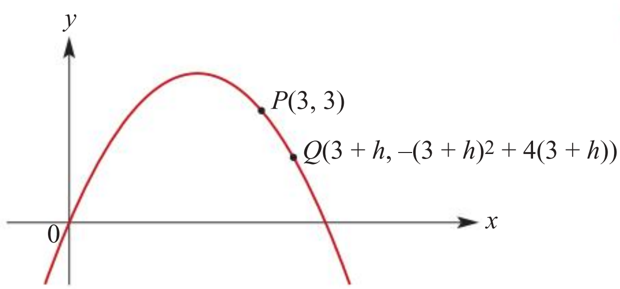
SF

Example 1

1 Let $f(x) = -x^2 + 4x$.
The graph of $y = f(x)$ is shown opposite.

a Determine the gradient of PQ.

b Determine the gradient of the tangent to the curve at P by considering what happens as h approaches 0.



2 Let $f(x) = x^2 - 3x$. Then the points $P(4, 4)$ and $Q(4 + h, (4 + h)^2 - 3(4 + h))$ are on the curve $y = f(x)$.

a Determine the gradient of the secant PQ.

b Determine the gradient of the tangent line to the curve at the point P by considering what happens as h approaches 0.

Example 2

3 The points $P(x, x^2 - 2x)$ and $Q(x + h, (x + h)^2 - 2(x + h))$ are on the curve $y = x^2 - 2x$. Determine the gradient of PQ and hence determine the derivative of $x^2 - 2x$.

Example 3

4 By first considering the gradient of the secant through $P(2, 16)$ and $Q(2 + h, (2 + h)^4)$ for the curve $y = x^4$, determine the gradient of the tangent to the curve at the point $(2, 16)$.
Hint: $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

 ISBN 978-1-009-53719-3

CF

5 A space vehicle moves so that the distance travelled over its first minute of motion is given by $y = 4t^4$, where y is the distance travelled in metres and t the time in seconds. By determining the gradient of the secant through the points where $t = 5$ and $t = 5 + h$, calculate the speed of the space vehicle when $t = 5$.

6 A population of insects grows so that the size of the population, P, at time t (days) is given by $P = 1000 + t^2 + t$. By determining the gradient of the secant through the points where $t = 3$ and $t = 3 + h$, calculate the rate of growth of the insect population at time $t = 3$.

SF

Example 4

7 Determine:

a $\lim\limits_{h\to 0}(10x^2 - 5xh)$ **b** $\lim\limits_{h\to 0}(20 - 10h)$ **c** $\lim\limits_{h\to 0}\dfrac{2x^2h^3 + xh^2 + h}{h}$

d $\lim\limits_{h\to 0}\dfrac{3x^2h - 2xh^2 + h}{h}$ **e** $\lim\limits_{h\to 0}\dfrac{30hx^2 + 2h^2 + h}{h}$ **f** $\lim\limits_{h\to 0}5$

8 Determine:

a $\lim\limits_{h\to 0}\dfrac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h}$ i.e. the derivative of $y = x^2 + 2x$

b $\lim\limits_{h\to 0}\dfrac{(5+h)^2 + 3(5+h) - 40}{h}$ i.e. the gradient of $y = x^2 + 3x$ at $x = 5$

c $\lim\limits_{h\to 0}\dfrac{(x+h)^3 + 2(x+h)^2 - (x^3 + 2x^2)}{h}$ i.e. the derivative of $y = x^3 + 2x^2$

CF

9 For the curve with equation $y = 3x^2 - x$:

a Determine the gradient of the secant PQ, where P is the point $(1, 2)$ and Q is the point $(1 + h,\ 3(1 + h)^2 - (1 + h))$.

b Determine the gradient of PQ when $h = 0.1$.

c Determine the gradient of the tangent to the curve at P.

10 For the curve with equation $y = \dfrac{2}{x}$:

a Determine the gradient of the chord AB, where $A = (2, 1)$ and $B = \left(2 + h,\ \dfrac{2}{2 + h}\right)$.

b Determine the gradient of AB when $h = 0.1$.

c Determine the gradient of the tangent to the curve at A.

11 For the curve with equation $y = x^2 + 2x - 3$:

a Determine the gradient of the secant PQ, where P is the point $(2, 5)$ and Q is the point $(2 + h,\ (2 + h)^2 + 2(2 + h) - 3)$.

b Determine the gradient of PQ when $h = 0.1$.

c Determine the gradient of the tangent to the curve at P.

Example 5, 6

12 For each of the following, determine $f'(x)$ by determining $\lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$:

a $f(x) = 3x^2$ **b** $f(x) = 4x$ **c** $f(x) = 3$

d $f(x) = 3x^2 + 4x + 3$ **e** $f(x) = 2x^3 - 4$ **f** $f(x) = 4x^2 - 5x$

g $f(x) = 3 - 2x + x^2$ **h** $f(x) = 2x - x^3$ **i** $f(x) = 2x - 3x^2$

 ISBN 978-1-009-53719-3 Cambridge University Press

13 Consider the following two approximations for $f'(a)$, where h is small:

i $f'(a) \approx \dfrac{f(a+h) - f(a)}{h}$ **ii** $f'(a) \approx \dfrac{f(a+h) - f(a-h)}{2h}$

Compare these two approximations for each of the following:

a $f(x) = x^2,\ a = 2$ **b** $f(x) = x^3,\ a = 2$ **c** $f(x) = x^3 + 2x - 4,\ a = 2$

14 By first considering the gradient of the secant through $P(x, f(x))$ and $Q(x+h, f(x+h))$ for the curve $f(x) = x^4$, determine the derivative of x^4.

Hint: $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

14B Rules for differentiation

Learning intentions

▶ To be able to differentiate polynomials using the rule $f(x) = x^n$, $f'(x) = nx^{n-1}$.

The derivative of x^n where n is a positive integer

From your work in the previous section, you may have noticed that differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

Proof We can prove this result using the binomial expansion, which was introduced in Section 7C.

Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1} \end{aligned}$$

$$\therefore \quad \begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

Hence $f'(x) = nx^{n-1}$.

 ISBN 978-1-009-53719-3

The derivative of a polynomial function

The following results are very useful when determining the derivative of a polynomial function. We will prove two of these results in the final section of this chapter.

- **Constant function:** If $f(x) = c$, then $f'(x) = 0$.
- **Linear function:** If $f(x) = mx + c$, then $f'(x) = m$.
- **Multiple:** If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- **Sum:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- **Difference:** If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivative of products and quotients in Chapter 16.

The process of determining the derivative function is called **differentiation**.

Example 7

Determine the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $f(x) = x^5 - 2x^3 + 2$

Then $f'(x) = 5x^4 - 2(3x^2) + 0$

$= 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 8

Determine the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus determine $f'(1)$.

Solution

Let $f(x) = 3x^3 - 6x^2 + 1$

Then $f'(x) = 3(3x^2) - 6(2x) + 0$

$= 9x^2 - 12x$

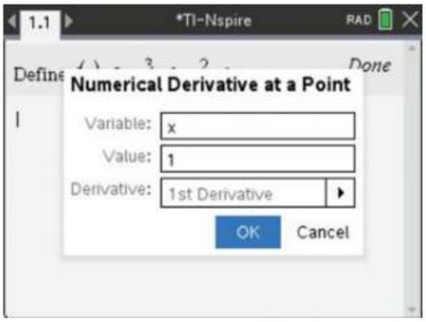
$\therefore\ f'(1) = 9 - 12$

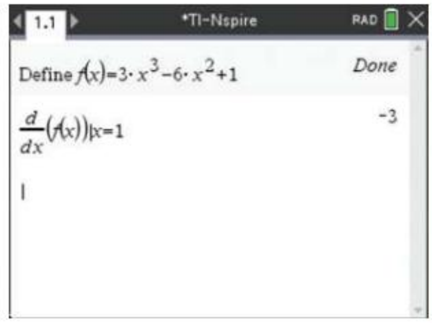
$= -3$

 ISBN 978-1-009-53719-3

Using the TI-Nspire CX non-CAS

- In a **Calculator** application, define $f(x) = 3x^3 - 6x^2 + 1$. Alternatively, the assign command := can be used (ctrl, ⊞, (:=))
- To determine the value of the derivative at $x = 1$, use menu > **Calculus** > **Numerical Derivative at a Point** and complete as shown.





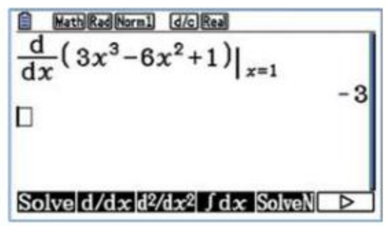
Using the Casio

To determine the derivative of $3x^3 - 6x^2 + 1$ at $x = 1$:

- Press MENU 1 to select **Run-Matrix** mode.
- Go to **Calculation** OPTN F4, then **d/dx** F2.
- Enter the expression $3x^3 - 6x^2 + 1$ and the x-value 1:

  3 X,θ,T ∧ 3 ► − 6 X,θ,T x² + 1

  ► 1 EXE



Determining the gradient of a tangent line

We defined the tangent line at a point on a graph in Section 14B. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 9

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, determine the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

 ISBN 978-1-009-53719-3

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, and so we write $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

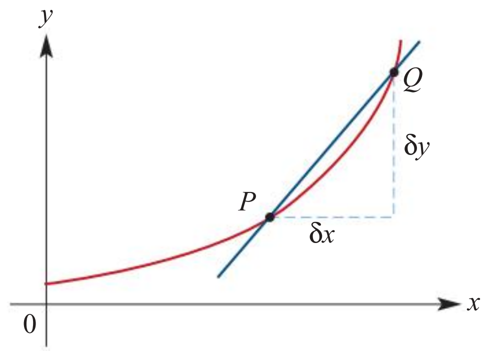
Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\frac{dz}{dt}$.

Warning: In this notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for determining the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



Example 10

a If $y = 2x^4$, determine $\frac{dy}{dx}$.

b If $y = t^2$, determine $\frac{dy}{dt}$.

c If $x = t^3 + t$, determine $\frac{dx}{dt}$.

d If $z = \frac{1}{3}x^3 + x^2$, determine $\frac{dz}{dx}$.

Solution

a $y = 2x^4$

$\frac{dy}{dx} = 8x^3$

b $y = t^2$

$\frac{dy}{dt} = 2t$

c $x = t^3 + t$

$\frac{dx}{dt} = 3t^2 + 1$

d $z = \frac{1}{3}x^3 + x^2$

$\frac{dz}{dx} = x^2 + 2x$

 ISBN 978-1-009-53719-3

Sometimes we need to expand or simplify an expression before we are able to determine its derivative. This is shown in the following example.

Example 11

a For $y = (x+3)^2$, determine $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, determine $\frac{dz}{dt}$.

c For $y = \frac{x^2+3x}{x}$, determine $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a It is first necessary to write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$

$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t+2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$

$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$

$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d

$$y = 2x^3 - 1$$

$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Determine the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'Determine $\frac{d}{dx}(2x^2 - 4x)$'. In general: $\frac{d}{dx}(f(x)) = f'(x)$.

Example 12

Determine:

a $\frac{d}{dx}(5x - 4x^3)$

b $\frac{d}{dz}(5z^2 - 4z)$

c $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 13

For each of the following curves, determine the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8

b $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = -6

Solution

a $y = x^3$ implies $\dfrac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\dfrac{8}{3}} = \dfrac{\pm 2\sqrt{6}}{3}$

The points are $\left(\dfrac{2\sqrt{6}}{3}, \dfrac{16\sqrt{6}}{9}\right)$ and $\left(\dfrac{-2\sqrt{6}}{3}, \dfrac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\dfrac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\dfrac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

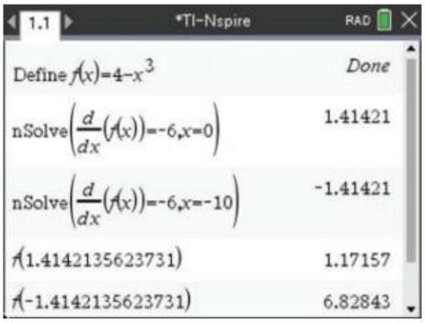
TI-84

Using the TI-Nspire CX non-CAS

The calculator can be used to obtain an approximate solution for Example 13c.

- Define $f(x) = 4 - x^3$.
- Solve the equation $\dfrac{d}{dx}(f(x)) = -6$.

Note: Use menu > **Algebra** > **Numerical Solve**.
To access the derivative template, use the template palette.



- To determine the y- coordinates, calculate the values of $f(-1.414\ldots)$ and $f(1.414\ldots)$ as shown.
- The points are $(-1.41, 6.83)$ and $(1.41, 1.17)$, correct to two decimal places.

 ISBN 978-1-009-53719-3

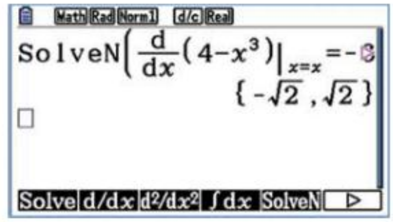
Using the Casio

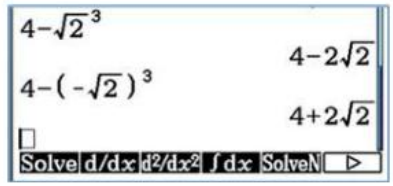
For Example 13c:

- Press MENU 1 to select **Run-Matrix** mode.
- Select the numerical solver by going to **Calculation** OPTN F4, then **SolveN** F5.
- Enter the equation $\dfrac{d}{dx}(4 - x^3) = -6$ as follows:

  F2 4 − X,θ,T ∧ 3 ► ► X,θ,T ►
  SHIFT • (−) 6) EXE

- Substitute each x-value into the rule $y = 4 - x^3$ to determine the corresponding y-value.
- The points are $(\sqrt{2}, 4 - 2\sqrt{2})$ and $(-\sqrt{2}, 4 + 2\sqrt{2})$.





Summary 14B

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- **Constant function:** If $f(x) = c$, then $f'(x) = 0$.
- **Linear function:** If $f(x) = mx + c$, then $f'(x) = m$.
- **Multiple:** If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- **Sum:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- **Difference:** If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.

For example, if $f(x) = 5x^3 - 10x^2 + 7$, then $f'(x) = 5(3x^2) - 10(2x) + 0 = 15x^2 - 20x$.

Exercise 14B

Example 7

SF

1 Determine the derivative of each of the following with respect to x:

a $x^2 + 4x$ **b** $2x + 1$ **c** $x^3 - x$

d $\dfrac{1}{2}x^2 - 3x + 4$ **e** $5x^3 + 3x^2$ **f** $-x^3 + 2x^2$

g $\dfrac{3x^3}{2} - 4x^2 + x$ **h** $\dfrac{5x^2}{3} - 4x + 2$ **i** $-2x - 3x^2 + 5x^4$

2 For each of the following, determine $f'(x)$:

a $f(x) = x^{12}$ **b** $f(x) = 3x^7$ **c** $f(x) = 5x$

d $f(x) = 5x + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^5 + 3x^4$ **h** $f(x) = 2x^4 - \dfrac{1}{3}x^3 - \dfrac{1}{4}x^2$ **i** $f(x) = 3 - 5x + 4x^2$

ISBN 978-1-009-53719-3

Example 8

3 For each of the following, determine $f'(1)$:

a $f(x) = x^6$ **b** $f(x) = 4x^5$ **c** $f(x) = 5x$

d $f(x) = 5x^2 + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^4 - 3x^3$ **h** $f(x) = 2x^4 - \frac{1}{3}x^3$ **i** $f(x) = -10x^3 - 2x^2 + 2$

4 For each of the following, determine $f'(-2)$:

a $f(x) = 5x^3$ **b** $f(x) = 4x^2$ **c** $f(x) = 5x^3 - 3x$ **d** $f(x) = -5x^4 - 2x^2$

Example 9

5 Determine the gradient of the tangent line to the graph of f at the given point:

a $f(x) = x^2 + 3x$, $(2, 10)$ **b** $f(x) = 3x^2 - 4x$, $(1, -1)$

c $f(x) = -2x^2 - 4x$, $(3, -30)$ **d** $f(x) = x^3 - x$, $(2, 6)$

Example 10

6 **a** If $y = t^2 - 7$, find $\frac{dy}{dt}$. **b** If $x = -5t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{2}x^4 - x^2$, find $\frac{dz}{dx}$.

Example 10, 11

7 For each of the following, determine $\frac{dy}{dx}$:

a $y = -x$ **b** $y = 10$ **c** $y = 4x^3 - 3x + 2$

d $y = \frac{1}{3}(x^3 - 3x + 6)$ **e** $y = (x + 1)(x + 2)$ **f** $y = 2x(3x^2 - 4)$

g $y = \frac{10x^5 + 3x^4}{2x^2}$, $x \neq 0$

8 **a** For $y = (x + 4)^2$, determine $\frac{dy}{dx}$.

b For $z = (4t - 1)^2(t + 1)$, determine $\frac{dz}{dt}$.

c For $y = \frac{x^3 + 3x}{x}$, determine $\frac{dy}{dx}$, $x \neq 0$.

9 **a** For the curve with equation $y = x^3 + 1$, find the gradient of the tangent line at points:

i $(1, 2)$ **ii** $(a, a^3 + 1)$

b Determine the derivative of $x^3 + 1$ with respect to x.

10 **a** Given that $y = x^3 - 3x^2 + 3x$, determine $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x, and interpret this in terms of the graph of $y = x^3 - 3x^2 + 3x$.

b Given that $y = \frac{x^2 + 2x}{x}$, for $x \neq 0$, determine $\frac{dy}{dx}$.

c Differentiate $y = (3x + 1)^2$ with respect to x.

11 For each of the following curves, determine the y-coordinate of the point on the curve with the given x-coordinate, and determine the gradient of the tangent line at that point:

a $y = x^2 - 2x + 1$, $x = 2$ **b** $y = x^2 + x + 1$, $x = 0$

c $y = x^2 - 2x$, $x = -1$ **d** $y = (x + 2)(x - 4)$, $x = 3$

e $y = 3x^2 - 2x^3$, $x = -2$ **f** $y = (4x - 5)^2$, $x = \frac{1}{2}$

SF

 ISBN 978-1-009-53719-3

12 **a** For each of the following, first determine $f'(x)$ and $f'(1)$. Then determine the values of x and $f(x)$ such that $f'(x) = 1$.

i $f(x) = 2x^2 - x$ **ii** $f(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2$

iii $f(x) = x^3 + x$ **iv** $f(x) = x^4 - 31x$

b In terms of the graph of $y = f(x)$, what is the interpretation of a value of x such that $f'(x) = 1$?

SF

Example 12 **13** Determine:

a $\frac{d}{dt}(3t^2 - 4t)$ **b** $\frac{d}{dx}(4 - x^2 + x^3)$ **c** $\frac{d}{dz}(5 - 2z^2 - z^4)$

d $\frac{d}{dy}(3y^2 - y^3)$ **e** $\frac{d}{dx}(2x^3 - 4x^2)$ **f** $\frac{d}{dt}(9.8t^2 - 2t)$

Example 13 **14** For each of the following curves, determine the coordinates of the points on the curve at which the gradient of the tangent line has the given value:

a $y = x^2$, gradient = 8 **b** $y = x^3$, gradient = 12

c $y = x(2 - x)$, gradient = 2 **d** $y = x^2 - 3x + 1$, gradient = 0

CF

14C Differentiating x^n where n is a negative integer

Learning intentions

- ▶ To be able to differentiate power functions with integer exponents and linear combinations of these.

In the previous section we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. For example,

- $f(x) = x^{-1}, \quad x \neq 0$
- $f(x) = 2x + x^{-1}, \quad x \neq 0$
- $f(x) = x + 3 + x^{-2} \quad x \neq 0$

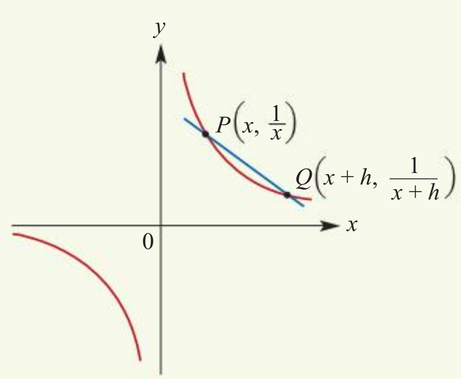
Example 14

Let $f(x) = \frac{1}{x}$. Determine $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

$$\frac{f(x+h) - f(x)}{x + h - x} = \left(\frac{1}{x+h} - \frac{1}{x}\right) \times \frac{1}{h}$$
$$= \frac{x - (x+h)}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-h}{(x+h)x} \times \frac{1}{h}$$
$$= \frac{-1}{(x+h)x}$$



So the gradient of the curve at P is

$$\lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = -x^{-2} \quad \text{and therefore } f'(x) = -x^{-2}.$$

Example 15

Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = x^{-3}$. Determine $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

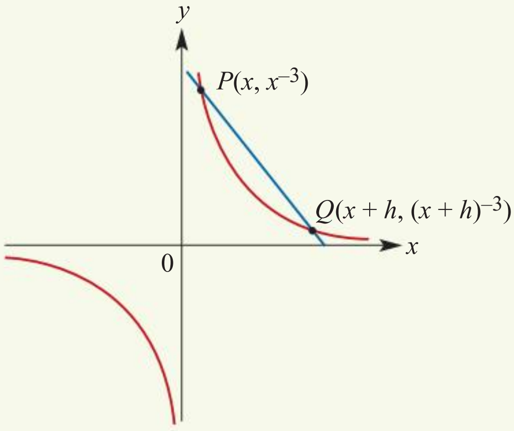
$$\frac{(x+h)^{-3} - x^{-3}}{h}$$

$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

We are now in a position to state the generalisation of the result we found in Section 14B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Note: We will consider rational indices in Chapter 16.

Example 16

Determine the derivative of:

a $2x^{-3} - x^{-1} + 2, \quad x \neq 0$

b $\dfrac{5x^3 + x}{x^2}, \quad x \neq 0$

Solution

a Let $f(x) = 2x^{-3} - x^{-1} + 2$

Then $f'(x) = 2(-3x^{-4}) - (-x^{-2}) + 0$

$= -6x^{-4} + x^{-2} \quad (\text{for } x \neq 0)$

b Let $f(x) = \dfrac{5x^3 + x}{x^2}$

$= 5x + x^{-1}$

Then $f'(x) = 5 + (-x^{-2})$

$= 5 - \dfrac{1}{x^2} \quad (\text{for } x \neq 0)$

 ISBN 978-1-009-53719-3 Cambridge University Press

Example 17

Determine the derivative f' of $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^2 - 6x^{-2} + 1$.

Solution

$$f'\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = 3(2x) - 6(-2x^{-3}) + 0$$
$$= 6x + 12x^{-3}$$

Example 18

Determine the gradient of the tangent to the curve determined by the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 + \dfrac{1}{x}$ at the point $(1, 2)$.

Solution

$$f'\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = 2x + (-x^{-2})$$
$$= 2x - x^{-2}$$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 19

Show that the derivative of the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$ is always negative.

Solution

$$f'\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = -3x^{-4}$$
$$= -\frac{3}{x^4}$$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

Summary 14C

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.
- For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Skill-sheet

Exercise 14C

Example 14

1 **a** Let $f(x) = \dfrac{1}{x-3}$, $x \neq 3$. Determine $f'(x)$ by first principles.

b Let $f(x) = \dfrac{1}{x+2}$ $x \neq -2$. Determine $f'(x)$ by first principles.

CF

Example 15

2 **a** Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-2}$. Find $f'(x)$ by first principles.

b Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-4}$. Find $f'(x)$ by first principles.

Hint: $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

 ISBN 978-1-009-53719-3

SF

Example 16

3 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\frac{3}{x^2} + 5x^2$ **c** $\frac{5}{x^3} + \frac{4}{x^2} + 1$

d $3x^2 + \frac{5}{3}x^{-4} + 2$ **e** $6x^{-2} + 3x$ **f** $\frac{3x^2 + 2}{x}$

4 Determine the derivative of each of the following:

a $\frac{3z^2 + 2z + 4}{z^2}, \ z \neq 0$ **b** $\frac{3 + z}{z^3}, \ z \neq 0$ **c** $\frac{2z^2 + 3z}{4z}, \ z \neq 0$

d $9z^2 + 4z + 6z^{-3}, \ z \neq 0$ **e** $9 - z^{-2}, \ z \neq 0$ **f** $\frac{5z - 3z^2}{5z}, \ z \neq 0$

Example 17

5 **a** Determine the derivative f' of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^4 - 6x^{-3} + x^{-1}$.

b Determine the derivative f' of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 5x^4 + 4x^{-2} + x^{-1}$.

6 Carefully sketch the graph of $f(x) = \frac{1}{x^2}, \ x \neq 0$.

a For points $P(1, f(1))$ and $Q(1 + h, f(1 + h))$, determine the gradient of the secant PQ.

b Hence determine the gradient of the tangent line to the curve $y = \frac{1}{x^2}$ at $x = 1$.

Example 18

7 For each of the following curves, determine the gradient of the tangent line to the curve at the given point:

a $y = x^{-2} + x^3, \ x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = \frac{x - 2}{x}, \ x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \frac{1}{x}, \ x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3}), \ x \neq 0$, at $(1, 1)$

CF

8 For the curve with equation $f(x) = x^{-2}$, determine the x-coordinate of the point on the curve at which the gradient of the tangent line is:

a 16 **b** -16

Example 19

9 Show that the derivative of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-1}$ is always negative.

14D Graphs of the derivative function

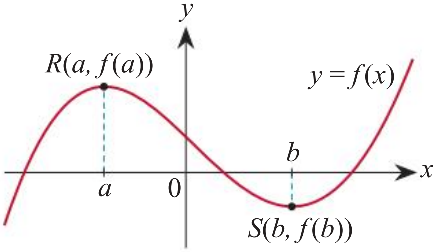
Learning intentions

- ▶ To understand the relationship between the graph of a function and the graph of the corresponding derivative function.

Sign of the derivative

Consider the graph of $y = f(x)$ shown here. At a point $(x, f(x))$ on the graph, the gradient is $f'(x)$.

By noting whether the curve is sloping upwards or downwards at a particular point, we can tell the sign of the derivative at that point:



Values of x	$x < a$	$x = a$	$a < x < b$	$x = b$	$x > b$
Sign of $f'(x)$	$f'(x) > 0$	$f'(a) = 0$	$f'(x) < 0$	$f'(b) = 0$	$f'(x) > 0$

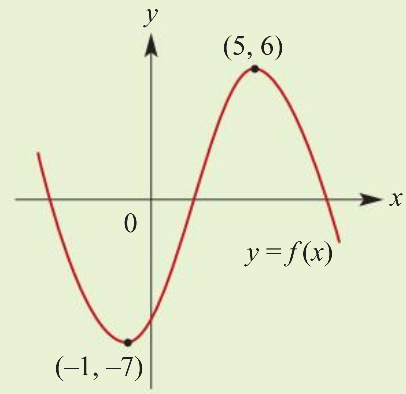
Example 20

For the graph of $f: \mathbb{R} \to \mathbb{R}$, determine:

a $\{x : f'(x) > 0\}$

b $\{x : f'(x) < 0\}$

c $\{x : f'(x) = 0\}$



Solution

a $\{x : f'(x) > 0\} = \{x : -1 < x < 5\} = (-1, 5)$

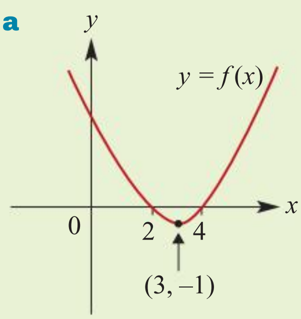
b $\{x : f'(x) < 0\} = \{x : x < -1\} \cup \{x : x > 5\} = (-\infty, -1) \cup (5, \infty)$

c $\{x : f'(x) = 0\} = \{-1, 5\}$

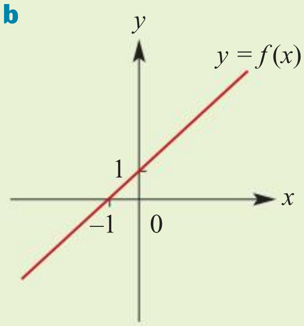
Example 21

Sketch the graph of $y = f'(x)$ for each of the following. (It is impossible to determine all features.)

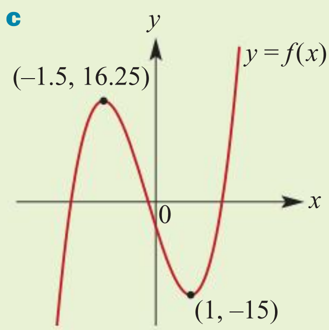
a



b



c

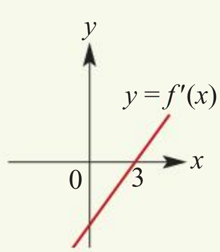


Solution

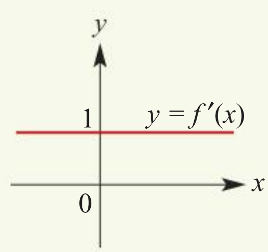
a $f'(x) > 0$ for $x > 3$

$f'(x) < 0$ for $x < 3$

$f'(x) = 0$ for $x = 3$



b $f'(x) = 1$ for all x

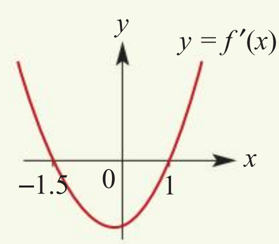


c $f'(x) > 0$ for $x > 1$

$f'(x) < 0$ for $-1.5 < x < 1$

$f'(x) > 0$ for $x < -1.5$

$f'(-1.5) = 0$ and $f'(1) = 0$



 ISBN 978-1-009-53719-3

If the rule for the function is given, then a graphics calculator can be used to plot the graph of its derivative function.

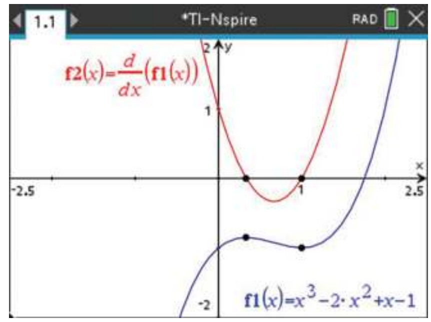
Using the TI-Nspire CX non-CAS

Plot the graphs of

$$f1(x) = x^3 - 2x^2 + x - 1$$

$$f2(x) = \frac{d}{dx}(f1(x))$$

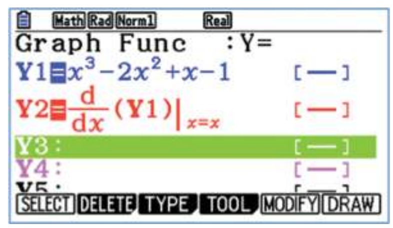
Note: Access the derivative template from .



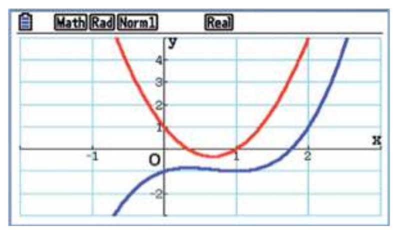
Using the Casio

- Press MENU 5 to select **Graph** mode.
- Enter the rule $y = x^3 - 2x^2 + x - 1$ in $Y1$.
- Enter the derivative of $Y1$ in $Y2$ as follows:

  OPTN F2 F1 F1 1 ► X,θ,T EXE

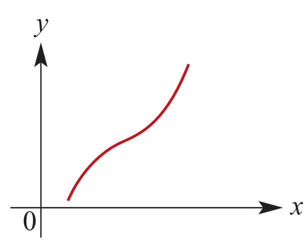


- Select **Draw** F6 to view the graphs.
- Adjust the View Window if required.



Increasing and decreasing functions

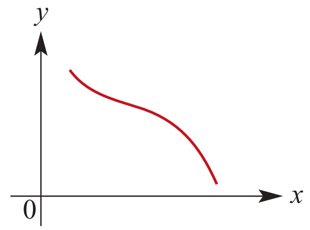
We say a function f is **strictly increasing** on an interval if $a < b$ implies $f(a) < f(b)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $a < b$ implies $f(a) > f(b)$.



For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^2$ is strictly decreasing.

Note: The word *strictly* refers to the use of the strict inequality signs $<, >$ rather than $\leq, \geq$.

Derivative tests

We can sometimes establish that a function is strictly increasing or strictly decreasing on an interval by checking the sign of its derivative on that interval.

- If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing. (Think of the tangents at each point – they each have positive gradient.)
- If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing. (Think of the tangents at each point – they each have negative gradient.)

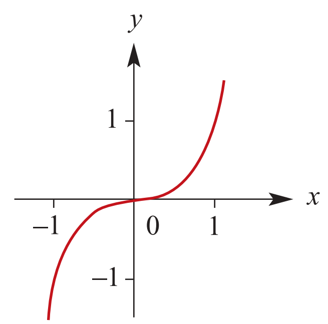
Warning: These two tests are not equivalent to the definitions of strictly increasing and strictly decreasing.

For example, the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

We can see that $f(x) = x^3$ is strictly increasing from its graph. Alternatively, consider

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ &= (a-b)\Big(a^2 + ab + \big(\tfrac{1}{2}b\big)^2 + b^2 - \big(\tfrac{1}{2}b\big)^2\Big) \\ &= (a-b)\Big(\big(a + \tfrac{1}{2}b\big)^2 + \tfrac{3}{4}b^2\Big) \end{aligned}$$

We can see that if $a < b$, then $a^3 < b^3$.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If θ is the angle that a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\theta$. That is:

$$m = \tan\theta$$

For example:

- If $\theta = 45^\circ$, then $\tan\theta = 1$ and the gradient is 1.
- If $\theta = 20^\circ$, then the gradient of the straight line is $\tan 20^\circ$.
- If $\theta = 135^\circ$, then $\tan\theta = -1$ and the gradient is -1.

 ISBN 978-1-009-53719-3

Example 22

Determine the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$$2x - 7 = 1 \qquad (\text{as } \tan 45° = 1)$$
$$2x = 8$$
$$\therefore \quad x = 4$$
$$y = 4^2 - 7 \times 4 + 8 = -4$$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$$2x - 7 = -2$$
$$2x = 5$$
$$\therefore \quad x = \frac{5}{2}$$

The coordinates are $\left(\frac{5}{2}, -\frac{13}{4}\right)$.

Example 23

The planned path for a flying saucer leaving a planet is defined by the equation

$$y = \frac{1}{4}x^4 + \frac{2}{3}x^3 \quad \text{for } x > 0$$

The units are kilometres. (The x-axis is horizontal and the y-axis vertical.)

a Determine the direction of motion when the x-value is:

i 2 **ii** 3

b Determine a point on the flying saucer's path where the path is inclined at 45° to the positive x-axis (i.e. where the gradient of the path is 1).

c Are there any other points on the path which satisfy the situation described in part **b**?

Solution

a $\dfrac{dy}{dx} = x^3 + 2x^2$

i When $x = 2$, $\dfrac{dy}{dx} = 8 + 8 = 16$

$\tan^{-1} 16 = 86.42°$ (to the x-axis)

ii When $x = 3$, $\dfrac{dy}{dx} = 27 + 18 = 45$

$\tan^{-1} 45 = 88.73°$ (to the x-axis)

b, c When the flying saucer is flying at 45° to the positive direction of the x-axis, the gradient of the curve of its path is given by $\tan 45°$. Thus to determine the point at which this happens we consider the equation

$$\frac{dy}{dx} = \tan 45°$$
$$x^3 + 2x^2 = 1$$
$$x^3 + 2x^2 - 1 = 0$$
$$(x + 1)(x^2 + x - 1) = 0$$

 ISBN 978-1-009-53719-3

$\therefore \quad x = -1$ or $x = \dfrac{-1 \pm \sqrt{5}}{2}$

The only acceptable solution is $x = \dfrac{-1 + \sqrt{5}}{2} \approx 0.62$, as the other two possibilities give negative values for x and we are only considering positive values for x.

Summary 14D

- A function f is **strictly increasing** on an interval if $a < b$ implies $f(a) < f(b)$.
- A function f is **strictly decreasing** on an interval if $a < b$ implies $f(a) > f(b)$.
- If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
- If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Skill-sheet

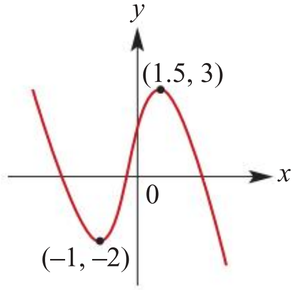
Exercise 14D

Example 20

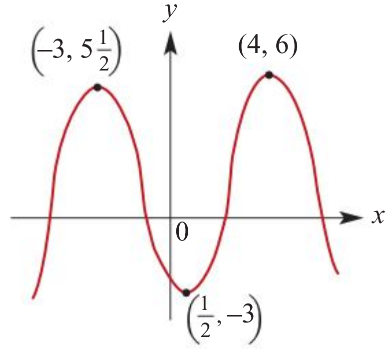
1 For each of the following, use the given graph of $y = f(x)$ to determine the sets:

i $\{x : f'(x) > 0\}$ **ii** $\{x : f'(x) < 0\}$ **iii** $\{x : f'(x) = 0\}$

a



b



2 For the function $f(x) = 2(x-1)^2$, determine the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -2$

3 For the function $f(x) = -3(x-2)^2 + 9$, determine the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -6$

4 For the function $f(x) = -3(x-2)^3 = -3x^3 + 18x^2 - 36x + 24$, determine the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) < 0$ **d** $f'(x) = -18$

5 For each of the following quadratic functions with rule $f(x)$, determine the rule of the derivative functions $f'(x)$ and in each case determine the values of x for which $f'(x) \geq 0$.

a $f(x) = x^2 - 4x + 6$ **b** $f(x) = -x^2 - 8x + 3$ **c** $f(x) = 3x^2 - 2x - 8$

d $f(x) = -2x^2 - 4x$ **e** $f(x) = -3x^2 - 11x + 3$ **f** $f(x) = 4x^2 - 2x - 5$

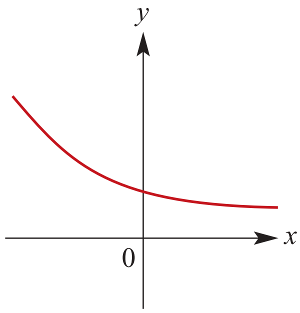
SF

 ISBN 978-1-009-53719-3

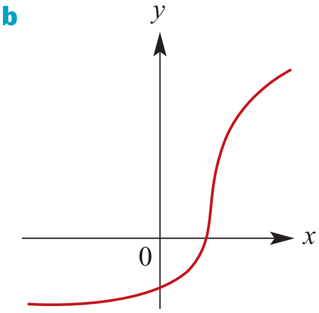
SF

6 For which of the following curves is $\dfrac{dy}{dx}$ positive for all values of x?

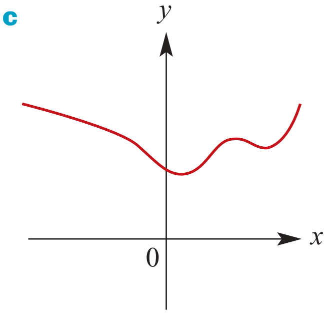
a



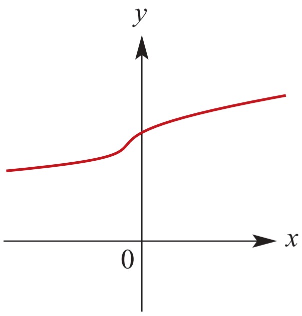
b



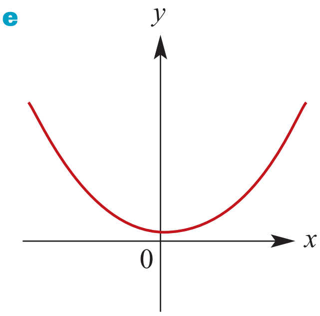
c



d

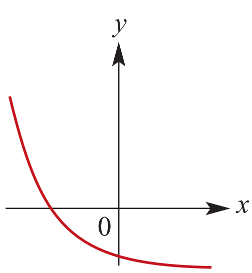


e

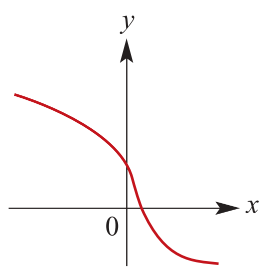


7 For which of the following curves is $\dfrac{dy}{dx}$ negative for all values of x?

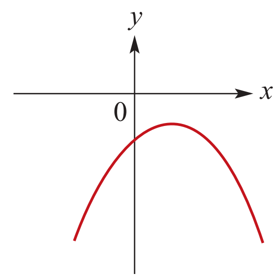
a



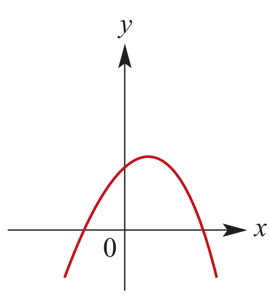
b



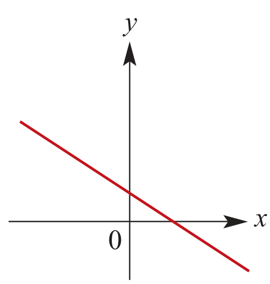
c



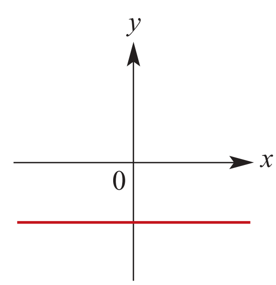
d



e



f



8 For each of the following $f(x)$ is the rule of a cubic function and $f'(x)$ is the rule of a quadratic function. Determine the values of x for which $f'(x) \le 0$.

a $f(x) = 2x^3 + 6x^2 - 48x$

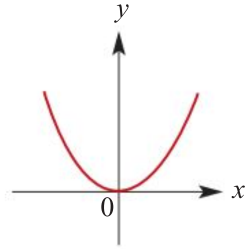
b $f(x) = \dfrac{x^3}{3} + 2x^2$

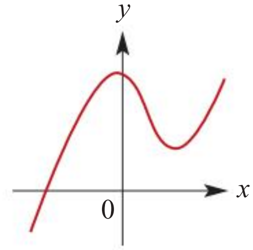
c $f(x) = \dfrac{x^3}{3} - 3x^2 + 4x - 5$

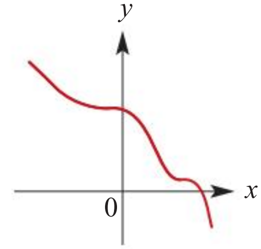
d $f(x) = x^3 - 6x^2 + 3x - 11$

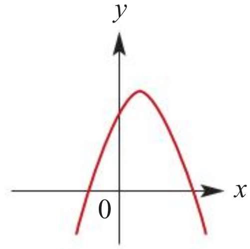
 ISBN 978-1-009-53719-3

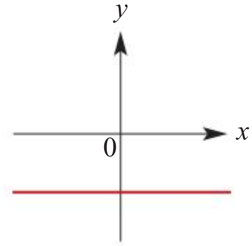
9 Which of the graphs labelled **A**–**F** correspond to each of the graphs labelled **a**–**f**?

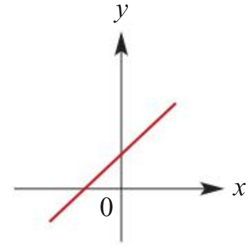
a


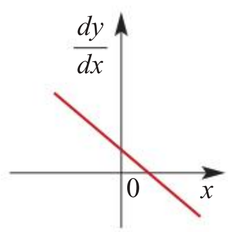
b


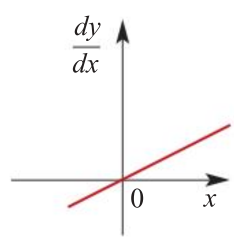
c


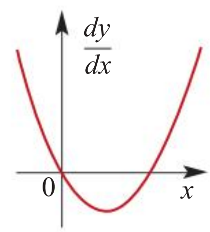
d


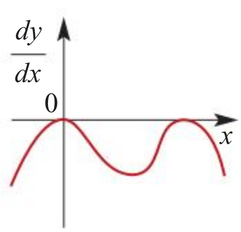
e


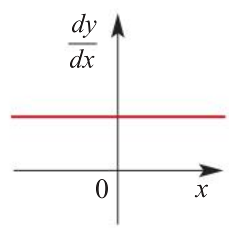
f


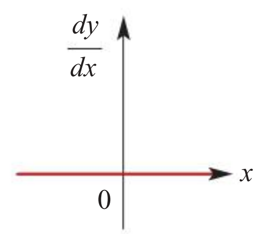
A


B


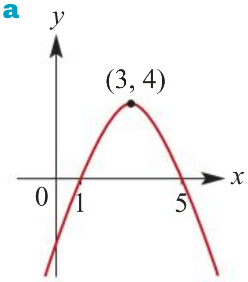
C


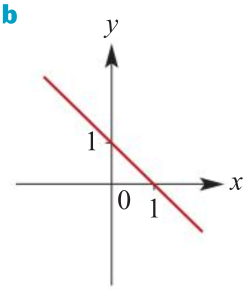
D


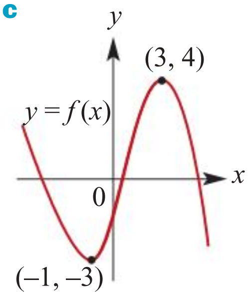
E


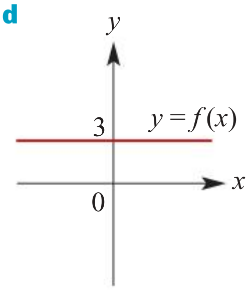
F


Example 21 **10** Sketch the graph of $y = f'(x)$ for each of the following:

a


b


c


d


Example 22 **11** Find the coordinates of the points on the curve $y = x^2 - 5x + 6$ at which the tangent:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 3x + 4$.

 ISBN 978-1-009-53719-3 Cambridge University Press

CF

12 Determine the coordinates of the points on the parabola $y = x^2 - x - 6$ at which:

a the gradient of the tangent is zero

b the tangent is parallel to the line $x + y = 6$.

13 Determine the coordinates of the points on the curve $y = -x^2 + 2$ at which the tangent

a makes an angle of 30° with the positive direction of the x-axis

b is parallel to the line $y = -x + 4$.

14 Determine the coordinates of the points on the curve $y = x^3 - 13x + 4$ at which the tangent

a makes an angle of 135° with the positive direction of the x-axis

b is parallel to the line $y = 2x - 3$.

15 Determine the coordinates of the points on the curve $y = (x-2)^3 + 6 = x^3 - 6x^2 + 12x - 2$ at which the tangent is parallel to the line $y = 3x + 4$.

16 Use a calculator to plot the graph of $y = f'(x)$ where:

a $f(x) = \sin x$ **b** $f(x) = \cos x$ **c** $f(x) = 2^x$

Example 23

17 The path of a particle is defined by the equation $y = \frac{1}{3}x^3 + \frac{2}{3}x^2$, for $x > 0$. The units are metres. (The x-axis is horizontal and the y-axis vertical.)

a Determine the direction of motion relative to the x-axis when the x-value is:

i 1 **ii** 0.5

b Determine a point on the particle's path where the path is inclined at 45° to the positive direction of the x-axis.

c Are there any other points on the path which satisfy the situation described in part **b**?

18 A car moves away from a set of traffic lights so that the distance, $S(t)$ metres, covered after t seconds is modelled by $S(t) = 0.2 \times t^3$.

a Determine its speed after t seconds. **b** What will its speed be when $t = 1, 3, 5$?

19 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Determine the values of a and b.

b Determine the coordinates of the point where the gradient is 0.

20 The curve with equation $y = ax^2 + \frac{b}{x}$ has a gradient of 4 at the point $(1, 5)$.

a Determine the values of a and b.

b Determine the coordinates of the points where the gradient is 0.

21 A rocket is launched from Cape York Peninsula so that after t seconds its height, $h(t)$ metres, is given by $h(t) = 20t^2$, $0 \le t \le 150$. After $2\frac{1}{2}$ minutes this model is no longer appropriate.

a Determine the height and the speed of the rocket when $t = 150$.

b After how long will its speed be 1000 m/s?

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

14E Limits and continuity

Learning intentions

- To be able to work with limits of a function at a point and to consider continuity.

Limits

It is not the intention of this course to provide a formal introduction to limits. We require only an intuitive understanding of limits and some fairly obvious rules for how to handle them.

> The notation $\lim_{x \to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'
>
> This means that we can make the value of $f(x)$ as close as we like to p, provided we choose x-values close enough to a.

We have met a similar idea earlier in the course. For example, we have seen that $\lim_{x \to \infty} f(x) = 4$ for the function with rule $f(x) = \frac{1}{x} + 4$. The graph of $y = f(x)$ can get as close as we like to the line $y = 4$, just by taking larger and larger values of x.

As we will see, for many functions (in particular, for polynomial functions), the limit at a particular point is simply the value of the function at that point.

Example 24

Determine $\lim_{x \to 2} 3x^2$.

Solution

$\lim_{x \to 2} 3x^2 = 3(2)^2 = 12$

Explanation

As x gets closer and closer to 2, the value of $3x^2$ gets closer and closer to 12.

If the function is not defined at the value for which the limit is to be found, a different procedure is used.

Example 25

For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, determine $\lim_{x \to 2} f(x)$.

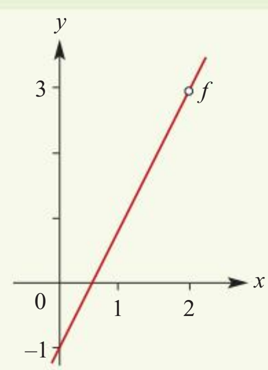
Solution

Observe that

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 2}{x - 2} \\ &= \frac{(2x - 1)(x - 2)}{x - 2} \\ &= 2x - 1 \qquad \text{(for } x \neq 2\text{)} \end{aligned}$$

Hence $\lim_{x \to 2} f(x) = 3$.

The graph of $f(x) = 2x - 1$, $x \neq 2$, is shown.



 ISBN 978-1-009-53719-3

We can investigate Example 25 further by looking at the values of the function as we take x-values closer and closer to 2.

Observe that $f(x)$ is defined for $x \in \mathbb{R} \setminus \{2\}$. Examine the behaviour of $f(x)$ for values of x close to 2.

$x < 2$	$x > 2$
$f(1.7) = 2.4$	$f(2.3) = 3.6$
$f(1.8) = 2.6$	$f(2.2) = 3.4$
$f(1.9) = 2.8$	$f(2.1) = 3.2$
$f(1.99) = 2.98$	$f(2.01) = 3.02$
$f(1.999) = 2.998$	$f(2.001) = 3.002$

From the table, it is apparent that, as x takes values closer and closer to 2 (regardless of whether x approaches 2 from the left or from the right), the values of $f(x)$ become closer and closer to 3. That is, $\lim\limits_{x \to 2} f(x) = 3$.

Note that the limit exists, but the function is not defined at $x = 2$.

Algebra of limits

The following important results are useful for the evaluation of limits.

Suppose that $f(x)$ and $g(x)$ are functions and that a is a real number. Assume that both $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist.

- **Sum:** $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- **Multiple:** $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$, where k is a given real number.
- **Product:** $\lim\limits_{x \to a} (f(x)\, g(x)) = \lim\limits_{x \to a} f(x) \ \lim\limits_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- **Quotient:** $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, provided $\lim\limits_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

Example 26

Determine:

a $\lim\limits_{h \to 0} (3h + 4)$ **b** $\lim\limits_{x \to 2} 4x(x + 2)$ **c** $\lim\limits_{x \to 3} \dfrac{5x + 2}{x - 2}$

Solution

a $\lim\limits_{h \to 0} (3h + 4) = \lim\limits_{h \to 0} (3h) + \lim\limits_{h \to 0} (4)$
$= 0 + 4$
$= 4$

b $\lim\limits_{x \to 2} 4x(x + 2) = \lim\limits_{x \to 2} (4x) \ \lim\limits_{x \to 2} (x + 2)$
$= 8 \times 4$
$= 32$

c $\lim\limits_{x \to 3} \dfrac{5x + 2}{x - 2} = \lim\limits_{x \to 3} (5x + 2) \div \lim\limits_{x \to 3} (x - 2)$
$= 17 \div 1$
$= 17$

 ISBN 978-1-009-53719-3

Example 27

Determine:

a $\lim\limits_{x\to3}\dfrac{x^2-3x}{x-3}$ **b** $\lim\limits_{x\to2}\dfrac{x^2-x-2}{x-2}$ **c** $\lim\limits_{x\to3}\dfrac{x^2-7x+10}{x^2-25}$

Solution

a $\lim\limits_{x\to3}\dfrac{x^2-3x}{x-3}=\lim\limits_{x\to3}\dfrac{x(x-3)}{x-3}=\lim\limits_{x\to3}x=3$

b $\lim\limits_{x\to2}\dfrac{x^2-x-2}{x-2}=\lim\limits_{x\to2}\dfrac{(x-2)(x+1)}{x-2}=\lim\limits_{x\to2}(x+1)=3$

c $\lim\limits_{x\to3}\dfrac{x^2-7x+10}{x^2-25}=\lim\limits_{x\to3}\dfrac{(x-2)(x-5)}{(x+5)(x-5)}=\dfrac{\lim\limits_{x\to3}(x-2)}{\lim\limits_{x\to3}(x+5)}=\dfrac{1}{8}$

Left and right limits

An idea which is useful in the following discussion is the existence of limits from the left and from the right. This is particularly useful when talking about piecewise-defined functions.

If the value of $f(x)$ approaches the number p as x approaches a from the right-hand side, then it is written as $\lim\limits_{x\to a^+}f(x)=p$.

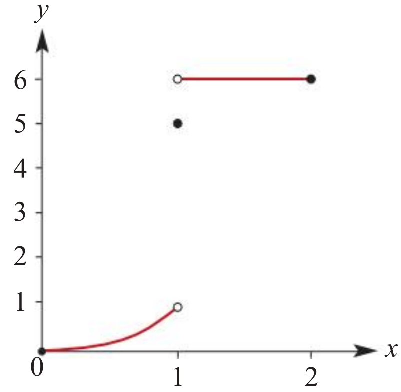
If the value of $f(x)$ approaches the number p as x approaches a from the left-hand side, then it is written as $\lim\limits_{x\to a^-}f(x)=p$.

The limit as x approaches a exists only if both the limit from the left and the limit from the right exist and are equal. Then $\lim\limits_{x\to a}f(x)=p$.

Piecewise-defined function

The following is an example of a piecewise-defined function where the limit does not exist for a particular value.

Let $f(x)=\begin{cases}x^3 & \text{if } 0\le x<1\\ 5 & \text{if } x=1\\ 6 & \text{if } 1<x\le 2\end{cases}$



It is clear from the graph of f that $\lim\limits_{x\to1}f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand, if x is allowed to approach 1 from the right, then $f(x)$ approaches 6. Also note that $f(1)=5$.

 ISBN 978-1-009-53719-3 Cambridge University Press

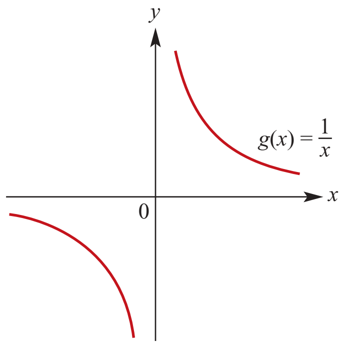
Rectangular hyperbola

As mentioned at the start of this section, the notation of limits is used to describe the asymptotic behaviour of graphs.

Consider the function defined by $g(x) = \frac{1}{x}$ for $x \neq 0$. The behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x\to 0^-} g(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} g(x) = \infty$$



Now examine this function as the magnitude of x becomes very large. It can be seen that, as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise, as x decreases without bound through negative values, the corresponding values of $g(x)$ also approach zero.

Symbolically this is written as:

$$\lim_{x\to\infty} g(x) = 0^+ \qquad \text{and} \qquad \lim_{x\to-\infty} g(x) = 0^-$$

Many functions approach a limiting value or limit as x approaches $\pm\infty$.

Continuity at a point

We only require an intuitive understanding of continuity.

A function with rule $f(x)$ is said to be continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise, there is said to be a discontinuity at $x = a$.

We can give a more formal definition of continuity using limits. A function f is continuous at the point $x = a$ provided $f(a)$, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ all exist and are equal.

We can state this equivalently as follows:

A function f is **continuous** at the point $x = a$ if the following conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x\to a} f(x) = f(a)$

The function is **discontinuous** at a point if it is not continuous at that point.

A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere. In contrast, the function

$$h(x) = \begin{cases} x^3 & \text{if } x < 1 \\ 6 & \text{if } x \geq 1 \end{cases}$$

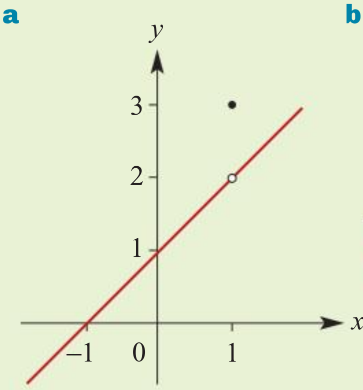
is defined for all real numbers but is not continuous at $x = 1$.

 ISBN 978-1-009-53719-3 Cambridge University Press

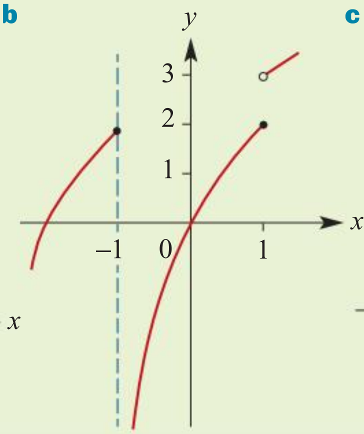
Example 28

State the values for x for which the functions shown below have a discontinuity:

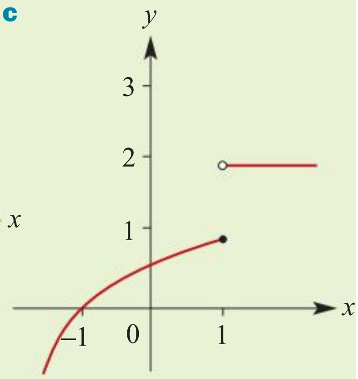
a



b



c



Solution

a There is a discontinuity at $x = 1$, since $f(1) = 3$ but $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$.

b There is a discontinuity at $x = 1$, since $f(1) = 2$ and $\lim_{x \to 1^-} f(x) = 2$ but $\lim_{x \to 1^+} f(x) = 3$.
There is also a discontinuity at $x = -1$, since $f(-1) = 2$ and $\lim_{x \to -1^-} f(x) = 2$ but $\lim_{x \to -1^+} f(x) = -\infty$.

c There is a discontinuity at $x = 1$, since $f(1) = 1$ and $\lim_{x \to 1^-} f(x) = 1$ but $\lim_{x \to 1^+} f(x) = 2$.

Example 29

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why:

a $f(x) = \begin{cases} 2x & \text{if } x \ge 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \ge 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \le -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x + 1 & \text{if } x \ge 0 \end{cases}$

d $f(x) = \begin{cases} x^2 + 1 & \text{if } x \ge 0 \\ -2x + 1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \ge 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

b $f(0) = 0$ but $\lim_{x \to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

c $f(-1) = -1$ but $\lim_{x \to -1^+} f(x) = 1$, therefore there is a discontinuity at $x = -1$.
$f(0) = 1$ but $\lim_{x \to 0^-} f(x) = 0$, therefore there is a discontinuity at $x = 0$.

d No discontinuity. **e** No discontinuity.

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 14E

- A function f is **continuous** at the point $x = a$ if the following conditions are met:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x \to a} f(x) = f(a)$
- The function is **discontinuous** at a point if it is not continuous at that point.
- A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere.
- **Algebra of limits** Suppose that $f(x)$ and $g(x)$ are functions and that a is a real number. Assume that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.
  - $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
    That is, the limit of the sum is the sum of the limits.
  - $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$, where k is a given real number.
  - $\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
    That is, the limit of the product is the product of the limits.
  - $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.
    That is, the limit of the quotient is the quotient of the limits.

Exercise 14E

SF

Example 24, 25

1 Determine the following limits:

a $\lim_{x \to 3} 15$ **b** $\lim_{x \to 6} (x - 5)$ **c** $\lim_{x \to \frac{1}{2}} (3x - 5)$

Example 26, 27

d $\lim_{t \to -3} \dfrac{t - 2}{t + 5}$ **e** $\lim_{t \to -1} \dfrac{t^2 + 2t + 1}{t + 1}$ **f** $\lim_{x \to 0} \dfrac{(x + 2)^2 - 4}{x}$

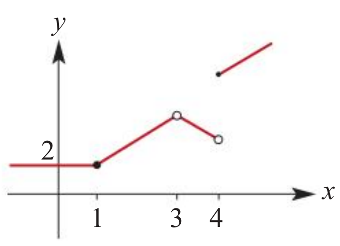
g $\lim_{t \to 1} \dfrac{t^2 - 1}{t - 1}$ **h** $\lim_{x \to 9} \sqrt{x + 3}$ **i** $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

j $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$ **k** $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}$ **l** $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

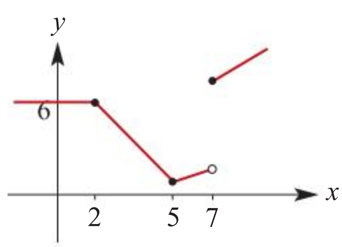
Example 28

2 For each of the following graphs, give the values of x at which a discontinuity occurs. Give reasons.

a



b



Example 29

3 For the following functions, state each value of x at which there is a discontinuity. Use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity.

a $f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \geq 0 \end{cases}$

4 The rule of a particular function is given below. For what values of x is the graph of this function discontinuous?

$$y = \begin{cases} 2 & \text{if } x < 1 \\ (x-4)^2 - 9 & \text{if } 1 \leq x < 7 \\ x - 7 & \text{if } x \geq 7 \end{cases}$$

SF

14F When is a function differentiable?

Learning intentions

- To be able to consider if certain functions are differentiable at a point.

A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

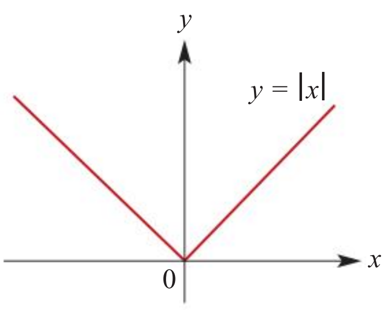
The polynomial functions considered in this chapter are differentiable for all real numbers. However, this is not true for all functions.

For example, consider the function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

This function is called the modulus function (or the absolute-value function) and is denoted by $f(x) = |x|$.

The graph of $y = f(x)$ is shown on the right.



Now consider the gradient of the secant through the points $(0, 0)$ and $(h, f(h))$ on the graph of $y = f(x)$:

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

The gradient of the secant does not approach a unique value as $h \to 0$.

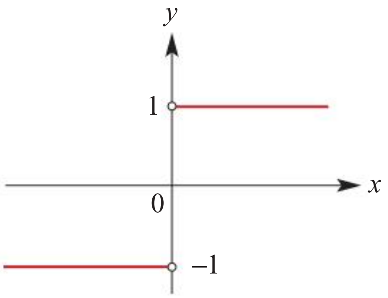
We say that $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

 ISBN 978-1-009-53719-3

The gradient of the curve $y = f(x)$ is 1 to the right of 0, and -1 to the left of 0. Therefore the derivative function is given by

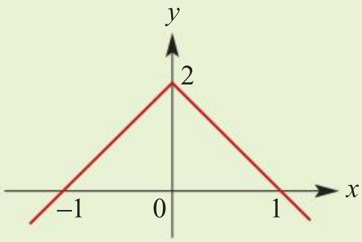
$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The graph of $y = f'(x)$ is shown on the right.



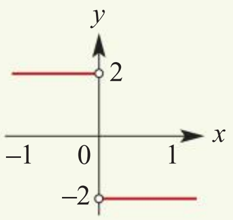
Example 30

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$,
i.e. the function is not differentiable at $x = 0$.



It was shown in the previous section that some piecewise-defined functions are continuous everywhere. Similarly, there are piecewise-defined functions which are differentiable everywhere. The smoothness of the 'joins' determines whether this is the case.

Example 31

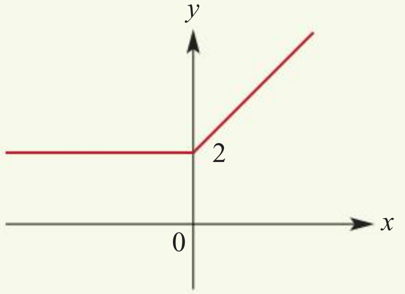
For the function with following rule, determine $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

In particular, $f'(0)$ is defined and is equal to 2. The two sections of the graph of $y = f(x)$ join smoothly at the point $(0, 1)$.



Example 32

For the function with rule

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

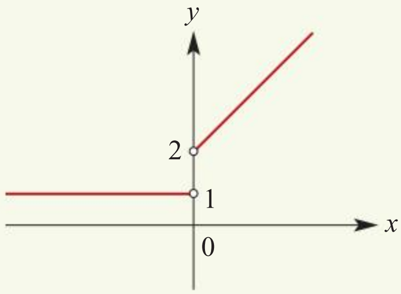
state the set of values for which the derivative is defined, determine $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$.

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$f'(0)$ is not defined as the limits from the left and right are not equal.

The function f is differentiable for $\mathbb{R} \setminus \{0\}$.



Proofs of two results for derivatives

Derivative of a constant function

If f is a constant function, $f(x) = c$, then $f'(a) = 0$ for all real numbers a.

Proof $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \dfrac{c - c}{h} = 0$

Derivative of a sum

If f and g are differentiable at a, then $f + g$ is also differentiable at a, and

$$(f+g)'(a) = f'(a) + g'(a)$$

Proof

$$\begin{aligned}(f+g)'(a) &= \lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h) + g(a+h) - f(a) - g(a)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\ &= f'(a) + g'(a)\end{aligned}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 14F

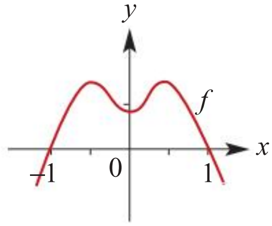
A function f is said to be **differentiable** at $x = a$ if $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

Exercise 14F

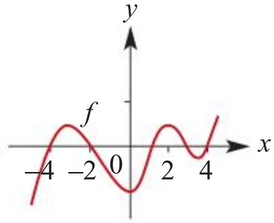
Example 30

1 In each of the figures below, the graph of a function f is given. Sketch the graph of f'. Obviously your sketch of f' cannot be exact, but $f'(x)$ should be zero at values of x for which the gradient of f is zero, and $f'(x)$ should be negative where the original graph slopes downwards, and so on.

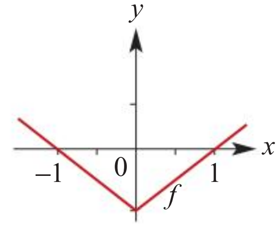
a



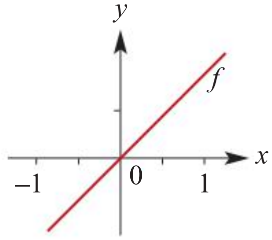
b



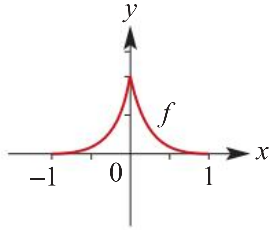
c



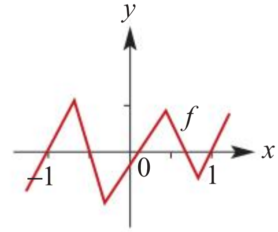
d



e



f



Example 31

2 For the function with the following rule, determine $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$$

Example 32

3 For the function with the following rule, state the set of values for which the derivative is defined, determine $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$$

4 For the function with the following rule, state the set of values for which the derivative is defined, determine $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 - 3x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$$

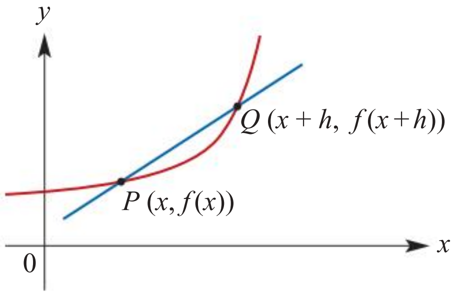
 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Chapter summary

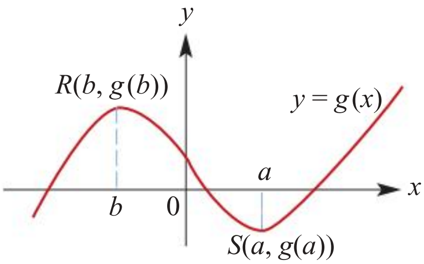
The derivative

- The notation for the limit as h approaches 0 is $\lim_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
    $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
    $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
  $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Rules for differentiation

- For $f(x) = c$, $f'(x) = 0$. That is, the derivative of a constant function is zero.

  For example:
  - $f(x) = 1$, $f'(x) = 0$
  - $f(x) = 27.3$, $f'(x) = 0$
- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

  For example:
  - $f(x) = x^2$, $f'(x) = 2x$
  - $f(x) = x^3$, $f'(x) = 3x^2$
  - $f(x) = x^{-2}$, $f'(x) = -2x^{-3}$
- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.

  For example:
  - $f(x) = 3x^2$, $f'(x) = 3(2x) = 6x$
  - $f(x) = 5x^3$, $f'(x) = 5(3x^2) = 15x^2$
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.

  For example:
  - $f(x) = x^2 + x^3$, $f'(x) = 2x + 3x^2$
  - $f(x) = 3x^2 + 5x^3$, $f'(x) = 6x + 15x^2$

 ISBN 978-1-009-53719-3

Review

Algebra of limits

- $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$, where k is a real number.
- $\lim_{x \to a} (f(x)\,g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

Continuity

- A function f is **continuous** at the point $x = a$ if:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x \to a} f(x) = f(a)$
- The function is **discontinuous** at a point if it is not continuous at that point.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

14A **1** I can determine the gradient of a secant determined by two points on a curve and use this to determine the gradient of a tangent line at a point. ☐

See Example 1, Example 2 and Questions 1 and 3

14A **2** I can use limit notation and determine limits of suitable expressions. ☐

See Example 3, Example 4 and Questions 4 and 7

14A **3** I can differentiate polynomials by first principles using $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. ☐

See Example 5, Example 6 and Question 12

14B **4** I can differentiate polynomial expressions by using the rule, $f(x) = x^n, f'(x) = nx^{n-1}$. ☐

See Example 7, Example 8 and Questions 1 and 3.

14B **5** I can determine the gradient of the tangent at a given point on the graph of a polynomial function. ☐

See Example 9 and Question 5.

14B **6** I can determine the points on suitable graph at which the gradient is known. ☐

See Example 13 and Question 14

 ISBN 978-1-009-53719-3

14C **7** I can differentiate x^n where n is a negative integer. ☐

See Example 16 and Question 3

14D **8** I can determine the intervals for which $f'(x) > 0, f'(x) < 0$ given sufficient information on the graph of $y = f(x)$. ☐

See Example 20 and Question 1

14D **9** I can sketch the graph of the derivative function from the graph of the function. ☐

See Example 21 and Question 10

14D **10** I can determine the angle a tangent line makes with the positive direction of the x-axis. ☐

See Example 22 and Question 11

14E **11** I can determine the limit value of simple functions for a given value of x. ☐

See Example 24, Example 25 and Question 1

14E **12** I can apply algebra of limits to evaluate limits. ☐

See Example 26, Example 27 and Question 1

14E **13** I can investigate whether a function is continuous or not at a given point. ☐

See Example 28, Example 29 and Questions 2 and 3

14F **14** I can sketch the graph of the derivative function including those of functions which are not differentiable for all of their domain. ☐

See Example 30, Example 31 and Questions 1 and 2

14F **15** I can investigate differentiability of a function. ☐

See Example 32 and Question 3

Short-response questions

Technology-free short-response questions

SF

1 Determine the derivative of each of the following by first principles:

a $y = 3x + 1$ **b** $y = 4 - x^2$ **c** $y = x^2 + 5x$
d $y = x^3 + x$ **e** $y = x^2 + 2x + 1$ **f** $y = 3x^2 - x$

2 Determine $\dfrac{dy}{dx}$ when:

a $y = 3x^2 - 2x + 6$ **b** $y = 5$ **c** $y = 2x(2 - x)$
d $y = 4(2x - 1)(5x + 2)$ **e** $y = (x + 1)(3x - 2)$ **f** $y = (x + 1)(2 - 3x)$
g $y = \dfrac{(x + 3)(2x + 1)}{4}$ **h** $y = \dfrac{2x^3 - x^2}{3x}$ **i** $y = \dfrac{x^4 + 3x^2}{2x^2}$

 ISBN 978-1-009-53719-3

Review

SF

3 For each of the following functions, determine the y-coordinate and the gradient of the tangent at the point on the curve for the given value of x:

a $y = x^2 - 2x + 1, \quad x = 2$
b $y = x^2 - 2x, \quad x = -1$
c $y = (x + 2)(x - 4), \quad x = 3$
d $y = 3x^2 - 2x^3, \quad x = -2$

4 Determine the coordinates of the points on the curves given by the following equations at which the gradient of the tangent at that point has the given value:

a $y = x^2 - 3x + 1, \quad \dfrac{dy}{dx} = 0$
b $y = x^3 - 6x^2 + 4, \quad \dfrac{dy}{dx} = -12$
c $y = x^2 - x^3, \quad \dfrac{dy}{dx} = -1$
d $y = x^3 - 2x + 7, \quad \dfrac{dy}{dx} = 1$
e $y = x^4 - 2x^3 + 1, \quad \dfrac{dy}{dx} = 0$
f $y = x(x - 3)^2, \quad \dfrac{dy}{dx} = 0$

5 For the function with rule $f(x) = 3(2x - 1)^2$, determine the values of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f(x) > 0$
f $f'(x) = 3$

6 Determine the derivative of each of the following with respect to x:

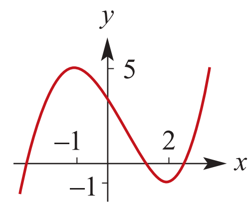
a x^{-4}
b $2x^{-3}$
c $-\dfrac{1}{3x^2}$
d $-\dfrac{1}{x^4}$
e $\dfrac{3}{x^5}$
f $\dfrac{x^2 + x^3}{x^4}$
g $\dfrac{3x^2 + 2x}{x^2}$
h $5x^2 - \dfrac{2}{x}$

CF

7 The curve with equation $y = ax^2 + bx$ has a tangent with gradient 3 at the point $(1, 1)$.

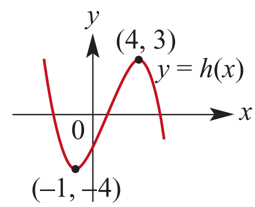
a Determine the values of a and b.
b Determine the coordinates of the points on the curve where the gradient is 0.

8 The graph of $y = f(x)$ is shown. Sketch the graph of $y = f'(x)$. (Not all details can be determined, but the x-axis intercepts and the shape of the graph can be determined.)



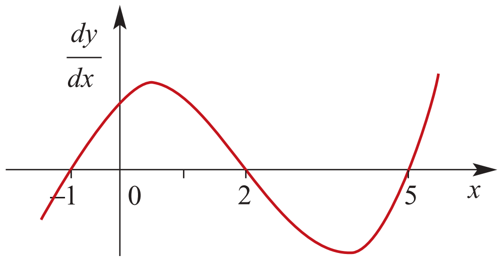
9 For the graph of $y = h(x)$, determine the values of x for which:

a $h'(x) > 0$
b $h'(x) < 0$
c $h'(x) = 0$



10 The diagram to the right shows part of the graph of $\frac{dy}{dx}$ against x.
Sketch a possible shape for the graph of y against x over the same interval if:

- $y = -1$ when $x = -1$
- $y = 0$ when $x = 0$
- $y = 1$ when $x = 2$.



Review

Technology-active short-response questions

11 Consider the function $f(x) = 2x^3 - 3x^2 + 2x + 8$.

a Determine the gradient of the chord joining the points $A(2.2, f(2.2))$ and $B(2.25, f(2.25))$

b Determine the derivative function $f'(x)$ and the value of $f'(2.2)$

12 Determine the values of x for which the derivative of $y = x^3 - 4x^2 + 2x - 3$ is equal to 1. Give the exact answers and also the answers correct to 3 decimal places.

13 Given that the curve with equation $y = ax^4 + bx^3$ has gradient 4 at the point on the curve with coordinates $(1, 5)$ calculate the values of a and b.

14 A tadpole begins to swim vertically upwards in a pond and after t seconds it is $25 - 0.1t^3$ cm below the surface.

a How long does the tadpole take to reach the surface, and what is its speed then?

b What is the average speed over this time?

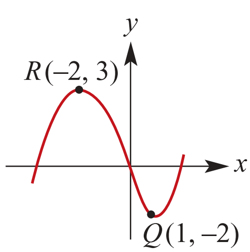
15 The curve with equation $y = 2x^4 - 3x^2 - x + 6$ has gradient -1 at the point $(0, 6)$. It also has gradient -1 at the points with coordinates (a, b), $a > 0$ and (c, d). Determine the exact values of a and c and the values of b and d correct to three decimal places.

16 The graph of a cubic polynomial $P(x) = ax^3 + bx^2 + cx$ passes through the points with coordinates $(-1, 6)$ and $(1, 2)$. The gradient of the graph when $x = 1$ is 1. Determine the values of a, b and c.

17 The tangent line to the graph of $y = ax^2 + bx$ at the point on the graph with coordinates $(1, 2)$ makes an angle of 120° with the positive direction of the x-axis. Determine the values of a and b correct to three decimal places.

18 The graph shown is that of a polynomial function of the form $P(x) = ax^3 + bx^2 + cx + d$. Determine the values of a, b, c and d.

Note: $Q(1, -2)$ is not a turning point.



.

 ISBN 978-1-009-53719-3

Review

CU

19 A body moves in a path described by the equation $y = \frac{1}{5}x^5 + \frac{1}{2}x^4$, for $x \geq 0$.
Units are in kilometres, and x and y are the horizontal and vertical axes respectively.

a What will be the direction of motion (give your answer as the angle between the direction of motion and the x-axis) when the x-value is:

i 1 km **ii** 3 km?

b Determine a value of x for which the gradient of the path is 32.

20 A trail over a mountain pass can be modelled by the curve $y = 2 + 0.12x - 0.01x^3$, where x and y are the horizontal and vertical distances respectively, measured in kilometres, and $0 \leq x \leq 3$.

a Determine the gradients at the beginning and the end of the trail.

b Calculate the point where the gradient is zero, and calculate also the height of the pass.

21 **a** Show that the gradients of the curve $y = x(x-2)$ at the points $(0, 0)$ and $(2, 0)$ only differ in sign. What is the geometrical interpretation for this?

b If the gradients of the curve $y = x(x-2)(x-5)$ at the points $(0, 0)$, $(2, 0)$ and $(5, 0)$ are ℓ, m and n respectively, show that $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 0$.

Multiple-choice questions

Technology-free multiple-choice questions

1 The gradient of the curve $y = x^3 + 4x$ at the point where $x = 2$ is

A 12 **B** 4 **C** 10 **D** 16

2 The gradient of the chord of the curve $y = 2x^2$ between the points where $x = 1$ and $x = 1 + h$ is given by

A $2(x+h)^2 - 2x^2$ **B** $4 + 2h$ **C** 4 **D** $4x$

3 If $y = 2x^4 - 5x^3 + 2$, then $\frac{dy}{dx}$ equals

A $8x^3 - 15x^2$ **B** $4x^4 - 15x^2 + 2$ **C** $4x^4 - 10x^2$ **D** $8x^3 - 15x + 2$

4 If $f(x) = x^2(x+1)$, then $f'(-1)$ equals

A -1 **B** 1 **C** 2 **D** -2

5 If $f(x) = (x-3)^2$, then $f'(x)$ equals

A $x - 3$ **B** $x - 6$ **C** $2x - 6$ **D** $2x + 9$

ISBN 978-1-009-53719-3

6 If $y = \dfrac{2x^4 + 9x^2}{3x}$, $\quad x \neq 0$, then $\dfrac{dy}{dx}$ equals

A $\dfrac{2x^4}{3} + 6x$ **B** $2x + 3$ **C** $2x^2 + 3$ **D** $\dfrac{8x^3 + 18x}{3}$

7 Given that $y = x^2 - 6x + 9$, the values of x for which $\dfrac{dy}{dx} \geq 0$ are

A $x \geq 3$ **B** $x > 3$ **C** $x \geq -3$ **D** $x \leq -3$

8 If $y = -2x^3 + 3x^2 - x + 1$, then $\dfrac{dy}{dx}$ equals

A $6x^2 + 6x - 1$ **B** $-6x^2 + 6x$
C $-6x^2 + 3x - 1$ **D** $-6x^2 + 6x - 1$

Technology-active multiple-choice questions

9 The coordinates of the point on the graph of $y = x^2 + 6x - 5$ at which the tangent is parallel to the line $y = 4x$ are

A $(-1, -10)$ **B** $(-1, -2)$ **C** $(1, 2)$ **D** $(-1, 4)$

10 If $y = 2x^4 - 36x^2$, the points at which the tangent to the curve is parallel to the x-axis are

A 0 and 3 **B** -3 and 3 **C** 0 and -3 **D** -3, 0 and 3

11 The gradient of the tangent of the curve $y = 4x^3 - 2x^2 + 4x$ at the point where $x = 2.5$ is

A 33.5 **B** 32.75 **C** 69 **D** 68.25

12 The values of x for which the corresponding points on the graph of $y = 4x^3 - 2.5x^2 - 3$ have tangent lines perpendicular to the line $y = 2x$ are

A $x = \dfrac{1}{3}$ or $x = \dfrac{1}{2}$ **B** $x = \dfrac{5}{6}$ or $x = \dfrac{3}{4}$
C $x = 0.251$ or $x = 0.16$ **D** $x = \dfrac{1}{6}$ or $x = \dfrac{1}{4}$

13 The angle that the tangent line to the graph of the function with rule $f(x) = 2x^3 - x^2 - 4$, at the point with coordinates $(1, -3)$ makes with the positive direction of the x-axis, is closest to

A 72.57 **B** 108.43 **C** 75.96° **D** 104.04°

15

Applications of differentiation of polynomials

Chapter contents

In this chapter we continue our study of calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The new techniques for sketching graphs of polynomial functions are a useful addition to the skills that were introduced in Chapter 7. At that stage, rather frustratingly, we were only able to determine the coordinates of turning points of cubic and quartic functions using technology. The new techniques are also used for determining maximum or minimum values for problems set in a 'real world' context.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. The problems of kinematics were the motivation for Newton's work on calculus.

15A Tangents and normals

Learning intentions

- To be able to determine the equation of the **tangent** and the **normal** at a given point.

The derivative of a function is a new function which gives the measure of the gradient of the tangent at each point on the curve. If the gradient is known, it is possible to determine the equation of the tangent for a given point on the curve.

Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Example 1

Determine the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $(1, \frac{3}{2})$ is a point on the tangent.

Since $\dfrac{dy}{dx} = 3x^2 + x$, the gradient of the tangent to the curve at $x = 1$ is 4.

Hence the equation of the tangent is

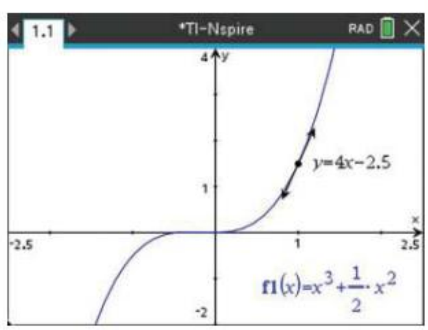
$$y - \tfrac{3}{2} = 4(x - 1)$$

which becomes

$$y = 4x - \tfrac{5}{2}$$

Using the TI-Nspire CX non-CAS

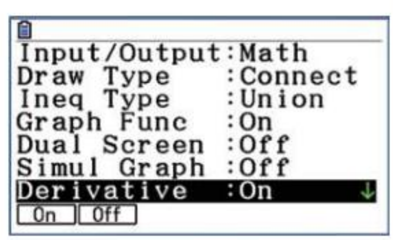
- In a **Graphs** application, plot the graph of $f1(x) = x^3 + \frac{1}{2}x^2$.
- To add a tangent line to the graph, use menu > **Geometry** > **Points & Lines** > **Tangent**. Click on the curve and press (. Enter 1 for the x-value 1 and press enter.
- Hence the tangent line at $x = 1$ has the equation $y = 4x - 2.5$.



Using the Casio

To change the derivative setting:

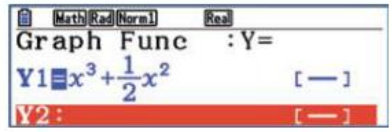
- In **Graph** mode, go to the set-up screen by pressing SHIFT MENU.
- Use the cursor key ▼ to move down to **Derivative**. Press F1 to change the setting to **On**. Then press EXIT.

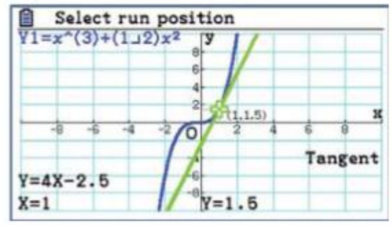


 ISBN 978-1-009-53719-3

To draw a tangent line on a graph:

- Enter the rule $y = x^3 + \frac{1}{2}x^2$ in $Y1$.
- Select **Draw** F6 to view the graph.
- To add a tangent line to the graph, select **Sketch** SHIFT F4, then **Tangent** F2.
- Enter the x-value 1 and press EXE twice.
- Hence the tangent line at $x = 1$ has the equation $y = 4x - 2.5$.





The **normal** to a curve at a point on the curve is the line which passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\dfrac{1}{m}$.

Example 2

Determine the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ lies on the normal.

Since $\dfrac{dy}{dx} = 3x^2 - 4x$, the gradient of the tangent at $x = 1$ is -1.

Thus the gradient of the normal at $x = 1$ is $\dfrac{-1}{-1} = 1$.

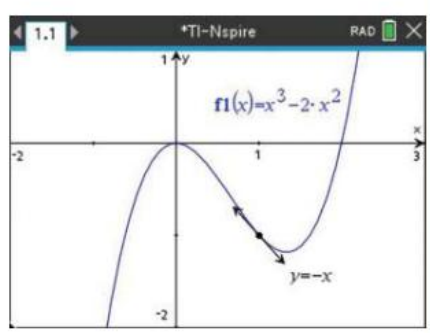
Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. the equation of the normal is $y = x - 2$.

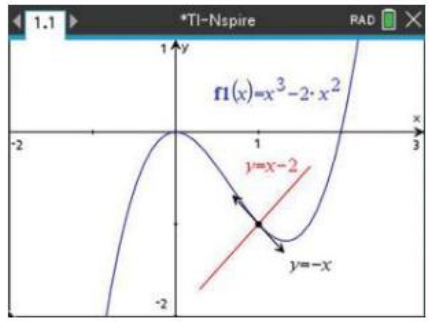
Using the TI-Nspire CX non-CAS

- In a **Graphs** application, plot the graph of $f1(x) = x^3 - 2x^2$.
- Add a tangent line at $x = 1$ as shown earlier.



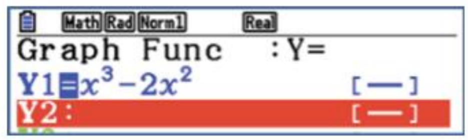
 ISBN 978-1-009-53719-3

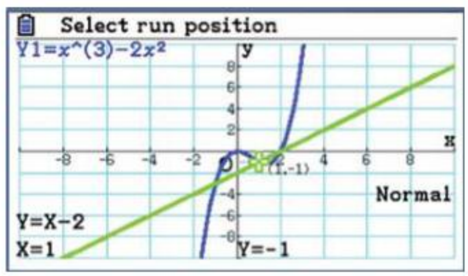
- To add a normal line, use (menu) > **Geometry** > **Construction** > **Perpendicular**, and double click on the point at $x = 1$.
- To display the equation of the normal, use (menu) > **Actions** > **Coordinates & Equations**, and click on the perpendicular.
- Hence the normal line at $x = 1$ has the equation $y = x - 2$.



Using the Casio

- In **Graph** mode, ensure that the derivative setting is On. (Go to the set-up screen (SHIFT) (MENU).)
- Enter the rule $y = x^3 - 2x^2$ in $Y1$.
- Select **Draw** (F6) to view the graph.
- To add a normal line to the graph, select **Sketch** (SHIFT) (F4), then **Normal** (F3).
- Enter the x-value 1 and press (EXE) twice.
- Hence the normal line at $x = 1$ has the equation $y = x - 2$.





Summary 15A

- **Equation of a tangent line**
  Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- **Gradient of a normal line**
  If a tangent has gradient m, the normal has gradient $-\dfrac{1}{m}$.

Skill-sheet

Exercise 15A

Example 1, 2

CF

1 Determine the equation of the tangent and the normal at the given point for:

a $f(x) = x^2$, $(2, 4)$

b $f(x) = (2x - 1)^2$, $(2, 9)$

c $f(x) = 3x - x^2$, $(2, 2)$

d $f(x) = 9x - x^3$, $(1, 8)$

2 Determine the equation of the tangent to the curve with equation $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

3 Determine the equation of the tangent to $y = x^2$ at the point $(1, 1)$ and the equation of the tangent to $y = \frac{1}{6}x^3$ at the point $\left(2, \frac{4}{3}\right)$.
Show that these tangents are parallel and determine the perpendicular distance between them.

CF

4 Determine the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

5 The curve with the equation $y = (x - 2)(x - 3)(x - 4)$ cuts the x-axis at the points $P = (2, 0)$, $Q = (3, 0)$ and $R = (4, 0)$.

- **a** Prove that the tangents at P and R are parallel.
- **b** At what point does the normal to the curve at Q cut the y-axis?

6 For the curve with equation $y = x^2 + 3$, show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point $(a, a^2 + 3)$.
Hence determine the coordinates of the two points on the curve, the tangents of which pass through the point $(2, 6)$.

CU

7 **a** Determine the equation of the tangent at the point $(2, 4)$ to the curve $y = x^3 - 2x$.
b Determine the coordinates of the point where the tangent meets the curve again.

CF

8 **a** Determine the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$.
b At what points on the curve is the tangent parallel to the line $4x + y - 3 = 0$?

15B Rates of change

Learning intentions

- ▶ To be able to use the derivative of a polynomial in **rate of change** problems.

The derivative of a function was defined geometrically in Section 14A. But the process of differentiation may be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is defined to be $f'(a)$.

Average rate of change has been discussed in Chapter 13.

The instantaneous rate of change of y with respect to x is given by $\dfrac{dy}{dx}$, that is, by the derivative of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, the change is an increase in the value of y corresponding to an increase in x.
- If $\dfrac{dy}{dx} < 0$, the change is a decrease in the value of y corresponding to an increase in x.

ISBN 978-1-009-53719-3

Example 3

For the function with rule $f(x) = x^2 + 2x$, determine:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b Average rate of change
$$= \frac{f(2 + h) - f(2)}{2 + h - 2}$$
$$= \frac{(2 + h)^2 + 2(2 + h) - 8}{h}$$
$$= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h}$$
$$= \frac{6h + h^2}{h}$$
$$= 6 + h$$

c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part b.

Example 4

A balloon develops a microscopic leak and gradually decreases in volume.
Its volume, V (cm^3), at time t (seconds) is $V = 600 - 10t - \dfrac{1}{100}t^2,\ t > 0$.

a Determine the rate of change of volume after:

i 10 seconds **ii** 20 seconds.

b For how long could the model be valid?

Solution

a
$$V = 600 - 10t - \frac{1}{100}t^2$$
$$\frac{dV}{dt} = -10 - \frac{t}{50}$$

i When $t = 10$, $\dfrac{dV}{dt} = -10 - \dfrac{1}{5} = -10\tfrac{1}{5}$

i.e. the volume is decreasing at a rate of $10\tfrac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\dfrac{dV}{dt} = -10 - \dfrac{2}{5} = -10\tfrac{2}{5}$

i.e. the volume is decreasing at a rate of $10\tfrac{2}{5}$ cm^3 per second.

 ISBN 978-1-009-53719-3

b The model will not be meaningful when $V < 0$.

Consider $V = 0$:

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = \frac{10 \pm \sqrt{100 + 4 \times 0.01 \times 600}}{-0.02}$$

$$\therefore \quad t = -1056.78 \quad \text{or} \quad t = 56.78 \quad \text{(to two decimal places)}$$

Hence the model may be suitable for $0 < t < 56.78$.

TI-84

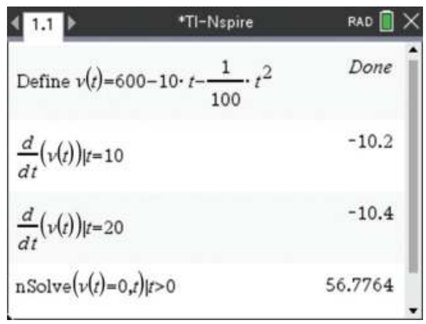
Using the TI-Nspire CX non-CAS

Define $v(t) = 600 - 10t - (1/100)t^2$.

a To determine the derivative of $v(t)$ at $t = 10$ and $t = 20$, use (menu) > **Calculus** > **Numerical Derivative at a Point** and complete the pop-up screen for $t = 10$ and $t = 20$, in turn. Press (ctrl) (enter) each time to obtain a decimal answer.

b To determine the domain, first solve the equation $v(t) = 0$ for $t > 0$ as shown.

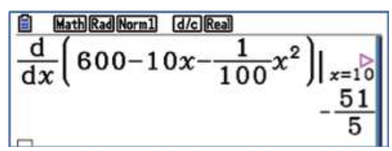
Hence the domain is $0 < t < 56.78$.

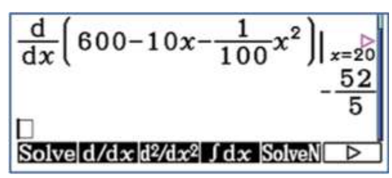


Using the Casio

a To determine the rate of change of volume after 10 seconds and 20 seconds:

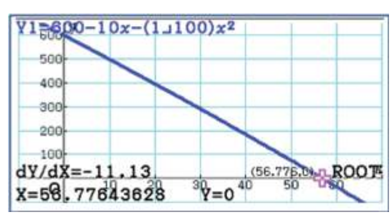
- In **Run-Matrix** mode, select **Calculation** (OPTN) (F4), then **d/dx** (F2).
- Enter the expression $600 - 10x - \frac{1}{100}x^2$ and the x-value 10.
- Repeat with the x-value 20.





b To determine when the model is valid:

- In **Graph** mode, draw the graph of $y = 600 - 10x - \frac{1}{100}x^2$.
- Set an appropriate View Window.
- Determine the x-axis intercept by going to **G-Solve** (SHIFT) (F5) and selecting **Root** (F1).



From the graph, the model is valid for $0 < t < 56.78$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary 15B

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is $f'(a)$.

Exercise 15B

SF

Example 3

1 Let $y = 35 + 12x^2$.

a Determine the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?

b Determine the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?

c Determine the rate of change of y with respect to x when $x = 2$.

CF

Example 4

2 According to a business magazine, the expected assets, $\$A$, of a proposed new company will be given by $A = 200\,000 + 600t^2 - \dfrac{200}{3}t^3$, where t is the number of months after the business is set up.

a Determine the rate of growth of assets at time t months.

b Determine the rate of growth of assets at time $t = 3$ months.

c When will the rate of growth of assets be zero?

3 As a result of a survey, the marketing director of a company found that the revenue, $\$R$, from pricing 100 items at $\$P$ each is given by the rule $R = 30P - 2P^2$.

a Determine $\dfrac{dR}{dP}$ and explain what it means.

b Calculate $\dfrac{dR}{dP}$ when $P = 5$ and $P = 10$.

c For what selling prices is revenue rising?

4 The population, P, of a new housing estate t years after 30 January 2012 is given by the rule $P = 100(5 + t - 0.25t^2)$. Determine the rate of change of the population after:

a 1 year **b** 2 years **c** 3 years.

5 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \dfrac{5}{8}\left(10t^2 - \dfrac{t^3}{3}\right)$, $0 \le t \le 20$.

a Determine the volume of water in the flask at time:

i $t = 0$ **ii** $t = 20$.

b Determine the rate of flow of water into the flask at time t.

c Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

6 The area, A km^2, of an oil slick is growing according to the rule $A = \dfrac{t}{2} + \dfrac{t^2}{10}$, where t is the time in hours since the leak started.

a Determine the area covered at the end of 1 hour.

b Determine the rate of increase of the area after 1 hour.

 ISBN 978-1-009-53719-3

15C Stationary points

Learning intentions

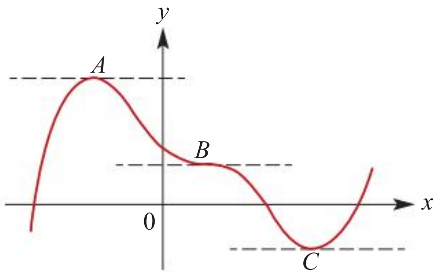
- To be able to determine the **stationary points** of certain polynomial functions.

In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, for $y = f(x)$, if $\dfrac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

In the graph shown, there are stationary points at A, B and C. At such points the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary points* becomes clear later in this chapter when we look at the application to the motion of a particle.

Example 5

Determine the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $p = 2t^3 - 5t^2 - 4t + 13$ for $t > 0$ **c** $y = 4 + 3x - x^3$

Solution

a $y = 9 + 12x - 2x^2$

$$\frac{dy}{dx} = 12 - 4x$$

The stationary points occur when $\dfrac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$, i.e. at $x = 3$.

When $x = 3$, $y = 9 + 12 \times 3 - 2 \times 3^2 = 27$.

Thus the stationary point is at $(3, 27)$.

b $p = 2t^3 - 5t^2 - 4t + 13 \quad (t > 0)$

$$\frac{dp}{dt} = 6t^2 - 10t - 4 \qquad (t > 0)$$

Thus, $\dfrac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$$(3t + 1)(t - 2) = 0$$

$$\therefore t = -\frac{1}{3} \text{ or } t = 2$$

But $t > 0$, therefore the only acceptable solution is $t = 2$.

When $t = 2$, $p = 16 - 20 - 8 + 13 = 1$.

So the corresponding stationary point is at $(2, 1)$.

 ISBN 978-1-009-53719-3

c $y = 4 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

Thus, $\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$\therefore \quad x = \pm 1$

The stationary points occur at $(1, 6)$ and $(-1, 2)$.

Example 6

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through $(0, 5)$ and has a stationary point at $(2, 7)$. Determine a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $c = 5$.

We have $\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$.

Therefore

$$0 = 12 + 4a + b \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$7 = 2^3 + 2^2a + 2b + 5$$

$$2 = 8 + 4a + 2b$$

$$4a + 2b + 6 = 0 \qquad (2)$$

Subtract (2) from (1):

$$-b + 6 = 0$$

$$\therefore \quad b = 6$$

Substitute in (1):

$$0 = 12 + 4a + 6$$

$$-18 = 4a$$

$$\therefore \quad -\frac{9}{2} = a$$

Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

Summary 15C

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, for $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Exercise 15C

Example 5

1 Determine the coordinates of the stationary points of each of the following functions:

a $f(x) = x^2 - 6x + 3$

b $y = x^3 - 4x^2 - 3x + 20$ for $x > 0$

c $z = x^4 - 32x + 50$

d $q = 8t + 5t^2 - t^3$ for $t > 0$

e $y = 2x^2(x - 3)$

f $y = 3x^4 - 16x^3 + 24x^2 - 10$

Example 6

2 The curve with equation $y = ax^2 + bx + c$ passes through $(0, -1)$ and has a stationary point at $(2, -9)$. Determine a, b and c.

3 The curve with equation $y = ax^2 + bx + c$ has a stationary point at $(1, 2)$. When $x = 0$, the slope of the curve is $45°$. Determine a, b and c.

4 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Determine the values of a and b.

b Determine the coordinates of the turning point.

5 The curve with equation $y = x^2 + ax + 3$ has a stationary point when $x = 4$. Determine a.

6 The curve with equation $y = x^2 - ax + 4$ has a stationary point when $x = 3$. Determine a.

7 Determine the coordinates of the stationary points of each of the following:

a $y = x^2 - 5x - 6$

b $y = (3x - 2)(8x + 3)$

c $y = 2x^3 - 9x^2 + 27$

d $y = x^3 - 3x^2 - 24x + 20$

e $y = (x + 1)^2(x + 4)$

f $y = (x + 1)^2 + (x + 2)^2$

8 The curve with equation $y = ax^2 + bx + 12$ has a stationary point at $(1, 13)$. Determine a and b.

9 The curve with equation $y = ax^3 + bx^2 + cx + d$ has a gradient of -3 at $(0, 7\frac{1}{2})$ and a turning point at $(3, 6)$. Determine a, b, c and d.

CF

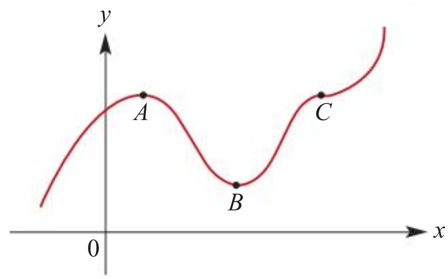
 ISBN 978-1-009-53719-3

15D Types of stationary points

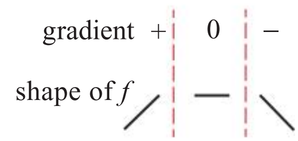
Learning intentions

- To be able to determine the nature of the **stationary points** on the curves of certain polynomial functions.

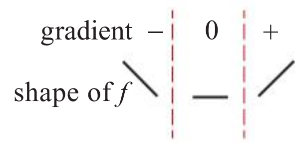
The graph of $y = f(x)$ below has three stationary points $A(a, f(a))$, $B(b, f(b))$, $C(c, f(c))$.



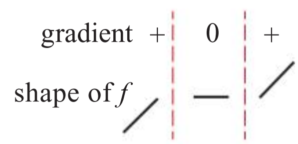
A Point A is called a **local maximum** point.
Notice that $f'(x) > 0$ immediately to the left of A, and that $f'(x) < 0$ immediately to the right of A. This means that f is strictly increasing immediately to the left of A, and strictly decreasing immediately to the right of A.



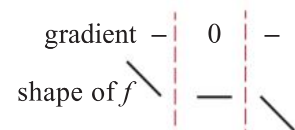
B Point B is called a **local minimum** point.
Notice that $f'(x) < 0$ immediately to the left of B, and that $f'(x) > 0$ immediately to the right of B. This means that f is strictly decreasing immediately to the left of B, and strictly increasing immediately to the right of B.



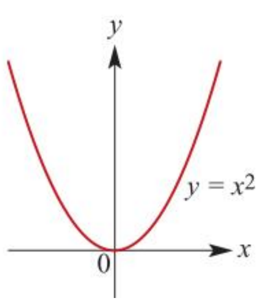
C The point C is called a **stationary point of inflection**.
Notice that $f'(x) > 0$ immediately to the left and right of C.



Clearly it is also possible to have stationary points of inflection with $f'(x) < 0$ immediately to the left and right.

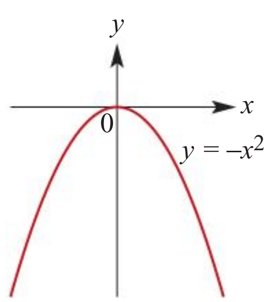


Stationary points of types A and B are referred to as **turning points**. Before proceeding with some more complicated functions, it is worth referring back to some of the functions we met earlier in this book.



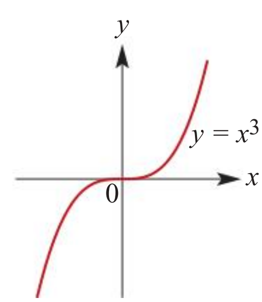
$f(x) = x^2$

Local minimum at $(0, 0)$.



$f(x) = -x^2$

Local maximum at $(0, 0)$.



$f(x) = x^3$

Stationary point of inflection at $(0, 0)$.

 ISBN 978-1-009-53719-3

Example 7

For the function $f(x) = 3x^3 - 4x + 1$:

a Determine the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$

$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $(-\frac{2}{3}, f(-\frac{2}{3}))$ and $(\frac{2}{3}, f(\frac{2}{3}))$, that is, at $(-\frac{2}{3}, 2\frac{7}{9})$ and $(\frac{2}{3}, -\frac{7}{9})$. So $f'(x)$ is of constant sign for each of

$$x < -\tfrac{2}{3}, \quad -\tfrac{2}{3} < x < \tfrac{2}{3} \quad \text{and} \quad x > \tfrac{2}{3}$$

To calculate the sign of $f'(x)$ for each of these intervals, simply choose a representative number in the interval.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	$+$	0	$-$	0	$+$
shape of f	/	—	\	—	/

There is a local maximum at $(-\frac{2}{3}, 2\frac{7}{9})$ and a local minimum at $(\frac{2}{3}, -\frac{7}{9})$.

b To sketch the graph of this function we need to determine the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.
To determine the x-axis intercepts use the factor theorem or your calculator.
Using the factor theorem and division Consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.
Using the factor theorem, we determine that $x - 1$ is a factor of $3x^3 - 4x + 1$.
By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $(x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$. We have

$$\begin{aligned} 3x^2 + 3x - 1 &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right] \\ &= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right) \end{aligned}$$

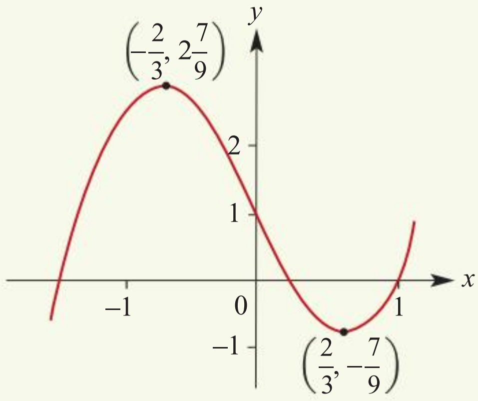
 ISBN 978-1-009-53719-3

Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.



Note: With your calculator you will determine approximate values for the x-axis intercepts.

Summary 15D

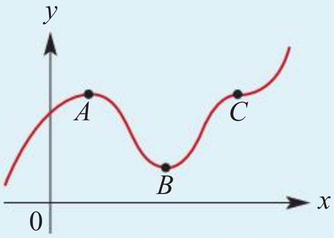
A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points

A Point A is a **local maximum**:
- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:
- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.



Stationary points of types A and B are called **turning points**.

Skill-sheet

Exercise 15D

Example 7

1 For each of the following, determine all stationary points and state their nature. Sketch the graph of each function. SF

a $y = 9x^2 - x^3$ **b** $y = x^3 - 3x^2 - 9x$ **c** $y = x^4 - 4x^3$

2 Determine the stationary points (and state their type) for each of the following functions: CF

a $y = x^2(x - 4)$ **b** $y = x^2(3 - x)$ **c** $y = x^4$
d $y = x^5(x - 4)$ **e** $y = x^3 - 5x^2 + 3x + 2$ **f** $y = x(x - 8)(x - 3)$

3 Sketch the graph of each of the following functions:

a $y = 2 + 3x - x^3$ **b** $y = 2x^2(x - 3)$ **c** $y = x^3 - 3x^2 - 9x + 11$

 ISBN 978-1-009-53719-3 Cambridge University Press

CF

4 The graph corresponding to each of the following equations has a stationary point at $(-2, 10)$. For each graph, determine the nature of the stationary point at $(-2, 10)$.

a $y = 2x^3 + 3x^2 - 12x - 10$

b $y = 3x^4 + 16x^3 + 24x^2 - 6$

5 For the function $y = x^3 - 6x^2 + 9x + 10$:

a Determine the values of x for which $\frac{dy}{dx} > 0$.

b Determine the stationary points on the curve corresponding to $y = x^3 - 6x^2 + 9x + 10$.

c Sketch the curve carefully between $x = 0$ and $x = 4$.

6 For the function $f(x) = 1 + 12x - x^3$, determine the values of x for which $f'(x) > 0$.

7 Let $f(x) = 3 + 6x - 2x^3$.

a Determine the values of x such that $f'(x) > 0$.

b Determine the values of x such that $f'(x) < 0$.

8 Let $f(x) = x(x + 3)(x - 5)$.

a Determine the values of x for which $f'(x) = 0$.

b Sketch the graph of $y = f(x)$ for $-5 \leq x \leq 6$, giving the coordinates of the intersections with the axes and the coordinates of the turning points.

9 Sketch the graph of $y = x^3 - 6x^2 + 9x - 4$. State the coordinates of the axis intercepts and the turning points.

10 Determine the coordinates of the points on the curve $y = x^3 - 3x^2 - 45x + 2$ where the tangent is parallel to the x-axis.

11 Let $f(x) = x^3 - 3x^2$.

a **i** Determine the values of x for which $f'(x) < 0$.

ii Determine the values of x for which $f'(x) > 0$.

iii Determine the values of x for which $f'(x) = 0$.

b Sketch the graph of $y = f(x)$.

12 Sketch the graph of $y = x^3 - 9x^2 + 27x - 19$ and state the coordinates of the stationary points.

13 Sketch the graph of $y = x^4 - 8x^2 + 7$. All axis intercepts and all turning points should be identified and their coordinates given.

 ISBN 978-1-009-53719-3 Cambridge University Press

15E Applications to maximum and minimum problems (Optional)

Learning intentions

- To be able to apply differentiation to the solution of **maximum and minimum** problems.

Many practical problems involve determining a maximum or minimum value of a function. We have solved some of these in Chapters 4 and 7. In the case of quadratic functions, we wrote the quadratic in turning point form and hence determined the maximum or minimum value. In the case of cubic functions, we used a graphics calculator to determine the maximum or minimum values.

In this section we use calculus to solve problems which involve determining a local maximum or local minimum.

Example 8

A loop of string of length 100 cm is to be formed into a rectangle. Determine the maximum area of this rectangle.

Solution

Let the length of the rectangle be x cm and the width y cm.

Then $2x + 2y = 100$. Thus $x + y = 50$ and hence

$$y = 50 - x \qquad (1)$$

It is clear that, for this problem, we must have $0 \le x \le 50$.

The area, A cm^2, is given by the formula $A = xy$.

Substituting from (1) gives

$$\begin{aligned} A &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

Differentiating with respect to x:

$$\frac{dA}{dx} = 50 - 2x$$

Thus $\frac{dA}{dx} = 0$ implies $x = 25$.

Since the coefficient of x^2 is negative, this stationary point is a local maximum. (Alternatively, we could check the sign of $A'(x)$ immediately to the left and the right of $x = 25$.)

The maximum area is formed when the rectangle is a square with side length 25 cm, and so the maximum area is 625 cm^2.

Note: It is clear that we could have completed this question without calculus by using our knowledge of quadratic functions.

 ISBN 978-1-009-53719-3

Example 9

Given that $x + 2y = 4$, calculate the minimum value of $x^2 + xy - y^2$.

Solution

Rearranging $x + 2y = 4$, we have $x = 4 - 2y$.

Let $P = x^2 + xy - y^2$.

Substituting for x gives

$$\begin{aligned} P &= (4 - 2y)^2 + (4 - 2y)y - y^2 \\ &= 16 - 16y + 4y^2 + 4y - 2y^2 - y^2 \\ &= 16 - 12y + y^2 \end{aligned}$$

$$\therefore \quad \frac{dP}{dy} = -12 + 2y$$

Stationary values occur when $\frac{dP}{dy} = 0$:

$$\begin{aligned} -12 + 2y &= 0 \\ y &= 6 \end{aligned}$$

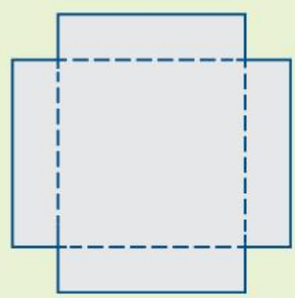
y		6	
$\frac{dP}{dy}$	$-$	0	$+$
shape of P	$\diagdown$	—	$\diagup$

From the diagram, there is a minimum when $y = 6$.

When $y = 6$, $x = -8$. Thus the minimum value of $x^2 + xy - y^2$ is -20.

Example 10

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Determine the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d Determine the value(s) of x for which $V = 0.5$ m^3.

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2 - 2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

 ISBN 978-1-009-53719-3

b Let $V = 4x^3 - 8x^2 + 4x$. The maximum volume will occur when $\dfrac{dV}{dx} = 0$.

We have $\dfrac{dV}{dx} = 12x^2 - 16x + 4$, and so $\dfrac{dV}{dx} = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \\ \therefore \quad x = \frac{1}{3} &\text{ or } x = 1 \end{aligned}$$

But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

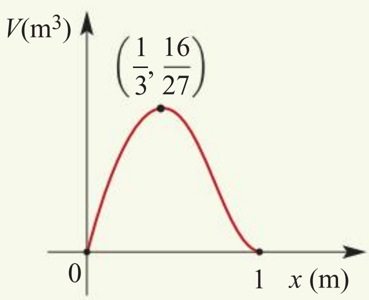
A maximum occurs when $x = \frac{1}{3}$.

$$\therefore \text{ Maximum volume } = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3}$$

$$= \frac{16}{27} \text{ m}^3$$

x		$\frac{1}{3}$		1
$\frac{dV}{dx}$	$+$	0	$-$	0
shape	/	—	\	—

c

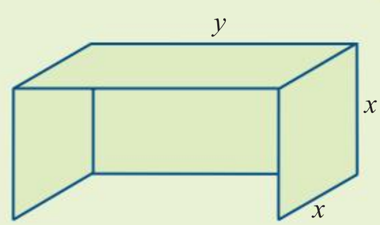


d To determine the value(s) of x for which $V = 0.5$ m^3, we need to solve the equation $V = 0.5$, i.e. $4x^3 - 8x^2 + 4x = 0.5$.

Using a calculator gives $x = 0.19$, $x = 0.5$ or $x = 1.31$, correct to two decimal places. But the domain of V is $[0, 1]$. Hence $x = 0.19$ m or $x = 0.5$ m.

Example 11

A canvas shelter is made up with a back, two square sides and a top. The area of canvas available is 24 m^2. Let V m^3 be the volume enclosed by the shelter.



a Determine the dimensions of the shelter that will create the largest possible enclosed volume.

b Sketch the graph of V against x for a suitable domain.

c Determine the values of x and y for which $V = 10$ m^3.

 ISBN 978-1-009-53719-3

Solution

a The volume $V = x^2y$. One of the variables must be eliminated.
We know that the area is 24 m^2.

$$\therefore \quad 2x^2 + 2xy = 24$$

Rearranging gives $y = \dfrac{24 - 2x^2}{2x}$, i.e. $y = \dfrac{12}{x} - x$.
Substituting in the formula for volume gives

$$V = 12x - x^3$$

Differentiation now gives

$$\frac{dV}{dx} = 12 - 3x^2$$

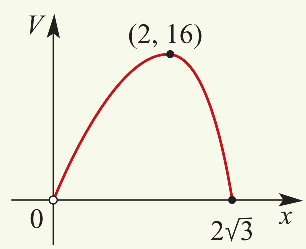
Stationary points occur when $\dfrac{dV}{dx} = 0$, which implies $12 - 3x^2 = 0$.

So stationary points occur when $x^2 = 4$, i.e. when $x = \pm 2$. But negative values have no meaning in this problem, so the only solution is $x = 2$.

Hence the maximum is at $x = 2$. The dimensions are 2 m, 2 m, 4 m.

x		2	
$\dfrac{dV}{dx}$	+	0	−
shape of V	/	—	\

b Note that $x > 0$ and $y \geq 0$.
This implies $x > 0$ and $12 - x^2 \geq 0$, i.e. $0 < x \leq 2\sqrt{3}$.



c Using a calculator, solve the equation $12x - x^3 = 10$ for $0 < x \leq 2\sqrt{3}$.
The solutions are $x = 0.8926$ and $x = 2.9305$, correct to four decimal places.

Possible dimensions to the nearest centimetre are 0.89 m, 0.89 m, 12.55 m and 2.93 m, 2.93 m, 1.16 m.

Maximum or minimum at an endpoint

Calculus can be used to determine a local maximum or local minimum, but these are often not the actual maximum or minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

Example 12

Define the function $f(x) = x^2 + 2$ for $x \in [-2, 4]$. Determine the absolute maximum value and the absolute minimum value of the function.

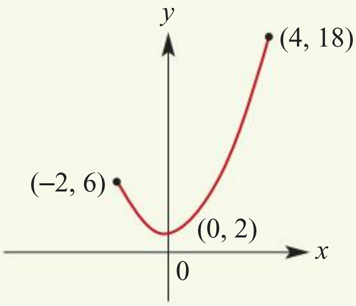
Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

The minimum value occurs at a stationary point of the graph, but the endpoint $(4, 18)$ is not a stationary point.

The absolute maximum value is 18 and the absolute minimum value is 2.



Example 13

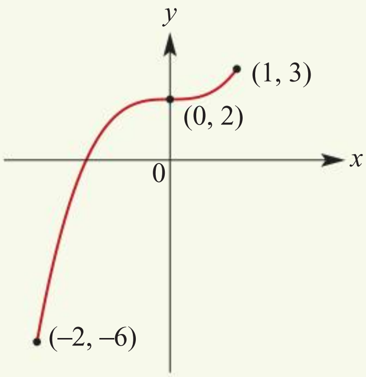
Define the function $f(x) = x^3 + 2$ for $x \in [-2, 1]$. Determine the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

The absolute maximum and the absolute minimum do not occur at stationary points.



Example 14

In Example 10, the maximum volume of a box was found. The maximum value corresponded to a local maximum of the graph of $V = 4x^3 - 8x^2 + 4x$. This was also the absolute maximum value.

If the height of the box must be at most 0.3 m (i.e. $x \le 0.3$), what will be the maximum volume of the box?

Solution

The local maximum of $V(x)$ for $x \in [0, 1]$ was at $x = \frac{1}{3}$. But $\frac{1}{3}$ is greater than 0.3.

For the new problem, we have $V'(x) > 0$ for all $x \in [0, 0.3]$, and so $V(x)$ is strictly increasing on the interval $[0, 0.3]$.

Therefore the maximum volume occurs when $x = 0.3$ and is $0.588\ \text{m}^3$.

Summary 15E

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, determine the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Skill-sheet

Exercise 15E

Example 8

1 A loop of string of length 200 cm is to be formed into a rectangle. Determine the maximum area of this rectangle.

CF

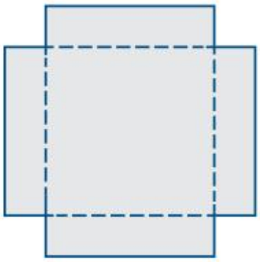
2 Determine the maximum value of the product of two numbers x and $10 - x$.

Example 9

3 Given that $x + y = 2$, calculate the minimum value of $x^2 + y^2$.

Example 10

4 From a square piece of metal of side length 6 m, four squares are removed as shown in the diagram. The metal is folded along the dashed lines to form an open box with height x m.



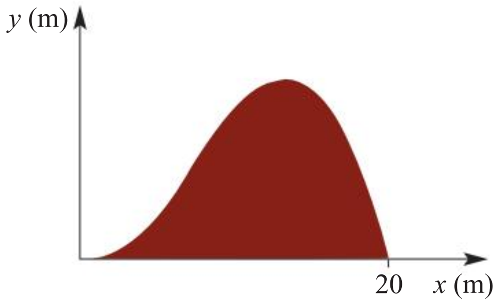
a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 24x^2 + 36x$.

b Determine the value of x that gives the box its maximum volume and determine the maximum volume.

5 A bank of earth has cross-section as shown in the diagram. The curve defining the bank has equation

CU

$$y = \frac{x^2}{400}(20 - x) \quad \text{for } x \in [0, 20]$$



a Determine the height of the bank where:

i $x = 5$ **ii** $x = 10$ **iii** $x = 15$

b Determine the value of x for which the height is a maximum and state the maximum height of the bank.

c Determine the values of x for which:

i $\dfrac{dy}{dx} = \dfrac{1}{8}$ **ii** $\dfrac{dy}{dx} = -\dfrac{1}{8}$

ISBN 978-1-009-53719-3

Example 11 **6** A cuboid has a total surface area of 150 cm^2 and a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

d If the maximum side length of the square base of the cuboid is 4 cm, what is the maximum volume possible?

CU

7 The volume of a cylinder is given by the formula $V = \pi r^2 h$. dDetermine the maximum value of V if $r + h = 12$.

8 A rectangular sheet of metal measures 50 cm by 40 cm. Congruent squares of side length x cm are cut from each of the corners and not used further. The sheet is then folded up to make a tray of depth x cm. Determine the value of x for which the volume of the tray is a maximum.

CF

Example 12 **9** Define the function $f(x) = 2 - 8x^2$ for $x \in [-2, 2]$. Determine the absolute maximum value and the absolute minimum value of the function.

Example 13 **10** Define the function $f(x) = x^3 + 2x + 3$ for $x \in [-2, 1]$. Determine the absolute maximum value and the absolute minimum value of the function for its domain.

11 Let $f(x) = 2x^3 - 6x^2$, where $0 \le x \le 4$. Determine the absolute maximum and the absolute minimum values of $f(x)$.

12 Let $f(x) = 2x^4 - 8x^2$, where $-2 \le x \le 5$. Determine the absolute maximum and the absolute minimum values of $f(x)$.

Example 14 **13** A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

b Determine the derivative $\dfrac{dV}{dx}$.

c Determine the local maximum for the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, determine the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, determine the absolute maximum value of V and the value of x for which this occurs.

CU

14 For the variables x, y and z, it is known that $x + y = 20$ and $z = xy$.

a If $x \in [2, 5]$, determine the possible values of y.

b Determine the maximum and minimum values of z.

CF

15 For the variables x, y and z, it is known that $z = x^2 y$ and $2x + y = 50$. Determine the maximum value of z if:

a $x \in [0, 25]$ **b** $x \in [0, 10]$ **c** $x \in [5, 20]$.

CU

 ISBN 978-1-009-53719-3

16 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Determine $\dfrac{dA}{dx}$.

c Determine the value of x that makes A a minimum.

d What is the minimum total area of the two squares?

15F Applications of differentiation to kinematics

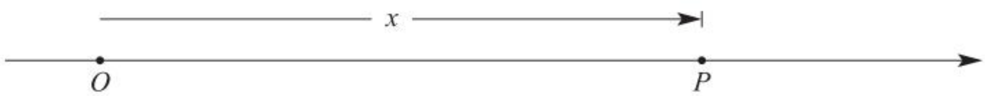
Learning intentions

- To be able to use derivatives in problems involving motion in a straight line.

In this section we continue our study of motion in a straight line from Section 13E.

Position

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.



Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O, while if $x = 3$, the position is 3 m to the right of O.

Sometimes there is a rule that enables the position at any instant to be calculated. In this case, we can view x as being a function of t. Hence $x(t)$ is the position at time t.

For example, imagine that a stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, then an approximate model for the position is

$$x(t) = 5t^2 \quad \text{for } 0 \le t \le 3$$

Example 15

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \ge 0$.

a Determine its initial position.

b Determine its position at $t = 4$.

Solution

a At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.

b At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

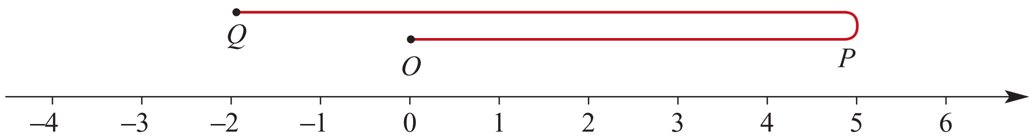
 ISBN 978-1-009-53719-3

Distance and displacement

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction).

For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

Velocity

In this section we focus on the instantaneous rates of change which arise when studying the motion of a particle travelling in a straight line. In particular, we define the velocity and acceleration of a particle.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

 ISBN 978-1-009-53719-3 Cambridge University Press

Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned} &1 \text{ metre per second} &&= 1 \text{ m/s} &&= 1 \text{ m s}^{-1} \\ &1 \text{ centimetre per second} &&= 1 \text{ cm/s} &&= 1 \text{ cm s}^{-1} \\ &1 \text{ kilometre per hour} &&= 1 \text{ km/h} &&= 1 \text{ km h}^{-1} \end{aligned}$$

The first and third units are connected in the following way:

$$\begin{aligned} 1 \text{ km/h} &= 1000 \text{ m/h} \\ &= \frac{1000}{60 \times 60} \text{ m/s} \\ &= \frac{5}{18} \text{ m/s} \\ \therefore \quad 1 \text{ m/s} &= \frac{18}{5} \text{ km/h} \end{aligned}$$

Example 16

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Determine its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$v = \dfrac{dx}{dt} = 2t - 7$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\dfrac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$

$= -6.25$

So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

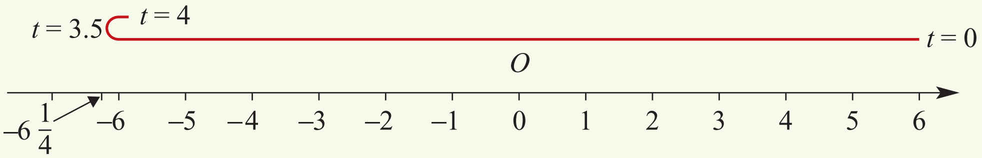
 ISBN 978-1-009-53719-3 Cambridge University Press

c Average velocity $= \dfrac{\text{change in position}}{\text{change in time}}$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$\therefore$ Average velocity $= \dfrac{-6-6}{4} = -3$ cm/s

d Average speed $= \dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled $= 12.25 + 0.25 = 12.5$

$\therefore$ Average speed $= \dfrac{12.5}{4} = 3.125$ cm/s

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Note: The second derivative $\dfrac{d^2x}{dt^2}$ is just the derivative of the derivative. It will be discussed further in Units 3 & 4.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

Example 17

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Determine its initial position, velocity and acceleration, and hence describe its motion.

b Determine the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a $x = t^3 - 6t^2 + 5$

$$v = \frac{dx}{dt} = 3t^2 - 12t$$

$$a = \frac{dv}{dt} = 6t - 12$$

So when $t = 0$, we have $x = 5$, $v = 0$ and $a = -12$.

Initially, the particle is instantaneously at rest 5 cm to the right of O, with an acceleration of -12 cm/s^2.

b $v = 0$ implies $3t^2 - 12t = 0$

$$3t(t-4) = 0$$

$$\therefore \quad t = 0 \text{ or } t = 4$$

The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.

At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds, the particle's position is 27 cm to the left of O, and its acceleration is 12 cm/s^2.

Example 18

A car starts from rest and moves a distance s metres in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

What is the initial acceleration and the acceleration when $t = 2$?

Solution

We are given

$$s = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

The car's velocity is given by

$$v = \frac{ds}{dt} = \frac{1}{2}t^2 + \frac{1}{2}t$$

The car's acceleration is given by

$$a = \frac{dv}{dt} = t + \frac{1}{2}$$

When $t = 0$, $a = \frac{1}{2}$, and when $t = 2$, $a = 2\frac{1}{2}$.

Hence the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

 ISBN 978-1-009-53719-3

Summary 15F

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is positive.
- **Average velocity** for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_2 is the position at time t_2 and x_1 is the position at time t_1.
- The instantaneous rate of change of position with respect to time is called the **instantaneous velocity**, or simply the **velocity**.
  If x is the position of the particle at time t, then its velocity is $v = \dfrac{dx}{dt}$.
- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is $\dfrac{\text{distance travelled}}{t_2 - t_1}$.
- **Average acceleration** for a time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$.

Exercise 15F

Example 15

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 12t + 11$, $t \geq 0$.

a Determine its initial position.

b Determine its position at $t = 3$.

Example 16

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 12t + 11$, $t \geq 0$.

a Determine its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 3 seconds?

d Determine its average speed for the first 3 seconds.

3 The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \frac{1}{3}t^3 - 12t + 6$.

a Determine the rate of change of position with respect to time at $t = 3$.

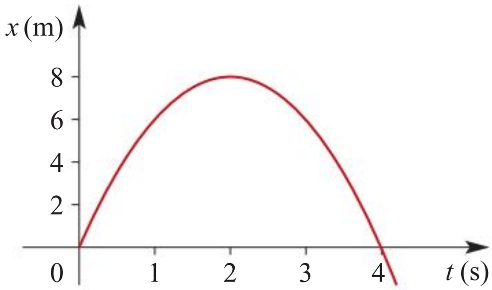
b Determine the time at which the velocity is zero.

CF

ISBN 978-1-009-53719-3

CU

4 A particle moves along a horizontal straight line, starting at a point O. The position–time graph for the motion of the particle is as shown.



a For which values of t is the particle stationary?

b For which values of t is the particle moving to the right?

c How far from O does the particle go to the right?

d From the start of its journey, how long does it take to return to O?

e The position of the particle, x m, at time t seconds is given by a rule of the form $x = pt^2 + qt + r$ for $t \geq 0$. From the graph, determine the values of p, q and r.

f Determine the velocity of the particle at time $t = 3$.

CF

Example 17

5 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 4t^3 - 6t^2 + 5$, $t \geq 0$.

a Determine its initial position, velocity and acceleration, and hence describe its motion.

b Determine the times when it is instantaneously at rest and determine its position and acceleration at those times.

Example 18

6 A car starts from rest and moves a distance s metres in t seconds, where $s = t^4 + t^2$.

a What is the acceleration when $t = 0$?

b What is the acceleration when $t = 2$?

7 Let $s = 10 + 15t - 4.9t^2$ be the height (in metres) of an object at time t (in seconds).

a Determine the velocity at time t.

b Determine the acceleration at time t.

8 The position, x metres, at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$.

a When does its velocity equal zero?

b Determine its acceleration at this time.

c Determine the distance travelled in the first 5 seconds.

d When does its velocity equal -2 m/s, and what is its position at this time?

9 A particle moves along a straight line so that after t seconds its position, s m, relative to a fixed point O on the line is given by $s = t^3 - 3t^2 + 2t$.

a When is the particle at O?

b What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

 ISBN 978-1-009-53719-3

10 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 7t + 12$.

- **a** Determine its initial position.
- **b** What is its position at $t = 5$?
- **c** Determine its initial velocity.
- **d** When does its velocity equal zero, and what is its position at this time?
- **e** What is its average velocity in the first 5 seconds?
- **f** What is its average speed in the first 5 seconds?

CF

11 A particle moves in a straight line so that after t seconds its position, s metres, is given by $s = t^4 + 3t^2$.

- **a** Determine the acceleration when $t = 1, t = 2, t = 3$.
- **b** Determine the average acceleration between $t = 1$ and $t = 3$.

12 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$.

- **a** Determine its initial position and velocity.
- **b** Determine its velocity at any time t.
- **c** At what times is the particle stationary?
- **d** What is the position of the particle when it is stationary?
- **e** For how long is the particle's velocity negative?
- **f** Determine its acceleration at any time t.
- **g** When is the particle's acceleration zero? What is its velocity and its position at that time?

13 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = 2t^3 - 5t^2 + 4t - 5$.

- **a** When is its velocity zero, and what is its acceleration at that time?
- **b** When is its acceleration zero, and what is its velocity at that time?

14 A particle is moving in a straight line in such a way that its position, x cm, relative to the point O at time t seconds ($t \geq 0$) satisfies $x = t^3 - 13t^2 + 46t - 48$. When does the particle pass through O, and what is its velocity and acceleration at those times?

15 Two particles are moving along a straight path so that their positions, x_1 cm and x_2 cm, relative to a fixed point P at any time t seconds are given by $x_1 = t + 2$ and $x_2 = t^2 - 2t - 2$.

CU

- **a** Determine the times when the particles are at the same position.
- **b** Determine the time when the particles are moving with the same velocity.

ISBN 978-1-009-53719-3

Review

Chapter summary

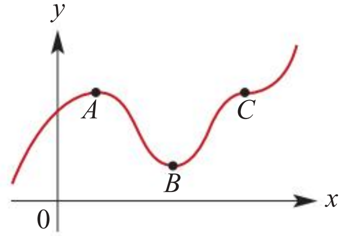
- **Tangents and normals**

  Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then
  - the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
  - the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

- **Stationary points**

  A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.

  The graph shown has three stationary points: A, B and C.

  A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

  B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

  C Point C is a **stationary point of inflection**.

  Stationary points of types A and B are referred to as **turning points**.

- **Maximum and minimum values**

  For a continuous function f defined on an interval $[a, b]$:
  - if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
  - if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

- **Motion in a straight line**

  For an object moving in a straight line with position x at time t:

  $$\text{velocity } v = \frac{dx}{dt} \qquad \text{acceleration } a = \frac{dv}{dt}$$

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills. ☑

15A **1** I can determine the equation of the tangent to the graph of a polynomial at a given point. ☐

See Example 1 and Question 1

15A **2** I can determine the equation of the normal to the graph of a polynomial at a given point. ☐

See Example 2 and Question 1

Review

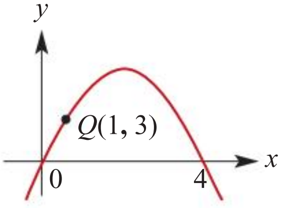
15B	3	I can determine the average rate of change of a polynomial function over a given interval. See Example 3 and Question 1	☐
15B	4	I can determine the average rate of change of a polynomial function based on a real-life context. See Example 4 and Question 2	☐
15C	5	I can determine the stationary points of a polynomial function. See Example 5 and Question 1	☐
15C	6	I can determine the equation of a polynomial function given sufficient information including information about stationary points. See Example 6 and Question 2	☐
15D	7	I can determine the nature of stationary points and sketch the graphs of cubic functions. See Example 7 and Question 1	☐
15E	8	I can determine local maxima and minima for functions defined to model real-life contexts. See Example 8, Example 9, Example 10, Example 11 and Questions 1, 3, 4 and 6	☐
15E	9	I can determine absolute maxima and minima for functions by considering endpoints and local maxima and minima. See Example 12, Example 13, Example 14 and Questions 9, 10 and 14	☐
15F	10	I can use distance versus time functions to determine the position of a particle at time *t*. See Example 15 and Question 1	☐
15F	11	I can use differentiation and a distance in terms of time function to determine the velocity and acceleration of a particle at time *t*. See Example 16, Example 17, Example 18 and Questions 2, 5 and 6	☐

Short-response questions

Technology-free short-response questions

SF

1 The graph of $y = 4x - x^2$ is shown.



a Determine $\dfrac{dy}{dx}$.

b Determine the gradient of the tangent to the curve at $Q(1, 3)$.

c Determine the equation of the tangent at Q.

Review

2 Determine the equations of the tangent and normal to each of the following at the given value of x.

a $y = x^2 - 7x \quad x = 2$

b $y = x^3 - 2x^2 \quad x = 1$

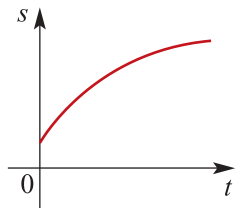
c $y = -x^3 + x^2 + 2x \quad x = -1$

d $y = 2x^3 - 2x^2 + 3x \quad x = -1$

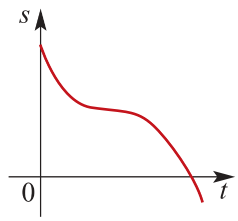
SF

3 Match each position–time graph shown with its velocity–time graph:

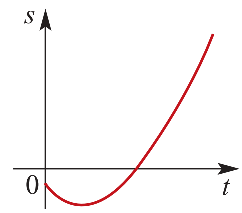
a



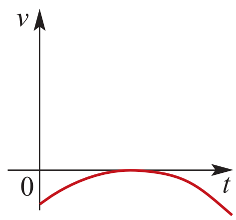
b



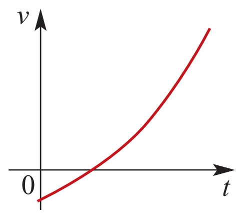
c



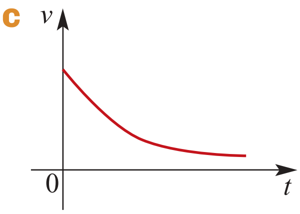
A



B

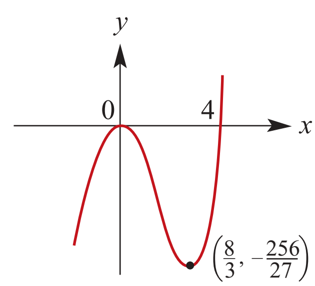


C



CF

4 The graph of $y = x^3 - 4x^2$ is shown.



a Determine $\dfrac{dy}{dx}$.

b Determine the gradient of the tangent to the curve at the point $(2, -8)$.

c Determine the equation of the tangent at the point $(2, -8)$.

d Determine the coordinates of the point Q where the tangent crosses the curve again.

5 Let $y = x^3 - 12x + 2$.

a Determine $\dfrac{dy}{dx}$ and the value(s) of x for which $\dfrac{dy}{dx} = 0$.

b State the nature of each of these stationary points.

c Determine the corresponding y-value for each of these.

6 Write down the values of x for which each of the following derivative functions are zero. For each of the corresponding stationary points, determine whether it is a local maximum, local minimum or stationary point of inflection.

a $\dfrac{dy}{dx} = 3x^2$

b $\dfrac{dy}{dx} = -3x^3$

c $f'(x) = (x - 2)(x - 3)$

d $f'(x) = (x - 2)(x + 2)$

e $f'(x) = (2 - x)(x + 2)$

f $f'(x) = -(x - 1)(x - 3)$

g $\dfrac{dy}{dx} = -x^2 + x + 12$

h $\dfrac{dy}{dx} = 15 - 2x - x^2$

 ISBN 978-1-009-53719-3

7 For each of the following, determine all stationary points and state the nature of each:

a $y = 4x - 3x^3$ **b** $y = 2x^3 - 3x^2 - 12x - 7$ **c** $y = x(2x - 3)(x - 4)$

8 Sketch the graph of each of the following. Give the coordinates of the stationary points and the axis intercepts.

a $y = 3x^2 - x^3$ **b** $y = x^3 - 6x^2$ **c** $y = (x + 1)^2(2 - x)$

d $y = 4x^3 - 3x$ **e** $y = x^3 - 12x^2$

9 Let $f(x) = x^2 - 7$. Determine the equation of each tangent line to the graph of $y = f(x)$ that passes through $P(3, -2)$.

10 A boy stands on the edge of a cliff of height 60 m. He throws a stone vertically upwards so that its distance, h m, above the cliff top is given by $h = 20t - 5t^2$.

a Calculate the maximum height reached by the stone above the cliff top.

b Calculate the time which elapses before the stone hits the beach (vertically below).

c Calculate the speed with which the stone hits the beach.

11 A particle P moves in a straight line so that its position, s m, with respect to a fixed point O is given by $s = t^3 - 9t^2 + 24t + 2$ where t is the time in seconds after the start of the motion. Determine:

a the initial velocity and acceleration.

b the values of t when P is instantaneously at rest.

c the distance travelled in the first four seconds.

d the average speed of P during the first five seconds.

12 A small metallic ball rolls along a straight track such that its position, s m, with respect to a fixed point O is given by $s = t^3 - 7t^2 + 15t - 9$ where t is the time in seconds after the start of the motion. Determine:

a the times at which the ball is at O.

b the velocity when $t = 2$

c the magnitude of the acceleration when the velocity is 4 m/s

d the distance travelled by the ball when it first comes to rest.

Maximum and minimum questions

13 Given that $x + y = 5$ and $P = xy$, determine:

a y in terms of x

b P in terms of x

c the maximum value of P and the corresponding values of x and y.

14 A rectangular patch of ground is to be enclosed with 100 metres of fencing wire. Determine the dimensions of the rectangle so that the area enclosed will be a maximum.

15 The sum of two numbers is 24. If one number is x, determine the value of x such that the product of the two numbers is a maximum.

16 Determine the least possible value of $x^2 + y^2$ given that $x + y = 12$.

Review

Technology-active short-response questions

SF

17 Let $f(x) = x^2 - 3x$. Determine the positive value of x, correct to two decimal places, for which the tangent to the curve $y = f(x)$ at the point $(x, f(x))$ passes through the point $(2, -8)$

18 Let $f(x) = x^3 - 9x^2 + 6x$. There are two points on the graph of $y = f(x)$ which have tangents with gradient -9. Determine the x-coordinates of these two points.

19 Let $f(x) = x^3 - 3x^2 + 4x$. The tangents to the graph of $y = f(x)$ at the points $(2, 4)$ and $(1, 2)$ intersect at the point with coordinates (a, b). Determine the values of a and b.

20 The normal to the graph of $y = 2x^3 - 3x^2 - 4x$ at $x = -1$ crosses this graph at two other points. Determine the x-coordinates of these points.

CF

21 The tangent to the graph of $y = 2x^2 - 3x + 8$ at the point with coordinates $(a, f(a))$ where $a > 0$ passes through the origin. Determine the possible values of a.

22 Let $f(x) = 2x^3 - 6x^2 - 4x$. The tangent to the graph of $y = f(x)$ at the point $P(1, -8)$ intersects the x-axis at a point $A(a, 0)$ and the normal to the graph at this point intersects the x-axis at $B(b, 0)$. Determine the area of triangle PAB.

23 Let $f(x) = x^3 - 2x^2 - 5x - 1, -2 \le x \le 4$.

a Determine the maximum value of $f(x)$ for this stated domain.

b State the values of x for which stationary points occur giving your answers correct to one decimal place.

24 For each of the following determine the coordinates of any stationary points, giving values correct to two decimal places when appropriate.

a $y = 2x^3 - 4x^2 - 6x + 3$ **b** $y = -x^3 + 2x^2 - 3x + 5$ **c** $y = 4x^3 - 3x^2 - 6x$

25 Let $f(x) = x^3 - ax^2 + bx$.

a Determine the values of a and b if there is a local minimum at $(3, -2)$.

b Using the values of a and b found in part **a** determine the coordinates of the local maximum of the graph of $y = f(x)$, correct to two decimal places.

26 The height, in metres, of a stone thrown vertically upwards from the surface of a planet is $2 + 10t - 4t^2$ after t seconds.

a Calculate the velocity of the stone after 3 seconds.

b Determine the acceleration due to gravity.

 ISBN 978-1-009-53719-3

27 A dam is being emptied. The quantity of water, V litres, remaining in the dam at any time t minutes after it starts to empty is given by

$$V(t) = 1000(30 - t)^3 \qquad \text{for } t \geq 0$$

a Sketch the graph of V against t.

b Determine the time at which there are:

i 2 000 000 litres of water in the dam

ii 20 000 000 litres of water in the dam.

c At what rate is the dam being emptied at any time t?

d How long does it take to empty the dam?

e At what time is the water flowing out at 8000 litres per minute?

f Sketch the graphs of $y = V(t)$ and $y = V'(t)$ on the one set of axes.

28 In a certain area of Australia the quantity of blackberries, W tonnes, ready for picking x days after 1 September is given by

$$W = \frac{x}{4000}\left(48\,000 - 2600x + 60x^2 - \frac{x^3}{2}\right) \qquad \text{for } 0 \leq x \leq 60$$

a Sketch the graph of W against x for $0 \leq x \leq 60$.

b After how many days will there be 50 tonnes of blackberries ready for picking?

c Determine the rate of increase of W, in tonnes per day, when $x = 20$, 40 and 60.

d Determine the value of W when $x = 30$.

29 A newly installed central heating system has a thermometer which shows the water temperature as it leaves the boiler (y°C). It also has a thermostat which switches off the system when $y = 65$.

The relationship between y and t, the time in minutes, is given by $y = 15 + \dfrac{1}{80}t^2(30 - t)$.

a Determine the temperature at $t = 0$.

b Determine the rate of increase of y with respect to t, when $t = 0$, 5, 10, 15 and 20.

c Sketch the graph of y against t for $0 \leq t \leq 20$.

30 The sweetness, S, of a pineapple t days after it begins to ripen is found to be given by $S = 4000 + (t - 16)^3$ units.

a At what rate is S increasing when $t = 0$?

b Determine $\dfrac{dS}{dt}$ when $t = 4$, 8, 12 and 16.

c The pineapple is said to be unsatisfactory when our model indicates that the rate of increase of sweetness is zero. When does this happen?

d Sketch the graph of S against t up to the moment when the pineapple is unsatisfactory.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

31 A slow train which stops at every station passes a certain signal box at 12 p.m. The motion of the train between the two stations on either side of the signal box is such that it is s km past the signal box at t minutes past 12 p.m., where CU

$$s = \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3$$

(Note that, before the train reaches the signal box, both s and t will be negative.)

a Use a calculator to help sketch the graphs of s against t and $\frac{ds}{dt}$ against t on the one set of axes. Sketch for $t \in [-2, 5]$.

b Determine the time of departure from the first station and the time of arrival at the second.

c Determine the distance of each station from the signal box.

d Determine the average velocity between the stations.

e Determine the velocity with which the train passes the signal box.

32 Water is draining from a tank. The volume, V L, of water at time t (hours) is given by $V(t) = 1000 + (2 - t)^3$, for $t \geq 0$ and $V(t) \geq 0$.

a What are the possible values of t?

b Determine the rate of draining when:

i $t = 5$ **ii** $t = 10$

33 A mountain path can be approximately described by the following rule, where y is the elevation, in metres above sea level, and x is the horizontal distance travelled in kilometres:

$$y = \frac{1}{5}(4x^3 - 8x^2 + 192x + 144) \qquad \text{for } 0 \leq x \leq 7$$

a How high above sea level is the start of the track, i.e. $x = 0$?

b When $x = 6$, what is the value of y?

c Use a calculator to draw a graph of the path. Sketch this graph.

d Does this model for the path make sense for $x > 7$?

e Determine the gradient of the graph for the following distances (be careful of units):

i $x = 0$ **ii** $x = 3$ **iii** $x = 7$

Maximum and minimum problems

34 **a** On the one set of axes sketch the graphs of $y = x^3$ and $y = 2 + x - x^2$. CF

b Note that $2 + x - x^2 \geq x^3$ for $x \leq 0$. Determine the value of x, with $x \leq 0$, for which the vertical distance between the two curves is a minimum and determine the minimum distance.

Hint: Consider the function with rule $y = 2 + x - x^2 - x^3$ for $x \leq 0$.

35 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Determine the rainfall that will produce the maximum and the minimum number of mosquitos. (First plot the graph of $y = M(x)$ using a calculator.)

36 Given that $2x + y = 10$ and $A = x^2y$, where $0 \le x \le 5$, determine:

a y in terms of x

b A in terms of x

c the maximum value of A and the corresponding values of x and y.

37 Given that $xy = 10$ and $T = 3x^2y - x^3$, determine the maximum value of T for $0 < x < \sqrt{30}$.

38 The sum of two numbers x and y is 8.

a Write down an expression for y in terms of x.

b Write down an expression for s, the sum of the squares of these two numbers, in terms of x.

c Determine the least value of the sum of their squares.

39 Determine two positive numbers whose sum is 4, such that the sum of the cube of the first and the square of the second is as small as possible.

40 A factory which produces n items per hour is found to have overhead costs of \$$C$ per hour, where

$$C = 400 - 16n + \frac{1}{4}n^2$$

How many items should be produced every hour to keep the overhead costs to a minimum?

41 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$, and determine the maximum value of P.

42 A farmer has 4 km of fencing wire and wishes to fence in a rectangular piece of land through which a straight river flows. The river is to form one side of the enclosure. How can this be done to enclose as much land as possible?

43 A beam has a rectangular cross-section of depth x cm and width y cm. The perimeter of the cross-section of the beam is 120 cm. The strength, S, of the beam is given by $S = 5x^2y$.

a Determine y in terms of x.

b Express S in terms of x.

c What are the possible values for x?

d Sketch the graph of S against x.

e Determine the values of x and y which give the strongest beam.

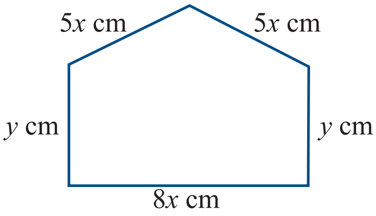
f If the cross-sectional depth of the beam must be less than or equal to 19 cm, determine the maximum strength of the beam.

44 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$, with x representing the temperature of the water in degrees (°C). (This model is valid only if $6 \le x \le 20$.) Determine the water temperature that results in the maximum number of salmon swimming upstream.

45 A piece of wire 360 cm long is used to make the twelve edges of a rectangular box for which the length is twice the breadth.

- **a** Denoting the breadth of the box by x cm, show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$.
- **b** Determine the domain of the function $V(x) = 180x^2 - 6x^3$ which describes the situation.
- **c** Sketch the graph of the function with rule $y = V(x)$.
- **d** Determine the dimensions of the box that has the greatest volume.
- **e** Determine the values of x (correct to two decimal places) for which $V = 20\,000$.

46 A piece of wire of length 90 cm is bent into the shape shown in the diagram.

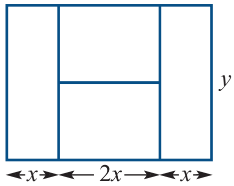


- **a** Show that the area, A cm^2, enclosed by the wire is given by $A = 360x - 60x^2$.
- **b** Determine the values of x and y for which A is a maximum.

47 A piece of wire 100 cm in length is to be cut into two pieces, one piece of which is to be shaped into a circle and the other into a square.

- **a** How should the wire be cut if the sum of the enclosed areas is to be a minimum? (Give your answer to the nearest centimetre.)
- **b** How should the wire be used to obtain a maximum area?

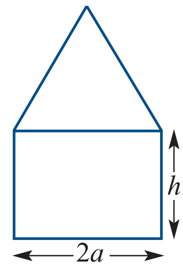
48 A roll of tape 36 metres long is to be used to mark out the edges and internal lines of a rectangular court of length $4x$ metres and width y metres, as shown in the diagram. Determine the length and width of the court for which the area is a maximum.



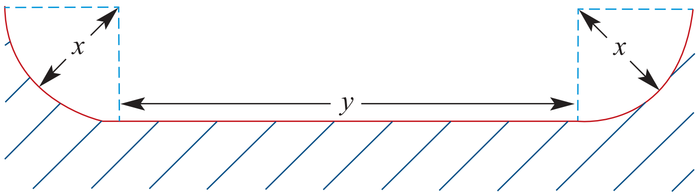
49 A rectangular chicken run is to be built on flat ground. A 16-metre length of chicken wire will be used to form three of the sides; the fourth side, of length x metres, will be part of a straight wooden fence.

- **a** Let y be the width of the rectangle. Determine an expression for A, the area of the chicken run, in terms of x and y.
- **b** Determine an expression for A in terms of x.
- **c** Determine the possible values of x.
- **d** Sketch the graph of A against x for these values of x.
- **e** What is the largest area of ground the chicken run can cover?

50 The diagram illustrates a window that consists of an equilateral triangle and a rectangle. The amount of light that comes through the window is directly proportional to the area of the window.
If the perimeter of such a window must be 8000 mm, determine the values of h and a (correct to the nearest mm) which allow the maximum amount of light to pass.



51 The diagram shows a cross-section of an open drainage channel. The flat bottom of the channel is y metres across and the sides are quarter circles of radius x metres. The total length of the bottom plus the two curved sides is 10 metres.



a Express y in terms of x.
b State the possible values that x can take.
c Determine an expression for A, the area of the cross-section, in terms of x.
d Sketch the graph of A against x, for possible values of x.
e Determine the value of x which maximises A.
f Comment on the cross-sectional shape of the drain.

52 A cylinder closed at both ends has a total surface area of 1000 cm^2. The radius of the cylinder is x cm and the height h cm. Let V cm^3 be the volume of the cylinder.
a Determine h in terms of x.
b Determine V in terms of x.
c Determine $\frac{dV}{dx}$.
d Determine the values of x for which $\frac{dV}{dx} = 0$.
e Sketch the graph of V against x for a suitable domain.
f Determine the maximum volume of the cylinder.
g Determine the value(s) of x and h for which $V = 1000$, correct to two decimal places.

53 A cylindrical aluminium can able to contain half a litre of drink is to be manufactured. The volume of the can must therefore be 500 cm^3.
a Determine the radius and height of the can which will use the least aluminium and therefore be the cheapest to manufacture.
b If the radius of the can must be no greater than 5 cm, determine the radius and height of the can that will use the least aluminium.

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

Multiple-choice questions

Technology-free multiple-choice questions

1 The equation of the tangent to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = x$ **B** $y = 5x - 2$
C $y = 5x$ **D** $y = 5x + 2$

2 The equation of the normal to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = -5x + 2$ **B** $y = \frac{1}{5}x + \frac{12}{5}$
C $y = -\frac{1}{5}x + \frac{12}{5}$ **D** $y = -\frac{1}{5}x + \frac{16}{5}$

3 The equation of the tangent to the curve $y = 2x - 3x^3$ at the origin is

A $y = -2x$ **B** $y = x$ **C** $y = 2x$ **D** $y = -x$

4 The average rate of change of the function $f(x) = 4x - x^2$ between $x = 0$ and $x = 1$ is

A 3 **B** −3 **C** 4 **D** −4

5 A particle moves in a straight line so that its position, S m, relative to O at a time t seconds $(t \geq 0)$ is given by $S = 4t^3 + 3t - 7$. The initial velocity of the particle is

A 0 m/s **B** −7 m/s **C** 3 m/s **D** −4 m/s

6 The function $y = x^3 - 12x$ has stationary points at $x =$

A 0 and 12 **B** −4 and 4 **C** −2 and 4 **D** −2 and 2

7 The curve $y = 2x^3 - 6x$ has a gradient of 6 at $x =$

A 2 **B** $\sqrt{2}$ **C** −2 and 2 **D** $-\sqrt{2}$ and $\sqrt{2}$

8 The rate of change of the function $f(x) = 2x^3 - 5x^2 + x$ at $x = 2$ is

A 5 **B** −2 **C** 2 **D** −5

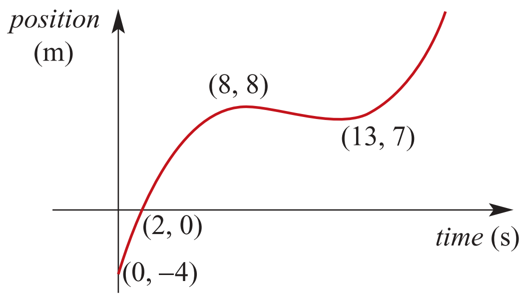
9 The average rate of change of the function $y = \frac{1}{2}x^4 + 2x^2 - 5$ between $x = -2$ and $x = 2$ is

A 0 **B** 5.5 **C** 11 **D** 22

10 The minimum value of the function $y = x^2 - 8x + 1$ is

A 1 **B** 4 **C** −15 **D** 0

A particle moves along a horizontal line. The graph of the particle's position relative to the origin over time is shown.



11 The particle has a velocity of zero at

A 8 s and 13 s **B** 2 s **C** 0 s **D** 8 s and 7 s

12 The time interval(s) during which the particle has a negative velocity are

A $8 < t < 13$ **B** $0 < t < 2$ and $8 < t < 13$
C $0 < t < 2$ **D** $7 < t < 8$

Technology-active multiple-choice questions

13 Let f be the function given by $f(x) = x^2 - 3x + 4$. The tangent line to the graph of f at $x = 2$ has equation $y = g(x)$.
$f(x) - g(x) < 1$ for

A $x > 0$ **B** $1 < x < 3$ **C** $x > 2$ **D** $0 < x < 2$

14 Two points A and B on the graph of $f(x) = x^3 + 3x$ have tangent lines with gradient 6. The distance between the x-intercepts of the tangent lines at A and B is

A 6 **B** 4 **C** $\frac{2}{3}$ **D** $\frac{1}{3}$

15 Let $f(x) = x^3 + 3x^2$ and $g(x) = x^2 + 20x - 16$. For which positive value of x are the tangent lines to the graphs of $y = f(x)$ and $y = g(x)$ parallel?

A 3 **B** 2 **C** $\frac{1}{2}$ **D** $\frac{10}{3}$

16 Let $f(x) = x^3 - 3x^2 - bx + 6$. The value of b for which there is only one stationary point of the graph of $y = f(x)$ is

A 2 **B** -2 **C** 0 **D** -3

17 Let $f(x) = 5x^3 - bx^2 - bx - 12$. The values of b for which there are no stationary points of the graph of $y = f(x)$ are

A $b \leq -15$ **B** $-0 \leq b \leq 15$ **C** $b > 15$ **D** $-15 < b < 0$

18 The tangent to the curve $y = 2x^2 + ax + b$ at the point $(2, 11)$ is perpendicular to the line $2y = x + 7$. The values of a and b are.

A $a = -10, b = 23$ **B** $a = 6, b = 15$ **C** $a = 4, b = 12$ **D** $a = 5, b = 5$

16

Further differentiation

Chapter contents

In Chapter 14 we have seen how to differentiate polynomial functions. In this chapter we add to the collection of functions that we can differentiate.

We start by extending our rule for differentiating powers of x to negative integer powers. We then introduce three rules for differentiating functions. The first is the chain rule.

What is a **composition** of functions?

If $f(x) = x^5$ and $g(x) = x^2 + 3x$, then the function with rule $f(g(x)) = (x^2 + 3x)^5$ is called the composition of f with g. The other composition from these two functions has rule $g(f(x)) = x^{10} + 3x^5$. We write the new functions as $f \circ g$ and $g \circ f$ respectively.

The chain rule gives a simple method for differentiating expressions such as $(x^2 + 3x)^5$.

By applying the chain rule, we are able to extend further our rule for differentiating powers of x to rational powers.

We also introduce the product rule and the quotient rule. By combining the chain, product and quotient rules, we are able to differentiate complicated expressions such as

$$(2x + 1)(x^2 + 2)^{10}, \quad (2x + 1)\sqrt{x^2 + 2} \quad \text{and} \quad \frac{2x + 1}{x^2 + 2}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

16A The chain rule

Learning intentions

- To be able to use the **chain rule**.

An expression such as $(x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term. This method is a great deal more tiresome for an expression such as $(x^3 + 1)^{30}$.

Instead, we can transform $y = (x^3 + 1)^2$ into two simpler functions defined by

$$u = x^3 + 1 \quad \text{and} \quad y = u^2$$

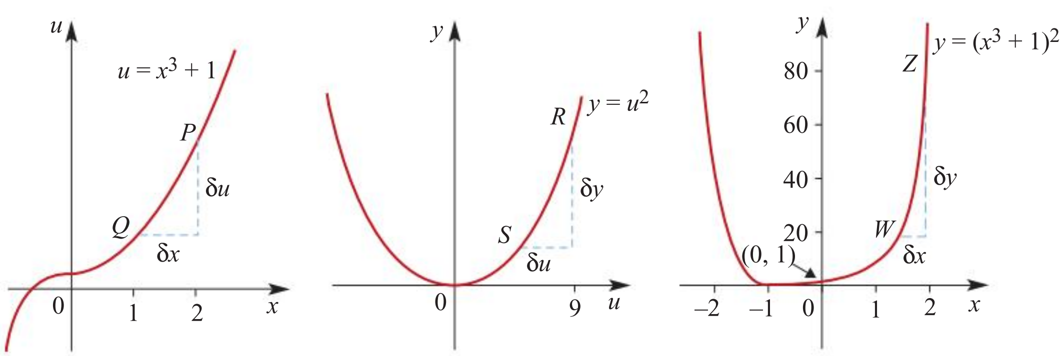
which are 'chained' together:

$$x \xrightarrow{h} u \xrightarrow{g} y$$

where $h(x) = x^3 + 1 = u$ and $g(u) = u^2 = y$.

We may exploit this connection to differentiate. We use Leibniz notation to explore this idea.

Consider $y = (x^3 + 1)^2$ at the point where $x = 2$.



When $x = 2$, $u = 9$
P is the point $(2, 9)$

When $u = 9$, $y = 81$
R is the point $(9, 81)$

When $x = 2$, $y = 81$
Z is the point $(2, 81)$

A section of a spreadsheet (shown below) illustrates the connection between the gradients of the secants PQ, RS and ZW.

x	u	y	δx	δu	δy	$\dfrac{\delta u}{\delta x}$	$\dfrac{\delta y}{\delta u}$	$\dfrac{\delta u}{\delta x} \times \dfrac{\delta y}{\delta u}$	$\dfrac{\delta y}{\delta x}$
1.60000	5.09600	25.96922	0.40000	3.90400	55.03078	9.76000	14.09600	137.57696	137.57696
1.80000	6.83200	46.67622	0.20000	2.16800	34.32378	10.84000	15.83200	171.61888	171.61888
1.90000	7.85900	61.76388	0.10000	1.14100	19.23612	11.41000	16.85900	192.36119	192.36119
1.99000	8.88060	78.86504	0.01000	0.11940	2.13496	11.94010	17.88060	213.49614	213.49614
1.99900	8.98801	80.78425	0.00100	0.01199	0.21575	11.99400	17.98801	215.74816	215.74816
1.99990	8.99880	80.97840	0.00010	0.00120	0.02160	11.99940	17.99880	215.97480	215.97480
1.99999	8.99988	80.99784	0.00001	0.00012	0.00216	11.99994	17.99988	215.99748	215.99748

It can be seen that, as δx gets smaller, so does δu, and that $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$.

By considering δx becoming smaller, and hence δu becoming smaller, we arrive at the following result. We will prove this result at the end of the section.

 ISBN 978-1-009-53719-3

The chain rule (using Leibniz notation)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

From the spreadsheet it can be seen that the gradient of $u = x^3 + 1$ at $x = 2$ is 12, and the gradient of $y = u^2$ at $u = 9$ is 18. The gradient of $y = (x^3 + 1)^2$ at $x = 2$ is 216. The chain rule can be used to confirm this:

$$\frac{du}{dx} = 3x^2 \quad \text{and, at } x = 2, \quad \frac{du}{dx} = 12$$

$$\frac{dy}{du} = 2u \quad \text{and, at } u = 9, \quad \frac{dy}{du} = 18$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 18 \times 12 = 216$$

Example 1

Determine the derivative of $y = (3x + 4)^{20}$.

Solution

Let $\quad u = 3x + 4 \quad$ then $\quad y = u^{20}$

So $\quad \dfrac{du}{dx} = 3 \quad$ and $\quad \dfrac{dy}{du} = 20u^{19}$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 20u^{19} \cdot 3 \\ &= 60(3x + 4)^{19} \end{aligned}$$

Example 2

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

Let $\quad u = 4x^3 - 5x \quad$ then $\quad y = u^{-2}$

So $\quad \dfrac{du}{dx} = 12x^2 - 5 \quad$ and $\quad \dfrac{dy}{du} = -2u^{-3}$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (-2u^{-3}) \cdot (12x^2 - 5) \\ &= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3} \end{aligned}$$

 ISBN 978-1-009-53719-3

Example 3

Determine the gradient of the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point $(1, 4)$.

Solution

Let $u = 3x^2 + 1$ then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = -16u^{-2}$

The chain rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -16u^{-2} \cdot 6x \\ &= \frac{-96x}{(3x^2+1)^2}\end{aligned}$$

$\therefore$ at $x = 1$, the gradient is $\dfrac{-96}{16} = -6$.

We can state the chain rule using function notation as follows.

The chain rule (using function notation)

$$(f \circ g)'(x) = f'(g(x))\, g'(x)$$

where $(f \circ g)(x) = f(g(x))$

Example 4

Given that $f(x) = (x^2 + 1)^3$, determine $f'(x)$.

Solution

We can write $f(x) = k(g(x))$, where

$$k(x) = x^3 \quad \text{and} \quad g(x) = x^2 + 1$$

We have

$$k'(x) = 3x^2 \quad \text{and} \quad g'(x) = 2x$$

Therefore, by the chain rule:

$$\begin{aligned}f'(x) &= k'(g(x))\, g'(x) \\ &= 3(g(x))^2 \times 2x \\ &= 6x(x^2+1)^2\end{aligned}$$

 ISBN 978-1-009-53719-3

Proof of the chain rule

Let f and g be differentiable functions. To determine the derivative of their composition $f \circ g$, we must evaluate the limit

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

Multiply and divide the expression inside the limit by an extra term:

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \frac{g(x+h) - g(x)}{h}$$

Now we can rewrite the limit as

$$(f \circ g)'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

The second limit is just $g'(x)$. Notice that the expression inside the first limit is the gradient of the secant of the graph of f through the two points $(g(x), f(g(x)))$ and $(g(x+h), f(g(x+h)))$. As $h \to 0$, this approaches the gradient of f at $g(x)$. So the first limit is $f'(g(x))$.

Hence we have shown that

$$(f \circ g)'(x) = f'(g(x))\, g'(x)$$

This proof is not without fault: for example, if there are values of h near zero for which $g(x+h) - g(x) = 0$, then we have division by zero in the first limit. However, a completely rigorous proof is beyond the scope of this course.

Summary 16A

- **The chain rule (using Leibniz notation)**

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- **The chain rule (using function notation)**

$$(f \circ g)'(x) = f'(g(x))\, g'(x), \quad \text{where } (f \circ g)(x) = f(g(x))$$

Exercise 16A

SF

Example 1, 2

1 Differentiate each of the following with respect to x:

a $(2x-1)^7$ **b** $(5x+2)^{12}$ **c** $(1-x)^4$

d $(x-1)^{30}$ **e** $(x^5 - x^{10})^{20}$ **f** $(x - x^3 - x^5)^4$

g $(x^2 + 2x + 1)^4$ **h** $(x^2 + 2x)^{-2}, \ x \neq -2, 0$ **i** $\left(x^2 - \dfrac{2}{x}\right)^{-3}, \ x \neq 0$

 ISBN 978-1-009-53719-3

Example 3

2 **a** Determine the derivative of $y = (2x^3 + 1)^4$.

b Determine the gradient of the curve $y = (2x^3 + 1)^4$ at the point $(1, 81)$.

SF

Example 4

3 For each of the following, determine $f'(x)$:

a $f(x) = (2x - 3)^3$

b $f(x) = (2x^2 + 7)^7$

c $f(x) = -4(3x^2 + 2)^5$

d $f(x) = 2(x - 3x^2)^{-2}$

e $f(x) = -(x^2 - x + 3)^4$

f $f(x) = -3(1 - x^3)^{-5}$

4 **a** Determine the gradient of the curve with equation $y = \dfrac{1}{x+3}$ at the point $(1, \frac{1}{4})$.

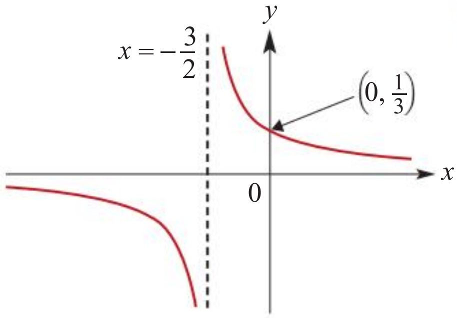
b Determine the gradient of the curve with equation $y = \dfrac{1}{(x+3)^3}$ at the point $(1, \frac{1}{64})$.

CF

5 The diagram is a sketch graph of the function $f(x) = \dfrac{1}{2x+3}$.

a Determine the gradient of the curve at the point $(0, \frac{1}{3})$.

b Determine the coordinates of the points on
the curve at which the gradient is $-\dfrac{2}{9}$.



6 Consider the curve with equation $y = 2(x^2 - 1)^3$.

a Determine $\dfrac{dy}{dx}$.

b Determine the value of $\dfrac{dy}{dx}$ at:

i $x = 0$ **ii** $x = 2$ **iii** $x = -1$

c Determine the equation of the tangent to the curve at $x = 0$.

d Determine the equation of the normal to the curve at $x = 0$.

e Determine the coordinates of the stationary points on the curve.

7 Consider the family of functions with rules of the form

$$f(x) = a(4 - x^2)^2$$

where a is a non-zero real number.

a Determine $f'(x)$ in terms of x and a.

b Determine in terms of a:

i $f'(1)$

ii the x-coordinates of the stationary points on the curve $y = f(x)$

iii the equation of the tangent to the curve $y = f(x)$ at $x = 1$.

 ISBN 978-1-009-53719-3 Cambridge University Press

16B Differentiating rational powers

Learning intentions

- To be able to differentiate **rational powers.**

Before using the chain rule to differentiate rational powers, we will show how to differentiate $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ by first principles.

Example 5

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}},\ x > 0$

b $g(x) = x^{\frac{1}{3}},\ x \neq 0$

Solution

a

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\therefore\quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

b We use the identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ and by observing that $\left(a^{\frac{1}{3}}\right)^3 = a$ and $\left(b^{\frac{1}{3}}\right)^3 = b$, we obtain

$$a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

and therefore

$$a^{\frac{1}{3}} - b^{\frac{1}{3}} = \frac{a-b}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$$

We now have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}\\ &= \frac{x+h-x}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}\\ &= \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\end{aligned}$$

Hence

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3x^{\frac{2}{3}}}$$

 ISBN 978-1-009-53719-3

Note: We can prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ for $n \geq 2$. We could use this result to determine the derivative of $x^{\frac{1}{n}}$ by first principles, but instead we will use the chain rule.

Using the chain rule

Using the chain rule in the form $\dfrac{dy}{du} = \dfrac{dy}{dx} \cdot \dfrac{dx}{du}$ with $y = u$, we have

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Thus $$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} \qquad \text{for } \frac{dx}{dy} \neq 0$$

Now let $y = x^{\frac{1}{n}}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

We have $y^n = x$ and so $\dfrac{dx}{dy} = ny^{n-1}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\dfrac{dy}{dx} = \dfrac{1}{n}x^{\frac{1}{n}-1}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

This result may now be extended to rational powers.

Let $y = x^{\frac{p}{q}}$, where $p, q \in \mathbb{Z} \setminus \{0\}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$. The chain rule yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-1}
\end{aligned}$$

Thus the result for integer powers has been extended to rational powers. In fact, the analogous result holds for any non-zero real power:

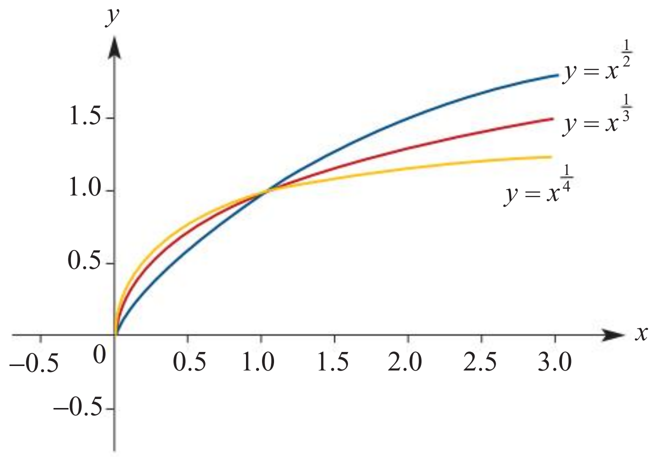
For $f(x) = x^a$, $f'(x) = ax^{a-1}$, where $a \in \mathbb{R} \setminus \{0\}$ and $x > 0$.

ISBN 978-1-009-53719-3

This result is stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined, although $(-2)^{\frac{1}{3}}$ is defined.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

The domain of each has been taken to be $\mathbb{R}^+$.

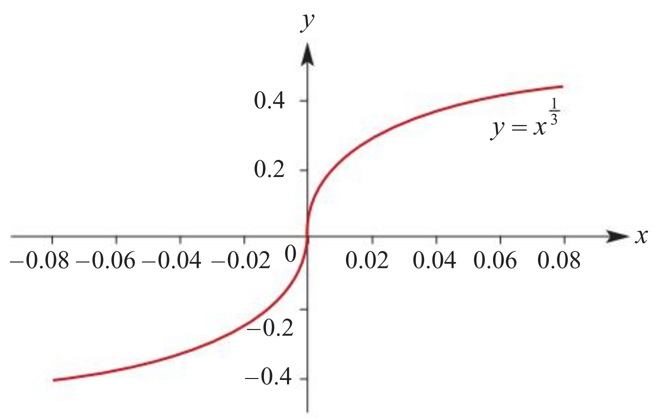


The figure to the right is the graph of the function $f(x) = x^{\frac{1}{3}}$ for $x \in \mathbb{R}$.

Note that the values shown here are $-0.08 \le x \le 0.08$.

From this it can be seen that the tangent to $y = x^{\frac{1}{3}}$ at the origin is on the y-axis.

Use a calculator to investigate graphs of this type further.



Example 6

Determine the derivative of each of the following with respect to x:

a $4x^{\frac{2}{3}}$

b $x^{\frac{1}{5}} - 2x^{-3}$

Solution

a Let $y = 4x^{\frac{2}{3}}$

Then $$\frac{dy}{dx} = 4 \times \frac{2}{3} \times x^{\frac{2}{3}-1}$$
$$= \frac{8}{3}x^{-\frac{1}{3}}$$

b Let $y = x^{\frac{1}{5}} - 2x^{-3}$

Then $$\frac{dy}{dx} = \frac{1}{5}x^{\frac{1}{5}-1} - 2 \times (-3x^{-3-1})$$
$$= \frac{1}{5}x^{-\frac{4}{5}} + 6x^{-4}$$

Summary 16B

For any non-zero rational number $r = \frac{p}{q}$, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

 ISBN 978-1-009-53719-3

Skill-sheet

Exercise 16B

Example 6

1 Determine the derivative of each of the following with respect to x:

a $x^{\frac{1}{3}}$ **b** $x^{\frac{3}{2}},\ x > 0$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}},\ x > 0$

d $2x^{\frac{1}{2}} - 3x^{\frac{5}{3}},\ x > 0$ **e** $x^{-\frac{5}{6}},\ x > 0$ **f** $x^{-\frac{1}{2}} - 4,\ x > 0$

SF

2 Using the chain rule, determine the derivative of each of the following with respect to x:

a $\sqrt{1 + x^2}$ **b** $\sqrt[3]{x + x^2}$ **c** $(1 + x^2)^{-\frac{1}{2}}$ **d** $(1 + x)^{\frac{1}{3}}$

3 **a** Determine the gradient of $y = x^{\frac{1}{3}}$ at each of the following points:

i $\left(\frac{1}{8}, \frac{1}{2}\right)$ **ii** $\left(-\frac{1}{8}, -\frac{1}{2}\right)$ **iii** $(1, 1)$ **iv** $(-1, -1)$

b Comment on your results.

CF

4 Consider the graphs of $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ for $x > 0$.

a Determine the values of x for which $x^{\frac{1}{2}} < x^{\frac{1}{3}}$.

b Determine the values of x for which the gradient of $y = x^{\frac{1}{2}}$ is greater than the gradient of $y = x^{\frac{1}{3}}$.

SF

5 Differentiate each of the following with respect to x:

a $(2 - 5\sqrt{x})^2$ **b** $(3\sqrt{x} + 2)^2$ **c** $\dfrac{2 + \sqrt{x}}{x^2}$ **d** $\dfrac{x^2 + 2}{\sqrt{x}}$ **e** $3\sqrt{x}\,(x^2 + 2)$

16C Sketch graphs

Learning intentions

- To be able to apply differentiation techniques to **sketching graphs**.

In Chapter 15 we used calculus to help us sketch the graphs of polynomial functions. In this section we show how the same techniques may be applied to non-polynomial functions.

We will sketch the graph of the function

$$f(x) = \frac{1}{x^2} + x \quad \text{for } x \neq 0$$

Consider the behaviour of $f(x)$ for very large x

This section of a spreadsheet shows the behaviour of $f(x)$ as the value of x becomes very large positive. This has been done using increasing powers of 2.

a

x	$f(x)$
1	2.0000000000
2	2.2500000000
4	4.0625000000
8	8.0156250000
16	16.0039062500
32	32.0009765625
64	64.0002441406
128	128.0000610352

We see that as $x \to \infty$, $f(x) \to x$.

This shows us that the line $y = x$ is an **oblique asymptote** for the graph.

It is clear also that as $x \to -\infty$, $f(x) \to x$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Consider vertical asymptotes

The function is not defined at $x = 0$. The behaviour of the function as x approaches 0 is demonstrated below. The values of x considered are $2^0, 2^{-1}, 2^{-2}, 2^{-3}$, etc.

b

x	$f(x)$
1.00000000	2.00000000
0.50000000	4.50000000
0.25000000	16.25000000
0.12500000	64.12500000
0.06250000	256.06250000
0.03125000	1024.03125000
0.01562500	4096.01562500
0.00781250	16384.00781250

Spreadsheet **b** shows the behaviour as x approaches 0 from the right: $f(x)$ becomes increasingly large. We write $\lim_{x\to 0^+} f(x) = \infty$.

c

x	$f(x)$
−1.00000000	0.00000000
−0.50000000	3.50000000
−0.25000000	15.75000000
−0.12500000	63.87500000
−0.06250000	255.93750000
−0.03125000	1023.96875000
−0.01562500	4095.98437500
−0.00781250	16383.99218750

Spreadsheet **c** shows the behaviour as x approaches 0 from the left: again $f(x)$ becomes increasingly large. We write $\lim_{x\to 0^-} f(x) = \infty$.

Determine the axis intercepts

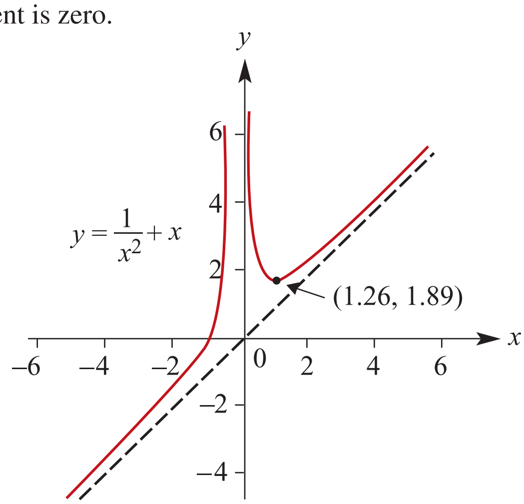
The graph does not have a y-axis intercept, as the function is not defined at $x = 0$. The graph crosses the x-axis when $\dfrac{1}{x^2} + x = 0$, i.e. when $x^3 = -1$, which implies $x = -1$.

Consider the stationary points

For $f(x) = \dfrac{1}{x^2} + x$, we have $f'(x) = \dfrac{-2}{x^3} + 1$, and so $f'(x) = 0$ implies $x = 2^{\frac{1}{3}}$. Thus there is a point on the curve at $x = 2^{\frac{1}{3}}$ at which the gradient is zero.

As the graph is not continuous at $x = 0$, we use a gradient chart for $x > 0$ only.

x		$2^{\frac{1}{3}}$	
$f'(x)$	$-$	0	$+$
shape of f	\	—	/



Thus there is a local minimum at the point (1.26, 1.89), with coordinates correct to two decimal places.

Spreadsheet **a** shows how rapidly the graph of $y = f(x)$ moves towards the line $y = x$. At $x = 4$, $f(x) = 4.0625$ and at $x = 8$, $f(x) = 8.015625$.

 ISBN 978-1-009-53719-3

Example 7

Sketch the graph of the function with rule $f(x) = \dfrac{x^2+3}{x+1}\left(= x - 1 + \dfrac{4}{x+1}\right)$, where $x \neq -1$.

Solution

By division, $f(x) = x - 1 + \dfrac{4}{x+1}$.

Behaviour for very large x

As $x \to \infty$, $f(x) \to x - 1$ from above.
As $x \to -\infty$, $f(x) \to x - 1$ from below.
There is an oblique asymptote with equation $y = x - 1$.

Vertical asymptote

$\lim\limits_{x \to -1^+} f(x) = \infty$ and $\lim\limits_{x \to -1^-} f(x) = -\infty$

There is a vertical asymptote with equation $x = -1$.

Axis intercepts

When $x = 0$, $f(x) = 3$. There is no x-axis intercept, as $x^2 + 3 \neq 0$ for all $x \in \mathbb{R}$.

Stationary points

For $\quad f(x) = x - 1 + \dfrac{4}{x+1}$

$$f'(x) = 1 - \frac{4}{(x+1)^2}$$

Determine where $f'(x) = 0$:

$$\begin{aligned} 1 - \frac{4}{(x+1)^2} &= 0 \\ (x+1)^2 &= 4 \\ x + 1 &= \pm 2 \end{aligned}$$

Therefore $x = 1$ or $x = -3$.

We consider two gradient charts.

First for $x > -1$:

x		1	
$f'(x)$	$-$	0	$+$
shape of f	\	—	/

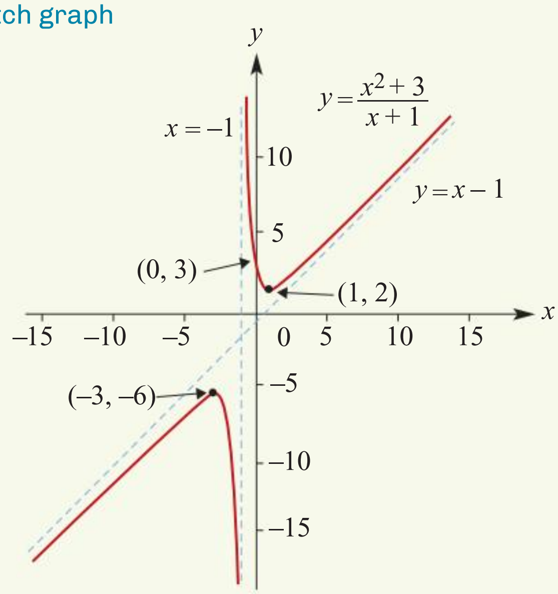
$\therefore$ Local minimum at $(1, 2)$

Next for $x < -1$:

x		-3	
$f'(x)$	$+$	0	$-$
shape of f	/	—	\

$\therefore$ Local maximum at $(-3, -6)$

Sketch graph



Summary 16C

Steps for sketching graphs

- Determine the x-axis and y-axis intercepts and the stationary points.
- Determine where the graph is strictly increasing and where it is strictly decreasing.
- Determine the nature of each stationary point (local maximum, local minimum or stationary point of inflection).
- Identify vertical asymptotes.
- Understand the behaviour as x becomes very large positive and very large negative.

Exercise 16C

Example 7

1 The equation of a curve is $y = 4x + \frac{1}{x}$.

a Determine the coordinates of the turning points.

b Determine the equation of the tangent to the curve at the point where $x = 2$.

2 Determine the x-coordinates of the points on the curve $y = \frac{x^2 - 1}{x}$ at which the gradient is 5.

3 Determine the gradient of $y = \frac{2x - 4}{x^2}$ at the point where the curve crosses the x-axis.

4 Sketch the curve $y = x - 5 + \frac{4}{x}$ by first determining:

a the coordinates of the points of intersection with the axes

b the equations of all asymptotes

c the coordinates of all turning points.

CF

 ISBN 978-1-009-53719-3 Cambridge University Press

5 If x is positive, determine the least value of $x + \frac{4}{x^2}$.

6 For positive values of x, sketch the graph of $y = x + \frac{4}{x}$ and find the least value of y.

7 Sketch the graph of each of the following, indicating the coordinates of axis intercepts and turning points, and the equations of asymptotes:

a $y = x + \frac{1}{x}, \quad x \neq 0$

b $y = \frac{1}{x^2} - x, \quad x \neq 0$

c $y = x + 1 + \frac{1}{x+3}, \quad x \neq -3$

d $y = x^3 + \frac{243}{x}, \quad x \neq 0$

e $y = x - 5 + \frac{1}{x}, \quad x \neq 0$

f $y = \frac{x^2 - 4}{x + 2}, \quad x \neq -2$

CF

16D The product rule

Learning intentions

- To be able to understand and use the **product rule**.

The product rule can be used to determine the derivative of a function of the form $f(x) = u(x) \cdot v(x)$. For example: $f(x) = (2x - 1)(x^2 + 3)$ and $f(x) = (5 - x^2)(3x + 7)$.

The product rule (using function notation)

Let $f(x) = u(x) \cdot v(x)$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

We will prove the product rule at the end of this section.

Example 8

Use the product rule to determine the derivative of $f(x) = (x^2 + 3x)(4x + 5)$.

Solution

Let $u(x) = x^2 + 3x$ and $v(x) = 4x + 5$. Then $f(x) = u(x) \cdot v(x)$.

The product rule gives

$$\begin{aligned} f'(x) &= u(x) \cdot v'(x) + v(x) \cdot u'(x) \\ &= (x^2 + 3x) \cdot 4 + (4x + 5) \cdot (2x + 3) \\ &= 4x^2 + 12x + 8x^2 + 22x + 15 \\ &= 12x^2 + 34x + 15 \end{aligned}$$

Note: This could also have been found by expanding the brackets and then differentiating.

We can state the product rule in Leibniz notation and give a geometric interpretation.

The product rule (using Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In the following figure, the white region represents $y = uv$ and the shaded region δy, as explained below.

δv	$u\delta v$	$\delta u \delta v$
v	uv	$v\delta u$
	u	δu

$$\begin{aligned}\delta y &= (u + \delta u)(v + \delta v) - uv \\ &= uv + u\delta v + v\delta u + \delta u\delta v - uv \\ &= u\delta v + v\delta u + \delta u\delta v\end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

In the limit, as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} = \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \text{and} \qquad \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 9

Differentiate $y = 3(x - 1)(x^2 - 3)$ using the product rule.

Solution

Let $u = 3(x - 1) = 3x - 3$ and $v = x^2 - 3$.

Then $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = 2x$.

The product rule gives

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (3x - 3)\cdot 2x + (x^2 - 3)\cdot 3 \\ &= 6x^2 - 6x + 3x^2 - 9 \\ &= 9x^2 - 6x - 9 \\ &= 3(3x^2 - 2x - 3)\end{aligned}$$

 ISBN 978-1-009-53719-3

Example 10

Use the product rule and the chain rule to differentiate $y = (x^2 + 2)^2(4x + 1)$.

Solution

Let $u = (x^2 + 2)^2$ and $v = 4x + 1$.

Then $\dfrac{du}{dx} = 2(x^2 + 2) \cdot 2x = 4x(x^2 + 2)$ (by the chain rule)

and $\dfrac{dv}{dx} = 4$

The product rule gives

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= (x^2 + 2)^2 \cdot 4 + (4x + 1) \cdot 4x(x^2 + 2)\\ &= 4(x^2 + 2)^2 + 4x(4x + 1)(x^2 + 2)\\ &= 4(x^2 + 2)[(x^2 + 2) + x(4x + 1)]\\ &= 4(x^2 + 2)(5x^2 + x + 2)\end{aligned}$$

Proof of the product rule

Let $f(x) = u(x) \cdot v(x)$, where u and v are differentiable functions. By the definition of the derivative of f, we have

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\\ &= \lim_{h\to 0}\frac{u(x+h)\,v(x+h) - u(x)\,v(x)}{h}\end{aligned}$$

Adding and subtracting $u(x + h)\,v(x)$:

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{u(x+h)\,v(x+h) - u(x)\,v(x) + [u(x+h)\,v(x) - u(x+h)\,v(x)]}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\cdot\left(\frac{v(x+h) - v(x)}{h}\right) + v(x)\cdot\left(\frac{u(x+h) - u(x)}{h}\right)\right]\end{aligned}$$

Since u and v are differentiable, we obtain

$$\begin{aligned}f'(x) &= \lim_{h\to 0}u(x+h)\cdot\lim_{h\to 0}\left(\frac{v(x+h) - v(x)}{h}\right) + \lim_{h\to 0}v(x)\cdot\lim_{h\to 0}\left(\frac{u(x+h) - u(x)}{h}\right)\\ &= u(x)\cdot v'(x) + v(x)\cdot u'(x)\end{aligned}$$

This completes the proof of the product rule.

 ISBN 978-1-009-53719-3

Summary 16D

- **The product rule (using function notation)**

  If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.

- **The product rule (using Leibniz notation)**

  If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Skill-sheet

Exercise 16D

Example 8

1 Determine the derivative of each of the following functions using the product rule:

a $f(x) = (3x + 2)(x^2 - 1)$
b $f(x) = (x + 4)(x^3 - 2)$
c $f(x) = (2 - x^2)(3x + 4)$
d $f(x) = 2(x - 2)(3 - x^3)$

Example 9

2 Determine the derivative of each of the following functions using the product rule:

a $y = (x + 3)(5x - 1)$
b $y = (x^2 + 7)(1 - x)$
c $y = 2(x - 2)(5x + 7)$
d $y = -3(2x + 1)(x^2 - 3)$

3 Differentiate each of the following expressions with respect to x. Only use the product rule if it is the simplest method.

a $(3x - 2)(x^3 + 4)$
b $3x^2(4x + 5)$
c $(x^2 + 5)(2 - x)$
d $(x^2 + 4)(x^2 - 4)$
e $5x^3(x^2 - 1)$
f $4(x^2 - 1)(x^2 + 2)$

Example 10

4 Combine the product rule and the chain rule to differentiate each of the following functions:

a $y = (x - 1)^2(2x + 3)$
b $y = (2x + 1)^2(x + 5)$
c $y = (1 - 2x)(x - 2)^3$
d $y = (3x + 1)^{\frac{3}{2}}(2x + 4)$
e $y = (x^2 + 1)\sqrt{2x - 4}$
f $y = x^3(3x^2 + 2x + 1)^{-1}$

5 For the function $f(x) = (5 - 3x)(3x + 1)^2$, determine:

a $f'(x)$ **b** $f'(1)$ **c** $f(1)$ **d** the equation of the tangent at $x = 1$.

6 For $f(x) = 2(x + 2)(x - 1)^2$, determine $f'(-3)$.

7 Determine the equation of the normal to the curve $y = (2x - 3)^2(1 - x)$ at $x = 0$.

8 Show that the derivative of $(1 - 2x)^3(x + 1)$ is equal to $-(8x + 5)(1 - 2x)^2$.

9 If the tangent to the curve $y = kx^4(3x - 5)^3$ at $x = 1$ has gradient 2, determine the value of k.

SF

CF

 ISBN 978-1-009-53719-3

16E The quotient rule

Learning intentions

- To be able to understand and use the **quotient rule** for differentiation.

The quotient rule can be used to determine the derivative of a function of the form $f(x) = \dfrac{u(x)}{v(x)}$.

For example: $f(x) = \dfrac{3x}{x^2 - 1}$ and $f(x) = \dfrac{(3-x)^2}{4x^3}$.

The quotient rule (using function notation)

Let $f(x) = \dfrac{u(x)}{v(x)}$, where $v(x) \neq 0$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

We will prove the quotient rule at the end of this section.

Example 11

Determine the derivative of $f(x) = \dfrac{x^2 + 1}{2x - 3}$.

Solution

Let $u(x) = x^2 + 1$ and $v(x) = 2x - 3$. Then $f(x) = \dfrac{u(x)}{v(x)}$.

We have $u'(x) = 2x$ and $v'(x) = 2$.

Therefore the quotient rule gives

$$\begin{aligned} f'(x) &= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} \\ &= \frac{(2x-3) \cdot 2x - (x^2+1) \cdot 2}{(2x-3)^2} \\ &= \frac{4x^2 - 6x - 2x^2 - 2}{(2x-3)^2} \\ &= \frac{2(x^2 - 3x - 1)}{(2x-3)^2} \end{aligned}$$

We can state the quotient rule using Leibniz notation as follows.

The quotient rule (using Leibniz notation)

If $y = \dfrac{u}{v}$, where u and v are functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

 ISBN 978-1-009-53719-3

Example 12

Differentiate the function $y = \dfrac{-3(x^2 - 2)}{5x + 2}$ using the quotient rule.

Solution

Let $u = -3(x^2 - 2)$ and $v = 5x + 2$. Then $\dfrac{du}{dx} = -6x$ and $\dfrac{dv}{dx} = 5$.

The quotient rule gives

$$\begin{aligned}\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} &= \frac{(5x+2)\cdot(-6x) + 3(x^2-2)\cdot 5}{(5x+2)^2} \\ &= \frac{-30x^2 - 12x + 15x^2 - 30}{(5x+2)^2} \\ &= \frac{-15x^2 - 12x - 30}{(5x+2)^2} \\ &= \frac{-3(5x^2 + 4x + 10)}{(5x+2)^2}\end{aligned}$$

Example 13

Use the quotient rule and the chain rule to differentiate $y = \dfrac{(2x-3)^4}{4x}$.

Solution

Let $u = (2x - 3)^4$ and $v = 4x$.

Then $\dfrac{du}{dx} = 4(2x-3)^3 \cdot 2 = 8(2x-3)^3$ (by the chain rule)

and $\dfrac{dv}{dx} = 4$

The quotient rule gives

$$\begin{aligned}\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} &= \frac{4x\cdot 8(2x-3)^3 - (2x-3)^4\cdot 4}{(4x)^2} \\ &= \frac{32x(2x-3)^3 - 4(2x-3)^4}{16x^2} \\ &= \frac{4(2x-3)^3[8x - (2x-3)]}{16x^2} \\ &= \frac{(2x-3)^3(6x+3)}{4x^2} \\ &= \frac{3(2x-3)^3(2x+1)}{4x^2}\end{aligned}$$

Proof of the quotient rule

The quotient rule can be proved from first principles, but instead we will use the product rule and the chain rule.

Let $f(x) = \dfrac{u(x)}{v(x)}$. Then we can write $f(x) = u(x) \cdot g(x)$, where $g(x) = [v(x)]^{-1}$.

Using the chain rule, we have

$$g'(x) = -[v(x)]^{-2} \cdot v'(x)$$

Therefore, using the product rule, we obtain

$$\begin{aligned} f'(x) &= u(x) \cdot g'(x) + g(x) \cdot u'(x) \\ &= -u(x) \cdot [v(x)]^{-2} \cdot v'(x) + [v(x)]^{-1} \cdot u'(x) \\ &= [v(x)]^{-2}\big(-u(x) \cdot v'(x) + v(x) \cdot u'(x)\big) \\ &= \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} \end{aligned}$$

This completes the proof of the quotient rule.

Summary 16E

- **The quotient rule (using function notation)**

  If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$.

- **The quotient rule (using Leibniz notation)**

  If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

Skill-sheet

Exercise 16E

SF

Example 11

1 Determine the derivative of each of the following functions using the quotient rule:

a $f(x) = \dfrac{x^2}{2x-1}$ **b** $f(x) = \dfrac{5x^2}{2x-3}$ **c** $f(x) = \dfrac{2-3x}{5x+1}$

Example 12

2 Determine the derivative of each of the following functions using the quotient rule:

a $y = \dfrac{4x^2-1}{2x-5}$ **b** $y = \dfrac{x^2+4x}{1-x}$ **c** $y = -\dfrac{5(x^2+1)}{2x-1}$

3 Differentiate each of the following expressions with respect to x. Only use the quotient rule if it is the simplest method.

a $\dfrac{3-x^2}{4x}$ **b** $\dfrac{3-4x}{2x^2}$ **c** $\dfrac{11x+7}{6x-5}$ **d** $-\dfrac{3x^2}{2x-9}$ **e** $\dfrac{4x^2+2x}{4x+2}$

ISBN 978-1-009-53719-3

Example 13

SF

4 Combine the quotient rule and the chain rule to differentiate each of the following functions:

a $y = \dfrac{(3x-1)^2}{2x}, \ x \neq 0$

b $y = \dfrac{(5-2x)^3}{3x^2}, \ x \neq 0$

c $y = \dfrac{7x^2}{(1-x)^2}, \ x \neq 1$

d $y = \dfrac{2-3x}{(4x+1)^2}, \ x \neq -\dfrac{1}{4}$

e $y = -\dfrac{2(3x-2)^2}{x+1}, \ x \neq -1$

f $y = -\dfrac{5x^3}{(3x+1)^2}, \ x \neq -\dfrac{1}{3}$

5 For the function $f(x) = \dfrac{3x-1}{3-x^2}$, determine:

a $f'(x)$ **b** $f'(2)$ **c** $f(2)$ **d** the equation of the tangent at $x = 2$.

6 For $f(x) = -\dfrac{4(x-3)^2}{3x^2+1}$, determine $f'(-1)$.

7 Determine the equation of the normal to the curve $y = \dfrac{(3-2x)^2}{5x+2}$ at $x = 1$.

8 Show that the derivative of $\dfrac{(2x+3)^3}{2x^2}$ is equal to $\dfrac{(x-3)(2x+3)^2}{x^3}$.

CF

9 If the tangent to the curve $y = \dfrac{k(x+2)^2}{5x+1}$ at $x = 1$ has gradient 2, determine the value of k.

10 **a** Determine the derivative of $\sqrt{4x+1}$.

b Given that $y = \dfrac{x}{\sqrt{4x+1}}$ show that $\dfrac{dy}{dx} = \dfrac{2x+1}{(4x+1)\sqrt{4x+1}}$.

11 **a** Determine the derivative of $\sqrt{2x^2-3}$.

b Determine the derivative of $y = \dfrac{3x^2}{\sqrt{2x^2-3}}$.

12 Let $h(x) = \dfrac{f(x)}{g(x)}$.

a If $f(3) = 6, g(3) = 12, f'(3) = 1, g'(3) = 2$, determine $h'(3)$.

b If $h'(5) = 1, f(5) = 7, g(5) = 8, g'(5) = 1$ then determine $f'(5)$.

13 Use the quotient rule and chain rule to determine the derivative of $\sqrt{\dfrac{x+4}{x^2+2}}$.

14 If $f(x) = \sqrt{\dfrac{x-a}{b-x}}$ where $a < x < b$ where a and b are constants, show that

$$f'\left(\frac{a+b}{2}\right) = \frac{2}{b-a}$$

 ISBN 978-1-009-53719-3

Chapter summary

- **Derivatives of powers**

  If $f(x) = x^r$, then $f'(x) = rx^{r-1}$, for any non-zero rational number r.

- **Chain rule**
  - The chain rule is often used to differentiate a complicated function by transforming it into two simpler functions, which are 'chained' together.

    For example: $y = \sqrt{3x^2 + 1}$ can be transformed into $y = \sqrt{u}$ and $u = 3x^2 + 1$.
  - Using Leibniz notation, the chain rule is stated as $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.
  - Using function notation, the chain rule is stated as $(f \circ g)'(x) = f'(g(x))\, g'(x)$, where $(f \circ g)(x) = f(g(x))$.

- **Product rule**
  - If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$.
  - If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

- **Quotient rule**
  - If $f(x) = \dfrac{u(x)}{v(x)}$, then $f'(x) = \dfrac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$.
  - If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

- **Sketch graphs** When sketching the graph of a function, consider:
  - the behaviour of the function as x becomes large positive and large negative
  - vertical asymptotes
  - axis intercepts
  - stationary points (i.e. where the gradient is zero).

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

16A **1** I can apply the chain rule to determine the derivative of functions covered in this course using the notation $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. ☐

See Example 1, Example 2, Example 3 and Questions 1 and 2

16A **2** I can apply the chain rule to determine the derivative of functions covered in this course using the notation $(f \circ g)'(x) = f'(g(x))g'(x)$. ☐

See Example 4 and Question 3

 ISBN 978-1-009-53719-3

16B **3** I can differentiate functions which involve rational exponents. ☐

See Example 5, Example 6 and Question 1

16C **4** I can sketch the graph of simple functions with oblique asymptotes. ☐

See Example 7 and Question 1

16D **5** I can apply the product rule to determine a derivative, ☐

See Example 8, Example 9, Example 10 and Questions 1, 2 and 4

16E **6** I can apply the quotient rule to determine a derivative ☐

See Example 11, Example 12, Example 13 and Questions 1, 2 and 4

Short-response questions

Technology-free short-response questions

SF

1 Determine the derivative of each of the following:

a $2x^{-3}$ **b** $-\frac{1}{3x^2}$ **c** $\frac{3}{x^5}$ **d** $\frac{x^2+x^3}{x^4}$ **e** $\frac{3x^2+2x}{x^2}$ **f** $5x^2-\frac{2}{x}$

g $x^{\frac{1}{2}}$ **h** $\sqrt[3]{x}$ **i** $-\frac{2}{\sqrt{x}}$ **j** $x^{\frac{4}{3}}$ **k** $x^{-\frac{1}{3}}$ **l** $x^{-\frac{1}{3}}+2x^{\frac{3}{5}}$

2 Differentiate each of the following with respect to x:

a $(2x+3)^2$ **b** $2(3x+4)^4$ **c** $(3-2x)^{-\frac{1}{2}}$ **d** $(x^2-x+1)^3$

e $\frac{1}{3+2x}$ **f** $\frac{1}{(2x-1)^{\frac{2}{3}}}$ **g** $\frac{3}{\sqrt{2+x^2}}$ **h** $\left(2x^2-\frac{3}{x^2}\right)^{\frac{1}{3}}$

3 Determine the gradient of the tangent to each of the following curves at the given point:

a $y=\sqrt{x}$, $(9,3)$ **b** $y=\frac{1}{2x+1}$, $(0,1)$ **c** $y=\frac{2}{x^2}$, $(4,\frac{1}{8})$

d $y=3+\frac{2}{x}$, $(1,5)$ **e** $y=\sqrt{x+1}$, $(8,3)$ **f** $y=(x^2-7x-8)^3$, $(8,0)$

4 **a** Determine the coordinates of the point(s) on the curve with equation $y=\frac{1}{x}$ for which the gradient is -4.

b Determine the coordinates of the point(s) on the curve with equation $y=\sqrt{x}$ for which the gradient is 2.

5 Using the product rule, determine the derivative of each of the following functions:

a $f(x)=(5x-1)(2x+1)$ **b** $f(x)=(2-3x^2)(4x+5)$

c $y=2(x-1)(x^2-5)$ **d** $y=(x^2+4)(5-3x^2)$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 © Evans et al. 2024 Cambridge University Press

6 Using the quotient rule, determine the derivative of each of the following functions:

a $f(x) = \dfrac{2x}{3x-4}$ **b** $f(x) = -\dfrac{2x^2}{5x-2}$ **c** $y = \dfrac{7x+2}{5-x}$ **d** $y = \dfrac{4x^2-1}{x+2}$

7 Combine the chain rule with either the product rule or the quotient rule to differentiate each of the following with respect to x:

a $(2x+1)^2(3x-1)$ **b** $-7x^2(1-x)^3$ **c** $\dfrac{(5x+2)^3}{6-5x}$

d $(x^2+1)\sqrt{x^3+1}$ **e** $\dfrac{-2x^2}{(8x+1)^2}$ **f** $\dfrac{-3(3x+7)^2}{(6x-7)^2}$

8 For each of the following curves, determine the equation of the tangent at $x = 1$:

a $y = 2x^2(5x-1)$ **b** $y = \dfrac{(x-3)^2}{3-2x}$

9 A rectangle has sides of length x cm and y cm, and the area of the rectangle is 16 cm^2.

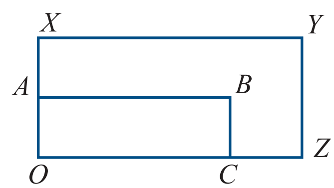
a Determine y in terms of x.

b Show that the perimeter, P cm, is given by $P = 2x + \dfrac{32}{x}$.

c Determine the value of x for which the value of P is a minimum and determine this value of P.

d Sketch the graph of P against x for $x > 0$.

10 The area of rectangle $OABC$ is 120 cm^2, the length of CZ is 5 cm and the length of AX is 7 cm. Let the length of OC be x cm.



a Determine the length of OA in terms of x.

b Determine the length of OX in terms of x.

c Determine the length of OZ in terms of x.

d Determine the area, y cm^2, of rectangle $OXYZ$ in terms of x.

e Determine the value of x for which the area, y cm^2, is a minimum.

Technology-active short-response questions

11 **a** Let $f(x) = \dfrac{1}{x^2}$. Calculate correct to two decimal places the value of a such that $f'(a) = -4$.

b Let $f(x) = x^{\frac{7}{3}}$. Calculate correct to two decimal places the values of a such that $f'(a) = 4$.

12 Let $f(x) = (4x^3 + 5x)^5$. Calculate correct to two decimal places the values of a such that $f'(a) = 1$.

13 A solid cylinder has radius r cm and height h cm. It has a fixed volume of 400 cm^3.

a Determine h in terms of r.

b Show that the total surface area, A cm^2, of the cylinder is given by $A = 2\pi r^2 + \dfrac{800}{r}$.

c Determine $\dfrac{dA}{dr}$.

d Solve the equation $\dfrac{dA}{dr} = 0$ for r.

 ISBN 978-1-009-53719-3

Review

CF

e Determine, correct to three significant figures, the minimum surface area of the cylinder.

f Sketch the graph of A against r.

14 An open rectangular box of height h cm has a horizontal rectangular base with side lengths x cm and $2x$ cm. The volume of the box is 36 cm^3.

a Express h in terms of x.

b Show that the total surface area, A cm^2, of the box is given by $A = 2x^2 + \dfrac{108}{x}$.

c Calculate the values of x and h which make the total surface area a minimum.

d Sketch the graph of A against x for $x > 0$.

15 The curve with equation $y = \sqrt{x+2}$ meets the x-axis at A and the y-axis at B.

a Determine the coordinates of A and B.

b By using the chain rule, determine $\dfrac{dy}{dx}$.

c **i** Determine the gradient of the curve where $x = -1$.

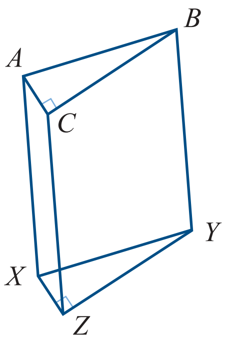
ii Determine the equation of the tangent at the point where $x = -1$.

iii If this tangent meets the x-axis at C and the y-axis at D, determine the distance CD.

d Determine the values of x for which $\dfrac{dy}{dx} < 1$.

16 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Determine values of p and q that make z a minimum.

17 The prism shown in the diagram has a triangular cross-section. The 'ends' of the prism shown are congruent right-angled triangles with the right angles at C and Z.
The lengths of some edges are $AX = CZ = BY = y$ cm, $AC = XZ = 3x$ cm and $CB = ZY = 4x$ cm.
The volume of the prism is 1500 cm^3.



a Express y in terms of x.

b Show that the total surface area, S cm^2, is given by
$S = 12x^2 + \dfrac{3000}{x}$.

c Determine $\dfrac{dS}{dx}$.

d Determine the minimum value of S.

Multiple-choice questions

Technology-free multiple-choice questions

1 The graph of $f(x) = x + \dfrac{1}{x}$ has a local minimum at $(a, f(a))$. The value of a is

A 2 **B** $-\dfrac{5}{2}$ **C** $\dfrac{5}{2}$ **D** 1

 ISBN 978-1-009-53719-3 Cambridge University Press

2 The derivative of $(5x^2 + 2x)^n$ is

A $n(10x + 2)(5x^2 + 2x)^{n-1}$ **B** $(5x^2 + 2x)^{n-1}$
C $(10x + 2)^n$ **D** $n(5x^2 + 2x)^{n-1}$

3 The equation of the normal to the curve $y = (2x - 7)^3$ at $x = 4$ is

A $2y - x = -3$ **B** $y = -6x + 25$
C $x - 6y = 2$ **D** $x + 6y = 10$

4 Define $f(x) = 4 + \dfrac{4}{2 - x}$ for $x \neq 2$. Then $f'(x) > 0$ for

A $x \neq 2$ **B** $x \in \mathbb{R}$ **C** $x < 2$ **D** $x > 2$

5 If $f(x) = 2x^{\frac{p}{q}}$, where p and q are integers, then $f'(x)$ equals

A $2x^{\frac{p-q}{q}}$ **B** $2px^{\frac{p}{q}-1}$ **C** $\dfrac{2p}{q}x^{\frac{p-q}{q}}$ **D** 2

6 Let $y = f(g(x))$, where $g(x) = x^3$. Then $\dfrac{dy}{dx}$ equals

A $3x^2 f'(x^3)$ **B** $3x^2 f(x^3)$
C $2x\, f(x)\, f'(x^3)$ **D** $2f(x)\, f'(x^3)$

7 The graph of $y = \dfrac{k}{2(x^2 + 1)}$ has gradient 1 when $x = 1$. The value of k is

A 1 **B** -1 **C** 4 **D** -4

8 Which of the following is *not true* for the curve $y = f(x)$, where $f(x) = x^{\frac{1}{5}}$?

A The gradient is defined for all x. **B** For $x > 0$, the gradient is positive.
C For $x > 0$, the gradient is decreasing. **D** The curve passes through the origin.

9 Which of the following is *not true* for $y = f(x)$, where $f(x) = x^{\frac{3}{4}}$?

A The maximal domain of the function is $\mathbb{R}^+ \cup \{0\}$.
B $f(x) > x$ for all $x > 1$.
C For $x > 0$, the gradient of the curve is positive.
D For $x > 0$, the gradient of the curve is decreasing.

10 If $f(x) = (x + 2)(x - 5)^{\frac{3}{2}}$, then $f'(6)$ is equal to

A 8 **B** 9 **C** 12 **D** 13

11 The derivative of $\dfrac{2x}{(3x + 2)^2}$ is

A $\dfrac{2(x + 2)}{(3x + 2)^3}$ **B** $\dfrac{-2(3x - 2)}{(3x + 2)^3}$ **C** $\dfrac{6x - 4}{(3x + 2)^4}$ **D** $\dfrac{6(x + 2)}{(3x + 2)^3}$

12 If $f(x) = 2x^{-2} + 3x^{-1} + 4x$ then $f'(x)$ is equal to

A $-4x^{-3} - 3x^{-2} + 4$ **B** $4x^{-3} + 3x^2 + 4x$
C $-4x^{-1} - 3x^2 + 4x$ **D** $-2x^{-1} + 3x^2 + 4x$

 ISBN 978-1-009-53719-3

Review

Technology-active multiple-choice questions

13 The tangent to the curve of $y = \frac{1}{x}$ at the point $\left(a, \frac{1}{a}\right)$ crosses the x-axis at $(4, 0)$. The gradient of the curve at the point $\left(a, \frac{1}{a}\right)$ is

A -0.25 **B** -0.5 **C** -0.75 **D** -1

14 The tangent to the curve of $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$ crosses the x-axis at an angle of θ with the positive direction of the x-axis. The value of θ correct to the nearest whole degree is

A $6°$ **B** $125°$ **C** $174°$ **D** $153°$

15 The normal to the curve $y = 5x^{\frac{2}{3}}$ when $x = 2$ crosses the y-axis at the point $(0, b)$. The value of b correct to one decimal place is

A 7.6 **B** 8.7 **C** 8.6 **D** 9.2

16 Let $f(x) = (3x^2 + 4x)^4$. At the point $(a, f(a)$ where $a > 0$, $f'(a) = 1$. The value of a correct to two decimal places is

A 0.15 **B** 1.90 **C** 0.09 **D** 0.86

 ISBN 978-1-009-53719-3

17 Revision of Unit 2

Revision

17A Short-response questions

Technology-free short-response questions

SF

1 Simplify the following, expressing the answers in positive-index form:

a $(-2a^2)^3 \times 3a^4$ **b** $\dfrac{5a^4 \times 2ab^2}{20a^2b^4}$ **c** $\dfrac{(xy^{-2})^{-1}}{y} \times \dfrac{3x^{-1}y^2}{4(xy)^3}$

d $\left(\dfrac{4a^2}{ab}\right)^3 \div (2ab^{-1})^3$ **e** $\sqrt{x^{-1}y^2} \times \left(\dfrac{y}{x}\right)^{-\frac{1}{3}}$ **f** $\sqrt{2x-1} \times (2x-1)^{-1}$

2 Simplify and evaluate:

a $\left(\dfrac{3}{5}\right)^{-2}$ **b** $\left(\dfrac{4^2}{2^6}\right)^{-2}$ **c** $\dfrac{27^2 \times 9^3}{81^2}$ **d** $(-27)^{-\frac{1}{3}}$

3 Simplify:

a $\dfrac{9^{2n} \times 8^n \times 16^n}{6^n}$ **b** $3\log_2(16)$

c $2\log_{10} 3 + \log_{10} 4$ **d** $\log_3\left(\dfrac{1}{27}\right)$

4 Sketch graphs of the following functions. Give equations of asymptotes and y-axis intercepts, and state the range of each.

a $f(x) = 2^x - 3,\ x \in \mathbb{R}$ **b** $f(x) = -3 \times 2^x + 1,\ x \in \mathbb{R}$

5 Solve these equations for x:

a $4^x = 8^{x-1}$ **b** $4^x = 5 \times 2^x - 4$ **c** $5^{x-1} > 125$

d $\log_2(x + 1) = 3$ **e** $\log_4(2x) - \log_4(x + 1) = 0$

6 Solve using logarithms:

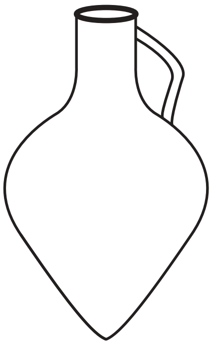
a $2^x = 5$ **b** $5^{3x+1} = 10$ **c** $0.6^x < 0.2$

 ISBN 978-1-009-53719-3

Revision

SF

7 Water is being poured steadily into the vessel shown on the right. Sketch a graph showing the relationship between the height of water and the volume that has been poured in.



8 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x(t) = 2t^3 - t$, $t \geq 0$.

a Determine the average velocity in the first second.

b Determine the average velocity in the next three seconds.

9 A curve has rule $y = 9 - x^2$.

a Determine the average rate of change over the interval:

i $[1, 3]$ **ii** $[1, 2]$

b Determine an expression for the average rate of change over the interval $[1, 1 + h]$.

c Use your result from part b to determine the instantaneous rate of change at $x = 1$.

10 By first considering the gradient of the secant through the points $P(x, f(x))$ and $Q(x + h, f(x + h))$ on the curve $f(x) = \frac{1}{2}x^2 - x$, determine the derivative of $\frac{1}{2}x^2 - x$.

11 Determine the derivative of each of the following:

a $2x^3 - x + 1$ **b** $(x - 1)(x + 2)$ **c** $\dfrac{x^2 + 5x}{x}$

12 Determine the gradient of the tangent to each of the following curves at the given point:

a $y = 3x^4 + x$, $(1, 4)$ **b** $y = 2x(1 - x)$, $(-2, -12)$

13 For the function $f(x) = x - 2x^2$, determine the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -10$

14 Determine the equations of the tangent and the normal to the curve $y = x^2 - 5x$ at $x = 1$.

15 For the graph of $y = 2(x^3 - 4x)$, determine all stationary points and state their nature.

16 Determine:

a $\dfrac{d}{dx}\left(2x^{-3} - x^{-1}\right)$ **b** $\dfrac{d}{dz}\left(\dfrac{3 - z}{z^3}\right)$

17 Determine the derivative of each of the following with respect to x:

a $3x^{\frac{3}{2}}$ **b** $\sqrt[5]{x}$ **c** $-\dfrac{2}{x^{\frac{5}{3}}}$ **d** $6x^{\frac{5}{3}}$ **e** $x^{-\frac{1}{5}}$ **f** $x^{-\frac{2}{3}} - 2x^{\frac{3}{2}}$

18 Differentiate each of the following with respect to x:

a $(3x + 5)^2$ **b** $-(2x + 7)^4$ **c** $2(x^2 + 2x)^3$ **d** $(5 - 2x)^{-\frac{1}{3}}$

e $\dfrac{4}{5 + 3x}$ **f** $\dfrac{1}{(x - 1)^{\frac{2}{3}}}$ **g** $\dfrac{3}{\sqrt{2 + 3x^2}}$ **h** $\left(2x^3 - \dfrac{5}{x}\right)^{\frac{1}{3}}$

19 Determine the gradient of the tangent to each of the following curves at the given point:

a $y = \sqrt[3]{x}$, $(27, 3)$ **b** $y = \dfrac{1}{3x + 1}$, $(0, 1)$ **c** $y = \dfrac{2}{x^3}$, $(2, \frac{1}{4})$

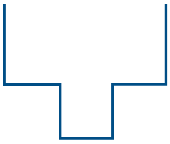
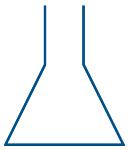
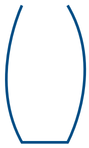
 ISBN 978-1-009-53719-3 Cambridge University Press

20 Determine the coordinates of the point(s) on the curve with equation $y = \frac{1}{x^2}$ for which the gradient is 4.

21 Use the product rule or the quotient rule to differentiate each of the following:

a $(x^2 + 1)(4x - 7)$ **b** $(2x - 1)(x + 2)^2$ **c** $-\frac{4x}{2x + 5}$ **d** $\frac{(3x - 1)^2}{(x + 3)^2}$

22 Water is being poured into the following flasks. Draw a graph of the height h (vertical axis) against the volume V (horizontal axis).

a  **b**  **c** 

23 A student runs a 100 m race in 12 seconds. Determine the student's average speed in:

a m/s **b** km/h

24 Determine the average rate of change of each function over the given interval:

a $f(x) = \sqrt{x + 1},\ 0 \le x \le 3$ **b** $f(x) = \frac{1}{x^2 + 1},\ -1 \le x \le 2$

25 Determine each of the following limits:

a $\lim\limits_{h \to 0} (2xh + x - 1)$ **b** $\lim\limits_{h \to 0} \left(\frac{3h^2 - h}{h}\right)$

26 Let $f(x) = x - x^2$.

a Determine $f(2 + h)$.

b Determine the gradient of the secant of the graph of f through the points $P(2, f(2))$ and $Q(2 + h, f(2 + h))$.

c Hence, determine the gradient of the tangent to the graph of f at the point $P(2, f(2))$.

27 Determine the derivative by first principles:

a $f(x) = 1 - x^2$ **b** $f(x) = 2(x - 1)^2 + 3$

28 Determine $\frac{dy}{dx}$ for each of the following polynomial functions:

a $y = 3x^4 - x + 1$ **b** $y = (x - 1)(x + 3)$

29 Determine the equations of the tangent and the normal to the curve $y = x^3 - x^2$ at $x = -1$.

30 A particle is moving in a straight line according to the rule $s = 2t^2 - t$ for $t \ge 0$, where s metres is the position of the particle (relative to a point O) at time t seconds.

a Determine the velocity and acceleration of the particle after 2 seconds.

b Determine the average velocity of the particle over the first 2 seconds.

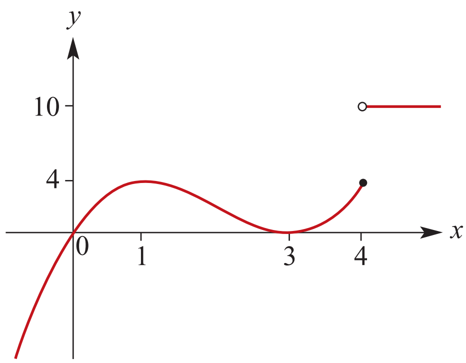
31 Consider the curve $y = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x - 1$. Determine the turning points on this curve and state their nature.

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

SF

32 The graph of a function f is shown on the right. Sketch a graph of the derivative function f'.



33 Determine the derivative of each function. (Do not use the quotient rule for part b.)

a $y = -3x^{-2} - \frac{1}{x} + 3$ **b** $y = \frac{x-1}{x^5}$ **c** $y = \sqrt{x} - 2x^{-\frac{1}{3}}$

34 Using the chain rule, determine the derivative of each function:

a $y = (1-2x)^6$ **b** $y = \frac{3}{(2x-1)^3}$ **c** $y = \sqrt{1-x^2}$

35 Using the product and quotient rules, determine the derivative of each function:

a $y = (2x-3)(x^2+1)$ **b** $y = \frac{3x-1}{x^2-1}$

36 A skydiver jumps from a hovering helicopter and begins falling. The skydiver reaches constant (terminal) speed. After a short while, the skydiver deploys her parachute and her speed reduces very quickly. She then maintains a constant speed until landing on the ground. Sketch a graph of the skydiver's speed (v) against time (t).

37 Determine the derivative of $f(x) = 1 - x^3$ using first principles.

38 Determine the gradient of the tangent line at $x = -1$ for each of the following graphs:

a $y = \frac{x^2 - x}{x}$ **b** $y = \frac{3}{2}x^2 - x$

39 Determine the coordinates of the points on the curve $y = 2x^3 - 1$ where the gradient is 6.

CF

40 Solve for x:

a $\log_2(x-2) + \log_2(x-4) = 3$ **b** $\log_2(x+1) + \log_2(x-1) = 4$

c $\log_{29} x = 1 - \log_{29}(x - 0.4)$

41 Solve for x:

a $2^{2x} - 6 \times 2^x + 8 = 0$ **b** $2 \times 3^{2x+1} + 3^x - 1 = 0$

42 A particular bushfire, which started from a controlled burn, is modelled by $A = 2^{0.25t}$, where A km^2 is the total area of the burned land at time t hours after the containment lines were breached.

a Determine the area burned after 5 hours.

b After how many hours will the area burned be equal to 20 km^2?

 ISBN 978-1-009-53719-3

43 Let x, y and z be real numbers greater than 1 such that $4^x = 3^y = 12^z$. Show that

$$z = \frac{xy}{x + y}$$

44 The tangent to the curve $y = kx^2 - 2$ at $x = 1$ has gradient 4. Determine the value of k.

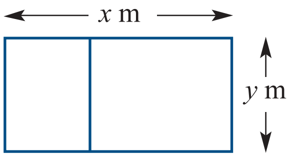
45 The graph of $y = ax^3 - bx$ has zero gradient at the point $(1, 3)$. Determine the values of a and b.

46 Let $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 7$. Determine the values of x for which $f'(x) < 0$.

47 Consider the curve with equation $y = (2x - 3)^2$. Determine the coordinates of the point of intersection of the tangents to this curve at $x = 1$ and $x = 2$.

48 The graph of $y = ax^2 + bx - 2$ has a stationary point at $(-1, 3)$. Determine the values of a and b.

49 A rectangular frame with a cross-strut is made from a 10-metre length of steel. Determine the frame's maximum possible area.



50 Determine the derivative of $y = \frac{\sqrt{2x - 1}}{x}$. Express your answer as a single fraction.

51 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = \frac{1}{6}t^3 - \frac{1}{2}t^2 + 1$.

a At what times does the particle have zero velocity?

b Determine its acceleration at those instants.

c Determine the velocity when the acceleration is zero.

Technology-active short-response questions

52 Let $f(x) = ax^3 + bx^2 + cx + d$. The graph of $y = f(x)$ passes through the points with coordinates $(0, 6), (-1, 4), (1, 6), (2, -6)$. Determine the values of a, b, c and d.

53 Let $f(x) = -\frac{5x^3}{3} - x^2 + 7x + 6$. Determine the values of x for which:

a $f'(x) = 0$ **b** $f'(x) > 0$ **c** $f'(x) < 0$

54 A function is given by $f(x) = \frac{1}{x^2}$, $0 < x \le 2$.

a Determine the gradient, correct to two decimal places, of the secant line through the points $(1, f(1))$ and:

i $(2, f(2))$ **ii** $(1.1, f(1.1))$ **iii** $(1.01, f(1.01))$

b Estimate the instantaneous rate of change of the function f at $x = 1$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

SF

55 A particle moves in a straight line so that its position from a point O on that line at time t seconds is given by $x = 11 + 6t^2 - t^3$, where t is the time in seconds from the start of the motion. Calculate:

a the velocity of the particle when its acceleration is 0 m/s^2.

b the average velocity of the particle over the first two seconds of motion.

CF

56 The position of an object moving in a straight line is given by $s(t) = kt - t^2$ for $t \geq 0$, where s is in metres, t is in seconds and k is a constant.

a If $k = 2$, determine the object's average velocity over the first 3 seconds.

b Determine the value of k if the average velocity over the first 3 seconds is equal to 1 m/s.

c If $k = 2$, estimate the object's instantaneous velocity when $t = 3$ by considering the average velocity between $t = 3$ and $t = 3.1$.

57 An object follows a path (in a vertical plane) described by the equation $y = x - 0.01x^2$, where x is the horizontal distance travelled and y is the height above ground level of the object when it has travelled a horizontal distance x. The object travels from $(0, 0)$, a point at ground level.

a What is the horizontal distance travelled by the object before it returns to ground level?

b Determine $\dfrac{dy}{dx}$.

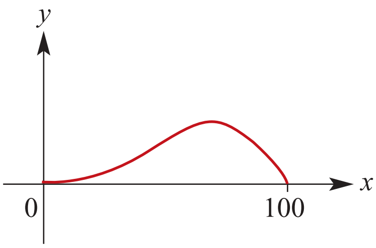
c Determine the value of x for which $\dfrac{dy}{dx} = 0$ and the corresponding y-value.

d Sketch the graph of y against x.

e State the coordinates of the point on the path for which the gradient is:

i $\dfrac{1}{2}$ **ii** $-\dfrac{1}{2}$

58 A designer of roller coasters decides to use a polynomial model for a new construction, and tries the polynomial $y = -0.0001(x^3 - 100x^2)$, where $x \in [0, 100]$. The graph is as shown.



a What are the coordinates of the highest point reached?

b Determine the gradient of the curve at:

i $x = 20$ **ii** $x = 80$ **iii** $x = 100$

c Describe the ride the roller coaster would provide.

d What alteration would you suggest?

59 An estimate for the population of the Earth, P in billions, is

$$P = 4 \times 2^{\frac{(t-1975)}{35}}$$

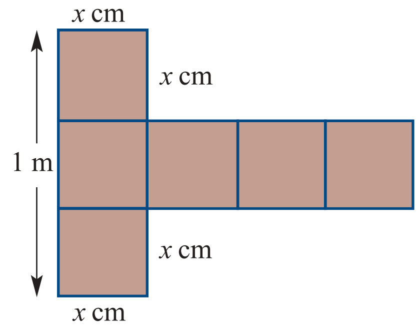
where t is the year.

a Evaluate P for:

i $t = 1975$ **ii** $t = 1995$ **iii** $t = 2005$

b When will the population of the Earth be twice that in 1997?

60 A box is made from the net shown.



a Write down an expression for the surface area of the net in terms of x.

b Write down an expression for the volume, V cm^3, of the made-up box in terms of x.

c Sketch the graph of V against x.

d Determine the dimensions and the volume of a box with maximum volume.

61 A bowl of water is initially at boiling point (i.e. 100°C). The temperature, θ°C, of the water t minutes after beginning to cool is $\theta = 80(2^{-t}) + 20$.

a Copy and complete this table:

t	0	1	2	3	4	5
θ						

b Draw a graph of θ against t.

c At what time is the temperature 60°C?

d Determine θ when $t = 3.5$.

62 A, B and C are three countries. Each of them now has a population of 70 million.

- Country A's population grows uniformly by 30 million in every period of 10 years.
- Country B's population grows uniformly by 50 million in every period of 10 years.
- Country C's population is multiplied by 1.3 every 10 years.

a Give an equation for the population of each country at time t.

b On the same set of axes, carefully draw the graphs of the three equations.

c From your graph determine out when the population of C overtakes:

i the population of A

ii the population of B.

63 Two tanks simultaneously start to leak. Tank A contains $V_1(t)$ litres of water and tank B contains $V_2(t)$ litres of water, where

$$V_1(t) = 1000 \times 3^{\frac{-t}{10}} \qquad t \geq 0$$

$$V_2(t) = 1000 - 40t \qquad 0 \leq t \leq 25$$

and t is the time after the tanks start to leak.

a Determine $V_1(0)$ and $V_2(0)$.

b Sketch the graphs of $y = V_1(t)$ and $y = V_2(t)$ for $0 \leq t \leq 25$.

c How much water is in tank A when tank B is empty?

d Determine the times at which the two tanks have equal amounts of water in them.

 ISBN 978-1-009-53719-3

Revision

64 A rectangular block, the length of whose base is twice its width, has a total surface area of 300 m^2. The width of the base is x m and the height of the block is h m. CF

a Determine h in terms of x, with h the subject of the formula.

b Determine V, the volume of the block, in terms of x.

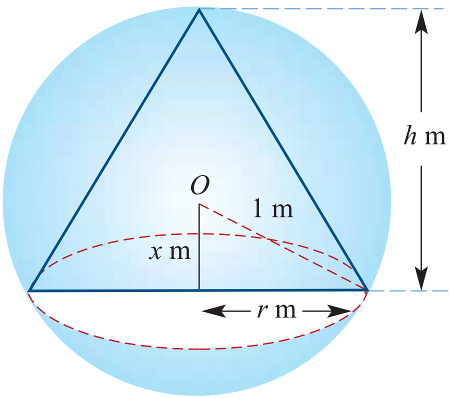
c Determine $\frac{dV}{dx}$.

d State the positive values of x for which $V > 0$.

e Determine the maximum value of V and the value of x for which this occurs.

f For the values of x established in part d, sketch the graph of V against x.

65 A right circular cone lies inside a sphere of radius 1 m as shown. The centre of the sphere, O, lies x m from the base of the cone. The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$. CU



a Determine:

i r in terms of x

ii h in terms of x

b Show that $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

c State a suitable domain for the function with rule $V = \frac{\pi}{3}(1 + x - x^2 - x^3)$.

d **i** Determine $\frac{dV}{dx}$.

ii Determine the values of x for which $\frac{dV}{dx} = 0$.

iii State the maximum possible volume of the cone.

e Sketch the graph of V against x.

66 The number of insects in a colony at time t days after 1 January is approximated by the function with rule $P(t) = 1000 \times 2^{\frac{t}{20}}$, where $t = 0$ corresponds to 1 January. This rule for the population is valid for the entire year.

a Determine the approximate number of insects in the colony on 1 January.

b Determine the approximate number of insects on 10 January (i.e. when $t = 9$).

c For what values of t is $P(t)$ equal to:

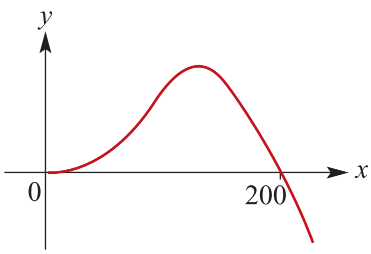
i 4000 **ii** 6000? (Give answers correct to two decimal places.)

d Determine $P(20)$ and $P(15)$. Hence calculate the average rate of change of P with respect to t for the time interval $[15, 20]$, giving your answer correct to two decimal places.

e **i** Determine the average rate of change of P with respect to t for the interval $[15, 15 + h]$, in terms of h.

ii Explain how the instantaneous rate of change of P with respect to t, for $t = 15$, could be found by numerical methods.

67 The equation of the curve is of the form $y = kx^2(a - x)$.



a Determine the values of a and k given that when $x = 170$, $y = 8.67$, and when $x = 200$, $y = 0$.

b **i** Determine the value of x for which there is a local maximum.

ii State the value of y which corresponds to this value of x.

c When $x = 105$, determine:

i the y-value **ii** the gradient of the curve at this point.

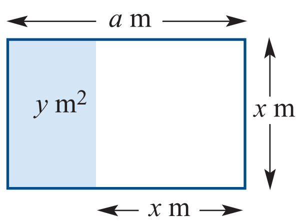
d **i** Determine the equation of the tangent at the point where $x = 105$.

ii State the y-axis intercept of the tangent.

e Determine the average rate of change of y with respect to x for the interval $[0, 105]$.

f Carefully sketch the graph of $y = kx^2(a - x)$, for $x \in [0, 200]$, and on the same set of axes sketch the graph of the tangent at $x = 105$.

68 The diagram shows a rectangle with sides a m and x m. The area of the shaded region is y m^2. In the following, a is a constant.



a Determine an expression for y in terms of x.

b Determine the set of possible values for x in terms of a.

c Determine the maximum value of y and the corresponding value of x (in terms of a).

d Explain briefly why this value of y is a maximum rather than a minimum.

e **i** Sketch the graph of y against x when $a = 9$.

ii State the set of possible values for y.

69 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = 0.6\left(20t^2 - \dfrac{2t^3}{3}\right)$, $0 \le t \le 20$.

a Determine the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Determine the rate of flow of water into the flask, i.e. Determine $\dfrac{dV}{dt}$.

c Sketch the graph of V against t for $t \in [0, 20]$.

d Sketch the graph of $\dfrac{dV}{dt}$ against t for $t \in [0, 20]$.

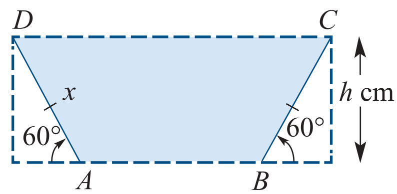
70 The function $y = ax^3 + bx^2$, where a and b are constants, has a stationary point at $(1, -1)$.

a Using the fact that $(1, -1)$ lies on the curve, form an equation involving a and b.

b Using the fact that $(1, -1)$ is also a stationary point, form a second equation and solve the two equations simultaneously to determine a and b.

c Sketch the graph of y against x.

71 A trough is to be made by bending a long sheet of metal 80 cm wide to give a trapezoid cross-section with sides of equal length x cm inclined at 60° to the horizontal as shown. The trough is shown end-on, with the width of the metal sheet running D-A-B-C.

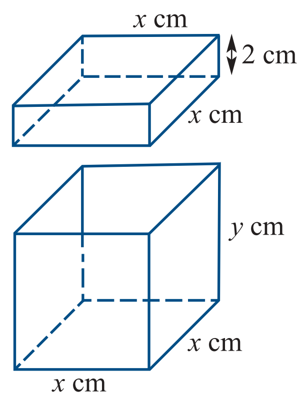


a **i** Determine the length AB in terms of x. **ii** Determine h in terms of x.

b Determine the cross-sectional area (shaded) in terms of x.

c Determine the value of x for which the cross-sectional area is a maximum.

72 The diagrams opposite show an open cardboard box of dimensions x cm, x cm, y cm and the lid for the box of dimensions x cm, x cm, 2 cm. The total amount of cardboard used for the box and the lid is 1400 cm^2. Let V cm^3 be the volume of the box.



a Determine y in terms of x.

b Determine V in terms of x.

c Determine $\frac{dV}{dx}$.

d Determine the values of x for which $\frac{dV}{dx} = 0$.

e Sketch the graph of V against x.

f Determine the maximum volume of the box.

g Determine the values of x and y such that $V = 1000$.

73 Charlie bought a painting for \$5000. The painting's value increases by 5% each year.

a Write a rule that gives the value of the painting, \$$V$, after n years.

b Determine the average rate of change in the value of the painting over the first 3 years.

c Determine n if the average rate of change in the value of the painting over the first n years is \$300 per year. (Round your answer to one decimal place.)

d Estimate the instantaneous rate of change in the value of the painting after 2 years. (Assume a continuous model. Round your answer to the nearest whole number.)

74 A cross-section of an aeroplane wing has its upper boundary described by the curve $y = ax(x - 3)^2$, $0 \le x \le 3$, where a is a positive constant. The height of the wing is y m at a distance of x m across the wing (relative to a reference point on the leading edge of the wing).

 ISBN 978-1-009-53719-3 Cambridge University Press

a Show that the gradient of the curve is given by $\frac{dy}{dx} = 3a(x - 3)(x - 1)$.

b Hence, determine the value of x at which this cross-section of the wing reaches its maximum height.

c If the maximum height is 40 cm, determine the value of a.

d Using the value of a found in part c, locate the point on the upper boundary of the wing where the gradient is -0.3.

75 A cylindrical drum is made using plate steel so that its surface area is exactly 1 m^2. Let r m denote the radius of the drum.

a Show that the volume of the drum, V m^3, is given by $V(r) = \frac{r}{2} - \pi r^3$.

b Determine the value of r for which the volume of the drum is a maximum.

c Determine this maximum volume, giving your answer in cubic metres correct to three decimal places.

d State the practical domain for the function V of r.

e Sketch the graph of V against r, using the domain from part d.

f Now assume that the radius of the drum must be at least 0.3 m.

i What is the maximum possible volume of the drum?

ii What is the range of possible heights of the drum?

76 Let $f(x) = \frac{x^2 + 2}{1 - x}$, $x \neq 1$.

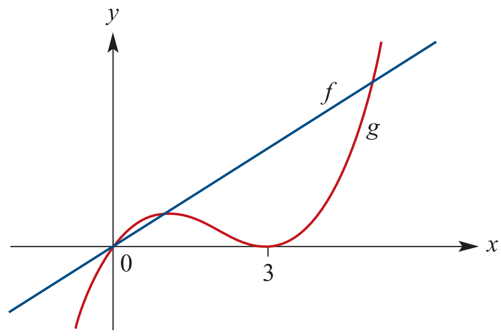
a Show that $f(x) = -x - 1 - \frac{3}{x - 1}$.

b Sketch the graph of $y = f(x)$, showing the axis intercepts and the coordinates of the stationary points.

c State the coordinates of the stationary points on the graph of:

i $y = 2f(x)$ **ii** $y = f(x - 3)$ **iii** $y = f(x + 3) - 2$ **iv** $y = f(x - 3) + 4$

77 The graphs of $f(x) = 4x$ and $g(x) = x(x - 3)^2$ are shown.



a Determine the coordinates of the turning points on the graph of g.

b Determine the points of intersection of the graphs of f and g.

c Find the x-values at which the tangent to the graph of g is parallel to the graph of f.

d Determine the equation of the tangent to the graph of g at the point $(a, g(a))$.

e Determine the coordinates of the point $A(a, g(a))$, where $a < 1$, such that the tangent to the graph of g at A passes through $(5, 20)$.

Hint: It may be useful to note that $a = 5$ must be a solution to the polynomial equation in a used to determine the point A.

17B Multiple-choice questions

Technology-free multiple-choice questions

1 $\dfrac{\sqrt{1.21 \times 10^{-6}}}{2 \times 10^{-4}}$ is equal to

A 1.74×10^{-5} **B** 5.5×10^{-2} **C** 1.74 **D** 5.5

2 If $\log_a 8 = 3$, then a is equal to

A 1 **B** 2 **C** 3 **D** 4

3 $5^{n-1} \times 5^{n+1}$ is equal to

A $5n^2$ **B** 5^{2n} **C** 10^{2n} **D** 25^{2n}

4 If $2^x = \dfrac{1}{64}$, then x is equal to

A 6 **B** -6 **C** 5 **D** -5

5 $125^a \times 5^b$ is equal to

A 625^{ab} **B** 125^{a+3b} **C** 5^{3a+b} **D** 5^{a+3b}

6 The solution of the equation $4^x = 10 - 4^{x+1}$ is

A $x = 4$ **B** $x = 2$ **C** $x = \dfrac{1}{4}$ **D** $x = \dfrac{1}{2}$

7 $\dfrac{7^{n+2} - 35(7^{n-1})}{44(7^{n+2})}$ is equal to

A $\dfrac{1}{49}$ **B** $\dfrac{1}{44}$ **C** $\dfrac{1}{28}$ **D** $\dfrac{1}{7}$

8 If $f(x) = 2 + 3^x$, then $f(2x) - f(x)$ is equal to

A 3^x **B** 3^{2x} **C** $2 + 3^x$ **D** $3^x(3^x - 1)$

9 If $7^{2x} \times 49^{2x-1} = 1$, then x is equal to

A -1 **B** $-\dfrac{1}{3}$ **C** $\dfrac{1}{3}$ **D** 1

10 The graphs of $y = 2^x$ and $y = (\frac{1}{2})^x$ have

A the same x-axis intercept **B** the same y-axis intercept

C no point in common **D** two points in common

11 If $f(x) = (2x)^0 + x^{-\frac{2}{3}}$, then $f(8)$ is equal to

A $\frac{5}{4}$ **B** $\frac{65}{4}$ **C** 5 **D** 20

12 If $C^d = 3$, then $C^{4d} - 5$ equals

A 76 **B** 22 **C** 86 **D** 35

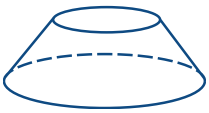
13 The value of $\log_2 56 - \log_2 7 + \log_2 2$ is

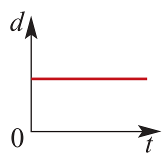
A $\log_2 51$ **B** 1 **C** 2 **D** 4

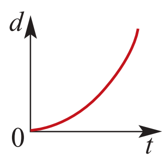
14 If $\log_b a = c$ and $\log_x b = c$, then $\log_a x$ equals

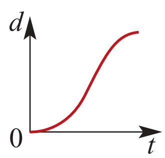
A a **B** c^{-2} **C** b^2 **D** b

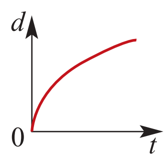
15 The bowl shown in the diagram is being filled with water from a tap flowing at a constant rate. The graph which shows how the depth, d, of water changes with time, t, is



A


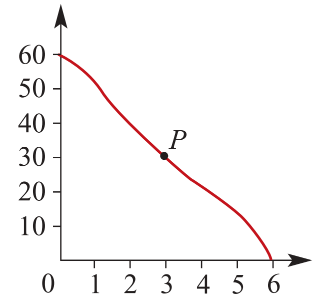
B


C


D


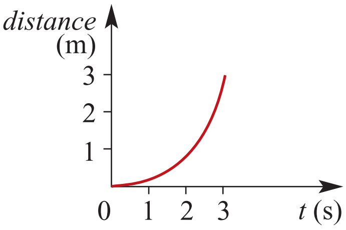
16 The point P has coordinates $(3, 30)$.
The gradient of the graph at P is closest to

A -10 **B** -0.1
C 0 **D** 1



17 For the distance–time graph shown, the average speed from $t = 0$ to $t = 3$ is

A 0.1 m/s **B** 1 m/s
C $\frac{1}{3}$ m/s **D** -3 m/s



18 Given $f(x) = 3x^2 + 1$, the average rate of change of $f(x)$ with respect to x for the interval $[0, 2]$ is

A 6 **B** 6.5 **C** 12 **D** 13

19 On a particular day, the temperature at 7 a.m. was 10°C, and the temperature at noon was 23.5°C. The average rate of increase of temperature, °C per hour, with respect to time is

A 13.5 **B** 2.7 **C** 4.7 **D** 4

 ISBN 978-1-009-53719-3

20 If $y = 5x^2 + 1$, then $\dfrac{dy}{dx}$ is equal to

A $10x$ **B** $10x + 1$ **C** $10x^2$ **D** $5x$

21 If $f(x) = x^2$, then the value of $f(5 + h) - f(5)$ is

A $2x$ **B** 25 **C** h^2 **D** $10h + h^2$

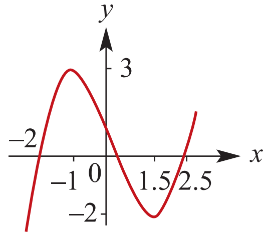
22 The graph shown has zero gradient for

A $x = -2$ or $x = 0$ or $x = 2.5$

B $x = -1$ or $x = 1.5$

C $-1 \le x \le 1.5$

D $x = 3$ or $x = -2$



23 Water is flowing into a tank. The volume, V m^3, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$. The instantaneous rate of change of the volume (m^3/min) of water at time $t = 2$ is

A $6t + 4$ **B** 22 **C** 16 **D** 10

24 Given that $\dfrac{f(3 + h) - f(3)}{h} = 2h^2 + 2h$, the value of $f'(3)$ is

A 0 **B** 14 **C** 2 **D** 5

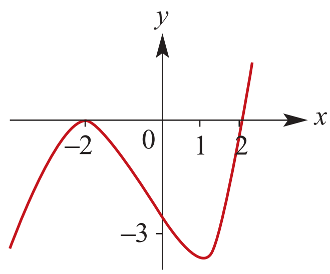
25 For the graph shown, the gradient is positive for

A $x < 2$

B $-2 < x < 2$

C $x < -2$ or $x > 1$

D $x < -2$ or $x > 2$



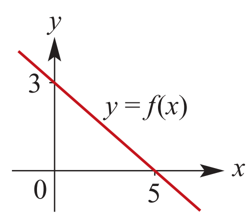
26 The graph of the function $f(x) = x^3 - x^2 - 5$ has stationary points at x equals

A $\dfrac{2}{3}$ only **B** 0 and $\dfrac{2}{3}$ **C** 0 and $-\dfrac{2}{3}$ **D** $-\dfrac{1}{3}$ and 1

27 For the graph shown, $f'(2)$ is equal to

A $-\dfrac{3}{5}$ **B** $\dfrac{5}{3}$

C -1.5 **D** 5



28 The gradient of the curve with equation $y = 2x^3 - 3x^2$ at the point $(1, -1)$ is

A -3 **B** -1 **C** 0 **D** 1

29 If $y = 7 + 2x - x^2$, then the maximum value of y is

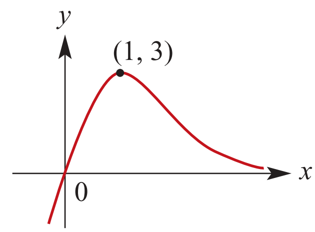
A $-4\frac{1}{2}$ **B** $4\frac{1}{2}$ **C** 8 **D** $8\frac{1}{2}$

 ISBN 978-1-009-53719-3 Cambridge University Press

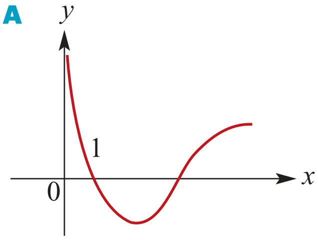
30 A ball was thrown vertically upwards and the height, s metres, it reached after t seconds was given by $s = 28t - 16t^2$. The greatest height reached by the ball was

A $12\frac{1}{4}$ m **B** $14\frac{1}{8}$ m **C** $16\frac{3}{4}$ m **D** $10\frac{1}{2}$ m

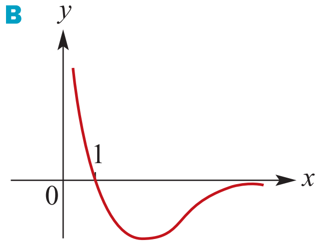
31 The graph of f is shown on the right. The graph of f' is best represented by



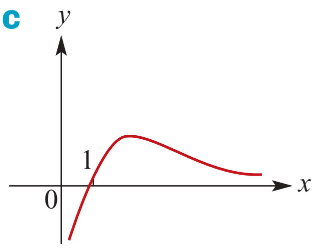
A



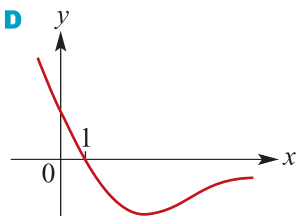
B



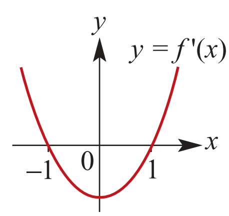
C



D



32 The graph of the gradient function for $f(x) = 2x^3 - 6x$ is shown. This means that $f(x)$ has a local maximum at



A $(-1, 4)$ **B** $(0, -6)$

C $(-3, 0)$ **D** $(0, 0)$

33 If $y = \dfrac{x^2}{2}(x^2 + 2x - 4)$, then $\dfrac{dy}{dx}$ equals

A $4x^3 + 6x^2 - 8x$ **B** $2x^2 + 2x$ **C** $2x^3 + 3x^2 - 4x$ **D** $2x^3 + 4x^2 - 8x$

34 The derivative of $5 + 3x^2$ is

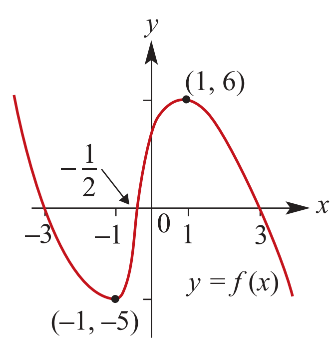
A $5x + x^3$ **B** $6x$ **C** $5x + 6x$ **D** $6x^3$

35 The graph of $y = f(x)$ shown has stationary points at $(1, 6)$ and $(-1, -5)$.

$f'(x) < 0$ for

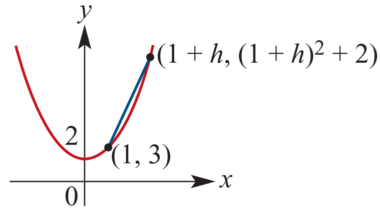
A $-1 < x < 1$ **B** $-3 < x < -\frac{1}{2}$

C $x < -5$ or $x > 1$ **D** $x < -1$ or $x > 1$



 ISBN 978-1-009-53719-3

36 The gradient of the chord AB joining points $A(1, 3)$ and $B(1 + h, (1 + h)^2 + 2)$ is

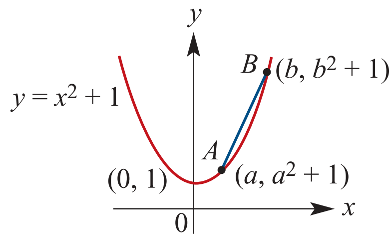


A $1 + h$ **B** $h^2 - 2h$
C $h^2 + 2h$ **D** $2 + h$

37 The gradient at the point $(1, -1)$ on the curve with equation $y = x^2(2x - 3)$ is

A 0 **B** $\frac{2}{3}$ **C** $\frac{4}{3}$ **D** $\frac{\pi}{2}$

38 The gradient of the chord AB is



A $b + a$ **B** $2a$
C $b - a$ **D** $a - b$

39 The derivative function of $f(x) = 3x^3 + 6x^2 - x + 1$ has rule

A $f'(x) = \frac{3x^4}{4} + 2x^3 - \frac{x^2}{2} + x$ **B** $f'(x) = 1 + 12x + 9x^2$
C $f'(x) = 9x^2 + 12x - 1$ **D** $f'(x) = 6x^2 + 12x - 1$

40 Suppose $A = 4xy$ and $y + 3x = 10$. Which one of the following equations can be used to directly calculate the value of x so that the value of A is a maximum?

A $10x - x^2 = 0$ **B** $10 - 3x^2 = 0$ **C** $5 - 3x = 0$ **D** $10 - 3x = 0$

41 The gradient of the graph of $y = x^2 + 3$ at the point with coordinates $(3, 12)$ is

A 24 **B** 6 **C** 3 **D** 4

42 The function $y = x^3 + 5x^2 - 8x$ has

A a local minimum when $x = -4$ and a local maximum when $x = \frac{2}{3}$
B a local maximum when $x = -4$ and a local minimum when $x = \frac{2}{3}$
C a local maximum when $x = 4$ and a local minimum when $x = -\frac{2}{3}$
D a local minimum when $x = 4$ and a local maximum when $x = -\frac{2}{3}$

43 The gradient of the tangent to the curve $y = f(x)$ at the point where $x = 1$ is

A $\lim_{h \to 0} \frac{(1 + h) - 1}{h}$ **B** $\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h}$
C $\frac{f(1 + h) - f(1)}{h}$ **D** $\frac{f(1)}{h}$

44 The minimum value of the function $y = x^2 + 4x - 3$ is

A -2 **B** 2 **C** -7 **D** -15

45 The gradient of the normal to the curve $y = x^2$ at the point where $x = 2$ is equal to

A 2 **B** 4 **C** $-\frac{1}{2}$ **D** $-\frac{1}{4}$

 ISBN 978-1-009-53719-3 Cambridge University Press

46 The function $y = x^2 - 3x - 4$ has negative gradient for

A $x < \frac{3}{2}$ **B** $x > -\frac{3}{2}$

C $-1 < x < 4$ **D** $x < 1$ or $x > 4$

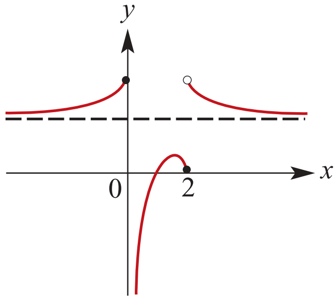
47 The graph shown in the diagram is

A discontinuous at $x = 2$ only

B discontinuous at $x = 0$ only

C discontinuous at $x = 0$ and $x = 2$

D continuous for all real values of x



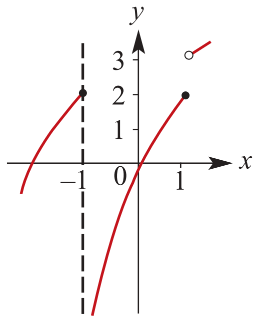
48 The function with the graph as shown in the diagram is

A continuous for all real values of x

B discontinuous for all real values of x

C discontinuous at $x = -1$ and $x = 1$

D discontinuous at $x = -1$ but continuous elsewhere



49 $\lim_{x \to 0} \frac{x^2 - x}{x}$ is

A -1 **B** 0 **C** 1 **D** ∞

50 If $y = \frac{2x + 5}{x}$, then $\frac{dy}{dx}$ equals

A 2 **B** $\frac{2}{x}$ **C** $2 + \frac{5}{x^2}$ **D** $\frac{-5}{x^2}$

51 The derivative of $y = \frac{1}{x^5}$ is equal to

A $\frac{1}{5x^4}$ **B** $-\frac{4}{x^4}$ **C** $\frac{5}{x^4}$ **D** $-\frac{5}{x^6}$

52 If $f(x) = 4x^3 - 3x + 7 - \frac{2}{x}$, then $f'(1)$ equals

A -2 **B** 18 **C** 11 **D** 7

53 If $f(x) = (3x^2 - 7)^4$, then $f'(x)$ is equal to

A $6x(3x^2 - 7)^3$ **B** $(6x - 7)^4$

C $24x(3x^2 - 7)^3$ **D** $24x(6x - 7)^3$

54 If $f(x) = \sqrt{9x^2 + 4}$, then $f'(x)$ is equal to

A $3x + 2$ **B** $\sqrt{18x}$ **C** $-9x(9x^2 + 4)^{-\frac{1}{2}}$ **D** $9x(9x^2 + 4)^{-\frac{1}{2}}$

 ISBN 978-1-009-53719-3 Cambridge University Press

55 The derivative of $\dfrac{2}{3+x}$ is

A $\dfrac{2}{3}$ **B** $\dfrac{1}{2}$ **C** $\dfrac{2}{3-x}$ **D** $\dfrac{-2}{(x+3)^2}$

56 The derivative of $\dfrac{x-1}{\sqrt{x}}$ is

A $2\sqrt{x}$ **B** $\dfrac{x+1}{x\sqrt{x}}$ **C** $\dfrac{3x-1}{2\sqrt{x}}$ **D** $\dfrac{x+1}{2x\sqrt{x}}$

57 If $f(x) = x \cdot g(x)$, where g is a differentiable function, then $f'(x) =$

A $x \cdot g'(x) + g(x)$ **B** $x \cdot g'(x)$ **C** $g'(x) + 1$ **D** $g'(x) + g(x)$

58 If $f(x) = \dfrac{g(x)}{x}$, where g is a differentiable function, then $f'(x) =$

A $g'(x) - 1$ **B** $\dfrac{g(x) - x \cdot g'(x)}{x^2}$

C $\dfrac{x \cdot g'(x) - g(x)}{x^2}$ **D** $\dfrac{g(x) - x \cdot g'(x)}{(g(x))^2}$

59 The graph of $y = -x^3 + ax^2 + bx$ has a local minimum at $x = 1$ and a local maximum at $x = 2$. The values of a and b are

A $a = \dfrac{9}{2},\ b = -6$ **B** $a = -6,\ b = -\dfrac{9}{2}$

C $a = 6,\ b = \dfrac{9}{2}$ **D** $a = \dfrac{9}{2},\ b = -6$

60 The function f has rule $f(x) = (x-3)(x^2+x+3)$ and domain $x \geq 0$. Let L be the line that passes through the x-axis intercept and the y-axis intercept of the graph of f. The tangent to the graph of f is parallel to the line L where x is equal to

A 2 **B** $\dfrac{3}{2}$ **C** $\dfrac{2+\sqrt{13}}{3}$ **D** $\dfrac{2+\sqrt{3}}{3}$

61 If the tangent to the graph of $y = \dfrac{1}{3}x^3$ at $x = c$ passes through the point $\left(0, -\dfrac{9}{4}\right)$, then c is equal to

A 1 **B** 3 **C** $\dfrac{9}{4}$ **D** $\dfrac{3}{2}$

Technology-active multiple-choice questions

62 The average rate of change of the function with rule $f(x) = x^{\frac{2}{3}} + x$ for $0.5 \leq x \leq 2$ correct to two decimal places is

A 1.64 **B** 2.46 **C** 1.89 **D** 1.54

63 If the tangent to the graph of $y = \dfrac{2}{3}x^3 + \dfrac{1}{3}x$ at $x = c$ passes through the point $(0, -0.65)$, then the value of c correct to two decimal places is

A 0.61 **B** 1.34 **C** 0.68 **D** 0.79

 ISBN 978-1-009-53719-3 Cambridge University Press

64 If the curve $y = ax^2 + bx + c, x \in \mathbb{R}$ passes through the points $(2, 8)$ and $(0, 0)$ and the tangent line to this curve at the origin is $y = x$, then the possible values of a, b and c are

A $a = \frac{3}{2}, b = 1, c = 0$ **B** $a = 1, b = 2, c = 0$

C $a = -\frac{1}{2}, b = 2, c = 1$ **D** $a = -\frac{5}{2}, b = 2, c = 0$

65 Let $f(x) = x^{15}(1 - x)^{55}$ where $0 \le x \le 1$. The maximum value of $f(x)$ occurs when

A $x = 0$ **B** $x = 1$ **C** $x = \frac{3}{14}$ **D** $x = \frac{1}{2}$

66 Let $f(x) = \frac{5}{2}x^3 - 4x^2 - \frac{16}{5}x + 1$. Then $f'(x) < 0$ for

A $x > \frac{4}{15}(\sqrt{10} + 2)$ **B** $x < \frac{4}{15}(2 - \sqrt{10})$

C $\frac{4}{15}(2 - \sqrt{10}) < x < \frac{4}{15}(\sqrt{10} + 2)$ **D** $x > \frac{4}{15}\sqrt{10} + 2$

67 The rule for a cubic function is $f(x) = ax^3 + bx^2 + cx$. The curve passes through the points $(2, 6)$ and $(1, 3)$ and the gradient of the tangent at the origin is -1.The values of a, b and c are

A $a = -1, b = 5, c = -1$ **B** $a = -1, b = 3, c = 1$

C $a = -2, b = 6, c = -1$ **D** $a = 2, b = -6, c = -1$

68 A particle P travels in a straight line through a fixed point O. Its distance x m from O is given by $x = t^3 - 9t^2 + 24t$ where t is time in seconds after passing O. The average **velocity** for $t = 1$ to $t = 2.5$ is

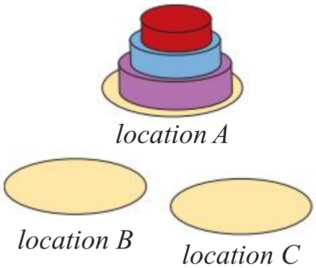
A 2.75 m/s **B** 2.25 m/s **C** 1.75 m/s **D** 3.25 m/s

69 A particle P travels in a straight line through a fixed point O. Its distance x m from O is given by $x = t^3 - 9t^2 + 24t$ where t is time in seconds after passing O. The average **speed** for $t = 1$ to $t = 2.5$ is

A $\frac{9}{4}$ m/s **B** $\frac{5}{8}$ m/s **C** $\frac{8}{5}$ m/s **D** $\frac{37}{12}$ m/s

17C Problem-solving and modelling questions

1 This problem is based on the so-called ‘Tower of Hanoi’ puzzle. Given a number of different sized discs, the problem is to move a pile of discs to a second location (if starting at A, then either to B or C) according to the following rules:



- Only one disc can be moved at a time.
- A total of only three locations (A, B and C) can be used to ‘rest’ the discs.
- A larger sized disc cannot be placed on top of a smaller disc.

 ISBN 978-1-009-53719-3 Cambridge University Press

- The task must be completed in the smallest possible number of moves.

a First complete the puzzle using two coins. Repeat with three coins and then with four coins, and thus complete the following table:

Number of discs, n	1	2	3	4
Minimum number of moves, M	1			

b Work out the formula which shows the relationship between M and n. Use your formula to extend the table of values for $n = 5, 6, 7$.

c Plot the graph of M against n.

d Investigate, for both $n = 3$ and $n = 4$, to determine whether there is a pattern for the number of times each particular disc is moved.

2 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements were taken.

Time (minutes)	3	6	9	12	15	18	21
Temperature (°C)	71.5	59	49	45.5	34	28	23.5

a Determine an exponential model to fit the data collected.

b Use this model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid after 25 minutes.

It is suspected that one of the temperature readings was incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

i the initial temperature of the liquid

ii the temperature of the liquid at $t = 12$.

e If the room temperature is 15°C, determine the approximate time at which the cooling of the liquid ceased.

3 **a** On the same coordinate axes, carefully sketch the graphs of $f(x) = 2^x$ and $g(x) = x^2$ for $-1 \le x \le 5$.

b The two graphs have three points of intersection.

i Determine the exact coordinates of the two points of intersection in the first quadrant.

ii Determine approximate coordinates (correct to two decimal places) for the point of intersection in the second quadrant.

c Solve the inequality $f(x) > g(x)$ for x. (You will have to use one approximate value.)

d Show that, for $x > 0$, the equation $2^x = x^2$ is equivalent to $\dfrac{\log_2 x}{x} = \dfrac{1}{2}$.

e Show that, for $x > 0$, the equation $4^x = x^4$ is equivalent to $\dfrac{\log_2 x}{x} = \dfrac{1}{2}$.

f Hence show that the points of intersection of $y = f(x)$ and $y = g(x)$ have the same x-coordinates as the points of intersection of $y = 4^x$ and $y = x^4$.

g Use your calculator to plot the graph of $y = \dfrac{\log_2 x}{x}$.

h Investigate points of intersection of similar graphs.

4 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Determine an expression in terms of x for the height of the block.

b Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

c What values of x should be considered? (That is, determine the domain of the function V.)

d Determine $\dfrac{dV}{dx}$.

e Determine the values of x for which $\dfrac{dV}{dx} = 0$. Hence determine the maximum volume possible.

f Sketch the graph of V against x for the values of x determined in part c.

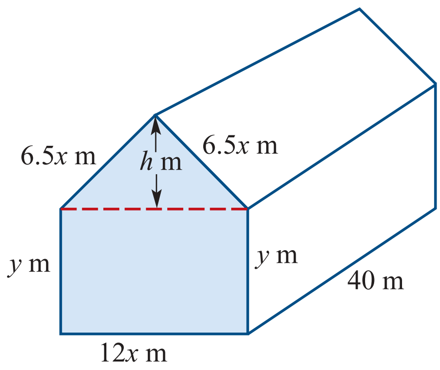
5 A stone is projected vertically upwards with a speed of 30 m/s. Its height, h m, above the ground after t seconds ($t < 6$) is given by $h = 30t - 5t^2$.

a Determine $\dfrac{dh}{dt}$.

b Determine the maximum height reached.

c Sketch the graph of h against t.

6 The metal frame of the front face of a building is as shown. Each length represents a steel girder. The total length of girder used for the front face is 70 metres. The building is 40 metres long.



a If $x = 2$, determine the value of y.

b Show that $y = \dfrac{70 - 25x}{2}$.

c **i** Determine the height h (marked on the diagram) in terms of x.

ii Show that the area of the front face of the building (shaded in the diagram) in terms of x and y is $A = 15x^2 + 12xy$.

d Determine the volume, V m^3, of the building in terms of x.

e **i** For what value of x is the volume of the building a maximum? What is the corresponding value of y?

ii State the corresponding maximum volume of the building.

 ISBN 978-1-009-53719-3 Cambridge University Press

7 The population density (number of residents per unit area) of many cities depends on the distance from the city centre. For a particular city, the population density P (in thousands of people per square kilometre) at a distance of r kilometres from the centre is given approximately by $P = 10 + 40r - 20r^2$.

a What is the population density in the centre of the city?

b What are the possible values for r?

c Sketch a graph of P against r.

d **i** Determine $\dfrac{dP}{dr}$.

ii Evaluate $\dfrac{dP}{dr}$ when $r = 0.5$, 1 and 2.

iii Sketch a graph of $\dfrac{dP}{dr}$ against r.

e Where is the population density greatest?

8 **a** The graph of a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$ has a stationary point at $(1, 2)$ and passes through the point $(2, 4)$. Determine b, c and d in terms of a.

b Describe the family of cubic functions $f(x) = ax^3 + bx^2 + cx + d$ such that the curve $y = f(x)$ has one stationary point, passes through $(1, 1)$ and has gradient 2 at $x = 1$.

c How many pieces of information are needed to determine a cubic function given that it has one stationary point?

d For a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, determine b and c in terms of a if it is known that the curve $y = f(x)$ has one stationary point and this occurs at $x = 1$. Hence describe the family of cubics with this property.

9 Consider the family of functions with rules of the form

$$f(x) = (x(a - x))^n$$

where a is a positive real number and n is a natural number with $n \geq 2$.

a Determine the coordinates of the points where the graph of $y = f(x)$ intersects the x-axis.

b Determine the coordinates of the three stationary points on the graph of $y = f(x)$.

c By considering the value of $f'(x)$ at $x = -a$, $x = \dfrac{a}{4}$, $x = \dfrac{3a}{4}$ and $x = 2a$, state the nature of these stationary points when:

i n is odd **ii** n is even.

d Use your calculator to generate several examples for n odd and n even.

Now consider the family of functions with rules of the form

$$g(x) = (x(2 - x))^n$$

where n is a natural number with $n \geq 2$.

e Determine the coordinates of the stationary points on the graph of $y = g(x)$.

f When n is odd, for what values of $k \geq 0$ does the equation $g(x) = k$ have:

i no solutions **ii** one solution **iii** two solutions?

g When n is even, for what values of $k \geq 0$ does the equation $g(x) = k$ have:

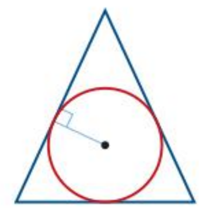
i two solutions **ii** three solutions **iii** four solutions?

10 **a** Show that, out of all the rectangles with a given perimeter P, the one with the shortest diagonal is a square.

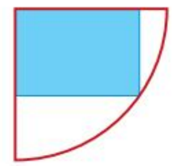
b **i** The perimeter of an isosceles triangle is 24 cm. Determine its maximum possible area.

ii Show that, out of all the isosceles triangles with a given perimeter P, the one with the maximum area is equilateral.

c An isosceles triangle circumscribes a circle of radius r. Determine the minimum area of such a triangle.



d A rectangle is inscribed in a quarter circle of radius r. Two sides of the rectangle are on the sides of the quarter circle. Determine the maximum area of such a rectangle.



e A wire of length ℓ is bent to form a sector of a circle of radius r. Determine the maximum area of such a sector.

17D Problem-solving and modelling investigations

For each of the following questions, there are different approaches and directions that you can take. Suggestions are given, but you should develop your solution on an individual basis.

1 **Graphs of cubic functions**

a Consider functions with rules of the form $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$. Determine the conditions under which the graph of f has:

i no stationary points **ii** exactly one stationary point **iii** two stationary points.

b Now consider functions with rules of the form $f(x) = ax^3 - 3px + d$, where $a, p, d \in \mathbb{R}$ and $a \neq 0$. Investigate the graphs of functions in this family.

i When does the graph of f have zero, one or two stationary points?

ii When does the graph of f have one, two or three x-axis intercepts? Illustrate the different cases.

c Assume that the graph of a cubic function has x-axis intercepts at $(\alpha, 0)$ and $(\beta, 0)$, where $\alpha \neq \beta$. Let m and n be the gradients of the tangents at $(\alpha, 0)$ and $(\beta, 0)$, respectively. Determine the coordinates $(\gamma, 0)$ of the third x-axis intercept, if it exists, in terms of α, β, m and n. Investigate the relationship between the three intercepts.

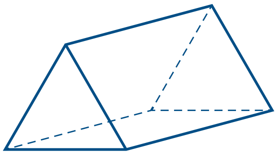
2 **Bicycle race** Caleb is riding in a bicycle race at a constant speed of k m/s. He passes a checkpoint at time $t = 0$. At this time, his support car starts from rest. The distance, d m, of the car from the checkpoint at time t seconds is given by $d = \frac{1}{3}(10t^2 - t^3)$.

a The rule for d has been chosen to model the situation in which, when the car catches up to Caleb, it is travelling at the same speed as him. That is, their distances and speeds will be the same when they meet. Afterwards, the car will fall behind. What is the value of k and at what time do they meet?

b Now consider different values of k. Determine an expression, in terms of k, for the times that the car could meet the bike. For what values of k will the car meet the bike? Illustrate the different cases graphically.

c Construct other possible functions to describe the motion of the car. Make sure that they are realistic.

3 **Designing tents 1** You wish to make a tent in the shape of a right prism whose cross-section is an equilateral triangle. Assume that the volume of the tent should be 2.2 m^3 and that the tent should be able to sleep two to three people.



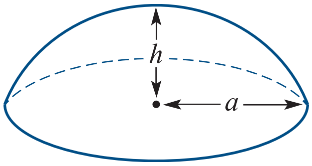
a Suppose that the material used for the top and two ends of the tent is 1.4 times as expensive per square metre as the material used for the floor of the tent. What should be the dimensions of the tent to minimise the total cost of material? How practical is this tent? If you think it is necessary, restrict the dimensions.

b Repeat part a, but now suppose that the material for the floor of the tent is 1.4 times as expensive per square metre as the material for the top and two ends.

4 **Designing tents 2** You now wish to make a tent in the shape of a spherical cap. As in the previous question, the volume of the tent should be 2.2 m^3 and the tent should be able to sleep two to three people.

The volume, V, and the curved surface area, A, of a spherical cap are given by

$$V = \frac{\pi h}{6}(3a^2 + h^2) \quad \text{and} \quad A = \pi(a^2 + h^2)$$



where a is the radius of the circular base and h is the height of the cap.

a Suppose that the material used for the top of the tent is 1.4 times as expensive per square metre as the material used for the floor of the tent. What should be the dimensions of the tent to minimise the total cost of material? How practical is this tent? If you think it is necessary, restrict the dimensions.

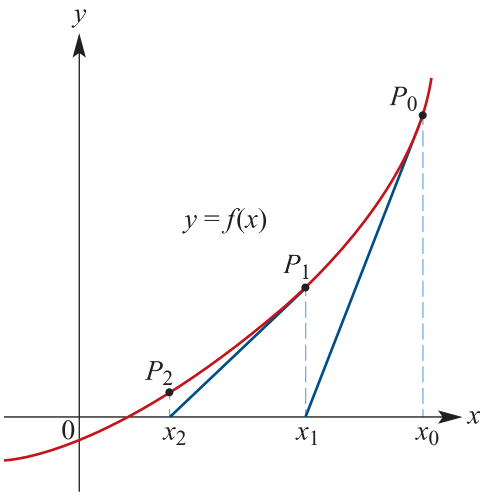
b Repeat part a, but now suppose that the material for the floor of the tent is 1.4 times as expensive per square metre as the material for the top.

5 Newton's method

This method can be used for determining approximate solutions to equations.

Suppose that the equation $f(x) = 0$ has a solution at $x = \alpha$. Choose x_0 close to α.

- Start with the point $P_0(x_0, f(x_0))$ on the curve $y = f(x)$.
- Now consider the tangent to the curve $y = f(x)$ at the point P_0.
  Let x_1 be the x-axis intercept of this tangent. In general, x_1 will be a better approximation to the solution α.
- Next consider the point $P_1(x_1, f(x_1))$.



This process is repeated to give a sequence of values $x_1, x_2, x_3, \ldots$ that are getting closer and closer to α. We can go from x_n to x_{n+1} by using the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{where } n = 0, 1, 2, \ldots$$

a Derive this formula for x_{n+1} in terms of x_n.

b Use Newton's method to solve the equation $x^2 - 2 = 0$ for $x > 0$, correct to 10 decimal places. Check your answer against the exact solution.

c Experiment with using Newton's method to solve other equations.

d Consider the function $f(x) = x^3 - 5x$ and the starting point $x_0 = 1$. What goes wrong? Can you determine other examples of functions and starting points for which Newton's method does not work?

Revision

18

Revision of Units 1 and 2

18A Short-response questions

Technology-free short-response questions

SF

1 Solve the linear equation $2x + 3(4 - x) = 8$ for x.

2 Solve the equation $\dfrac{at + b}{ct + d} = 2$ for t.

3 Solve the inequality $\dfrac{4x}{3} - 4 \le 2x - 3$.

4 Differentiate each of the following with respect to x:

a $f(x) = 8x^3 - 2x^2 + 3x + 4$

b $f(x) = 5x - 3$

c $f(x) = -x^2 + 6x - 3$

d $f(x) = 3x + 4x^2$

5 Solve the equation $-2x^2 - 3x + 7 = 0$ for x.

6 If $f(x) = x^3 + 5x^2$, evaluate $f'(-2)$.

7 Give a sequence of transformations that will take the graph of $f(x) = \cos x$ to the graph of $g(x) = 2\cos\left(\dfrac{\pi x}{3}\right) - 5$.

8 A curve has an equation of the form $y = a\sin(bx + c)^\circ + d$ where a, b and d are positive real constants and c is a real constant. The curve has a maximum at $(34, 5)$ and a minimum at $(70, 2)$, with no other turning points between these two points. Determine the values of a, b and d. Determine a possible value for c.

9 A team of five people is to be selected from a group of 12 people. In how many ways can this be done?

10 Expand $(2 + 3x)^4$.

 ISBN 978-1-009-53719-3 Cambridge University Press

11 Solve the equation $2^{3x-2} = 1024$ for x.

12 Determine the equation of the tangent to the curve $y = x^3 - 2x^2 + 3$ at the point where $x = 1$.

13 Solve the inequality $\dfrac{1-5x}{3} \geq -12$.

14 If $a = \dfrac{y^2 - xz}{10}$, determine a when $x = -5$, $y = 7$ and $z = 6$.

15 **a** Determine the coordinates of the midpoint of the line segment joining $(8, 14)$ and (a, b) in terms of a and b.

b If $(5, 10)$ is the midpoint, determine the values of a and b.

16 Differentiate each of the following with respect to x:

a $(3x^2 - 4)^5$ **b** $(4x^2 + 6x + 3)(x - 5)$ **c** $\dfrac{3x^2 + 4x + 2}{x^3 - 2x}$

17 The line passing through the points $A(-2, 6)$ and $B(10, 15)$ meets the axes at P and Q.

a Determine the equation of AB.

b Determine the length of PQ.

18 The graph of the function $y = f(x)$ is a parabola that has turning point $(2, -6)$ and passes through the point $(6, 12)$. Determine the rule for the function.

19 Determine the value of a in the polynomial $P(x) = ax^3 + 4x^2 + 3$ if $P(2) = 3$.

20 Determine the maximal domains of the functions with the following rules:

a $f(x) = \dfrac{1}{x-3}$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{2-x}$

d $f(x) = \sqrt{x-4}$ **e** $f(x) = \dfrac{1}{\sqrt{5-x}}$

21 Differentiate each of the following with respect to x:

a $\dfrac{x^2+7}{x}$ **b** $(x^2 - 4)^3$ **c** $\sqrt{x^2+1}$

d $(x-4)(x^3+1)^2$ **e** $\dfrac{x^2+7}{4x^3+1}$

22 Given that $-4 \leq x \leq 6$ and $2 \leq y \leq 8$, calculate:

a the smallest possible value of $x - y$

b the largest possible value of $\dfrac{x}{y}$

c the largest possible value of $x^2 + y^2$.

23 A man bought 20 books. Some of the books cost \$72 each and the other books cost \$24 each. If the man spent a total of \$720 on these books, how many of the \$24 books did he buy?

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

SF

24 If $A = (-7, 6)$ and $B = (11, -5)$, determine:

a the coordinates of the midpoint of AB

b the distance between A and B

c the equation of AB

d the equation of the perpendicular bisector of AB.

25 Sketch the graph of $y = -x^2 + 4x + 2$, clearly labelling the x-axis and y-axis intercepts and the turning point.

26 A piece of wire 6000 cm long is used to make the edges of a cuboid. The length is $5x$ cm, the width is $4x$ cm and the height is w cm.

a Determine w in terms of x.

b Determine the volume, V cm^3, of the cuboid in terms of x.

c State the possible values of x.

d Determine the volume if $x = 100$.

27 A box contains 8 red cups and 10 blue cups.

a If a cup is chosen at random, the colour observed, the cup replaced and a second cup withdrawn, what is the probability that both cups drawn are red?

b If the first cup is not replaced, what is the probability that both cups drawn are red?

28 Box A contains three pieces of paper numbered 1, 3, 5.
Box B contains three pieces of paper numbered 2, 4, 6.
One piece of paper is removed at random from each box.
Determine the probability that the two numbers obtained have a sum that is divisible by 3.

29 A letter is chosen at random from the word GOOGLE.

a What is the probability that the letter is a vowel?

b What is the probability that the letter is an O?

30 David has either a sandwich or fruit salad for lunch. If he has a sandwich for lunch one day, the probability he has a sandwich for lunch the next day is 0.4. If he has fruit salad for lunch one day, the probability he has fruit salad for lunch the next day is 0.3. Suppose he has a sandwich for lunch on a Monday. What is the probability that he has fruit salad for lunch on the following Wednesday?

31 Solve the equation $\cos(3x) = \frac{1}{2}$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

 ISBN 978-1-009-53719-3 Cambridge University Press

32 The graph of $y = ax^3 + bx + c$ has axis intercepts at $(0, 6)$ and $(-2, 0)$ and has a stationary point where $x = 1$.

a Determine the value of c.

b Write down two simultaneous equations in a and b from the given information.

c Hence determine the values of a and b.

33 A tangent to the graph of $y = x^4$ has equation $y = -32x + a$. Determine the value(s) of a.

34 Define the function f by $f(x) = 4\cos(2x)$ for $-\pi \le x \le \pi$.

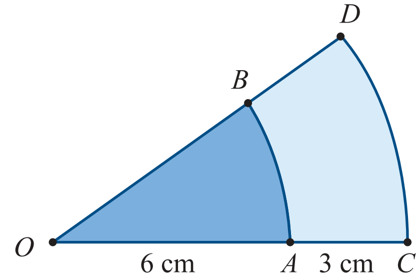
a Write down the amplitude and period of the function.

b Sketch the graph of the function f on a set of axes. Label the axis intercepts and the endpoints with their coordinates.

35 In the figure, the arcs AB and CD are arcs of concentric circles with centre O. Given that $OA = 6$ cm, $AC = 3$ cm and the area of sector AOB is 12 cm^2, calculate:

a $\angle AOB$ in radians

b the area and perimeter of the region $ABDC$.



36 Four identical balls are numbered 1, 3, 5 and 7 and put into a box. A ball is randomly drawn from the box, and not returned to the box. A second ball is then randomly drawn from the box.

a What is the probability that the second ball drawn is numbered 1?

b What is the probability that the sum of the numbers on the two balls is 8?

c Given that the sum of the numbers on the two balls is 8, what is the probability that the second ball drawn is numbered 1?

37 The line $y = x + 1$ cuts the circle $x^2 + y^2 + 2x - 4y + 1 = 0$ at the points A and B.

a Determine the midpoint of line segment AB.

b Sketch the graphs of $y = x + 1$ and $x^2 + y^2 + 2x - 4y + 1 = 0$ on the same axes.

38 Solve the following equations:

a $4^x - 5 \times 2^x - 24 = 0$ **b** $2^{5-3x} - 4^{x^2} = 0$

39 The curve $y = ax^3 - 2x^2 - x + 7$ has a gradient of 4 when $x = -1$. Determine the value of a.

40 Let $P(x) = 3x^2 + x + 10$. If $P(-b) = P(2b)$, where $b \ne 0$, calculate the value of b.

41 Show that the curves $y = x^3$ and $y = x^3 + x^2 + 6x + 9$ intersect at one point and determine the gradient of each of these curves at that point.

42 **a** Determine the coordinates of the stationary points on the graph of $y = x^3 - 75x - 10$ and sketch the graph.

b Determine the values of p for which the equation $x^3 - 75x - 10 = p$ has more than one real solution.

43 Let $f(x) = (x-2)^2(x-4)^3$.

a Determine $f'(x)$.

b Determine the values of x at which there is a stationary point on the curve $y = f(x)$.

c Determine the equation of the tangent to the curve $y = f(x)$ at:

i $x = 4$ **ii** $x = 3$ **iii** $x = \frac{5}{2}$

44 The height, h cm, of a bicycle pedal above the ground at time t seconds is given by

$$h(t) = 30 + 16\sin(2t), \quad t \ge 0$$

a Determine the height at time $t = 0$.

b What are the maximum and minimum heights of the pedal above the ground?

c Determine the first time that the pedal reaches its maximum height.

d Determine the first time that the pedal reaches its minimum height.

e How long does it take for the pedal to complete one revolution?

45 Let $P(x) = x^3 + 2kx^2 + k^2x + 11$. Given that $P(1) = 2P(-1)$ determine the possible value(s) of k.

46 The graph of $y = ax^2 + bx + c$ cuts the y-axis at $(0, 6)$ and the gradient of the graph is 12 when $x = 0$.

a Determine the values of b and c.

b Determine a if the point $(1, 0)$ is on the graph.

c Determine a if the point $(-1, 0)$ is on the graph.

47 Prove that, for each real number k, the quadratic equation $3x^2 - 2(k-1)x + k^2 + 1 = 0$ has no solutions.

48 A circle has equation $x^2 + y^2 + 4x - 6y = 23$.

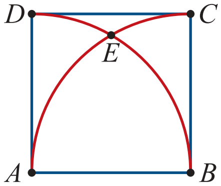
a Determine the centre and the radius of the circle.

b The line $y = 1$ cuts the circle at points A and B.

i Determine the coordinates of the points A and B.

ii Determine the length of the chord AB.

49 The figure shows a square $ABCD$. The arcs BD and AC have centres A and B, respectively, and they intersect at the point E. Determine the ratio of arc length DE to arc length EB.



Revision

50 A jar contains five red marbles, three white marbles and two blue marbles. Two marbles are drawn from the jar at random, one after the other, without replacement. Consider the following events:

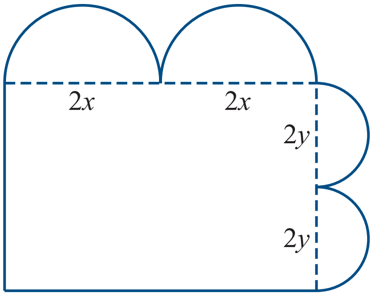
- R_1 the first marble is red
- W_1 the first marble is white
- B_1 the first marble is blue
- R_2 the second marble is red
- W_2 the second marble is white
- B_2 the second marble is blue.

Determine:

a $P(R_2 \mid R_1)$ **b** $P(B_2 \mid R_1)$ **c** $P(W_2 \mid R_1)$ **d** $P(R_1 \cap R_2)$
e $P(R_2)$ **f** $P(B_2)$ **g** $P(W_1 \cap R_2)$ **h** $P(R_1 \mid R_2)$

CF

51 A piece of wire is bent into the shape shown in the diagram. There are two semicircles of radius x and two semicircles of radius y.



a Determine an expression for the perimeter of the shape, P, in terms of x and y.
b Determine an expression for the area of the shape, A, in terms of x and y.

Now assume that the length of the wire is $2\pi + 4$.

c Determine y in terms of x.
d Determine an expression for the area, A, in terms of x.
e Determine the values of x and y for which this area is a maximum.
f Determine this maximum area.

CU

52 A curve has equation $y = ax^3 + bx^2 - 9x$, where a and b are real constants with $a \neq 0$. The curve has a stationary point at $x = -1$.

a Determine b in terms of a.
b Determine the coordinates of the stationary points in terms of a.
c Determine the equation of the tangent to the curve at $x = 1$ in terms of a.
d Determine the values of a and b such that the curve has only one stationary point.
e Determine the values of a and b such that the tangent to the curve at $x = 1$ cuts the y-axis at the point $(0, 1)$.

Technology-active short-response-questions

53 A circle has radius 10 cm. An arc subtends an angle of 55° at the centre of the circle. Determine the length of the arc correct to two decimal places.

SF

54 Determine the solution of the cubic equation $2x^3 - x^2 + 6x - 11 = 0$. Give your answer correct to two decimal places.

55 Determine the gradient of the tangent to the curve the curve with equation $y = 2x(x-3)^2$ at the point where $x = 3.01$.

 ISBN 978-1-009-53719-3 Cambridge University Press

56 A piece of wire 40 cm long is bent into the shape of a sector of a circle. If the length of the arc is 16 cm, determine:

a the angle of the sector **b** the area of the sector.

57 Let $f(x) = x^3 - 6x + 5$ for $0 \le x \le 4$.

a Determine all stationary points on the graph of $y = f(x)$ and determine their nature.

b Sketch the graph of $y = f(x), 0 \le x \le 4$.

c Determine the range of the function f.

d What are the maximum and minimum values of the function f?

58 A particle starts from a point O and moves in a straight line. After t seconds its position, x cm, relative to O is given by $x = 2t^2 - 6t$. Determine:

a its velocity, v cm/s, at time t seconds

b its velocity after 3 seconds

c its average velocity in the first 3 seconds

d the distance travelled in the first 3 seconds

e its average speed in the first 3 seconds.

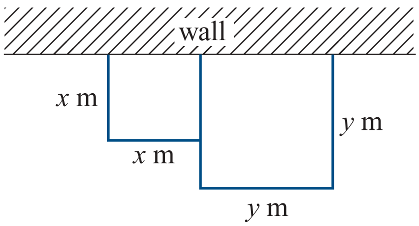
59 A piece of wire of length 52 cm is bent to form a trapezium in which the lengths of the two parallel sides are $2y$ cm and $(16x + 2y)$ cm. The other two sides both have length $10x$ cm.

a Determine y in terms of x.

b Determine the area of the trapezium, A cm^2, in terms of x.

c Determine the values of x and y for which A is a maximum.

60 Kerrie has 5200 m of fencing to make two square enclosures using an existing wall as one side of the enclosure. The squares have side lengths x m and y m with $x \le y$, as shown.



a Show that the total area, A m^2, of the two enclosures is given by

$$A = x^2 + \frac{(5200 - 2x)^2}{9}$$

b Determine the minimum area and the values of x and y for which this occurs.

c Sketch the graph of A against x.

61 A die has six faces numbered 0, 2, 4, 6, 8 and 10. A second die has faces numbered 1, 3, 5, 7, 9 and 11. Both dice are tossed and the results recorded.

a List the sample space of the experiment as a set of ordered pairs.

b Determine the probability that the sum of the values from the two dice is:

i 1 **ii** 13 **iii** 9

c Find the probability of obtaining a sum of 15 given that the sum is greater than 7.

CF

62 For the function $f(x) = -x^3 + ax^2$, where a is a positive real number:

a Determine the intervals on which f has:

i negative gradient

ii positive gradient.

b Determine the equation of the tangent to the graph of f at the point $(a, 0)$.

c Determine the equation of the normal to the graph of f at the point $(a, 0)$.

CU

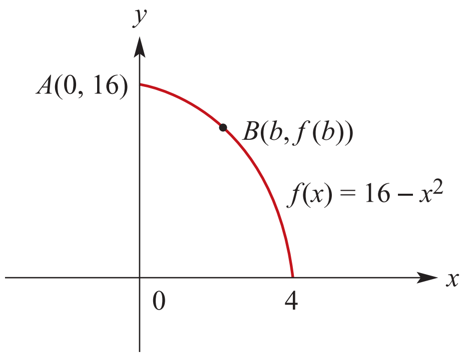
63 Suppose that the moon's orbit around Earth is a circle of radius 385 000 km and that the time taken for one revolution is 27.3 days.

a Consider two points on the moon's orbit that are 9 days apart. Determine the angle between the two radii from the centre of Earth to these points.

b Determine the area of the sector formed by these two radii.

c Comment on the accuracy of the assumptions.

64 The diagram shows the graph of the function $f(x) = 16 - x^2$ for $0 \le x \le 4$.



The line segment AB is drawn from the point $A(0, 16)$ to the point $B(b, f(b))$, where $0 < b < 4$.

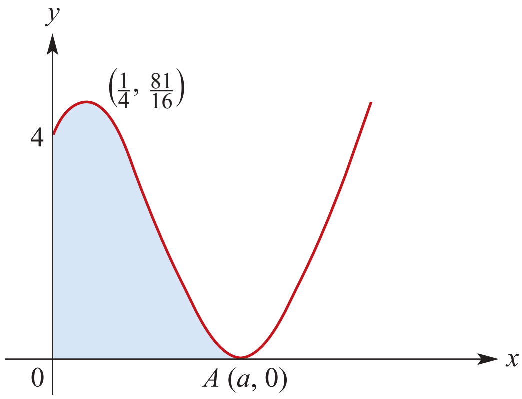
a **i** What is the gradient of AB?

ii At what value of x between 0 and b does the tangent to the graph of f have the same gradient as AB?

b **i** What is the area, $S(b)$, of the trapezium bounded by line segment AB, the x-axis, the line $x = 0$ and the line $x = b$?

ii For what value of b does this area equal 28?

65 Consider the function given by $f(x) = (\sqrt{x} - 2)^2(\sqrt{x} + 1)^2$ for $x > 0$. The graph of $y = f(x)$ is shown.



a Point $A(a, 0)$ is the x-axis intercept of $y = f(x)$. Determine the value of a.

b There is a stationary point at $x = \dfrac{1}{4}$.

State the interval for which the graph of f has negative gradient.

c The area of the shaded region of the graph is $\dfrac{136}{15}$.

Let O be the origin, C a point on the positive y-axis and B the point such that $OABC$ is a rectangle. Determine the length OC such that the area of the rectangle $OABC$ is the same as the area of the shaded region.

Revision

66 Consider the family of functions with rules of the form $f(x) = \sqrt{x - 2a}$, where a is a positive number.

- **a** Determine the maximal domain of such a function.
- **b** Solve the equation $f(x) = x$ for x.
- **c** For what value of a does the equation $f(x) = x$ have one solution?
- **d** Sketch the graphs of $f(x) = \sqrt{x - \frac{1}{4}}$ and $y = x$ on the one set of axes and label the point at which they touch with its coordinates.

67 Each night Frederick goes to the library or the study hall. If he goes to the library one night, the probability he goes to the library the next night is 0.7, and if he goes to the study hall one night, the probability he goes to the study hall the next night is 0.4. Suppose he goes to the library one Wednesday night.

- **a** What is the probability that he goes to the library on each of the next three nights?
- **b** What is the probability that he goes to the library on exactly two of the next three nights?

68 Platypus Sports sells hockey sticks directly to hockey clubs on a yearly basis. If a hockey club buys its sticks from Platypus one year, there is a 75% chance it will buy sticks from Platypus the next year. If a hockey club does not buy its sticks from Platypus one year, there is a 20% chance it will buy its sticks from Platypus the next year.
Suppose the Cross-Sticks club buys its hockey sticks from Platypus this year ($n = 0$).

- **a** What is the probability, correct to four decimal places, that Cross-Sticks buys its sticks from Platypus for the next three years?
- **b** What is the probability, correct to four decimal places, that it buys its sticks from Platypus for exactly two of the next three years?
- **c** What is the probability, correct to four decimal places, that it buys its sticks from Platypus in the third year ($n = 2$)?

69 Consider the family of quadratics with rules of the form $f(x) = (p - 1)x^2 + 4x + (p - 4)$, where p is an arbitrary constant.

- **a** Sketch the graph of f when:
  i $p = 0$ **ii** $p = 2$
- **b** Determine the coordinates of the turning point of the graph of $y = f(x)$ in terms of p.
- **c** For which values of p does the turning point of the graph of $y = f(x)$ lie on the x-axis?
- **d** For which values of p are there two distinct solutions to the equation $f(x) = 0$?
- **e** Let $f(x) = 2x^2 + 4x - 1$. (This is when $p = 3$.) Now let $y = g(x)$ be the image of the graph of $y = f(x)$ under a reflection in the y-axis. Determine the equation for $y = g(x)$ and sketch the graphs of both $y = f(x)$ and $y = g(x)$ on the same set of axes.

 ISBN 978-1-009-53719-3

70 Consider the function f given by

$$f(x) = -mx + 3, \quad x \in [a, b]$$

where $m > 0$ and $0 \le a < b$. The function f defines the line segment with endpoints $A(a, -ma + 3)$ and $B(b, -mb + 3)$.

a Determine the range of f.

b Determine the coordinates of the midpoint of AB.

c Determine the equation of the line that passes through the midpoint of AB and is perpendicular to the line through A and B.

d Determine the equation of the image of the line through A and B under a translation of 3 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis. Give the coordinates of the endpoints of the line segment under this transformation.

e Determine the equation of the image of the line through A and B under a reflection in the y-axis. Give the coordinates of the endpoints of the line segment under this transformation.

f If $a = 0$, determine the values of b and m if the midpoint of AB has coordinates $(6, -4)$.

71 A road near the shoreline is sometimes covered by water, and then the road is closed until it is uncovered again. The height, h metres, of the tide above mean sea level is modelled by the equation $h(t) = 2.3\cos(kt)$, where t hours is the time measured from 1 p.m. on a particular day. It is assumed that high tides occur every 12 hours.

a Determine the value of k.

b If on this day the road was closed for 1 hour 30 minutes after 1 p.m., determine the height of the road above mean sea level. Give your answer in centimetres.

c A raised footpath along the road is only under water for 1 hour after 1 p.m. Determine the height of the footpath above mean sea level. Give your answer in centimetres.

72 A company produces electronic components at two factories, one in Brisbane and the other in Toowoomba. Assume that 60% of all components made by the company are produced in the Brisbane factory, that 10% of the Brisbane-produced components are defective and that 5% of the Toowoomba-produced components are defective.

a Determine the overall proportion of defective components produced by the company.

b If a component is selected at random from the company's production and found to be defective, what is the probability that it was produced in Toowoomba?

c If two components are randomly selected from the company's production, what is the probability that:

  i at least one of them is defective

  ii they are both defective?

 ISBN 978-1-009-53719-3 Cambridge University Press

Revision

18B Multiple-choice questions

Technology-free multiple-choice questions

1 The function $f(x) = 5\sin\left(\frac{x}{4}\right)$ has period

A $\frac{\pi}{4}$ **B** 8π **C** 5 **D** 4π

2 For the function $f(x) = x^2 + 2x$, the average rate of change of $f(x)$ with respect to x for the interval $[0, 3]$ is

A 5 **B** 8 **C** 10 **D** 15

3 Let $y = (x-2)^2 + 3$ for $1 \le x < 4$. The range of this function is

A $y \ge 2$ **B** $y > 0$ **C** $y \ge 3$ **D** $3 \le y < 7$

4 The simultaneous equations $(m-2)x + y = 0$ and $2x + (m-3)y = 0$ have infinitely many solutions for

A $m \in \mathbb{R}$ **B** $m \ne 1$ **C** $m \in \{1, 4\}$ **D** $m \ne 4$

5 The function f has rule $f(x) = 2\log_{10}(3x)$. If $f(5x) = \log_{10}(y)$, then y is equal to

A $30x$ **B** $12x$ **C** $225x^2$ **D** $50x^2$

6 A bag contains 2 white balls and 4 black balls. Three balls are drawn from the bag without replacement. The probability they are all black is

A $\frac{1}{6}$ **B** $\frac{27}{125}$ **C** $\frac{1}{5}$ **D** $\frac{8}{216}$

7 Let $f(x) = \frac{1}{3}x^3 - 2x^2 + 1$. The graph of $y = f(x)$ has negative gradient for

A $0 < x < 4$ **B** $x > 0$ or $x < 4$ **C** $x > \sqrt{3}$

D $x < -\sqrt{2}$ or $x > \sqrt{2}$

8 The maximal domain of the function with rule $f(x) = \sqrt{2x+1}$ is

A $x \ne -\frac{1}{2}$ **B** $x \ge -\frac{1}{2}$ **C** $x \in \mathbb{R}$ **D** $x > 0$

9 In algebraic notation, 11 is four times the number that is 9 more than x can be written as

A $11 = 4(x+9)$ **B** $4x + 9 = 11$

C $11 = 9(x+4)$ **D** $\frac{4}{x+9} = 11$

10 A car covers a distance of 120 km at a speed of a km/h, and a train covers the same distance at $(a-4)$ km/h. If the time taken by the train is one hour more than the time taken by the car, then a is equal to

A 20 **B** 24 **C** 10 **D** 15

11 The equation of the parabola that passes through the point $(-3, 12)$ and has its vertex at $(-2, 8)$ is

A $y = 4(x+2)^2 + 8$ **B** $y = (x+2)^2 + 4$

C $y = 4(x-2)^2 + 4$ **D** $y = (x-2)^2 - 4$

ISBN 978-1-009-53719-3

12 Let $f(x) = 5 - 2x$. If the domain of f is $[-3, 5)$, then the range of f is

A $(-5, 11]$ **B** $[-5, 11)$ **C** $(-5, \infty)$ **D** $(-5, 11)$

13 Define the function f by $f(x) = 2x^2 + 7$ for $x \in [-3, 2)$. The range of f is

A $x \geq 7$ **B** $0 \leq x < 25$ **C** $15 \leq x \leq 25$ **D** $7 \leq x \leq 25$

14 $\dfrac{\sqrt{0.144 \times 10^5}}{2 \times 10^4}$ is equal to

A 7.2×10^{-3} **B** 6×10^{-3} **C** 6×10^4 **D** 7.2×10^5 **E** 12×10^6

15 Let $P(x) = x^2 + ax + b$. If $P(-1) = 0$, then $-a + b + 7$ equals

A -7 **B** 6 **C** 1 **D** 7

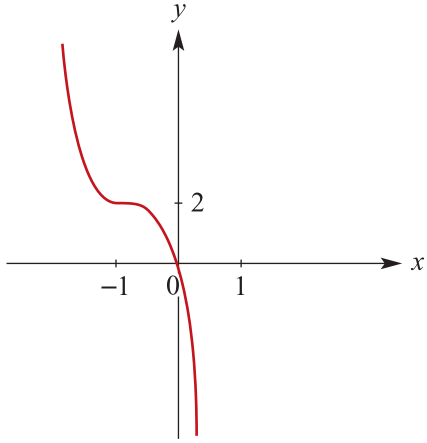
16 A possible equation of the graph shown is

A $y = -2(x + 1)^3 + 2$

B $y = -(x + 1)^3 + 2$

C $y = -(x - 1)^3 + 2$

D $y = -(x - 1)^3 - 1$



17 Let $f(x) = x$ for $x \in \mathbb{R}$. Which one of the following is *not* true?

A $f(xy) = f(x)f(y)$ **B** $f(x) - f(-x) = 0$

C $f(7x) = 7f(x)$ **D** $f(x - y) = f(x) - f(y)$

18 Under a transformation, the image of the curve with equation $y = f(x)$ is the curve with equation $y = 3f(2x)$. The transformation could be

A a dilation of factor 2 from the x-axis and a dilation of factor 3 from the y-axis

B a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 3 from the y-axis

C a dilation of factor 2 from the x-axis and a dilation of factor $\frac{1}{3}$ from the y-axis

D a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis

19 The set of solutions to the equation $25^x - 7 \times 5^x + 12 = 0$ is

A $\{3, 4\}$ **B** $\{-3, -4\}$ **C** $\{\log_5 4, \log_5 3\}$ **D** $\{1, \log_5 2\}$

20 A particle moves in a straight line so that its position, s m, relative to O at a time t seconds $(t > 0)$ is given by $s = 4t^3 - 5t - 10$. The acceleration at $t = 1$ is

A 12 m/s^2 **B** 24 m/s^2 **C** -11 m/s^2 **D** -10 m/s^2

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Revision

21 The average rate of change of the function $y = 2x^4 + x^3 - 1$ between $x = -1$ and $x = 1$ is

A 1 **B** 0 **C** 4 **D** 3

22 A function f is such that:

- $f'(x) = 0$ for $x = 3$ and for $x = 5$
- $f'(x) > 0$ for $3 < x < 5$
- $f'(x) < 0$ for $x < 3$ and for $x > 5$.

Which one of the following is true?

A The graph of f has a local maximum point where $x = 3$.

B The graph of f has a stationary point of inflection where $x = 5$.

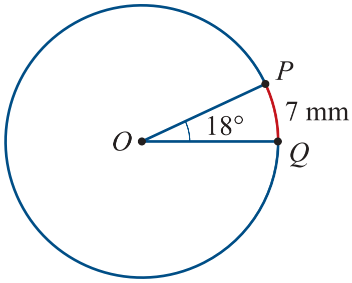
C The graph of f has a local minimum point where $x = 5$.

D The graph of f has a local minimum point where $x = 3$.

23 In the expansion of $(2x - 1)^5$, the coefficient of x^2 is

A 2^2 **B** -2^2 **C** $2^2 \times \binom{5}{3}$ **D** $-2^2 \times \binom{5}{3}$

24 Arc PQ of length 7 mm subtends an angle of 18° at the centre, O, of the circle. The radius of the circle is



A $\dfrac{\pi}{35}$ mm **B** $\dfrac{\pi}{70}$ mm

C $\dfrac{35}{\pi}$ mm **D** $\dfrac{70}{\pi}$ mm

25 The tangent at the point $(1, 5)$ on the curve $y = f(x)$ has equation $y = x + 4$. The tangent at the point $(3, 6)$ on the curve $y = f(x - 2) + 1$ has equation

A $y = 2x - 4$ **B** $y = x + 3$

C $y = -2x + 14$ **D** $y = 2x + 4$

26 For a cubic function with rule $y = f(x)$, the graph of the derivative function f' crosses the x-axis at $(1, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 12. The value of x for which the graph of $y = f(x)$ has a local maximum is

A -3 **B** 0 **C** -1 **D** 1

27 The tangent to the curve $y = \sqrt[3]{x}$ has gradient $\dfrac{1}{6}$ at

A $x = \pm 2\sqrt{2}$ **B** $x = \pm 6\sqrt{6}$ **C** $x = 8$ **D** $x = \pm\dfrac{\sqrt{2}}{4}$

 ISBN 978-1-009-53719-3

28 A curve has equation $y = f(x)$. The following transformations are applied to the curve in the given order:

- a reflection in the y-axis
- a dilation of factor 2 from the x-axis
- a translation of 2 units in the positive direction of the y-axis.

The equation of the resulting curve is

A $y = 2f(-x) + 2$ **B** $y = -2f(x) - 2$
C $y + 2 = -2f(x)$ **D** $y = -2f(-x) + 2$

29 If $f(x) = x(x-1)^3$, then $f'(x)$ equals

A $-x(x-2)^2$ **B** $2x(x-1)^2$
C $(x-1)^2(2x-1)$ **D** $(x-1)^2(4x-1)$

30 If $f(x) = (g(x))^4$, where g is a differentiable function, then $f'(x)$ equals

A $(g'(x))^4$ **B** $4(g'(x))^3$
C $4(g(x))^3$ **D** $4g'(x)\,(g(x))^3$

31 In Kiana's swimming team there are 15 swimmers who compete in freestyle, 10 who compete in butterfly. Eight of these swimmers compete in both. Determine the probability that a swimmer chosen at random from the team competes in freestyle, given that they compete in butterfly.

A $\dfrac{8}{15}$ **B** $\dfrac{4}{5}$ **C** $\dfrac{8}{17}$ **D** $\dfrac{15}{22}$

32 Given that for two events A and B, $P(A) = 0.25$, $P(B) = 0.4$, and $P(A \cap B) = 0.15$, determine $P(B|A)$.

A 0.325 **B** 0.4 **C** 0.625 **D** 0.6

Technology-active multiple-choice questions

33 The values of x for which the corresponding points on the graph of $y = \dfrac{1}{3}x^3 - 4x^2 - 6x + 3$ have tangent lines perpendicular to the line $y = \dfrac{x}{18}$ are:

A $x = \dfrac{2}{3}$ or $x = 2$ **B** $x = \dfrac{5}{6}$ or $x = \dfrac{3}{4}$
C $x = 2$ or $x = 6$ **D** $x = -1$ or $x = 5$

34 The angle that the tangent line to the graph of the function with rule $f(x) = 2x^3 - 3x^2 - x - 4$ at the point with coordinates $(1.1, f(1.1))$ makes with the positive direction of the x-axis is closest to:

A $161.22°$ **B** $63.40°$ **C** $99.36°$ **D** $116.57°$

35 Let $f(x) = 2x^3 - 9x^2 - 2x + 24$. Then $f(x)$ is negative for:

A $2 < x < 4$ or $x < -\dfrac{3}{2}$ **B** $x > -\dfrac{3}{2}$
C $x < -\dfrac{1}{2}$ **D** $x > 4$ or $-\dfrac{3}{2} < x < 2$

 ISBN 978-1-009-53719-3

Revision

36 Let $f(x) = -2x^3 + 4x - 2$ for $-2 < x < 2$. The range of f is

A $-10 < x < 6$ **B** $x > -10$ **C** $-5 < x < 5$ **D** $-1 < x < 5$

37 Let $f(x) = \begin{cases} -x^3 + 3x + 30 & x > 4 \\ x^3 + 2x - 6 & -2 < x \le 4 \\ 2x - 1 & x \le -2 \end{cases}$.

The number of solutions of the equation $f(x) = 6$ is

A 0 **B** 1 **C** 2 **D** 3

38 A chord of length 4 cm is drawn in a circle of radius 6 cm. The area of the smaller region inside the circle cut off by the chord, correct to one decimal place, is:

A 0.9 cm^2 **B** 0.2 cm^2 **C** 1.8 cm^2 **D** 10.6 cm^2

39 The number of solutions of the equation $\cos x = \frac{x}{3}$ is:

A 1 **B** 2 **C** 3 **D** 4

The following information relates to Questions 40 *and* 41.

The probability that Teale goes to the gym on Monday is 0.7. If she goes to the gym on Monday, then the probability that she will go again on Tuesday is 0.75. If she doesn't go to the gym on Monday, then the probability that Teale will go on Tuesday is only 0.5.

40 Determine the probability that Teale goes to the gym on both Monday and Tuesday.

A 0.49 **B** 0.525 **C** 0.175 **D** 0.5625

41 Determine the probability that Teale goes to the gym on Tuesday.

A 0.625 **B** 0.7 **C** 0.75 **D** 0.675

42 If A and B are independent events such that $P(A) = 0.16$ and $P(B) = 0.23$, then $P(A \cup B)$ is equal to:

A 0.39 **B** 0.3532 **C** 0.4268

D Cannot be determined from the information given.

 ISBN 978-1-009-53719-3

Appendix

A

Counting methods and applications to probability

Chapter contents

When determining the probability of an event occurring, we often need to know the number of outcomes contained in the event and in the sample space. To do this in Chapter 9 we listed the sample space using a tree diagram or a table and counted the number of outcomes.

When dealing with more complicated probability problems, listing the sample space and the event becomes too difficult. There may be hundreds of outcomes for a particular experiment, and even if they were comparatively easy to list we would soon tire of the task. In this chapter we will look at ways of counting the number of outcomes for various experiments and this will enable us to deal with more complicated probability problems.

For example, suppose there are 25 students in a class. The number of different ways of choosing a group of 5 students from the class is 53 130, if it doesn't matter about the order of choice. The number of ways of choosing 5 students in order from the class is 6 375 600. If it takes you 1 minute to choose one ordered group of 5 students, then it would take you over 12 years to obtain every possible ordered group. It is evident that we need to develop some methods to be able to 'count' in these situations.

The material in this chapter builds on the the ideas introduced in chapter 7 in your study of the Binomial theorem.

 ISBN 978-1-009-53719-3

A1 Addition and multiplication principles

Learning intentions

- To be able to use addition and multiplication principles to help count the number of possible outcomes.

The addition rule

Some people find the decision about what to wear when they get up in the morning to be very difficult, and the more clothes they own, the more complex the decision becomes! Let us consider the number of choices they might have by looking at some examples.

Example 1

Sandi can't decide whether to wear a T-shirt or a tank top. She has four T-shirts and two tank tops. How many choices does she have?

Solution

As Sandi is going to wear a T-shirt *or* a tank top, she has a total of six choices from among these two items.

Example 2

Sandi's next choice is whether to wear shorts or a skirt. She has three pairs of shorts and four skirts. How many choices does she have?

Solution

Once again, as Sandi will wear shorts *or* a skirt, she has a total of seven choices from these two items.

Addition rule

In general, to choose between alternatives the total number of possibilities is found by adding the number of choices available for each alternative.

For example, for two alternatives, if there are m choices for the first alternative and n choices for the second alternative, then there are $m + n$ choices altogether.

This rule is generally associated with the word 'or'.

Example 3

At the library Alan is having trouble deciding which book to borrow. He has a choice of three mystery novels, three biographies or two science fiction books. How many choices of book does he have?

Solution

As he is choosing between alternatives (mystery novels *or* biographies *or* science fiction), he has a total of $3 + 3 + 2 = 8$ choices.

 ISBN 978-1-009-53719-3 Cambridge University Press

The multiplication rule

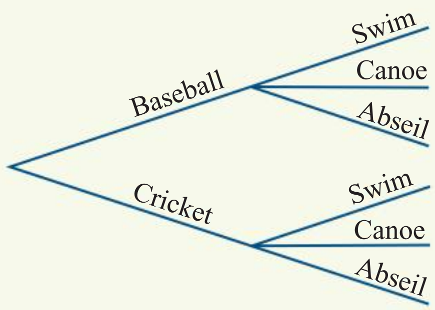
Sometimes the question arises of determining the number of possibilities when making successive choices, as in multi-stage experiments.

Example 4

Maya is taking part in an adventure camp. For this morning's activity, she can choose baseball or cricket. For this afternoon, she can choose swimming, canoeing or abseiling. How many choices does Maya have for a full day's program?

Solution

A tree diagram may help to answer this question.



By counting the branches of the tree diagram, it is found that there are six choices.

This answer could also be found by noting that there are two choices for the morning and three choices for the afternoon, and $2 \times 3 = 6$.

Multiplication rule

When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

For example, for two stages, if there are m choices for the first stage and n choices for the second stage, then there are $m \times n$ choices altogether.

This rule is generally associated with the word 'and'.

Example 5

Bobbie will wear either a shirt and a skirt, or a dress. She has seven different shirts, four different skirts and six different dresses. How many different outfits does she have?

Solution

If Bobbie wears a shirt *and* skirt, using the multiplication rule the number of choices is

$7 \times 4 = 28.$

Bobbie wears either a shirt and skirt (28 choices) *or* a dress (6 choices). Using the addition rule, in total she has

$28 + 6 = 34$ choices.

 ISBN 978-1-009-53719-3 Cambridge University Press

Summary A1

There are two simple rules which greatly assist in determining the number of possibilities in a sample space.

- **Addition rule** When choosing between alternatives, the total number of possibilities is found by adding the number of options available for each alternative. For example, for two alternatives, if there are m choices for the first alternative and n choices for the second alternative, then there are $m + n$ choices altogether.This rule is generally associated with the use of the word 'or' in the question.
- **Multiplication rule** When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage. For example, for two stages, if there are m choices for the first stage and n choices for the second stage, then there are $m \times n$ choices altogether.

Exercise A1

Example 3

SF

1 Determine how many choices of book are possible if one book is to be selected from the following:

a eight novels, three dictionaries

b three mysteries, two dramas, seven science fiction

c twenty-two romances, fourteen mysteries, one autobiography

d ten novels, three biographies, twelve encyclopaedias, four atlases.

Example 4

2 Lacey is planning a day out with friends, a movie followed by lunch at a cafe. She can choose between five different movies, and three different cafes. How many choices does Lacey have for the day out?

3 Sam is trying to decide what to wear to to the beach. They have a choice of four different swimsuits and six different hats. How many choices of outfit does Sam have if both a swimsuit and a hat must be worn?

4 Determine how many different meals are possible if three courses (one entree, one main course and one dessert) are to be selected from a menu that lists:

a three entrees, four main courses, five desserts

b ten entrees, ten main courses, five desserts

c five entrees, seven main courses, ten desserts

d eight entrees, eight main courses, eight desserts.

5 The menu in a restaurant lists four choices of entree, eight of main course and four of dessert. Determine the number of choices of meal possible:

a if one of each of the three courses must be chosen

b if you can choose to omit the entree.

 ISBN 978-1-009-53719-3

Example 5

6 Tom will wear either a t-shirt and jeans, or a suit and shirt to the interview. He has four t-shirts, three pairs of jeans, two suits and five shirts. How many different outfits does he have?

SF

7 John can either go to Bob's Diner, which has a choice of three entrees, five main courses and three desserts, or Kate's Cafe, which has a choice of two entrees, six main courses and four desserts. How many different meals are possible if he eats three courses?

8 A student has to select a two-unit study for her course. She has to choose one unit in each semester. In semester one she has to choose one of two mathematics units, three language units and four science units. In semester two she has a choice of two history units, three geography units and two art units. How many choices does she have for the complete study?

9 Dominic is travelling from Adelaide to Brisbane. He could fly directly from Adelaide to Brisbane on one of three airlines, or he could fly from Adelaide to Sydney on one of four airlines and then travel from Sydney to Brisbane with one of five bus lines, or he could go on one of three bus lines directly from Adelaide to Brisbane. In how many ways could Dominic travel from Adelaide to Brisbane?

CF

10 Andre cannot decide how to spend his evening. He can read a book at home and order a takeaway meal from one of five different local restaurants which deliver, or go to the movies with friends. If he does go to the the movie with his friends, then he can either choose to have dinner with them afterwards at one of three cafes, or he could go straight home. How many different choices does he have?

11 Miller is buying accessories for his new phone. There are five different cases, three different screen covers, and four different head sets. Miller is going to buy two of these items (that is a case and a screen cover, a case and a head set, or screen cover and a head set). How many different choices does he have?

12 A particular new model of car is available in five choices of colour, three choices of transmission, four types of interior and two types of engine. Air conditioning is optional. How many different types of car are possible?

13 A company uses one letter followed by four digits for product codes. If any of the letters A–Z is allowed in the first position, and any of the digits 0–9 in the next four positions, how many different product codes are possible? (The letters and digits may be used more than once.)

14 Suppose that a licence plate must consist of three letters followed by three numbers. If any of the letters A–Z is allowed in the first three positions, and any of the digits 0–9 in the second three positions, how many different licence plates are possible? (The letters and digits may be used more than once.)

15 Morse code consists of a succession of dots and dashes. The symbols formed by the code may consist of one, two, three or four dots or dashes. How many different symbols may be represented by this code?

A2 Arrangements

Learning intentions

- To be able to calculate the number of arrangements (or permutations) of r objects from a set of n objects.

The number of **arrangements** of a set of objects is the number of different ways these objects can be arranged in a distinct order. Consider, for example, arrangements of the letters A, B, C. The possibilities are ABC, ACB, BAC, BCA, CAB, CBA. Each of these is considered a different arrangement (even though they all use the same letters), and counting them we can see that there are six arrangements in total.

While it is quite simple to determine the number of arrangements by listing the possibilities when there are only three objects, this is not a good strategy when there are more than three. However, we can work out the number of arrangements by the following logic.

Consider that we have three boxes to be filled:

- We have 3 choices of letter for the first box (A, B or C).
- We only have 2 choices for the second box (because we have already used one letter).
- We only have 1 choice for the third box (because we have already used two letters).

3	2	1

Using the multiplication rule, we see that the total number of arrangements (or choices) is

$$3 \times 2 \times 1 = 6$$

Example 6

How many ways are there of arranging four different books on a shelf?

Solution

Consider the bookshelf as having four possible positions in which books can be placed:

As we have four books:

- There are 4 choices of book to place in position 1.
- There are 3 choices of book to place in position 2.
- There are 2 choices of book to place in position 3.
- There is 1 choice of book to place in position 4.

4	3	2	1

Using the multiplication rule, we know that the total number of choices will be the product of these individual choices:

The number of arrangements of four books in a row is

$$4 \times 3 \times 2 \times 1 = 24$$

 ISBN 978-1-009-53719-3

In general, if n objects are arranged in a row, then there are n choices for the first position, $n - 1$ choices for the second position, $n - 2$ choices for the third position, and so on, until 1 is reached.

Thus, the number of ways of arranging n objects in a row is

$$n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

Example 7

A photo is to be taken of a group of 12 students. How many ways are there of arranging the group if they all sit in a row?

Solution

As there are 12 students, the number of arrangements is

$$12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 479\,001\,600$$

which is rather a large number of choices!

Continuing to write out the expression $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ can be rather cumbersome. So for convenience this is written as $n!$, which is read as 'n factorial'.

The notation $n!$ (read as 'n factorial') is an abbreviation for the product of all the integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

A calculator can be used to evaluate $n!$. See the appropriate appendix in the Interactive Textbook for directions.

Sometimes we wish to arrange objects, but not use all of them every time. Suppose, for example, we wanted to determine how many arrangements there are of the letters A, B, C when taken in groups of two. The possibilities are AB, BA, AC, CA, BC, CB; six in total.

Once again, we can use our 'box' approach to work out the number of arrangements without listing them. This time we have two boxes to be filled:

- We have 3 choices of letter for the first box (A, B or C).
- We have 2 choices for the second box (because we have already used one letter).

3	2

Using the multiplication rule again, we see that the total number of arrangements (or choices) is $3 \times 2 = 6$.

 ISBN 978-1-009-53719-3

Example 8

A painter is to paint the five circles of the Olympic flag. He cannot remember the colours to use for any of the circles, but he knows they should all be different. He has eight colours of paint available. In how many ways can he paint the circles on the flag?

Solution

We represent the painter's choices with five boxes:

- For the first box (paint colour) there are 8 choices.
- For the second box (paint colour) there are 7 choices.
- For the third box (paint colour) there are 6 choices.
- For the fourth box (paint colour) there are 5 choices.
- For the fifth box (paint colour) there are 4 choices.

8	7	6	5	4

Thus the total number of arrangements possible is $8 \times 7 \times 6 \times 5 \times 4 = 6720$.

Could the factorial notation be used to express the answer to Example 8? In that example, the number of arrangements of eight objects in groups of five was determined to be

$$8 \times 7 \times 6 \times 5 \times 4$$

Multiplying this answer by 1, we can write

$$8 \times 7 \times 6 \times 5 \times 4 = (8 \times 7 \times 6 \times 5 \times 4) \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{8!}{3!}$$

Thus, the number of ways of choosing and arranging five objects from eight objects can be written as

$$\frac{8!}{3!} \quad \text{or} \quad \frac{8!}{(8-5)!}$$

In general, the number of arrangements of r objects from n possible objects is given by

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Arrangements are also called **permutations** in mathematics, and so this expression for the number of arrangements of r objects from n possible objects is often denoted by the symbol ${}^nP_r$.

Example 9

Determine the number of different four-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, if each digit:

a can only be used once **b** can be used more than once

Solution

a As we are arranging nine objects (n) in groups of four (r):

$$\frac{9!}{(9-4)!} = \frac{9!}{5!} = 9 \times 8 \times 7 \times 6 = 3024$$

b There are nine choices for each of the four positions, so the total number of choices is $9 \times 9 \times 9 \times 9 = 9^4 = 6561$.

Definition of 0!

For us to be able to use our formula for the number of arrangements in all situations, we need to define a value for $0!$. Consider the number of arrangements of n objects in a group of size n. From first principles, we have found that this is equal to $n!$. Using our formula for the number of arrangements of n objects in a group of size n gives us the answer

$$^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$$

For this to make sense, we need to define

$$\boxed{0! = 1}$$

Arrangements with restrictions

If a more complicated arrangement is required, then it is usually best to deal with the restriction first, as shown in the following example.

Example 10

How many different even four-digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, if each digit may be used only once?

Solution

We have four boxes to be filled. Dealing with the restriction first, there are 4 choices for the final (fourth) box:

			4

Having now selected the last digit, we have seven digits remaining:

- 7 choices for the first box
- 6 choices for the second box
- 5 choices for the third box.

7	6	5	4

Multiplying gives $7 \times 6 \times 5 \times 4 = 840$.

Explanation

The restriction is that the number must be even, so it ends in 2, 4, 6 or 8.

The number of choices is reduced by one each time we use a digit.

ISBN 978-1-009-53719-3

Summary A2

- An **arrangement** is a grouping of objects, such that the order of the objects is important. That is, the arrangements ABC and CBA are different.
- The number of ways of arranging n objects in a row is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

- The number of ways of arranging n objects in a group of size r is

$$^{n}P_{r} = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Skill-sheet

Exercise A2

SF

1 Evaluate:

a $3!$ **b** $5!$ **c** $7!$ **d** $2!$ **e** $0!$ **f** $1!$

2 Evaluate:

a $\dfrac{5!}{3!}$ **b** $\dfrac{9!}{7!}$ **c** $\dfrac{3!}{0!}$ **d** $\dfrac{8!}{6!}$ **e** $\dfrac{5!}{0!}$ **f** $\dfrac{10!}{7!}$

Example 6

3 In how many ways can five books be arranged in a row on a shelf?

Example 7

4 In how many ways can seven students be seated on a row of seven chairs?

5 In how many ways can four pictures be hung in a row on a wall?

6 In how many ways can six cups be hung on six hooks?

Example 8

7 In how many ways can three pictures be hung along a wall if there are ten pictures available?

8 If there are eight swimmers in the final of the 1500 m freestyle event, in how many ways can the first three places be filled?

9 Determine the number of ways in which the letters of the word TROUBLE can be arranged:

a if they are all used **b** if they are used three at a time.

10 Determine the number of ways in which the letters of the word PANIC can be arranged:

a if they are all used **b** if they are used four at a time.

Example 9

11 Determine the number of four-letter code words that can be made from the letters of the word COMPLEX:

a if no letter can be used more than once **b** if the letters can be re-used.

 ISBN 978-1-009-53719-3

CF

12 Determine how many code words of three or four letters can be made from the letters of the word NUMBER:

a if no letter can be used more than once **b** if the letters can be re-used.

Example 10 **13** If no digit can be used more than once, determine how many numbers can be formed from the digits 3, 4, 5, 6, 7:

a three-digit numbers **b** even three-digit numbers

c greater than 700.

14 If no digit can be used more than once, determine how many numbers can be formed from the digits 3, 4, 5, 6, 7, 8:

a two- or three-digit numbers **b** even six-digit numbers

c greater than 7000.

15 Four boys and two girls sit in a line on stools in front of a counter. Determine the number of ways in which they can arrange themselves:

a if there are no restrictions **b** if the two girls wish to be at the ends.

A3 Selections

Learning intentions

- To be able to calculate the number of selections (or combinations) of r objects from a set of n objects.

In the previous section, we discussed methods for counting the number of ways in which objects can be chosen and then arranged. Sometimes we are only interested in the number of different groups of objects that can be selected. That is, the order is unimportant.

Consider again the letters A, B, C when taken in groups of two. We saw that there are six arrangements: AB, BA, AC, CA, BC, CB. However, there are only three distinct selections: $\{A, B\}$, $\{A, C\}$, $\{B, C\}$.

Example 11

Four flavours of ice-cream (vanilla, chocolate, strawberry and caramel) are available at the school tuck shop. How many different double-scoop selections are possible if two different flavours must be used?

Solution

The possibilities are:

- vanilla and chocolate
- vanilla and strawberry
- vanilla and caramel
- chocolate and strawberry
- chocolate and caramel
- strawberry and caramel

giving a total of six different selections.

In this example, the selection 'vanilla and chocolate' is considered to be the same as 'chocolate and vanilla', and so is counted only once. Such choices without regard to order are called **selections** or **combinations**.

 ISBN 978-1-009-53719-3

Both the notations $\binom{n}{r}$ and ${}^nC_r$ are used to represent the number of different ways in which groups of size r can be chosen from a total of n objects when order is unimportant. We will use $\binom{n}{r}$ for the remainder of this chapter, but you may find that the notation ${}^nC_r$ is used in your calculator. When the total group size n is not large, the combinations can be listed. But obviously a more efficient method is preferable. Consider again Example 8 concerning the colours on the Olympic flag. Suppose that first the five colours to be used are chosen, and then they are arranged on the flag. This is shown as:

Choose the colours	Arrange them	Possible arrangements
$\binom{8}{5}$	$\times\, 5!$	$= \dfrac{8!}{3!}$

So, since $\binom{8}{5} \times 5! = \dfrac{8!}{3!}$, we can find an expression for $\binom{8}{5}$ by dividing both sides by 5!.

$$\binom{8}{5} = \frac{8!}{3!\,5!}$$

Note that the two figures on the bottom line (3 and 5) add to 8.

In general, the number of combinations of r objects from n possible objects is

$$\binom{n}{r} = \frac{{}^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

A commonly used alternative notation for $\binom{n}{r}$ is ${}^nC_r$.

A calculator can be used to determine values of $\binom{n}{r}$. See Section 7B of this book or the appropriate appendix in the Interactive Textbook for directions.

Example 12

Consider the situation from Example 11 again: If four flavours of ice-cream are available, how many double-scoop selections are possible if two different flavours must be used?

Solution

The number of combinations of four flavours in groups of size two is

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

This is the same as the answer we found before by listing the combinations.

Once again, not all combination problems are so straightforward, as shown by the following example.

Example 13

A team of three boys and three girls is to be chosen from a group of eight boys and five girls. How many different teams are possible?

Solution

Three boys can be chosen from eight in $\binom{8}{3}$ ways, and three girls from five in $\binom{5}{3}$ ways.

 ISBN 978-1-009-53719-3

Since we are choosing boys and girls, the total number of possible teams is

$$\binom{8}{3} \times \binom{5}{3} = 56 \times 10 = 560$$

Selections of any size

The following result is useful when asked to count all the combinations of any size from a group of n objects.

For n objects,

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

We can see why it is true using the following argument for three objects.

There are three different books A, B and C, and David can choose whichever ones he likes (including choosing none of them). How many ways can this choice be made?

Let $S = \{A, B, C\}$. The subsets are

$$\varnothing, \quad \{A\}, \quad \{B\}, \quad \{C\}, \quad \{A, B\}, \quad \{A, C\}, \quad \{B, C\}, \quad \{A, B, C\}$$

So there are eight choices.

We can use two different methods to count these choices. First we focus on how many books David chooses:

0 He can choose no books in $\binom{3}{0} = 1$ way. (This corresponds to $\varnothing$.)

1 He can choose one book in $\binom{3}{1} = 3$ ways. (This corresponds to $\{A\}, \{B\}, \{C\}$.)

2 He can choose two books in $\binom{3}{2} = 3$ ways. (This corresponds to $\{A, B\}, \{A, C\}, \{B, C\}$.)

3 He can choose three books in $\binom{3}{3} = 1$ way. (This corresponds to $\{A, B, C\}$.)

By the addition rule, the total number of choices is

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

Now, instead, we focus on each book one at a time:

A Book A can be chosen or not (2 ways).

B Book B can be chosen or not (2 ways).

C Book C can be chosen or not (2 ways).

By the multiplication rule, there are a total of $2 \times 2 \times 2 = 2^3$ choices. This gives

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 2^3$$

This argument can easily be extended to prove the result for n objects.

 ISBN 978-1-009-53719-3

Example 14

Nick is making an invitation list for his party, and has seven friends to choose from. If he may choose to invite any number of friends (from one to all seven), how many possible party lists does he have? (Assume he will invite at least one person to his party.)

Solution

Nick may invite one person to the party, and he has $\binom{7}{1} = 7$ ways of doing this. If he invites two people to the party, he has $\binom{7}{2} = 21$ ways of doing this.

Continuing in this way, we can see that Nick's total number of choices is

$$\binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$

Since we know that

$$\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 2^7$$

the required answer is

$$\begin{aligned} 2^7 - \binom{7}{0} &= 2^7 - 1 \\ &= 128 - 1 \\ &= 127 \end{aligned}$$

Summary A3

- A **selection** (or **combination**) is a grouping of objects, such that the order of the objects is not important. That is, the selections ABC and CBA are the same.
- Knowing the number of distinct selection of objects, when taking groups of a particular size, may enable us to determine the number of elements in the sample space without having to list them.
- In general, the number of combinations of r objects from n possible objects is

$$\binom{n}{r} = \frac{^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- A commonly used alternative notation for $\binom{n}{r}$ is ^nC_r.
- For n objects, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

 ISBN 978-1-009-53719-3 Cambridge University Press

Skill-sheet

Exercise A3

Example 11

1 For each of the following examples, determine the number of selections possible by listing the possibilities:

a An ice-cream with two different scoops is selected from a choice of vanilla, chocolate and strawberry.

b Two students from the group of Jaime, Georgia and Wey are chosen to represent the class in debating.

c Two students from the group of Thomas, William, Jackson and Phillip are chosen for the tennis team.

d Three scarves are chosen from a blue scarf, a green scarf, a red scarf and a white scarf.

Example 12

2 Evaluate:

a $\binom{5}{3}$ **b** $\binom{5}{2}$ **c** $\binom{7}{4}$ **d** $\binom{7}{3}$

Compare your answers for parts a and b, and for parts c and d.

3 Evaluate:

a $\binom{20}{18}$ **b** $\binom{100}{99}$ **c** $\binom{100}{2}$ **d** $\binom{250}{248}$

4 Evaluate:

a $\binom{6}{3}$ **b** $\binom{7}{1}$ **c** $\binom{8}{2}$ **d** $\binom{50}{48}$

5 How many netball teams of seven can be chosen from 13 players?

6 An ice-cream parlour has 25 different flavours of ice-cream available. How many different three-scoop ice-cream sundaes are available if three different flavours are to be used and the order of the scoops does not matter?

7 How many different hands of seven cards can be dealt from a normal pack of 52 cards?

8 In Tattslotto six numbers are selected from 45. How many different possible selections are there? (Do not attempt to consider supplementary numbers.)

SF

Example 13

9 A student has the choice of three mathematics subjects and four science subjects. In how many ways can they choose to study one mathematics and two science subjects?

10 A survey is to be conducted, and eight people are to be chosen from a group of 30.

a In how many different ways could the eight be chosen?

b If the group contains 10 men and 20 women, how many groups containing exactly two men are possible?

11 From a standard 52-card deck, how many 7-card hands have 5 spades and 2 hearts?

12 In how many ways can a committee of five be selected from eight women and four men:

a without restriction **b** if there must be exactly three women on the committee?

CF

ISBN 978-1-009-53719-3

CF

13 Six females and five males are interviewed for five positions. If all are acceptable for any position, in how many ways could the following combinations be selected?

a three males and two females
b four females and one male
c five females
d five people regardless of sex
e at least four females

14 The selectors of a sporting team need to choose 10 athletes from the 15 track and 12 field athletes who have qualified to compete.

a How many groups are possible?
b How many groups would contain track athletes only?
c How many groups would contain field athletes only?
d How many groups would contain half track and half field athletes?

15 A student representative committee of five is to be chosen from four male and six female students. How many committees could be selected which contain more female than male students?

Example 14
16 Joanne is offered a selection of five different sweets. She can choose to pass or to select any number of them. In total how many choices does she have?

17 Eight people have auditioned for roles in a play. The director can choose none, or any number of them for the production. In how many ways can selections be made from these eight people, any number at a time?

18 How many colours can be obtained by mixing a set volume of red, blue, green, orange and white paints if any number of paints can be used at a time?

19 How many varieties of fruit salad, using at least two fruits, can be obtained from apples, oranges, pears, passionfruit, kiwifruit and nectarines, taken any number at a time?

20 In how many ways can a group of six people be divided into:

a two equal groups
b two unequal groups, if there must be at least one person in each group?

 ISBN 978-1-009-53719-3

A4 Applications to probability

Learning intentions

- To be able to apply counting methods to probability.

As discussed in Chapter 9, for a sample space with equally likely outcomes, the probability of an event occurring is determined by dividing the number of outcomes in the event by the total number of possible outcomes. Establishing the number of outcomes in the event and the total number of outcomes can be achieved by using permutations and combinations.

Example 15

Four-letter 'words' are to be made by arranging letters of the word SPECIAL. What is the probability that the 'word' will start with a vowel?

Solution

Since we are interested in words starting with a vowel, order is important.

There are 7 letters to be arranged in groups of 4

Total number of possible arrangements $= 7 \times 6 \times 5 \times 4 = 840$.

Now consider words which start with a vowel.

Since there are three vowels, we have 3 choices for the first letter. Having done this, we have six letters remaining, any of which can be placed in the three remaining positions, and this can be done in $6 \times 5 \times 4 = 120$ ways.

Thus the number of arrangements which start with a vowel $3 \times 6 \times 5 \times 4 = 360$

Hence, the probability of the word starting with a vowel is

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}} = \frac{360}{840} = \frac{3}{7}$$

Example 16

Three students are to be chosen to represent the class in a debate. If the class consists of six boys and four girls, what is the probability that the team will contain:

a exactly one girl **b** at least two girls?

Solution

Since there are 10 students, the number of possible teams is $\binom{10}{3} = 120$.

a One girl can be chosen for the team in $\binom{4}{1} = 4$ different ways. Having placed one girl, the other two places must be filled by boys, and this can be done in $\binom{6}{2} = 15$ different ways. Thus the total number of teams containing one girl and two boys is $4 \times 15 = 60$, and the probability that the team contains exactly one girl is $\dfrac{60}{120} = \dfrac{1}{2}$.

b If the team is to contain at least two girls, then it may contain two *or* three girls.

The number of teams containing exactly two girls is $\binom{6}{1} \times \binom{4}{2} = 36$.

The number of teams containing exactly three girls is $\binom{6}{0} \times \binom{4}{3} = 4$.

Thus the total number of teams containing at least two girls is 40, and the probability of this is $\dfrac{40}{120} = \dfrac{1}{3}$.

 ISBN 978-1-009-53719-3

Summary A4

Using our knowledge of arrangements and selections, we can calculate probabilities for sample spaces with equally like outcomes:

- First determine the total number of possible outcomes.
- Then determine the number of outcomes in the event of interest.
- The required probability is equal to

$$\frac{\text{number of outcomes in the event}}{\text{total number of outcomes}}$$

Skill-sheet

Exercise A4

SF

Example 15

1 Three letter 'words' are to be made by arranging the letters of the word COMPUTER. What is the probability that the 'word' will start with a vowel?

2 Three letter 'words' are to be made by arranging letters of the word SMILE. What is the probability that the 'word' will start with a vowel?

3 A three-digit number is to be formed from the set of digits $\{1, 2, 3, 4, 5\}$. If each digit can be used only once, determine the probability that the number:

a is even **b** is odd.

4 A four-digit number is to be formed from the set of digits $\{1, 2, 3, 4, 5, 6\}$. If each digit can be used only once, determine the probability that the number:

a is even **b** is odd.

Example 16

5 Three lollies are chosen at random from a box containing four jelly beans and five peppermints. What is the probability that:

a exactly one jelly bean is chosen

b at least two jelly beans are chosen?

6 Arlo selects two apples from a box containing 3 green apples and 5 red apples. What is the probability that he selects one green apple and one red apple?

7 A committee of 4 people is to be chosen at random from the members of a school council which consists of 5 students and 8 teachers. Determine the probability that the committee will contain:

a no teachers **b** at least 2 students.

8 Five pieces of fruit are chosen at random from a box containing ten apples and 8 bananas. What is the probability that:

a at least one banana is chosen **b** at least one apple is chosen?

9 A team of six is to be chosen from a group of eight boys and five girls. What is the probability that the team consists of three boys and three girls?

 ISBN 978-1-009-53719-3

SF

10 A committee of six is to be chosen from a group of eight primary students and nine secondary students. What is the probability that the committee consists of:

a three primary students and three secondary students

b at least two secondary students?

CF

11 Three letters are chosen at random from the word HEART and arranged in a row. Determine the probability that:

a the letter H is first **b** the letter H is chosen **c** both vowels are chosen.

12 Three Year 10 and three Year 11 students are to be randomly seated in a row. Determine the probability that both the end places will be filled by Year 11 students.

13 A netball team of seven players is to be chosen from six men and seven women. Determine the probability that the selected team contains more men than women.

14 Bill is making a sandwich. He may choose any combination of the following: lettuce, tomato, carrot, cheese, cucumber, beetroot, onion, ham and must choose at least one. Determine the probability that:

a the sandwich contains ham

b the sandwich contains three ingredients

c the sandwich contains at least three ingredients.

15 A bag contains five white, six red and seven blue balls. If three balls are selected at random, without replacement, determine the probability they are:

a all red **b** all different colours.

16 Susie chooses four pieces of bubble gum from a jar containing five red, two blue and three green pieces of bubble gum. Calculate the probability that Susie selects:

a no green bubble gum

b at least one green bubble gum and at least one red bubble gum

c at least one red bubble gum, given that there is at least one green bubble gum.

17 A four-digit number (with no repetitions) is to be formed from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. Determine the probability that the number:

a is even **b** is less than 4000

c is less than 4000, given that it is greater than 3000.

18 An exam paper has two sections. Section A has 9 questions of which a student must answer 6 questions and section B has 6 questions of which a student must answer 4 questions. A student does not read the instructions properly and randomly chooses 10 questions in the second paper to answer. Determine the probability that they picked 6 questions from section A and 4 questions from section B.

19 A committee of four is to be chosen from a group of five men and six women. The selection rules state that the committee must have at least one member from each gender on the committee. Determine the probability that the committee contains:

a exactly one woman **b** exactly two men.

 ISBN 978-1-009-53719-3

Review

Chapter summary

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ and $0! = 1$, where the notation $n!$ is read as 'n factorial'.
- An **arrangement** (or **permutation**) is a grouping of objects, such that the order of the objects is important. That is, the arrangements ABC and CBA are different.
- The number of ways of arranging n objects in a row is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

- The number of ways of arranging r objects from n possible objects is

$$^nP_r = \frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

- A **selection** (or **combination**) is a grouping of objects, such that the order of the objects is not important. That is, the selections ABC and CBA are the same.
- The number of combinations of r objects from n possible objects is

$$\binom{n}{r} = \frac{^nP_r}{r!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)}{r!} = \frac{n!}{r!\,(n-r)!}$$

- A commonly used alternative notation for $\binom{n}{r}$ is ^nC_r.
- For n objects, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.
- Permutations and combinations may be used when determining probabilities. The probability is determined by dividing the number of outcomes in an event by the total number of outcomes.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

A1 **1** I can determine a number of choices using the addition and multiplication rules. ☐

See Example 3, Example 4, Example 5 and Question 2

A2 **2** I can determine the number of arrangements of n objects. ☐

See Example 6, Example 7 and Question 3

A2 **3** I can determine the number of arrangements of n objects in groups of size r. ☐

See Example 8, Example 9 and Question 7

A2 **4** I can determine the number of arrangements when there are restrictions. ☐

See Example 10 and Question 13

 ISBN 978-1-009-53719-3

A3 **5** I can understand the notion of a combination as an unordered set of r objects taken from a set of n distinct objects ☐

See Example 11, Example 12 and Questions 2 and 6

A4 **6** I can use my knowledge of combinations to determine probability when outcomes are equally likely. ☐

See Example 15, Example 16 and Question 1

Short-response questions

Technology-free short-response questions

SF

1 Evaluate:

a $\binom{6}{1}$ **b** $\binom{6}{2}$ **c** $\binom{6}{4}$ **d** $\binom{6}{0}$

2 Evaluate:

a $\binom{4}{2}$ **b** $\binom{5}{2}$ **c** $\binom{10}{2}$ **d** $\binom{7}{2}$

3 Evaluate:

a $\binom{1000}{998}$ **b** $\binom{1\,000\,000}{999\,999}$ **c** $\binom{1\,000\,000}{1}$

4 How many different three-digit house numbers can be constructed from six brass numerals 1, 2, 3, 4, 5, 6?

5 A committee of two students is to be selected from a class of 20 students. How many different committees are possible?

6 A group of three parents is to be selected from the nine parents who volunteered for a student excursion. How many different groups are possible?

7 A bushwalking club has 80 members: 50 men and 30 women. A committee consisting of two men and one woman is to be selected. How many different committees are possible?

CF

8 A supermarket sells n different brands of dishwashing liquid. Each brand offers four different sized bottles (small, medium, large and economy), and each is available as either lemon-scented or pine-scented. How many different types of dishwashing liquid bottles are available at this supermarket?

9 Determine the value of n if $\binom{n}{2} = 36$.

10 Determine the value of n if $\binom{n+1}{n-1} = 45$.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Review

CF

11 There are five vowels and 21 consonants in the English alphabet. How many different four-letter 'words' can be formed that contain two different vowels and two different consonants?

12 A pizza restaurant offers the following toppings: onions, green peppers, mushrooms, anchovies and pepperoni.

a How many different kinds of pizza with three different toppings can be ordered?

b How many different kinds with any number of toppings (between none and all five) can be ordered?

Technology-active short-response questions

SF

13 A committee of three students is to be selected from 7 Year 11 students and 9 Year 12 students. What is the probability that there are two Year 11 students on the committee?

14 There are 12 goldfish and 8 black fish in a pond. Taj dips his net into the pond and catches four fish. What is the probability that two or three of them are black?

15 There are 48 jelly beans in a jar, 25% of which are red and the rest yellow. Scarlett selects 6 jelly beans from the jar. What is the probability that she selects three red and three yellow jelly beans?

CF

16 There are 10 chairs in a row.

a In how many ways can three people be seated?

b In how many of these will the two end chairs be occupied?

c What is the probability that the two end chairs will be empty?

17 Three letters are chosen at random from the word OLYMPICS and arranged in a row. What is the probability that:

a the letter O is first

b the letter Y is chosen

c both vowels are chosen?

18 All possible three-digit numbers are formed from the odd digits $\{1, 3, 5, 7, 9\}$.

a How many such numbers are possible if each digit is used only once?

b How many of the numbers from part a are larger than 350?

19 Min Woo has five flat batteries and ten charged batteries. Unfortunately his little brother mixes them up, and he can't tell them apart. He selects four batteries at random for his calculator.

a How many different combinations of the 15 batteries could he select?

b What is the probability that all four batteries are charged?

c What is the probability that at least one battery is flat?

 ISBN 978-1-009-53719-3 Cambridge University Press

Review

CF

20 There are seven mints and 11 jubes in the lolly jar. Steve puts his hand in the jar and selects four lollies at random.

a How many different combinations of the lollies are there?

b In how many of these are there no mints?

c What is the probability he selects two mints and two jubes?

CU

21 Suppose that eight students, consisting of g girls and b boys, have nominated for a committee of three. If the committee is selected at random from this group:

a Determine an expression for the probability that there is exactly one female on the committee in terms of g.

b If the probability of the committee including exactly one female is $\frac{15}{28}$, use this expression to determine the possible values of g.

22 In a certain school n students are eligible for the school debating team, which consists of three speakers. Of this group of n there are 6 Year 12 students and the rest are Year 11 students. If the team is selected at random from the group:

a Determine an expression the probability that there will be exactly one Year 11 student on the team in terms of n.

b Use this expression to determine the value of n, if the probability that there will be exactly one Year 11 student on the team is greater than 0.5.

23 A restaurant offers three courses on their menu: entree, main course and dessert. Suppose that all three courses must be chosen, and that there are m choices for the entree, $\frac{m}{2}$ choices for the main course, and 25-m choices for the dessert. If the total number of choices is more than 1000, what are the possible values of m?

24 If the number of ways of seating three people on n chairs is between 1000 and 1500, what are the possible values for the number of chairs?

25 Suppose a committee of four is to be chosen from a group of n men and m women.

a Determine an expression for the number of ways choosing the committee.

b Determine expression for the number of ways of choosing the committee, if there should be exactly two men and two women on the committee.

c If there are twice as many men as women to choose from, and the number of ways of choosing a committee with two men and two women is less than 720, what are the possible values for the number of men?

Review

Multiple-choice questions

Technology-free multiple-choice questions

1 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** $\dfrac{10!}{2!}$ **C** $\dfrac{10!}{2!\,8!}$ **D** $\dbinom{10}{1}$

2 $\dbinom{21}{3}$ is equal to

A $21!$ **B** $\dfrac{21!}{3!}$ **C** $\dfrac{21!}{18!\,3!}$ **D** $\dfrac{21!}{18!}$

3 $\dbinom{n}{2}$ is equal to

A $\dfrac{n!}{2!}$ **B** $\dfrac{n(n-1)}{2}$ **C** $\dfrac{n}{2}$ **D** $\dfrac{n-2}{2}$

4 $\dbinom{n-1}{n-3}$ is equal to

A $\dfrac{(n-1)!}{(n-3)!}$ **B** $\dfrac{n-1}{n-3}$ **C** $\dfrac{(n-1)(n-2)}{6}$ **D** $\dfrac{(n-1)(n-2)}{2}$

5 In how many ways can a hand of six cards be dealt from a pack of 52 cards?

A $6!$ **B** $\dbinom{52}{6}$ **C** $\dbinom{46}{6}$ **D** $\dfrac{52!}{6!}$

6 The number of ways of arranging a group of three pictures from a possible 12 is equal to:

A $3!$ **B** $\dbinom{12}{9}$ **C** $\dfrac{12!}{3!}$ **D** $\dfrac{12!}{9!}$

7 A class consists of 10 girls and 14 boys. In how many ways could a committee of two girls and two boys be chosen?

A $\dbinom{10}{2}\times\dbinom{14}{2}$ **B** $\dbinom{24}{4}$ **C** $10!\times 14!$ **D** $\dfrac{10!\,14!}{4!}$

8 A class consists of 8 girls and 6 boys. A committee of three students is chosen. The probability that it contains two girls can be determined by evaluating:

A $\dfrac{\binom{8}{2}\binom{6}{1}}{\binom{14}{3}}$ **B** $\dfrac{\binom{8}{2}}{\binom{14}{3}}$ **C** $\dfrac{\binom{14}{3}}{\binom{8}{2}\binom{6}{1}}$ **D** $\dfrac{8!\times 6!}{14!}$

9 There are 56 year 11 students and 37 Year 12 students at a school. Five students are chosen to attend a conference. What is the probability that exactly two year 11 students are chosen?

A $\dfrac{\binom{56}{11}\binom{37}{12}}{\binom{93}{5}}$ **B** $\dfrac{2}{56}$ **C** $\dfrac{\binom{56}{2}}{\binom{93}{5}}$ **D** $\dfrac{\binom{56}{2}\binom{37}{3}}{\binom{93}{5}}$

10 Three letters are chosen at random from the word SMART and arranged in a row. What is the probability that the words starts and ends with either S or T?

A $\dfrac{2}{5}$ **B** $\dfrac{1}{10}$ **C** $\dfrac{2}{25}$ **D** $\dfrac{2}{10}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Technology-active multiple-choice questions

11 For his holiday reading Matt has selected eight detective novels, three biographies and four science fiction books, but he only has room in his case for three books. If he selects one book from each group, how many combinations of books are possible?

A 15 **B** 28 **C** 56 **D** 96

12 Georgia is choosing six subjects for Year 12. She has already chosen four subjects. For the remaining two she will choose one of three mathematics subjects, and either one of five languages or one of three science subjects. How many different subject combinations are possible?

A 11 **B** 15 **C** 24 **D** 45

13 How many different licence plates can be made using two letters followed by four digits, if neither the letters nor the digits can be repeated?

A 720 **B** 5690 **C** 3 276 000 **D** 6 760 000

14 How many different teams of 7 can be selected from a squad of 12 players?

A 792 **B** 84 **C** 5040 **D** 120

15 Three-letter ‘words’ are to be made by arranging the letters of the word METHODS. What is the probability that the word consists of a vowel followed by two consonants?

A $\frac{1}{105}$ **B** $\frac{4}{21}$ **C** $\frac{5}{26}$ **D** $\frac{2}{7}$

16 What is the probability that a team of four chosen at random from a group of eight friends, four males and four females, would consist of three women and one man?

A $\frac{1}{2}$ **B** $\frac{3}{4}$ **C** $\frac{3}{8}$ **D** $\frac{8}{35}$

17 A committee of three students is to be selected from 7 Year 11 students and 9 Year 12 students. What is the probability that there are at least two Year 11 students on the committee?

A $\frac{2}{5}$ **B** $\frac{27}{80}$ **C** $\frac{3}{16}$ **D** $\frac{3}{20}$

18 There are 14 goldfish and 9 black fish in a pond. Maisie dips her net into the pond and catches four fish. What is the probability that no more than one of them is black?

A $\frac{611}{1265}$ **B** $\frac{13}{115}$ **C** $\frac{468}{1265}$ **D** $\frac{182}{529}$

Appendix

B

Discrete probability distributions

Chapter contents

In Appendix A we introduced some strategies for helping us to determine the number of outcomes in an event. This allows us to determine probabilities without listing all the possible outcomes. In this chapter we extend our ability to calculate probabilities for complex events even further. We do this by introducing useful rules for calculating probabilities in some very commonly encountered situations.

We will again concentrate on finite sample spaces. Therefore we start with a sample space ε and the outcomes are the elements of this sample space. We can assign a probability to each of these outcomes from the interval [0, 1], and the sum of the probabilities of these outcomes is 1. An event is a subset of the sample space, and the probability of an event is equal to the sum of the probabilities of the outcomes in that event.

We will see that a distribution for a random experiment is a description of the probabilities of the outcomes. This is sometimes done by a table and sometimes by a rule.

B1 Discrete random variables

Learning intentions

- To understand the concept of a **discrete random variable**.
- To be able to determine a **probability distribution** for a discrete random variable.

An important next step in our study of probability is to define a random variable and assign a numerical value to each outcome in the sample space.

Random variables

Consider the sample space obtained when a coin is tossed three times:

$$\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Suppose we are particularly interested in the number of heads associated with each outcome. We let X represent the number of heads observed when a coin is tossed three times. Then each outcome in the sample space can be associated with a value of X, as shown in the table.

Outcome	Number of heads
HHH	$X = 3$
HHT	$X = 2$
HTH	$X = 2$
THH	$X = 2$
HTT	$X = 1$
THT	$X = 1$
TTH	$X = 1$
TTT	$X = 0$

From the table we can see that the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we call X a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one which can take only a countable number of distinct values, such as 0, 1, 2, 3, 4. Discrete random variables are usually (but not necessarily) generated by counting. The number of children in a family, the number of brown eggs in a carton of a dozen eggs, and the number of times we roll a die before we observe a 'six' are all examples of discrete random variables.
- A **continuous random variable** is one which can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

In this chapter we are interested in understanding more about discrete random variables.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

Discrete probability distributions

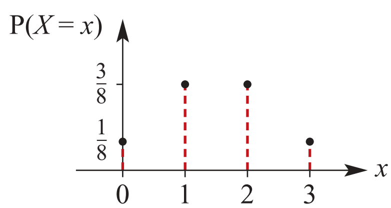
Because the values of a random variable are associated with outcomes in the sample space, we can determine the probability of each value of the random variable occurring.

Let's look again at the results obtained when a coin is tossed three times. Assuming that the coin is fair, we can add probabilities to the previous table.

Outcome	Number of heads	Probability	
HHH	$X = 3$	$\frac{1}{8}$	$P(X = 3) = \dfrac{1}{8}$
HHT	$X = 2$	$\frac{1}{8}$	
HTH	$X = 2$	$\frac{1}{8}$	$P(X = 2) = \dfrac{3}{8}$
THH	$X = 2$	$\frac{1}{8}$	
HTT	$X = 1$	$\frac{1}{8}$	
THT	$X = 1$	$\frac{1}{8}$	$P(X = 1) = \dfrac{3}{8}$
TTH	$X = 1$	$\frac{1}{8}$	
TTT	$X = 0$	$\frac{1}{8}$	$P(X = 0) = \dfrac{1}{8}$

The list of all possible values of the random variable X, together with the probability associated with each value, is known as the probability distribution of X. More usually, we would summarise the probability distribution associated with the number of heads observed when a fair coin is tossed three times in a table or a graph as follows.

x	0	1	2	3
$P(X = x)$	$\dfrac{1}{8}$	$\dfrac{3}{8}$	$\dfrac{3}{8}$	$\dfrac{1}{8}$



Note that, since every possible value of the random variable is included, the probabilities must add to 1.

The **probability distribution** of a discrete random variable X is a function

$$p(x) = P(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

For any discrete probability distribution, the following two conditions must hold:

1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \le p(x) \le 1$ for all x.

2 The sum of all the values of $p(x)$ is 1.

Example 1

Consider the function:

x	1	2	3	4	5
$P(X = x)$	$2c$	$3c$	$4c$	$5c$	$6c$

For what value of c is this a probability distribution?

Solution

$$2c + 3c + 4c + 5c + 6c = 1$$
$$20c = 1$$
$$c = \frac{1}{20}$$

Explanation

For a probability distribution, we require that the probabilities are such that $0 \le p \le 1$, and that they add to 1.

Example 2

Consider the probability distribution:

x	1	2	3	4	5	6
$P(X = x)$	0.2	0.3	0.1	0.2	0.15	0.05

Use the table to calculate:

a $P(X = 3)$
b $P(X < 3)$
c $P(X \ge 4)$
d $P(3 \le X \le 5)$
e $P(X \ne 5)$

Solution

a $P(X = 3) = 0.1$

b $P(X < 3) = 0.2 + 0.3$
$= 0.5$

c $P(X \ge 4) = 0.2 + 0.15 + 0.05$
$= 0.4$

d $P(3 \le X \le 5) = 0.1 + 0.2 + 0.15$
$= 0.45$

e $P(X \ne 5) = 1 - P(X = 5)$
$= 1 - 0.15$
$= 0.85$

Explanation

b If X is less than 3, then from the table we see that X can take the value 1 or 2.

c If X is greater than or equal to 4, then X can take the value 4, 5 or 6.

d Here X can take the value 3, 4 or 5.

e The sum of all the probabilities must be 1.

 ISBN 978-1-009-53719-3

Example 3

The table shows a probability distribution for a random variable X.

x	1	2	3	4	5	6
$P(X = x)$	0.2	0.2	0.07	0.17	0.13	0.23

Calculate the following probabilities:

a $P(X > 4)$ **b** $P(2 < X < 5)$ **c** $P(X \geq 5 \mid X \geq 3)$

Solution

a $P(X > 4) = P(X = 5) + P(X = 6)$
$= 0.13 + 0.23 = 0.36$

b $P(2 < X < 5) = P(X = 3) + P(X = 4)$
$= 0.07 + 0.17 = 0.24$

c
$$\begin{aligned} P(X \geq 5 \mid X \geq 3) &= \frac{P(X \geq 5 \text{ and } X \geq 3)}{P(X \geq 3)} \\ &= \frac{P(X \geq 5)}{P(X \geq 3)} \qquad \text{(since } X \geq 5 \text{ implies } X \geq 3\text{)} \\ &= \frac{P(X = 5) + P(X = 6)}{P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)} \\ &= \frac{0.13 + 0.23}{0.07 + 0.17 + 0.13 + 0.23} \\ &= \frac{0.36}{0.6} = \frac{3}{5} \end{aligned}$$

Example 4

The following distribution table gives the probabilities for the number of people on a carnival ride at a particular time of day.

Number of people (t)	0	1	2	3	4	5
$P(T = t)$	0.05	0.2	0.3	0.2	0.1	0.15

Determine:

a $P(T > 4)$ **b** $P(1 < T < 5)$ **c** $P(T < 3 \mid T < 4)$

Solution

a $P(T > 4) = P(T = 5) = 0.15$

b $P(1 < T < 5) = P(T = 2) + P(T = 3) + P(T = 4) = 0.6$

c
$$\begin{aligned} P(T < 3 \mid T < 4) &= \frac{P(T < 3 \text{ and } T < 4)}{P(T < 4)} \\ &= \frac{P(T < 3)}{P(T < 4)} = \frac{0.55}{0.75} = \frac{11}{15} \end{aligned}$$

 ISBN 978-1-009-53719-3

Summary B1

- The **probability distribution** of a discrete random variable X is a function

  $$p(x) = P(X = x)$$

  that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For any discrete probability distribution, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \le p(x) \le 1$ for all x.
  2. The sum of all the values of $p(x)$ is 1.

Skill-sheet

Exercise B1

SF

Example 1

1 A random variable X can take the values $x = 1, 2, 3, 4$. Indicate whether or not each of the following is a probability distribution for such a variable, and if not, give reasons:

a $p(1) = 0.05$ $p(2) = 0.35$ $p(3) = 0.55$ $p(4) = 0.15$

b $p(1) = 0.125$ $p(2) = 0.5$ $p(3) = 0.25$ $p(4) = 0.0625$

c $p(1) = 13\%$ $p(2) = 69\%$ $p(3) = 1\%$ $p(4) = 17\%$

d $p(1) = 51$ $p(2) = 12$ $p(3) = 34$ $p(4) = 3$

e $p(1) = 0.66$ $p(2) = 0.32$ $p(3) = -0.19$ $p(4) = 0.2$

2 For each of the following, write a probability statement in terms of the discrete random variable X showing the probability that:

a X is equal to 2
b X is greater than 2
c X is at least 2
d X is less than 2
e X is 2 or more
f X is less than or equal to 2
g X is greater than 2 and less than 5

3 A random variable X can take the values 0, 1, 2, 3, 4, 5. List the set of values that X can take for each of the following probability statements:

a $P(X = 2)$
b $P(X > 2)$
c $P(X \ge 2)$
d $P(X < 2)$
e $P(X \le 2)$
f $P(2 \le X \le 5)$
g $P(2 < X \le 5)$
h $P(2 \le X < 5)$
i $P(2 < X < 5)$

Example 2

4 Consider the probability distribution:

x	0	1	2	3	4	5
$P(X = x)$	0.3	0.2	0.2	0.1	0.05	0.15

Use the table to determine:

a $P(X = 2)$ **b** $P(X < 2)$ **c** $P(X \ge 3)$ **d** $P(2 \le X \le 4)$ **e** $P(X \ne 3)$

 ISBN 978-1-009-53719-3

SF

5 Consider the following function:

x	1	2	3	4	5
$P(X = x)$	k	$2k$	$3k$	$4k$	$5k$

a For what value of k is this a probability distribution?

b Determine $P(2 \leq X \leq 4)$.

6 Let R be the number of people who miss their airline flight. The probability distribution of R is shown below. Determine:

r	0	1	2	3	4	5	6	7
$p(r)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	0.01

a $P(R > 4)$ **b** $P(R \geq 2)$.

7 Suppose Y is a random variable with the distribution given in the table.

y	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$P(Y = y)$	0.08	0.13	0.09	0.19	0.20	0.03	0.10	0.18

Determine:

a $P(Y \leq 0.5)$ **b** $P(Y > 0.5)$ **c** $P(0.3 \leq Y \leq 0.8)$.

Example 3

8 The table shows a probability distribution for a random variable X.

x	1	2	3	4	5	6
$P(X = x)$	0.1	0.13	0.17	0.27	0.20	0.13

Determine the following probabilities:

a $P(X > 3)$ **b** $P(3 < X < 6)$ **c** $P(X \geq 4 \mid X \geq 2)$

CF

9 When a pair of dice is rolled, there are 36 equally likely outcomes. Let Y denote the sum of the dice.

a Determine $P(Y = 7)$.

b Determine the probability distribution of the random variable Y.

10 When a pair of dice is rolled, there are 36 equally likely outcomes. Let X denote the larger of the values showing on the dice. If both the values are the same, then X denotes the common value. Determine:

a $P(X = 4)$. **b** The probability distribution of X.

11 Suppose that a fair coin is tossed three times. You lose \$3 if three heads appear and lose \$2 if two heads appear. You win \$1 if one head appears and win \$3 if no heads appear. Let \$$Y$ be the amount you win or lose. Determine:

a The probability distribution of Y. **b** $P(Y \leq 1)$.

 ISBN 978-1-009-53719-3

B2 The binomial distribution

Learning intentions

- To use a **Bernoulli random variable** as a model for two-outcome situations.
- To understand the concepts of **Bernoulli trials** and the concept of a **binomial random variable** as the number of 'successes' in independent Bernoulli trials, with the same probability of success in each trial.
- To determine and use the probabilities associated with the binomial distribution with parameters n and p.

Bernoulli trials

Suppose we are interested in a random experiment which has only two possible outcomes. For example, when tossing a coin, the random experiment might be 'head' or 'tail'. When tossing a dice the random experiment might be a 'six' or 'not a six'. Suppose further that the probability of observing a particular outcome is the same each time we undertake the random experiment. Such random experiments are called **Bernoulli trials**.

A **Bernoulli trial** is a random experiment with the following properties:

- The outcomes take one of two values, which are usually designated either a success S, or a failure F.
- The probability of success each time the random experiment is conducted is a constant p, and thus the probability of failure is $(1 - p)$.

Bernoulli random variables

The outcome from a Bernoulli trial is represented by a **Bernoulli random variable**, which is a random variable that takes only the values 0 (indicating a failure) and 1 (indicating a success). Thus a Bernoulli random variable X has a probability distribution of the form

x	0	1
$P(X = x)$	$(1 - p)$	p

Example 5

Suppose that a hockey player has a probability of 0.3 of scoring a goal from a penalty shot each time that they take the shot. If the random variable X represents the outcome from a shot, write down the probability distribution of X.

Solution

x	0	1
$P(X = x)$	0.7	0.3

Explanation

X will take the value 0 if goal is missed, and the value 1 if the goal is scored.

 ISBN 978-1-009-53719-3

The binomial distribution

Consider now a sequence of n Bernoulli trials. The total number of successes in a sequence of n Bernoulli trials is called a **binomial random variable** and is said to have a **binomial distribution**.

Suppose for example the hockey player from Example 5 takes three penalty shots, where each time the probability of scoring a goal is 0.3. The random variable X is defined as the total number of goals they score. The possible values that X can take are $\{0,1,2,3\}$. We can list the possible outcomes from the three shots, and determine the probability of each, as follows (G represents making the goal, M represents missing the goal):

Outcome	Score	Probability	
GGG	$X = 3$	$0.3 \times 0.3 \times 0.3$	$P(X = 3) = (0.3)^3 = 0.027$
GGM	$X = 2$	$0.3 \times 0.3 \times 0.7$	
GMG	$X = 2$	$0.3 \times 0.7 \times 0.3$	$P(X = 2) = 3 \times (0.3)^2 \times (0.7) = 0.189$
MGG	$X = 2$	$0.7 \times 0.3 \times 0.3$	
MMG	$X = 1$	$0.7 \times 0.7 \times 0.3$	
MGM	$X = 1$	$0.7 \times 0.3 \times 0.7$	$P(X = 1) = 3 \times (0.3) \times (0.7)^2 = 0.441$
GMM	$X = 1$	$0.3 \times 0.7 \times 0.7$	
MMM	$X = 0$	$0.7 \times 0.7 \times 0.7$	$P(X = 0) = (0.7)^3 = 0.343$

Thus the entire probability distribution can be defined by the following table.

x	0	1	2	3
$P(X = x)$	0.027	0.189	0.441	0.343

Listing all of the possible outcomes in order to determine the probability each time is very time consuming, and difficult when the number of Bernoulli trials is large. However, we can use our knowledge of selections from Appendix A to make listing all of the outcomes unnecessary.

Consider for example the probability that $X = 1$, that is, when exactly one goal is scored. We can see from the table that there are three ways this can occur. Since the goal could be scored on on the first, second or third attempt, we can consider this as selecting one object from a group of three, which can be done in $\binom{3}{1}$ ways.

Consider the probability that $X = 2$, that is, when exactly two goals are scored. Again from the table there are three ways this can occur. Since the two goals could occur on any two of the three attempts, we can consider this as selecting two objects from a group of three, which can be done in $\binom{3}{2}$ ways.

Thus we could write a rule of this probability distribution as follows:

$$P(X = x) = \binom{3}{x}(0.3)^x 0.7^{3-x} \qquad x = 0, 1, 2, 3$$

This is an example of the binomial probability distribution, which we can define thus:

 ISBN 978-1-009-53719-3

Consider a sequence of n identical trials such that:

- Each trial results in one of two outcomes, which are usually designated as either a **success**, S, or a **failure**, F.
- The probability of success on a single trial, p, is constant for all trials (and thus the probability of failure on a single trial is $(1 - p)$).
- The trials are independent (so that the outcome of any trial is not affected by the outcome of any previous trial).

If the random variable, X, is the number of successes in n such trials then X has a **binomial distribution** and

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\dbinom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

Note: n and p are referred to as **parameters** of the binomial distribution.

Example 6

Suppose that in a certain town the probability of rain falling on any one day is 0.4. Assuming that the occurrence of rain on any day is independent of whether or not rain falls on any other day, determine the probability that rain will fall on exactly three days in a week.

Solution

Let X represent the number of days on which it rains in a week. Since there are only two possible outcomes on each day (rain or no rain), and the probability of rain on any day is constant (0.4), regardless of previous outcomes, then X has a binomial distribution with parameters $n = 7$ and $p = 0.4$.

Thus $\quad P(X = x) = \binom{7}{x}(0.4)^x(0.6)^{7-x} \quad x = 0, 1, \dots, 7$

and so

$$\begin{aligned} P(X = 3) &= \binom{7}{3}(0.4)^3(0.6)^{7-3} \\ &= \frac{7!}{3!\,4!} \times 0.064 \times 0.1296 \\ &= 0.290304 \end{aligned}$$

 ISBN 978-1-009-53719-3

Example 7

For the situation described in Example 6, use a calculator to determine the probability that:

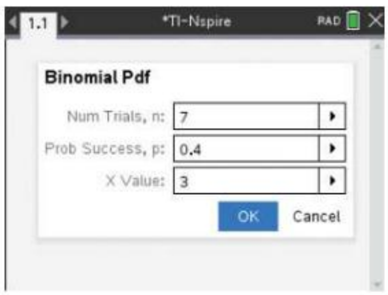
a rain will fall on exactly three days of a chosen week

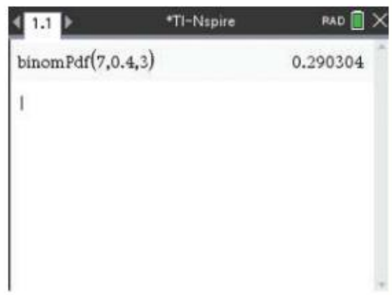
b rain will fall on no more than three days of a chosen week

c rain will fall on at least three days of a chosen week.

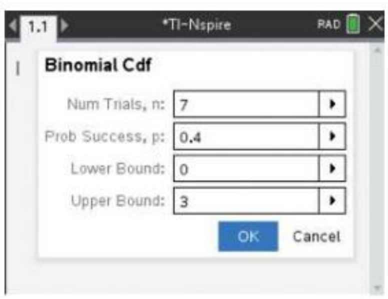
Using the TI-Nspire CX non-CAS

a As shown in Example 6, we want to determine $P(X = 3)$, where $n = 7$ and $p = 0.4$. Use menu > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown to determine that $P(X = 3) = 0.290304$.

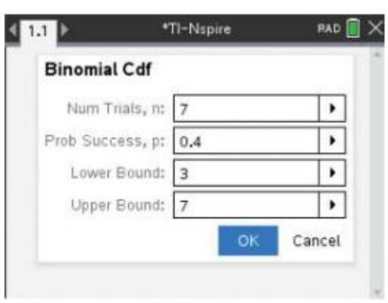


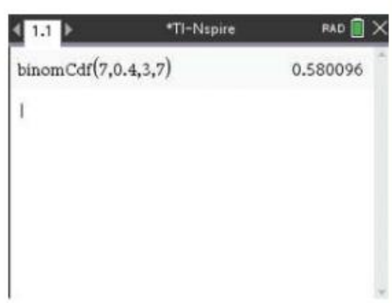


b Here $P(X \leq 3)$ is required. Use menu > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown to determine that $P(X \leq 3) = 0.710208$.



c Here $P(X \geq 3)$ is required. Use menu > **Probability** > **Distributions** > **Binomial Cdf** and complete as shown to determine that $P(X \geq 3) = 0.580096$.



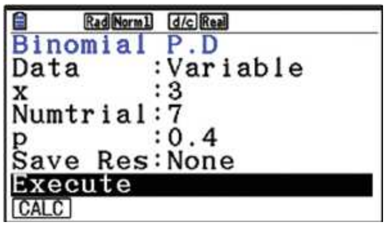


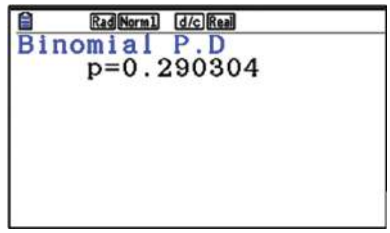
 ISBN 978-1-009-53719-3 Cambridge University Press

Using the Casio

a As shown in Example 6, we want to determine $P(X = 3)$, where $n = 7$ and $p = 0.4$.

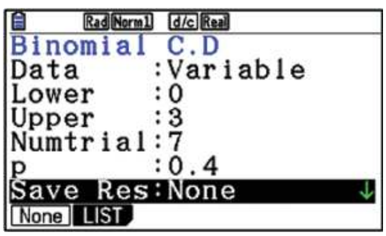
- In the **Statistics** mode, go to Menu 2.
- For the binomial probability distribution, select **Distributions** F5, **Binomial** F5, **Bpd** F1.
- Enter the values of $n = 3$, $p = 0.4$, and $x = 3$ as shown.
- Press EXE to determine that $P(X = 3) = 0.290304$.

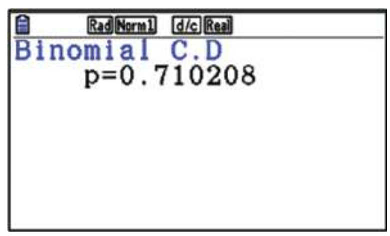




b Here $P(X \leq 3)$ is required.

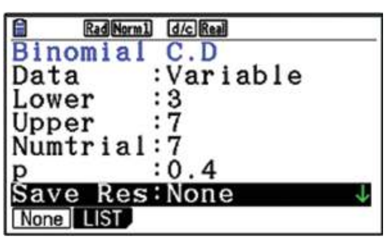
- For the binomial cumulative distribution, select **Distributions** F5, **Binomial** F5, **Bcd** F2.
- Enter the values of $n = 3$, $p = 0.4$, and Lower= 0, Upper= 3 as shown.
- Press EXE to determine that $P(X \leq 3) = 0.710208$.

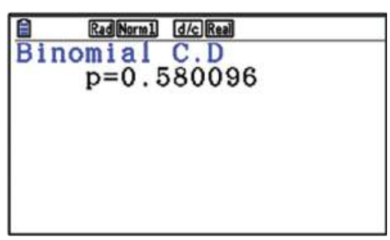




c Here $P(X \geq 3)$ is required.

- For the binomial cumulative distribution, select **Distributions** F5, **Binomial** F5, **Bcd** F2.
- Enter the values of $n = 3$, $p = 0.4$, and Lower= 3, Upper= 7 as shown.
- Press EXE to determine that $P(X \geq 3) = 0.580096$.





 ISBN 978-1-009-53719-3

Example 8

The probability of an archer obtaining a maximum score from a shot is 0.4. Determine the probability that out of five shots the archer obtains the maximum score:

a three times

b three times, given that she obtains the maximum score at least once.

Solution

Let X be the number of maximum scores from five shots. Then X has a binomial distribution with $n = 5$ and $p = 0.4$.

a
$$\begin{aligned} P(X = 3) &= \binom{5}{3}(0.4)^3(0.6)^2 \\ &= 10 \times 0.064 \times 0.36 \\ &= 0.2304 \quad \left(= \frac{144}{625}\right) \end{aligned}$$

b
$$\begin{aligned} P(X = 3 \mid X > 0) &= \frac{P(X = 3)}{P(X > 0)} \\ &= \frac{0.2304}{1 - P(X = 0)} \\ &= \frac{0.2304}{1 - 0.6^5} \\ &= 0.2498 \end{aligned}$$

correct to four decimal places.

Summary B2

A **Bernoulli trial** is a random experiment with the following properties:

- The outcomes take one of two values, which are usually designated either a success S, or a failure F.
- The probability of success each time the random experiment is conducted is a constant p, and thus the probability of failure is $(1 - p)$.

If the random variable X is the number of successes in a sequence of n Bernoulli trials, each with probability of success p, then X has a **binomial distribution** and probabilities can be determined using the formula

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

ISBN 978-1-009-53719-3

Exercise B2

Give probabilities correct to three decimal places unless otherwise specified.

Example 5

SF

1 Suppose that the probability of winning a game of chance is $\frac{1}{50}$, and this probability remains unchanged each time the game is played. Suppose further that a 'success' is defined as winning the game. The random variable X represents number of successes in a single play of the game.

a Explain why X can be considered a Bernoulli random variable.

b Write down the probability distribution of X.

2 The random variable X has the following binomial probability distribution:

$$P(X = x) = \binom{3}{x}(0.3)^x(0.7)^{3-x} \quad x = 0, 1, 2, 3$$

a Write down the values of the parameters n and p for this binomial distribution.

b Evaluate:

i $P(X = 1)$ **ii** $P(X = 2)$

3 The random variable X has the following binomial probability distribution:

$$P(X = x) = \binom{2}{x}\left(\tfrac{1}{3}\right)^x\left(\tfrac{2}{3}\right)^{2-x} \quad x = 0, 1, 2$$

a Write down the values of the parameters n and p for this binomial distribution.

b Evaluate:

i $P(X = 0)$ **ii** $P(X = 1)$

Example 6

4 Suppose that a fair coin is tossed six times. Let the random variable X represent the number of heads observed.

a Explain why X can be considered a binomial random variable.

b Write down a general rule for the probability distribution of X.

c Determine:

i $P(X = 3)$ **ii** $P(X = 0)$

5 Suppose that a ball is selected from a bag containing 3 red balls and 5 red balls. After the ball is selected and its colour is noted it is returned to the bag. Let X represent the number of red balls obtained if five balls are selected from the bag:

a Explain why X can be considered a binomial random variable.

b Write down a general rule for the probability distribution of X.

c Determine:

i $P(X = 0)$ **ii** $P(X = 2)$

Example 7

CF

6 Suppose that in a certain town the probability of rain falling on any one day is 0.3. Assuming independence, determine the probability that:

a rain will fall on exactly three days of a given week

b rain will fall on at least three days of a given week.

 ISBN 978-1-009-53719-3

CF

7 A fair dice is rolled 10 times and the number of times a six is observed is noted. Determine the probability of observing:

a exactly two 6s **b** fewer than two 6s **c** at least two 6s.

8 A five-sided dice is rolled seven times and the number of 2s that occur in the seven rolls is noted. Determine the probability of observing:

a exactly one 2. **b** at least one 2.

9 A breakfast cereal manufacturer places a coupon in every tenth packet of cereal entitling the buyer to a free packet of cereal. Over a period of two months a family purchases five packets of cereal.

a Write down the rule for the probability distribution for X, the number of coupons in the five packets.

b Determine $P(X = 1)$.

10 Assume there is a probability of 0.2 that a person will be left-handed. If there are seven players in a netball team, determine:

a the probability none of them will be left-handed

b the probability that one of them will be left-handed

c the probability that at least one of them will be left-handed.

11 An insurance company examines its records and notes that 30% of accident claims are made by drivers aged under 21. If there are 100 accident claims in the next 12 months, determine the probability that 40 or more of them are made by drivers aged under 21.

12 A restaurant is able to seat 80 customers inside, and many more at outside tables. Generally, 80% of their customers prefer to sit inside. If 100 customers arrive one day, determine the probability that the restaurant will seat inside all those who make this request.

13 A supermarket has four checkouts. A customer in a hurry decides to leave without making a purchase if all the checkouts are busy. At that time of day the probability of each checkout being free is $\frac{1}{4}$. Assuming that whether or not a checkout is busy is independent of any other checkout, determine the probability that the customer will make a purchase. (Give an exact answer.)

14 An aircraft has four engines. The probability that any one of them will fail on a flight is 0.003. Assuming the four engines operate independently, determine the probability that on a particular flight:

a no engine failure occurs

b not more than one engine failure occurs (correct to four decimal places)

c all four engines fail (correct to two significant figures).

 ISBN 978-1-009-53719-3

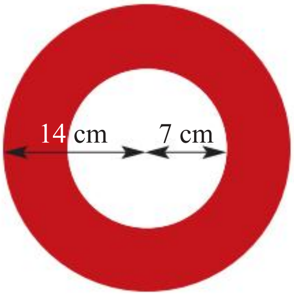
CF

15 An examination consists of 10 multiple-choice questions. Each question has four possible answers. Suppose a student guesses the answer to each question. What is the probability the student will make:

a at least three correct guesses
b at least four correct guesses
c at least five correct guesses?

16 An examination consists of 20 multiple-choice questions. Each question has four possible answers. At least 10 correct answers are required to pass the examination. Suppose a student guesses the answer to each question. What is the probability that the student passes?

17 A dart is thrown at random at a board that has the shape of a circle as shown.



a Calculate the probability that the dart will hit the shaded region.
b Mia throws three darts at the board. If the probability of hitting the shaded region remains the same for each throw, calculate the probability that she hits the shaded region:

i exactly once **ii** at least once.

Example 8

18 The probability of a darts player scoring a bullseye from one throw is 0.7. Determine the probability that out of five throws, the darts player scores a bullseye:

a exactly three times
b exactly three times, given that he scores a bullseye at least once.

19 An examination consists of 12 multiple-choice questions. Each question has four possible answers. At least 6 correct answers are required to pass the examination. Suppose a student guesses the answer to each question.

a What is the probability that the student passes?
b Given that the student passes, what is the probability that they had seven answers correct?
c Given that the student passes, what is the probability that they had seven or more answers correct?

CU

20 The probability of a darts player scoring a bullseye from one throw is p. Determine the maximum possible value of the probability that he scores exactly three bullseyes from five throws, and give the value of p for which this maximum occurs.

 ISBN 978-1-009-53719-3

Review

Chapter summary

- A **random variable** associates a number with each outcome of a random experiment.
- A **discrete** random variable is one which can take only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of a discrete random variable X is a function

$$p(x) = P(X = x)$$

that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For any discrete probability distribution, the following two conditions must hold:
  - Each value of $p(x)$ belongs to the interval $[0, 1]$. That is, $0 \leq p(x) \leq 1$ for all x.
  - The sum of all the values of $p(x)$ is 1.
- A **Bernoulli trial** is a random experiment with the following properties:
  - The outcomes take one of two values, which are usually designated either a success S, or a failure F.
  - The probability of success each time the random experiment is conducted is a constant p, and thus the probability of failure is $(1 - p)$.
- If the random variable X is the number of successes in n Bernoulli trials then X has a **binomial distribution** and

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, \dots, n$$

where $\binom{n}{x} = \dfrac{n!}{x!\,(n - x)!}$

Note: n and p are referred to as **parameters** of the binomial distribution.

Skills checklist

Check-list

Download this checklist from the Interactive Textbook, then print it and fill it out to check your skills.

B1	1	I can understand the definition of a probability distribution.	☐
		See Example 1 and Questions 1 and 2	
B1	2	I can interpret a probability distribution table and use it to determine probabilities.	☐
		See Example 2 and Question 5	
B1	3	I can use a probability distribution table to determine conditional probabilities.	☐
		See Example 3 and Question 9	

 ISBN 978-1-009-53719-3

B2 **4** I can determine the probability distribution for a Bernoulli random variable. ☐

See Example 5 and Question 1

B2 **5** I can evaluate probabilities using the binomial distribution formula. ☐

See Question 2

B2 **6** I can identify a binomial random variable and determine the relevant values of the parameters n and p. ☐

See Example 6 and Question 4

B2 **7** I can use a calculator to determine binomial probabilities. ☐

See Example 7 and Question 7

Review

Short-response questions

Technology-free short-response questions

SF

1 For the probability distribution

x	0	1	2	3	4
$P(X = x)$	0.12	0.25	0.43	0.12	0.08

calculate:

a $P(X \le 2)$ **b** $P(X \ge 3)$ **c** $P(1 \le X \le 3)$

2 A box contains 100 cards. Twenty-five cards are numbered 1, 28 are numbered 2, 30 are numbered 3 and 17 are numbered 4. One card will be drawn from the box and its number X observed. Give the probability distribution of X.

3 Two of the integers $\{1, 2, 3, 6, 7, 9\}$ are chosen at random. (An integer can be chosen twice.) Let X denote the sum of the two integers.

a List all choices and the corresponding values of X.

b What values can the random variable X take?

c Determine the probability distribution of X.

4 Let X represent the number of bullseyes scored by a player in a game of darts. The probability distribution of X is given by

$$P(X = x) = \binom{6}{x}(0.3)^x(0.7)^{6-x} \qquad x = 0, 1, \dots, 6$$

a What is the probability that this player will score a bullseye on any single throw?

b How many darts are thrown by the player in this game?

Review

CF

5 Twenty-five per cent of trees in a forest have severe leaf damage from air pollution. If three trees are selected at random, determine the probability that:

a two of the selected trees have severe leaf damage

b at least one has severe leaf damage.

6 In a large batch of eggs, one in three is found to be bad. What is the probability that of four eggs there will be:

a no bad egg

b exactly one bad egg

c more than one bad egg?

7 Previous experience indicates that, of the students entering a particular diploma course, $p\%$ will successfully complete it. One year, 15 students commence the course. Calculate, in terms of p, the probability that:

a all 15 students successfully complete the course

b only one student fails

c no more than two students fail.

CU

8 The probability of winning a particular game is $\frac{3}{5}$. (Assume all games are independent.)

a Determine the probability of winning at least one game when the game is played three times.

b Given that, when the game is played m times, the probability of winning exactly two games is three times the probability of winning exactly one game, determine the value of m.

9 For two events A and B, suppose that $P(A \mid B) = 0.6$, $P(A \mid B') = 0.1$ and $P(B) = 0.4$. The random variable X takes the value 4 if both A and B occur, 3 if only A occurs, 2 if only B occurs, and 1 if neither A nor B occur.

a Specify the probability distribution of X.

b Determine $P(X \geq 2)$.

Technology-active short-response questions

SF

10 A telephone salesperson has a probability of 0.05 of making a sale on any phone call.

a What is the probability that they will make one sale in the next 10 calls?

b What is the probability that they will make at least one sale in the next 20 calls?

11 The probability that a certain brand of light globe will last less than 1000 hours is 0.1.

a Determine the probability that at least one light globe in a box of 10 will last less than 100 hours.

b Determine the probability that at least two light globes in a box of 10 will last less than 100 hours.

SF

12 Each week, a security firm transports a large sum of money between two locations. The day on which the journey is made is varied at random and, in any given week, each of the five days from Monday to Friday is equally likely to be chosen. (In the following, give your answers correct to four decimal places.)
Calculate the probability in a 10-week period.

a Friday will be chosen exactly three times.

b Friday will be chosen exactly three times, given that it is chosen at least twice.

13 Jolanta is playing a game of chance. Her probability of winning a game is constant each time she plays the game. First suppose that Jolanta's probability of winning any one game is 0.6.

a What is the probability that she wins at least once in five plays of the game?

b What is the probability that she wins at least five times in 10 plays of the game, given that she wins at least four times?

CF

14 The number of times that Tom cooks his dinner at home in a randomly selected week is given by the random variable X, which can take values 0, 1, 2, 3, 4, 5, 6 and 7. The probability distribution for X is given in the table.

x	0	1	2	3	4	5	6	7
$P(X = x)$	0	k	0.1	0.2	0.2	0.3	0.1	0

a **i** Determine the value of k.

ii Determine the probability that Tom cooks his dinner at home more than three times in the week.

b Suppose the number of times that Tom cooks at home in a given week is independent of the number of times that he cooks at home in any other week.
Determine the probability (correct to four decimal places) that, in the 10-week period, there are four weeks where Tom cooks at home exactly five times in the week.

15 A test consists of 20 multiple-choice questions, where each question has five possible answers. At least 10 correct answers are required to pass the test. Suppose that a student guesses the answer to every question. Let X be the number of questions that the student gets correct.

a Determine the probability that the student passes the test.

b Given that the student has passed the test, what is the probability that they scored at least 80%?

CU

16 Hannah is playing a game of chance.

a Suppose that the probability of her winning at least once in every five games is 0.99968. What is probability of her winning in any one game?

b Suppose that the probability of her winning exactly once in five games is more than 0.3. What are the possible values for the probability of her winning one game?

Multiple-choice questions

Technology-free multiple-choice questions

1 Consider the following table, which represents the probability distribution of a random variable X.

x	0	1	2	3	4
$P(X = x)$	k	$2k$	$3k$	$2k$	k

For the table to represent a probability distribution, the value of k is

A $\frac{1}{10}$ **B** $\frac{1}{9}$ **C** $\frac{1}{5}$ **D** $\frac{1}{7}$

2 Suppose that the random variable X has the probability distribution given in the table:

x	1	2	3	4	5	6
$P(X = x)$	0.05	0.23	0.18	0.33	0.14	0.10

Then $P(X \geq 5)$ is equal to

A 0.24 **B** 0.10 **C** 0.90 **D** 0.14

3 Which one of the following random variables has a binomial distribution?

A the number of tails observed when a fair coin is tossed 10 times
B the number of times a player rolls a die before a 6 is observed
C the number of text messages a students sends in a day
D the number of accidents which occur per day at a busy intersection

4 Suppose that X is the number of male children born into a family of six children. If the distribution of X is binomial, with probability of success of 0.48, the probability that a family with six children will have exactly three male children is

A $(0.48)^3$ **B** $(0.48)^6$ **C** $\binom{6}{3}(0.48)^3$ **D** $\binom{6}{3}(0.48)^3(0.52)^3$

5 The probability that a student will be left-handed is known to be 0.23. If nine students are selected at random for the softball team, then the probability that at least one of these students is left-handed is given by

A $(0.23)^9$ **B** $9 \times 0.23 \times (0.77)^8$ **C** $1 - (0.77)^9$
D $1 - (0.77)^9 - 9 \times 0.23 \times (0.77)^8$

6 A bag contains 5 green marbles and 15 red marbles. Three marbles are drawn at random from the basket with replacement. The probability that exactly two of the marbles are green is:

A $\frac{3}{64}$ **B** $\frac{9}{64}$ **C** $\frac{10}{64}$ **D** $\frac{1}{19}$

 ISBN 978-1-009-53719-3

Technology-active multiple-choice questions

7 The probability of winning a single game of chance is 0.1 each time it is played. If the game is played eight times, the probability of winning three times is closest to:

A 0.995 **B** 0.033 **C** 0.005 **D** 0.001

8 The probability of observing five or more heads when a fair coin is tossed ten times is closest to:

A 0.377 **B** 0.246 **C** 0.623 **D** 0.754

9 The probability of snow falling on any day in July at a certain ski resort is 0.3, and whether it snows or not is independent of whether or not it has snowed on any other day. The probability of it snowing on at least one day in a randomly chosen week is closest to:

A 0.061 **B** 0.939 **C** 0.918 **D** 0.082

The following information relates to Questions 10 *and* 11

The probability of rain falling on any day in October in a certain city is 0.6, and whether it rains or not is independent of whether or not it has rained on any other day. Alex has tennis lessons on Mondays, and if it rains the lessons are cancelled. Assume that there are four Mondays in October this particular year.

10 The probability that exactly three of the tennis lessons in October will be cancelled is closest to:

A 0.346 **B** 0.194 **C** 0.870 **D** 0.654

11 The probability that more than one of the tennis lessons will be cancelled, given that at least one is cancelled, is closest to:

A 0.821 **B** 0.158 **C** 0.974 **D** 0.842

12 It is known that 3% of the items produced by a certain machine may be faulty. If 20 of the items are selected at random from the machine for testing then the probability that less than 2 will be faulty is closest to:

A 0.997 **B** 0.099 **C** 0.979 **D** 0.880

13 If 56% of the students in a certain state have part-time jobs, and 60 students are chosen at random from that state, then the probability that at least 30 of those chosen have part-time jobs is closest to:

A 0.066 **B** 0.210 **C** 0.857 **D** 0.143

14 Tom is choosing lucky numbers from a box. The probability of winning a prize with any one of the lucky numbers is 0.1, and whether or not a prize is won on a single draw is independent of any other draw. Suppose Tom draws 10 lucky numbers. The probability he wins three or four times is closest to:

A 0.0686 **B** 0.0112 **C** 0.0702 **D** 0.0574

 ISBN 978-1-009-53719-3 Cambridge University Press

Appendix C

The problem-solving and modelling task

Joel Speranza

Chapter contents

C1 About the problem-solving and modelling task

Mathematisation is the process of taking a real-world problem, translating it to a mathematically purposeful representation, and solving that problem. In Mathematical Methods you will be assessed on your ability to do this through an assessment item called a problem-solving and modelling task (PSMT).

This chapter outlines how to plan, solve and present your PSMT at a high level and provides real examples of high-level student work. Each section of this chapter is accompanied by a video lesson with additional advice, accessible via the included QR codes.

How is the PSMT marked?

Before you begin any assessment, you should consider how teachers will make judgements of your work. Your teacher will use the Instrument-specific Marking Guide (ISMG) from the syllabus to determine your mark. The ISMG is broken into four criteria (Formulate, Solve, Evaluate and Communicate) and each criterion is assessed using three to five descriptors. Throughout this chapter, we will be focusing on the top descriptor in each criterion. These top descriptors will be displayed in this appendix where needed. The full ISMG with all descriptors can accessed through the Interactive Textbook.

 ISBN 978-1-009-53719-3 Cambridge University Press

C2 A content guide for a PSMT report

A suggested set of headings for writing the PSMT report has been provided in a downloadable Word document in the Interactive Textbook. The rest of this appendix provides notes on what to include under these headings, which are reproduced here in black ('Introduction to the task', 'Formulating a solution', 'Developing a solution', and so on). Extracts from the ISMG display the criteria and descriptors covered by each heading of the report.

Video C1

You also need to adhere to the required word limit and conventions of the mathematical report genre. You may have seen another report genre, the scientific report, and the two have similarities.

Word length guides of each section of the report are indicative only and will be dependent on your specific PSMT. Setting out your PSMT in this way is not compulsory but does help you to achieve this highlighted descriptor from the ISMG:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

1. Introduction to the task (200 words)

Video C2

These are the criteria and descriptors to be covered by the introduction:

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

Note that, as long as you understand the task, and have written down what you need to do, you do not have to finalise the introduction at the beginning of the process. You may find it better to write the introduction alongside writing the conclusion, ensuring that the conclusion addresses the goal of the task which was stated in the introduction.

The purpose of the introduction is twofold:

1 To introduce a report which 'can be read independently' of the task sheet. This means that a person who has never seen the task sheet will be able to understand what the task is,

simply by reading your introduction. No reference to the task sheet should be made here. You should also begin with providing some context on what the problem is to be solved, and why solving the problem is important.

2 To show 'justified mathematical translation of important aspects of the task'. Mathematical translation is the process of taking a real-world problem and moving it into the mathematical world – a process known as **mathematisation**. In this part of an introduction, you should give a brief description of the mathematical techniques you will use to solve the problem. This description can include:
- Applicable mathematical or statistical principles
- Mathematical concepts and techniques
- Technology that you will use throughout the task

The student sample below shows:

- how the task can be read independently of the task sheet
- justified mathematical translation of important aspects of the task.

Introduction

As the population of Australia continues to grow, it is crucial that we plan ahead. Important aspects in life such as resources, housing, and medical products and services need to account for this growing population before the demand for necessary requirements of life have such a demand that cannot be catered for (Sommerfeld, 2018). By knowing predictions for a future population count, we can plan ahead and make choices that will benefit the future now.

The purpose of this task is to determine a prediction for the rate of population growth in Australia for 2061 by exploring multiple mathematical models. A multitude of mathematical techniques will be utilised throughout this task, including; modelling of exponential, logarithmic, logistic, and sinusoidal functions, as well as deriving to find a rate of change, determining percentage error, and using technology such as GeoGebra, a TI-84 Plus CE Graphics Calculator and Excel to model equations and manage data.

Student sample taken from the 2020 Mathematical Methods subject report

Using mathematical language?

In the introduction and throughout the PSMT, you must demonstrate correct use of mathematical language.

Video C3

These are the criteria and descriptors to be covered:

Communicate	Marks
The student response has the following characteristics:	
• correct use of appropriate mathematical language • logical organisation of the response, which can be read independently of the task sheet • justification of decisions using mathematical reasoning	3–4

The syllabus objectives elaborate on mathematical language as ‘terminology, symbols, conventions and representations’. What follows is a non-exhaustive list of ways to demonstrate each aspect of mathematical language.

Terminology

- Procedural mathematical language (mathematical verbs) e.g. determine, solve, verify, calculate integrate, derive.
- Technical mathematical language (mathematical nouns) e.g. search the unit outline for terminology specific to the content being assessed.

Conventions

- Equations have a left and right hand side
- Equal signs are aligned
- Define variables before using them

Symbols

- Use mathematical symbols correctly
- If typing mathematics, use equation editor

Representations: Graphs

- An appropriate title
- Axes labelled with appropriate units
- An appropriate scale for each axis
- A legend if appropriate

Representations: Diagrams

- An appropriate title
- Labelled
- Drawn to scale or a label indicating otherwise
- Vertices of shapes labelled with capital letters
- Angles labelled with Greek letters

2. Formulating a solution (400 words)

In the section titled ‘Formulating a solution’, you are aiming to demonstrate the descriptors of the ISMG shown below.

Formulate	Marks
The student response has the following characteristics:	
• justified statements of important assumptions • justified statements of important observations • justified mathematical translation of important aspects of the task	3–4

It is vital that your assumptions are both important and expressed as justified statements. This is the most important section of the PSMT, as many sections of the PSMT cannot be completed to a high-standard without them. This is demonstrated in the following flowchart.

Observations are considered and assumptions made in order to 'mathematise' the problem.

If the solution does not rely on these, then they are not 'important' and therefore do not meet the criteria. If they are not backed by justified statements, then the evidence for your solution being valid is low.

Once a solution is found, 'justified statements about the reasonableness of the solution by considering observations and assumptions' must be made. If these observations and assumptions are poorly justified or are not important, then it will be difficult to do this.

At the end of the PMST, 'justified statements of relevant strengths and limitations of the solution' are made. These are often found by examining the assumptions and observations that have been made. If assumptions and observations are not important, this section becomes more difficult to complete.

Observations vs assumptions

Students can often be confused about the difference between an observation and an assumption. Both are vital to creating a mathematical model but serve different purposes. The table below outlines the key differences between them.

Aspect	Observations	Assumptions
Definition	Data or information required to solve a mathematical problem and/or develop a mathematical model.	Conditions that are stated to be true when beginning to solve a mathematical problem and/or develop a mathematical model.
Nature and verifiability	Factual, based on actual data or information, and can be empirically verified.	While not directly verifiable, are rational and necessary for solving the problem. They are based on plausible reasoning or existing knowledge.
Role in modelling	Provides data and empirical evidence for creation of the model.	Simplifies complexity and fills data gaps. Dictates the strengths, limitations, and applicability of the model.
Flexibility	Generally rigid; they are facts that can't be altered. However, the interpretation of observations can evolve with new data.	More flexible and can be adjusted or replaced as new information becomes available or as the model evolves.

 ISBN 978-1-009-53719-3 Cambridge University Press

Aspect	Observations	Assumptions
Examples	**1** Measuring the temperature and humidity levels in various regions over a year. **2** Recording the frequency and intensity of rainfall in a specific area. **3** The observable fact that warm air rises and cool air sinks, affecting weather patterns.	**1** Assuming that future weather patterns will reflect past trends due to climate consistency. **2** Assuming a certain level of accuracy in satellite data used for cloud cover analysis. **3** Assuming that ocean currents will remain relatively stable over the short-term forecasting period.
Role in validation	Used for validating the model by comparing its predictions with actual observations.	Validated indirectly through the model's performance and its ability to make accurate predictions within the defined framework of these assumptions.

2.1 Observations

Video C4

To ensure that your observations reach the level of 'justified statements' they should include:

- a discussion of how the observations affect the mathematical model/solution
- in-text referencing to a reputable source (while it is possible to justify statements without this, consider the inclusion of a reference the 'gold-standard').

The following student sample demonstrates justified statements of observations. Each observation contains:

- a statement of what the observation is
- a reference that provides justification for the observation being made
- justification for how this observation impacts the mathematical model/solution.

Structuring your observations in this way provide a high likelihood of them being considered justified statements on the ISMG.

Observations

It was observed that the male-to-female ratio in Western Australia is 102 males for every 100 females (McCrindle, 2014). This observation directly impacts the mathematical model as the initial total population must divide by this ratio only to consider the female Western Australian population. This is relevant as males do not reproduce any offspring and cannot be factored into the Leslie matrix.

It was observed that Western Australia takes 30% of immigrants into Australia each year (Australian Bureau of Statistic, 2021). Further, 12,706 to 18,200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). These observations impacted the mathematical model as an increase in immigrants will impact the projected populations, causing an increase or decrease in total population growth rate.

 ISBN 978-1-009-53719-3

Humans live to approximately 100 years (Vaupel, 2010); therefore, a 21 × 21 matrix is reasonable. From this, the age classes for the Leslie matrix were split into 5-year categories. Hence, each new generation produced by the Leslie matrix represents a 5-year gap. This observation is relevant as it impacts the scope mathematical model by investigating only eight new generations.

Student sample taken from the 2022 Specialist Mathematics subject report

2.2 Assumptions

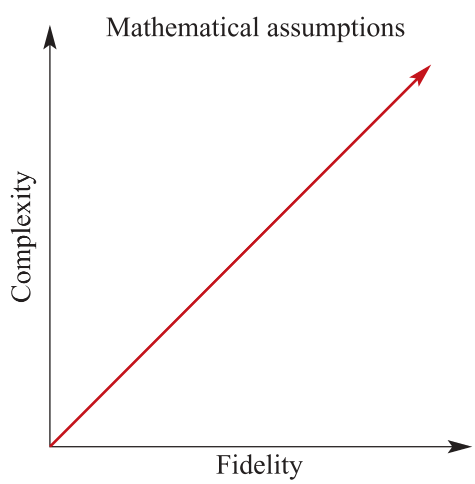
Video C5

As with observations, to justify the assumptions you make you should include discussion of how the assumptions affect the mathematical model/solution; and in-text referencing to a reputable source. While it is possible to justify statements without this, it is considered best practice to include one.

Referencing should be used to justify an assumption and it can be done in two different ways:

- use a source that brings your mathematical model closer to representing the real world
- use a source that takes your mathematical model further from representing the real world but is required to reduce the complexity of the model.

To illustrate this further, each assumption can be thought of as existing on the graph shown below, with mathematical complexity increasing as fidelity (how closely the assumption models the real world) increases. Therefore, each assumption can be thought of as a trade-off between complexity and fidelity. Adopting this approach from the beginning makes future sections of the PSMT (evaluating the reasonableness of the solution and strengths and limitations) easier to complete.



The student sample which follows demonstrates justified statements of assumptions. Each assumption contains:

- a statement of what the assumption is
- a reference that provides justification for the assumption being made
- justification for how the assumption impacts the mathematical model/solution.

Assumptions

It was assumed that birth rates would only impact women aged 15–44 as in Australia. The average woman's reproductive years are between ages 15 and 44 (Watson, 2018). This assumption restricts birth rates to only six of twenty-one age classes. The assumption was made to reduce the anomalies to develop clean data.

It was assumed that the new immigrant population introduced into Western Australia was equally divided into the age classes from 18 to 34. Most migrants to Australia are young adults, with 61.2% aged between 18 and 34 years (abs.gov.au, 2018). Further, 12 706 to 18 200 immigrants settled in Australia during 2018 and 2020. (Lawrence, 2018). This assumption was made to create a realistic data spread that included the possible impact immigrants' survival and or birth rate would have on the total population.

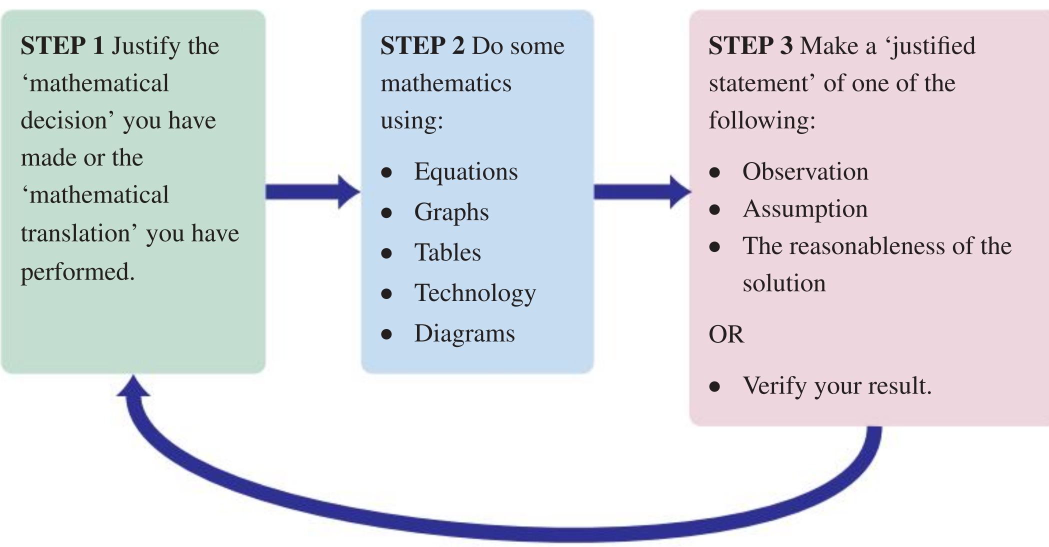
It was assumed that the investigation started during 2016 as the female population data collected was from 2016 (abs.gov.au, 2016); therefore, a 21×21 matrix is reasonable. This assumption impacts the investigation as the potential increasing or decreasing growth rate and total population can be compared to secondary data to determine the model's validity.

Student sample taken from the 2022 Specialist Mathematics subject report.

3. Developing a Solution (700 words)

Video C6

In this section of the PSMT, you are attempting to demonstrate almost all the descriptors of the ISMG, as shown in the following example. This is also a section where students can get a little confused about how to set things out. Below is a flowchart you can use throughout this section to ensure that you demonstrate the full range of ISMG descriptors.



The sample following is a simple example of how this flowchart can be put into practice. It is from a PSMT in which the task is to develop a flying fox. The mathematics has been simplified to allow us to focus on the structure of the response, rather than the mathematics. To guide you, the example has been annotated with:

- in the left margin, the numbered steps from the flowchart above
- in the right margin, the matching ISMG descriptors from the syllabus.

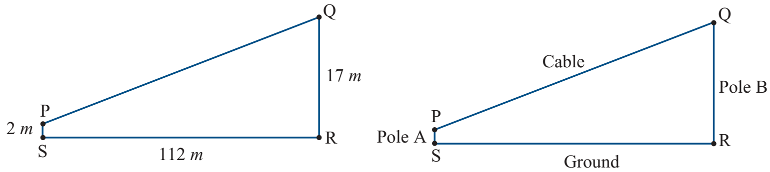
Use them to guide your solution, but do not annotate your own report with them.

3.1 Calculating the length of the cable

STEP 1

The flying fox cable, two supporting poles and the ground can be modelled as the quadrilateral, $PQRS$, as shown below.

Formulate
- Justified mathematical translation of important aspects of the task



STEP 2

Communicate
- Correct use of appropriate mathematical language

STEP 3

It is assumed that the cable is perfectly taut and has no sag. While the cable in a real flying fox will have a sag (Evans, 2017), this assumption allows the cable's length to be calculated.

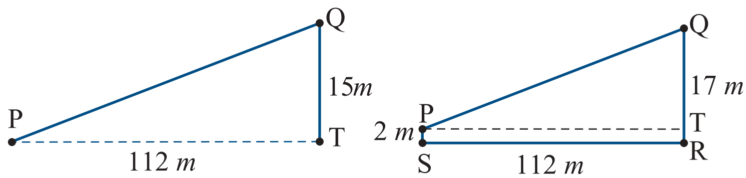
Formulate
- Justified statements of important assumptions

STEP 1

A horizontal line drawn through point P creates a right triangle PQT. Creating a right triangle will allow the length of the cable to be calculated using Pythagoras' theorem.

Communicate
- Justification of decisions using mathematical reasoning

STEP 2



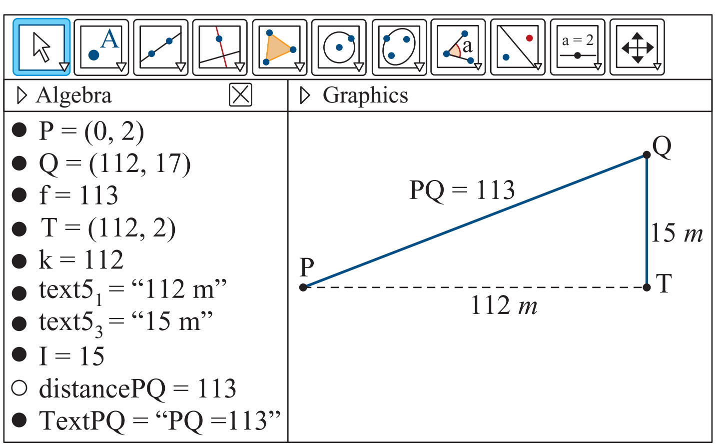
$$\begin{aligned}\text{Length of cable} &= \sqrt{112^2 + 15^2}\\ &= \sqrt{12544 + 225}\\ &= \sqrt{12769}\\ &= 113\end{aligned}$$

Solve
- Accurate use of mathematical knowledge for important aspects of the task

STEP 3

The length of the cable is calculated to be 113 metres.

This result can be verified using a scale diagram drawn in Geogebra, as shown in this screenshot.



Solve
- Efficient use of technology

 ISBN 978-1-009-53719-3

3.2 Calculating the total cost of the cable

STEP 1

The total cost of cable can now be calculated. In construction, it is common to assume that you require an additional 10% of materials to allow for wastage (Jones, 2017). 8 mm aircraft-grade galvanised cable is perfect for ziplines up to 150 metres in length and has a current cost of $8 per metre. (cable-ride.com, n.d)

STEP 2

$$\begin{aligned}\text{Total cost} &= 113 \times 8 \times 1.1\\ &= \$994.40\end{aligned}$$

Adding an additional 10% to materials is a reasonable solution when considering the assumption made that there is no sag in the cable. In reality there will be sag, and this will increase the amount of the cable required.

STEP 3

The $994.40 total cost for the cable can be verified by comparing it to a 90 metre zipline kit available online for $1387. (cable-ride.com, n.d). While this is $392.60 more expensive than our cable, it is for a complete kit rather than just a cable. These two prices are close enough to support our result.

Formulate

- Justified statements of important assumptions
- Justified statements of important observations

Solve

- Accurate use of mathematical knowledge for important aspects of the task

Evaluate

- Justified statements about the reasonableness of the solution by considering the assumptions

Evaluate

- Verified results

Efficient use of technology

Solve

The student response has the following characteristics:

- Efficient use of technology

Video C7

Throughout your assignment, you should be looking for opportunities to demonstrate the efficient use of technology, as required by the ISMG descriptor shown. The key descriptor here is 'efficient', which can be understood as using technology in a way that minimises wasted effort and/or time. An example of efficiency would be using Excel formulas to perform repeated calculations, rather than performing each calculation by hand or on a calculator.

It is often the case that while students use quite a bit of technology throughout their PSMT, they often don't provide the evidence that they have used it. Here are some types of technology students use and the evidence that they can provide to demonstrate that they have used it.

Types of technology	Evidence provided
Spreadsheet software (e.g. Microsoft Excel)	■ Screenshots of graphs or tables ■ Samples of spreadsheet formulas used
Calculator	■ Photos or screenshots of the calculator screen ■ Descriptions of how the calculator was used
Graphing software (e.g. Desmos or Geogebra)	■ Screenshots ■ Descriptions of how the software was used
Logging software (e.g. data collectors)	■ Sample of the data collected ■ Screenshots

 ISBN 978-1-009-53719-3

The following examples which evidence efficient use of technology are taken from the 2022 Mathematics subject reports.

To verify this a larger Excel spreadsheet was created to solve the parametric equations for a variety of t values (also extending to ten years):

t	Xleft	Xright	Difference	Intersect?	Yleft	Yright	Difference	Intersect?	Zleft	Zright	Difference	Intersect?
0.5	0.999963	1.999997	1.0000337	FALSE	0.0086	0.001805	0.0067944	FALSE	0	-2	1.999997	FALSE
1	0.999852	1.999987	1.0001349	FALSE	0.017199	0.003611	0.0135882	FALSE	0	-1.99999	1.999987	FALSE

Figure 12: Spreadsheet

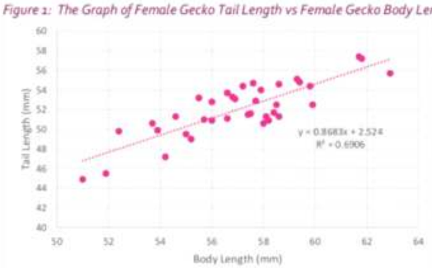
The Excel equation inserted in the 'Intersect?' column (Figure 13), calculated if the difference between the two sides of the parametric equation was less than 4.2587×10^{-5}, the conversion of Earth's radius to AU, to ensure constant units.

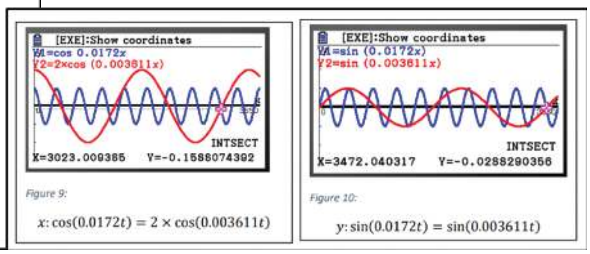
=IF(D2<(0.0000425875),"YES")

Student sample taken from the 2022 Specialist Mathematics subject report

The data without outliers was utilised for further analysis and extrapolation.

Figure 1: The Graph of Female Gecko Tail Length vs Female Gecko Body Length without Outlier





Student sample taken from the 2022 General Mathematics subject report

Student sample taken from the 2022 Specialist Mathematics subject report

4. Evaluating and verifying the solution (600 words)

Video C8

4.1 Verifying results

Evaluate
The student response has the following characteristics:
• Verified results

In this section you are aiming to demonstrate the descriptor shown here, verifying the overall solution. If you have been following the advice from the previous section, you will have been verifying some of your results in the process of coming to your solution. Below, we look at four techniques students can use for verifying results.

1 **Verifying through estimation:** This method simplifies the problem before calculating an approximate solution. For instance, if you have calculated the area of a composite shape, you can verify your solution by simplifying it into a basic rectangle and recalculating the area. If the estimated area is close to your calculated area, it helps confirm the accuracy of your solution.

2 **Verifying through research:** This method involves comparing results with reliable sources. This method is particularly useful for tasks related to historical data or correlation studies. Findings can be matched against data from textbooks, academic journals, or credible online sources. For example, in a task exploring the correlation between car weight and fuel consumption, you can verify the solution against existing research on this topic.

 ISBN 978-1-009-53719-3

3 **Verifying through technology:** Using technology to redo algebraic calculations can aid in verifying results. For example, when calculating the area under a curve, employing graphing calculators or software for an approximation and comparing it to manual calculations can serve as a verification method.

4 **Verifying through an alternative method:** This method is effective in problems with multiple solutions and involves employing various techniques to solve the same issue. For example, you may initially solve an algebraic equation by factoring, then recheck the solution using the quadratic formula. If both methods produce identical results, it strongly suggests that the solution is correct.

4.2 Evaluating reasonableness of the solution

Video C9

In this section you are aiming to demonstrate the content descriptor below:

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

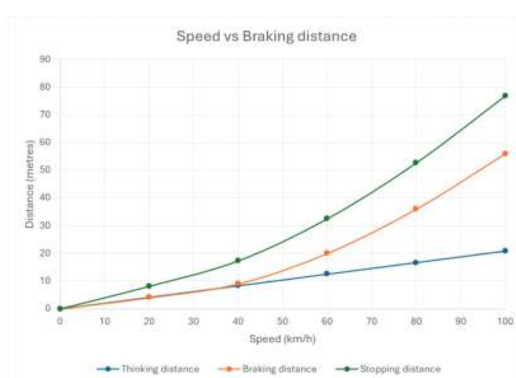
When evaluating the reasonableness of your solution by considering assumptions and observations you should:

- consider the solution found
- consider how it is affected by the observation or assumption
- consider how the solution might be different if the observation or assumption was altered (often with some mathematical working included).

Below is an example of evaluating the reasonableness of a solution by considering an assumption from a PSMT investigating the braking distance of a car dependent on the speed at which it is travelling.

Example

The distance calculated for a car to come to a complete stop is underestimated because of the assumption that driver's response times were instantaneous. If driver response time were factored in, the braking distance would be greater than the distances calculated in this report.



Assuming average driver response time of 1.5 seconds (Muttart, 2004), we can see how the solution would change if this time was taken into account in the graph pictured. At the top speed of 100 km/h, braking distance is increased from the initial solution of 56 m to 77 m, an increase of 37.5%.

Note: Students often make the mistake of evaluating the reasonableness of their assumption, rather than evaluating the reasonableness of their solution by considering their assumption. The distinction is subtle but important. The justified statements made must refer to the solution and how it is affected by the assumption, not just the assumption itself.
Correct: The solution is reasonable because. . .

Incorrect: The assumption is reasonable because. . .

A subtle but important difference.

4.3 Strengths and limitations of the solution

Video C10

In this section you are aiming to demonstrate the content descriptor below:

Evaluate	Marks
The student response has the following characteristics:	
• verified results • justified statements about the reasonableness of the solution by considering the assumptions • justified statements about the reasonableness of the solution by considering the observations • justified statements of relevant strengths of the solution • justified statements of relevant limitations of the solution	4–5

Strengths and limitations of the solution can be thought of in the following way.

- Strengths – aspects of the model that make it useful
- Weaknesses – aspects of the model that limit its usefulness

A series of questions to help identify these strengths and limitations is below. Do not aim to answer all these questions but use them as prompts to generate ideas.

Strengths	**Limitations**
What assumptions were made that closely align to the real world?	What assumptions were made that do not align closely with the real world?
What aspects of the real world does the solution consider?	What aspects of the real world does the solution not consider?
Could the method used to create this solution be easily adapted and used to solve other, related problems?	Are there other, related problems that the method used could not be easily adapted to solve?
What aspects of the solution can be verified using other observational data?	What aspects of the solution cannot be verified using observational data?
What are the potential, positive consequences of using this solution in the real world?	What are the potential, negative consequences of using this solution in the real world?
What aspects of the solution will continue to be accurate into the future?	What aspects of the solution will cease to be accurate into the future?

To make justified statements of strengths/limitations:

- state the strength/limitation
- justify why it is a strength/limitation.

The student sample of limitations below provides examples of this.

The following limitations were observed

There was a limited amount of data points that were used as a sample.This means the findings were less reliable as it may not be an accurate representation of all rugby games.

Another limitation is that when using extrapolation with regards to the regression line, it may not be accurate to predict further outcomes because the prediction is outside the sample data range.

One final limitation is that the R2 value found is not considerably strong, therefore a smaller percentage of the points scored per game can be attributed to the line breaks achieved per game, decreasing the reliability of the study.

Student sample taken from the 2022 General Mathematics subject report

5. Conclusion (100 words)

The conclusion is another opportunity to show logical organisation of your response. In the conclusion you should:

- restate the purpose of the mathematical report
- provide a summary of your solution, stating an appropriate answer to the task.

Conclusion

The purpose of this report was to use functions and derivatives to create a reasonable prediction for the rate of change for the Australian population in 2061. It was found that the most reasonable model was solution three as it produces a reasonable population and somewhat reasonable rates of change. Therefore, using this logistic function, it is predicted that the population will reach approximately 39.6 million in 2061, with a percentage rate of change of 0.95% and an instantaneous rate of change of approximately 370 000 addition people per year.

Student sample taken from the 2022 Mathematical Methods subject report

6. Reference List (Not included in page or word count)

Use a standard referencing style. Ask your teacher for guidance on this if you need it.

7. Appendix (Not included in page or word count)

An appendix is for supporting material such as data, diagrams, calculations and screenshots or print-outs from technology, that don't form a direct part of the solution or evaluation. The appendix is not marked so don't include important items that you want a mark for. If you haven't already included your use of technology in the report, you could put a small sample into the body of your assignment to get marks for efficient use of technology.

 ISBN 978-1-009-53719-3

Glossary

A

Acceleration [p. 635] The acceleration of a particle is defined as the rate of change of its velocity with respect to time.

Acceleration, average [p. 635] The average acceleration of a particle for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.

Acceleration, instantaneous [p. 635] $a = \dfrac{dv}{dt}$

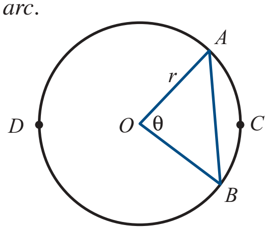
Addition rule for choices [p. 720] To determine the total number of choices from disjoint alternatives, simply add up the number of choices available for each alternative.

Addition rule for probability [p. 385] The probability of A or B or both occurring is given by
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Amplitude of trigonometric functions [p. 320] The distance between the mean position and the maximum position is called the amplitude. The graph of $y = \sin x$ has an amplitude of 1.

Arc [p. 342] Two points on a circle divide the circle into arcs; the shorter is the *minor arc*, and the longer is the *major arc*.

Arc length, ‘ [p. 343] The length of arc ACB is given by $\ell = r\theta$, where $\theta^c = \angle AOB$.



Arrangements [pp. 250, 724] counted when order is important. The number of ways of selecting and arranging r objects from a total of n objects is

$$\frac{n!}{(n-r)!} = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$$

Asymptote [pp. 167, 661] A straight line is an asymptote of the graph of a function $y = f(x)$ if the graph of $y = f(x)$ gets arbitrarily close to the straight line. An asymptote can be horizontal, vertical or oblique.

B

Binomial expansions [p. 256] In the expansion of $(a + b)^n$, each term is of the form ${}^nC_r\, a^{n-r} b^r$.
For example:

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

C

Chain rule [p. 654] The chain rule can be used to differentiate a complicated function $y = f(x)$ by transforming it into two simpler functions, which are ‘chained’ together:

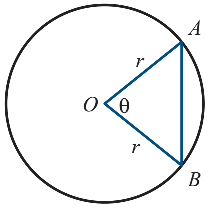
$x \xrightarrow{h} u \xrightarrow{g} y$

Using Leibniz notation, the chain rule is stated as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

 ISBN 978-1-009-53719-3 Cambridge University Press

Chord [p. 342]
a line segment with endpoints on a circle



Chord length [p. 343]

$$AB = 2r\sin\left(\frac{\theta}{2}\right)$$

where $\theta^c = \angle AOB$

Circle, general equation [p. 176] The general equation for a circle is

$$(x - h)^2 + (y - k)^2 = r^2$$

where the centre of the circle is the point (h, k) and the radius is r.

Coefficient [pp. 101, 246] the number that multiplies a power of x in a polynomial. E.g. for $2x^5 - 7x^2 + 4$, the coefficient of x^2 is -7.

Combinations [pp. 251, 729] *see* selections

Complement, A' [pp. 362, 370] the set of outcomes that are in the sample space, ε, but not in A. The probability of the event A' is

$$P(A') = 1 - P(A)$$

Complementary relationships [p. 340]

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Composition of functions [p. 652] For two functions f and g, the function with rule $h(x) = f(g(x))$ is the composition of f with g. We write $h = f \circ g$. For example, if $f(x) = x^4$ and $g(x) = x + 1$, then $(f \circ g)(x) = f(g(x)) = (x + 1)^4$.

Conditional probability [p. 392] the probability of an event A occurring when it is known that some event B has occurred, given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Constant function [p. 205] a function with a rule of the form $f(x) = a$; e.g. $f(x) = 7$

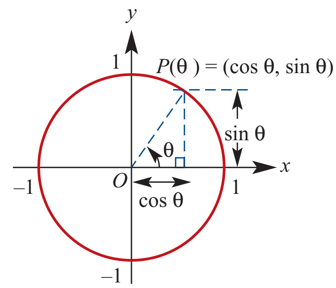
Continuous function [p. 596] A function f is continuous at the point $x = a$ if the following three conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$

We say that a function is continuous everywhere if it is continuous for all real numbers.

Coordinates [p. 35] an ordered pair of numbers that identifies a point in the Cartesian plane; the first number identifies the position with respect to the x-axis, and the second number identifies the position with respect to the y-axis

Cosine function [p. 311] cosine θ is defined as the x-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Cubic function [p. 245] A polynomial of degree 3 is called a cubic, and is a function f with rule $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$.

D

Degree of a polynomial [pp. 101, 246] given by the highest power of x with a non-zero coefficient. For example, the polynomial $2x^5 - 7x^2 + 4$ has degree 5.

Dependent variable [p. 215] If one variable, y, can be expressed as a function of another variable, x, then the value of y depends on the value of x. We say that y is the *dependent variable* and that x is the *independent variable*.

Derivative function [p. 570] also called the gradient function. The derivative f' of a function f is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

Diameter [p. 342] a chord of a circle that passes through the centre

Difference of two cubes [p. 293]

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Difference of two squares [pp. 104, 107]

$$x^2 - y^2 = (x - y)(x + y)$$

Differentiable [p. 599] A function f is said to be differentiable at the point $x = a$ if $\lim_{h\to 0} \dfrac{f(a + h) - f(a)}{h}$ exists.

 ISBN 978-1-009-53719-3 Cambridge University Press

Differentiation rules [pp. 573, 659]

- Power: $f(x) = x^n$, $f'(x) = nx^{n-1}$, for $n \in \mathbb{Q} \setminus \{0\}$
- Constant: $f(x) = c$, $f'(x) = 0$
- Sum: $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$
- Multiple: $f(x) = k\,g(x)$, $f'(x) = k\,g'(x)$

see also chain rule, product rule, quotient rule

Dilation from the x-axis [p. 228] A dilation of factor b from the x-axis is described by the rule $(x, y) \to (x, by)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = bf(x)$.

Dilation from the y-axis [p. 228] A dilation of factor a from the y-axis is described by the rule $(x, y) \to (ax, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f\left(\frac{x}{a}\right)$.

Discontinuity [p. 596] A function is said to be discontinuous at a point if it is not continuous at that point.

Discrete random variable [p. 745] a random variable X which can take only a countable number of values, usually whole numbers

Discriminant, Δ, of a quadratic [p. 135] the expression $b^2 - 4ac$, which is part of the quadratic formula. For the quadratic equation $ax^2 + bx + c = 0$:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Disjoint sets [pp. 84, 362] If sets A and B have no elements in common, we say A and B are disjoint and write $A \cap B = \varnothing$.

Displacement [p. 633] The displacement of a particle moving in a straight line is defined as the change in position of the particle.

Distance between two points [p. 37] The distance between the given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Division of polynomials [p. 285] When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the quotient and $R(x)$ the remainder, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Domain [p. 196] the set of all the first coordinates of the ordered pairs in a relation

E

Element [pp. 84, 361] a member of a set.

- If x is an element of a set A, we write $x \in A$.
- If x is *not* an element of a set A, we write $x \notin A$.

Empty set, $\varnothing$ [pp. 84, 361] the set that has no elements

Even function [p. 281] A function f is even if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.

Event [p. 367] a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes.

Exponential function [p. 463] a function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1

F

Factor [pp. 285, 286] a number or expression that divides another number or expression without remainder

Factor theorem [p. 291] If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$. Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Factorise [p. 106] express as a product of factors

Formula [p. 23] an equation containing symbols that states a relationship between two or more quantities; e.g. $A = \ell w$ (area = length × width). The value of A, the subject of the formula, can be found by substituting given values of ℓ and w.

Function [p. 202] a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$.

Function, vertical-line test [p. 203] used to identify whether a relation is a function or not. If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a *function*.

G

Gradient function *see* derivative function

Gradient of a line [p. 39] The gradient is

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of two points on the line.

Gradient of a vertical line [p. 40] The gradient of a vertical line (parallel to the y-axis) is undefined.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

I

Implied domain *see* maximal domain

Independence [p. 401] Two events A and B are independent if $P(A \cap B) = P(A) \times P(B)$ or $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

Independent variable [p. 215] If one variable, y, can be expressed as a function of another variable, x, then the value of y depends on the value of x. We say that y is the *dependent variable* and that x is the *independent variable*.

Index laws [p. 464]

- To multiply two powers with the same base, add the indices: $a^m \times a^n = a^{m+n}$
- To divide two powers with the same base, subtract the indices: $a^m \div a^n = a^{m-n}$
- To raise a power to another power, multiply the indices: $(a^m)^n = a^{m \times n}$
- Rational indices: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$
- For base $a \in \mathbb{R}^+ \setminus \{1\}$, if $a^x = a^y$, then $x = y$.

Inequality [p. 19] a mathematical statement that contains an inequality symbol rather than an equals sign; e.g. $2x + 1 < 4$

Integers [p. 85] the elements of $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

Intersection of sets [pp. 84, 362] The intersection of two sets A and B, written $A \cap B$, is the set of all elements common to A and B.

Interval [p. 86] a subset of the real numbers of the form $[a, b]$, $[a, b)$, (a, ∞), etc.

Irrational number [p. 85] a real number that is not rational; e.g. π and $\sqrt{2}$

L

Law of total probability [p. 395] In the case of two events, A and B: $P(A) = P(A \mid B)P(B) + P(A \mid B')P(B')$

Leading term [p. 246] The leading term, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient. E.g. the leading term of $2x^5 - 7x^2 + 4$ is $2x^5$.

Limits, properties [p. 594]

- Sum: $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
- Multiple: $\lim\limits_{x \to a} (kf(x)) = k \lim\limits_{x \to a} f(x)$
- Product: $\lim\limits_{x \to a} (f(x)\,g(x)) = \lim\limits_{x \to a} f(x) \; \lim\limits_{x \to a} g(x)$
- Quotient: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, if $\lim\limits_{x \to a} g(x) \neq 0$

Linear equation [p. 2] a polynomial equation of degree 1; e.g. $2x + 1 = 0$

Linear function [p. 205] a function with a rule of the form $f(x) = mx + c$; e.g. $f(x) = 3x + 1$

Literal equation [p. 5] an equation for the variable x in which the coefficients of x, including the constants, are pronumerals; e.g. $ax + b = c$

Logarithm [p. 504] If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = y$ and $\log_a y = x$ are equivalent.

Logarithm laws [p. 504]

- $\log_a(mn) = \log_a m + \log_a n$
- $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$
- $\log_a\left(\dfrac{1}{n}\right) = -\log_a n$
- $\log_a(m^p) = p \log_a m$

M

Maximal domain [pp. 200, 208] When the rule for a relation is given and no domain is specified, then the domain taken is the largest for which the rule has meaning.

Maximum and minimum value [p. 628] For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the *absolute maximum* value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the *absolute minimum* value of the function.

Midpoint of a line segment [p. 36] If $P(x, y)$ is the midpoint of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, then

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Monic polynomial [p. 246] a polynomial whose leading term has coefficient 1; e.g. $x^3 + 4x^2 + 5$

Multiplication rule for choices [p. 721] When sequential choices are involved, the total number of possibilities is found by multiplying the number of options at each successive stage.

Multiplication rule for probability [p. 394] the probability of events A and B both occurring is $P(A \cap B) = P(A \mid B) \times P(B)$

Multi-stage experiment [p. 380] an experiment that could be considered to take place in more than one stage; e.g. tossing two coins

Mutually exclusive [p. 362] Two events are said to be mutually exclusive if they have no outcomes in common.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

N

***n*!** [pp. 250, 725] The notation $n!$ (read as 'n factorial') is an abstraction for the product of all the integers from n down to 1:

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

Natural numbers [p. 85] the elements of $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

${}^nC_r$ [pp. 251, 729] the number of combinations of n objects in groups of size r:

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for ${}^nC_r$ is $\dbinom{n}{r}$.

Normal line, equation [p. 612] Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, the equation of the normal at (x_1, y_1) is

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

Null factor theorem [pp. 111, 126] If $ab = 0$, then $a = 0$ or $b = 0$.

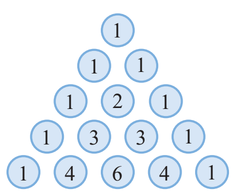
O

Odd function [p. 281] A function f is odd if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry about the origin.

Ordered pair [pp. 35, 196] An ordered pair, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

P

Pascal's triangle [p. 253] a triangular pattern of numbers formed by the binomial coefficients ${}^nC_r$



Period of a function [p. 320] A function f with domain $\mathbb{R}$ is periodic if there is a positive constant a such that $f(x + a) = f(x)$ for all x. The smallest such a is called the period of f. For example, the period of the sine function is 2π, as $\sin(x + 2\pi) = \sin x$.

Permutations [pp. 250, 724] *see* arrangements

Piecewise-defined function [p. 212] a function which has different rules for different subsets of its domain

Polynomial function [pp. 101, 246] A polynomial has a rule of the type

$$y = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \quad n \in \mathbb{N} \cup \{0\}$$

where $a_0, a_1, \dots, a_n$ are numbers called coefficients.

Position [pp. 551, 632] For a particle moving in a straight line, the position of the particle relative to a point O on the line is determined by its distance from O and whether it is to the right or left of O. The direction to the right of O is positive.

Power function [p. 281] a function of the form $f(x) = x^r$, where r is a non-zero real number

Probability [p. 360] a numerical value assigned to the likelihood of an event occurring. If the event A is impossible, then $P(A) = 0$; if the event A is certain, then $P(A) = 1$; otherwise $0 < P(A) < 1$.

Probability distribution [p. 746] a function, denoted $p(x)$ or $P(X = x)$, which assigns a probability to each value of a discrete random variable X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.

Probability table [p. 388] a table used for illustrating a probability problem diagrammatically

Product rule [p. 665] Let $f(x) = u(x) \cdot v(x)$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = u(x) \cdot v'(x) + v(x) \cdot u'(x)$$

In Leibniz notation:

If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

Pythagorean identity [p. 341]

$$\cos^2\theta + \sin^2\theta = 1$$

Q

$\mathbb{Q}$ [p. 85] the set of all rational numbers

Quadratic formula [p. 132] The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic function [p. 101] A quadratic has a rule of the form $y = ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

Quadratic, turning point form [p. 116] The turning point form of a quadratic function is $y = a(x - h)^2 + k$, where (h, k) is the turning point.

Quartic function [p. 245] A polynomial of degree 4 is called a quartic, and is a function f with rule $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a \neq 0$.

Quotient rule [p. 669] Let $f(x) = \frac{u(x)}{v(x)}$, $v(x) \neq 0$. If $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

In Leibniz notation:

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

R

$\mathbb{R}^+$ [p. 85] $\{ x : x > 0 \}$, positive real numbers

$\mathbb{R}^-$ [p. 85] $\{ x : x < 0 \}$, negative real numbers

$\mathbb{R} \setminus \{0\}$ the set of real numbers excluding 0

$\mathbb{R}^2$ [p. 224] $\{ (x, y) : x, y \in \mathbb{R} \}$; i.e. $\mathbb{R}^2$ is the set of all ordered pairs of real numbers

Radian [p. 308] One radian (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit:

$$1^c = \frac{180^\circ}{\pi} \quad \text{and} \quad 1^\circ = \frac{\pi^c}{180}$$

Random experiment [p. 366] an experiment, such as the rolling of a die, in which the outcome of a single trial is uncertain but observable

Random variable [p. 745] a variable that takes its value from the outcome of a random experiment; e.g. the number of heads observed when a coin is tossed three times

Range [p. 196] the set of all the second coordinates of the ordered pairs in a relation

Rational number [p. 85] a number that can be written as $\frac{p}{q}$, for some integers p and q with $q \neq 0$

Rectangular hyperbola [p. 167] The basic rectangular hyperbola has equation $y = \frac{1}{x}$.

Reflection in the x-axis [p. 229] A reflection in the x-axis is described by the rule $(x, y) \to (x, -y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = -f(x)$.

Reflection in the y-axis [p. 229] A reflection in the y-axis is described by the rule $(x, y) \to (-x, y)$. The curve with equation $y = f(x)$ is mapped to the curve with equation $y = f(-x)$.

Relation [p. 196] a set of ordered pairs; e.g. $\{ (x, y) : y = x^2 \}$. A relation may be written as a listed set of ordered pairs or as a rule with a specified or implied domain. A relation may be represented by a graph or by an arrow diagram.

Remainder theorem [p. 290] When a polynomial $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Repeated factor of a polynomial [p. 269] a factor that occurs more than once in the factorised form of a polynomial. For example, $x - a$ is a repeated factor of $P(x) = (x - a)^3(x - b)$.

S

Sample space [p. 366] the set of all possible outcomes for a random experiment, sometimes denoted ε

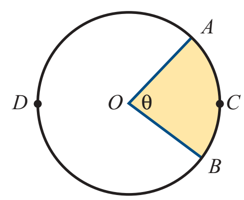
Scientific notation [p. 473] A number is in standard form when written as a product of a number between 1 and 10 and an integer power of 10; e.g. 6.626×10^{-34}.

Secant [p. 541] a straight line that passes through two points $(a, f(a))$ and $(b, f(b))$ on the graph of a function $y = f(x)$

Sector [pp. 343, 344] Two radii and an arc define a region called a sector. In this diagram, the shaded region is a *minor sector* and the unshaded region is a *major sector*.

Area of sector $= \frac{1}{2}r^2\theta$

where $\theta^c = \angle AOB$

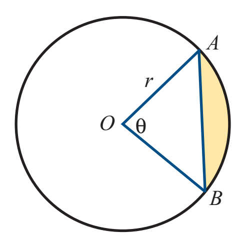


Segment [pp. 342, 345] Every chord divides the interior of a circle into two regions called segments; the smaller is the *minor segment* (shaded), and the larger is the *major segment*.

Area of segment

$= \frac{1}{2}r^2(\theta - \sin\theta)$

where $\theta^c = \angle AOB$



Selections [pp. 251, 729] counted when order is not important. The number of ways of selecting r objects from a total of n objects is

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

An alternative notation for $\binom{n}{r}$ is ${}^nC_r$.

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

Set difference The set $A \setminus B$ contains all the elements of A that are not in B. For example, $\mathbb{R} \setminus \{0\}$ is the set of all real numbers excluding 0.

Set notation [pp. 84, 361]

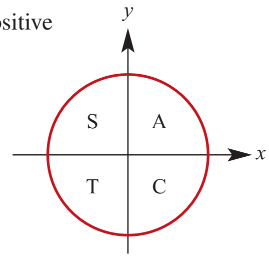
$\in$ means 'is an element of'
$\notin$ means 'is not an element of'
$\subseteq$ means 'is a subset of'
$\cap$ means 'intersection'
$\cup$ means 'union'
$\varnothing$ is the empty set, containing no elements

Sets of numbers [p. 85]

$\mathbb{N}$ is the set of natural numbers
$\mathbb{Z}$ is the set of integers
$\mathbb{Q}$ is the set of rational numbers
$\mathbb{R}$ is the set of real numbers

Signs of trigonometric functions [p. 315]

1st quadrant	all are positive
2nd quadrant	sin is positive
3rd quadrant	tan is positive
4th quadrant	cos is positive

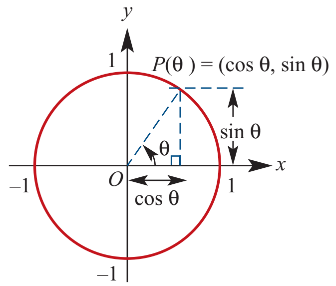


Simulation [p. 407] the process of finding an approximate solution to a probability problem by repeated trials using a simulation model

Simulation model [p. 407] a simple model which is analogous to a real-world situation. For example, the outcomes from a toss of a coin (head, tail) could be used as a simulation model for the sex of a child (male, female) under the assumption that in both situations the probabilities are 0.5 for each outcome.

Simultaneous equations [p. 10] equations of two or more lines or curves in the Cartesian plane, the solutions of which are the points of intersection of the lines or curves

Sine function [p. 311] sine θ is defined as the y-coordinate of the point P on the unit circle where OP forms an angle of θ radians with the positive direction of the x-axis.



Speed [p. 634] the magnitude of velocity

Speed, average [pp. 540, 634]

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

Standard form [p. 473] *see* scientific notation

Stationary point [pp. 618, 621] A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is said to be a stationary point if $f'(a) = 0$.

Straight line, equation given two points [p. 48] $y - y_1 = m(x - x_1)$, where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Straight line, gradient–intercept form [p. 45] The gradient–intercept form of the equation of a straight line is $y = mx + c$, where m is the gradient and c is the y-axis intercept.

Straight lines, parallel [p. 56] Two non-vertical straight lines are parallel to each other if and only if they have the same gradient.

Straight lines, perpendicular [p. 57] Two straight lines are perpendicular to each other if and only if the product of their gradients is -1 (or if one is horizontal and the other vertical).

Strictly decreasing [p. 587] A function f is strictly decreasing on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

Strictly increasing [p. 586] A function f is strictly increasing on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.

Subset [pp. 84, 361] A set B is called a subset of set A if every element of B is also an element of A. We write $B \subseteq A$.

Sum of two cubes [p. 293]

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Surd of order *n* [p. 89] a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power

T

Tangent function [p. 313] The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Tangent line, equation [pp. 575, 611] Let (x_1, y_1) be a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Translation [p. 224] A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule $(x, y) \to (x + h, y + k)$, where h and k are positive numbers. The curve with equation $y = f(x)$ is mapped to the curve with equation $y - k = f(x - h)$.

Tree diagram [pp. 382, 395] a diagram representing the outcomes of a multi-stage experiment

Trigonometric functions [pp. 311, 313] the sine, cosine and tangent functions

Trigonometric functions, exact values [p. 318]

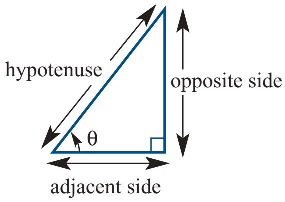
θ^c	$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined

Trigonometric ratios [p. 314]

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$



U

Union of sets [pp. 84, 361] The union of two sets A and B, written $A \cup B$, is the set of all elements which are in A or B or both.

Universal set, ε [p. 361] the set of all elements which are being considered in the current context

V

Velocity [pp. 554, 633] The velocity of a particle is defined as the rate of change of its position with respect to time.

Velocity, average [pp. 552, 633]

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Velocity, instantaneous [pp. 554, 633]

$$v = \frac{dx}{dt}$$

Venn diagram [pp. 84, 361] a diagram showing sets and the relationships between sets

Vertical-line test [p. 203] *see* function

Z

$\mathbb{Z}$ [p. 85] the set of all integers

Zero polynomial [p. 246] The number 0 is called the zero polynomial.

 ISBN 978-1-009-53719-3

Answers

Chapter 1

Exercise 1A

1 a 3 **b** 9 **c** 1 **d** -8
e 5 **f** 2 **g** $\frac{5}{3}$ **h** $\frac{-7}{2}$
i $\frac{7}{3}$ **j** $\frac{20}{3}$ **k** $\frac{-10}{3}$ **l** $\frac{14}{5}$

2 a $a+b$ **b** $a-b$ **c** $\frac{b}{a}$ **d** ab
e $\frac{bc}{a}$

3 a $y=5$ **b** $t=5$ **c** $y=-\frac{3}{2}$ **d** $x=2$
e $a=\frac{11}{2}$ **f** $a=\frac{8}{3}$ **g** $y=136$ **h** $t=1$
i $x=12$ **j** $y=-\frac{9}{5}$ **k** $x=-7$ **l** $y=2$

4 a $\frac{4}{3}$ **b** -5 **c** 2

5 a -1 **b** 18 **c** $\frac{6}{5}$ **d** 23
e 0 **f** 10 **g** 12 **h** 8
i $-\frac{14}{5}$ **j** $\frac{12}{5}$ **k** $\frac{7}{2}$

6 a $\frac{-b}{a}$ **b** $\frac{e-d}{c}$ **c** $\frac{c}{a}-b$ **d** $\frac{b}{c-a}$
e $\frac{ab}{b+a}$ **f** $a+b$ **g** $\frac{b-d}{a-c}$ **h** $\frac{bd-c}{a}$

7 a -18 **b** -78.2 **c** 16.75 **d** 28
e 34 **f** $\frac{3}{26}$

8 $x=\frac{a^2+b^2+2ab}{ac+bc}=\frac{a+b}{c}$

9 $x=\frac{ab}{a-b-c}$

Exercise 1B

1 a $x+2=6$, 4 **b** $3x=10$, $\frac{10}{3}$
c $3x+6=22$, $\frac{16}{3}$ **d** $3x-5=15$, $\frac{20}{3}$
e $6(x+3)=56$, $\frac{19}{3}$ **f** $\frac{x+5}{4}=23$, 87

2 $A=\$8$, $B=\$24$, $C=\$16$ **3** 14 and 28
4 8 kg **5** 1.3775 m^2 **6** 49, 50, 51
7 17, 19, 21, 23 **8** 4200 L **9** 21
10 3 km **11** 9 and 12 dozen
12 7.5 km/h **13** 3.6 km **14** 30, 6

Exercise 1C

1 a $x=-1, y=-1$ **b** $x=5, y=21$
c $x=-1, y=5$ **d** $x=5, y=19$
e $x=-4, y=-13$ **f** $x=-\frac{8}{5}, y=-\frac{2}{5}$

2 a $x=8, y=-2$ **b** $x=-1, y=4$
c $x=7, y=\frac{1}{2}$

3 a $x=2, y=-1$ **b** $x=2.5, y=-1$
c $m=2, n=3$ **d** $x=2, y=-1$
e $s=2, t=5$ **f** $x=10, y=13$
g $x=\frac{4}{3}, y=\frac{7}{2}$ **h** $p=1, q=-1$
i $x=-1, y=\frac{5}{2}$

4 a No solutions **b** Infinitely many solutions
c One solution **d** One solution

Exercise 1D

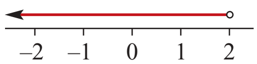
1 25, 113 **2** 22.5, 13.5
3 a \$70 **b** \$12 **c** \$3
4 a \$168 **b** \$45 **c** \$15
5 17 and 28 **6** 44 and 12
7 5 pizzas, 25 hamburgers
8 Started with 60 and 50; finished with 30 each
9 134 adults, 16 children **10** $\frac{7}{10}$
11 26 **12** 420 adults, 540 children
13 \$17 000 **14** 120 shirts, 300 ties
15 360 Outbacks, 300 Bush Walkers

16 2800 in Brisney, 3200 in Sydbane
17 20 kg at \$10, 40 kg at \$11 and 40 kg at \$12

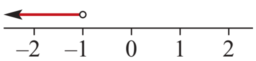
Exercise 1E

1 a $x < 1$ **b** $x > 13$ **c** $x \geq 3$ **d** $x \leq 12$
e $x \leq -6$ **f** $x > 3$ **g** $x < -2$ **h** $x \geq -8$
i $x \leq \frac{3}{2}$

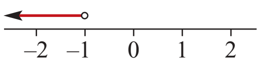
2 a $x < 2$



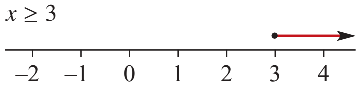
b $x < -1$



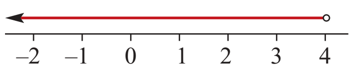
c $x < -1$



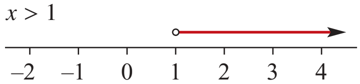
d $x \geq 3$



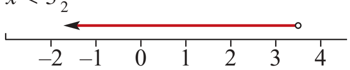
e $x < 4$



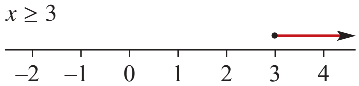
f $x > 1$



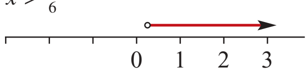
g $x < 3\frac{1}{2}$



h $x \geq 3$



i $x > \frac{1}{6}$



3 a $x > \frac{-1}{2}$ **b** $x < 2$ **c** $x > -5$

4 $3x < 20,\ x < \frac{20}{3}$, 6 pages **5** 87

Exercise 1F

1 a 18 **b** 9 **c** 3 **d** −18
e 3 **f** 81 **g** 5 **h** 20

2 a $S = a + b + c$ **b** $P = xy$ **c** $C = 5p$
d $T = dp + cq$ **e** $T = 60a + b$

3 a 15 **b** 31.4 **c** 1000 **d** 12
e 314 **f** 720

4 a $V = \frac{c}{P}$ **b** $a = \frac{F}{m}$ **c** $P = \frac{I}{rt}$
d $r = \frac{w - H}{C}$ **e** $t = \frac{S - P}{Pr}$ **f** $r = \frac{R(V-2)}{V}$

5 a $T = 48$ **b** $b = 8$ **c** $h = 3.82$ **d** $b = 10$

6 a $(4a + 3w)$ m **b** $(h + 2b)$ m
c $3wh$ m^2 **d** $(4ah + 8ab + 6wb)$ m^2

7 a i $T = 2\pi(p + q) + 4h$ **ii** $88\pi + 112$
b $p = \frac{A}{\pi h} - q$

8 a $D = \frac{2}{3}$ **b** $b = 2$ **c** $n = \frac{60}{29}$ **d** $r = 4.8$

9 a $D = \frac{1}{2}bc(1 - k^2)$ **b** $k = \sqrt{1 - \frac{2D}{bc}}$
c $k = \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$

10 a $P = 4b$ **b** $A = 2bc - c^2$ **c** $b = \frac{A + c^2}{2c}$

11 a $b = \frac{a^2 - a}{2}$ **b** $x = \frac{-ay}{b}$
c $r = \pm\sqrt{3q - p^2x^2}$ **d** $v = \pm\sqrt{u^2\left(1 - \frac{x^2}{y^2}\right)}$

Chapter 1 review

Short-response questions

Technology-free

1 a 1 **b** $\frac{-3}{2}$ **c** $\frac{-2}{3}$ **d** −27
e 12 **f** $\frac{44}{13}$ **g** $\frac{1}{8}$ **h** 31

2 a $t = a - b$ **b** $\frac{cd - b}{a}$ **c** $\frac{d}{a} + c$
d $\frac{cb - a}{c - 1}$ **e** $\frac{2b}{c - a}$ **f** $\frac{1 - cd}{ad}$

3 a $x < \frac{2}{3}$ **b** $x \leq -148\frac{1}{2}$ **c** $x < \frac{22}{29}$ **d** $x \geq \frac{-7}{17}$

4 $x = 2(z + 3t),\ -10$

5 a $a = \frac{28}{11}, b = -\frac{9}{11}$ **b** $a = -\frac{11}{5}, b = -\frac{33}{5}$

6 a $d = e^2 + 2f$ **b** $f = \frac{d - e^2}{2}$ **c** $f = \frac{1}{2}$

7 400π cm^2

8 a 196π **b** $\frac{975\pi}{2}$

9 a $r = \frac{A}{\pi s}$ **b** $w = \frac{T - P}{Pr}$
c $r = \frac{n - p}{v^2}$ **d** $x = \frac{ac - b^2}{b}$

10 a $s = 75$ **b** $t = 8$

11 $5\sqrt{2}$ cm

12 12 m and 17 m

13 $m = 2$ and $n = 15$

14 Mr Apollo earns \$100 000, Mr Adonis earns \$107 200 and Ms Aphrodite earns \$96 000

15 5 hours travelling on highways

16 a $C = \frac{-10}{9}$ **b** $F = 86$ **c** $x = -40$
d $x = -62.5$ **e** $x = \frac{-160}{13}$ **f** $k = 5$

17 a $r = \frac{2uv}{u + v}$ **b** $m = \frac{v}{u}$

18 a $T = 6w + 6\ell$
b i $T = 8w$ **ii** $\ell = \frac{25}{6}, w = 12\frac{1}{2}$

 ISBN 978-1-009-53719-3

c **i** $y = \frac{L - 6x}{8}$ **ii** $y = 22$
d $x = 10, y = 5$

Technology-active

19 **a** 18.662 **b** 3.43 **c** 7.6 **d** −23.5
e 4.05 **f** 712.89 **g** 2.6 **h** 64.4

20 **a** 6.16 **b** 6.2 **c** 11.2 **d** 3.56
e 497.6 **f** −0.564 **g** 2.3 **h** −9

21 **a** $x = \frac{5}{6}, y = \frac{1}{3}$ **b** $x = \frac{180}{181}, y = -\frac{290}{181}$
c $x = \frac{500}{53}, y = -\frac{110}{53}$

22 435.42

23 \$7366

24 The three lines intersect at the point (4, 3)

25 **a** Distance Tom travelled = ut km
Distance Julie travelled = vt km
b **i** $t = \frac{d}{u+v}$ h
ii Distance from $A = \frac{ud}{u+v}$ km
c $t = 1.25$ h, distance from $A = 37.5$ km

26 **a** Average speed $= \frac{2uv}{u+v}$
b **i** $\frac{uT}{v}$ **ii** $\frac{vT + uT}{v}$

Multiple-choice questions

1 C **2** C **3** B **4** A **5** B **6** B
7 A **8** A **9** A **10** A **11** D **12** A
13 C **14** C **15** B **16** D **17** A

Chapter 2

Exercise 2A

1 **a** (5, 8) **b** $(\frac{1}{2}, \frac{1}{2})$ **c** (1.6, 0.7)
d (−0.7, 0.85)

2 $M_{AB}(3, 3)$, $M_{BC}(8, 3\frac{1}{2})$, $M_{AC}(6, 1\frac{1}{2})$

3 Coordinates of C are (6, 8.8)

4 **a** (4, 4) **b** (2, −0.2) **c** (−2, 5) **d** (−4, −3)

5 $\left(\frac{1+a}{2}, \frac{4+b}{2}\right)$, $a = 9$, $b = -6$

6 **a** $5\sqrt{2} \approx 7.07$ **b** $\sqrt{17} \approx 4.12$
c $\sqrt{34} \approx 5.83$ **d** 13

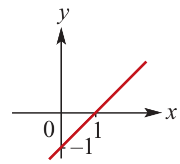
7 $\sqrt{97} + \sqrt{85} + \sqrt{104} \approx 29.27$

8 $PM = \sqrt{145} \approx 12.04$ **9** DN

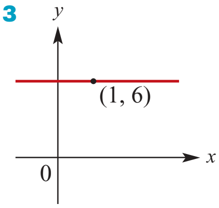
Exercise 2B

1 **a** 4 **b** 2 **c** $\frac{1}{4}$
d −4 **e** 1 **f** −1
g $\frac{5}{4}$ **h** −2 **i** $\frac{-5}{4}$
j $\frac{4}{3}$ **k** 0

2 Any line parallel to the one shown $y = x - 1$



3



4 **a** $-\frac{1}{4}$ **b** $-\frac{5}{2}$ **c** −2 **d** −8 **e** 0 **f** −1
g 7 **h** 11 **i** −13 **j** 11 **k** 111 **l** 61

5 **a** −2 **b** $\frac{2}{5}$

6 **a** 54 **b** $\frac{5}{6}$

7 **a** 45° **b** 45° **c** 26.57° **d** 135°

8 **a** 45° **b** 26.57° **c** 161.57° **d** 49.40°
e 161.57° **f** 135°

9 **a** 1 **b** −1 **c** $\sqrt{3}$ **d** $-\sqrt{3}$

Exercise 2C

1 **a** $m = 3, c = 6$ **b** $m = -6, c = 7$
c $m = 3, c = -6$ **d** $m = -1, c = -4$

2 **a** $y = 3x + 5$ **b** $y = -4x + 6$ **c** $y = 3x - 4$

3 **a** $m = 3, c = -6$ **b** $m = 2, c = -4$
c $m = \frac{1}{2}, c = -2$ **d** $m = \frac{1}{3}, c = -\frac{5}{3}$

4 **a** $m = 2, c = -9$ **b** $m = -\frac{3}{4}, c = \frac{5}{2}$
c $m = -\frac{1}{3}, c = -2$ **d** $m = \frac{5}{2}, c = -2$

5 **a** $y = 3x - 11$ **b** $y = -2x + 9$

6 **a** $y = -\frac{1}{3}x + \frac{11}{3}$ **b** $y = -\frac{7}{5}x + 4$
c $y = -2x + 4$ **d** $y = \frac{11}{3}x - \frac{61}{3}$

7 **a** 2 **b** $y = 2x + 6$

8 **a** $y = 2x + 4$ **b** $y = -2x + 8$

9 **a** $y = 2x + 6$ **b** $y = -2x + 4$
c $y = -5x + 15$

10 **a** $y = -\frac{2}{3}x + 4$ **b** $y = -2x - 6$
c $y = -x + 4$ **d** $y = -\frac{3}{2}x + 3$

11 **a** $y = \frac{2}{3}x + 4$ **b** $y = \frac{2}{3}x - \frac{2}{3}$
c $y = \frac{1}{2}x + 1\frac{1}{2}$ **d** $y = -\frac{1}{2}x + 2$
e $y = x + 3.5$ **f** $y = -0.5x + 0.25$

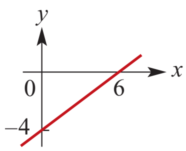
12 **a** $y = 4x + 4$ **b** $y = -\frac{2}{3}x$ **c** $y = -x - 2$
d $y = \frac{1}{2}x - 1$ **e** $y = 3\frac{1}{2}$ **f** $x = -2$

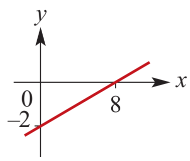
13 Yes **14** Only c

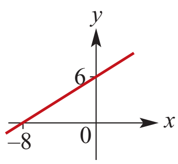
15 **a** $x = 4$ **b** $y = 11$ **c** $x = 11$ **d** $y = -1$

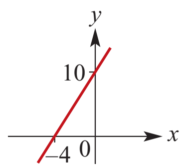
Exercise 2D

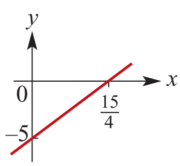
1 a (0, 4), (4, 0) **b** (0, −4), (4, 0)
c (0, −6), (−6, 0) **d** (0, 8), (−8, 0)

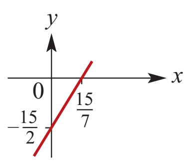
2 a


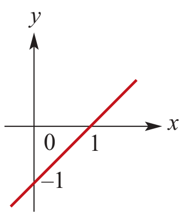
b


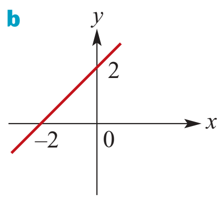
c


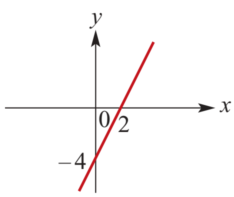
d


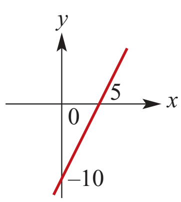
e


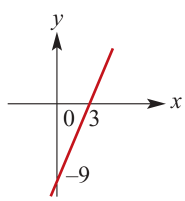
f


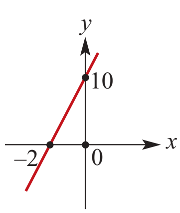
3 a


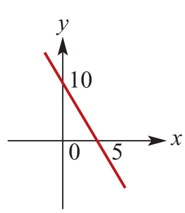
b


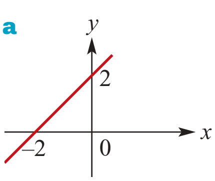
c


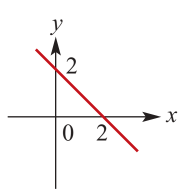
4 a


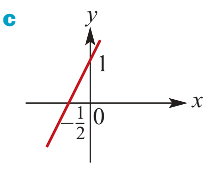
b


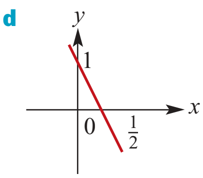
c


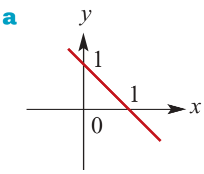
d


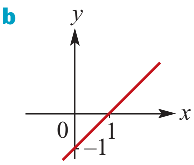
5 a


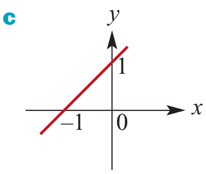
b


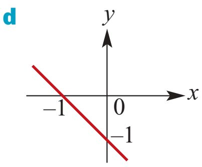
c


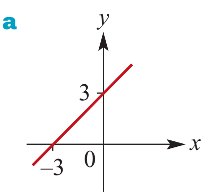
d


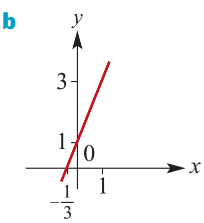
6 a


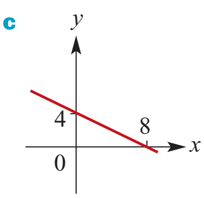
b


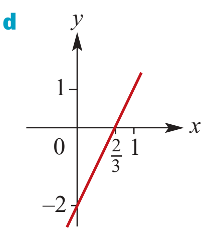
c


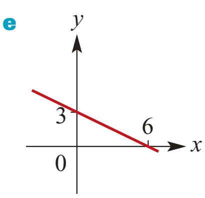
d


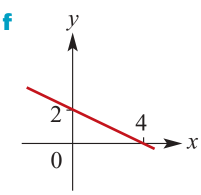
7 a


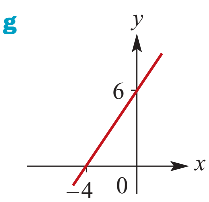
b


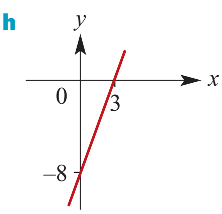
c


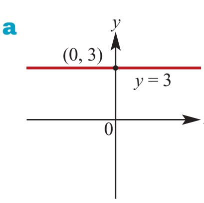
d


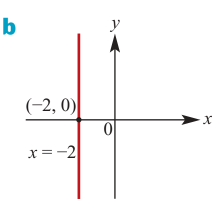
e


f


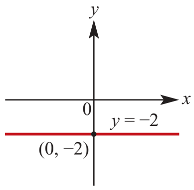
g


h


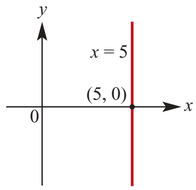
8 a


b


c



d



9 a 45° **b** 135° **c** 45° **d** 135°
e 63.43° (to 2 d.p.) **f** 116.57° (to 2 d.p.)

10 a 71.57° **b** 135° **c** 45° **d** 161.57°

11 $a = -4,\ b = \frac{4}{3},\ d = -1,\ e = \frac{14}{3}$

Exercise 2E

1 a $y = 2x - 10$ **b** $y = -\frac{1}{2}x$
c $y = -2x + 6$ **d** $y = \frac{1}{2}x - 4$
e $y = \frac{2}{3}x - \frac{14}{3}$ **f** $y = -\frac{3}{2}x + 4$
g $y = -\frac{1}{3}x - \frac{2}{3}$ **h** $y = 3x - 14$

2 Parallel lines: a, b, c; non-parallel lines: d

3 a $y = 4$ **b** $x = 2$ **c** $y = 4$ **d** $x = 3$

4 $y = 2x + 2$

5 Midpoint of AB is $(-1, 6)$; $y = 2x + 8$

6 $m_{BC} = -\frac{3}{5},\ m_{AB} = \frac{5}{3}$
$\therefore m_{BC} \times m_{AB} = -\frac{3}{5} \times \frac{5}{3} = -1$
$\therefore \triangle ABC$ is a right-angled triangle

7 $m_{AB} = -2,\ m_{BC} = \frac{1}{2}$

8 $m_{RS} = -\frac{1}{2},\ m_{ST} = 2 \therefore RS \perp ST$
$m_{UT} = -\frac{1}{2},\ m_{ST} = 2 \therefore UT \perp ST$
(Also need to show $SR = UT$.)
$\therefore RSTU$ is a rectangle

9 $\ell = -\frac{16}{3},\ m = \frac{80}{3}$

10 a $y = -\frac{1}{2}x + \frac{11}{2}$ **b** $B(1, 5)$ **c** $C(2, 7)$

Exercise 2F

1 $m = 5$

2 $c = 5$

3 a $y = -\frac{1}{m}x + 3$ **b** $m = \frac{1}{7}$

4 $m = 2$

5 a $x = \frac{3}{m}$ **b** $m = \frac{9}{5}$ **c** $m \geq 3$
d $y = -\frac{1}{m}x - 3$

6 a $x = -\frac{c}{2}$ **b** $c = -4$ **c** $c \geq -2$
d $y = -\frac{1}{2}x + c$

7 a $x = 4a$ **b** $m = \frac{12}{a}$ **c i** $a = 6$ **ii** $a = -6$

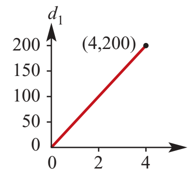
8 a $x = \frac{c}{2}$ **b** $c = 9$ **c** $c \leq 2$ **d** $y = \frac{1}{2}x + c$
e i $c = 12$ **ii** $c = 4$ **iii** $c = 8$

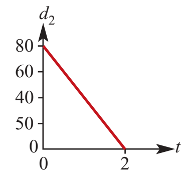
9 a $\frac{12}{b}$ **b** $-\frac{3}{b}$ **c i** $b = -3$ **ii** $b = \frac{3}{2}$
d $y = \frac{b}{3}x - \frac{4b}{3}$

Exercise 2G

1 $w = 20n + 350$ for $n \in \mathbb{N} \cup \{0\}$

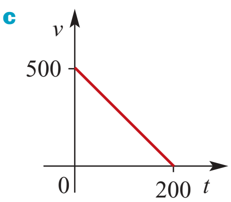
2 a $d_1 = 50t$ **b** $d_2 = 80 - 40t$
c Gradient = 50 Gradient = −40





3 a $V = 5t$ **b** $V = 10 + 5t$

4 a $v = 500 - 2.5t$
b Domain $0 \leq t \leq 200$; Range $0 \leq v \leq 500$
c

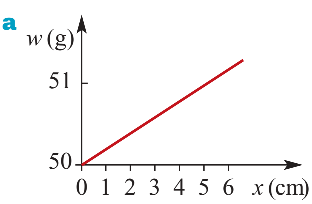


5 $C = 1.5n + 2.6$

6 a $C = 0.24x + 85$ **b** \$145

7 $d = 200 - 5t$

8 a



b $w = 0.2x + 50$ **c** $x = 12.5$ cm

9 a $C = 0.06n - 1$ **b** \$59

10 a $C = 5n + 175$ **b** Yes **c** \$175

Exercise 2H

1 Both lines have gradient −1, but their y-axis intercepts are 6 and $\frac{13}{2}$

2 $(t, 6 - t)$ where t is any real number

3 a $m = 4$ **b** $m \neq 4$ **c** $m = \frac{9}{5}$

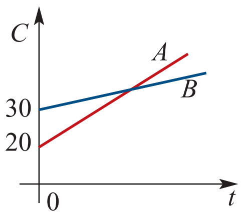
4 $k = 2,\ m = 5$

5 $k = 24,\ m = 0$

6 $m = -3$

7 a $m = -5$ **b** $m = 3$

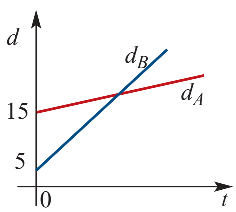
8 a



b $t = 5$

9 $b = 0.28$ and $a = 0.3$, $\dfrac{25}{7}$ m/s

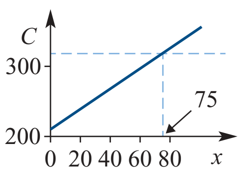
10 a



b 2 p.m.

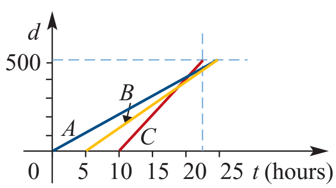
11 a $C_1 = 210 + 1.6x$, $C_2 = 330$

b



c Fixed-charge method is cheaper for $x > 75$

12 a



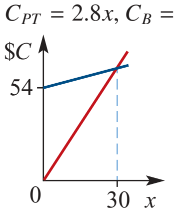
b C wins the race

d C leaves 5 hours after B, and overtakes B $13\frac{1}{3}$ hours after B had started, then overtakes A 20 hours after A had started. C wins with a total time of $22\frac{1}{2}$ hours ($12\frac{1}{2}$ hours for journey + 10 hours handicap), with A and B deadheating for second place, each with a total time of 25 hours.

13 Both craft will pass over the point $(5\frac{1}{3}, -4)$

14 a $C_{PT} = 2.8x$, $C_B = 54 + x$

b

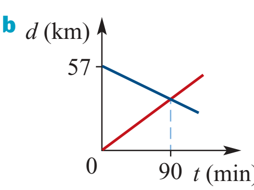


c More than 30

15 a $d_A = \dfrac{1}{3}t$

$d_M = 57 - \dfrac{3}{10}t$

b



c 10:30 a.m. **d** Anne 30 km, Maureen 27 km

Chapter 2 review

Short-response questions

Technology-free

1 a Midpoint = $(3, 2)$; Length = 4

b Midpoint = $\left(-\dfrac{1}{2}, -\dfrac{9}{2}\right)$; Length = $\sqrt{74}$

c Midpoint = $\left(5, \dfrac{5}{2}\right)$; Length = 5

2 a $\dfrac{9}{4}$ **b** $-\dfrac{10}{11}$ **c** Undefined

d -1 **e** $\dfrac{b}{a}$ **f** $\dfrac{-b}{a}$

3 a $y = 4x$ **b** $y = 4x + 5$

c $y = 4x + 2$ **d** $y = 4x - 5$

4 a $a = -2$ **b** $b = \dfrac{20}{3}$

5 $4y + 3x = -7$

6 $3y + 2x = -5$

7 a $y = 11$ **b** $y = 6x - 10$ **c** $3y + 2x = -3$

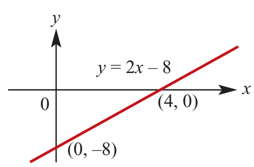
8 $y = x + 1$

9 $y + x = 1$

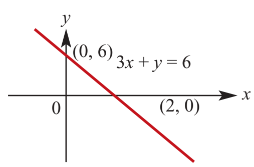
10 $y = \dfrac{1}{3}x + \dfrac{20}{3}$

11 $a = 1, b = -\dfrac{1}{2}, d = 5, e = 3$

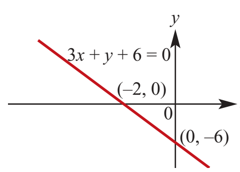
12 a



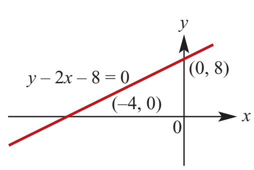
b



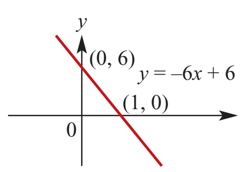
c



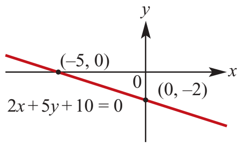
d



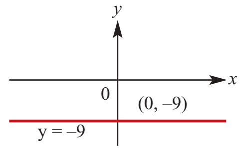
e



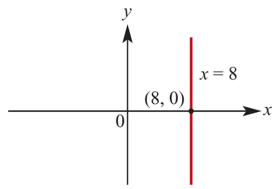
f



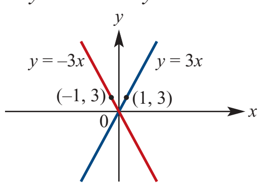
13 a $y = -9$



b $x = 8$



c i $y = 3x$ **ii** $y = -3x$



14 a $d = 60t$ **b** $m = 60$

15 $S = 800 + 500n$

16 a $y = 2x + 2$ **b i** $\frac{-2}{a}$ **ii** $-2 < a < 0$

c $\left(\frac{1}{a-1}, \frac{1}{a-1} + 3\right)$

17 a $C = 550 + 190n$ **b** 12 days

c Fewer than 5 days

18 a Cost of the plug

b Cost per metre of cable **c** 1.8 **d** $11\frac{1}{9}$ m

19 a The maximum profit (when $x = 0$)

b 43 seats

c Profit reduces by \$24 for each empty seat

20 a $y = 3x + 2$ **b** $(0, 2)$

c $y = 3x - 8$ **d** $(2, -2)$

e Area = 10 square units

f Area = 40 square units

Technology-active

21 a 7.07 **b** 10.30 **c** 9.22

22 a 168.69° **b** 21.80° **c** 50.19°

23 a $x = 0.40, y = -2.18$ **b** $x = -5.29, y = 7.01$

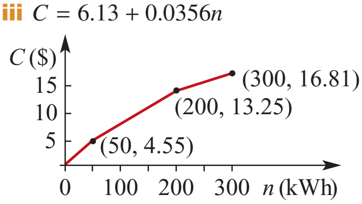
c $x = 3.86, y = 5.97$

24 $11x + 3y = 29$, 105.26°

25 a i $C = 0.091n$ **ii** $C = 1.65 + 0.058n$

iii $C = 6.13 + 0.0356n$

b



i For 30 kWh, $C = 2.73$

ii For 90 kWh, $C = 6.87$

iii For 300 kWh, $C = 16.81$

c 389.61 kWh

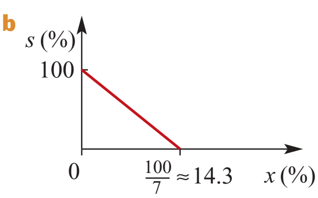
26 a $\frac{128}{19}$

b $y = -\frac{199}{190}x + \frac{128}{19}$

c No, since gradient of AB is $\frac{20}{19} \approx 1.053$, whereas the gradient of VC is -1.047

27 a $s = 100 - 7x$

b



c $\frac{5}{7}\%$

d $14\frac{2}{7}\%$

e Probably not a realistic model when $s = 0$

f $0 \le x \le 14\frac{2}{7}$

28 a AB, $y = x + 2$; CD, $y = 2x - 6$

b Intersection is at (8, 10), on the near bank

29 a No **b** $1\frac{41}{71}$ km to the east of H

30 a $y = x - 38$ **b** $B(56, 18)$

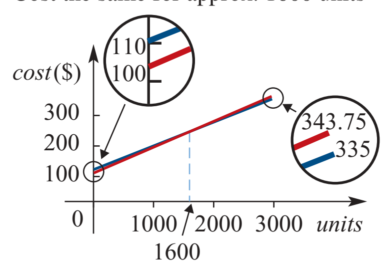
c $y = -2x + 166$ **d** $(78, 10)$

31 a Cost with method 1 is \$226.75; cost with method 2 is \$227; so method 1 is cheaper

b

	0	1000	2000	3000
Method 1	100	181.25	262.50	343.75
Method 2	110	185	260	335

c Cost the same for approx. 1600 units



d $C_1 = 0.08125x + 100$, $C_2 = 0.075x + 110$, $x = 1600$

Multiple-choice questions

1 C **2** D **3** A **4** D **5** B **6** C
7 A **8** D **9** B **10** D **11** B **12** D
13 B **14** B **15** D

Chapter 3

Exercise 3A

1 a {7, 13} **b** {3, 4, 5, 6, 7, 11, 13, 19}

2 a $\{2, 3, 5, 7, 11, 13\}$ **b** $\{1, 3, 5, 7, 9, 11, 13\}$
c $\{3, 5, 7, 11, 13\}$
d $\{1, 2, 3, 5, 7, 9, 11, 13\}$

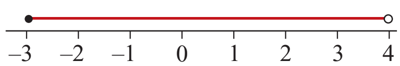
3 a $\{7, 11\}$ **b** $\{7, 11\}$
c $\{1, 2, 3, 5, 7, 11, 15, 25, 30\}$
d $\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$
e $\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$
f $\{1, 7, 11, 25, 30\}$

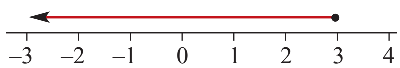
4 a $\{11, 17, 25\}$ **b** $\{11, 25\}$
c $\{11, 14, 15, 17, 23, 25, 36, 37, 60\}$
d $\{11, 14, 17, 23, 25, 30, 32, 36, 37\}$
e $\{11, 14, 15, 17, 23, 25, 30, 32, 36, 37, 60\}$
f $\{11, 14, 15, 17, 25, 60\}$

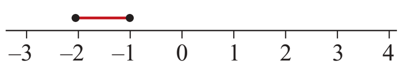
5 $\{-1, \frac{3}{4}, 1, \frac{11}{8}\}$

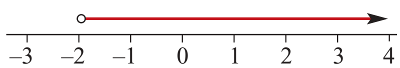
6 a $A \cap B \cap C = \varnothing; A \cup B \cup C =$ $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
b $A \cap B \cap C = \{0\}; A \cup B \cup C =$ $\{0, 3, 4, 6, 8, 9, 10, 12, 15, 16, 20\}$

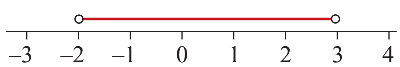
7 $\{-1, 1, 2, 100\}$

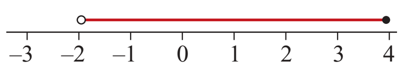
8 a 

b 

c 

d 

e 

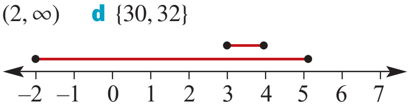
f 

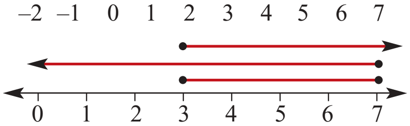
9 a $-2 < x \le 1$ or $(-2, 1]$
b $-3 \le x \le 3$ or $[-3, 3]$
c $-3 \le x < 2$ or $[-3, 2)$
d $-1 < x < 2$ or $(-1, 2)$

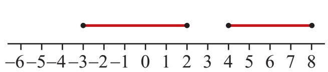
10 a $[-1, 2]$ **b** $(-4, 2]$ **c** $(0, \sqrt{2})$
d $[-3, 2)$ **e** $(-1, \infty)$ **f** $(-\infty, -2]$
g $(-\infty, \infty)$ **h** $[0, \infty)$ **i** $(-\infty, 0]$

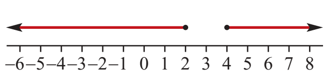
11 a $\{x : 3 < x \le 8\}$ **b** $\{x : -2 \le x < 11\}$

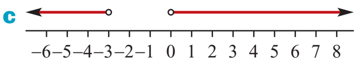
12 a $\{7\}$ **b** B, i.e. $\{7, 11, 25, 30, 32\}$
c $(2, \infty)$ **d** $\{30, 32\}$

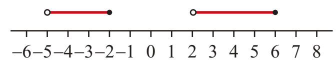
13 a 

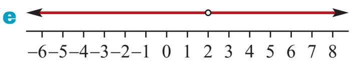
b 

14 a 

b 

c 

d 

e 

f 

15 a $(-6, -3)$ **b** $\varnothing$ **c** $[-6, 0]$
d $[-1, 2]$ **e** $\{1\}$ **f** $(-10, -1]$

Exercise 3B

1 a $28\sqrt{6}$ **b** $-24\sqrt{77}$ **c** $28\sqrt{10}$

2 a 11 **b** 50 **c** 44 **d** 147

3 a $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $3\sqrt{3}$ **d** $5\sqrt{2}$
e $3\sqrt{5}$ **f** $11\sqrt{10}$ **g** $7\sqrt{2}$ **h** $6\sqrt{3}$
i 5 **j** $5\sqrt{3}$ **k** $16\sqrt{2}$

4 a $\frac{2\sqrt{6}}{5}$ **b** $\frac{3\sqrt{7}}{8}$ **c** $\frac{4\sqrt{3}}{7}$ **d** $\frac{4\sqrt{5}}{9}$

5 a $14\sqrt{3}$ **b** $5\sqrt{5}$ **c** $10\sqrt{2} - \sqrt{13}$
d $2\sqrt{11} + 6\sqrt{13}$ **e** $6\sqrt{7} + 4\sqrt{6} - 2\sqrt{17}$
f $9\sqrt{3} - 4\sqrt{6}$

6 a $3\sqrt{2}$ **b** $6\sqrt{3}$ **c** $4\sqrt{7}$
d $5\sqrt{10}$ **e** $28\sqrt{2}$ **f** 0

7 a $\sqrt{14} + 11\sqrt{3}$ **b** $5\sqrt{7}$ **c** 0
d $\sqrt{3} + \sqrt{2}$ **e** $15\sqrt{3} + 5\sqrt{2}$
f $\sqrt{5} + \sqrt{2}$

8 a $27 - 10\sqrt{2}$ **b** $6\sqrt{2} + 5$ **c** 7
d $6 - 4\sqrt{2}$ **e** $10\sqrt{5} + 6$ **f** 2
g $2\sqrt{6} + 5$ **h** $13 - \sqrt{22}$ **i** 18

9 a $17 + 11\sqrt{5}$ **b** 2
c 30 **d** 27

10 a $4a - 4\sqrt{a} + 1$ **b** $2\sqrt{(x+1)(x+2)} + 2x + 3$

11 $3\sqrt{5}, 4\sqrt{3}, 7, 5\sqrt{2}$

12 a $5 - 3\sqrt{2}$ **b** $7 - 2\sqrt{6}$

13 a $4\sqrt{13}$ **b** $5\sqrt{5}$ **c** $\sqrt{39}$ **d** $3\sqrt{5}$
e $3\sqrt{14}$ **f** $\sqrt{161}$

14 $\sqrt{2}$

15 a $\sqrt{5}$ **b** $\sqrt{6}$

16 $7\sqrt{5}$

17 a $\sqrt{14}$ **b** $4 + \sqrt{14}$ **c** $\frac{1}{2}$

18 a $2 + \sqrt{2}$ **b** $-\sqrt{2}$ **c** $-5\sqrt{2}$

19 a 4 **b** $8 - 4\sqrt{3}$ **c** $12 - 6\sqrt{3}$

Exercise 3C

1 a $\frac{\sqrt{5}}{5}$ **b** $\sqrt{7}$ **c** $-\frac{\sqrt{2}}{2}$ **d** $\frac{2\sqrt{3}}{3}$
e $\frac{\sqrt{6}}{8}$ **f** $\frac{\sqrt{2}}{4}$ **g** $\frac{\sqrt{33}}{11}$ **h** $\frac{5\sqrt{7}}{14}$

i $\frac{\sqrt{3}}{2}$ **j** $\frac{\sqrt{20}}{30} = \frac{\sqrt{5}}{15}$

2 a $\frac{2-\sqrt{2}}{2}$ **b** $\sqrt{2}-1$
c $\sqrt{3}+2$ **d** $\frac{4+\sqrt{10}}{6}$
e $\sqrt{6}-2$ **f** $\frac{\sqrt{5}+\sqrt{3}}{2}$
g $3(\sqrt{6}+\sqrt{5})$ **h** $2\sqrt{2}+3$
i $\frac{6\sqrt{10}+23}{13}$ **j** $\frac{28\sqrt{11}+93}{5}$

3 a $2(2\sqrt{2}+3)$ **b** $4\sqrt{5}+9$
c $\sqrt{2}-1$ **d** $4-2\sqrt{3}$
e $\frac{2\sqrt{3}}{9}$ **f** $\frac{5\sqrt{3}+8}{11}$
g $\frac{\sqrt{5}+3}{2}$ **h** $\frac{5\sqrt{2}+6}{14}$

4 a $\sqrt{5}+\sqrt{2}-3$ **b** $\sqrt{5}+\sqrt{2}+3$
c $-\sqrt{5}+\sqrt{2}-1$

5 a $\sqrt{2}$ **b** 4 **c** 6
d $4(\sqrt{5}+2)$ **e** $2(\sqrt{3}+\sqrt{2})$ **f** 1
g $\sqrt{7}+\sqrt{6}$ **h** $\frac{\sqrt{5}+\sqrt{3}}{2}$
i $\frac{\sqrt{5}+\sqrt{2}}{3}$

6 a $\frac{\sqrt{3}}{3}$ **b** $2-\sqrt{3}$ **c** $\sqrt{3}-\sqrt{2}$

7 a $EG = \frac{\sqrt{17}}{2}, EC = \frac{\sqrt{26}}{2}$
b $EG = \sqrt{14}, EC = 4\sqrt{7}$
c $EG = \frac{8\sqrt{30}}{15}, EC = \frac{2\sqrt{4530}}{15}$
d $EG = \sqrt{14}, EC = \sqrt{19}$

8 a $1+\sqrt{2}$
b $-1+\sqrt{3}$
c i $\sqrt{2}-1$ **ii** $\sqrt{3}+1$

Chapter 3 review

Short-response questions

1 a $-2 \le x < 4$: Interval notation $[-2, 4)$
b $-2 \le x \le 4$: Interval notation $[-2, 4]$
c $1 \le x < 8$: Interval notation $[1, 8)$
d $-1 < x \le 6$: Interval notation $(-1, 6]$
e $-4 < x \le 2$ or $1 < x \le 5$: Interval notation $(-4, -2] \cup (1, 5]$
f $-4 < x \le 2$ or $x > 2$: Interval notation $(-4, -2] \cup (2, \infty)$
g $x \le -3$ or $x > 1$: Interval notation $(-\infty, -3] \cup (1, \infty)$

2 a $8\sqrt{7}$ **b** $11\sqrt{3}-3\sqrt{2}$ **c** $7\sqrt{5}$
d $3\sqrt{7}$ **e** $3\sqrt{11}$ **f** $9\sqrt{6}$

3 a 216 **b** 30 **c** $4\sqrt{6}$
d 48 **e** $6\sqrt{5}+15\sqrt{2}$ **f** $6\sqrt{2}$
g $7+4\sqrt{3}$ **h** $15-11\sqrt{2}$
i $3-2\sqrt{2}$ **j** -1 **k** 29
l $\sqrt{6}+2\sqrt{2}$

4 $a = 6, b = 2$
5 $a = 6, b = 12$
6 $\sqrt{195}$
7 $17\sqrt{7}$
8 a $8+6\sqrt{2}$ **b** $7+7\sqrt{2}$
9 $3+2\sqrt{3}$
10 a $\sqrt{13}$ **b** $2\sqrt{13}$ **c** $3\sqrt{10}$ **d** $2\sqrt{17}$
12 $\frac{2\sqrt{6}+2\sqrt{3}+9\sqrt{2}}{12}$

Multiple-choice questions

1 C **2** D **3** C **4** D
5 C **6** B **7** C **8** D
9 D **10** B **11** B

Chapter 4

Exercise 4A

1 a $2x-8$ **b** $-2x+8$ **c** $6x-12$
d $-12+6x$ **e** x^2-x **f** $2x^2-10x$

2 a $6x+1$ **b** $3x-6$ **c** $x+1$ **d** $5x-3$

3 a $14x-32$ **b** $2x^2-11x$
c $32-16x$ **d** $6x-11$

4 a $2x^2-11x$ **b** $3x^2-15x$
c $-20x-6x^2$ **d** $6x-9x^2+6x^3$
e $2x^2-x$ **f** $6x-6$

5 a $6x^2-2x-28$ **b** $x^2-22x+120$
c $36x^2-4$ **d** $8x^2-22x+15$
e $x^2-(\sqrt{3}+2)x+2\sqrt{3}$
f $2x^2+\sqrt{5}x-5$ **g** $3x^2+\sqrt{7}x-14$
h $5x^2+(10\sqrt{2}-3)x-6\sqrt{2}$
i $5x^2-(3\sqrt{5}+32\sqrt{10})x+96\sqrt{2}$

6 a $6x^3-5x^2-14x+12$ **b** x^3-1
c $24-20x-8x^2+6x^3$ **d** $3x^2+4x+3$
e $-10x^2+5x-2$

7 a $x^2-8x+16$ **b** $4x^2-12x+9$
c $36-24x+4x^2$ **d** $x^2-x+\frac{1}{4}$
e $x^2-2\sqrt{5}x+5$ **f** $x^2-4\sqrt{3}x+12$

8 a x^2-9 **b** $4x^2-16$ **c** $81x^2-121$
d $4x^2-9$ **e** $4x^2-25$ **f** x^2-5
g $4x^2-27$ **h** $3x^2-7$

9 a $x^2+y^2-z^2-2xy$ **b** $4a^2-4ab+b^2-c^2$
c $9w^2+8uz-16z^2-u^2$
d $4a^2-5b^2+4ac+c^2$

10 a i x^2+2x+1 **ii** $(x+1)^2$
b i $(x-1)^2+2(x-1)+1$ **ii** x^2

Exercise 4B

1 a $2(x+2)$ **b** $4(a-2)$ **c** $3(2-x)$
d $2(x-5)$ **e** $6(3x+2)$ **f** $8(3-2x)$

2 a $2x(2x-y)$ **b** $8x(a+4y)$ **c** $6b(a-2)$
d $2xy(3+7x)$ **e** $x(x+2)$ **f** $5x(x-3)$
g $-4x(x+4)$ **h** $7x(1+7x)$ **i** $x(2-x)$

3 a $6x^2y^2(x+2)$ **b** $xy(7x-6y)$
c $2xy^2(4x+3)$

4 a $(x^2+1)(x+5)$ **b** $(x+3)(y+2)$
c $(x-1)(x+1)(y-1)(y+1)$
d $(a+b)(x+y)$ **e** $(a^2+1)(a-3)$
f $(2a-5)(b-6)$ **g** $(2x+5)(x-1)$
h $(x+2)(x-2)(x+2)$
i $(x-a)(x+a)(x-b)$

5 a $(x-6)(x+6)$ **b** $(x-9)(x+9)$
c $(x-a)(x+a)$ **d** $(2x-9)(2x+9)$
e $(3x-4)(3x+4)$ **f** $(5x-y)(5x+y)$
g $3(x-4)(x+4)$ **h** $2(x-7)(x+7)$
i $3a(x-3)(x+3)$ **j** $(a-\sqrt{7})(a+\sqrt{7})$
k $(\sqrt{2}a-\sqrt{5})(\sqrt{2}a+\sqrt{5})$
l $(x-2\sqrt{3})(x+2\sqrt{3})$

6 a $(x-6)(x+2)$ **b** $(7+x)(3-x)$
c $3(x-1)(x+3)$ **d** $-5(2x+1)$
e $-24x$ **f** $-5(x+7)(x+1)$

7 a $(x-9)(x+2)$ **b** $(y-16)(y-3)$
c $(a-2)(a-12)$ **d** $(a+9)^2$
e $(x-8)(x+3)$ **f** $(x-12)(x+10)$

8 a $(3x-1)(x-2)$ **b** $(2x+1)(3x+2)$
c $(5x+3)(x+4)$ **d** $(2x+1)(x+4)$
e $(3x-2)(2x-5)$ **f** $(3x+1)(2x-3)$
g $(3x-2)(4x-3)$ **h** $(x-2)(5x+6)$
i $x(5x-6)(x-2)$

9 a $3(y-6)(y+2)$ **b** $2(x-7)(x-2)$
c $4(x-3)(x-6)$ **d** $3(x+2)(x+3)$
e $a(x+3)(x+4)$ **f** $3x(4-x)^2$

10 a $x(x+2)$ **b** $(2x-3)(x+2)$
c $2(2x+5)(x+2)$

Exercise 4C

1 a 2 or 3 **b** 0 or 2 **c** 4 or 3 **d** 4 or 3
e 3 or −4 **f** 0 or 1 **g** $\frac{5}{2}$ or 6 **h** −4 or 4

2 a −0.65, 4.65 **b** −0.58, 2.58 **c** −2.58, 0.58

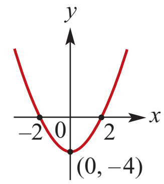
3 a 9, −8 **b** 4, 2 **c** 11, −3 **d** 4, −16
e 2, −7 **f** −3, 8

4 a $-\frac{3}{2}, -1$ **b** $\frac{1}{2}, \frac{3}{2}$ **c** $-\frac{2}{3}, -\frac{3}{2}$ **d** $-\frac{3}{2}, 2$
e $\frac{5}{6}, 3$ **f** $-\frac{3}{2}, 3$ **g** $\frac{1}{2}, \frac{3}{5}$ **h** $-\frac{3}{4}, \frac{2}{3}$
i $\frac{1}{2}$ **j** −5, 1 **k** 0, 3 **l** −5, −3
m $\frac{1}{5}, 2$

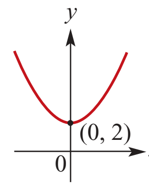
5 3 **6** 4 or 9 **7** 2, $2\frac{3}{8}$
8 13 **9** 50 **10** 6 cm, 2 cm
11 5 cm **12** \$90, \$60 **13** 42

Exercise 4D

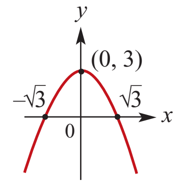
1 a i (0, −4)
ii $x = 0$
iii (−2, 0), (2, 0)



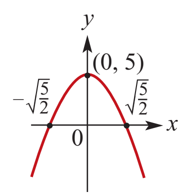
b i (0, 2)
ii $x = 0$
iii None



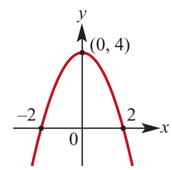
c i (0, 3)
ii $x = 0$
iii $(-\sqrt{3}, 0), (\sqrt{3}, 0)$



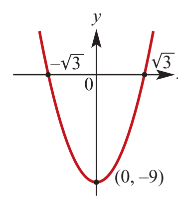
d i (0, 5)
ii $x = 0$
iii $\left(-\sqrt{\frac{5}{2}}, 0\right)$, $\left(\sqrt{\frac{5}{2}}, 0\right)$



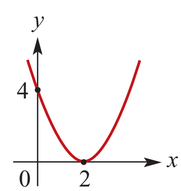
e i (0, 4)
ii $x = 0$
iii (−2, 0), (2, 0)



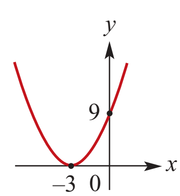
f i (0, −9)
ii $x = 0$
iii $(-\sqrt{3}, 0), (\sqrt{3}, 0)$



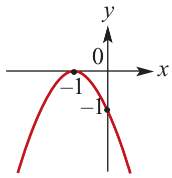
2 a i (2, 0)
ii $x = 2$
iii (2, 0)



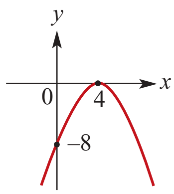
b i (−3, 0)
ii $x = -3$
iii (−3, 0)



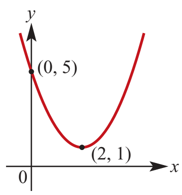
c **i** $(-1, 0)$
ii $x = -1$
iii $(-1, 0)$



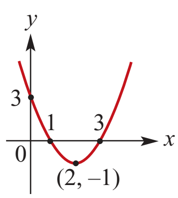
d **i** $(4, 0)$
ii $x = 4$
iii $(4, 0)$



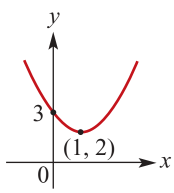
3 a **i** $(2, 1)$
ii $x = 2$
iii None



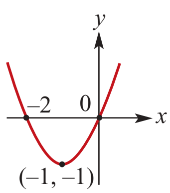
b **i** $(2, -1)$
ii $x = 2$
iii $(1, 0), (3, 0)$



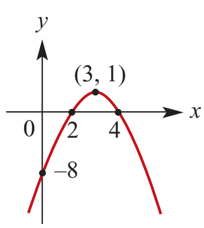
c **i** $(1, 2)$
ii $x = 1$
iii None



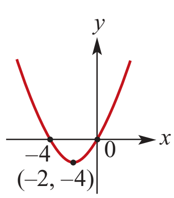
d **i** $(-1, -1)$
ii $x = -1$
iii $(-2, 0), (0, 0)$



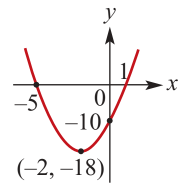
e **i** $(3, 1)$
ii $x = 3$
iii $(2, 0), (4, 0)$



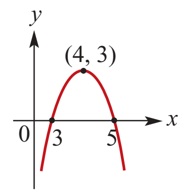
f **i** $(-2, -4)$
ii $x = -2$
iii $(-4, 0), (0, 0)$



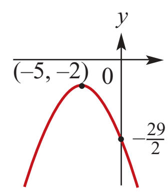
g **i** $(-2, -18)$
ii $x = -2$
iii $(-5, 0), (1, 0)$



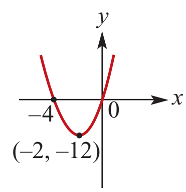
h **i** $(4, 3)$
ii $x = 4$
iii $(3, 0), (5, 0)$



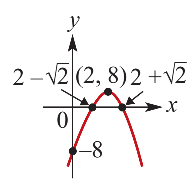
i **i** $(-5, -2)$
ii $x = -5$
iii None



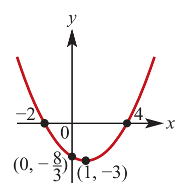
j **i** $(-2, -12)$
ii $x = -2$
iii $(0, 0), (-4, 0)$



k **i** $(2, 8)$
ii $x = 2$
iii $(2 - \sqrt{2}, 0)$, $(2 + \sqrt{2}, 0)$



l **i** $(1, -3)$
ii $x = 1$
iii $(-2, 0), (4, 0)$



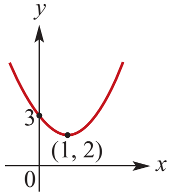
Exercise 4E

1 a $x^2 - 2x + 1$ **b** $x^2 + 4x + 4$
c $x^2 - 6x + 9$ **d** $x^2 - 6x + 9$
e $x^2 + 4x + 4$ **f** $x^2 - 10x + 25$
g $x^2 - x + \frac{1}{4}$ **h** $x^2 - 3x + \frac{9}{4}$

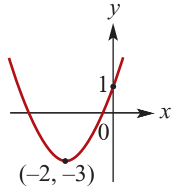
2 a $(x - 2)^2$ **b** $(x - 6)^2$ **c** $-(x - 2)^2$
d $2(x - 2)^2$ **e** $-2(x - 3)^2$ **f** $\left(x - \frac{1}{2}\right)^2$
g $\left(x - \frac{3}{2}\right)^2$ **h** $\left(x + \frac{5}{2}\right)^2$

3 a $1 \pm \sqrt{2}$ **b** $2 \pm \sqrt{6}$ **c** $3 \pm \sqrt{7}$
d $\frac{5 \pm \sqrt{17}}{2}$ **e** $\frac{2 \pm \sqrt{2}}{2}$ **f** $-\frac{1}{3}, 2$
g $-1 \pm \sqrt{1 - k}$ **h** $\frac{-1 \pm \sqrt{1 - k^2}}{k}$
i $\frac{3k \pm \sqrt{9k^2 - 4}}{2}$

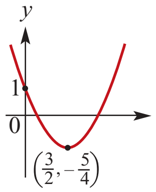
4 a $y = (x-1)^2 + 2$
t. pt $(1, 2)$



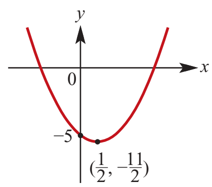
b $y = (x+2)^2 - 3$
t. pt $(-2, -3)$



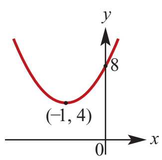
c $y = \left(x - \dfrac{3}{2}\right)^2 - \dfrac{5}{4}$
t. pt $\left(\dfrac{3}{2}, -\dfrac{5}{4}\right)$



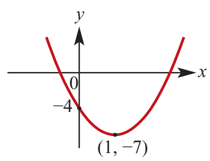
5 a $y = 2\left(x - \dfrac{1}{2}\right)^2 - \dfrac{11}{2}$
t. pt $\left(\dfrac{1}{2}, -\dfrac{11}{2}\right)$



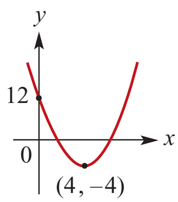
b $y = 4(x+1)^2 + 4$
t. pt $(-1, 4)$



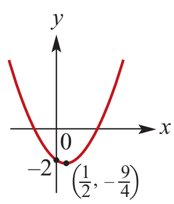
c $y = 3(x-1)^2 - 7$
t. pt $(1, -7)$



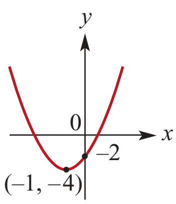
6 a $y = (x-4)^2 - 4$
t. pt $(4, -4)$



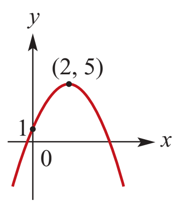
b $y = \left(x - \dfrac{1}{2}\right)^2 - \dfrac{9}{4}$
t. pt $\left(\dfrac{1}{2}, -\dfrac{9}{4}\right)$



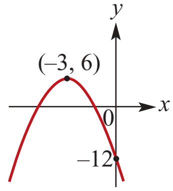
c $y = 2(x+1)^2 - 4$
t. pt $(-1, -4)$



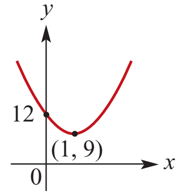
d $y = -(x-2)^2 + 5$
t. pt $(2, 5)$



e $y = -2(x+3)^2 + 6$
t. pt $(-3, 6)$



f $y = 3(x-1)^2 + 9$
t. pt $(1, 9)$

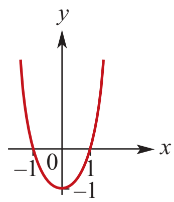


Exercise 4F

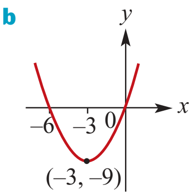
1 a 7 **b** 7 **c** 1

2 a -2 **b** 8 **c** 4

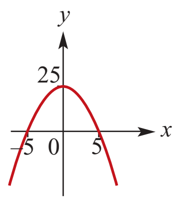
3 a



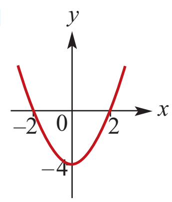
b



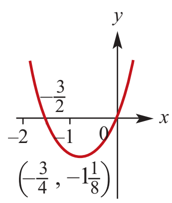
c



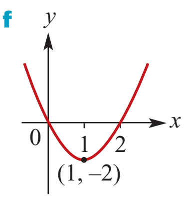
d



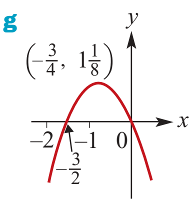
e



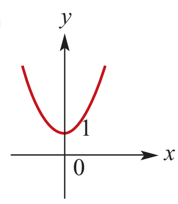
f



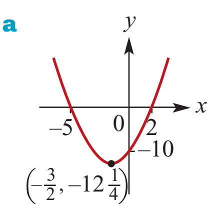
g



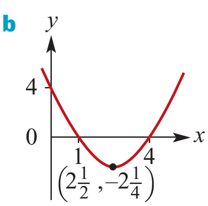
h



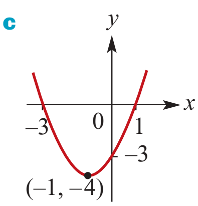
4 a



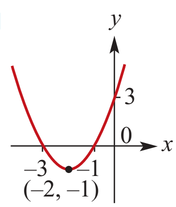
b



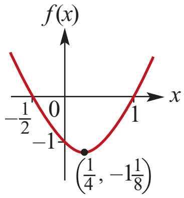
c



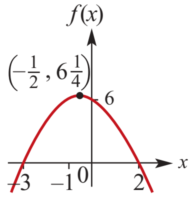
d



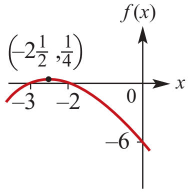
e



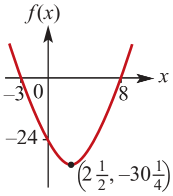
f



g

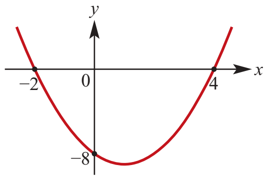


h



Exercise 4G

1 a $-2, 4$ **b**



c $-2 \le x \le 4$ **d** $x < -2$ or $x > 4$

2 a $x \le -2$ or $x \ge 3$ **b** $-4 < x < -3$

c $-4 \le x \le \frac{1}{2}$ **d** $x < 2$ or $x > 6$

e $2 < x < 3$ **f** $\frac{3}{2} \le x \le \frac{7}{2}$

g $-\frac{7}{2} < x < 2$ **h** $-2 \le x \le \frac{5}{2}$

i $x < -5$ or $x > \frac{5}{2}$ **j** $-2 \le x \le \frac{7}{2}$

k $x < \frac{2}{5}$ or $x > \frac{7}{2}$ **l** $x \le \frac{5}{2}$ or $x \ge \frac{11}{2}$

3 a $x < -5$ or $x > 5$ **b** $-\frac{2}{3} \le y \le \frac{2}{3}$

c $y < -4$ or $y > 4$ **d** $-\frac{6}{5} \le x \le \frac{6}{5}$

e $y \le -\frac{1}{4}$ or $y \ge \frac{1}{4}$ **f** $y < -\frac{5}{6}$ or $y > \frac{5}{6}$

4 a $x \ge 2$ or $x \le -4$ **b** $-3 < x < 8$

c $-2 \le x \le 6$ **d** $x > 3$ or $x < -\frac{3}{2}$

e $-\frac{3}{2} < x < -\frac{2}{3}$ **f** $-3 \le x \le -2$

g $x > \frac{2}{3}$ or $x < -\frac{3}{4}$ **h** $\frac{1}{2} \le x \le \frac{3}{5}$

i $-4 \le x \le 5$

j $\frac{1}{2}(5 - \sqrt{41}) \le p \le \frac{1}{2}(5 + \sqrt{41})$

k $y < -1$ or $y > 3$ **l** $x \le -2$ or $x \ge -1$

5 a $x \le \dfrac{-3 - \sqrt{29}}{2}$ or $x \ge \dfrac{-3 + \sqrt{29}}{2}$

b $\dfrac{5 - \sqrt{17}}{2} < x < \dfrac{5 + \sqrt{17}}{2}$

c $\dfrac{3 - \sqrt{17}}{4} \le x \le \dfrac{3 + \sqrt{17}}{4}$

d $\dfrac{-3 - \sqrt{41}}{2} < x < \dfrac{-3 + \sqrt{41}}{2}$

e $\dfrac{-7 - \sqrt{41}}{4} < x < \dfrac{-7 + \sqrt{41}}{4}$

f $x \le \dfrac{4 - \sqrt{6}}{2}$ or $x \ge \dfrac{4 + \sqrt{6}}{2}$

6 The square of any number is greater than or equal to zero.

7 The negative of the square of any number is less than or equal to zero.

8 $x^2 + 2x + 7 = (x + 1)^2 + 6$. For all x, we have $(x + 1)^2 \ge 0$ and so $(x + 1)^2 + 6 \ge 6$

9 $-x^2 - 2x - 7 = -(x + 1)^2 - 6$. For all x, we have $-(x + 1)^2 \le 0$ and so $-(x + 1)^2 - 6 \le -6$

Exercise 4H

1 a i 40 **ii** $2\sqrt{10}$

b i 28 **ii** $2\sqrt{7}$

c i 172 **ii** $2\sqrt{43}$

d i 96 **ii** $4\sqrt{6}$

e i 189 **ii** $3\sqrt{21}$

2 a $1 + \sqrt{5}$ **b** $\dfrac{3 - \sqrt{5}}{2}$ **c** $\dfrac{1 + \sqrt{5}}{2}$ **d** $1 + 2\sqrt{2}$

3 a $-3 \pm \sqrt{13}$ **b** $\dfrac{7 \pm \sqrt{61}}{2}$ **c** $\dfrac{1}{2}, 2$

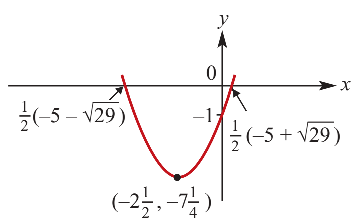
d $-1 \pm \frac{3}{2}\sqrt{2}$ **e** $-2 \pm \frac{3}{2}\sqrt{2}$ **f** $1 \pm \dfrac{\sqrt{30}}{5}$

g $1 \pm \dfrac{\sqrt{2}}{2}$ **h** $1, \dfrac{-3}{2}$ **i** $\dfrac{-3 \pm \sqrt{6}}{5}$

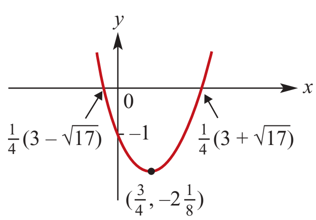
j $\dfrac{-13 \pm \sqrt{145}}{12}$ **k** $\dfrac{2 \pm \sqrt{4 - 2k^2}}{2k}$

l $\dfrac{2k \pm \sqrt{6k^2 - 2k}}{2(1 - k)}$

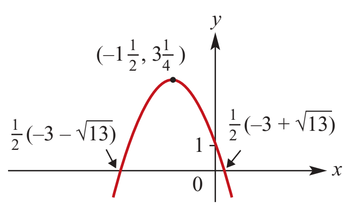
4 a



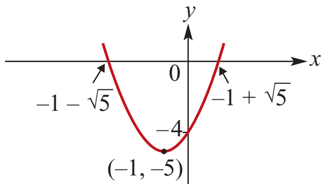
b



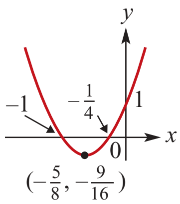
c



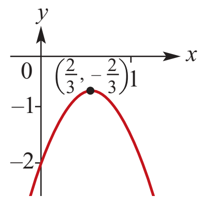
d



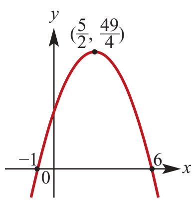
e



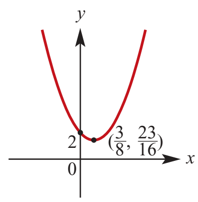
f



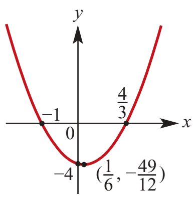
g



h



i



Exercise 4I

1 a 20 **b** -12 **c** 25 **d** 41 **e** 41

2 a Crosses the x-axis **b** Does not touch
c Just touches the x-axis
d Crosses the x-axis **e** Does not touch
f Does not touch

3 a Two real solutions **b** No real solutions
c Two real solutions **d** Two real solutions
e Two real solutions **f** No real solutions

4 a $\Delta = 0$, one rational solution
b $\Delta = 1$, two rational solutions
c $\Delta = 17$, two irrational solutions
d $\Delta = 0$, one rational solution
e $\Delta = 57$, two irrational solutions
f $\Delta = 1$, two rational solutions

5 a i $-\sqrt{5} < m < \sqrt{5}$ **ii** $m = \pm\sqrt{5}$
iii $m > \sqrt{5}$ or $m < -\sqrt{5}$
b i $0 \le m < \dfrac{4}{3}$ **ii** $m = \dfrac{4}{3}$
iii $m > \dfrac{4}{3}$ or $m < 0$
c i $-\dfrac{4}{5} < m < 0$ **ii** $m = 0$ or $m = -\dfrac{4}{5}$
iii $m < -\dfrac{4}{5}$ or $m > 0$
d i $-2 < m < 1$ **ii** $m = -2$ or $m = 1$
iii $m > 1$ or $m < -2$

6 $\Delta = (2m - n)^2$, a perfect square **7** $p > \dfrac{4}{3}$

8 $p = \dfrac{-1}{2}$

9 a $p = \pm 3$ **b** $p > 1$ **c** $p > \dfrac{2}{3}$ **d** $p > 1$

10 $-2 < p < 8$

11 $\Delta = -4q^2 < 0$ for all values of q

12 a $\Delta = 16m^2 - 96m + 176 = 16(m-3)^2 + 32$
b $\Delta \ge 32$; therefore two solutions

13 a $\Delta = 16$
b $\Delta > 0$; therefore two solutions

14 $\Delta = (m + 4)^2$, a perfect square; therefore rational solutions

15 $\Delta = (m - 2n)^2$, a perfect square; therefore rational solutions

16 The graph will cross the x-axis twice

17 The graph will cross the x-axis twice

Exercise 4J

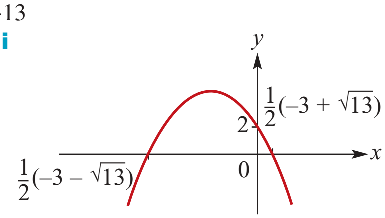
1 a $(1-\sqrt{5}, -1-\sqrt{5}), (1+\sqrt{5}, -1+\sqrt{5})$
b $(-3, 9), (2, 4)$ **c** $(-3, 9), \left(\dfrac{7}{4}, \dfrac{49}{16}\right)$
d $(1, 3), (2, 5)$

2 a $(2, 0), (-5, 7)$ **b** $(1, -3), (4, 9)$
c $(1, -3), (-3, 1)$ **d** $(-1, 1), (-3, -3)$
e $\left(\dfrac{1+\sqrt{33}}{2}, -3-\sqrt{33}\right), \left(\dfrac{1-\sqrt{33}}{2}, -3+\sqrt{33}\right)$
f $\left(\dfrac{5+\sqrt{33}}{2}, 23+3\sqrt{33}\right), \left(\dfrac{5-\sqrt{33}}{2}, 23-3\sqrt{33}\right)$

3 a Touch at $(2, 0)$ **b** Touch at $(3, 9)$
c Touch at $(-2, -4)$ **d** Touch at $(-4, -8)$

4 a $x = 8, y = 16$ and $x = -1, y = 7$
b $x = -\dfrac{16}{3}, y = 37\frac{1}{3}$ and $x = 2, y = 30$
c $x = \dfrac{4}{5}, y = 10\frac{2}{5}$ and $x = -3, y = 18$
d $x = 10\frac{2}{3}, y = 0$ and $x = 1, y = 29$
e $x = 0, y = -12$ and $x = \frac{3}{2}, y = -7\frac{1}{2}$
f $x = 1.14, y = 14.19$ and $x = -1.68, y = 31.09$

5 a -13
b i



ii $m = -6 \pm \sqrt{32} = -6 \pm 4\sqrt{2}$

6 a $c = \dfrac{-1}{4}$ **b** $c > \dfrac{-1}{4}$

7 $a = 3$ or $a = -1$

8 $b = 1$

9 $y = (2 + 2\sqrt{3})x - 4 - 2\sqrt{3}$ and
$y = (2 - 2\sqrt{3})x - 4 + 2\sqrt{3}$

Exercise 4K

1 $a = -4, c = 6$

2 a $\Delta = b^2 - 16a$ **b** $a = \frac{b^2}{16}$ **c** $a = \frac{1}{4}, b = 2$

3 a $y = 2(x+2)(x-6)$ **b** $y = -2(x+2)^2 + 4$
c $y = -x^2 + 2x - 3$

4 2 **5** $a = \frac{4}{7}, b = \frac{-24}{7}$

6 $a = -2, b = 1, c = 6$

7 a $y = -\frac{5}{16}x^2 + 5$ **b** $y = x^2$
c $y = \frac{1}{11}x^2 + \frac{7}{11}x$ **d** $y = x^2 - 4x + 3$
e $y = -\frac{5}{4}x^2 - \frac{5}{2}x + \frac{15}{4}$ **f** $y = x^2 - 4x + 6$

8 $y = \frac{5}{16}(x+1)^2 + 3$ **9** $y = -\frac{1}{2}(x^2 - 3x - 18)$

10 $y = (x+1)^2 + 3$ **11** $y = \frac{1}{180}x^2 - x + 75$

12 $y = 2x^2 - 4x$ **13** $y = x^2 - 2x - 1$

14 a C **b** B **c** D **d** A

15 a $y = a\left(x + \frac{1}{a}\right)^2 + a - \frac{1}{a}$ **b** $\left(-\frac{1}{a}, a - \frac{1}{a}\right)$
c $a = \pm 1$ **d** $-1 < a < 1$

16 $y = -2x^2 + 8x - 6$

17 a $y = ax(x-10), a > 0$
b $y = a(x+4)(x-10), a < 0$
c $y = \frac{1}{18}(x-6)^2 + 6$ **d** $y = a(x-8)^2, a < 0$

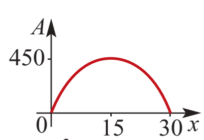
18 a $y = -\frac{1}{4}x^2 + x + 2$ **b** $y = x^2 + x - 5$

19 $r = -\frac{1}{8}t^2 + 2\frac{1}{2}t - 6\frac{3}{8}$ **20 a** B **b** D

21 a $y = -2x^2 - x + 5$ **b** $y = 2x^2 - x - 5$
c $y = 2x^2 + \frac{5}{2}x - \frac{11}{2}$

Exercise 4L

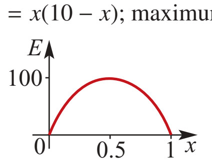
1 a $A = 60x - 2x^2$ **b**



c Maximum area = 450 m^2

2 $A = x(10 - x)$; maximum area = 25 m^2

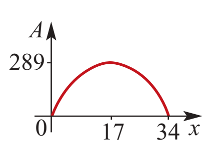
3 a



b 0 and 1 **c** 0.5

d 0.23 and 0.77

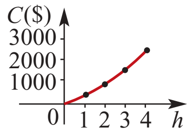
4 a $A = 34x - x^2$ **b**



c 289 cm^2

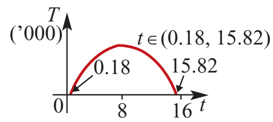
5 a $4x + 10y = 80$
b i $A = 1.64x^2 - 25.6x + 256$
ii 31.22 and 48.78

6 a



The domain depends on the height of the alpine area. In Australia, the highest mountain is approx. 2 km high and the minimum alpine height would be approx. 1 km. Thus, for Australia, domain = [1, 2].
b Theoretically, no. But of course there is a practical maximum.
c \$1225

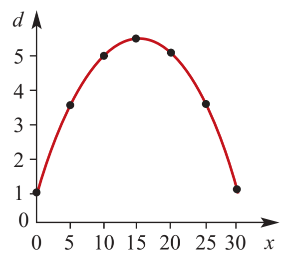
7 a



b 8874 units

8 a

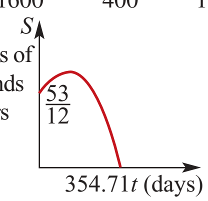
x	0	5	10	15	20	25	30
d	1	3.5	5	5.5	5	3.5	1



b i 5.5 m
ii $15 - 5\sqrt{7}$ m or $15 + 5\sqrt{7}$ m from the bat
iii 1 m above the ground

9 $a = -\frac{16}{15}, b = \frac{8}{5}, c = 0$

10 a $a = -\frac{7}{21600}, b = \frac{41}{400}, c = \frac{53}{12}$
b



c i S = \$1 236 666 **ii** S = \$59 259

Chapter 4 review

Short-response questions

Technology-free

1 a $\left(x + \frac{9}{2}\right)^2$ **b** $(x+9)^2$ **c** $\left(x - \frac{2}{5}\right)^2$
d $(x+b)^2$ **e** $(3x-1)^2$ **f** $(5x+2)^2$

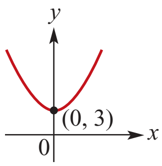
2 a $-3x + 6$ **b** $-ax + a^2$
c $49a^2 - b^2$ **d** $x^2 - x - 12$

e $2x^2 - 5x - 12$ **f** $x^2 - y^2$
g $a^3 - b^3$ **h** $6x^2 + 8xy + 2y^2$
i $3a^2 - 5a - 2$ **j** $4xy$
k $2u + 2v - uv$ **l** $-3x^2 + 15x - 12$

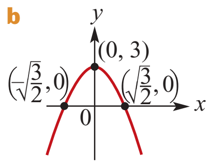
3 a $4(x - 2)$ **b** $x(3x + 8)$
c $3x(8a - 1)$ **d** $(2 - x)(2 + x)$
e $a(u + 2v + 3w)$ **f** $a^2(2b - 3a)(2b + 3a)$
g $(1 - 6ax)(1 + 6ax)$ **h** $(x + 4)(x - 3)$
i $(x + 2)(x - 1)$ **j** $(2x - 1)(x + 2)$
k $(3x + 2)(2x + 1)$ **l** $(3x + 1)(x - 3)$
m $(3x - 2)(x + 1)$ **n** $(3a - 2)(2a + 1)$
o $(3x - 2)(2x - 1)$

4 a $x = 5$ or $x = -3$ **b** $x = 9$ or $x = 0$
c $x = 2$ or $x = 3$ **d** $x = -1$ or $x = 25$
e $x = -3$ or $x = -2$ **f** $x = 6$
g $x = -\dfrac{1}{2}$ or $x = 3$ **h** $x = -\dfrac{5}{6}$ or $x = \dfrac{3}{2}$
i $x = -\dfrac{12}{5}$ or $x = 1$

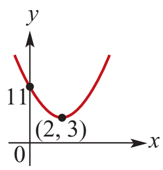
5 a



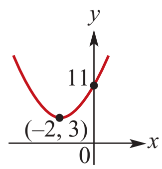
b



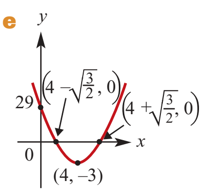
c



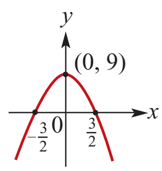
d



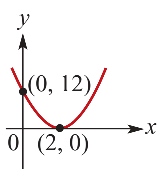
e



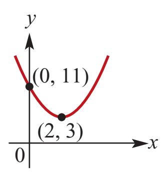
f



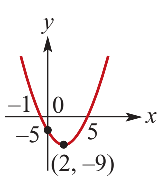
g



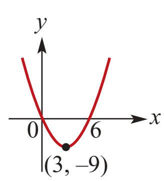
h



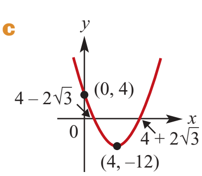
6 a



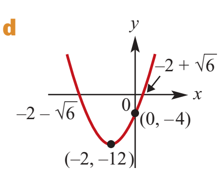
b



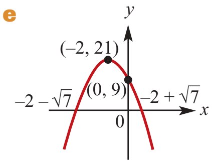
c



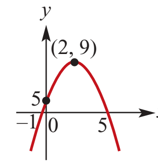
d



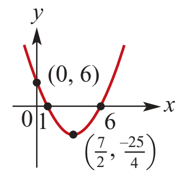
e

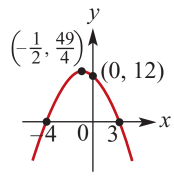


f

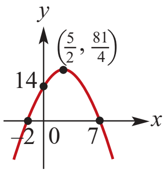


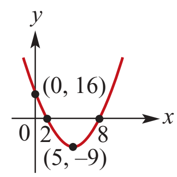
7 a $x = \dfrac{7}{2}$ **b** $x = -\dfrac{1}{2}$



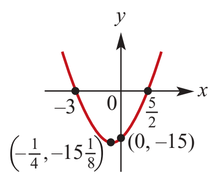


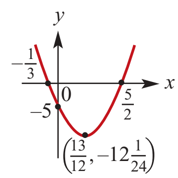
c $x = \dfrac{5}{2}$ **d** $x = 5$



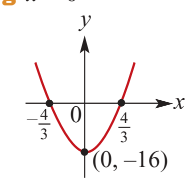


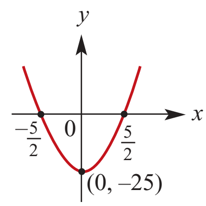
e $x = -\dfrac{1}{4}$ **f** $x = \dfrac{13}{12}$





g $x = 0$ **h** $x = 0$





8 a $x < 0$ or $x > 1$
b $-2 - \sqrt{34} \le x \le -2 + \sqrt{34}$
c $-2 \le x \le \dfrac{1}{3}$ **d** $\dfrac{3}{2} \le x \le 5$

9 a $-3 \pm \sqrt{6}$ **b** $\dfrac{-9 \pm \sqrt{33}}{2}$
c $2 \pm \sqrt{2}$ **d** $\dfrac{-7 \pm \sqrt{33}}{4}$
e $\dfrac{-7 \pm \sqrt{17}}{4}$ **f** $\dfrac{-9 \pm \sqrt{93}}{6}$

10 $p = 1$ or $p = -\dfrac{3}{10}$

11 $y = \dfrac{5}{3}x(x - 5)$

12 $y = 3(x - 5)^2 + 2$

13 $m < -21 - 4\sqrt{29}$ or $m > -21 + 4\sqrt{29}$, $m \ne \dfrac{3}{2}$

14 225

15 $y = 5(x-1)^2 + 5$

16 a $(3, 9), (-1, 1)$

b $\left(\frac{4-\sqrt{38}}{2}, 27 - 4\sqrt{38}\right)$, $\left(\frac{4+\sqrt{38}}{2}, 27 + 4\sqrt{38}\right)$

c $\left(\frac{-7-\sqrt{73}}{6}, 2\right), \left(\frac{-7+\sqrt{73}}{6}, 2\right)$

d $(\frac{1}{2}, \frac{1}{2}), (-2, 8)$

17 a $y = 2(x+4)(x-1)$ **b** $y = -2(x+1)^2 + 3$

c $y = 2x^2 - 2x - 3$

18 2.16 m

19 a $m = \pm\sqrt{8} = \pm 2\sqrt{2}$ **b** $m \le -\sqrt{5}$ or $m \ge \sqrt{5}$

20 a $x = 0$ and $x = -b$ **b** $\left(-\frac{b}{2}, -\frac{b^2}{4}\right)$

c i $(0, 0), (1-b, 1-b)$ **ii** $b = 1$ **iii** $b \neq 1$

Technology-active

21 a $-2.739, 1.293$ **b** $-1.246, 2.166$

22 $a = 0.731, b = -3.409, c = -1.559$

23 a 73.26 m **b** 33.21 m **c** 1.78 m

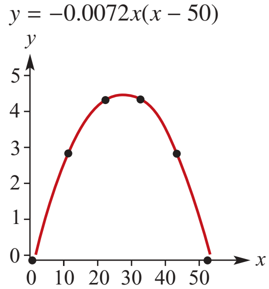
d 72.125 m

24 a 24.20 m **b** 5.39 seconds

c 4.44 seconds

25 a $y = -0.0072x(x-50)$

b



c 10.57 m and 39.43 m $\left(25 \pm \frac{25\sqrt{3}}{3}\text{ m}\right)$

d 3.2832 m **e** 3.736 m (correct to 3 d.p.)

26 a Width $= \frac{12-4x}{6}$ cm; length $= \frac{12-4x}{3}$ cm

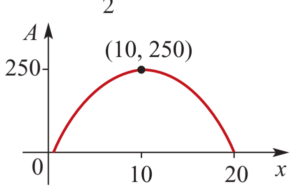
b $A = \frac{17}{9}x^2 - \frac{16}{3}x + 8$

c Length for square $= \frac{96}{17} \approx 5.65$ cm and length for rectangle $= \frac{108}{17} \approx 6.35$ cm

27 a $V = 0.72x^2 - 1.2x$ **b** 22 hours

28 a $\ell = 50 - \frac{5x}{2}$ **b** $A = 50x - \frac{5}{2}x^2$

c



d Maximum area $= 250$ m^2 when $x = 10$ m

29 $x = \frac{-1+\sqrt{5}}{2}$

30 a $\sqrt{25 + x^2}$

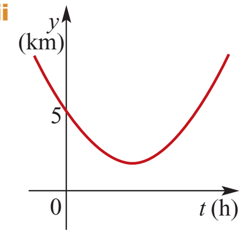
b i $16 - x$ **ii** $\sqrt{x^2 - 32x + 265}$

c 7.5 **d** 10.840 **e** 12.615

31 a i $y = \sqrt{64t^2 + 100(t-0.5)^2}$

$= \sqrt{164t^2 - 100t + 25}$

ii



iii $t = \frac{1}{2}$, 1:30 p.m.; $t = \frac{9}{82}$, 1:07 p.m.

iv 0.305; 1:18 p.m.; distance 3.123 km

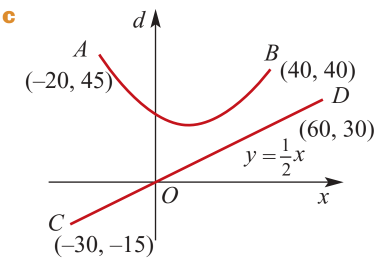
b i $0, \frac{25}{41}$ **ii** $\frac{25 \pm 2\sqrt{269}}{82}$

32 a $45\sqrt{5}$

b i $y = \frac{1}{600}(7x^2 - 190x + 20\,400)$

ii $\left(\frac{95}{7}, \frac{5351}{168}\right)$

c



d i The distance (measured parallel to the y-axis) between path and pond

ii Minimum value $= \frac{473}{24}$ when $x = 35$

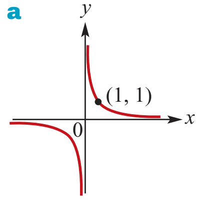
Multiple-choice questions

1 A	**2** B	**3** B	**4** D	**5** A
6 B	**7** D	**8** D	**9** C	**10** A
11 A	**12** D	**13** C	**14** B	**15** C
16 C	**17** D	**18** B	**19** D	

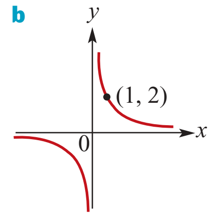
Chapter 5

Exercise 5A

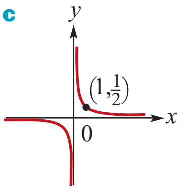
1 a



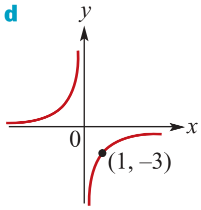
b



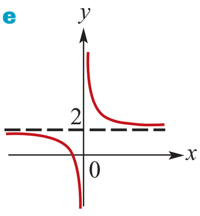
c



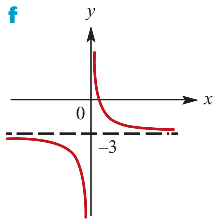
d



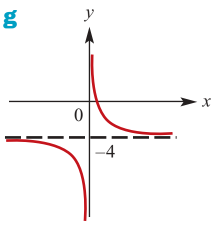
e



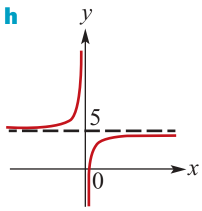
f



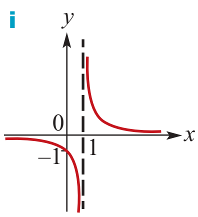
g



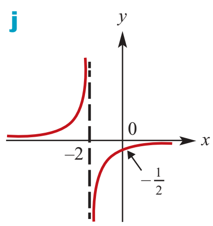
h



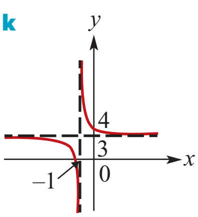
i



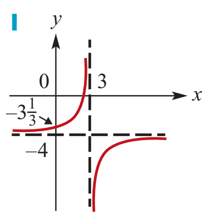
j



k

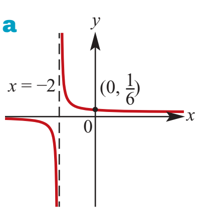


l

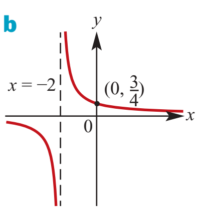


2 a $y = 0, x = 0$ **b** $y = 0, x = 0$
c $y = 0, x = 0$ **d** $y = 0, x = 0$
e $y = 2, x = 0$ **f** $y = -3, x = 0$
g $y = -4, x = 0$ **h** $y = 5, x = 0$
i $y = 0, x = 1$ **j** $y = 0, x = -2$
k $y = 3, x = -1$ **l** $y = -4, x = 3$

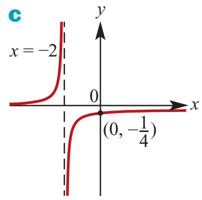
3 a



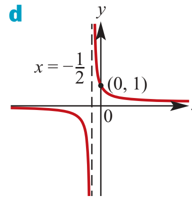
b



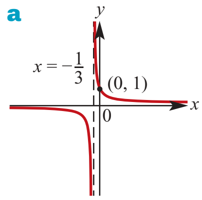
c



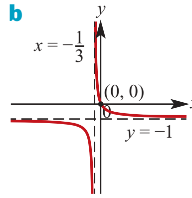
d



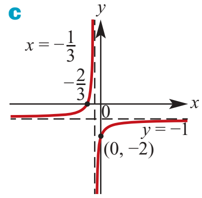
4 a



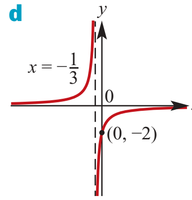
b



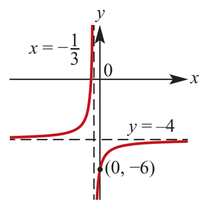
c



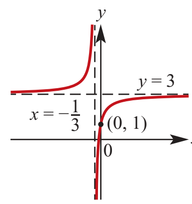
d



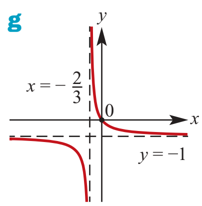
e x-axis intercept $-\frac{1}{2}$
y-axis intercept -6



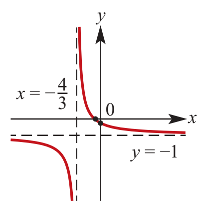
f x-axis intercept $-\frac{1}{9}$
y-axis intercept 1



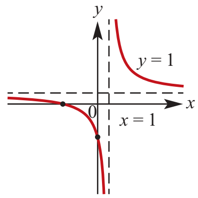
g



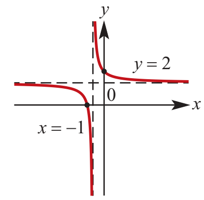
h x-axis intercept $-\frac{1}{3}$
y-axis intercept $-\frac{1}{4}$



5 x-axis intercept -3
y-axis intercept -3

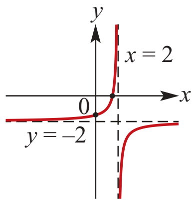


6 x-axis intercept $-\frac{3}{2}$
y-axis intercept 3



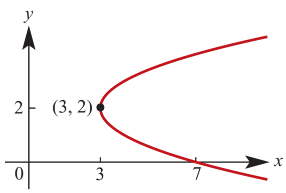
7 x-axis intercept $\frac{3}{2}$

y-axis intercept $-\frac{3}{2}$

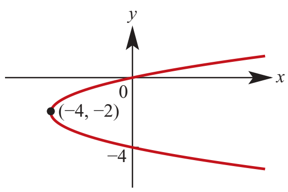


Exercise 5B

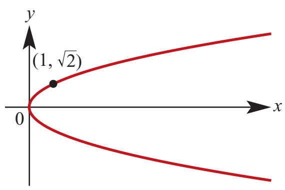
1 a



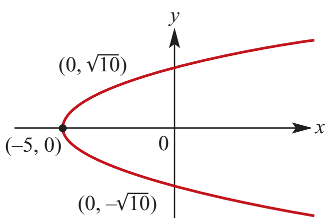
b



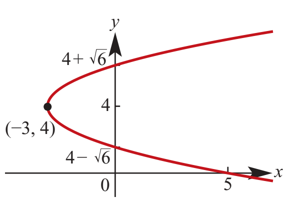
c



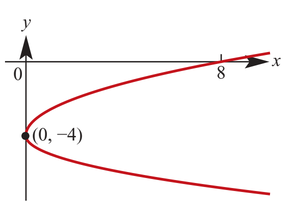
d



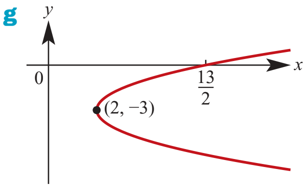
e



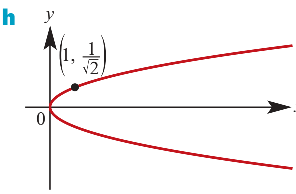
f



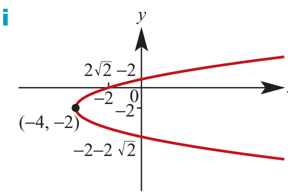
g



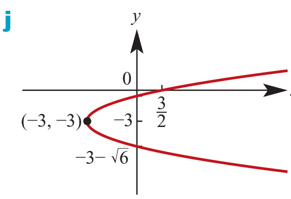
h



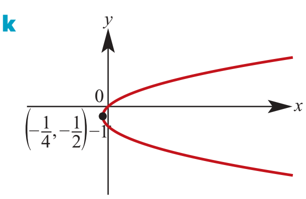
i



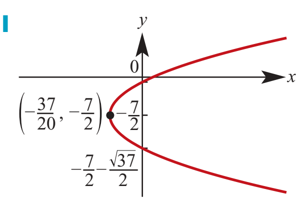
j



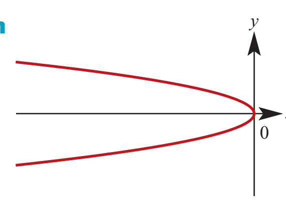
k



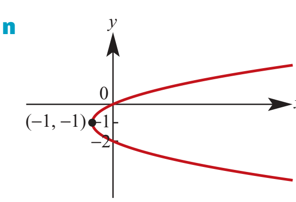
l



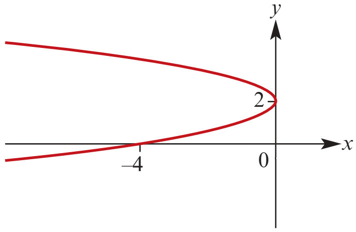
m



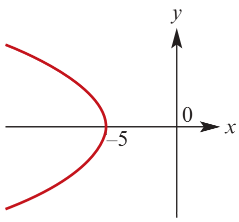
n



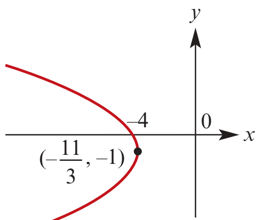
o



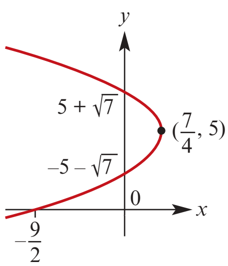
2 a



b

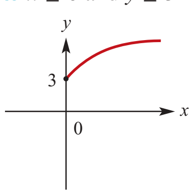


c

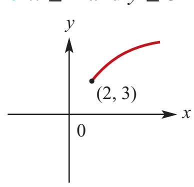


Exercise 5C

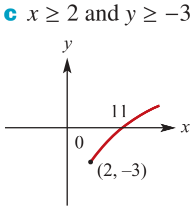
1 a $x \geq 0$ and $y \geq 3$



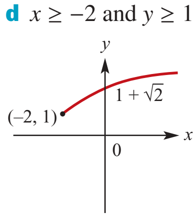
b $x \geq 2$ and $y \geq 3$



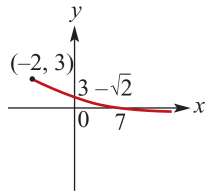
c $x \geq 2$ and $y \geq -3$



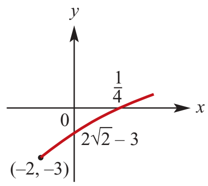
d $x \geq -2$ and $y \geq 1$



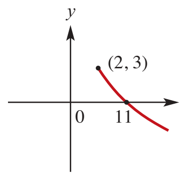
e $x \geq -2$ and $y \leq 3$



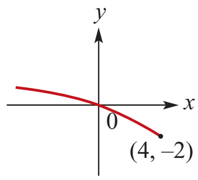
f $x \geq -2$ and $y \geq -3$



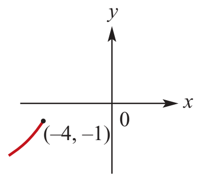
2 a $x \geq 2$ and $y \leq 3$



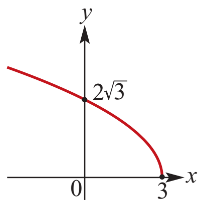
b $x \leq 4$ and $y \geq -2$



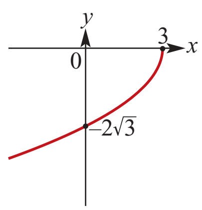
c $x \leq -4$ and $y \leq -1$



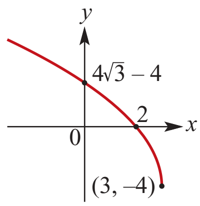
d $x \leq 3$ and $y \geq 0$



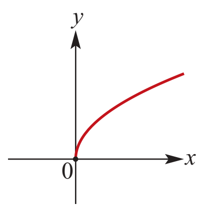
e $x \leq 3$ and $y \leq 0$



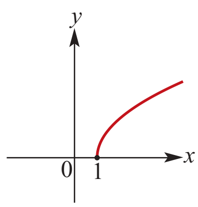
f $x \leq 3$ and $y \geq -4$



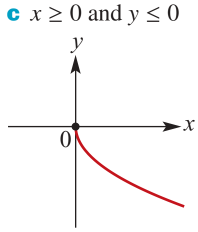
3 a $x \geq 0$ and $y \geq 0$



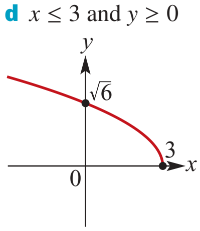
b $x \geq 1$ and $y \geq 0$



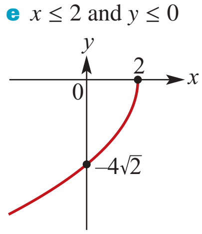
c $x \geq 0$ and $y \leq 0$



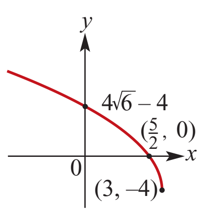
d $x \leq 3$ and $y \geq 0$



e $x \leq 2$ and $y \leq 0$



f $x \leq 3$ and $y \geq -4$



Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

Exercise 5D

1 a $x^2 + y^2 = 9$

b $x^2 + y^2 = 16$

c $(x-1)^2 + (y-3)^2 = 25$

d $(x-2)^2 + (y+4)^2 = 9$

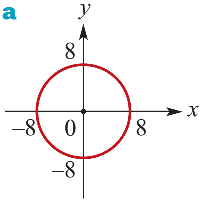
e $(x+3)^2 + (y-4)^2 = \dfrac{25}{4}$

f $(x+5)^2 + (y+6)^2 = (4.6)^2$

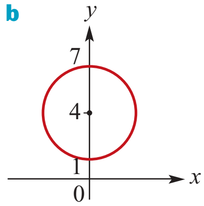
2 a $C(1,3),\ r = 2$ **b** $C(2,-4),\ r = \sqrt{5}$

c $C(-3,2),\ r = 3$ **d** $C(-5,4),\ r = \sqrt{8}$

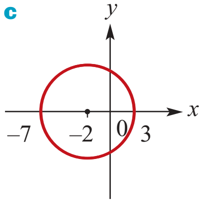
3 a



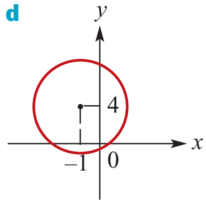
b



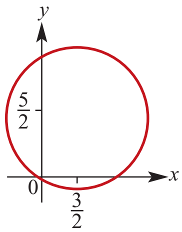
c



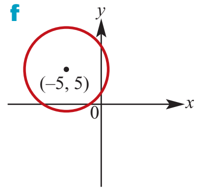
d



e



f

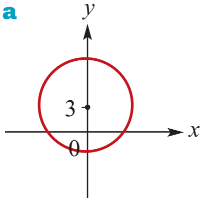


4 a $C(0,3),\ r = 5$ **b** $C(4,-6),\ r = \sqrt{42}$

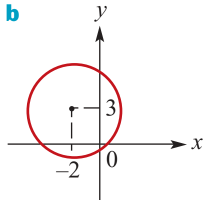
c $C(3,-2),\ r = 2$ **d** $C(-2,3),\ r = 5$

e $C(4,-2),\ r = \sqrt{19}$ **f** $C(\frac{1}{2},-2),\ r = \frac{3}{2}$

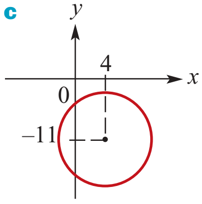
5 a



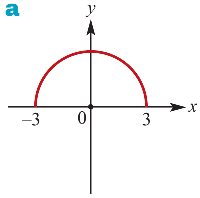
b



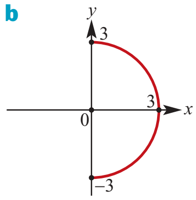
c



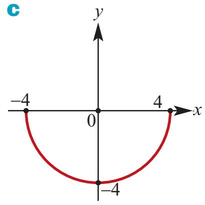
6 a



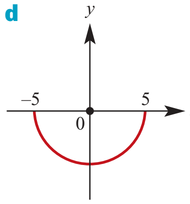
b



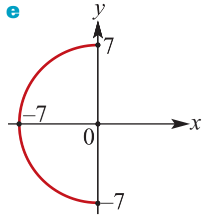
c



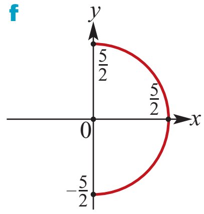
d



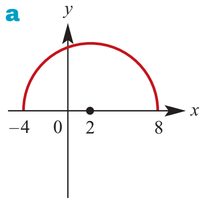
e



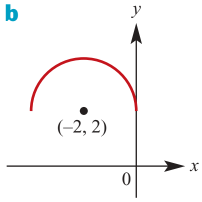
f



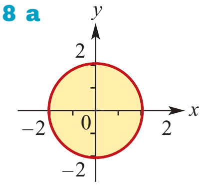
7 a



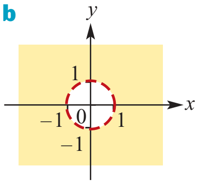
b



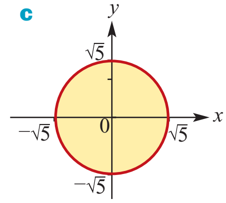
8 a



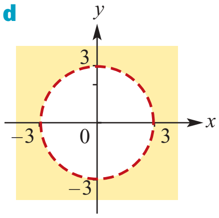
b



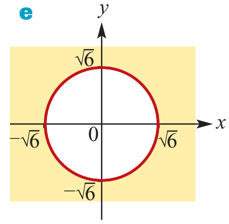
c



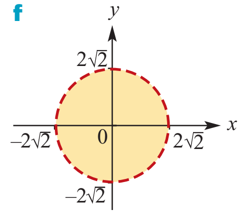
d



e



f



Exercise 5E

1 $a = 5$ **2** $a = -6, h = 3, k = 4$

3 $a = \frac{1}{2}, k = \frac{15}{2}$ **4** $a = -16, h = 2, k = -4$

5 $a = 4\sqrt{2}$ **6** $a = \frac{2\sqrt{3}}{3}, h = -2$

7 $(x-2)^2 + (y-1)^2 = 20$

8 $(x+2)^2 + (y-3)^2 = 1$

9 $(x+2)^2 + (y-3)^2 = 16$

10 $(x-2)^2 + (y+3)^2 = 9$

11 $(x-4)^2 + (y-4)^2 = 20$

12 $(x-4)^2 + (y-5)^2 = 25$ and $(x+4)^2 + (y-5)^2 = 25$

13 $(x+1)^2 + (y+1)^2 = 10$

14 a $(x-2)^2 + (y+2)^2 = 49$

b $y = 3\sqrt{x-1} - 2$ **c** $y = \frac{1}{x-2} + 2$

d $y = -\frac{2}{x-1} - 2$ **e** $y = \sqrt{2-x} + 1$

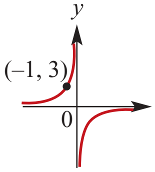
f $(y+2)^2 = 2x + 9$

Chapter 5 review

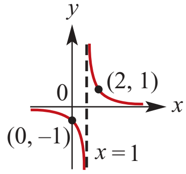
Short-response questions

Technology-free

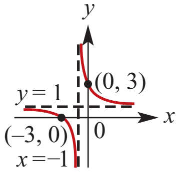
1 a



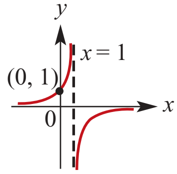
b



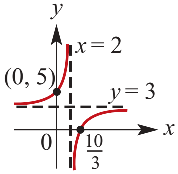
c



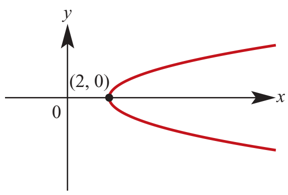
d



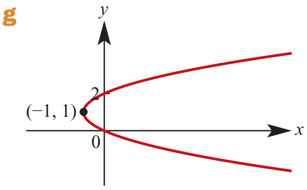
e



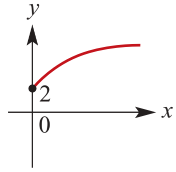
f



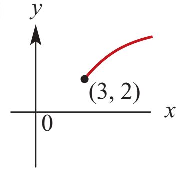
g



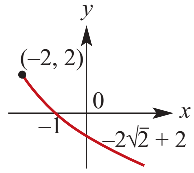
h



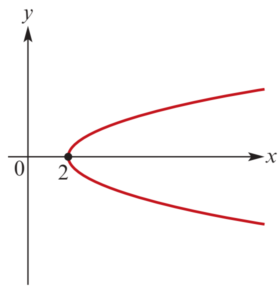
i



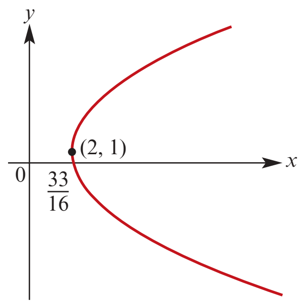
j



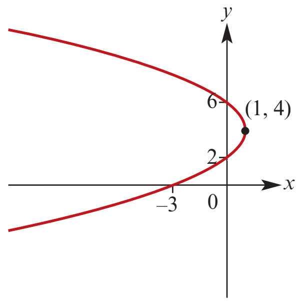
k



l



m



2 a $(x-3)^2 + (y+2)^2 = 25$

b $\left(x - \frac{3}{2}\right)^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{2}$

c $\left(x - \frac{1}{4}\right)^2 + \left(y + \frac{1}{4}\right)^2 = \frac{17}{8}$

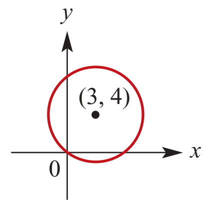
d $(x+2)^2 + (y-3)^2 = 13$

e $(x-3)^2+(y-3)^2=18$
f $(x-2)^2+(y+3)^2=13$

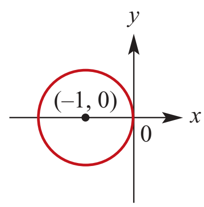
3 $2y+3x=0$

4 $2x+2y=1$ or $y=x-\dfrac{5}{2}$

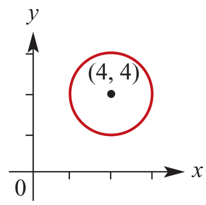
5 a $(x-3)^2+(y-4)^2=25$



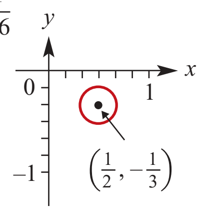
b $(x+1)^2+y^2=1$



c $(x-4)^2+(y-4)^2=4$



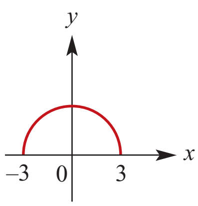
d $\left(x-\dfrac{1}{2}\right)^2+\left(y+\dfrac{1}{3}\right)^2=\dfrac{1}{36}$



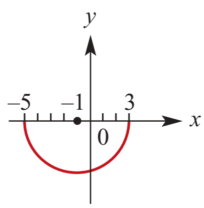
6 $C(-2,3)$, $r=6$

7 y-axis: $4\sqrt{6}$; x-axis: $2\sqrt{21}$

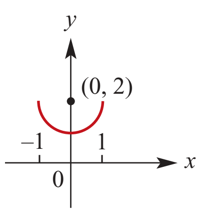
8 a



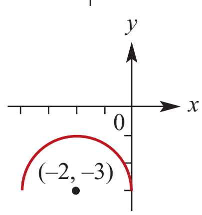
b



c



d



9 $a=-6, h=2, k=5$

10 a $(y+2)^2=x+6$; vertex $(-6,-2)$;
axis intercepts $(-2,0)$, $(0,-2\pm\sqrt{6})$

b $(y+3)^2=-2\left(x-\dfrac{5}{2}\right)$; vertex $\left(\dfrac{5}{2},-3\right)$;
axis intercepts $(-2,0)$, $(0,-3\pm\sqrt{5})$

c $(y+2)^2=\dfrac{5}{2}\left(x+\dfrac{2}{5}\right)$; vertex $\left(-\dfrac{2}{5},-2\right)$;
axis intercepts $\left(\dfrac{6}{5},0\right)$, $(0,-1)$, $(0,-3)$

11 Touches at $\left(2,\dfrac{1}{2}\right)$

12 a $B(0,3), A(-2,0)$ **b** $PB=AQ=\sqrt{13}$

Technology-active

13 $a=2.83, b=2, k=4$

14 13 knots

15 a $V=\dfrac{121.8}{P}$ **b** $9.6\ \text{kg/cm}^2$

16 a $\sqrt{35}\approx 5.92$ m/s **b** 119 m

17 a $w=\dfrac{3000}{d}$ **b** 600 kg **c** 333 kg

18 $h=3, k=6$

19 $a=-24, k=30$

20 $\left(x-\dfrac{1}{2}\right)^2+y^2=\dfrac{125}{4}$

21 a $(1,1)$, $(6,6)$

b $\left(\dfrac{-2a+7+\sqrt{25-12a}}{2},\dfrac{7+\sqrt{25-12a}}{2}\right)$,
$\left(\dfrac{-2a+7-\sqrt{25-12a}}{2},\dfrac{7-\sqrt{25-12a}}{2}\right)$

c i $a=\dfrac{25}{12}$ **ii** $a>\dfrac{25}{12}$

22 b $(-3+2\sqrt{2},-2+2\sqrt{2})$, $(-3-2\sqrt{2},-2-2\sqrt{2})$
c 8
d $(-3,-2)$

23 a $(x-10)^2+y^2=25$ **c** $m=\pm\dfrac{\sqrt{3}}{3}$

d $P\left(\dfrac{15}{2},\dfrac{\pm5\sqrt{3}}{2}\right)$ **e** $5\sqrt{3}$

24 a $x^2+y^2=16$

b ii $m=\pm\dfrac{\sqrt{3}}{3}$; $y=\dfrac{\sqrt{3}}{3}x-\dfrac{8\sqrt{3}}{3}$
$y=-\dfrac{\sqrt{3}}{3}x+\dfrac{8\sqrt{3}}{3}$

25 a $\dfrac{4}{3}$ **b** $\dfrac{-3}{4}$ **c** $4y+3x=25$ **d** $\dfrac{125}{12}$

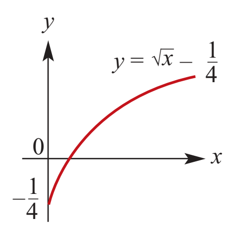
26 a i $\dfrac{y_1}{x_1}$ **ii** $\dfrac{-x_1}{y_1}$

c $\sqrt{2}x+\sqrt{2}y=8$ or $\sqrt{2}x+\sqrt{2}y=-8$

27 a $y=\dfrac{-\sqrt{3}}{3}x+\dfrac{2\sqrt{3}}{3}a$, $y=\dfrac{\sqrt{3}}{3}x-\dfrac{2\sqrt{3}}{3}a$

b $x^2+y^2=4a^2$

28 b ii $\left(\dfrac{1}{4},\dfrac{1}{4}\right)$



c **i** $\frac{-1}{4} \le k < 0$ **ii** $k = 0$ or $k < \frac{-1}{4}$

iii $k > 0$

29 **a** $0 < k < \frac{1}{4}$ **b** $k = \frac{1}{4}$ or $k \le 0$

30 **b** **i** $a = \frac{1}{2}, (0, 1)$ **ii** $a = \frac{3}{25}, \left(-\frac{7}{75}, \frac{8}{25}\right)$

c **i** $\frac{1}{2}x + \frac{1}{2}y = 1$ **ii** $\frac{1}{5}x - \frac{2}{5}y = 1$

31 **b** **i** $a = \frac{1}{4}, \left(\frac{1}{2}, \frac{1}{2}\right)$ **ii** $a = \frac{1}{16}, \left(\frac{1}{4}, \frac{1}{4}\right)$

c **i** $\frac{1}{4}x + y = 1$ **ii** $-\frac{1}{4}x - y = 1$

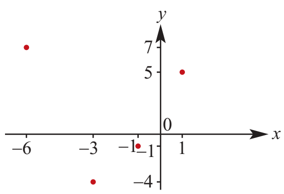
Multiple-choice questions

1 D	**2** A	**3** D	**4** A	**5** A
6 C	**7** C	**8** B	**9** D	**10** A
11 A	**12** B	**13** A	**14** A	**15** A
16 A	**17** C	**18** C	**19** A	**20** A

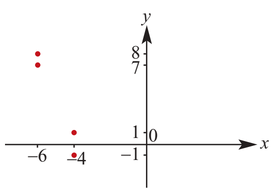
Chapter 6

Exercise 6A

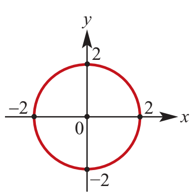
1 **a** Domain = $\{-3, -1, -6, 1\}$;
Range = $\{-4, -1, 7, 5\}$



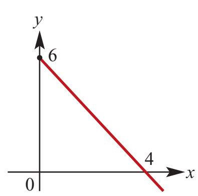
b Domain = $\{-4, -6\}$; Range = $\{-1, 1, 7, 8\}$



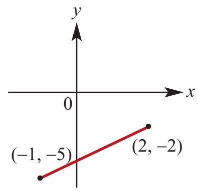
c Domain = $[-2, 2]$
Range = $[-2, 2]$



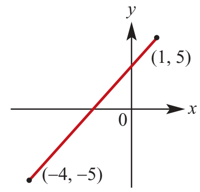
d Domain = $[0, \infty)$
Range = $(-\infty, 6]$



e Domain = $[-1, 2]$
Range = $[-5, -2]$



f Domain = $[-4, 1]$
Range = $[-5, 5]$



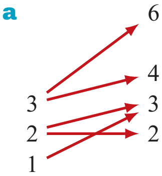
2 **a** Domain = $[-2, 2]$ i.e. $-2 \le x \le 2$;
Range = $[-1, 2]$ i.e. $-1 \le y \le 2$

b Domain = $[-2, 2]$ i.e. $-2 \le x \le 2$;
Range = $[-2, 2]$ i.e. $-2 \le y \le 2$

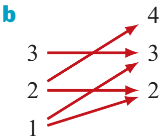
c Domain = $\mathbb{R}$;
Range = $[-1, \infty)$ i.e. $y \ge -1$

d Domain = $\mathbb{R}$;
Range = $(-\infty, 4]$ i.e. $y \le 4$

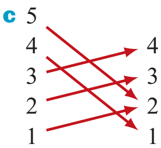
3 **a**



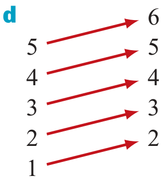
b



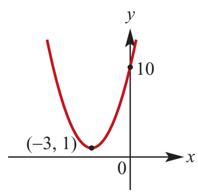
c



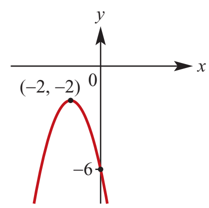
d



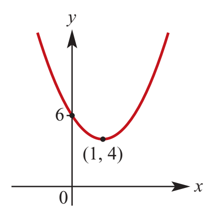
4 **a** $y = (x + 3)^2 + 1$
Range = $[1, \infty)$



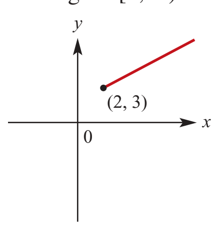
b $y = -(x + 2)^2 - 2$
Range = $(-\infty, -2]$



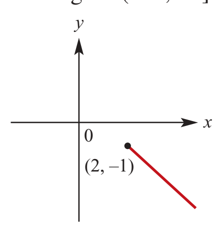
c $y = 2(x - 1)^2 + 4$
Range = $[4, \infty)$



5 **a** Range = $[3, \infty)$

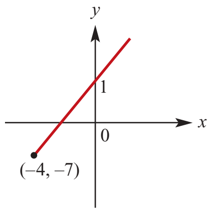


b Range = $(-\infty, -1]$

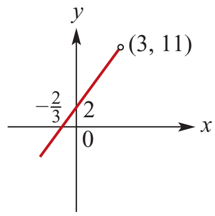


Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

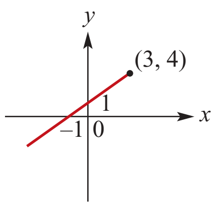
c Range $= [-7, \infty)$



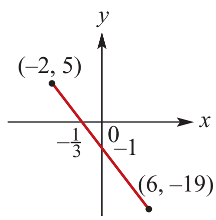
d Range $= (-\infty, 11)$



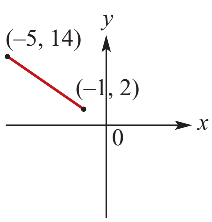
e Range $= (-\infty, 4]$



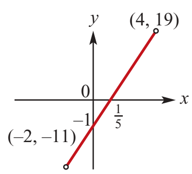
f Range $= [-19, 5]$



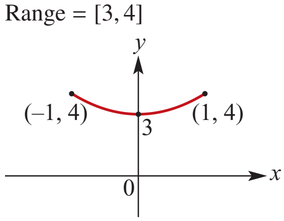
g Range $= [2, 14]$



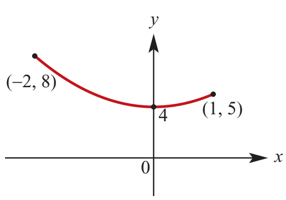
h Range $= (-11, 19)$



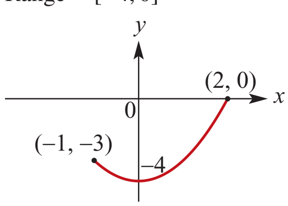
6 a Range $= [3, 4]$



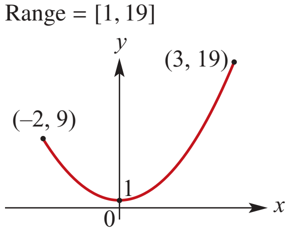
b Range $= [4, 8]$



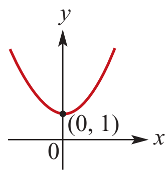
c Range $= [-4, 0]$



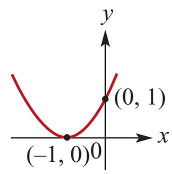
d Range $= [1, 19]$



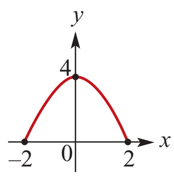
7 a Range $= [1, \infty)$



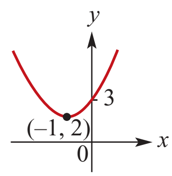
b Range $= [0, \infty)$



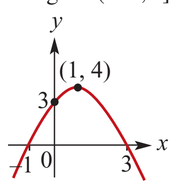
c Range $= [0, 4]$



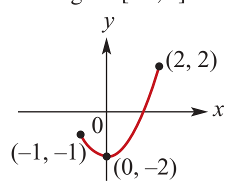
d Range $= [2, \infty)$



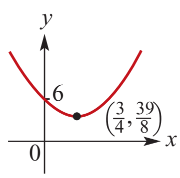
e Range $= (-\infty, 4]$



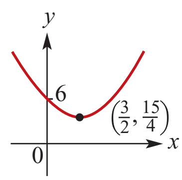
f Range $= [-2, 2]$



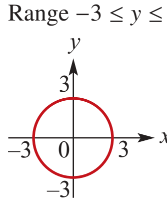
g Range $= \left[\frac{39}{8}, \infty\right)$



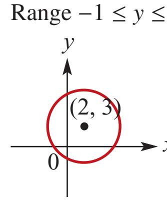
h Range $= \left[\frac{15}{4}, \infty\right)$



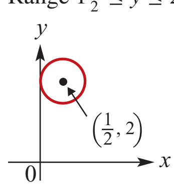
8 a Domain $-3 \le x \le 3$
Range $-3 \le y \le 3$



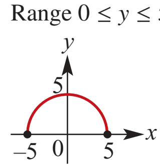
b Domain $-2 \le x \le 6$
Range $-1 \le y \le 7$



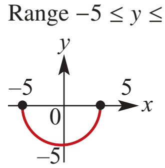
c Domain $0 \le x \le 1$
Range $1\frac{1}{2} \le y \le 2\frac{1}{2}$



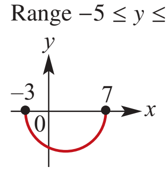
d Domain $-5 \le x \le 5$
Range $0 \le y \le 5$



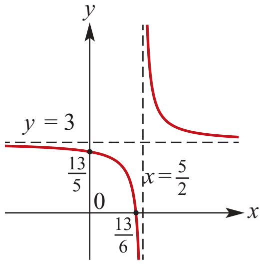
e Domain $-5 \le x \le 5$
Range $-5 \le y \le 0$



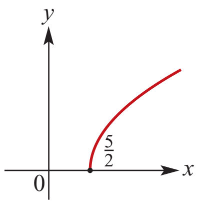
f Domain $-3 \le x \le 7$
Range $-5 \le y \le 0$



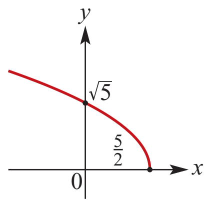
9 a Domain = $\mathbb{R} \setminus \left\{\frac{5}{2}\right\}$; Range = $\mathbb{R} \setminus \{3\}$



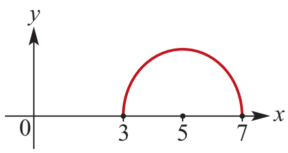
b Domain = $\left[\frac{5}{2}, \infty\right)$
Range = $[0, \infty)$



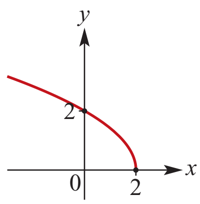
c Domain = $\left(-\infty, \frac{5}{2}\right]$
Range = $[0, \infty)$



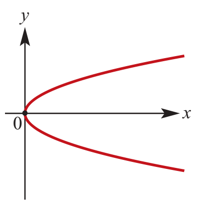
d Domain = $[3, 7]$; Range = $[0, 2]$



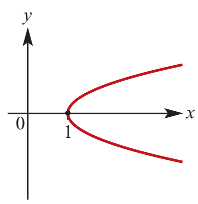
e Domain = $(-\infty, 2]$
Range = $[0, \infty)$



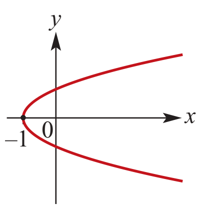
10 a Domain = $[0, \infty)$
Range = $\mathbb{R}$



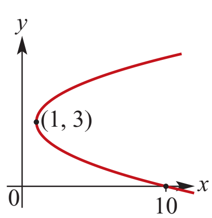
b Domain = $[1, \infty)$
Range = $\mathbb{R}$



c Domain = $[-1, \infty)$
Range = $\mathbb{R}$



d Domain = $[1, \infty)$
Range = $\mathbb{R}$



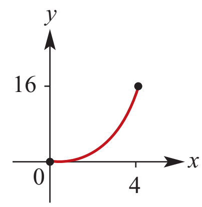
Exercise 6B

1 a Not a function; Domain = $\{0, 1, 2, 3\}$;
Range = $\{1, 2, 3, 4\}$
b A function; Domain = $\{-2, -1, 0, 1, 2\}$;
Range = $\{-5, -2, -1, 2, 4\}$
c Not a function; Domain = $\{-1, 0, 3, 5\}$;
Range = $\{1, 2, 4, 6\}$
d A function; Domain = $\{1, 2, 4, 5, 6\}$;
Range = $\{3\}$

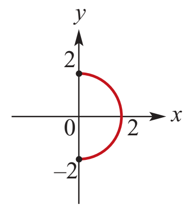
2 Functions: a, b, c, e

3 Functions: c, d, e

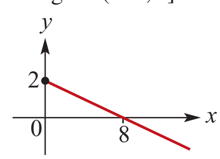
4 a A function
Domain = $[0, 4]$
Range = $[0, 16]$



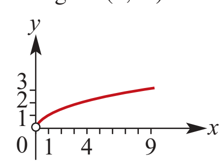
b Not a function
Domain = $[0, 2]$
Range = $[-2, 2]$



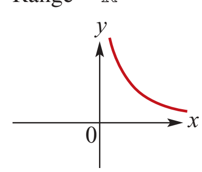
c A function
Domain = $[0, \infty)$
Range = $(-\infty, 2]$



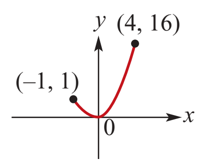
d A function
Domain = $(0, \infty)$
Range = $(0, \infty)$



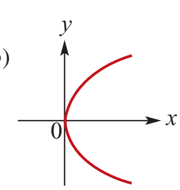
e A function
Domain = $\mathbb{R}^+$
Range = $\mathbb{R}^+$



f A function
Domain = $[-1, 4]$
Range = $[0, 16]$



g Not a function
Domain = $[0, \infty)$
Range = $\mathbb{R}$



5 a A function; Domain = $\mathbb{R}$; Range = $\{-2\}$
b Not a function; Domain = $\{3\}$; Range = $\mathbb{Z}$
c A function; Domain = $\mathbb{R}$; Range = $\mathbb{R}$
d A function; Domain = $\mathbb{R}$; Range = $[5, \infty)$
e Not a function; Domain = $[-3, 3]$;
Range = $[-3, 3]$

6 a i -3 **ii** 5 **iii** -5 **iv** $2x - 5$ **v** $\frac{2}{a} - 3$

b i 4 **ii** -4 **iii** $\frac{4}{3}$ **iv** 2

c **i** 4 **ii** 36 **iii** 36 **iv** $(a-2)^2$

d **i** 0 **ii** $\dfrac{a}{1+a}$ **iii** $\dfrac{-a}{1-a}$ **iv** $1-a$

7 **a** $5, 2t+1$ **b** $x=\dfrac{5}{2}$ **c** $x=-\dfrac{1}{2}$

d $t=-1$ **e** $x \geq -1$ **f** $x \geq 1$

8 **a** 1 **b** $\dfrac{1}{6}$ **c** ± 3

d $-1, 4$ **e** $-1, 3$ **f** $-2, 3$

9 **a** $g(-1)=-1, g(2)=8, g(-2)=0$

b $h(-1)=3, h(2)=18, h(-2)=-14$

c **i** $g(-3x)=9x^2-6x$

ii $g(x-5)=x^2-8x+15$

iii $h(-2x)=-16x^3-4x^2+6$

iv $g(x+2)=x^2+6x+8$

v $h(x^2)=2x^6-x^4+6$

10 **a** $f(2)=5, f(-4)=29$

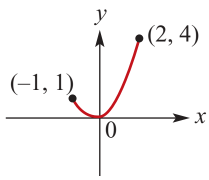
b Range $=[-3,\infty)$

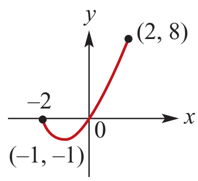
11 **a** $f(2)=7$ **b** $x=2$ **c** $x=-1$

12 **a** 2 **b** ± 1 **c** $x=\pm\sqrt{3}$

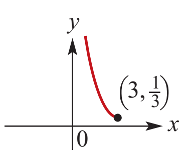
13 **a** $x=-1$ **b** $x>-1$ **c** $x=-\dfrac{6}{7}$

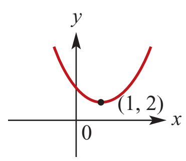
14 **a** Range $=[0,4]$ **b** Range $=[-1,8]$



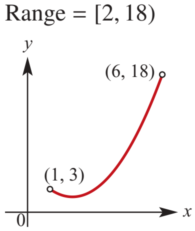


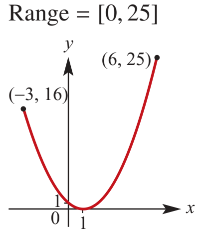
c Range $=\left[\dfrac{1}{3},\infty\right)$ **d** Range $=[2,\infty)$





e Range $=[2,18)$ **f** Range $=[0,25]$





15 **a** Domain $=\mathbb{R}$; Range $=\mathbb{R}$

b Domain $=[0,\infty)$; Range $=[0,\infty)$

c Domain $=\mathbb{R}$; Range $=[1,\infty)$

d Domain $=[-3,3]$; Range $=[-3,0]$

e Domain $=\mathbb{R}^+$; Range $=\mathbb{R}^+$

f Domain $=\mathbb{R}$; Range $=(-\infty,3]$

g Domain $=[2,\infty)$; Range $=[0,\infty)$

h Domain $=[\frac{1}{2},\infty)$; Range $=[0,\infty)$

i Domain $=(-\infty,\frac{3}{2}]$; Range $=[0,\infty)$

j Domain $=\mathbb{R}\setminus\{\frac{1}{2}\}$; Range $=\mathbb{R}\setminus\{0\}$

k Domain $=\mathbb{R}\setminus\{\frac{1}{2}\}$; Range $=(0,\infty)$

l Domain $=\mathbb{R}\setminus\{\frac{1}{2}\}$; Range $=\mathbb{R}\setminus\{2\}$

16 **a** Domain $=[4,\infty)$; Range $=[0,\infty)$

b Domain $=(-\infty,4]$; Range $=[0,\infty)$

c Domain $=[2,\infty)$; Range $=[3,\infty)$

d Domain $=\mathbb{R}\setminus\{4\}$; Range $=\mathbb{R}\setminus\{0\}$

e Domain $=\mathbb{R}\setminus\{4\}$; Range $=\mathbb{R}\setminus\{3\}$

f Domain $=\mathbb{R}\setminus\{-2\}$; Range $=\mathbb{R}\setminus\{-3\}$

17 **a** Domain $=\mathbb{R}$; Range $=\mathbb{R}$

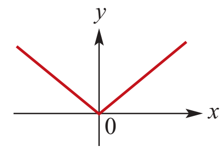
b Domain $=\mathbb{R}$; Range $=[2,\infty)$

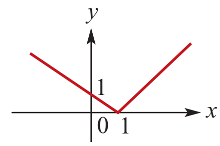
c Domain $=[-4,4]$; Range $=[-4,0]$

d Domain $=\mathbb{R}\setminus\{-2\}$; Range $=\mathbb{R}\setminus\{0\}$

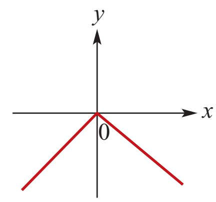
Exercise 6C

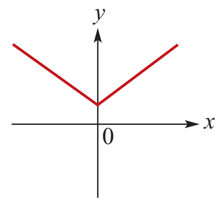
1 **a** Range $=[0,\infty)$ **b** Range $=[0,\infty)$



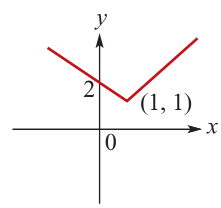


c Range $=(-\infty,0]$ **d** Range $=[1,\infty)$

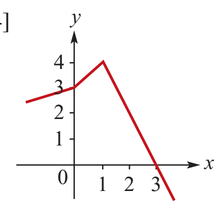




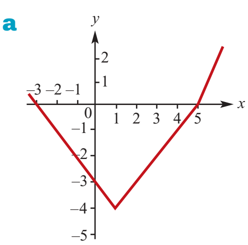
e Range $=[1,\infty)$



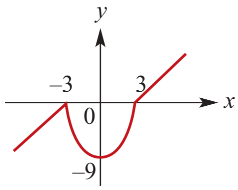
2 Range $=(-\infty,4]$



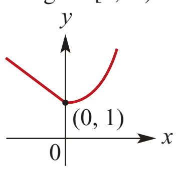
3 **a**



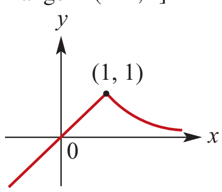
b



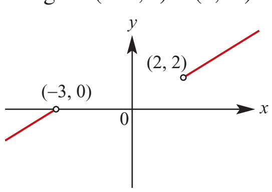
4 a Range = $[1, \infty)$



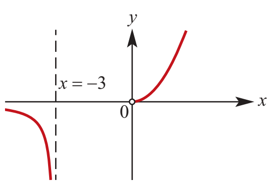
b Range = $(-\infty, 1]$



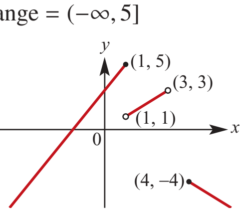
5 a Range = $(-\infty, 0) \cup (2, \infty)$



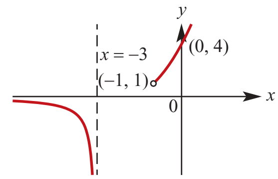
b Range = $\mathbb{R} \setminus \{0\}$



c Range = $(-\infty, 5]$



d Range = $(-\infty, 0) \cup (1, \infty)$



6 $f(x) = \begin{cases} x+3, & -3 \le x \le -1 \\ -x+1, & -1 < x \le 2 \\ -\frac{1}{2}x, & 2 < x \le 4 \end{cases}$

Exercise 6D

1 b i 25.06 **ii** 25.032 **iii** 25.2 **iv** 26

2 a $a = -3, b = \dfrac{1}{2}$ **b** 6

3 $f(x) = 7 - 5x$

4 a i $f(0) = -\dfrac{9}{2}$ **ii** $f(1) = -3$

b 3

5 $f(x) = -7(x-2)(x-4)$

6 $f(x) = (x-3)^2 + 7$, Range = $[7, \infty)$

7 $a = \dfrac{1}{10}, b = -\dfrac{9}{10}, c = 2$

8 $f(x) = -2(x-1)(x+5)$

$g(x) = -50(x-1)\left(x + \dfrac{1}{5}\right)$

9 a $k < \dfrac{-37}{12}$ **b** $k = -\dfrac{25}{12}$

10 a $f(n) = n + 3$

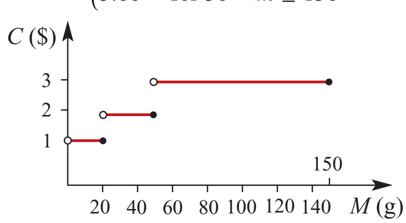
b Range = $\{4, 5, 6, \ldots\} = \mathbb{N} \setminus \{1, 2, 3\}$

Exercise 6E

1 $C = 0.15n + 45$ where n is the number of calls

2 a $C(m) = \begin{cases} 1.20 & \text{for } 0 < m \le 20 \\ 2.00 & \text{for } 20 < m \le 50 \\ 3.00 & \text{for } 50 < m \le 150 \end{cases}$

b



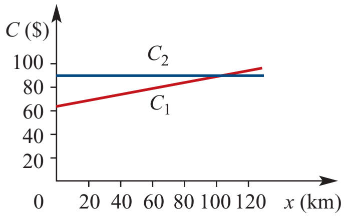
Domain = $(0, 150]$

Range = $\{1.20, 2.00, 3.00\}$

 ISBN 978-1-009-53719-3

3 a $C_1 = 64 + 0.25x$, $C_2 = 89$

b



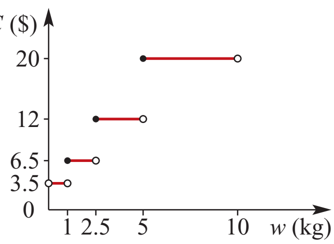
c $x > 100$ km

4 a Length $= (50 - x)$ cm

b $A(x) = x(50 - x)$ **c** $0 \le x \le 50$

d Maximum area $= 625$ cm^2 when $x = 25$

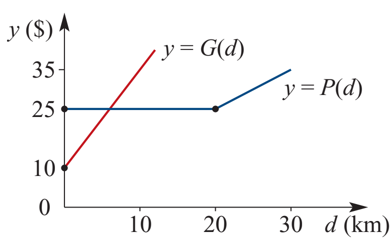
5 a



b i \$6.50 **ii** \$12 **iii** \$20

c Package them together

6 c

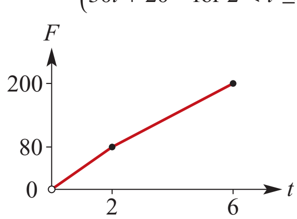


d i \$27.50 **ii** \$25

e Purple Taxi **f** Greater than 6 km

7 a $F(t) = \begin{cases} 40t & \text{for } 0 < t \le 2 \\ 30t + 20 & \text{for } 2 < t \le 6 \end{cases}$

b



c i \$60 **ii** \$95 **iii** \$125

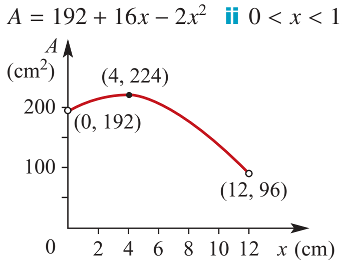
d \$35 per hour

8 a i $A = (8 + x)y - x^2$

ii $P = 2x + 2y + 16$

b i $A = 192 + 16x - 2x^2$ **ii** $0 < x < 12$

iii



iv 224 cm^2

9 a 40 mg/L **b** 45 mg/L **c** 36 mg/L

d At $t = 9$ hours; $C = 9$ mg/L

e 54 mg/L (quite a lot)

10 a 3 m

b i 8 m **ii** 4.8 m **iii** 4 m

c $5 - \sqrt{10} \approx 1.84$ m, 8.5 m

Exercise 6F

1 a $y = \frac{1}{4}\sqrt{x}$ **b** $y = 2x^{\frac{5}{4}}$ **c** $y = 3.5x^{0.4}$

d $y = 10x^{\frac{2}{3}}$ **e** $y = 2x^{-\frac{5}{2}}$ **f** $y = 3.2x^{-0.4}$

2 a $a = 100$, $b = 0.2$ **b** 158.49

3 a $a = 1500$, $b = -0.5$ **b** 474.34

Exercise 6G

1 a $(-1, 1)$ **b** $(-5, 8)$ **c** $(-6, 2)$

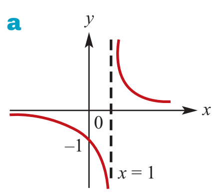
d $(-7, 9)$ **e** $(-5, 3)$

2 a $g(x) = \dfrac{1}{x-2} - 1$ **b** $g(x) = \dfrac{1}{x-4} + 3$

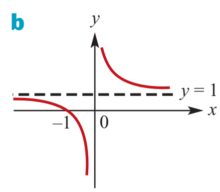
c $g(x) = (x + 2)^2 - 3$ **d** $g(x) = (x - 4)^2 - 2$

e $g(x) = \sqrt{x - 2} - 1$

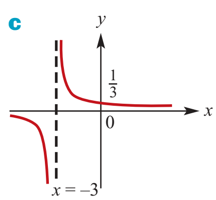
3 a



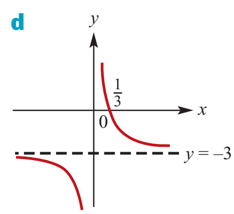
b



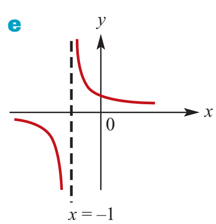
c



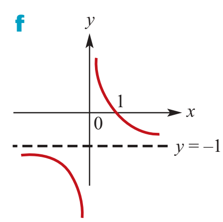
d



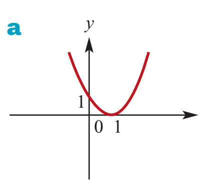
e



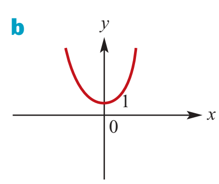
f

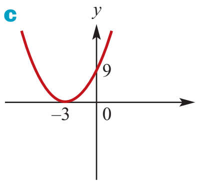


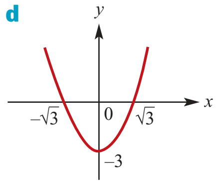
4 a

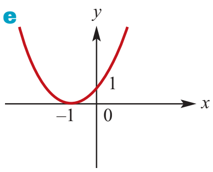


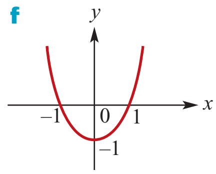
b

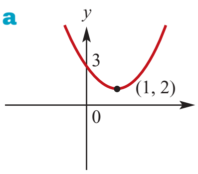


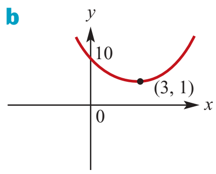
c


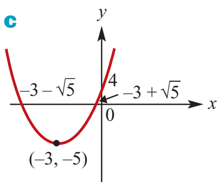
d


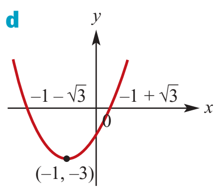
e


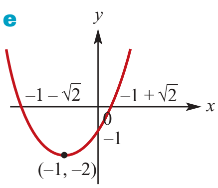
f


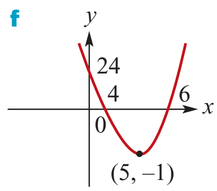
5 a


b


c


d


e


f


Exercise 6H

1 a $(-2, 3)$ **b** $(2, -3)$ **c** $(-2, -12)$ **d** $(-8, -3)$

2 a i $y = 4x^2$ **ii** $y = \dfrac{x^2}{25}$ **iii** $y = \dfrac{2x^2}{3}$

iv $y = 4x^2$ **v** $y = -x^2$ **vi** $y = x^2$

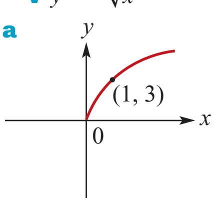
b i $y = \dfrac{1}{2x}$ **ii** $y = \dfrac{5}{x}$ **iii** $y = \dfrac{2}{3x}$

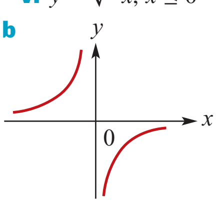
iv $y = \dfrac{4}{x}$ **v** $y = \dfrac{-1}{x}$ **vi** $y = \dfrac{-1}{x}$

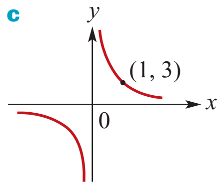
c i $y = \sqrt{2x}$ **ii** $y = \sqrt{\dfrac{x}{5}}$

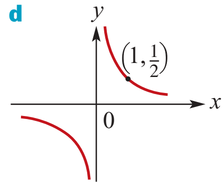
iii $y = \dfrac{2\sqrt{x}}{3}$ **iv** $y = 4\sqrt{x}$

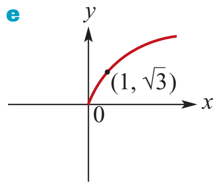
v $y = -\sqrt{x}$ **vi** $y = \sqrt{-x}, x \le 0$

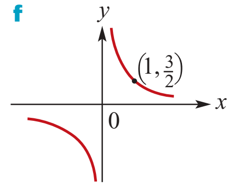
3 a


b


c


d


e


f


Exercise 6I

1 a $y = 3\sqrt{x-2}$ **b** $y = -\sqrt{x+3}$

c $y = -3\sqrt{x}$ **d** $y = -\sqrt{\dfrac{x}{2}}$

e $y = 2\sqrt{x-2} - 3$ **f** $y = \sqrt{\dfrac{x+2}{2}} - 3$

2 a $y = \dfrac{3}{x-2}$ **b** $y = \dfrac{-1}{x+3}$

c $y = -\dfrac{3}{x}$ **d** $y = -\dfrac{2}{x}$

e $y = \dfrac{2}{x-2} - 3$ **f** $y = \dfrac{2}{x+2} - 3$

Exercise 6J

1 a i A dilation of factor 2 from the x-axis, then a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

ii A reflection in the x-axis, then a translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii A dilation of factor $\frac{1}{2}$ from the y-axis, then a translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

b i A dilation of factor 2 from the x-axis, then a translation of 3 units in the negative direction of the x-axis

ii A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii A translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

c i A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

ii A dilation of factor $\frac{1}{3}$ from the y-axis, then a dilation of factor 2 from the x-axis

iii A reflection in the x-axis, then a translation of 2 units in the positive direction of the y-axis

2 a $(x, y) \to (3x + 2, y - 5)$

b $(x, y) \to \left(3x + 1, -\dfrac{y-7}{3}\right)$

c $(x, y) \to \left(-(x-4), \dfrac{y}{2}\right)$

d $(x, y) \to \left(-(x-4), \dfrac{15-y}{2}\right)$

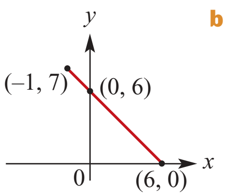
Chapter 6 review

Short-response questions

Technology-free

1 a -16 **b** 26 **c** $-\dfrac{2}{3}$

2 a



b Range $= [0, 7]$

3 a Range $= \mathbb{R}$ **b** Range $= [-5, 4]$

c Range $= [0, 4]$ **d** Range $= (-\infty, 9]$

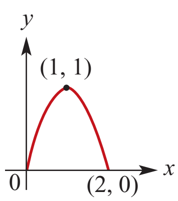
e Range $= [2, \infty)$ **f** Range $= \{-6, 2, 4\}$

g Range $= [0, \infty)$ **h** Range $= \mathbb{R} \setminus \{2\}$

i Range $= [-5, 1]$ **j** Range $= [-1, 3]$

4 a $a = -15, b = \dfrac{33}{2}$ **b** Domain $= \mathbb{R} \setminus \{0\}$

5 a



b Range $= [0, 1]$

6 $a = 3, b = -5$

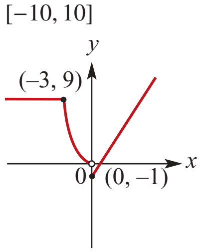
7 $a = -\frac{1}{2}, b = 2, c = 0$

8 a $\mathbb{R} \setminus \{2\}$ **b** $[2, \infty)$

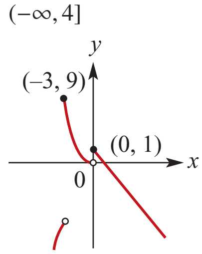
c $[-5, 5]$ **d** $\mathbb{R} \setminus \{\frac{1}{2}\}$

e $[-10, 10]$ **f** $(-\infty, 4]$

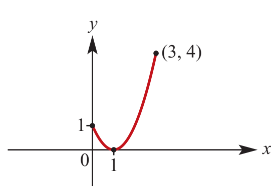
9 a



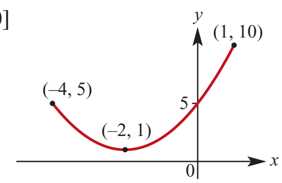
b



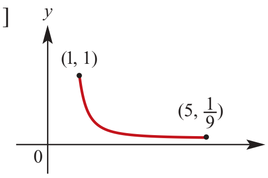
10 a Range $= [0, 4]$



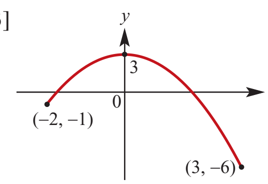
b Range $= [1, 10]$



c Range $= [\frac{1}{9}, 1]$



d Range $= [-6, 3]$



11 a Domain $= [1, \infty)$; Range $= [0, \infty)$

b Domain $= (-\infty, 1]$; Range $= [0, \infty)$

c Domain $= [0, \infty)$; Range $= (-\infty, 1]$

12 a Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{0\}$

b Domain $= \mathbb{R} \setminus \{-1\}$; Range $= \mathbb{R} \setminus \{0\}$

c Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{3\}$

13 a Domain $= [-1, 1]$; Range $= [0, 1]$

b Domain $= [-3, 3]$; Range $= [0, 3]$

c Domain $= [-1, 1]$; Range $= [3, 4]$

14 a $2p + 5$ **b** $2(p + h) + 5$

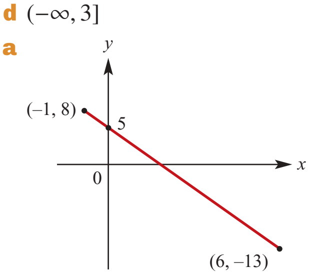
c $2h$ **d** 2

15 -2

16 a $\left(-\infty, -\dfrac{15}{8}\right]$ **b** $[3\frac{7}{8}, \infty)$ **c** $(-\infty, 20]$

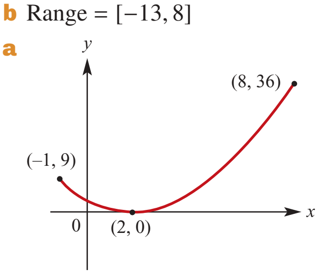
d $(-\infty, 3]$

17 a



b Range $= [-13, 8]$

18 a



b Range $= [0, 36]$

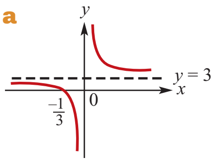
19 a Domain $-3 \le x \le 3$; Range $-3 \le y \le 3$

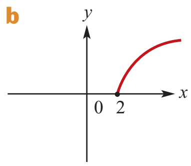
b Domain $1 \le x \le 3$; Range $-1 \le y \le 1$

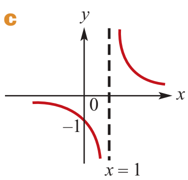
c Domain $0 \le x \le 1$; Range $0 \le y \le 1$

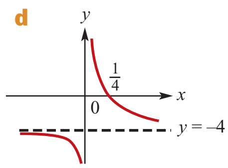
d Domain $-1 \le x \le 9$; Range $-5 \le y \le 5$
e Domain $-4 \le x \le 4$; Range $-2 \le y \le 6$

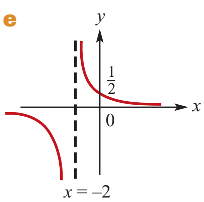
20 a $\{2, 4, 6, 8\}$ **b** $\{4, 3, 2, 1\}$
c $\{-3, 0, 5, 12\}$ **d** $\{1, \sqrt{2}, \sqrt{3}, 2\}$

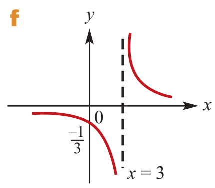
21 a


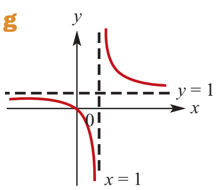
b


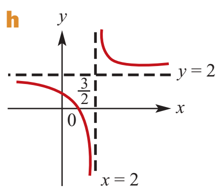
c


d


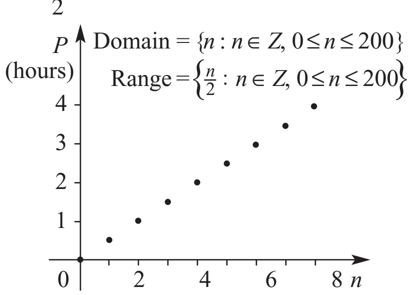
e


f


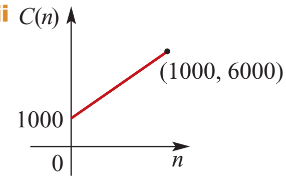
g


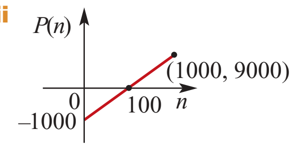
h


22 a **i** $(x, y) \to (x - 1, 3y + 2)$
ii $(x, y) \to (x - 2, -2y + 3)$
iii $(x, y) \to \left(\dfrac{x - 1}{3}, y - 1\right)$
b **i** $(x, y) \to (x - 2, 4y)$
ii $(x, y) \to (x - 6, y - 12)$
iii $(x, y) \to (x + 3, 4y - 5)$
c **i** $(x, y) \to (x + 4, y + 2)$
ii $(x, y) \to \left(\dfrac{x}{2}, 2y\right)$
iii $(x, y) \to (x, -2y + 3)$

23 a $P = \dfrac{1}{2}n$
b


Domain $= \{n : n \in Z, 0 \le n \le 200\}$
Range $= \left\{\frac{n}{2} : n \in Z, 0 \le n \le 200\right\}$

24 a **i** $C(n) = 1000 + 5n, n > 0$
ii


b **i** $P(n) = 15n - (1000 + 5n)$
$= 10n - 1000$
ii


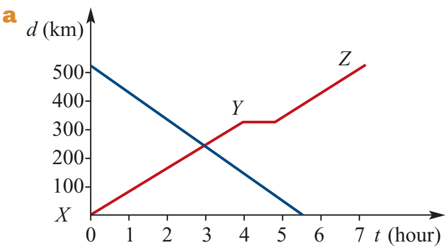
Technology-active

25 1.89 cm

26 a 1.648 **b** 2.167 **c** 1.903

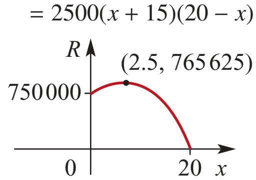
27 2.19

28 $a = 4.29, b = 0.32$

29 $w = 0.01\ell^{3.11}$
a


Coach starting from X:
$$d = \begin{cases} 80t & \text{for } 0 \le t \le 4 \\ 320 & \text{for } 4 < t \le 4\frac{3}{4} \\ 80t - 60 & \text{for } 4\frac{3}{4} < t \le 7\frac{1}{4} \end{cases}$$
Range $= [0, 520]$
Coach starting from Z:
$$d = 520 - \frac{1040t}{11} \text{ for } 0 \le t \le 5\frac{1}{2}$$
Range $= [0, 520]$
b The coaches pass $238\frac{1}{3}$ km from X

30 $V = 8000(1 - 0.05n) = 8000 - 400n$

31 a $R = (50000 - 2500x)(15 + x)$
$= 2500(x + 15)(20 - x)$
b


c Price for max revenue = \$17.50

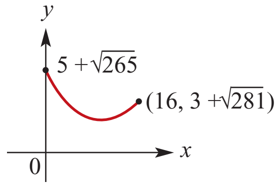
32 a $A(x) = \dfrac{x}{4}(2a - (6 - \sqrt{3})x)$ **b** $0 < x < \dfrac{a}{3}$

33 a $C(n) = \begin{cases} 24n + 45 & \text{for } 1 \le n \le 100 \\ 20n + 445 & \text{for } n > 100 \end{cases}$
b \$3405 **c** \$23

34 a **i** $d(x) = \sqrt{x^2 + 25} + \sqrt{(16 - x)^2 + 9}$

ii $0 \le x \le 16$

b i



ii 1.54 **iii** 3.40 or 15.04

c i Minimum is $8\sqrt{5}$, occurs when $x = 10$

ii Range $= [8\sqrt{5}, 5 + \sqrt{265}]$

35 a $k = \frac{1}{4}$ **b** $h = -\frac{1}{4}$

36 a $h = -1 \pm 2\sqrt{2}$ **b** $a = \pm 2\sqrt{2}$

c $a = -8, b = 16$

37 a $k = 10$

b i $h > 2 + \sqrt{10}$ **ii** $h < 2 - \sqrt{10}$

iii $2 - \sqrt{10} < h < 2 + \sqrt{10}$

Multiple-choice questions

1 A	**2** D	**3** C	**4** A	**5** D
6 D	**7** C	**8** B	**9** C	**10** B
11 B	**12** B	**13** C	**14** A	**15** A
16 C	**17** B	**18** C	**19** D	**20** C
21 B	**22** C	**23** A	**24** A	

Chapter 7

Exercise 7A

1 a -3 **b** -1 **c** -7 **d** -15

2 a 0 **b** 0

3 a 6 **b** 9 **c** 26 **d** 11

e $a^3 + 4a^2 - 2a + 6$ **f** $8a^3 + 16a^2 - 4a + 6$

4 a $a = 4$ **b** $a = 4$ **c** $c = 6$

d $a = -33, b = -15$ **e** $a = -9, b = 23$

5 a $x^3 - 2x^2 - 2x + 2$ **b** $x^3 - x^2 + 2x$

c $x^3 - 2x^2 + 4x - 2$ **d** $3x^3 - 6x^2 + 3x$

e $-3x^4 + 8x^3 - 7x^2 + 2x$

f $-3x^3 - x^2 + 2x$ **g** $x^3 - x^2 - x + 2$

h $x^5 - x^4 - x^3 + x^2$

6 a $x^3 - 4x^2 + 7x - 6$

b $x^3 - 6x^2 + 11x - 12$

c $2x^3 - 5x^2 - x + 4$

d $x^3 + (b - 2)x^2 + (c - 2b)x - 2c$

e $2x^3 - 7x^2 - 10x - 3$

7 a $x^3 - 6x^2 + 11x - 6$

b $2x^3 - 3x^2 - 2x + 3$

c $-2x^3 + 15x^2 - 31x + 12$

d $-4x^3 + 14x^2 - 8x - 8$

e $x^3 - 12x^2 + 45x - 50$

8 a $x^3 + (b + 1)x^2 + (c + b)x + c$

b $b = -8, c = 12$

c $(x + 1)(x - 6)(x - 2)$

Exercise 7B

1 1, 5, 10, 10, 5, 1

2 a 7 **b** 6 **c** 66 **d** 56

e 100 **f** 499 500

3 a n **b** $\frac{n(n-1)}{2}$ **c** n

4 $n = 10$

5 a 35 **b** 35

6 a $8 \times 7 \times 6 = 336$ **b** $\binom{8}{3} = 56$

7 2 598 960

8 a 10 **b** 45 **c** 45 **d** 10

9 126

10 126

11 120

12 45 379 620

13 a 1, 6, 15, 20, 15, 6, 1

b 1, 7, 21, 35, 35, 21, 7, 1

14 Pascal's identity $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$,

$n = 12, r = 6$ and $\binom{11}{5} = \binom{11}{6}$

15 1, 11, 55, 165, 330, 462, 462, 330, 165, 55, 11, 1

16 a 252 **b** 126 **c** 126

17 Proof

18 $n = 4$

Exercise 7C

1 a $x^4 + 8x^3 + 24x^2 + 32x + 16$

b $16x^4 + 32x^3 + 24x^2 + 8x + 1$

c $16x^4 - 96x^3 + 216x^2 - 216x + 81$

d $27x^3 - 27x^2 + 9x - 1$

e $16x^4 - 32x^3 + 24x^2 - 8x + 1$

f $-32x^5 + 80x^4 - 80x^3 + 40x^2 - 10x + 1$

g $-243x^5 + 405x^4 - 270x^3 + 90x^2 - 15x + 1$

h $16x^4 - 96x^3 + 216x^2 - 216x + 81$

2 a $16x^8 + 32x^6y + 24x^4y^2 + 8x^2y^3 + y^4$

b $625x^8 + 1500x^6y + 1350x^4y^2 + 540x^2y^3 + 81y^4$

c $32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + y^{10}$

d $x^9 - 3x^6y^4 + 3x^3y^8 - y^{12}$

3 a 180 **b** -448 **c** -540 **d** 720

4 $1, 24x, 240x^2, 1280x^3$

5 $1, -21x, 189x^2, -945x^3$

6 $1, 8x^2, 28x^4, 56x^6$

7 $8,\ x^7,\ 128y^7$

8 a $\binom{35}{15}x^{20}y^{15}$ **b** $-\binom{12}{9} \cdot 5^3 \cdot 3^9x^3y^9$

9 $c = 2$

10 $\binom{5}{r}(x^2)^r(-y)^{5-r}$

11 $k = 3$ or $k = -3$

12 $n = 8$

13 $n = 9$

Exercise 7D

1 a $1, -2, 4$ **b** $4, 6$ **c** $\frac{1}{2}, 3, -\frac{2}{3}$ **d** $0, -3, \frac{5}{2}$

2 a $-2, 0, 4$ **b** $0, -1 \pm 2\sqrt{3}$ **c** $-5, 0, 8$
d $0, -1 \pm \sqrt{17}$

3 a 1 **b** -1 **c** $5, \pm\sqrt{10}$ **d** $\pm 4, a$

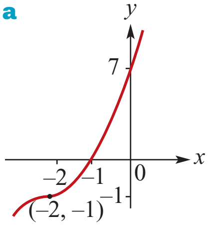
4 a $2, 3, -5$ **b** $-1, -\frac{2}{3}, 3$ **c** $1, -\sqrt{2}, \sqrt{2}$
d $-\frac{2}{5}, -4, 2$ **e** $-\frac{1}{2}, \frac{1}{3}, 1$ **f** $-2, -\frac{3}{2}, 5$

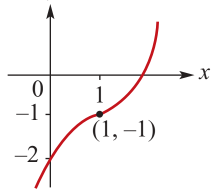
5 a $-6, 2, 3$ **b** $-2, -\frac{2}{3}, \frac{1}{2}$ **c** 3
d -1 **e** $-1, 3$ **f** $3, -2 \pm \sqrt{3}$

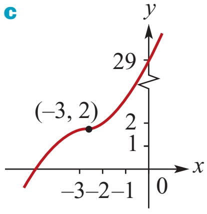
6 a $0, \pm 2\sqrt{2}$ **b** $1 + 2\sqrt[3]{2}$ **c** -2
d -5 **e** $\frac{1}{10}$

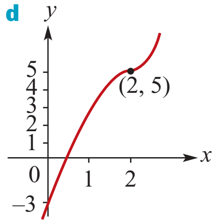
7 a $2(x-9)(x-13)(x+11)$
b $(x+11)(x+3)(2x-1)$
c $(x+11)(2x-9)(x-11)$
d $(2x-1)(x+11)(x+15)$

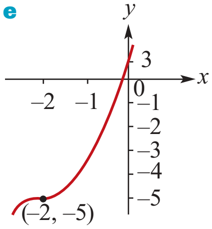
Exercise 7E

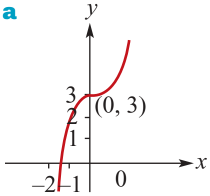
1 a


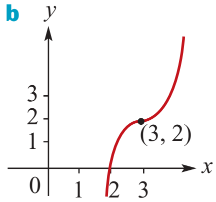
b


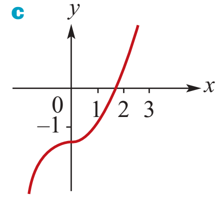
c


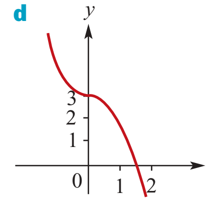
d


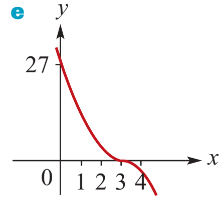
e


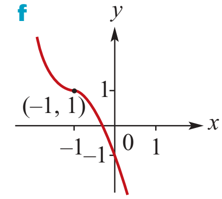
2 a


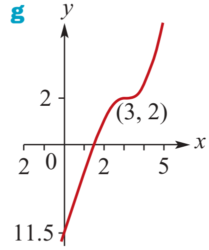
b


c


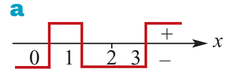
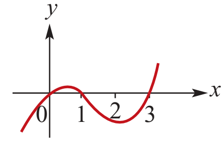
d


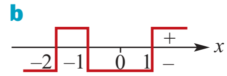
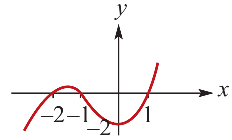
e


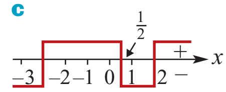
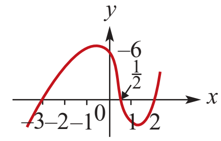
f


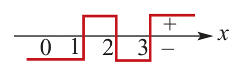
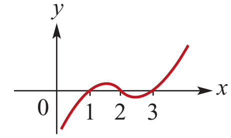
g


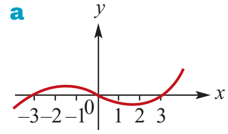
Exercise 7F

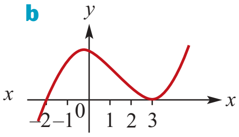
1 a



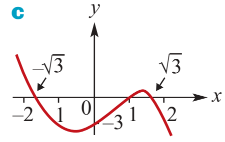
b



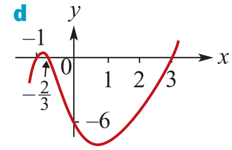
c



d



2 a


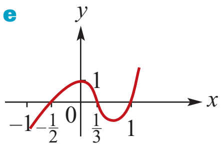
b


c


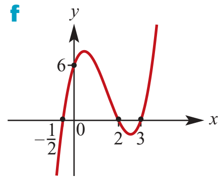
d


Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

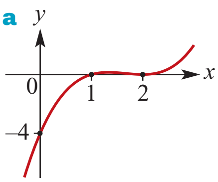
e



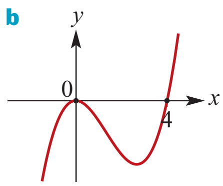
f



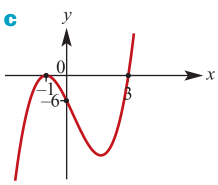
3 a



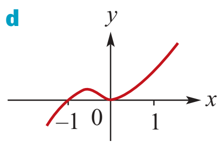
b



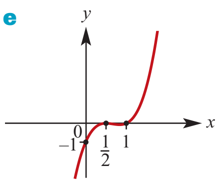
c



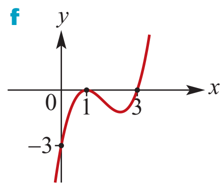
d



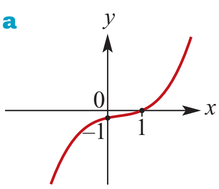
e



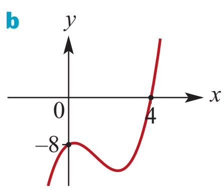
f



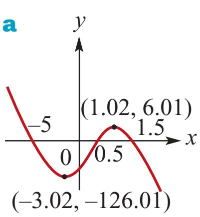
4 a



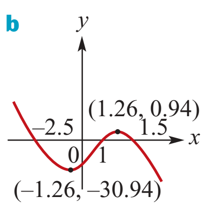
b



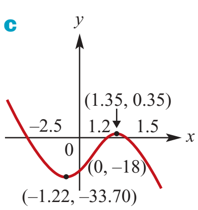
5 a



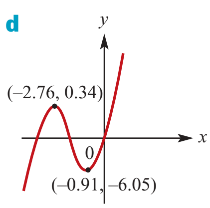
b



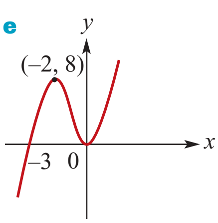
c



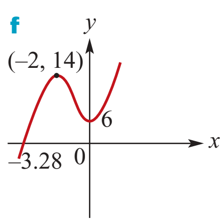
d



e



f



6 $f(x) = (x + 1)^2(x - 3)$, so graph just touches the x-axis at $x = -1$ and cuts it at $x = 3$

Exercise 7G

1 a $a = 11$ **b** $a = 2$ **c** $a = \frac{4}{3}, b = \frac{44}{3}$

2 a $y = -\frac{1}{8}(x + 2)^3$ **b** $y - 2 = -\frac{1}{4}(x - 3)^3$

3 $y = 2x(x - 2)^2$ **4** $y = -2x(x + 4)^2$

5 $y = -2(x - 1)(x - 3)(x + 1)$ **6** $a = 36$

7 a $y = (x - 3)^3 + 2$ **b** $y = \frac{23}{18}x^3 + \frac{67}{18}x^2$

c $y = 5x^3$

8 a $y = -\frac{1}{3}x^3 + \frac{4}{3}x$ **b** $y = \frac{1}{4}x(x^2 + 2)$

9 a $y = -4x^3 - 50x^2 + 96x + 270$

b $y = 4x^3 - 60x^2 + 80x + 26$

c $y = x^3 - 2x^2 + 6x - 4$

d $y = 2x^3 - 3x$

e $y = 2x^3 - 3x^2 - 2x + 1$

f $y = x^3 - 3x^2 - 2x + 1$

g $y = -x^3 - 3x^2 - 2x + 1$

Exercise 7H

1 a Length of each edge $= 20 - 2x$

b $V(x) = 4x(10 - x)^2$

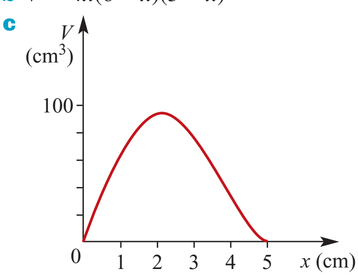
c $V(5) = 500$; Volume of box $= 500$ cm^3

d $x = 5$ or $x = \frac{5}{2}(3 - \sqrt{5})$

2 a $\ell = 12 - 2x$, $w = 10 - 2x$

b $V = 4x(6 - x)(5 - x)$

c



d $V = 80$

e $x = 3.56$ or $x = 0.51$

f $V_{max} = 96.8$ cm^3 when $x = 1.81$

3 a Surface area $= x^2 + 4xh$

b $h = \frac{75 - x^2}{4x}$

c $V = \frac{1}{4}(75x - x^3)$

d i $\frac{71}{2}$ **ii** $\frac{125}{2}$ **iii** 22

e $x = -2 + 3\sqrt{7}$

4 a $h = 60 - 20x$

b $V = 600x^2(3 - x)$

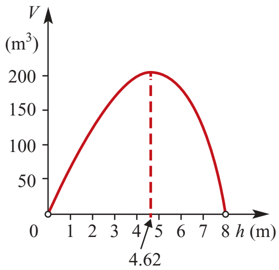
c 0

d $x = 1$ or $x = 1 + \sqrt{3}$

5 a $x = \sqrt{64 - h^2}$

b $V = \frac{\pi h}{3}(64 - h^2)$

c



d Domain $0 < h < 8$

e 64π

f $h = 2.48$ or $h = 6.47$

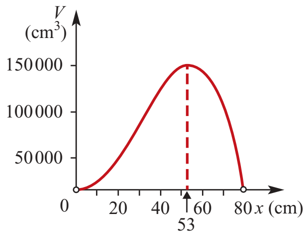
g $V_{max} \approx 206.37$ m^3 when $h = 4.62$

6 a $h = 160 - 2x$

b $V = x^2(160 - 2x)$

c Domain $0 < x < 80$

d

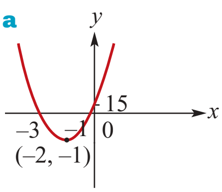


e $x = 20.498$ or $x = 75.63$

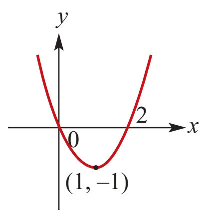
f $V_{max} \approx 151\,703.7$ cm^3 when $x \approx 53$

Exercise 7I

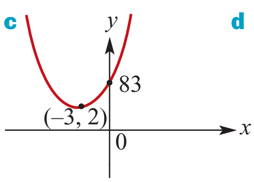
1 a



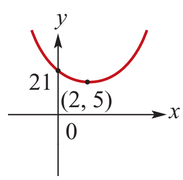
b



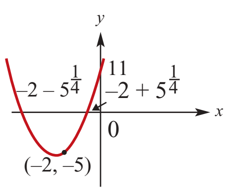
c



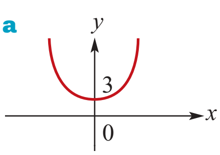
d



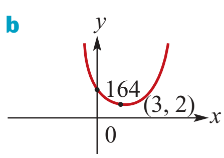
e



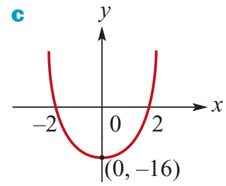
2 a



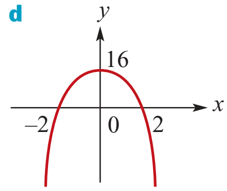
b



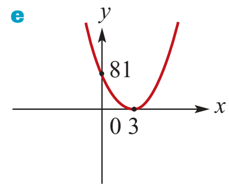
c



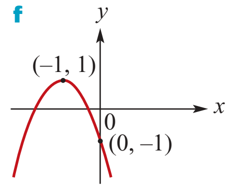
d



e



f



3 a $x = 0$ or $x = 3$

b $x = 2$ or $x = -1$ or $x = 5$ or $x = -3$

c $x = 0$ or $x = -2$ **d** $x = 0$ or $x = 6$

e $x = 0$ or $x = 3$ or $x = -3$

f $x = 3$ or $x = -3$

g $x = 0$ or $x = 4$ or $x = -4$

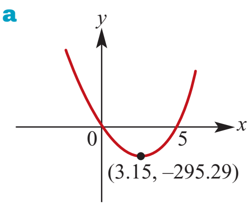
h $x = 0$ or $x = 4$ or $x = 3$

i $x = 0$ or $x = 4$ or $x = 5$

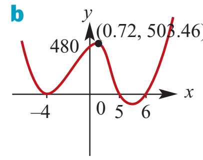
j $x = 2$ or $x = -2$ or $x = 3$ or $x = -3$

k $x = 4$ **l** $x = -4$ or $x = 2$

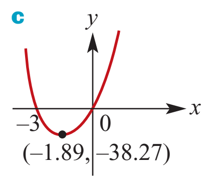
4 a



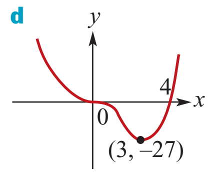
b



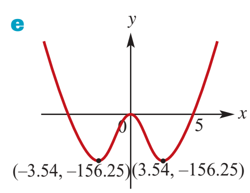
c



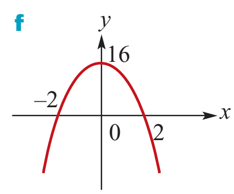
d



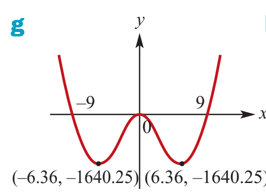
e



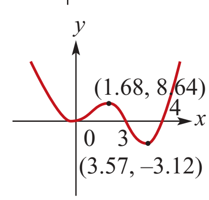
f



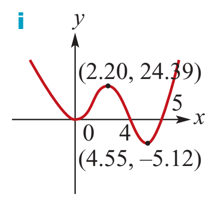
g



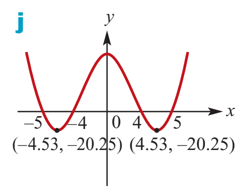
h



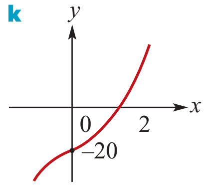
i



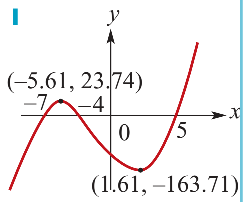
j



k

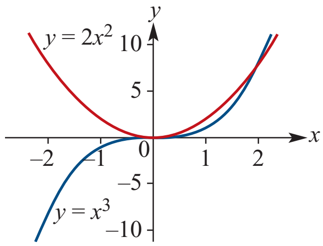


l



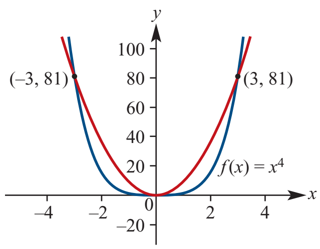
5 a Even **b** Odd **c** Even **d** Odd

6 a



b $x = 0, x = 2$ **c** $x \le 2$

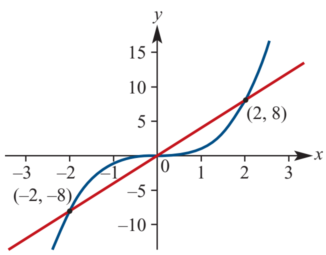
7 a



b $x = 0, x = 3, x = -3$

c $-3 \le x \le 3$

8 a



b $x = 0, x = 2, x = -2$

c $x \le -2$ or $0 \le x \le 2$

Exercise 7J

1 a $x^2 + 2x + \dfrac{3}{x-1}$ **b** $2x^2 - x - 3 + \dfrac{6}{x+1}$

c $3x^2 - 10x + 22 - \dfrac{43}{x+2}$

d $2x^2 + 3x + 10 + \dfrac{28}{x-3}$

2 a $x^2 - x + 4 - \dfrac{8}{x+1}$

b $2x^2 - 8x + 49 - \dfrac{181}{x+4}$

c $x^2 + x - 3 + \dfrac{11}{x+3}$

d $x^2 - x + 4 + \dfrac{8}{x-2}$

3 a $x^2 - 2x + 5$ **b** $2x^2 - 2x - 6$

c $x^2 - 2x - 6$ **d** $3x^2 - x - 6$

4 a Quotient $x^2 - 3$; Remainder 7

b Quotient $x^2 + 2x + 15$; Remainder 71

c Quotient $2x^2 - 3x$; Remainder -7

d Quotient $5x^2 + 20x + 77$; Remainder 315

5 a $\dfrac{1}{2}x^2 + \dfrac{7}{4}x - \dfrac{3}{8} + \dfrac{103}{8(2x+5)}$

b $x^2 + 2x - 3 - \dfrac{2}{2x+1}$ **c** $x^2 + 2x - 15$

d $\dfrac{1}{3}x^2 - \dfrac{8}{9}x - \dfrac{8}{27} + \dfrac{19}{27(3x-1)}$

6 a $x^2 + 3x + 8 + \dfrac{9}{x-1}$

b $x^2 - \dfrac{x}{2} + \dfrac{9}{4} + \dfrac{21}{4(2x-1)}$

7 a Quotient $2x - 6$; Remainder 0

b Quotient $x - 6$; Remainder -2

c Quotient $2x - 6$; Remainder 42

d Quotient $x^2 - 4x + 2$; Remainder $-x + 7$

e Quotient $x^2 - 3x + 7$; Remainder $-10x + 9$

f Quotient $x^2 + x - \dfrac{3}{2}$; Remainder $\dfrac{15}{2}x + 16$

Exercise 7K

1 a -2 **b** -29 **c** 15 **d** 4 **e** 7

f -12 **g** 0 **h** -5 **i** -8

2 a $a = -3$ **b** $a = 2$ **c** $a = 4$ **d** $a = -10$

3 a $P(1) = 0$ **b** $P(1) = 0$

c $P(-2) = 0$ **d** $P(\frac{3}{2}) = 0$

4 a 6 **b** 28 **c** $-\dfrac{1}{3}$

5 a $(x-1)(x+1)(2x+1)$ **b** $(x+1)^3$

c $(x-1)(6x^2 - 7x + 6)$

d $(x-1)(x+5)(x-4)$ **e** $(x+1)^2(2x-1)$

f $(x+1)(x-1)^2$

g $(x-2)(4x^2 + 8x + 19)$

h $(x+2)(2x+1)(2x-3)$

6 1

7 a $(x-1)(x^2 + x + 1)$

b $(x+4)(x^2 - 4x + 16)$

c $(3x-1)(9x^2 + 3x + 1)$

d $(4x-5)(16x^2 + 20x + 25)$

e $(1-5x)(1 + 5x + 25x^2)$

f $(3x+2)(9x^2 - 6x + 4)$

g $(4m - 3n)(16m^2 + 12mn + 9n^2)$

h $(3b + 2a)(9b^2 - 6ab + 4a^2)$

8 a $(x+2)(x^2 - x + 1)$

b $(3x+2)(x-1)(x-2)$

c $(x-3)(x+1)(x-2)$

d $(3x+1)(x+3)(2x-1)$

9 $a = 3, b = -3, P(x) = (x-1)(x+3)(x+1)$

10 b i n odd **ii** n even

11 a $a = 1, b = 1$

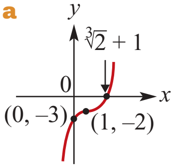
b i $P(x) = x^3 - 2x^2 + 3$

Chapter 7 review

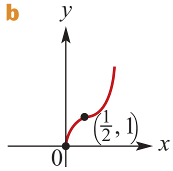
Short-response questions

Technology-free

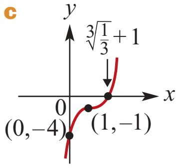
1 a



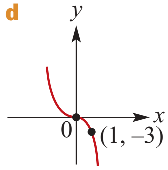
b



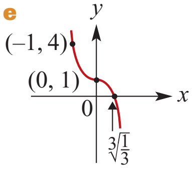
c



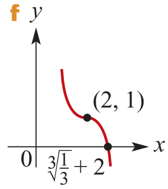
d



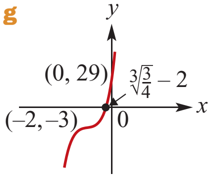
e



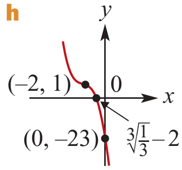
f



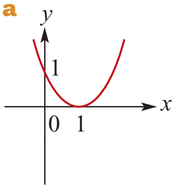
g



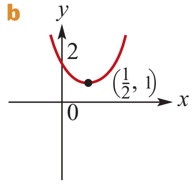
h



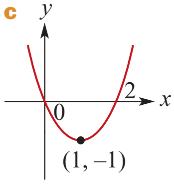
2 a



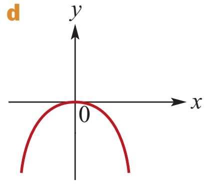
b



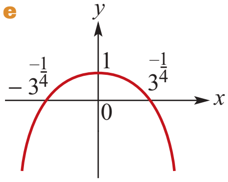
c



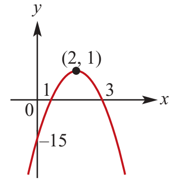
d



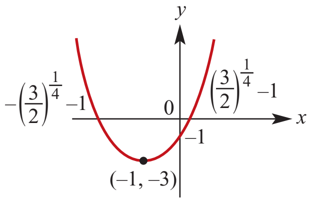
e



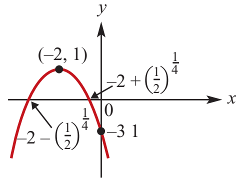
f



g



h



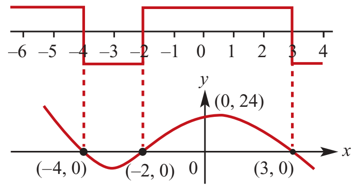
3 a $0, 4, -4$ **b** $-2, 2, 6$ **c** $-6, -2, 2$

4 $a = 4, b = -12, c = 0, d = 16$

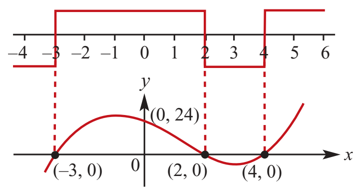
5 a $a = 1, b = -5$ **b** $x = -4$ or $x = 0$

6 $a = 3, b = -24$

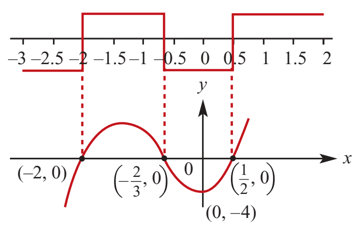
7 a



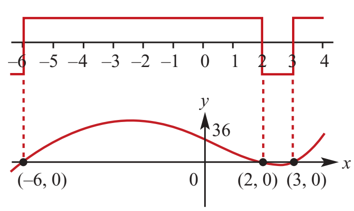
b



c



d



8 $y = -\dfrac{2}{5}(x+2)(x-1)(x-5)$

9 $y = \dfrac{2}{81}x(x+4)^2$

10 a Dilation of factor 2 from the x-axis, then translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

b Reflection in the x-axis, then translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

c Dilation of factor $\frac{1}{2}$ from the y-axis, then translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

11 $a = 1 - k, b = k + 1$

12 a $a = 3, b = 8$ **b** $(x + 3)(2x - 1)(x - 1)$

13 a $P\left(\frac{3}{2}\right) = 0$ and $P(-2) = 0$; $(3x + 1)$

b $x = -2, \frac{1}{2}, 3$

c $x = -1, -\sqrt{11}, +\sqrt{11}$

14 a $P\left(\frac{1}{3}\right) = 0$ **b** $(3x - 1)(x + 3)(x - 2)$

15 a -41 **b** 12 **c** $\frac{43}{9}$

Technology-active

16 a $x = 2, x = -\frac{1}{2}, x = -3$

b $x = 2, x = \frac{\sqrt{17} + 1}{4}, x = \frac{1 - \sqrt{17}}{4}$

c $x = -1, x = 2, x = 6$

17 $f(x) = x^3 - 3x^2 - 2x + 11$

18 a $V = \pi r^2(6 - r)$ **b** $0 < r < 6$

c $V(3) = 27\pi$ **d** $r = 3$ or $r = \frac{3}{2}(1 + \sqrt{5})$

e Maximum ≈ 100.53 (correct to 2 d.p.)

19 $c = 3\sqrt{3}$ or $c = -3\sqrt{3}$

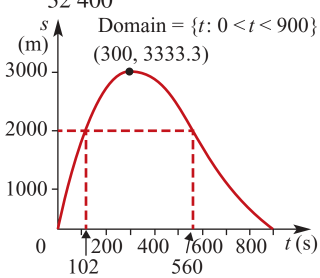
20 $n = 8, a = -\frac{3}{2}$

21 $n = 9, a = \frac{4}{3}$

22 a $v = \frac{1}{32\,400}(t - 900)^2$

b $s = \frac{t}{32\,400}(t - 900)^2$

c



d No, it is not feasible as the maximum range of the taxi is less than 3.5 km (≈ 3.33 km)

e Maximum speed ≈ $\frac{2000}{102} = 20$ m/s

Minimum speed ≈ $\frac{2000}{560} = 3.6$ m/s

23 a $a = \frac{-43}{15\,000}, b = 0.095, c = \frac{-119}{150}, d = 15.8$

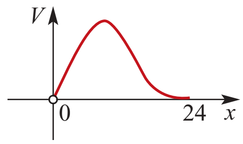
b i (5.59, 13.83) **ii** (0, 15.8)

24 a $R = a(x - 5)^3 + 10$ **b** $a = \frac{2}{25}$

c $R = \frac{12}{343}(x - 7)^3 + 12$

25 a $V = (96 - 4x)(48 - 2x)x = 8x(24 - x)^2$

b



i $0 < x < 24$

ii $V_{max} = 16\,384$ cm^3 when $x = 8.00$

c 15 680 cm^3 **d** 14 440 cm^3 **e** 9720 cm^3

Multiple-choice questions

1 A	**2** C	**3** A	**4** C	**5** A
6 B	**7** A	**8** A	**9** C	**10** C
11 A	**12** C	**13** A	**14** C	**15** A
16 D	**17** A	**18** D	**19** A	**20** B
21 D	**22** B	**23** A	**24** B	**25** B

Chapter 8

Exercise 8A

1 a $\frac{\pi}{3}$ **b** $\frac{4\pi}{5}$ **c** $\frac{4\pi}{3}$ **d** $\frac{11\pi}{6}$

e $\frac{7\pi}{3}$ **f** $\frac{8\pi}{3}$

2 a 120° **b** 150° **c** 210° **d** 162°

e 100° **f** 324° **g** 220° **h** 324°

3 a 34.38° **b** 108.29° **c** 166.16° **d** 246.94°

e 213.14° **f** 296.79° **g** 271.01° **h** 343.77°

4 a 0.66 **b** 1.27 **c** 1.87 **d** 2.81

e 1.47 **f** 3.98 **g** 2.38 **h** 5.74

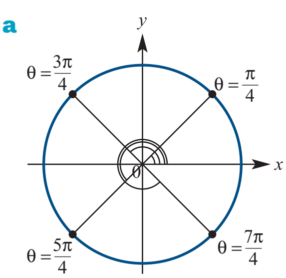
5 a −60° **b** −720° **c** −540° **d** −180°

e 300° **f** −330° **g** 690° **h** −690°

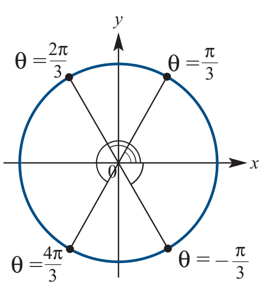
6 a -2π **b** -3π **c** $-\frac{4\pi}{3}$ **d** -4π

e $-\frac{11\pi}{6}$ **f** $-\frac{7\pi}{6}$

7 a

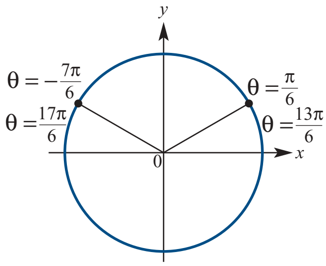


b



 ISBN 978-1-009-53719-3 Cambridge University Press

c



Exercise 8B

1 a 0, 1 **b** −1, 0 **c** 1, 0 **d** 1, 0
e 0, −1 **f** 1, 0 **g** −1, 0 **h** 0, 1

2 a 0.95 **b** 0.75 **c** −0.82 **d** 0.96
e −0.5 **f** −0.03 **g** −0.86 **h** 0.61

3 a 0, −1 **b** −1, 0 **c** −1, 0 **d** −1, 0
e −1, 0 **f** 0, −1 **g** 0, −1 **h** 0, −1

Exercise 8C

1 a 0 **b** 0 **c** Undefined
d 0 **e** Undefined **f** Undefined

2 a −34.23 **b** −2.57 **c** −0.97
d −1.38 **e** 0.95 **f** 0.75
g 1.66

3 a 0 **b** 0 **c** 0
d 0 **e** 0 **f** 0

Exercise 8D

1 a 67.98° **b** 4.5315 **c** 2.5357
d 6.4279 **e** 50.19° **f** 3.4202
g 2.3315 **h** 6.5778 **i** 6.5270

Exercise 8E

1 a −0.42 **b** −0.7 **c** −0.42 **d** −0.38
e 0.42 **f** −0.38 **g** −0.7 **h** 0.7

2 a −0.7 **b** −0.6 **c** −0.4 **d** −0.6
e −0.7 **f** −0.7 **g** 0.4 **h** 0.6

3 a $\frac{5\pi}{6}$ **b** $\frac{7\pi}{6}$ **c** $\frac{11\pi}{6}$

4 a $-\frac{1}{2}$ **b** $\frac{\sqrt{3}}{2}$ **c** $\frac{1}{2}$ **d** $\frac{-\sqrt{3}}{2}$
e $-\sqrt{3}$ **f** $-\sqrt{3}$

5 a $-\frac{\sqrt{3}}{2}$ **b** $\frac{1}{2}$ **c** $-\sqrt{3}$ **d** $-\frac{\sqrt{3}}{2}$ **e** $-\frac{1}{2}$

6 a $a = 0.7660$, $b = 0.6428$
b $c = -0.7660$, $d = 0.6428$
c i $\cos 140° = -0.7660$, $\sin 140° = 0.6428$
ii $\cos 140° = -\cos 40°$

7 a 120° **b** 240° **c** −60° **d** 120°
e 240° **f** 300°

Exercise 8F

1 a $\sin = \frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = -\sqrt{3}$

b $\sin = \frac{1}{\sqrt{2}}$, $\cos = -\frac{1}{\sqrt{2}}$, $\tan = -1$

c $\sin = -\frac{1}{2}$, $\cos = -\frac{\sqrt{3}}{2}$, $\tan = \frac{1}{\sqrt{3}}$

d $\sin = -\frac{\sqrt{3}}{2}$, $\cos = -\frac{1}{2}$, $\tan = \sqrt{3}$

e $\sin = -\frac{1}{\sqrt{2}}$, $\cos = \frac{1}{\sqrt{2}}$, $\tan = -1$

f $\sin = \frac{1}{2}$, $\cos = \frac{\sqrt{3}}{2}$, $\tan = \frac{1}{\sqrt{3}}$

g $\sin = \frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = \sqrt{3}$

h $\sin = -\frac{1}{\sqrt{2}}$, $\cos = -\frac{1}{\sqrt{2}}$, $\tan = 1$

i $\sin = \frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = \sqrt{3}$

j $\sin = -\frac{\sqrt{3}}{2}$, $\cos = \frac{1}{2}$, $\tan = -\sqrt{3}$

2 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{\sqrt{3}}$
d $-\frac{1}{2}$ **e** $-\frac{1}{\sqrt{2}}$ **f** $\sqrt{3}$
g $-\frac{\sqrt{3}}{2}$ **h** $\frac{1}{\sqrt{2}}$ **i** $-\frac{1}{\sqrt{3}}$

3 a $-\frac{\sqrt{3}}{2}$ **b** $-\frac{1}{\sqrt{2}}$ **c** $\frac{1}{\sqrt{3}}$
d Undefined **e** 0 **f** $-\frac{1}{\sqrt{2}}$
g $\frac{1}{\sqrt{2}}$ **h** −1

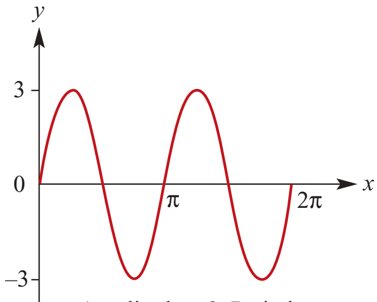
Exercise 8G

1 a i 2π **ii** 2 **b i** π **ii** 3 **c i** $\frac{2\pi}{3}$ **ii** $\frac{1}{2}$
d i 4π **ii** 3 **e i** $\frac{2\pi}{3}$ **ii** 4 **f i** $\frac{\pi}{2}$ **ii** $\frac{1}{2}$
g i 4π **ii** 2 **h i** 2 **ii** 2 **i i** 4 **ii** 3

2 a Dilation of factor 3 from the x-axis; Amplitude = 3; Period = 2π
b Dilation of factor $\frac{1}{5}$ from the y-axis; Amplitude = 1; Period $\frac{2\pi}{5}$
c Dilation of factor 3 from the y-axis; Amplitude = 1; Period = 6π
d Dilation of factor 2 from the x-axis and dilation of factor $\frac{1}{5}$ from the y-axis; Amplitude = 2; Period = $\frac{2\pi}{5}$

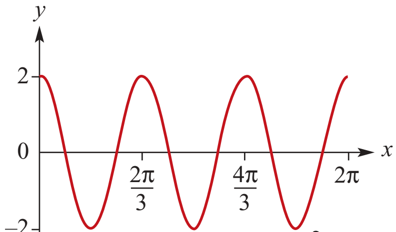
 ISBN 978-1-009-53719-3 Cambridge University Press

3 a



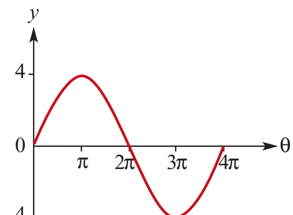
Amplitude = 3, Period = π

b



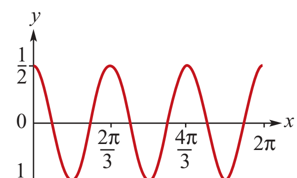
Amplitude = 2, Period = $\frac{2\pi}{3}$

c



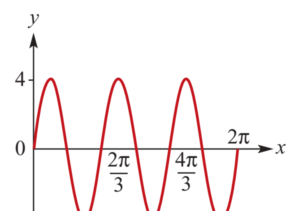
Amplitude = 4, Period = 4π

d



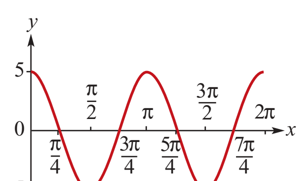
Amplitude = $\frac{1}{2}$, Period = $\frac{2\pi}{3}$

e



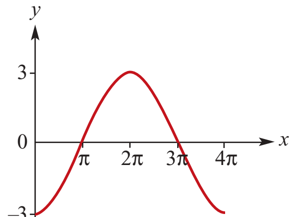
Amplitude = 4, Period = $\frac{2\pi}{3}$

f



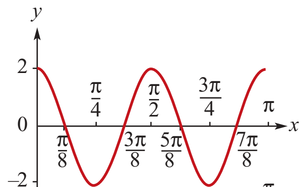
Amplitude = 5, Period = π

g



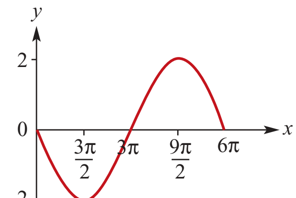
Amplitude = 3, Period = 4π

h



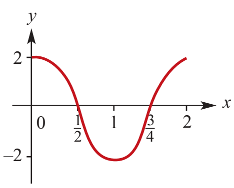
Amplitude = 2, Period = $\frac{\pi}{2}$

i

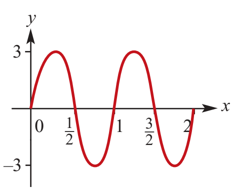


Amplitude = 2, Period = 6π

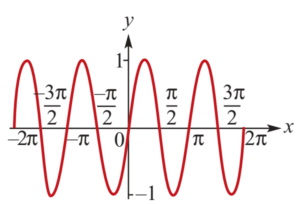
4 a



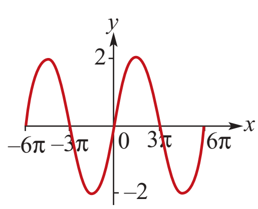
b



5 a

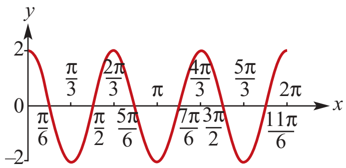


b

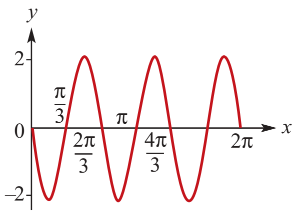


ISBN 978-1-009-53719-3

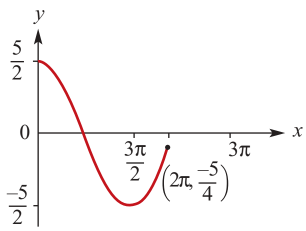
c



d



6



7 a Dilation of factor $\frac{1}{5}$ from the y-axis and reflection in the x-axis;
Amplitude = 1; Period = $\frac{2\pi}{5}$

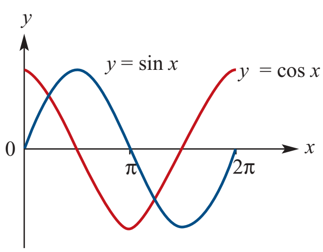
b Reflection in the y-axis;
Amplitude = 1; Period = 2π

c Dilation of factor 3 from the y-axis and dilation of factor 2 from the x-axis;
Amplitude = 2; Period = 6π

d Dilation of factor 2 from the y-axis, dilation of factor 4 from the x-axis and reflection in the x-axis; Amplitude = 4; Period = 4π

e Dilation of factor 3 from the y-axis, dilation of factor 2 from the x-axis and reflection in the y-axis; Amplitude = 2; Period = 6π

8 a



b $\frac{\pi}{4}, \frac{5\pi}{4}$

Exercise 8H

1 a $\frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ **b** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

c $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$

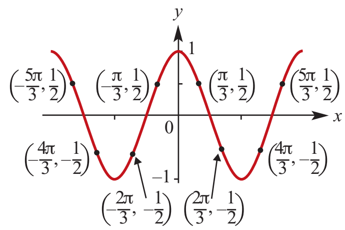
2 a 0.93, 2.21 **b** 4.30, 1.98 **c** 3.50, 5.93
d 0.41, 2.73 **e** 2.35, 3.94 **f** 1.77, 4.51

3 a 150, 210 **b** 30, 150 **c** 120, 240
d 120, 240 **e** 60, 120 **f** 45, 135

4 a $\frac{\pi}{6}, \frac{11\pi}{6}$ **b** $\frac{5\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$

5 a $\frac{3\pi}{4}, -\frac{3\pi}{4}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{2\pi}{3}, -\frac{2\pi}{3}$

6



7 a $\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

b $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$

c $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

d $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

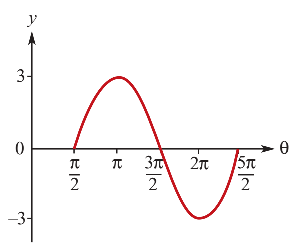
e $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}$

f $\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

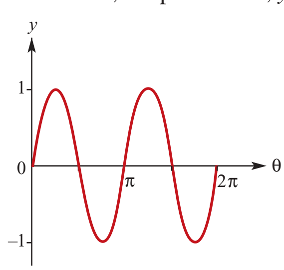
8 a 2.034, 2.678, 5.176, 5.820
b 1.893, 2.820, 5.034, 5.961
c 0.580, 2.562, 3.721, 5.704
d 0.309, 1.785, 2.403, 3.880, 4.498, 5.974

Exercise 8I

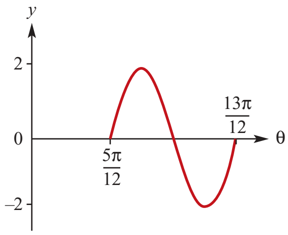
1 a Period = 2π, Amplitude = 3, $y = \pm 3$



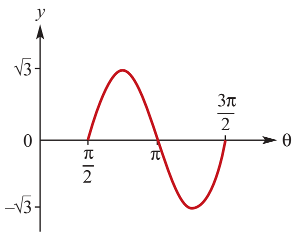
b Period = π, Amplitude = 1, $y = \pm 1$



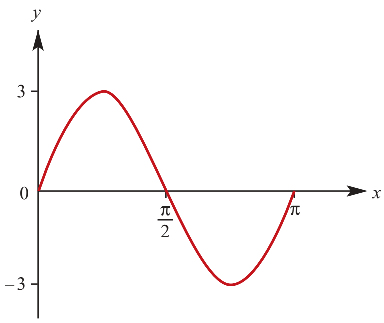
c Period $= \dfrac{2\pi}{3}$, Amplitude $= 2$, $y = \pm 2$



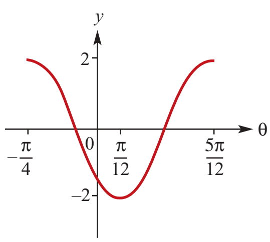
d Period $= \pi$, Amplitude $= \sqrt{3}$, $y = \pm\sqrt{3}$



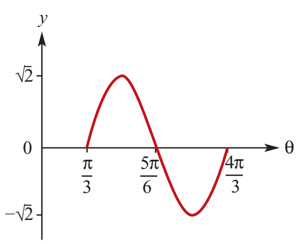
e Period $= \pi$, Amplitude $= 3$, $y = \pm 3$



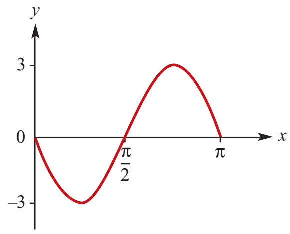
f Period $= \dfrac{2\pi}{3}$, Amplitude $= 2$, $y = \pm 2$



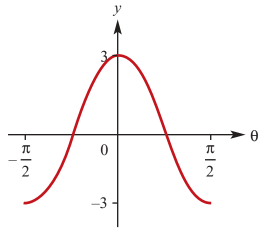
g Period $= \pi$, Amplitude $= \sqrt{2}$, $y = \pm\sqrt{2}$



h Period $= \pi$, Amplitude $= 3$, $y = \pm 3$

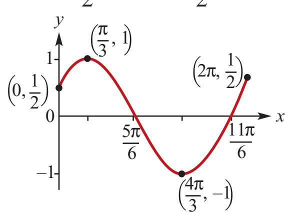


i Period $= \pi$, Amplitude $= 3$, $y = \pm 3$



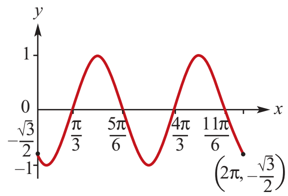
2 a $f(0) = \dfrac{1}{2}$, $f(2\pi) = \dfrac{1}{2}$

b



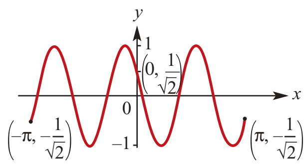
3 a $f(0) = -\dfrac{\sqrt{3}}{2}$, $f(2\pi) = -\dfrac{\sqrt{3}}{2}$

b



4 a $f(-\pi) = -\dfrac{1}{\sqrt{2}}$, $f(\pi) = -\dfrac{1}{\sqrt{2}}$

b



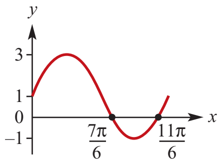
5 a $y = 3\sin\left(\dfrac{x}{2}\right)$ **b** $y = 3\sin(2x)$

c $y = 2\sin\left(\dfrac{x}{3}\right)$ **d** $y = \sin 2\left(x - \dfrac{\pi}{3}\right)$

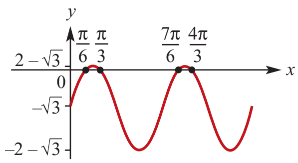
e $y = \sin\dfrac{1}{2}\left(x + \dfrac{\pi}{3}\right)$

Exercise 8J

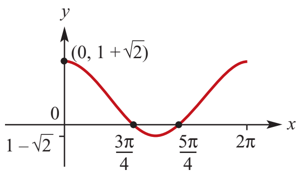
1 a



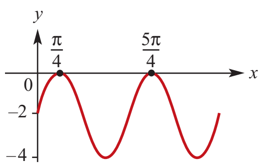
b



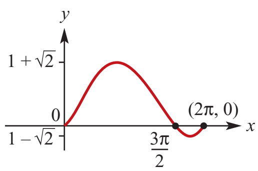
c



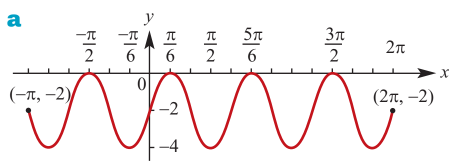
d



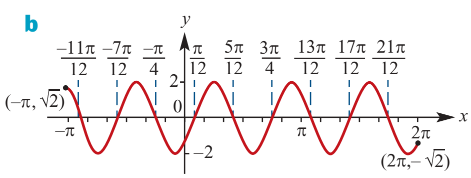
e



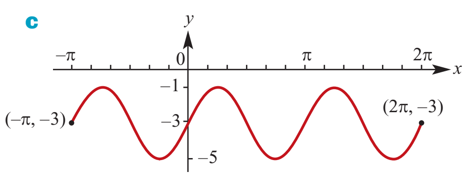
2 a



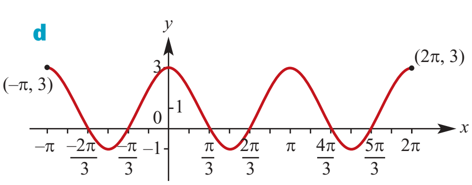
b



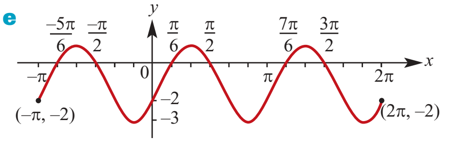
c



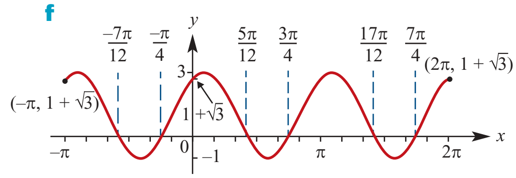
d



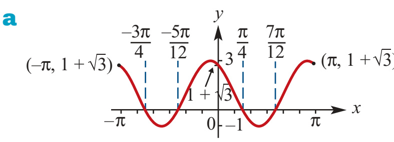
e



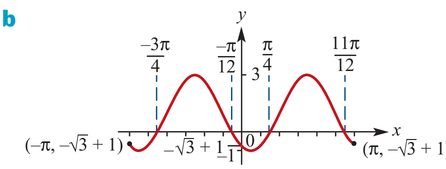
f



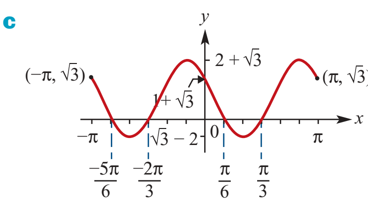
3 a



b



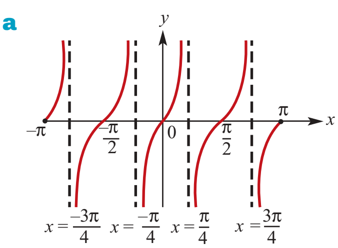
c



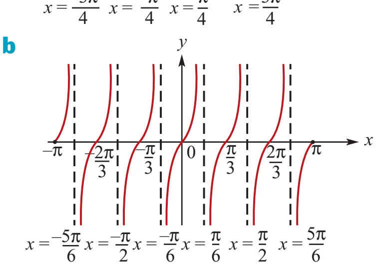
Exercise 8K

1 a $\frac{\pi}{4}$ **b** $\frac{3\pi}{2}$ **c** $\frac{\pi}{2}$

2 a

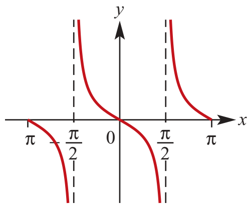


b



Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

c



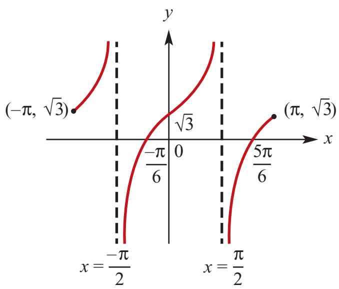
3 a $\frac{-7\pi}{8}, \frac{-3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}$

b $\frac{-17\pi}{18}, \frac{-11\pi}{18}, \frac{-5\pi}{18}, \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$

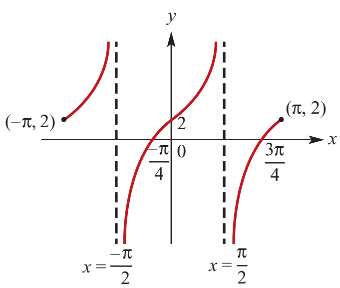
c $\frac{-5\pi}{6}, \frac{-\pi}{3}, \frac{\pi}{6}, \frac{2\pi}{3}$

d $\frac{-13\pi}{18}, \frac{-7\pi}{18}, \frac{-\pi}{18}, \frac{5\pi}{18}, \frac{11\pi}{18}, \frac{17\pi}{18}$

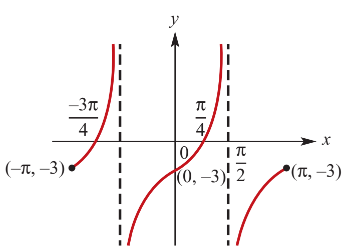
4 a



b



c



Exercise 8L

1 a 0.6 **b** 0.6 **c** −0.7 **d** 0.3 **e** −0.3 **f** $\frac{10}{7}$ **g** −0.3 **h** 0.6 **i** −0.6 **j** −0.3

2 a $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ **c** $\frac{5\pi}{12}$ **d** $\frac{\pi}{14}$

3 $\sin x = \frac{-4}{5}$, $\tan x = \frac{-4}{3}$

4 $\cos x = \frac{-12}{13}$, $\tan x = \frac{-5}{12}$

5 $\sin x = \frac{-2\sqrt{6}}{5}$, $\tan x = -2\sqrt{6}$

Exercise 8M

1 a i 5 cm **ii** $\frac{5\pi}{3}$ cm

b $\frac{25\pi}{6}$ cm^2

c 34.38°

2 45.81 cm

3 a 95.5° **b** 112.89°

4 a 6.20 cm **b** 2.73 cm^2

5 a 125.66 m **b** 41.96%

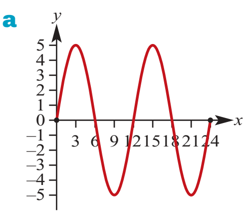
6 a 10.47 m **b** 20.94 m^2

7 $r = 7$ cm, $\theta = \left(\frac{18}{7}\right)^c$ or $r = 9$ cm, $\theta = \left(\frac{14}{9}\right)^c$

8 $16\sin\left(\frac{5}{4}\right) - 10 \approx 5.18$ cm^2

Exercise 8N

1 a



b $t = 3$ and $t = 15$

c 5 m above mean sea level

d $\frac{5\sqrt{3}}{2}$ m above mean sea level

e $\frac{5\sqrt{3}}{2}$ m above mean sea level

f $t \in [1, 5] \cup [13, 17]$

2 a 5 metres **b** 1 metre

c $t = 0.524, 2.618$ or 4.712 seconds

d $t = 0, 1.047$ or 2.094 seconds

e Particle oscillates between $x = 1$ and $x = 5$

3 a 7 m **b** 3 m

c $t = \frac{1}{4}, \frac{5}{4}, \frac{9}{4}, \frac{13}{4}$ or $\frac{17}{4}$

d $t = \frac{1}{12}, \frac{5}{12}, \frac{13}{12}, \frac{17}{12}, \frac{25}{12}$ or $\frac{29}{12}$

e Particle oscillates between $x = 3$ and $x = 7$

4 a i 10 **ii** $10 + 5\sqrt{3}$ **iii** $10 + 5\sqrt{3}$ **iv** $10 - 5\sqrt{3}$ **v** $10 - 5\sqrt{3}$

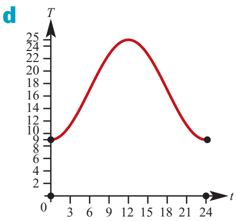
b 6 seconds **c** 20 metres

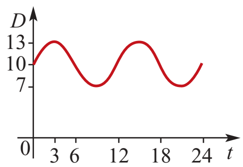
d $\frac{1}{2}, \frac{5}{2}, \frac{13}{2}, \frac{17}{2}$ s **e** $\frac{7}{2}, \frac{11}{2}, \frac{19}{2}, \frac{23}{2}$ s

5 a 9°C **b** Maximum = 25°C; Minimum = 9°C

c Between 7:28 and 16:32

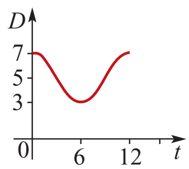
d



6 a 

b $0 \le t \le 7$ or $11 \le t \le 19$ or $23 \le t \le 24$

c 12.9 m

7 a $p = 5, q = 2$ **b** 

c The ship can enter 2 hours after low tide

Chapter 8 review

Short-response questions

Technology-free

1 a $\frac{11\pi}{6}$ **b** $\frac{9\pi}{2}$ **c** 6π **d** $\frac{23\pi}{4}$ **e** $\frac{3\pi}{4}$

f $\frac{9\pi}{4}$ **g** $\frac{13\pi}{6}$ **h** $\frac{7\pi}{3}$ **i** $\frac{4\pi}{9}$

2 a $150°$ **b** $315°$ **c** $495°$ **d** $45°$

e $1350°$ **f** $-135°$ **g** $-45°$ **h** $-495°$

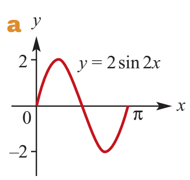
i $-1035°$

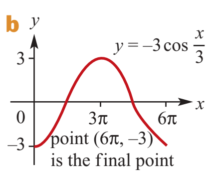
3 a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $-\frac{1}{2}$ **d** $-\frac{\sqrt{3}}{2}$

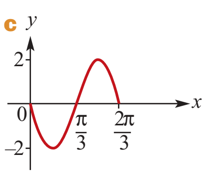
e $\frac{\sqrt{3}}{2}$ **f** $-\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $-\frac{1}{\sqrt{2}}$

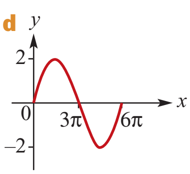
4 a $2, 4\pi$ **b** $3, \frac{\pi}{2}$ **c** $\frac{1}{2}, \frac{2\pi}{3}$ **d** $3, \pi$

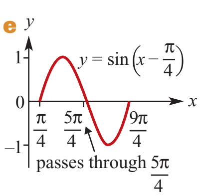
e $4, 6\pi$ **f** $\frac{2}{3}, 3\pi$

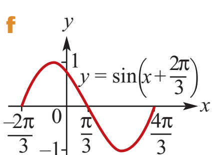
5 a 

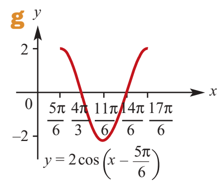
b 

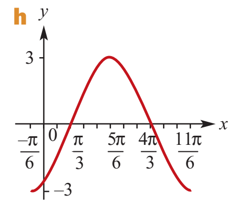
c 

d 

e 

f 

g 

h 

6 a $-\frac{2\pi}{3}, -\frac{\pi}{3}$ **b** $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}, \frac{5\pi}{6}$

c $\frac{\pi}{6}, \frac{3\pi}{2}$ **d** $\frac{7\pi}{6}$ **e** $\frac{\pi}{2}, \frac{7\pi}{6}$

7 $\left(\frac{11}{6}\right)^c$

8 180 cm^2

9 a 2 **b** 0.7^c

10 a $\frac{2\sqrt{3}\pi}{3}$ cm **b** $2\pi \text{ cm}^2$

Technology-active

11 a $\theta - \sin\theta = \frac{\pi}{2}$ **b** $\theta = 2.3$

12 a 4.77 cm **b** 180 cm^2 **c** 9.55 cm

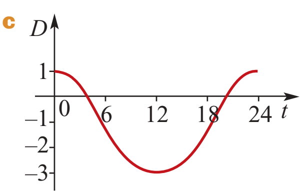
13 57%

14 1.9

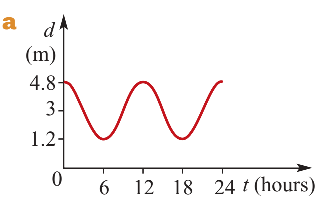
15 a i 1.83×10^{-3} hours **ii** 11.79 hours

b 26 April ($t = 3.86$), 15 August ($t = 7.48$)

16 a 19.5°C **b** $D = -1 + 2\cos\left(\frac{\pi t}{12}\right)$

c 

d $4 < t < 20$ (8 a.m. to midnight)

17 a 

b 3 a.m., 3 p.m., 3 a.m.

c 9 a.m., 9 p.m. **d** 10:03 a.m.

e i 6:12 p.m. **ii** 5 trips

Multiple-choice questions

1 C **2** D **3** B **4** D **5** C
6 D **7** D **8** B **9** C **10** A
11 A **12** D **13** C **14** D **15** D
16 D **17** A **18** D **19** A

Chapter 9

Exercise 9A

1 a {I, N, T, E, R} **b** {A, E, I}
c {P, R, B, L, T, Y}

2 a $\in$ **b** $\subseteq$ **c** $\in$ **d** $\subseteq$ **e** $\subseteq$

3 a $\{H, T\}$ **b** {1, 2, 3, 4, 5, 6}

4 a $\emptyset$ **b** {2}, so not the empty set

5 a T **b** F **c** F **d** T **e** F **f** T **g** T **h** T

6 a $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$,
$A \cap B = \{1, 3, 5, 7\}$
b $M \cup N = \{a, b, c, d, e, f, g\}$,
$M \cap N = \{a, c, e\}$
c $X \cup Y = \{\text{M, O, B, I, L, E, T, A}\}$,
$X \cap Y = \{\text{B, L, E}\}$

7 a {1, 2, 3, 4, 6} **b** {2, 4}
c {5, 6, 7, 8, 9, 10} **d** {1, 3}
e {1, 3, 5, 6, 7, 8, 9, 10}
f {5, 7, 8, 9, 10}

8 a {1, 2, 3, 5, 6, 7, 9, 10, 11}
b {1, 3, 5, 7, 9, 11} **c** {2, 4, 6, 8, 10, 12}
d {1, 3, 5, 7, 9, 11} **e** {1, 3, 5, 7, 9, 11}

9 a {E, H, M, S} **b** {C, H, I, M}
c {A, C, E, I, S, T} **d** {H, M}
e {C, E, H, I, M, S} **f** {H, M}

10 a 12 **b** 11 **c** 5

11 a 8 **b** 7 **c** 1 **d** 8 **e** 7 **f** 7

12 a {1, 2, 5, 7, 8, 11, 13, 26}
b {4, 16, 27, 31}
c {1, 4, 8, 13, 16, 27, 31}
d {1, 2, 4, 5, 7, 8, 11, 13, 16, 26, 27, 31}
e {1, 8, 13} **f** 8 **g** 4
h 7 **i** 12 **j** 0 **k** 3

13 a {1, 2, 4, 5, 7, 8, 11, 13, 19, 21, 28}
b {2, 5, 8, 11, 19}
c $A = \{1, 2, 4, 5, 7, 8, 11, 13, 19, 21, 28\}$
d $B = \{2, 5, 8, 11, 19\}$ **e** {1, 4, 7, 13, 21, 28}

14 3

15 a 35 **b** 135

16 40

17 63

18 30

19 a 34 **b** 22 **c** 64 **d** 36

20 a 36 **b** 11 **c** 15

Exercise 9B

1 $\{H, T\}$

2 {1, 2, 3, 4, 5, 6}

3 a {0, 1, 2, 3, 4} **b** {0, 1, 2, 3, 4, 5, 6}
c {0, 1, 2, 3}

4 a $\{0, 1, 2, 3, \dots\}$ **b** $\{0, 1, 2, 3, \dots, 41\}$
c $\{1, 2, 3, \dots\}$

5 a {2, 4, 6} **b** $\{FFF\}$ **c** $\emptyset$

6 a $\frac{1}{2}$ **b** $\frac{3}{10}$ **c** $\frac{3}{20}$

7 a $\frac{4}{15}$ **b** $\frac{2}{3}$ **c** $\frac{4}{15}$

8 a $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** $\frac{3}{13}$ **d** $\frac{3}{26}$

9 a $\frac{9}{13}$ **b** $\frac{10}{13}$ **c** $\frac{5}{13}$ **d** $\frac{1}{13}$

10 a $\frac{1}{365}$ **b** $\frac{6}{73}$ **c** $\frac{6}{73}$ **d** $\frac{18}{73}$

11 a $\frac{1}{9}$ **b** $\frac{1}{3}$ **c** $\frac{5}{9}$ **d** $\frac{4}{9}$

12 $\frac{1}{3}$

13 0.4

14 a $\frac{1}{3}$ **b** $\frac{1}{8}$ **c** $\frac{1}{4}$

15 a $\frac{1}{7}$ **b** $\frac{5}{7}$

16 a $\frac{3}{4}$ **b** $\frac{1}{2}$ **c** $\frac{10}{13}$ **d** $\frac{23}{26}$

17 $\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}$

18 a $\frac{1}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{2}{13}, \frac{4}{13}$ **b** $\frac{9}{13}$

Exercise 9C

1 a $\frac{17}{50}$ **b** $\frac{1}{10}$ **c** $\frac{4}{15}$ **d** $\frac{1}{200}$

2 a No **b** Answers will vary
c Answers will vary **d** Yes
e It will never be possible to find an exact probability via experiment, but as the number of trials approaches infinity the relative frequency approaches the exact probability.

3 $P(\text{a 6 from first die}) \approx \frac{78}{500} = 0.156$

$P(\text{a 6 from second die}) \approx \frac{102}{700} \approx 0.146$

Choose first die, as higher probability of a 6

4 a $\frac{17}{20}$ **b** $\frac{4}{5}$ **c** $\frac{9}{10}$ **d** 51

5 0.445

6 a 0.17 **b** 0.62 **c** 0.39

7 a 0.195 **b** 0.280 **c** 0.265

8 a $\frac{\pi}{4}$ **b** $\frac{\pi}{4} \approx 0.7854$

9 $\frac{3}{4}$

10 a $\frac{1}{3}$ **b** $\frac{1}{6}$ **c** $\frac{5}{6}$

11 a $\frac{4\pi}{25}$ **b** $1 - \frac{4\pi}{25} \approx 0.4973$

 ISBN 978-1-009-53719-3

Exercise 9D

1 a $\frac{1}{4}$ **b** $\frac{1}{4}$

2 a $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\frac{1}{4}$

3 a $\frac{1}{2}$ **b** $\frac{1}{18}$ **c** $\frac{5}{18}$

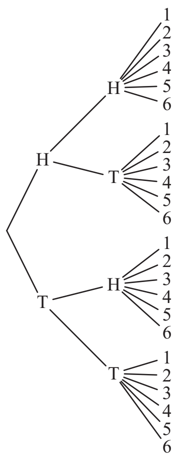
4 a $\frac{1}{12}$ **b** $\frac{1}{2}$ **c** $\frac{7}{12}$

5 a $\frac{3}{8}$ **b** $\frac{3}{8}$ **c** $\frac{1}{8}$ **d** $\frac{1}{8}$

6 a $\frac{1}{2}$ **b** $\frac{1}{4}$ **c** $\frac{3}{4}$

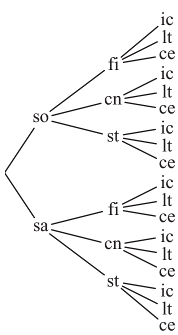
7 $\frac{1}{4}$

8 a



b i $\frac{1}{24}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{8}$ **iv** $\frac{1}{2}$

9 a



b i $\frac{1}{18}$ **ii** $\frac{1}{3}$ **iii** $\frac{1}{6}$ **iv** $\frac{2}{3}$

c i $\frac{1}{36}$ **ii** $\frac{1}{2}$ **iii** $\frac{5}{12}$ **iv** $\frac{1}{12}$

10 a

		2nd ball				
		1	2	3	4	5
1st ball	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)

b i $\frac{4}{25}$ **ii** $\frac{4}{5}$ **iii** $\frac{3}{25}$

Exercise 9E

1 a 0.2 **b** 0.45

2 a $\frac{2}{3}$ **b** 0 **c** $\frac{1}{2}$ **d** $\frac{5}{6}$

3 a $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{1}{6}, \frac{2}{3}$

4 a $\frac{7}{18}$ **b** $\frac{4}{18} = \frac{2}{9}$ **c** $\frac{2}{18} = \frac{1}{9}$ **d** $\frac{1}{2}$

5 a $\frac{3}{10}$ **b** $\frac{1}{5}$ **c** $\frac{1}{10}$ **d** $\frac{2}{5}$

6 $P(A \cup B) = 0.7$

7 $P(A \cup B) = 0.47$

8 a $P(A \cap B) = 0.28$ **b** $P(A \cup B) = 0.45$

9 a $P(A \cap B) = 0.45$ **b** $P(A \cup B) = 0.58$

10 a $P(A \cap B) = 0$ **b** $P(A \cup B) = 0.7$

11 a $P(A \cap B) = 0$ **b** $P(A \cup B) = 0.23$

12 $P(A \cap B) = 0.2$

13 $P(A \cap B) = 0.05$

14 $P(A \cup B') = 0.7$

15 0.32

16 a 0.43 **b** 0.29

Exercise 9F

1 a 0.2 **b** 0.5 **c** 0.3 **d** 0.7

2 a 0.75 **b** 0.4 **c** 0.87 **d** 0.48

3 a 0.63 **b** 0.23 **c** 0.22 **d** 0.77

4 a 0.45 **b** 0.40 **c** 0.25 **d** 0.70

5 a 0.9 **b** 0.6 **c** 0.1 **d** 0.9

6 a 95% **b** 5%

7 a $A = \{J♥, Q♥, K♥, J♠, Q♠, K♠, J♦, Q♦, K♦, J♣, Q♣, K♣\}$

$C = \{2♥, 3♥, 4♥, 5♥, 6♥, 7♥, 8♥, 9♥, 10♥, J♥, Q♥, K♥, A♥\}$

b i $P(\text{a picture card}) = \frac{3}{13}$

ii $P(\text{a heart}) = \frac{1}{4}$

iii $P(\text{a heart picture card}) = \frac{3}{52}$

iv $P(\text{a picture card or a heart}) = \frac{11}{26}$

v $P(\text{a picture card or a club, diamond or spade}) = \frac{21}{26}$

8 a $\frac{8}{15}$ **b** $\frac{7}{10}$ **c** $\frac{2}{15}$ **d** $\frac{1}{3}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

9 a 0.8 **b** 0.57 **c** 0.28 **d** 0.08
10 a 0.81 **b** 0.69 **c** 0.74 **d** 0.86
11 a 0 **b** 1 **c** $\frac{1}{5}$ **d** $\frac{1}{3}$
12 a 0.88 **b** 0.58 **c** 0.30 **d** 0.12

Exercise 9G

1 $\frac{1}{4}$ **2** $\frac{1}{3}$ **3** $\frac{1}{13}$ **4** $\frac{7}{12}$ **5** $\frac{1}{6}$
6 a $\frac{4}{17}$ **b** $\frac{4}{7}$
7 $\frac{7}{12}$
8 a $\frac{375}{500} = \frac{3}{4}$ **b** $\frac{225}{300} = \frac{3}{4}$
9 a $\frac{65}{284}$ **b** $\frac{137}{568}$ **c** $\frac{21}{65}$ **d** $\frac{61}{246}$
10 a 0.06 **b** 0.2
11 a $\frac{4}{7}$ **b** 0.3 **c** $\frac{15}{22}$
12 a 0.2 **b** 0.5 **c** 0.4
13 a 0.2 **b** $\frac{10}{27}$ **c** $\frac{1}{3}$
14 a 0.3 **b** 0.75
15 16%
16 $\frac{1}{5}$
17 a $\frac{1}{16}$ **b** $\frac{1}{169}$ **c** $\frac{1}{4}$ **d** $\frac{1}{2}$
18 a $\frac{1}{17}$ **b** $\frac{1}{221}$ **c** $\frac{13}{51}$ **d** $\frac{26}{51}$
19 $0.230\,808 \approx 0.231$
20 a $\frac{15}{28}$ **b** $\frac{1}{2}$ **c** $\frac{1}{2}$ **d** $\frac{2}{5}$
e $\frac{3}{7}$ **f** $\frac{8}{13}$ **g** $\frac{5}{28}$ **h** $\frac{3}{14}$
21 a 0.85 **b** 0.6 **c** 0.51 **d** 0.51
22 0.4, 68%
23 a i 0.44 **ii** 0.40 **iii** 0.35 **iv** 0.18 **v** 0.19
b 0.372
c i 0.478 **ii** 0.425
24 a i 0.56 **ii** 0.05 **iii** 0.12 **iv** 0.03 **v** 0.05
b 0.081
c 0.35
25 a $\frac{1}{6}$ **b** $\frac{53}{90}$ **c** $\frac{15}{53}$
26 a $B \subseteq A$ **b** $A \cap B = \emptyset$ **c** $A \subseteq B$

Exercise 9H

1 No
2 a Yes **b** Yes **c** No
3 $P(A) \times P(B) = \frac{3}{6} \times \frac{2}{6} = \frac{1}{6} = P(A \cap B)$
4 $P(A) \times P(B) = \frac{20}{36} \times \frac{9}{36} = \frac{5}{36} = P(A \cap B)$
5 $P(\text{male and support guns}) = 0.35$;
$P(\text{male}) \times P(\text{support guns}) = 0.6 \times 0.65 =$
$0.39 \neq 0.35$; therefore not independent
6 $P(\text{female and prefer sport}) = 0.3$;
$P(\text{female}) \times P(\text{prefer sport}) = 0.4 \times 0.75 = 0.3$;
therefore independent
7 $P(\text{speeding and serious}) = \frac{42}{376} \approx 0.112$;
$P(\text{speeding}) \times P(\text{serious}) = \frac{130}{376} \times \frac{103}{376} \approx$
$0.095 \neq 0.112$; therefore not independent
8 a 0.6 **b** 0.42 **c** 0.88
9 0.6
10 a 0.35 **b** 0.035 **c** 0.1225 **d** 0.025
11 a $\frac{4}{15}$ **b** $\frac{1}{15}$ **c** $\frac{133}{165}$ **d** $\frac{6}{11}$ **e** $\frac{4}{15}$
No, as $P(L \mid F) \neq P(L)$
12 a 0.35 **b** 0.875
13 a $\frac{18}{65}$ **b** $\frac{12}{65}$ **c** $\frac{23}{65}$ **d** $\frac{21}{65}$ **e** $\frac{4}{65}$
f $\frac{8}{65}$ **g** $\frac{2}{15}$ **h** $\frac{8}{21}$; No
14 a i 0.75 **ii** 0.32 **iii** 0.59 **b** No **c** No

Exercise 9I

1 Approx. 0.125
2 Approx. 0.5
3 Approx. 0.033
4 Approx. 29.29
5 a Approx. 1.75 **b** Approx. 2.19

Chapter 9 review

Short-response questions

Technology-free

1 a {A, E, I, O, U, N, V, R, S, T, Y} **b** {E, I, U}
c {A, O} **d** {N, V, R, S, T, Y}
2 a 15 **b** 76 **c** 139 **d** 61
3 a $\frac{1}{6}$ **b** $\frac{5}{6}$
4 0.007
5 a $\frac{1}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$
6 a 0.36 **b** $\frac{87}{245}$
7 $\frac{4}{15}$
8 a {156, 165, 516, 561, 615, 651} **b** $\frac{2}{3}$ **c** $\frac{1}{3}$
9 a $\frac{5}{12}$ **b** $\frac{1}{4}$
10 a 0.036 **b** 0.027 **c** 0.189 **d** 0.729
11 a $\frac{1}{27}$ **b** $\frac{4}{27}$ **c** $\frac{4}{9}$ **d** $\frac{20}{27}$
12 No
13 a 0.5 **b** 0 **c** 1
14 a $\frac{7}{18}$ **b** $\frac{1}{2}$

15 a $\frac{2}{7}$ **b** $\frac{32}{63}$ **c** $\frac{9}{16}$

16 a 0.2 **b** 0.4

17 a 0.7 **b** 0.3 **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

Technology-active

18 a 0.443 **b** 0.776 **c** 0.234

19 a 0.15 **b** 0.148

20 a $\frac{1}{8}$ **b** $\frac{9}{58}$

21 0.208

Multiple-choice questions

1 D **2** C **3** B **4** B **5** C **6** A
7 C **8** D **9** D **10** A **11** A **12** B
13 B **14** C **15** C **16** A **17** C **18** A
19 C **20** B **21** D **22** B

Chapter 10

Short-response questions

Technology-free

1 a $\left(-\frac{1}{2}, 5\right)$ **b** $-\frac{6}{11}$

2 $a = -2,\ a = \frac{1}{2}$ or $a = \frac{2}{3}$

3 a $(x+4)^2 - 7$ **b** $x = -4$

4 $[1, 9]$ i.e. $1 \le y \le 9$

5 $(x-2)^2 + (y+3)^2 = 25$

6 $[-5, \infty)$ i.e. $y \ge -5$

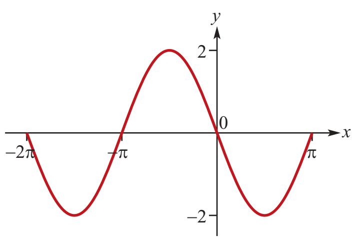
7 $\left[\frac{11}{4}, 15\right]$ i.e. $[\frac{11}{4} \le y \le 5$

8 $\left(\frac{y+3}{2}\right)^2 = x$

9 a Domain $= (-\infty, 1]$; Range $= \mathbb{R}$
b Domain $= [-1, 5]$; Range $= [-3, 3]$

10 $-\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

11



12 a −0.7 **b** 0.7 **c** −0.7 **d** 0.7

13 a Period $= \frac{2\pi}{3}$; Amplitude $= 2$
b Range $= [-1, 3]$

14 a $\frac{4\pi}{3}$ cm **b** $\frac{16\pi}{3}$ cm^2

15 a $M\left(\frac{1}{2}, -\frac{3}{2}\right), N\left(\frac{3}{2}, 4\right)$ **b** $m_{BC} = m_{MN} = \frac{11}{2}$

16 a −6 **b** −36
c $8x^3 + 24x^2 + 28x + 9$

17 a $y = (x+2)^2 - 13$ **b** $y = \left(x - \frac{3}{2}\right)^2 - \frac{53}{4}$
c $y = 2\left(x - \frac{3}{4}\right)^2 + \frac{79}{8}$

18 a $\left(\frac{1-\sqrt{41}}{2}, 3 - 2\sqrt{41}\right), \left(\frac{\sqrt{41}+1}{2}, 2\sqrt{41}+3\right)$
b $(2, 6)$ **c** $(-4, 14)$

19 a $\mathbb{R} \setminus \{\frac{5}{2}\}$ **b** $(-\infty, 5]$ **c** $\mathbb{R}$

20 a $[1, 4]$ **b** $[0, 3]$ **c** $(-4, \infty)$
d $[3, \infty)$ **e** $\mathbb{R}$

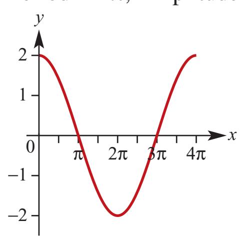
21 a $\frac{\pi}{3}$ **b** $\frac{3\pi}{2}$ **c** $\frac{7\pi}{9}$

22 a −1 **b** 0 **c** 0 **d** Undefined

23 a −0.3 **b** −0.5 **c** 1.6 **d** −0.6 **e** 0.1 **f** $\frac{4}{5}$

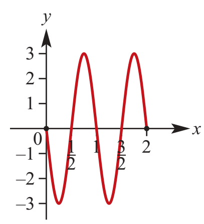
24 a $\frac{\sqrt{3}}{2}$ **b** $-\frac{\sqrt{3}}{2}$ **c** −1 **d** $\frac{1}{2}$ **e** $\frac{1}{\sqrt{2}}$ **f** $-\sqrt{3}$

25 a Period $= 4\pi$; Amplitude $= 2$
b



c Dilation of factor 2 from the x-axis and dilation of factor 2 from the y-axis

26



27 $16x^4 - 32x^3 + 24x^2 - 8x + 1$

28 a $\frac{-7\pi}{6}, \frac{-5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$
b $\frac{-7\pi}{4}, \frac{-5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
c $\frac{-17\pi}{12}, \frac{-13\pi}{12}, \frac{-5\pi}{12}, \frac{-\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$
d $\frac{-4\pi}{3}, \frac{-\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3}$

29 a $\{5, 7, 10, 14, 15\}$
b $\{1, 3, 5, 9, 11, 13, 15, 17, 19\}$
c $\{2, 4, 6, 8, 10, 12, 16, 18\}$

30 20

31 a $\frac{1}{9}$ **b** $\frac{8}{9}$

32 a $\{348, 384, 438, 483, 834, 843\}$, $n(\varepsilon) = 6$
b $\frac{2}{3}$ **c** $\frac{2}{3}$

33 a $\frac{1}{4}$ **b** $\frac{3}{8}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

34 **a** $\{11, 12, 13, 14, 15\}$
b $\{1, 3, 5, 7, 9, 11, 13, 14, 15\}$
c $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$
d $\{2, 4, 6, 8, 10\}$
e $\{11, 13, 14, 15\}$

35 $n(C \cup D) = 29$

36 **a** $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{1}{6}$

37 **a** $\frac{1}{4}$ **b** $\frac{7}{10}$ **c** $\frac{3}{10}$

38 **a** 0.4 **b** 0.2 **c** 0.7

39 **a** $12a^2 - 4$ **b** $3a^2 - 6a - 1$
c $12a$

40 $y = \frac{3}{2}x^2 + \frac{5}{2}x + 2$

41 **a** No **b** $-\frac{3}{7}$ **c** $x \leq -\frac{3}{7}$ $k = -\frac{3}{16}$

42 $x = 2$ and $y = 3$, or $x = 3$ and $y = 2$

43 **a** $y = \frac{-11}{x+1} + 4$ **b** $(0, -7)$, $\left(\frac{7}{4}, 0\right)$
c $x = -1$, $y = 4$

44 **a** $y = \frac{1}{2}x + 5$ **b** $c = 8$, $d = 9$

45 $0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

46 **a** $k = -2$ **b** $k \neq -2$ **c** $k < -1$ and $k \neq -2$

47 **a** $y = mx + (3 - 2m)$
b $(0, 3 - 2m)$, $\left(\frac{2m-3}{m}, 0\right)$
c $m = -\frac{9}{2}$ or $m = -\frac{1}{2}$

48 $(0, -2)$, $(0, -6)$, $(3, 0)$, $(4, 0)$

49 $b = 4$, $c = 1$

50 **a** **i** $x + 6$ cm **ii** $x(x + 6)$ cm^2
iii Two sides $5x$ cm^2; two sides $5(x + 6)$ cm^2
iv $x^2 + 26x + 60$ cm^2
b $x = 4$

51 $a = 2$, $h = 3$, $k = 4$

52 $AB = BC = CD = DA = 5\sqrt{2}$,

53 **a** $x < \frac{-3 - \sqrt{29}}{2}$ or $x > \frac{-3 + \sqrt{29}}{2}$
b $x \leq \frac{5 - \sqrt{65}}{4}$ or $x \geq \frac{5 + \sqrt{65}}{4}$

54 $a = -5$, $R = -35$

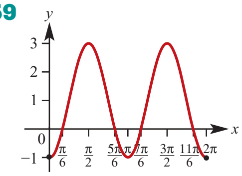
55 **a** $3b + 2f = 18.20$ **b** \$2.80

56 **a** $k = 1$ **b** $k = -16$

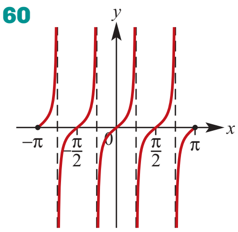
57 **a** $\frac{2}{5}$ **b** $2y + 5x - 17 = 0$

58 **a** $a\left(x + \frac{1}{a}\right)^2 + \frac{a^2 - 1}{a}$ **b** $\left(-\frac{1}{a}, \frac{a^2 - 1}{a}\right)$
c $a = \pm 1$ **d** $-1 < a < 1, a \neq 0$

59



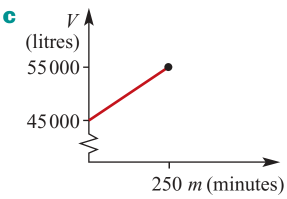
60



61 **a** First two terms: $2^{35}x^{35}$ and $35 \cdot 2^{34} \cdot 3x^{34}y$;
Last two terms: $35 \cdot 2 \cdot 3^{34}xy^{34}$ and $3^{35}y^{35}$
b First two terms: 2^{20} and $20 \cdot 2^{19}y^3$;
Last two terms: $20 \cdot 2y^{57}$ and y^{60}

62 $k = 3$

63 **a** $V = 45\,000 + 40m$ **b** 4 hours 10 minutes
c



64 **a** $P(1) = P(2) = P(3) = P(5) = \frac{2}{17}$,
$P(4) = \frac{8}{17}$, $P(6) = \frac{1}{17}$
b $\frac{9}{17}$

65 $\frac{1}{4}$

66 **a** 0.6 **b** $\frac{1}{3}$

67 **a** $\frac{1}{2}$ **b** $\frac{3}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{4}$

68 **a** 0.48 **b** 0.56

69 **a** $P(A \cap B) = 0.05$ **b** $P(A \mid B) = 0.25$

70 **a** **i** 36 **ii** 11 **iii** 15
b **i** $\frac{19}{36}$ **ii** $\frac{5}{12}$

71 **a**

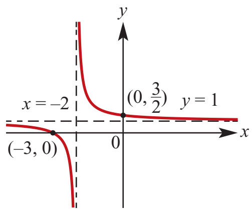
1	2	3	4	5	6
$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{1}{15}$

b $\frac{14}{15}$

72 **a** $\frac{1}{17}$ **b** $\frac{25}{102}$ **c** $\frac{13}{51}$ **d** $\frac{1}{2}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3

73 a



b $A\left(0, \frac{3}{2}\right), B(-3, 0)$ **c** $y = \frac{1}{2}x + \frac{3}{2}$

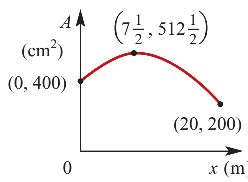
d $\left(-\frac{3}{2}, \frac{3}{4}\right)$ **e** $y = -2x - \frac{9}{4}$

74 a i $A = (10 + x)y - x^2$ **ii** $P = 2(y + x + 10)$

b i $A = 400 + 30x - 2x^2$

ii $512\frac{1}{2}$ m^2 **iii** $0 \le x \le 20$

iv



75 a $(9, 5)$ **b** $k = 4$

76 a $A = 10x^2 + 28x + 16$

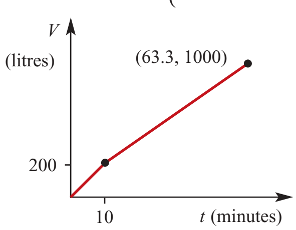
b i 54 cm^2 **ii** 112 cm^2

c 3 **d** ss **e** $2x^3 + 8x^2 + 8x$ **f** 3 **g** 6.66

77 a 20 m **b** 20 m **c** 22.5 m

78 a 200 L **b** $V = \begin{cases} 20t & 0 \le t \le 10 \\ 15t + 50 & 10 < t \le \frac{190}{3} \end{cases}$

c



79 a $x \le a$ **b** $\left(\frac{\sqrt{4a+1} - 1}{2}, \frac{\sqrt{4a+1} - 1}{2}\right)$

c $a = 2$ **d** $a = 6$ **e** $a = c^2 + c$

81 b i $\frac{1}{3}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{3}$

Technology-active

82 a 21.80° **b** 3.06 **c** 9.97

83 15 cm^2

84 $p = -38$, $(x - 3)(x + 4)(3x - 2)$

85 a 0.4695 **b** −0.7986 **c** −1.963

d 0.9205 **e** 0.9666 **f** −0.9728

86 $x = -1$, $x = 2$ or $x = 3$

87 $T = 0.186\ell^{0.528}$

88 64 cm^2

89 a 0.41 **b** 0.73 **c** 0.32

90 $P(A \cup B) = 0.55$

91 $P(A \cap B) = 0.2$

92 a

	B	B'	
A	0.3	0.35	0.65
A'	0.24	0.11	0.35
	0.54	0.46	1

b i 0.35 **ii** 0.65 **iii** 0.46

93 a $P(A \mid B) = 0.375$ **b** $P(B \mid A) = 0.3$

94 a $P(C \cap D) = 0.12$ **b** $P(C \mid D) = 0.3$

95 a $a = 3$, $b = 17$ **b** $x = -1, 2$ or $\frac{7}{2}$

c $x = -\frac{1}{3}, \frac{2}{3}$ or $\frac{7}{6}$

96 a 7 m **b** 3 m **c** $t = 0, \frac{\pi}{3}$ or $\frac{2\pi}{3}$

d $t = \frac{\pi}{6}$ or $t = \frac{5\pi}{6}$

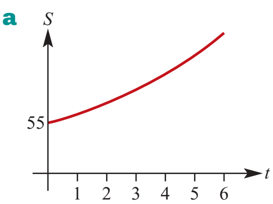
e Particle oscillates with period $\frac{2\pi}{3}$ s and amplitude 2 m about a point at a distance of 5 m from O

97 176.885 cm^2

98 a Min = −3; Max = 3 **b** Period = 4

c Five solutions

99 a



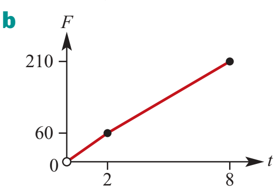
b 2010: \$55 000 000; 2011: \$63 822 000; 2015: \$127 330 000

c 2014

d \$270 210 000

100 a $F(t) = \begin{cases} 30t & \text{for } 0 < t \le 2 \\ 25t + 10 & \text{for } 2 < t \le 8 \end{cases}$

b



c i \$45 **ii** \$60 **iii** \$122.50

d $G(t) = -\frac{5}{8}t^2 + \frac{125}{4}t, \ 0 < t \le 8$

101 a $4b - 5c - d = 41$, $2b - 7c - d = 53$, $-4b + 3c - d = 25$

b $x^2 + y^2 - 2x - 4y - 29 = 0$

102 $\frac{1}{5}$

103 a 0.2475 **b** 0.4125

104 a 0.31 **b** 0.614 **c** 0.244

105 a 0.05 **b** 0.25 **c** 0.75

106 a $\frac{3}{11}$ **b** $\frac{3}{11}$ **c** $\frac{5}{11}$ **d** $\frac{1}{22}$

Mathematical Methods Units 1 & 2 ISBN 978-1-009-53719-3 Cambridge University Press

e $\frac{1}{3}$ **f** $\frac{1}{4}$ **g** $\frac{5}{33}$ **h** $\frac{3}{11}$

107 a $\frac{3}{8}$ **b** $\frac{2}{5}$ **c** $\frac{1}{2}$

108 a i $\frac{15}{28}$ **ii** $\frac{37}{56}$ **iii** $\frac{43}{49}$

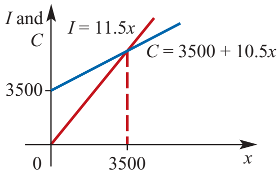
b i $\frac{9}{14}$ **ii** $\frac{135}{392}$

109 a $c = -b - 8$ **b** $x = 0$ or $x = -b$

c $y = 0$ or $y = b + 8$ **d** $b = -8$

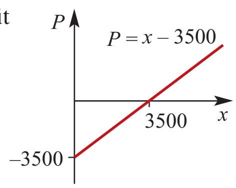
110 a $C = 3500 + 10.5x$ **b** $I = 11.5x$

c



d 3500

e Profit



f 5500

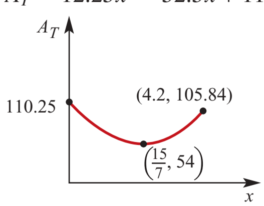
111 a $A_R = 6x^2$

b $A_S = (10.5 - 2.5x)^2$

c $0 \le x \le 4.2$

d $A_T = 12.25x^2 - 52.5x + 110.25$

e



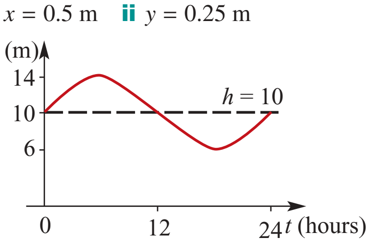
f $110.25\ \text{cm}^2$ (when area of rectangle = 0)

g Rectangle 9×6 and square 3×3, or rectangle $\frac{27}{7} \times \frac{18}{7}$ and square $\frac{51}{7} \times \frac{51}{7}$

112 a $A = 6x^2 + 7xy + 2y^2$

c i $x = 0.5$ m **ii** $y = 0.25$ m

113 a

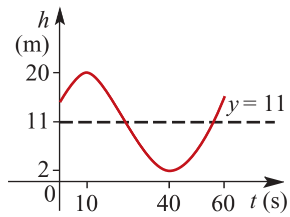


b $t = 3.2394$ and $t = 8.7606$

c $t \in [0.9652, 11.0348]$

114 a 60 seconds

b



c [2, 20]

d First at height 2 metres after 40 seconds; then every 60 seconds after this first time

e At $t = 0$, $t = 20$ and $t = 60$, for $t \in [0, 60]$

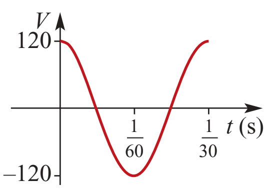
115 a $2y + 3x = 22$

b i $B(0, 11)$ **ii** $D(8, -1)$

c 52 units^2

d 6.45 units

116 a



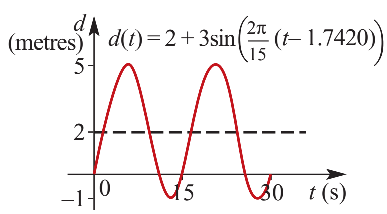
b $t = \frac{1}{180}$ s

c $t = \frac{k}{30}$ s, for $k = 0, 1, 2, \ldots$

117 a i 15 seconds **ii** 3 m **iii** $c = \frac{2\pi}{15}$

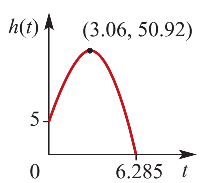
b $h = 1.74202$

c



118 a 50.9 m **b** $t = 6.12$ seconds

c

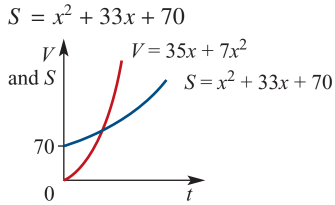


d 6.285 seconds

119 a $x + 5$ **b** $V = 35x + 7x^2$

c $S = x^2 + 33x + 70$

d



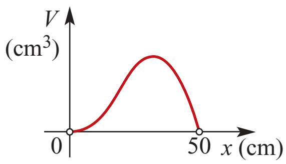
e 3.25 cm **f** 10 cm

120 a 3.2% **b** 0.625

121 a i $y = 250 - 5x$

ii $V = x^2(250 - 5x) = 5x^2(50 - x)$

b

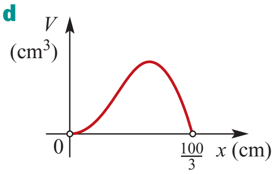


c $0 < x < 50$
d $x = 11.378$ or $x = 47.813$
e $V_{\max} = 92\,592.59\ \text{cm}^3$ when $x = 33.33$ and $y = 83.33$

122 a 25 km/h **b** Tap A 60 min; Tap B 75 min
c 4 cm

123 a $y = \frac{3}{4}x - 4$ **b** $y = -\frac{4}{3}x + \frac{38}{3}$
c $D(8, 2)$ **d** 5 units **e** 50 units^2

124 a $h = 100 - 3x$ **b** $V = 2x^2(100 - 3x)$
c $0 < x < \frac{100}{3}$ **d**



e i $x = 18.142$ or $x = 25.852$
ii $x = 12.715$ or $x = 29.504$
f $V_{\max} = 32\,921.811\ \text{cm}^3$ when $x = 22.222$
g i $S = 600x - 14x^2$
ii $S_{\max} = \frac{45000}{7}\ \text{cm}^2$ when $x = \frac{150}{7}$
h $x = 3.068$ or $x = 32.599$

125 a i 12 kg apples **ii** 15 kg bananas
b i 12 kg apples **ii** 15 kg bananas
c

a	15 kg apples	20 kg bananas
b	20 kg apples	30 kg bananas

126 a i 11.93°C **ii** 16.07°C
b 5 p.m.
c Between 2:14 p.m. and 7:46 p.m.
d Before 9:35 a.m. and after 7:51 p.m.

127 a $\frac{2}{5}$ **b** $\frac{1}{3}$ **c** $\frac{28}{75}$ **d** $\frac{38}{75}$

128 a i 1.15 radians **ii** 7559.63 cm^2
iii 96.25%
b 15.65 cm

129 a 0.214 **b** 0.096 **c** 0.536 **d** 0.524
e 0.601

130 a i 0.49 **ii** 0.91
b 0.933

131 a 0.0829
b The test has to be more reliable to give useful information

132 a i 0.0847 **ii** 0.0026 **iii** 0.000027
iv 0.0873
b 0.0788
c 78.2% of crates
d i 0.0864 **ii** 0.0882

133 a $\frac{1}{2}$ **b** $\frac{13}{36}$

134 a $\frac{59}{120}$ **b** $\frac{45}{59}$

135 a i $m = 30, q = 35, s = 25$
ii $m + q = 65$
b $\frac{3}{10}$
c $\frac{7}{12}$

136 a i $\frac{1}{9}$ **ii** $\frac{5}{18}$
b i $\frac{1}{81}$ **ii** $\frac{13}{324}$

137 a $\frac{1}{60}$ **b** $\frac{1}{5}$ **c** $\frac{3}{5}$ **d** $\frac{6}{13}$

138 a 0.084 **b** 0.52 **c** 0.68

139 a 60 **b** 8 **c** 0.1

140 a i 10 000 cm^2 **ii** 400 cm^2 **iii** 6400 cm^2
b i 0.04 **ii** 0.12 **iii** 0.64
c i 0.0016 **ii** 0.000 64

Multiple-choice questions

1 B	**2** D	**3** C	**4** A	**5** B
6 D	**7** C	**8** C	**9** C	**10** A
11 D	**12** B	**13** A	**14** D	**15** B
16 B	**17** D	**18** C	**19** B	**20** D
21 D	**22** D	**23** B	**24** D	**25** D
26 A	**27** B	**28** A	**29** C	**30** D
31 B	**32** B	**33** C	**34** A	**35** C
36 A	**37** B	**38** C	**39** D	**40** D
41 D	**42** C	**43** C	**44** C	**45** A
46 D	**47** A	**48** D	**49** A	**50** D
51 A	**52** D	**53** B	**54** D	**55** C
56 A	**57** D	**58** D	**59** D	**60** A
61 B	**62** D	**63** C	**64** D	**65** B
66 D	**67** C	**68** B	**69** D	**70** C
71 C	**72** D	**73** A	**74** D	**75** C
76 C	**77** B	**78** B	**79** C	**80** C
81 C	**82** B	**83** D	**84** A	**85** B
86 C	**87** B	**88** C	**89** C	**90** C

Problem-solving and modelling

See solutions supplement

Chapter 11

Exercise 11A

1 a x^5 **b** $8x^7$ **c** x^2 **d** $2x^3$
e a^6 **f** 2^6 **g** x^2y^2 **h** x^4y^6
i $\frac{x^3}{y^3}$ **j** $\frac{x^6}{y^4}$

2 a 3^{17} **b** x^7y^5 **c** 3^{4x+3} **d** $30a^5b^6$

3 a x^2y **b** b^{4x+1} **c** $4a^5b$

4 a $\frac{1}{49}$ **b** 64 **c** $\frac{8}{125}$

5 **a** b^{10} **b** 729 **c** b^4

6 **a** $\dfrac{27a^8b}{16}$ **b** $\dfrac{125b^6}{c^9}$

7 **a** 64 **b** $-27a^3$ **c** $-96a^3$

8 **a** 2^{-2n} **b** 2^4 **c** $\dfrac{5^{2n}}{2^{2n}}$

9 **a** x^9 **b** 2^{16} **c** 3^{17} **d** q^8p^9
e $a^{11}b^3$ **f** 2^8x^{18} **g** $m^{11}n^{12}p^{-2}$
h $2a^5b^{-2}$

10 **a** x^2y^3 **b** $8a^8b^3$ **c** x^5y^2 **d** $\dfrac{9}{2}x^2y^3$

11 **a** $\dfrac{1}{n^4p^5}$ **b** $\dfrac{2x^8z}{y^4}$ **c** $\dfrac{b^5}{a^5}$ **d** $\dfrac{a^3b}{c}$
e $a^{n+2}b^{n+1}c^{n-1}$

12 **a** 3^{17n} **b** 2^{3-n} **c** $\dfrac{3^{4n-11}}{2^2}$ **d** $2^{n+1}3^{3n-1}$
e 5^{3n-2} **f** $2^{3x-3} \times 3^{-4}$
g $3^{6-n} \times 2^{-5n}$ **h** $3^3 = 27$ **i** 6

13 **a** $2^{12} = 4096$ **b** $5^5 = 3125$ **c** $3^3 = 27$

14 **a** $-\dfrac{x}{y}$ **b** $\dfrac{x+y}{y-x}$ **c** $\dfrac{x^2y}{x^2+y}$

Exercise 11B

1 **a** 25 **b** 27 **c** $\dfrac{1}{9}$ **d** 16
e $\dfrac{1}{2}$ **f** $\dfrac{1}{4}$ **g** $\dfrac{1}{25}$ **h** 16
i $\dfrac{1}{10\,000}$ **j** 1000 **k** 27 **l** $\dfrac{3}{5}$
m -2 **n** $\dfrac{1}{625}$ **o** 16 **p** 343

2 **a** $a^{\frac{1}{6}}b^{-\frac{7}{6}}$ **b** $a^{-6}b^{\frac{9}{2}}$ **c** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$
d $\dfrac{1}{4}$ **e** x^6y^{-8} **f** $a^{\frac{14}{15}}$

3 **a** $(2x-1)^{\frac{3}{2}}$ **b** $(x-1)^{\frac{5}{2}}$ **c** $(x^2+1)^{\frac{3}{2}}$
d $(x-1)^{\frac{7}{2}}$ **e** $x(x-1)^{-\frac{1}{2}}$ **f** $(5x^2+1)^{\frac{4}{3}}$

Exercise 11C

1 **a** 4.78×10 **b** 6.728×10^3
c 7.923×10 **d** 4.358×10^4
e 2.3×10^{-3} **f** 5.6×10^{-7}
g $1.200\,034 \times 10$ **h** 5.0×10^7
i 2.3×10^{10} **j** 1.3×10^{-9}
k 1.65×10^5 **l** 1.4567×10^{-5}

2 **a** 1.0×10^{-8} **b** 1.67×10^{-24}
c 5×10^{-5} **d** $1.853\,18 \times 10^3$
e 9.461×10^{12} **f** 2.998×10^{10}

3 **a** 81 280 000 000 000 **b** 270 000 000
c 0.000 000 000 000 28

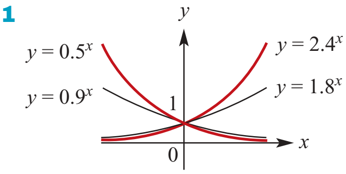
4 **a** 4.569×10^2 **b** 3.5×10^4
c 5.6791×10^3 **d** 4.5×10^{-2}
e 9.0×10^{-2} **f** 4.5682×10^3

5 **a** 0.000 0567 **b** $\dfrac{262}{2625}$

6 **a** 11.8 **b** 4.76×10^7

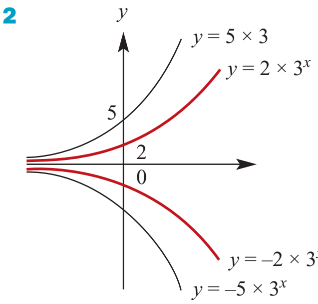
Exercise 11D

1



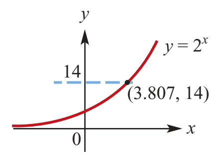
- All pass through (0, 1)
- Horizontal asymptote $y = 0$
- Increasing for base > 1
- Decreasing for base < 1

2

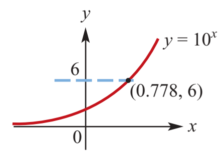


- $y = a \times b^x$ has y-axis intercept at $(0, a)$
- Horizontal asymptote $y = 0$
- Graphs c and d are reflections in the x-axis of graphs a and b

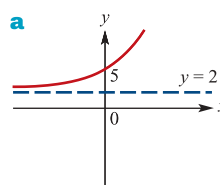
3 $x = 3.807$



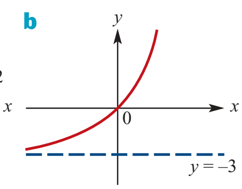
4 $x = 0.778$



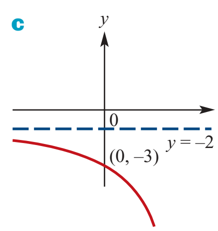
5 **a**



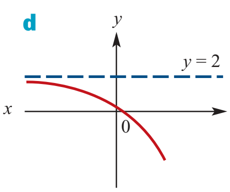
b



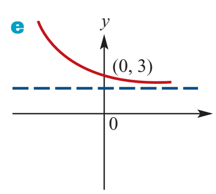
c



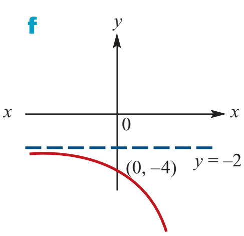
d



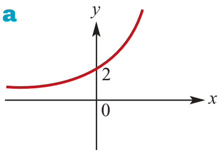
e



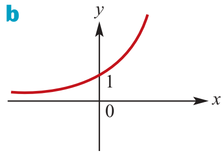
f



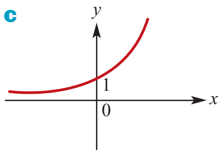
6 a



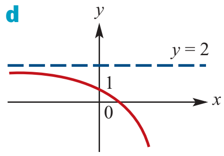
b



c



d



Exercise 11E

1 a 3 **b** 3 **c** $\frac{1}{2}$ **d** $\frac{3}{4}$ **e** $\frac{1}{3}$

f 4 **g** 2 **h** 3 **i** 3

2 a 1 **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{4}{3}$ **e** -1

f 8 **g** 3 **h** -4 **i** 8 **j** 4

k $3\frac{1}{2}$ **l** 6 **m** $7\frac{1}{2}$

3 a $\frac{4}{5}$ **b** $\frac{3}{2}$ **c** $5\frac{1}{2}$

4 a 0 **b** 0, -2 **c** 1, 2 **d** 0, 1

5 a 2.32 **b** 1.29 **c** 1.26 **d** 1.75

6 a $x > 2$ **b** $x > \frac{1}{3}$ **c** $x \le \frac{1}{2}$ **d** $x < 3$

e $x < \frac{3}{4}$ **f** $x > 1$ **g** $x \le 3$

Exercise 11F

1 a $N = 1000 \times 2^{\frac{t}{15}}$ **b** 50 minutes

2 79 726 years

3 7575 years

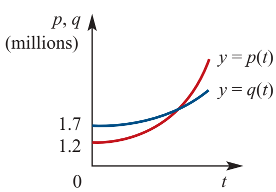
4 a 535 millibars **b** 7331 metres

5 22 hours

6 6.4°C

7 $t > 18.668\ldots$

8 a



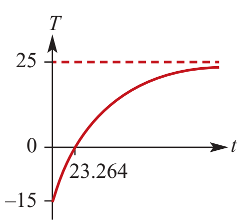
b i $t = 12.56\ldots$ (mid 1962)

ii $t = 37.56\ldots$ (mid 1987)

9 a $T \to 25$; room temperature is 25°C

b 23.264 minutes

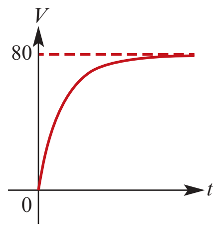
c



d The ice-cream approaches but never reaches room temperature

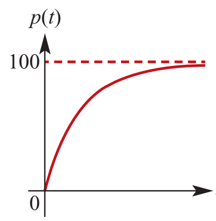
10 a 80 m/s **b** 1.577 seconds

c



11 a $p(t) \to 100$ **b** 25.237 days

c



12 a $y = 3 \times 5^x$ **b** $y = 4 \times \left(\frac{1}{2}\right)^x$

c $y = 5 \times \left(\frac{3}{2}\right)^x$

13 a $N = 1000 \times 10^{\frac{t}{5}}$ **b** 210 minutes

c 15 hours **d** 251 189 bacteria

14 $y = 1.5 \times 0.575^x$

15 $p = 2.5 \times 1.35^t$

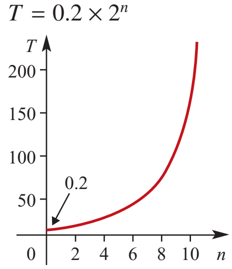
16 $d_0 = 41.92$, $m = 0.094$

17 a

Cuts, n	Sheets	Thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	1.6
4	16	3.2
5	32	6.4
6	64	12.8
7	128	25.6
8	256	51.2
9	512	102.4
10	1024	204.8

b $T = 0.2 \times 2^n$

c



d 214 748.4 m

 ISBN 978-1-009-53719-3 Cambridge University Press

Chapter 11 review

Short-response questions

Technology-free

1 a a^4 **b** $\frac{1}{b^2}$ **c** $\frac{1}{m^2n^2}$ **d** $\frac{1}{ab^6}$

e $\frac{3a^6}{2}$ **f** $\frac{5}{3a^2}$ **g** a^3 **h** $\frac{n^8}{m^4}$

i $\frac{1}{p^2q^4}$ **j** $\frac{8}{5a^{11}}$ **k** $2a$ **l** $a^2 + a^6$

2 3.84×10^8

3 a $\frac{1}{125}$ **b** 81 **c** -3

d 100 000 **e** 125

4 a $4x^4y^2$ **b** $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$ **c** 1

5 a $x = \frac{5}{2}$ **b** $x = 1$ **c** $x = -2$

d $x = -4$ **e** $x = \frac{8}{3}$ **f** $x = -1$

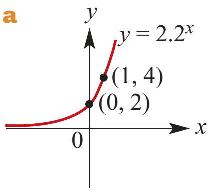
6 a $x = \frac{3}{2}$ **b** $x = \frac{5}{3}$

c $x = \frac{4}{3}$ **d** $x = \frac{3}{2}$

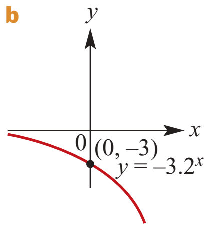
7 a $x = 3$ **b** $x = 3$ or $x = 0$

c $x = 1$ **d** $x = 2$ or $x = 3$

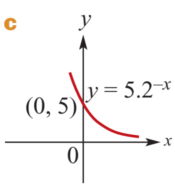
8 a



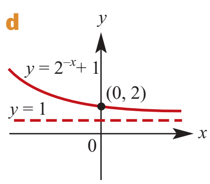
b



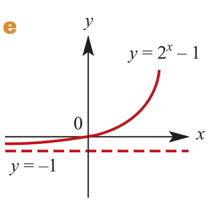
c



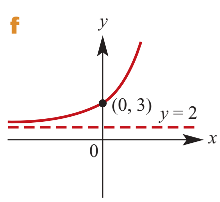
d



e



f



9 a $f(x) > 3$ **b** $f(x) < 12$

c $f(x) > 33$ **d** $f(x) > 12$

10 10^6 seconds or $11\frac{31}{54}$ days

11 50

12 a $x = 1$ or $x = 2$ **b** $x = -1$ **c** $x = \frac{1}{2}$

13 a $\frac{3}{8m^3}$ **b** $2 \times 6^{\frac{1}{4}} \times m^{\frac{5}{2}}$ **c** $\frac{1}{5}$

14 $\frac{375}{7}$

15 a $x = -1$ **b** $x = 2$ **c** $x = 2$ **d** $x = 4$

16 a $x = \frac{3}{2}, y = 1$ **b** $x = \frac{1}{4}, y = \frac{11}{4}$

c $x = 3, y = 7$

17 a $b = \frac{1}{3}$ **b** 80

Technology-active

18 a 3.86 **b** 0.11 **c** 1.77

19 $27.62086\ldots \approx 27.62$ years

20 a 64% **b** $7.2125\ldots \approx 7.21$ hours

c $20.6377\ldots \approx 20.64$ hours

21 a 128 000 **b** 2.95 hours

22 a $b = \frac{1}{3}$ **b** 50.4 hours

23 a $y = 2 \times 1.585^x$ **b** $y = 2 \times 10^{0.2x}$

24 $n = 2$

25 a 1.9×10^{-8} N **b** 9.8×10^{24} kg

26 a $\left(\frac{1}{2}\right)^{3n}$ **b** $\left(\frac{1}{2}\right)^{5n-2}$ **c** $n = 3$

27 a $729 \times \left(\frac{1}{4}\right)^n$ **b** $128 \times \left(\frac{1}{2}\right)^n$

c 4 times

28 a Batch 1: $15(0.95)^n$; Batch 2: $20(0.94)^n$

b 32 years

29 a 13.81 years **b** 7.38 years

Multiple-choice questions

1 C **2** A **3** C **4** B **5** C

6 D **7** A **8** B **9** C **10** A

11 C **12** B **13** D **14** A **15** D

16 D **17** C **18** A **19** C

Chapter 12

Exercise 12A

1 a 7 **b** 4 **c** 3 **d** -1

2 a $\log_2(10a)$ **b** 1 **c** $\log_2\left(\frac{9}{4}\right)$ **d** 1

e $3\log_2 a$ **f** 9 **g** $-\log_5 6$ **h** -2

3 a 3 **b** 4 **c** -7 **d** -3

e 4 **f** -3 **g** 4 **h** -6

i -9 **j** -1 **k** 4 **l** -2

4 a 2 **b** 7 **c** 9 **d** 1

e $\frac{5}{2}$ **f** $\log_x(a^5)$ **g** 3 **h** 1

5 a 2 **b** 27 **c** $\frac{1}{125}$ **d** 8

e 30 **f** $\frac{2}{3}$ **g** 8 **h** 64

i 4 **j** 10

6 a 5 **b** 32.5 **c** 22 **d** 20

e $\frac{3 \pm \sqrt{17}}{2}$ **f** 3 or 0

7 $2 + 3a - \dfrac{5c}{2}$ **9** 10

10 a 4 **b** $\dfrac{6}{5}$ **c** 3 **d** 10
e 9 **f** 2

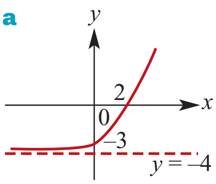
Exercise 12B

1 a 2.81 **b** −1.32 **c** 2.40 **d** 0.79
e −2.58 **f** −0.58

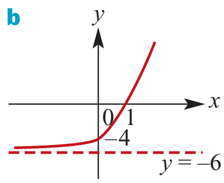
2 a 1.90 **b** 3.10 **c** −0.87

3 a $x > 3$ **b** $x < 1.46$ **c** $x \le 2.77$

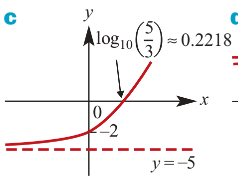
4 a



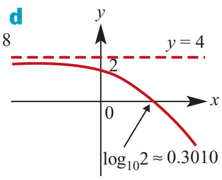
b



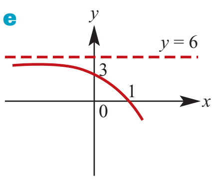
c



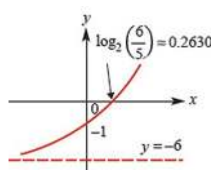
d



e



f



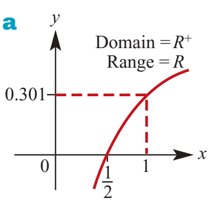
5 a $(\log_6(1080), 0), \left(0, \dfrac{1079}{216}\right)$ **b** (0, 0)

c $\left(\log_2\left(\dfrac{4}{3}\right), 0\right), (0, 2)$

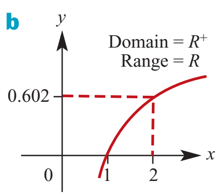
6 $a = 6 \times \left(\dfrac{10}{3}\right)^{-\frac{2}{3}}$ and $k = \dfrac{1}{3}\log_{10}\left(\dfrac{10}{3}\right)$

Exercise 12C

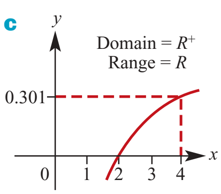
1 a



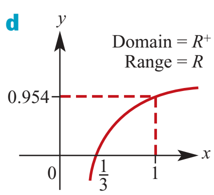
b



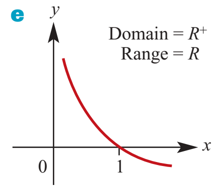
c



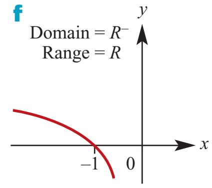
d



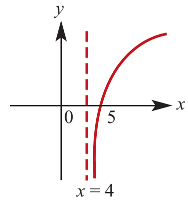
e



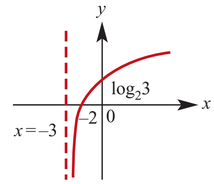
f



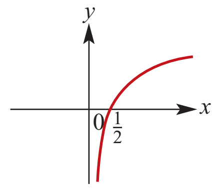
2 a Domain = $(4, \infty)$



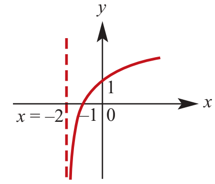
b Domain = $(-3, \infty)$



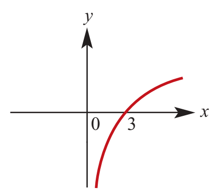
c Domain = $(0, \infty)$



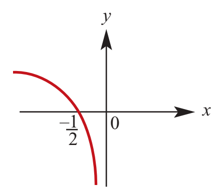
d Domain = $(-2, \infty)$



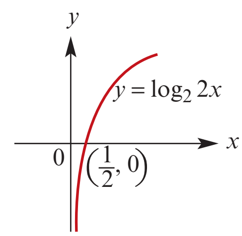
e Domain = $(0, \infty)$



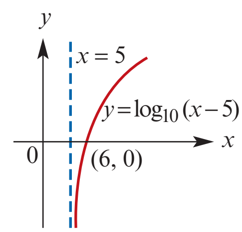
f Domain = $(-\infty, 0)$



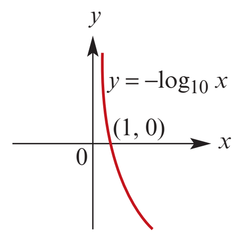
3 a Domain = $\mathbb{R}^+$



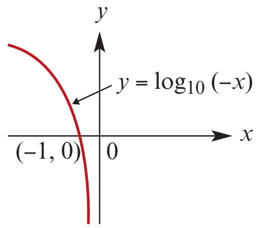
b Domain = $(5, \infty)$



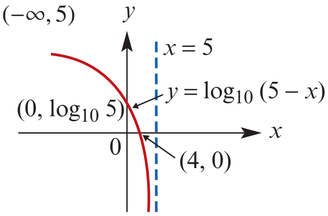
c Domain = $\mathbb{R}^+$



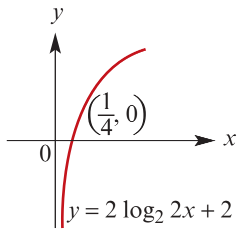
d Domain = $\mathbb{R}^-$



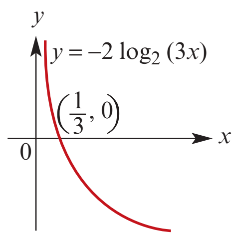
e Domain = $(-\infty, 5)$



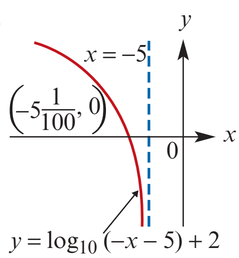
f Domain = $\mathbb{R}^+$



g Domain = $\mathbb{R}^+$

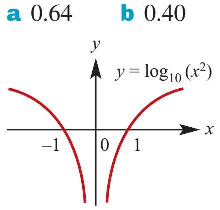


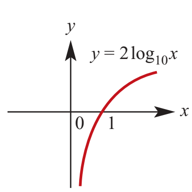
h Domain = $(-\infty, -5)$



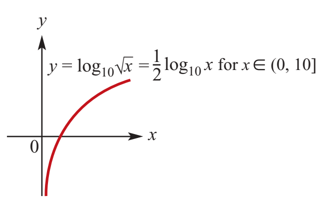
4 a 0.64 **b** 0.40

5

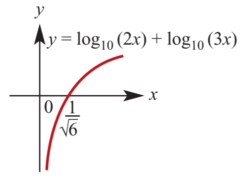


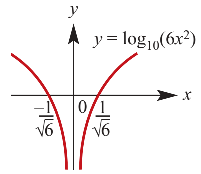


6



7





8 A dilation of $\dfrac{1}{\log_2 3}$ from the y-axis

9 A dilation of $\dfrac{1}{\log_{10} 2}$ from the y-axis

Exercise 12D

1 a $x = \dfrac{1}{2} \times 5^y$ **b** $x = \dfrac{1}{2} \times 5^{\frac{y-1}{3}}$

c $x = \log_2(y - 2)$ **d** $x = \log_2(y) - 2$

e $x = \dfrac{1}{2}(7^y - 1)$ **f** $x = \dfrac{1}{3}\left(7^{\frac{y}{4}} - 2\right)$

g $x = 10^y - 1$ **h** $x = \log_5\left(\dfrac{y}{2}\right) + 1$

2 $t = \dfrac{-1}{k} \log_2\left(\dfrac{P - b}{A}\right)$

3 a $x = 2^{\frac{y-5}{2}}$ **b** $x = -\dfrac{1}{6} \log_5\left(\dfrac{P}{A}\right)$

c $n = \dfrac{\log_{10}\left(\dfrac{y}{a}\right)}{\log_{10} x}$ **d** $x = \log_{10}\left(\dfrac{y}{5}\right)$

e $x = \dfrac{1}{2} \times 3^{\frac{5-y}{3}}$ **f** $n = \dfrac{\log_{10}\left(\dfrac{y}{6}\right)}{2 \log_{10} x}$

g $x = \dfrac{1}{2}(4^y + 1)$ **h** $x = \log_6\left(\dfrac{5}{5 - y}\right)$

4 a $a \approx 0.544$ **b** $k \approx 549.3$

6 a 9^u **b** $u + \dfrac{1}{2}$ **c** $\dfrac{2}{u}$

7 $x = 625$ or $x = \dfrac{1}{625}$

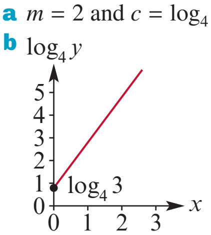
8 $\dfrac{2}{p}$

9 b $x = 4$

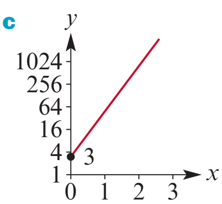
Exercise 12E

1 a $m = 2$ and $c = \log_4 3$

b

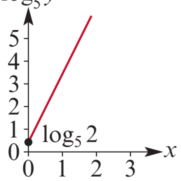


c

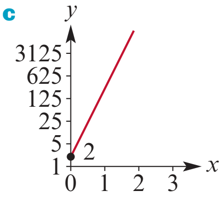


2 a $m = 3$ and $c = \log_5 2$

b



c



3 Street 10^{-9} watt/cm^2; Quiet car 10^{-11} watt/cm^2

4 a Increases by $10 \log_{10} 2 \approx 3$ dB
b Increases by $10 \log_{10} 10 = 10$ dB
c $P_{\text{new}} = (P_{\text{old}})^3 \times 10^{32}$
d $P = 10^{-16}$ **e** $P = 10^{-6}$

5 $P_1 = 10^{\frac{\lambda}{10}} \times P_2$ **6** $5 + \log_{10} 5 \approx 5.7$

7 $7.3 - \log_{10} 4 \approx 6.7$ **8** $[10^{-4}, 10^{-2}]$

9 a 10^{-7} moles per litre **b** 2.3
c 1.48 litres **d** 2.66

10 a $R = \frac{2}{3}(\log_{10} E - 4.4)$
b i 7.943×10^{11} J **ii** 2.512×10^{13} J
iii 7.943×10^{14} J **iv** 2.512×10^{16} J
v 1.995×10^{17} J
c 5.73

11 $100 + 10 \log_{10} 2 \approx 103$ dB

12 a i \$4024.23 **ii** \$6500.00
iii \$11 903.09 **iv** \$17 647.05
b $1.5 \log_{10} x + \frac{2000}{x}$
c i \$8.05 **ii** \$6.50 **iii** \$5.95 **iv** \$5.88

13 a 7.3 **b** 63.55

14 a i 14.33 days **ii** 16.95 days
iii 27.67 days **iv** 31.67 days
b $n = \dfrac{2000}{1 + 1999 \times 3^{-\frac{t}{4}}}$

15 a $30 \le P \le 99.9649\ldots$
b 73.63% **c** 30.97 minutes
d $t = 3^{\frac{P-30}{16}}$

Chapter 12 review

Short-response questions

Technology-free

1 a $\log_2 7$ **b** $\frac{1}{2}\log_2 7$ **c** $\log_{10} 2$
d $\log_{10}\left(\frac{18}{5}\right)$ **e** $1 + \log_{10} 11$
f $1 + \log_{10} 101$ **g** $\frac{1}{5}\log_2 100$
h $-\log_2 10$

2 a 2 **b** 3 **c** 3 **d** -1

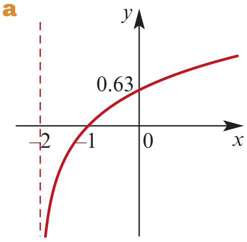
3 a 6 **b** 7 **c** 2 **d** 0
e 3 **f** -2 **g** -3 **h** 4

4 a $\log_a(xy)$ **b** $\log_2(x(x+3))$
c $\log_b\left(\frac{2x}{3y}\right)$ **d** $\log_a 8$
e -2 **f** $6 \log_3 x$

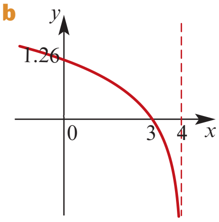
5 a 9 **b** 8 **c** 2 **d** 26

6 a $\log_{10} 6$ **b** $\log_{10} 6$
c $\log_{10}\left(\frac{a^2}{b}\right)$ **d** $\log_{10}\left(\frac{a^2}{25000}\right)$
e $\log_{10} y$ **f** $\log_{10}\left(\frac{a^2 b^3}{c}\right)$

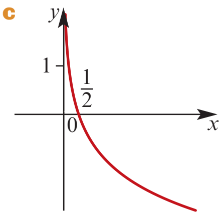
7 a



b



c



8 a $x = 6$ **b** $x = \sqrt{17}$

9 a $x = 1$ **b** $x = -4$ or $x = 20$

10 3

11 a $k = \frac{1}{7}$ **b** $q = \frac{3}{2}$

13 a $y = 4 \times 2^{-\frac{2}{3}x}$ **b** $y = 4 \times \left(2^{-\frac{2}{3}}\right)^x$

Technology-active

14 a 5.6 **b** 5 **c** $\frac{4}{9}$

15 a $a = 0.2$ and $b = 5$
b i $z = x \log_{10} b$
ii $a = 0.2$ and $k = \log_{10} 5$

16 a X \$1.82 Y \$1.51 Z \$2.62
b X \$4.37 Y \$4.27 Z \$3.47
c Intersect at $t = 21.784\ldots$ and $t = 2.090\ldots$; therefore Mar 2019 until Oct 2020
d Mar 2020 until Oct 2020; approx. 8 months

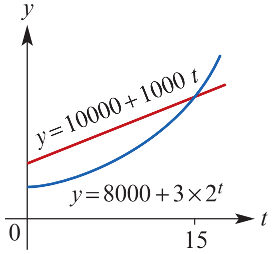
17 a $A = 10^{\frac{10R+75}{23}} - 4800$
b 2 010 537.68 km
c 7.69
d i 291.7 km **ii** 800.0 km **iii** 1321 km

18 a $\alpha = -0.25$, $\beta = 3$

d	0	1	3	7	15	31
A	3	2.75	2.5	2.25	2	1.75

b 4095 cm
c 146 cm

19 a



b i (12.210, 22 209.62)
ii $t = 12.21$ **iii** 22 210
c ii (12.21, 12.21)
d $c = 0.52$

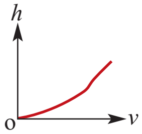
Multiple-choice questions

1 A **2** C **3** A **4** D **5** A
6 A **7** B **8** A **9** C **10** C
11 C **12** C **13** B **14** D **15** A
16 B **17** A **18** C **19** D

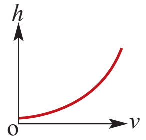
Chapter 13

Exercise 13A

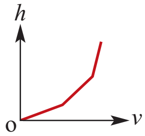
1 a



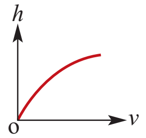
b



c



d

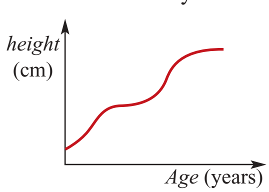


2 For the first 2 minutes, the particle travels a distance of 4 m with its speed increasing. For the next 4 minutes, it travels 4 m in the same direction at constant speed. Then it turns back and returns to its starting point O, travelling at a constant speed and taking 8 minutes to reach O.

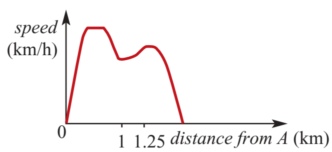
Note: For questions 3–6, there may be more than one correct answer.

3 C is the most likely

4

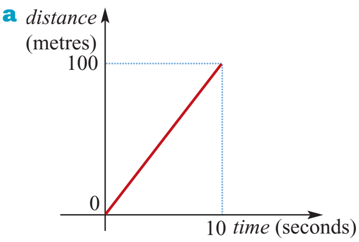


5

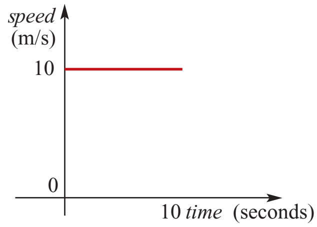


6 C and B are the most likely

7 a



b



8 D

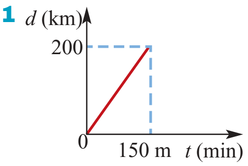
9 C

10 a $(-4, 0)$ **b** $[-7, -4) \cup (0, 3]$

11 a $(-3, 0)$ **b** $[-5, -3) \cup (0, 2]$

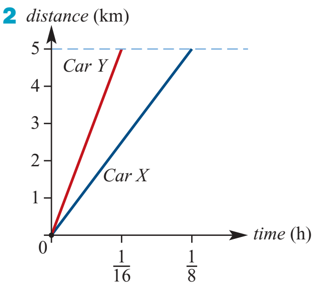
Exercise 13B

1

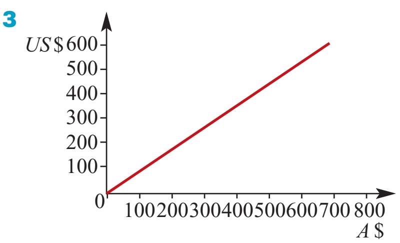


$\frac{4}{3}$ km/min = 80 km/h

2



3

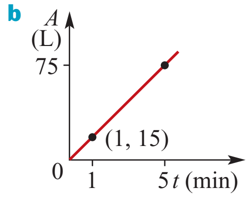


4 a 60 km/h **b** 3 m/s
c 400 m/min = 24 km/h = $6\frac{2}{3}$ m/s
d 35.29 km/h (correct to 2 d.p.)
e 20.44 m/s (correct to 2 d.p.)

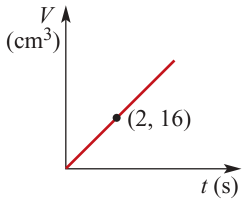
5 a 8 L/min **b** 50 L/min **c** $\frac{135}{13}$ L/min

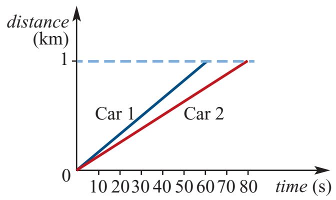
6 a

t	0	0.5	1	1.5	2	3	4	5
A	0	7.5	15	22.5	30	45	60	75

b 

7 $\frac{\$200}{13}$ per hour = \$15.38 per hour

8 

9 

Exercise 13C

1 3.2 m/s

2 a 2 **b** 7 **c** $\frac{-1}{2}$ **d** $\frac{1-\sqrt{5}}{4}$

3 a $\frac{-25}{7}$ **b** $\frac{-18}{7}$ **c** 4 **d** $\frac{4b}{3a}$

4 a 4 m/s **b** 32 m/s

5 a \$2450.09 **b** \$150.03 per year

6 3.125 cm/min

7 C

Exercise 13D

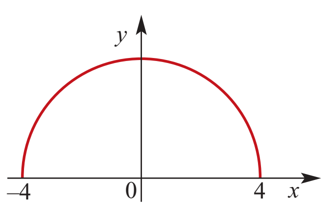
1 7.19

2 a 0.015 **b** $\frac{1}{60} \approx 0.0167$

3 a i 9 **ii** 4.3246 **iii** 2.5893
b 2.30

4 a 25°C at 16:00 **b** ≈ 3°C/h **c** −2.5°C/h

5 −0.5952

6 

a 0 **b** −0.6 **c** −1.1

7 4

8 a 16 m^3/min
b 10 m^3/min

9 a 18 million/min
b 8.3 million/min

10 a 620 m^3/min flowing out
b 4440 m^3/min flowing out
c 284 000 m^3/min flowing out

11 a 7 **b** 9 **c** 2 **d** 35

12 a 28 **b** 12

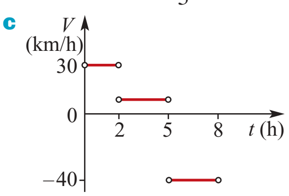
13 a 10 **b** 4

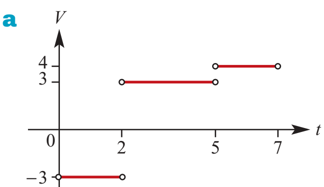
14 a i $\frac{2}{\pi} \approx 0.637$ **ii** $\frac{2\sqrt{2}}{\pi} \approx 0.9003$
iii 0.959 **iv** 0.998
b 1

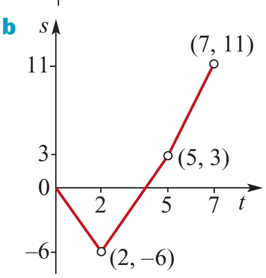
Exercise 13E

1 a 4 m/s **b** 1.12 m/s

2 a i 30 km/h **ii** $\frac{20}{3}$ km/h **iii** −40 km/h

c 

3 a 

b 

4 a $t = 2.5$ **b** $0 \le t < 2.5$ **c** 6 m
d 5 s **e** 3 m/s

5 a $t = 6$ **b** 15 m/s **c** 17.5 m/s
d 20 m/s **e** −10 m/s **f** −20 m/s

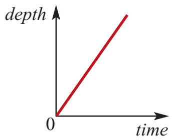
6 a 11 m/s **b** 15 m **c** 1 s
d 2.8 s **e** 15 m/s

7 a $t = 2$, $t = 3$ and $t = 8$
b $0 < t < 2.5$ and $t > 6$
c $t = 2.5$ and $t = 6$

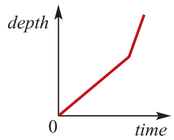
Chapter 13 review

Short-response questions

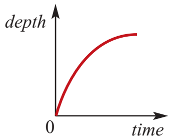
1 a



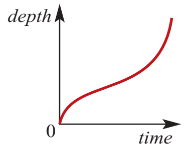
b



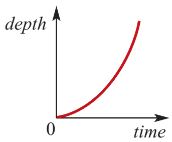
c



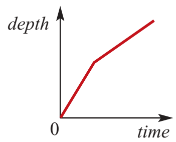
d



e

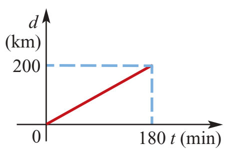


f



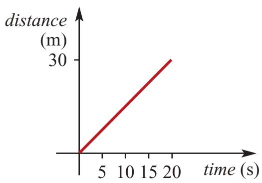
2 36 cm^2/cm

3 a

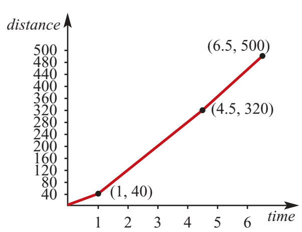


$$\text{Speed} = \frac{200}{3}\text{ km/h} = \frac{10}{9}\text{ km/min}$$

b



4

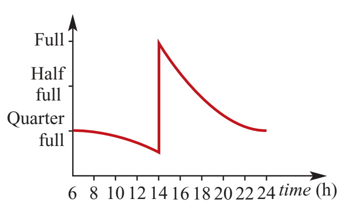


5 10 m/s

6 a 1 **b** 13

7 a 2
b Positive: $-1 < x < 1$
Negative: $x < -1$ and $x > 1$

8



9 a m **b** cm **c** $-m$

10 a $\approx 3\frac{1}{3}$ kg/year **b** ≈ 4.4 kg/year
c $(0, 5) \cup (10, 12)$ **d** $(2, 7) \cup (11, 17\frac{1}{2})$

Technology-active

11 a 1.58 **b** 0.74 **c** 0.49

12 a 1.05 **b** 1.27 **c** 2.05

13 a -2 m/s **b** -12.26 m/s **c** -14 m/s

14 a Yes, the relation is linear
b 0.05 ohm/°C

15 a $b + a \quad (a \neq b)$ **b** 3 **c** 4.01

16 a i 9.8 m/s **ii** 29.4 m/s
b i $4.9(8h - h^2)$
ii $4.9(8 - h)$
iii 38.22 m/s, 38.71 m/s, 38.955 m/s, 39.151 m/s, 39.1951 m/s

17 a $2\frac{2}{3}$, $1\frac{3}{5}$; Gradient $= -1\frac{1}{15}$
b 2.1053, 1.9048; Gradient $= -1.003$
c -1.000025 **d** -1.0000003

18 a i 1049.1 **ii** 1164.3 **iii** 1297.7 **iv** 1372.4
b At 2.8 the gradient is 1452.8

19 a 0 **b** 6 **c** -6

20 $v(t) = \begin{cases} 2 & 0 \le t < 3 \\ -\dfrac{3}{2} & 3 < t \le 7 \end{cases}$

21 a $a^2 + ab + b^2$ **b** 7 **c** 12.06 **d** $3b^2$

22 a i 2.5×10^8 **ii** 5×10^8
b 0.006 billion/year
c i 0.004 billion/year **ii** 0.015 billion/year
d 25 years after 2020

23 a B **b** A **c** 15 m **d** 45 s
e 0.98 m/s, 1.724 m/s, 1.136 m/s

Multiple-choice questions

1 B	**2** B	**3** D	**4** B	**5** D
6 D	**7** C	**8** C	**9** C	**10** A
11 D	**12** B	**13** C	**14** D	**15** B

Chapter 14

Exercise 14A

1 a $-2 - h$ **b** -2

2 a $5 + h$ **b** 5

3 $2x - 2$ **4** 32

5 2000 m/s **6** 7 per day

7 a $10x^2$ **b** 20 **c** 1
d $3x^2 + 1$ **e** $30x^2 + 1$ **f** 5

8 a $2x + 2$ **b** 13 **c** $3x^2 + 4x$

9 a $5 + 3h$ **b** 5.3 **c** 5

10 a $-\dfrac{1}{2 + h}$ **b** -0.48 **c** $-\dfrac{1}{2}$

11 a $6 + h$ **b** 6.1 **c** 6

12 a $6x$ **b** 4 **c** 0
d $6x + 4$ **e** $6x^2$ **f** $8x - 5$
g $-2 + 2x$ **h** $2 - 3x^2$ **i** $2 - 6x$

13 a i $f'(2) \approx 4 + h$ **ii** $f'(2) \approx 4$
b i $f'(2) \approx 12 + 6h + h^2$ **ii** $f'(2) \approx 12 + h^2$
c i $f'(2) \approx 14 + 6h + h^2$ **ii** $f'(2) \approx 14 + h^2$
In each case, the approximation given by **ii** is better than the approximation given by **i**

14 $4x^3$

Exercise 14B

1 a $2x+4$ **b** 2 **c** $3x^2-1$ **d** $x-3$
e $15x^2+6x$ **f** $-3x^2+4x$
g $\dfrac{9x^2}{2}-8x+1$ **h** $\dfrac{10x}{3}-4$
i $-2-6x+20x^3$

2 a $12x^{11}$ **b** $21x^6$ **c** 5 **d** 5
e 0 **f** $10x-3$ **g** $50x^4+12x^3$
h $8x^3-x^2-\frac{1}{2}x$ **i** $-5+8x$

3 a 6 **b** 20 **c** 5 **d** 10 **e** 0
f 7 **g** 31 **h** 7 **i** -34

4 a 60 **b** -16 **c** 57 **d** 168

5 a 7 **b** 2 **c** -16 **d** 11

6 a $2t$ **b** $-15t^2+1$ **c** $2x^3-2x$

7 a -1 **b** 0 **c** $12x^2-3$
d x^2-1 **e** $2x+3$ **f** $18x^2-8$
g $8x-4$ **h** -1 **i** $15x^2+3x$

8 a $2(x+4)$ **b** $48t^2+16t-7$ **c** $2x$

9 a i 3 **ii** $3a^2$ **b** $3x^2$

10 a $\dfrac{dy}{dx}=3(x-1)^2\geq 0$ for all x;
therefore gradient of graph ≥ 0 for all x
b $\dfrac{dy}{dx}=1$ for $x\neq 0$
c $18x+6$

11 a 1, Gradient $=2$ **b** 1, Gradient $=1$
c 3, Gradient $=-4$ **d** -5, Gradient $=4$
e 28, Gradient $=-36$ **f** 9, Gradient $=-24$

12 a i $4x-1$, 3, $(\frac{1}{2},0)$
ii $\dfrac{1}{2}+\dfrac{2}{3}x$, $\dfrac{7}{6}$, $\left(\dfrac{3}{4},\dfrac{25}{16}\right)$
iii $3x^2+1$, 4, $(0,0)$
iv $4x^3-31$, -27, $(2,-46)$
b Coordinates of the point where gradient is 1

13 a $6t-4$ **b** $-2x+3x^2$ **c** $-4z-4z^3$
d $6y-3y^2$ **e** $6x^2-8x$ **f** $19.6t-2$

14 a $(4,16)$ **b** $(2,8)$, $(-2,-8)$ **c** $(0,0)$
d $\left(\dfrac{3}{2},-\dfrac{5}{4}\right)$

Exercise 14C

1 a $-\dfrac{1}{(x-3)^2}$ **b** $-\dfrac{1}{(x+2)^2}$

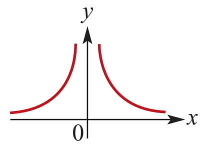
2 a $-\dfrac{2}{x^3}$ **b** $-\dfrac{4}{x^5}$

3 a $-6x^{-3}-5x^{-2}$ **b** $-6x^{-3}+10x$
c $-15x^{-4}-8x^{-3}$ **d** $6x-\dfrac{20}{3}x^{-5}$
e $-12x^{-3}+3$ **f** $3-2x^{-2}$

4 a $-2z^{-2}-8z^{-3}$, $z\neq 0$ **b** $-9z^{-4}-2z^{-3}$, $z\neq 0$
c $\dfrac{1}{2}$, $z\neq 0$ **d** $18z+4-18z^{-4}$, $z\neq 0$
e $2z^{-3}$, $z\neq 0$ **f** $-\dfrac{3}{5}$, $z\neq 0$

5 a $f'(x)=12x^3+18x^{-4}-x^{-2}$
b $f'(x)=20x^3-8x^{-3}-x^{-2}$

6



a Gradient of $PQ=\dfrac{-2-h}{(1+h)^2}$ **b** -2

7 a $11\frac{3}{4}$ **b** $\dfrac{1}{8}$ **c** -1 **d** 5

8 a $-\dfrac{1}{2}$ **b** $\dfrac{1}{2}$

9 $f'(x)=-\dfrac{1}{x^2}<0$ for all $x\neq 0$

Exercise 14D

1 a i $-1<x<1.5$
ii $x<-1$ or $x>1.5$
iii $x=-1$ or $x=1.5$
b i $x<-3$ or $\frac{1}{2}<x<4$
ii $-3<x<\frac{1}{2}$ or $x>4$
iii $x=-3$, $x=\frac{1}{2}$ or $x=4$

2 a $x=1$ **b** $x=1$ **c** $x>1$ **d** $x<1$ **e** $x=\frac{1}{2}$

3 a $x=2+\sqrt{3}$ or $x=2-\sqrt{3}$ **b** $x=2$
c $x<2$ **d** $x>2$ **e** $x=3$

4 a $x=2$ **b** $x=2$ **c** $x\neq 2$
d $x=2-\sqrt{2}$ or $x=2+\sqrt{2}$

5 a $f'(x)=2x-4, x\geq 2$
b $f'(x)=-2x-8, x\leq -4$
c $f'(x)=6x-2, x\geq\dfrac{1}{3}$
d $f'(x)=-4x-4, x\leq -1$
e $f'(x)=-6x-11, x\leq -\dfrac{11}{6}$
f $f'(x)=8x-2, x\geq\dfrac{1}{4}$

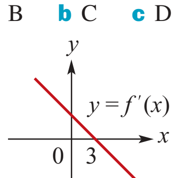
6 Graphs b and d

7 Graphs a, b and e

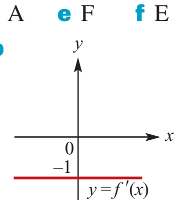
8 a $-4\leq x\leq 2$ **b** $-4\leq x\leq 0$
c $3-\sqrt{5}\leq x\leq 3+\sqrt{5}$
d $2-\sqrt{3}\leq x\leq 2+\sqrt{3}$

9 a B **b** C **c** D **d** A **e** F **f** E

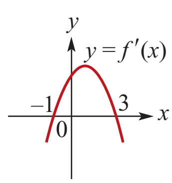
10 a



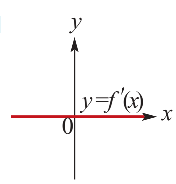
b



c



d



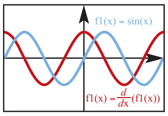
11 a $(3,0)$ **b** $(4,2)$

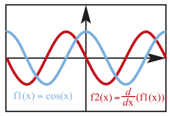
12 a $\left(\frac{1}{2}, -6\frac{1}{4}\right)$ **b** $(0, -6)$

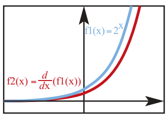
13 a $\left(-\frac{\sqrt{3}}{6}, \frac{23}{12}\right)$ **b** $\left(\frac{1}{2}, \frac{7}{4}\right)$

14 a $(2, -14), (-2, 22)$
b $(\sqrt{5}, 4 - 8\sqrt{5}), (-\sqrt{5}, 4 + 8\sqrt{5})$

15 $(1, 5), (3, 7)$

16 a


b


c


17 a i $66.80°$ **ii** $42.51°$
b $(0.5352, 0.2420)$
c No

18 a $(0.6)t^2$ **b** 0.6 m/s, 5.4 m/s, 15 m/s

19 a $a = 2, b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

20 a $a = 3, b = 2$ **b** $\left(3^{-\frac{1}{3}}, 3^{\frac{4}{3}}\right)$

21 a Height = 450 000 m; Speed = 6000 m/s
b $t = 25$ s

Exercise 14E

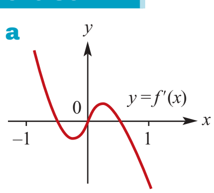
1 a 15 **b** 1 **c** $-3\frac{1}{2}$ **d** $-2\frac{1}{2}$
e 0 **f** 4 **g** 2 **h** $2\sqrt{3}$
i -2 **j** 12 **k** $\frac{11}{9}$ **l** $\frac{1}{4}$

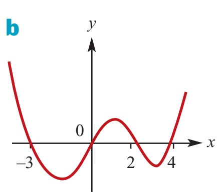
2 a 3, 4 **b** 7

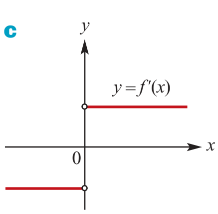
3 a Discontinuity at 0, as $f(0) = 0$, $\lim_{x \to 0^+} f(x) = 0$ but $\lim_{x \to 0^-} f(x) = 2$
b Discontinuity at 1, as $f(1) = 3$, $\lim_{x \to 1^+} f(x) = 3$ but $\lim_{x \to 1^-} f(x) = -1$
c Discontinuity at 0, as $f(0) = 1$, $\lim_{x \to 0^+} f(x) = 1$ but $\lim_{x \to 0^-} f(x) = 0$

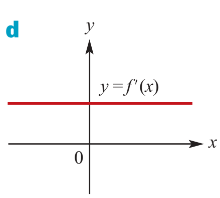
4 $x = 1$

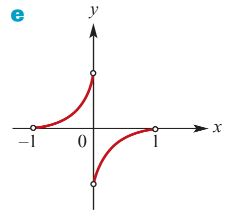
Exercise 14F

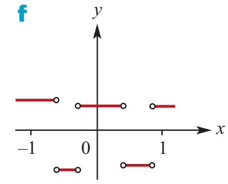
1 a


b


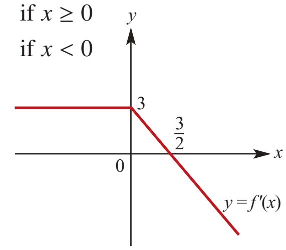
c


d


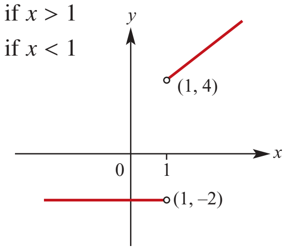
e


f


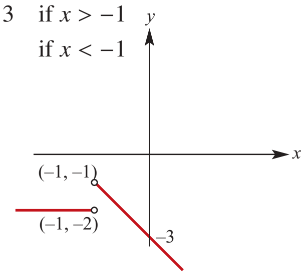
2 $f'(x) = \begin{cases} -2x + 3 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$



3 $f'(x) = \begin{cases} 2x + 2 & \text{if } x > 1 \\ -2 & \text{if } x < 1 \end{cases}$



4 $f'(x) = \begin{cases} -2x - 3 & \text{if } x > -1 \\ -2 & \text{if } x < -1 \end{cases}$



Chapter 14 review

Short-response questions

Technology-free

1 a 3 **b** $-2x$ **c** $2x + 5$
d $3x^2 + 1$ **e** $2x + 2$ **f** $6x - 1$

2 a $6x - 2$ **b** 0 **c** $4 - 4x$
d $4(20x - 1)$ **e** $6x + 1$ **f** $-6x - 1$
g $\frac{4x + 7}{4}$ **h** $\frac{4x - 1}{3}$ **i** x

3 a 1; 2 **b** 3; −4 **c** −5; 4 **d** 28; −36

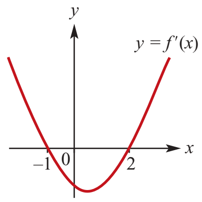
4 a $\left(\frac{3}{2}, -\frac{5}{4}\right)$ **b** $(2, -12)$
c $\left(-\frac{1}{3}, \frac{4}{27}\right), (1, 0)$ **d** $(-1, 8), (1, 6)$
e $(0, 1), \left(\frac{3}{2}, -\frac{11}{16}\right)$ **f** $(3, 0), (1, 4)$

5 a $x = \frac{1}{2}$ **b** $x = \frac{1}{2}$ **c** $x > \frac{1}{2}$ **d** $x < \frac{1}{2}$
e $x \in \mathbb{R} \setminus \left\{\frac{1}{2}\right\}$ **f** $x = \frac{5}{8}$

6 a $-4x^{-5}$ **b** $-6x^{-4}$ **c** $\frac{2}{3x^3}$ **d** $\frac{4}{x^5}$
e $-\frac{15}{x^6}$ **f** $-\frac{2}{x^3} - \frac{1}{x^2} = -\frac{2+x}{x^3}$
g $-\frac{2}{x^2}$ **h** $10x + \frac{2}{x^2}$

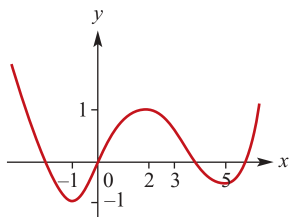
7 a $a = 2, b = -1$ **b** $\left(\frac{1}{4}, -\frac{1}{8}\right)$

8



9 a $-1 < x < 4$
b $x < -1$ or $x > 4$
c $x = -1$ or $x = 4$

10



Technology-active

11 a 18.355 **b** $6x^2 - 6x + 2, 17.84$

12 $x = \frac{4+\sqrt{13}}{3}$ or $x = \frac{4-\sqrt{13}}{3}$
$x = 0.131$ or $x = 2.535$

13 $a = -11, b = 16$

14 a $t = \sqrt[3]{250}$, 11.9 cm/s **b** 3.97 cm/s

15 $a = \frac{\sqrt{6}}{2}, c = -\frac{\sqrt{6}}{2}, b \approx 4.775, d \approx 7.225$

16 $a = -\frac{5}{2}, b = 4, c = \frac{1}{2}$

17 $a = -3.732, b = 5.732$

18 $y = \frac{7}{36}x^3 + \frac{1}{36}x^2 - \frac{20}{9}x$

19 a i 71.57° **ii** 89.58° **b** 2 km

20 a 0.12, −0.15
b $x = 2, y = 2.16$; Height 2.16 km

21 a At $x = 0$, gradient is −2; at $x = 2$, gradient is 2. Angles of inclination to the positive direction of the x-axis are supplementary.

Multiple-choice questions

1 D **2** B **3** A **4** B **5** C
6 C **7** A **8** D **9** A **10** D
11 C **12** D **13** C

Chapter 15

Exercise 15A

1 a $y = 4x - 4, \quad 4y + x = 18$
b $y = 12x - 15, \quad 12y + x = 110$
c $y = -x + 4, \quad y = x$
d $y = 6x + 2, \quad 6y + x = 49$

2 $y = 2x - 10$

3 $y = 2x - 1, \quad y = 2x - \frac{8}{3}$;
Both have gradient 2; Distance apart $= \frac{\sqrt{5}}{3}$

4 $y = 3x + 2, \quad y = 3x + 6$

5 a Tangents both have gradient 2 **b** $(0, -3)$

6 $(3, 12), (1, 4)$

7 a $y = 10x - 16$ **b** $(-4, -56)$

8 a $y = 5x - 1$ **b** $(2, 4), (4, -8)$

Exercise 15B

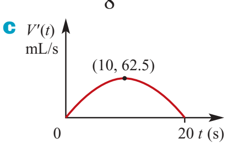
1 a $36; \frac{36}{1} = 36$ **b** $48h - 12h^2$ **c** 48

2 a $1200t - 200t^2$ **b** \$1800 per month
c At $t = 0$ and $t = 6$

3 a $30 - 4P$
b 10, −10
c For $P < 7.5$ revenue increases as P increases

4 a 50 people per year **b** 0 people per year
c Decreasing by 50 people per year

5 a i 0 mL **ii** $833\frac{1}{3}$ mL
b $V'(t) = \frac{5}{8}(20t - t^2)$
c



6 a 0.6 km² **b** 0.7 km²/h

Exercise 15C

1 a $(3, -6)$ **b** $(3, 2)$ **c** $(2, 2)$ **d** $(4, 48)$
e $(0, 0), \ (2, -8)$ **f** $(0, -10), \ (2, 6)$

2 $a = 2, b = -8, c = -1$

3 $a = -\frac{1}{2}, b = 1, c = 1\frac{1}{2}$

4 a $a = 2, b = -5$ **b** $\left(\frac{5}{4}, -\frac{25}{8}\right)$

5 $a = -8$

6 $a = 6$

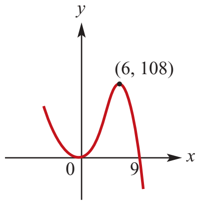
7 a $(2.5, -12.25)$ **b** $\left(\frac{7}{48}, -\frac{625}{96}\right)$
c $(0, 27), \ (3, 0)$ **d** $(-2, 48), \ (4, -60)$
e $(-3, 4), \ (-1, 0)$ **f** $(-1.5, 0.5)$

8 $a = -1, b = 2$

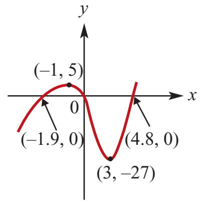
9 $a = -\frac{2}{9}, b = \frac{3}{2}, c = -3, d = 7\frac{1}{2}$

Exercise 15D

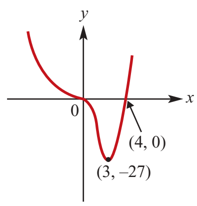
1 a min $(0, 0)$
max $(6, 108)$



b min $(3, -27)$
max $(-1, 5)$



c inflection $(0, 0)$
min $(3, -27)$



2 a $(0, 0)$ max; $\left(\frac{8}{3}, -\frac{256}{27}\right)$ min

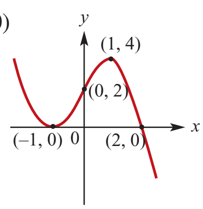
b $(0, 0)$ min; $(2, 4)$ max **c** $(0, 0)$ min

d $\left(\frac{10}{3}, \frac{-200\,000}{729}\right)$ min; $(0, 0)$ inflection

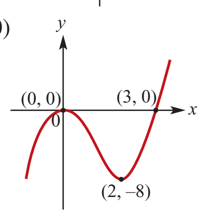
e $(3, -7)$ min; $\left(\frac{1}{3}, \frac{67}{27}\right)$ max

f $(6, -36)$ min; $\left(\frac{4}{3}, \frac{400}{27}\right)$ max

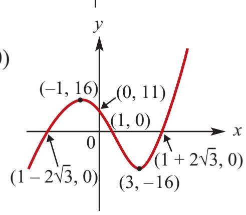
3 a intercepts $(2, 0)$, $(-1, 0)$
max at $(1, 4)$
min at $(-1, 0)$



b intercepts $(3, 0)$, $(0, 0)$
min at $(2, -8)$
max at $(0, 0)$

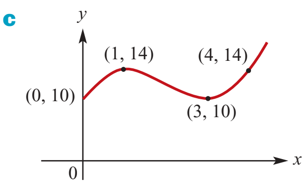


c intercepts $(0, 11)$,
$(1 \pm 2\sqrt{3}, 0)$, $(1, 0)$
min at $(3, -16)$
max at $(-1, 16)$



4 a Local maximum
b Stationary point of inflection

5 a $x < 1$ or $x > 3$
b $(1, 14)$ max; $(3, 10)$ min
c

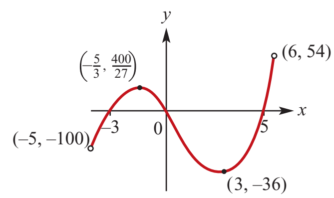


6 $-2 < x < 2$

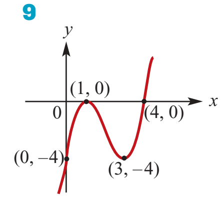
7 a $-1 < x < 1$ **b** $x < -1$ or $x > 1$

8 a $x = -\frac{5}{3}$, $x = 3$

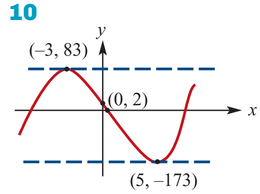
b max at $\left(-\frac{5}{3}, \frac{400}{27}\right)$, min at $(3, -36)$
intercepts $(5, 0)$, $(0, 0)$, $(-3, 0)$



9

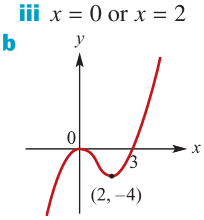


10

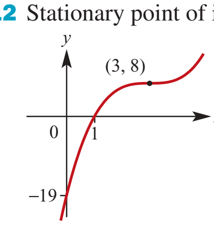


11 a i $0 < x < 2$ **ii** $x < 0$ or $x > 2$
iii $x = 0$ or $x = 2$

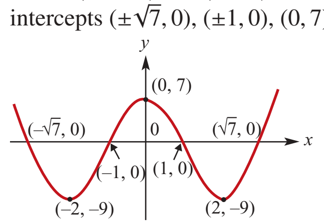
b



12 Stationary point of inflection at $(3, 8)$



13 min at $(-2, -9)$ and $(2, -9)$; max at $(0, 7)$
intercepts $(\pm\sqrt{7}, 0)$, $(\pm 1, 0)$, $(0, 7)$



Exercise 15E

1 2500 cm^2

2 25

3 2

4 a $V(x) = (6 - 2x)^2 x$
b $V_{max} = 16$ m^3 when $x = 1$

5 a i 0.9375 m **ii** 2.5 m **iii** 2.8125 m
b $x = \dfrac{40}{3}, y = \dfrac{80}{27}$
c i $x = 11.937$, $x = 1.396$ **ii** $x = 14.484$

6 b $V = \dfrac{75x - x^3}{2}$ **c** 125 cm^3 **d** 118 cm^3

7 256π

8 $x = \dfrac{5}{3}(9 - \sqrt{21})$

9 Absolute max = 2; Absolute min = −30

10 Absolute max = 6; Absolute min = −9

11 Absolute max = 32; Absolute min = −8

12 Absolute max = 1050; Absolute min = −8

13 b $\dfrac{dV}{dx} = 30x - 36x^2$ **c** $\dfrac{125}{36}$ **d** $\dfrac{432}{125}$
e $\dfrac{125}{36}$ when $x = \dfrac{5}{6}$

14 a $15 \le y \le 18$ **b** Max 75, min 36

15 a $\dfrac{125\,000}{27}$ **b** 3000 **c** $\dfrac{125\,000}{27}$

16 b $\dfrac{dA}{dx} = \dfrac{1}{4}(x - 5)$ **c** $x = 5$ **d** $\dfrac{25}{8}$ m^2

Exercise 15F

1 a $x = 11$ **b** $x = -16$

2 a −12 cm/s **b** $t = 6$, $x = -25$
c −9 cm/s **d** 9 cm/s

3 a −3 cm/s **b** $2\sqrt{3}$ s

4 a $t = 2$ **b** $0 \le t < 2$ **c** 8 m
d 4 s **e** $p = -2$, $q = 8$, $r = 0$
f −4 m/s

5 a $x = 5$ cm, $v = 0$ cm/s, $a = -12$ cm/s^2
b $t = 0, x = 5, a = -12$; $t = 1, x = 3, a = 12$

6 a 2 m/s^2 **b** 50 m/s^2

7 a $(15 - 9.8t)$ m/s **b** −9.8 m/s^2

8 a 3.5 s **b** 2 m/s^2 **c** 14.5 m
d When $t = 2.5$ s; the particle is 1.25 m to the left of O

9 a 0 s, 1 s, 2 s
b 2 m/s, −1 m/s, 2 m/s; −6 m/s^2, 0 m/s^2, 6 m/s^2
c 0 m/s

10 a 12 cm to the right of O
b 2 cm to the right of O
c Moving to the left at 7 cm/s
d $t = 3.5$ s; particle is 0.25 cm to the left of O
e −2 cm/s **f** 2.9 cm/s

11 a 18 m/s^2, 54 m/s^2, 114 m/s^2 **b** 58 m/s^2

12 a 3 cm to the left of O, moving to the right at 24 cm/s
b $v = 3t^2 - 22t + 24$
c After $\frac{4}{3}$ s and 6 s
d $11\frac{22}{27}$ cm to the right of O and 39 cm to the left of O
e $4\frac{2}{3}$ s
f $a = 6t - 22$
g When $t = \frac{11}{3}$ s and the particle is $13\frac{16}{27}$ cm left of O moving to the left at $16\frac{1}{3}$ cm/s

13 a When $t = \frac{2}{3}$ s and $a = -2$ cm/s^2, and when $t = 1$ and $a = 2$ cm/s^2
b When $t = \frac{5}{6}$ and the particle is moving to the left at $\frac{1}{6}$ cm/s

14 When $t = 2$ s, $v = 6$ cm/s, $a = -14$ cm/s^2
When $t = 3$ s, $v = -5$ cm/s, $a = -8$ cm/s^2
When $t = 8$ s, $v = 30$ cm/s, $a = 22$ cm/s^2

15 a $t = 4$ s and $t = -1$ s **b** $t = \frac{3}{2}$ s

Chapter 15 review

Short-response questions

Technology-free

1 a $\dfrac{dy}{dx} = 4 - 2x$ **b** 2 **c** $y = 2x + 1$

2 a Tangent: $y = -3x - 4$, Normal: $y = \dfrac{x}{3} - \dfrac{32}{3}$
b Tangent: $y = -x$, Normal: $y = x - 2$
c Tangent: $y = -3x - 3$, Normal: $y = \dfrac{x}{3} + \dfrac{1}{3}$
d Tangent: $y = 13x + 6$,
Normal: $y = -\dfrac{x}{13} - \dfrac{92}{13}$

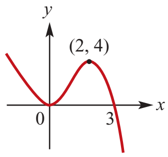
3 a C **b** A **c** B

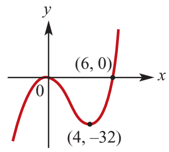
4 a $3x^2 - 8x$ **b** −4 **c** $y = -4x$ **d** (0, 0)

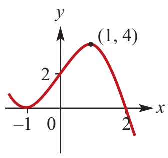
5 a $3x^2 - 12$; $x = \pm 2$
b Local minimum when $x = 2$
Local maximum when $x = -2$
c $x = 2, y = -14$; $x = -2, y = 18$

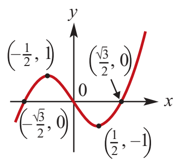
6 a Stationary point of inflection at $x = 0$
b Maximum at $x = 0$
c Min at $x = 3$, max at $x = 2$
d Min at $x = 2$, max at $x = -2$
e Max at $x = 2$, min at $x = -2$
f Max at $x = 3$, min at $x = 1$
g Max at $x = 4$, min at $x = -3$
h Max at $x = 3$, min at $x = -5$

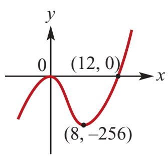
7 a $\left(-\dfrac{2}{3}, -\dfrac{16}{9}\right)$ minimum, $\left(\dfrac{2}{3}, \dfrac{16}{9}\right)$ minimum
b (−1, 0) maximum, (2, −27) minimum
c $\left(\dfrac{2}{3}, \dfrac{100}{27}\right)$ maximum, (3, −9) minimum

8 a


b


c


d


e


9 $y = 2x - 8, y = 10x - 32$

10 a 20 m **b** 6 s **c** 40 m/s

11 a 24 m/s, -18 m/s^2 **b** 2 seconds, 4 seconds
c 24 m **d** 5.6 m/s

12 a $t = 1, t = 3$ **b** -1 m/s **c** 8 m/s^2
d $\dfrac{275}{27}$ m

13 a $y = 5 - x$ **b** $P = x(5 - x)$
c Max value is 6.25 when $x = 2.5$ and $y = 2.5$

14 Maximum area is 625 m^2 for 25 m × 25 m

15 $x = 12$

16 72

Technology-active

17 4.45

18 $x = 1$ or $x = 5$

19 $a = \dfrac{5}{3}, b = \dfrac{8}{3}$

20 $x = \dfrac{1}{4}, x = \dfrac{9}{4}$

21 $a = 2$

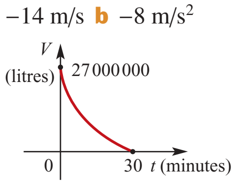
22 $\dfrac{1616}{5}$

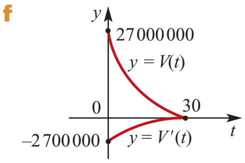
23 a 11 **b** $x = -0.8$ or $x = 2.1$

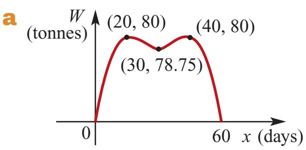
24 a local minimum (1.87, –9.13);
local maximum (–0.54, 4.76)
b no stationary points, gradient always negative
c local minimum (1, –5);
local maximum (–0.5, 1.75)

25 a $a = \dfrac{52}{9}, b = \dfrac{23}{3}$
b local maximum, (0.85, 2.96)

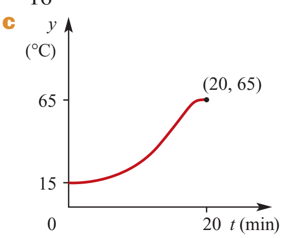
26 a -14 m/s **b** -8 m/s^2

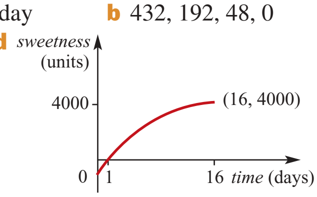
27 a


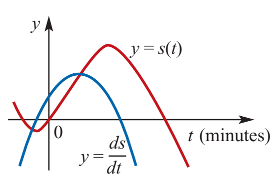
b i 17.4 minutes **ii** 2.9 minutes
c $\dfrac{dV}{dt} = -3000(30 - t)^2$
d 30 minutes
e 28.37 minutes
f


28 a


b From 5.71 days until 54.29 days
c When $x = 20$ and when $x = 40$, $\dfrac{dW}{dx} = 0$;
When $x = 60$, $\dfrac{dW}{dx} = -12$ tonnes per day
d When $x = 30$, $W = 78.75$

29 a 15°C
b 0°C/min, $\dfrac{45}{16}$ °C/min, $\dfrac{15}{4}$ °C/min, $\dfrac{45}{16}$ °C/min, 0°C/min
c


30 a 768 units/day **b** 432, 192, 48, 0
c $t = 16$ **d**


31 a


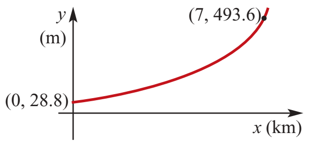
b 11:59 a.m., 12:03 p.m.
c $\dfrac{5}{27}$ km, 1 km
d $\dfrac{8}{27}$ km/min $= 17\frac{7}{9}$ km/h
e $\dfrac{1}{3}$ km/min = 20 km/h

32 a $0 \le t \le 12$
b i 27 L/h **ii** 192 L/h

33 a 28.8 m

b 374.4

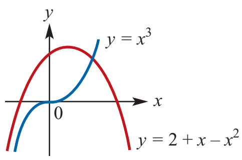
c



d Path gets too steep after 7 km

e **i** 0.0384 **ii** 0.0504 **iii** 0.1336

34 **a**



b For $x \le 0$, the minimum vertical distance occurs when $x = -1$; Min distance = 1 unit

35 8 mm for maximum and $\frac{4}{3}$ mm for minimum

36 **a** $y = 10 - 2x$ **b** $A = x^2(10 - 2x)$

c Max value is $\dfrac{1000}{27}$ when $x = \dfrac{10}{3}$ and $y = \dfrac{10}{3}$

37 $20\sqrt{10}$

38 **a** $y = 8 - x$ **b** $s = x^2 + (8 - x)^2$ **c** 32

39 $\dfrac{4}{3}, \dfrac{8}{3}$

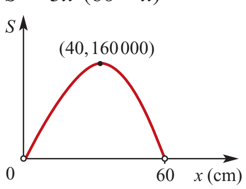
40 32

41 Maximum value of P is 2500

42 Maximum area is 2 km^2 for 2 km × 1 km

43 **a** $y = 60 - x$ **b** $S = 5x^2(60 - x)$

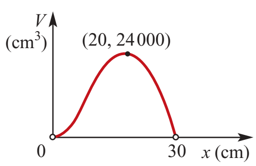
c $0 < x < 60$ **d**



e $x = 40, y = 20$ **f** 74 005

44 12°C

45 **b** $0 < x < 30$ **c**



d 20 cm, 40 cm, 30 cm

e $x = 14.82$ or $x = 24.0$

46 **b** Maximum when $x = 3$ and $y = 18$

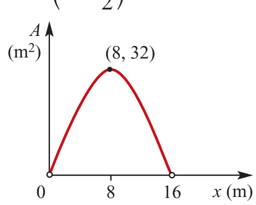
47 **a** Use 44 cm for circle and 56 cm for square

b Use all the wire for the circle

48 Length 7.2 metres, width 4.5 metres

49 **a** $A = xy$ **b** $A = \left(8 - \dfrac{x}{2}\right)x$

c $0 < x < 16$ **d**

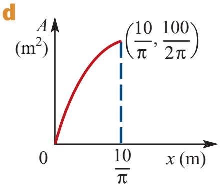


e 32 m^2

50 $h = 1188, a = 937$

51 **a** $y = 10 - \pi x$ **b** $0 \le x \le \dfrac{10}{\pi}$

c $A = \dfrac{x}{2}(20 - \pi x)$ **d**

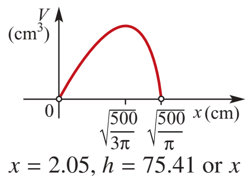


e Maximum at $x = \dfrac{10}{\pi}$ **f** A semicircle

52 **a** $h = \dfrac{500}{\pi x} - x$ **b** $V = 500x - \pi x^3$

c $\dfrac{dV}{dx} = 500 - 3\pi x^2$ **d** $x = 10\sqrt{\dfrac{5}{3\pi}} \approx 7.28$

e



f 2427.89 cm^3

g $x = 2.05, h = 75.41$ or $x = 11.46, h = 2.42$

53 **a** $r = 4.3$ cm, $h = 8.6$ cm

b $r = 4.3$ cm, $h = 8.6$ cm

Multiple-choice questions

1 B **2** D **3** C **4** A **5** C **6** D
7 D **8** A **9** A **10** C **11** A **12** A
13 B **14** C **15** B **16** D **17** D **18** A

Chapter 16

Exercise 16A

1 **a** $14(2x - 1)^6$ **b** $60(5x + 2)^{11}$

c $-4(1 - x)^3$ **d** $30(x - 1)^{29}$

e $100(x^4 - 2x^9)(x^5 - x^{10})^{19}$

f $4(1 - 3x^2 - 5x^4)(x - x^3 - x^5)^3$ **g** $8(x + 1)^7$

h $-4(x + 1)(x^2 + 2x)^{-3}$

i $-6(x + x^{-2})(x^2 - 2x^{-1})^{-4}$

2 **a** $24x^2(2x^3 + 1)^3$ **b** 648

3 **a** $6(2x - 3)^2$ **b** $28x(2x^2 + 7)^6$

c $-120x(3x^2 + 2)^4$ **d** $4(x - 3x^2)^{-3}(6x - 1)$

e $-4(x^2 - x + 3)^3(2x - 1)$

f $-45x^2(1 - x^3)^{-6}$

4 **a** $-\dfrac{1}{16}$ **b** $-\dfrac{3}{256}$

5 **a** $-\dfrac{2}{9}$ **b** $\left(-3, -\dfrac{1}{3}\right), \left(0, \dfrac{1}{3}\right)$

6 **a** $12x(x^2 - 1)^2$

b **i** 0 **ii** 216 **iii** 0

c $y = -2$ **d** $x = 0$

e $(-1, 0), (0, -2), (1, 0)$

7 **a** $4ax(x^2 - 4)$

b **i** $-12a$ **ii** $x = -2, x = 0, x = 2$

iii $y = -12ax + 21a$

Exercise 16B

1 a $\frac{1}{3}x^{-\frac{2}{3}}$ **b** $\frac{3}{2}x^{\frac{1}{2}},\ x > 0$

c $\frac{5}{2}x^{\frac{3}{2}} - \frac{3}{2}x^{\frac{1}{2}},\ x > 0$ **d** $x^{-\frac{1}{2}} - 5x^{\frac{2}{3}},\ x > 0$

e $-\frac{5}{6}x^{-\frac{11}{6}},\ x > 0$ **f** $-\frac{1}{2}x^{-\frac{3}{2}},\ x > 0$

2 a $x(1 + x^2)^{-\frac{1}{2}}$ **b** $\frac{1}{3}(1 + 2x)(x + x^2)^{-\frac{2}{3}}$

c $-x(1 + x^2)^{-\frac{3}{2}}$ **d** $\frac{1}{3}(1 + x)^{-\frac{2}{3}}$

3 a i $\frac{4}{3}$ **ii** $\frac{4}{3}$ **iii** $\frac{1}{3}$ **iv** $\frac{1}{3}$

4 a $0 < x < 1$ **b** $x > \left(\frac{2}{3}\right)^6$

5 a $-5x^{-\frac{1}{2}}(2 - 5\sqrt{x})$ **b** $3x^{-\frac{1}{2}}(3\sqrt{x} + 2)$

c $-4x^{-3} - \frac{3}{2}x^{-\frac{5}{2}}$ **d** $\frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

e $\frac{15}{2}x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

Exercise 16C

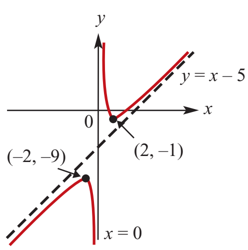
1 a $(\frac{1}{2}, 4), (-\frac{1}{2}, -4)$ **b** $y = \frac{15}{4}x + 1$

2 $\pm\frac{1}{2}$

3 $\frac{1}{2}$

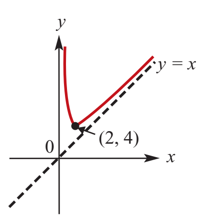
4 a $(4, 0), (1, 0)$ **b** $y = x - 5,\ x = 0$

c $(2, -1)$ min
$(-2, -9)$ max

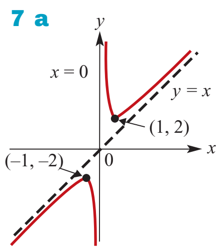


5 3

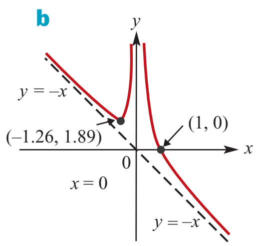
6 4



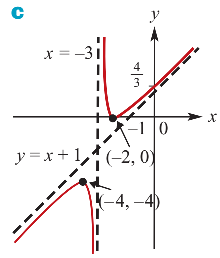
7 a



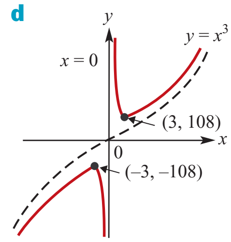
b



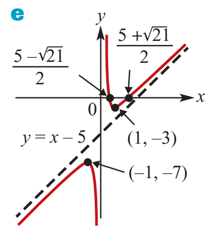
c



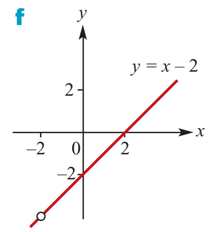
d



e



f



Exercise 16D

1 a $9x^2 + 4x - 3$ **b** $4x^3 + 12x^2 - 2$

c $-9x^2 - 8x + 6$ **d** $-8x^3 + 12x^2 + 6$

2 a $10x + 14$ **b** $-3x^2 + 2x - 7$

c $20x - 6$ **d** $-18x^2 - 6x + 18$

3 a $12x^3 - 6x^2 + 12$ **b** $6x(6x + 5)$

c $-3x^2 + 4x - 5$ **d** $4x^3$

e $5x^2(5x^2 - 3)$ **f** $8x(2x^2 + 1)$

4 a $2(x - 1)(3x + 2)$ **b** $3(2x + 1)(2x + 7)$

c $(7 - 8x)(x - 2)^2$ **d** $5(3x + 1)^{\frac{1}{2}}(3x + 4)$

e $\frac{5x^2 - 8x + 1}{\sqrt{2x - 4}}$

f $x^2(3x^2 + 4x + 3)(3x^2 + 2x + 1)^{-2}$

5 a $27(3x + 1)(1 - x)$ **b** 0

c 32 **d** $y = 32$

6 48 **7** $y = \frac{1}{21}x + 9$ **9** $k = \frac{1}{2}$

Exercise 16E

1 a $\frac{2x(x - 1)}{(2x - 1)^2}$ **b** $\frac{10x(x - 3)}{(2x - 3)^2}$ **c** $-\frac{13}{(5x + 1)^2}$

2 a $\frac{2(4x^2 - 20x + 1)}{(2x - 5)^2}$ **b** $\frac{-x^2 + 2x + 4}{(x - 1)^2}$

c $-\frac{10(x^2 - x - 1)}{(2x - 1)^2}$

3 a $\frac{-x^2 - 3}{4x^2}$ **b** $\frac{2x - 3}{x^3}$ **c** $-\frac{97}{(6x - 5)^2}$

d $-\frac{6x(x - 9)}{(2x - 9)^2}$ **e** $1\ \ (x \neq -\frac{1}{2})$

4 a $\frac{9}{2} - \frac{1}{2x^2}$ **b** $-\frac{2(x + 5)(2x - 5)^2}{3x^3}$

c $\frac{14x}{(1 - x)^3}$ **d** $\frac{12x - 19}{(4x + 1)^3}$

e $-\frac{2(3x - 2)(3x + 8)}{(x + 1)^2}$ **f** $-\frac{15x^2(x + 1)}{(3x + 1)^3}$

5 a $\dfrac{3x^2 - 2x + 9}{(x^2 - 3)^2}$ **b** 17 **c** -5

d $y = 17x - 39$

6 -16

7 $343x - 231y = 310$

9 $k = -8$

10 a $\dfrac{2}{\sqrt{4x + 1}}$

11 a $\dfrac{2x}{\sqrt{2x^2 - 3}}$

b $\dfrac{6x^3 - 18x}{(2x^2 - 3)^{\frac{3}{2}}}$

12 a 0 **b** $\dfrac{71}{8}$

13 $\dfrac{-(x^2 + 8x - 2)}{2\sqrt{x + 4}(x^2 + 2)^{\frac{3}{2}}}$

Chapter 16 review

Short-response questions

Technology-free

1 a $-6x^{-4}$ **b** $\dfrac{2}{3x^3}$ **c** $-\dfrac{15}{x^6}$

d $-\dfrac{2}{x^3} - \dfrac{1}{x^2} = -\dfrac{2 + x}{x^3}$ **e** $-\dfrac{2}{x^2}$

f $10x + \dfrac{2}{x^2}$ **g** $\dfrac{1}{2x^{\frac{1}{2}}}$ **h** $\dfrac{1}{3x^{\frac{2}{3}}}$

i $\dfrac{1}{x^{\frac{3}{2}}}$ **j** $\dfrac{4}{3}x^{\frac{1}{3}}$ **k** $-\dfrac{1}{3x^{\frac{4}{3}}}$

l $-\dfrac{1}{3x^{\frac{4}{3}}} + \dfrac{6}{5x^{\frac{2}{5}}}$

2 a $8x + 12$ **b** $24(3x + 4)^3$ **c** $\dfrac{1}{(3 - 2x)^{\frac{3}{2}}}$

d $3(2x - 1)(x^2 - x + 1)^2$ **e** $\dfrac{-2}{(3 + 2x)^2}$

f $\dfrac{-4}{3(2x - 1)^{\frac{5}{3}}}$ **g** $\dfrac{-3x}{(2 + x^2)^{\frac{3}{2}}}$

h $\dfrac{1}{3}\left(4x + \dfrac{6}{x^3}\right)\left(2x^2 - \dfrac{3}{x^2}\right)^{-\frac{2}{3}}$

3 a $\dfrac{1}{6}$ **b** -2 **c** $-\dfrac{1}{16}$ **d** -2 **e** $\dfrac{1}{6}$ **f** 0

4 a $\left(\dfrac{1}{2}, 2\right)$, $\left(-\dfrac{1}{2}, -2\right)$ **b** $\left(\dfrac{1}{16}, \dfrac{1}{4}\right)$

5 a $20x + 3$ **b** $-36x^2 - 30x + 8$

c $6x^2 - 4x - 10$ **d** $-2x(6x^2 + 7)$

6 a $-\dfrac{8}{(3x - 4)^2}$ **b** $-\dfrac{2x(5x - 4)}{(5x - 2)^2}$

c $\dfrac{37}{(x - 5)^2}$ **d** $\dfrac{4x^2 + 16x + 1}{(x + 2)^2}$

7 a $(2x + 1)(18x - 1)$ **b** $7x(x - 1)^2(5x - 2)$

c $\dfrac{-50(x - 2)(5x + 2)^2}{(5x - 6)^2}$

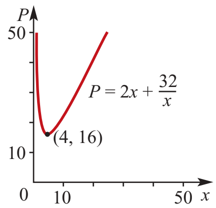
d $\dfrac{x(7x^3 + 3x + 4)}{2\sqrt{x^3 + 1}}$ **e** $-\dfrac{4x}{(8x + 1)^3}$

f $\dfrac{378(3x + 7)}{(6x - 7)^3}$

8 a $y = 26x - 18$ **b** $y = 4x$

9 a $y = \dfrac{16}{x}$ **c** $x = 4$, $P = 16$

d



10 a $OA = \dfrac{120}{x}$ **b** $OX = \dfrac{120}{x} + 7$

c $OZ = x + 5$ **d** $y = 7x + \dfrac{600}{x} + 155$

e $x = \dfrac{10\sqrt{42}}{7} \approx 9.26$ cm

Technology-active

11 a $a = 0.79$

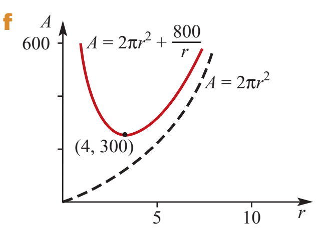
b $a = -1.50$ or $a = 1.50$

12 a $a = -0.09$ or $a = 0.09$

13 a $h = \dfrac{400}{\pi r^2}$ **c** $\dfrac{dA}{dr} = 4\pi r - \dfrac{800}{r^2}$

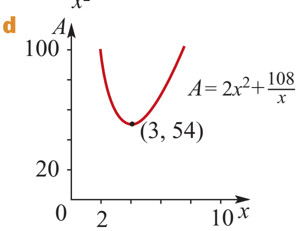
d $r = \left(\dfrac{200}{\pi}\right)^{\frac{1}{3}} \approx 3.99$ **e** $A = 301$ cm^2

f



14 a $h = \dfrac{18}{x^2}$ **c** $x = 3$, $h = 2$

d



15 a $A(-2, 0)$, $B(0, \sqrt{2})$ **b** $\dfrac{1}{2\sqrt{x + 2}}$

c i $\dfrac{1}{2}$ **ii** $2y - x = 3$ **iii** $\dfrac{3\sqrt{5}}{2}$

d $x > -\dfrac{7}{4}$

16 $p = \dfrac{3}{2}$, $q = \dfrac{8}{3}$

17 a $y = \frac{250}{x^2}$ **c** $\frac{dS}{dx} = 24x - \frac{3000}{x^2}$

d $S_{\min} = 900 \text{ cm}^2$

Multiple-choice questions

1 D **2** A **3** D **4** A **5** C
6 A **7** D **8** A **9** B **10** D
11 B **12** A **13** A **14** C **15** B
16 C

Chapter 17

Short-response questions

Technology-free

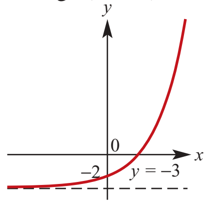
1 a $-24a^{10}$ **b** $\frac{a^3}{2b^2}$ **c** $\frac{3}{4x^5}$ **d** 8

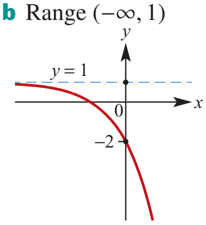
e $\frac{y^{\frac{2}{3}}}{x^{\frac{1}{6}}}$ **f** $\frac{1}{(2x-1)^{\frac{1}{2}}}$

2 a $\frac{25}{9}$ **b** 16 **c** 81 **d** $-\frac{1}{3}$

3 a $2^{6n} \times 3^{3n}$ **b** 12 **c** $\log_{10} 36$ **d** -3

4 a Range $(-3, \infty)$ **b** Range $(-\infty, 1)$



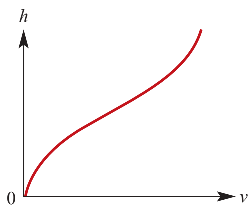


5 a $x = 3$ **b** $x = 0, 2$ **c** $x > 4$ **d** $x = 7$ **e** $x = 1$

6 a $x = \log_2 5$ **b** $x = \frac{1}{3}\log_5(2)$

c $x > \frac{\log_{10} 0.2}{\log_{10} 0.6}$

7



8 a 1 m/s **b** 41 m/s

9 a i -4 **ii** -3 **b** $-2 - h$ **c** -2

10 $x - 1$ **11 a** $6x^2 - 1$ **b** $2x + 1$ **c** 1

12 a 13 **b** 10

13 a $x = 0$ or $x = \frac{1}{2}$ **b** $x = \frac{1}{4}$ **c** $x < \frac{1}{4}$

d $x > \frac{1}{4}$ **e** $x = \frac{11}{4}$

14 Tangent $y = -3x - 1$; Normal $y = \frac{1}{3}x - \frac{13}{3}$

15 Local minimum $\left(\frac{2}{\sqrt{3}}, -\frac{32}{3\sqrt{3}}\right)$

Local maximum $\left(-\frac{2}{\sqrt{3}}, \frac{32}{3\sqrt{3}}\right)$

16 a $-6x^{-4} + x^{-2}$ **b** $\frac{2z - 9}{z^4}$

17 a $\frac{9}{2}x^{\frac{1}{2}}$ **b** $\frac{1}{5}x^{-\frac{4}{5}}$ **c** $\frac{10}{3}x^{-\frac{8}{3}}$ **d** $10x^{\frac{2}{3}}$

e $-\frac{1}{5}x^{-\frac{6}{5}}$ **f** $-\frac{2}{3}x^{-\frac{5}{3}} - 3x^{\frac{1}{2}}$

18 a $6(3x + 5)$ **b** $-8(2x + 7)^3$

c $12x^2(x + 1)(x + 2)^2$ **d** $\frac{2}{3}(5 - 2x)^{-\frac{4}{3}}$

e $\frac{-12}{(5 + 3x)^2}$ **f** $-\frac{2}{3}(x - 1)^{-\frac{5}{3}}$

g $-9x(2 + 3x^2)^{-\frac{3}{2}}$

h $\frac{1}{3}\left(2x^3 - \frac{5}{x}\right)^{-\frac{2}{3}}\left(6x^2 + \frac{5}{x^2}\right)$

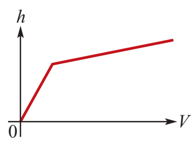
19 a $\frac{1}{27}$ **b** -3 **c** $-\frac{3}{8}$

20 $\left(-\left(\frac{1}{2}\right)^{\frac{1}{3}}, 4^{\frac{1}{3}}\right)$

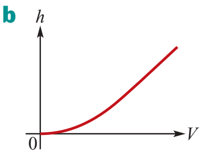
21 a $12x^2 - 14x + 4$ **b** $2(x + 2)(3x + 1)$

c $-\frac{20}{(2x + 5)^2}$ **d** $\frac{20(3x - 1)}{(x + 3)^3}$

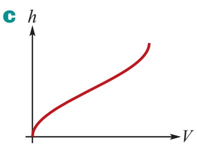
22 a



b



c



23 a 8.33 m/s **b** 30 km/h

24 a $\frac{1}{3}$ **b** $-\frac{1}{10}$

25 a $x - 1$ **b** -1

26 a $-2 - 3h - h^2$ **b** $-3 - h$ **c** -3

27 a $-2x$ **b** $4x - 4$

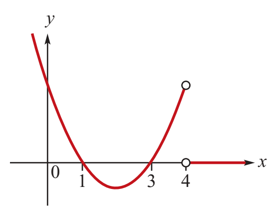
28 a $12x^3 - 1$ **b** $2x + 2$

29 Tangent $y = 5x + 3$; Normal $y = -\frac{1}{5}x - \frac{11}{5}$

30 a 7 m/s, 4 m/s^2 **b** 3 m/s

31 Local max $\left(2, \frac{11}{3}\right)$, local min $\left(3, \frac{7}{2}\right)$

32



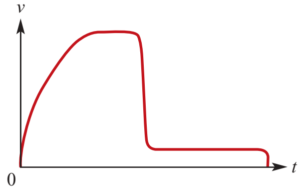
33 a $\frac{6 + x}{x^3}$ **b** $\frac{5 - 4x}{x^6}$ **c** $\frac{1}{2}x^{-\frac{1}{2}} + \frac{2}{3}x^{-\frac{4}{3}}$

34 a $-12(1-2x)^5$ **b** $-\dfrac{18}{(2x-1)^4}$

c $-\dfrac{x}{\sqrt{1-x^2}}$

35 a $6x^2-6x+2$ **b** $\dfrac{-3x^2+2x-3}{(x^2-1)^2}$

36



37 $f'(x)=-3x^2$

38 a 1 **b** -4

39 $(1,1),\ (-1,-3)$

40 a $x=6$ **b** $x=\sqrt{17}$

c $x=\dfrac{1+11\sqrt{6}}{5}$

41 a $x=1$ or $x=2$ **b** $x=-1$

42 a 2.38 km^2 **b** 17.29 hours

44 $k=2$

45 $a=-\dfrac{3}{2},\ b=-\dfrac{9}{2}$ **46** $-1<x<2$

47 $\left(\dfrac{3}{2},-1\right)$ **48** $a=-5,\ b=-10$

49 $\dfrac{25}{6}\text{ m}^2$ **50** $\dfrac{1-x}{x^2\sqrt{2x-1}}$

51 a $t=0$ and $t=2$

b $t=0,\ a=-1\text{ cm/s}^2;\ t=2,\ a=1\text{ cm/s}^2$

c $-\frac{1}{2}$ cm/s

Technology-active

52 $a=-\dfrac{5}{3}, b=-1, c=\dfrac{8}{3}, d=6$

53 a $x=-\dfrac{7}{5}$ or $x=1$ **b** $-\dfrac{7}{5}<x<1$

c $x<-\dfrac{7}{5}$ or $x>1$

54 a i -0.75 **ii** -1.74 **iii** -1.97

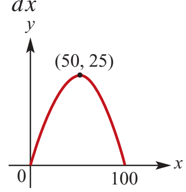
b -2

55 a 12 m/s **b** 8 m/s

56 a -1 m/s **b** $k=4$ **c** -4 m/s

57 a 100 **b** $\dfrac{dy}{dx}=1-0.02x$

c $x=50,\ y=25$ **d**



e i $(25, 18.75)$ **ii** $(75, 18.75)$

58 a $(66\frac{2}{3}, 14\frac{22}{27})$

b i 0.28 **ii** -0.32 **iii** -1

c A gradual rise to the turning point and a descent which becomes increasingly steep (in fact, alarmingly steep)

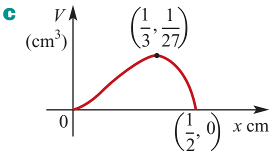
d Smooth out the end of the trip

59 a i 4 billion **ii** 5.944 billion **iii** 7.25 billion

b 2032

60 a $A=4x-6x^2$ **b** $V=x^2-2x^3$

c

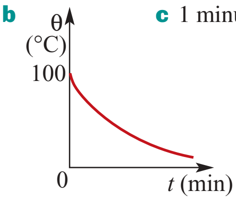


d $\dfrac{1}{3}\text{ cm}\times\dfrac{1}{3}\text{ cm}\times\dfrac{1}{3}\text{ cm}$; Volume $=\dfrac{1}{27}\text{ cm}^3$

61 a

t	0	1	2	3	4	5
θ	100	60	40	30	25	22.5

b



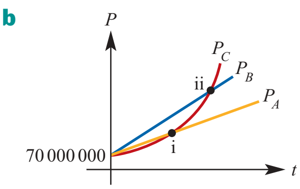
c 1 minute **d** 27.071

62 a $P_A=70\,000\,000+3\,000\,000t$

$P_B=70\,000\,000+5\,000\,000t$

$P_C=70\,000\,000\times1.3^{\frac{t}{10}}$

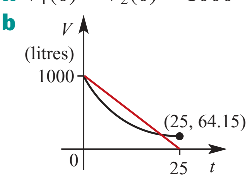
b



c i 35 years **ii** 67 years

63 a $V_1(0)=V_2(0)=1000$

b



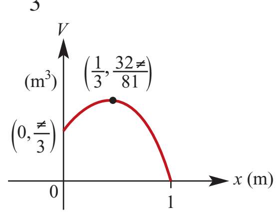
c 64.15 litres **d** $t=0$ and $t=23.00$

64 a $h=\dfrac{150-2x^2}{3x}$ **b** $V=\dfrac{2}{3}(150x-2x^3)$

c $\dfrac{dV}{dx}=2(50-2x^2)$ **d** $0<x<5\sqrt{3}$

e $\dfrac{1000}{3}\text{ m}^3$ when $x=5$

f



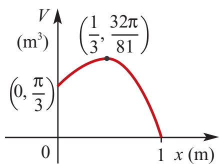
65 a i $r=\sqrt{1-x^2}$ **ii** $h=1+x$

c $0<x<1$

d i $\dfrac{dV}{dx}=\dfrac{\pi}{3}(1-2x-3x^2)$

ii $x=\dfrac{1}{3}$ **iii** $\dfrac{32\pi}{81}\text{ m}^3$

e



66 a 1000 insects

b 1366 insects

c i $t = 40$ **ii** $t = 51.70$

d 63.64

e i $\dfrac{1000 \times 2^{\frac{3}{4}}\left(2^{\frac{h}{20}} - 1\right)}{h}$

ii Consider h decreasing and approaching zero; instantaneous rate of change $= 58.286$ insects/day

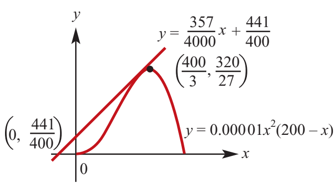
67 a $a = 200$, $k = 0.000\,01$

b i $\dfrac{400}{3}$ **ii** $\dfrac{320}{27}$ **c i** $\dfrac{8379}{800}$ **ii** $\dfrac{357}{4000}$

d i $y = \dfrac{357}{4000}x + \dfrac{441}{400}$ **ii** $\dfrac{441}{400}$

e 0.09975

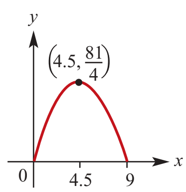
f



68 a $y = ax - x^2$ **b** $0 < x < a$ **c** $\dfrac{a^2}{4}, \dfrac{a}{2}$

d Negative coefficient of x^2 for quadratic function

e i

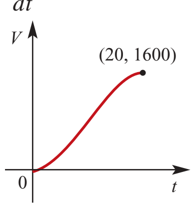


ii $\left(0, \dfrac{81}{4}\right]$

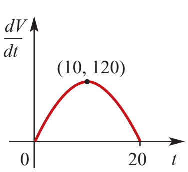
69 a i 0 **ii** 1600 mL

b $\dfrac{dV}{dt} = 0.6(40t - 2t^2)$

c

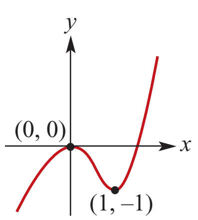


d



70 a $-1 = a + b$ **b** $0 = 3a + 2b$, $a = 2$, $b = -3$

c



71 a i $80 - 2x$ **ii** $h = \dfrac{\sqrt{3}}{2}x$

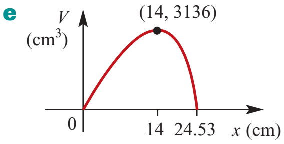
b $A = \dfrac{\sqrt{3}}{4}x(160 - 3x)$ **c** $x = \dfrac{80}{3}$

72 a $y = \dfrac{1400 - 2x^2 - 8x}{4x}$

b $V = -\dfrac{x^3}{2} - 2x^2 + 350x$

c $\dfrac{dV}{dx} = -\dfrac{3}{2}x^2 - 4x + 350$ **d** $x = 14$

e



f Maximum volume is 3136 cm³

g $x = 22.83$ and $y = 1.92$, or $x = 2.94$ and $y = 115.45$

73 a $V = 5000 \times 1.05^n$ **b** \$262.71 per year

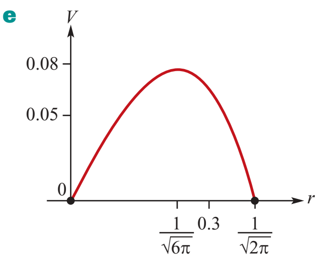
c $n = 8.2$ years **d** \$269 per year

74 b $x = 1$ **c** $a = 0.1$ **d** $(2, 0.2)$

75 b $r = \sqrt{\dfrac{1}{6\pi}}$ **c** 0.077 m³

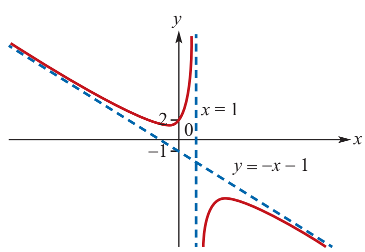
d $0 \le r \le \sqrt{\dfrac{1}{2\pi}}$

e



f i 0.065 m³ **ii** $0 \le h \le 0.23$

76 b



Local min $(1 - \sqrt{3}, 2\sqrt{3} - 2)$;
Local max $(1 + \sqrt{3}, -2\sqrt{3} - 2)$

c i Local min $(1 - \sqrt{3}, 4\sqrt{3} - 4)$;
Local max $(1 + \sqrt{3}, -4\sqrt{3} - 4)$

ii Local min $(4 - \sqrt{3}, 2\sqrt{3} - 2)$;
Local max $(4 + \sqrt{3}, -2\sqrt{3} - 2)$

iii Local min $(-2 - \sqrt{3}, 2\sqrt{3} - 4)$;
Local max $(-2 + \sqrt{3}, -2\sqrt{3} - 4)$

iv Local min $(4 - \sqrt{3}, 2\sqrt{3} + 2)$;
Local max $(4 + \sqrt{3}, -2\sqrt{3} + 2)$

77 a $(1, 4), (3, 0)$ **b** $(0, 0), (1, 4), (5, 20)$

c $x = \dfrac{6 \pm \sqrt{21}}{3}$

d $y = (a-3)(3(a-1)x - 2a^2)$ **e** $\left(\dfrac{1}{2}, \dfrac{25}{8}\right)$

Multiple-choice questions

1 D	**2** B	**3** B	**4** B	**5** C
6 D	**7** A	**8** D	**9** C	**10** B
11 A	**12** A	**13** D	**14** B	**15** B
16 A	**17** B	**18** A	**19** B	**20** A
21 D	**22** B	**23** C	**24** A	**25** C
26 B	**27** A	**28** C	**29** C	**30** A
31 D	**32** A	**33** C	**34** B	**35** D
36 D	**37** A	**38** A	**39** C	**40** C
41 B	**42** B	**43** B	**44** C	**45** D
46 A	**47** C	**48** C	**49** A	**50** D
51 D	**52** C	**53** C	**54** D	**55** D
56 D	**57** A	**58** D	**59** A	**60** C
61 D	**62** A	**63** D	**64** A	**65** C
66 C	**67** C	**68** B	**69** D	

Problem-solving and modelling

See solutions supplement

Chapter 18

Short-response questions

Technology-free

1 $x = 4$ **2** $t = \dfrac{2d - b}{a - 2c}$ **3** $x \geq -\dfrac{3}{2}$

4 a $24x^2 - 4x + 3$ **b** 5

c $-2x + 6$ **d** $3 + 8x$

5 $x = \dfrac{-3 \pm \sqrt{65}}{4}$

6 $f'(-2) = -8$

7
- Dilation of factor 2 from the x-axis
- Dilation of factor $\dfrac{3}{\pi}$ from the y-axis
- Translation of 5 units in the negative direction of the y-axis

8 $a = \dfrac{3}{2},\ b = 5,\ d = \dfrac{7}{2},\ c = -80$

(Other values for c are possible)

9 792

10 $16 + 96x + 216x^2 + 216x^3 + 81x^4$

11 $x = 4$

12 $y = -x + 3$

13 $x \leq \dfrac{37}{5}$ **14** $a = 7.9$

15 a $\left(\dfrac{a+8}{2}, \dfrac{b+14}{2}\right)$ **b** $a = 2, b = 6$

16 a $30x(3x^2 - 4)^4$ **b** $12x^2 - 28x - 27$

c $\dfrac{-3x^4 - 8x^3 - 12x^2 + 4}{(x^3 - 2x)^2}$

17 a $4y - 3x = 30$ **b** $\dfrac{25}{2}$

18 $f(x) = \dfrac{9}{8}(x-2)^2 - 6$

19 $a = -2$

20 a $\mathbb{R} \setminus \{3\}$ i.e. $x \neq 3$

b $\mathbb{R} \setminus \{2\}$ i.e. $x \neq 2$

c $(-\infty, 2]$ i.e. $x \leq 2$

d $[4, \infty)$ i.e. $x \geq 4$

e $(-\infty, 5)$ i.e. $x < 5$

21 a $1 - 7x^{-2}$ **b** $6x(x^2 - 4)^2$ **c** $x(x^2 + 1)^{-\frac{1}{2}}$

d $(x^3 + 1)(7x^3 - 24x^2 + 1)$

e $-\dfrac{2x(2x^3 + 42x - 1)}{(4x^3 + 1)^2}$

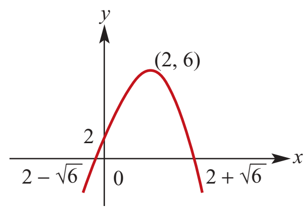
22 a -12 **b** 3 **c** 100

23 15

24 a $(2, \frac{1}{2})$ **b** $\sqrt{445}$ **c** $11x + 18y = 31$

d $22y - 36x + 61 = 0$

25



26 a $w = 1500 - 9x$ **b** $V = 20x^2(1500 - 9x)$

c $0 \leq x \leq \dfrac{500}{3}$ **d** $120\,000\,000$ cm^3

27 a $\dfrac{16}{81}$ **b** $\dfrac{28}{153}$

28 $\dfrac{1}{3}$

29 a $\dfrac{1}{2}$ **b** $\dfrac{1}{3}$

30 0.42

31 $-\dfrac{\pi}{9}, \dfrac{\pi}{9}$

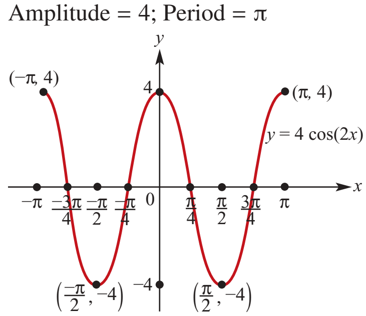
32 a $c = 6$ **b** $0 = -8a - 2b + 6,\ 0 = 3a + b$

c $a = 3, b = -9$

33 $a = -48$

34 a Amplitude = 4; Period = π

b



35 a $\left(\dfrac{2}{3}\right)^c$

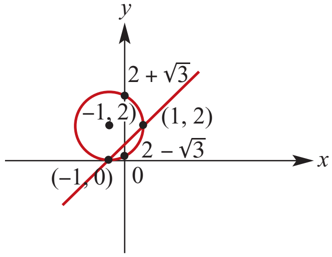
b Area = 15 cm^2; Perimeter = 16 cm

36 a $\dfrac{1}{4}$ **b** $\dfrac{1}{3}$ **c** $\dfrac{1}{4}$

 ISBN 978-1-009-53719-3 Cambridge University Press

37 a $(0, 1)$

b

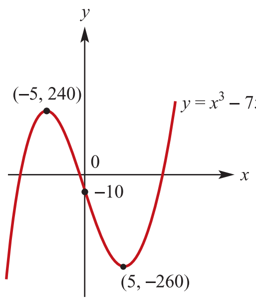


38 a $x = 3$ **b** $x = -\frac{5}{2}$ or $x = 1$

39 $a = \frac{1}{3}$ **40** $b = -\frac{1}{3}$

41 Intersect at $(-3, -27)$; both curves have gradient 27 at this point

42 a $(5, -260)$ and $(-5, 240)$



b $-260 \leq p \leq 240$

43 a $(x - 2)(x - 4)^2(5x - 14)$

b $x = 2,\ x = 4,\ x = \frac{14}{5}$

c i $y = 0$ **ii** $y = x - 4$ **iii** $27x + 16y = 54$

44 a 30 cm **b** Max = 46 cm; Min = 14 cm

c $t = \frac{\pi}{4}$ s **d** $t = \frac{3\pi}{4}$ s **e** π s

45 $k = -\frac{4}{3}$ or $k = 2$

46 a $b = 12,\ c = 6$ **b** $a = -18$ **c** $a = 6$

47 $\Delta = -8((k + \frac{1}{2})^2 + \frac{3}{4}) < 0$ for all k

48 a Centre $= (-2, -3)$; Radius $= 6$

b i $(-2 + 4\sqrt{2}, 1),\ (-2 - 4\sqrt{2}, 1)$ **ii** $8\sqrt{2}$

49 $DE : EB = 1 : 2$

50 a $\frac{4}{9}$ **b** $\frac{2}{9}$ **c** $\frac{1}{3}$ **d** $\frac{2}{9}$

e $\frac{1}{2}$ **f** $\frac{1}{5}$ **g** $\frac{1}{6}$ **h** $\frac{4}{9}$

51 a $P = 4x + 4y + 2\pi x + 2\pi y$

b $A = 16xy + \pi x^2 + \pi y^2$ **c** $y = 1 - x$

d $A = (2\pi - 16)x^2 + (16 - 2\pi)x + \pi$

e $x = y = \frac{1}{2}$ **f** $\frac{\pi}{2} + 4$

52 a $b = \frac{3(a - 3)}{2}$

b $\left(-1, \frac{a + 9}{2}\right), \left(\frac{3}{a}, -\frac{27(a + 1)}{2a^2}\right)$

c $y = (6a - 18)x - \frac{7a - 9}{2}$

d $a = -3,\ b = -9$ **e** $a = 1,\ b = -3$

Technology-active

53 9.60 cm

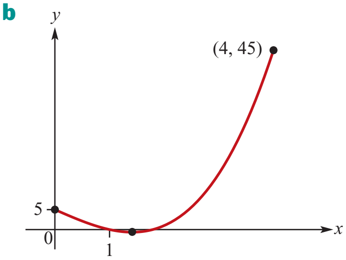
54 $x = 1.34$

55 0.1206

56 a $\frac{4}{3}$ radians **b** 96 cm^2

57 a Local min $(\sqrt{2}, 5 - 4\sqrt{2})$

b



c Range $= [5 - 4\sqrt{2}, 45]$

d Max = 45; Min $= 5 - 4\sqrt{2}$

58 a $v = 4t - 6$ **b** 6 cm/s **c** 0 cm/s

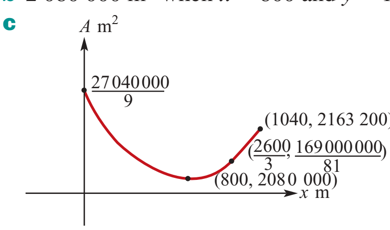
d 9 cm **e** 3 cm/s

59 a $y = 13 - 9x$ **b** $A = 156x - 60x^2$

c $x = \frac{13}{10}$ and $y = \frac{13}{10}$

60 b 2 080 000 m^2 when $x = 800$ and $y = 1200$

c



61 a $\{(0, 1), (0, 3), (0, 5), (0, 7), (0, 9), (0, 11),$
$(2, 1), (2, 3), (2, 5), (2, 7), (2, 9), (2, 11),$
$(4, 1), (4, 3), (4, 5), (4, 7), (4, 9), (4, 11),$
$(6, 1), (6, 3), (6, 5), (6, 7), (6, 9), (6, 11),$
$(8, 1), (8, 3), (8, 5), (8, 7), (8, 9), (8, 11),$
$(10, 1), (10, 3), (10, 5), (10, 7), (10, 9), (10, 11)\}$

b i $\frac{1}{36}$ **ii** $\frac{5}{36}$ **iii** $\frac{5}{36}$

c $\frac{2}{13}$

62 a i $(-\infty, 0) \cup \left(\frac{2a}{3}, \infty\right)$ **ii** $\left(0, \frac{2a}{3}\right)$

b $y = -a^2(x - a)$

c $y = \frac{x}{a^2} - \frac{1}{a}$

63 a 118.68° **b** 1.535×10^{12} km^2

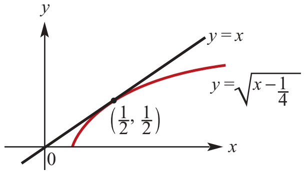
64 a i $-b$ **ii** $x = \frac{b}{2}$

b i $S(b) = \frac{b}{2}(32 - b^2)$ **ii** $b = 2$

65 a $a = 4$ **b** $\left(\frac{1}{4}, 4\right)$ **c** $\frac{34}{15}$

66 a $x \geq 2a$ **b** $x = \frac{1 \pm \sqrt{1 - 8a}}{2}$ **c** $a = \frac{1}{8}$

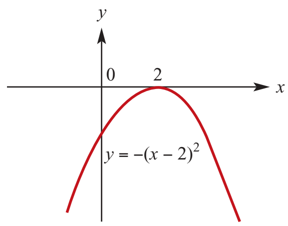
d



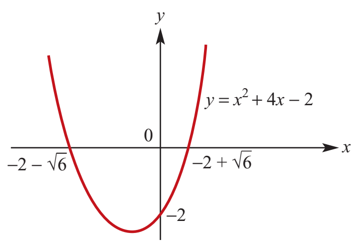
67 a 0.343 **b** 0.399

68 a 0.4219 **b** 0.2156 **c** 0.6125

69 a i



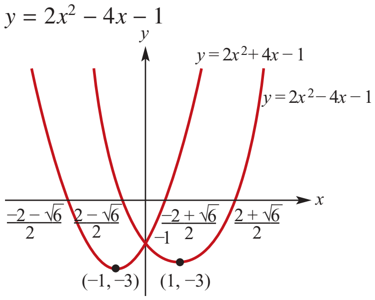
ii



b $\left(\frac{2}{1 - p}, \frac{p^2 - 5p}{p - 1}\right)$ **c** $p = 0$ or $p = 5$

d $0 < p < 5$ and $p \neq 1$

e $y = 2x^2 - 4x - 1$



70 a $[-mb + 3, -ma + 3]$

b $\left(\frac{a + b}{2}, \frac{-m(a + b) + 6}{2}\right)$

c $2ym - 2x = -m^2(a + b) + 6m - (a + b)$

d $y = -mx - 3m + 8$

$A'(a - 3, -ma + 8)$, $B'(b - 3, -mb + 8)$

e $y = mx + 3$

$A'(-a, -ma + 3)$, $B'(-b, -mb + 3)$

f $b = 12$, $m = \frac{7}{6}$

71 a $\frac{\pi}{6}$ **b** $115\sqrt{2}$ cm **c** $115\sqrt{3}$ cm

72 a 0.08

b 0.25

c i 0.1536 **ii** 0.0064

Multiple-choice questions

1 B	**2** A	**3** D	**4** C	**5** C
6 C	**7** A	**8** B	**9** A	**10** B
11 A	**12** A	**13** D	**14** B	**15** B
16 A	**17** B	**18** D	**19** C	**20** B
21 A	**22** D	**23** D	**24** D	**25** B
26 D	**27** A	**28** A	**29** D	**30** D
31 B	**32** D	**33** C	**34** A	**35** A
36 A	**37** B	**38** A	**39** C	**40** B
41 D	**42** B			

Appendix A

Exercise A1

1 a 11 **b** 12 **c** 37 **d** 29

2 15

3 24

4 a 60 **b** 500 **c** 350 **d** 512

5 a 128 **b** 160

6 22

7 93

8 63

9 26

10 9

11 47

12 240

13 260 000

14 17 576 000

15 30

Exercise A2

1 a 6 **b** 120 **c** 5040 **d** 2 **e** 1 **f** 1

2 a 20 **b** 72 **c** 6 **d** 56 **e** 120 **f** 720

3 120 **4** 5040 **5** 24

6 720 **7** 720 **8** 336

9 a 5040 **b** 210

10 a 120 **b** 120

11 a 840 **b** 2401

12 a 480 **b** 1512

13 a 60 **b** 24 **c** 252

14 a 150 **b** 360 **c** 1560

15 a 720 **b** 48

Exercise A3

1 a 3 **b** 3 **c** 6 **d** 4

2 a 10 **b** 10 **c** 35 **d** 35

3 a 190 **b** 100 **c** 4950 **d** 31 125

4 a 20 **b** 7 **c** 28 **d** 1225

5 1716 **6** 2300

7 133 784 560 **8** 8 145 060 **9** 18

10 a 5 852 925 **b** 1 744 200

11 100 386

12 a 792 **b** 336

13 a 150 **b** 75 **c** 6 **d** 462
e 81

14 a 8 436 285 **b** 3003 **c** 66
d 2 378 376

15 186 **16** 32 **17** 256

18 31 **19** 57

20 a 10 **b** 21

Exercise A4

1 $\frac{3}{8}$

2 $\frac{2}{5}$

3 a $\frac{2}{5}$ **b** $\frac{3}{5}$

4 a $\frac{1}{2}$ **b** $\frac{1}{2}$

5 a $\frac{10}{21}$ **b** $\frac{17}{42}$

6 $\frac{15}{28}$

7 a $\frac{1}{143}$ **b** $\frac{73}{143}$

8 a $\frac{33}{34}$ **b** $\frac{152}{153}$

9 $\frac{140}{429}$

10 a $\frac{84}{221}$ **b** $\frac{423}{442}$

11 a 0.2 **b** 0.6 **c** 0.3

12 0.2

13 $\frac{329}{858}$

14 a $\frac{2^7}{2^8 - 1} \approx 0.502$ **b** $\frac{56}{255}$ **c** $\frac{73}{85}$

15 a $\frac{5}{204}$ **b** $\frac{35}{136}$

16 a $\frac{1}{6}$ **b** $\frac{17}{21}$ **c** $\frac{34}{35}$

17 a $\frac{25}{49}$ **b** $\frac{3}{7}$ **c** $\frac{1}{5}$

18 $\frac{60}{143}$

19 a $\frac{6}{31}$ **b** $\frac{15}{31}$

Appendix A review

Short-response questions

Technology-free

1 a 6 **b** 15 **c** 15 **d** 1

2 a 6 **b** 10 **c** 45 **d** 21

3 a 499 500 **b** 1 000 000 **c** 1 000 000

4 120 **5** 190 **6** 84

7 5416 **8** $8n$ **9** 9

10 9 **11** 50 400

12 a 10 **b** 32

Technology-active

13 $\frac{27}{80}$

14 $\frac{168}{323}$

15 0.128

16 a 720 **b** 48 **c** $\frac{7}{15}$

17 a $\frac{1}{8}$ **b** $\frac{3}{8}$ **c** $\frac{3}{28}$

18 a 60 **b** 45

19 a 1365 **b** $\frac{2}{13}$ **c** $\frac{11}{13}$

20 a 3060 **b** 330 **c** $\frac{77}{204}$

21 a $\frac{g(8-g)(7-g)}{112}$ **b** $g = 2$ or 3

22 a $\frac{90(n-6)}{n(n-1)(n-2)}$ **b** $n = 8$ or 9

23 $m = 14, 16, 18$

24 $n = 12$

25 a $\frac{1}{24}(n+m)(n+m-1)(n+m-2)(n+m-3)$
b $\frac{1}{4}nm(n-1)(m-1)$ **c** $n = 4, 6, 8$ or 10

Multiple-choice questions

1 A **2** C **3** B **4** D **5** B
6 D **7** A **8** A **9** D **10** B
11 D **12** C **13** C **14** A **15** B
16 D **17** A **18** A

Appendix B

Exercise B1

1 a No **b** No **c** Yes **d** No **e** No

2 a $P(X = 2)$ **b** $P(X > 2)$ **c** $P(X \geq 2)$
d $P(X < 2)$ **e** $P(X \geq 2)$ **f** $P(X \leq 2)$
g $P(2 < X < 5)$

3 a $\{2\}$ **b** $\{3, 4, 5\}$ **c** $\{2, 3, 4, 5\}$
d $\{0, 1\}$ **e** $\{0, 1, 2\}$ **f** $\{2, 3, 4, 5\}$
g $\{3, 4, 5\}$ **h** $\{2, 3, 4\}$ **i** $\{3, 4\}$

4 a 0.2 **b** 0.5 **c** 0.3 **d** 0.35 **e** 0.9

5 a $\frac{1}{15}$ **b** $\frac{3}{5}$

6 a 0.09 **b** 0.69

7 a 0.49 **b** 0.51 **c** 0.74

8 a 0.6 **b** 0.47 **c** $\frac{2}{3}$

9 a $\frac{1}{6}$

 ISBN 978-1-009-53719-3 Cambridge University Press

b

y	2	3	4	5	6	7	8	9	10	11	12
$p(y)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

10 a $\frac{7}{36}$

b

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

11 a

y	−3	−2	1	3
$p(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b $\frac{7}{8}$

Exercise B2

1 a Outcome takes one of two values, and the probability of success each trial is constant.

b

x	0	1
$P(X = x)$	$\frac{49}{50}$	$\frac{1}{50}$

2 a $n = 3$, $p = 0.3$
b i 0.441 **ii** 0.189

3 a $n = 2$, $p = \frac{1}{3}$
b i 0.444 **ii** 0.444

4 a X is the number of successes in n trials, the outcome of each trial takes one of two values, the probability of success each trial is constant, and the trials are independent.
b $P(X = x) = \binom{6}{x}(0.5)^x(0.5)^{6-x}$ $x = 0, 1, \ldots, 6$
c i 0.313 **ii** 0.016

5 a X is the number of successes in n trials, the outcome of each trial takes one of two values, the probability of success each trial is constant, and the trials are independent.
b $P(X = x) = \binom{5}{x}\left(\frac{3}{8}\right)^x\left(\frac{5}{8}\right)^{5-x}$ $x = 0, 1, \ldots, 5$
c i 0.095 **ii** 0.343

6 a 0.227 **b** 0.353

7 a 0.291 **b** 0.485 **c** 0.515

8 a 0.367 **b** 0.790

9 a $P(X = x) = \binom{5}{x}(0.1)^x(0.9)^{5-x}$ $x = 0, 1, \ldots, 5$
b 0.328

10 a 0.210 **b** 0.367 **c** 0.790

11 0.021

12 0.540

13 $\frac{175}{256}$

14 a 0.988 **b** 0.9999 **c** 8.1×10^{-11}

15 a i 0.474 **ii** 0.224 **iii** 0.078
b answers will vary, about 5 or more.

16 0.014

17 a 0.75
b i 0.141 **ii** 0.984

18 a 0.309 **b** 0.310

19 a 0.054 **b** 0.211 **c** 0.262

20 Maximum 0.346 when $p = 0.6$

Appendix B review

Short-response questions

1 a 0.8 **b** 0.2 **c** 0.8

2

x	1	2	3	4
$P(X = x)$	0.25	0.28	0.30	0.17

3 a

		2nd choice					
		1	2	3	6	7	9
1st choice	1	2	3	4	7	8	10
	2	3	4	5	8	9	11
	3	4	5	6	9	10	12
	6	7	8	9	12	13	15
	7	8	9	10	13	14	16
	9	10	11	12	15	16	18

b $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18\}$

c

x	2	3	4	5	6	7	8	9
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$	$\frac{4}{36}$

x	10	11	12	13	14	15	16	18
$p(x)$	$\frac{4}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

4 a 0.3 **b** 6

5 a $\frac{9}{64}$ **b** $\frac{37}{64}$

6 a $\frac{16}{81}$ **b** $\frac{32}{81}$ **c** $\frac{11}{27}$

7 a $\left(\frac{p}{100}\right)^{15}$ **b** $15\left(\frac{p}{100}\right)^{14}\left(1 - \frac{p}{100}\right)$
c $\left(\frac{p}{100}\right)^{15} + 15\left(\frac{p}{100}\right)^{14}\left(1 - \frac{p}{100}\right)$
$+ 105\left(\frac{p}{100}\right)^{13}\left(1 - \frac{p}{100}\right)^2$

8 a $\frac{117}{125}$ **b** $m = 5$

9 a

x	1	2	3	4
$p(x)$	0.54	0.16	0.06	0.24

b 0.46

Technology-active

10 a 0.315 **b** 0.642

11 a 0.651 **b** 0.264

12 a 0.2013 **b** 0.3225

13 a 0.990 **b** 0.882

14 a i 0.1 **ii** 0.6
b 0.2001

15 a 0.003 **b** 5.32×10^{-6}

16 a 0.8 **b** $0.0860 < p < 0.3619$

Multiple-choice questions

1 B **2** A **3** A **4** D **5** C **6** B
7 B **8** C **9** C **10** A **11** D **12** D
13 C **14** A