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Specialist Mathematics

Year 12 QCE
Queensland Curriculum

2026 Edition

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Apex Exam Guidebook
Specialist Mathematics
Year 12 QCE

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Acknowledgements
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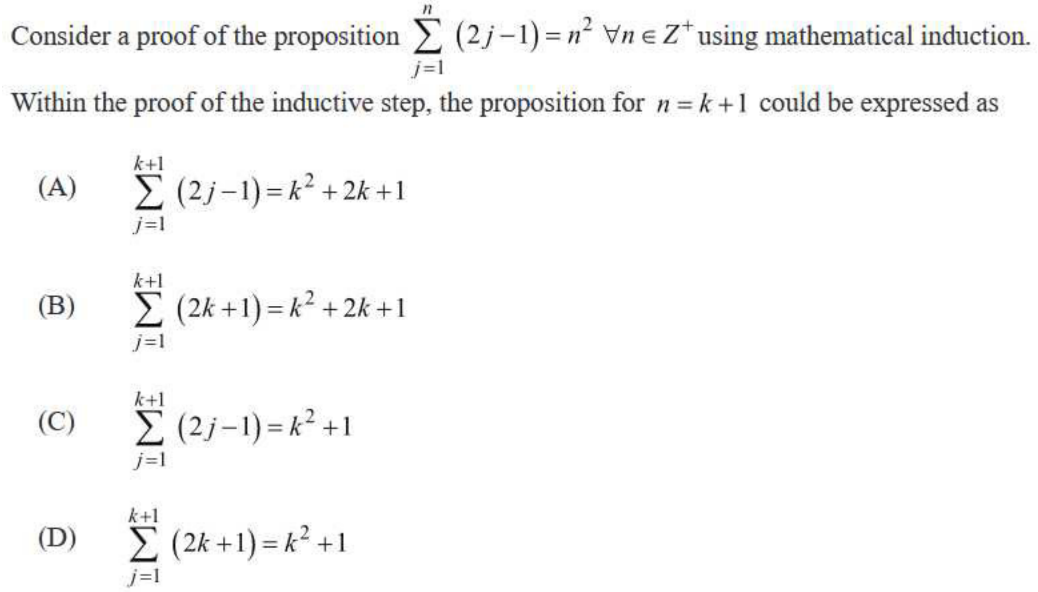
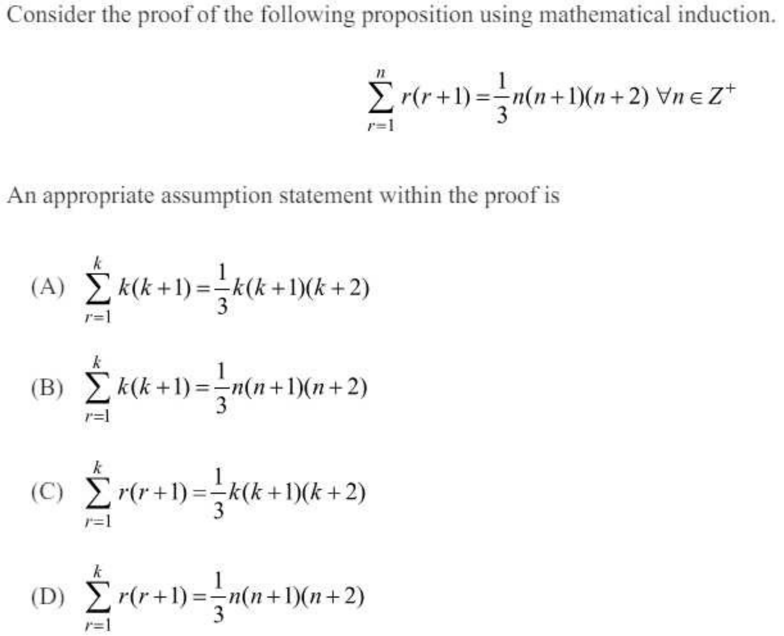
Unit 3 Mathematical induction, and further vectors, matrices and complex numbers

Unit 3 – Topic 1: Proof by mathematical induction

Paper 1 Section 1

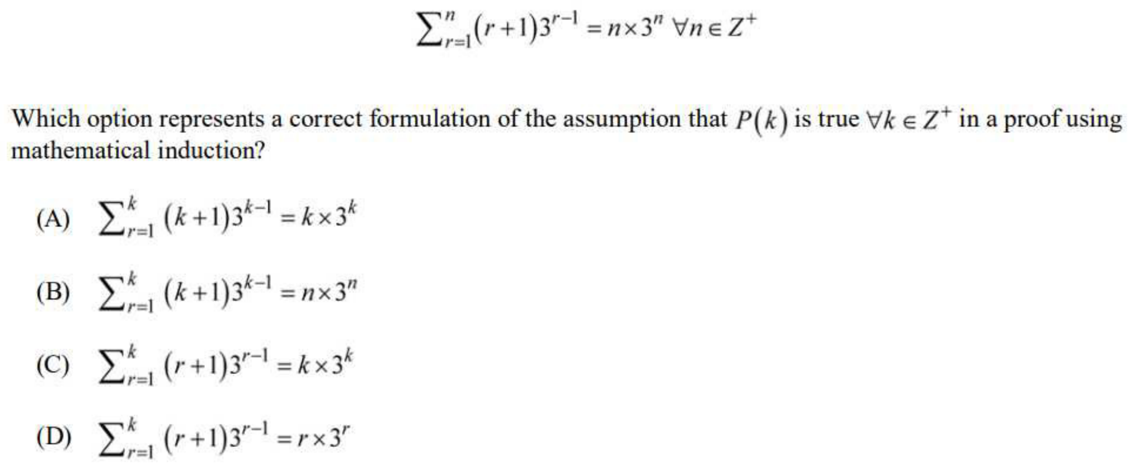
2024 Paper 1 Section 1 Question 3 **Proof by mathematical induction**	 Consider a proof of the proposition $\sum_{j=1}^{n}(2j-1)=n^2\ \forall n\in Z^+$ using mathematical induction. Within the proof of the inductive step, the proposition for $n=k+1$ could be expressed as (A) $\sum_{j=1}^{k+1}(2j-1)=k^2+2k+1$ (B) $\sum_{j=1}^{k+1}(2k+1)=k^2+2k+1$ (C) $\sum_{j=1}^{k+1}(2j-1)=k^2+1$ (D) $\sum_{j=1}^{k+1}(2k+1)=k^2+1$

2023 Paper 1 Section 1 Question 2 **Proof by mathematical induction**	 Consider the proof of the following proposition using mathematical induction. $\sum_{r=1}^{n}r(r+1)=\frac{1}{3}n(n+1)(n+2)\ \forall n\in Z^+$ An appropriate assumption statement within the proof is (A) $\sum_{r=1}^{k}k(k+1)=\frac{1}{3}k(k+1)(k+2)$ (B) $\sum_{r=1}^{k}k(k+1)=\frac{1}{3}n(n+1)(n+2)$ (C) $\sum_{r=1}^{k}r(r+1)=\frac{1}{3}k(k+1)(k+2)$ (D) $\sum_{r=1}^{k}r(r+1)=\frac{1}{3}n(n+1)(n+2)$

2021
Paper 1
Section 1
Question 8

Proof by mathematical induction

Let $P(n)$ be the proposition that



$$\sum_{r=1}^{n}(r+1)3^{r-1} = n\times 3^{n}\ \forall n\in Z^{+}$$

Which option represents a correct formulation of the assumption that $P(k)$ is true $\forall k \in Z^{+}$ in a proof using mathematical induction?

(A) $\sum_{r=1}^{k}(k+1)3^{k-1} = k\times 3^{k}$

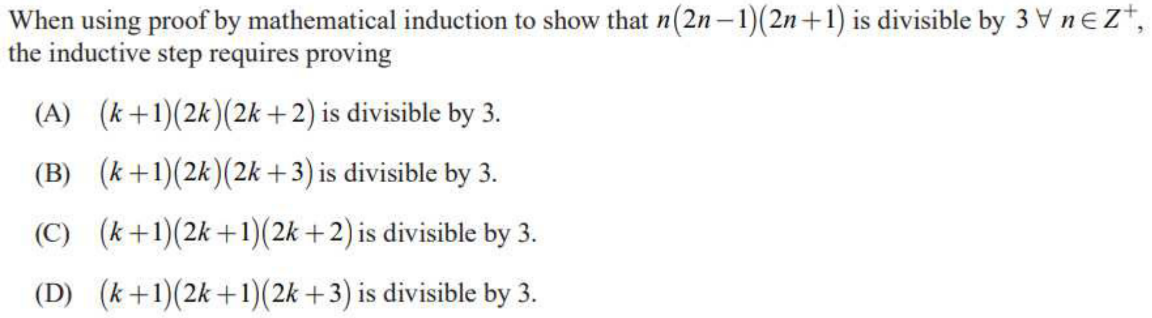
(B) $\sum_{r=1}^{k}(k+1)3^{k-1} = n\times 3^{n}$

(C) $\sum_{r=1}^{k}(r+1)3^{r-1} = k\times 3^{k}$

(D) $\sum_{r=1}^{k}(r+1)3^{r-1} = r\times 3^{r}$

2020
Paper 1
Section 1
Question 2

Proof by mathematical induction



When using proof by mathematical induction to show that $n(2n-1)(2n+1)$ is divisible by $3\ \forall\ n\in Z^{+}$, the inductive step requires proving

(A) $(k+1)(2k)(2k+2)$ is divisible by 3.

(B) $(k+1)(2k)(2k+3)$ is divisible by 3.

(C) $(k+1)(2k+1)(2k+2)$ is divisible by 3.

(D) $(k+1)(2k+1)(2k+3)$ is divisible by 3.

2024 Paper 1 Section 2 Question 16 Proof by mathematical induction	Use mathematical induction to prove that $12^n + 2(5^{n-1})$ is a multiple of 7 for $n \in Z^+$. [6 marks]

2023
Paper 1
Section 2
Question 15

Proof by mathematical induction

The sum of a geometric progression with n terms, where the first term is 1 and the common ratio is r, is given by

$$1+r+r^2+r^3+\ldots+r^{n-1}=\frac{r^n-1}{r-1}\quad (\text{for } r\neq 1).$$

Prove that this rule is true $\forall n\in Z^+$ using mathematical induction by completing the steps of the proof as indicated.

a) Initial statement: *[1 mark]*

Assuming the rule is true for $n=k$,

$$1+r+r^2+r^3+\ldots+r^{k-1}=\frac{r^k-1}{r-1}\quad (r\neq 1).$$

b) Inductive step: *[3 marks]*

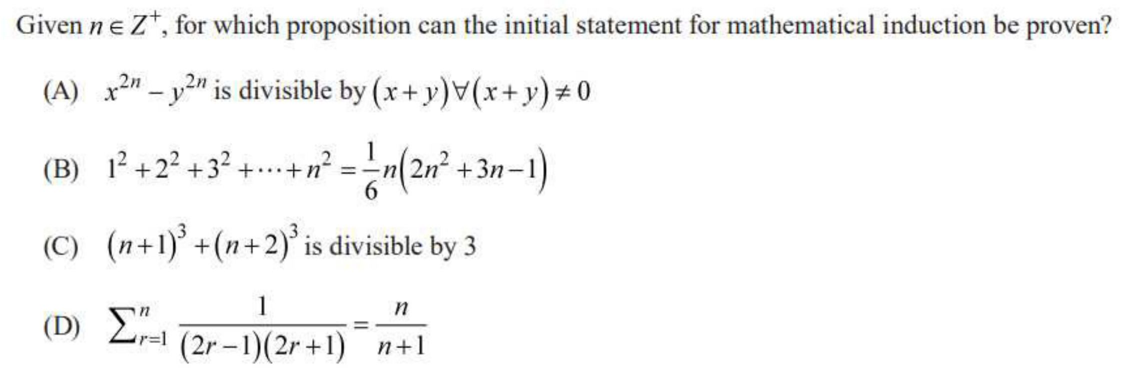
c) Conclusion: *[1 mark]*

2022 **Paper 1** **Section 2** **Question 18** **Proof by mathematical induction**	It is proposed that the following expression is divisible by $\left(1+\text{cis}(\theta)\right)$ for $n \in Z^+$, $\left(1+\text{cis}(\theta)\right) \neq 0$. $$\sum_{r=0}^{2n+1} \text{cis}(r\theta)$$ Evaluate the reasonableness of the proposition. [5 marks]

2021 Paper 1 Section 2 Question 16 Proof by mathematical induction	Use mathematical induction to prove that $2^{2n} + 3n - 1$ is divisible by 3 $\forall n \in Z^+$. [6 marks]

2020 **Paper 1** **Section 2** **Question 16** **Proof by mathematical induction**	Given $\cos(\theta) \neq 0 \ \forall \ n \in Z^{+}$, use mathematical induction to prove $$\cos(\theta)-\cos(3\theta)+\cos(5\theta)-\ldots+(-1)^{n+1}\cos((2n-1)\theta)=\frac{1-(-1)^{n}\cos(2n\theta)}{2\cos(\theta)}$$

Paper 2 Section 1

2021 Paper 2 Section 1 Question 3 Proof by mathematical induction	 Given $n \in Z^+$, for which proposition can the initial statement for mathematical induction be proven? (A) $x^{2n} - y^{2n}$ is divisible by $(x+y)\forall(x+y) \neq 0$ (B) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n\left(2n^2 + 3n - 1\right)$ (C) $(n+1)^3 + (n+2)^3$ is divisible by 3 (D) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{n+1}$

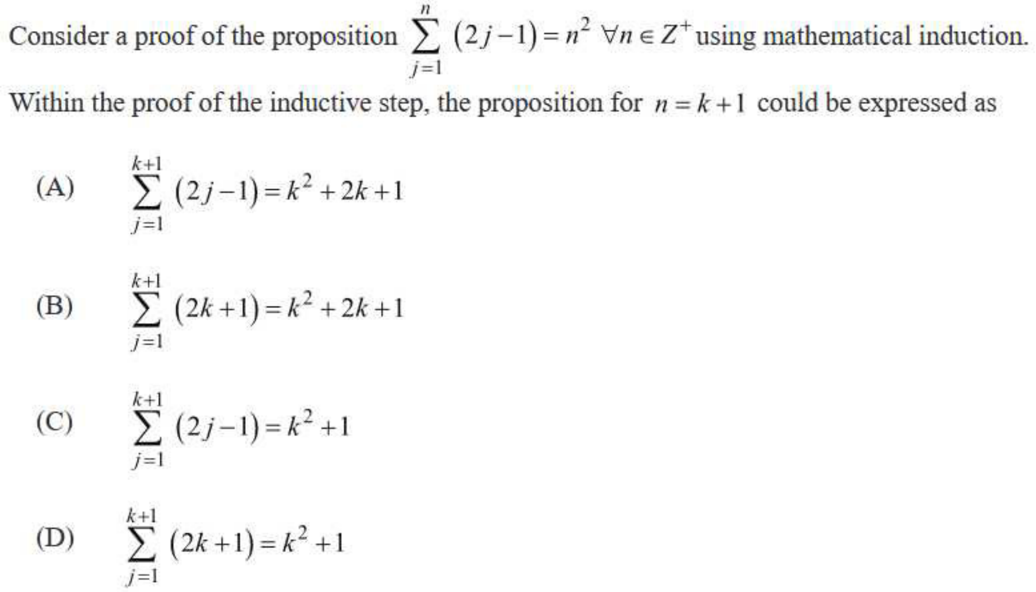
Paper 2 Section 2

There have been no questions on this topic for this section in the exams of recent years.

Marking Guide – Paper 1 Section 1

2024
Paper 1
Section 1
Question 3

Proof by mathematical induction



Consider a proof of the proposition $\sum_{j=1}^{n}(2j-1)=n^2\ \forall n\in Z^+$ using mathematical induction.

Within the proof of the inductive step, the proposition for $n=k+1$ could be expressed as

(A) $\sum_{j=1}^{k+1}(2j-1)=k^2+2k+1$

(B) $\sum_{j=1}^{k+1}(2k+1)=k^2+2k+1$

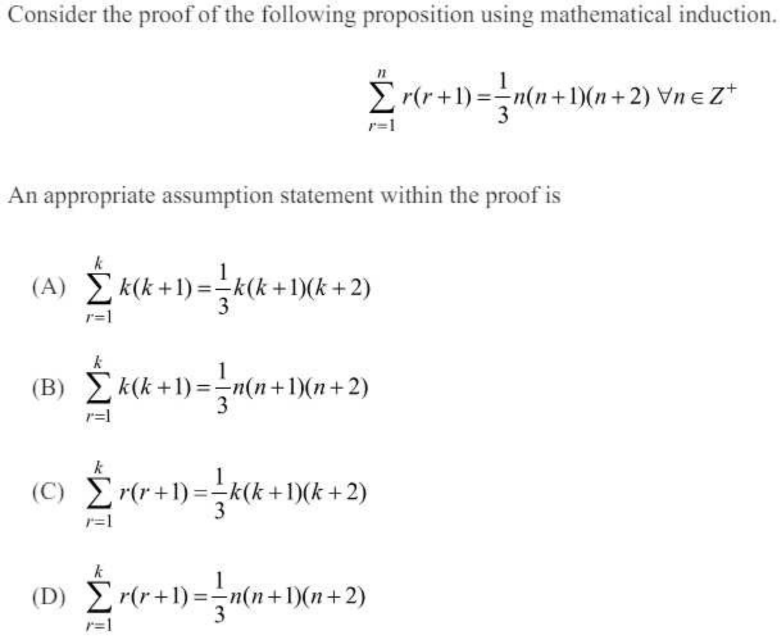
(C) $\sum_{j=1}^{k+1}(2j-1)=k^2+1$

(D) $\sum_{j=1}^{k+1}(2k+1)=k^2+1$

Answer is A.

2023
Paper 1
Section 1
Question 2

Proof by mathematical induction



Consider the proof of the following proposition using mathematical induction.

$$\sum_{r=1}^{n}r(r+1)=\frac{1}{3}n(n+1)(n+2)\ \forall n\in Z^+$$

An appropriate assumption statement within the proof is

(A) $\sum_{r=1}^{k}k(k+1)=\frac{1}{3}k(k+1)(k+2)$

(B) $\sum_{r=1}^{k}k(k+1)=\frac{1}{3}n(n+1)(n+2)$

(C) $\sum_{r=1}^{k}r(r+1)=\frac{1}{3}k(k+1)(k+2)$

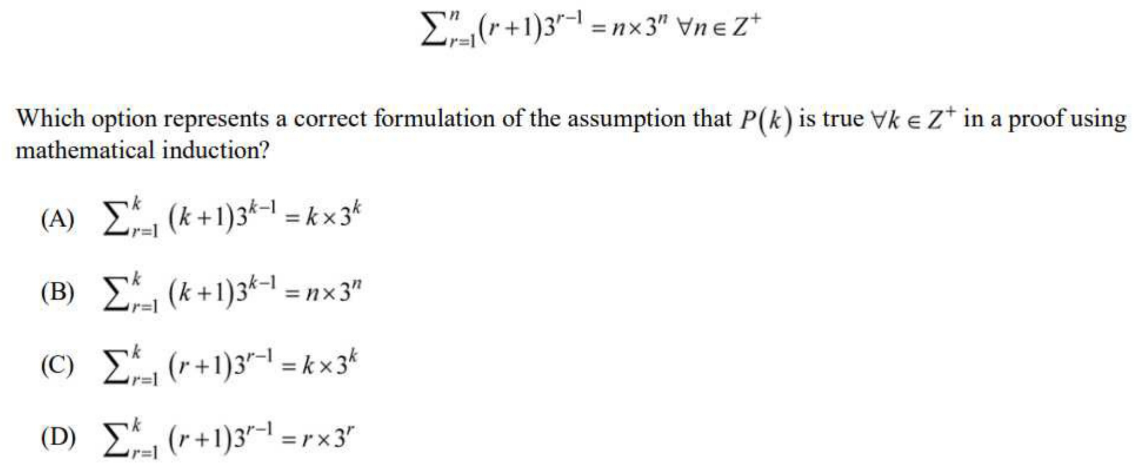
(D) $\sum_{r=1}^{k}r(r+1)=\frac{1}{3}n(n+1)(n+2)$

Answer is C.

2021
Paper 1
Section 1
Question 8

Proof by mathematical induction

Let $P(n)$ be the proposition that



$$\sum_{r=1}^{n}(r+1)3^{r-1} = n\times 3^{n}\ \forall n\in Z^{+}$$

Which option represents a correct formulation of the assumption that $P(k)$ is true $\forall k\in Z^{+}$ in a proof using mathematical induction?

(A) $\sum_{r=1}^{k}(k+1)3^{k-1} = k\times 3^{k}$

(B) $\sum_{r=1}^{k}(k+1)3^{k-1} = n\times 3^{n}$

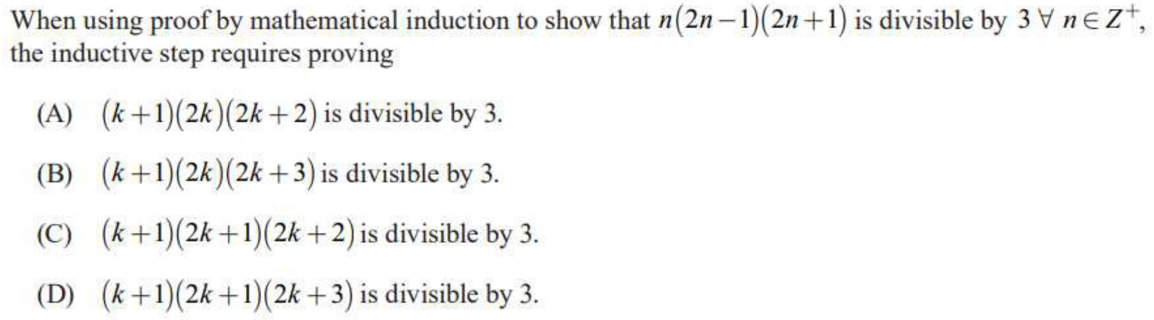
(C) $\sum_{r=1}^{k}(r+1)3^{r-1} = k\times 3^{k}$

(D) $\sum_{r=1}^{k}(r+1)3^{r-1} = r\times 3^{r}$

Answer is C.

2020
Paper 1
Section 1
Question 2

Proof by mathematical induction



When using proof by mathematical induction to show that $n(2n-1)(2n+1)$ is divisible by $3\ \forall\ n\in Z^{+}$, the inductive step requires proving

(A) $(k+1)(2k)(2k+2)$ is divisible by 3.

(B) $(k+1)(2k)(2k+3)$ is divisible by 3.

(C) $(k+1)(2k+1)(2k+2)$ is divisible by 3.

(D) $(k+1)(2k+1)(2k+3)$ is divisible by 3.

Answer is D.

Marking Guide – Paper 1 Section 2

2024
Paper 1
Section 2
Question 16

Proof by mathematical induction

Use mathematical induction to prove that $12^n+2\left(5^{n-1}\right)$ is a multiple of 7 for $n\in Z^+$.

[6 marks]

Sample response	The response
Method 1 Prove $12^n+2\left(5^{n-1}\right)$ is a multiple of 7, $n\in Z^+$. i.e. $12^n+2\left(5^{n-1}\right)=7m,\ m\in Z^+$ Initial statement: Let $n=1$ $12^1+2\left(5^{1-1}\right)=14$ $=7\times 2$ Proposition is true for $n=1$.	• correctly proves the initial statement **[1 mark]**
Assume proposition is true for $n=k$ $12^k+2\left(5^{k-1}\right)=7m$ for some $m\in Z^+$ … (1)	• correctly formulates an appropriate assumption **[1 mark]**
Inductive step: Let $n=k+1$ RTP $12^{k+1}+2\left(5^k\right)$ is a multiple of 7 $12^{k+1}+2\left(5^k\right)$	• correctly establishes an appropriate expression representing the LHS of the inductive step **[1 mark]**
$=12\left(12^k\right)+2\times 5\left(5^{k-1}\right)$ $=12\left(12^k\right)+10\left(5^{k-1}\right)$ $=12\left(12^k\right)+24\left(5^{k-1}\right)-14\left(5^{k-1}\right)$ $=12\left(\left(12^k\right)+2\left(5^{k-1}\right)\right)-14\left(5^{k-1}\right)$ $=12(7m)-14\left(5^{k-1}\right)$ … using (1)	• uses assumption within the inductive step **[1 mark]**
$=7\left(12m-2\left(5^{k-1}\right)\right)$ $=7p$ where $p\in Z^+$	• completes proof by determining an expression with a factor of 7 representing the RHS of the inductive step **[1 mark]**
The proposition is true for $n=k+1$. By mathematical induction, the proposition is true for $n=1, 2, 3\ldots$	• shows logical organisation, having attempted all steps of the proof, **including** a suitable conclusion **[1 mark]**
Method 2 Prove $12^n+2\left(5^{n-1}\right)$ is a multiple of 7, $n\in Z^+$ i.e. $12^n+2\left(5^{n-1}\right)=7m,\ m\in Z^+$ Initial statement: Let $n=1$ $12^1+2\left(5^{1-1}\right)=14$ $=7\times 2$ Proposition is true for $n=1$.	• correctly proves the initial statement **[1 mark]**
Assume proposition is true for $n=k$ $12^k+2\left(5^{k-1}\right)=7m$ for some $m\in Z^+$ $12^k=7m-2\left(5^{k-1}\right)$ for some $m\in Z^+$ … (1)	• correctly formulates an appropriate assumption **[1 mark]**
Inductive step: Let $n=k+1$ RTP $12^{k+1}+2\left(5^k\right)$ is a multiple of 7 $12^{k+1}+2\left(5^k\right)$	• correctly establishes an appropriate expression representing the LHS of the inductive step **[1 mark]**

	$=12\left(12^k\right)+10\left(5^{k-1}\right)$ $=12\left(7m-2\left(5^{k-1}\right)\right)+10\left(5^{k-1}\right)$ ... using (1) $=84m-24\left(5^{k-1}\right)+10\left(5^{k-1}\right)$ $=7\left(12m-2\left(5^{k-1}\right)\right)$ $=7p$ where $p\in Z^+$ The proposition is true for $n=k+1$. By mathematical induction, the proposition is true for $n=1, 2, 3 \ldots$	• uses assumption within the inductive step **[1 mark]** • completes proof by determining an expression with a factor of 7 representing the RHS of the inductive step **[1 mark]** • shows logical organisation, having attempted all steps of the proof, **including** a suitable conclusion **[1 mark]**
	Method 3 Prove $12^n+2\left(5^{n-1}\right)$ is a multiple of 7, $n\in Z^+$ i.e. $12^n+2\left(5^{n-1}\right)=7m,\ m\in Z^+$ Initial statement: Let $n=1$ $12^1+2\left(5^{1-1}\right)=14$ $=7\times2$ Proposition is true for $n=1$. Assume proposition is true for $n=k$ $12^k+2\left(5^{k-1}\right)=7m$ for some $m\in Z^+$ $2\left(5^{k-1}\right)=7m-12^k$ for some $m\in Z^+$ ... (1) Inductive step: Let $n=k+1$ RTP $12^{k+1}+2\left(5^k\right)$ is a multiple of 7 $12^{k+1}+2\left(5^k\right)$ $=12^{k+1}+5\times2\left(5^{k-1}\right)$ $=12^{k+1}+5\left(7m-12^k\right)$ ... using (1) $=12^{k+1}+35m-5\times12^k$ $=12^k(12-5)+35m$ $=7\times12^k+35m$ $=7\left(12^k+5m\right)$ $=7p$ where $p\in Z^+$ The proposition is true for $n=k+1$. By mathematical induction, the proposition is true for $n=1, 2, 3 \ldots$	• correctly proves the initial statement **[1 mark]** • correctly formulates an appropriate assumption **[1 mark]** • correctly establishes an appropriate expression representing the LHS of the inductive step **[1 mark]** • uses assumption within the inductive step **[1 mark]** • completes proof by determining an expression with a factor of 7 representing the RHS of the inductive step **[1 mark]** • shows logical organisation, having attempted all steps of the proof, **including** a suitable conclusion **[1 mark]**

2023
Paper 1
Section 2
Question 15

Proof by mathematical induction

The sum of a geometric progression with n terms, where the first term is 1 and the common ratio is r, is given by

$$1+r+r^2+r^3+\ldots+r^{n-1}=\frac{r^n-1}{r-1}\ \left(\text{for } r\neq 1\right).$$

Prove that this rule is true $\forall n\in Z^+$ using mathematical induction by completing the steps of the proof as indicated.

a) Initial statement: *[1 mark]*

Sample response	The response
Initial statement Prove the rule is true for $n=1$. $\text{LHS}=1$ $\text{RHS}=\frac{r-1}{r-1}$ $=1=\text{LHS}$	• correctly proves the initial statement **[1 mark]**

Assuming the rule is true for $n=k$,

$$1+r+r^2+r^3+\ldots+r^{k-1}=\frac{r^k-1}{r-1}\ \left(r\neq 1\right).$$

b) Inductive step: *[3 marks]*

Sample response	The response
Given assumption $1+r+r^2+r^3+\ldots+r^{k-1}=\frac{r^k-1}{r-1}\ \left(r\neq 1\right)$ Inductive step Prove the rule is true for $n=k+1$ for $r\neq 1$ $1+r+r^2+r^3+\ldots+r^{k-1}+r^k=\frac{r^{k+1}-1}{r-1}$ $\text{LHS}=\frac{r^k-1}{r-1}+r^k$ $=\frac{r^k-1+r^{k+1}-r^k}{r-1}$ $=\frac{r^{k+1}-1}{r-1}$ $=\text{RHS}$	• correctly establishes an expression representing the left-hand side requirement of the inductive step proof **[1 mark]** • uses the given assumption within the inductive step proof **[1 mark]** • shows mathematical reasoning to complete the inductive step proof **[1 mark]**

c) Conclusion: *[1 mark]*

Sample response	The response
Conclusion The rule is proven true for $n = k + 1$. By mathematical induction, the rule is true for $n = 1, 2, \ldots$	• states a suitable conclusion to the proof **[1 mark]**

2022
Paper 1
Section 2
Question 18

Proof by mathematical induction

It is proposed that the following expression is divisible by $\left(1+\text{cis}(\theta)\right)$ for $n \in Z^+$, $\left(1+\text{cis}(\theta)\right) \neq 0$.

$$\sum_{r=0}^{2n+1} \text{cis}(r\theta)$$

Evaluate the reasonableness of the proposition. [5 marks]

Sample Response	The response
Mathematical induction can be used to prove the proposition $\sum_{r=0}^{2n+1} \text{cis}(r\theta)$ is divisible by $\left(1+\text{cis}\ (\theta)\right)$ for $n \in Z^+$ Let $n = 1$ $\sum_{r=0}^{3} \text{cis}(r\theta) = \text{cis}(0) + \text{cis}(\theta) + \text{cis}(2\theta) + \text{cis}(3\theta)$ $= 1 + \text{cis}(\theta) + \left(\text{cis}(\theta)\right)^2 + \left(\text{cis}(\theta)\right)^3$ $= \left(1+\text{cis}(\theta)\right)(1 + (\text{cis}\theta)^2)$	• correctly proves the initial statement [1 mark]
This expression is divisible by $\left(1+\text{cis}\ (\theta)\right)$, so the proposition is true for $n = 1$. Assume $n = k$ is true for $k \in Z^+$: $\sum_{r=0}^{2k+1} \text{cis}(r\theta) = \left(1+\text{cis}(\theta)\right)Q(\theta)$ where $Q(\theta)$ is a function of θ	• correctly establishes an appropriate assumption for $n = k$ [1 mark]
Let $n = k + 1$ $\sum_{r=0}^{2k+3} \text{cis}(r\theta) = \sum_{r=0}^{2k+1} \text{cis}(r\theta) + \text{cis}\left((2k+2)\theta\right) + \text{cis}\left((2k+3)\theta\right)$	• expresses the sum based on $n = k + 1$ in terms of the assumption [1 mark]
$= \left(1+\text{cis}(\theta)\right)Q(\theta) + \left(\text{cis}(\theta)\right)^{2k+2} + \left(\text{cis}(\theta)\right)^{2k+3}$ $= \left(1+\text{cis}(\theta)\right)Q(\theta) + \left(\text{cis}(\theta)\right)^{2k+2}\left(1+\text{cis}(\theta)\right)$	• expresses a result using a common factor of $(1 + \text{cis}(\theta))$ [1 mark]
$= \left(1+\text{cis}(\theta)\right)\left(Q(\theta) + \left(\text{cis}(\theta)\right)^{2k+2}\right)$ $= \left(1+\text{cis}(\theta)\right)R(\theta)$ where $R(\theta)$ is a function of θ The proposition is true for $n = k + 1$. By mathematical induction, the formula is true for $n = 1, 2, \ldots$	• proves the inductive step [1 mark]

2021
Paper 1
Section 2
Question 16

Proof by mathematical induction

Use mathematical induction to prove that $2^{2n}+3n-1$ is divisible by 3 $\forall n \in Z^+$.

[6 marks]

Sample Response	The response
RTP $2^{2n}+3n-1$ is always divisible by 3 for $n \in Z^+$, i.e. $2^{2n}+3n-1=3m$, for some $m \in Z^+$ Initial statement: Let $n=1$: $2^2+3-1=6$ $=3\times 2$ Proposition is true for $n=1$	• correctly proves the initial statement [1 mark]
$=3\times 2$ Proposition is true for $n=1$ Assume proposition is true for $n=k\ \forall\ k \in Z^+$ $2^{2k}+3k-1=3m$ for some $m \in Z^+$	• correctly formulates an assumption for $n=k$ [1 mark]
Inductive step: Let $n=k+1$ $2^{2(k+1)}+3(k+1)-1=2^{2k+2}+3k+3-1$ $=2^2\times 2^{2k}+3k-1+3$	• correctly establishes an expression representing the LHS of the inductive step proof and uses an index law to establish a term with a factor of 2^{2k} [1 mark]
$=2^{2k}+3k-1+3\times 2^{2k}+3$ $=3m+3\times 2^{2k}+3$	• uses previous assumption in the inductive step [1 mark]
$=3(m+2^{2k}+1)$ $=3p$ for some $p \in Z^+$	• completes proof by determining an expression with a factor of 3 representing the RHS of the inductive step proof [1 mark]
The proposition is true for $n=k+1$ By mathematical induction, the proposition is true for $n=1,2,\ldots$	• shows logical organisation, having attempted all steps of the proof, including the use of a suitable conclusion [1 mark]

2020
Paper 1
Section 2
Question 16

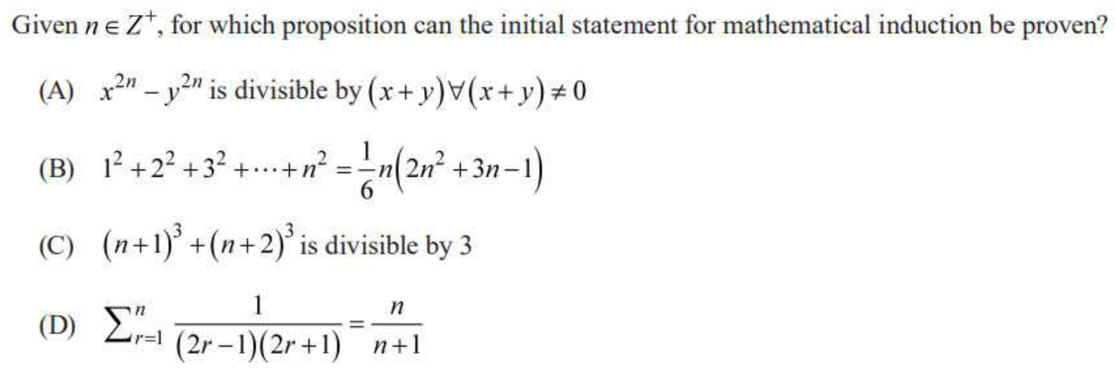
Proof by mathematical induction

Given $\cos(\theta)\neq 0\ \forall\ n\in Z^{+}$, use mathematical induction to prove

$$\cos(\theta)-\cos(3\theta)+\cos(5\theta)-\ldots+(-1)^{n+1}\cos((2n-1)\theta)=\frac{1-(-1)^{n}\cos(2n\theta)}{2\cos(\theta)}$$

Sample Response	The response
$\cos(\theta)-\cos(3\theta)+\cos(5\theta)-\cdots+(-1)^{n+1}\cos((2n-1)\theta)=\frac{1-(-1)^{n}\cos(2n\theta)}{2\cos(\theta)}$ Let $P(n)$ represent the proposition. R.T.P. $P(1)$ is true. $\text{LHS}=\cos\theta$ $\text{RHS}=\frac{1-(-1)\cos(2\theta)}{2\cos\theta}$ $=\frac{1+\cos(2\theta)}{2\cos\theta}$ $=\frac{1+2\cos^{2}\theta-1}{2\cos\theta}$ $=\cos\theta$ $=\text{RHS}$ $\therefore P(1)$ is true.	• correctly proves the initial statement [1 mark]
Assume $P(k)$ is true. $\cos(\theta)-\cos(3\theta)+\cos(5\theta)-\cdots+(-1)^{k+1}\cos((2k-1)\theta)=\frac{1-(-1)^{k}\cos(2k\theta)}{2\cos(\theta)}$ R.T.P. $P(k+1)$ is true. $\cos(\theta)-\cos(3\theta)+\cos(5\theta)-\cdots+(-1)^{k+1}\cos((2k-1)\theta)+(-1)^{(k+1)+1}\cos((2(k+1)-1)\theta)=\frac{1-(-1)^{k+1}\cos(2(k+1)\theta)}{2\cos(\theta)}$	• correctly states the assumption and the proof requirement for the inductive step [1 mark]
$\text{LHS}=\frac{1-(-1)^{k}\cos(2k\theta)}{2\cos(\theta)}+(-1)^{k+2}\cos((2k+1)\theta)$	• uses assumption in the proof of the inductive step [1 mark]
$=\frac{1-(-1)^{k}\cos(2k\theta)+(-1)^{k+2}\cos((2k+1)\theta)\,2\cos(\theta)}{2\cos(\theta)}$ $=\frac{1-(-1)^{k}\cos(2k\theta)+(-1)^{k+2}(\cos[(2k+1)\theta-\theta]+\cos[(2k+1)\theta+\theta])}{2\cos(\theta)}$ $=\frac{1-(-1)^{k}\cos(2k\theta)+(-1)^{k+2}\cos(2k\theta)+(-1)^{k+2}\cos((2k+2)\theta)}{2\cos(\theta)}$	• determines a simplified expression based on the use of a common denominator and a suitable trigonometric product identity [1 mark]
$=\frac{1+(-1)^{k+2}\cos((2k+2)\theta)}{2\cos(\theta)}$	• determines a simplified expression based on the recognition that $(-1)^{k}=(-1)^{k+2}$ [1 mark]
$=\frac{1+(-1)^{k+1}(-1)^{1}\cos(2(k+1)\theta)}{2\cos(\theta)}$ $=\frac{1-(-1)^{k+1}\cos(2(k+1)\theta)}{2\cos(\theta)}$ $=\text{RHS}$ So $P(k+1)$ is true. By mathematical induction, the formula is true for $n=1,2,\ldots$	• completes proof and communicates a suitable conclusion [1 mark]

Marking Guide – Paper 2 Section 1

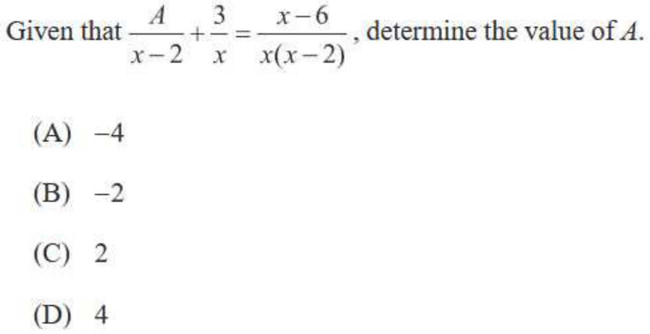
2021 Paper 2 Section 1 Question 3 Proof by mathematical induction	Given $n \in Z^+$, for which proposition can the initial statement for mathematical induction be proven? (A) $x^{2n} - y^{2n}$ is divisible by $(x+y) \forall (x+y) \neq 0$ (B) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(2n^2 + 3n - 1)$ (C) $(n+1)^3 + (n+2)^3$ is divisible by 3 (D) $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{n+1}$ **Answer is A.**

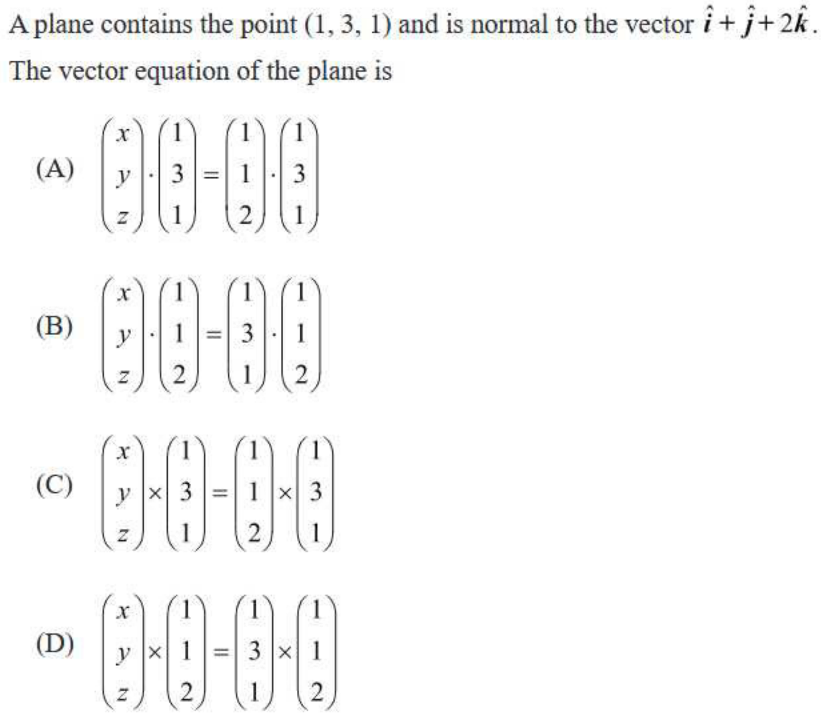
Marking Guide – Paper 2 Section 2

There have been no questions on this topic for this section in the exams of recent years.

Unit 3 – Topic 2: Vectors and matrices

Paper 1 Section 1

2024 Paper 1 Section 1 Question 2 Vectors and matrices	 Given that $\frac{A}{x-2}+\frac{3}{x}=\frac{x-6}{x(x-2)}$, determine the value of A. (A) -4 (B) -2 (C) 2 (D) 4

2024 Paper 1 Section 1 Question 4 Vectors and matrices	 A plane contains the point (1, 3, 1) and is normal to the vector $\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$. The vector equation of the plane is (A) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}$ (B) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}$ (C) $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\1\end{pmatrix}$ (D) $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}$

2024
Paper 1
Section 1
Question 5

Vectors and matrices

The augmented matrix shown is produced when a Gaussian elimination technique is used to solve a certain system of equations with three variables.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -10 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

Given that row 1 values of the matrix represent $x+4y+2z=-10$, the unique solution for y is

(A) $\frac{2}{5}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{2}$

2024
Paper 1
Section 1
Question 6

Vectors and matrices

Players P, Q, R and S played each other once in a competition where there were no draws.

Only the following results are known.

- Player P defeated players Q and R.
- Player Q defeated two players.
- Players R and S each defeated one player.

Based on these results, a dominance matrix N was partially constructed as shown.

$$N = \begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \end{array} \begin{array}{c} \begin{array}{cccc} \text{P} & \text{Q} & \text{R} & \text{S} \end{array} \\ \left[\begin{array}{cccc} 0 & 1 & 1 & 0 \\ \square & \square & \square & \square \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{array}\right] \end{array}$$

The completed matrix N is

(A) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

2024
Paper 1
Section 1
Question 7

Vectors and matrices

A, B and C are points in three-dimensional space.

If $2\overrightarrow{AB} = \overrightarrow{BC}$, then

(A) $\left|\overrightarrow{AB}\right|$ is twice the value of $\left|\overrightarrow{BC}\right|$.

(B) $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular.

(C) only one plane contains A, B and C.

(D) a straight line passes through A, B and C.

2023
Paper 1
Section 1
Question 1

Vectors and matrices

The position of a particle is given by $\boldsymbol{r} = (t+2)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$ for $t \geq 0$.

Determine the corresponding Cartesian equation.

(A) $y = x^2 - 4$

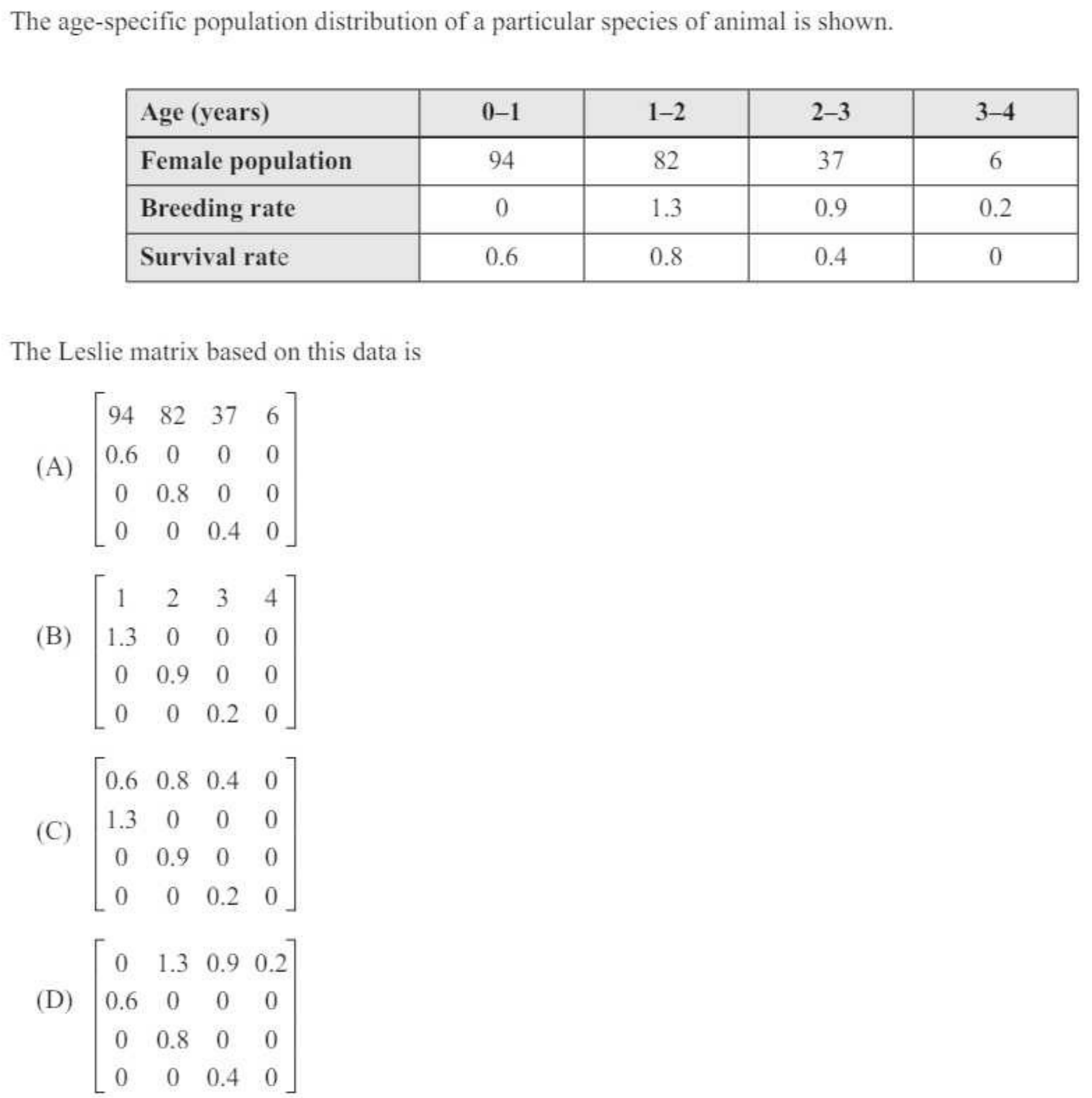
(B) $y = x^2 + 4$

(C) $y = x^2 - 4x + 4$

(D) $y = x^2 + 4x + 4$

2023
Paper 1
Section 1
Question 4

Vectors and matrices



The age-specific population distribution of a particular species of animal is shown.

Age (years)	**0–1**	**1–2**	**2–3**	**3–4**
Female population	94	82	37	6
Breeding rate	0	1.3	0.9	0.2
Survival rate	0.6	0.8	0.4	0

The Leslie matrix based on this data is

(A) $\begin{bmatrix} 94 & 82 & 37 & 6 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

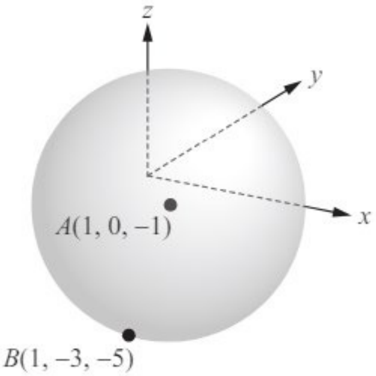
(C) $\begin{bmatrix} 0.6 & 0.8 & 0.4 & 0 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1.3 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

2023
Paper 1
Section 1
Question 8

Vectors and matrices

Point A is the centre of a sphere and point B lies on its surface as shown.



The equation of the sphere is

(A) $x^2 - 2x + y^2 + z^2 + 2z = 23$

(B) $x^2 + 2x + y^2 + z^2 - 2z = 23$

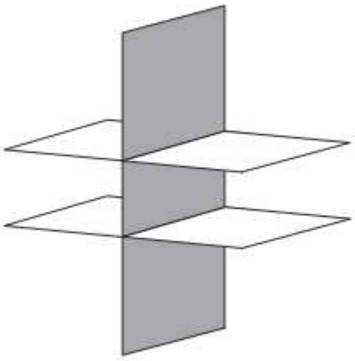
(C) $x^2 - 2x + y^2 + z^2 + 2z = 25$

(D) $x^2 + 2x + y^2 + z^2 - 2z = 25$

2023
Paper 1
Section 1
Question 9

Vectors and matrices

The geometric interpretation of a certain system of three equations with no solution is shown.



Given two of the equations are $x + y - z = 0.5$ and $x - y - z = 0.5$, the third equation could be

(A) $2x - 2y - 2z = 1$

(B) $2x + 2y - 2z = 1$

(C) $2x - 2y + 2z = 3$

(D) $2x + 2y - 2z = 3$

2022 **Paper 1** **Section 1** **Question 6** **Vectors and matrices**	The Cartesian equation for a sphere with centre (−2, 3, −4) and radius 9 is (A) $(x-2)^2+(y+3)^2+(z-4)^2=9$ (B) $(x+2)^2+(y-3)^2+(z+4)^2=9$ (C) $(x-2)^2+(y+3)^2+(z-4)^2=81$ (D) $(x+2)^2+(y-3)^2+(z+4)^2=81$

2022 **Paper 1** **Section 1** **Question 10** **Vectors and matrices**	A plane is represented by the equation $x-2z\ =5$. A vector normal to this plane is (A) $\begin{pmatrix}1\\-2\\5\end{pmatrix}$ (B) $\begin{pmatrix}1\\0\\2\end{pmatrix}$ (C) $\begin{pmatrix}1\\0\\-2\end{pmatrix}$ (D) $\begin{pmatrix}1\\-2\\-5\end{pmatrix}$

2021 **Paper 1** **Section 1** **Question 3** **Vectors and matrices**	An object has a velocity $\boldsymbol{v}(t)=e^{-2t}\hat{\boldsymbol{i}}+\left(\frac{1}{t}\right)\hat{\boldsymbol{k}}$, where t represents time $(t>0)$. The displacement $\boldsymbol{r}(t)$ of the object could be (A) $-2e^{-2t}\hat{\boldsymbol{i}}+\ln(t)\hat{\boldsymbol{k}}$ (B) $-2e^{-2t}\hat{\boldsymbol{i}}-\frac{1}{t^2}\hat{\boldsymbol{k}}$ (C) $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{i}}+\ln(t)\hat{\boldsymbol{k}}$ (D) $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{i}}-\frac{1}{t^2}\hat{\boldsymbol{k}}$

2021
Paper 1
Section 1
Question 5

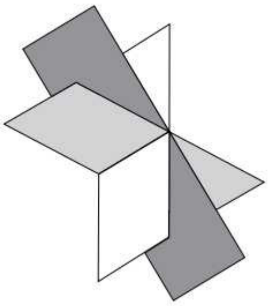
Vectors and matrices

The augmented matrix shown is produced when a Gaussian elimination technique is used to solve a certain system of equations with three variables.

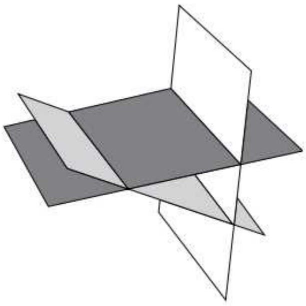
$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The geometric interpretation of the solution to this system of equations is best represented by

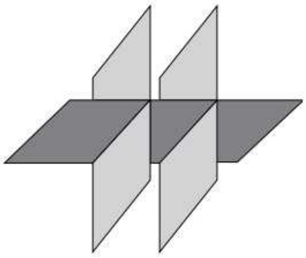
(A)



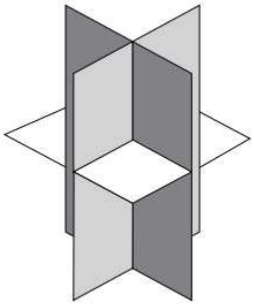
(B)



(C)



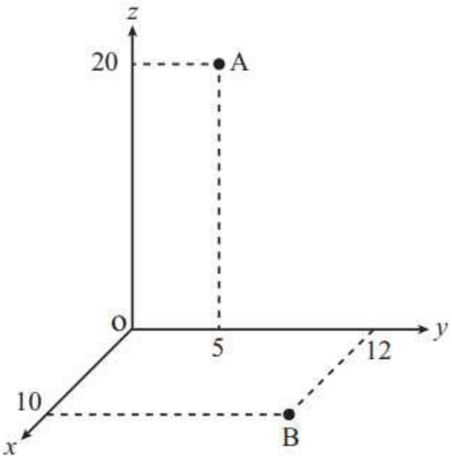
(D)



2020
Paper 1
Section 1
Question 4

Vectors and matrices

Consider points A and B as shown.



The position vector representing the midpoint of AB is

(A) $\begin{pmatrix} 5 \\ 8.5 \\ 10 \end{pmatrix}$

(B) $\begin{pmatrix} 5 \\ 10 \\ 8.5 \end{pmatrix}$

(C) $\begin{pmatrix} 10 \\ 8.5 \\ 5 \end{pmatrix}$

(D) $\begin{pmatrix} 10 \\ 5 \\ 8.5 \end{pmatrix}$

2020
Paper 1
Section 1
Question 8

Vectors and matrices

An equation of the line passing through the points A(2, 4, 5) and B(3, –2, 1) is

(A) $2\hat{\imath} + 4\hat{\jmath} + 5\hat{k} + t(3\hat{\imath} - 2\hat{\jmath} + \hat{k}), t \in R$

(B) $-3\hat{\imath} + 2\hat{\jmath} - \hat{k} + t(\hat{\imath} - 6\hat{\jmath} - 4\hat{k}), t \in R$

(C) $\frac{x-1}{2} = \frac{y+6}{4} = \frac{z+4}{5}$

(D) $\frac{x-3}{-1} = \frac{y+2}{6} = \frac{z-1}{4}$

Paper 1 Section 2

2024
Paper 1
Section 2
Question 11

Vectors and matrices

The vector equation of a straight line is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + k\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, where k is a scalar.

a) Express the equation of the line as a pair of parametric equations. *[1 mark]*

b) Use your result from Question 11a) to express the equation of the line as a Cartesian equation. *[1 mark]*

c) Determine the coordinates of the point that the line passes through when $k = 5$. *[1 mark]*

d) Determine the value of k when the line intersects the y-axis. *[1 mark]*

2024
Paper 1
Section 2
Question 14

Vectors and matrices

The displacement (cm) of a particle from the origin as it travels in two-dimensional space at time t for $0 \leq t < \frac{\pi}{2}$ seconds is given by

$$\boldsymbol{r} = \left(2\sec(t) - 1\right)\hat{\boldsymbol{i}} + \tan(t)\,\hat{\boldsymbol{j}}$$

a) Express the path of the particle as a pair of parametric equations. *[1 mark]*

A general Cartesian form of a hyperbola with centre (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\text{, where } a, b \neq 0.$$

b) Use a suitable Pythagorean identity to show that the path of the particle can be expressed in this general Cartesian form. *[3 marks]*

c) Determine the centre of the hyperbolic path of the particle. *[1 mark]*

2024
Paper 1
Section 2
Question 17

Vectors and matrices

The acceleration (m s^{-2}) of an object that moves in a straight line in an easterly direction over time t for $0 \le t \le \frac{\pi}{6}$ seconds is given by $a = 2\left(1+v^2\right)$, where v is its velocity (m s^{-1}).

The object is initially at rest at a position that is $\ln\left(\sqrt{2}\right)$ metres west of the origin.

A student uses this information to calculate that the object is positioned at the origin when $t = \frac{\pi}{6}$ seconds.

Evaluate the reasonableness of the student's calculation.

[6 marks]

2023
Paper 1
Section 2
Question 12

Vectors and matrices

Given $\boldsymbol{A}=\begin{pmatrix}1 & -2\\ 1 & 2\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$ and $\boldsymbol{C}=\begin{pmatrix}-1 & -1\\ 0 & 3\end{pmatrix}$, determine $\boldsymbol{X}$ in the matrix equation $\boldsymbol{XA}-\boldsymbol{XC}=\boldsymbol{B}$.

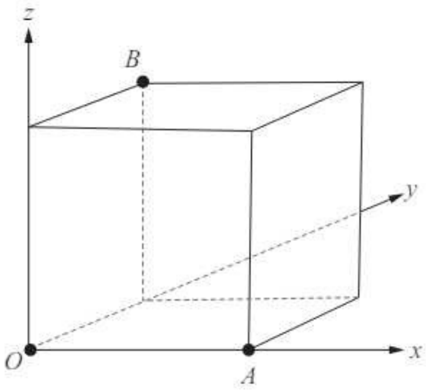
(5 marks)

2023
Paper 1
Section 2
Question 14

Vectors and matrices

Consider a cube with three edges positioned along the x-, y- and z-axes on the Cartesian plane as shown. Points O, A and B are vertices of the cube.

Not to scale



a) Given $\overrightarrow{OA} = 2\hat{\boldsymbol{i}}$, determine $\overrightarrow{OB}$. Express your answer in terms of $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. *[1 mark]*

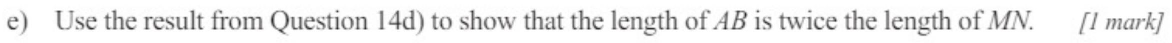
b) Calculate $\overrightarrow{OA} \times \overrightarrow{OB}$. *[1 mark]*

Consider the triangle formed by joining points O, A and B.

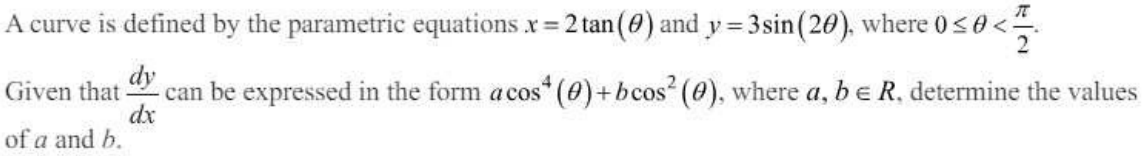
c) Use the result from Question 14b) to determine the area of the triangle. *[2 marks]*

Let points M and N be the midpoints of the triangle's sides OA and OB respectively.

d) Determine $\overrightarrow{MN}$. *[1 mark]*



e) Use the result from Question 14d) to show that the length of AB is twice the length of MN. *[1 mark]*

2023 **Paper 1** **Section 2** **Question 16** **Vectors and matrices**	 A curve is defined by the parametric equations $x = 2\tan(\theta)$ and $y = 3\sin(2\theta)$, where $0 \le \theta < \frac{\pi}{2}$. Given that $\frac{dy}{dx}$ can be expressed in the form $a\cos^4(\theta) + b\cos^2(\theta)$, where $a, b \in R$, determine the values of a and b. (5 marks)

2023
Paper 1
Section 2
Question 19

Vectors and matrices

Object A is released from the origin with constant velocity, $\boldsymbol{v}_A$, such that its position after t seconds is given by

$$\boldsymbol{r}_A = 2\sqrt{3}\,t\,\hat{\boldsymbol{i}} + 3t\,\hat{\boldsymbol{j}} + 2t\,\hat{\boldsymbol{k}},\ t \geq 0.$$

At a later time, object B is released from point $P\left(3\sqrt{3},\ 6,\ 0\right)$ and travels towards point $Q\left(5\sqrt{3},\ 8,\ 4\right)$ with constant velocity, $\boldsymbol{v}_B$, such that $\left|\boldsymbol{v}_B\right| = \sqrt{2}\left|\boldsymbol{v}_A\right|$.

Given that objects A and B collide, determine the time between the release of the two objects.

Assume all positions are given in metres and all velocities are given in metres per second.

(6 marks)

2022
Paper 1
Section 2
Question 11

Vectors and matrices

The position vector of a particle, r_1 (cm), over time, t(s), is given by

$$\boldsymbol{r}_1(t) = (2t+1)\hat{\boldsymbol{i}} + (t+3)\hat{\boldsymbol{j}} - (2t-3)\hat{\boldsymbol{k}}$$

a) Determine the velocity vector of the particle. [1 mark]

b) Determine the time when the position vector of the particle is perpendicular to its velocity vector. [2 marks]

The position vector of a second particle, r_2(cm), over time, t(s), is given by

$$\boldsymbol{r}_2(t) = (16 - 4t)\hat{\boldsymbol{i}} - (3t - 13)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

c) Determine whether the two particles collide. [3 marks]

2022 **Paper 1** **Section 2** **Question 16** **Vectors and matrices**	Consider this system of equations that corresponds to three planes. $x+5y=1+2z$ $x+z=3y+3$ $8y-\lambda=3z$ a) Use a Gaussian technique to determine the value of λ for which this system of equations has infinitely many solutions. [4 marks]

b) Use the result from Question 16a) to determine the infinitely many solutions. Express your answer in the form of a vector equation of a line. [3 marks]

2021
Paper 1
Section 2
Question 12

Vectors and matrices

Consider the plane $x - y - 2z = 15$.

a) Determine a vector n that is perpendicular to the plane. [1 mark]

b) Determine the vector equation of the line l that is perpendicular to the plane and contains the point A (–2, 1, 3). [1 mark]

c) Use the result from Question 12b) to express the equation of the line *l* in parametric form. [1 mark]

The line *l* and the plane intersect at point S.

d) Show that the coordinates of S are (2, – 3, – 5). [3 marks]

e) Determine $\overrightarrow{AS}$.

[1 mark]

f) Use a property of parallel vectors to verify that $\overrightarrow{AS}$ and $\boldsymbol{n}$ are parallel.

[1 mark]

2021
Paper 1
Section 2
Question 13

Vectors and matrices

Use $z = a + bi$ and $w = c + di$, where $a, b, c, d \in R$, to prove

$$|z-w|^2 = |z|^2 + |w|^2 - 2Re(z\overline{w})$$

[6 marks]

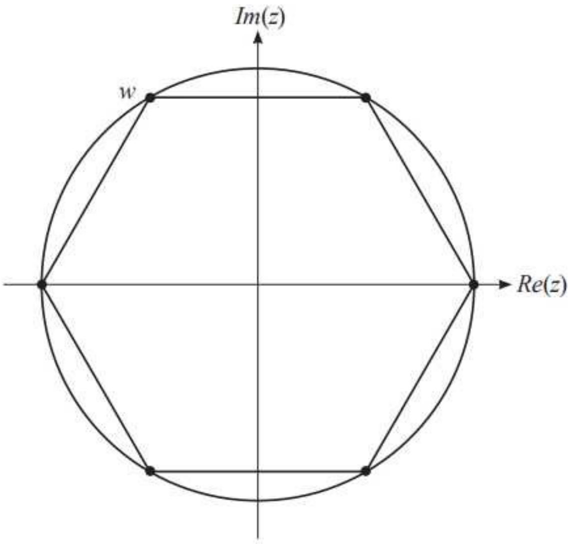
2021 **Paper 1** **Section 2** **Question 14** **Vectors and matrices**	An object is projected vertically upwards from ground level. After the object has been in motion for t seconds, its position vector through the air, in metres, is modelled by $$\boldsymbol{r}(t) = 5t(8-t)\hat{\boldsymbol{j}}$$ a) Determine the velocity of the object through the air, $v(t)$, in metres per second. [2 marks] b) Determine the number of seconds until the object reaches its maximum height. [2 marks] c) Determine the maximum height that the object reaches, in metres. [2 marks]

2021 Paper 1 Section 2 Question 19 Vectors and matrices	The velocity vectors of two objects A and B $\left(\text{in m s}^{-1}\right)$ at time t $\left(\text{in s}\right)$ are given respectively by $$\boldsymbol{v}_{\text{A}} = 6\sin(3t)\hat{\boldsymbol{i}} + 6\cos(3t)\hat{\boldsymbol{j}}$$ $$\boldsymbol{v}_{\text{B}} = \cos(t)\hat{\boldsymbol{i}} - \sin(t)\hat{\boldsymbol{j}}$$ Objects A and B are initially at $(-2, 0, 2)$ and $(0, 1, -1)$ respectively. Determine the position of Object A when it is 4 metres away from Object B for the first time. [7 marks]

2020
Paper 1
Section 2
Question 11

Vectors and matrices

The vertices of a regular hexagon are positioned on the circumference of a unit circle as shown on the Argand plane.



Consider the complex number w, as shown on the plane.

a) Determine w, expressing your answer in the form $r\ \text{cis}(\theta)$. [1 mark]

b) Convert w into Cartesian form. [2 marks]

Each vertex of the hexagon is a solution of an equation of the form $z^n = a$ where $z \in C$.

c) State the value of n. [1 mark]

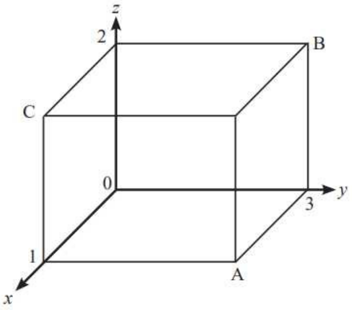
d) State the value of a. [1 mark]

e) Verify that w satisfies the equation $z^n = a$ using the results from 11c) and 11d). [2 marks]

2020
Paper 1
Section 2
Question 12

Vectors and matrices

Consider the vertices A, B and C of the rectangular prism as shown.



a) State the coordinates of A, B and C. [1 mark]

b) Determine a unit vector, $\hat{n}$, that is normal to the plane containing A, B and C. [3 marks]

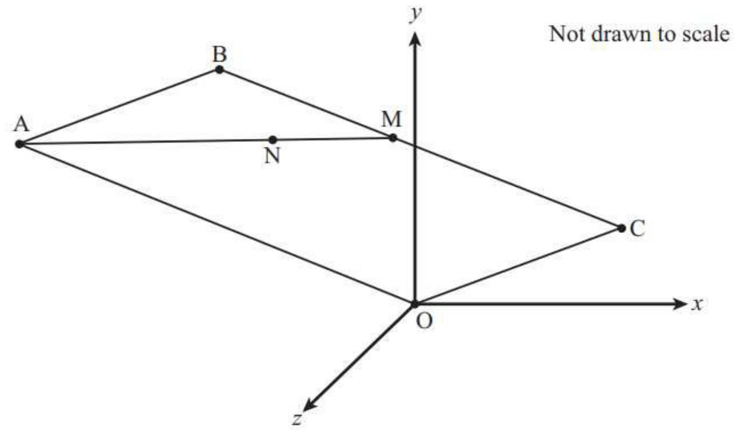
c) Verify that $\hat{n}$ is perpendicular to $\overrightarrow{AB}$. [2 marks]

d) Determine the Cartesian equation of the plane that contains A, B and C. [2 marks]

2020
Paper 1
Section 2
Question 15

Vectors and matrices

The points O(0, 0, 0), A(–6, 2, –2) and C(3, 1, 2) are represented in three-dimensional space in the diagram.



OABC forms a parallelogram in three-dimensional space.

a) Determine the coordinates of B. [1 mark]

M is the midpoint of BC.

b) Determine the vector that represents $\overrightarrow{OM}$. [1 mark]

N divides AM in the ratio 2:1.

c) Determine the vector that represents $\overrightarrow{ON}$. [2 marks]

d) Use a vector method to show that O, B and N lie on a straight line. [2 marks]

Paper 2 Section 1

2024
Paper 2
Section 1
Question 3

Vectors and matrices

Given $\boldsymbol{a} = \hat{j} + \hat{k}$ and $\boldsymbol{b} = 2\hat{i} + \hat{k}$, determine $\boldsymbol{a} \times \boldsymbol{b}$.

(A) $\hat{i} - 2\hat{j} - 2\hat{k}$

(B) $\hat{i} - 2\hat{j} + 2\hat{k}$

(C) $\hat{i} + 2\hat{j} - 2\hat{k}$

(D) $\hat{i} + 2\hat{j} + 2\hat{k}$

2024
Paper 2
Section 1
Question 5

Vectors and matrices

The equation of a plane is $2x - 4z - 8 = 0$.

Determine the point where the plane intersects the z-axis.

(A) (0, 0, −4)

(B) (0, 0, −2)

(C) (0, 0, 2)

(D) (0, 0, 4)

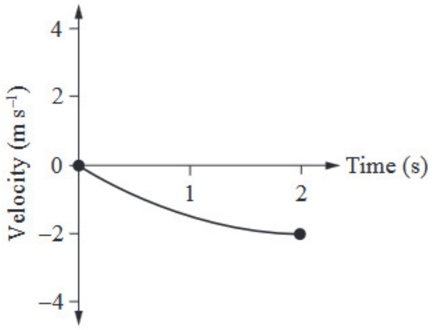
2024
Paper 2
Section 1
Question 10

Vectors and matrices

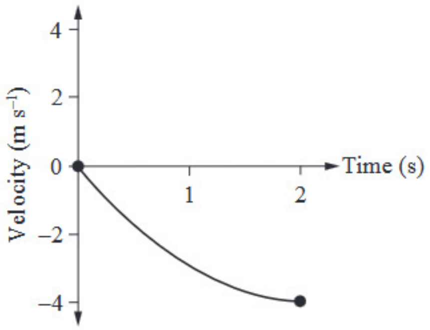
The acceleration (m s^{-2}) of an object at time, t, for $0 \le t \le 2$ seconds is given by $a = \dfrac{2}{t+1}$.

Given that the object is initially at rest, its velocity–time graph is

(A)



(B)



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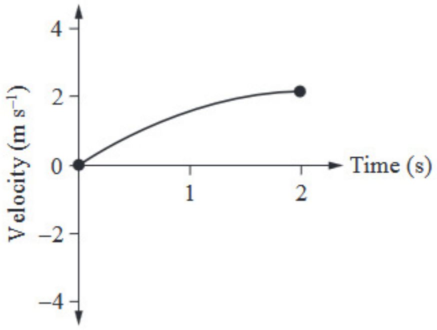

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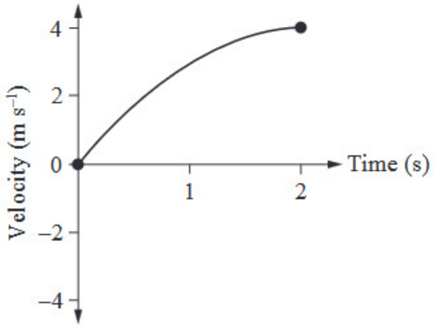


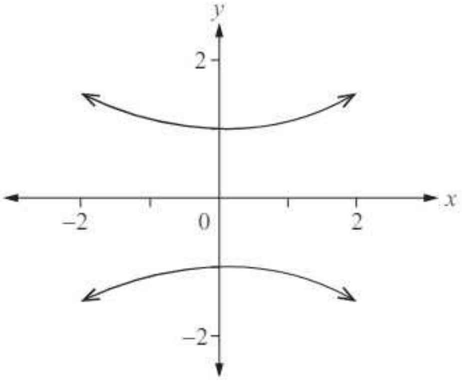
2023
Paper 2
Section 1
Question 4

Vectors and matrices

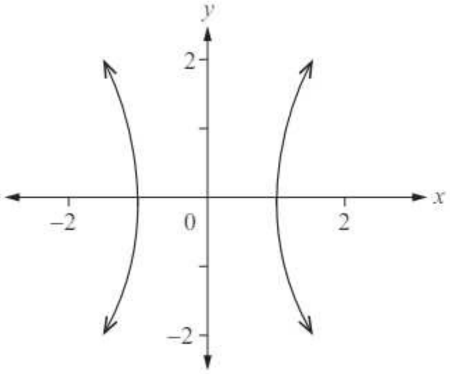
The position of a particle can be modelled using $\boldsymbol{r} = \cos(t)\hat{\boldsymbol{i}} - 2\sin(t)\hat{\boldsymbol{j}},\ t \geq 0$.

Which curve best represents the path of the particle?

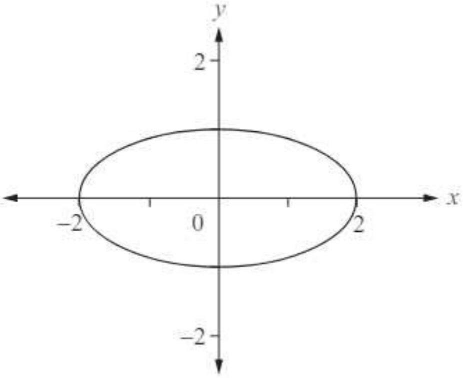
(A)



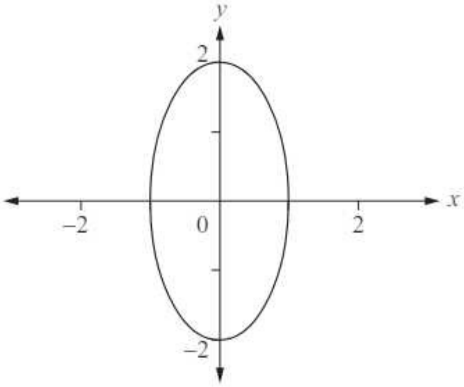
(B)



(C)



(D)



2023
Paper 2
Section 1
Question 5

Vectors and matrices

A plane contains the origin and the points (1, 2, 3) and (3, 2, 1).

A vector normal to the plane is

(A) $\begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$

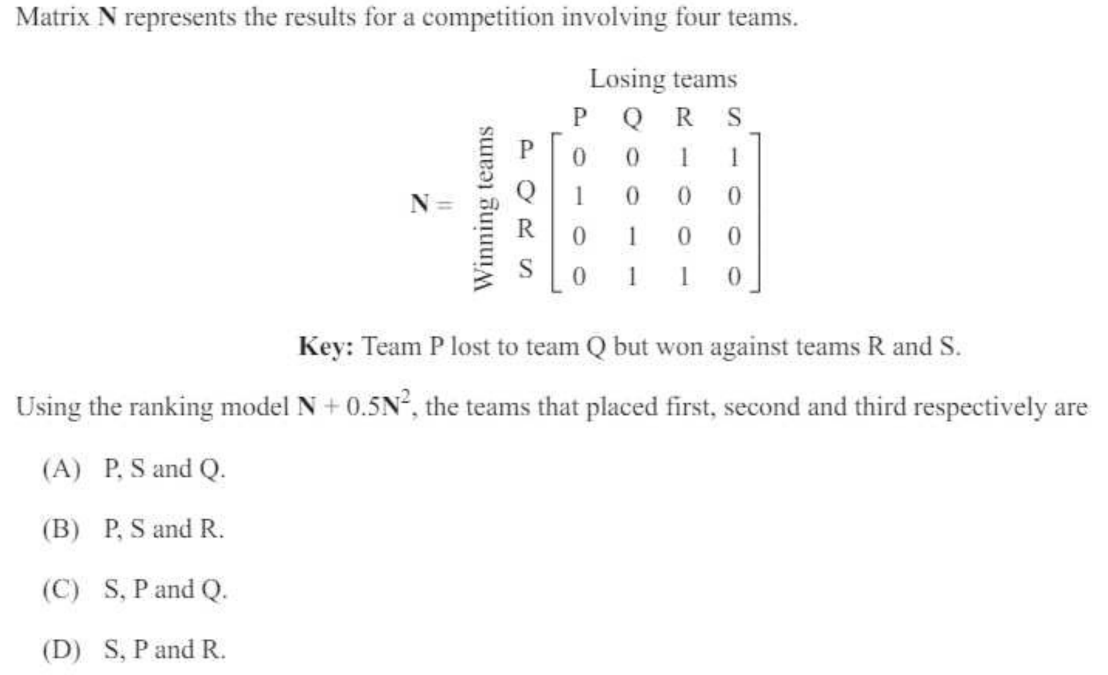
(B) $\begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$

(C) $\begin{pmatrix} -4 \\ -8 \\ -4 \end{pmatrix}$

(D) $\begin{pmatrix} -4 \\ -8 \\ 4 \end{pmatrix}$

2023
Paper 2
Section 1
Question 7

Vectors and matrices



Matrix **N** represents the results for a competition involving four teams.

Losing teams

$$\mathbf{N} = \text{Winning teams}\ \begin{array}{c|cccc} & P & Q & R & S \\ \hline P & 0 & 0 & 1 & 1 \\ Q & 1 & 0 & 0 & 0 \\ R & 0 & 1 & 0 & 0 \\ S & 0 & 1 & 1 & 0 \end{array}$$

Key: Team P lost to team Q but won against teams R and S.

Using the ranking model $\mathbf{N} + 0.5\mathbf{N}^2$, the teams that placed first, second and third respectively are

(A) P, S and Q.

(B) P, S and R.

(C) S, P and Q.

(D) S, P and R.

2022
Paper 2
Section 1
Question 2

Vectors and matrices

The win/draw/loss results after a netball competition involving five teams is represented in matrix M.

Losing teams

$$\mathbf{M} = \text{Winning teams}\ \begin{array}{c|ccccc} & P & Q & R & S & T \\ \hline P & 0 & 1 & 2 & 0 & 2 \\ Q & 1 & 0 & 0 & 1 & 1 \\ R & 0 & 2 & 0 & 0 & 0 \\ S & 2 & 1 & 2 & 0 & 2 \\ T & 0 & 1 & 2 & 0 & 0 \end{array}$$

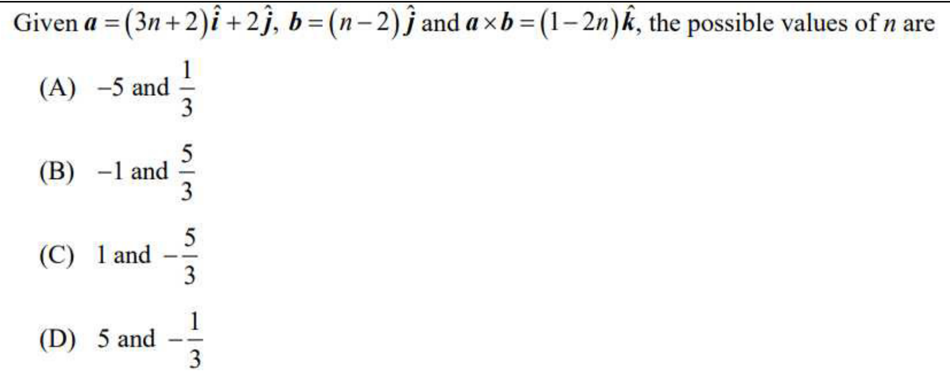
Key: Team P drew with Team Q, defeated Team R and Team T, and lost to Team S

The model $M + M^2 + M^3$ is used to rank the teams. The final positions from first to fifth are

(A) S, Q, P, R, T
(B) S, Q, P, T, R
(C) S, P, Q, T, R
(D) S, P, Q, R, T

2022
Paper 2
Section 1
Question 7

Vectors and matrices



Given $\boldsymbol{a} = (3n+2)\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = (n-2)\hat{\boldsymbol{j}}$ and $\boldsymbol{a} \times \boldsymbol{b} = (1-2n)\hat{\boldsymbol{k}}$, the possible values of n are

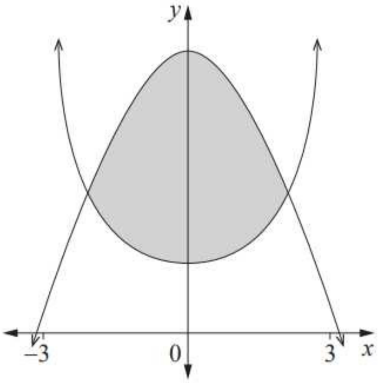
(A) -5 and $\frac{1}{3}$

(B) -1 and $\frac{5}{3}$

(C) 1 and $-\frac{5}{3}$

(D) 5 and $-\frac{1}{3}$

2022 **Paper 2** **Section 1** **Question 9** **Vectors and matrices**	Consider the matrix equation. $$\mathbf{X}\begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$$ Matrix **X** is (A) $\begin{bmatrix}0&1&1\\1&-1&2\\-1&0&2\end{bmatrix}$ (B) $\begin{bmatrix}0&1&-1\\1&-1&0\\1&2&2\end{bmatrix}$ (C) $\begin{bmatrix}2&2&1\\4&3&3\\5&5&5\end{bmatrix}$ (D) $\begin{bmatrix}2&4&5\\2&3&5\\1&3&5\end{bmatrix}$

2021 **Paper 2** **Section 1** **Question 2** **Vectors and matrices**	Determine the area of the shaded region between the graphs of the functions $y=\frac{1}{3}\sec\left(\frac{x}{3}\right)$ and $y=2\cos\left(\frac{x}{2}\right)$, as shown.  Not to scale (A) 5.29 units^2 (B) 5.51 units^2 (C) 5.65 units^2 (D) 5.71 units^2

2021 **Paper 2** **Section 1** **Question 5** **Vectors and matrices**	A vector normal to the plane that contains the vectors $\begin{pmatrix}1\\3\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\2\end{pmatrix}$ is (A) $6\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}$ (B) $6\hat{\boldsymbol{i}}+2\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}}$ (C) $6\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}+3\hat{\boldsymbol{k}}$ (D) $6\hat{\boldsymbol{i}}-2\hat{\boldsymbol{j}}-3\hat{\boldsymbol{k}}$

2021
Paper 2
Section 1
Question 6

Vectors and matrices

The Cartesian equation of a sphere is given by $x^2 + y^2 + z^2 + 2x - 2y = 7$.

The centre and radius of the sphere are

(A) $(-1, 1, 0)$ and 3 respectively.

(B) $(-1, 1, 0)$ and 9 respectively.

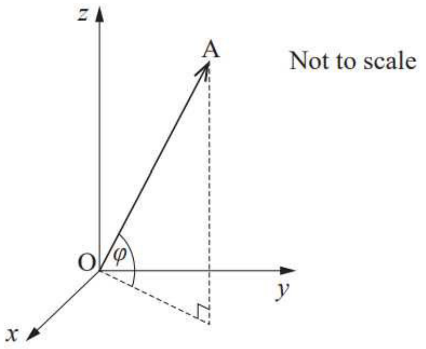
(C) $(1, -1, 0)$ and 3 respectively.

(D) $(1, -1, 0)$ and 9 respectively.

2021
Paper 2
Section 1
Question 7

Vectors and matrices

The altitude angle of $\overrightarrow{OA}$ is represented as φ.



Given the coordinates of A are $(3, 4, 6)$, the altitude angle of $\overrightarrow{OA}$ in radians is

(A) 0.93

(B) 0.88

(C) 0.69

(D) 0.66

2020
Paper 2
Section 1
Question 1

Vectors and matrices

The position x (m) at time t (s) of a 7 kg particle moving in a straight line is given by

$$x = 3t^3 - 5t^2 + 2t - 4 \text{ for } 0 \leq t \leq 10$$

Determine the time when the particle has a momentum of 620 kg m s^{-1}.

(A) 1.73 s
(B) 2.60 s
(C) 3.66 s
(D) 3.71 s

2020
Paper 2
Section 1
Question 2

Vectors and matrices

The Leslie matrix for a certain endangered species is given.

$$\mathbf{L}=\begin{bmatrix}0.8 & 2.4 & 0.3\\ 0.4 & 0 & 0\\ 0 & 0.55 & 0\end{bmatrix}$$

A group of the species was moved into a secure property at the start of 2018. The initial female population is given.

$$\mathbf{N}_0=\begin{bmatrix}150\\ 80\\ 40\end{bmatrix}$$

The best estimate of the total female population at the start of 2025 is

(A) 3000
(B) 4000
(C) 5000
(D) 6000

2020
Paper 2
Section 1
Question 6

Vectors and matrices

Solve the matrix equation for **X**.

$$\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\mathbf{X}\begin{bmatrix}4 & 5\\ 6 & 7\end{bmatrix}=\begin{bmatrix}8 & 9\\ 0 & 1\end{bmatrix}$$

(A) $\begin{bmatrix}-9 & -9\\ 4 & 4\end{bmatrix}$

(B) $\begin{bmatrix}3 & -4\\ -1 & 2\end{bmatrix}$

(C) $\begin{bmatrix}13 & -14\\ -11 & 12\end{bmatrix}$

(D) $\begin{bmatrix}54 & 56\\ -28 & -29\end{bmatrix}$

2020
Paper 2
Section 1
Question 9

Vectors and matrices

Two objects, P and Q, move in three-dimensional space such that their positions, *r*, over time, *t*, are described by the following vectors until they collide.

$$\boldsymbol{r}_P=(t^2-4t)\hat{\boldsymbol{\imath}}+(2t^2-t+3)\hat{\boldsymbol{\jmath}}-(6-5t)\hat{\boldsymbol{k}}$$
$$\boldsymbol{r}_Q=(-t^2+2t)\hat{\boldsymbol{\imath}}+(3t+t^2)\hat{\boldsymbol{\jmath}}+t^2\hat{\boldsymbol{k}}$$

The objects will collide at

(A) $t=0$
(B) $t=1$
(C) $t=2$
(D) $t=3$

<table>
<tr>
<td>2024
Paper 2
Section 2
Question 12

Vectors and matrices</td>
<td>
A system of linear equations is given by

$$x - 2y - 2z = -6$$
$$-3x - y + z = 2$$
$$2x + 3y - 5z = 10$$

a) Express the system of equations as a matrix equation of the form $\boldsymbol{AX} = \boldsymbol{B}$, where $\boldsymbol{A}$ is a 3×3 matrix and both $\boldsymbol{X}$ and $\boldsymbol{B}$ are 3×1 column vectors. *[1 mark]*

b) Use matrix algebra to express $\boldsymbol{X}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{B}$. *[1 mark]*

c) Use your result from Question 12b) to determine the solution of the system of equations. *[1 mark]*

d) Verify your result from Question 12c) using one of the given linear equations. *[1 mark]*
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2024
Paper 2
Section 2
Question 15

Vectors and matrices

The vectors representing the position (m) of particles A and B are given by $\boldsymbol{r}_{\mathrm{A}} = (4t-9)\hat{\boldsymbol{i}} - 2(5-t)\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}$ and $\boldsymbol{r}_{\mathrm{B}} = (t^2+1)\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + (4-at^2)\hat{\boldsymbol{k}}$ respectively, where t is the time of motion for $0 \le t \le 10$ seconds.

a) Show that particle A passes through the point $P(5, -3, -8)$. *[2 marks]*

b) Given that particle B also passes through point P, determine the value of a. *[2 marks]*

c) Determine the vector that represents the displacement of particle B relative to particle A during the given time of motion. Express your answer in simplest form. *[1 mark]*

d) Use your result from Question 15c) to determine the shortest distance between particles A and B during the given time of motion. *[3 marks]*

2024
Paper 2
Section 2
Question 17

Vectors and matrices

An object moves with a constant speed of v in a circular path.

The position vector of the object is given by

$$\boldsymbol{r} = r\cos(\omega t)\hat{\boldsymbol{i}} + r\sin(\omega t)\hat{\boldsymbol{j}}$$

where

- r is the radius (metres) of the circle
- ω is the angular velocity (radians per second)
- t is the time (seconds) of motion for $t \geq 0$.

Use vector calculus to prove that the magnitude of the acceleration of the object is $|\boldsymbol{a}| = \frac{v^2}{r}$.

[6 marks]

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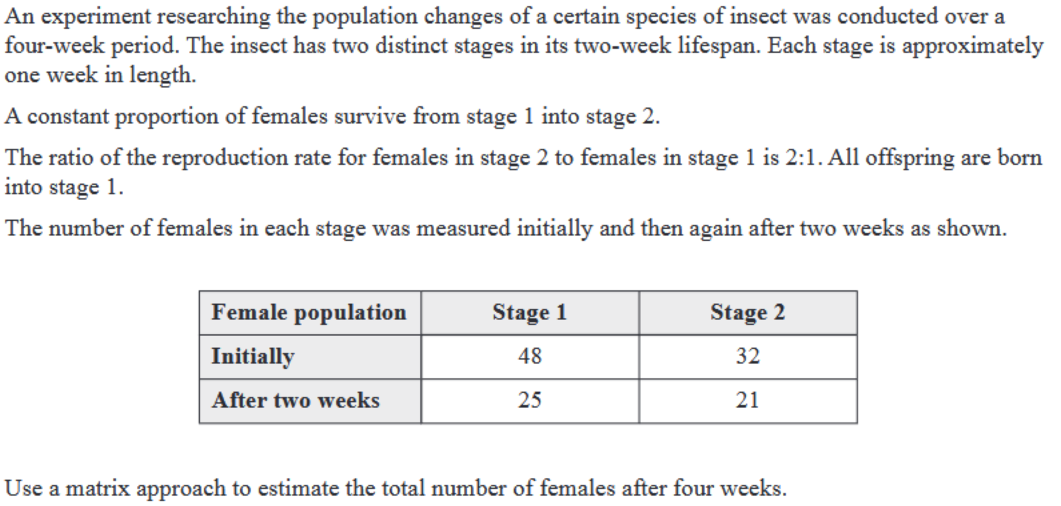
Section 2
Question 19

Vectors and matrices



An experiment researching the population changes of a certain species of insect was conducted over a four-week period. The insect has two distinct stages in its two-week lifespan. Each stage is approximately one week in length.

A constant proportion of females survive from stage 1 into stage 2.

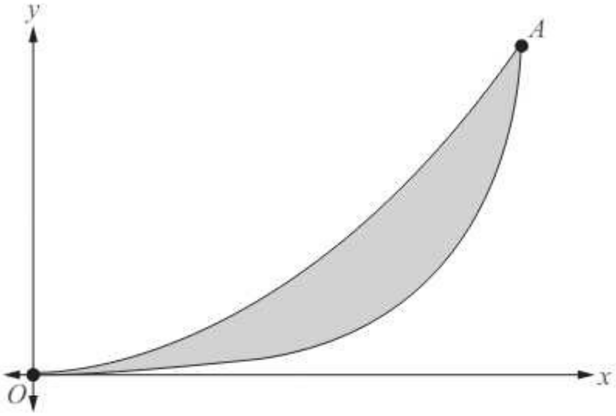
The ratio of the reproduction rate for females in stage 2 to females in stage 1 is 2:1. All offspring are born into stage 1.

The number of females in each stage was measured initially and then again after two weeks as shown.

Female population	Stage 1	Stage 2
Initially	48	32
After two weeks	25	21

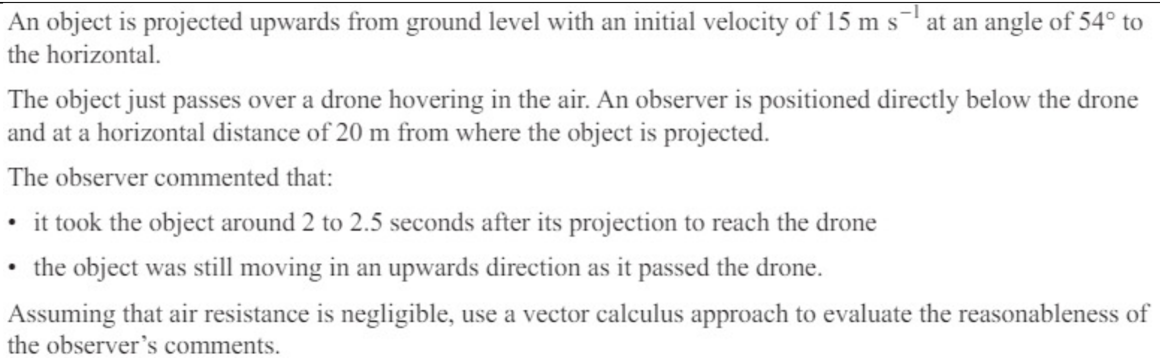
Use a matrix approach to estimate the total number of females after four weeks.

[6 marks]

2023 Paper 2 Section 2 Question 11 Vectors and matrices	The bounded region between the graphs of the functions $y=-1+\sec\left(\frac{x}{5}\right)$ and $y=0.1x^2$ over a certain domain is shaded as shown. The two functions intersect at the origin and point A.  a) Determine the coordinates of point A. *[1 mark]* b) Calculate the area of the shaded region. *[1 mark]* The shaded region is rotated about the x-axis to form a solid of revolution. c) Determine the volume of the solid formed. *[2 marks]*

2023
Paper 2
Section 2
Question 17

Vectors and matrices



An object is projected upwards from ground level with an initial velocity of 15 m s^{-1} at an angle of 54° to the horizontal.

The object just passes over a drone hovering in the air. An observer is positioned directly below the drone and at a horizontal distance of 20 m from where the object is projected.

The observer commented that:

- it took the object around 2 to 2.5 seconds after its projection to reach the drone
- the object was still moving in an upwards direction as it passed the drone.

Assuming that air resistance is negligible, use a vector calculus approach to evaluate the reasonableness of the observer's comments.

2022
Paper 2
Section 2
Question 12

Vectors and matrices

A scientist collects data for a species of tree frog in a protected area. Details for the female tree frog population are shown in the table.

Age (years)	0–1	1–2	2–3	3–4
Population in Year 1	150	101	84	62
Birth (breeding) rate	0.4	0.7	0.5	0.1
Survival rate	0.6	0.3	0.2	0

The scientist uses a Leslie matrix model to make predictions about the female tree frog population.

a) State the initial population matrix. [1 mark]

b) Determine the Leslie matrix. [1 mark]

A species is considered to be endangered if the female population in a restricted area is predicted to fall to less than 125 in the next 20 years.

c) Determine whether this species of tree frog is considered to be endangered. [3 marks]

2022
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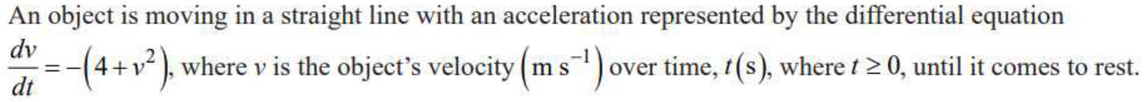
Section 2
Question 14

Vectors and matrices



An object is moving in a straight line with an acceleration represented by the differential equation $\frac{dv}{dt} = -\left(4+v^2\right)$, where v is the object's velocity $\left(\mathrm{m\ s^{-1}}\right)$ over time, $t(\mathrm{s})$, where $t \geq 0$, until it comes to rest.

a) Determine the general solution of the differential equation. [3 marks]

The initial velocity of the object is 1.5 m s^{-1}.

b) Determine the time when the particle comes to rest. [2 marks]

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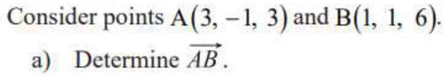
Section 2
Question 15

Vectors and matrices

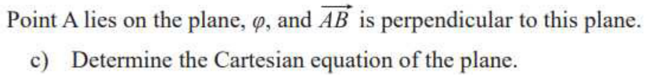


Consider points $A(3, -1, 3)$ and $B(1, 1, 6)$.

a) Determine $\overrightarrow{AB}$.

[1 mark]

b) Determine the Cartesian equation of the line that passes through points A and B. [2 marks]



Point A lies on the plane, φ, and $\overrightarrow{AB}$ is perpendicular to this plane.

c) Determine the Cartesian equation of the plane.

[2 marks]

2022 **Paper 2** **Section 2** **Question 19** **Vectors and matrices**	A research organisation plans to use a drone to drop a scientific instrument vertically from a stationary position above the ocean surface. The acceleration (ms^{-2}) of the falling instrument can be modelled by $9.8 – 0.1v$, where v is its velocity (ms^{-1}). In order for the instrument sensors to activate, its speed as it hits the ocean surface must reach at least 20 m s^{-1}. However, if it hits with a speed above 50 ms^{-1} , the sensors will be damaged. Determine the range of the drone's flying height above the ocean surface to ensure that the sensors are activated but not damaged. [7 marks]

2021
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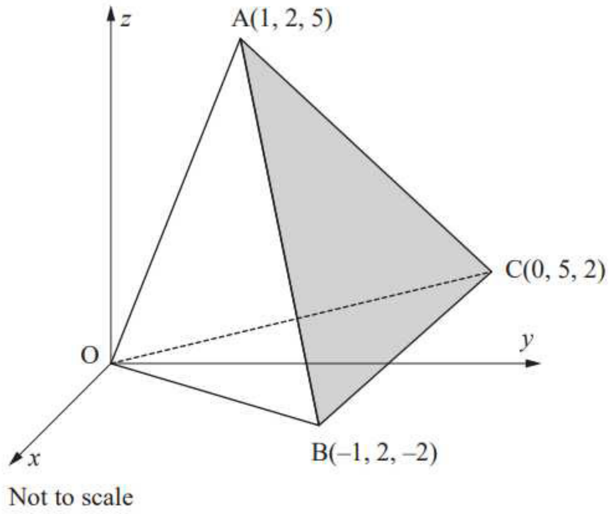
Section 2
Question 11

Vectors and matrices

OABC is a triangular-based pyramid, as shown.



Use a vector method to determine the area of the shaded face of the pyramid. [4 marks]

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Paper 2
Section 2
Question 14

Vectors and matrices

The Tasmanian thornbill is a species of bird that has an average life span of three years. Female thornbills do not reproduce in their first year, but produce an average of four female offspring in each of their second and third years. The survival rate of each age group is estimated as 25% in their first year and 30% in their second year.

A Leslie matrix, L, modelling the population distribution of the Tasmanian thornbill, has been partially completed.

$$\mathbf{L} = \begin{bmatrix} 0 & 4 & 4 \\ x & 0 & 0 \\ 0 & y & 0 \end{bmatrix}$$

a) State the values of x and y. [1 mark]

At the start of 2021, a study began into the population of Tasmanian thornbills. The study:

- estimated that the initial female population was 510 in their first year, 480 in their second year and 420 in their third year
- found that the ratio of male to female was approximately 1:2.

b) Estimate the total population of Tasmanian thornbills at the start of 2025. [4 marks]

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Paper 2
Section 2
Question 16

Vectors and matrices

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

Given $\mathbf{A}^4 = p\mathbf{A}^3 + q\mathbf{A}^2 + r\mathbf{A} + 3\mathbf{I}$, use matrix algebra to determine the value of the scalars p, q and r.

[6 marks]

2021 **Paper 2** **Section 2** **Question 17** **Vectors and matrices**	An object with a mass of 2 kg is released from rest at the top of a 1 metre long frictionless plane inclined at 30° to the horizontal. A force of ***P*** newtons acting parallel to the plane opposes the motion of the object as it travels down the plane. When the object is x metres from the top of the plane, its velocity is v m s^{-1}. Given $\lvert \boldsymbol{P} \rvert = \dfrac{4}{\sqrt{4-x^2}}$, determine x when $v = 2$. [7 marks]

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Paper 2
Section 2
Question 11

Vectors and matrices

Teams A, B, C, D and E participated in a competition with the following results:

- A defeated D.
- B defeated A, C and E.
- C defeated A and E.
- D defeated B, C and E.
- E defeated A.

To rank the teams at the end of the competition, the organisers constructed a dominance matrix, N, that is partially completed.

a) By allocating 1 to represent ‘defeated’ and 0 to represent either ‘was defeated by’ or ‘no result’, complete matrix N. [1 mark]

Losing teams

$$\mathbf{N} = \text{Winning teams} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{ccccc} A & B & C & D & E \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{array}$$

The organisers need to rank the teams into individual places from first to fifth place.

They decide to use the ranking model $N + N^2$ to achieve this.

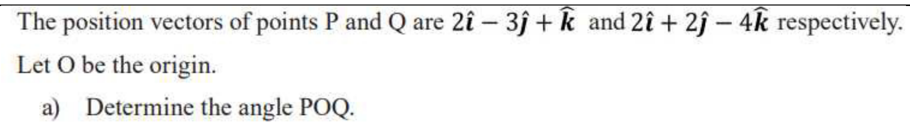
b) Use the model $N + N^2$ to rank the teams. [2 marks]

c) Use the result from 11b) to identify a limitation of the organisers' ranking model. [1 mark]

d) State a mathematical refinement the organisers could consider to overcome the limitation of the ranking model identified in 11c). [1 mark]

2020
Paper 2
Section 2
Question 15

Vectors and matrices



The position vectors of points P and Q are $2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $2\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ respectively.

Let O be the origin.

a) Determine the angle POQ.

[2 marks]

Points O, P and Q are joined to form a triangle.

b) Determine the area of triangle POQ. [2 marks]

2020 Paper 2 Section 2 Question 17 Vectors and matrices	An object is released from rest at a height of 100 m above the ground. The motion of the vertical descent of the object is modelled by $v\dfrac{dv}{dx} = 9.8 - 0.004v^2 \ \ (v \geq 0)$ where v is the velocity (m s^{-1}) and x is the displacement from the ground (m). Determine the velocity of the object when it strikes the ground. [7 marks]

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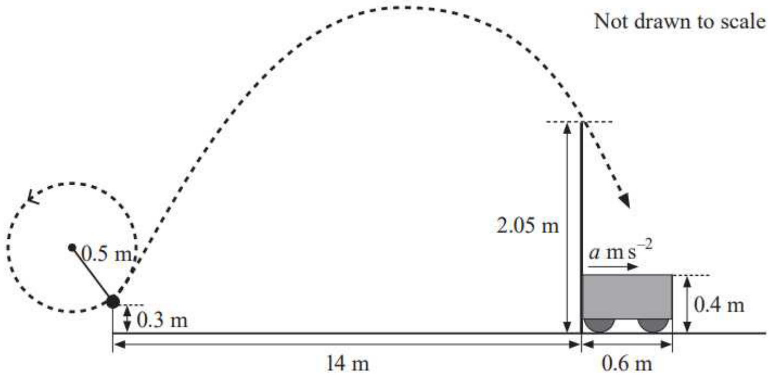
Paper 2
Section 2
Question 19

Vectors and matrices

An object is swinging at the end of a 0.5 m length of string in a vertical circular path with a constant angular speed, completing each revolution in 0.24 seconds.

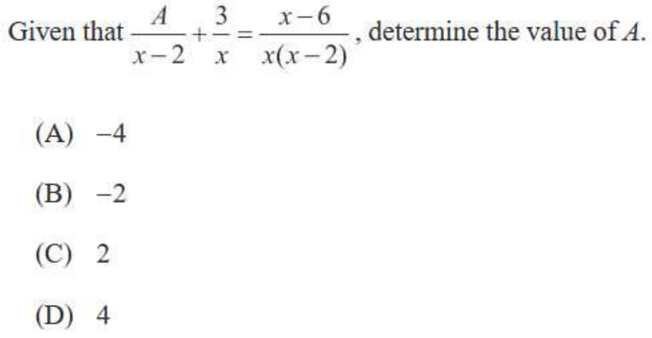
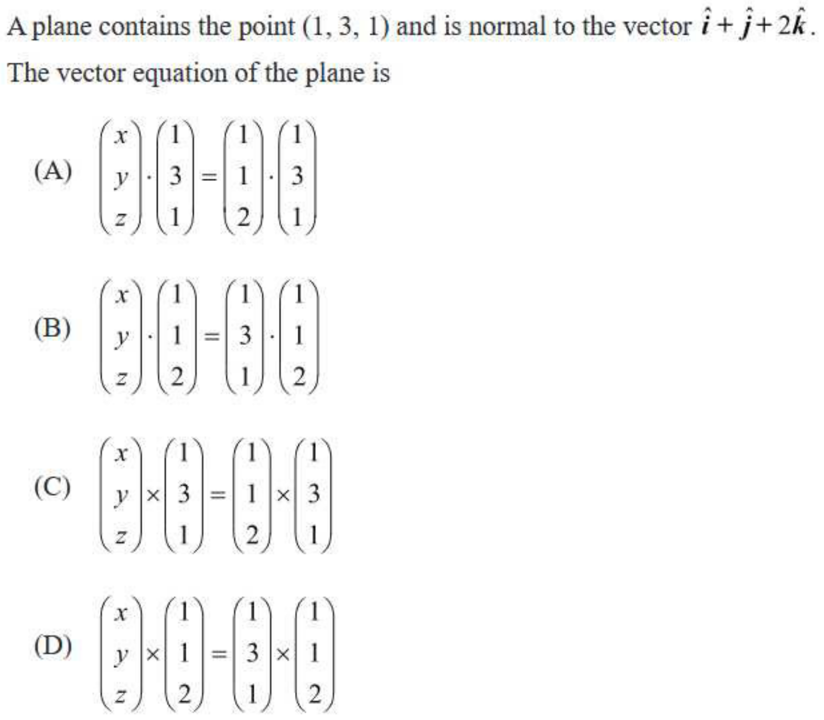
The object is projected from a height of 0.3 m above the ground in a vertical plane and just passes over a narrow pole as shown in the diagram. The pole is 2.05 m high and its base is 14 m horizontally from where the object was projected.



A flat-topped vehicle of length 0.6 m and height 0.4 m is initially at rest against the pole as shown in the diagram. At the instant that the object is projected, the vehicle moves in a horizontal direction away from the pole in the same vertical plane with an acceleration of magnitude of a m s^{-2}. The object strikes the middle of the top of the vehicle.

Assuming that air resistance is negligible, use vector calculus to model the motion of the projectile in order to determine the value of *a*. [7 marks]

Marking Guide – Paper 1 Section 1

2024 **Paper 1** **Section 1** **Question 2** **Vectors and matrices**	 Given that $\frac{A}{x-2}+\frac{3}{x}=\frac{x-6}{x(x-2)}$, determine the value of A. (A) -4 (B) -2 (C) 2 (D) 4 **Answer is B.**
2024 **Paper 1** **Section 1** **Question 4** **Vectors and matrices**	 A plane contains the point (1, 3, 1) and is normal to the vector $\hat{i}+\hat{j}+2\hat{k}$. The vector equation of the plane is (A) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}$ (B) $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\2\end{pmatrix}$ (C) $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}1\\3\\1\end{pmatrix}$ (D) $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}\times\begin{pmatrix}1\\1\\2\end{pmatrix}$ **Answer is B.**

2024
Paper 1
Section 1
Question 5

Vectors and matrices

The augmented matrix shown is produced when a Gaussian elimination technique is used to solve a certain system of equations with three variables.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & -10 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 3 & 2 \end{array}\right]$$

Given that row 1 values of the matrix represent $x+4y+2z=-10$, the unique solution for y is

(A) $\frac{2}{5}$

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) $\frac{5}{2}$

Answer is D.

2024
Paper 1
Section 1
Question 6

Vectors and matrices

Players P, Q, R and S played each other once in a competition where there were no draws.

Only the following results are known.

- Player P defeated players Q and R.
- Player Q defeated two players.
- Players R and S each defeated one player.

Based on these results, a dominance matrix N was partially constructed as shown.

$$N = \begin{array}{c} \\ P \\ Q \\ R \\ S \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ \square & \square & \square & \square \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The completed matrix N is

(A) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Answer is B.

2024
Paper 1
Section 1
Question 7

Vectors and matrices

A, B and C are points in three-dimensional space.

If $2\overrightarrow{AB} = \overrightarrow{BC}$, then

(A) $|\overrightarrow{AB}|$ is twice the value of $|\overrightarrow{BC}|$.

(B) $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular.

(C) only one plane contains A, B and C.

(D) a straight line passes through A, B and C.

Answer is D.

2023
Paper 1
Section 1
Question 1

Vectors and matrices

The position of a particle is given by $\boldsymbol{r} = (t+2)\hat{\boldsymbol{i}} + t^2\hat{\boldsymbol{j}}$ for $t \geq 0$.

Determine the corresponding Cartesian equation.

(A) $y = x^2 - 4$

(B) $y = x^2 + 4$

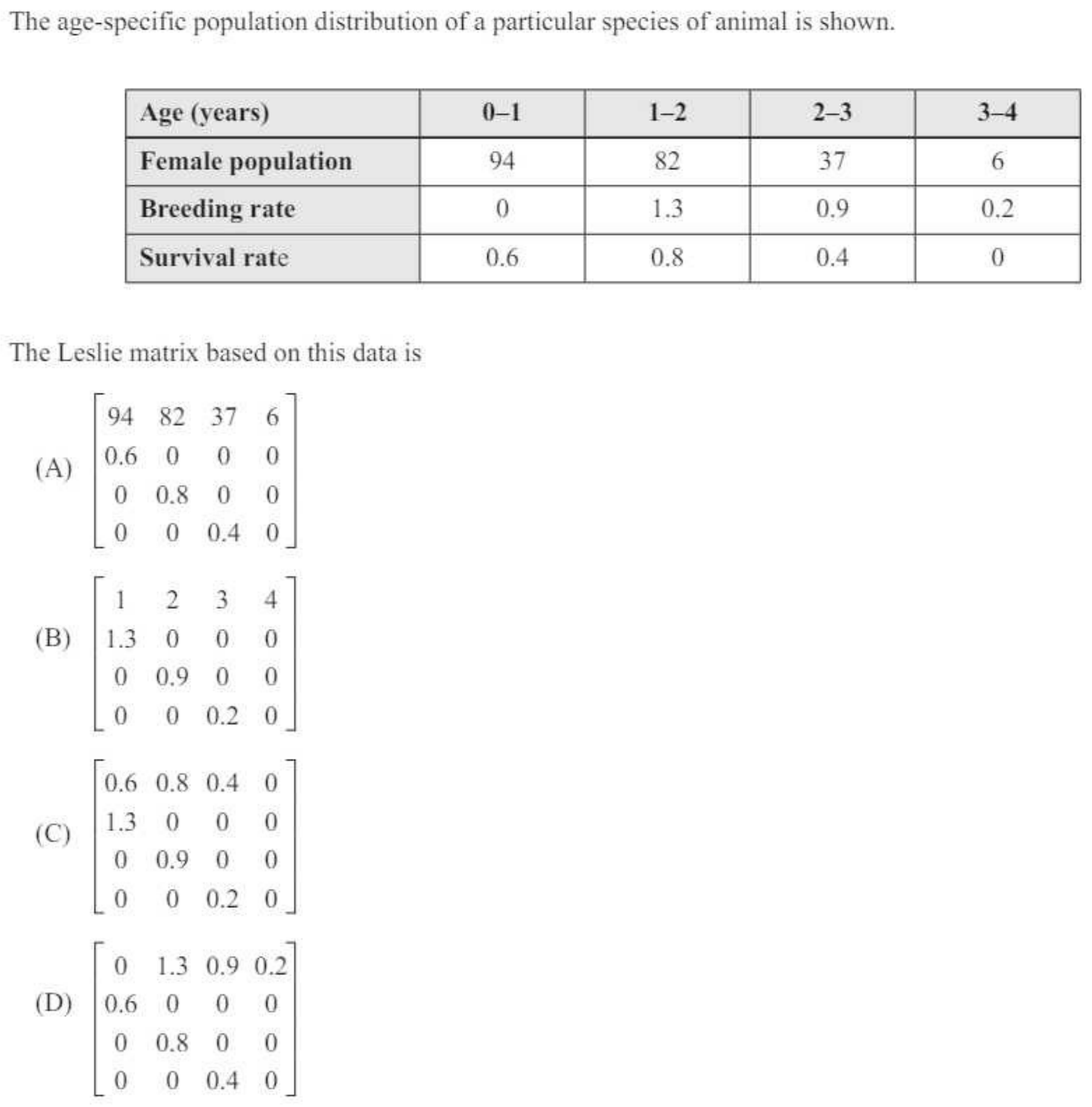
(C) $y = x^2 - 4x + 4$

(D) $y = x^2 + 4x + 4$

Answer is C.

2023
Paper 1
Section 1
Question 4

Vectors and matrices



The age-specific population distribution of a particular species of animal is shown.

Age (years)	**0–1**	**1–2**	**2–3**	**3–4**
Female population	94	82	37	6
Breeding rate	0	1.3	0.9	0.2
Survival rate	0.6	0.8	0.4	0

The Leslie matrix based on this data is

(A) $\begin{bmatrix} 94 & 82 & 37 & 6 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0.6 & 0.8 & 0.4 & 0 \\ 1.3 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$

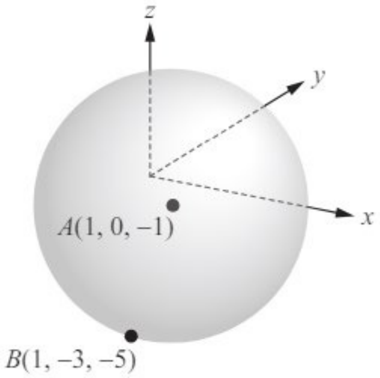
(D) $\begin{bmatrix} 0 & 1.3 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{bmatrix}$

Answer is D.

2023
Paper 1
Section 1
Question 8

Vectors and matrices

Point A is the centre of a sphere and point B lies on its surface as shown.



The equation of the sphere is

(A) $x^2 - 2x + y^2 + z^2 + 2z = 23$

(B) $x^2 + 2x + y^2 + z^2 - 2z = 23$

(C) $x^2 - 2x + y^2 + z^2 + 2z = 25$

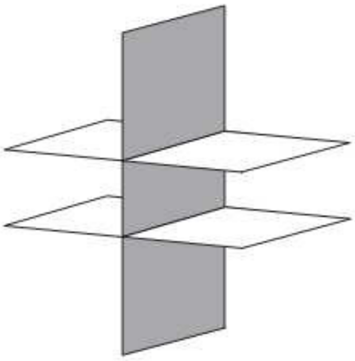
(D) $x^2 + 2x + y^2 + z^2 - 2z = 25$

Answer is A.

2023
Paper 1
Section 1
Question 9

Vectors and matrices

The geometric interpretation of a certain system of three equations with no solution is shown.



Given two of the equations are $x + y - z = 0.5$ and $x - y - z = 0.5$, the third equation could be

(A) $2x - 2y - 2z = 1$

(B) $2x + 2y - 2z = 1$

(C) $2x - 2y + 2z = 3$

(D) $2x + 2y - 2z = 3$

Answer is D.

2022
Paper 1
Section 1
Question 6

Vectors and matrices

The Cartesian equation for a sphere with centre (−2, 3, −4) and radius 9 is

(A) $(x-2)^2+(y+3)^2+(z-4)^2=9$

(B) $(x+2)^2+(y-3)^2+(z+4)^2=9$

(C) $(x-2)^2+(y+3)^2+(z-4)^2=81$

(D) $(x+2)^2+(y-3)^2+(z+4)^2=81$

Answer is D.

2022
Paper 1
Section 1
Question 10

Vectors and matrices

A plane is represented by the equation $x - 2z = 5$. A vector normal to this plane is

(A) $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$

(B) $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

(C) $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

(D) $\begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

Answer is C.

2021
Paper 1
Section 1
Question 3

Vectors and matrices

An object has a velocity $\boldsymbol{v}(t)=e^{-2t}\hat{\boldsymbol{i}}+\left(\frac{1}{t}\right)\hat{\boldsymbol{k}}$, where t represents time $(t>0)$.

The displacement $\boldsymbol{r}(t)$ of the object could be

(A) $-2e^{-2t}\hat{\boldsymbol{i}}+\ln(t)\hat{\boldsymbol{k}}$

(B) $-2e^{-2t}\hat{\boldsymbol{i}}-\frac{1}{t^2}\hat{\boldsymbol{k}}$

(C) $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{i}}+\ln(t)\hat{\boldsymbol{k}}$

(D) $-\frac{1}{2}e^{-2t}\hat{\boldsymbol{i}}-\frac{1}{t^2}\hat{\boldsymbol{k}}$

Answer is C.

2021
Paper 1
Section 1
Question 5

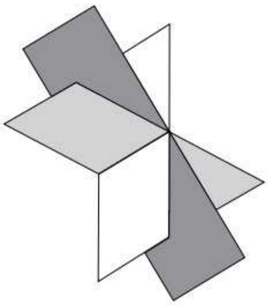
Vectors and matrices

The augmented matrix shown is produced when a Gaussian elimination technique is used to solve a certain system of equations with three variables.

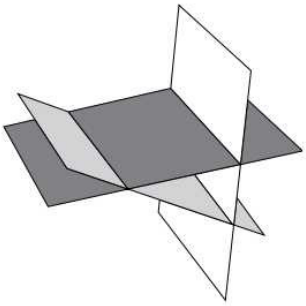
$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 4 \\ 0 & -1 & 5 & -6 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The geometric interpretation of the solution to this system of equations is best represented by

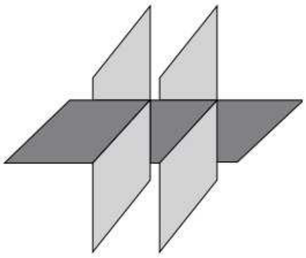
(A)



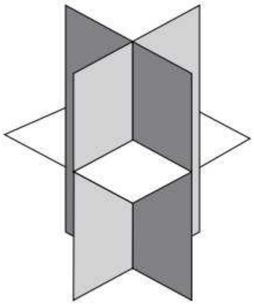
(B)



(C)



(D)

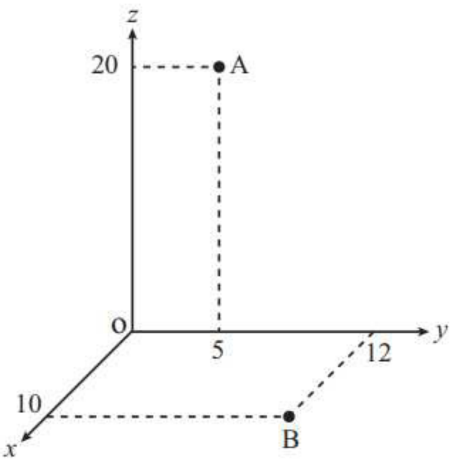


Answer is D.

2020
Paper 1
Section 1
Question 4

Vectors and matrices

Consider points A and B as shown.



The position vector representing the midpoint of AB is

(A) $\begin{pmatrix} 5 \\ 8.5 \\ 10 \end{pmatrix}$

(B) $\begin{pmatrix} 5 \\ 10 \\ 8.5 \end{pmatrix}$

(C) $\begin{pmatrix} 10 \\ 8.5 \\ 5 \end{pmatrix}$

(D) $\begin{pmatrix} 10 \\ 5 \\ 8.5 \end{pmatrix}$

Answer is A.

2020
Paper 1
Section 1
Question 8

Vectors and matrices

An equation of the line passing through the points A(2, 4, 5) and B(3, –2, 1) is

(A) $2\hat{\imath} + 4\hat{\jmath} + 5\hat{k} + t(3\hat{\imath} - 2\hat{\jmath} + \hat{k}), t \in R$

(B) $-3\hat{\imath} + 2\hat{\jmath} - \hat{k} + t(\hat{\imath} - 6\hat{\jmath} - 4\hat{k}), t \in R$

(C) $\frac{x-1}{2} = \frac{y+6}{4} = \frac{z+4}{5}$

(D) $\frac{x-3}{-1} = \frac{y+2}{6} = \frac{z-1}{4}$

Answer is D.

Marking Guide – Paper 1 Section 2

2024
Paper 1
Section 2
Question 11

Vectors and matrices

The vector equation of a straight line is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + k\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, where k is a scalar.

a) Express the equation of the line as a pair of parametric equations. *[1 mark]*

Sample response	The response
Given $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + k\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ In parametric form, $x = 2 - k$ $y = 2k$	• correctly express the equation of the line as a pair of parametric equations **[1 mark]**

b) Use your result from Question 11a) to express the equation of the line as a Cartesian equation. *[1 mark]*

Sample response	The response
$k = 2 - x$ $y = 2(2 - x)$	• expresses equation of line as a Cartesian equation **[1 mark]**

c) Determine the coordinates of the point that the line passes through when $k = 5$. *[1 mark]*

Sample response	The response
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $= \begin{pmatrix} -3 \\ 10 \end{pmatrix}$ Required point is $(-3, 10)$.	• determines coordinates of point **[1 mark]**

d) Determine the value of k when the line intersects the y-axis. *[1 mark]*

Sample response	The response
At the y-intercept, $x = 0$. $0 = 2 - k$ $k = 2$	• determines value of k **[1 mark]**

2024
Paper 1
Section 2
Question 14

Vectors and matrices

The displacement (cm) of a particle from the origin as it travels in two-dimensional space at time t for $0 \le t < \frac{\pi}{2}$ seconds is given by

$$\boldsymbol{r} = (2\sec(t) - 1)\hat{\boldsymbol{i}} + \tan(t)\hat{\boldsymbol{j}}$$

a) Express the path of the particle as a pair of parametric equations. *[1 mark]*

Sample response	The response
$r = (2\sec(t) - 1)\hat{i} + \tan(t)\hat{j}$ $x = 2\sec(t) - 1 \quad \ldots(1)$ $y = \tan(t) \quad \ldots(2)$	• correctly expresses the path of the particle as a pair of parametric equations **[1 mark]**

A general Cartesian form of a hyperbola with centre (h, k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1, \text{ where } a, b \neq 0.$$

b) Use a suitable Pythagorean identity to show that the path of the particle can be expressed in this general Cartesian form. *[3 marks]*

Sample response	The response
From (1), $\sec(t)=\dfrac{x+1}{2} \qquad \ldots(3)$	• expresses sec(t) as the subject of the formula **[1 mark]**
Using the Pythagorean identity $\tan^2(A)+1=\sec^2(A)$	• uses a suitable Pythagorean identity **[1 mark]**
From (2) and (3), $y^2+1=\left(\dfrac{x+1}{2}\right)^2$ $\dfrac{(x+1)^2}{4}-y^2=1$	• expresses path of the particle in general Cartesian form **[1 mark]**

c) Determine the centre of the hyperbolic path of the particle. *[1 mark]*

Sample response	The response
Centre is $(-1, 0)$.	• determines the centre of the path of the particle **[1 mark]**

2024
Paper 1
Section 2
Question 17

Vectors and matrices

The acceleration (m s^{-2}) of an object that moves in a straight line in an easterly direction over time t for $0 \le t \le \frac{\pi}{6}$ seconds is given by $a = 2\left(1+v^2\right)$, where v is its velocity (m s^{-1}).

The object is initially at rest at a position that is $\ln\left(\sqrt{2}\right)$ metres west of the origin.

A student uses this information to calculate that the object is positioned at the origin when $t = \frac{\pi}{6}$ seconds.

Evaluate the reasonableness of the student's calculation.

[6 marks]

Sample response	The response
Method 1 $a = 2\left(1+v^2\right)$ $\frac{dv}{dt} = 2\left(1+v^2\right)$ $\int \frac{1}{1+v^2}\,dv = \int 2\,dt$ $\tan^{-1}(v) = 2t + c$	• correctly determines a general solution to the differential equation in terms of v and time **[1 mark]**
At $t=0, v=0$: $\tan^{-1}(0) = 0 + c$ $\therefore c = 0$	• determines an appropriate constant of integration **[1 mark]**
$\therefore \tan^{-1}(v) = 2t$ $v = \tan(2t)$ $x = \int v\,dt$ $= \int \tan(2t)\,dt$ $= -\frac{1}{2}\int \frac{-2\sin(2t)}{\cos(2t)}\,dt$ $= -\frac{1}{2}\ln\left|\left(\cos(2t)\right)\right| + c$	• determines a general solution for displacement in terms of time **[1 mark]**
Given $x = -\ln\left(\sqrt{2}\right)$ when $t=0$ $-\ln\left(\sqrt{2}\right) = -\frac{1}{2}\ln\left(\cos(0)\right) + c$ $c = -\ln\left(\sqrt{2}\right)$	• determines an appropriate constant of integration **[1 mark]**
$x = -\frac{1}{2}\ln\left(\cos(2t)\right) - \ln\left(\sqrt{2}\right)$ When $t = \frac{\pi}{6}$ $x = -\frac{1}{2}\ln\left(\cos\left(\frac{\pi}{3}\right)\right) - \ln\left(\sqrt{2}\right)$ $= -\frac{1}{2}\ln\left(\frac{1}{2}\right) - \ln\left(\sqrt{2}\right)$	• determines a result for displacement when $t = \frac{\pi}{6}$ without a trigonometric term **[1 mark]**
$x = \ln\left(\frac{1}{2}\right)^{-\frac{1}{2}} - \ln\left(\sqrt{2}\right)$ $= \ln\left(\sqrt{2}\right) - \ln\left(\sqrt{2}\right)$ $= 0$ So the calculation is reasonable.	• provides an appropriate statement of reasonableness based on mathematical reasoning **[1 mark]**

	Method 2 $a=2\left(1+v^2\right)$ $\frac{dv}{dt}=2\left(1+v^2\right)$ $\int\frac{1}{1+v^2}\,dv=\int 2\,dt$ $\tan^{-1}(v)=2t+c$	• correctly determines a general solution to the differential equation in terms of v and time **[1 mark]**
	At $t=0, v=0$: $\tan^{-1}(0)=0+c$ $\therefore c=0$ $\tan^{-1}(v)=2t$ $v=\tan(2t)\quad \ldots(1)$	• determines an appropriate constant of integration **[1 mark]**
	$a=2\left(1+v^2\right)$ $v\frac{dv}{dx}=2\left(1+v^2\right)$ $\int\frac{v}{1+v^2}\,dv=\int 2\,dx$ $\frac{1}{2}\ln\left(1+v^2\right)=2x+c$	• determines a general solution to the differential equation in terms of v and displacement **[1 mark]**
	At $x=-\ln(\sqrt{2}), v=0$ $\frac{1}{2}\ln(1)=-2\ln(\sqrt{2})+c$ $\therefore c=\ln(2)$ $\therefore \frac{1}{2}\ln\left(1+v^2\right)=2x+\ln(2)\quad \ldots(2)$	• determines an appropriate constant of integration **[1 mark]**
	When $t=\frac{\pi}{6}$ using (1) $v=\tan\left(2\times\frac{\pi}{6}\right)$ $=\sqrt{3}$	• determines velocity when $t=\frac{\pi}{6}$ **[1 mark]**
	Using (2) $\frac{1}{2}\ln\left(1+v^2\right)=2x+\ln(2)$ $\frac{1}{2}\ln\left(1+\left(\sqrt{3}\right)^2\right)=2x+\ln(2)$ $2x=\frac{1}{2}\ln(4)-\ln(2)$ $x=0$ So the calculation is reasonable.	• provides an appropriate statement of reasonableness based on mathematical reasoning **[1 mark]**

Method 3	
$a=2\left(1+v^2\right)$ $\frac{dv}{dt}=2\left(1+v^2\right)$ $\int\frac{1}{1+v^2}\,dv=\int 2\,dt$ $\tan^{-1}(v)=2t+c$	- correctly determines a general solution to the differential equation in terms of v and time **[1 mark]**
At $t=0, v=0: \tan^{-1}(0)=0+c$ $\therefore c=0$ $\tan^{-1}(v)=2t$ $v=\tan(2t)\quad \ldots(1)$	- determines an appropriate constant of integration **[1 mark]**
$a=2\left(1+v^2\right)$ $v\frac{dv}{dx}=2\left(1+v^2\right)$ $\int\frac{v}{1+v^2}\,dv=\int 2\,dx$ $\frac{1}{2}\ln\left(1+v^2\right)=2x+c$	- determines a general solution to the differential equation in terms of v and displacement **[1 mark]**
At $x=-\ln(\sqrt{2}), v=0$ $\frac{1}{2}\ln(1)=-2\ln(\sqrt{2})+c$ $\therefore c=\ln(2)$ $\therefore \frac{1}{2}\ln\left(1+v^2\right)=2x+\ln(2)\quad \ldots(2)$	- determines an appropriate constant of integration **[1 mark]**
When $t=\frac{\pi}{6}$ (using (1) and (2)) $\frac{1}{2}\ln\left(1+\left(\tan\left(\frac{\pi}{3}\right)\right)^2\right)=2x+\ln(2)$ $\frac{1}{2}\ln\left(1+\left(\sqrt{3}\right)^2\right)=2x+\ln(2)$	- determines a result for the displacement when $t=\frac{\pi}{6}$ without a trigonometric term **[1 mark]**
$\frac{1}{2}\ln(4)=2x+\ln(2)$ $2x=\ln(2)-\ln(2)$ $x=0$ So the calculation is reasonable.	- provides an appropriate statement of reasonableness based on mathematical reasoning **[1 mark]**

2023
Paper 1
Section 2
Question 12

Vectors and matrices

Given $A=\begin{pmatrix}1 & -2\\ 1 & 2\end{pmatrix}$, $B=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}$ and $C=\begin{pmatrix}-1 & -1\\ 0 & 3\end{pmatrix}$, determine X in the matrix equation $XA-XC=B$.

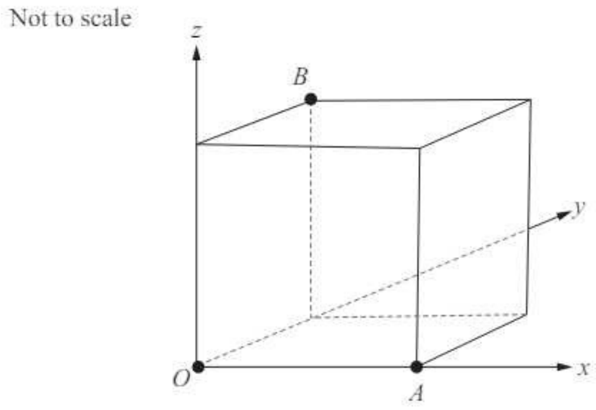
(5 marks)

Sample response	The response
$XA-XC=B$ $X(A-C)=B$	- correctly recognises the need to use X as a common factor **[1 mark]**
$X=B(A-C)^{-1}$	- expresses X as the subject of the equation **[1 mark]**
$X=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}\left[\begin{pmatrix}1 & -2\\ 1 & 2\end{pmatrix}-\begin{pmatrix}-1 & -1\\ 0 & 3\end{pmatrix}\right]^{-1}$ $=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\ 1 & -1\end{pmatrix}^{-1}$	- represents X in terms of two matrices **[1 mark]**
$=\begin{pmatrix}0 & 2\\ 1 & 3\end{pmatrix}\frac{1}{-1}\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}$	- calculates the inverse of an appropriate matrix **[1 mark]**
$=\begin{pmatrix}2 & -4\\ 4 & -7\end{pmatrix}$	- determines X **[1 mark]**

2023
Paper 1
Section 2
Question 14

Vectors and matrices

Consider a cube with three edges positioned along the x-, y- and z-axes on the Cartesian plane as shown. Points O, A and B are vertices of the cube.



a) Given $\overrightarrow{OA}=2\hat{\boldsymbol{i}}$, determine $\overrightarrow{OB}$. Express your answer in terms of $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$. *[1 mark]*

Sample response	The response
$\overrightarrow{OB}=2\hat{\boldsymbol{j}}+2\hat{\boldsymbol{k}}$	- correctly states $\overrightarrow{OB}$ **[1 mark]**

b) Calculate $\overrightarrow{OA}\times\overrightarrow{OB}$. *[1 mark]*

Sample response	The response
$\overrightarrow{OA}\times\overrightarrow{OB}=2\hat{i}\times\left(2\hat{j}+2\hat{k}\right)$ $=-4\hat{\boldsymbol{j}}+4\hat{\boldsymbol{k}}$	- calculates $\overrightarrow{OA}\times\overrightarrow{OB}$ **[1 mark]**

Consider the triangle formed by joining points O, A and B.

c) Use the result from Question 14b) to determine the area of the triangle. *[2 marks]*

Sample response	The response
$\text{Area}=\frac{1}{2}\left\lvert\overrightarrow{OA}\times\overrightarrow{OB}\right\rvert$ $=\frac{1}{2}\sqrt{(-4)^2+(4)^2}$ $=2\sqrt{2}\text{ units}^2$	- states an expression representing area of ΔOAB **[1 mark]** - calculates area **[1 mark]**

Let points M and N be the midpoints of the triangle's sides OA and OB respectively.

d) Determine $\overrightarrow{MN}$. *[1 mark]*

Sample response	The response
$m = \frac{1}{2}\overrightarrow{OA} = \hat{i}$ $n = \frac{1}{2}\overrightarrow{OB} = \hat{j} + \hat{k}$ $\overrightarrow{MN} = -\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$	• determines $\overrightarrow{MN}$ **[1 mark]**

e) Use the result from Question 14d) to show that the length of AB is twice the length of MN. *[1 mark]*

Sample response	The response
$\overrightarrow{AB} = -2\hat{i} + 2\hat{j} + 2\hat{k}$ $= 2\left(-\hat{\boldsymbol{i}} + \hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}\right)$ $= 2\overrightarrow{MN}$ So, the length of AB is twice the length of MN.	• shows $\overrightarrow{AB} = 2\overrightarrow{MN}$ **[1 mark]**

2023
Paper 1
Section 2
Question 16

Vectors and matrices

A curve is defined by the parametric equations $x = 2\tan(\theta)$ and $y = 3\sin(2\theta)$, where $0 \le \theta < \frac{\pi}{2}$.

Given that $\frac{dy}{dx}$ can be expressed in the form $a\cos^4(\theta) + b\cos^2(\theta)$, where $a, b \in R$, determine the values of a and b.

(5 marks)

Sample response	The response
$x = 2\tan(\theta)$ $\frac{dx}{d\theta} = 2\sec^2(\theta)$	• correctly determines $\frac{dx}{d\theta}$ **[1 mark]**
$y = 3\sin(2\theta)$ $\frac{dy}{d\theta} = 6\cos(2\theta)$	• correctly determines $\frac{dy}{d\theta}$ **[1 mark]**
$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \text{ or } \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ $= \frac{6\cos(2\theta)}{2\sec^2(\theta)}$	• determines an expression for $\frac{dy}{dx}$ **[1 mark]**
$= 3\left(2\cos^2(\theta) - 1\right)\cos^2(\theta)$ $= 6\cos^4(\theta) - 3\cos^2(\theta)$	• determines an expression for $\frac{dy}{dx}$ in the required form **[1 mark]**
$\therefore a = 6,\ b = -3$	• states the values of a and b **[1 mark]**

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Question 19

Vectors and matrices

Object A is released from the origin with constant velocity, $\boldsymbol{v}_A$, such that its position after t seconds is given by

$$\boldsymbol{r}_A = 2\sqrt{3}\,t\,\hat{\boldsymbol{i}} + 3t\,\hat{\boldsymbol{j}} + 2t\,\hat{\boldsymbol{k}},\ t \geq 0.$$

At a later time, object B is released from point $P\left(3\sqrt{3},\,6,\,0\right)$ and travels towards point $Q\left(5\sqrt{3},\,8,\,4\right)$ with constant velocity, $\boldsymbol{v}_B$, such that $\left|\boldsymbol{v}_B\right| = \sqrt{2}\left|\boldsymbol{v}_A\right|$.

Given that objects A and B collide, determine the time between the release of the two objects.

Assume all positions are given in metres and all velocities are given in metres per second.

(6 marks)

Sample response	The response
$\dfrac{dy}{dx} = \dfrac{x}{\left(x^2+1\right)\tan(y)}$	• correctly separates the variables **[1 mark]**
$\displaystyle\int \tan(y)\,dy = \int \frac{x}{x^2+1}\,dx$	• applies suitable integration methods **[1 mark]**
$\displaystyle -\int \frac{-\sin(y)}{\cos(y)}\,dy = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$	• determines a value for the constant of integration **[1 mark]**
$-\ln\left\|\cos(y)\right\| = \dfrac{1}{2}\ln\left\|x^2+1\right\| + c$	
Given $y = 0$ when $x = 0$,	
$-\ln\left\|\cos(0)\right\| = \dfrac{1}{2}\ln\left\|1\right\| + c \Rightarrow c = 0$	
$\therefore \dfrac{1}{2}\ln\left\|x^2+1\right\| = -\ln\left\|\cos(y)\right\|$	• determines an expression for a solution that does not contain logarithms **[1 mark]**
$\ln\sqrt{x^2+1} = \ln\left\|\dfrac{1}{\cos(y)}\right\|$	
$\sqrt{x^2+1} = \left\|\sec(y)\right\|$	• expresses x in terms of y **[1 mark]**
$x^2+1 = \sec^2(y)$	• evaluates the reasonableness of the results and expresses the solution in the form of $x = f(y)$ in simplified form **[1 mark]**
$x^2 = \tan^2(y)$	
$x = \pm\tan(y)$	
As $x \geq 0$, $-\dfrac{\pi}{2} < y \leq 0$	
$x = -\tan(y)$	

	Method 1 $r_A = 2\sqrt{3}t\hat{i} + 3t\hat{j} + 2t\hat{k}, t \ge 0$ $v_A = 2\sqrt{3}\hat{i} + 3\hat{j} + 2\hat{k}$ $\lvert v_A \rvert = \sqrt{(2\sqrt{3})^2 + 3^2 + 2^2} = 5$ $\therefore \lvert \boldsymbol{v}_B \rvert = 5\sqrt{2}\text{ ms}^{-1}$ Given v_B is constant, the position of object B from the origin as it moves can be represented along the line $r_B = b + ld, l \in R$, where $b = 3\sqrt{3}\hat{i} + 6\hat{j}$ $d = 5\sqrt{3}\hat{i} + 8\hat{j} + 4\hat{k} - (3\sqrt{3}\hat{i} + 6\hat{j})$ $= 2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k}$ $\boldsymbol{r}_B = (3\sqrt{3} + 2\sqrt{3}l)\hat{\boldsymbol{i}} + (6 + 2l)\hat{\boldsymbol{j}} + 4l\hat{\boldsymbol{k}}$ Let the objects collide when $t = t_1$ $r_A(t_1) = 2\sqrt{3}t_1\hat{i} + 3t_1\hat{j} + 2t_1\hat{k}$ Collision occurs when $r_A = r_B$ Equating $\hat{i}$, $\hat{j}$ and $\hat{k}$ components: $3\sqrt{3} + 2\sqrt{3}l = 2\sqrt{3}t_1 \quad ...(1)$ $6 + 2l = 3t_1 \quad ...(2)$ $4l = 2t_1 \quad ...(3)$	• correctly determines the value of $\lvert \boldsymbol{v}_B \rvert$ **[1 mark]** • determines a vector in terms of a parameter representing the position of object B from the origin as it moves **[1 mark]** • determines at least two simultaneous equations based on the collision of the objects **[1 mark]**
	From (3), $l = \frac{t_1}{2}$ Substituting into (1): $3\sqrt{3} + 2\sqrt{3}\left(\frac{t_1}{2}\right) = 2\sqrt{3}t_1$ $t_1 = 3\text{ s}$ The collision occurs 3 seconds after object A is released. Collision point is $2\sqrt{3}(3)\hat{i} + 3(3)\hat{j} + 2(3)\hat{k}$ $= 6\sqrt{3}\hat{i} + 9\hat{j} + 6\hat{k}$ Distance object B travels from P to collision point is $\sqrt{(6\sqrt{3} - 3\sqrt{3})^2 + (6 - 9)^2 + 6^2} = 6\sqrt{2}$ Time taken for object B to reach collision point is $\frac{d}{s} = \frac{6\sqrt{2}}{5\sqrt{2}}$ $= 1.2\text{ s}$ Time between the release of the two objects is $3 - 1.2 = 1.8\text{ s}$	• determines time from release to collision for object A **[1 mark]** • determines distance that object B travels to reach collision point **[1 mark]** • determines time between the release of the two objects **[1 mark]**

	Method 2 $r_A = 2\sqrt{3}t\hat{i} + 3t\hat{j} + 2t\hat{k}, t \geq 0$ $v_A = 2\sqrt{3}\hat{i} + 3\hat{j} + 2\hat{k}$ $\lvert v_A \rvert = \sqrt{\left(2\sqrt{3}\right)^2 + 3^2 + 2^2} = 5$ $\therefore \lvert \boldsymbol{v_B} \rvert = 5\sqrt{2}\ \mathrm{ms}^{-1}$ $\overrightarrow{PQ} = q - p = 2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k}a$ $\lvert \overrightarrow{PQ} \rvert = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2 + 4^2}$ $= \sqrt{32}$ $\widehat{PQ} = \frac{\overrightarrow{PQ}}{\lvert \overrightarrow{PQ} \rvert} = \frac{1}{4\sqrt{2}}\left(2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k}\right)$ Given v_B is in the direction of $\overrightarrow{PQ}$, $\boldsymbol{v_B} = 5\sqrt{2}\,\widehat{PQ}$ $= \frac{5\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{5}{2}\hat{j} + 5\hat{\boldsymbol{k}}$ Given v_A and v_B are constant, the position of object B from the origin as it moves can be represented along the line $r_B = b + lv_B, l \in R$, where l is the time of motion $b = 3\sqrt{3}\hat{i} + 6\hat{j}$.	- correctly determines the value of $\lvert v_B \rvert$ **[1 mark]** - determines the velocity vector of object B **[1 mark]**

	$\boldsymbol{r_B} = \left(3\sqrt{3} + \frac{5\sqrt{3}}{2}l\right)\hat{\boldsymbol{i}} + \left(6 + \frac{5}{2}l\right)\hat{\boldsymbol{j}} + 5l\,\hat{\boldsymbol{k}}$	• determines a vector in terms of a parameter representing the position of B from the origin as it moves **[1 mark]**
	Let the objects collide when $t = t_1$. $r_A(t_1) = 2\sqrt{3}t_1\,\hat{\boldsymbol{i}} + 3t_1\hat{\boldsymbol{j}} + 2t_1\hat{\boldsymbol{k}}$ Collision occurs when $\boldsymbol{r_A} = \boldsymbol{r_B}$ Equating $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{k}}$ components: $3\sqrt{3} + \frac{5\sqrt{3}}{2}l = 2\sqrt{3}\,t_1 \quad \ldots(1)$ $6 + \frac{5}{2}l = 3t_1 \quad \ldots(2)$ $5l = 2t_1 \quad \ldots(3)$	• determines at least two simultaneous equations based on the collision of the objects **[1 mark]**
	From (3), $l = \frac{2t_1}{5}$ Substituting into (2): $6 + \frac{5}{2}\left(\frac{2t_1}{5}\right) = 3t_1$ $t_1 = 3\text{ s}$	• determines time from release to collision for object A **[1 mark]**
	$\therefore l = 1.2\text{ s}$ Time between the release of the two objects is $3 - 1.2 = 1.8\text{ s}$.	• determines time between the release of the two objects **[1 mark]**

	Method 3 $r_A = 2\sqrt{3}t\hat{i} + 3t\hat{j} + 2t\hat{k}, t \ge 0$ $v_A = 2\sqrt{3}\hat{i} + 3\hat{j} + 2\hat{k}$ $\lvert v_A \rvert = \sqrt{(2\sqrt{3})^2 + 3^2 + 2^2} = 5$ $\therefore \lvert \mathbf{v}_B \rvert = 5\sqrt{2}\text{ ms}^{-1}$ $\overrightarrow{PQ} = q - p = 2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k}$ $\lvert \overrightarrow{PQ} \rvert = \sqrt{(2\sqrt{3})^2 + 2^2 + 4^2} = \sqrt{32}$ $\widehat{PQ} = \frac{\overrightarrow{PQ}}{\lvert \overrightarrow{PQ} \rvert} = \frac{1}{4\sqrt{2}}(2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k})$ Given that v_B is in the direction of $\overrightarrow{PQ}$ $v_B = 5\sqrt{2}\,\widehat{PQ}$ $= \frac{5\sqrt{3}}{2}\hat{i} + \frac{5}{2}\hat{j} + 5\hat{k}$ Given r_B is in the direction of $\overrightarrow{PQ}$ and the object is released from B. Let the time between the time of release of the two objects be t_0. Displacement of object B from P to collision point is: $d = v_B t = 5\sqrt{2}\,\widehat{PQ}(t - t_0)$ $= \frac{5\sqrt{2}}{4\sqrt{2}}(2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k})(t - t_0)$	• correctly determines the value of $\lvert \mathbf{v}_B \rvert$ **[1 mark]** • determines the velocity vector of object B **[1 mark]** • determines displacement of object B from P to the collision point in terms of time between release of the two objects **[1 mark]**
	The position of object B from the origin as it moves can be represented by $r_B = b + 5\sqrt{2}\,\widehat{PQ}(t - t_0)$ $= 3\sqrt{3}\hat{i} + 6\hat{j} + \frac{5}{4}(2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k})(t - t_0)$ $= 3\sqrt{3}\hat{i} + 6\hat{j} + \frac{5(t - t_0)}{4}(2\sqrt{3}\hat{i} + 2\hat{j} + 4\hat{k})$ Collision occurs when $r_A = r_B$ Equating $\hat{i}$, $\hat{j}$ and $\hat{k}$ components: $2\sqrt{3}t = 3\sqrt{3} + \frac{5\sqrt{3}}{2}t - \frac{5\sqrt{3}}{2}t_0 \quad \ldots (1)$ $3t = 6 + \frac{5}{2}t - \frac{5}{2}t_0 \quad \ldots (2)$ $2t = 5t - 5t_0 \quad \ldots (3)$ From (3), $t = \frac{5}{3}t_0$ Substituting into (2) $\frac{5}{6}t_0 = 6 - \frac{5}{2}t_0$ $t_0 = 1.8$ Time between the release of the two objects is 1.8 s.	• determines a vector representing the position of B from the origin as it moves **[1 mark]** • determines at least two simultaneous equations based on the time between release of the two objects and original position of object B **[1 mark]** • determines time between the release of the two objects **[1 mark]**

	Method 4 $\mathbf{r}_A = 2\sqrt{3}t\hat{i} + 3t\hat{j} + 2t\hat{k}, t \geq 0$ $v_A = 2\sqrt{3}\hat{i} + 3\hat{j} + 2\hat{k}$ $\lvert \mathbf{v}_A \rvert = \sqrt{\left(2\sqrt{3}\right)^2 + 3^2 + 2^2} = 5$ $\therefore \lvert \mathbf{v}_B \rvert = 5\sqrt{2}\ \mathrm{ms}^{-1}$	• correctly determines the value of $\lvert \mathbf{v}_B \rvert$ **[1 mark]**
	$\overrightarrow{PQ} = q - p = 2\sqrt{3}\,\hat{i} + 2\,\hat{j} + 4\,\hat{k}a$ $\lvert \overrightarrow{PQ} \rvert = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2 + 4^2}$ $= \sqrt{32}$ $\widehat{PQ} = \frac{\overrightarrow{PQ}}{\lvert \overrightarrow{PQ} \rvert} = \frac{1}{4\sqrt{2}}\left(2\sqrt{3}\,\hat{i} + 2\,\hat{j} + 4\,\hat{k}\right)$ Given v_B is in the direction of $\overrightarrow{PQ}$ $\boldsymbol{v_B} = 5\sqrt{2}\,\widehat{PQ}$ $= \frac{5\sqrt{3}}{2}\hat{\boldsymbol{i}} + \frac{5}{2}\hat{\boldsymbol{j}} + 5\hat{\boldsymbol{k}}$	• determines velocity vector of object B **[1 mark]**
	$\boldsymbol{r_B} = \int v_B\ dt$ $= \left(\frac{5\sqrt{3}}{2}t + c_1\right)\hat{\boldsymbol{i}} + \left(\frac{5}{2}t + c_2\right)\hat{\boldsymbol{j}} + (5t + c_3)\hat{\boldsymbol{k}}$	• determines a general expression for the position of object B **[1 mark]**
	Let object B be released from $\left(3\sqrt{3}, 6, 0\right)$ at b seconds after object A is released.	

$r_B(b)=\left(\frac{5\sqrt{3}}{2}b+c_1\right)\hat{i}+\left(\frac{5}{2}b+c_2\right)\hat{j}+(5b+c_3)\hat{k}$ $\frac{5\sqrt{3}}{2}b+c_1=3\sqrt{3}\Rightarrow c_1=3\sqrt{3}-\frac{5\sqrt{3}}{2}b \quad ...(1)$ $\frac{5}{2}b+c_2=6\Rightarrow c_2=6-\frac{5}{2}b \quad ...(2)$ $5b+c_3=0\Rightarrow c_3=-5b \quad ...(3)$	- determines at least two simultaneous equations based on the time between release of the two objects and original position of object B **[1 mark]**
Let the objects collide a seconds after object A is released. $r_A(a)=r_B(a)$ $2\sqrt{3}a=\frac{5\sqrt{3}}{2}a+c_1\Rightarrow c_1=-\frac{\sqrt{3}a}{2}$ $3a=\frac{5}{2}a+c_2\Rightarrow c_2=\frac{a}{2}$ $2a=5a+c_3\Rightarrow c_3=-3a$ Equating parts: $3\sqrt{3}-\frac{5\sqrt{3}}{2}b=-\frac{\sqrt{3}a}{2} \quad ...(4)$ $6-\frac{5}{2}b=\frac{a}{2} \quad ...(5)$ $-5b=-3a \quad ...(6)$	- determines at least two simultaneous equations based on time between release of the two objects and the time that the two objects collide **[1 mark]**
Using $6\times(5)+(6)$ $36-20b=0$ $b=1.8$ Time between the release of the two objects is **1.8** s.	- determines time between the release of the two objects **[1 mark]**

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Question 11

Vectors and matrices

The position vector of a particle, r_1 (cm), over time, t(s), is given by

$$\boldsymbol{r}_1(t) = (2t+1)\hat{\boldsymbol{i}} + (t+3)\hat{\boldsymbol{j}} - (2t-3)\hat{\boldsymbol{k}}$$

a) Determine the velocity vector of the particle. [1 mark]

Sample Response	The response
$\boldsymbol{v}_1(t) = 2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$	• correctly determines the velocity vector [1 mark]

b) Determine the time when the position vector of the particle is perpendicular to its velocity vector. [2 marks]

Sample Response	The response
Require $\boldsymbol{r_1}.\boldsymbol{v_1} = 0$ $\begin{pmatrix} 2t+1 \\ t+3 \\ -(2t-3) \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 0$ $2(2t+1) + 1(t+3) + 2(2t-3) = 0$ $9t - 1 = 0$ $t = \frac{1}{9}$ s	• establishes an equation in the form • solves equation to determine the time [1 mark]

The position vector of a second particle, r_2(cm), over time, t(s), is given by

$$\boldsymbol{r}_2(t) = (16-4t)\hat{\boldsymbol{i}} - (3t-13)\hat{\boldsymbol{j}} + 2\hat{\boldsymbol{k}}$$

c) Determine whether the two particles collide. [3 marks]

Sample Response	The response
Let $\boldsymbol{r_1}(t) = \boldsymbol{r_2}(t)$ Equating $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ components $2t + 1 = 16 - 4t \quad \ldots(1)$ $t + 3 = -3t + 13 \quad \ldots(2)$ $-2t + 3 = 2 \quad \ldots(3)$ From (1), $t = 2.5$ s From (2), $t = 2.5$ s From (3), $t = 0.5$ s	• correctly equates 2 pairs of components for the particles, one of which must be the $\hat{k}$ component, to form 2 equations [1 mark] • solves equations to determine two different time values [1 mark]
The particles do not collide as the solutions are not consistent.	• uses mathematical reasoning to make an appropriate conclusion whether the two particles collide [1 mark]

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Question 16

Vectors and matrices

Consider this system of equations that corresponds to three planes.

$$x+5y=1+2z$$
$$x+z=3y+3$$
$$8y-\lambda=3z$$

a) Use a Gaussian technique to determine the value of λ for which this system of equations has infinitely many solutions. [4 marks]

Sample Response	The response
Rearranging equations: $x+5y-2z=1$ $x-3y+z=3$ $8y-3z=\lambda$	• correctly rearranges the three equations [1 mark]
Expressing in matrix form: $\left[\begin{array}{ccc\|c}1&5&-2&1\\1&-3&1&3\\0&8&-3&\lambda\end{array}\right]$	• establishes an augmented matrix [1 mark]
$\left[\begin{array}{ccc\|c}1&5&-2&1\\0&-8&3&2\\0&8&-3&\lambda\end{array}\right]$ $R_2'=R_2-R_1$ $\left[\begin{array}{ccc\|c}1&5&-2&1\\0&-8&3&2\\0&0&0&\lambda+2\end{array}\right]$ $R_3'=R_3+R_2$	• establishes a row of zeros in the row containing λ [1 mark]
For $\lambda=-2$ there are infinitely many solutions.	• determines a value of λ [1 mark]

b) Use the result from Question 16a) to determine the infinitely many solutions. Express your answer in the form of a vector equation of a line. [3 marks]

Sample Response	The response
Method 1 using $\left[\begin{array}{ccc\|c}1&5&-2&1\\0&-8&3&2\\0&0&0&0\end{array}\right]$ Letting $z=k$ $(k\in R)$ Row 2: $-8y+3z=2$ $8y-3k=-2\Rightarrow y=\dfrac{3k-2}{8}$	• expresses y in terms of a parameter [1 mark]
$8y-3k=-2\Rightarrow y=\dfrac{3k-2}{8}$ Row 1: $x+5y-2z=1$ $x+5\left(\dfrac{3k-2}{8}\right)-2k=1$ $x+\dfrac{15k-10}{8}-\dfrac{16k}{8}=1\Rightarrow x=\dfrac{k}{8}+\dfrac{9}{4}$	• expresses x in terms of a parameter [1 mark]
The solutions in vector form are: $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{1}{8}\\\frac{3}{8}\\1\end{bmatrix}k+\begin{bmatrix}\frac{9}{4}\\-\frac{1}{4}\\0\end{bmatrix}$	• determines the infinite solutions expressed in the form of a vector equation of a line [1 mark]

Method 2 using $\left[\begin{array}{ccc\|c} 1 & 5 & -2 & 1 \\ 0 & -8 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\left[\begin{array}{ccc\|c} 1 & 5 & -2 & 1 \\ 0 & 1 & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_2' = \frac{R_2}{-8}$ $\left[\begin{array}{ccc\|c} 1 & 0 & -\frac{1}{8} & \frac{9}{4} \\ 0 & 1 & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$ $R_1' = R_1 - 5R_2$ Letting $z = k$ $(k \in R)$ $x = \frac{k}{8} + \frac{9}{4}$	• expresses x in terms of a parameter [1 mark]
$y = \frac{3k}{8} - \frac{1}{4}$	• expresses y in terms of a parameter [1 mark]
The solutions in vector form are: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 1 \end{bmatrix} k + \begin{bmatrix} \frac{9}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix}$	• determines the infinite solutions expressed in the form of a vector equation of a line [1 mark]
Method 3 using $\left[\begin{array}{ccc\|c} 1 & 5 & -2 & 1 \\ 1 & -3 & 1 & 3 \\ 0 & 8 & -3 & -2 \end{array}\right]$ Letting $y = k$ $(k \in R)$ Row 3: $-8y + 3z = 2$ $-8k + 3z = 2$ $z = \frac{8k}{3} + \frac{2}{3}$	• expresses z in terms of a parameter [1 mark]
Row 2: $x - 3y + z = 3$ $x - 3k + \frac{8k}{3} + \frac{2}{3} = 3$ $x = \frac{k}{3} + \frac{7}{3}$	• expresses x in terms of a parameter [1 mark]
The solutions in vector form are: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 1 \\ \frac{8}{3} \end{bmatrix} k + \begin{bmatrix} \frac{7}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$	• determines the infinite solutions expressed in the form of a vector equation of a line [1 mark]

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Question 12

Vectors and matrices

Consider the plane $x - y - 2z = 15$.

a) Determine a vector *n* that is perpendicular to the plane. [1 mark]

Sample Response	The response
A vector perpendicular to $x - y - 2z = 15$ is $n = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$	• correctly determines a suitable vector $\boldsymbol{n}$ [1 mark]

b) Determine the vector equation of the line l that is perpendicular to the plane and contains the point A (–2, 1, 3). [1 mark]

Sample Response	The response
Vector equation of line l is $\boldsymbol{r} = \boldsymbol{a} + k\boldsymbol{d}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $k \in R$	• determines vector equation of the line [1 mark]

c) Use the result from Question 12b) to express the equation of the line *l* in parametric form. [1 mark]

Sample Response	The response
Equation of line l in parametric form $x = -2 + k$ $y = 1 - k$ $z = 3 - 2k \quad k \in R$	• expresses equation of the line in parametric form [1 mark]

The line *l* and the plane intersect at point S.

d) Show that the coordinates of S are (2, – 3, – 5). [3 marks]

Sample Response	The response
Method 1 Given S lies on the plane $x - y - 2z = 15$ $(-2 + k) - (1 - k) - 2(3 - 2k) = 15$	• substitutes result from 12c) into the equation of the plane [1 mark]
$k = 4$	• determines value of the parameter [1 mark]
The coordinates of S are $(-2 + 4, 1 - 4, 3 - 8) = (2, -3, -5)$	• determines coordinates of S [1 mark]
Method 2 Substituting $(2, -3, -5)$ into $x - y - 2z = 15$ LHS $= 2 - (-3) - 2(-5) = 15 =$ RHS, so S lies on the plane.	• verifies that S lies on the plane [1 mark]
Substituting $(2, -3, -5)$ into line l result from 12c) $2 = -2 + k \Rightarrow k = 4$	• uses x-coordinate of S to determine value of the parameter [1 mark]
$-3 = 1 - k \Rightarrow k = 4$ $-5 = 3 - 2k \Rightarrow k = 4$ $\therefore$ S lies on line l	• shows that parameter value is consistent across the set of parametric equations by considering the remaining two coordinates of S [1 mark]
Method 3 Substituting $(2, -3, -5)$ into $x - y - 2z = 15$ LHS $= 2 - (-3) - 2(-5) = 15 =$ RHS, so S lies on the plane.	• verifies that S lies on the plane [1 mark]
Substituting $(2, -3, -5)$ into line l result from 12b) $\begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ $2 = -2 + k \Rightarrow k = 4$	• uses x-coordinate of S to determine the value of the parameter [1 mark]
$-3 = 1 - k \Rightarrow k = 4$ $-5 = 3 - 2k \Rightarrow k = 4$ $\therefore$ S lies on line l	• shows that parameter value is consistent across the set of parametric equations by considering the remaining two coordinates of S [1 mark]

e) Determine $\overrightarrow{AS}$.

[1 mark]

Sample Response	The response
$\overrightarrow{AS} = s - a = \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix}$	• determines $\overrightarrow{AS}$ [1 mark]

f) Use a property of parallel vectors to verify that $\overrightarrow{AS}$ and $\boldsymbol{n}$ are parallel.

[1 mark]

Sample Response	The response
$\begin{pmatrix} 4 \\ -4 \\ -8 \end{pmatrix} = 4\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \Rightarrow$ result is verified	• shows that $\overrightarrow{AS}$ is a scalar multiple of ***nn*** [1 mark]

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Question 13

Vectors and matrices

Use $z = a + bi$ and $w = c + di$, where $a, b, c, d \in R$, to prove

$$|z - w|^2 = |z|^2 + |w|^2 - 2Re(z\overline{w})$$

[6 marks]

Sample Response	The response
RTP $|z - w|^2 = |z|^2 + |w|^2 - 2Re(z\,\overline{w})$ LHS $= |z - w|^2$ $= |(a + bi) - (c + di)|^2$ $= |(a - c) + (b - d)i|^2$	• correctly expresses $z - w$ in terms of a, b, c and d in Cartesian form [1 mark]
$= (a - c)^2 + (b - d)^2$ $= a^2 - 2ac + c^2 + b^2 - 2bd + d^2$	• expresses |$z - w$| 2 in terms of a, b, c and d in expanded form [1 mark]
RHS $= |z|^2 + |w|^2 - 2Re(z\,\overline{w})$ $= |a + bi|^2 + |c + di|^2 \ldots$ $\ldots -2Re((a + bi)\,(c - di))$ $= a^2 + b^2 + c^2 + d^2 \ldots$ $\ldots -2Re((ac + bd) + (bc - ad)i)$	• correctly expresses |z| 2 + |w| 2 in terms of a, b, c and d in expanded form [1 mark] • correctly expresses ($z\,\overline{w}$) in terms of a, b, c and d [1 mark]
$= a^2 + b^2 + c^2 + d^2 - 2(ac + bd)$ $= a^2 + b^2 + c^2 + d^2 - 2ac - 2bd$ $= a^2 - 2ac + c^2 + b^2 - 2bd + d^2$ $=$ LHS	• completes proof [1 mark] • shows logical organisation, communicating key steps [1 mark]

2021
Paper 1
Section 2
Question 14

Vectors and matrices

An object is projected vertically upwards from ground level. After the object has been in motion for t seconds, its position vector through the air, in metres, is modelled by

$$\boldsymbol{r}(t) = 5t(8-t)\hat{\boldsymbol{j}}$$

a) Determine the velocity of the object through the air, $v\ (t)$, in metres per second. [2 marks]

Sample Response	The response
Method 1 $\boldsymbol{r}(t) = 5t(8-t)\hat{\jmath}$ $= (40t - 5t^2)\hat{\jmath}$	• correctly expands the given expression [1 mark]
$\boldsymbol{v}(t) = (40 - 10t)\,\hat{\boldsymbol{\jmath}}$	• determines $\boldsymbol{v}(t)$ [1 mark]
Method 2 $\boldsymbol{r}(t) = 5t(8-t)\hat{\jmath}$ Using the product rule $\boldsymbol{v}(t) = \big(5t(-1) + 5(8-t)\big)\hat{\jmath}$	• correctly uses the product rule to determine $\boldsymbol{v}(t)$ [1 mark]
$= (-5t + 40 - 5\text{t})\hat{\jmath}$ $= (40 - 10t)\hat{\boldsymbol{\jmath}}$	• determines $\boldsymbol{v}(t)$ [1 mark]

b) Determine the number of seconds until the object reaches its maximum height. [2 marks]

Sample Response	The response
Maximum height occurs when $\boldsymbol{v}(t) = 0\hat{\jmath}$ $40 - 10t = 0$	• establishes equation in terms of t [1 mark]
$10t = 40$ $t = 4\text{ s}$	• determines t [1 mark]

c) Determine the maximum height that the object reaches, in metres. [2 marks]

Sample Response	The response
Maximum height $= 5 \times 4(8-4)$ $= 20 \times 4$ $= 80\text{ m}$	• substitutes result from 14b) into expression for position [1 mark] • determines maximum height [1 mark]

2021
Paper 1
Section 2
Question 19

Vectors and matrices

The velocity vectors of two objects A and B $\left(\text{in m s}^{-1}\right)$ at time t (in s) are given respectively by

$$\boldsymbol{v}_{\text{A}} = 6\sin(3t)\hat{\boldsymbol{i}} + 6\cos(3t)\hat{\boldsymbol{j}}$$
$$\boldsymbol{v}_{\text{B}} = \cos(t)\hat{\boldsymbol{i}} - \sin(t)\hat{\boldsymbol{j}}$$

Objects A and B are initially at $(-2, 0, 2)$ and $(0, 1, -1)$ respectively. Determine the position of Object A when it is 4 metres away from Object B for the first time.

[7 marks]

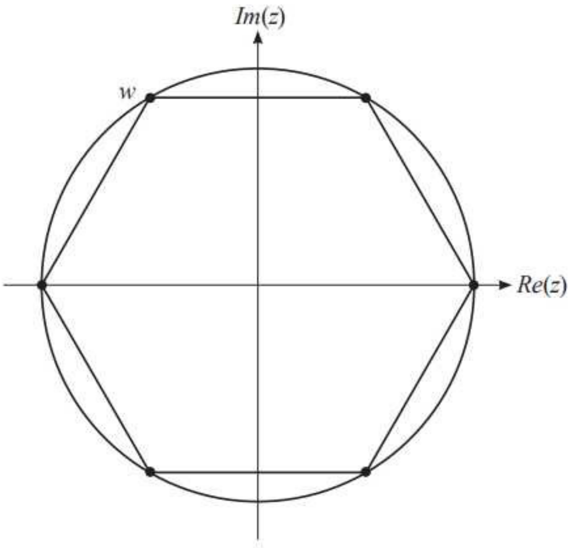
Sample Response	The response
$\boldsymbol{v}_A = 6\sin(3t)\hat{\imath} + 6\cos(3t)\hat{\jmath}$ $\boldsymbol{v}_B = \cos(t)\hat{\imath} - \sin(t)\hat{\jmath}$ $\boldsymbol{r}_A = \int \boldsymbol{v}_A\, dt = -2\cos(3t)\hat{\imath} + 2\sin(3t)\hat{\jmath} + \boldsymbol{c}_A$ When $t = 0$ $-2\hat{\imath} + 2\hat{\boldsymbol{k}} = -2\cos(0)\hat{\imath} + 2\sin(0)\hat{\jmath} + \boldsymbol{c}_A \Rightarrow \boldsymbol{c}_A = 2\hat{\boldsymbol{k}}$ $\therefore \boldsymbol{r}_A = -2\cos(3t)\hat{\imath} + 2\sin(3t)\hat{\jmath} + 2\hat{\boldsymbol{k}}$	• correctly determines the expression for the position of Object A [1 mark]

$\boldsymbol{r}_B = \int \boldsymbol{v}_B\,dt = \sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} + \boldsymbol{c}_B$ When $t = 0$ $\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}} = \sin(0)\hat{\boldsymbol{\imath}} + \cos(0)\hat{\boldsymbol{\jmath}} + \boldsymbol{c}_B \Rightarrow \boldsymbol{c}_B = -\hat{\boldsymbol{k}}$ $\therefore \boldsymbol{r}_B = \sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}}$	• correctly determines the expression for the position of Object B [1 mark]
$\boldsymbol{r}_B - \boldsymbol{r}_A$ $= (\sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{k}})\ldots$ $\ldots -(-2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}})$ $= (\sin(t) + 2\cos(3t))\hat{\boldsymbol{\imath}} + (\cos(t) - 2\sin(3t))\hat{\boldsymbol{\jmath}} - 3\hat{\boldsymbol{k}}$	• determines an expression to represent the relative position of Objects A and B [1 mark]
$\lvert\boldsymbol{r}_B - \boldsymbol{r}_A\rvert$ $= \sqrt{\sin^2(t) + 4\sin(t)\cos(3t) + 4\cos^2(3t) + \cdots}$ $\overline{\ldots\cos^2(t) - 4\cos(t)\sin(3t) + 4\sin^2(3t) + 9}$ $= \sqrt{14 - 4(\sin(3t)\cos(t) - \cos(3t)\sin(t))}$ $= \sqrt{14 - 4(\sin(3t - t))}$	• determines an expression to represent the distance (or square of the distance) between the objects [1 mark]
$= \sqrt{14 - 4\sin(2t)}$ Given $\lvert\boldsymbol{r}_B - \boldsymbol{r}_A\rvert = 4$ $\sqrt{14 - 4\sin(2t)} = 4$ $\sin(2t) = -\frac{1}{2}$ $2t = \frac{7\pi}{6}$	• uses a trigonometric identity to determine an expression in terms of a single trigonometric function that represents the distance (or square of the distance) between the objects [1 mark]
$t = \frac{7\pi}{12}$ s (first positive solution)	• determines the first time that Object A is 4 metres away from Object B [1 mark]
Position of A $\boldsymbol{r}_A = -2\cos(3t)\hat{\boldsymbol{\imath}} + 2\sin(3t)\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ $= -2\cos\left(\frac{7\pi}{4}\right)\hat{\imath} + 2\sin\left(\frac{7\pi}{4}\right)\hat{\jmath} + 2\hat{\boldsymbol{k}}$ $= -\sqrt{2}\hat{\boldsymbol{\imath}} - \sqrt{2}\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$ (m)	• determines position of Object A [1 mark]

2020
Paper 1
Section 2
Question 11

Vectors and matrices

The vertices of a regular hexagon are positioned on the circumference of a unit circle as shown on the Argand plane.



Consider the complex number w, as shown on the plane.

a) Determine w, expressing your answer in the form $r\ \text{cis}(\theta)$. [1 mark]

Sample Response	The response
$\lvert w \rvert = 1$ $\arg(w) = \frac{2\pi}{3}$ $w = \text{cis}\left(\frac{2\pi}{3}\right)$	• correctly expresses w in $r\text{cis}(\theta)$ form [1 mark]

b) Convert w into Cartesian form. [2 marks]

Sample Response	The response
$w = \cos\left(\frac{2\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)i$ $= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$	• converts w into $r\cos(\theta) + r\cos(\theta)i$ form [1 mark] • uses exact values for $Re(w)$ and $Im(w)$ [1 mark]

Each vertex of the hexagon is a solution of an equation of the form $z^n = a$ where $z \in C$.

c) State the value of n. [1 mark]

Sample Response	The response
There are 6 roots of unity so the equation is $z^6 = 1$. $n = 6$	• correctly states the value of n [1 mark]

d) State the value of a. [1 mark]

Sample Response	The response
$a = 1$	• correctly states the value of a [1 mark]

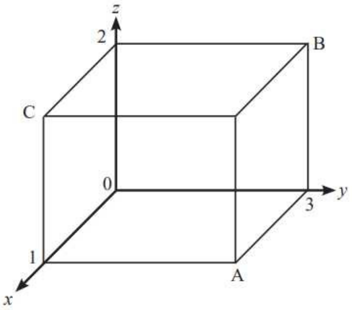
e) Verify that w satisfies the equation $z^n = a$ using the results from 11c) and 11d). [2 marks]

Sample Response	The response
Verify that $w^6 = 1$ $\left(\text{cis}\left(\frac{2\pi}{3}\right)\right)^6 = \text{cis}\left(6 \times \frac{2\pi}{3}\right)$ $= \text{cis}(4\pi)$ $= \text{cis}(0)$ $= 1$	• uses De Moivre's theorem in the calculation of w^n for the value of n [1 mark] • verifies the result by showing that the result of the calculation is the value of a [1 mark]

2020
Paper 1
Section 2
Question 12

Vectors and matrices

Consider the vertices A, B and C of the rectangular prism as shown.



a) State the coordinates of A, B and C. [1 mark]

Sample Response	The response
A(1, 3, 0), B(0, 3, 2) and C(1, 0, 2)	• correctly states the coordinates of A, B and C [1 mark]

b) Determine a unit vector, $\hat{n}$, that is normal to the plane containing A, B and C. [3 marks]

Sample Response	The response
$\overrightarrow{AB} = b - a = \begin{pmatrix}0\\3\\2\end{pmatrix} - \begin{pmatrix}1\\3\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$ $\overrightarrow{AC} = c - a = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}1\\3\\0\end{pmatrix} = \begin{pmatrix}0\\-3\\2\end{pmatrix}$	• determines 2 vectors in the plane containing A, B and C [1 mark]
$\boldsymbol{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}0\\-3\\2\end{pmatrix}$ $= \begin{pmatrix}0 \times 2 - 2 \times -3\\2 \times 0 - -1 \times 2\\-1 \times -3 - 0 \times 0\end{pmatrix}$ $= \begin{pmatrix}6\\2\\3\end{pmatrix}$	• determines a vector normal to the plane [1 mark]
$\hat{\boldsymbol{n}} = \frac{1}{\|n\|} n = \frac{1}{\sqrt{6^2 + 2^2 + 3^2}} n$ $= \frac{1}{7}\begin{pmatrix}6\\2\\3\end{pmatrix}$	• determines a unit vector normal to the plane [1 mark]

c) Verify that $\hat{n}$ is perpendicular to $\overrightarrow{AB}$. [2 marks]

Sample Response	The response
Verify that $\hat{n}\,.\,\overrightarrow{AB} = 0$ $\hat{n}\,.\,\overrightarrow{AB} = \frac{1}{7}\begin{pmatrix}6\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\0\\2\end{pmatrix}$	• substitutes the results from b) into $\hat{n}.\overrightarrow{AB}$ [1 mark]
$=\frac{1}{7}(-6+0+6)$ $= 0$ The required result is verified.	• verifies the result by showing that the result of the calculation is 0 [1 mark]

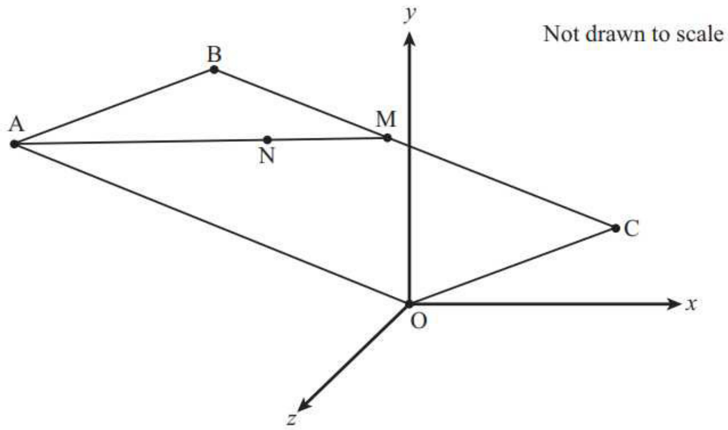
d) Determine the Cartesian equation of the plane that contains A, B and C. [2 marks]

Sample Response	The response
Method 1 **Determining equation of plane:** Using $\boldsymbol{n} = \begin{pmatrix}6\\2\\3\end{pmatrix}$ and $\boldsymbol{a} = \begin{pmatrix}1\\3\\0\end{pmatrix}$ Vector equation of plane $\boldsymbol{r}\,.\,\boldsymbol{n} = \boldsymbol{a}\,.\,\boldsymbol{n}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}6\\2\\3\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix}.\begin{pmatrix}6\\2\\3\end{pmatrix}$	• determines the vector equation of the plane [1 mark]
Cartesian equation of plane $6x + 2y + 3z = 12$	• determines the Cartesian equation of the plane [1 mark]
Method 2 **Determining equation of plane:** Cartesian equation of plane $ax + by + cz + d = 0$ where $\boldsymbol{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ Using $\boldsymbol{n} = \begin{pmatrix}6\\2\\3\end{pmatrix}$ and A $(1, 3, 0)$ to determine d $6(1) + 2(3) + 3(0) + d = 0$	• substitutes a vector normal to the plane and a suitable set of coordinates into the Cartesian equation of plane formula to form an equation in terms of d [1 mark]
$d = -12$ Cartesian equation of plane $6x + 2y + 3z - 12 = 0$	• determines the Cartesian equation of the plane [1 mark]

2020
Paper 1
Section 2
Question 15

Vectors and matrices

The points O(0, 0, 0), A(–6, 2, –2) and C(3, 1, 2) are represented in three-dimensional space in the diagram.



OABC forms a parallelogram in three-dimensional space.

a) Determine the coordinates of B. [1 mark]

Sample Response	The response
$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC}$ $= \begin{pmatrix}-6\\2\\-2\end{pmatrix} + \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}-3\\3\\0\end{pmatrix}$ The coordinates of B are $(-3, 3, 0)$.	• correctly determines the coordinates of B [1 mark]

M is the midpoint of BC.

b) Determine the vector that represents $\overrightarrow{OM}$. [1 mark]

Sample Response	The response
Midpoint of BC is $\left(\frac{-3+3}{2}, \frac{3+1}{2}, \frac{0+2}{2}\right) = (0, 2, 1)$ $\overrightarrow{OM} = \begin{pmatrix}0\\2\\1\end{pmatrix}$	• determines vector $\overrightarrow{OM}$ [1 mark]

N divides AM in the ratio 2:1.

c) Determine the vector that represents $\overrightarrow{ON}$. [2 marks]

Sample Response	The response
$\overrightarrow{ON} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AM}$ $= \overrightarrow{OA} + \frac{2}{3}(\overrightarrow{OM} - \overrightarrow{OA})$	• determines an expression for $\overrightarrow{ON}$ involving $\overrightarrow{ON}$ and the 1:2 ratio [1 mark]
$= \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OM}$ $= \frac{1}{3}\begin{pmatrix}-6\\2\\-2\end{pmatrix} + \frac{2}{3}\begin{pmatrix}0\\2\\1\end{pmatrix}$ $= \begin{pmatrix}-2\\2\\0\end{pmatrix}$	• determines vector $\overrightarrow{ON}$ [1 mark]

d) Use a vector method to show that O, B and N lie on a straight line. [2 marks]

Sample Response	**The response**
Method 1 To show O, B and N lie on a straight line: Using parallel vectors with a common point $\overrightarrow{ON} = k\overrightarrow{OB}$	• correctly communicates a requirement that O, B and N lie on a straight line [1 mark]
Using the results from a) and c) $\begin{bmatrix}-2\\2\\0\end{bmatrix} = \frac{2}{3}\begin{bmatrix}-3\\3\\0\end{bmatrix}$ It is shown that O, B and N lie on a straight line.	• correctly demonstrates that the requirement has been met [1 mark]
Method 2 To show O, B and N lie on a straight line: Using vector product to show vectors are parallel with a common point $\overrightarrow{ON} \times \overrightarrow{OB} = \mathbf{0}$	• correctly communicates a requirement that O, B and N lie on a straight line [1 mark]
$\overrightarrow{ON} \times \overrightarrow{OB} = \begin{bmatrix}-2\\2\\0\end{bmatrix} \times \begin{bmatrix}-3\\3\\0\end{bmatrix}$ $= \begin{bmatrix}0\\0\\0\end{bmatrix}$ It is shown that O, B and N lie on a straight line.	• correctly demonstrates that the requirement has been met [1 mark]

Marking Guide – Paper 2 Section 1

2024
Paper 2
Section 1
Question 3

Vectors and matrices

Given $\boldsymbol{a} = \hat{j} + \hat{k}$ and $\boldsymbol{b} = 2\hat{i} + \hat{k}$, determine $\boldsymbol{a} \times \boldsymbol{b}$.

(A) $\hat{i} - 2\hat{j} - 2\hat{k}$

(B) $\hat{i} - 2\hat{j} + 2\hat{k}$

(C) $\hat{i} + 2\hat{j} - 2\hat{k}$

(D) $\hat{i} + 2\hat{j} + 2\hat{k}$

Answer is C.

2024
Paper 2
Section 1
Question 5

Vectors and matrices

The equation of a plane is $2x - 4z - 8 = 0$.

Determine the point where the plane intersects the z-axis.

(A) (0, 0, –4)

(B) (0, 0, –2)

(C) (0, 0, 2)

(D) (0, 0, 4)

Answer is B.

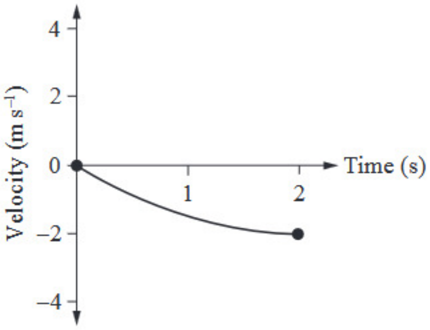
2024
Paper 2
Section 1
Question 10

Vectors and matrices

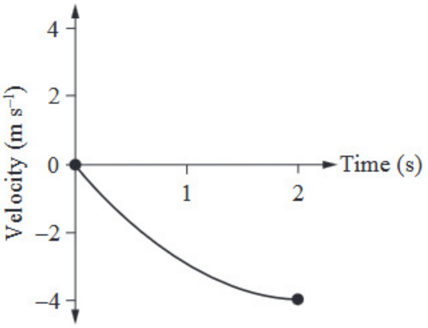
The acceleration (m s^{-2}) of an object at time, t, for $0 \le t \le 2$ seconds is given by $a = \dfrac{2}{t+1}$.

Given that the object is initially at rest, its velocity–time graph is

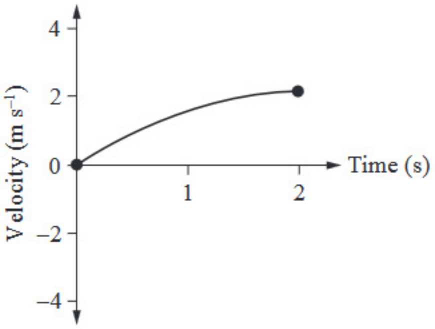
(A)



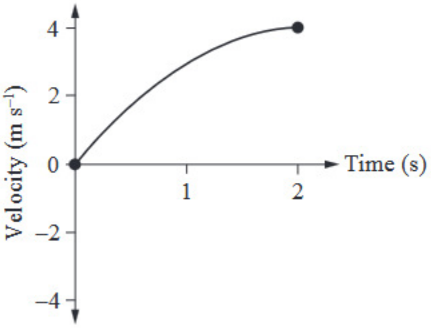
(B)



(C)



(D)



Answer is C.

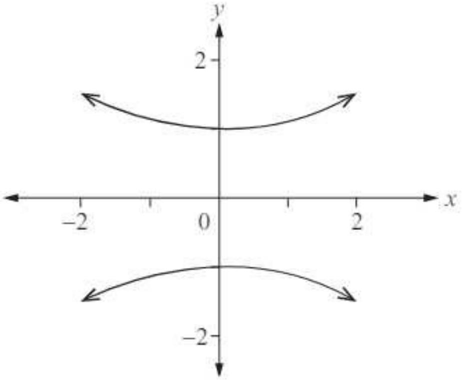
2023
Paper 2
Section 1
Question 4

Vectors and matrices

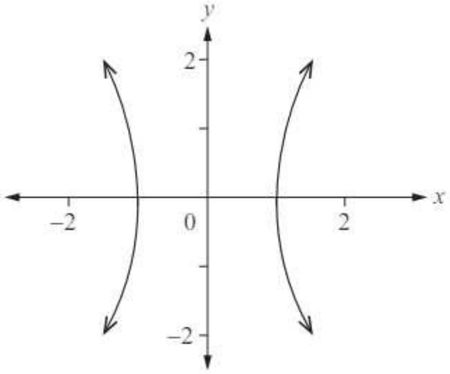
The position of a particle can be modelled using $\boldsymbol{r} = \cos(t)\hat{\boldsymbol{i}} - 2\sin(t)\hat{\boldsymbol{j}},\ t \geq 0$.

Which curve best represents the path of the particle?

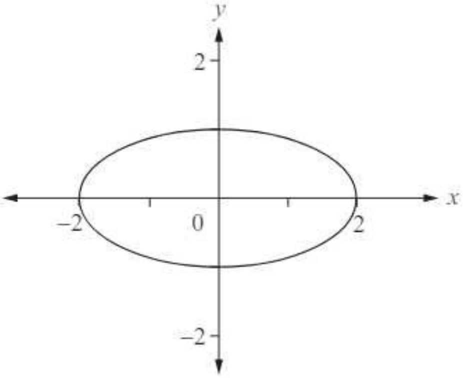
(A)



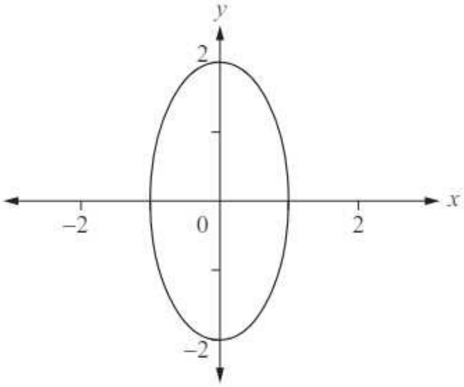
(B)



(C)



(D)



Answer is D.

2023
Paper 2
Section 1
Question 5

Vectors and matrices

A plane contains the origin and the points (1, 2, 3) and (3, 2, 1).

A vector normal to the plane is

(A) $\begin{pmatrix} 4 \\ -8 \\ 4 \end{pmatrix}$

(B) $\begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}$

(C) $\begin{pmatrix} -4 \\ -8 \\ -4 \end{pmatrix}$

(D) $\begin{pmatrix} -4 \\ -8 \\ 4 \end{pmatrix}$

Answer is A.

2023
Paper 2
Section 1
Question 7

Vectors and matrices

Matrix **N** represents the results for a competition involving four teams.

$$\mathbf{N} = \begin{array}{cc} & \text{Losing teams} \\ & \begin{array}{cccc} P & Q & R & S \end{array} \\ \text{Winning teams} \begin{array}{c} P \\ Q \\ R \\ S \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Key: Team P lost to team Q but won against teams R and S.

Using the ranking model $\mathbf{N} + 0.5\mathbf{N}^2$, the teams that placed first, second and third respectively are

(A) P, S and Q.

(B) P, S and R.

(C) S, P and Q.

(D) S, P and R.

Answer is A.

2022
Paper 2
Section 1
Question 2

Vectors and matrices

The win/draw/loss results after a netball competition involving five teams is represented in matrix M.

$$\mathbf{M} = \text{Winning teams}\quad \begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \overset{\text{Losing teams}}{\begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \\ \hline 0 & 1 & 2 & 0 & 2 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 \end{array}}$$

Key: Team P drew with Team Q, defeated Team R and Team T, and lost to Team S

The model M + M^2 + M^3 is used to rank the teams. The final positions from first to fifth are

(A) S, Q, P, R, T – Answer
(B) S, Q, P, T, R
(C) S, P, Q, T, R
(D) S, P, Q, R, T

2022
Paper 2
Section 1
Question 7

Vectors and matrices

Given $\boldsymbol{a} = (3n+2)\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}}$, $\boldsymbol{b} = (n-2)\hat{\boldsymbol{j}}$ and $\boldsymbol{a} \times \boldsymbol{b} = (1-2n)\hat{\boldsymbol{k}}$, the possible values of n are

(A) -5 and $\frac{1}{3}$

(B) -1 and $\frac{5}{3}$

(C) 1 and $-\frac{5}{3}$

(D) 5 and $-\frac{1}{3}$

Answer is B.

2022
Paper 2
Section 1
Question 9

Vectors and matrices

Consider the matrix equation.

$$\mathbf{X}\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$$

Matrix **X** is

(A) $\begin{bmatrix}0 & 1 & 1\\ 1 & -1 & 2\\ -1 & 0 & 2\end{bmatrix}$

(B) $\begin{bmatrix}0 & 1 & -1\\ 1 & -1 & 0\\ 1 & 2 & 2\end{bmatrix}$

(C) $\begin{bmatrix}2 & 2 & 1\\ 4 & 3 & 3\\ 5 & 5 & 5\end{bmatrix}$

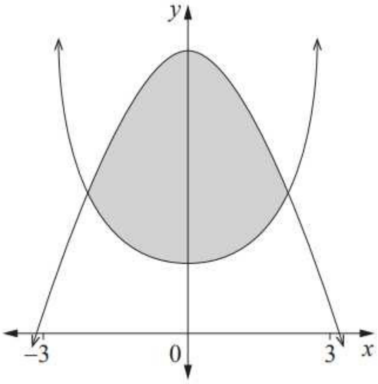
(D) $\begin{bmatrix}2 & 4 & 5\\ 2 & 3 & 5\\ 1 & 3 & 5\end{bmatrix}$

Answer is A.

2021
Paper 2
Section 1
Question 2

Vectors and matrices

Determine the area of the shaded region between the graphs of the functions $y=\frac{1}{3}\sec\left(\frac{x}{3}\right)$ and $y=2\cos\left(\frac{x}{2}\right)$, as shown.



Not to scale

(A) 5.29 units^2
(B) 5.51 units^2
(C) 5.65 units^2
(D) 5.71 units^2 – Answer

2021
Paper 2
Section 1
Question 5

Vectors and matrices

A vector normal to the plane that contains the vectors $\begin{pmatrix}1\\3\\0\end{pmatrix}$ and $\begin{pmatrix}1\\0\\2\end{pmatrix}$ is

(A) $6\hat{i}+2\hat{j}+3\hat{k}$

(B) $6\hat{i}+2\hat{j}-3\hat{k}$

(C) $6\hat{i}-2\hat{j}+3\hat{k}$

(D) $6\hat{i}-2\hat{j}-3\hat{k}$

Answer is D.

2021
Paper 2
Section 1
Question 6

Vectors and matrices

The Cartesian equation of a sphere is given by $x^2+y^2+z^2+2x-2y=7$.

The centre and radius of the sphere are

(A) $(-1, 1, 0)$ and 3 respectively.

(B) $(-1, 1, 0)$ and 9 respectively.

(C) $(1, -1, 0)$ and 3 respectively.

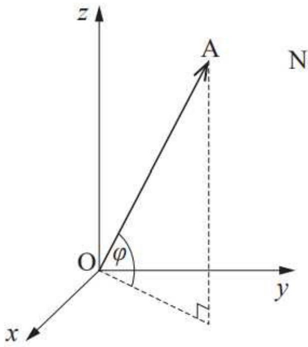
(D) $(1, -1, 0)$ and 9 respectively.

Answer is A.

2021
Paper 2
Section 1
Question 7

Vectors and matrices

The altitude angle of $\overrightarrow{OA}$ is represented as φ.



Not to scale

Given the coordinates of A are $(3, 4, 6)$, the altitude angle of $\overrightarrow{OA}$ in radians is

(A) 0.93

(B) 0.88

(C) 0.69

(D) 0.66

Answer is B.

2020 Paper 2 Section 1 Question 1 Vectors and matrices	The position x (m) at time t (s) of a 7 kg particle moving in a straight line is given by $x = 3t^3 - 5t^2 + 2t - 4$ for $0 \leq t \leq 10$ Determine the time when the particle has a momentum of 620 kg m s^{-1}. (A) 1.73 s (B) 2.60 s (C) 3.66 s **(D) 3.71 s – Answer**

2020 Paper 2 Section 1 Question 2 Vectors and matrices	The Leslie matrix for a certain endangered species is given. $\mathbf{L} = \begin{bmatrix} 0.8 & 2.4 & 0.3 \\ 0.4 & 0 & 0 \\ 0 & 0.55 & 0 \end{bmatrix}$ A group of the species was moved into a secure property at the start of 2018. The initial female population is given. $\mathbf{N}_0 = \begin{bmatrix} 150 \\ 80 \\ 40 \end{bmatrix}$ The best estimate of the total female population at the start of 2025 is (A) 3000 **(B) 4000 – Answer** (C) 5000 (D) 6000

2020 Paper 2 Section 1 Question 6 Vectors and matrices	Solve the matrix equation for **X**. $\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \mathbf{X} \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 0 & 1 \end{bmatrix}$ (A) $\begin{bmatrix} -9 & -9 \\ 4 & 4 \end{bmatrix}$ (B) $\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 13 & -14 \\ -11 & 12 \end{bmatrix}$ (D) $\begin{bmatrix} 54 & 56 \\ -28 & -29 \end{bmatrix}$ **Answer is B.**

2020 **Paper 2** **Section 1** **Question 9** **Vectors and matrices**	Two objects, P and Q, move in three-dimensional space such that their positions, *r*, over time, *t*, are described by the following vectors until they collide. $\boldsymbol{r}_P = (t^2 - 4t)\hat{\boldsymbol{\imath}} + (2t^2 - t + 3)\hat{\boldsymbol{\jmath}} - (6 - 5t)\hat{\boldsymbol{k}}$ $\boldsymbol{r}_Q = (-t^2 + 2t)\hat{\boldsymbol{\imath}} + (3t + t^2)\hat{\boldsymbol{\jmath}} + t^2\hat{\boldsymbol{k}}$ The objects will collide at (A) $t = 0$ (B) $t = 1$ (C) $t = 2$ **(D) $t = 3$ – Answer**

Marking Guide – Paper 2 Section 2

2024
Paper 2
Section 2
Question 12

Vectors and matrices

A system of linear equations is given by

$$\begin{aligned} x - 2y - 2z &= -6 \\ -3x - y + z &= 2 \\ 2x + 3y - 5z &= 10 \end{aligned}$$

a) Express the system of equations as a matrix equation of the form $\boldsymbol{AX} = \boldsymbol{B}$, where $\boldsymbol{A}$ is a 3×3 matrix and both $\boldsymbol{X}$ and $\boldsymbol{B}$ are 3×1 column vectors. *[1 mark]*

Sample response	The response
$\begin{bmatrix} 1 & -2 & -2 \\ -3 & -1 & 1 \\ 2 & 3 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 10 \end{bmatrix}$	• correctly expresses the equations of the form ***AX* = *B* [1 mark]**

b) Use matrix algebra to express $\boldsymbol{X}$ in terms of $\boldsymbol{A}$ and $\boldsymbol{B}$. *[1 mark]*

Sample response	The response
$X = \boldsymbol{A}^{-1}\boldsymbol{B}$	• correctly expresses ***X*** in terms of ***A*** and ***B*** **[1 mark]**

c) Use your result from Question 12b) to determine the solution of the system of equations. *[1 mark]*

Sample response	The response
Using GDC $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$	• determines solution to equations **[1 mark]**

d) Verify your result from Question 12c) using one of the given linear equations. *[1 mark]*

Sample response	The response
Substituting $x = -2, y = 3, z = -1$ into $x - 2y - 2z = -6$ $(-2) - 2(3) - 2(-1) = -6$ So result is verified.	• verifies result by substituting result from 12c) into one of the given linear equations **[1 mark]**

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Question 15

Vectors and matrices

The vectors representing the position (m) of particles A and B are given by $r_A = (4t-9)\hat{i} - 2(5-t)\hat{j} - 8\hat{k}$ and $r_B = (t^2+1)\hat{i} - 3\hat{j} + (4-at^2)\hat{k}$ respectively, where t is the time of motion for $0 \le t \le 10$ seconds.

a) Show that particle A passes through the point $P(5, -3, -8)$. *[2 marks]*

Sample response	The response
Method 1 If $r_A = (4t-9)\hat{i} - 2(5-t)\hat{j} - 8\hat{k}$ passes through $P(5, -3, -8)$. Consider x_A $4t-9=5$ $t=3.5$ s	• correctly determines time taken for particle A to reach *P* by considering the $\hat{i}$ component of $\mathbf{r}_A$ **[1 mark]**
Consider y_A $-2(5-t)=-3$ $t=3.5$ s So particle A passes through point *P*.	• correctly determines time taken for particle A to reach *P* by considering the $\hat{j}$ component of $\mathbf{r}_A$ **[1 mark]**
Method 2 If $r_A = (4t-9)\hat{i} - 2(5-t)\hat{j} - 8\hat{k}$ passes through $P(5, -3, -8)$. Consider x_A $4t-9=5$ $t=3.5$ s	• correctly determines time taken for particle *A* to reach *P* by considering a suitable component of r_A **[1 mark]**
$y_A(t=3.5) = -2(5-3.5)$ $=-3$ $\therefore r_A(3.5) = 5\hat{i} - 3\hat{j} - 8\hat{k}$ So particle A passes through point *P*.	• correctly uses time taken to show particle *A* passes through *P* **[1 mark]**

b) Given that particle B also passes through point *P*, determine the value of *a*. *[2 marks]*

Sample response	The response
Given $r_B = (t^2+1)\hat{i} - 3\hat{j} + (4-at^2)\hat{k}$ also passes through $P(5, -3, -8)$. Consider x_B $t^2+1=5$ $t^2=4$ $t=2$ s (as $0 \le t \le 10$)	• correctly determines time for object *B* to reach *P* **[1 mark]**
Consider z_B $4-a(2)^2=-8$ $4a=12$ $a=3$	• determines value of *a* **[1 mark]**

c) Determine the vector that represents the displacement of particle B relative to particle A during the given time of motion. Express your answer in simplest form. *[1 mark]*

Sample response	The response
Displacement vector of particle B relative to particle A is $\boldsymbol{r}_B - \boldsymbol{r}_A$ $= (t^2+1)\hat{\boldsymbol{i}} - 3\hat{\boldsymbol{j}} + (4-3t^2)\hat{\boldsymbol{k}}$ $-\left((4t-9)\hat{\boldsymbol{i}} - 2(5-t)\hat{\boldsymbol{j}} - 8\hat{\boldsymbol{k}}\right)$ $= \left(t^2-4t+10\right)\hat{\boldsymbol{i}} + (7-2t)\hat{\boldsymbol{j}} + (12-3t^2)\hat{\boldsymbol{k}}$ m	- determines the required displacement vector in simplest form **[1 mark]**

d) Use your result from Question 15c) to determine the shortest distance between particles A and B during the given time of motion. *[3 marks]*

Sample response	The response
Distance between particles is $\lvert \boldsymbol{r}_B - \boldsymbol{r}_A \rvert$ $= \sqrt{\left(t^2-4t+10\right)^2 + (7-2t)^2 + (12-3t^2)^2}$ m Particles are closest when the distance between the particles is a minimum. Using GDC Shortest distance $= 6.69$ m	- determines an expression that represents the distance between the particles **[1 mark]** - justifies when shortest distance between the particles occurs **[1 mark]** - determines shortest distance between particles **[1 mark]**

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Question 17

Vectors and matrices

An object moves with a constant speed of v in a circular path.

The position vector of the object is given by

$$\boldsymbol{r} = r\cos(\omega t)\hat{\boldsymbol{i}} + r\sin(\omega t)\hat{\boldsymbol{j}}$$

where

- r is the radius (metres) of the circle
- ω is the angular velocity (radians per second)
- t is the time (seconds) of motion for $t \geq 0$.

Use vector calculus to prove that the magnitude of the acceleration of the object is $|\boldsymbol{a}| = \dfrac{v^2}{r}$.

[6 marks]

Sample response	The response
Given $\boldsymbol{r} = r\cos(\omega t)\boldsymbol{i} + r\sin(\omega t)\boldsymbol{j}$ $\boldsymbol{v} = \dfrac{dr}{dt} = -r\omega\sin(\omega t)\hat{\boldsymbol{i}} + r\omega\cos(\omega t)\hat{\boldsymbol{j}}$	correctly determines the velocity vector **[1 mark]**
$\boldsymbol{a} = \dfrac{dv}{dt} = -r\omega^2\cos(\omega t)\hat{\boldsymbol{i}} - r\omega^2\sin(\omega t)\hat{\boldsymbol{j}}$	determines an acceleration vector **[1 mark]**
$v = \lvert \boldsymbol{v} \rvert$ $= \sqrt{(-r\omega\sin(\omega t))^2 + (r\omega\cos(\omega t))^2}$ $= \sqrt{r^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))}$ $= r\omega$	determines a simplified expression for the modulus of ***v*** **[1 mark]**
$\lvert \boldsymbol{a} \rvert = \sqrt{(-r\omega^2\cos(\omega t))^2 + (-r\omega^2\sin(\omega t))^2}$	determines an expression for the modulus of ***a*** **[1 mark]**
$\lvert \boldsymbol{a} \rvert = \sqrt{r^2\omega^4(\cos^2(\omega t) + \sin^2(\omega t))}$ $= r\omega^2 = \dfrac{r^2\omega^2}{r}$ $= \dfrac{v^2}{r}$	correctly completes the proof based on prior evidence **[1 mark]** shows logical organisation of a fully attempted proof, communicating key steps **[1 mark]**

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Question 19

Vectors and matrices

An experiment researching the population changes of a certain species of insect was conducted over a four-week period. The insect has two distinct stages in its two-week lifespan. Each stage is approximately one week in length.

A constant proportion of females survive from stage 1 into stage 2.

The ratio of the reproduction rate for females in stage 2 to females in stage 1 is 2:1. All offspring are born into stage 1.

The number of females in each stage was measured initially and then again after two weeks as shown.

Female population	Stage 1	Stage 2
Initially	48	32
After two weeks	25	21

Use a matrix approach to estimate the total number of females after four weeks.

[6 marks]

Sample response	The response
Let the birth rate for Stage 1 females be x and the survival rates for Stage 1 females to Stage 2 females be y. Let $\mathbf{L}$ be the Leslie matrix for this species. $\mathbf{L}=\begin{bmatrix} x & 2x \\ y & 0 \end{bmatrix}$	• correctly determines an appropriate Leslie matrix **[1 mark]**
Let $\mathbf{P}_n$ represent the population of the species after n weeks. $\mathbf{P}_0=\begin{bmatrix} 48 \\ 32 \end{bmatrix}$ $\mathbf{P}_2=\mathbf{L}^2\mathbf{P}_0$ $\begin{bmatrix} 25 \\ 21 \end{bmatrix}=\begin{bmatrix} x & 2x \\ y & 0 \end{bmatrix}^2\begin{bmatrix} 48 \\ 32 \end{bmatrix}$ $=\begin{bmatrix} x^2+2xy & 2x^2 \\ xy & 2xy \end{bmatrix}\begin{bmatrix} 48 \\ 32 \end{bmatrix}$ $=\begin{bmatrix} 48x^2+96xy+64x^2 \\ 48xy+64xy \end{bmatrix}$	• determines a matrix equation linking the initial population with the population after two weeks **[1 mark]**
Equating parts $112x^2+96xy=25 \quad \ldots(1)$ $112xy=21 \quad \ldots(2)$	• determines two simultaneous equations in terms of the relevant birth and survival rates **[1 mark]**
From (2) $xy=\frac{21}{112} \quad \ldots(3)$ Substituting into (1) $112x^2+96\left(\frac{21}{112}\right)=25$ $x=0.25$ (reject $-$ive solution as $x>0$) Using 3, $y=0.75$	• determines appropriate values of x and y **[1 mark]**
$\mathbf{P}_4=\mathbf{L}^4\mathbf{P}_0$ $=\begin{bmatrix} 0.25 & 0.5 \\ 0.75 & 0 \end{bmatrix}^4\begin{bmatrix} 48 \\ 32 \end{bmatrix}$ $=\begin{bmatrix} 13.6 \\ 12.6 \end{bmatrix}$ Approximate **total number of females** at the conclusion of the experiment **is 26.**	• determines the total number of females at the conclusion of the experiment **[1 mark]**
	• shows logical organisation of a fully attempted solution, communicating key steps **[1 mark]**

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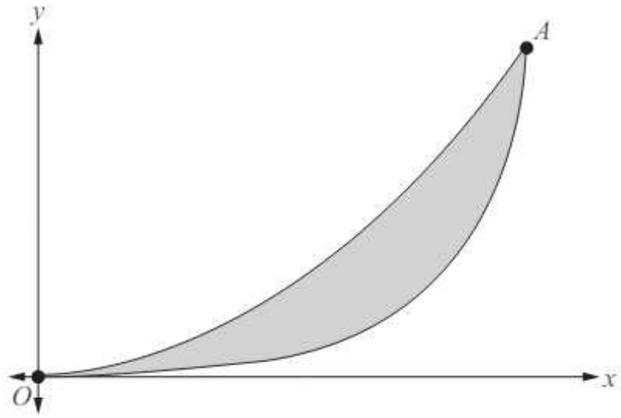
Question 11

Vectors and matrices

The bounded region between the graphs of the functions $y=-1+\sec\left(\frac{x}{5}\right)$ and $y=0.1x^2$ over a certain domain is shaded as shown. The two functions intersect at the origin and point A.



a) Determine the coordinates of point A. *[1 mark]*

Sample response	The response
$(7, 4.9)$	• correctly determines point A **[1 mark]**

b) Calculate the area of the shaded region. *[1 mark]*

Sample response	The response
Using GDC Area = 6.14 units^2	• calculates area **[1 mark]**

The shaded region is rotated about the x-axis to form a solid of revolution.

c) Determine the volume of the solid formed. *[2 marks]*

Sample response	The response
$\left\lvert\pi\int_a^b\left([f(x)]^2-[g(x)]^2\right)dx\right\rvert$ $=\pi\int_0^7\left(\left[0.1x^2\right]^2-\left[\left(-1+\sec\left(\frac{x}{5}\right)\right)\right]^2\right)dx$	• determines a definite integral representing a value related to the volume **[1 mark]**
Using GDC Volume $=69.76$ units^3	• calculates volume **[1 mark]**

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Question 17

Vectors and matrices

An object is projected upwards from ground level with an initial velocity of 15 m s^{-1} at an angle of 54° to the horizontal.

The object just passes over a drone hovering in the air. An observer is positioned directly below the drone and at a horizontal distance of 20 m from where the object is projected.

The observer commented that:

- it took the object around 2 to 2.5 seconds after its projection to reach the drone
- the object was still moving in an upwards direction as it passed the drone.

Assuming that air resistance is negligible, use a vector calculus approach to evaluate the reasonableness of the observer's comments.

Sample response	The response
Method 1 Let $\hat{i}$ and $\hat{j}$ be the horizontal and vertical unit vectors respectively. Let t represent the time in seconds after the projection of the object. $\boldsymbol{a}(t) = -9.8\hat{j}$ $\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\, dt = -9.8t\,\hat{j} + c$ Given $\boldsymbol{v}(0) = 15\cos\left(54^\circ\right)\hat{i} + 15\sin\left(54^\circ\right)\hat{j}$ $\boldsymbol{v}(t) = 15\cos\left(54^\circ\right)\hat{i} + \left(15\sin\left(54^\circ\right) - 9.8t\right)\hat{j}$	• correctly determines the velocity function of the object using vector calculus **[1 mark]**
$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\, dt$ $= 15\cos\left(54^\circ\right)t\,\hat{i} + \left(15\sin\left(54^\circ\right)t - 4.9t^2\right)\hat{j} + c$ Let origin be at the release point: $\boldsymbol{r}(0) = 0\hat{i} + 0\hat{j}$ $\boldsymbol{r}(t) = 15\cos\left(54^\circ\right)t\,\hat{i} + \left(15\sin\left(54^\circ\right)t - 4.9t^2\right)\hat{j}$	• determines displacement function of the object **[1 mark]**
When $r_x = 20 \Rightarrow 15\cos\left(54^\circ\right)t = 20$ Time object just passes drone: $t = 2.27\text{s}$	• determines time when the object just passes drone **[1 mark]**
Finding maximum value of $r_y : 15\sin\left(54^\circ\right)t - 4.9t^2$ Using GDC Time object reaches maximum height: $t = 1.24\text{s}$	• determines time when the object reaches maximum height **[1 mark]**
While the estimation of the time taken for the object to reach the drone is reasonable, the comment regarding the direction of the object as it passed the drone is not reasonable as it would have been moving in a downward direction at that time.	• uses mathematical justification to evaluate the reasonableness of both comments based on prior mathematical reasoning **[1 mark]** • shows logical organisation, communicating key steps **[1 mark]**

	Method 2 Let $\hat{i}$ and $\hat{j}$ be the horizontal and vertical unit vectors respectively. Let t represent the time in seconds after the projection of the object. $\boldsymbol{a}(t) = -9.8\hat{j}$ $\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\,dt$ $= -9.8t\,\hat{j} + c$ Given $\boldsymbol{v}(0) = 15\cos(54^\circ)\hat{i} + 15\sin(54^\circ)\hat{j}$ $\boldsymbol{v}(t) = 15\cos(54^\circ)\hat{\boldsymbol{i}} + \left(15\sin(54^\circ) - 9.8t\right)\hat{\boldsymbol{j}}$ Considering horizontal component of velocity: $v_x = 15\cos(54^\circ) = 8.82$ At $t = 2$, $x = vt = 2\times 8.82 = 17.64$ m At $t = 2.5$, $x = 2.5\times 8.82 = 22.05$ m Considering vertical component of velocity: At $t = 2, v_y = 12.14 - 9.8\times 2 = -7.46\ \text{ms}^{-1}$ While the estimation of the time taken for the object to reach the drone is reasonable, the comment regarding the direction of the object as it passed the drone is not reasonable as it would have been moving in a downward direction at that time.	• correctly determines the velocity function of the object using vector calculus **[1 mark]** • determines horizontal distance travelled after 2 seconds **[1 mark]** • determines horizontal distance travelled after 2.5 seconds **[1 mark]** • determines vertical velocity after 2 seconds **[1 mark]** • uses mathematical justification to evaluate the reasonableness of both comments based on prior mathematical reasoning **[1 mark]** • shows logical organisation, communicating key steps **[1 mark]**

	Method 3 Let $\hat{i}$ and $\hat{j}$ be the horizontal and vertical unit vectors respectively. Let t represent the time in seconds after the projection of the object. $\boldsymbol{a}(t) = -9.8\hat{\boldsymbol{j}}$ $\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\,dt = -9.8t\,\hat{\boldsymbol{j}} + c$ Given $\boldsymbol{v}(0) = 15\cos\left(54^\circ\right)\hat{i} + 15\sin\left(54^\circ\right)\hat{j}$ $\boldsymbol{v}(t) = 15\cos\left(54^\circ\right)\hat{\boldsymbol{i}} + \left(15\sin\left(54^\circ\right) - 9.8t\right)\hat{\boldsymbol{j}}$	- correctly determines the velocity function of the object using vector calculus **[1 mark]**
	$\boldsymbol{r}(t) = \int \boldsymbol{v}(t)\,dt$ $= 15\cos\left(54^\circ\right)t\,\hat{\boldsymbol{i}} + \left(15\sin\left(54^\circ\right)t - 4.9t^2\right)\hat{\boldsymbol{j}} + c$ Let origin be at the release point: $\boldsymbol{r}(0) = 0\hat{\boldsymbol{i}} + 0\hat{\boldsymbol{j}}$ $\boldsymbol{r}(t) = 15\cos\left(54^\circ\right)t\,\hat{\boldsymbol{i}} + \left(15\sin\left(54^\circ\right)t - 4.9t^2\right)\hat{\boldsymbol{j}}$	- determines displacement function of the object **[1 mark]**
	At $t = 2$, $x = 17.64$ m and $y = 4.67$ m	- determines horizontal and vertical distances travelled after 2 seconds **[1 mark]**
	At $t = 2.5$, $x = 22.05$ m and $y = -0.29$ m	- determines horizontal and vertical distances travelled after 2.5 seconds **[1 mark]**
	While the estimation of the time taken for the object to reach the drone is reasonable, the comment regarding the direction of the object as it passed the drone is not reasonable as it would have been moving in a downward direction at that time.	- uses mathematical justification to evaluate the reasonableness of both comments based on prior mathematical reasoning **[1 mark]** - shows logical organisation, communicating key steps **[1 mark]**

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Vectors and matrices

A scientist collects data for a species of tree frog in a protected area. Details for the female tree frog population are shown in the table.

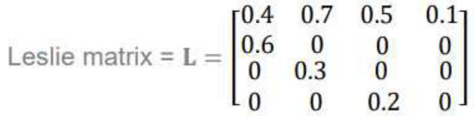
Age (years)	0–1	1–2	2–3	3–4
Population in Year 1	150	101	84	62
Birth (breeding) rate	0.4	0.7	0.5	0.1
Survival rate	0.6	0.3	0.2	0

The scientist uses a Leslie matrix model to make predictions about the female tree frog population.

a) State the initial population matrix. [1 mark]

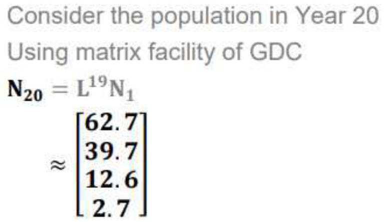
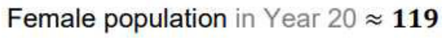
Sample Response	The response
 Initial population matrix = $N_1 = \begin{bmatrix}150\\101\\84\\62\end{bmatrix}$	• correctly states the initial population matrix [1 mark]

b) Determine the Leslie matrix. [1 mark]

Sample Response	The response
 Leslie matrix = $L = \begin{bmatrix}0.4 & 0.7 & 0.5 & 0.1\\0.6 & 0 & 0 & 0\\0 & 0.3 & 0 & 0\\0 & 0 & 0.2 & 0\end{bmatrix}$	• correctly determines the Leslie matrix [1 mark]

A species is considered to be endangered if the female population in a restricted area is predicted to fall to less than 125 in the next 20 years.

c) Determine whether this species of tree frog is considered to be endangered. [3 marks]

Sample Response	The response
 Consider the population in Year 20 Using matrix facility of GDC $\mathbf{N_{20}} = L^{19}N_1$ $\approx \begin{bmatrix}\mathbf{62.7}\\\mathbf{39.7}\\\mathbf{12.6}\\\mathbf{2.7}\end{bmatrix}$	• calculates a matrix representing the female population within a 20-year period [1 mark]
 Female population in Year 20 ≈ **119**	• calculates female population for a year within a 20-year period [1 mark]
 The female population is less than 125 within the 20-year period so the species is considered to be endangered.	• makes a suitable decision whether the species is considered endangered [1 mark]

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Question 14

Vectors and matrices

An object is moving in a straight line with an acceleration represented by the differential equation $\frac{dv}{dt} = -\left(4+v^2\right)$, where v is the object's velocity $\left(\text{m s}^{-1}\right)$ over time, $t\left(\text{s}\right)$, where $t \geq 0$, until it comes to rest.

a) Determine the general solution of the differential equation. [3 marks]

Sample Response	**The response**
$\frac{dv}{dt} = -(4+v^2)$ $\frac{1}{4+v^2}\frac{dv}{dt} = -1$ $\int \frac{1}{4+v^2}\, dv = \int -1\, dt$	• correctly establishes a suitable integration result based on the separation of variables technique [1 mark]
$\frac{1}{2}\int \frac{2}{4+v^2} dv = \int -1\, dt$ $\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right)$ $= -t + c$	• determines one side of the general solution in terms of v [1 mark] • determines the other side of the general solution in terms of t [1 mark]

The initial velocity of the object is 1.5 m s^{-1}.

b) Determine the time when the particle comes to rest. [2 marks]

Sample Response	**The response**
Given $v(0) = 1.5$ $\frac{1}{2}\tan^{-1}\left(\frac{1.5}{2}\right) = c$	• determines an expression that represents the integration constant [1 mark]
$c \approx 0.32$ When $v = 0$: $\frac{1}{2}\tan^{-1}(0) \approx -t + 0.32$ $t \approx 0.32$ s The particle comes to rest after 0.32 s.	• determines the time when the particle is at rest [1 mark]

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Question 15

Vectors and matrices

Consider points $A(3, -1, 3)$ and $B(1, 1, 6)$.

a) Determine $\overrightarrow{AB}$.

[1 mark]

Sample Response	The response
$\overrightarrow{AB} = \begin{pmatrix}1\\1\\6\end{pmatrix} - \begin{pmatrix}3\\-1\\3\end{pmatrix} = \begin{pmatrix}-2\\2\\3\end{pmatrix}$	• correctly determines $\overrightarrow{AB}$ [1 mark]

b) Determine the Cartesian equation of the line that passes through points A and B. [2 marks]

Sample Response	The response
Cartesian equation of the line is $\frac{x-3}{-2} = \frac{y+1}{2} = \frac{z-3}{3}$	• correctly substitutes the coordinates of A into the numerator of the equation [1 mark] • substitutes direction vector of the line into the denominator [1 mark]

Point A lies on the plane, φ, and $\overrightarrow{AB}$ is perpendicular to this plane.

c) Determine the Cartesian equation of the plane.

[2 marks]

Sample Response	The response
From above, normal of the plane is $\begin{pmatrix}-2\\2\\3\end{pmatrix}$ Equation of plane using point A $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}-2\\2\\3\end{pmatrix} = \begin{pmatrix}3\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}-2\\2\\3\end{pmatrix}$	• substitutes values into a suitable mathematical rule that represents an equation of the plane [1 mark]
$-2x + 2y + 3z = 1$	• determines a Cartesian equation of the plane [1 mark]

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Vectors and matrices

A research organisation plans to use a drone to drop a scientific instrument vertically from a stationary position above the ocean surface. The acceleration (ms^{-2}) of the falling instrument can be modelled by 9.8 – 0.1v, where v is its velocity (ms^{-1}).

In order for the instrument sensors to activate, its speed as it hits the ocean surface must reach at least 20 m s^{-1}. However, if it hits with a speed above 50 ms^{-1} , the sensors will be damaged.

Determine the range of the drone's flying height above the ocean surface to ensure that the sensors are activated but not damaged. [7 marks]

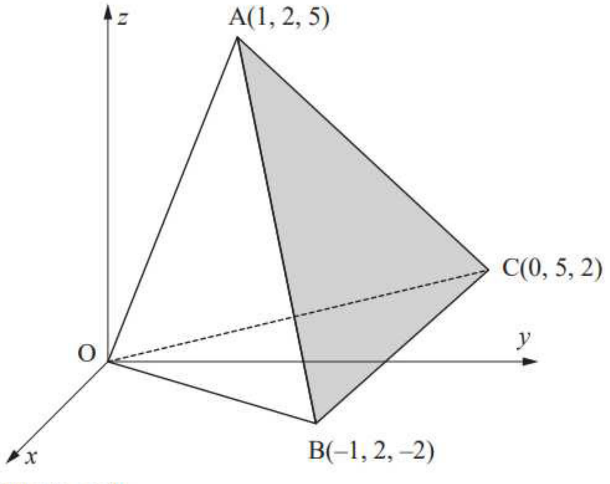
Sample Response	The response
Method 1 – Considers a differential equation in terms of v and x and assumes the origin is at the point of release. Assume downwards as the positive direction $a = 9.8 - 0.1v$ $v\frac{dv}{dx} = 9.8 - 0.1v$	• correctly establishes a differential equation in terms of v and x [1 mark]
$\int \frac{v}{9.8 - 0.1v}\,dv = \int dx$ $\int \frac{v}{v - 98}\,dv = \int \frac{-1}{10}\,dx$ $\int 1 + \frac{98}{v - 98}\,dv = \int \frac{-1}{10}\,dx$ $v + 98\ln\lvert v - 98\rvert = \frac{-x}{10} + c$	• determines general solution of the differential equation [1 mark]
Assume the origin is at the point of release. Given $v(0) = 0 \Rightarrow c = 98\ln(98)$	• determines value for the constant [1 mark]
$v + 98\ln\lvert v - 98\rvert = \frac{-x}{10} + 98\ln(98)$	• determines model for the velocity in terms of displacement [1 mark]
Let the distance to the ocean surface be h metres. Consider time of drop for each required velocity When $v = 20$ m s^{-1}, $x = h$ $20 + 98\ln\lvert -78\rvert = \frac{-h}{10} + 98\ln(98)$ $h = 23.7$ m	• determines displacement of the drop for the minimum acceptable speed [1 mark]
When $v = 50$ m s^{-1}, $x = h$ $50 + 98\ln\lvert -48\rvert = 98\ln(98) - \frac{h}{10}$ $h = 199.5$ m	• determines displacement of the drop for the maximum acceptable speed [1 mark]
The range of the drone's flying height above the ocean surface should be between 23.7 m and 199.5 m.	• communicates range of the drone's flying height including units [1 mark]
Method 2 – Considers a differential equation in terms of v and x and assumes the origin is at the ocean surface. Assume downwards as the positive direction $a = 9.8 - 0.1v$ $v\frac{dv}{dx} = 9.8 - 0.1v$	• correctly establishes a differential equation in terms of v and x [1 mark]
$\int \frac{v}{9.8 - 0.1v}\,dv = \int 1\,dx$ $\int \frac{v}{v - 98}\,dv = \int \frac{-1}{10}\,dx$ $\int 1 + \frac{98}{v - 98}\,dv = \int \frac{-1}{10}\,dx$ $v + 98\ln\lvert v - 98\rvert = \frac{-x}{10} + c$	• determines a general solution of the differential equation [1 mark]
Assume the origin is the point directly below the drone on the ocean surface and let the height of release be h metres above the origin. Given $v(h) = 0 \Rightarrow c = 98\ln(98) + \frac{h}{10}$	• determines a value for the constant [1 mark]

$v + 98\ln\lvert v-98\rvert = \frac{-x}{10} + 98\ln(98) + \frac{h}{10}$	• determines a model for the velocity in terms of its displacement and initial height [1 mark]
Consider time of drop for each required velocity When $v = 20\ \text{m s}^{-1}, x = 0$ $20 + 98\ln\lvert -78\rvert = 98\ln(98) + \frac{h}{10}$ $h = -23.7\ \text{m}$	• determines displacement of the drop for the minimum acceptable speed [1 mark]
When $v = 50\ \text{m s}^{-1}, x = 0$ $50 + 98\ln\lvert -48\rvert = 98\ln(98) + \frac{h}{10}$ $h = -199.5\ \text{m}$	• determines displacement of the drop for the maximum acceptable speed [1 mark]
The range of the drone's flying height above the ocean surface should be between 23.7 m and 199.5 m.	• communicates range of the drone's flying height including units [1 mark]
Method 3 – Considers a differential equation in terms of v and t and assumes release time at $t = 0$. Assume downwards as the positive direction $a = 9.8 - 0.1v$ $\frac{dv}{dt} = 9.8 - 0.1v$	• correctly establishes a differential equation in terms of v and t [1 mark]
$\int \frac{dv}{9.8 - 0.1v} = \int dt$ $\frac{\ln(9.8 - 0.1v)}{-0.1} = t + c$	• determines a general solution of the differential equation [1 mark]
$9.8 - 0.1v = Ae^{-0.1t}$ $v = 10(9.8 - Ae^{0.1t})$ Given $v(0) = 0 \Rightarrow A = 9.8$ $\therefore\ v = 98(1 - e^{-0.1t})$	• determines a model for the velocity using the initial boundary condition [1 mark]
Consider time of drop for each required velocity When $v = 20\ \text{m s}^{-1}$ and $v = 50\ \text{m s}^{-1}$ $20 = 98(1 - e^{-0.1t})$ and $50 = 98(1 - e^{-0.1t})$ Using solve facility of GDC $t = 2.283$ s and $t = 7.138$ s	• uses this model to determine the time of the drop for both the minimum and maximum acceptable speeds [1 mark]
Determining the change in displacement of drop using calculus facility of GDC: $v = 20\ \text{m s}^{-1}, \Delta x = \int_0^{2.283} 98(1 - e^{-0.1t})\,dt \approx 23.7\ \text{m}$ $v = 50\ \text{m s}^{-1}, \Delta x = \int_0^{7.138} 98(1 - e^{-0.1t})\,dt \approx 199.5\ \text{m}$	• determines displacement of the drop for the minimum acceptable speed [1 mark] • determines displacement of the drop for the maximum acceptable speed [1 mark]
The range of the drone's flying height above the ocean surface should be between 23.7 m and 199.5 m.	• communicates range of the drone's flying height including units [1 mark]

2021
Paper 2
Section 2
Question 11

Vectors and matrices

OABC is a triangular-based pyramid, as shown.



Not to scale

Use a vector method to determine the area of the shaded face of the pyramid. [4 marks]

Sample Response	The response
$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a} = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -7 \end{pmatrix}$ $\overrightarrow{AC} = \boldsymbol{c} - \boldsymbol{a} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -3 \end{pmatrix}$	• correctly determines the two vectors representing any two sides of the triangle ABC [1 mark]
$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 21 \\ 1 \\ -6 \end{pmatrix}$	• determines cross product of two vectors representing any two sides of the triangle ABC [1 mark]
$\text{Area} = \frac{1}{2}\lvert\overrightarrow{AB} \times \overrightarrow{AC}\rvert$	• uses vector expression to represent the area of a triangle [1 mark]
$= \frac{1}{2}\left\lvert\begin{pmatrix} 21 \\ 1 \\ -6 \end{pmatrix}\right\rvert$ $= \frac{1}{2}\sqrt{21^2 + 1^2 + (-6)^2}$ $= 10.93 \text{ units}^2$	• determines area [1 mark]

2021
Paper 2
Section 2
Question 14

Vectors and matrices

The Tasmanian thornbill is a species of bird that has an average life span of three years. Female thornbills do not reproduce in their first year, but produce an average of four female offspring in each of their second and third years. The survival rate of each age group is estimated as 25% in their first year and 30% in their second year.

A Leslie matrix, L, modelling the population distribution of the Tasmanian thornbill, has been partially completed.

$$\mathbf{L} = \begin{bmatrix} 0 & 4 & 4 \\ x & 0 & 0 \\ 0 & y & 0 \end{bmatrix}$$

a) State the values of x and y. [1 mark]

Sample Response	The response
$x = 0.25$ $y = 0.3$	• correctly states the values of x and y [1 mark]

At the start of 2021, a study began into the population of Tasmanian thornbills. The study:

- estimated that the initial female population was 510 in their first year, 480 in their second year and 420 in their third year
- found that the ratio of male to female was approximately 1:2.

b) Estimate the total population of Tasmanian thornbills at the start of 2025. [4 marks]

Sample Response	The response
Number of females at the start of 2021 $\mathrm{N}_0 = \begin{bmatrix} 510 \\ 480 \\ 420 \end{bmatrix}$	• correctly identifies the initial matrix [1 mark]
Number of females at the start of 2025 $\mathrm{N}_4 = \mathrm{L}^4\,\mathrm{N}_0$ $= \begin{bmatrix} 2166 \\ 938.25 \\ 81.45 \end{bmatrix}$	• determines female population in their first, second and third years at the start of 2025 [1 mark]
Total female population $\approx 2166 + 938 + 81$ ≈ 3185	• determines total female population at the start of 2025 [1 mark]
Total population $\approx \frac{3}{2} \times 3186 \approx \mathbf{4778}$	• determines total population at the start of 2025, rounded to a whole number [1 mark]

2021
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Section 2
Question 16

Vectors and matrices

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

Given $\mathbf{A}^4 = p\mathbf{A}^3 + q\mathbf{A}^2 + r\mathbf{A} + 3\mathbf{I}$, use matrix algebra to determine the value of the scalars p, q and r.
[6 marks]

Sample Response	**The response**
Given $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$ $\mathbf{A}^4 = p\mathbf{A}^3 + q\mathbf{A}^2 + r\mathbf{A} + 3\mathbf{I}$ Using GDC and substituting into given equation $\begin{bmatrix} 177 & 176 & 144 \\ 144 & 145 & 120 \\ 176 & 176 & 145 \end{bmatrix} = p\begin{bmatrix} 37 & 38 & 32 \\ 32 & 31 & 25 \\ 38 & 38 & 31 \end{bmatrix} + q\begin{bmatrix} 9 & 8 & 6 \\ 6 & 7 & 6 \\ 8 & 8 & 7 \end{bmatrix}$ $+ r\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	• correctly substitutes results into given equation to form an equation based on 3 × 3 matrices [1 mark]
Equating elements in Row 1: $177 = 37p + 9q + r + 3 \Rightarrow 37p + 9q + r = 174$ $176 = 38p + 8q + 2r + 0 \Rightarrow 38p + 8q + 2r = 176$ $144 = 32p + 6q + 2r + 0 \Rightarrow 32p + 6q + 2r = 144$	• uses equation based on 3 × 3 matrices to establish 3 equations in 3 unknowns [1 mark]
Expressing the 3 equations in matrix form: $\begin{bmatrix} 37 & 9 & 1 \\ 38 & 8 & 2 \\ 32 & 6 & 2 \end{bmatrix}\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 174 \\ 176 \\ 144 \end{bmatrix}$ $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 37 & 9 & 1 \\ 38 & 8 & 2 \\ 32 & 6 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 174 \\ 176 \\ 144 \end{bmatrix}$ $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ 10 \\ 10 \end{bmatrix}$	• expresses 3 equations in matrix form [1 mark] • demonstrates the use of matrix algebra in solving the matrix equation [1 mark] • determines values of p, q and r [1 mark] • shows logical organisation, communicating key steps to at least the start of expressing the 3 equations in matrix form [1 mark]

2021
Paper 2
Section 2
Question 17

Vectors and matrices

An object with a mass of 2 kg is released from rest at the top of a 1 metre long frictionless plane inclined at 30° to the horizontal.

A force of $\boldsymbol{P}$ newtons acting parallel to the plane opposes the motion of the object as it travels down the plane.

When the object is x metres from the top of the plane, its velocity is v m s^{-1}.

Given $|\boldsymbol{P}| = \dfrac{4}{\sqrt{4-x^2}}$, determine x when $v = 2$.

[7 marks]

Sample Response	The response
Method 1 Resolving net forces along the plane $F_{\text{net}} = 2g\sin(30°) - \dfrac{4}{\sqrt{4-x^2}}$	• correctly determines the net forces along the plane [1 mark]
$F_{\text{net}} = ma$ $g - \dfrac{4}{\sqrt{4-x^2}} = 2a$	• determines equation for acceleration along the plane [1 mark]
$g - \dfrac{4}{\sqrt{4-x^2}} = 2v\dfrac{dv}{dx}$	• determines differential equation in terms of velocity and displacement [1 mark]
$v\dfrac{dv}{dx} = 4.9 - \dfrac{2}{\sqrt{4-x^2}}$ $\int v\ dv = \int 4.9 - \dfrac{2}{\sqrt{4-x^2}}\ dx$ $\dfrac{v^2}{2} = 4.9x - 2\sin^{-1}\left(\dfrac{x}{2}\right) + c$	• determines general solution to a differential equation [1 mark]
Given $v = 0$ when $x = 0$ $0 = 0 - 2\sin^{-1}(0) + c$ $c = 0$ $\therefore v^2 = 9.8x - 4\sin^{-1}\left(\dfrac{x}{2}\right)$	• determines value of arbitrary constant [1 mark]
When $v = 2$ $4 = 9.8x - 4\sin^{-1}\left(\dfrac{x}{2}\right)$	• establishes equation to solve for x when $v = 2$ [1 mark]
Solving for x using GDC $x = 0.51$ m	• determines x [1 mark]
Method 2 Resolving net forces along the plane $F_{\text{net}} = 2g\sin(30°) - \dfrac{4}{\sqrt{4-x^2}}$	• correctly determines the net forces along the plane [1 mark]
$F_{\text{net}} = ma$ $g - \dfrac{4}{\sqrt{4-x^2}} = 2a$	• determines equation for acceleration along the plane [1 mark]
$g - \dfrac{4}{\sqrt{4-x^2}} = 2v\dfrac{dv}{dx}$	• determines differential equation in terms of velocity and displacement [1 mark]
$v\dfrac{dv}{dx} = 4.9 - \dfrac{2}{\sqrt{4-x^2}}$ $\int v\ dv = \int 4.9 - \dfrac{2}{\sqrt{4-x^2}}\ dx$ $\dfrac{v^2}{2} = 4.9x + 2\cos^{-1}\left(\dfrac{x}{2}\right) + c$	• determines general solution to a differential equation [1 mark]

Given $v = 0$ when $x = 0$ $0 = 0 + 2\cos^{-1}(0) + c$ $c = -\pi$ $\therefore v^2 = 9.8x + 4\cos^{-1}\left(\frac{x}{2}\right) - 2\pi$	• determines value of arbitrary constant [1 mark]
When $v = 2$ $4 = 9.8x + 4\cos^{-1}\left(\frac{x}{2}\right) - 2\pi$	• establishes equation to solve for x when $v = 2$ [1 mark]
Solving for x using GDC $x = 0.51$ m	• determines x [1 mark]

2020
Paper 2
Section 2
Question 11

Vectors and matrices

Teams A, B, C, D and E participated in a competition with the following results:

- A defeated D.
- B defeated A, C and E.
- C defeated A and E.
- D defeated B, C and E.
- E defeated A.

To rank the teams at the end of the competition, the organisers constructed a dominance matrix, N, that is partially completed.

a) By allocating 1 to represent 'defeated' and 0 to represent either 'was defeated by' or 'no result', complete matrix N. [1 mark]

$$\mathbf{N} = \text{Winning teams} \quad \begin{array}{c|ccccc} & \multicolumn{5}{c}{\text{Losing teams}} \\ & A & B & C & D & E \\ \hline A & \square & \square & \square & \square & \square \\ B & \square & \square & \square & \square & \square \\ C & \square & \square & \square & \square & \square \\ D & \square & \square & \square & \square & \square \\ E & \square & \square & \square & \square & \square \end{array}$$

The organisers need to rank the teams into individual places from first to fifth place.

They decide to use the ranking model $\mathrm{N} + \mathrm{N}^2$ to achieve this.

Sample Response	The response
$\mathbf{N} = \text{Winning teams} \quad \begin{array}{c\|ccccc} & \multicolumn{5}{c}{\text{Losing teams}} \\ & A & B & C & D & E \\ \hline A & 0 & 0 & 0 & 1 & 0 \\ B & 1 & 0 & 1 & 0 & 1 \\ C & 1 & 0 & 0 & 0 & 1 \\ D & 0 & 1 & 1 & 0 & 1 \\ E & 1 & 0 & 0 & 0 & 0 \end{array}$	• correctly completes matrix **N** [1 mark]

b) Use the model N + N^2 to rank the teams. [2 marks]

<table>
<tr><th>Sample Response</th><th>The response</th></tr>
<tr><td>$\mathbf{N} + \mathbf{N}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 & 2 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 1 & 2 & 0 & 3 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$</td><td>• calculates $\mathbf{N} + \mathbf{N}^2$ [1 mark]</td></tr>
<tr><td>Ranking teams using the model $\mathbf{N} + \mathbf{N}^2$
<table>
<tr><th>Team</th><th>Model value</th><th>Rank position</th></tr>
<tr><td>A</td><td>4</td><td>3</td></tr>
<tr><td>B</td><td>7</td><td>2</td></tr>
<tr><td>C</td><td>4</td><td>3</td></tr>
<tr><td>D</td><td>9</td><td>1</td></tr>
<tr><td>E</td><td>2</td><td>5</td></tr>
</table></td><td>• ranks the teams to show that teams A and C are tied [1 mark]</td></tr>
</table>

c) Use the result from 11b) to identify a limitation of the organisers' ranking model. [1 mark]

Sample Response	The response
The limitation of the ranking model is that it does not provide individual positions from first to fifth.	• identifies a limitation of the organiser's model based on the model $\mathbf{N} + \mathbf{N}^2$ [1 mark]

d) State a mathematical refinement the organisers could consider to overcome the limitation of the ranking model identified in 11c). [1 mark]

Sample Response	The response
The ranking model could be improved by including weightings in the calculations (e.g. $\mathrm{N} + \frac{1}{2}\mathrm{N}^2$)	• correctly describes a suitable mathematical refinement [1 mark]

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Paper 2
Section 2
Question 15

Vectors and matrices

The position vectors of points P and Q are $2\hat{\imath} - 3\hat{\jmath} + \hat{k}$ and $2\hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ respectively.

Let O be the origin.

a) Determine the angle POQ.

[2 marks]

Sample Response	The response
Determining angle POQ $\cos(\theta) = \frac{\overrightarrow{OP} \cdot \overrightarrow{OQ}}{\lvert\overrightarrow{OP}\rvert\lvert\overrightarrow{OQ}\rvert} = \frac{-6}{\sqrt{14}\sqrt{24}}$	• correctly determines a numerical expression for $\cos(\theta)$ [1 mark]
$\theta \approx 1.90$	• determines angle POQ [1 mark]

Points O, P and Q are joined to form a triangle.

b) Determine the area of triangle POQ. [2 marks]

Sample Response	The response
Method 1 Finding area of triangle POQ: $A = \frac{1}{2}\lvert\overrightarrow{OP} \times \overrightarrow{OQ}\rvert$ Using vector facility of GDC or otherwise $= \frac{1}{2}\lvert 10\hat{\imath} + 10\hat{\jmath} + 10\hat{k}\rvert$	• correctly determines $\overrightarrow{OP} \times \overrightarrow{OQ}$ [1 mark]
$= 8.66$ units^2	• determines area [1 mark]
Method 2 Finding area of triangle POQ: $A = \frac{1}{2}\lvert\overrightarrow{OP}\rvert\lvert\overrightarrow{OQ}\rvert\sin(\theta)$ $= \frac{1}{2}\sqrt{14}\sqrt{24}\sin(1.90)$	• correctly substitutes values into suitable rule [1 mark]
$= 8.66$ units^2	• determines area [1 mark]

2020
Paper 2
Section 2
Question 17

Vectors and matrices

An object is released from rest at a height of 100 m above the ground.

The motion of the vertical descent of the object is modelled by

$$v\frac{dv}{dx} = 9.8 - 0.004v^2 \quad (v \geq 0)$$

where v is the velocity (m s^{-1}) and x is the displacement from the ground (m).

Determine the velocity of the object when it strikes the ground. [7 marks]

Sample Response	The response
$v\frac{dv}{dx} = 9.8 - 0.004v^2\ , v > 0$ $\int \frac{v}{9.8 - 0.004v^2} dv = \int dx$	• correctly uses separation of variables [1 mark]
$\frac{-1}{0.008} \int \frac{-0.008v}{9.8 - 0.004v^2} dv = \int dx$ $-125 \ln\lvert 9.8 - 0.004v^2\rvert = x + c$	• correctly develops the general solution of the differential equation [1 mark]
Given $v = 0$ when $x = -100$	• correctly uses the given position of the origin [1 mark]
$-125 \ln\lvert 9.8\rvert = -100 + c$ $c \approx -185.298$	• uses the given condition to determine value for c [1 mark]
$-125 \ln\lvert 9.8 - 0.004v^2\rvert = x - 185.298$ Determining v when $x = 0$ $-125 \ln\lvert 9.8 - 0.004v^2\rvert = -185.298$	• substitutes the displacement at impact to form an equation in terms of v [1 mark]
Using graph facility of GDC $v \approx -36.7$ ms^{-1} or $v \approx 36.7$ ms^{-1} As $v > 0$, the negative solution is rejected $\therefore v \approx 36.7$ ms^{-1}	• determines one reasonable solution of v [1 mark] • shows logical organisation communicating key steps [1 mark]

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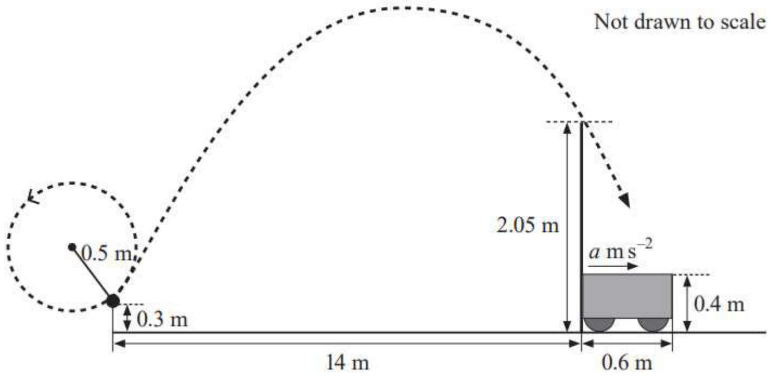
Question 19

Vectors and matrices

An object is swinging at the end of a 0.5 m length of string in a vertical circular path with a constant angular speed, completing each revolution in 0.24 seconds.

The object is projected from a height of 0.3 m above the ground in a vertical plane and just passes over a narrow pole as shown in the diagram. The pole is 2.05 m high and its base is 14 m horizontally from where the object was projected.



A flat-topped vehicle of length 0.6 m and height 0.4 m is initially at rest against the pole as shown in the diagram. At the instant that the object is projected, the vehicle moves in a horizontal direction away from the pole in the same vertical plane with an acceleration of magnitude of a m s^{-2}. The object strikes the middle of the top of the vehicle.

Assuming that air resistance is negligible, use vector calculus to model the motion of the projectile in order to determine the value of *a*. [7 marks]

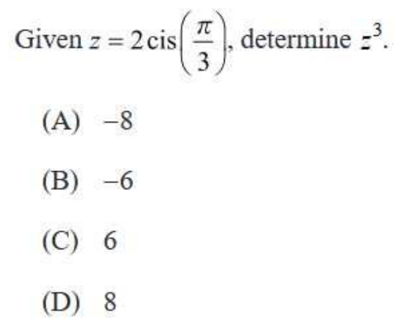
Sample Response	The response
Method 1 Finding the angular velocity during the swing: $\omega = \frac{2\pi}{T}$ $= \frac{2\pi}{0.24}$ $= 26.18 \text{ rad } s^{-1}$ Finding the magnitude of the velocity at release: $v = r\omega$ $= 0.5 \times 26.18$ $= 13.09 \text{ m s}^{-1}$	• correctly determines the tangential velocity of object [1 mark]
Let the origin in the vertical plane of motion be at the ground directly below the point of release and the angle of release to the horizontal be θ. Assume the unit  vectors: Equations of motion for the projectile: $\boldsymbol{a} = -9.8\,\hat{\jmath}$ $\boldsymbol{v} = 13.09\ \cos(\theta)\,\hat{\imath} + (13.09\ \sin(\theta) - 9.8t)\hat{\jmath}$ $\boldsymbol{r} = 13.09\ \cos(\theta)\,t\,\hat{\imath}$ $+ (0.3 + 13.09\ \sin(\theta)\,t - 4.9t^2)\hat{\jmath}$	• identifies the parametric form of the projectile path [1 mark]
The parametric form of the projectile path: $x = 13.09\ \cos(\theta)\,t \qquad \ldots(1)$ $y = 0.3 + 13.09\ \sin(\theta)\,t - 4.9t^2 \qquad \ldots(2)$ From (1) $t = \frac{x}{13.09\ \cos(\theta)} \qquad \ldots(3)$	• determines the Cartesian form of the projectile path [1 mark]
Substituting into (2) to change into Cartesian form: $y = 0.3 + \frac{13.09\sin(\theta)x}{13.09\cos(\theta)} - \frac{4.9x^2}{(13.09)^2\cos^2(\theta)}$	• determines the angle of release of the object [1 mark]

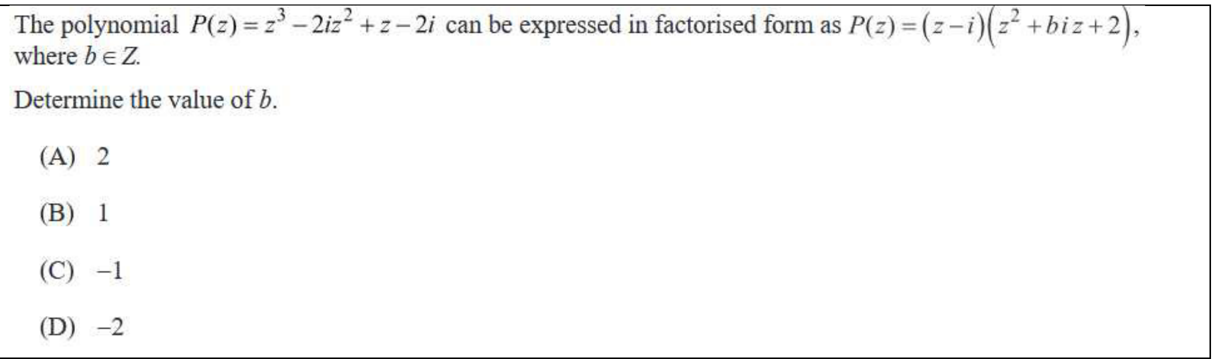
	$y = 0.3 + \tan(\theta)x - \frac{0.0286x^2}{\cos^2(\theta)}$ GIVEN THE POINT **(14, 2.05)** LIES ON THE PATH: $2.05 = 0.3 + 14\,\mathrm{TAN}(\theta) - \frac{0.0286(14)^2}{\cos^2(\theta)}$ SOLVING THE EQUATION USING GDC: $\theta = 0.6444$	
	The Cartesian path of the projectile is $y = 0.3 + \tan(0.6444)x - \frac{0.0286x^2}{\cos^2(0.6444)}$ At impact at point A, $y = 0.4$ m $0.4 = 0.3 + \tan(0.6444)x - \frac{0.0286x^2}{\cos^2(0.6444)}$ SOLVING THE QUADRATIC EQUATION USING GDC: $x = 0.134$ or $x = 16.660$ m As $x >$ 14 m, point of impact is (**16.660**, 0.4)	• determines the horizontal displacement at impact [1 mark]
	Time until impact (using (3): $t = \frac{16.660}{13.09\ \cos(0.6444)}$ $= 1.592$ s	• determines the time of flight [1 mark]
	Equations of motion for the cart: $\boldsymbol{a} = a\,\hat{\imath}$ $\boldsymbol{v} = at\,\hat{\imath}$ $\boldsymbol{r} = \left(\frac{at^2}{2} + 14.3\right)\hat{\imath} + 0.4\hat{\jmath}$ Equating $\hat{\imath}$ components when $t = 1.592$ s: $\frac{a(1.592)^2}{2} + 14.3 = 16.660 \Rightarrow \boldsymbol{a} \approx \mathbf{1.86}\ \mathrm{m\ s^{-2}}$	• determines the acceleration of the trolley [1 mark]
	Method 2 Finding the angular velocity during the swing: $\omega = \frac{2\pi}{T}$ $= \frac{2\pi}{0.24}$ $= 26.18\ \mathrm{rad}\ s^{-1}$ Finding the magnitude of the velocity at release: $\boldsymbol{v} = r\omega$ $= 0.5 \times 26.18$ $= 13.09\ \mathrm{m\ s^{-1}}$	• correctly determines the tangential velocity of object [1 mark]
	Let the origin in the vertical plane of motion be the point of release and the angle of release to the horizontal be θ. Assume the unit  vectors: Equations of motion for the projectile: $\boldsymbol{a} = -9.8\,\hat{\jmath}$ $\boldsymbol{v} = 13.09\ \cos(\theta)\,\hat{\imath} + (13.09\ \sin(\theta) - 9.8t)\hat{\jmath}$ $\boldsymbol{r} = 13.09\ \cos(\theta)\,t\,\hat{\imath} + (13.09\ \sin(\theta)\,t - 4.9t^2)\hat{\jmath}$ The parametric form of the projectile path: $x = 13.09\ \cos(\theta)\,t \quad \ldots(1)$ $y = 13.09\ \sin(\theta)\,t - 4.9t^2 \quad \ldots(2)$	• identifies the parametric form of the projectile path [1 mark]

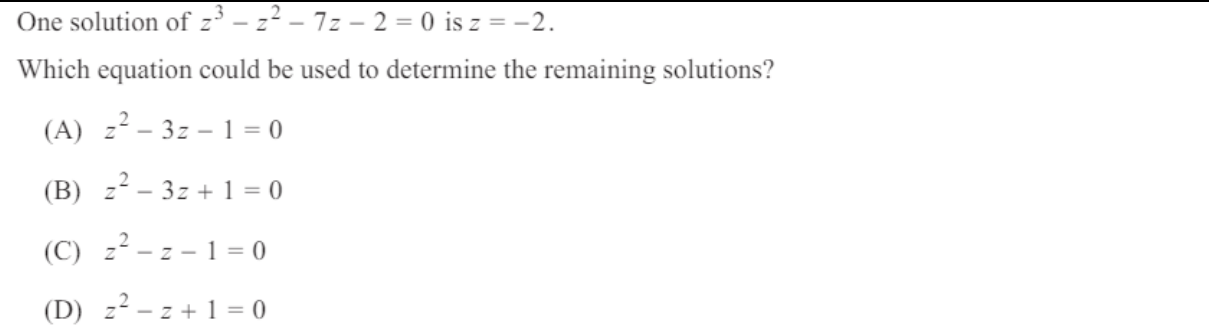
From (1) $t = \frac{x}{13.09\ \cos(\theta)}$ ... (3) Substituting into (2) to change into Cartesian form: $y = \frac{13.09 \sin(\theta) x}{13.09 \cos(\theta)} - \frac{4.9x^2}{(13.09)^2 \cos^2(\theta)}$ $y = \tan(\theta)x - \frac{0.0286x^2}{\cos^2(\theta)}$	• determines the Cartesian form of the projectile path [1 mark]
GIVEN THE POINT $(14, 1.75)$ LIES ON THE PATH: $1.75 = 14\,\mathrm{TAN}(\theta) - \frac{0.0286(14)^2}{\cos^2(\theta)}$ SOLVING THE EQUATION USING GDC: $\theta = 0.6444$	• determines the angle of release of the object [1 mark]
The Cartesian path of the projectile is $y = \tan(0.6444)x - \frac{0.0286x^2}{\cos^2(0.6444)}$ At impact at point A, $y = 0.1$ m $0.1 = \tan(0.6444)x - \frac{0.0286x^2}{\cos^2(0.6444)}$ SOLVING THE QUADRATIC EQUATION USING GDC: $x = 0.134$ or $x = 16.660$ m As $x > 14$ m, point of impact is $(16.660, 0.1)$	• determines the horizontal displacement at impact [1 mark]
Time until impact (using (3): $t = \frac{16.660}{13.09\ \cos(0.6444)}$ $= 1.592$ s	• determines the time of flight [1 mark]
Equations of motion for the cart: $\boldsymbol{a} = a\,\hat{\imath}$ $\boldsymbol{v} = at\,\hat{\imath}$ $\boldsymbol{r} = \left(\frac{at^2}{2} + 14.3\right)\hat{\imath} + 0.1\hat{\jmath}$ Equating $\hat{\imath}$ components when $t = 1.592$ s: $\frac{a(1.592)^2}{2} + 14.3 = 16.660 \Rightarrow \boldsymbol{a} = 1.86\ \mathrm{m\ s^{-2}}$	• determines the acceleration of the trolley [1 mark]

Unit 3 – Topic 3: Complex numbers 2

Paper 1 Section 1

2024 Paper 1 Section 1 Question 8 Complex numbers 2	 Given $z = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$, determine z^3. (A) -8 (B) -6 (C) 6 (D) 8

2024 Paper 1 Section 1 Question 10 Complex numbers 2	 The polynomial $P(z) = z^3 - 2iz^2 + z - 2i$ can be expressed in factorised form as $P(z) = (z-i)\left(z^2 + biz + 2\right)$, where $b \in Z$. Determine the value of b. (A) 2 (B) 1 (C) -1 (D) -2

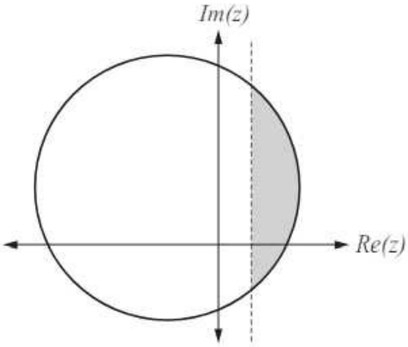
2023 Paper 1 Section 1 Question 3 Complex numbers 2	 One solution of $z^3 - z^2 - 7z - 2 = 0$ is $z = -2$. Which equation could be used to determine the remaining solutions? (A) $z^2 - 3z - 1 = 0$ (B) $z^2 - 3z + 1 = 0$ (C) $z^2 - z - 1 = 0$ (D) $z^2 - z + 1 = 0$

2023
Paper 1
Section 1
Question 6

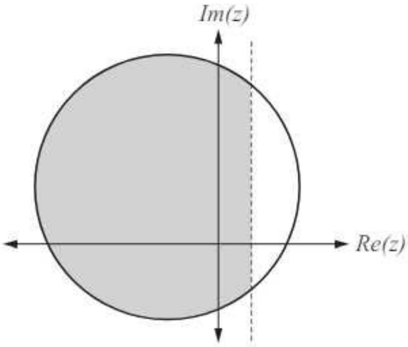
Complex numbers 2

The shaded region defined as $\{z:|z+2-i|\le 5\}\cap\{z:Re(z)<1\}, z\in C$ is best represented by

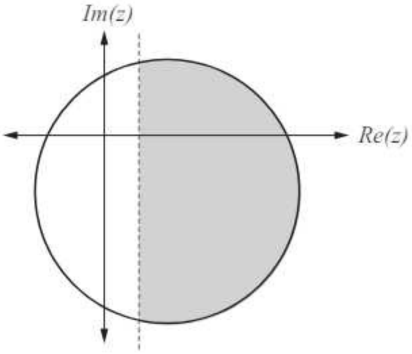
(A)



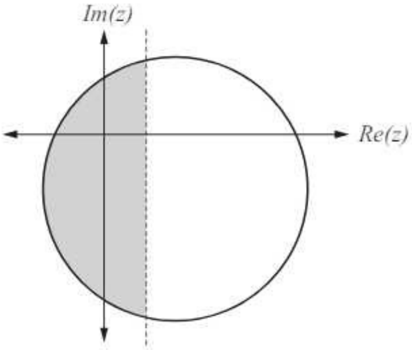
(B)



(C)



(D)



2022
Paper 1
Section 1
Question 4

Complex numbers 2

When using proof by mathematical induction to prove De Moivre's theorem expressed as $(r\text{cis}(\theta))^n = r^n\text{cis}(n\theta)\forall n\in Z^+$, which statement would be correct in the proof of the inductive step?

(A) $(r\text{cis}(\theta))^k = r^k\text{cis}(k\theta)$

(B) $(r\text{cis}(\theta))^k = r^{k+1}\text{cis}(k+\theta)$

(C) $(r\text{cis}(\theta))^{k+1} = r^{k+1}\text{cis}(k\theta+1)$

(D) $(r\text{cis}(\theta))^{k+1} = r^{k+1}\text{cis}((k+1)\theta)$

2021
Paper 1
Section 1
Question 2

Complex numbers 2

When the polynomial $P(z)=z^3-iz^2-z-i$ is divided by $z-i$, the remainder is

(A) $-2i$

(B) 0

(C) $2i$

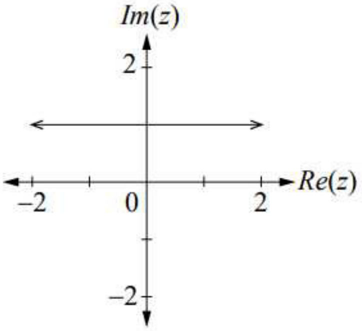
(D) $4i$

2021
Paper 1
Section 1
Question 6

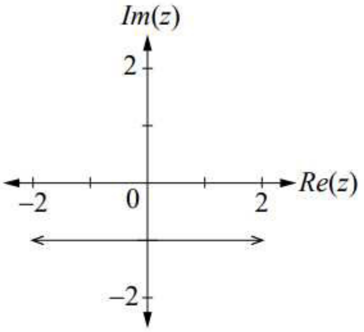
Complex numbers 2

The subset of the complex plane that represents $|z| = |z-2|$ for $z \in C$ is

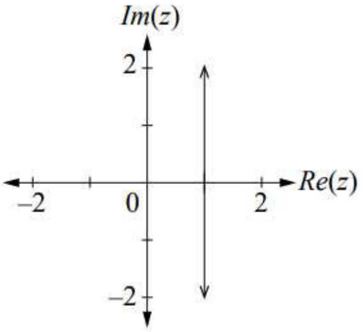
(A)



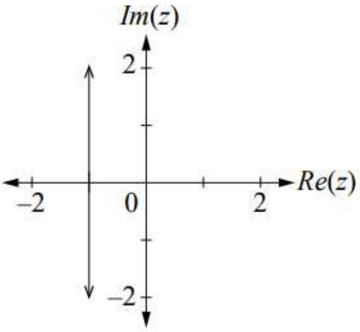
(B)



(C)



(D)



2020
Paper 1
Section 1
Question 6

Complex numbers 2

Given $z = 2 - 2i$ and $w = -3 + i$, calculate $z^2 - \bar{w}$

(A) $3 - 9i$

(B) $3 - 7i$

(C) $11 - 9i$

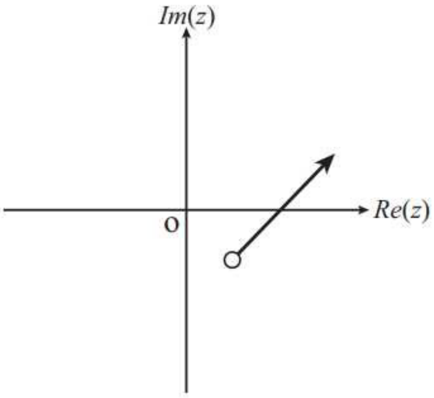
(D) $11 - 7i$

2020
Paper 1
Section 1
Question 10

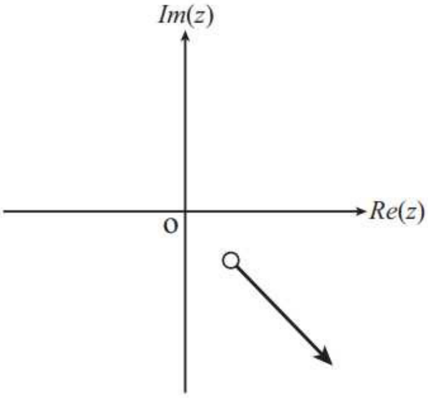
Complex numbers 2

The subset of the complex plane that represents $\arg[z+i-1]+\frac{\pi}{4}=0$ for $z \in C$ is

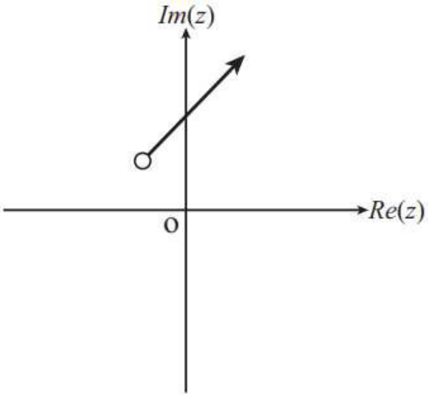
(A)



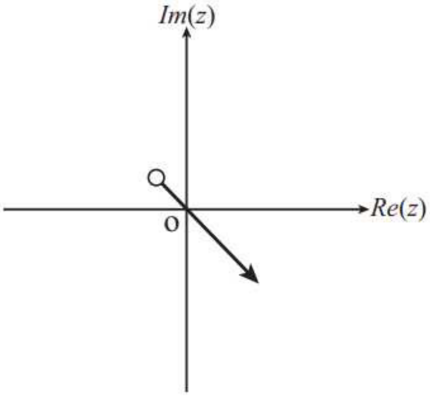
(B)



(C)



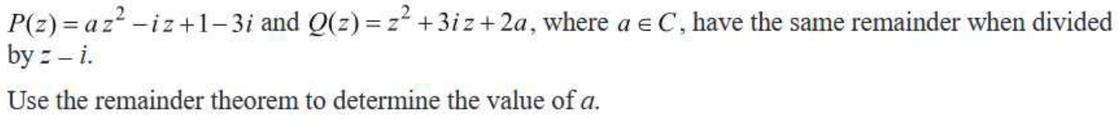
(D)



Paper 1 Section 2

2024
Paper 1
Section 2
Question 13

Complex numbers 2

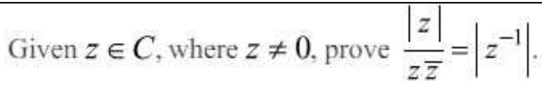


$P(z) = a\,z^2 - i\,z + 1 - 3i$ and $Q(z) = z^2 + 3i\,z + 2a$, where $a \in C$, have the same remainder when divided by $z - i$.

Use the remainder theorem to determine the value of a.

[4 marks]

2024 Paper 1 Section 2 Question 19 Complex numbers 2	Consider complex numbers of the form $w = x + i$, where x is a positive real number. If $Re\left(w^7\right) = 0$, determine all possible values of x. `[6 marks]

2023 **Paper 1** **Section 2** **Question 13** **Complex** **numbers 2**	 Given $z \in C$, where $z \neq 0$, prove $\frac{\lvert z \rvert}{z\overline{z}} = \lvert z^{-1} \rvert$. (5 marks)

2022
Paper 1
Section 2
Question 12

Complex
numbers 2

Given $z_1 = a + bi$, $z_2 = c + di$ $\forall a,\ b,\ c,\ d \in R$, and $z_2 \neq 0$, prove the identity

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

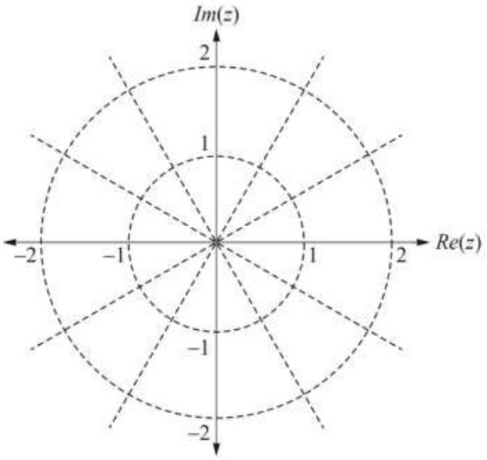
[6 marks]

2022
Paper 1
Section 2
Question 15

Complex numbers 2

Consider the equation $z^3 = 1$ where $z \in C$.

a) Sketch the solutions to $z^3 = 1$ on the Argand diagram. [2 marks]



The solutions to $z^3 = 1$ can be expressed in the form $z = a + bi$, where $a, b \in R$.

b) Determine the largest possible positive value of ab. [2 marks]

2020
Paper 1
Section 2
Question 18

Complex numbers 2

Consider the function $P(z) = 2z^4 + az^3 + 6z^2 + az + b$ where $a, b \in Z^+$

One of the roots of $P(z)$ is $z = -i$

Determine the possible value/s for a and b such that all remaining roots of $P(z)$ have an imaginary component.

Paper 2 Section 1

2024
Paper 2
Section 1
Question 1

Complex numbers 2

Given that $z = -2+3i$ is a root of $z^3 + az + b = 0$, where $a, b \in R$, another root is

(A) $-2-3i$

(B) $-2+3i$

(C) $2-3i$

(D) $2+3i$

2024
Paper 2
Section 1
Question 7

Complex numbers 2

Determine the number of roots of $w^8 = 1$ that can be expressed in the form $a + bi$, where $a, b \in R^+$.

(A) 0

(B) 1

(C) 2

(D) 3

2023
Paper 2
Section 1
Question 3

Complex numbers 2

Given that $2i$ is a root of $z^2 - pz - q = 0$, where $p, q \in R$, determine the values of p and q.

(A) $p=-4$ and $q=-4$

(B) $p=-4$ and $q=4$

(C) $p=0$ and $q=-4$

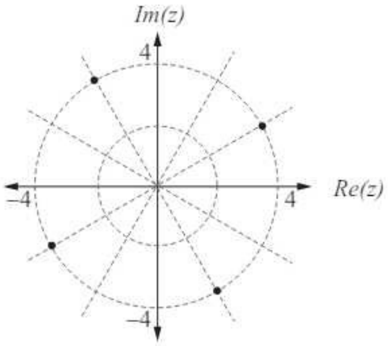
(D) $p=0$ and $q=4$

2023
Paper 2
Section 1
Question 10

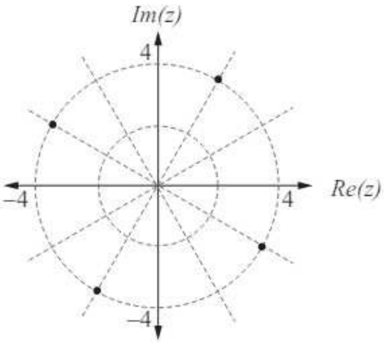
Complex numbers 2

The Argand diagram that represents the solutions to $z^4 = 16 \text{ cis}\left(\frac{2\pi}{3}\right)$, $z \in C$ is

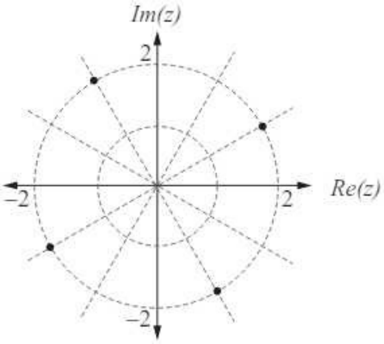
(A)



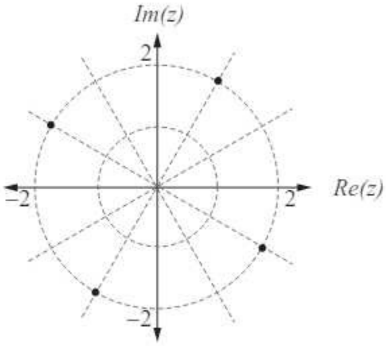
(B)



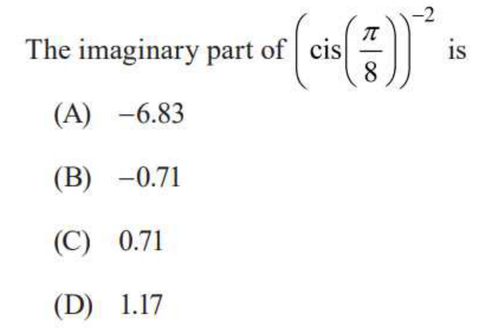
(C)

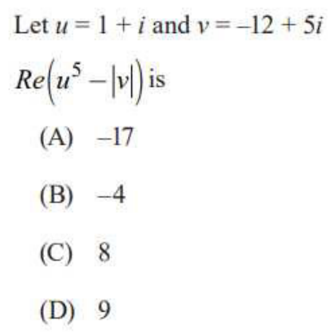


(D)



2022 **Paper 2** **Section 1** **Question 1** **Complex numbers 2**	A solution of the equation $z^2 = ai$, where $a \in R$, is $z = -2 - 2i$. The other solution is (A) $-8i$ (B) $-2 + 2i$ (C) $2 + 2i$ (D) $8i$

2021 **Paper 2** **Section 1** **Question 8** **Complex numbers 2**	 The imaginary part of $\left(\text{cis}\left(\frac{\pi}{8}\right)\right)^{-2}$ is (A) -6.83 (B) -0.71 (C) 0.71 (D) 1.17

2020 **Paper 2** **Section 1** **Question 8** **Complex numbers 2**	 Let $u = 1 + i$ and $v = -12 + 5i$ $Re\left(u^5 - \lvert v \rvert\right)$ is (A) -17 (B) -4 (C) 8 (D) 9

Paper 2 Section 2

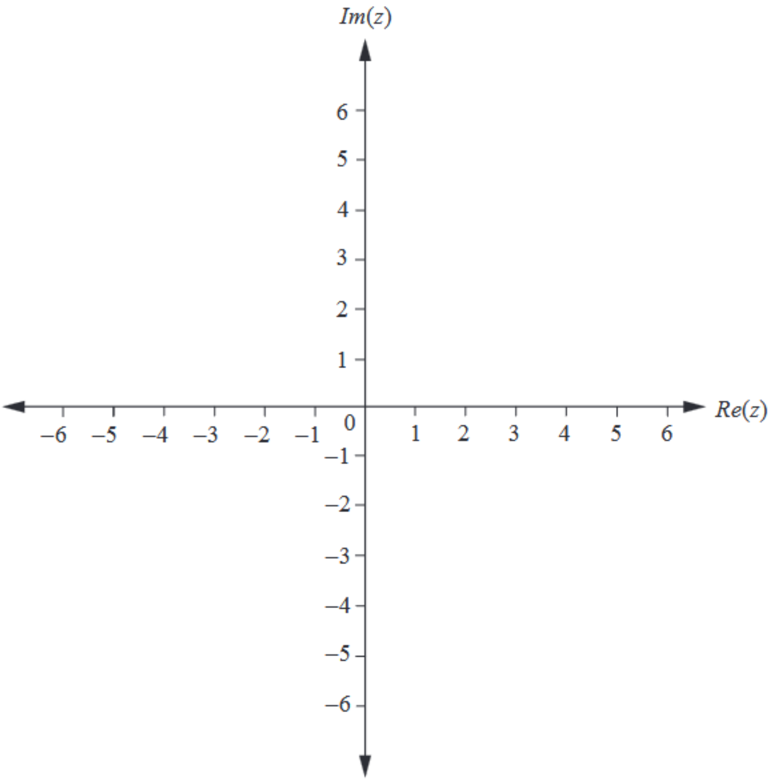
2024
Paper 2
Section 2
Question 16

Complex numbers 2

Two subsets of the complex plane are $S = \{z : |z-1| = 4\}$ and $T = \left\{z : \arg(z+i) = \frac{\pi}{3}\right\}$, where $z \in C$.

Determine the complex number/s where S and T intersect. Leave your answer/s in Cartesian form.

Provide an Argand diagram with a sketch of subsets S and T as part of your solution.



[6 marks]

2023
Paper 2
Section 2
Question 12

Complex numbers 2

Consider the complex number $z = -3 + 2i$.

a) Determine z^3 using the binomial theorem. Leave your answer in the form $a + bi$, where $a, b \in R$. *[2 marks]*

b) Convert z into the form of $r\,\text{cis}(\theta)$, where $-\pi < \theta \le \pi$. *[1 mark]*

c) Use the result from Question 12b) to determine z^3 using De Moivre's theorem.
Leave your answer in the form of $r\,\text{cis}(\theta)$, where $-\pi < \theta \leq \pi$. *[2 marks]*

d) Evaluate the reasonableness of your results from Questions 12a) and 12c), noting that the two methods to determine z^3 should produce the same result. *[2 marks]*

2023
Paper 2
Section 2
Question 18

Complex numbers 2

Consider the complex solutions to the following equation, where $0 < \arg(z) < \pi$.

$$(z+1)\left(z^{14}-z^{13}+z^{12}-z^{11}+\ldots+z^{4}-z^{3}+z^{2}-z\right)=1-z$$

Let w_1 be the solution with the maximum possible real part and w_2 be the solution with the maximum possible imaginary part.

Show that $\dfrac{{w_1}^4}{w_2} \in Z$.

(5 marks)

2022 Paper 2 Section 2 Question 18 Complex numbers 2	Consider the polynomials $P(z)=z^3+(i-a)z^2-2biz+3i$ and $Q(z)=z-2i$, where $a,\ b\in R$. Given $\frac{P(z)}{Q(z)}$ has a remainder of $a-bi$, evaluate the reasonableness that $\left(z-(a-bi)\right)$ is a factor of $P(z)$. [5 marks]

2021
Paper 2
Section 2
Question 18

Complex numbers 2

Consider the polynomial $P(z) = z^3 + az^2 + bz + c$, where $a, b, c \in R$ and $z \in C$.

Two of the roots of $P(z)$ are also roots of $z^4 + z^3 + z^2 + z + 1$. The remaining root of $P(z)$ is $z = 2$.

Given $z^5 - 1 = (z-1)\left(z^4 + z^3 + z^2 + z + 1\right)$, determine a possible expression for $P(z)$.

Leave your answer in expanded form.

[6 marks]

2020
Paper 2
Section 2
Question 16

Complex
numbers 2

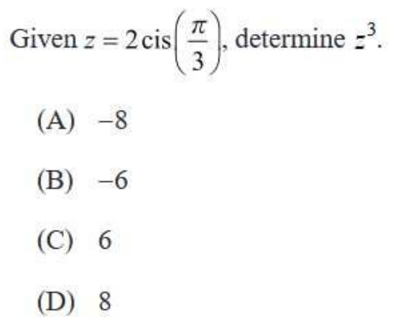
Consider the identity

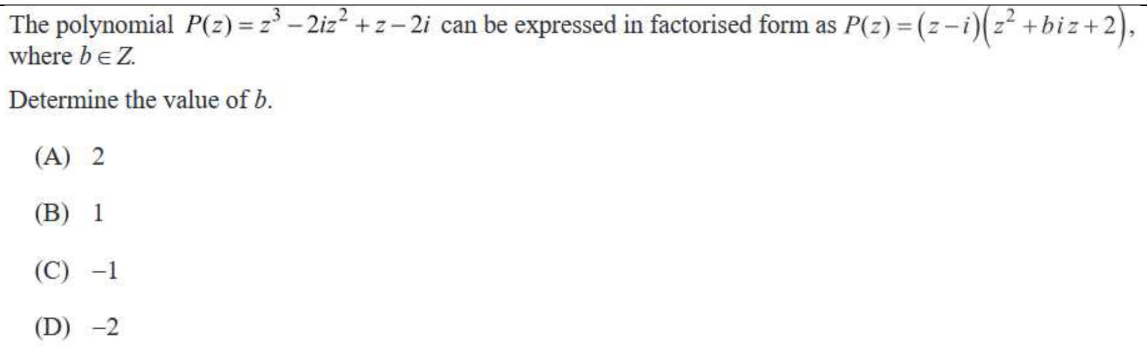
$$\cos(4\theta) = A\cos^4(\theta) + B\sin^2(\theta) + C \text{ where } A, B \text{ and } C \in Z$$

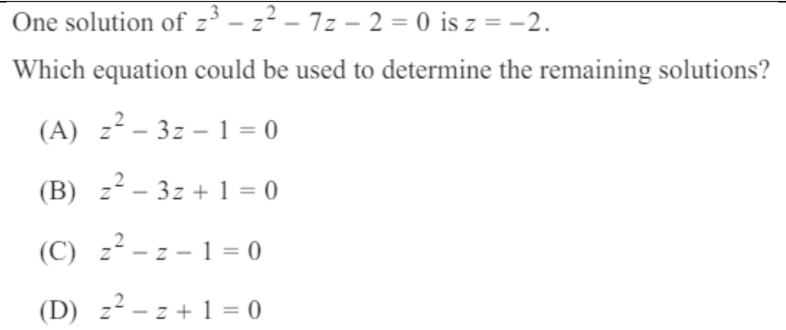
a) Determine the values of A, B and C using De Moivre’s theorem. [5 marks]

	b) State an appropriate method of verifying your results from 16a). [1 mark]

Marking Guide – Paper 1 Section 1

2024 Paper 1 Section 1 Question 8 Complex numbers 2	 Given $z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$, determine z^3. (A) -8 (B) -6 (C) 6 (D) 8 **Answer is A.**

2024 Paper 1 Section 1 Question 10 Complex numbers 2	 The polynomial $P(z) = z^3 - 2iz^2 + z - 2i$ can be expressed in factorised form as $P(z) = (z-i)(z^2 + biz + 2)$, where $b \in Z$. Determine the value of b. (A) 2 (B) 1 (C) -1 (D) -2 **Answer is C.**

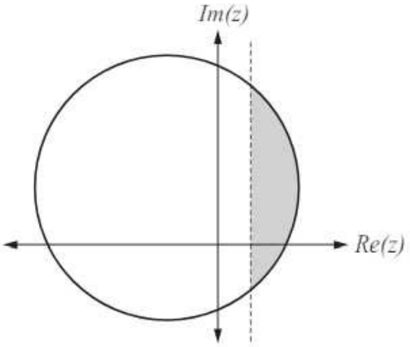
2023 Paper 1 Section 1 Question 3 Complex numbers 2	 One solution of $z^3 - z^2 - 7z - 2 = 0$ is $z = -2$. Which equation could be used to determine the remaining solutions? (A) $z^2 - 3z - 1 = 0$ (B) $z^2 - 3z + 1 = 0$ (C) $z^2 - z - 1 = 0$ (D) $z^2 - z + 1 = 0$ **Answer is A.**

2023
Paper 1
Section 1
Question 6

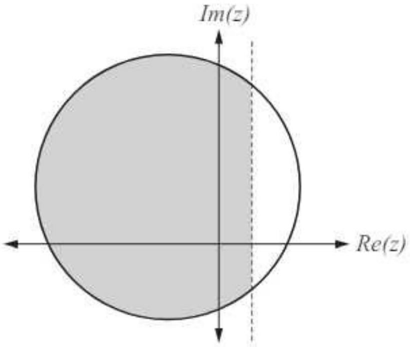
Complex numbers 2

The shaded region defined as $\{z : |z+2-i| \le 5\} \cap \{z : Re(z) < 1\}, z \in C$ is best represented by

(A)



(B)



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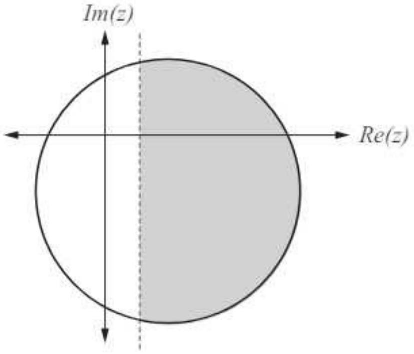
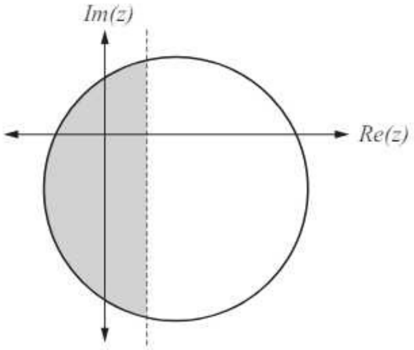


(D)



Answer is B.

2022
Paper 1
Section 1
Question 4

Complex numbers 2

When using proof by mathematical induction to prove De Moivre's theorem expressed as $(r\text{cis}(\theta))^n = r^n\text{cis}(n\theta)\,\forall n \in Z^+$, which statement would be correct in the proof of the inductive step?

(A) $(r\text{cis}(\theta))^k = r^k\text{cis}(k\theta)$

(B) $(r\text{cis}(\theta))^k = r^{k+1}\text{cis}(k+\theta)$

(C) $(r\text{cis}(\theta))^{k+1} = r^{k+1}\text{cis}(k\theta+1)$

(D) $(r\text{cis}(\theta))^{k+1} = r^{k+1}\text{cis}((k+1)\theta)$

Answer is D.

2021
Paper 1
Section 1
Question 2

Complex numbers 2

When the polynomial $P(z) = z^3 - iz^2 - z - i$ is divided by $z - i$, the remainder is

(A) $-2i$

(B) 0

(C) $2i$

(D) $4i$

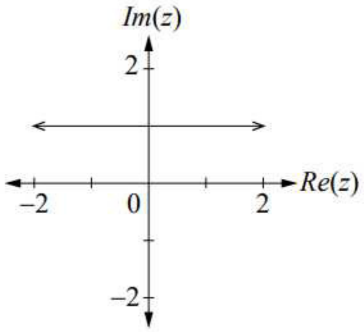
Answer is A.

2021
Paper 1
Section 1
Question 6

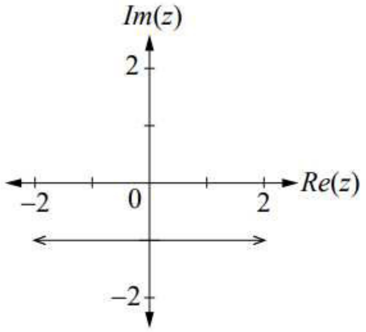
Complex numbers 2

The subset of the complex plane that represents $|z| = |z-2|$ for $z \in C$ is

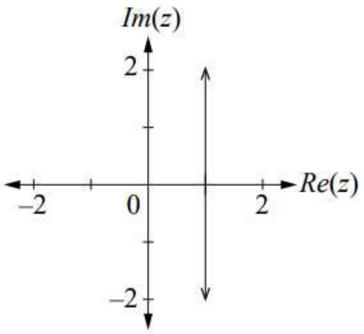
(A)



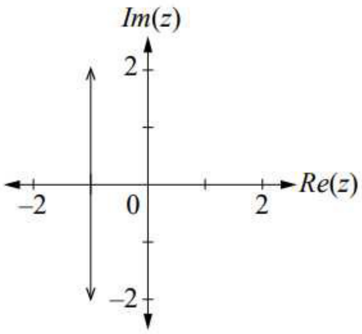
(B)



(C)



(D)



Answer is C.

2020
Paper 1
Section 1
Question 6

Complex numbers 2

Given $z = 2 - 2i$ and $w = -3 + i$, calculate $z^2 - \overline{w}$

(A) $3 - 9i$

(B) $3 - 7i$

(C) $11 - 9i$

(D) $11 - 7i$

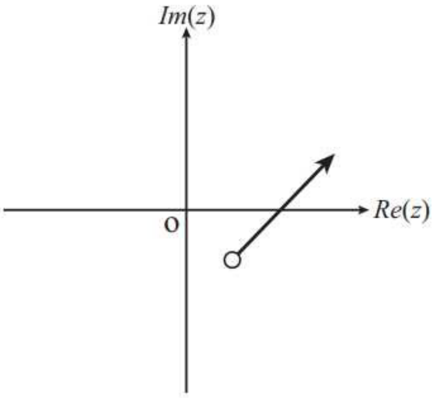
Answer is B.

2020
Paper 1
Section 1
Question 10

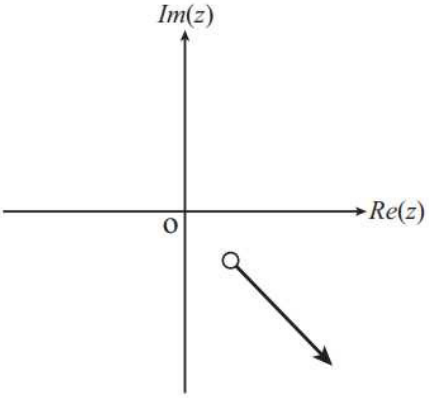
Complex numbers 2

The subset of the complex plane that represents $\arg[z+i-1]+\frac{\pi}{4}=0$ for $z\in C$ is

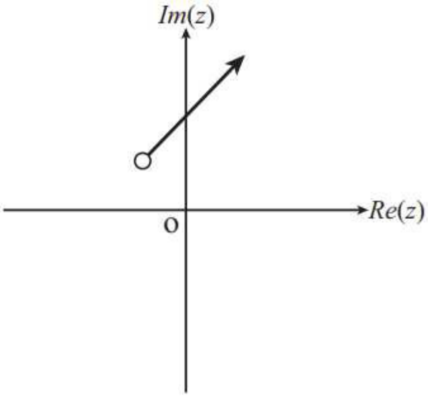
(A)



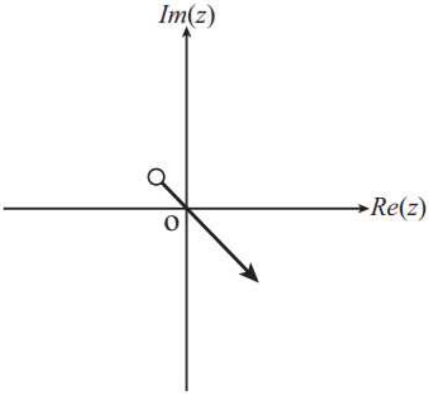
(B)



(C)



(D)



Answer is B.

Marking Guide – Paper 1 Section 2

<table>
<tr>
<td>2024
Paper 1
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Question 13

Complex numbers 2</td>
<td>
$P(z) = az^2 - iz + 1 - 3i$ and $Q(z) = z^2 + 3iz + 2a$, where $a \in C$, have the same remainder when divided by $z - i$.

Use the remainder theorem to determine the value of a.

[4 marks]
<table>
<tr><th>Sample response</th><th>The response</th></tr>
<tr><td>$P(z)$ and $Q(z)$ have the same remainder when $z = i$.</td><td>• correctly uses the remainder theorem [1 mark]</td></tr>
<tr><td>$P(i) = a(i)^2 - i(i) + 1 - 3i$
$= -a + 2 - 3i$</td><td>• determines simplified expression representing $P(i)$ [1 mark]</td></tr>
<tr><td>$Q(i) = (i)^2 + 3i(i) + 2a$
$= -1 - 3 + 2a$
$= 2a - 4$</td><td>• determines simplified expression representing $Q(i)$ [1 mark]</td></tr>
<tr><td>Using the remainder theorem with the given information,
$P(i) = Q(i)$
$-a + 2 - 3i = 2a - 4$
$3a = 6 - 3i$
$a = 2 - i$</td><td>• determines value of a [1 mark]</td></tr>
</table>
</td>
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</table>

2024
Paper 1
Section 2
Question 19

Complex numbers 2

Consider complex numbers of the form $w = x + i$, where x is a positive real number.

If $Re\left(w^7\right) = 0$, determine all possible values of x.

[6 marks]

Sample response	The response
Method 1 Given $w = x + i$ Let $w = r\,\text{cis}(\theta)$ $w^7 = r^7\,\text{cis}(7\theta)$	• correctly uses De Moivre's theorem **[1 mark]**
$\arg(w) = \theta$ $= \tan^{-1}\left(\frac{y}{x}\right), x \neq 0$ $= \tan^{-1}\left(\frac{1}{x}\right)$ where $x \in R^+, x \neq 0$	• correctly determines an expression representing $\arg(w)$ in terms of x **[1 mark]**
$\arg(w^7) = 7\theta = 7\tan^{-1}\left(\frac{1}{x}\right)$ Given $Re\left(w^7\right) = 0$, then $\arg(w^7) = (2n+1)\frac{\pi}{2}$ where $n \in Z$	• determines a relationship involving $\arg\left(w^7\right)$ using the condition $Re\left(w^7\right) = 0$ **[1 mark]**
$7\tan^{-1}\left(\frac{1}{x}\right) = (2n+1)\frac{\pi}{2}$ $\frac{1}{x} = \tan\left(\frac{(2n+1)\pi}{14}\right)$ $x = \cot\left(\frac{(2n+1)\pi}{14}\right)$	• determines a general expression representing possible values of x **[1 mark]**
As $x \in R^+$, consider $\frac{(2n+1)\pi}{14}$ for angles that lie in quadrant 1. $n = 0 : x = \cot\left(\frac{\pi}{14}\right)$	• determines one value of x **[1 mark]**
$n = 1 : x = \cot\left(\frac{3\pi}{14}\right)$ $n = 2 : x = \cot\left(\frac{5\pi}{14}\right)$	• evaluates the reasonableness of solution by determining the remaining two values of x **[1 mark]**
Method 2 Given $Re\left(w^7\right) = 0$, consider solutions of the equation $w^7 = ai$ where $w = x + i$ and $a \in R, a \neq 0$.	• correctly considers $Re\left(w^7\right) = 0$ **[1 mark]**
When $a > 0$, the 7 possible solutions would have arguments that are separated by $\frac{2\pi}{7}$ from the positive imaginary axis. Possible principal arguments for w are $\frac{3\pi}{14}, \frac{7\pi}{14}, \frac{11\pi}{14}, -\frac{\pi}{14}, -\frac{5\pi}{14}, -\frac{9\pi}{14}, -\frac{13\pi}{14}$ When $a < 0$, the 7 possible solutions would have arguments that are separated by $\frac{2\pi}{7}$ from the negative imaginary axis. Possible principal arguments for w are $-\frac{3\pi}{14}, -\frac{7\pi}{14}, -\frac{11\pi}{14}, \frac{\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{13\pi}{14}$ $\arg(w) = \theta$ $= \tan^{-1}\left(\frac{y}{x}\right), x \neq 0$ $= \tan^{-1}\left(\frac{1}{x}\right)$ where $x \in R^+, x \neq 0$	• determines distinct arguments of the solution of $w^7 = ai$ where $a > 0$ **[1 mark]**

		As $x \in R^+$, the only suitable principal argument for w when $a>0$ is $\frac{3\pi}{14}=\tan^{-1}\left(\frac{1}{x}\right)$ $\frac{3\pi}{14}. \ \frac{1}{x}=\tan\left(\frac{3\pi}{14}\right)$ $x=\cot\left(\frac{3\pi}{14}\right)$ As $x \in R^+$, the only suitable principal arguments for w when $a<0$ are $\frac{\pi}{14}$ and $\frac{5\pi}{14}$. Similarly, $x=\cot\left(\frac{\pi}{14}\right)$ $x=\cot\left(\frac{5\pi}{14}\right)$	• determines distinct arguments of the solution of $w^7=ai$ where $a<0$ **[1 mark]** • correctly determines an expression representing arg(w) in terms of x **[1 mark]** • determines one possible value of x **[1 mark]** • evaluates the reasonableness of solution by determining the remaining two possible values of x **[1 mark]**
		Method 3 Given $Re\left(w^7\right)=0$, consider solutions of the equation $z^7=ai$ where $z=r\operatorname{cis}(\theta)$ and $a \in R, a \neq 0$. When $a>0$, the 7 possible solutions would have arguments that are separated by $\frac{2\pi}{7}$ from the positive imaginary axis. So solutions of $z^7=ai$ have the form $z=r\operatorname{cis}\left(\frac{\pi}{2}+\frac{2n\pi}{7}\right)$ where $n \in Z$. When $a<0$, the 7 possible solutions would have arguments that are separated by $\frac{2\pi}{7}$ from the negative imaginary axis. So solutions of $z^7=ai$ have the form $z=r\operatorname{cis}\left(-\frac{\pi}{2}+\frac{2n\pi}{7}\right)$ where $n \in Z$. $z=r\cos\left(\pm\frac{\pi}{2}+\frac{2n\pi}{7}\right)+r\sin\left(\pm\frac{\pi}{2}+\frac{2n\pi}{7}\right)i$ Expressing z in the form $w=x+i$ $\therefore w=\dfrac{r\cos\left(\pm\frac{\pi}{2}+\frac{n\pi}{7}\right)}{r\sin\left(\pm\frac{\pi}{2}+\frac{n\pi}{7}\right)}+i$ $x=\cot\left(\pm\frac{\pi}{2}+\frac{n\pi}{7}\right)$	• correctly considers $Re\left(w^7\right)=0$ **[1 mark]** • recognises the relationship between the arguments of the solutions of $z^7=ai$ where $a>0$ **[1 mark]** • recognises the relationship between the arguments of the solutions of $z^7=ai$ where $a<0$ **[1 mark]** • determines an expression representing possible values of x **[1 mark]**

As $x \in R^+$, the possible value of x when $a > 0$ is $x = \cot\left(\frac{\pi}{2} - \frac{2\pi}{7}\right)$ $= \cot\left(\frac{3\pi}{14}\right)$	• determines one possible value of x **[1 mark]**
Similarly, as $x \in R^+$, the possible values of x when $a < 0$ are $x = \cot\left(-\frac{\pi}{2} + \frac{8\pi}{7}\right) = \cot\left(\frac{\pi}{14}\right)$ $x = \cot\left(-\frac{\pi}{2} + \frac{12\pi}{7}\right) = \cot\left(\frac{5\pi}{14}\right)$	• evaluates the reasonableness of solution by determining the remaining two possible values of x **[1 mark]**

2023
Paper 1
Section 2
Question 13

Complex numbers 2

Given $z \in C$, where $z \neq 0$, prove $\frac{\lvert z \rvert}{z\bar{z}} = \lvert z^{-1} \rvert$. (5 marks)

Sample response	The response
Method 1 Prove $\frac{\lvert z \rvert}{z\bar{z}} = \lvert z^{-1} \rvert$ Let $z = a + bi$, where $a, b \in R$ $\text{LHS} = \frac{\lvert z \rvert}{z\bar{z}}$ $= \frac{\lvert a+bi \rvert}{(a+bi)(a-bi)}$	• correctly represents $\bar{z}$ in terms of a and b **[1 mark]**
$= \frac{\sqrt{a^2+b^2}}{a^2+b^2}$	• correctly represents $\lvert z \rvert$ as a radical expression in terms of a and b **[1 mark]** • correctly simplifies $z\bar{z}$ in terms of a and b **[1 mark]**
$= \frac{1}{\sqrt{a^2+b^2}}$ $= \frac{1}{\lvert a+bi \rvert}$ $= \left\lvert \frac{1}{a+bi} \right\rvert$	• determines an expression without an index that represents $\lvert z \rvert^{-1}$ **[1 mark]**
$= \lvert z^{-1} \rvert$ $= \text{RHS}$	• correctly shows mathematical reasoning to complete proof **[1 mark]**

Method 2	
Prove $\frac{\lvert z\rvert}{z\overline{z}}=\lvert z^{-1}\rvert$	
Let $z=a+bi$ where $a,b\in R$	
$\text{LHS}=\frac{\lvert z\rvert}{z\overline{z}}$	
$=\frac{\lvert a+bi\rvert}{(a+bi)(a-bi)}$	- correctly represents $\overline{z}$ in terms of a and b **[1 mark]**
$=\frac{\sqrt{a^2+b^2}}{a^2+b^2}$	- correctly represents $\lvert z\rvert$ as a radical expression in terms of a and b **[1 mark]** - correctly simplifies $z\overline{z}$ in terms of a and b **[1 mark]**
$\text{RHS}=\lvert z^{-1}\rvert$	
$=\lvert (a+bi)^{-1}\rvert$	
$=\left\lvert\frac{1}{a+bi}\right\rvert$	- determines an expression without an index that represents $\lvert z\rvert^{-1}$ **[1 mark]**
$=\frac{1}{\sqrt{a^2+b^2}}$	
$=\frac{\sqrt{a^2+b^2}}{a^2+b^2}$ $=\text{LHS}$	- correctly shows mathematical reasoning to establish the intermediate position to complete proof **[1 mark]**
Method 3	
Prove $\frac{\lvert z\rvert}{z\overline{z}}=\lvert z^{-1}\rvert$	
$\text{LHS}=\frac{\lvert z\rvert}{z\overline{z}}$	
$=\frac{\lvert z\rvert}{\lvert z\rvert^2}$	- correctly simplifies $z\overline{z}$ **[1 mark]**
$=\frac{1}{\lvert z\rvert}$	- correctly simplifies LHS **[1 mark]**
$\text{RHS}=\lvert z^{-1}\rvert$	
$=\frac{\lvert z\rvert\lvert z^{-1}\rvert}{\lvert z\rvert}$	- correctly uses a suitable method to form the product $\lvert z\rvert\lvert z^{-1}\rvert$ **[1 mark]**
$=\frac{\lvert z\,z^{-1}\rvert}{\lvert z\rvert}$	- recognises that the product of the moduli equals the modulus of the product **[1 mark]**
$=\frac{1}{\lvert z\rvert}$ $=\text{LHS}$	- correctly shows mathematical reasoning to establish the intermediate position to complete proof **[1 mark]**

Method 4 Let $z = r\,\text{cis}(\theta)$ Prove $\dfrac{\lvert z \rvert}{z\bar{z}} = \lvert z^{-1} \rvert$ $\text{LHS} = \dfrac{\lvert z \rvert}{z\bar{z}}$	
$= \dfrac{\lvert a+bi \rvert}{(a+bi)(a-bi)}$	• correctly represents $\bar{z}$ **[1 mark]**
$= \dfrac{r}{r\,\text{cis}(\theta)r\,\text{cis}(-\theta)}$	
$= \dfrac{r}{r^2}$	• correctly expresses the numerator in terms of r **[1 mark]**
$= \dfrac{1}{r}$	• simplifies the denominator **[1 mark]**
$\text{RHS} = \lvert z^{-1} \rvert$	
$= \lvert r^{-1}\text{cis}(-\theta) \rvert$	• correctly uses De Moivre's theorem **[1 mark]**
$= r^{-1}$ $= \dfrac{1}{r}$ $= \text{LHS}$	• correctly shows mathematical reasoning to establish the intermediate position to complete proof **[1 mark]**

2022
Paper 1
Section 2
Question 12

Complex numbers 2

Given $z_1 = a+bi,\ z_2 = c+di\ \forall a,\ b,\ c,\ d \in R$, and $z_2 \neq 0$, prove the identity

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

[6 marks]

Sample Response	The response
Method 1 Prove $\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$, given $z_1 = a+bi,\ z_2 = c+di,$ $z_2 \neq 0$ $\text{LHS} = \left\lvert\frac{a+bi}{c+di}\right\rvert$ $= \left\lvert\frac{(a+bi)}{(c+di)}\cdot\frac{(c-di)}{(c-di)}\right\rvert$ $= \left\lvert\frac{ac-adi+bci-bdi^2}{c^2+d^2}\right\rvert$	• correctly multiplies the numerator and denominator by the complex conjugate of z_2 [1 mark] • realises denominator (in simplest form) and expands numerator [1 mark]
$= \left\lvert\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\right\rvert$ $= \sqrt{\frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)^2}}$	• determines modulus of the expression [1 mark]
$= \sqrt{\frac{(ac)^2+2abcd+(bd)^2+(bc)^2-2abcd+(ad)^2}{(c^2+d^2)^2}}$ $= \sqrt{\frac{(ac)^2+(bd)^2+(bc)^2+(ad)^2}{(c^2+d^2)^2}}$	• simplifies numerator [1 mark]
$= \sqrt{\frac{a^2(c^2+d^2)+b^2(c^2+d^2)}{(c^2+d^2)^2}}$ $= \sqrt{\frac{(a^2+b^2)\,(c^2+d^2)}{(c^2+d^2)^2}}$	• factorises numerator [1 mark]
$= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$ $= \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$ $= \text{RHS}$ QED	• completes the proof using mathematical reasoning [1 mark]
Method 2 $z_1 = a+bi = \sqrt{a^2+b^2}\,\text{cis}(\theta_1)$, $Arg(z_1)=\theta_1$ $z_2 = c+di = \sqrt{c^2+d^2}\,\text{cis}(\theta_2)$, $z_2 \neq 0$, $Arg(z_2)=\theta_2$	• correctly determines the magnitude of z_1 and z_2 [1 mark] • correctly expresses the argument of z_1 and z_2 [1 mark]
$\text{LHS} = \left\lvert\frac{\sqrt{a^2+b^2}\,\text{cis}(\theta_1)}{\sqrt{c^2+d^2}\,\text{cis}(\theta_2)}\right\rvert$ $= \left\lvert\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}\text{cis}(\theta_1-\theta_2)\right\rvert$	• uses De Moivre’s formula to express quotient of modulus [1 mark] • uses De Moivre’s formula to express difference of arguments [1 mark]
$= \left\lvert\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}\right\rvert$ $= \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$	• determines magnitude of quotient [1 mark]

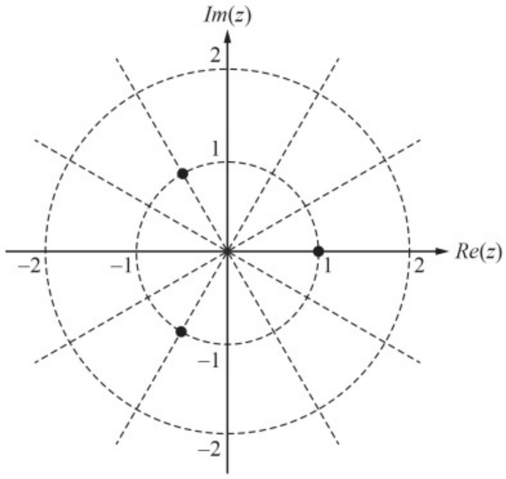
$\text{RHS} = \dfrac{\left\lvert\sqrt{a^2+b^2}\,\text{Arg}(z_1)\right\rvert}{\left\lvert\sqrt{c^2+d^2}\,\text{Arg}(z_2)\right\rvert}$ $= \dfrac{\left\lvert\sqrt{a^2+b^2}\right\rvert}{\left\lvert\sqrt{c^2+d^2}\right\rvert}$ $= \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$ $= \text{LHS}$	• completes the proof using mathematical reasoning [1 mark]

2022
Paper 1
Section 2
Question 15

Complex numbers 2

Consider the equation $z^3 = 1$ where $z \in C$.

a) Sketch the solutions to $z^3 = 1$ on the Argand diagram. [2 marks]

Sample Response	The response
	• correctly sketches 3 solutions, each with a modulus of 1 [1 mark] • correctly sketches 3 solutions with arguments of 0 and $\pm\frac{2\pi}{3}$ **[1 mark]**

The solutions to $z^3 = 1$ can be expressed in the form $z = a + bi$, where $a, b \in R$.

b) Determine the largest possible positive value of ab. [2 marks]

Sample Response	The response
The only positive product of a and b occurs for the solution $z = \text{cis}\left(\frac{-2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$	• recognises a solution from which its a and b values produce the largest possible positive product [1 mark]
Determining the largest value of ab $ab = \frac{-1}{2} \times \frac{-\sqrt{3}}{2}$ $= \frac{\sqrt{3}}{4}$	• determines a positive product of ab [1 mark]

2020
Paper 1
Section 2
Question 18

Complex numbers 2

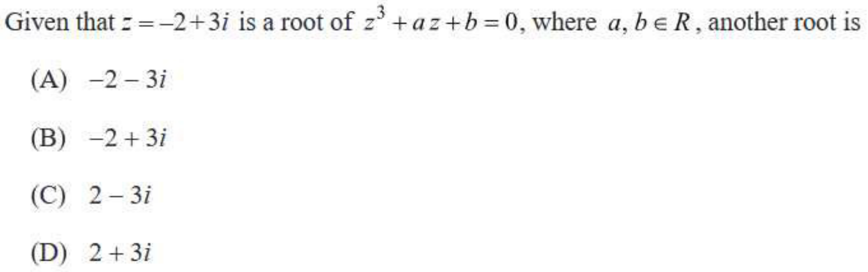
Consider the function $P(z)=2z^4+az^3+6z^2+az+b$ where $a,b\in Z^+$

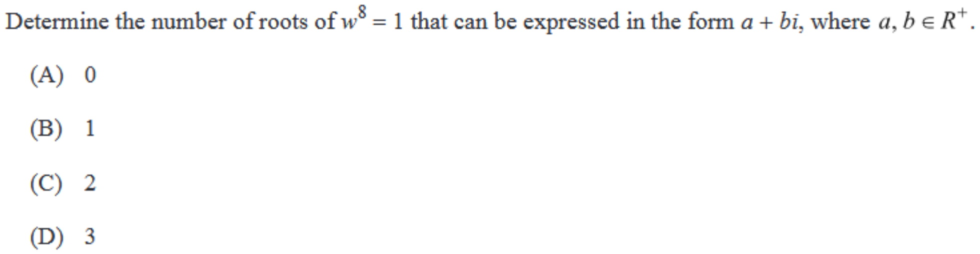
One of the roots of $P(z)$ is $z=-i$

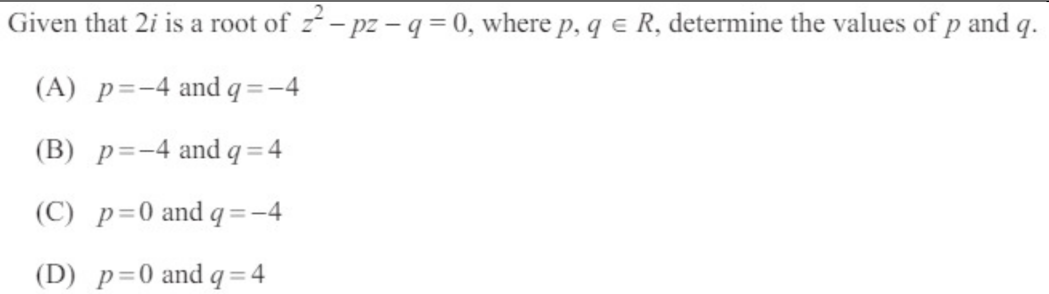
Determine the possible value/s for a and b such that all remaining roots of $P(z)$ have an imaginary component.

Sample Response	The response
$P(z)=2z^4+az^3+6z^2+az+b$ where $a,b\in Z^+$ Given $z=-i$ is a root of $P(z)$, then $\mathbf{P(-i)=0}$ $\therefore\ 2(-i)^4+a(-i)^3+6(-i)^2+a(-i)+b=0$ $2+ai-6-ai+b=0$ $-4+b=0$ $b\ =4$	• correctly applies the factor theorem to determine b [1 mark]
$\therefore P(z)=2z^4+az^3+6z^2+az+4$ Given that the coefficients of the polynomial are real, another root is $z=i$, another **factor of $P(z)$ is $(z-i)$.**	• correctly uses the conjugate root of the given root to identify another factor of $P(z)$ [1 mark]
$(z-i)(z+i)=(z^2+1)$ is a factor of $P(z)$	• correctly identifies that (z^2+1) is a factor of $P(z)$ [1 mark]
By inspection, $P(z)=(z^2+1)(2z^2+az+4)$	• determines the remaining quadratic factor in terms of a [1 mark]
Given all roots of $P(z)$ have an imaginary component, $2z^2+az+4$ must have only complex roots. For complex roots, $b^2-4ac<0$ $a^2-4\times2\times4<0$	• applies the complex root requirement to the remaining quadratic factor [1 mark]
$a<\sqrt{32}$ So $a=1,2,3,4$ or 5 and $b\ =\ 4$	• determines the possible values for a given $a,b\in Z^+$ [1 mark]

Marking Guide – Paper 2 Section 1

2024 Paper 2 Section 1 Question 1 Complex numbers 2	 Given that $z=-2+3i$ is a root of $z^3+az+b=0$, where $a, b \in R$, another root is (A) $-2-3i$ (B) $-2+3i$ (C) $2-3i$ (D) $2+3i$ **Answer is A.**

2024 Paper 2 Section 1 Question 7 Complex numbers 2	 Determine the number of roots of $w^8=1$ that can be expressed in the form $a+bi$, where $a, b \in R^+$. (A) 0 (B) 1 (C) 2 (D) 3 **Answer is B.**

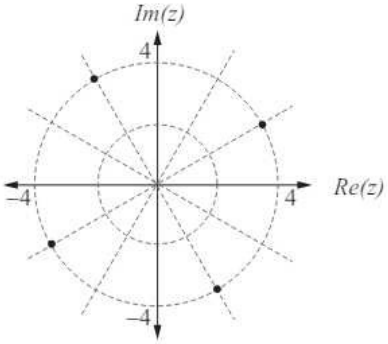
2023 Paper 2 Section 1 Question 3 Complex numbers 2	 Given that $2i$ is a root of $z^2-pz-q=0$, where $p, q \in R$, determine the values of p and q. (A) $p=-4$ and $q=-4$ (B) $p=-4$ and $q=4$ (C) $p=0$ and $q=-4$ (D) $p=0$ and $q=4$ **Answer is C.**

2023
Paper 2
Section 1
Question 10

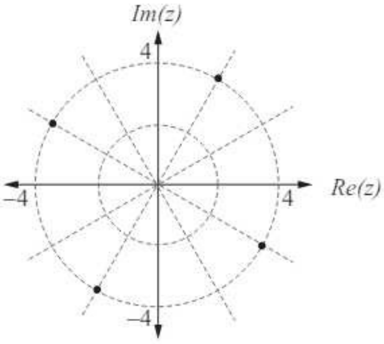
Complex numbers 2

The Argand diagram that represents the solutions to $z^4 = 16\,\text{cis}\left(\frac{2\pi}{3}\right)$, $z \in C$ is

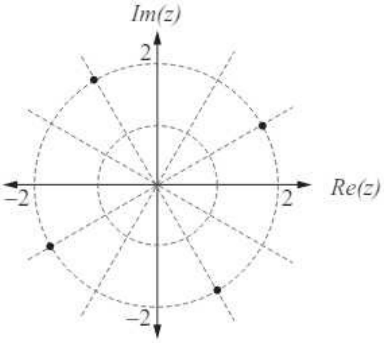
(A)



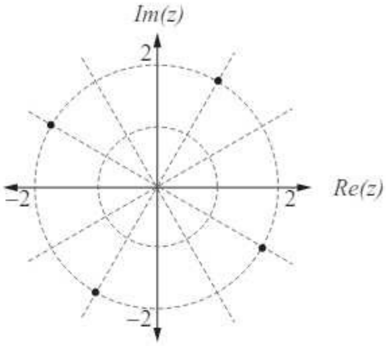
(B)



(C)



(D)



Answer is C.

2022
Paper 2
Section 1
Question 1

Complex numbers 2

A solution of the equation $z^2 = ai$, where $a \in R$, is $z = -2 - 2i$.

The other solution is

(A) $-8i$
(B) $-2 + 2i$
(C) $2 + 2i$ – Answer
(D) $8i$

2021
Paper 2
Section 1
Question 8

Complex numbers 2

The imaginary part of $\left(\text{cis}\left(\frac{\pi}{8}\right)\right)^{-2}$ is

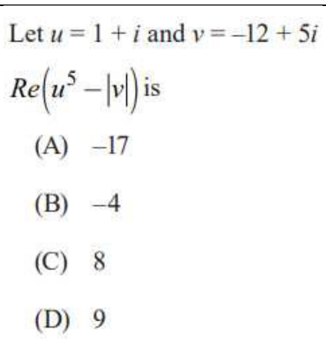
(A) -6.83

(B) -0.71

(C) 0.71

(D) 1.17

Answer is B.

2020 **Paper 2** **Section 1** **Question 8** **Complex numbers 2**	 Let $u = 1 + i$ and $v = -12 + 5i$ $Re\left(u^5 - \lvert v \rvert\right)$ is (A) -17 (B) -4 (C) 8 (D) 9 **Answer is A.**

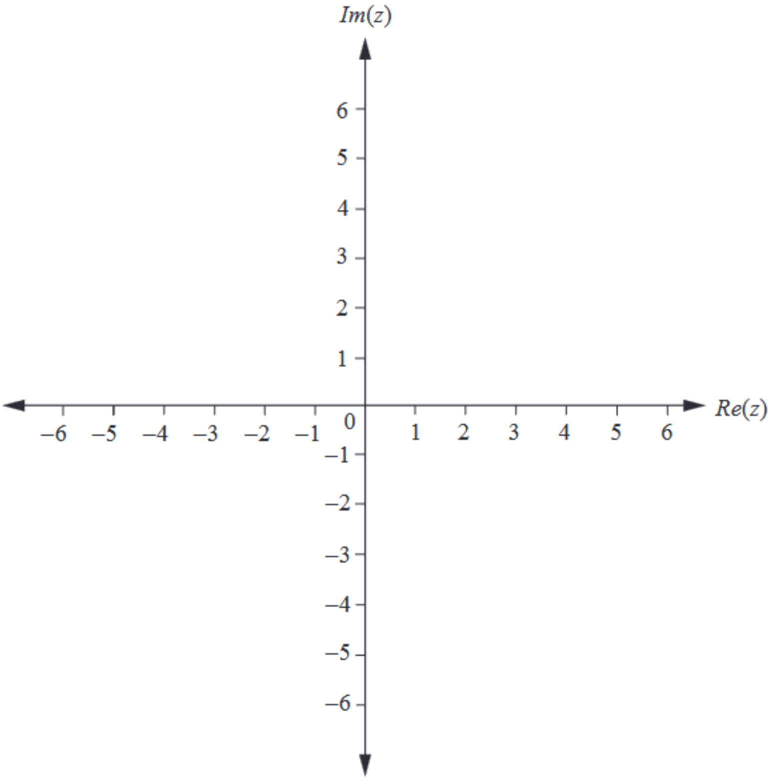
**2024
Paper 2
Section 2
Question 16**

**Complex
numbers 2**

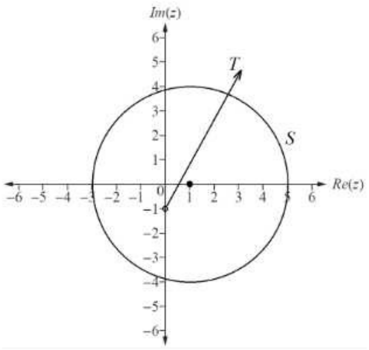
Two subsets of the complex plane are $S = \{z : |z-1| = 4\}$ and $T = \left\{z : \arg(z+i) = \frac{\pi}{3}\right\}$, where $z \in C$.

Determine the complex number/s where S and T intersect. Leave your answer/s in Cartesian form.

Provide an Argand diagram with a sketch of subsets S and T as part of your solution.



[6 marks]

Sample response	The response
S and T can be represented on an Argand diagram as shown. 	• correctly represents subset S on the Argand diagram as a circle of radius 4 centred at (1, 0) **[1 mark]** • correctly represents subset T on the Argand diagram as a ray starting at (0, –1) and moving through quadrants 4 and 1 **[1 mark]**
Let $z = x + yi$ represent the points in subsets S and T. $S : (x-1)^2 + y^2 = 16 \quad \ldots(1)$	• determines an equation to represent subset S **[1 mark]**
$T : y = \tan\left(\frac{\pi}{3}\right)x - 1, x > 0 \quad \ldots(2)$	• determines an equation to represent subset T **[1 mark]**

Substituting (2) into (1): $(x-1)^2+\left(\sqrt{3}x-1\right)^2=16$ Using GDC $x\approx-1.31$ or $x\approx 2.67$ As the point of intersection of the circle and the ray lies in quadrant 1, the only reasonable solution is $x\approx 2.67$ From (2), $y\approx\sqrt{3}(2.67)-1\approx 3.63$ The complex number where *S* and *T* intersect is $2.67+3.63i$.	• determines reasonable solution for *x* **[1 mark]** • determines where *S* and *T* intersect expressed as a complex number in Cartesian form **[1 mark]**

2023
Paper 2
Section 2
Question 12

Complex numbers 2

Consider the complex number $z=-3+2i$.

a) Determine z^3 using the binomial theorem. Leave your answer in the form $a+bi$, where $a,\ b\in R$. *[2 marks]*

Sample response	The response
Using binomial theorem: $z^3=(-3+2i)^3$ $=(-3)^3+3(-3)^2(2i)+3(-3)(2i)^2+(2i)^3$	• correctly uses the binomial theorem **[1 mark]**
$=-27+54i+36-8i$ $=9+46i$	• expresses z^3 in the form of $a+bi$ **[1 mark]**

b) Convert z into the form of $r\,\text{cis}(\theta)$, where $-\pi<\theta\leq\pi$. *[1 mark]*

Sample response	The response
$z=3.61\text{cis}(2.55)$	• correctly converts z into $r\,\text{cis}(\theta)$ form where $-\pi<\theta\leq\pi$ **[1 mark]**

c) Use the result from Question 12b) to determine z^3 using De Moivre's theorem. Leave your answer in the form of $r\,\text{cis}(\theta)$, where $-\pi<\theta\leq\pi$. *[2 marks]*

Sample response	The response
Using De Moivre's theorem $z^3=(3.61\text{cis}(2.55))^3$ $=3.61^3\,\text{cis}(3\times2.55)$	• uses De Moivre's theorem to calculate z^3 **[1 mark]**
$=46.87\text{cis}(7.66)$ $=46.87\text{cis}(1.38)$	• determines z^3 in the form of $r\,\text{cis}(\theta)$ where $-\pi<\theta\leq\pi$ **[1 mark]**

d) Evaluate the reasonableness of your results from Questions 12a) and 12c), noting that the two methods to determine z^3 should produce the same result. *[2 marks]*

Sample response	The response
$46.87\text{cis}(1.38) = 46.87\left(\cos(1.38) + i\sin(1.38)\right)$ $= 8.89 + 46.02i$ $\approx 9 + 46i$	• shows mathematical reasoning to convert from $r\,\text{cis}(\theta)$ form to $a+bi$ form **[1 mark]**
The two methods produce approximately the same z^3 value. The small variation is a result of rounding used in earlier calculations.	• states a decision regarding the reasonableness **[1 mark]**

2023
Paper 2
Section 2
Question 18

Complex numbers 2

Consider the complex solutions to the following equation, where $0 < \arg(z) < \pi$.

$$(z+1)\left(z^{14} - z^{13} + z^{12} - z^{11} + \ldots + z^4 - z^3 + z^2 - z\right) = 1 - z$$

Let w_1 be the solution with the maximum possible real part and w_2 be the solution with the maximum possible imaginary part.

Show that $\dfrac{{w_1}^4}{w_2} \in Z$.

(5 marks)

Sample response	The response
$(z+1)\left(z^{14} - z^{13} + z^{12} - z^{11} + \ldots + z^4 - z^3 + z^2 - z\right) = 1 - z$ $z^{15} + z^{14} - z^{14} + z^{13} - z^{13} + \ldots$ $+ z^4 - z^4 - z^3 + z^3 + z^2 - z^2 - z = 1 - z$ $z^{15} = 1$	• correctly simplifies the original equation **[1 mark]**
The solutions are $z = \text{cis}\left(\dfrac{2n\pi}{15}\right)$ where $n \in Z$ and $0 < arg(z) < \pi$.	• describes the location of the solutions **[1 mark]**
Solution with the maximum possible real part has its argument closest to 0. $w_1 = \text{cis}\left(\dfrac{2\pi}{15}\right)$	• determines w_1 **[1 mark]**
Solution with the maximum possible imaginary part has its argument closest to $\dfrac{\pi}{2}$. $w_2 = \text{cis}\left(\dfrac{8\pi}{15}\right)$	• determines w_2 **[1 mark]**
$\dfrac{{w_1}^4}{w_2} = \dfrac{\text{cis}\left(4 \times \dfrac{2\pi}{15}\right)}{\text{cis}\left(\dfrac{8\pi}{15}\right)} = 1 \in Z$	• shows that $\dfrac{{w_1}^4}{w_2}$ is an integer **[1 mark]**

2022
Paper 2
Section 2
Question 18

Complex numbers 2

Consider the polynomials $P(z)=z^3+(i-a)z^2-2biz+3i$ and $Q(z)=z-2i$, where $a,\ b\in R$.

Given $\dfrac{P(z)}{Q(z)}$ has a remainder of $a-bi$, evaluate the reasonableness that $\left(z-(a-bi)\right)$ is a factor of $P(z)$.

[5 marks]

Sample Response	The response
$P(z)=z^3+(i-a)z^2-2biz+3i$ Using the remainder theorem $P(2i)=a-bi$	• correctly determines an expression for $P(2i)$ using the remainder theorem [1 mark]
By substitution $P(2i)=(2i)^3+(i-a)(2i)^2-2bi(2i)+3i$ $=-8i-4(i-a)+4b+3i$ $=-8i-4i+4a+4b+3i$ $=4a+4b-9i$	• correctly determines an expression for $P(2i)$ using substitution into $P(z)$ [1 mark]
Equating parts: $a=4a+4b$ $-b=-9$	• forms two simultaneous equations by equating parts [1 mark]
So $a=-12$ and $b\ =9$ $\therefore\ P(z)=z^3+(i+12)z^2-18iz+3i$ $P(a-bi)=P(-12-9i)\approx 1566-285i$	• determines $P(a-bi)$ using the values for a and b [1 mark]
Since $P(-12-9i)\neq 0$, it is not reasonable that $(z-(a-bi))$ is a factor of $P(z)$.	• evaluates the reasonableness of the statement using mathematical reasoning [1 mark]

2021
Paper 2
Section 2
Question 18

Complex numbers 2

Consider the polynomial $P(z)=z^3+az^2+bz+c$, where $a, b, c\in R$ and $z\in C$.

Two of the roots of $P(z)$ are also roots of $z^4+z^3+z^2+z+1$. The remaining root of $P(z)$ is $z=2$.

Given $z^5-1=(z-1)\left(z^4+z^3+z^2+z+1\right)$, determine a possible expression for $P(z)$.

Leave your answer in expanded form.

[6 marks]

Sample Response	The response
Method 1 The roots of $z^5\ =\ 1$ are $z\ =\ \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k\in Z$	• correctly determines the roots of $z^5=1$ [1 mark]
Given $z^5-1\ =\ (z-1)(z^4+z^3+z^2+z+1)$, the four roots of $z^4+z^3+z^2+z+1$ must be the four complex roots of the five 5th roots of unity, $z^5\ =\ 1$. By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5\ =\ 1$. One possible pair of roots is $\text{cis}\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(-\frac{2\pi}{5}\right)$	• correctly recognises one possible pair of roots [1 mark]
Determining a quadratic factor of $P(z)$ $\left(z-\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{5}\right)\right)$	• determines a quadratic factor of $P(z)$ in factorised form [1 mark]
$=\ z^2-\left(\text{cis}\left(\frac{2\pi}{5}\right)+\ \text{cis}\left(-\frac{2\pi}{5}\right)\right)z+\ \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)$ $=\ z^2-\left(\cos\left(\frac{2\pi}{5}\right)+\ i\sin\left(\frac{2\pi}{5}\right)+\ \cos\left(-\frac{2\pi}{5}\right)+\right.$ $\left.i\sin\left(-\frac{2\pi}{5}\right)\right)z+\ \text{cis}(0)$ $=\ z^2-2\cos\left(\frac{2\pi}{5}\right)z+\ 1$	• expresses determined quadratic factor of $P(z)$ in expanded form [1 mark]
$P(z)\ =\ (z-2)\left(z^2-2\cos\left(\frac{2\pi}{5}\right)z+\ 1\right)$	• uses the factor of $z=2$ to express $P(z)$ in factorised form [1 mark]
$=\ z^3-2\left(\cos\left(\frac{2\pi}{5}\right)+\ 1\right)z^2+\ \left(4\cos\left(\frac{2\pi}{5}\right)+\ 1\right)z-2$	• determines $P(z)$ in expanded form [1 mark]

Method 2 The roots of $z^5 = 1$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k \in Z$	• correctly determines the roots of $z^5 = 1$ [1 mark]
Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the four roots of $z^4 + z^3 + z^2 + z + 1$ must be the four complex roots of the five 5th roots of unity, $z^5 = 1$. By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5 = 1$. One possible pair of roots is $\text{cis}\left(\frac{4\pi}{5}\right)$ and $\text{cis}\left(-\frac{4\pi}{5}\right)$.	• correctly determines one possible pair of roots [1 mark]
Determining a quadratic factor of $P(z)$ $\left(z - \text{cis}\left(\frac{4\pi}{5}\right)\right)\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$	• determines a quadratic factor of $P(z)$ in factorised form [1 mark]
$= z^2 - \left(\text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)$ $= z^2 - \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)\right)z + \text{cis}(0)$ $= z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1$	• expresses determined quadratic factor of $P(z)$ in expanded form [1 mark]
$P(z) = (z-2)\left(z^2 - 2\cos\left(\frac{4\pi}{5}\right)z + 1\right)$	• uses the factor of $z = 2$ to express $P(z)$ in factorised form [1 mark]
$= z^3 - 2\left(\cos\left(\frac{4\pi}{5}\right) + 1\right)z^2 + \left(4\cos\left(\frac{4\pi}{5}\right) + 1\right)z - 2$	• determines $P(z)$ in expanded form [1 mark]
Method 3 Given $z = 2$ is a root of $P(z)$ $\therefore P(2) = 0 \Rightarrow (2)^3 + a(2)^2 + 2b + c = 0$ $8 + 4a + 2b + c = 0 \quad \ldots(1)$	• correctly determines equation using $z = 2$ [1 mark]
The roots of $z^5 = 1$ are $z = \text{cis}\left(\frac{2k\pi}{5}\right)$ where $k \in Z$	• correctly determines the roots of $z^5 = 1$ [1 mark]
Given $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the four roots of $z^4 + z^3 + z^2 + z + 1$ must be the four complex roots of the five 5th roots of unity, $z^5 = 1$. By the conjugate root theorem, the two remaining roots of $P(z)$ must be a conjugate pair of roots of $z^5 = 1$. One possible pair of roots is $\text{cis}\left(\frac{2\pi}{5}\right)$ and $\text{cis}\left(-\frac{2\pi}{5}\right)$	• determines one possible pair of roots [1 mark]
$P(z) = z^3 + az^2 + bz + c$ $P\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) = 0$ $\therefore \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^3 + a\left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^2 + b\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) + c = 0$ $\text{cis}\left(\frac{6\pi}{5}\right) + a\,\text{cis}\left(\frac{4\pi}{5}\right) + b\,\text{cis}\left(\frac{2\pi}{5}\right) + c = 0 \quad \ldots(2)$ $P\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right) = 0$ $\therefore \left(\text{cis}\left(-\frac{2\pi}{5}\right)\right)^3 + a\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right)^2 + \ldots$ $\ldots b\left(\text{cis}\left(-\frac{2\pi}{5}\right)\right) + c = 0$ $\text{cis}\left(-\frac{6\pi}{5}\right) + a\,\text{cis}\left(-\frac{4\pi}{5}\right) + b\,\text{cis}\left(-\frac{2\pi}{5}\right) + c = 0 \quad \ldots(3)$	• determines two equations using two possible conjugate pairs [1 mark]
Solving the three simultaneous equations using GDC $a \approx -2.62, b \approx 2.236, c = -2$	• solves the simultaneous equations to determine a, b and c [1 mark]
$P(z) \approx z^3 - 2.62z^2 + 2.24z - 2$	• determines $P(z)$ in expanded form [1 mark]

2020
Paper 2
Section 2
Question 16

Complex numbers 2

Consider the identity

$$\cos(4\theta) = A\cos^4(\theta) + B\sin^2(\theta) + C \text{ where } A, B \text{ and } C \in Z$$

a) Determine the values of A, B and C using De Moivre's theorem. [5 marks]

Sample Response	**The response**
Method 1 Using De Moivre's theorem: $(\text{cis}(\theta))^4 = \text{cis}(4\theta)$	• correctly uses De Moivre's theorem [1 mark]
Equating real parts $\boldsymbol{\cos(4\theta) = Re(\cos(\theta) + i\sin(\theta))^4}$ $\boldsymbol{= \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)}$	• uses binomial expansion with the real parts and simplifies the expression [1 mark]
$= \cos^4(\theta) - 6\cos^2(\theta)(1 - \cos^2(\theta)) + (1 - \cos^2(\theta))^2$ $= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + 1 - 2\cos^2(\theta) + \cos^4(\theta)$ $\boldsymbol{= 8\cos^4(\theta) - 8\cos^2(\theta) + 1}$	• establishes a simplified expression following the use of a suitable Pythagorean identity [1 mark]
$\boldsymbol{= 8\cos^4(\theta) - 8(1 - \sin^2(\theta)) + 1}$ $= 8\cos^4(\theta) - 8 + 8\sin^2(\theta) + 1$ $\boldsymbol{= 8\cos^4(\theta) + 8\sin^2(\theta) - 7}$	• establishes a simplified expression in the form of $A\cos^4(\theta) + B\sin^2(\theta) + C$ [1 mark]
So $\boldsymbol{A = 8, B = 8, C = -7}$	• communicates the values of A, B and C [1 mark]
Method 2 Using De Moivre's theorem: $(\text{cis}(\theta))^4 = \text{cis}(4\theta)$	• correctly uses De Moivre's theorem [1 mark]
Using the binomial expansion $(\text{cis}(\theta))^4 = (\cos(\theta) + i\sin(\theta))^4$ $\boldsymbol{= \cos^4(\theta) + 4\cos^3(\theta)\sin(\theta)i}$ $\boldsymbol{- 6\cos^2(\theta)\sin^2(\theta) - 4\cos(\theta)\sin^3(\theta)i + \sin^4(\theta)}$	• uses binomial expansion and simplifies the expression [1 mark]
Equating real parts $\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)(1 - \cos^2(\theta) + (1 - \cos^2(\theta))^2$ $= \cos^4(\theta) - 6\cos^2(\theta) + 6\cos^4(\theta) + 1 - 2\cos^2(\theta) + \cos^4(\theta)$ $\boldsymbol{= 8\cos^4(\theta) - 8\cos^2(\theta) + 1}$	• establishes a simplified expression following the use of a suitable Pythagorean identity [1 mark]
$\boldsymbol{= 8\cos^4(\theta) - 8(1 - \sin^2(\theta)) + 1}$ $= 8\cos^4(\theta) - 8 + 8\sin^2(\theta) + 1$ $\boldsymbol{= 8\cos^4(\theta) + 8\sin^2(\theta) - 7}$	• establishes a simplified expression in the form of $A\cos^4(\theta) + B\sin^2(\theta) + C$ [1 mark]
So $\boldsymbol{A = 8, B = 8, C = -7}$	• communicates the values of A, B and C [1 mark]

b) State an appropriate method of verifying your results from 16a). [1 mark]

Sample Response	**The response**
A verification strategy would be to graph $\boldsymbol{y = \cos(4\theta)}$ and $\boldsymbol{y = 8\cos^4(\theta) + 8\sin^2(\theta) - 7}$ to confirm that the two graphs are the same.	• describes an appropriate verification strategy [1 mark]

Unit 4: Further calculus and statistical inference

Unit 4 – Topic 1: Integration and applications of integration

Paper 1 Section 1

2024
Paper 1
Section 1
Question 9

Integration and applications of integration

Use a suitable double-angle identity to determine $\int 2\sin^2(x)\,dx$.

(A) $x-2\sin(2x)+c$

(B) $x+2\sin(2x)+c$

(C) $x-\dfrac{\sin(2x)}{2}+c$

(D) $x+\dfrac{\sin(2x)}{2}+c$

2022
Paper 1
Section 1
Question 1

Integration and applications of integration

Let $z = a + 3i$ and $w = -3 + bi$, where $a, b \in R$.

If $z = w$, then

(A) $a = -3$, $b = -3$
(B) $a = -3$, $b = 3$
(C) $a = 3$, $b = -3$
(D) $a = 3$, $b = 3$

2022
Paper 1
Section 1
Question 8

Integration and applications of integration

Use the substitution $u = \tan(x)$ to determine $\int \tan(x)\sec^2(x)dx$.

(A) $\frac{1}{2}\tan(x)+c$

(B) $\frac{1}{2}\tan^2(x)+c$

(C) $\tan(x)+c$

(D) $\tan^2(x)+c$

2020
Paper 1
Section 1
Question 1

Integration and applications of integration

The indefinite integral $\int\frac{3x-A}{1-x^2}\,dx$ can be determined using the partial fractions $\frac{-1}{1+x}+\frac{2}{1-x}$

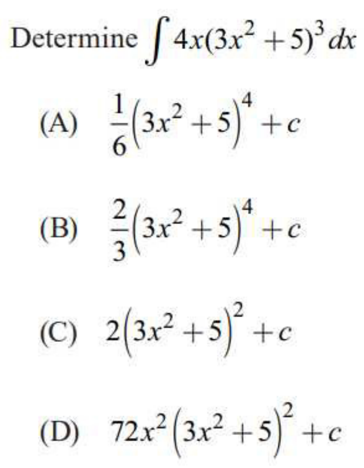
The value of A is

(A) –3

(B) –1

(C) 1

(D) 3

2020 **Paper 1** **Section 1** **Question 5** **Integration and applications of integration**	 Determine $\int 4x(3x^2+5)^3\,dx$ (A) $\frac{1}{6}\left(3x^2+5\right)^4+c$ (B) $\frac{2}{3}\left(3x^2+5\right)^4+c$ (C) $2\left(3x^2+5\right)^2+c$ (D) $72x^2\left(3x^2+5\right)^2+c$

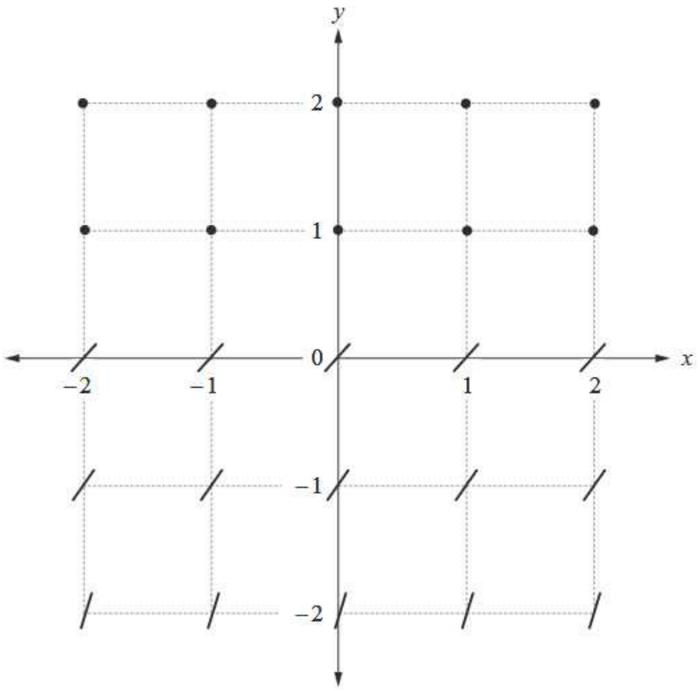
Paper 1 Section 2

2024
Paper 1
Section 2
Question 15

Integration and applications of integration

A sketch of a partially completed slope field for the differential equation $\frac{dy}{dx} = 1 - y$ is shown.

a) Complete the slope field by sketching the slopes at the 10 points indicated. *[2 marks]*



b) Use the completed slope field to sketch the solution curve for $\frac{dy}{dx} = 1 - y$, given the point (–1, 0) lies on this curve. *[1 mark]*

The differential equation can be solved by rearranging it into the form $\frac{1}{1-y}\frac{dy}{dx} = 1$.

c) Determine the equation of the solution curve sketched in Question 15b) by solving the differential equation, given the point (–1, 0) lies on this curve. Leave your answer in the form $y = f(x)$. *[3 marks]*

2024
Paper 1
Section 2
Question 18

Integration and applications of integration

A random variable X has a probability density function given by

$$f(x)=\begin{cases} k\sin^{-1}(x), & 0\le x\le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

where k is a positive constant.

Determine the value of k.

[6 marks]

2023 **Paper 1** **Section 2** **Question 11** **Integration and applications of integration**	Determine the following definite integrals. a) $\int_0^1 \frac{1}{1+x^2}\,dx$ (2 marks) b) $\int_0^{\frac{\pi}{4}} 2\sin^2(x)\,dx$ (3 marks)

2022
Paper 1
Section 2
Question 13

Integration and applications of integration

a) Use partial fractions to determine $\int \frac{22}{(2x-3)(x+4)}\,dx$

[6 marks]

b) Use the result from Question 13a) to determine $\int_{-3}^{0} \frac{22}{(2x-3)(x+4)}\,dx$

Express your answer in simplest form.

[2 marks]

2022
Paper 1
Section 2
Question 17

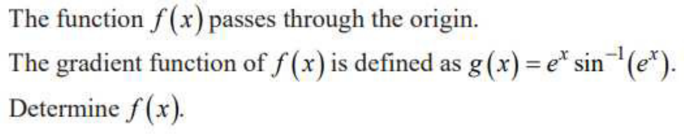
Integration and applications of integration

The region between the x-axis and the curve of the function $y = 1 + \sin(2x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the x-axis to form a solid of revolution.

Determine the volume of this solid. Express your answer in simplest form. [5 marks]

2022
Paper 1
Section 2
Question 19

Integration and applications of integration



The function $f(x)$ passes through the origin.

The gradient function of $f(x)$ is defined as $g(x) = e^x \sin^{-1}(e^x)$.

Determine $f(x)$.

[7 marks]

2021 **Paper 1** **Section 2** **Question 15** **Integration and applications of integration**	Use partial fractions to determine $\int \frac{4x-17}{x^2-x-6}\,dx$, where $x \in R, x \neq -2, x \neq 3$. Express your answer in the form $\ln\lvert f(x)\rvert + c$. [4 marks]

2021 **Paper 1** **Section 2** **Question 17** **Integration and applications of integration**	The area between the graphs of the functions $y = 4x$ and $y = 2x^2$ is rotated about the y-axis to form a solid of revolution with a volume of V units^3. Determine the exact value of V. [7 marks]

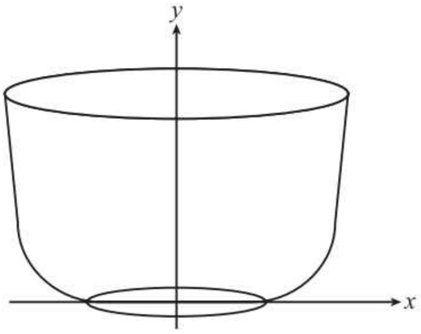
2020 **Paper 1** **Section 2** **Question 13** **Integration and applications of integration**	The expected value of an exponential random variable X with parameter $\lambda > 0$ can be determined using the rule $E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx$ Use integration by parts to determine $\mathrm{E}(\mathrm{X})$. Express your answer in simplest form. [4 marks]

2020
Paper 1
Section 2
Question 17

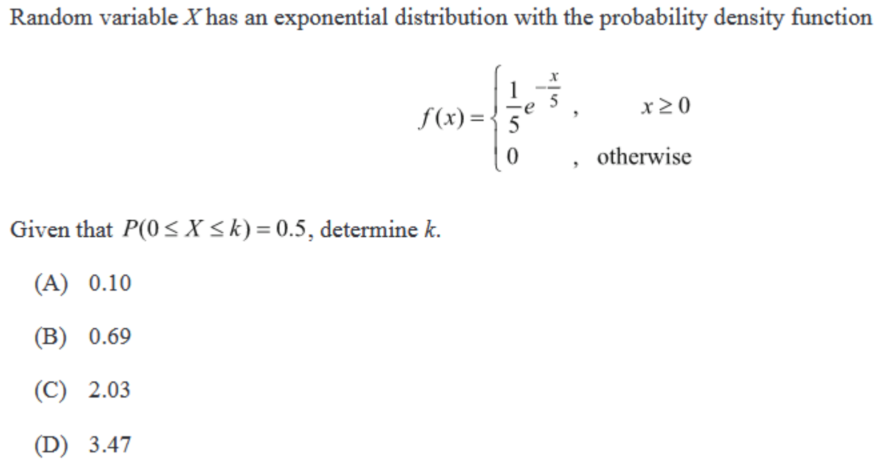
Integration and applications of integration

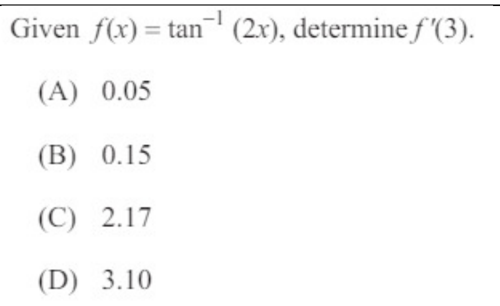
Determine the smallest positive value of a given

$$\int_{-a}^{a} 1+\left(\frac{\sec(2x)+\tan(2x)}{\operatorname{cosec}(2x)+1}\right)^{2} dx = 1$$

2020 Paper 1 Section 2 Question 19 **Integration and applications of integration**	A circular-based bowl has been positioned symmetrically on a Cartesian plane as shown in the diagram.  The bowl has a shape that can be generated by rotating the curve $y=\dfrac{4}{8-x}-1$ about the y-axis for $4 \le x \le 7.6$ cm. The bowl is being filled with a liquid at the rate of 7π cm^3 s^{-1}. Determine the rate at which the depth of liquid is increasing when the depth of liquid reaches one-third of the height of the bowl. [7 marks]

Paper 2 Section 1

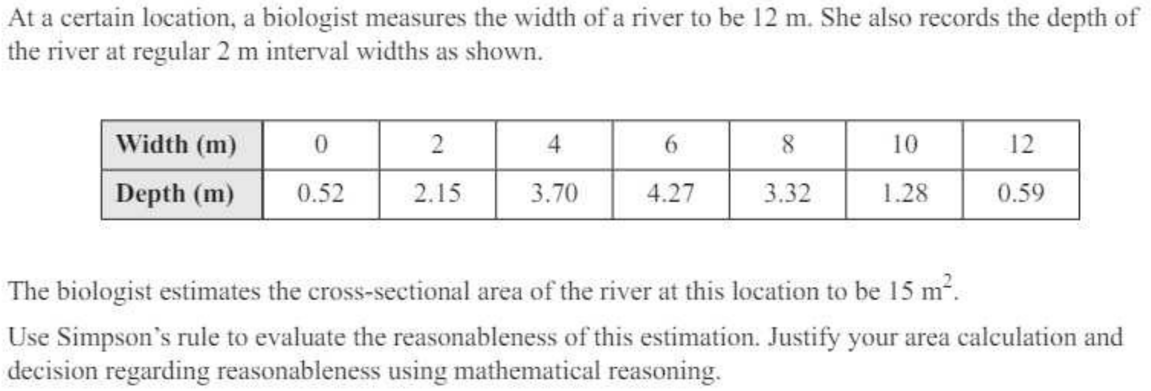
2024 Paper 2 Section 1 Question 9 Integration and applications of integration	 Random variable X has an exponential distribution with the probability density function $f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}}, & x\geq 0\\ 0, & \text{otherwise}\end{cases}$ Given that $P(0\leq X\leq k)=0.5$, determine k. (A) 0.10 (B) 0.69 (C) 2.03 (D) 3.47

2023 Paper 2 Section 1 Question 8 Integration and applications of integration	 Given $f(x)=\tan^{-1}(2x)$, determine $f'(3)$. (A) 0.05 (B) 0.15 (C) 2.17 (D) 3.10

Paper 2 Section 2

2023
Paper 2
Section 2
Question 14

Integration and applications of integration



At a certain location, a biologist measures the width of a river to be 12 m. She also records the depth of the river at regular 2 m interval widths as shown.

Width (m)	0	2	4	6	8	10	12
Depth (m)	0.52	2.15	3.70	4.27	3.32	1.28	0.59

The biologist estimates the cross-sectional area of the river at this location to be 15 m^2.

Use Simpson's rule to evaluate the reasonableness of this estimation. Justify your area calculation and decision regarding reasonableness using mathematical reasoning.

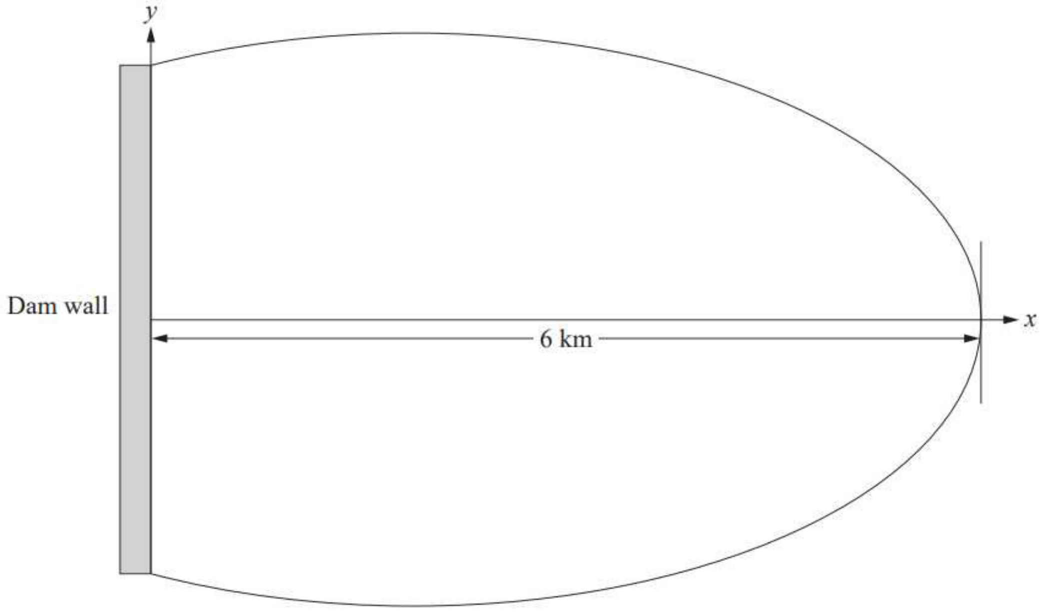
(4 marks)

2022
Paper 2
Section 2
Question 11

Integration and applications of integration

An aerial view of the surface of a dam, 6 km in length, is symmetrically positioned on a Cartesian plane as shown. A dam wall is located along the y-axis.

The surrounding edge of the dam can be modelled by the ellipse $\frac{(x-2)^2}{16}+\frac{y^2}{9}=1$, for $0 \le x \le 6$.



Not to scale

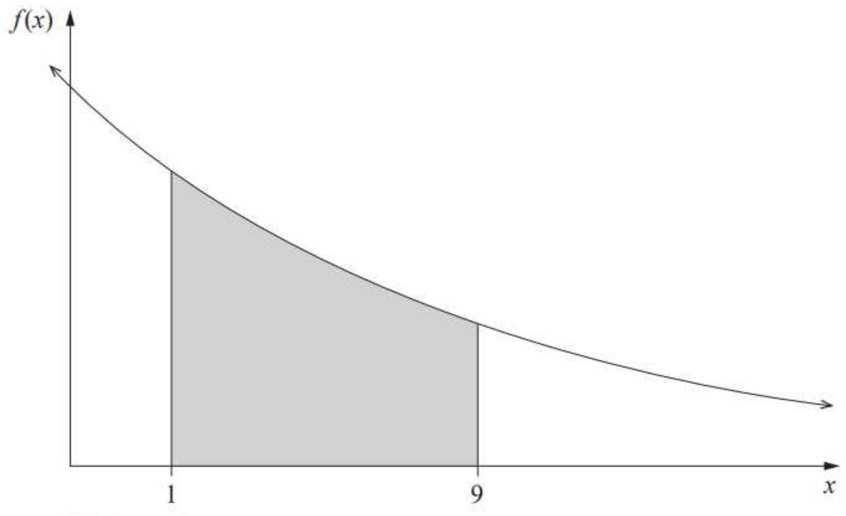
a) Use Simpson's rule with four strips to determine an approximate area of the surface of the dam. [4 marks]

b) Evaluate the reasonableness of this approximation. [2 marks]

2021
Paper 2
Section 2
Question 13

Integration and applications of integration

The area under the graph of the function $f(x) = 0.2e^{-0.2x}$ for $1 \leq x \leq 9$ is shaded.



Not to scale

a) Use Simpson’s rule with four intervals to determine an approximation for this area. [4 marks]

b) Use a calculus approach to evaluate the reasonableness of your area approximation from Question 13a). [2 marks]

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Question 14

Integration and applications of integration

The time, t, (months) that it takes before a phone owner cracks the screen on their phone can be modelled by an exponentially distributed random variable

$$f(t) = \begin{cases} 0.16e^{-0.16t}, & t \geq 0 \\ 0 \quad\quad\quad\quad , & \text{otherwise} \end{cases}$$

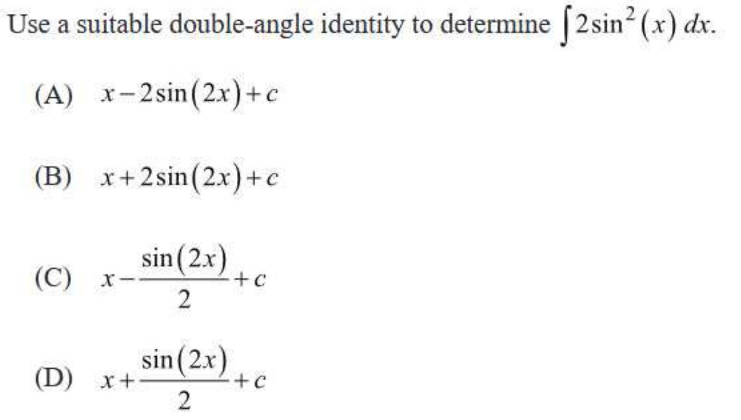
a) Show that $f(t)$ is a probability density function. [1 mark]

b) Determine the probability that a phone owner cracks the screen on their phone within 1 year. [2 marks]

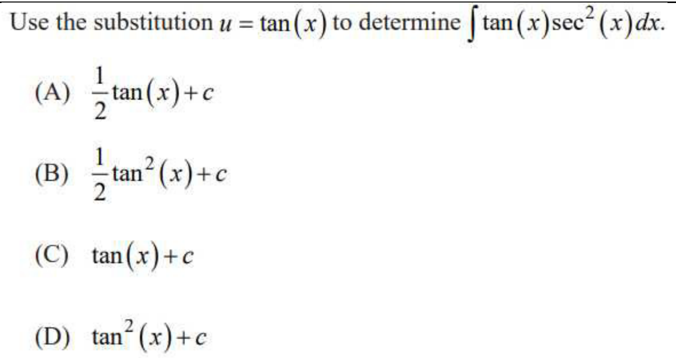
Three-quarters of phone owners take between 1 and m months before they crack the screen on their phone.

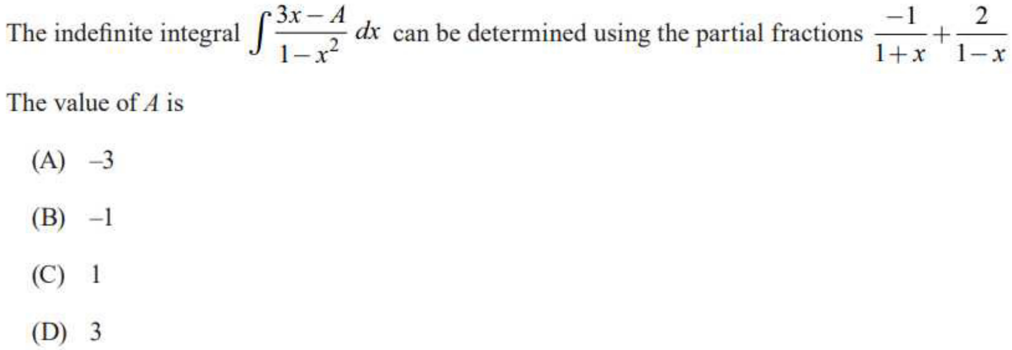
c) Determine the value of m. [2 marks]

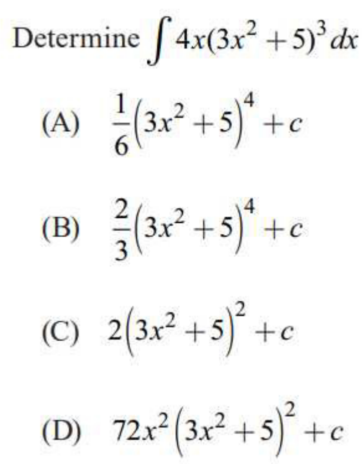
Marking Guide – Paper 1 Section 1

2024 Paper 1 Section 1 Question 9 Integration and applications of integration	 Use a suitable double-angle identity to determine $\int 2\sin^2(x)\,dx$. (A) $x-2\sin(2x)+c$ (B) $x+2\sin(2x)+c$ (C) $x-\frac{\sin(2x)}{2}+c$ (D) $x+\frac{\sin(2x)}{2}+c$ **Answer is C.**

2022 Paper 1 Section 1 Question 1 Integration and applications of integration	Let $z = a + 3i$ and $w = -3 + bi$, where $a, b \in R$. If $z = w$, then (A) $a = -3, b = -3$ **(B) $a = -3, b = 3$ – Answer** (C) $a = 3, b = -3$ (D) $a = 3, b = 3$

2022 Paper 1 Section 1 Question 8 Integration and applications of integration	 Use the substitution $u = \tan(x)$ to determine $\int \tan(x)\sec^2(x)\,dx$. (A) $\frac{1}{2}\tan(x)+c$ (B) $\frac{1}{2}\tan^2(x)+c$ (C) $\tan(x)+c$ (D) $\tan^2(x)+c$ **Answer is B.**

2020 Paper 1 Section 1 Question 1 Integration and applications of integration	 The indefinite integral $\int \frac{3x-A}{1-x^2}\,dx$ can be determined using the partial fractions $\frac{-1}{1+x}+\frac{2}{1-x}$ The value of A is (A) -3 (B) -1 (C) 1 (D) 3 **Answer is B.**

2020 **Paper 1** **Section 1** **Question 5** **Integration and applications of integration**	Determine $\int 4x(3x^2+5)^3 dx$ (A) $\frac{1}{6}\left(3x^2+5\right)^4+c$ (B) $\frac{2}{3}\left(3x^2+5\right)^4+c$ (C) $2\left(3x^2+5\right)^2+c$ (D) $72x^2\left(3x^2+5\right)^2+c$ **Answer is A.**

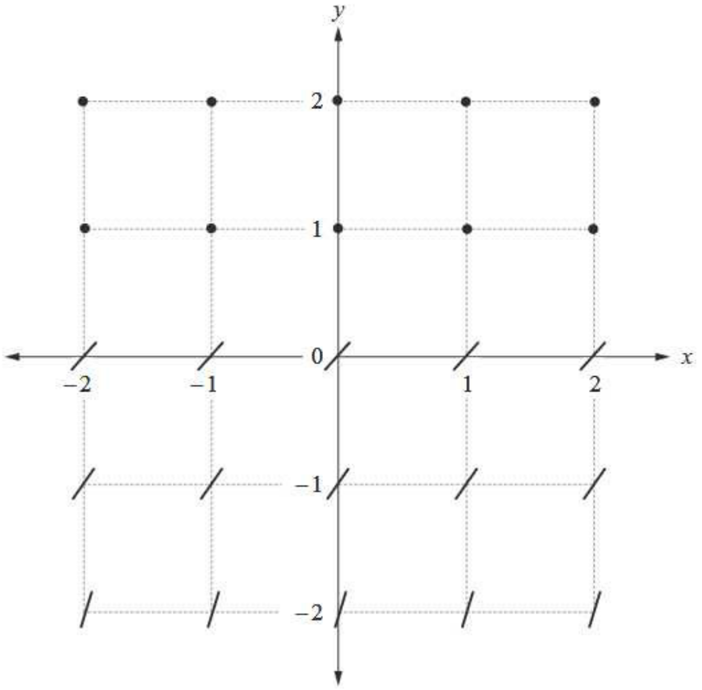
Marking Guide – Paper 1 Section 2

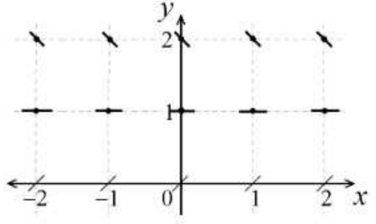
2024
Paper 1
Section 2
Question 15

Integration and applications of integration

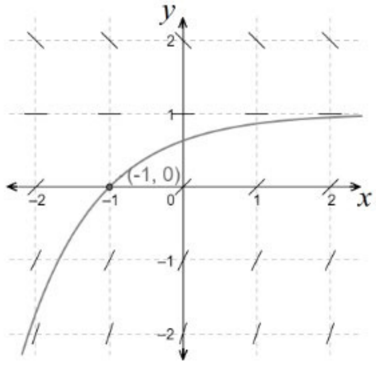
A sketch of a partially completed slope field for the differential equation $\frac{dy}{dx} = 1 - y$ is shown.

a) Complete the slope field by sketching the slopes at the 10 points indicated. *[2 marks]*



Sample response	The response
	• correctly sketches the slope as 0 for all five points where $y = 1$ **[1 mark]** • correctly sketches the slope as –1 for all five points where $y = 2$ **[1 mark]**

b) Use the completed slope field to sketch the solution curve for $\frac{dy}{dx}=1-y$, given the point (–1, 0) lies on this curve. *[1 mark]*

Sample response	The response
	• sketches a solution curve that passes through (–1, 0) and is asymptotic to the line $y=1$ **[1 mark]**

The differential equation can be solved by rearranging it into the form $\frac{1}{1-y}\frac{dy}{dx}=1$.

c) Determine the equation of the solution curve sketched in Question 15b) by solving the differential equation, given the point (–1, 0) lies on this curve. Leave your answer in the form $y=f(x)$. *[3 marks]*

Sample response	The response
Method 1 $\frac{1}{1-y}\frac{dy}{dx}=1$ $\int\frac{1}{1-y}\,dy=\int 1\,dx$ $-\ln\lvert 1-y\rvert=x+c$ At $(-1, 0)$, $-\ln(1)=-1+c \Rightarrow c=1$ $\ln(\lvert 1-y\rvert)=-x-1$ $y=1-e^{-x-1}$	• correctly determines a general solution to the equation **[1 mark]** • determines an appropriate constant of integration **[1 mark]** • expresses solution in the form $y=f(x)$ **[1 mark]**
Method 2 $\frac{1}{1-y}\frac{dy}{dx}=1$ $\int\frac{1}{1-y}\,dy=\int 1\,dx$ $-\ln\lvert 1-y\rvert=x+c$ $1-y=e^{-x-c} \Rightarrow y=1-Ae^{-x}$ At $(-1, 0)$, $0=1-Ae \Rightarrow A=\frac{1}{e}$ $y=1-\frac{1}{e}e^{-x}$	• correctly determines a general solution to the equation **[1 mark]** • determines an appropriate constant of integration **[1 mark]** • expresses solution in the form $y=f(x)$ **[1 mark]**

2024
Paper 1
Section 2
Question 18

Integration and applications of integration

A random variable X has a probability density function given by

$$f(x)=\begin{cases} k\sin^{-1}(x), & 0\le x\le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

where k is a positive constant.

Determine the value of k.

[6 marks]

Sample response	The response
Method 1	
Using a property of a pdf $\int_0^1 k\sin^{-1}(x)\ dx\ =1 \qquad \ldots(1)$	• correctly uses a suitable pdf property **[1 mark]**
Consider $\int k\sin^{-1}(x)\ dx$ Using integration by parts $\int u\frac{dv}{dx}\ dx=uv-\int v\frac{du}{dx}\ dx$	
Let $u=\sin^{-1}(x) \qquad \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$	• correctly determines both results required to progress integration by parts **[1 mark]**
Let $\frac{dv}{dx}=k \qquad v=kx$	• uses integration by parts **[1 mark]**
$\int k\sin^{-1}(x)\ dx=kx\sin^{-1}(x)-\int\frac{kx}{\sqrt{1-x^2}}\ dx$	

Using substitution for the developed integral $\int \frac{kx}{\sqrt{1-x^2}}\,dx$ Let $u=1-x^2$ $\frac{du}{dx}=-2x$ $\int \frac{kx}{\sqrt{1-x^2}}\,dx=-\frac{k}{2}\int\frac{-2x}{\sqrt{1-x^2}}\,dx$ $=-\frac{k}{2}\int\frac{1}{\sqrt{u}}\,du$ $\int \frac{kx}{\sqrt{1-x^2}}\,dx=-\frac{k}{2}\int u^{-\frac{1}{2}}\,du$ $=-ku^{\frac{1}{2}}$ $=-k\sqrt{1-x^2}+c$ $\therefore \int k\sin^{-1}x\,dx=kx\sin^{-1}(x)+k\sqrt{1-x^2}+c$ Substituting into (1) $\int_0^1 k\sin^{-1}(x)\,dt=1$ $k\left(x\sin^{-1}(x)+\sqrt{1-x^2}\right)\Big\|_0^1=1$ $k\left(\left(\sin^{-1}(1)+\sqrt{1-1}\right)-\left(0+\sqrt{1-0}\right)\right)=1$ $k\left(\frac{\pi}{2}-1\right)=1$ $k=\frac{2}{\pi-2}$	• uses a suitable substitution method to progress the developed integral within the use of integration by parts **[1 mark]** • determines a general result for the required integral **[1 mark]** • determines value of k **[1 mark]**
Method 2 Using a property of a pdf $\int_0^1 k\sin^{-1}(x)\,dx=1$ $\int_0^1 \sin^{-1}(x)\,dx=\frac{1}{k}$ The area between $y=\sin^{-1}(x)$ and the x-axis for $0\le x\le 1$ is Area 1: $\int_0^1 \sin^{-1}(x)\,dx=\frac{1}{k}$ Consider the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le\frac{\pi}{2}$ Area 2 = $\int_0^{\frac{\pi}{2}}\sin(y)\,dy$ Area 2 $=-\cos(y)\Big\|_0^{\frac{\pi}{2}}$ $=-\cos\left(\frac{\pi}{2}\right)+\cos(0)$ $=1$ Area 1 + Area 2 = Area of rectangle $\frac{1}{k}+1=\frac{\pi}{2}\times 1$ $\frac{1}{k}=\frac{\pi-2}{2}$ $k=\frac{2}{\pi-2}$	• correctly uses a suitable pdf property **[1 mark]** • correctly represents the area between $y=\sin^{-1}(x)$ and the x-axis for $0\le x\le 1$ **[1 mark]** • represents the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le\frac{\pi}{2}$ **[1 mark]** • determines the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le\frac{\pi}{2}$ **[1 mark]** • determines an equation using the two areas **[1 mark]** • determines value of k **[1 mark]**

2023
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Section 2
Question 11

Integration and applications of integration

Determine the following definite integrals.

a) $\int_0^1 \frac{1}{1+x^2}\,dx$

(2 marks)

Sample response	**The response**
$\int_0^1 \frac{1}{1+x^2}\,dx$	• correctly uses the required integration rule **[1 mark]**
$=\tan^{-1}(x)\Big\|_0^1$	• calculates value of the definite integral **[1 mark]**
$=\tan^{-1}(1)-\tan^{-1}(0)$	
$=\frac{\pi}{4}$	

b) $\int_0^{\frac{\pi}{4}} 2\sin^2(x)\,dx$

(3 marks)

Sample response	**The response**
$\int_0^{\frac{\pi}{4}} 2\sin^2(x)\,dx$	
$=\int_0^{\frac{\pi}{4}} 2\times\frac{1}{2}\left(1-\cos(2x)\right)dx$	• correctly uses the required trigonometric substitution **[1 mark]**
$=\int_0^{\frac{\pi}{4}} 1-\cos(2x)\,dx$	
$=x-\frac{\sin(2x)}{2}\Big\|_0^{\frac{\pi}{4}}$	• uses suitable integration rules **[1 mark]**
$=\left(\frac{\pi}{4}-\frac{\sin\left(\frac{\pi}{2}\right)}{2}\right)-\left(0-\frac{\sin(0)}{2}\right)$	
$=\frac{\pi}{4}-\frac{1}{2}$	• calculates value of the definite integral **[1 mark]**

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Question 18

Integration and applications of integration

A random variable X has a probability density function given by

$$f(x)=\begin{cases} k\sin^{-1}(x), & 0\le x\le 1 \\ 0 \quad , & \text{otherwise} \end{cases}$$

where k is a positive constant.

Determine the value of k.

[6 marks]

Sample response	The response
Method 1 Using a property of a pdf $\int_0^1 k\sin^{-1}(x)\;dx=1 \quad \ldots(1)$	• correctly uses a suitable pdf property **[1 mark]**
Consider $\int k\sin^{-1}(x)\;dx$ Using integration by parts $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$ Let $u=\sin^{-1}(x)$ $\quad \frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$	• correctly determines both results required to progress integration by parts **[1 mark]**
Let $\frac{dv}{dx}=k$ $\quad v=kx$	• uses integration by parts **[1 mark]**
$\int k\sin^{-1}(x)\;dx=kx\sin^{-1}(x)-\int\frac{kx}{\sqrt{1-x^2}}\,dx$ Using substitution for the developed integral $\int\frac{kx}{\sqrt{1-x^2}}\,dx$ Let $u=1-x^2$ $\frac{du}{dx}=-2x$ $\int\frac{kx}{\sqrt{1-x^2}}\,dx=-\frac{k}{2}\int\frac{-2x}{\sqrt{1-x^2}}\,dx$ $=-\frac{k}{2}\int\frac{1}{\sqrt{u}}\,du$	• uses a suitable substitution method to progress the developed integral within the use of integration by parts **[1 mark]**
$\int\frac{kx}{\sqrt{1-x^2}}\,dx=-\frac{k}{2}\int u^{-\frac{1}{2}}\,du$ $=-ku^{\frac{1}{2}}$ $=-k\sqrt{1-x^2}+c$ $\therefore \int k\sin^{-1}x\;dx=kx\sin^{-1}(x)+k\sqrt{1-x^2}+c$ Substituting into (1)	• determines a general result for the required integral **[1 mark]**
$\int_0^1 k\sin^{-1}(x)\;dt=1$ $k\left(x\sin^{-1}(x)+\sqrt{1-x^2}\right)\Big\|_0^1=1$ $k\left(\left(\sin^{-1}(1)+\sqrt{1-1}\right)-\left(0+\sqrt{1-0}\right)\right)=1$ $k\left(\frac{\pi}{2}-1\right)=1$ $k=\frac{2}{\pi-2}$	• determines value of k **[1 mark]**

Method 2	
Using a property of a pdf $\int_0^1 k\sin^{-1}(x)\,dx = 1$ $\int_0^1 \sin^{-1}(x)\,dx = \frac{1}{k}$	• correctly uses a suitable pdf property **[1 mark]**
The area between $y=\sin^{-1}(x)$ and the x-axis for $0\le x\le 1$ is Area 1: $\int_0^1 \sin^{-1}(x)\,dx = \frac{1}{k}$	• correctly represents the area between $y=\sin^{-1}(x)$ and the x-axis for $0\le x\le 1$ **[1 mark]**
Consider the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le \frac{\pi}{2}$ Area $2 = \int_0^{\frac{\pi}{2}} \sin(y)\,dy$	• represents the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le \frac{\pi}{2}$ **[1 mark]**
Area $2 = -\cos(y)\Big\|_0^{\frac{\pi}{2}}$ $= -\cos\left(\frac{\pi}{2}\right)+\cos(0)$ $=1$	• determines the area between $y=\sin^{-1}(x)$ and the y-axis for $0\le y\le \frac{\pi}{2}$ **[1 mark]**
Area 1 + Area 2 = Area of rectangle $\frac{1}{k}+1=\frac{\pi}{2}\times 1$	• determines an equation using the two areas **[1 mark]**
$\frac{1}{k}=\frac{\pi-2}{2}$ $k=\frac{2}{\pi-2}$	• determines value of k **[1 mark]**

2022
Paper 1
Section 2
Question 13

Integration and applications of integration

a) Use partial fractions to determine $\int \frac{22}{(2x-3)(x+4)}\,dx$

[6 marks]

Sample Response	**The response**
Using partial fractions $\frac{22}{(2x-3)(x+4)} = \frac{A}{2x-3} + \frac{B}{x+4}$ $\therefore 22 = A(x+4) + B(2x-3)$	• correctly sets up the partial fractions [1 mark]
$\therefore 22 = A(x+4) + B(2x-3)$ Let $x = \frac{3}{2}$: $A\left(\frac{3}{2}+4\right) = 2 \Rightarrow A = 4$ Let $x = -4$: $B(2 \times -4 - 3) = 22 \Rightarrow B = -2$	• determines value of A [1 mark] • determines value of B [1 mark]
$\int \frac{22}{2x^2+5x-12}\,dx = \int \frac{4}{2x-3}\,dx + \int \frac{-2}{x+4}\,dx$ $= 2\int \frac{2}{2x-3}\,dx - 2\int \frac{1}{x+4}\,dx$ $= 2\ln\lvert 2x-3\rvert - 2\ln\lvert x+4\rvert + c$	• determines an expression for the indefinite integral [1 mark]

b) Use the result from Question 13a) to determine $\int_{-3}^{0} \frac{22}{(2x-3)(x+4)}\,dx$

Express your answer in simplest form.

[2 marks]

Sample Response	**The response**
$\int_{-3}^{0} \frac{22}{(2x-3)(x+4)}\,dx$ $= \left(2\ln\lvert 2x-3\rvert - 2\ln\lvert x+4\rvert\right)\Big\rvert_{-3}^{0}$ $= \left((2\ln\lvert -3\rvert - 2\ln\lvert 4\rvert) - (2\ln\lvert -9\rvert - 2\ln\lvert 1\rvert)\right)$	• substitutes limits of integration into the result from 13a) [1 mark]
$= 2(\ln(3) - \ln(4) - \ln(9)) = 2\ln\left(\frac{3}{9\times 4}\right)$ $= 2\ln\left(\frac{1}{12}\right)$	• expresses a definite integral value in simplest form [1 mark]

2022
Paper 1
Section 2
Question 17

Integration and applications of integration

The region between the x-axis and the curve of the function $y = 1 + \sin(2x)$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the x-axis to form a solid of revolution.

Determine the volume of this solid. Express your answer in simplest form. [5 marks]

Sample Response	The response
Volume can be found using $V = \pi \int_a^b y^2 dx$ $= \pi \int_0^{\frac{\pi}{2}} (1 + \sin(2x))^2 \, dx$	• correctly substitutes into the appropriate volume of a solid of revolution rule [1 mark]
$= \pi \int_0^{\frac{\pi}{2}} 1 + 2\sin(2x) + \sin^2(2x) \, dx$	• expands an integrand [1 mark]
$= \pi \int_0^{\frac{\pi}{2}} 1 + 2\sin(2x) + \frac{1}{2}(1 - \cos(4x)) \, dx$	• uses a suitable double-angle identity to enable an integration process to be completed [1 mark]
$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} + 2\sin(2x) - \frac{1}{2}\cos(4x)\right) dx$ $V = \pi \left(\frac{3}{2}x - \cos(2x) - \frac{1}{8}\sin(4x)\right)\Big\|_0^{\frac{\pi}{2}}$	• integrates an expression [1 mark]
$= \pi \left[\left(\frac{3\pi}{4} - \cos(\pi) - \frac{1}{8}\sin(2\pi)\right) - (0 - \cos(0) - 0)\right]$ $= \pi \left(\frac{3\pi}{4} + 2\right)$ units^3	• determines volume in simplest form [1 mark]

2022
Paper 1
Section 2
Question 19

Integration and applications of integration

The function $f(x)$ passes through the origin.

The gradient function of $f(x)$ is defined as $g(x)=e^x\sin^{-1}(e^x)$.

Determine $f(x)$.

[7 marks]

Sample Response	The response
Method 1 $\boldsymbol{f(x)=\int e^x\sin^{-1}(e^x)\,dx}$ Using integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$ Let $u=\sin^{-1}(e^x)\Rightarrow\frac{du}{dx}=\frac{e^x}{\sqrt{1-e^{2x}}}$ Let $\frac{dv}{dx}=e^x\Rightarrow v=e^x$	• correctly determines the expressions for $\frac{du}{dx}$ and v in preparation for the use of the integration by parts rule [1 mark]
$f(x)=e^x\sin^{-1}(e^x)-\int\frac{e^x}{\sqrt{1-e^{2x}}}.e^x\,dx$ $=e^x\sin^{-1}(e^x)-\int\frac{e^{2x}}{\sqrt{1-e^{2x}}}\,dx$	• applies the integration by parts rule and simplifies an integrand [1 mark]
Using the substitution $u=1-e^{2x}$ $du=-2e^{2x}dx\Rightarrow dx=\frac{du}{-2e^{2x}}$	• determines a suitable substitution variable in preparation for using a substitution method of integration [1 mark]
$f(x)=e^x\sin^{-1}(e^x)-\int\frac{e^{2x}}{\sqrt{u}}\frac{du}{-2e^{2x}}$ $=e^x\sin^{-1}(e^x)+\frac{1}{2}\int\frac{1}{\sqrt{u}}du$	• expresses an integrand in terms of the substitution variable [1 mark]
$=e^x\sin^{-1}(e^x)+\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c$ $=e^x\sin^{-1}(e^x)+\sqrt{1-e^{2x}}+c$	• determines a general solution for $f(x)$ [1 mark]
Substituting (0, 0) to determine the c value $f(0)=e^0\sin^{-1}(e^0)+\sqrt{1-e^0}+c$ $0=\frac{\pi}{2}+c\Rightarrow c=-\frac{\pi}{2}$	• determines a value of the constant of integration and communicates a solution [1 mark]
$f(x)=e^x\sin^{-1}(e^x)+\sqrt{1-e^{2x}}-\frac{\pi}{2}$	• shows logical organisation communicating key steps up to the stage where an integration method using substitution is considered [1 mark]
Method 2 $\boldsymbol{f(x)=\int e^x\sin^{-1}(e^x)\,dx}$ Using the substitution $w=e^x$ $dw=e^x\,dx\Rightarrow dx=\frac{dw}{e^x}$ $\boldsymbol{f(w)=\int\sin^{-1}(w)\,dw}$	• correctly determines the equivalent substitution variable in preparation for using a substitution method of integration [1 mark] • expresses an integrand in terms of the substitution variable [1 mark]
Using integration by parts: $\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx$ Let $u=\sin^{-1}(w)\Rightarrow\frac{du}{dw}=\frac{1}{\sqrt{1-w^2}}$ Let $\frac{dv}{dw}=1\Rightarrow v=w$	•determines expressions for u' and v in preparation for using the integration by parts rule [1 mark]

$f(w) = w\sin^{-1}(w) - \int \frac{w}{\sqrt{1-w^2}}\,dw$ $= w\sin^{-1}(w) - \int w(1-w^2)^{\frac{-1}{2}}\,dw$ $= w\sin^{-1}(w) - \left(-\frac{1}{2}\right)(2)(1-w^2)^{\frac{1}{2}}$ $= w\sin^{-1}(w) + \sqrt{1-w^2} + c$	• applies the integration by parts rule [1 mark]
Using the substitution $w = e^x$ $f(x) = e^x\sin^{-1}(e^x) + \sqrt{1-e^{2x}} + c$	• determines a general solution for $f(x)$ [1 mark]
Substituting (0, 0) to determine the c value $f(0) = e^0\sin^{-1}(e^0) + \sqrt{1-e^0} + c$ $0 = \frac{\pi}{2} + c \Rightarrow c = -\frac{\pi}{2}$ $f(x) = e^x\sin^{-1}(e^x) + \sqrt{1-e^{2x}} - \frac{\pi}{2}$	• determines a value of the constant of integration and communicates a solution [1 mark] • shows logical organisation communicating key steps up to the stage where an integration method using integration by parts is considered [1 mark]

2021
Paper 1
Section 2
Question 15

Integration and applications of integration

Use partial fractions to determine $\int \frac{4x-17}{x^2-x-6}\,dx$, where $x \in R, x \neq -2, x \neq 3$.

Express your answer in the form $\ln\left|f(x)\right| + c$.

[4 marks]

Sample Response	The response
$\frac{4x-17}{x^2-x-6} = \frac{A}{(x+2)} + \frac{B}{(x-3)}$	• correctly factorises the denominator to establish the form of the partial fraction decomposition [1 mark]
$= \frac{A(x-3)+B(x+2)}{(x+2)(x-3)}$ $x = 3: \ -5 = 5B \Rightarrow B = -1$ $x = -2: \ -25 = -5A \Rightarrow A = 5$	• determines values of A and B [1 mark]
$\int \frac{4x-17}{x^2-x-6}\,dx = \int \frac{5}{(x+2)} + \frac{-1}{(x-3)}\,dx$ $= 5\ln\vert x+2\vert - \ln\vert x-3\vert$	• determines indefinite integral of the fraction [1 mark]
$= \ln\vert x+2\vert^5 - \ln\vert x-3\vert$ $= \ln\left\vert\frac{(x+2)^5}{x-3}\right\vert + c$	• determines expression in the form $\ln\vert f(x)\vert$ [1 mark]

2021
Paper 1
Section 2
Question 17

Integration and applications of integration

The area between the graphs of the functions $y = 4x$ and $y = 2x^2$ is rotated about the y-axis to form a solid of revolution with a volume of V units^3.

Determine the exact value of V.

[7 marks]

Sample Response	The response
Finding the points of intersection of the two functions $y = 4x$ and $y = 2x^2$ $4x = 2x^2$	• correctly uses simultaneous equations to establish an equation in one unknown [1 mark]
$2x^2 - 4x = 0 \Rightarrow 2x(x-2) = 0 \Rightarrow x = 0$ and $x = 2$ When $x = 0, y = 0$ When $x = 2, y = 8$	• correctly determines y-coordinates of the points of intersection [1 mark]
Rearranging the two functions in the form $x = f(y)$ and $x = g(y)$ $x = \frac{y}{4}$ and $x = \pm\sqrt{\frac{y}{2}}$	• correctly determines functions in the form $x = f(y)$ [1 mark]
Finding volume of revolution between curves $V = \left\lvert \pi \int_a^b [f(y)]^2 - [g(y)]^2 \, dy \right\rvert$ $= \left\lvert \pi \int_0^8 \frac{y}{2} - \frac{y^2}{16} \, dy \right\rvert$	• determines expression to represent the volume between the two curves [1 mark]
$= \pi \left\lvert \frac{y^2}{4} - \frac{y^3}{48} \right\rvert_0^8$	• integrates expression [1 mark]
$= \pi \left\lvert \left(16 - \frac{32}{3}\right) - (0) \right\rvert$ $= \frac{16\pi}{3}$	• determines (positive) value of V in terms of π [1 mark] • shows logical organisation, communicating key steps to at least the start of finding the volume of revolution [1 mark]

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Paper 1
Section 2
Question 13

Integration and applications of integration

The expected value of an exponential random variable X with parameter λ > 0 can be determined using the rule

$$E(X) = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx$$

Use integration by parts to determine E(X).

Express your answer in simplest form. [4 marks]

Sample Response	**The response**
$E(X) = \int_0^{\infty} x\,\lambda e^{-\lambda x}\,dx$ $\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$ $u = x \quad \frac{du}{dx} = 1$ $\frac{dv}{dx} = \lambda e^{-\lambda x} \quad v = -e^{-\lambda x}$	• correctly determines $\frac{du}{dx}$ and v [1 mark]
$E(X) = -xe^{-\lambda x}\Big\vert_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$	• substitutes into the integration by parts rule [1 mark]
$= 0 + \int_0^{\infty} e^{-\lambda x}\,dx$	• calculates $-xe^{-\lambda x}\Big\vert_0^{\infty}$ to equal 0 **[1 mark]**
$= \frac{e^{-\lambda x}}{-\lambda}\Big\vert_0^{\infty}$ $= 0 - \frac{1}{-\lambda}$ $= \frac{1}{\lambda}$	• shows that $E(X) = \frac{1}{\lambda}$ **[1 mark]**

2020
Paper 1
Section 2
Question 17

Integration and applications of integration

Determine the smallest positive value of a given

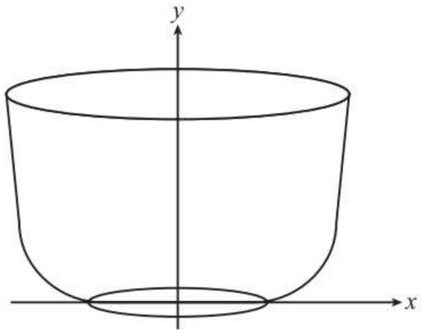
$$\int_{-a}^{a} 1+\left(\frac{\sec(2x)+\tan(2x)}{\operatorname{cosec}(2x)+1}\right)^2 dx=1$$

Sample Response	The response
$\int_{-a}^{a} 1+\left(\frac{\sec(2x)+\tan(2x)}{\operatorname{cosec}(2x)+1}\right)^2 dx=1$ $\int_{-a}^{a} 1+\left(\frac{\frac{1}{\cos(2x)}+\frac{\sin(2x)}{\cos(2x)}}{\frac{1}{\sin(2x)}+1}\right)^2 dx=1$	• correctly expresses the expression inside the brackets of the integrand in terms of $\sin(2x)$and $\cos(2x)$ [1 mark]
$\int_{-a}^{a} 1+\left(\frac{\frac{1+\sin(2x)}{\cos(2x)}}{\frac{1+\sin(2x)}{\sin(2x)}}\right)^2 dx=1$ $\int_{-a}^{a} 1+\left(\frac{\sin(2x)}{\cos(2x)}\right)^2 dx=1$ $\int_{-a}^{a} 1+(\tan(2x))^2\,dx=1$	• correctly simplifies the expression inside the brackets of the integrand [1 mark]
$\int_{-a}^{a} \sec^2(2x)dx=1$	• uses a suitable Pythagorean identity to express the integrand as a single trigonometric expression [1 mark]
$\frac{1}{2}\tan(2x)\Big\vert_{-a}^{a}=1$	• determines the definite integral equation [1 mark]
$\frac{1}{2}(\tan(2a)-\tan(-2a))=1$ $\frac{1}{2}(\tan(2a)+\tan(2a))=1$	• substitutes limits of integration [1 mark]
$\frac{1}{2}(2\tan(2a))=1 \Rightarrow 2a=\tan^{-1}(1)$ $2a=\frac{\pi}{4}\Rightarrow a=\frac{\pi}{8}$	• solves equation to determine the smallest positive value of a [1 mark] • shows logical organisation communicating key steps [1 mark]

2020
Paper 1
Section 2
Question 19

Integration and applications of integration

A circular-based bowl has been positioned symmetrically on a Cartesian plane as shown in the diagram.



The bowl has a shape that can be generated by rotating the curve $y=\dfrac{4}{8-x}-1$ about the y-axis for $4 \le x \le 7.6$ cm.

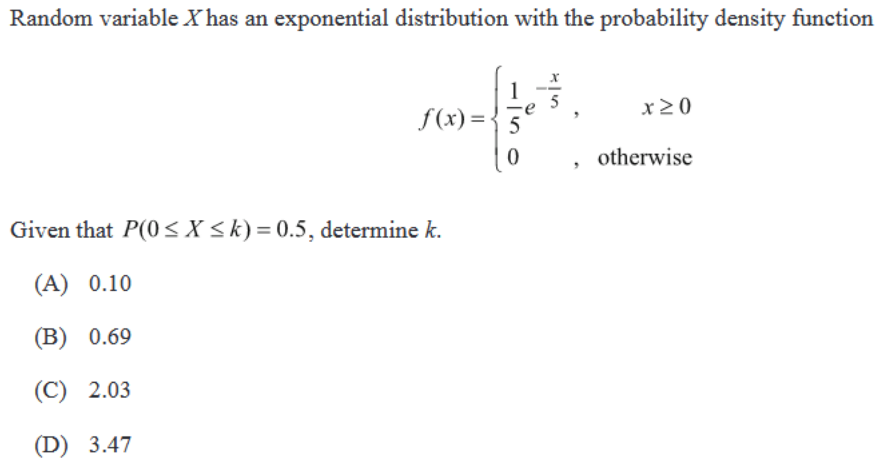
The bowl is being filled with a liquid at the rate of 7π cm^3 s^{-1}.

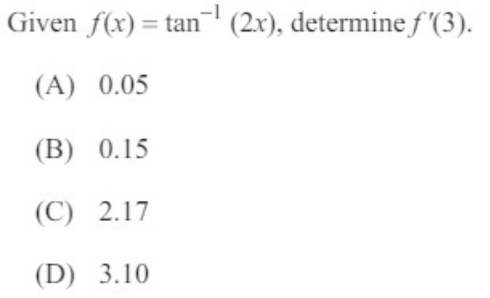
Determine the rate at which the depth of liquid is increasing when the depth of liquid reaches one-third of the height of the bowl.

[7 marks]

Sample Response	The response
Volume of revolution $y=\frac{4}{8-x}-1 \Rightarrow 8-x=\frac{4}{y+1}$ $x=8-\frac{4}{y+1}$	• correctly expresses x as the subject of the given relationship [1 mark]
$V=\pi\int_a^b [f(y)]^2\,dy=\pi\int_0^h\left(8-\frac{4}{y+1}\right)^2 dy$	• establishes volume of bowl as a definite integral [1 mark]
$\frac{dV}{dh}=\pi\left(8-\frac{4}{h+1}\right)^2$	• determines an expression for $\frac{dV}{dh}$[1 mark]
Related rate of change $\frac{dh}{dt}=\frac{dh}{dV}\frac{dV}{dt}\Rightarrow\frac{dh}{dt}=\frac{1}{\frac{dV}{dh}}\frac{dV}{dt}$ $\frac{dh}{dt}=\frac{1}{\pi\left(8-\frac{4}{h+1}\right)^2}\times 7\pi$	• uses a related rate of change equation to determine a relationship between $\frac{dh}{dt}$ and h [1 mark]
$=\frac{7}{\left(8-\frac{4}{h+1}\right)^2}$ Required depth $=\frac{1}{3}\times f(7.6)=\frac{1}{3}\times 9=3$ cm	• correctly determines the instantaneous depth of liquid for the required calculation [1 mark]
$\therefore \left.\frac{dh}{dt}\right\|_{h=3}=\frac{7}{\left(8-\frac{4}{3+1}\right)^2}=\frac{1}{7}$ cm s^{-1}	• determines the required rate [1 mark] • shows logical organisation communicating key steps [1 mark]

Marking Guide – Paper 2 Section 1

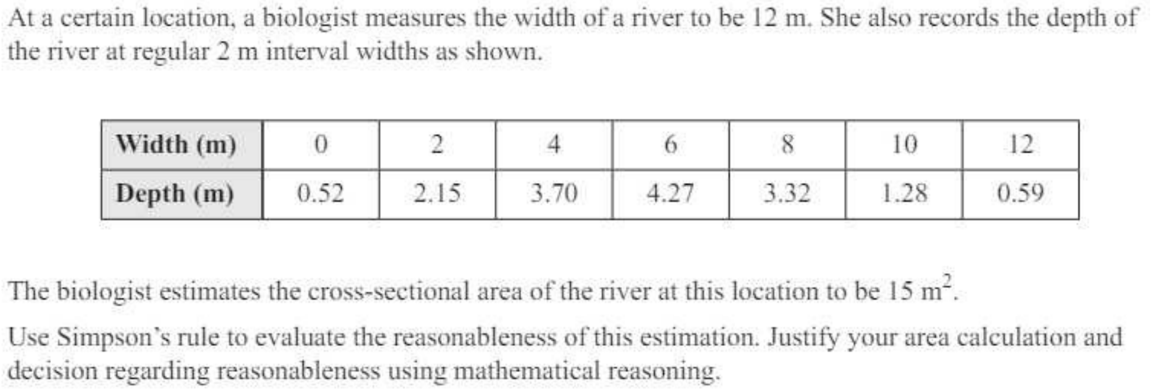
2024 Paper 2 Section 1 Question 9 Integration and applications of integration	 Random variable X has an exponential distribution with the probability density function $f(x)=\begin{cases}\frac{1}{5}e^{-\frac{x}{5}}, & x\geq 0\\ 0, & \text{otherwise}\end{cases}$ Given that $P(0\leq X\leq k)=0.5$, determine k. (A) 0.10 (B) 0.69 (C) 2.03 (D) 3.47 **Answer is D.**

2023 Paper 2 Section 1 Question 8 Integration and applications of integration	 Given $f(x)=\tan^{-1}(2x)$, determine $f'(3)$. (A) 0.05 (B) 0.15 (C) 2.17 (D) 3.10 **Answer is A.**

Marking Guide – Paper 2 Section 2

2023
Paper 2
Section 2
Question 14

Integration and applications of integration



At a certain location, a biologist measures the width of a river to be 12 m. She also records the depth of the river at regular 2 m interval widths as shown.

Width (m)	0	2	4	6	8	10	12
Depth (m)	0.52	2.15	3.70	4.27	3.32	1.28	0.59

The biologist estimates the cross-sectional area of the river at this location to be 15 m^2.

Use Simpson's rule to evaluate the reasonableness of this estimation. Justify your area calculation and decision regarding reasonableness using mathematical reasoning.

(4 marks)

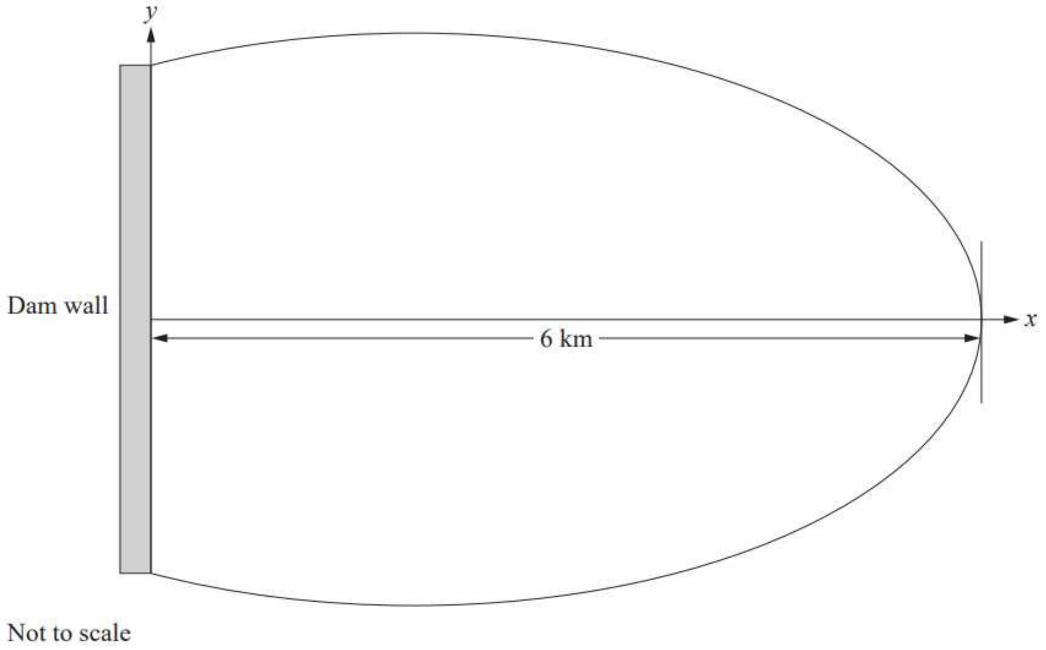
Sample response	The response
Use Simpson's rule to estimate the cross-sectional area of the river. Using the given data: $w = 2$.	- correctly identifies the interval width **[1 mark]**
$\text{Area} \approx \frac{w}{3}\left(y_0 + 4(y_1 + y_3 + ...) + 2(y_2 + y_4 + ...) + y_n\right)$ $\approx \frac{2}{3}(0.52 + 4(2.15 + 4.27 + 1.28) + ...$ $2(3.70 + 3.32) + 0.59)$	- justifies the area calculation by substituting the depth values of the data into Simpson's rule **[1 mark]**
$\text{Area} \approx 30.63\,\text{m}^2$ The estimate of the area is less than half of the value obtained using Simpson's rule, so it is not reasonable.	- calculates area **[1 mark]** - states and justifies a decision regarding the reasonableness of the estimation using mathematical reasoning **[1 mark]**

2022
Paper 2
Section 2
Question 11

Integration and applications of integration

An aerial view of the surface of a dam, 6 km in length, is symmetrically positioned on a Cartesian plane as shown. A dam wall is located along the y-axis.

The surface edge of the dam can be modelled by the ellipse $\frac{(x-2)^2}{16}+\frac{y^2}{9}=1$, for $0 \le x \le 6$.



a) Use Simpson's rule with four strips to determine an approximate area of the surface of the dam. [4 marks]

<table>
<tr><th>Sample Response</th><th>The response</th></tr>
<tr><td>$w=\frac{b-a}{n}=\frac{6}{4}=1.5$</td><td>• correctly determines the width of each interval [1 mark]</td></tr>
<tr><td>Considering values in Quadrant 1:
$\frac{(x-2)^2}{16}+\frac{y^2}{9}=1 \Rightarrow y=\sqrt{9\left(1-\frac{(x-2)^2}{16}\right)}$
Table of values:
<table>
<tr><th>x</th><th>y</th></tr>
<tr><td>0</td><td>2.598</td></tr>
<tr><td>1.5</td><td>2.976</td></tr>
<tr><td>3</td><td>2.905</td></tr>
<tr><td>4.5</td><td>2.342</td></tr>
<tr><td>6</td><td>0</td></tr>
</table>
</td><td>• correctly determines the 5 required y-values [1 mark]</td></tr>
<tr><td>Determining the area in the upper half of the ellipse
$A \approx \frac{w}{3}[f(x_0)+4[f(x_1)+f(x_3)+\cdots] + 2[f(x_2)+f(x_4)+\cdots]+f(x_n)]$
$\approx \frac{1.5}{3}(2.598+0+4(2.976+2.342)+2\times 2.905)$
$\approx 14.84 \text{ km}^2$</td><td>• substitutes y-values into Simpson's rule [1 mark]</td></tr>
<tr><td>Using symmetry of the ellipse
Required area $\approx 2 \times 14.84$
$\approx 29.68 \text{ km}^2$</td><td>• determines an approximate area [1 mark]</td></tr>
</table>

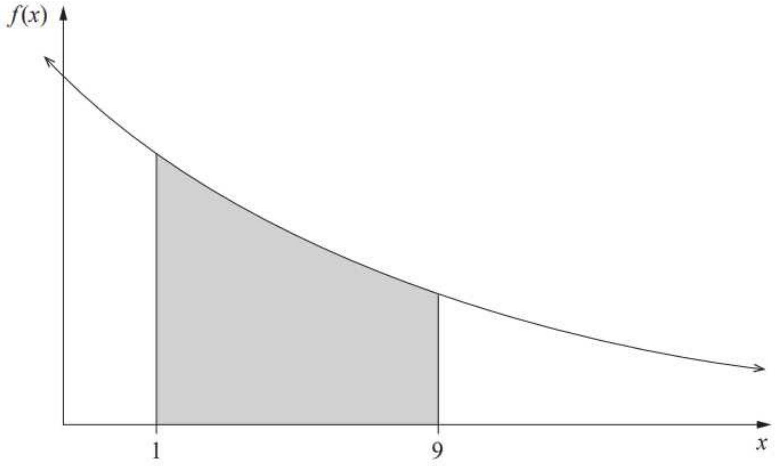
b) Evaluate the reasonableness of this approximation. [2 marks]

Sample Response	The response
$\text{Area} = 2\int_0^6 \sqrt{9\left(1 - \frac{(x-2)^2}{16}\right)}\,dx$	• uses a suitable mathematical representation to communicate the approach [1 mark]
Using integration facility of GDC, Area ≈ $\mathbf{30.32}\ \text{km}^2$ The approximation is reasonable as the error is only around 2%.	• determines area and evaluates the reasonableness of the approximation [1 mark]

2021
Paper 2
Section 2
Question 13

Integration and applications of integration

The area under the graph of the function $f(x) = 0.2e^{-0.2x}$ for $1 \le x \le 9$ is shaded.



Not to scale

a) Use Simpson's rule with four intervals to determine an approximation for this area. [4 marks]

Sample Response	The response
Given $a = 1,\ b = 9,\ n = 4$ $w = \frac{b-a}{n} = 2$	• correctly determines the interval width required for the calculations [1 mark]
Table of values x $f(x) = 0.2e^{-0.2x}$ 1 $f(1) = 0.2e^{-0.2} = 0.164$ 3 $f(3) = 0.2e^{-0.6} = 0.110$ 5 $f(5) = 0.2e^{-1} = 0.074$ 7 $f(7) = 0.2e^{-1.4} = 0.049$ 9 $f(9) = 0.2e^{-1.8} = 0.033$	• correctly determines all required values of $f(x)$ [1 mark]
Using Simpson's rule: Area ≈ $\frac{2}{3}\big(f(1) + 4(f(3) + f(7)) + 2(f(5)) + f(9)\big)$	• substitutes values into Simpson's rule [1 mark]
$\approx \frac{2}{3}(0.9803)$ $\approx 0.65\ \text{units}^2$	• approximates the area using Simpson's rule [1 mark]

b) Use a calculus approach to evaluate the reasonableness of your area approximation from Question 13a). [2 marks]

Sample Response	The response
$\text{Area} = \int_1^9 0.2e^{-0.2x}\ dx$	• correctly expresses the exact area as a definite integral [1 mark]
$\approx 0.65\ \text{units}^2$ The approximation is reasonable.	• calculates the value of the definite integral and provides a suitable comment on the reasonableness of the response [1 mark]

2020
Paper 2
Section 2
Question 14

Integration and applications of integration

The time, *t*, (months) that it takes before a phone owner cracks the screen on their phone can be modelled by an exponentially distributed random variable

$$f(t)=\begin{cases}0.16e^{-0.16t}, & t\geq 0\\ 0\quad , & \text{otherwise}\end{cases}$$

a) Show that $f(t)$ is a probability density function. [1 mark]

Sample Response	The response
Using integration facility of GDC $\int_0^{\infty} 0.16e^{-0.16t}dt=1$	• correctly substitutes the given information into a definite integral to show that $f(t)$ is a probability density function [1 mark]

b) Determine the probability that a phone owner cracks the screen on their phone within 1 year. [2 marks]

Sample Response	The response
1 year = 12 months $P(0<x<12)=\int_0^{12} 0.16e^{-0.16t}dt$	• correctly represents the required probability as a definite integral [1 mark]
Using integration facility of GDC ≈ 0.85	• determines the probability [1 mark]

Three-quarters of phone owners take between 1 and *m* months before they crack the screen on their phone.

c) Determine the value of *m*. [2 marks]

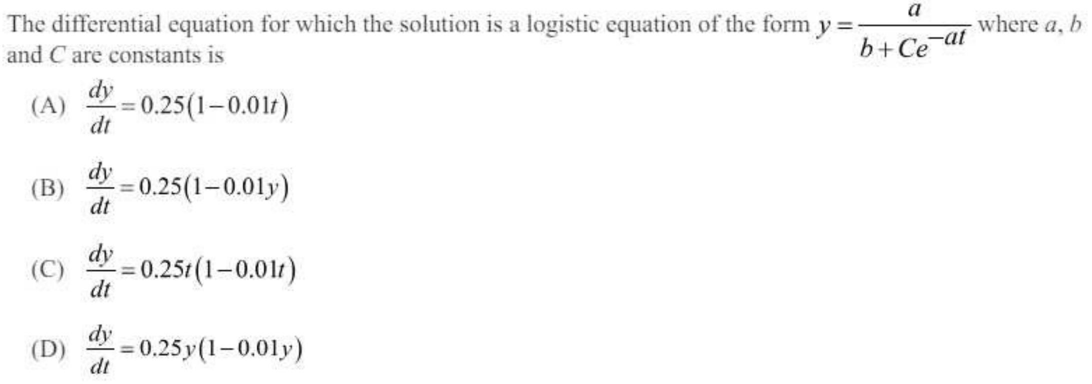
Sample Response	The response
$P(1<x<m)=0.75$ $\int_1^{m} 0.16e^{-0.16t}\ dt=0.75$	• correctly establishes definite integral equation [1 mark]
Using solve facility of GDC or otherwise $m\approx 14.26$ months	• solves equation to determine m [1 mark]

Unit 4 – Topic 2: Rates of change and differential equations

Paper 1 Section 1

2023
Paper 1
Section 1
Question 7

Rates of change and differential equations



The differential equation for which the solution is a logistic equation of the form $y = \dfrac{a}{b + Ce^{-at}}$ where a, b and C are constants is

(A) $\dfrac{dy}{dt} = 0.25(1 - 0.01t)$

(B) $\dfrac{dy}{dt} = 0.25(1 - 0.01y)$

(C) $\dfrac{dy}{dt} = 0.25t(1 - 0.01t)$

(D) $\dfrac{dy}{dt} = 0.25y(1 - 0.01y)$

2022
Paper 1
Section 1
Question 3

Rates of change and differential equations

A particle travels in a straight line over time, t, with a constant acceleration, $a(t)$.

Which function could represent the particle's displacement, $x(t)$?

(A) $x(t) = t^3$

(B) $x(t) = t^2$

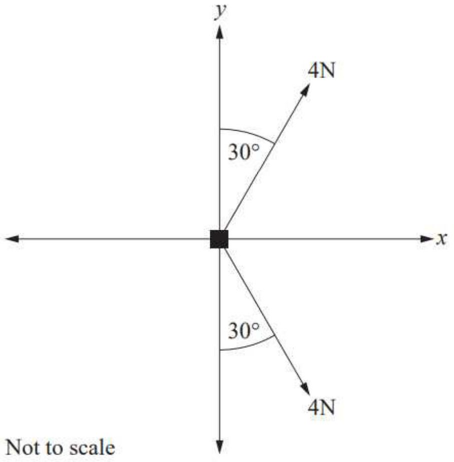
(C) $x(t) = \dfrac{1}{t}$

(D) $x(t) = \sqrt{t}$

2022
Paper 1
Section 1
Question 7

Rates of change and differential equations

Two forces act concurrently on a 2 kg object placed at the origin.



The magnitude of the acceleration of the object is

(A) $2\ \text{m s}^{-2}$

(B) $2\sqrt{3}\ \text{m s}^{-2}$

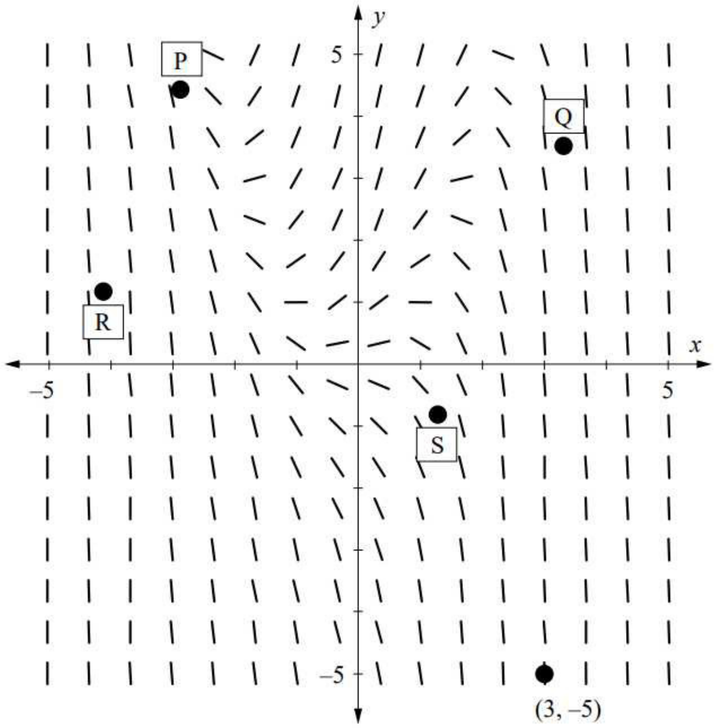
(C) $4\ \text{m s}^{-2}$

(D) $4\sqrt{3}\ \text{m s}^{-2}$

2021
Paper 1
Section 1
Question 9

Rates of change and differential equations

The slope field for the differential equation $\frac{dy}{dx} = y - x^2$ is shown.



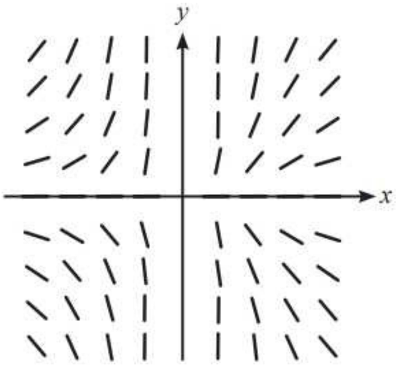
The solution curve to the differential equation that passes through the point $(3,-5)$ also passes through point

(A) P
(B) Q
(C) R
(D) S

2020
Paper 1
Section 1
Question 7

Rates of change and differential equations

The diagram shows a slope field.



The differential equation represented by the slope field is

(A) $\dfrac{dy}{dx} = \dfrac{5y}{x}$

(B) $\dfrac{dy}{dx} = \dfrac{5y^2}{x}$

(C) $\dfrac{dy}{dx} = \dfrac{5y}{x^2}$

(D) $\dfrac{dy}{dx} = \dfrac{5y^2}{x^2}$

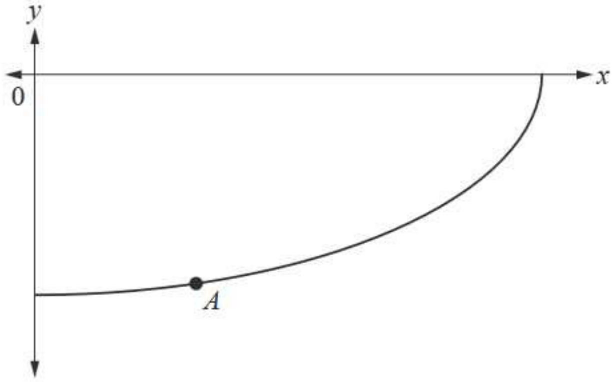
Paper 1 Section 2

2024
Paper 1
Section 2
Question 12

Rates of change and differential equations

Point A lies on a section of the ellipse $3x^2 + y^2 = 10$ as shown.

The coordinates of A are $\left(\sqrt{2}, y_1\right)$.



a) Determine the value of y_1. *[2 marks]*

b) Use implicit differentiation to determine an expression for $\frac{dy}{dx}$ in terms of x and y. *[2 marks]*

c) Use your results from Questions 12a) and 12b) to determine the gradient of the tangent to the curve at A. *[1 mark]*

Consider the region between the given section of the ellipse, the x-axis and the lines $x = 0$ and $x = \sqrt{2}$.

d) Determine the volume of the solid of revolution formed by rotating this region about the x-axis. Express your answer in simplest form. *[2 marks]*

2023
Paper 1
Section 2
Question 17

Rates of change and differential equations

An object of mass 2 kg is moving with a constant velocity ($\mathrm{m\,s^{-1}}$) of $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$.

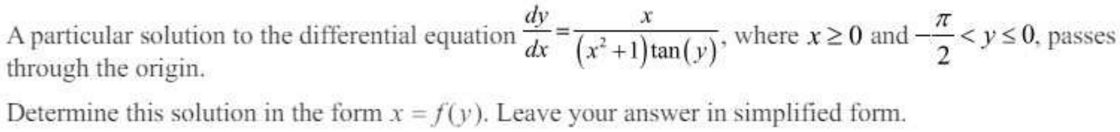
At an instant, two forces (N), $\boldsymbol{F}_1 = 5t\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{F}_2 = -t\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, act simultaneously on the object for t seconds, where $0 \le t \le 2$.

Determine the magnitude of the momentum of the object when $t = 1$.

(7 marks)

2023
Paper 1
Section 2
Question 18

Rates of change and differential equations



A particular solution to the differential equation $\frac{dy}{dx}=\frac{x}{\left(x^2+1\right)\tan\left(y\right)}$, where $x \geq 0$ and $-\frac{\pi}{2} < y \leq 0$, passes through the origin.

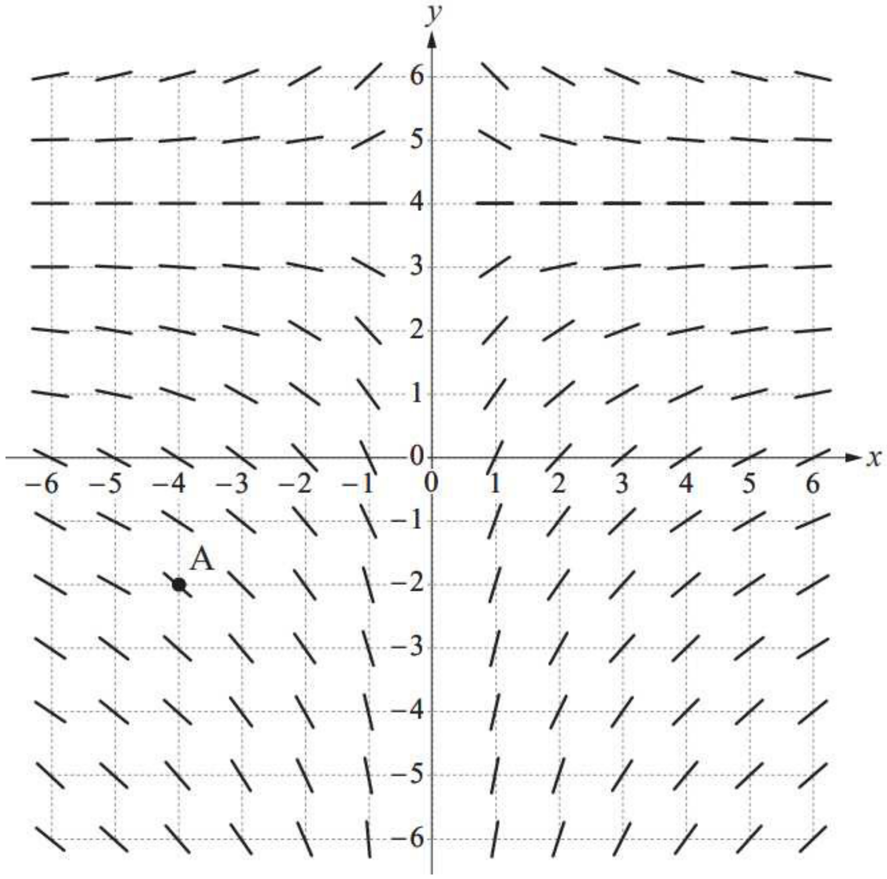
Determine this solution in the form $x = f(y)$. Leave your answer in simplified form.

(6 marks)

2022
Paper 1
Section 2
Question 14

Rates of change and differential equations

The slope field for the differential equation $\frac{dy}{dx} = \frac{-0.5(y-4)}{x}$, $x \neq 0$ using $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$ is shown.



a) Determine the value of the slope at point A. [2 marks]

b) Use the slope field to sketch the solution curve for $\frac{dy}{dx} = \frac{-0.5(y-4)}{x}$ given that when $x = -6$, $y = 3.5$

[2 marks]

2021 **Paper 1** **Section 2** **Question 11** **Rates of change and differential equations**	Let $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$ for suitable values of x where $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. a) Determine $f(2)$. [1 mark] a) Determine $f'(2)$. [1 mark] c) Use the results from Questions 11a) and 11b) to determine the equation of the tangent to the graph of $y = f(x)$ at $x = 2$. [2 marks]

2021
Paper 1
Section 2
Question 18

Rates of change and differential equations

This differential equation can be used to determine the current I (amperes) at time t (seconds) with voltage V (volts) in an electric circuit containing a resistance R (ohms):

$$k\frac{dI}{dt} + RI = V$$

where k, R and V are positive constants and $t \geq 0$.

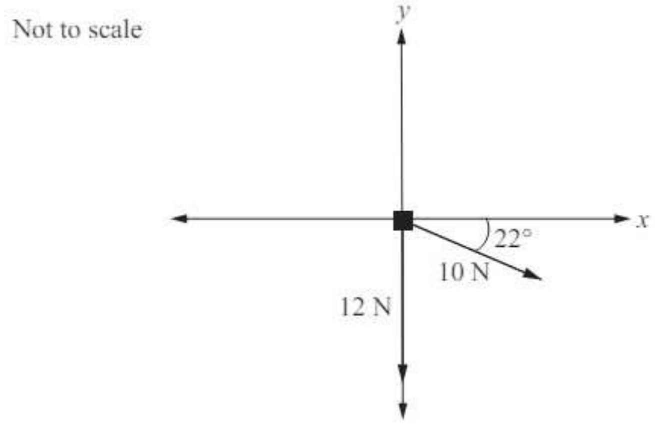
Assuming that there is no current in the electric circuit initially, show that the size of the current can never be greater than $\frac{V}{R}$. [6 marks]

2020 **Paper 1** **Section 2** **Question 14** **Rates of change and differential equations**	The motion of an object that moves in a straight line is given by $v(x) = \cos^{-1}(2x)$ where v is the velocity (m s^{-1}) and x is the displacement (m) from the origin. a) Determine $a(x)$ where a is the acceleration (m s^{-2}) of the object. [2 marks] b) Use the result from 14a) to determine $a(0)$, given $-2\pi \le a(0) \le 0$. Express your answer in simplest form. [2 marks]

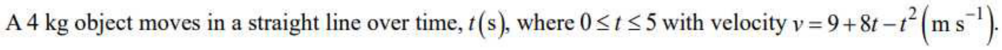
Paper 2 Section 1

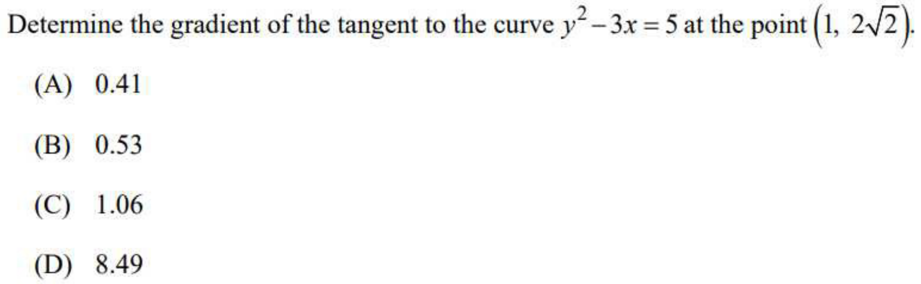
2024 Paper 2 Section 1 Question 6 Rates of change and differential equations	Two concurrent forces represented in the polar form of $\boldsymbol{F}_1 = (1.21\text{ N}, 120°)$ and $\boldsymbol{F}_2 = (1.30\text{ N}, -160°)$ act on an object. Determine the magnitude of the resultant force. (A) 0.50 N (B) 1.92 N (C) 2.51 N (D) 3.70 N

2023 Paper 2 Section 1 Question 1 Rates of change and differential equations	The acceleration (m s^{-2}) of an object moving with simple harmonic motion is modelled by $a = -2.95x$, where x is its displacement (m) from the origin. Determine the period of the motion in seconds. (A) 1.72 (B) 2.13 (C) 2.95 (D) 3.66

2023 Paper 2 Section 1 Question 6 Rates of change and differential equations	Two coplanar forces of magnitudes 12 N and 10 N act on an object in the directions shown.  Determine the magnitude of the resultant force acting on the object. (A) 12.41 N (B) 15.55 N (C) 15.69 N (D) 18.27 N

2022 Paper 2 Section 1 Question 3 Rates of change and differential equations	Determine the solution of the differential equation $\frac{dy}{dx}=\frac{\sin(2x)}{\cos(2x)}$ given $y=0$ when $x=\frac{\pi}{5}$. (A) $y=-2\ln\lvert\cos(2x)\rvert-2.35$ (B) $y=-2\ln\lvert\cos(2x)\rvert+2.35$ (C) $y=-\frac{1}{2}\ln\lvert\cos(2x)\rvert-0.59$ (D) $y=-\frac{1}{2}\ln\lvert\cos(2x)\rvert+0.59$

2022 **Paper 2** **Section 1** **Question 6** **Rates of change and differential equations**	 A 4 kg object moves in a straight line over time, $t\,(\mathrm{s})$, where $0 \le t \le 5$ with velocity $v = 9 + 8t - t^2\ (\mathrm{m\ s^{-1}})$. Determine the momentum of the object when $t = 3$. (A) 24 kg m s^{-1} (B) 27 kg m s^{-1} (C) 96 kg m s^{-1} (D) 100 kg m s^{-1}

2022 **Paper 2** **Section 1** **Question 8** **Rates of change and differential equations**	 Determine the gradient of the tangent to the curve $y^2 - 3x = 5$ at the point $\left(1,\ 2\sqrt{2}\right)$. (A) 0.41 (B) 0.53 (C) 1.06 (D) 8.49

2021
Paper 2
Section 1
Question 9

Rates of change and differential equations

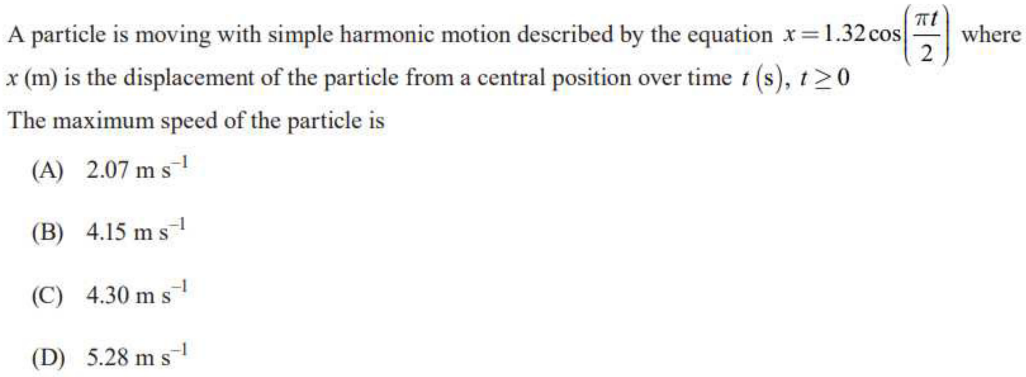
Two vertical forces act on a skydiver with a mass of 85 kg, as shown.

When the magnitude of the air resistance is 62 N, the magnitude of the acceleration of the skydiver is

(A) 0.73 m s^{-2}
(B) 2.65 m s^{-2}
(C) 9.07 m s^{-2}
(D) 12.44 m s^{-2}

2020
Paper 2
Section 1
Question 4

Rates of change and differential equations



A particle is moving with simple harmonic motion described by the equation $x = 1.32\cos\left(\frac{\pi t}{2}\right)$ where x (m) is the displacement of the particle from a central position over time $t\,(\mathrm{s})$, $t \geq 0$

The maximum speed of the particle is

(A) 2.07 m s^{-1}

(B) 4.15 m s^{-1}

(C) 4.30 m s^{-1}

(D) 5.28 m s^{-1}

2020
Paper 2
Section 1
Question 5

Rates of change and differential equations

The gradient of the tangent at point A on the curve $y^2 = 4x$ is 1.36

The x-coordinate of A is

(A) 0.12
(B) 0.46
(C) 0.54
(D) 1.47

2024
Paper 2
Section 2
Question 13

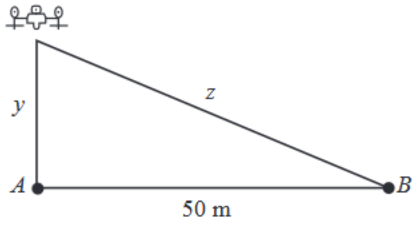
Rates of change and differential equations

A drone travels vertically from point A at a constant speed of 8 m s^{-1} over time t for $t \geq 0$ seconds.

Observation of the drone is made from point B, which is 50 m horizontally from point A.

When the drone is y metres above point A, it is z metres from point B as shown.

Not to scale



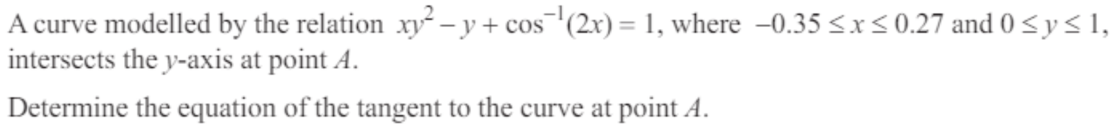
a) Determine an equation expressing z^2 in terms of y^2. *[1 mark]*

b) State the value of $\frac{dy}{dt}$. *[1 mark]*

c) Use your results from Questions 13a) and 13b) to determine the rate at which z is increasing with respect to time when the drone is 20 metres above point A. *[3 marks]*

2023
Paper 2
Section 2
Question 16

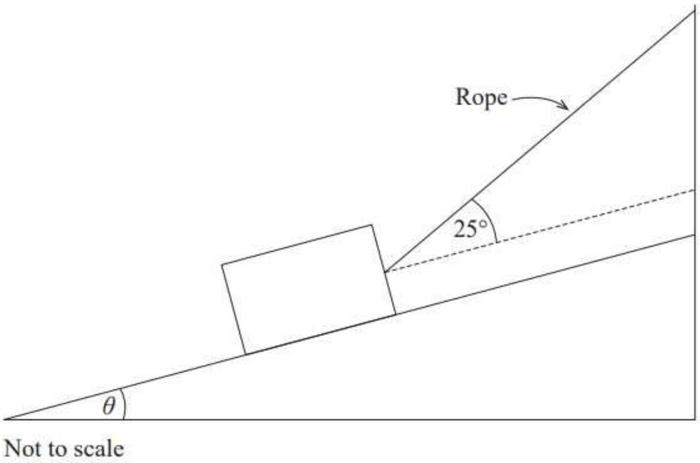
Rates of change and differential equations



A curve modelled by the relation $xy^2 - y + \cos^{-1}(2x) = 1$, where $-0.35 \leq x \leq 0.27$ and $0 \leq y \leq 1$, intersects the y-axis at point A.

Determine the equation of the tangent to the curve at point A.

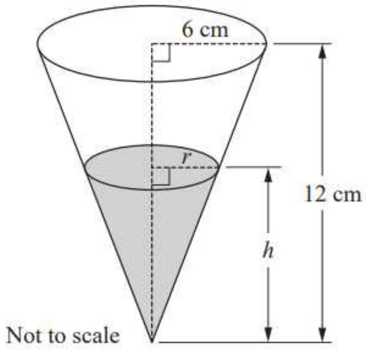
(6 marks)

2022 Paper 2 Section 2 Question 16 Rates of change and differential equations	An object with a mass of 12 kg lies on a frictionless inclined plane. A rope is attached to the object at an angle of 25° above the plane, as shown.  The force of the rope, T N, prevents the object from moving. When the rope is detached, the object moves down the plane with an acceleration of 5.6 m s^{-2}. Determine the magnitude of T. [6 marks]

2021
Paper 2
Section 2
Question 15

Rates of change and differential equations

Water is poured into a cone-shaped cup at a rate of 2 $\text{cm}^3\ \text{s}^{-1}$. The cup has a height of 12 cm and a radius of 6 cm, as shown.



Not to scale

As the cup fills, the ratio of the height of the water h to the surface radius of the water r remains constant.

a) Given that $h = 2r$, determine a function for the volume of water in the cup, V, in terms of h.

Express your answer in simplified form. [1 mark]

b) Use the results from Question 15a) to show that the rate at which the height of water in the cup is increasing with respect to time is given by $\frac{8}{\pi r^2}$. [3 marks]

	c) Determine the rate at which the height of water in the cup is increasing with respect to time when the volume of water in the cup reaches half of the total capacity of the cup. [4 marks]

2020
Paper 2
Section 2
Question 12

Rates of change and differential equations

For a certain experiment, the number of yeast cells, N, after t hours in a test tube can be modelled by the differential equation

$$\frac{dN}{dt}=\frac{1}{1000}N\left(1000-N\right) \text{ for } t\geq 0$$

a) Given $\dfrac{1000}{N\left(1000-N\right)}=\dfrac{1}{N}+\dfrac{1}{1000-N}$, show that the general solution of the differential equation can be expressed as

$$\ln\left|\frac{N}{1000-N}\right|=t+c$$

[2 marks]

A scientist commenced this experiment at 9:00 am on a certain day and observed that 100 yeast cells were present at this time.

b) Show that the solution of the differential equation can be expressed as

$$N = \frac{1000}{1+9e^{-t}}$$

[3 marks]

c) Determine the time of day when 900 yeast cells were present. [2 marks]

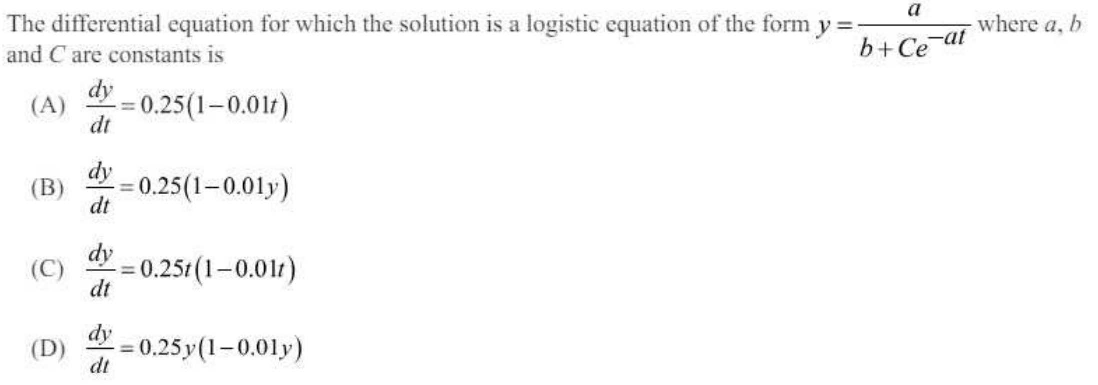
The scientist predicted that the number of yeast cells would eventually exceed 1200.

d) Evaluate the reasonableness of the scientist's prediction. [2 marks]

Marking Guide – Paper 1 Section 1

2023
Paper 1
Section 1
Question 7

Rates of change and differential equations



The differential equation for which the solution is a logistic equation of the form $y=\frac{a}{b+Ce^{-at}}$ where a, b and C are constants is

(A) $\frac{dy}{dt}=0.25(1-0.01t)$

(B) $\frac{dy}{dt}=0.25(1-0.01y)$

(C) $\frac{dy}{dt}=0.25t(1-0.01t)$

(D) $\frac{dy}{dt}=0.25y(1-0.01y)$

Answer is D.

2022
Paper 1
Section 1
Question 3

Rates of change and differential equations

A particle travels in a straight line over time, *t*, with a constant acceleration, *a* (*t*).

Which function could represent the particle's displacement, *x* (*t*)?

(A) $x(t)=t^3$

(B) $x(t)=t^2$

(C) $x(t)=\frac{1}{t}$

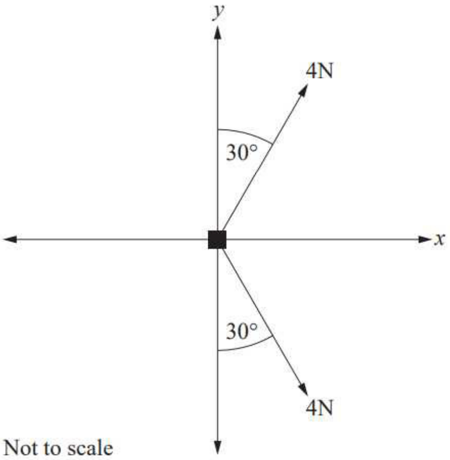
(D) $x(t)=\sqrt{t}$

Answer is B.

2022
Paper 1
Section 1
Question 7

Rates of change and differential equations

Two forces act concurrently on a 2 kg object placed at the origin.



The magnitude of the acceleration of the object is

(A) 2 m s^{-2}

(B) $2\sqrt{3} \text{ m s}^{-2}$

(C) 4 m s^{-2}

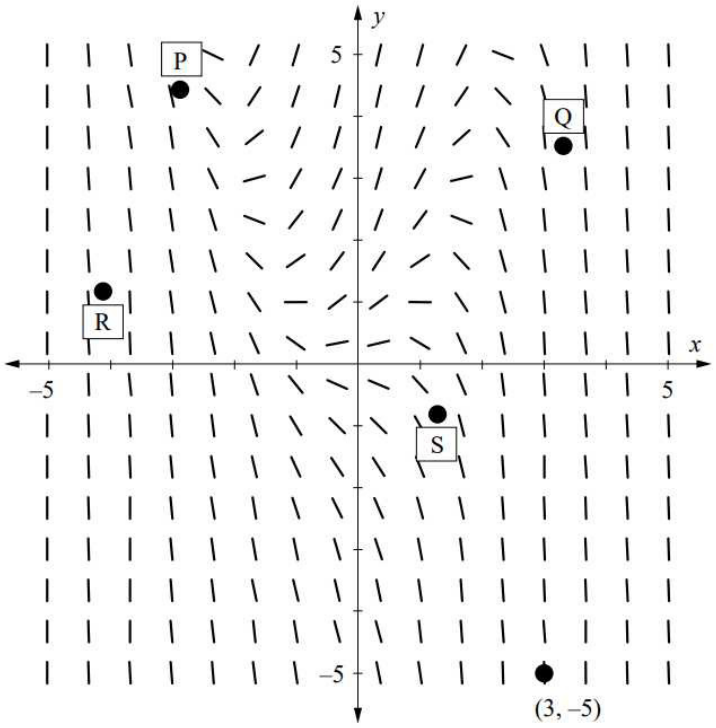
(D) $4\sqrt{3} \text{ m s}^{-2}$

Answer is A.

2021
Paper 1
Section 1
Question 9

Rates of change and differential equations

The slope field for the differential equation $\frac{dy}{dx} = y - x^2$ is shown.



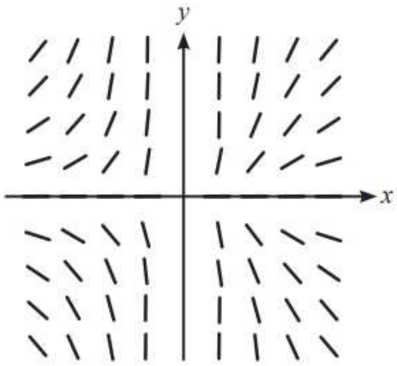
The solution curve to the differential equation that passes through the point $(3, -5)$ also passes through point

(A) P – Answer
(B) Q
(C) R
(D) S

2020
Paper 1
Section 1
Question 7

Rates of change and differential equations

The diagram shows a slope field.



The differential equation represented by the slope field is

(A) $\dfrac{dy}{dx}=\dfrac{5y}{x}$

(B) $\dfrac{dy}{dx}=\dfrac{5y^2}{x}$

(C) $\dfrac{dy}{dx}=\dfrac{5y}{x^2}$

(D) $\dfrac{dy}{dx}=\dfrac{5y^2}{x^2}$

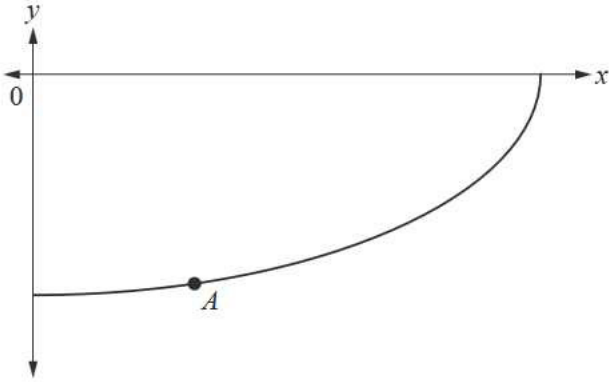
Answer is C.

2024
Paper 1
Section 2
Question 12

Rates of change and differential equations

Point A lies on a section of the ellipse $3x^2 + y^2 = 10$ as shown.

The coordinates of A are $\left(\sqrt{2}, y_1\right)$.



a) Determine the value of y_1. *[2 marks]*

Sample response	The response
Given A lies on $3x^2 + y^2 = 10$ at $x = \sqrt{2}$ $3\left(\sqrt{2}\right)^2 + {y_1}^2 = 10$ ${y_1}^2 = 4$ Given A lies in quadrant 4 $y_1 = -2$	• correctly substitutes into equation **[1 mark]** • evaluates reasonableness of solution to determine y-coordinate of A **[1 mark]**

b) Use implicit differentiation to determine an expression for $\dfrac{dy}{dx}$ in terms of x and y. *[2 marks]*

Sample response	The response
$3x^2 + y^2 = 10$ $6x + 2y\dfrac{dy}{dx} = 0$ $\dfrac{dy}{dx} = -\dfrac{3x}{y} \ (y \neq 0)$	• correctly uses implicit differentiation **[1 mark]** • expresses $\dfrac{dy}{dx}$ in terms of x and y **[1 mark]**

c) Use your results from Questions 12a) and 12b) to determine the gradient of the tangent to the curve at A. *[1 mark]*

Sample response	The response
At A, gradient of tangent is $\dfrac{dy}{dx} = \dfrac{3\sqrt{2}}{2}$	• determines gradient of tangent at A **[1 mark]**

Consider the region between the given section of the ellipse, the x-axis and the lines $x = 0$ and $x = \sqrt{2}$.

d) Determine the volume of the solid of revolution formed by rotating this region about the x-axis. Express your answer in simplest form. *[2 marks]*

Sample response	The response
Volume $= \pi \int_a^b [f(x)]^2 \, dx$ $= \pi \int_0^{\sqrt{2}} (10 - 3x^2) \, dx$	• correctly determines a definite integral that represents the required volume **[1 mark]**
Volume $= \pi (10x - x^3) \Big\|_0^{\sqrt{2}}$ $= \pi \left(\left(10\sqrt{2} - (\sqrt{2})^3 \right) - 0 \right)$ $= \pi (10\sqrt{2} - 2\sqrt{2})$ $= 8\sqrt{2}\pi$ units^3	• determines volume in simplest form **[1 mark]**

2023
Paper 1
Section 2
Question 17

Rates of change and differential equations

An object of mass 2 kg is moving with a constant velocity ($\mathrm{m\,s^{-1}}$) of $\boldsymbol{v} = 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$.

At an instant, two forces (N), $\boldsymbol{F}_1 = 5t\hat{\boldsymbol{j}} - 3\hat{\boldsymbol{k}}$ and $\boldsymbol{F}_2 = -t\hat{\boldsymbol{j}} + \hat{\boldsymbol{k}}$, act simultaneously on the object for t seconds, where $0 \le t \le 2$.

Determine the magnitude of the momentum of the object when $t = 1$.

(7 marks)

Sample response	The response
Method 1	
$\boldsymbol{F}_{net} = F_1 + F_2 = (5t\,\hat{j} - 3\hat{k}) + (-t\,\hat{j} + \hat{k})$ $= 4t\,\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}},\ 0 \le t \le 2$	• correctly determines an expression for the net force acting on the object **[1 mark]**
$= ma$ $4t\,\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}} = 2\boldsymbol{a}$ $\boldsymbol{a} = 2t\,\hat{j} - \hat{k}$	• determines an expression involving the object's acceleration after the forces act **[1 mark]**
$\boldsymbol{v} = \int a\,dt$ $= t^2\hat{\boldsymbol{j}} - t\,\hat{\boldsymbol{k}} + c$	• determines a general solution representing the object's velocity after the forces act **[1 mark]**
At $t = 0$, $\boldsymbol{v} = 3\hat{i} + \hat{k}$ $\therefore c = 3\hat{i} + \hat{k}$ $\boldsymbol{v} = t^2\hat{\boldsymbol{j}} - t\,\hat{\boldsymbol{k}} + 3\hat{\boldsymbol{i}} + \hat{\boldsymbol{k}}$ $= 3\hat{i} + t^2\hat{j} + (1-t)\hat{k}$	• determines a particular solution representing the velocity of the object after the forces act **[1 mark]**
Momentum $= m\boldsymbol{v}$ $= 2\left(3\hat{i} + t^2\hat{j} + (1-t)\hat{k}\right)$ At $t = 1$, $m\boldsymbol{v} = 2\left(3\hat{i} + \hat{j}\right)$	• determines an expression for the momentum of the object at $t = 1$ **[1 mark]**
$\lvert m\boldsymbol{v} \rvert = 2\sqrt{3^2 + 1^2} = 2\sqrt{10}\ \mathrm{kg\,m\,s^{-1}}$	• determines a magnitude of the momentum of the object at $t = 1$ **[1 mark]**
	• shows logical organisation, communicating key steps to at least the start of determining the momentum of the object **[1 mark]**

Method 2	
$\boldsymbol{F}_{net} = F_1 + F_2 = (5t\,\hat{j} - 3\hat{k}) + (-t\,\hat{j} + \hat{k})$ $= 4t\,\hat{\boldsymbol{j}} - 2\hat{\boldsymbol{k}},\ 0 \le t \le 2$	- correctly determines an expression for the net force acting on the object **[1 mark]**
$= ma = m\dfrac{dv}{dt}$ $4t\,\hat{j} - 2\hat{k} = 2\dfrac{dv}{dt}$	- determines an expression involving the object's acceleration after the forces act **[1 mark]**
$2t\,\hat{j} - \hat{k} = \dfrac{dv_x}{dt} + \dfrac{dv_y}{dt} + \dfrac{dv_z}{dt}$ $\dfrac{dv_x}{dt} = 0 \Rightarrow v_x = c_1$ $\dfrac{dv_y}{dt} = 2t \Rightarrow v_y = t^2 + c_2$ $\dfrac{dv_z}{dt} = -1 \Rightarrow v_z = -t + c_3$	- determines a general solution representing the object's velocity for each component after the forces act **[1 mark]**
At $t = 0,\ v = 3\hat{i} + \hat{k}$ $\therefore c_1 = 3,\ c_2 = 0,\ c_3 = 1$ $v_x = 3,\ v_y = t^2,\ v_z = 1 - t$	- determines a particular solution representing the velocity of the object after the forces act in component form **[1 mark]**
At $t = 1$: $v = 3\hat{i} + \hat{j}$ Momentum $= mv = 2\left(3\hat{i} + \hat{j}\right)$	- determines an expression for the momentum of the object at $t = 1$ **[1 mark]**
$\lvert m\boldsymbol{v}\rvert = \sqrt{6^2 + 2^2} = 2\sqrt{10}$ kg ms^{-1}	- determines a magnitude of the momentum of the object at $t = 1$ **[1 mark]**
	- shows logical organisation, communicating key steps to at least the start of determining the momentum of the object **[1 mark]**

2023
Paper 1
Section 2
Question 18

Rates of change and differential equations

A particular solution to the differential equation $\frac{dy}{dx}=\frac{x}{\left(x^2+1\right)\tan(y)}$, where $x \geq 0$ and $-\frac{\pi}{2}<y \leq 0$, passes through the origin.

Determine this solution in the form $x=f(y)$. Leave your answer in simplified form.

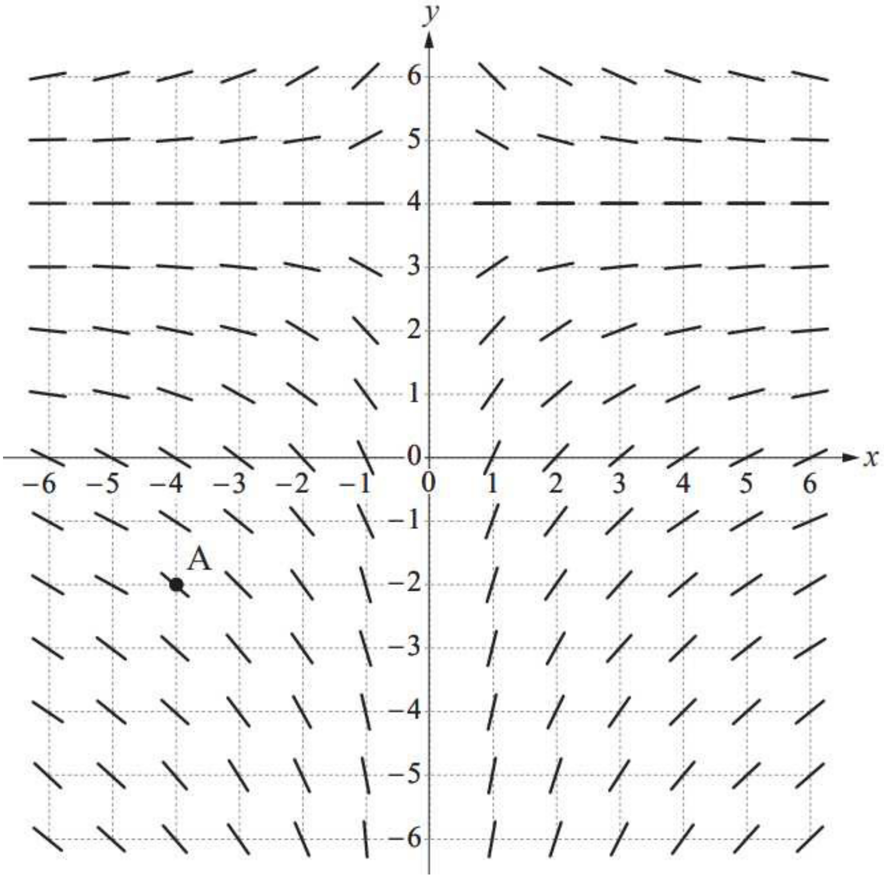
(6 marks)

Sample response	The response
$\frac{dy}{dx}=\frac{x}{\left(x^2+1\right)\tan(y)}$	
$\int \tan(y)\,dy=\int \frac{x}{x^2+1}\,dx$	• correctly separates the variables **[1 mark]**
$-\int \frac{-\sin(y)}{\cos(y)}\,dy=\frac{1}{2}\int \frac{2x}{x^2+1}\,dx$	
$-\ln\lvert\cos(y)\rvert=\frac{1}{2}\ln\lvert x^2+1\rvert+c$	• applies suitable integration methods **[1 mark]**
Given $y=0$ when $x=0$,	
$-\ln\lvert\cos(0)\rvert=\frac{1}{2}\ln\lvert 1\rvert+c \Rightarrow c=0$	• determines a value for the constant of integration **[1 mark]**
$\therefore \frac{1}{2}\ln\lvert x^2+1\rvert=-\ln\lvert\cos(y)\rvert$	
$\ln\sqrt{x^2+1}=\ln\left\lvert\frac{1}{\cos(y)}\right\rvert$	
$\sqrt{x^2+1}=\lvert\sec(y)\rvert$	• determines an expression for a solution that does not contain logarithms **[1 mark]**
$x^2+1=\sec^2(y)$	
$x^2=\tan^2(y)$	• expresses x in terms of y **[1 mark]**
$x=\pm\tan(y)$	
As $x \geq 0$, $-\frac{\pi}{2}<y \leq 0$	• evaluates the reasonableness of the results and expresses the solution in the form of $x=f(y)$ in simplified form **[1 mark]**
$x=-\tan(y)$	

2022
Paper 1
Section 2
Question 14

Rates of change and differential equations

The slope field for the differential equation $\frac{dy}{dx}=\frac{-0.5(y-4)}{x}$, $x \neq 0$ using $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$ is shown.

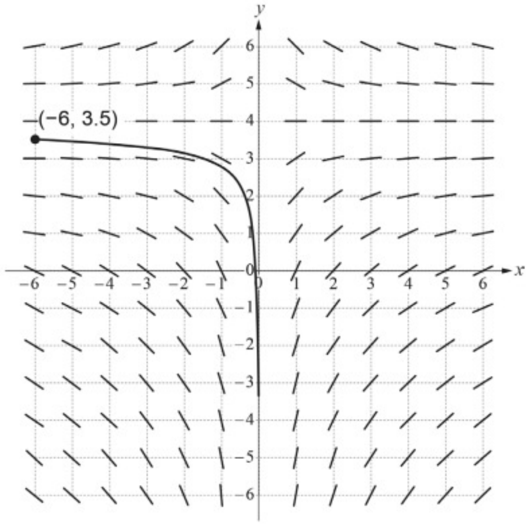


a) Determine the value of the slope at point A. [2 marks]

Sample Response	The response
Substituting A(−4, −2) $\frac{dy}{dx}=\frac{-0.5(-2-4)}{-4}$ $=-\frac{3}{4}$	• correctly substitutes the coordinates of A into the differential equation [1 mark] • determines a value of the slope at A [1 mark]

b) Use the slope field to sketch the solution curve for $\frac{dy}{dx} = \frac{-0.5(y-4)}{x}$ given that when $x = -6$, $y = 3.5$

[2 marks]

Sample Response	The response
	• correctly shows a solution curve that starts at (−6, 3.5) and has a negative slope in the 2nd quadrant [1 mark] • correctly shows a solution curve with an asymptotic nature in the 3rd quadrant as $xx \to 0-$ and there is no solution curve in the 1st or 4th quadrant [1 mark]

2021
Paper 1
Section 2
Question 11

Rates of change and differential equations

Let $f(x) = \tan^{-1}\left(\frac{x}{2}\right)$ for suitable values of x where $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

a) Determine $f(2)$. [1 mark]

Sample Response	The response
$f(2) = \tan^{-1}(1)$ $= \frac{\pi}{4}$ as $f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	• correctly determines the required value [1 mark]

a) Determine $f'(2)$. [1 mark]

Sample Response	The response
$f'(x) = \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{2}\right)\right)$ $= \frac{2}{4+x^2}$	• correctly determines the gradient function [1 mark]
$f'(2) = \frac{1}{4}$	• determines gradient of the tangent [1 mark]

c) Use the results from Questions 11a) and 11b) to determine the equation of the tangent to the graph of $y = f(x)$ at $x = 2$. [2 marks]

Sample Response	The response
Equation of the tangent has the form $y = mx + c$ From 11a) $x = 2, y = \frac{\pi}{4}$ From 11b) $m = \frac{1}{4}$ $\frac{\pi}{4} = \frac{1}{4}(2) + c \Rightarrow c = \frac{\pi}{4} - \frac{1}{2}$	• determines y-intercept of the tangent [1 mark]
Equation of the tangent is $y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$	• determines equation of the tangent [1 mark]

2021
Paper 1
Section 2
Question 18

Rates of change and differential equations

This differential equation can be used to determine the current I (amperes) at time t (seconds) with voltage V (volts) in an electric circuit containing a resistance R (ohms):

$$k\frac{dI}{dt}+RI=V$$

where k, R and V are positive constants and $t \geq 0$.

Assuming that there is no current in the electric circuit initially, show that the size of the current can never be greater than $\frac{V}{R}$. [6 marks]

Sample Response	The response
$k\frac{dI}{dt}+RI=V \Rightarrow k\frac{dI}{dt}=V-RI$ $\int \frac{k}{V-RI}\,dI=\int 1\,dt$	• correctly uses the separation of variables method to set up indefinite integrals [1 mark]
$-\frac{k}{R}\ln\lvert V-RI\rvert=t+c$ Given $I=0$ when $t=0$	• develops a general solution of the differential equation [1 mark]
$c=-\frac{k}{R}\ln(V)$ (as $V>0$)	• uses the given condition to determine expression for the constant of integration [1 mark]
$-\frac{k}{R}\ln\lvert V-RI\rvert=t-\frac{k}{R}\ln(V)$ $\ln\lvert V-RI\rvert=-\frac{R}{k}t+\ln(V)$	• rearranges relationship to express $\ln\lvert V-RI\rvert$ as the subject of the equation [1 mark]
$V-RI=e^{-\frac{R}{k}t+\ln(V)}$ $V-RI=V\,e^{-\frac{R}{k}t}$	• expresses relationship as an exponential function [1 mark]
For all t, $e^{-\frac{R}{k}t}>0 \Rightarrow V-RI>0 \Rightarrow V>RI$ $\Rightarrow I<\frac{V}{R}$ So, the size of the current can never be greater than $\frac{V}{R}$.	• considers value of I over time to determine the required limit [1 mark]

2020
Paper 1
Section 2
Question 14

Rates of change and differential equations

The motion of an object that moves in a straight line is given by $v(x) = \cos^{-1}(2x)$ where v is the velocity (m s^{-1}) and x is the displacement (m) from the origin.

a) Determine $a(x)$ where a is the acceleration (m s^{-2}) of the object.

[2 marks]

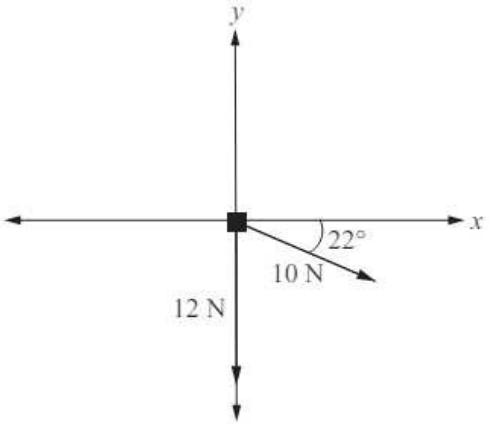
Sample Response	The response
$v(x) = \cos^{-1}(2x)$ $a = v\dfrac{dv}{dx}$ $\dfrac{dv}{dx} = \dfrac{-1}{\sqrt{0.25 - x^2}}$	• correctly determines $\frac{dv}{dx}$ [1 mark]
$a = \dfrac{-\cos^{-1}(2x)}{\sqrt{0.25 - x^2}}$	• determines an expression for acceleration as a function of displacement [1 mark]

b) Use the result from 14a) to determine $a(0)$, given $-2\pi \le a(0) \le 0$. Express your answer in simplest form.

[2 marks]

Sample Response	The response
When $x = 0$ $a = \dfrac{-\cos^{-1}(0)}{0.5}$ $\cos^{-1}(0) = \dfrac{\pi}{2}$	• determines a correct exact value for the inverse trigonometric expression on the numerator [1 mark]
$a(0) = \dfrac{-\left(\frac{\pi}{2}\right)}{0.5}$ $= -\pi \quad (\text{m s}^{-2})$	• determines a reasonable solution for $a(0)$ (based on the given range $(-2\pi \le a(0) \le 0)$ [1 mark]

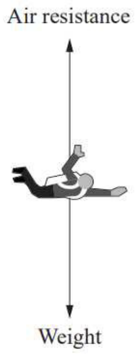
Marking Guide – Paper 2 Section 1

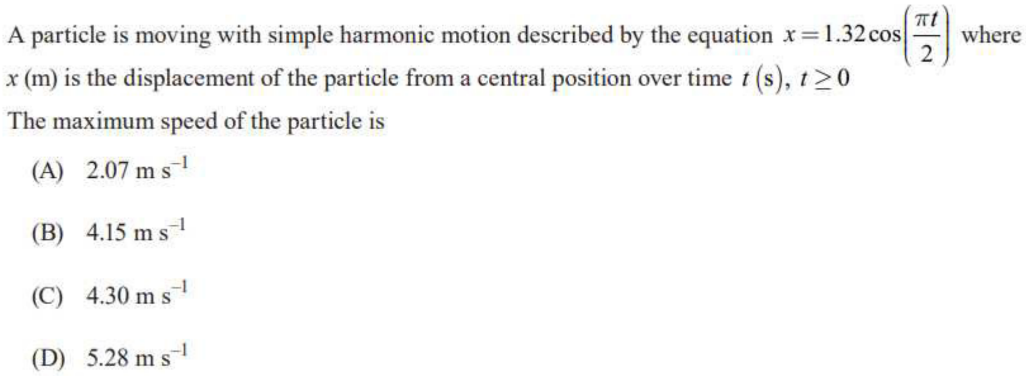
2024 Paper 2 Section 1 Question 6 **Rates of change and differential equations**	Two concurrent forces represented in the polar form of $\boldsymbol{F}_1 = (1.21\text{ N}, 120°)$ and $\boldsymbol{F}_2 = (1.30\text{ N}, -160°)$ act on an object. Determine the magnitude of the resultant force. (A) 0.50 N (B) 1.92 N (C) 2.51 N (D) 3.70 N **Answer is B.**
2023 Paper 2 Section 1 Question 1 **Rates of change and differential equations**	The acceleration (m s^{-2}) of an object moving with simple harmonic motion is modelled by $a = -2.95x$, where x is its displacement (m) from the origin. Determine the period of the motion in seconds. (A) 1.72 (B) 2.13 (C) 2.95 (D) 3.66 **Answer is D.**
2023 Paper 2 Section 1 Question 6 **Rates of change and differential equations**	Two coplanar forces of magnitudes 12 N and 10 N act on an object in the directions shown. Not to scale  Determine the magnitude of the resultant force acting on the object. (A) 12.41 N (B) 15.55 N (C) 15.69 N (D) 18.27 N **Answer is D.**

2022 Paper 2 Section 1 Question 3 Rates of change and differential equations	Determine the solution of the differential equation $\frac{dy}{dx}=\frac{\sin(2x)}{\cos(2x)}$ given $y=0$ when $x=\frac{\pi}{5}$. (A) $y=-2\ln\lvert\cos(2x)\rvert-2.35$ (B) $y=-2\ln\lvert\cos(2x)\rvert+2.35$ (C) $y=-\frac{1}{2}\ln\lvert\cos(2x)\rvert-0.59$ (D) $y=-\frac{1}{2}\ln\lvert\cos(2x)\rvert+0.59$ **Answer is C.**

2022 Paper 2 Section 1 Question 6 Rates of change and differential equations	A 4 kg object moves in a straight line over time, $t(\text{s})$, where $0 \le t \le 5$ with velocity $v=9+8t-t^2\left(\text{m s}^{-1}\right)$. Determine the momentum of the object when $t = 3$. (A) 24 kg m s^{-1} (B) 27 kg m s^{-1} **(C) 96 kg m s^{-1} – Answer** (D) 100 kg m s^{-1}

2022 Paper 2 Section 1 Question 8 Rates of change and differential equations	Determine the gradient of the tangent to the curve $y^2-3x=5$ at the point $\left(1,\ 2\sqrt{2}\right)$. (A) 0.41 (B) 0.53 (C) 1.06 (D) 8.49 **Answer is B.**

2021 Paper 2 Section 1 Question 9 Rates of change and differential equations	Two vertical forces act on a skydiver with a mass of 85 kg, as shown.  When the magnitude of the air resistance is 62 N, the magnitude of the acceleration of the skydiver is (A) 0.73 m s^{-2} (B) 2.65 m s^{-2} **(C) 9.07 m s^{-2} – Answer** (D) 12.44 m s^{-2}

2020 **Paper 2** **Section 1** **Question 4** **Rates of change and differential equations**	 A particle is moving with simple harmonic motion described by the equation $x = 1.32\cos\left(\frac{\pi t}{2}\right)$ where x (m) is the displacement of the particle from a central position over time $t\,(\mathrm{s})$, $t \geq 0$ The maximum speed of the particle is (A) $2.07\ \mathrm{m\ s^{-1}}$ (B) $4.15\ \mathrm{m\ s^{-1}}$ (C) $4.30\ \mathrm{m\ s^{-1}}$ (D) $5.28\ \mathrm{m\ s^{-1}}$ **Answer is A.**

2020 **Paper 2** **Section 1** **Question 5** **Rates of change and differential equations**	The gradient of the tangent at point A on the curve $y^2 = 4x$ is 1.36 The x-coordinate of A is (A) 0.12 (B) 0.46 **(C) 0.54 – Answer** (D) 1.47

2024
Paper 2
Section 2
Question 13

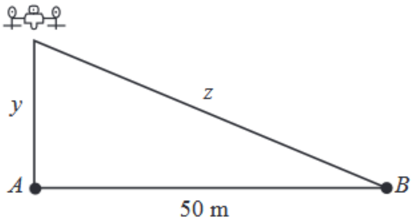
Rates of change and differential equations

A drone travels vertically from point A at a constant speed of 8 m s^{-1} over time t for $t \geq 0$ seconds.

Observation of the drone is made from point B, which is 50 m horizontally from point A.

When the drone is y metres above point A, it is z metres from point B as shown.

Not to scale



a) Determine an equation expressing z^2 in terms of y^2. *[1 mark]*

Sample response	The response
$z^2 = y^2 + 50^2$ ... (1)	• correctly expresses z^2 in terms of y^2 **[1 mark]**

b) State the value of $\frac{dy}{dt}$. *[1 mark]*

Sample response	The response
$\frac{dy}{dt} = 8$ m s^{-1}	• correctly states the value of $\frac{dy}{dt}$ **[1 mark]**

c) Use your results from Questions 13a) and 13b) to determine the rate at which z is increasing with respect to time when the drone is 20 metres above point A. *[3 marks]*

Sample response	The response
Method 1 From (1) $z = \left(y^2 + 2500\right)^{\frac{1}{2}}$ (as $z > 0$) $\frac{dz}{dy} = \frac{1}{2}\left(y^2 + 2500\right)^{\frac{-1}{2}} \times 2y$ ... (2)	• determines a mathematical equation in terms of $\frac{dz}{dy}$ **[1 mark]**
Rate at which z is increasing with respect to time is $\frac{dz}{dt}$ $\frac{dz}{dt} = \frac{dz}{dy} \times \frac{dy}{dt}$	• states an equation using related rates **[1 mark]**
Using (2) When $y = 20$ $\frac{dz}{dt} = \frac{1}{2}\left(20^2 + 2500\right)^{\frac{-1}{2}} \times 2 \times 20 \times 8$ ≈ 2.97 m s^{-1}	• determines value of $\frac{dz}{dt}$ when $y = 20$ **[1 mark]**

	Method 2 From (1) When $y = 20$ $z^2 = 20^2 + 50^2$ $z \approx 53.85$ m $(\text{as } z > 0)$ Rate at which z is increasing with respect to time is $\frac{dz}{dt}$ From (1) $2z\frac{dz}{dt} = 2y\frac{dy}{dt}$ When $y = 20$ $(53.85)\frac{dz}{dt} \approx (20)(8)$ $\frac{dz}{dt} \approx 2.97\,\text{m}\,\text{s}^{-1}$	• determines value of z when $y = 20$ **[1 mark]** • determines the implicit derivative of the result from 13a) **[1 mark]** • determines value of $\frac{dz}{dt}$ when $y = 20$ **[1 mark]**
	Method 3 From 13b) $\frac{dy}{dt} = 8 \Rightarrow y = 8t + c$ At $t = 0, y = 0 \Rightarrow c = 0$ $y = 8t$ When $y = 20, t = 2.5$ s From 13b) $z^2 = y^2 + 2500$ $z = \sqrt{y^2 + 2500}$ $= \sqrt{64t^2 + 2500}$ Using GDC When $t = 2.5$ $\frac{dz}{dt} \approx 2.97\,\text{m}\,\text{s}^{-1}$	• determines t when $y = 20$ **[1 mark]** • determines equation representing z in terms of t **[1 mark]** • determines value of $\frac{dz}{dt}$ when $y = 20$ **[1 mark]**

2023
Paper 2
Section 2
Question 16

Rates of change and differential equations

A curve modelled by the relation $xy^2 - y + \cos^{-1}(2x) = 1$, where $-0.35 \le x \le 0.27$ and $0 \le y \le 1$, intersects the y-axis at point A.

Determine the equation of the tangent to the curve at point A.

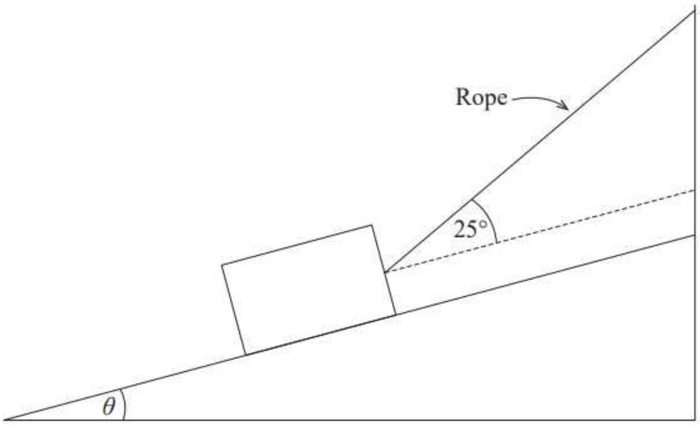
(6 marks)

Sample response	The response
Given $xy^2 - y + \cos^{-1}(2x) = 1$ Determining y-coordinate of A $0 - y + \cos^{-1}(0) = 1 \Rightarrow y = 0.57$	• correctly determines y-intercept **[1 mark]**
$\frac{d}{dx}\cos^{-1}(2x) = \frac{-1}{\sqrt{0.25 - x^2}}$	• correctly determines $\frac{d}{dx}\cos^{-1}(2x)$ **[1 mark]**
$\frac{d}{dx}\left(xy^2\right) = y^2 + 2xy\frac{dy}{dx}$	• correctly determines $\frac{d}{dx}xy^2$ **[1 mark]**
Determining $\frac{dy}{dx}$ $\frac{d}{dx}\left(xy^2 - y + \cos^{-1}(2x)\right) = \frac{d}{dx}(1)$ $y^2 + 2xy\frac{dy}{dx} - \frac{dy}{dx} + \frac{-1}{\sqrt{0.25 - x^2}} = 0$ $\frac{dy}{dx}(2xy - 1) = \frac{1}{\sqrt{0.25 - x^2}} - y^2$ $\frac{dy}{dx} = \frac{\frac{1}{\sqrt{0.25 - x^2}} - y^2}{2xy - 1}$	• determines an expression for $\frac{dy}{dx}$ using a common factor **[1 mark]**
Determining $\frac{dy}{dx}$ at A. $\frac{dy}{dx} = -\left(\frac{1}{\sqrt{0.25}} - (0.571)^2\right) = -1.67$	• determines a value for $\frac{dy}{dx}$ at A **[1 mark]**
Determining equation of tangent at A $y = mx + c$ $y = -1.67x + 0.57$	• determines equation of the tangent at A **[1 mark]**

2022
Paper 2
Section 2
Question 16

Rates of change and differential equations

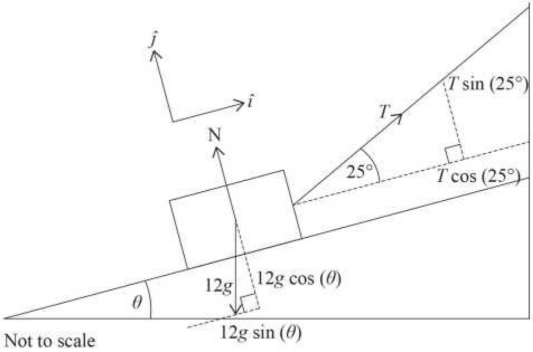
An object with a mass of 12 kg lies on a frictionless inclined plane. A rope is attached to the object at an angle of 25° above the plane, as shown.



Not to scale

The force of the rope, T N, prevents the object from moving. When the rope is detached, the object moves down the plane with an acceleration of 5.6 m s^{-2}.

Determine the magnitude of T. [6 marks]

<table>
<tr><th>Sample Response</th><th>The response</th></tr>
<tr><td>


Not to scale

Assume the direction of the unit vectors as shown.

Consider rope attached: $|F_{net}| = 0$

Consider the $\hat{\imath}$ components

$|\boldsymbol{T}|\cos(25°) = 12g\sin(\boldsymbol{\theta})$ ...[1]
</td><td>• correctly determines an equation in terms of $|\boldsymbol{T}|$ (or T) and θ when the rope is attached [1 mark]</td></tr>
<tr><td>Consider rope detached:

$|\boldsymbol{F}_{\text{net}}| = ma = 12 \times 5.6 = \mathbf{67.2}$</td><td>• correctly calculates the magnitude of the net force acting when the rope is detached [1 mark]</td></tr>
<tr><td>Consider the $\hat{\imath}$ components

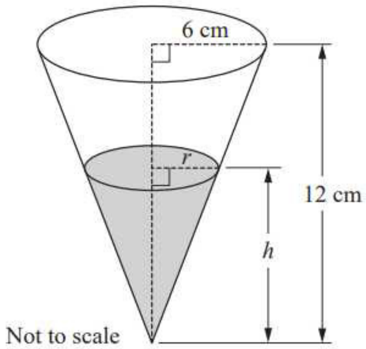
$-12g\sin(\boldsymbol{\theta}) = 12 \times (-5.6)$</td><td>• determines an equation in terms of θ when the rope is detached [1 mark]</td></tr>
<tr><td>$\boldsymbol{\theta} = 34.85°$</td><td>• determines a value representing θ [1 mark]</td></tr>
<tr><td>Substituting into [1]

$|\boldsymbol{T}|\cos(25°) = 12g\sin(34.85°)$</td><td>• determines an equation in terms of $|\boldsymbol{T}|$ (or T) [1 mark]</td></tr>
<tr><td>$|T| \approx 74.15$ N</td><td>• determines the magnitude of $\boldsymbol{T}$ [1 mark]</td></tr>
</table>

2021
Paper 2
Section 2
Question 15

Rates of change and differential equations

Water is poured into a cone-shaped cup at a rate of 2 cm^3 s^{-1}. The cup has a height of 12 cm and a radius of 6 cm, as shown.



As the cup fills, the ratio of the height of the water *h* to the surface radius of the water *r* remains constant.

a) Given that $h = 2r$, determine a function for the volume of water in the cup, *V*, in terms of *h*.

Express your answer in simplified form. [1 mark]

Sample Response	The response
$V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $= \frac{\pi h^3}{12}$	• correctly determines a function for the volume in simplified form [1 mark]

b) Use the results from Question 15a) to show that the rate at which the height of water in the cup is increasing with respect to time is given by $\frac{8}{\pi r^2}$. [3 marks]

Sample Response	The response
$\frac{dV}{dh} = \frac{\pi h^2}{4}$	• determines an expression for $\frac{dV}{dh}$ [1 mark]
$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$	• correctly states a related rates equation in terms of *V*, *h* and *t* [1 mark]
$2 = \frac{\pi h^2}{4}\frac{dh}{dt}$ $\frac{dh}{dt} = \frac{8}{\pi h^2}$	• demonstrates suitable working leading to the required result [1 mark]

c) Determine the rate at which the height of water in the cup is increasing with respect to time when the volume of water in the cup reaches half of the total capacity of the cup. [4 marks]

Sample Response	The response
Half capacity $= \frac{1}{2}\left(\frac{1}{3}\pi r^2 h\right) = \frac{1}{6}\pi(6)^2(12)$ $\approx 226.195 \text{ cm}^3$ Solving $\frac{\pi h^3}{12} = 226.195$	• correctly uses a suitable equation to determine *h* when the cup is half full [1 mark]
$\therefore h \approx 9.52$ cm	• determines *h* when *V* reaches half capacity [1 mark]
$\frac{dh}{dt} \approx \frac{8}{\pi(9.52)^2} \approx 0.03$	• determines required rate [1 mark]
Required rate is 0.03 cm s^{-1}	• correctly communicates relevant units [1 mark]

2020
Paper 2
Section 2
Question 12

Rates of change and differential equations

For a certain experiment, the number of yeast cells, N, after t hours in a test tube can be modelled by the differential equation

$$\frac{dN}{dt}=\frac{1}{1000}N(1000-N)\text{ for }t\geq 0$$

a) Given $\frac{1000}{N(1000-N)}=\frac{1}{N}+\frac{1}{1000-N}$, show that the general solution of the differential equation can be expressed as

$$\ln\left|\frac{N}{1000-N}\right|=t+c$$

[2 marks]

Sample Response	The response
$\frac{dN}{dt}=\frac{1}{1000}N(1000-N)$ $\frac{1000}{N(1000-N)}\frac{dN}{dt}=1$ $\left(\frac{1}{N}+\frac{1}{1000-N}\right)\frac{dN}{dt}=1$	• correctly uses separation of variables technique and substitutes the given result into the differential equation [1 mark]
$\int\frac{1}{N}+\frac{1}{1000-N}\,dN=\int 1\,dt$ $\ln\lvert N\rvert-\ln\lvert 1000-N\rvert=t+c$ $\ln\left\lvert\frac{N}{1000-N}\right\rvert=t+c$	• correctly develops the required general solution [1 mark]

A scientist commenced this experiment at 9:00 am on a certain day and observed that 100 yeast cells were present at this time.

b) Show that the solution of the differential equation can be expressed as

$$N=\frac{1000}{1+9e^{-t}}$$

[3 marks]

Sample Response	The response
Let t = time after 9:00am (in hours) When $t=0,\ N=100$ $\ln\left\lvert\frac{100}{1000-100}\right\rvert=0+c$ $c=\ln\left(\frac{1}{9}\right)$	• correctly determines c [1 mark]
$\ln\left(\frac{N}{1000-N}\right)=t+\ln\left(\frac{1}{9}\right)$ $\frac{N}{1000-N}=e^{t+\ln\left(\frac{1}{9}\right)}$ $\frac{N}{1000-N}=\frac{e^t}{9}$	• substitutes the value of c into the general equation and simplifies sufficiently to produce a function that includes the term e^t [1 mark]
$9N=1000e^t-Ne^t$ $N(9+e^t)=1000e^t$ $N=\frac{1000e^t}{(9+e^t)}$ $N=\frac{1000}{1+9e^{-t}}$	• develops the solution for N in the required form [1 mark]

c) Determine the time of day when 900 yeast cells were present. [2 marks]

Sample Response	The response
Given $N = 900$ $900 = \frac{1000}{1 + 9e^{-t}}$ Using solve facility of GDC $t \approx 4.394$ (hours)	• correctly determines the value of t when $N = 900$ [1 mark]
The time of day is **1:24 pm.**	• communicates the time of day [1 mark]

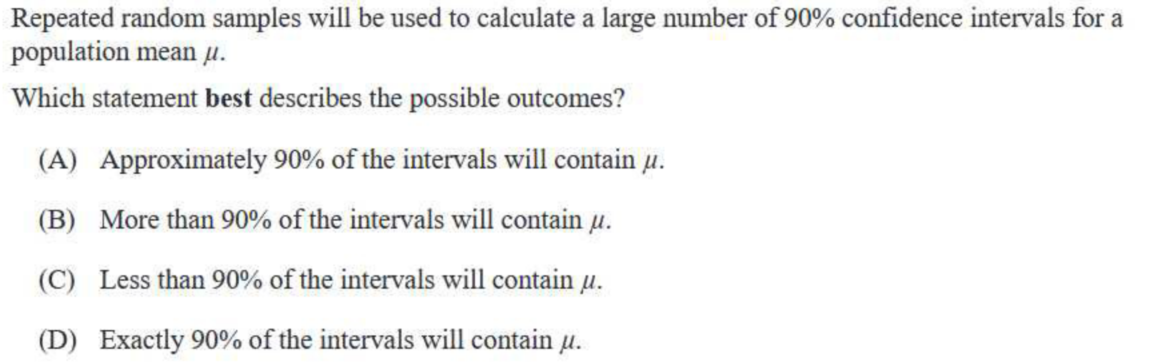
The scientist predicted that the number of yeast cells would eventually exceed 1200.

d) Evaluate the reasonableness of the scientist's prediction. [2 marks]

Sample Response	The response
Method 1 Given $N = 1200$: As $t \to \infty, N \to 1000$ The number of yeast cells has a limit of 1000.	• correctly recognises that the number of yeast cells will never exceed 1000 [1 mark]
As N will never reach 1200, the scientist's prediction is **not reasonable.**	• comments that the prediction is not reasonable [1 mark]
Method 2 Given $N = 1200$: $1200 = \frac{1000}{1+9e^{-t}}$ Using GDC to solve the equation No solution.	• correctly recognises that the number of yeast cells will never reach 1200 [1 mark]
As N will never reach 1200, **the scientist's prediction is not reasonable.**	• comments that the prediction is not reasonable [1 mark]

Unit 4 – Topic 3: Statistical inference

Paper 1 Section 1

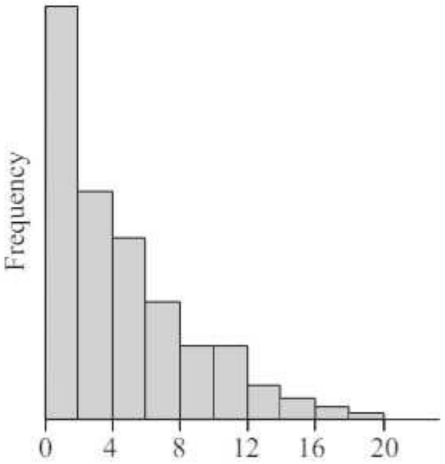
2024 **Paper 1** **Section 1** **Question 1** **Statistical inference**	 Repeated random samples will be used to calculate a large number of 90% confidence intervals for a population mean μ. Which statement **best** describes the possible outcomes? (A) Approximately 90% of the intervals will contain μ. (B) More than 90% of the intervals will contain μ. (C) Less than 90% of the intervals will contain μ. (D) Exactly 90% of the intervals will contain μ.

2023 **Paper 1** **Section 1** **Question 5** **Statistical inference**	A confidence interval for a parameter is a range of values within which the (A) sample estimate of the parameter always lies. (B) sample estimate of the parameter never lies. (C) parameter always lies. (D) parameter never lies.

2023
Paper 1
Section 1
Question 10

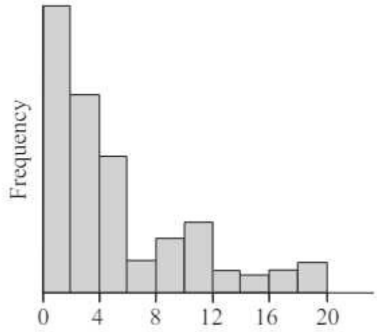
Statistical inference

A random variable is drawn from a population with the distribution shown in the histogram.

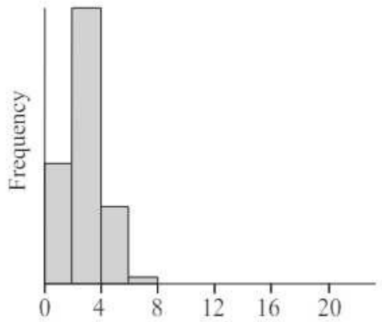


A number of samples of size 10 were randomly selected from this distribution and the sample means, $\overline{x}$, were recorded. The histogram that most likely represents the distribution of the sample means is

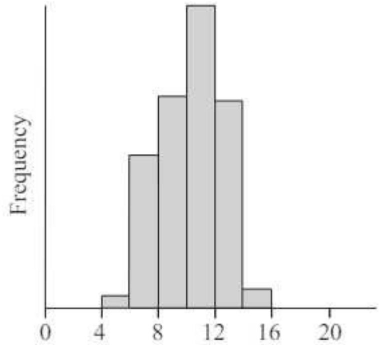
(A)



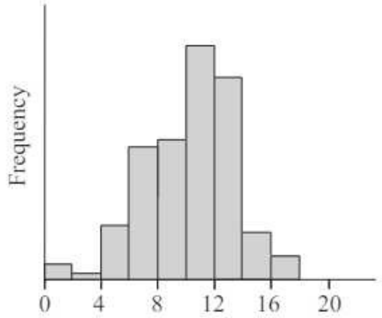
(B)



(C)



(D)

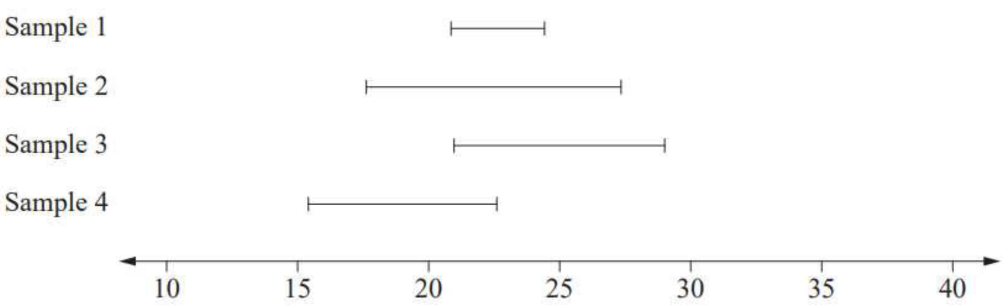


2022
Paper 1
Section 1
Question 2

Statistical inference

Which statement regarding sample means is true?

(A) The distribution of X is always normally distributed.
(B) The distribution of $\bar{X}$ is always normally distributed.
(C) The value of $\bar{x}$ changes when different samples are selected.
(D) The value of μ changes when different samples are selected

2022 Paper 1 Section 1 Question 5 Statistical inference	Four random samples of different sizes were taken to estimate a certain population mean, given a known population standard deviation. A 95% confidence interval was calculated for each sample.  Which sample used the largest sample size? (A) Sample 1 (B) Sample 2 (C) Sample 3 (D) Sample 4

2022 Paper 1 Section 1 Question 9 Statistical inference	A random variable X is normally distributed with a mean of 36 and a standard deviation of 4. The respective mean and standard deviation of the distribution of $\bar{X}$ from repeated random samples of size 9 are (A) 4 and $\frac{4}{9}$ (B) 4 and $\frac{4}{3}$ (C) 36 and $\frac{4}{9}$ (D) 36 and $\frac{4}{3}$

2021 Paper 1 Section 1 Question 1 Statistical inference	Which of the following is a population parameter? (A) s (B) μ (C) $\overline{x}$ (D) z

2021 Paper 1 Section 1 Question 4 Statistical inference	The number of sunflower seeds in each packet produced by a company is known to be normally distributed with a standard deviation of 100. A worker counts the number of seeds in a random sample of four packets and calculates the sample mean. Based on this sampling, the standard deviation of the distribution of the sample mean is (A) 25 (B) 50 (C) 75 (D) 100

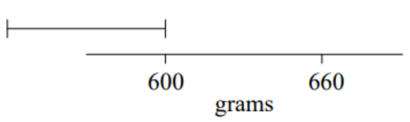
2021
Paper 1
Section 1
Question 7

Statistical inference

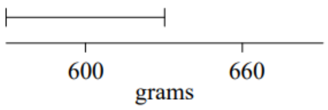
The mass of a particular variety of cake is claimed to be normally distributed with a mean of 660 grams. A random sample of five of these cakes is found to have a mean mass of 600 grams.

Which option represents an approximate confidence interval for μ based on this sample?

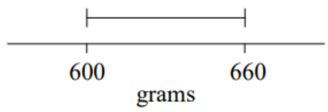
(A)



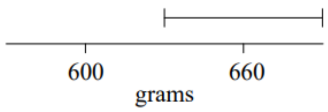
(B)



(C)



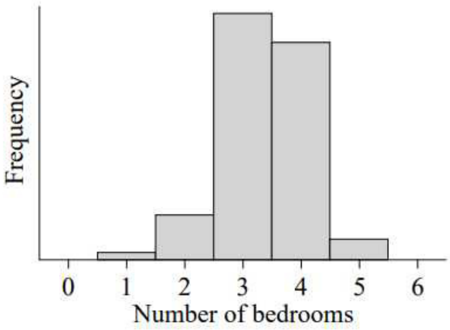
(D)



2021
Paper 1
Section 1
Question 10

Statistical inference

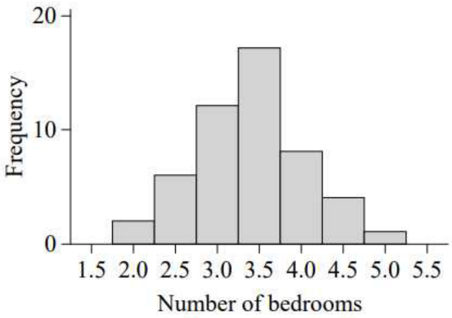
The 2016 Australian census recorded the number of bedrooms per household. The results are summarised in the histogram, as shown. Based on this data, the mean number of bedrooms per household was calculated to be 3.5.



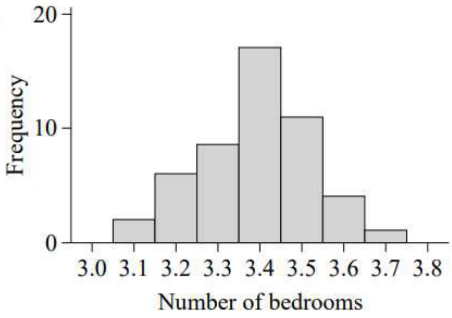
Fifty samples of size 40 were randomly selected from the census data and the sample means recorded.

The histogram that most likely represents the distribution of the sample means is

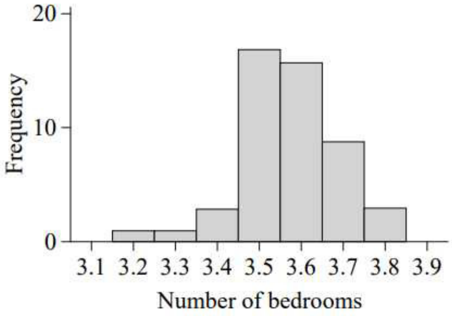
(A)



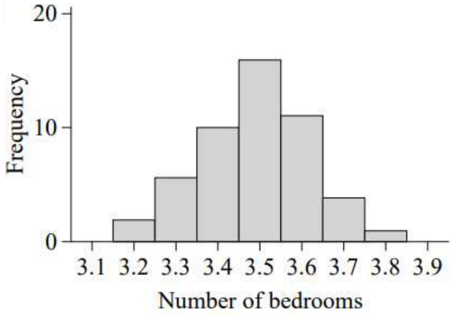
(B)

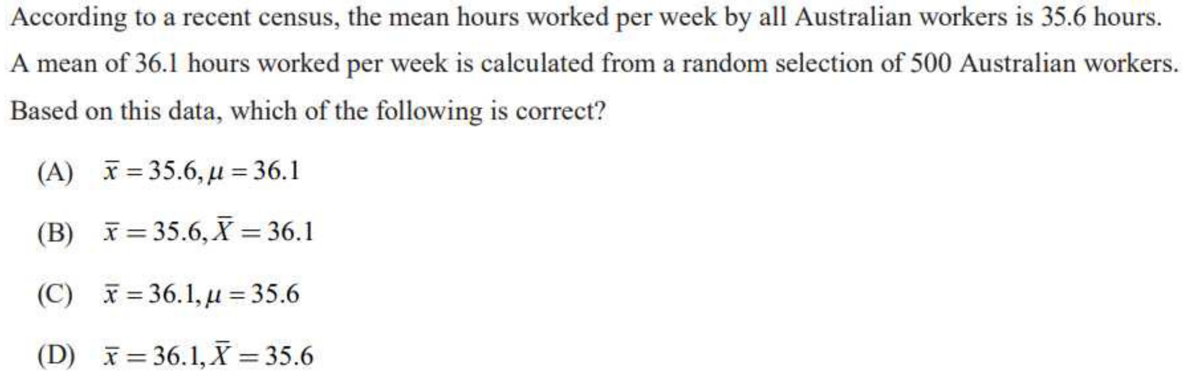


(C)



(D)



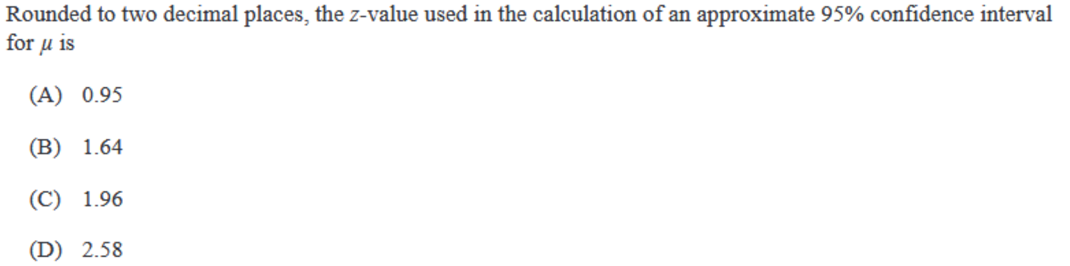
2020 **Paper 1** **Section 1** **Question 3** **Statistical inference**	According to a recent census, the mean hours worked per week by all Australian workers is 35.6 hours. A mean of 36.1 hours worked per week is calculated from a random selection of 500 Australian workers. Based on this data, which of the following is correct? (A) $\bar{x} = 35.6, \mu = 36.1$ (B) $\bar{x} = 35.6, \bar{X} = 36.1$ (C) $\bar{x} = 36.1, \mu = 35.6$ (D) $\bar{x} = 36.1, \bar{X} = 35.6$

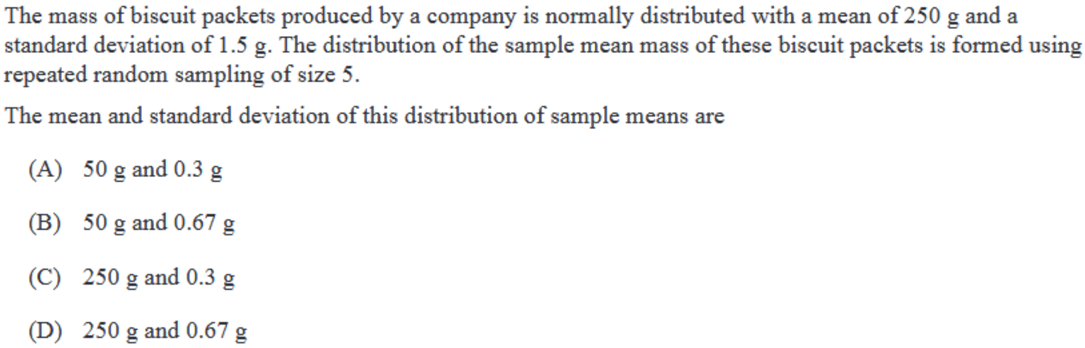
2020 **Paper 1** **Section 1** **Question 9** **Statistical inference**	The scores on a test are assumed to be normally distributed. Researchers use the results from a random sample of scores to calculate a confidence interval for the population mean. However, a shorter confidence interval width is required so the researchers decide to use a second sample for their calculations. Assuming that the standard deviations for both samples are the same, the researchers can ensure that a shorter confidence interval width is produced by (A) decreasing the sample size and decreasing the confidence level. (B) decreasing the sample size and increasing the confidence level. (C) increasing the sample size and decreasing the confidence level. (D) increasing the sample size and increasing the confidence level.

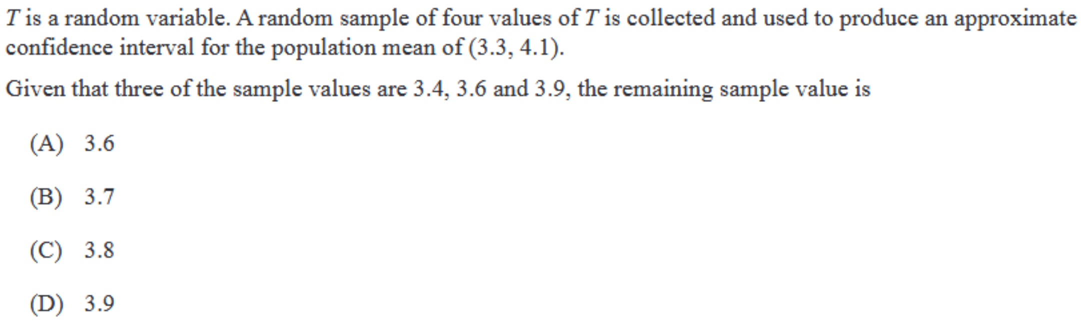
Paper 1 Section 2

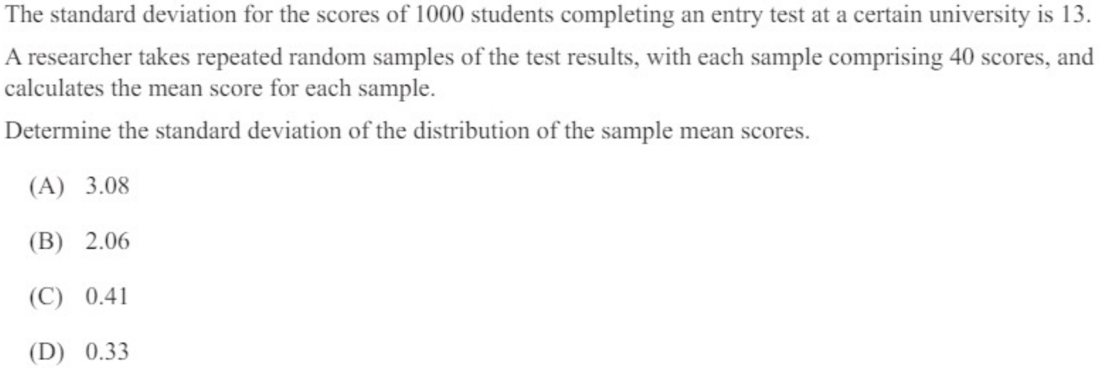
There have been no questions on this topic for this section in the exams of recent years.

Paper 2 Section 1

2024 Paper 2 Section 1 Question 2 Statistical inference	 Rounded to two decimal places, the z-value used in the calculation of an approximate 95% confidence interval for μ is (A) 0.95 (B) 1.64 (C) 1.96 (D) 2.58

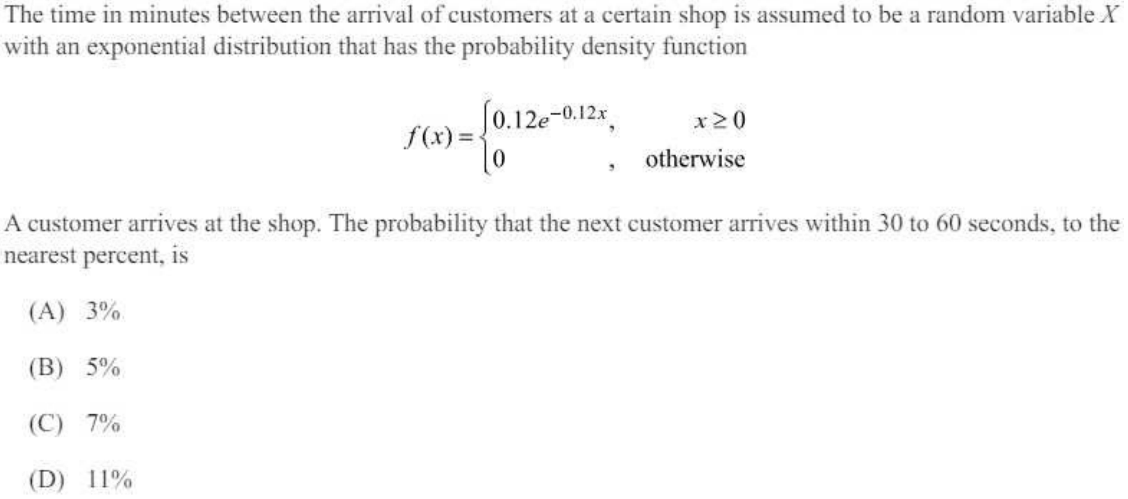
2024 Paper 2 Section 1 Question 4 Statistical inference	 The mass of biscuit packets produced by a company is normally distributed with a mean of 250 g and a standard deviation of 1.5 g. The distribution of the sample mean mass of these biscuit packets is formed using repeated random sampling of size 5. The mean and standard deviation of this distribution of sample means are (A) 50 g and 0.3 g (B) 50 g and 0.67 g (C) 250 g and 0.3 g (D) 250 g and 0.67 g

2024 Paper 2 Section 1 Question 8 Statistical inference	 T is a random variable. A random sample of four values of T is collected and used to produce an approximate confidence interval for the population mean of (3.3, 4.1). Given that three of the sample values are 3.4, 3.6 and 3.9, the remaining sample value is (A) 3.6 (B) 3.7 (C) 3.8 (D) 3.9

2023 Paper 2 Section 1 Question 2 Statistical inference	 The standard deviation for the scores of 1000 students completing an entry test at a certain university is 13. A researcher takes repeated random samples of the test results, with each sample comprising 40 scores, and calculates the mean score for each sample. Determine the standard deviation of the distribution of the sample mean scores. (A) 3.08 (B) 2.06 (C) 0.41 (D) 0.33

2023
Paper 2
Section 1
Question 9

Statistical
inference



The time in minutes between the arrival of customers at a certain shop is assumed to be a random variable X with an exponential distribution that has the probability density function

$$f(x)=\begin{cases}0.12e^{-0.12x}, & x\geq 0\\ 0\quad , & \text{otherwise}\end{cases}$$

A customer arrives at the shop. The probability that the next customer arrives within 30 to 60 seconds, to the nearest percent, is

(A) 3%

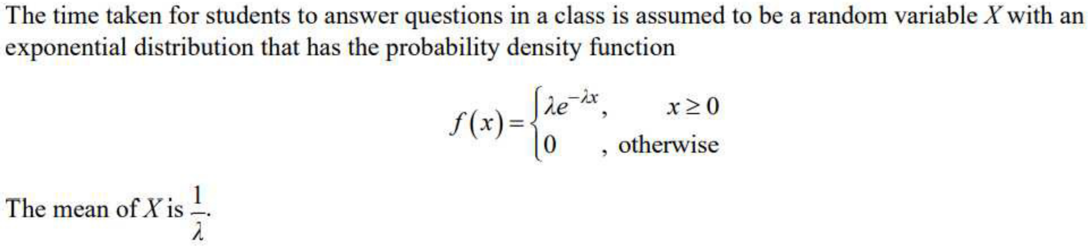
(B) 5%

(C) 7%

(D) 11%

2022
Paper 2
Section 1
Question 4

Statistical
inference



The time taken for students to answer questions in a class is assumed to be a random variable X with an exponential distribution that has the probability density function

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0\quad , & \text{otherwise}\end{cases}$$

The mean of X is $\frac{1}{\lambda}$.

The mean length of time taken for students to answer questions in this class is 15 seconds.

The probability that the next question in this class is answered between 8 seconds and 17 seconds is

(A) 0.05
(B) 0.12
(C) 0.22
(D) 0.26

2022
Paper 2
Section 1
Question 5

Statistical
inference

A random sample of the petrol price per litre at 50 petrol stations produced a sample mean of \$1.52 and a standard deviation of \$0.14.

Based on this sample and using a z-value of 1.5, an approximate confidence interval for μ is

(A) (\$1.47, \$1.57)
(B) (\$1.48, \$1.56)
(C) (\$1.49, \$1.55)
(D) (\$1.50, \$1.54)

2022
Paper 2
Section 1
Question 10

Statistical
inference

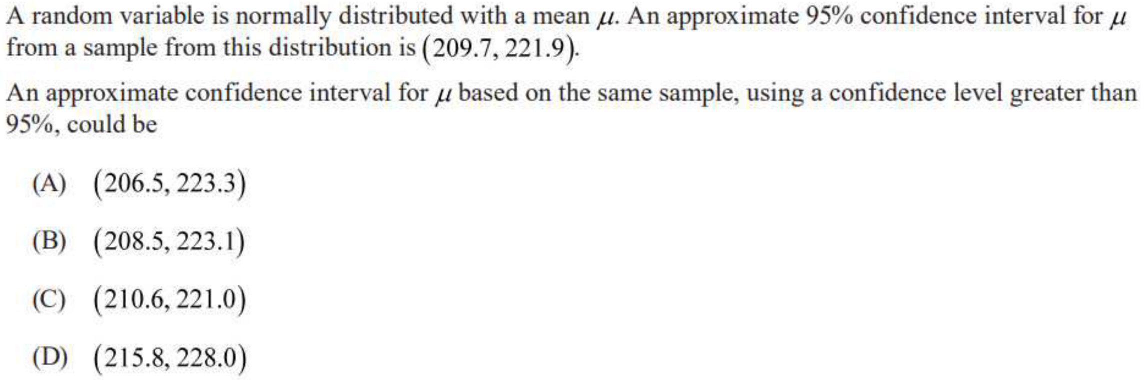
In a town, the mean number of residents per household is 3.79 people with a standard deviation of 1.47 people.

Using a random sample of 45 households from the town, determine the probability that the mean number of residents per household will be more than 4.

(A) 0.17
(B) 0.33
(C) 0.83
(D) 0.96

2021 **Paper 2** **Section 1** **Question 1** **Statistical inference**	The time taken to complete orders at a pizza store is normally distributed with a mean time (μ) of 10 minutes. The owner of the pizza store records the time taken to complete orders for a random sample of 20 pizzas each day over a 30-day period. From this data, an approximate 90% confidence interval for μ is calculated at the end of each day. How many of these confidence intervals would be expected to contain μ? (A) 3 (B) 18 (C) 27 (D) 30

2021 **Paper 2** **Section 1** **Question 4** **Statistical inference**	The mean time that visitors spend at an art exhibition is 39 minutes and the standard deviation is 6 minutes. Determine the approximate probability that the mean time spent at the exhibition by a random sample of 35 visitors is between 38 and 40 minutes. (A) 0.13 (B) 0.16 (C) 0.68 (D) 0.84

2021 **Paper 2** **Section 1** **Question 10** **Statistical inference**	 A random variable is normally distributed with a mean μ. An approximate 95% confidence interval for μ from a sample from this distribution is $(209.7, 221.9)$. An approximate confidence interval for μ based on the same sample, using a confidence level greater than 95%, could be (A) $(206.5, 223.3)$ (B) $(208.5, 223.1)$ (C) $(210.6, 221.0)$ (D) $(215.8, 228.0)$

2020 **Paper 2** **Section 1** **Question 3** **Statistical inference**	The masses of packages of cheese produced by a company are assumed to be normally distributed with a known mean of μ grams and a standard deviation of 7.37 grams. The packages of cheese are labelled to contain 500 grams. Given there is a 25% probability that the mean mass of 20 randomly selected packages will be less than the labelled amount, the value of μ is (A) 498.89 (B) 500.25 (C) 501.11 (D) 504.98

2020 **Paper 2** **Section 1** **Question 7** **Statistical inference**	The heights of all students at a school were measured. A mean height of 157.0 cm was calculated from this data. A random sample of 35 students from this school was selected. The mean height of this sample was 159.7 cm with a standard deviation of 8.7 cm. The smallest confidence level that could be used to produce a confidence interval that contains μ, based on this sample, is (A) 85% (B) 90% (C) 95% (D) 99%

2020 **Paper 2** **Section 1** **Question 10** **Statistical inference**	The time taken by the Year 7 students at a particular school to complete a standardised test is known to be normally distributed. A researcher claims that the population mean is 8.2 minutes. The mean time taken to complete this test by a sample of 10 of these students is 8.1 minutes with a standard deviation of 1.2 minutes. The 95% confidence interval for μ based on this sample is (A) (7.36, 8.84) minutes (B) (7.46, 8.94) minutes (C) (7.86, 8.33) minutes (D) (7.96, 8.44) minutes

Paper 2 Section 2

2024
Paper 2
Section 2
Question 11

Statistical inference

A company claims that the mean battery life of their latest model of smartphone is 9.5 hours.

To test this claim, the battery lives of a random sample of 40 of the smartphones were measured.

A sample mean of 9.31 hours and a standard deviation of 0.52 hours were calculated from this data.

a) Determine an approximate 95% confidence interval for μ. Give your answer to at least two decimal places. *[1 mark]*

b) Determine an approximate 99% confidence interval for μ. Give your answer to at least two decimal places. *[1 mark]*

A manager comments that either confidence interval could be used to support the company's claim.

c) Use your results from Questions 11a) and 11b) to evaluate the reasonableness of the manager's comment. Justify your decision using mathematical reasoning. *[2 marks]*

2024
Paper 2
Section 2
Question 14

Statistical inference

The height of Year 12 students at a school is normally distributed, with a mean height of 168.6 cm and standard deviation of 12.7 cm.

The heights of a random sample of 20 of these students are recorded.

a) Explain why it can be assumed that the sample means for random samples of the heights of students from this school are normally distributed. *[1 mark]*

b) Determine the probability that the mean height of this sample will be greater than 170 cm. *[2 marks]*

There is a 75% probability that the mean height of this sample will lie within ±h cm of the population mean.

c) Determine $P\left(\overline{X} \geq 168.6 + h\right)$. *[1 mark]*

d) Use your result from Question 14c) to determine the value of h. *[1 mark]*

2024 **Paper 2** **Section 2** **Question 18** **Statistical inference**

A random variable X is normally distributed, with a known mean μ and standard deviation σ.
In figure 1, the shaded region between 4 and μ represents 30% of the distribution of X.

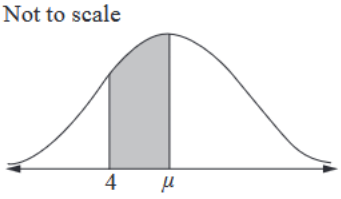


Figure 1

Consider the distribution of $\bar{X}$ based on repeated random sampling of X using a certain sample size.
In figure 2, the shaded region between μ and 6 represents 30% of the distribution of $\bar{X}$.

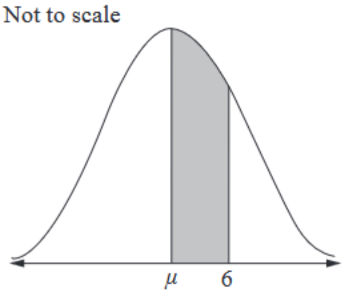


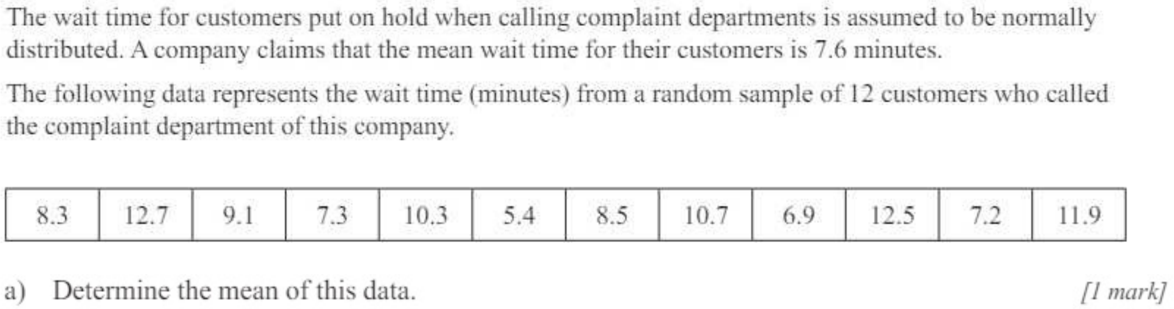
Figure 2

Given $P\left(4 \le \bar{X} \le 6\right) \approx 0.77$, determine $P\left(4 \le X \le 6\right)$.

[6 marks]

2023
Paper 2
Section 2
Question 13

Statistical inference

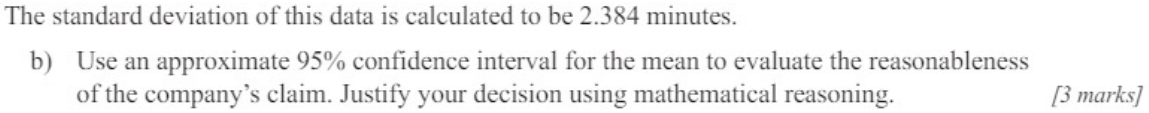


The wait time for customers put on hold when calling complaint departments is assumed to be normally distributed. A company claims that the mean wait time for their customers is 7.6 minutes.

The following data represents the wait time (minutes) from a random sample of 12 customers who called the complaint department of this company.

8.3	12.7	9.1	7.3	10.3	5.4	8.5	10.7	6.9	12.5	7.2	11.9

a) Determine the mean of this data. *[1 mark]*

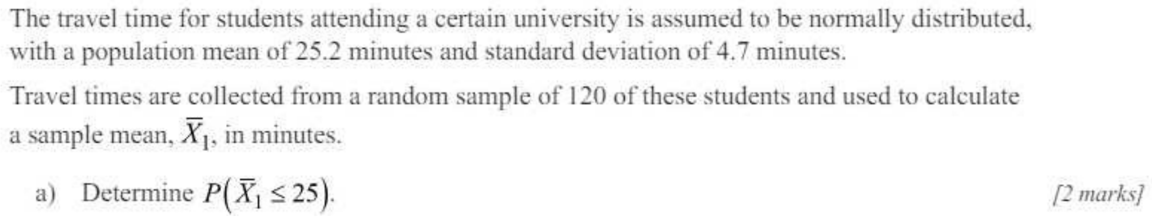


The standard deviation of this data is calculated to be 2.384 minutes.

b) Use an approximate 95% confidence interval for the mean to evaluate the reasonableness of the company's claim. Justify your decision using mathematical reasoning. *[3 marks]*

2023
Paper 2
Section 2
Question 15

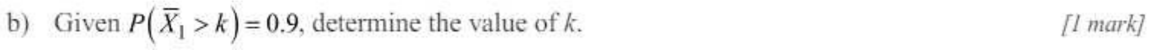
Statistical inference



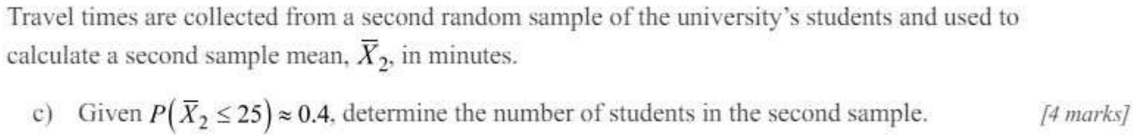
The travel time for students attending a certain university is assumed to be normally distributed, with a population mean of 25.2 minutes and standard deviation of 4.7 minutes.

Travel times are collected from a random sample of 120 of these students and used to calculate a sample mean, $\overline{X}_1$, in minutes.

a) Determine $P\left(\overline{X}_1 \le 25\right)$. *[2 marks]*



b) Given $P\left(\overline{X}_1 > k\right) = 0.9$, determine the value of k. *[1 mark]*

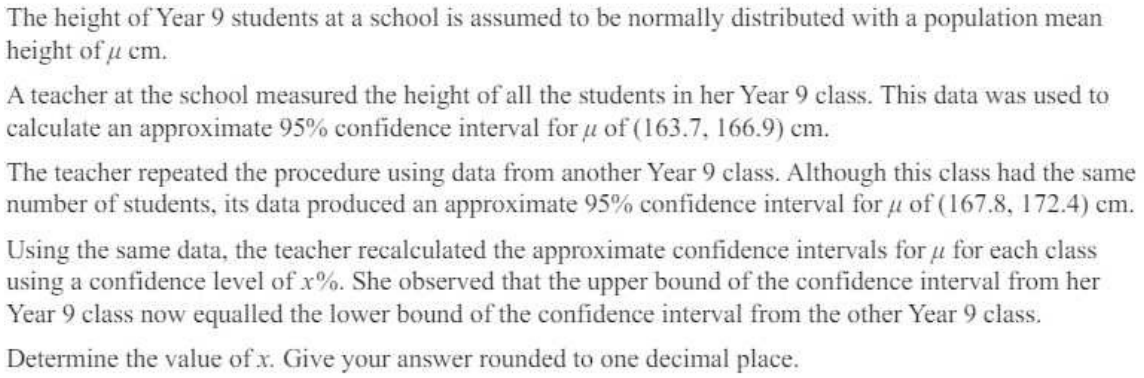


Travel times are collected from a second random sample of the university's students and used to calculate a second sample mean, $\overline{X}_2$, in minutes.

c) Given $P\left(\overline{X}_2 \le 25\right) \approx 0.4$, determine the number of students in the second sample. *[4 marks]*

2023
Paper 2
Section 2
Question 19

Statistical inference



The height of Year 9 students at a school is assumed to be normally distributed with a population mean height of μ cm.

A teacher at the school measured the height of all the students in her Year 9 class. This data was used to calculate an approximate 95% confidence interval for μ of (163.7, 166.9) cm.

The teacher repeated the procedure using data from another Year 9 class. Although this class had the same number of students, its data produced an approximate 95% confidence interval for μ of (167.8, 172.4) cm.

Using the same data, the teacher recalculated the approximate confidence intervals for μ for each class using a confidence level of x%. She observed that the upper bound of the confidence interval from her Year 9 class now equalled the lower bound of the confidence interval from the other Year 9 class.

Determine the value of x. Give your answer rounded to one decimal place.

(7 marks)

2022
Paper 2
Section 2
Question 13

Statistical inference

An article claims that the mean starting salary of graduates in Australia is currently $64 800 with a standard deviation of $4500.

To check the validity of this claim, an employment agent intends to collect data on the starting salaries of a random sample of 360 graduates.

a) Determine the probability that the sample mean starting salary will be between $64 000 and $65 000. [2 marks]

From the data, the agent calculates a confidence interval for the population mean starting salary of ($64 589, $65 811).

c) Comment on the reasonableness of the article's claim based on this confidence interval. [2 marks]

2022
Paper 2
Section 2
Question 17

Statistical inference

The mass of a population of elephants is known to be normally distributed.

A biologist randomly selects a number of elephants from this population and measures their masses. The mean mass of the sample is 5206 kg with a standard deviation of 356 kg.

The biologist uses the data to calculate a 90% confidence interval for the population mean mass of (5159.1, 5252.9) kg.

Determine a 99% confidence interval for the population mean mass based on the same data. [6 marks]

2021
Paper 2
Section 2
Question 12

Statistical inference

The life span of batteries manufactured by a company is assumed to be normally distributed with an unknown mean and standard deviation.

A supervisor at the company randomly selects n batteries and uses the life spans from this sample to calculate an approximate 95% confidence interval for the population mean of (2321.4, 2423.6) hours.

a) Determine the mean life span for this sample of batteries. [1 mark]

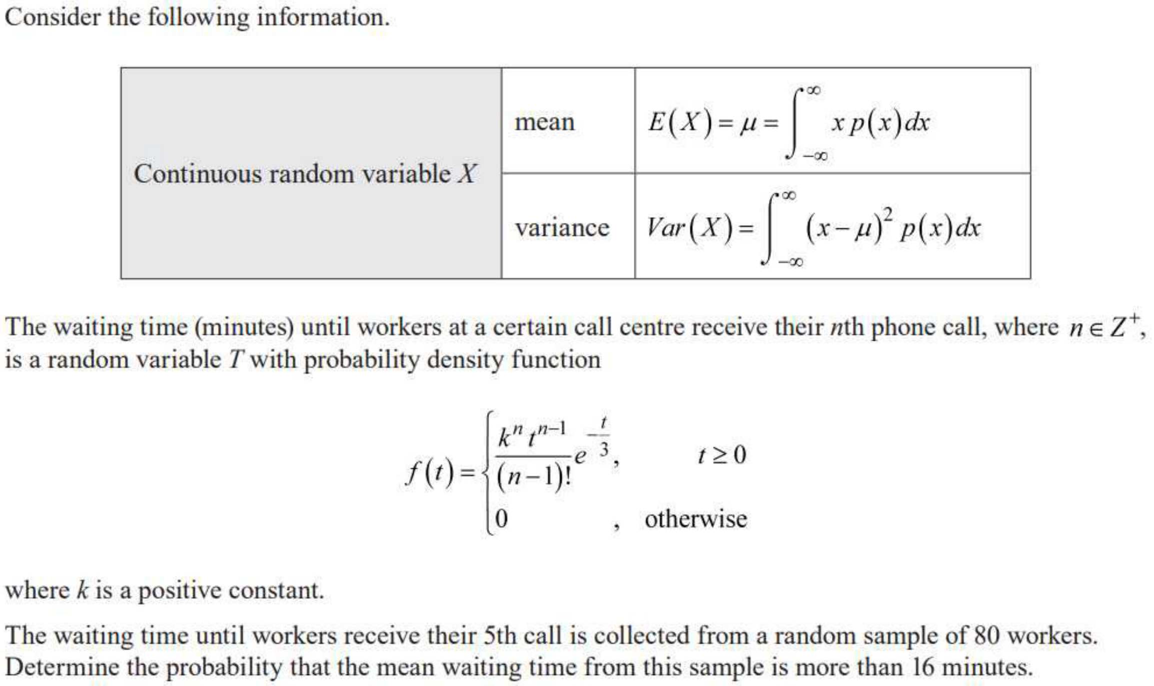
The standard deviation of the life spans of batteries in this sample is 125 hours.

b) Determine n. [3 marks]

c) Use the result from Question 12b) to explain whether the assumption that the life span of batteries is normally distributed is required to support the supervisor's calculations. [2 marks]

2021
Paper 2
Section 2
Question 19

Statistical inference



Consider the following information.

Continuous random variable X	mean	$E(X)=\mu=\int_{-\infty}^{\infty} x\,p(x)\,dx$
	variance	$Var(X)=\int_{-\infty}^{\infty} (x-\mu)^2\,p(x)\,dx$

The waiting time (minutes) until workers at a certain call centre receive their nth phone call, where $n \in Z^+$, is a random variable T with probability density function

$$f(t)=\begin{cases} \dfrac{k^n\,t^{n-1}}{(n-1)!}e^{-\frac{t}{3}}, & t \ge 0 \\ 0 \quad\quad\quad\quad , & \text{otherwise} \end{cases}$$

where k is a positive constant.

The waiting time until workers receive their 5th call is collected from a random sample of 80 workers. Determine the probability that the mean waiting time from this sample is more than 16 minutes.

[7 marks]

2020
Paper 2
Section 2
Question 13

Statistical inference

Data records show that the speeds of cars at a particular location on a highway are normally distributed with a mean of 98.7 km h^{-1} and a standard deviation of 4.1 km h^{-1}. The speed limit at this location is 100 km h^{-1}.

A police officer plans to record the speeds of 20 randomly selected cars at this location.

a) Determine the expected number of cars in the sample that will be travelling within ±1 km h^{-1} of the population mean. [2 marks]

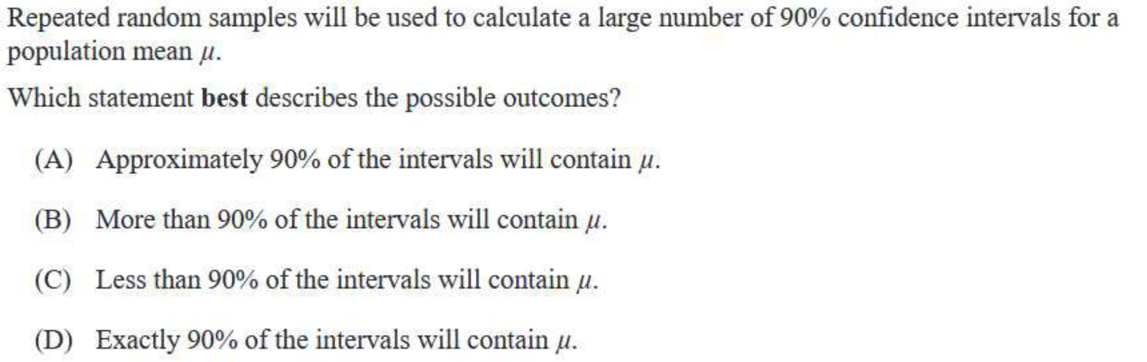
b) Determine the probability that the mean speed of this sample will exceed the speed limit. [2 marks]

There is a 5% probability that the mean speed of this sample will exceed *k*.

c) Determine the value of *k*. [2 marks]

2020 **Paper 2** **Section 2** **Question 18** **Statistical inference**	The mass of a certain species of kangaroo is known to be normally distributed with a mean mass of μ kg and standard deviation of σ kg. When one of the kangaroos is randomly selected, the probability that its mass is greater than 83.2 kg is 0.145. When a sample of 12 kangaroos is randomly selected, the probability that the sample mean mass is less than 74.1 kg is 0.079. A 90% approximate confidence interval for μ is calculated using a random sample of n of the kangaroos that has a sample mean mass of 79.1 kg and a sample standard deviation equal to σ. Determine the possible range of values that n could have been, given that the confidence interval did not contain μ. [6 marks]

Marking Guide – Paper 1 Section 1

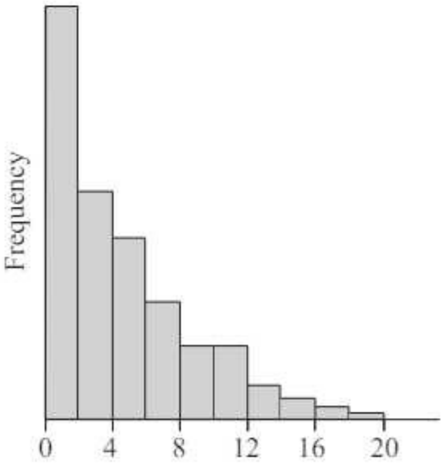
2024 **Paper 1** **Section 1** **Question 1** **Statistical inference**	 Repeated random samples will be used to calculate a large number of 90% confidence intervals for a population mean μ. Which statement **best** describes the possible outcomes? (A) Approximately 90% of the intervals will contain μ. (B) More than 90% of the intervals will contain μ. (C) Less than 90% of the intervals will contain μ. (D) Exactly 90% of the intervals will contain μ. **Answer is A.**

2023 **Paper 1** **Section 1** **Question 5** **Statistical inference**	A confidence interval for a parameter is a range of values within which the (A) sample estimate of the parameter always lies. (B) sample estimate of the parameter never lies. (C) parameter always lies. **(D) parameter never lies. – Answer is A.**

2023
Paper 1
Section 1
Question 10

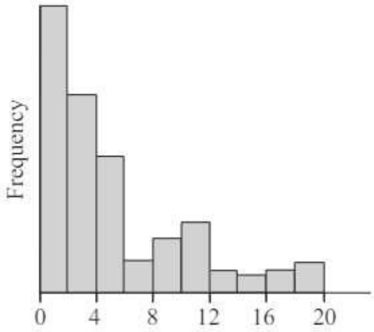
Statistical inference

A random variable is drawn from a population with the distribution shown in the histogram.

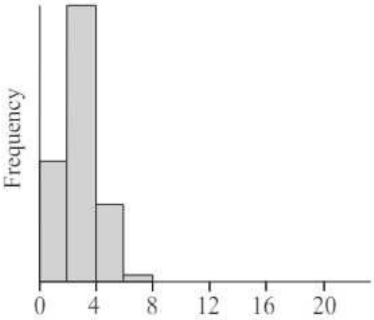


A number of samples of size 10 were randomly selected from this distribution and the sample means, $\bar{x}$, were recorded. The histogram that most likely represents the distribution of the sample means is

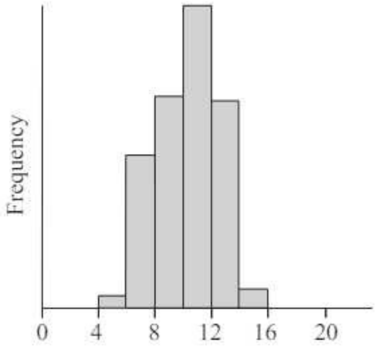
(A)



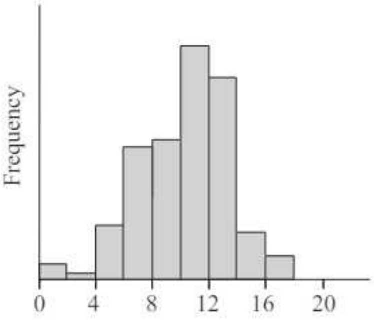
(B)



(C)



(D)



Answer is B.

2022
Paper 1
Section 1
Question 2

Statistical inference

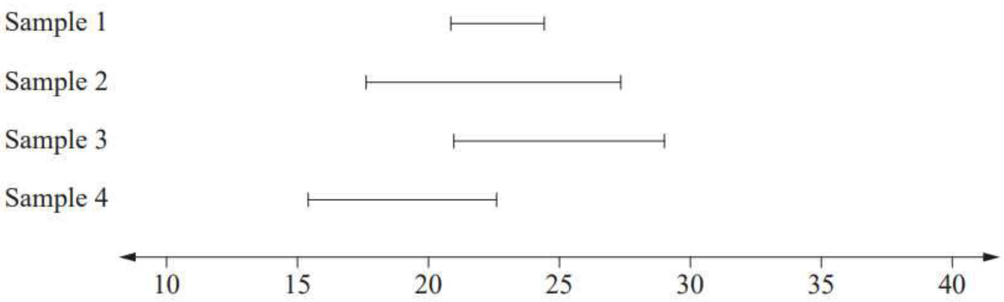
Which statement regarding sample means is true?

(A) The distribution of X is always normally distributed.
(B) The distribution of $\bar{X}$ is always normally distributed.
(C) The value of $\bar{x}$ changes when different samples are selected. – Answer
(D) The value of μ changes when different samples are selected

2022
Paper 1
Section 1
Question 5

Statistical inference

Four random samples of different sizes were taken to estimate a certain population mean, given a known population standard deviation. A 95% confidence interval was calculated for each sample.



Which sample used the largest sample size?

(A) Sample 1 – Answer
(B) Sample 2
(C) Sample 3
(D) Sample 4

2022
Paper 1
Section 1
Question 9

Statistical inference

A random variable X is normally distributed with a mean of 36 and a standard deviation of 4.

The respective mean and standard deviation of the distribution of $\bar{X}$ from repeated random samples of size 9 are

(A) 4 and $\frac{4}{9}$

(B) 4 and $\frac{4}{3}$

(C) 36 and $\frac{4}{9}$

(D) 36 and $\frac{4}{3}$

Answer is D.

2021
Paper 1
Section 1
Question 1

Statistical inference

Which of the following is a population parameter?

(A) s

(B) μ

(C) $\overline{x}$

(D) z

Answer is B.

2021
Paper 1
Section 1
Question 4

Statistical inference

The number of sunflower seeds in each packet produced by a company is known to be normally distributed with a standard deviation of 100. A worker counts the number of seeds in a random sample of four packets and calculates the sample mean.

Based on this sampling, the standard deviation of the distribution of the sample mean is

(A) 25
(B) 50 – Answer
(C) 75
(D) 100

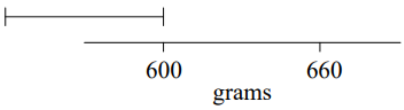
2021
Paper 1
Section 1
Question 7

Statistical inference

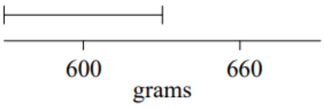
The mass of a particular variety of cake is claimed to be normally distributed with a mean of 660 grams. A random sample of five of these cakes is found to have a mean mass of 600 grams.

Which option represents an approximate confidence interval for μ based on this sample?

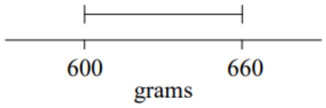
(A)



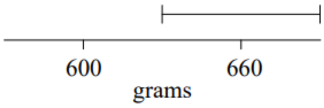
(B)



(C)



(D)

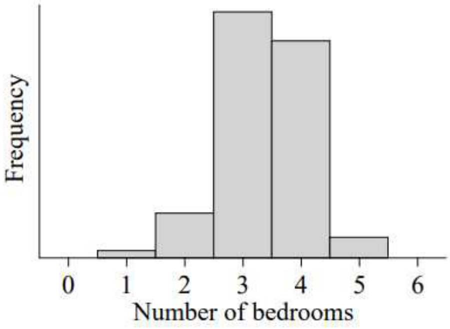


Answer is B.

2021
Paper 1
Section 1
Question 10

Statistical inference

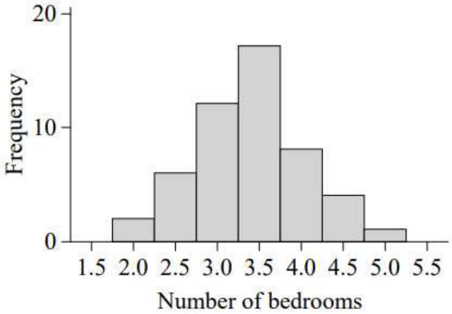
The 2016 Australian census recorded the number of bedrooms per household. The results are summarised in the histogram, as shown. Based on this data, the mean number of bedrooms per household was calculated to be 3.5.



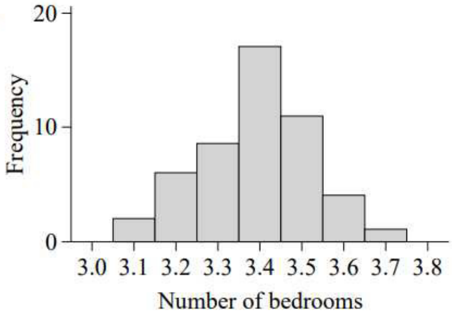
Fifty samples of size 40 were randomly selected from the census data and the sample means recorded.

The histogram that most likely represents the distribution of the sample means is

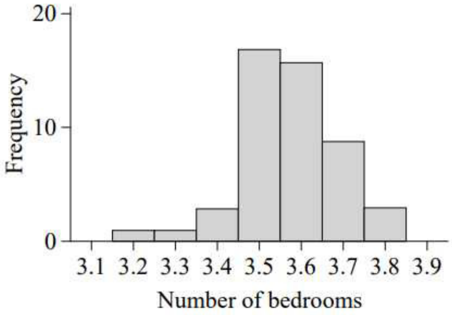
(A)



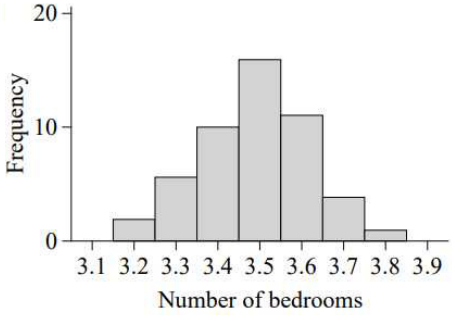
(B)



(C)



(D)

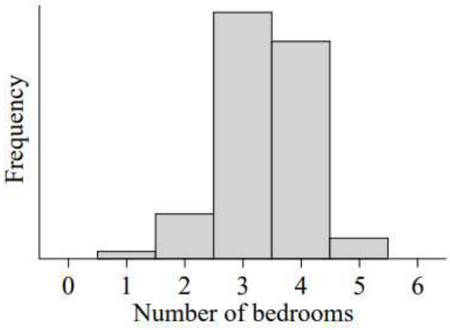


Answer is D.

2021
Paper 1
Section 1
Question 10

Statistical inference

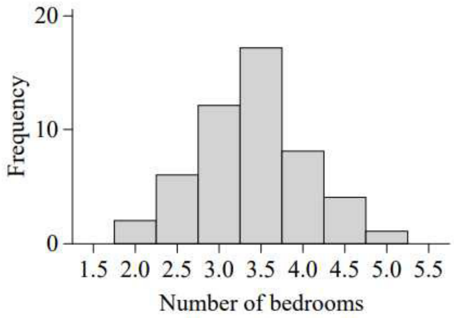
The 2016 Australian census recorded the number of bedrooms per household. The results are summarised in the histogram, as shown. Based on this data, the mean number of bedrooms per household was calculated to be 3.5.



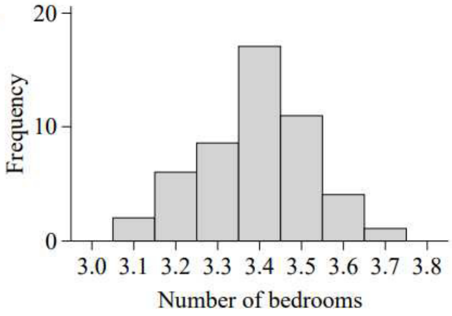
Fifty samples of size 40 were randomly selected from the census data and the sample means recorded.

The histogram that most likely represents the distribution of the sample means is

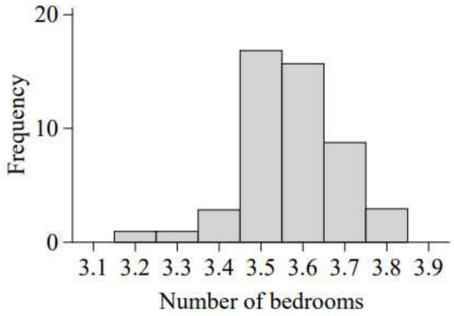
(A)



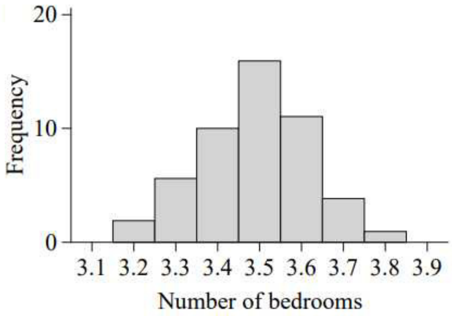
(B)



(C)



(D)



Answer is D.

2020
Paper 1
Section 1
Question 3

Statistical inference

According to a recent census, the mean hours worked per week by all Australian workers is 35.6 hours.

A mean of 36.1 hours worked per week is calculated from a random selection of 500 Australian workers.

Based on this data, which of the following is correct?

(A) $\bar{x} = 35.6, \mu = 36.1$

(B) $\bar{x} = 35.6, \bar{X} = 36.1$

(C) $\bar{x} = 36.1, \mu = 35.6$

(D) $\bar{x} = 36.1, \bar{X} = 35.6$

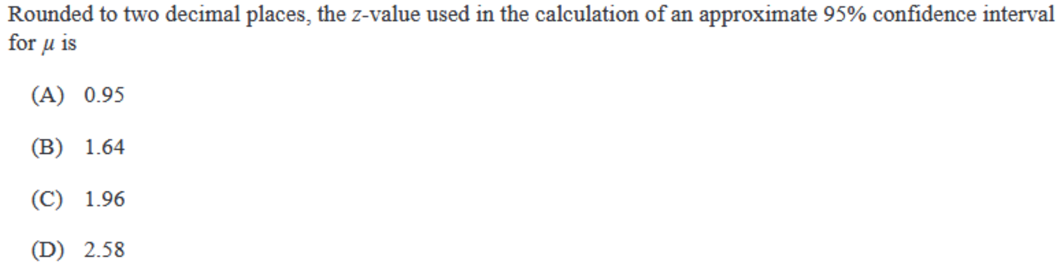
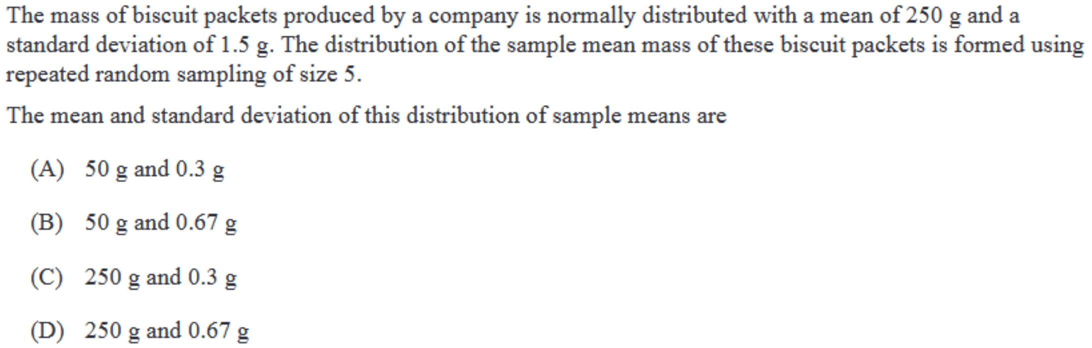
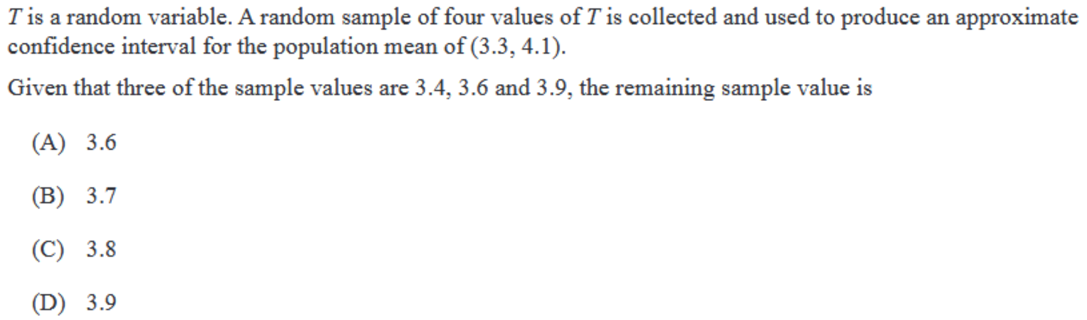
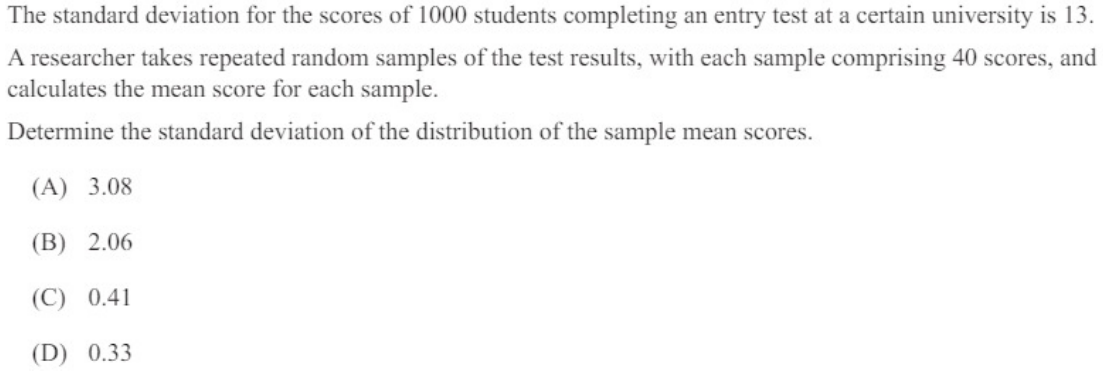
Answer is C.

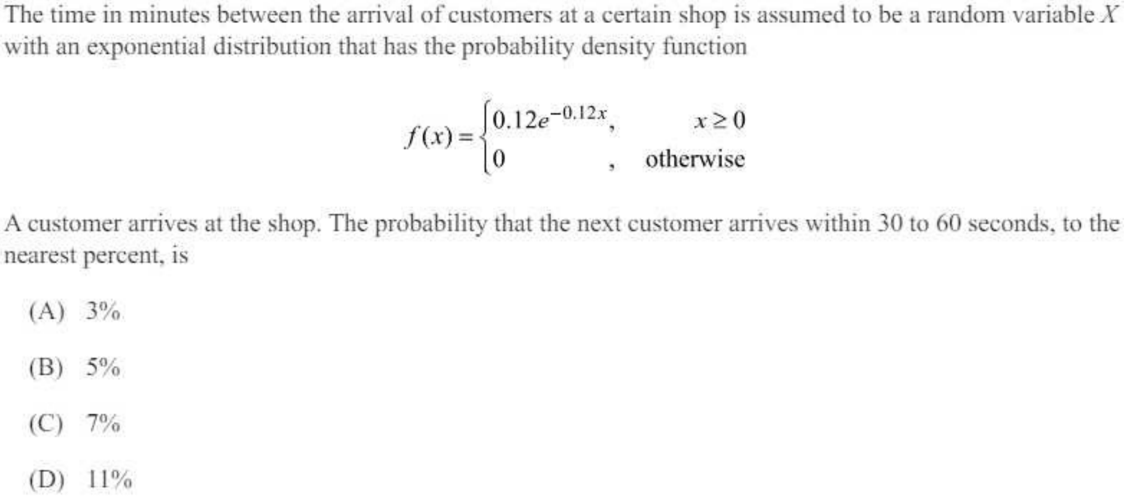
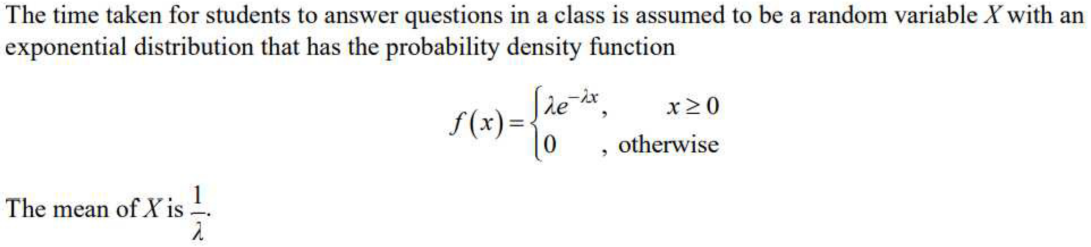
2020 Paper 1 Section 1 Question 9 Statistical inference	The scores on a test are assumed to be normally distributed. Researchers use the results from a random sample of scores to calculate a confidence interval for the population mean. However, a shorter confidence interval width is required so the researchers decide to use a second sample for their calculations. Assuming that the standard deviations for both samples are the same, the researchers can ensure that a shorter confidence interval width is produced by (A) decreasing the sample size and decreasing the confidence level. (B) decreasing the sample size and increasing the confidence level. **(C) increasing the sample size and decreasing the confidence level. – Answer** (D) increasing the sample size and increasing the confidence level.

Marking Guide – Paper 1 Section 2

There have been no questions on this topic for this section in the exams of recent years.

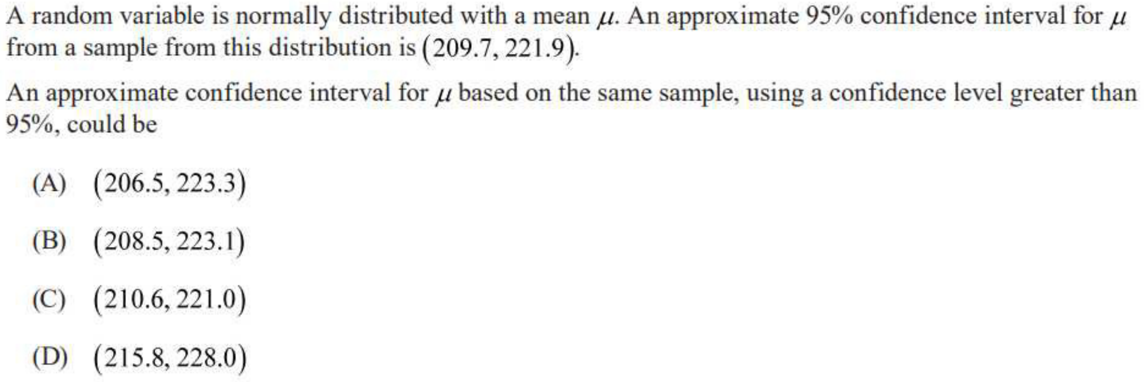
Marking Guide – Paper 2 Section 1

2024 Paper 2 Section 1 Question 2 Statistical inference	 Rounded to two decimal places, the z-value used in the calculation of an approximate 95% confidence interval for μ is (A) 0.95 (B) 1.64 (C) 1.96 (D) 2.58 **Answer is C.**
2024 Paper 2 Section 1 Question 4 Statistical inference	 The mass of biscuit packets produced by a company is normally distributed with a mean of 250 g and a standard deviation of 1.5 g. The distribution of the sample mean mass of these biscuit packets is formed using repeated random sampling of size 5. The mean and standard deviation of this distribution of sample means are (A) 50 g and 0.3 g (B) 50 g and 0.67 g (C) 250 g and 0.3 g (D) 250 g and 0.67 g **Answer is D.**
2024 Paper 2 Section 1 Question 8 Statistical inference	 T is a random variable. A random sample of four values of T is collected and used to produce an approximate confidence interval for the population mean of (3.3, 4.1). Given that three of the sample values are 3.4, 3.6 and 3.9, the remaining sample value is (A) 3.6 (B) 3.7 (C) 3.8 (D) 3.9 **Answer is D.**
2023 Paper 2 Section 1 Question 2 Statistical inference	 The standard deviation for the scores of 1000 students completing an entry test at a certain university is 13. A researcher takes repeated random samples of the test results, with each sample comprising 40 scores, and calculates the mean score for each sample. Determine the standard deviation of the distribution of the sample mean scores. (A) 3.08 (B) 2.06 (C) 0.41 (D) 0.33 **Answer is B.**

2023 Paper 2 Section 1 Question 9 Statistical inference	 The time in minutes between the arrival of customers at a certain shop is assumed to be a random variable X with an exponential distribution that has the probability density function $f(x)=\begin{cases}0.12e^{-0.12x}, & x\geq 0\\ 0 \quad , & \text{otherwise}\end{cases}$ A customer arrives at the shop. The probability that the next customer arrives within 30 to 60 seconds, to the nearest percent, is (A) 3% (B) 5% (C) 7% (D) 11% **Answer is B.**
2022 Paper 2 Section 1 Question 4 Statistical inference	 The time taken for students to answer questions in a class is assumed to be a random variable X with an exponential distribution that has the probability density function $f(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0\\ 0 & , \text{otherwise}\end{cases}$ The mean of X is $\frac{1}{\lambda}$. The mean length of time taken for students to answer questions in this class is 15 seconds. The probability that the next question in this class is answered between 8 seconds and 17 seconds is (A) 0.05 (B) 0.12 (C) 0.22 **(D) 0.26 – Answer**
2022 Paper 2 Section 1 Question 5 Statistical inference	A random sample of the petrol price per litre at 50 petrol stations produced a sample mean of \$1.52 and a standard deviation of \$0.14. Based on this sample and using a z-value of 1.5, an approximate confidence interval for μ is (A) (\$1.47, \$1.57) (B) (\$1.48, \$1.56) **(C) (\$1.49, \$1.55) – Answer** (D) (\$1.50, \$1.54)
2022 Paper 2 Section 1 Question 10 Statistical inference	In a town, the mean number of residents per household is 3.79 people with a standard deviation of 1.47 people. Using a random sample of 45 households from the town, determine the probability that the mean number of residents per household will be more than 4. **(A) 0.17 – Answer** (B) 0.33 (C) 0.83 (D) 0.96

2021 **Paper 2** **Section 1** **Question 1** **Statistical inference**	The time taken to complete orders at a pizza store is normally distributed with a mean time (μ) of 10 minutes. The owner of the pizza store records the time taken to complete orders for a random sample of 20 pizzas each day over a 30-day period. From this data, an approximate 90% confidence interval for μ is calculated at the end of each day. How many of these confidence intervals would be expected to contain μ? (A) 3 (B) 18 **(C) 27 – Answer** (D) 30

2021 **Paper 2** **Section 1** **Question 4** **Statistical inference**	The mean time that visitors spend at an art exhibition is 39 minutes and the standard deviation is 6 minutes. Determine the approximate probability that the mean time spent at the exhibition by a random sample of 35 visitors is between 38 and 40 minutes. (A) 0.13 (B) 0.16 **(C) 0.68 – Answer** (D) 0.84

2021 **Paper 2** **Section 1** **Question 10** **Statistical inference**	 A random variable is normally distributed with a mean μ. An approximate 95% confidence interval for μ from a sample from this distribution is $(209.7, 221.9)$. An approximate confidence interval for μ based on the same sample, using a confidence level greater than 95%, could be (A) $(206.5, 223.3)$ (B) $(208.5, 223.1)$ (C) $(210.6, 221.0)$ (D) $(215.8, 228.0)$ **Answer is B.**

2020 **Paper 2** **Section 1** **Question 3** **Statistical inference**	The masses of packages of cheese produced by a company are assumed to be normally distributed with a known mean of μ grams and a standard deviation of 7.37 grams. The packages of cheese are labelled to contain 500 grams. Given there is a 25% probability that the mean mass of 20 randomly selected packages will be less than the labelled amount, the value of μ is (A) 498.89 (B) 500.25 **(C) 501.11 – Answer** (D) 504.98

2020 **Paper 2** **Section 1** **Question 7** **Statistical inference**	The heights of all students at a school were measured. A mean height of 157.0 cm was calculated from this data. A random sample of 35 students from this school was selected. The mean height of this sample was 159.7 cm with a standard deviation of 8.7 cm. The smallest confidence level that could be used to produce a confidence interval that contains μ, based on this sample, is (A) 85% (B) 90% **(C) 95% – Answer** (D) 99%

2020 **Paper 2** **Section 1** **Question 10** **Statistical inference**	The time taken by the Year 7 students at a particular school to complete a standardised test is known to be normally distributed. A researcher claims that the population mean is 8.2 minutes. The mean time taken to complete this test by a sample of 10 of these students is 8.1 minutes with a standard deviation of 1.2 minutes. The 95% confidence interval for μ based on this sample is **(A) (7.36, 8.84) minutes – Answer** (B) (7.46, 8.94) minutes (C) (7.86, 8.33) minutes (D) (7.96, 8.44) minutes

2024
Paper 2
Section 2
Question 11

Statistical inference

A company claims that the mean battery life of their latest model of smartphone is 9.5 hours.

To test this claim, the battery lives of a random sample of 40 of the smartphones were measured.

A sample mean of 9.31 hours and a standard deviation of 0.52 hours were calculated from this data.

a) Determine an approximate 95% confidence interval for μ. Give your answer to at least two decimal places. *[1 mark]*

Sample response	The response
Given $n = 40$, $\overline{x} = 9.31$ and $s = 0.52$ Using GDC $\text{CI}(95\%) = (9.15, 9.47)$ hours	- correctly calculates 95% confidence interval to at least two decimal places **[1 mark]**

b) Determine an approximate 99% confidence interval for μ. Give your answer to at least two decimal places. *[1 mark]*

Sample response	The response
Using GDC $\text{CI}(99\%) = (9.10, 9.52)$ hours	- correctly calculates 99% confidence interval to at least two decimal places **[1 mark]**

A manager comments that either confidence interval could be used to support the company's claim.

c) Use your results from Questions 11a) and 11b) to evaluate the reasonableness of the manager's comment. Justify your decision using mathematical reasoning. *[2 marks]*

Sample response	The response
The 95% confidence interval does not include the claimed mean battery life of 9.5 hours, although the 99% CI does. So the comment is not reasonable.	- justifies decision using mathematical reasoning **[1 mark]** - provides appropriate statement of reasonableness **[1 mark]**

2024
Paper 2
Section 2
Question 14

Statistical inference

The height of Year 12 students at a school is normally distributed, with a mean height of 168.6 cm and standard deviation of 12.7 cm.

The heights of a random sample of 20 of these students are recorded.

a) Explain why it can be assumed that the sample means for random samples of the heights of students from this school are normally distributed. *[1 mark]*

Sample response	The response
The distribution of sample means is normally distributed as the population from which a random sample is taken is normally distributed.	- correctly explains the assumption based on the normality of the population distribution **[1 mark]**

b) Determine the probability that the mean height of this sample will be greater than 170 cm. *[2 marks]*

Sample response	The response
$\mu_{\overline{X}} = 168.6$ cm $\sigma_{\overline{X}} = \dfrac{12.7}{\sqrt{20}} \approx 2.84$ cm Using GDC $P(\overline{X} > 170) \approx 0.31$	- correctly determines the value for $\sigma_{\overline{X}}$ **[1 mark]** - determines probability **[1 mark]**

There is a 75% probability that the mean height of this sample will lie within $\pm h$ cm of the population mean.

c) Determine $P(\overline{X} \geq 168.6 + h)$. *[1 mark]*

Sample response	The response
$P(\overline{X} \geq 168.6 + h) = 0.125$	• correctly determines the probability **[1 mark]**

d) Use your result from Question 14c) to determine the value of h. *[1 mark]*

Sample response	The response
Using GDC $168.6 + h \approx 171.867$ $h \approx 3.27$ cm	• determines *h* **[1 mark]**

2024
Paper 2
Section 2
Question 18

Statistical inference

A random variable X is normally distributed, with a known mean μ and standard deviation σ.
In figure 1, the shaded region between 4 and μ represents 30% of the distribution of X.

Not to scale

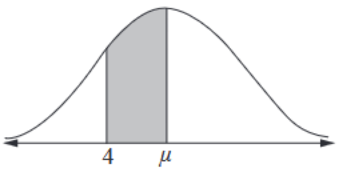


Figure 1

Consider the distribution of $\overline{X}$ based on repeated random sampling of X using a certain sample size.
In figure 2, the shaded region between μ and 6 represents 30% of the distribution of $\overline{X}$.

Not to scale

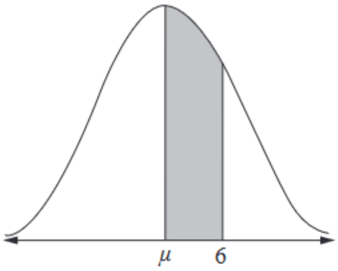


Figure 2

Given $P\left(4 \le \overline{X} \le 6\right) \approx 0.77$, determine $P\left(4 \le X \le 6\right)$.

[6 marks]

Sample response	The response
Method 1	
From Figure 1 Using GDC $\mu - 4 = 0.8416\sigma$ ... (1)	• correctly determines an equation from Figure 1 in terms of μ and σ **[1 mark]**
Let the sample size be n. From Figure 2 Using GDC $6-\mu = \dfrac{0.8416\sigma}{\sqrt{n}}$ ... (2)	• correctly determines an equation from Figure 2 in terms of μ, σ and n **[1 mark]**
Given $P\left(4 \le \overline{X} \le 6\right) \approx 0.77$, then $P\left(4 \le \overline{X} \le \mu\right) \approx 0.77 - 0.3 \approx 0.47$ Using GDC $\mu - 4 = \dfrac{1.8808\sigma}{\sqrt{n}}$ ... (3)	• correctly determines an equation using $P\left(4 \le \overline{X} \le 6\right) \approx 0.77$ in terms of μ, σ and n **[1 mark]**
Dividing (2) by (3) $\dfrac{6-\mu}{\mu-4} = \dfrac{0.8416}{1.8808}$ Using GDC $\mu = 5.382$	• determines μ **[1 mark]**
Using (1): $5.382 - 4 = 0.8416\sigma$ $\sigma = 1.642$	• determines σ **[1 mark]**
Using GDC $P(4 \le X \le 6) \approx 0.45$	• determines required probability **[1 mark]**

	Method 2 From Figure 1 Using GDC $\mu - 4 = 0.8416\sigma$ ...(1) Let the sample size be n. From Figure 2 Using GDC $6 - \mu = \frac{0.8416\sigma}{\sqrt{n}}$ ...(2) Given $P\left(4 \le \bar{X} \le 6\right) \approx 0.77$, then $P\left(4 \le \bar{X} \le \mu\right) \approx 0.77 - 0.3 \approx 0.47$ Using GDC $\mu - 4 = \frac{1.8808\sigma}{\sqrt{n}}$ ...(3) Equating (1) and (3) $0.8416\sigma = \frac{1.8808\sigma}{\sqrt{n}}$ Using GDC $n = 5$ Using 2: $6 - \mu = \frac{0.8416\sigma}{\sqrt{5}} \Rightarrow 6 - \mu = 0.376\sigma$ Using GDC $P(\mu \le X \le 6) \approx 0.147$ $P(4 \le X \le 6) \approx 0.3 + 0.147 \approx 0.45$	• correctly determines an equation from Figure 1 in terms of μ and σ **[1 mark]** • correctly determines an equation from Figure 2 in terms of μ, σ and n **[1 mark]** • correctly determines an equation using $P\left(4 \le \bar{X} \le 6\right) \approx 0.77$ in terms of μ, σ and n **[1 mark]** • determines n **[1 mark]** • determines $P(\mu \le X \le 6)$ **[1 mark]** • determines required probability **[1 mark]**
	Method 3 From Figure 1 Using GDC $\mu - 4 = 0.8416\sigma$ ...(1) Let the sample size be n. From Figure 2 Using GDC $6 - \mu = \frac{0.8416\sigma}{\sqrt{n}}$ ...(2) Given $P\left(4 \le \bar{X} \le 6\right) \approx 0.77$, then $P\left(4 \le \bar{X} \le \mu\right) \approx 0.77 - 0.3 \approx 0.47$ Using GDC $\mu - 4 = \frac{1.8808\sigma}{\sqrt{n}}$ ...(3) Equating (1) and (3) $0.8416\sigma = \frac{1.8808\sigma}{\sqrt{n}}$ Using GDC $n = 5$	• correctly determines an equation from Figure 1 in terms of μ and σ **[1 mark]** • correctly determines an equation from Figure 2 in terms of μ, σ and n **[1 mark]** • correctly determines an equation using $P\left(4 \le \bar{X} \le 6\right) \approx 0.77$ in terms of μ, σ and n **[1 mark]** • determines n **[1 mark]**

Using 1: $\frac{4-\mu}{\sigma}=-0.8416$ Using 2: $\frac{6-\mu}{\sigma}=0.3766$ $P(4 \le X \le 6)=P(-0.8416 \le z \le 0.3766)$	• expresses required probability in terms of z-scores **[1 mark]**
Using GDC $P(4 \le X \le 6) \approx 0.45$	• determines required probability **[1 mark]**

2023
Paper 2
Section 2
Question 13

Statistical inference

The wait time for customers put on hold when calling complaint departments is assumed to be normally distributed. A company claims that the mean wait time for their customers is 7.6 minutes.

The following data represents the wait time (minutes) from a random sample of 12 customers who called the complaint department of this company.

8.3	12.7	9.1	7.3	10.3	5.4	8.5	10.7	6.9	12.5	7.2	11.9

a) Determine the mean of this data. *[1 mark]*

Sample response	The response
Using GDC $\bar{x}=9.23$ minutes	• correctly determines the mean of the data **[1 mark]**

The standard deviation of this data is calculated to be 2.384 minutes.

b) Use an approximate 95% confidence interval for the mean to evaluate the reasonableness of the company's claim. Justify your decision using mathematical reasoning. *[3 marks]*

Sample response	The response
Using result fom 13a) and given $s=2.384$ and $n=12$. Using GDC $\text{CI}\,(95\%)=(7.88, 10.58)$ minutes	• calculates 95% confidence interval **[1 mark]**
The company's **claim is not reasonable.**	• states an appropriate evaluation of the reasonableness of the claim **[1 mark]**
This decision is justified as **the claimed** population **mean wait time lies outside of the confidence interval.**	• uses mathematical reasoning to justify the decision **[1 mark]**

2023
Paper 2
Section 2
Question 15

Statistical inference

The travel time for students attending a certain university is assumed to be normally distributed, with a population mean of 25.2 minutes and standard deviation of 4.7 minutes.

Travel times are collected from a random sample of 120 of these students and used to calculate a sample mean, $\overline{X}_1$, in minutes.

a) Determine $P\left(\overline{X}_1 \le 25\right)$. *[2 marks]*

Sample response	The response
Given $\mu_{\overline{X}} = 25.2$ $\sigma_{\overline{X}_1} = \frac{\sigma}{\sqrt{n}} = \frac{4.7}{\sqrt{120}}$ $= 0.429$ minutes	• correctly calculates $\sigma_{\overline{X}}$ for the first sample **[1 mark]**
Using GDC $P(\overline{X}_1 \le 25) = 0.32$	• calculates required probability **[1 mark]**

b) Given $P\left(\overline{X}_1 > k\right) = 0.9$, determine the value of k. *[1 mark]*

Sample response	The response
$P(\overline{X}_1 > k) = 0.9$ Using GDC $k = 24.65$ minutes	• calculates k **[1 mark]**

Travel times are collected from a second random sample of the university's students and used to calculate a second sample mean, $\overline{X}_2$, in minutes.

c) Given $P\left(\overline{X}_2 \le 25\right) \approx 0.4$, determine the number of students in the second sample. *[4 marks]*

Sample response	The response
$P(z \le z_1) \approx 0.4 \Rightarrow z_1 = -0.253$ $z = \frac{\overline{X}_2 - \mu}{\frac{\sigma}{\sqrt{n}}}$	• correctly calculates the z-value based on given probability **[1 mark]**
$-0.253 = \frac{25 - 25.2}{\frac{4.7}{\sqrt{n}}}$	• determines an equation in terms of the sample size (n) **[1 mark]**
Using GDC $n \approx 35.3$	• determines an approximate value of n **[1 mark]**
The sample size is 35.	• evaluates the reasonableness of the solution by rounding n to an integer value **[1 mark]**

2023
Paper 2
Section 2
Question 19

Statistical inference

The height of Year 9 students at a school is assumed to be normally distributed with a population mean height of μ cm.

A teacher at the school measured the height of all the students in her Year 9 class. This data was used to calculate an approximate 95% confidence interval for μ of (163.7, 166.9) cm.

The teacher repeated the procedure using data from another Year 9 class. Although this class had the same number of students, its data produced an approximate 95% confidence interval for μ of (167.8, 172.4) cm.

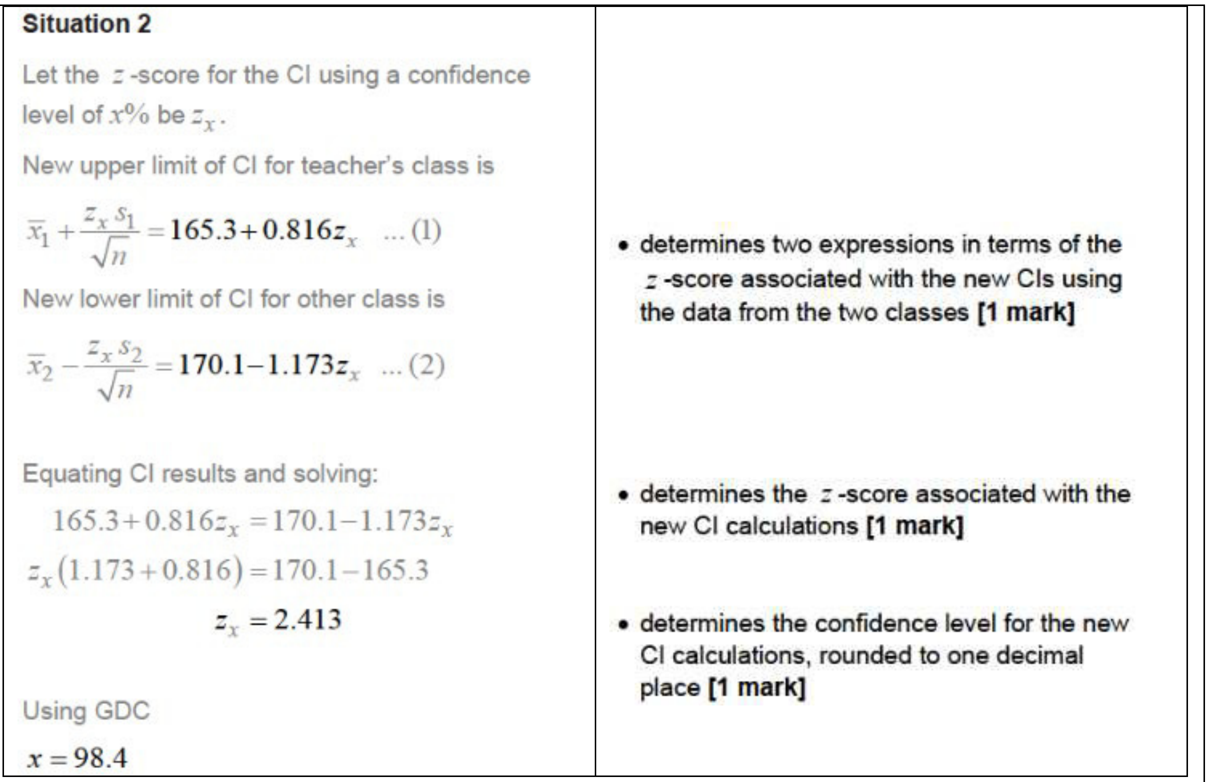
Using the same data, the teacher recalculated the approximate confidence intervals for μ for each class using a confidence level of x%. She observed that the upper bound of the confidence interval from her Year 9 class now equalled the lower bound of the confidence interval from the other Year 9 class.

Determine the value of x. Give your answer rounded to one decimal place.

(7 marks)

Sample response	The response
Method 1 **Situation 1** z-score for $95\%\text{CI}=1.96$	• correctly determines the z-score associated with a 95% CI **[1 mark]**
Teacher's class: $\bar{x}_1=\dfrac{166.9+163.7}{2}=165.3$ Other class: $\bar{x}_2=\dfrac{172.4+167.8}{2}=170.1$	• correctly determines the sample means for both classes **[1 mark]**
Teacher's class: Let s_1 be the sample standard deviation for Class 1 of sample size n. $165.3+\dfrac{1.96s_1}{\sqrt{n}}=166.9$ $\dfrac{s_1}{\sqrt{n}}=0.816$	• determines a relationship between the sample standard deviation and the sample size for the teacher's class **[1 mark]**
Other class: $\bar{x}_2=\dfrac{172.4+167.8}{2}=170.1$ Let s_2 be the sample standard deviation for Class 2 of sample size n. $170.1+\dfrac{1.96s_2}{\sqrt{n}}=172.4$ $\dfrac{s_2}{\sqrt{n}}=1.173$	• determines a relationship between the sample standard deviation and the sample size for the other class **[1 mark]**

Situation 2 Let the z-score for the CI using a confidence level of $x\%$ be z_x. New upper limit of CI for teacher's class equals new lower limit of CI for other class. $\bar{x}_1 + \frac{z_x s_1}{\sqrt{n}} = \bar{x}_2 - \frac{z_x s_2}{\sqrt{n}}$ $z_x\left(\frac{s_1}{\sqrt{n}} + \frac{s_2}{\sqrt{n}}\right) = \bar{x}_2 - \bar{x}_1$	- determines an equation in terms of the z-score associated with the new CIs using the data from the two classes **[1 mark]**
Using earlier results $z_x(0.816 + 1.173) = 170.1 - 165.3$ $z_x = 2.413$	- determines the z-score associated with the new CI calculations **[1 mark]**
Using GDC $x = 98.4$	- determines the confidence level for the new CI calculations, rounded to one decimal place **[1 mark]**
Method 2 **Situation 1** z-score for 95% CI$=1.96$	- correctly determines the z-score associated with a 95% CI **[1 mark]**
Teacher's class: $\bar{x}_1 = \frac{166.9 + 163.7}{2} = 165.3$ Other class: $\bar{x}_2 = \frac{172.4 + 167.8}{2} = 170.1$	- correctly determines the sample means for each class **[1 mark]**
Teacher's class: Let s_1 be the sample standard deviation for Class 1 of sample size n. $165.3 + \frac{1.96 s_1}{\sqrt{n}} = 166.9$ $\frac{s_1}{\sqrt{n}} = 0.816$	- determines a relationship between the sample standard deviation and the sample size for the teacher's class **[1 mark]**
Other class: $\bar{x}_2 = \frac{172.4 + 167.8}{2} = 170.1$ Let s_2 be the sample standard deviation for Class 2 of sample size n. $170.1 + \frac{1.96 s_2}{\sqrt{n}} = 172.4$ $\frac{s_2}{\sqrt{n}} = 1.173$	- determines a relationship between the sample standard deviation and the sample size for the other class **[1 mark]**



Situation 2 Let the z-score for the CI using a confidence level of $x\%$ be z_x. New upper limit of CI for teacher's class is $\bar{x}_1 + \frac{z_x s_1}{\sqrt{n}} = 165.3 + 0.816z_x \quad \dots(1)$ New lower limit of CI for other class is $\bar{x}_2 - \frac{z_x s_2}{\sqrt{n}} = 170.1 - 1.173z_x \quad \dots(2)$	• determines two expressions in terms of the z-score associated with the new CIs using the data from the two classes **[1 mark]**
Equating CI results and solving: $165.3 + 0.816z_x = 170.1 - 1.173z_x$ $z_x(1.173 + 0.816) = 170.1 - 165.3$ $z_x = 2.413$	• determines the z-score associated with the new CI calculations **[1 mark]**
Using GDC $x = 98.4$	• determines the confidence level for the new CI calculations, rounded to one decimal place **[1 mark]**

2022
Paper 2
Section 2
Question 13

Statistical inference

An article claims that the mean starting salary of graduates in Australia is currently \$64 800 with a standard deviation of \$4500.

To check the validity of this claim, an employment agent intends to collect data on the starting salaries of a random sample of 360 graduates.

a) Determine the probability that the sample mean starting salary will be between \$64 000 and \$65 000. [2 marks]

Sample Response	The response
As $n \geq 30$, the sample mean distribution can be assumed to be normal. $\mu_{\bar{X}} = 64\,800, \sigma_{\bar{X}} = \frac{4500}{\sqrt{360}}$	• correctly calculates $\sigma_{\bar{X}}$ [1 mark]
$P(64\,000 \leq \bar{X} \leq 65\,000) = 0.8$	• calculates required probability [1 mark]

From the data, the agent calculates a confidence interval for the population mean starting salary of (\$64 589, \$65 811).

Sample Response	The response
$\bar{x} = \frac{65\,811 + 64589}{2} = \$65\,200$	• correctly determines the value of $\bar{x}$ [1 mark]

c) Comment on the reasonableness of the article's claim based on this confidence interval. [2 marks]

Sample Response	The response
μ lies within the confidence interval. The website's claim is reasonable.	• correctly recognises that the population mean lies within the confidence interval [1 mark] • correctly evaluates the reasonableness of the claim using a suitable comment [1 mark]

2022
Paper 2
Section 2
Question 17

Statistical inference

The mass of a population of elephants is known to be normally distributed.

A biologist randomly selects a number of elephants from this population and measures their masses. The mean mass of the sample is 5206 kg with a standard deviation of 356 kg.

The biologist uses the data to calculate a 90% confidence interval for the population mean mass of (5159.1, 5252.9) kg.

Determine a 99% confidence interval for the population mean mass based on the same data. [6 marks]

Sample Response	The response
Consider the sample: $\bar{x} = 5206$ kg $s = 356$ kg For a 90% CI, $z =$ 1.645 Given the 90% CI is (5159.1, 5252.9) kg Lower value of the CI $= \bar{x} + z\frac{s}{\sqrt{n}}$	• correctly determines the required value of z for a 90% CI [1 mark]
$5159.1 = 5206 - 1.645 \times \frac{356}{\sqrt{n}}$	• establishes an equation in terms of n [1 mark]
Using solve facility of GDC $n \approx 155.9$	• solves the equation to determine n [1 mark]
Sample size is 156	• rounds n to an integer value [1 mark]
Determining a 99% CI for this sample Using statistics facility of GDC CI is (5132.6, 5279.4) kg	• determines a 99% confidence interval [1 mark] • shows logical organisation communicating key steps [1 mark]

2021
Paper 2
Section 2
Question 12

Statistical inference

The life span of batteries manufactured by a company is assumed to be normally distributed with an unknown mean and standard deviation.

A supervisor at the company randomly selects n batteries and uses the life spans from this sample to calculate an approximate 95% confidence interval for the population mean of (2321.4, 2423.6) hours.

a) Determine the mean life span for this sample of batteries. [1 mark]

Sample Response	The response
$\bar{x} = \frac{2321.4+2423.6}{2} =$ 2372.5 hours	• correctly determines the mean lifetime of the sample [1 mark]

The standard deviation of the life spans of batteries in this sample is 125 hours.

b) Determine n. [3 marks]

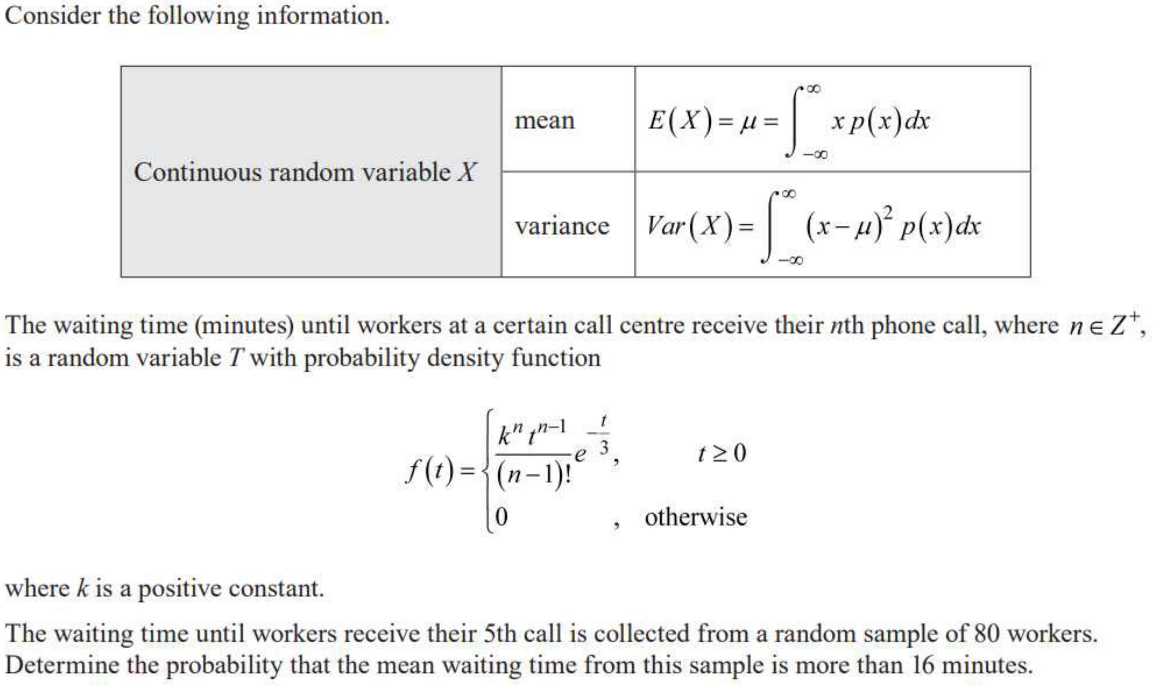
Sample Response	The response
Approximate confidence interval for μ is $\left(\bar{x} - z\frac{\sigma}{\sqrt{n}}, \bar{x} + z\frac{\sigma}{\sqrt{n}}\right)$ Using the upper value of the 95% confidence interval $2372.5 + 1.96\frac{125}{\sqrt{n}} = 2423.6$	• determines an equation in terms of n [1 mark]
Solving equation: $n \approx 22.98$	• solves equation for n [1 mark]
Sample size used was 23.	• expresses sample size rounded to a whole number [1 mark]

c) Use the result from Question 12b) to explain whether the assumption that the life span of batteries is normally distributed is required to support the supervisor's calculations. [2 marks]

Sample Response	The response
Yes, the assumption is required. Because the sample size is less than 30, the distribution of the sample mean cannot be assumed to be normal unless the population is normally distributed.	• makes a suitable statement [1 mark] • discusses the sample size and its relation to a normal distribution [1 mark]

2021
Paper 2
Section 2
Question 19

Statistical inference



Consider the following information.

Continuous random variable X	mean	$E(X)=\mu=\int_{-\infty}^{\infty} x\,p(x)\,dx$
	variance	$Var(X)=\int_{-\infty}^{\infty}(x-\mu)^2\,p(x)\,dx$

The waiting time (minutes) until workers at a certain call centre receive their nth phone call, where $n \in Z^+$, is a random variable T with probability density function

$$f(t)=\begin{cases}\dfrac{k^n t^{n-1}}{(n-1)!}e^{-\frac{t}{3}}, & t \geq 0\\ 0 \quad , & \text{otherwise}\end{cases}$$

where k is a positive constant.

The waiting time until workers receive their 5th call is collected from a random sample of 80 workers. Determine the probability that the mean waiting time from this sample is more than 16 minutes.

[7 marks]

Sample Response	The response
Using the property of a PDF $\int_{-\infty}^{\infty} p(x)dx = 1$ Using $n = 5$ in the given PDF $\int_{0}^{\infty} \frac{k^5\, t^4}{4!}\, e^{-\frac{t}{3}}\, dt = 1$	• correctly determines equation in terms of k [1 mark]
Solving the equation: $k = \frac{1}{3}$	• solves equation to determine k [1 mark]
Mean of distribution for waiting time until 5th call, μ $E(X) = \int_{-\infty}^{\infty} x\,p(x)dx$ $\mu = \int_{0}^{\infty} t\, \frac{\left(\frac{1}{3}\right)^5 t^4}{4!}\, e^{-\frac{t}{3}}\, dt = \int_{0}^{\infty} \frac{\left(\frac{1}{3}\right)^5 t^5}{4!}\, e^{-\frac{t}{3}}\, dt$ = 15 minutes	• determines population mean [1 mark]
Variance of distribution for 5th call $Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2\, p(x)\, dx$ $= \int_{0}^{\infty} (t-15)^2 \frac{\left(\frac{1}{3}\right)^5 t^4}{4!} e^{-\frac{t}{3}} dt = 45 \text{ minutes}^2$ $\therefore \sigma = \sqrt{45}$ minutes	• determine population variance [1 mark]

Consider the distribution of the sample mean of the waiting time until the 5th phone call is received, $\overline{T}$. As the sample size is large, the distribution of $\overline{T}$ can be considered normal.	• justifies that the distribution of $\overline{T}$ can be considered normal [1 mark]
$\mu_{\overline{T}} = 15$ and $\sigma_{\overline{T}} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{45}}{\sqrt{80}} = 0.75$ Using normal cdf on GDC: $P(\overline{T} > 16) \approx 0.09$	• determines mean and standard deviation of the sample mean [1 mark] • determines required probability [1 mark]

2020
Paper 2
Section 2
Question 13

Statistical inference

Data records show that the speeds of cars at a particular location on a highway are normally distributed with a mean of 98.7 km h^{-1} and a standard deviation of 4.1 km h^{-1}. The speed limit at this location is 100 km h^{-1}.

A police officer plans to record the speeds of 20 randomly selected cars at this location.

a) Determine the expected number of cars in the sample that will be travelling within ±1 km h^{-1} of the population mean. [2 marks]

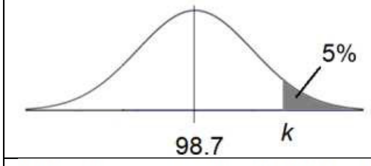
Sample Response	The response
$\mu_X = 98.7$ $\sigma_X = 4.1$ Using normal cdf facility of GDC $P(97.7 \leq X \leq 99.7) \approx 0.193$	• correctly calculates required probability [1 mark]
Expected number $= 20 \times 0.193$ $= 3.85$ ≈ 4	• calculates the expected number [1 mark]

b) Determine the probability that the mean speed of this sample will exceed the speed limit. [2 marks]

Sample Response	The response
$\mu_{\overline{X}} = 98.7$ $\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{4.1}{\sqrt{20}} \approx 0.917$	• correctly calculates $\sigma_{\bar{X}}$ [1 mark]
Using normal cdf facility of GDC $P(\overline{X} > 100) \approx 0.08$	• calculates required probability [1 mark]

There is a 5% probability that the mean speed of this sample will exceed *k*.

c) Determine the value of *k*. [2 marks]

Sample Response	The response
	• correctly uses appropriate mathematical representation [1 mark]
Using inverse normal facility of GDC $k \approx 100.2$ km per hour	• correctly determines the value of *k* [1 mark]

2020
Paper 2
Section 2
Question 18

Statistical inference

The mass of a certain species of kangaroo is known to be normally distributed with a mean mass of μ kg and standard deviation of σ kg.

When one of the kangaroos is randomly selected, the probability that its mass is greater than 83.2 kg is 0.145.

When a sample of 12 kangaroos is randomly selected, the probability that the sample mean mass is less than 74.1 kg is 0.079.

A 90% approximate confidence interval for μ is calculated using a random sample of n of the kangaroos that has a sample mean mass of 79.1 kg and a sample standard deviation equal to σ.

Determine the possible range of values that n could have been, given that the confidence interval did not contain μ. [6 marks]

Sample Response	The response
Sample 1: $n = 1$ $P(X > 83.2) = 0.145$ $P\left(z > \frac{83.2-\mu}{\sigma}\right) = 0.145$ $\frac{83.2 - \mu}{\sigma} = 1.058$ $\mu = 83.2 - 1.058\sigma \quad \ldots(1)$	• correctly uses the sample of 1 to determine an equation in terms of μ and σ [1 mark]
Sample 2: $n = 12$ $P(\bar{X} < 74.1) = 0.079$ $P\left(z < \frac{74.1-\mu}{\frac{\sigma}{\sqrt{12}}}\right) = 0.079$ $\frac{74.1 - \mu}{\frac{\sigma}{\sqrt{12}}} = -1.412$ $\mu = 74.1 + \frac{1.412\sigma}{\sqrt{12}} \quad \ldots(2)$	• correctly uses the sample of 12 to determine an equation in terms of μ and σ [1 mark]
Using graph facility of GDC to solve (1) and (2) $\mu = 76.63$ kg , $\sigma = 6.21$ kg Sample 3: Consider the 90% CI Since $\bar{x} = 79.1$, $\mu = 76.63$ can only lie in an interval below the lower bound of CI. Determining n where the lower bound of CI $= \mu$ $\bar{x} - z\frac{s}{\sqrt{n}} = 76.63$ $79.1 - 1.64 \times \frac{6.21}{\sqrt{n}} = 76.63$ Using solve facility of GDC, $n \approx 17.1$	• solves simultaneous equations to determine the values of μ and σ [1 mark] • determines solution of n [1 mark]
As μ must lie in an interval below the lower bound of CI, the range of values is $n \geq 18$ where $n \in Z$.	• evaluates the reasonableness of the solution to the equation to determine suitable integer values of n [1 mark] shows logical organisation communicating key steps [1 mark]